25/11/2022

Eigen values & vectors

$A \rightarrow$ Square matrix of order n

$X \rightarrow$ column vector of order n

A real no. λ such that $AX = \lambda X$ is called Eigen value, X is called Eigen vector

$$AX - \lambda IX = 0$$

preserves the direction

$$(A - \lambda I)X = 0$$

changes magnitude

$\boxed{\text{multiple of } X}$

$|A - \lambda I| = 0$ is called characteristic equation, which gives eigen values. Such eigen values in (1), we get eigen vector.

Ex: (1) $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$ $I_{2\times2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\lambda I = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{bmatrix}$$

$$\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix}$$

$$|A - \lambda I| = (5-\lambda)(2-\lambda) - 4$$

$$= 10 - \lambda 5 - 2\lambda + \lambda^2 - 4$$

$$= 10 - 7\lambda + \lambda^2 - 4$$

$$= 6 - 7\lambda + \lambda^2$$

$$= \lambda^2 - 7\lambda + 6 = 0$$

$$(\lambda - 1)(\lambda - 6) = 0$$

$\lambda = 1, 6 \leftarrow$ Eigen values.

$$(A - \lambda I)X = 0.$$

$$\begin{bmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{bmatrix}_{2\times 2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{2\times 1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}_{2\times 1}$$

for $\lambda = 1 \rightarrow \begin{bmatrix} 5-1 & 4 \\ 1 & 2-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$n = 2$

$\rho = 1$

$a = 1$

$x_1 + x_2 = 0.$

$x_2 = K.$

$x_1 = -K.$

$X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -K \\ K \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ K cannot be 0.

$\lambda = 6 \rightarrow \begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$R_1 + R_2 \begin{bmatrix} -1 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$-x_1 + 4x_2 = 0$

$x_2 = K$

$-x_1 + 4K = 0$

$x_1 = 4K$

$\boxed{\begin{array}{l} \because AX = \lambda X \\ \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix} = 6 \begin{bmatrix} 4 \\ 1 \end{bmatrix} \\ \text{acts on vector} \\ \text{increases by 6 times} \end{array}}$

$X_6 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4K \\ K \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ for $K = 1$.

(2) $A = \begin{bmatrix} 4 & -6 & -6 \\ 0 & -2 & 0 \\ 1 & -1 & -1 \end{bmatrix}$

$= \begin{bmatrix} 4 & -6 & -6 \\ 0 & -2 & 0 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$

$= \begin{bmatrix} 4-\lambda & -6 & -6 \\ 0 & -2-\lambda & 0 \\ \phi & -1 & -1-\lambda \end{bmatrix}$

$|A-\lambda I| = \begin{vmatrix} 4-\lambda & -6 & -6 \\ 0 & -2-\lambda & 0 \\ 1 & -1 & -1-\lambda \end{vmatrix}$

$4-\lambda\left((\lambda+1)(\lambda+2)\right) - 6\left(-\lambda-2\right))$

$\cancel{4-\lambda(\lambda^2+2\lambda+\lambda+2)+6\lambda+12}$

$4-\lambda\left((\lambda+1)(\lambda+2) + 6(\lambda+2)\right)$

$(\lambda+2)\left((4-\lambda)(\lambda+1) + 6\right)$

$(\lambda+2)\left((4\lambda+4-\lambda^2-\lambda+6\right)$

$(\lambda+2)(-\lambda^2+3\lambda -2) = 0$

$(\lambda+2)(\lambda^2-3\lambda+2) = 0$

$(\lambda+2)(\lambda-2)(\lambda-1) = 0$

$\lambda = -2, 2, 1.$

for $\lambda = -2$ $\begin{bmatrix} 6 & -6 & -6 \\ 0 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix}$

$R1/6 =$ $\begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix}$

$R1 - R3 =$

$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

$n - r = 1$

$x_2 = K$ $\quad x_1 - x_2 - x_3 = 0$

$-2x_3 = 0$ $\quad x_1 - K = 0$

$x_3 = 0$ $\quad x = K$ $\quad = \begin{pmatrix} K \\ K \\ 0 \end{pmatrix}$

for $K = 1$

$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

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for $\lambda = 1$ $\begin{bmatrix} 3 & -6 & -6 \\ 0 & -3 & 0 \\ 1 & -1 & -2 \end{bmatrix}$

$R_1/3$ $\begin{bmatrix} 1 & -2 & -2 \\ 0 & -3 & 0 \\ 1 & -1 & -2 \end{bmatrix}$

$R_1 - R_3$ $\begin{bmatrix} 1 & -2 & -2 \\ 0 & -3 & 0 \\ 0 & -1 & 0 \end{bmatrix}$

$R2/3 =$ $\begin{bmatrix} 1 & -2 & -2 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \end{bmatrix}$

$R3 - R2 =$ $\begin{bmatrix} 1 & -2 & -2 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$x_3 = K.$ $\quad$ Rank = 2 $\quad n = 3.$ $\quad n - r = 1$

$-x_2 = 0$ $\quad$ for $K = 1$

$x_1 - 2x_2 - 2x_3 = 0$ $\quad \lambda_1 = \begin{pmatrix} 2K \\ 0 \\ K \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$

$x_1 - 2K = 0$ $\quad x_1 = 2K$

$\begin{pmatrix}3\\0\\1\end{pmatrix}$

for $\lambda = 2$

$$\begin{bmatrix} 2 & -6 & -6 \\ 0 & -4 & 0 \\ 1 & -1 & -3 \end{bmatrix}$$

$$R1/2 = \begin{pmatrix} 1 & -3 & -3 \\ 0 & -4 & 0 \\ 1 & -1 & -3 \end{pmatrix}$$

$$R1-R3 = \begin{bmatrix} 1 & -3 & -3 \\ 0 & -4 & 0 \\ 0 & -2 & 0 \end{bmatrix} /2 - R2$$

$$\begin{pmatrix} 1 & -3 & -3 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$x_3 = K.$

$-4x_2 = 0.$

$x_2 = 0$

$x_1 - 3x_2 - 3x_3 = 0$

$x_1 - 0 - 3K = 0$

$x_1 = 3K$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3K \\ 0 \\ K \end{pmatrix} \text{ for } K=1 \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}.$$

28/11/2022

$$A = \begin{bmatrix} 6 & 5 & 0 \\ 0 & 7 & 2 \\ 0 & 0 & 9 \end{bmatrix}$$

for upper, lower, diagonal matrices, diagonal elements are eigen values. $\lambda = 6, 7, 9$. find eigen vectors.

$$\begin{bmatrix} 6-\lambda & \cancel{5} & 0 \\ 0 & 7-\lambda & 2 \\ 0 & 0 & 9-\lambda \end{bmatrix}$$

for $\lambda = 6$.

$$\begin{bmatrix} 0 & 5 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 2 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & -10 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

rank $= 2$ $n = 3$ assume k.

$x_1 = k$.

$5x_2 = 0 \Rightarrow x_2 = 0$

$x_2 + 2x_3 = 0$

$3x_3 = 0$

$x_1 = k$

$$\cancel{X_6} = \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\lambda = 7.$

$$\begin{bmatrix} -1 & 5 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 5 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x_1 + 5x_2 = 0.$$

$$x_3 = 0$$

$$x_2 = K$$

$$x_1 = 5K.$$

$$X_7 = \begin{pmatrix} 5K \\ K \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix}$$

$\lambda = 9 =$

$$\begin{bmatrix} -3 & 5 & 0 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_2 = x_3.$ $\quad -3x_1 + 5x_2 = 0$

$x_3 = K.$ $\quad -3x_1 + 5K = 0$

$x_2 = K.$ $\quad x_1 = 5K/3.$

$K = 3$

$$X_9 = \begin{pmatrix} 5K/3 \\ K \\ K \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ 3 \end{pmatrix}$$

Rank of (0) matrix
= 0.

(2) $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$\lambda = 2, 2, 2.$

$M_\lambda = M_2 = 3$ Algebraic multiplicity of λ.

No. of times λ value is repeated.

$X_2 = (A - \lambda I)X = 0$

$$\begin{bmatrix} 2-2 & 0 & 0 \\ 0 & 2-2 & 0 \\ 0 & 0 & 2-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$x_1 = a$

$x_2 = b$

$x_3 = c.$

$X_2 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$a = 1, b = 0, c = 0$; $a = 0, b = 1, c = 0$; $a = b = 0, c = 1$; $a = b = 1, c = 0$.

3 linearly independant Eigen vectors can be generated, all values cannot be '0'.

(3) $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$X_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\lambda = 2, 2, 2.$

$\Delta_2 = M_2 - m_2 = 3 - 2 = 1.$

$m_2 = 2$

Geometric multiplicity

defect of eigenvalue.

$x_1 = a \quad x_2 = b.$

$1 x_3 = 0$

x_3 is a bounded variable

$X_2 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a \\ b \\ 0 \end{pmatrix}$

$a = 1, b = 1$: $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ ✗ linearly dependent

$a = 1, b = 0$: $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$a = 0, b = 1$: $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Geometric multiplicity: L.I eigen vectors for a repeated eigen value.

(4) $A = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$\lambda = 2, 2, 2.$

$$\begin{bmatrix} 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$n - r = 2$

$x_1 = a.$

$x_3 = b.$

$-x_2 + x_3 = 0$

$x_2 + b = 0$

$x_2 = b.$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a \\ b \\ b \end{pmatrix}$$

$a = 0$, $b = 1$

$a = 1$, $b = 1$ X

$a = 1$, $b = 0$.

$m_2 = 2.$

$\Delta_2 = M_2 - m_2 = 1.$

This is a defect in Eigen vectors.

Defect

$\boxed{\Delta_\lambda \geq 0}$

$M_\lambda \geq m_\lambda.$

1) $A = \begin{bmatrix} 4 & 0 \\ 0 & -6 \end{bmatrix}$

$\lambda = 4, -6.$

for $\lambda = 4$ $\quad A - \lambda I = 0 \quad (A - 4I)X = 0$

$$\begin{bmatrix} 0 & 0 \\ 0 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_1 = K.$

$-10 x_2 = 0.$:Bounded.:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} K \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

for $\lambda = -6$ $\quad A - \lambda I = 0.$

$$\begin{bmatrix} 10 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$10 x_1 = 0 \qquad x_2 = K$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ K \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

2) $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$\lambda = 0, 0$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_1 = K. \qquad x_2 = 0 \qquad m_2 = 1$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} K \\ 0 \end{pmatrix} \overset{K=1}{=} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \cancel{K=0}$$

$\Delta_2 = M_2 - m_2$

$= 2 - 1 = 1$ Defect.

2/12/2022.

Eigen Basis:

if an nxn matrix has n independant eigen vectors ~~an~~ then, they form a basis called Eigen Basis. if the n eigen valus are distinct then we have 'n' linearly independant eigen vectors, which forms the eigen basis. if the matrix has repeated eigen values then we may or may not have an eigen basis, depending upon the defect of each eigen value,

If the defect of any eigen value, is >0 then we cannot have an eigen basis.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Spectrum: $\{1, 2, 3\}$.

$X_1 \; X_2 \; X_3$.

$$X_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 = k$

$x_2 = 0$

$2x_3 = 0$

$x_3 = 0$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} k \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ for } k=1$$

$$X_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_2 = K.$ $K = 1$

$x_1 = 0$ $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ K \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$x_3 = 0$

$$X_3 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$K = 1.$

$x_3 = K$ $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ K \end{pmatrix} \Rightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$x_1 = 0$

$x_2 = 0$

Eigen vector $= \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$

Eigen Basis.

(2) $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 1 & -2 & 1 \end{bmatrix}$

Spectrum $= \{1, 0, 1\}.$

$(A - 0I) = 0$

$$X_0 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_3 = a.$

$x_1 = 0$

$-2x_2 + x_3 = 0$

$-2x_2 + a = 0$

$x_2 = a/2$

for $a = 2$

$\begin{pmatrix} 0 \\ a/2 \\ a \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$

$$X_1 = \begin{bmatrix} 0 & 0 & 0 \\ -1 & -1 & 0 \\ 1 & -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_3 = a.$

$x_1 - 2x_2 = 0$

$x_2 = 0$

$x_1 = 0$

$$X_1 = \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad a \neq 0.$$

$M_1 = 2.$

$m_1 = 1.$

$\Delta_1 = 1 \leftarrow$ As defect is existing, it cannot have eigen basis.

③ $A = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$

Symmetric ~~values~~ matrix will always have orthonormal basis of eigen ~~basis~~ vectors for $\mathbb{R}^n$.

$\cancel{\lambda = 5}.$

$$\cancel{\begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}}$$

$\cancel{3x_2 = 0}$

$\cancel{3x_1 = 0}.$

$$\begin{bmatrix} 5-\lambda & 3-0 \\ 3-0 & 5-\lambda \end{bmatrix}$$

$(5-\lambda)^2 - 9 = 0 \qquad \lambda \{2, 8\}$

$\lambda^2 - 10\lambda + 25 - 9 = 0$

$\lambda^2 - 10\lambda + 16 = 0$

Eigen vectors -

$$X_2 = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_2 = k.$

$x_1 + x_2 = 0$

$x_1 + K = 0$

$x_1 = -K.$ $\qquad \begin{matrix} x_1 \\ x_2 \end{matrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$X_8 = \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_2 = a.$

$-x_1 + x_2 = 0$

$-x_1 = -a$

$EV = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}.$

→ if an $n \times n$ matrix has a basis of eigen vectors, then $D = X^{-1}AX$ is diagonal with diagonal entries as the eigen values of A. X is the matrix with eigen vectors as its columns, the process of getting D is called diagonalisation.

① $A = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$

$A - \lambda I = 0$

$\left[\quad \right]_{2\times 2} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\left[\quad \right] - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$

$\begin{bmatrix} 4-\lambda & 1 \\ 3 & 2-\lambda \end{bmatrix}$

$(4-\lambda)(2-\lambda) - 3$

$= 8 - 4\lambda - 2\lambda + \lambda^2 - 3$

$= 8 - 6\lambda + \lambda^2 - 3$

$= \lambda^2 - 6\lambda + 5 = 0$

$\lambda = (5, 1)$

$X_5 = \begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix}$

$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$

$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$-x_1 + x_2 = 0$

$-x_1 + a = 0$

$x_1 = a$

$\begin{matrix} x_1 \\ x_2 \end{matrix} = \begin{pmatrix} a \\ a \end{pmatrix}$

$= \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$X_1 = \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix}$

$\begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$3x_1 + a = 0$

$x_1 = -a/3$

$x_2 = a$

for $a = 3$,

$\begin{bmatrix} -a/3 \\ a \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

E. vectors $= \left\{ \begin{bmatrix} -1 \\ 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

$$X = \begin{bmatrix} -1 & 1 \\ 3 & 1 \end{bmatrix} \qquad -1 - 3 = -4.$$

$$X^{-1} = \underbrace{\frac{1}{-4}}_{\text{determinant}} \begin{bmatrix} 1 & -1 \\ -3 & -1 \end{bmatrix}$$

change the sign.

principal diagonal ~~as it is~~ exchange

$$X^{-1} = \begin{bmatrix} -1/4 & 1/4 \\ 3/4 & 1/4 \end{bmatrix}$$

$$D = X^{-1} A X.$$

$$X^{-1}AX = \begin{bmatrix} -1/4 & 1/4 \\ 3/4 & 1/4 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1/\cancel{4} \times \cancel{4} + 1/4 \times 3 & -1/4 \times 1 + 1/4 \times 2 \\ 3/\cancel{4} \times \cancel{4} + \frac{1}{4} \times 3 & 3/4 \times 1 + \frac{1}{4} \times 2 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 + \frac{3}{4} & -\frac{1}{4} + \frac{1}{2} \\ 3 + 3/4 & \frac{3}{4} + \frac{2}{4} \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1/4 & \frac{+1}{4} \\ 15/4 & \frac{5}{4} \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 3 & 1 \end{bmatrix}$$

$\frac{1}{4} - \frac{3}{4}$ $\quad -\frac{2}{4}$ $\quad \frac{3}{4}+\frac{1}{4}$

$$= \begin{bmatrix} -\frac{1}{4} \times -1 + \frac{+1}{4} \times 3 & -\frac{1}{4} \times 1 + \frac{+1}{4} \times 1 \\ \frac{15}{4} \times -1 + \frac{5}{4} \times 3 & \frac{15}{4} \times 1 + \frac{5}{4} \times 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}.$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$$

Diagonalization: $A = XDX^{-1}$

$$A^m = (XDX^{-1})(XDX^{-1})\cdots(XDX^{-1})$$

$$= XD^mX^{-1}$$

$$\boxed{X\begin{bmatrix}1^m & 0\\ 0 & 5^m\end{bmatrix}X^{-1}}$$

(2) $\begin{bmatrix}5 & 3\\ 3 & 5\end{bmatrix}$

Eigenvector=

$$X = \begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}$$

$$X^{-1} = \frac{1}{2}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$$

$$= \begin{bmatrix}1/2 & -1/2\\ 1/2 & 1/2\end{bmatrix}$$

$$D = X^{-1}AX = \begin{bmatrix}1/2 & -1/2\\ 1/2 & 1/2\end{bmatrix}\begin{bmatrix}5 & 3\\ 3 & 5\end{bmatrix}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}$$

$$= \begin{bmatrix}\frac{5}{2}-\frac{3}{2} & \frac{3}{2}-\frac{5}{2}\\ \frac{5}{2}+\frac{3}{2} & \frac{3}{2}+\frac{5}{2}\end{bmatrix}\begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix}$$

$$= \begin{bmatrix}1 & -1\\ 4 & 4\end{bmatrix}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}$$

$$= \begin{bmatrix}1+1 & 1-1\\ 4-4 & 4+4\end{bmatrix} = \begin{bmatrix}2 & 0\\ 0 & 8\end{bmatrix}$$

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$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ -2 & 2 & 3 \end{bmatrix}$$

$\lambda = \{1, 2, 3\}$.

$$\lambda_1 = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 1 & 0 \\ -2 & 2 & 2 \end{bmatrix} \qquad X_2 = \begin{bmatrix} -1 & 0 & 0 \\ 2 & 0 & 0 \\ -2 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} -2 & 2 & 1 \\ 2 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 3 & 2 \end{bmatrix} \qquad \begin{bmatrix} -2 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_2 = a \quad x_3 = b$.

$-2x_1 + a + b = 0$

$x_1 = \frac{a+b}{2} \qquad \begin{pmatrix} \frac{a+b}{2} \\ a \\ b \end{pmatrix}$

~~$X_2 = K$.~~

~~$3x_2 + 2x_3 = 0$.~~

~~$2x_3 = -K$~~

~~$x_3 = -K/2$~~

$$\begin{bmatrix} 0 & \cancel{3} & \cancel{2} \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad X_3 = \begin{bmatrix} -2 & 0 & 0 \\ 2 & 1 & 0 \\ -2 & 2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 1 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad = \begin{bmatrix} -2 & 2 & 0 \\ 2 & 1 & 0 \\ -2 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 0 \end{bmatrix} \qquad = \begin{bmatrix} -1 & 1 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$x_3 = K$.

$$= \begin{bmatrix} -1 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$3x_2 + 2x_3 = 0$

$3x_2 = -2K$

$x_2 = -2K/3$

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$-x_1 + x_2 + x_3 = 0$

$-x_1 + -\frac{2K}{3} + K = 0$

$-3x_1 + 2K + 3K = 0$

$x_1 = [illegible]K/3$.

$$\begin{pmatrix} \frac{[illegible]K}{3} \\ \frac{-2K}{3} \\ K \end{pmatrix}$$

$x_3 = K$. $\qquad -x_1 + 0 =$

$3x_2 = 0 \qquad (1 \; -2 \; 3)$

$$X = \begin{bmatrix} 1 & 0 & 0 \\ -2 & -1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

$$X^{-1} = \frac{\operatorname{adj} A}{|A|}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -2 & -1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

$$X^{-1}AX = \begin{bmatrix} 1 & 0 & 0 \\ -2 & -1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ -2 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & -1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -2-2 & -2 & 0 \\ 1+4-2 & 4+\cancel{2} & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & -1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -4+4 & 2 & 0 \\ & & \end{bmatrix}$$

$$X^{-1}AX = D$$

$$(XX^{-1})A(XX^{-1}) = XDX^{-1}$$

$$A = XDX^{-1}$$

$$A^{100} = XD^{100}X^{-1}$$

$$X \begin{bmatrix} 1^{10} & 0 & 0 \\ 0 & 2^{10} & 0 \\ 0 & 0 & 3^{10} \end{bmatrix}$$

$$X^{-1}AX = D.$$
$$(XX^{-1})A(XX^{-1}) = XDX^{-1}$$
$$A = XDX^{-1}.$$
$$A^{100} = XD^{100}X^{-1}$$

Application of LA to Markov chain

1) 3 boys A, B, C are throwing a ball to each other. A always throws the ball to B, B always throws to C, C throws to either A or B with 50% chances.

This is an ex. of markov chain where the states of the ball, at any nth step is dependant only on the $n-1$th step and not on the past or future. Such a property is called markov property. The above information is put in the form of matrix called Transition probability matrix (TPM). or

⇐ Application of Diagonalization

$P^{(1)}$ - one step TPM. — nth step.

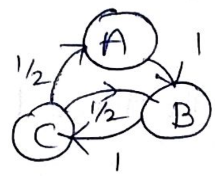
$$P^{(1)} = \begin{array}{c|c} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} \end{array}$$

$(n-1)^{th}$ step states.

stochastic matrix.

[markov property the state of one depends on previous or next state]

Steady State ~~An~~



state diagram

if we are interested in knowing the status of the ball, then after 100 steps, we need to raise the matrix P to the power of 100, which is computationally costly. using the idea of LA, P can be diagonalised and the diagonal matrix can be raised to 100 in a much less computational time.

Suppose we want to know the status of the ball in the long run, ~~then~~ or after I-m. steps, then we need to find the limiting state distribution denoted by π.

$$\pi = (\pi_1, \pi_2, \pi_3 \ldots \ldots \pi_n)$$

where n is the no. of states.

To find the limiting distribution, (steady state distribution) we use the fact that

$$\left.\begin{aligned} \pi P &= \pi \\ \sum_{i=1}^{n} \pi_i &= 1 \end{aligned}\right\} \circledast$$

Referring to the ball example again, we can find S.S.D as follows, since there are 3 states, $\pi = (\pi_1, \pi_2, \pi_3)$

from $\circledast$ we have $(\pi_1, \pi_2, \pi_3)\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} = (\pi_1, \pi_2, \pi_3)$

from $\circledast$ $\pi_1 + \pi_2 + \pi_3 = 1$.

$$\frac{\pi_3}{2} = \pi_1 \qquad \pi_2 = \pi_3$$

$$\pi_1 + \frac{\pi_3}{2} = \pi_2$$

also from ⊛ we have $\pi_1 + \pi_2 + \pi_3 = 1$

$$\frac{\pi_3}{2} + \pi_3 + \pi_3 = 1$$

$$\frac{5\pi_3}{2} = 1$$

$$5\pi_3 = 2$$

$$\pi_3 = 2/5$$

$$\pi_1 = 1/5$$

$$\pi = \left(1/5, \frac{2}{5}, \frac{2}{5}\right) = (0.2, 0.4, 0.4)$$

20% 40% 40%

$$= \frac{1}{5} + \frac{2}{5} + \frac{2}{5} = 1.$$

$$\pi P = \pi \Rightarrow P^T \pi^T = 1\, \pi^T$$

$$AX = 1 \cdot X$$

$\lambda = 1$

$$P^T = \begin{bmatrix} 0 & 0 & 1/2 \\ 1 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix} \quad \lambda = \{1, \ , \ \}$$

$$A - \lambda I$$

$$X_1 = \begin{bmatrix} -1 & 0 & 1/2 \\ 1 & -1 & 1/2 \\ 0 & 1 & -1 \end{bmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{bmatrix} -1 & 0 & 1/2 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 1/2 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\pi_1 = \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix} \begin{pmatrix} \frac{1}{2}K \\ K \\ K \end{pmatrix}$$

$$K = \frac{2}{5} = \begin{pmatrix} \frac{1}{2}\cdot\frac{2}{5} \\ 2/5 \\ 2/5 \end{pmatrix}$$

$$= \begin{pmatrix} 1/5 \\ 2/5 \\ 2/5 \end{pmatrix}.$$

$$\pi_3 = K$$

$$-\pi_2 + \pi_3 = 0$$

$$\pi_2 = K = \pi_3$$

$$-\pi_1 + \tfrac{1}{2}\pi_3 = 0$$

$$-\pi_1 + \tfrac{1}{2}K = 0$$

$$\pi_1 = \tfrac{1}{2}K$$

(2) A city's weather pattern is as follows, if today is rainy day, the probability that tomorrow is also a rainy day is 70%, if today is a dry day, the prob. that tomorrow is also a dry day is 80%. find the limiting state distribution for this markov chain using the

(1) definition of limiting state distribution

(2) using the concept of LA, compare the results.

$$P = \begin{array}{c c} & \begin{array}{cc} R & D \end{array} \\ \begin{array}{c} R \\ D \end{array} & \begin{bmatrix} 0.7 & 0.3 \\ 0.2 & 0.8 \end{bmatrix} \end{array}$$

$$\Pi P = \Pi.$$

$$(\Pi_1, \Pi_2)_{1\times 2} \begin{bmatrix} 0.7 & 0.3 \\ 0.2 & 0.8 \end{bmatrix}_{2\times 2} = (\Pi_1, \Pi_2)$$

$$0.7\Pi_1 + 0.2\Pi_2 = \Pi_1$$

$$0.3\Pi_1 + 0.8\Pi_2 = \Pi_2$$

$$\Pi_1 + \Pi_2 = 1.$$

$$\Pi_1 = 1 - \Pi_2$$

$$0.3(1-\Pi_2) + 0.8\Pi_2 = \Pi_2$$

$$0.3 + 0.5\Pi_2 = \Pi_2$$

$$0.3 = 0.5\Pi_2$$

$$\Pi_2 = 3/5$$

$$\Pi_1 = 1 - 3/5 = 2/5 \quad \Pi = (2/5, 3/5)$$

~~(π P)~~

Find eigen value of P^T.

$$P^T = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix}$$

$$(0.7-\lambda)(0.8-\lambda) - 0.06 = 0.$$

$$0.56 - 0.7\lambda - 0.8\lambda \overset{-\lambda^2}{} - 0.06 = 0$$

$$+\lambda^2 - 1.5\lambda + 0.5 = 0$$

$$\lambda^2 - 0.5\lambda - 1\lambda + 0.5 = 0 \qquad 0.5 \times 1.$$

$$\lambda = 0.5, 1.$$

$$X_1 = \begin{bmatrix} 0.7-1 & 0.2 \\ 0.3 & 0.8-1 \end{bmatrix}$$

$$= \begin{bmatrix} -0.3 & 0.2 \\ 0.3 & -0.2 \end{bmatrix}$$

$$= \begin{bmatrix} -0.3 & 0.2 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad n = 2, \ \rho = 1$$

$$\pi_2 = K$$

$$-0.3\pi_1 + 0.2K = 0$$

$$\pi_1 = \frac{0.2K}{0.3} = \frac{2}{3}K$$

$$\pi_2 = K$$

$$\frac{2K}{3} + K = 1$$

$$\frac{5K}{3} = 1$$

$$K = 3/5.$$

for $K = 3/5$

Same $\begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix} = \begin{pmatrix} \frac{2}{3}K \\ K \end{pmatrix} = \begin{pmatrix} 2/5 \\ 3/5 \end{pmatrix}$

9/12/2022 Fibonacci Series = petals of flower honey bee follow fibonacci

$f_0, f_1, f_2, f_3, f_4 \ldots$

$f_0 = 0 \quad f_1 = 1$

$f_{n+2} = f_{n+1} + f_n \quad n \geq 2$

$f_{n+1} = f_{n+1}$

$$\underbrace{\begin{bmatrix} f_{n+2} \\ f_{n+1} \end{bmatrix}}_{2\times1} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{2\times2} \underbrace{\begin{bmatrix} f_{n+1} \\ f_n \end{bmatrix}}_{2\times1}$$

any thing expressed as previous term = fib.

$$X_{n+1} = A \quad X_n.$$

$$X_0 = \begin{bmatrix} f_1 \\ f_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$X_1 = AX_0.$

$X_2 = AX_1 = A(AX_0) = A^2X_0$

$X_3 = AX_2 = A(A^2X_0) = A^3X_0$

$X^{-1}AX = D.$

$X_n = A^nX_0$

$$\begin{bmatrix} f_{n+1} \\ f_n \end{bmatrix} = A^nX_0$$

A is diagnolizable

$A^m = XD^mX^{-1}$

$$= (S\lambda^nS^{-1})X_0$$

$S, \lambda = X, D$

diagonal matrix

eigenvectors.

Assume $S^{-1}X_0 = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

$S\lambda^n S^{-1} \rightarrow = S\lambda^n \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

① what are the eigen values of matrix A.

$|A-\lambda I|$

$A = \begin{bmatrix} 1-\lambda & 1 \\ 1 & 0-\lambda \end{bmatrix}$

$= (1-\lambda)(0-\lambda) - 1$

$= -\lambda(1-\lambda) - 1$

$= -\lambda + \lambda^2 - 1$

$= +\lambda^2 - \lambda - 1$

$\lambda = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$ $\qquad \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$A_{\frac{1+\sqrt{5}}{2}} = \begin{bmatrix} 1 - \frac{1+\sqrt{5}}{2} & 1 \\ 1 & 0 - \frac{1+\sqrt{5}}{2} \end{bmatrix}$

$= \frac{2 - 1+\sqrt{5}}{2}$

14/12/22

$\lambda = \frac{1+\sqrt{5}}{2}$

$A = \begin{bmatrix} 1 - \frac{1+\sqrt{5}}{2} & 1 \\ 1 & 0 - \frac{1+\sqrt{5}}{2} \end{bmatrix}$

$S = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix}$ $\qquad S = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}$

$$x_n = A^n x_0$$
$$= S \lambda^n (S^{-1} x_0)$$
$$= S \lambda^n (S^{-1} x_0)$$

$$\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = S \lambda^n \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \qquad \lambda^n = \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix}$$

$$\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^n & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^n \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1\left(\frac{1+\sqrt{5}}{2}\right)^n \\ c_2\left(\frac{1-\sqrt{5}}{2}\right)^n \end{bmatrix}$$

we are interested in 2nd row so taken that only. 1st row not solved

$$F_n = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$n = 0 \quad f_0 = 0$

$n = 1 \quad f_1 = 1$

$$F_0 = c_1(-)^0 + c_2(-)^0$$

$$F_0 = c_1 + c_2$$

$$0 = c_1 + c_2$$

$$c_1 = -c_2$$

$$F_1 = c_1\left(\frac{1+\sqrt{5}}{2}\right)^1 + c_2\left(\frac{1-\sqrt{5}}{2}\right)^1$$

$$1 = \frac{c_1}{2} + \frac{\sqrt{5}c_1}{2} + \frac{c_2}{2} - \frac{\sqrt{5}c_2}{2}$$

$$1 = \frac{c_1 + c_2}{2} + \frac{\sqrt{5}(c_1 - c_2)}{2}$$

Substitute $c_1 = -c_2$.

$$1 = \frac{-c_2 + c_2}{2} + \frac{\sqrt{5}(-c_2 - c_2)}{2}$$

$$2 = 0 + \sqrt{5}(-c_2 - c_2)$$

$$\cancel{2} = -\cancel{2}\sqrt{5}\, c_2$$

$$c_2 = -1/\sqrt{5}$$

$$c_1 = -c_2$$

$$= -(-1/\sqrt{5}) = 1/\sqrt{5}$$

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

values get decreased, becomes negligible. So consider only first term

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n\right]$$

Ceil [Golden Ratio] gives rounding value

899.9 = 900

Towers of Hanoi

H_n = no. of moves.

$H_{n-1} + 1 + H_{n-1}$

$$H_n = 2H_{n-1} + 1$$

$$= 2(2H_{n-2} + 1) + 1$$

$$= 2^2 H_{n-2} + 2 + 1$$

$$= 2^2(2H_{n-3} + 1) + 2 + 1$$

$$= 2^3 \cdot H_{n-3} + 2^2 + 2 + 1$$

$= 2^{n-1} H_n - (n-1) + (2^{n-2} + \dots + 2^2 + 2 + 1)$

$= 2^{n-1} H_1 + (1 + 2 + 2^2 + \dots 2^{n-2})$

$= 2^{n-1}(1) + \frac{1 \times (2^{n-1} - 1)}{2-1}$ $\Downarrow$ $\frac{a(r^n - 1)}{r-1}$

$= 2^{n-1} + 2^{n-1} - 1$

$= 2 \times 2^{n-1} - 1$

$= 2^n - 1.$

$H_n = 2H_{n-1} + 1.$

19/12/22 **SVD**

$$A_{m \times n} = \underset{m \times m}{\overset{AA^T}{Q_1}} \; \underset{m \times n}{\Sigma} \; \underset{n \times n}{\overset{A^TA}{Q_2}}$$

$A = USV.$

$SVD(A)_{m \times n}$

Columns of Q_1 are normalized.
Eigen vectors of AA^T

Columns of Q_2 are normalized
Eigen vectors of A^TA.

$\Sigma \rightarrow$ diagonal matrix
on diagonal $\sqrt{\text{non zero eigen values of } AA^T \text{ or } A^TA}$ (~~and~~ or)

① $A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}_{2 \times 3}$ $A^T = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix}$

$A \cdot A^T = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix}$

$= \begin{bmatrix} 9+4+4 & 6+6-2 \\ 6+6-4 & 4+9+4 \end{bmatrix} = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}_{2 \times 2}$

Symmetric matrix = always diagonaizable will have different λ's

$$A^TA = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix}_{3\times2} \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}_{2\times3} = \begin{bmatrix} & & \\ & & \end{bmatrix}_{3\times3}.$$

$$= \begin{bmatrix} 9+4 & 6+6 & 6-4 \\ 6+6 & 4+9 & 4-6 \\ 6-4 & 4-6 & 4+4 \end{bmatrix}$$

$$= \begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{bmatrix}_{3\times3}.$$

$$AA^T = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix} \Rightarrow \begin{bmatrix} 17-\lambda & 8 \\ 8 & 17-\lambda \end{bmatrix}$$

$$(17-\lambda)^2 + 64 = 0$$

$$\lambda = 25, 9.$$

$$X_{25} = \begin{bmatrix} 17-25 & 8 \\ 8 & 17-25 \end{bmatrix}$$

$$= \begin{bmatrix} -8 & 8 \\ 8 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R2 = R1 + R2$

$$= \begin{bmatrix} -8 & 8 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$= -8x_1 + 8x_2 = 0$$

$$x_2 = K.$$

$$-8x_1 + 8K = 0 \qquad \sqrt{1^2+1^2} = \sqrt{2}$$

$$x_1 = 8K/8$$

$$x_1 = K.$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} K \\ K \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$X_9 = \begin{pmatrix} 17-9 & 8 \\ 8 & 17-9 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 8 \\ 8 & 8 \end{pmatrix}$$

$R2 = R1 - R2$

$$\begin{pmatrix} 8 & 8 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$8x_1 + 8x_2 = 0$$

$$8x_1 + 8K = 0 \qquad x_2 = K$$

$$x_1 = -K.$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -K \\ K \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$Q_1 = \begin{array}{c} \begin{array}{cc} x_{25} & x_9 \end{array} \\ \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \end{array}$$

$$A^T A = \begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 13-\lambda & 12 & 2 \\ 12 & 13-\lambda & -2 \\ 2 & -2 & 8-\lambda \end{bmatrix} = 0$$

$$((13-\lambda)((13-\lambda)(8-\lambda) - (4)) - 12(12(8-\lambda) - (-4)) + 2$$

$$(-24 - (2(13-\lambda))) = 0$$

$$\lambda = 25, 9, 0.$$

$$X_{25} = \begin{bmatrix} -12 & 12 & 2 \\ 12 & -12 & -2 \\ 2 & -2 & -17 \end{bmatrix}$$

$$\begin{bmatrix} -12 & 12 & 2 \\ 0 & 0 & 0 \\ 2 & -2 & -17 \end{bmatrix}$$

$$\begin{bmatrix} -12 & 12 & 2 \\ 2 & -2 & -17 \\ 0 & 0 & 0 \end{bmatrix} \times 6$$

$$\begin{bmatrix} -12 & 12 & 2 \\ 0 & 0 & -100 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

assume $x_2 = K$

$-100x_3 = 0$

$x_3 = 0$

$-12x_1 + 12x_2 + 2x_3 = 0$

$-12x_1 + 12K = 0$

$x_1 = \cancel{K}\ K.$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} K \\ K \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$x_9 = \begin{bmatrix} 4 & 12 & 2 \\ 12 & 4 & -2 \\ 2 & -2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 6 & 1 \\ 6 & 2 & -1 \\ 2 & -2 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 6 & 1 \\ 0 & 16 & 4 \\ 0 & 8 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 1 \\ 0 & 8 & 2 \\ 0 & 8 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 6 & 1 \\ 0 & 8 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 6 & 1 \\ 0 & 8 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$8x_2 + 2x_3 = 0$$

$$2x_1 + 6x_2 + x_3 = 0$$

$$x_3 = K$$

$$8x_2 + 2K = 0$$

$$x_2 = -\frac{\cancel{2}K}{\cancel{8}\,4} = -K/4$$

$$2x_1 + 6\left(\frac{-K}{4}\right) + K = 0$$

$$4 \cdot 2x_1 + (-6K) + 4K = 0$$

$$8x_1 - 2K = 0$$

$$x_1 = 2K/8 = K/4$$

$$\sqrt{16+1+1}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} K/4 \\ -K/4 \\ K \end{pmatrix} \text{ for } K = 4 = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{18} \\ -1/\sqrt{18} \\ 4/\sqrt{18} \end{pmatrix}$$

$$X_0 = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$$

$$Q_2^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{18} & -1/\sqrt{18} & 4/\sqrt{18} \\ -2/3 & 2/3 & 1/3 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}$$

Square root of eigen value of AA^T & A^TA will be the diagonal values of Σ & it will be arranged in decreasing order.

$$A = Q_1 \Sigma Q_2^T$$

$$= \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{18} & -1/\sqrt{18} & 4/\sqrt{18} \\ -2/3 & 2/3 & 1/3 \end{bmatrix}$$

1/12/22

Applying LA to ordinary differential Equations

Rabit problem leads to fibonacci series.

$$y' = 3y$$

$$\int \frac{dy}{y} = \int 3\,dx$$

$$\log_e y = 3x + C$$

$$y = e^{3x+C}$$

$$= e^{3x} \cdot e^{C}$$

$$y = K \cdot e^{3x}$$

1) System of equations we have seen.

$y_1' = 3y_1 \qquad y_1[0] = 1$

$y_2' = -2y_2 \qquad y_2[0] = 4$

$y_3' = 5y_3 \qquad y_3(0) = -2$

$$\begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

General solution

$y_1 = c_1 e^{3x} \qquad y_2 = c_2 e^{-2x} \qquad y_3 = c_3 e^{5x}.$

Particular Solution:

$x = 0$

$1 = C_1 \quad 4 = C_2 \quad -2 = C_3.$

② $y_1' = y_1 + y_2 \qquad y_1(0) = 1$

$y_2' = 4y_1 - 2y_2 \qquad y_2(0) = 6.$

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{bmatrix}$$

$$(1-\lambda)(-2-\lambda) - 4 = 0$$

$$\lambda = 2, -3.$$

$$X_2 = \begin{bmatrix} 1-2 & 1 \\ 4 & -2-2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_2 = K$

$x_1 = x_2 = K.$

$$X_2 = \begin{bmatrix} K \\ K \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$X_3 = \begin{bmatrix} 4 & 1 \\ 4 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 1 \\ 0 & 0 \end{bmatrix} \quad x_2 = K$$

$4x_1 + x_2 = 0 \qquad x_1 = -K/4$

$$\begin{bmatrix} -k/4 \\ k \end{bmatrix} = \quad \text{for } k = 4 \quad \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

$$X = \begin{array}{cc} X_2 & X_3 \end{array}$$
$$X = \begin{bmatrix} 1 & -1 \\ 1 & 4 \end{bmatrix}$$

$$X^{-1} = \frac{1}{5}\begin{bmatrix} 4 & 1 \\ -1 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$$

$$D = X^{-1}AX$$

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$y_1' = 2y_1 \Rightarrow y_1 = c_1 e^{2x}$$

$$y_2' = -3y_2 \Rightarrow y_2 = c_2 e^{-3x}$$

$$y = \cancel{x} \, x u$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} c_1 e^{2x} \\ c_2 e^{-3x} \end{bmatrix}$$

$$y_1 = c_1 e^{2x} - c_2 e^{-3x}$$

$$y_2 = c_1 e^{2x} + 4c_2 e^{-3x}$$

$$1 = c_1 e^{2x} - c_2 e^{-3x}$$

$$6 = c_1 e^{2x} + 4c_2 e^{-3x}.$$

$x = 0$.

$1 = c_1 - c_2$

$x = 0$

$6 = c_1 + 4c_2$

$$c_1 - c_2 = 1$$
$$c_1 + 4c_2 = 6$$
$$-5c_2 = -5$$

$c_2 = 1$

$c_1 - c_2 = 1$

$c_1 - 1 = 1 \quad c_1 = 2$

$$y_1 = 2e^{2x} - e^{-3x}$$

$$y_2 = 2e^{2x} + 4e^{-3x}$$

(3) $y_1' = y_1 + 4y_2$

$y_2' = 2y_1 + 3y_2$

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{bmatrix}$$

$$(1-x)(3-x) - 8 = 0$$

$$3 - \lambda - 3\lambda + x^2 - 8 = 0$$

$$x^2 - 4x - 5 = 0$$

$$x(x+1) - 5(x+1) = 0$$

$$x = 5 \quad \lambda = -1$$

$$x_5 = \begin{bmatrix} 1-5 & 4 \\ 2 & 3-5 \end{bmatrix}$$

$$\begin{array}{c} -4/ \\ 2/ \end{array} \begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_2 = k$$

$$x_1 - x_2 = 0$$

$$x_1 = x_2 \quad x_1 = k$$

$$x_5 = \begin{bmatrix} k \\ k \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$x_{-1} = \begin{bmatrix} 1+1 & 4 \\ 2 & 3+1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_2 = k$

$x_1 + 2x_2 = 0$

$x_1 = -2k.$

$$x_{-1} = \begin{bmatrix} 2k \\ k \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$$

$$X^{-1} = \frac{1}{3} \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$$

$$y' = Dy$$

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$y_1' = 5y_1 \Rightarrow y_1 = c_1 e^{5x}$

$y_2' = -y_2 \quad y_2 = c_2 e^{-x}.$

$y = XU$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} C_1 e^{5x} \\ C_2 e^{-x} \end{bmatrix}$$

$$y_1 = C_1 e^{5x} - 2C_2 e^{-x}$$

$$y_2 = C_1 e^{5x} + C_2 e^{-x}.$$

(4) $$\begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= \begin{bmatrix} 4-\lambda & 0 & 1 \\ -2 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{bmatrix}$$

$$(4-\lambda)(1-\lambda-\lambda+\lambda^2) + (2(1-\lambda)) = 0$$

$$(4-\lambda)(1-2\lambda+\lambda^2) + 2 - 2\lambda = 0$$

$$4 - 8\lambda + 4\lambda^2 - \lambda + 2\lambda^2 - \lambda^3 + 2 - 2\lambda = 0.$$

$$-\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0$$

$$(4-\lambda)((1-\lambda)(1-\lambda) + 2(1-\lambda) \quad (1-\lambda)[(4-\lambda)(1-\lambda) + 2] = 0$$

$$(1-\lambda)((4-\lambda)(1-\lambda)+2) = 0$$

$$(1-\lambda)(4-4\lambda-\lambda+\lambda^2+2) = 0$$

$$(1-\lambda)(6-5\lambda+\lambda^2) = 0$$

$$\lambda = 1, 2, 3.$$

$$X_3 = \begin{bmatrix} 4-3 & 0 & 1 \\ -2 & 1-3 & 0 \\ -2 & 0 & 1-3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ -2 & -2 & 0 \\ -2 & 0 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} /-1$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_3 = K.$

$x_1 + x_3 = 0$

$x_2 = 0 \qquad x_1 + K = 0$

$\qquad\qquad x_1 = -K$

$$X_3 = \begin{bmatrix} -K \\ 0 \\ K \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} 4-2 & 0 & 1 \\ -2 & 1-2 & 0 \\ -2 & 0 & 1-2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ -2 & -1 & 0 \\ -2 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} -K/2 \\ 0 \\ K \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} 4-1 & 0 & 1 \\ -2 & 1-1 & 0 \\ -2 & 0 & 1-1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 1 \\ -2 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} 0 \\ K \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$X = \begin{bmatrix} -1 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$y' = Dy$$

$$\begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$y_1' = 3y_1 \Rightarrow y_1 = C_1 e^{3x}$$

$$y_2' = 2y_1 \Rightarrow y_2 = C_2 e^{2x}.$$

$$y_3' = y_3 \Rightarrow y_3 = C_3 e^{x}.$$

$$y = xu.$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} C_1 e^{3x} \\ C_2 e^{2x} \\ C_3 e^{x} \end{bmatrix}$$

$$y_1 = -C_1 e^{3x} - C_2 e^{2x}$$

$$y_2 = C_2 e^{2x}$$

$$y_3 = C_1 e^{3x} + 2C_2 e^{2x}.$$