21DS636-**Statistical Modelling**

Probability Theory

Bayes' Theorem

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Bayes’ Theorem

- Suppose we have estimated **prior probabilities** for events we are concerned with, and then obtain new information.

- We would like to a sound method to computed *revised* or **posterior probabilities.**

- Bayes’ theorem gives us a way to do this.

Probability Revision using Bayes' Theorem

Prior Probabilities → New Information → Application of Bayes' Theorem → Posterior Probabilities

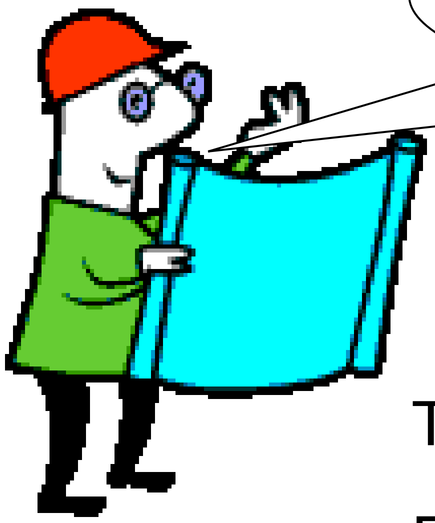
- **Bayes' theorem has an interesting interpretation:**
  - $P(A)$ represents the a-priori probability of the event A. Suppose B has occurred, and assume that A and B are not independent. How can this new information be used to update our knowledge about A? Bayes' rule takes into account the new information ("B has occurred") and gives out the a-posteriori probability of A given B.

    $P(A)$ $P(A|B)$

  - We can also view the event B as new knowledge obtained from a fresh experiment. We know something about A as $P(A)$. The new information is available in terms of B.
  - The new information should be used to improve our knowledge/understanding of A. Bayes' theorem gives the exact mechanism for incorporating such new information.

Understanding Bayes' Theorem Using Example

- Consider a manufacturing firm that receives shipment of parts from two suppliers.
  - Let S_1 denote the event that a part is received from supplier 1; S_1
  - *The* S_2 is the event the part is received from supplier 2 S_2



We get 65 % of our parts from supplier 1 and 35 % from supplier 2.

Thus:

$P(S_1) = .65$ and $P(S_2) = .35$

Quality Levels Differ between Suppliers

	Percentage Good Parts (%)	Percentage Bad Parts (%)
Supplier 1	98	2
Supplier 2	95	5

G

B

- Let *G* denote that a part is good, and *B* denote the event that a part is bad.
- Thus, we have the following conditional probabilities:

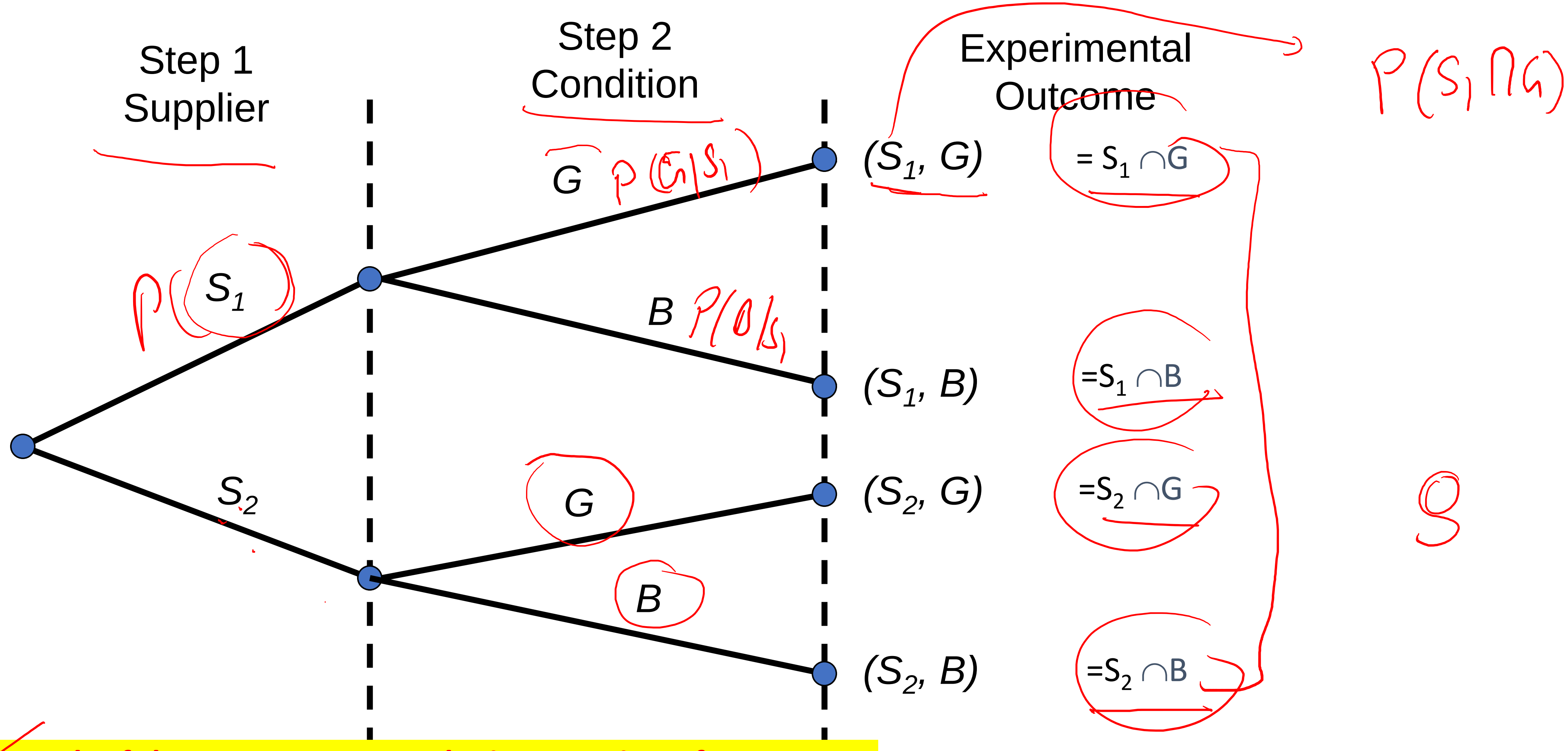
$P(S_1) = 0.65$

$P(S_2) = 0.35$

$$\boxed{P(G \mid S_1)} = .98 \quad \text{and} \quad \boxed{P(B \mid S_1)} = .02$$

$$\boxed{P(G \mid S_2)} = .95 \quad \text{and} \quad \boxed{P(B \mid S_2)} = .05$$

Tree Diagram for Two-Supplier Example



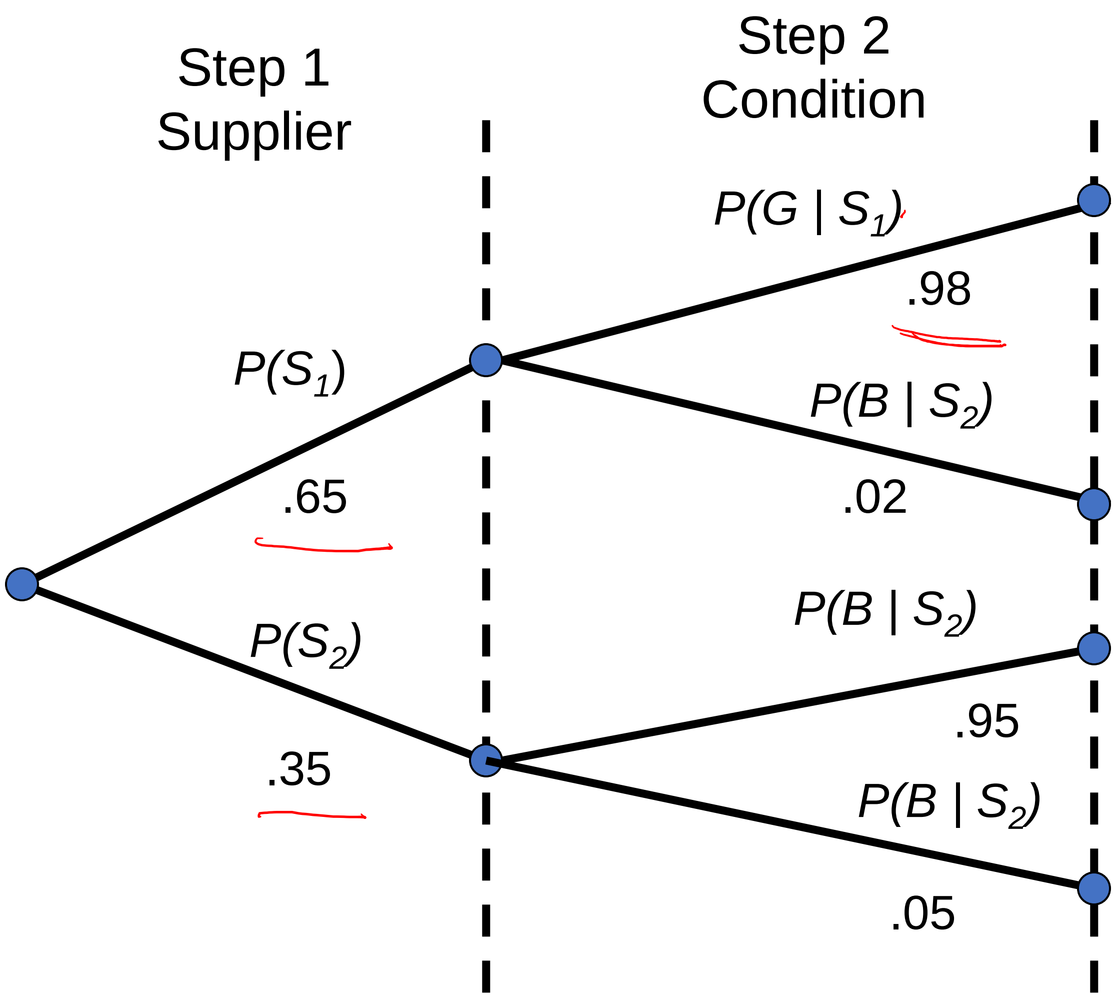
- **Each Branch of the tree represents the intersection of two events**
- **The last four branches represent Mutually Exclusive events**

- **Each of the experimental outcomes is the intersection of 2 events.**

- **For Example, the probability of selecting a part from supplier 1 and that is good is given by:**

$$P(S_1, G) = P(S_1 \cap G) = P(S_1)P(G \,|S_1)$$

Probability Tree for Two-Supplier Example



Probability of Outcome

$P(S_1 \cap G) = P(S_1)P(G|S_1) = .6370$

0.65 X 0.98

0.65 X 0.02

$P(S_1 \cap B) = P(S_1)P(B|S_1) = .0130$

$P(S_2 \cap G) = P(S_2)P(G|S_2) = .3325$

0.35 X 0.95

$P(S_2 \cap B) = P(S_2)P(G|S_2) = .0175$

0.35 X 0.05

- A bad part broke one of our machines—so we're through for the day.
- What is the probability the part came from suppler 1?

We know from the law of conditional probability that:

$$P(S_1|B) = \frac{P(S_1 \cap B)}{P(B)} \qquad (1)$$

P(B)?

Observe from the probability tree that:

$$P(S_1 \cap B) = P(S_1)P(B|S_1) \qquad (2)$$

The probability of selecting a bad part is found by adding together the probability of selecting a bad part from supplier 1 and the probability of selecting bad part from supplier 2.

That is:

$P(B) = P[(S_1 \cap B) \text{ OR } (S_2 \cap B)]$

$$P(B) = P(S_1 \cap B) + P(S_2 \cap B)$$
$$= P(S_1)P(B|S_1) + P(S_2)P(B|S_2) \quad (3)$$

Bayes' Theorem for 2 events

By substituting equations (2) and (3) into (1)

$$P(S_1|B) = \frac{P(S_1)P(B|S_1)}{P(S_1)P(B|S_1) + P(S_2)P(B|S_2)}$$

$$\frac{P(S_1 \cap B)}{P(B)}$$

and writing a similar result for *P(S₂|B)*, we obtain Bayes' theorem for the 2-event case:

$$P(S_2|B) = \frac{P(S_2)P(B|S_2)}{P(S_1)P(B|S_1) + P(S_2)P(B|S_2)}$$

Do the Math

$$P(S_1|B) = \frac{P(S_1)P(B|S_1)}{P(S_1)P(B|S_1) + P(S_2)P(B|S_2)}$$

$P(S_1) = 0.68$

$$= \frac{(.65)(.02)}{(.65)(.02) + (.35)(.05)} = \frac{.0130}{.0305} \quad = \boxed{}$$

$$P(S_2|B) = \frac{P(S_2)P(B|S_2)}{P(S_2)P(B|S_2) + P(S_2)P(B|S_2)}$$

$$= \frac{(.35)(.05)}{(.65)(.02) + (.35)(.05)} = \frac{.0175}{.0305} = .5738$$

$P(S_2) = 0.35$

57.38 to Bad is coming from S_2

Bayes' Theorem

$$P(A_i \mid B) = \frac{P(A_i)P(B \mid A_i)}{P(A_1)P(B \mid A_1) + P(A_2)P(B \mid A_2) + \ldots + P(A_n)P(B \mid A_n)}$$

$A \cap A_2 \cap A_3 - - \cap A_n = \phi$

$A_1 \cap A_2 = \phi$ $\quad A_1 \cap A_2 \cap A_3 = \phi$

- $S = A_1 \cup \cdots \cup A_n$ and A_i : mutually exclusive

$\Rightarrow B = (A_1 \cap B) \cup \cdots \cup (A_n \cap B)$ and $(A_i \cap B)$: mutually exclusive

$\Rightarrow P(B) = P(A_1 \cap B) + \cdots + P(A_n \cap B)$

$= P(A_1)P(B \mid A_1) + \cdots + P(A_n)P(B \mid A_n)$

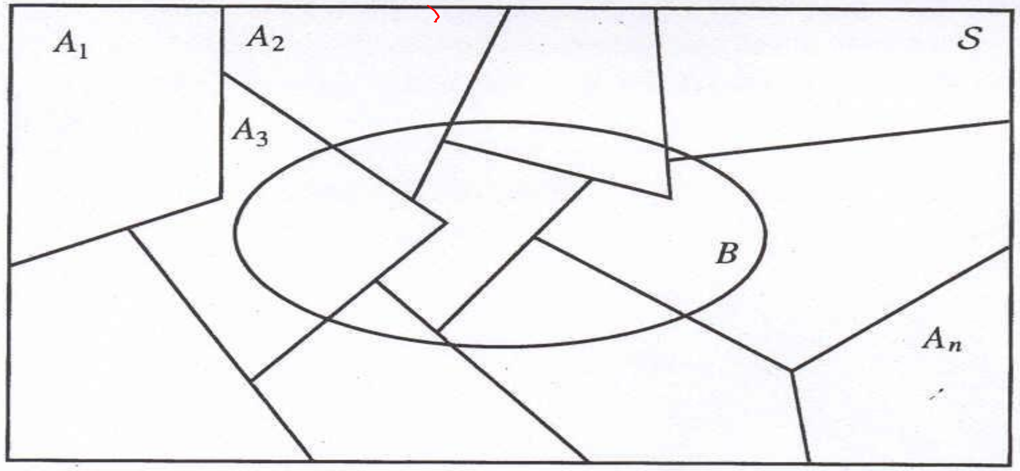
Law of total Probability

Prior

likelihood

S

FIGURE 1.67 ● A partition $A_1, \ldots, A_n$ and an event B



Law of Total Probability

- **Law of Total Probability**

If $A_1, A_2, \ldots, A_n$ is a partition of a sample space, then the probability of an event ***B*** can be obtained from the probabilities $P(A_i)$ and $P(B \mid A_i)$ using the formula

$$P(B) = P(A_1)P(B \mid A_1) + \cdots + P(A_n)P(B \mid A_n)$$

Calculation of Posterior Probabilities

- $P(A_i)$ and $P(B \mid A_i) \Rightarrow P(A_i \mid B) = ?$
- $P(A_1), \cdots, P(A_n)$: the prior probabilities
- $P(A_1 \mid B), \cdots, P(A_n \mid B)$: the posterior probabilities

$$\Rightarrow P(A_i \mid B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(A_i)P(B \mid A_i)}{P(B)} = \frac{P(A_i)P(B \mid A_i)}{\sum_{j=1}^{n} P(A_j)P(B \mid A_j)}$$

- **Bayes' Theorem**

If $A_1, A_2, \ldots, A_n$ is a partition of a sample space, then the **posterior probabilities** of the event conditional on an event ***B*** can be obtained from the probabilities $P(A_i)$ and $P(B \mid A_i)$ using the formula

$$P(A_i \mid B) = \frac{P(A_i)P(B \mid A_i)}{\sum_{j=1}^{n} P(A_j)P(B \mid A_j)}$$

Example: Car Warranties

A company sells a certain type of car, which it assembles in one of four possible locations. Plant I supplies 20%; plant II, 24%; plant III, 25%; and plant IV, 31%. A customer buying a car does not know where the car has been assembled, and so the probabilities of a purchased car being from each of the four plants can be thought of as being 0.20, 0.24, 0.25, and 0.31.

$P(I)=0.20$ $P(III)=0.25$

$P(II)=0.24$ $P(IV)=0.31$

Each new car sold carries a 1-year bumper-to-bumper warranty.

P(claim | plant I) = 0.05, P(claim | plant II) = 0.11

P(claim | plant III) = 0.03, P(claim | plant IV) = 0.08

- ***P(claim | plant I) = 0.05*** → a car assembled in plant I has a probability of 0.05 of receiving a claim on its warranty.

- Notice that **claims are clearly not independent of assembly location** because these four conditional probabilities are unequal.

1. What is the probability of a claim is made on the warranty of the car? $P(\text{Claim})$
2. If a claim is made on the warranty of the car, what is the probability that it is assembled on plant1? $\to P(I \mid \text{Claim})$
3. If a no claim is made on the warranty of the car, what is the probability that it is assembled on plant1? $P(I \mid C')$
4. Calculate 2. and 3. for plant2 and plant3 as well.

1) $\cup C$

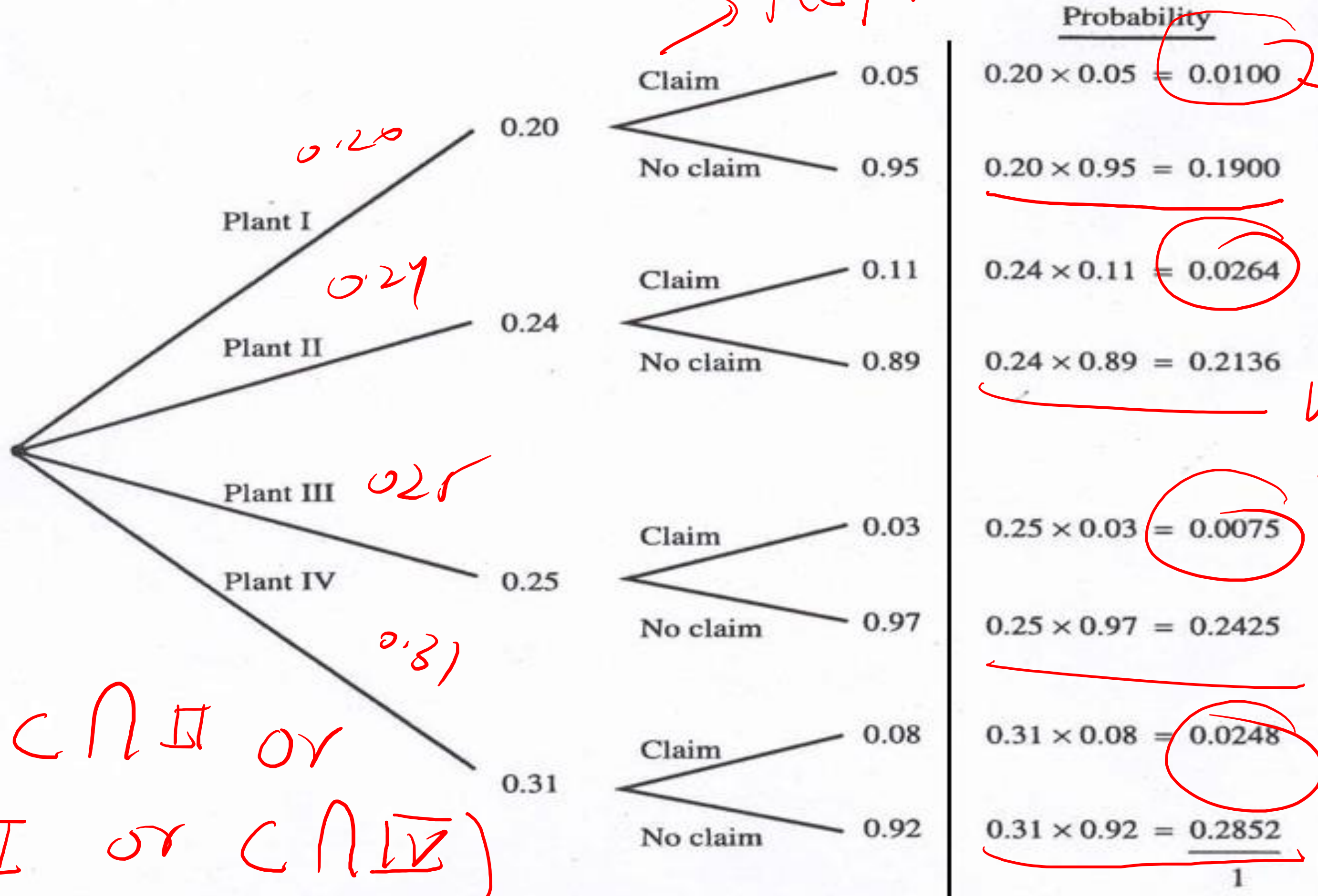
$P(\text{claim}) = P(\text{plant I, claim}) + P(\text{plant II, claim}) + P(\text{plant III, claim}) + P(\text{plant IV, claim})$

$= 0.0687$

$P(I) \cdot P(C|I)$

$P(C|\text{Plant } I)$

$\cap$



$\to P(I \cap C)$

$P(I \cap C')$

$P(II \cap C)$

$P(II \cap C')$

$= P(II) \cdot P(C'|II)$

$P(C)$

$= P(C \cap I \text{ or } C \cap II \text{ or } C \cap III \text{ or } C \cap IV)$

$= P(I) \cdot P(C|I) + P(II) \cdot P(C|II) + P(III) \cdot P(C|III) + P(IV) \cdot P(C|IV)$

$= 0.0100 + 0.0264 + 0.0075 + 0.0248$

$$\frac{P(I \cap C)}{P(C)}$$

Prior Prob

likelihood Prob

Total Prob

- The prior probabilities

$$P(plant\ \ I) = 0.20, \quad P(plant\ \ II) = 0.24$$

$$P(plant\ \ III) = 0.25, \quad P(plant\ \ IV) = 0.31$$

- **If a claim is made on the warranty of the car, how does this change these probabilities?**

Posteriori prob

$$P(plant\ \ I \mid claim) = \frac{P(plant\ \ I)P(claim \mid plant\ \ I)}{P(claim)} = \frac{0.20 \times 0.05}{0.0687} = 0.146$$

$$P(plant\ \ II \mid claim) = \frac{P(plant\ \ II)P(claim \mid plant\ \ II)}{P(claim)} = \frac{0.24 \times 0.11}{0.0687} = 0.384$$

$$P(plant\ \ III \mid claim) = \frac{P(plant\ \ III)P(claim \mid plant\ \ III)}{P(claim)} = \frac{0.25 \times 0.03}{0.0687} = 0.109$$

$$P(plant\ \ IV \mid claim) = \frac{P(plant\ \ IV)P(claim \mid plant\ \ IV)}{P(claim)} = \frac{0.31 \times 0.08}{0.0687} = 0.361$$

- **If a no claim is made on the warranty of the car, what is the probability that it is assembled on plant1?**
- *No* claim is made on the warranty

$$P(plant\ I \mid no\ claim) = \frac{P(plant\ I)P(no\ claim \mid plant\ I)}{P(no\ claim)}$$

$$= \frac{0.20 \times 0.95}{0.9313} = 0.204$$

$$P(plant\ II \mid no\ claim) = \frac{P(plant\ II)P(no\ claim \mid plant\ II)}{P(no\ claim)}$$

$$= \frac{0.24 \times 0.89}{0.9313} = 0.229$$

$P(C) =$

$$P(plant\ III \mid no\ claim) = \frac{P(plant\ III)P(no\ claim \mid plant\ III)}{P(no\ claim)}$$

$$= \frac{0.25 \times 0.97}{0.9313} = 0.261$$

$$P(plant\ IV \mid no\ claim) = \frac{P(plant\ IV)P(no\ claim \mid plant\ IV)}{P(no\ claim)}$$

$$= \frac{0.31 \times 0.92}{0.9313} = 0.306$$

$P(C') = 1 - P(C)$

$1 - 0.0687$

Example: Two boxes B_1 and B_2 contain 100 and 200 light bulbs respectively. The first box (B_1) has 15 defective bulbs and the second(B_2) 5. Suppose a box is selected at random and one bulb is picked out.

(a) What is the probability that it is defective?

(b) Suppose we test the bulb and it is found to be defective. What is the probability that it came from box 1?

Solution: Note that box B_1 has 85 good and 15 defective bulbs. Similarly box B_2 has 195 good and 5 defective bulbs.

Let D = “Defective bulb is picked out”.

Then $$P(D \mid B_1) = \frac{15}{100} = 0.15, \quad P(D \mid B_2) = \frac{5}{200} = 0.025.$$

Since a box is selected at random, they are equally likely.

$$P(B_1) = P(B_2) = \frac{1}{2}.$$

Thus B_1 and B_2 form a partition, therefore, we obtain

$$\begin{aligned} P(D) &= P(B_1)P(D|B_1) + P(B_2)P(D|B_2) \\ &= \frac{1}{2} \times 0.15 + \frac{1}{2} \times 0.025 = 0.0875. \end{aligned}$$

Thus, there is about 9% probability that a bulb picked at random is defective.

(b) Suppose we test the bulb and it is found to be defective. What is the probability that it came from box 1?

$$P(B_1 \mid D) = ?$$

$$P(B_1|D) = \frac{P(B_1)P(D|B_1)}{P(D)} = \frac{1/2 \times 0.15}{0.0875} = 0.8571. \qquad (1)$$

Notice that initially $P(B_1) = 0.5;$ then we picked out a box at random and tested a bulb that turned out to be defective. Can this information shed some light about the fact that we might have picked up box 1?

From (1), $P(B_1 \mid D) = 0.857 > 0.5,$ and indeed it is more likely at this point that we must have chosen box 1 in favor of box 2.

(Recall box1 has six times more defective bulbs compared to box2).

Example: A consulting firm submitted a bid for a large consulting contract. The firm’s management felt it had a 50-50 change of landing the project. However, the agency to which the bid was submitted subsequently asked for additional information. Past experience indicates that that for 75% of successful bids and 40% of unsuccessful bids the agency asked for additional information.

a. What is the prior probability of the bid being successful (that is, prior to the request for additional information).

b. What is the conditional probability of a request for additional information given that the bid will be ultimately successful.

c. Compute the posterior probability that the bid will be successful given a request for additional information.

Solution:

- Let S_1 denote the event of successfully obtaining the project.
- S_2 is the event of not obtaining the project.
- B is the event of being asked for additional information about a bid.

a. $P(S_1) = .5$

b. $P(B \mid S_1) = .75$

c. Use Bayes' theorem to compute the posterior probability that a request for information indicates a successful bid.

$$P(S_1 \mid B) = \frac{P(S_1 \cap B)}{P(S_1 \cap B) + P(S_2 \cap B)} = \frac{(.5)(.75)}{(.5)(.75) + (.5)(.4)}$$

$$= \frac{.375}{.575} \cong .652$$

Example: Your retail business is considering holding a sidewalk sale promotion next Saturday. Past experience indicates that the probability of a successful sale is 60%, if it does not rain. This drops to 30% if it does rain on Saturday. A phone call to the weather bureau finds an estimated probability of 20% for rain.

What is the probability that you have a successful sale?

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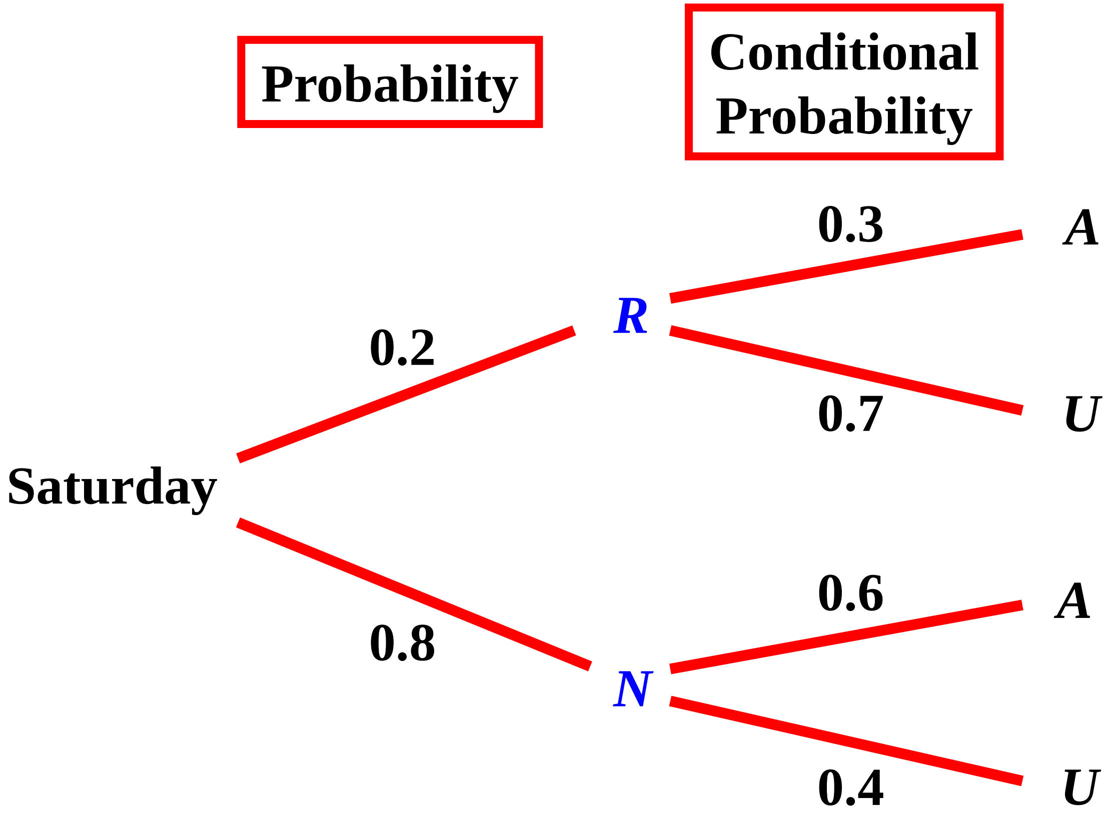
Solution: Tree Diagram



Event	Probability
$R \cap A$	$0.2 \cdot 0.3 = 0.06$ ($P(R)$, $P(A \mid R)$)
$R \cap U$	$0.2 \cdot 0.7 = 0.14$
$N \cap A$	$0.8 \cdot 0.6 = 0.48$ ($P(N)$, $P(A \mid N)$)
$N \cap U$	$0.8 \cdot 0.4 = 0.32$

Events

R- rains next Saturday

N -does not rain next Saturday.

A -sale is successful

U- sale is unsuccessful.

Using

$P(A) = P(R \cap A) + P(N \cap A)$

$=0.06+0.48=0.54$

***Each Branch of the tree represents the intersection of two events**
***The four branches represent Mutually Exclusive events**

Extension of Example:

What is the probability that it was raining given that sale was successful? i.e. find *P(R|A)?*

The conditional probability that it rains given that sale is successful the How do we calculate?

Using conditional probability formula

$$P(R|A)=\frac{P(R\cap A)}{P(A)}=\frac{P(A|R)\cdot P(R)}{P(A|R)\cdot P(R)+P(A|N)\cdot P(N)}$$

$$=\frac{0.3\cdot 0.2}{0.3\cdot 0.2+0.6\cdot 0.8}$$

$$= 0.1111$$

Example:

In a recent New York Times article, it was reported that light trucks, which include SUV's, pick-up trucks and minivans, accounted for 40% of all personal vehicles on the road in 2002. Assume the rest are cars. Of every 100,000 car accidents, 20 involve a fatality; of every 100,000 light truck accidents, 25 involve a fatality. If a fatal accident is chosen at random, what is the probability the accident involved a light truck?

Solution:

Events

C- Cars , T –Light truck

F –Fatal Accident , N- Not a Fatal Accident

Given

$P(F|C) = 20/10000$ and $P(F|T) = 25/100000$

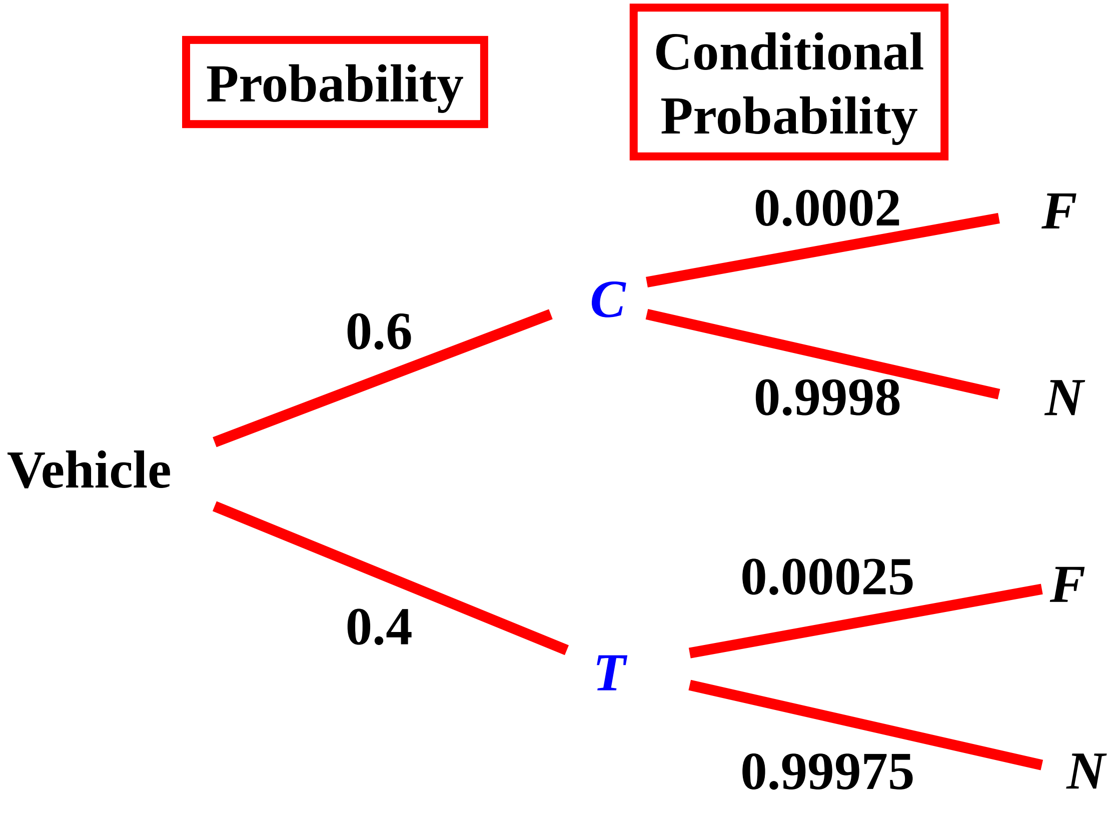
$P(T) = 0.4$

In addition we know C *and* T are complementary events : $P(C)=1-P(T)=0.6$

Our goal is to compute the conditional probability of a Light truck accident given that it is fatal $P(T/F)$.

Solution: Tree Diagram

Q: what is the probability of a Light truck accident given that it is fatal ? → P(T|F)



Event	Probability
$C \cap F$	$0.6 \cdot 0.0002 = .00012$
$C \cap N$	$0.6 \cdot 0.9998 = 0.59988$
$T \cap F$	$0.4 \cdot 0.00025 = 0.0001$
$T \cap N$	$0.4 \cdot 0.99975 = .3999$

$$P(T\,|\,F) = \frac{P(T \cap F)}{P(F)} = \frac{P(T \cap F)}{P(T \cap F) + P(C \cap F)}$$

$$= \frac{(0.00025)(0.4)}{(0.00025)(0.4) + (0.0002)(0.6)}$$

$$= 0.4545$$

Conditional Independence

w.k. A, B are ID

i.e. $P(A \cap B) = P(A) \cdot P(B)$

$P(A|B) = P(A)$, $P(B) \neq 0$

$P(B|A) = P(B)$, $P(A) \neq 0$

- Two events A and B are conditionally independent given an event C with P(C)>0 if

$$\mathbf{P(A \cap B|C)=P(A|C)P(B|C)}$$

- Thus, if A and B are conditionally independent given C, then

$$\mathbf{P(A|B,C)=P(A|C)}$$

$$P(B|A,C) = P(B|C)$$

Example: A box contains two coins: a regular coin (C) and one fake two-headed coin (C') ($P(H)=1$). I choose a coin at random and toss it twice. Define the following events.

A= First coin toss results in an Head. (time)

B= Second coin toss results in an Head. (time)

C= Coin-1 (regular) has been selected.

C' = fake two headed coin

Find $P(A|C)$,$P(B|C)$,$P(A \cap B|C)$,$P(A)$,$P(B)$, and $P(A \cap B)$. Note that A and B are NOT independent, but they are conditionally independent given C

Solution:

1) $P(A|C) = 1/2$

2) $P(B|C) = 1/2$

3) $P(A \cap B|C) = P(A|C) \cdot P(B|C) = 1/2 \cdot 1/2 = 1/4$

4) $P(A)$ = P(first time coin tossed resulted in a ~~head~~ head)

$= P(A \cap C \text{ or } A \cap C') = P(C) \cdot P(A|C) + P(C')\, P(A|C') = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot 1 = \frac{1}{4} + \frac{1}{2} = 3/4$

$P(B) = 3/4$

5) $P(A \cap B)$

$= P(A \cap B \cap C \text{ OR } A \cap B \cap C')$

$= P(A \cap B \cap C) + P(A \cap B \cap C')$

$= P(C) \cdot P(A \cap B|C) + P(C') \cdot P(A \cap B|C')$

$1/2 \cdot 1/4 + 1/2 \cdot 1$

Exercise Questions

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Q1: There are three true coins, one false coin with head on both sides. A coin is chosen randomly and tossed four times. If head occurs all the four times what is the probability that the false coin has been chosen and used?

Solution: Let T_c = True coin is selected $P(T_c) = 3/4$

F_c = False " " " $P(F_c) = 1/4$

H = head occurs all the four times of tossing.

To find $P(F_c/H)$

$$= \frac{P(F_c \cap H)}{P(H)} = \frac{\overbrace{P(F_c)}^{=1/4} \cdot \overbrace{P(H/F_c)}^{1}}{P(H)}$$

$$P(H) = P(H \cap T_c \text{ OR } H \cap F_c) = P(T_c) \cdot P(H/T_c) + P(F_c) \cdot P(H/F_c)$$

$$= 3/4 \left(\frac{1}{2}\right)^4 + 1/4 \cdot 1$$

Exercise Questions

Q2: For a certain binary communication channel, the probability that a transmitted “0” is received as “0” is 0.95 and the probability that a transmitted “1” is received as “1” is 0.90. The probability of transmitted “1” is 0.6. what is the probability that

1. a “1” is received
2. a “1” was transmitted given that a “”1” was received.

Solution:

Exercise Questions

Q3: A certain diseases occurs in mild or severe form. Three quarter of patients have the mild form. A new drug is available , probability that the mild case of the disease respond to the drug 0.9 and he probability that sever case respond is 0.5

- What is the probability that randomly chosen case respond to the drug.
- You are told that a certain patient has responded to the drug, what is the probability that the patient has the mild form of the disease.

Solution: