Practical 10 : Newton Interpolation

1

Divided Differences

Figure 1:

Definition. Let f be a function defined at the distinct points x_0, x_1, x_2, $\ldots$, x_n.

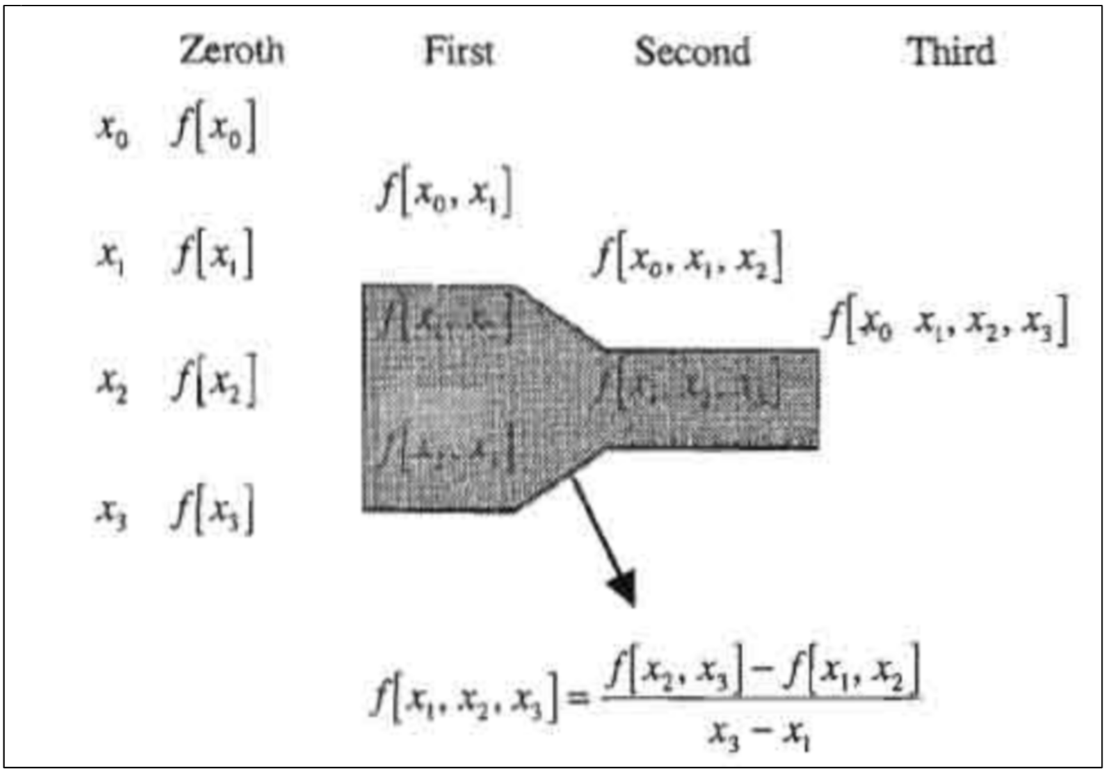
The ZEROTH DIVIDED DIFFERENCE of f with respect to the point x_i is $f[x_i] \equiv f(x_i)$.

For $0 < k \leq n$, the kTH DIVIDED DIFFERENCE of f with respect to the points x_i, x_{i+1}, x_{i+2}, $\ldots$, x_{i+k} is

$$f[x_i, x_{i+1}, x_{i+2}, \ldots, x_{i+k}] = (f[x_{i+1}, x_{i+2}, x_{i+3}, \ldots, x_{i+k}] - f[x_i, x_{i+1}, x_{i+2}, \ldots, x_{i+k-1}])/(x_{i+k} - x_i).$$

Organization of a Divided Difference Table

Figure 2:



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Divided Difference Table

(%i1) kill(all);

(%o0) *done*

(%i1)
```
dd_table(a, f):=block
(
   n:length(a),
   for i:1 thru n do A[i, 1]:a[i],
   for i:1 thru n do A[i, 2]:f[i],
   s:0,
   for j:3 thru n+1 do
   (
      for i:1 thru n+2−j do A[i, j]:(A[i+1, j−1]−A[i, j−1])/(A[i+1+s, 1]−A[i, 1]
      s:s+1
   ),
   table:genmatrix(A, n, n+1),
   table
);
```

(%o1) $\mathrm{dd\_table}(a,f):=\mathrm{block}(n:\mathrm{length}(a),\mathrm{for}\ i\ \mathrm{thru}\ n\ \mathrm{do}\ A_{i,1}:a_i,\mathrm{for}\ i\ \mathrm{thru}\ n\ \mathrm{do}\ A_{i,2}:f_i,s:0,\mathrm{for}\ j\ \mathrm{from}\ 3\ \mathrm{thru}\ n+1\ \mathrm{do}\left(\mathrm{for}\ i\ \mathrm{thru}\ n+2-j\ \mathrm{do}\ A_{i,j}:\frac{A_{i+1,j-1}-A_{i,j-1}}{A_{i+1+s,1}-A_{i,1}},s:s+1\right),table:\mathrm{genmatrix}(A,n,n+1),table)$

(%i2) ans:dd_table([1, 3/2, 0, 2], [3, 13/4, 3, 5/3]);

(%o2)
$$\begin{pmatrix} 1 & 3 & \frac{1}{2} & \frac{1}{3} & -2 \\ \frac{3}{2} & \frac{13}{4} & \frac{1}{6} & -\frac{5}{3} & A_{2,5} \\ 0 & 3 & -\frac{2}{3} & A_{3,4} & A_{3,5} \\ 2 & \frac{5}{3} & A_{4,3} & A_{4,4} & A_{4,5} \end{pmatrix}$$

(%i3) ans;

(%o3)
$$\begin{pmatrix} 1 & 3 & \frac{1}{2} & \frac{1}{3} & -2 \\ \frac{3}{2} & \frac{13}{4} & \frac{1}{6} & -\frac{5}{3} & A_{2,5} \\ 0 & 3 & -\frac{2}{3} & A_{3,4} & A_{3,5} \\ 2 & \frac{5}{3} & A_{4,3} & A_{4,4} & A_{4,5} \end{pmatrix}$$

(%i4) dd_table([−1, 0, 1, 2, −2, 3], [5, 1, 1, 11, 5, 35]);

(%o4)
$$\begin{pmatrix} -1 & 5 & -4 & 2 & 1 & -\frac{1}{12} & 0 \\ 0 & 1 & 0 & 5 & \frac{13}{12} & -\frac{1}{12} & A_{2,7} \\ 1 & 1 & 10 & \frac{17}{6} & \frac{5}{6} & A_{3,6} & A_{3,7} \\ 2 & 11 & \frac{3}{2} & \frac{9}{2} & A_{4,5} & A_{4,6} & A_{4,7} \\ -2 & 5 & 6 & A_{5,4} & A_{5,5} & A_{5,6} & A_{5,7} \\ 3 & 35 & A_{6,3} & A_{6,4} & A_{6,5} & A_{6,6} & A_{6,7} \end{pmatrix}$$

(%i5) dd_table([−1, 0, 1, 2], [5, 1, 1, 11]);

(%o5)
$$\begin{pmatrix} -1 & 5 & -4 & 2 & 1 \\ 0 & 1 & 0 & 5 & \frac{13}{12} \\ 1 & 1 & 10 & \frac{17}{6} & \frac{5}{6} \\ 2 & 11 & \frac{3}{2} & \frac{9}{2} & A_{4,5} \end{pmatrix}$$

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Newton form of the interpolating polynomial

Figure 3:

and, hence, the Newton form of the interpolating polynomial is

$$P_{0,1,2,\ldots,n}(x) = \sum_{k=0}^{n} f[x_0, x_1, x_2, \ldots, x_k] \left(\prod_{i=0}^{k-1} (x - x_i) \right) .$$

4

Newton form of the interpolating polynomial

(%i6) kill(all);

(%o0) *done*

```
(%i1) P(a,  f,  x):=block(


        n:length(a),
        for i:1 thru n do A[i, 1]:a[i],
        for i:1 thru n do A[i, 2]:f[i],
        s:0,
        for j:3 thru n+1 do
        (
            for i:1 thru n+2−j do A[i,  j]:(A[i+1,  j−1]−A[i,  j−1])/(A[i+1+s,  1]−A[i,
            s:s+1
        ),
        table:genmatrix(A,  n,  n+1),

        sum:table[1,  2],
        for i:3 thru n+1 do
        (
            sum:sum+( table[1,  i]·product((x−table[k,  1]),  k,   1,   i−2)   )
        ),
        return(sum)
    );
```

(%o1) $P(a,f,x):=\mathrm{block}\left(n:\mathrm{length}(a),\mathrm{for}\ i\ \mathrm{thru}\ n\ \mathrm{do}\ A_{i,1}:a_i,\mathrm{for}\ i\ \mathrm{thru}\ n\ \mathrm{do}\ A_{i,2}:f_i,s:0,\mathrm{for}\ j\ \mathrm{from}\ 3\ \mathrm{thru}\ n+1\ \mathrm{do}\ \left(\mathrm{for}\ i\ \mathrm{thru}\ n+2-j\ \mathrm{do}\ A_{i,j}:\frac{A_{i+1,j-1}-A_{i,j-1}}{A_{i+1+s,1}-A_{i,1}},s:s+1\right),table:\mathrm{genmatrix}(A,n,n+1),sum:table_{1,2},\mathrm{for}\ i\ \mathrm{from}\ 3\ \mathrm{thru}\ n+1\ \mathrm{do}\ sum:sum+table_{1,i}\prod_{k=1}^{i-2}x-table_{k,1},\mathrm{return}(sum)\right)$

4.1

(%i2) P([−1, 0, 1, 2], [5, 1, 1, 11], x);

(%o2) $(x-1)\,x\,(x+1)+2\,x\,(x+1)-4\,(x+1)+5$

(%i3) P([−1, 0, 1, 2], [5, 1, 1, 11], 1.5);

(%o3) 4.375

4.2

(%i4) P([−1, 0, 1, 2, −2, 3], [5, 1, 1, 11, 5, 35], x);

(%o4) $$-\frac{(x-2)(x-1)x(x+1)}{12}+(x-1)x(x+1)+2x(x+1)-4(x+1)+5$$

(%i5) P([−1, 0, 1, 2, −2, 3], [5, 1, 1, 11, 5, 35], 1.5);

(%o5) 4.453125

4.3

4

(%i6) P([0, 1, 2], [2, −1, 4], x);

(%o6) $$4(x-1)x-3x+2$$

4.4

5

(%i7) P([−1, 0, 1, 2], [3, 1, −3, 1], x);

(%o7) $$\frac{5(x-1)x(x+1)}{3}-x(x+1)-2(x+1)+3$$

4.5

6

(%i8) P([−7, −5, −4, −1], [10, 5, 2, 10], x);

(%o8) $$\frac{19(x+4)(x+5)(x+7)}{72}-\frac{(x+5)(x+7)}{6}-\frac{5(x+7)}{2}+10$$

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Exercise

Figure 4:

4. Construct the divided difference table for the following data set, and then write out the Newton form of the interpolating polynomial.

x	0	1	2
y	2	-1	4

5. Construct the divided difference table for the following data set, and then write out the Newton form of the interpolating polynomial.

x	-1	0	1	2
y	3	-1	-3	1

6. Construct the divided difference table for the following data set, and then write out the Newton form of the interpolating polynomial.

x	-7	-5	-4	-1
y	10	5	2	10

7. Write out the Newton form of the interpolating polynomial for $f(x) = \ln x$ that passes through the points $(1, \ln 1)$, $(2, \ln 2)$, and $(3, \ln 3)$.

8. Write out the Newton form of the interpolating polynomial for $f(x) = \sin x$ that passes through the points $(0, \sin 0)$, $(\pi/4, \sin \pi/4)$, and $(\pi/2, \sin \pi/2)$.

9. Write out the Newton form of the interpolating polynomial for $f(x) = e^x$ that passes through the points $(-1, e^{-1})$, $(0, e^0)$, and $(1, e^1)$.