Practical 11 : Trapezoidal Rule

1

1.1

Basic Trapezoidal Rule

Figure 1:

Therefore, the closed Newton-Cotes quadrature formula corresponding to $n = 1$ is

$$I(f) \approx I_{1,\text{closed}}(f) = \frac{\Delta x}{2}[f(a) + f(b)] = \frac{b-a}{2}[f(a) + f(b)].$$

Geometrically, this quadrature rule approximates the value of the definite integral as the area of a trapezoid (see Figure 6.8); hence, this rule is known as the trapezoidal rule.

1.2

Figure 2:

1. Approximate the value of each of the following integrals using the trapezoidal rule. Verify that the theoretical error bound holds in each case.
   (a) $\int_1^2 \frac{1}{x}\,dx$
   (b) $\int_0^1 e^{-x}\,dx$
   (c) $\int_0^1 \frac{1}{1+x^2}\,dx$
   (d) $\int_0^1 \tan^{-1} x\,dx$.

1.3

```
(%i1) kill(all);
(%o0) done
```

```
(%i1) T(g,  a,  b):=block
(
    define(f(x),  g),
    return(((b−a)/2)·(f(a)+f(b)))
);
```

(%o1) $T(g,a,b):=\text{block}\left(\text{define}(f(x),g),\text{return}\left(\frac{b-a}{2}(f(a)+f(b))\right)\right)$

1.4

(%i2) ratprint:false;

(%o2) ***false***

```
(%i8) f1(x):=1/x;
      a:1.0;
      b:2.0;
      t:T(f1(x),    a,    b);
      v:integrate(f1(x),    x,    a,   b);
      e:abs(v−t);
```

(%o3) $\mathrm{f1}(x):=\frac{1}{x}$

(%o4) 1.0

(%o5) 2.0

(%o6) 0.75

(%o7) 0.6931471805599453

(%o8) 0.05685281944005471

1.5

```
(%i14) f1(x):=exp(−x);
       a:0.0;
       b:1.0;
       t:T(f1(x),    a,    b);
       v:integrate(f1(x),    x,    a,   b);
       e:abs(v−t);
```

(%o9) $\mathrm{f1}(x):=\exp(-x)$

(%o10) 0.0

(%o11) 1.0

(%o12) 0.6839397205857212

(%o13) $\frac{17854706}{28245729}$

(%o14) 0.05181916175716339

1.6

```
(%i20) f1(x):=1/(1+x^2);
       a:0.0;
       b:1.0;
       t:T(f1(x),   a,   b);
       v:integrate(f1(x),   x,   a,   b);
       e:abs(v−t);
```

(%o15) $f1(x) := \frac{1}{1+x^2}$

(%o16) 0.0

(%o17) 1.0

(%o18) 0.75

(%o19) 0.7853981633974483

(%o20) 0.03539816339744828

1.7

```
(%i26) f1(x):=atan(x);
       a:0.0;
       b:1.0;
       t:T(f1(x),   a,   b);
       v:integrate(f1(x),   x,   a,   b);
       e:abs(v−t);
```

(%o21) $f1(x) := \operatorname{atan}(x)$

(%o22) 0.0

(%o23) 1.0

(%o24) 0.3926990816987241

(%o25) 0.4388245731174756

(%o26) 0.04612549141875149

1.8

Exercise

Do the above for the following

Figure 3:

6. $\int_1^2 \frac{1}{x}\ dx$

7. $\int_0^1 e^{-x}\ dx$

8. $\int_0^1 \tan^{-1} x\ dx$

9. $\int_1^2 \frac{\sin x}{x}\ dx$

10. $\int_0^1 \frac{1}{\sqrt{1+x^4}}\ dx$

2

2.1

Composite Trapezoidal Rule

Figure 4:

If the integration interval $[a, b]$ is split into n subintervals by defining $h = (b-a)/n$ and $x_j = a + jh$, $0 \le j \le n$, and then the trapezoidal rule formula is applied on each subinterval $[x_{j-1}, x_j]$, we obtain

$$\begin{aligned} I(f) &= \sum_{j=1}^{n} \int_{x_{j-1}}^{x_j} f(x)\ dx \\ &= \sum_{j=1}^{n} \frac{x_j - x_{j-1}}{2}\left[f(x_{j-1}) + f(x_j)\right] - \sum_{j=1}^{n} \frac{(x_j - x_{j-1})^3}{12} f''(\xi_j) \\ &= \underbrace{\frac{h}{2}\left[f(x_0) + 2\sum_{j=1}^{n-1} f(x_j) + f(x_n)\right]}_{\text{composite trapezoidal rule}} - \underbrace{\frac{h^3}{12}\sum_{j=1}^{n} f''(\xi_j)}_{\text{error}} \end{aligned}$$

where, for each j, $x_{j-1} < \xi_j < x_j$.

2.2

(%i27)
```
Tc(g1, a, b, n):=block
(
    define(f(x), g1),
    h:(b−a)/n,
    s:0.0,
    for i:1 thru n−1 do s:s+f(a+i·h),
    ans:(h/2)·(f(a)+2·s+f(b)),
    return(float(ans))
);
```

(%o27) $\mathrm{Tc}(g1,a,b,n):=\mathrm{block}\left(\mathrm{define}(\mathrm{f}(x),g1),h:\frac{b-a}{n},s:0.0,\mathrm{for}\ i\ \mathrm{thru}\ n-1\ \mathrm{do}\ s:s+\right.$

2.3

Figure 5: /home/jvr/Downloads/na_29_june_22/myTest/p11_trap_qn.png

Consider the integral

$$I(f) = \int_0^\pi \sin x \, dx,$$

(%i28)
```
Tc(sin(x), 0, %pi, 2);
```

(%o28) 1.570796326794896

(%i29)
```
num:makelist(2^j, j, 0, 7);
```

(%o29) [1,2,4,8,16,32,64,128]

(%i30)
```
for k in num do block(
    temp:Tc(sin(x), 0, %pi, k),
    print(k, "      ", temp)
    );
```

```
1       0.0
2       1.570796326794896
4       1.896118897937039
8       1.97423160194555
16        1.993570343772338
32        1.998393360970144
64        1.999598388640037
128        1.999899600184203
```

(%o30) *done*

check

Figure 6:

n	h	$T_h(f)$	$e_h = \lvert I(f) - T_h(f)\rvert$	e_{2h}/e_h
1	π	0.0000000	2.0000000	
2	$\frac{\pi}{2}$	1.5707963	0.4292036	4.659792
4	$\frac{\pi}{4}$	1.8961188	0.1038811	4.131681
8	$\frac{\pi}{8}$	1.9742316	0.0257683	4.031337
16	$\frac{\pi}{16}$	1.9935703	0.0064296	4.007741
32	$\frac{\pi}{32}$	1.9983933	0.0016066	4.001929
64	$\frac{\pi}{64}$	1.9995983	0.0004016	4.000482
128	$\frac{\pi}{128}$	1.9998996	0.0001004	4.000120

2.4

Figure 7:

Consider the definite integral

$$I(f) = \int_0^1 \sqrt{1+x^3}\, dx.$$

(%i31) ;

```
for k in num do block(
    temp:Tc(sqrt(1+x^3),   0.0,     1.0,   k),
    print(k, "      ", temp)
    );
```

```
1        1.207106781186547
2        1.133883476483184
4        1.116993293318717
8        1.112830349496382
16        1.111793319381881
32        1.111534292393827
64        1.111469550038714
128        1.111453365349166
```

(%o31) *done*

check

Figure 8:

n	h	$T_h(f)$	$\frac{T_h(f)-T_{h/2}(f)}{T_{h/2}(f)-T_{h/4}(f)}$
1	1	1.207106781186	4.335258
2	$\frac{1}{2}$	1.133883476483	4.057269
4	$\frac{1}{4}$	1.116993293318	4.014294
8	$\frac{1}{8}$	1.112830349496	4.003560
16	$\frac{1}{16}$	1.111793319381	4.000889
32	$\frac{1}{32}$	1.111534292393	4.000222
64	$\frac{1}{64}$	1.111469550038	
128	$\frac{1}{128}$	1.111453365349	

3

Exercise

Do the above for the following

Figure 9:

6. $\int_1^2 \frac{1}{x}\, dx$

7. $\int_0^1 e^{-x}\, dx$

8. $\int_0^1 \tan^{-1} x\, dx$

9. $\int_1^2 \frac{\sin x}{x}\, dx$

10. $\int_0^1 \frac{1}{\sqrt{1+x^4}}\, dx$