Practical 8 : Gauss-Seidel Method

1

1.1

Gauss_seidel Method

Figure 1:

An obvious improvement that can be made to the Jacobi method is to use the value of $x_i^{(k+1)}$ as soon as it has been calculated in the computation of all subsequent entries in the vector $\mathbf{x}^{(k+1)}$, rather than waiting until the next iteration. After all, $x_i^{(k+1)}$ is supposed to be a better approximation to x_i than $x_i^{(k)}$. This modification amounts to changing equation (3) to

$$x_i^{(k+1)} = \frac{1}{a_{i,i}}\left[b_i - \sum_{j=1}^{i-1} a_{i,j}x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{i,j}x_j^{(k)}\right]; \qquad (4)$$

Figure 2:

the only difference between the equations is that the superscript on x in the first summation is now $k+1$. The iteration scheme corresponding to equation (4) is known as the Gauss-Seidel method. Note that the Gauss-Seidel method is not vectorizable. The entries in $\mathbf{x}^{(k+1)}$ must be computed in succession. Hence, the Gauss-Seidel method is also known as Successive Relaxation.

Figure 3:

Working backward from equation (4), we find that the splitting upon which the Gauss-Seidel method is based is

$$M = D - L \quad \text{and} \quad N = U.$$

Thus, the iteration matrix for the Gauss-Seidel method is given by

$$T_{gs} = (D-L)^{-1}U,$$

and the vector $\mathbf{c}$ is given by

$$\mathbf{c}_{gs} = (D-L)^{-1}\mathbf{b}.$$

The necessary and sufficient condition for the matrix M to be nonsingular is the same as above: for each i, we must have $d_{i,i} \equiv a_{i,i} \neq 0$.

1.2

Figure 4:

Reconsider the system of equations

$$\begin{array}{rcrcrcr} 5x_1 & + & x_2 & + & 2x_3 & = & 10 \\ -3x_1 & + & 9x_2 & + & 4x_3 & = & -14 \\ x_1 & + & 2x_2 & - & 7x_3 & = & -33\,. \end{array}$$

The Gauss-Seidel method, when applied to this system, will produce the sequence of approximations $\{\mathbf{x}^{(k)}\}$ according to the rules

$$\begin{aligned} x_1^{(k+1)} &= \frac{1}{5}\left[10 - x_2^{(k)} - 2x_3^{(k)}\right] \\ x_2^{(k+1)} &= \frac{1}{9}\left[-14 + 3x_1^{(k+1)} - 4x_3^{(k)}\right] \\ x_3^{(k+1)} &= -\frac{1}{7}\left[-33 - x_1^{(k+1)} - 2x_2^{(k+1)}\right]\,. \end{aligned}$$

1.3

function for Gauss-Seidel method

```
kill(all);
```

(%o0) *done*

```
→   gauss_seidel(A,  b,  iter):=block(
      n:matrix_size(A)[1],
      for i:1 thru n do
      (
        for j:1 thru n do
        (
          if(i > j) then (l[i,  j]:A[i,  j])
          else ( l[i,  j]:0)
        )
      ),
      L:genmatrix(l,  n,  n),

      for i:1 thru n do
      (
        for j:1 thru n do
        (
          if(i < j) then (u[i,  j]:A[i,  j])
          else ( u[i,  j]:0)
        )
      ),
      U:genmatrix(u,  n,  n),

      for i:1 thru n do
      (
        for j:1 thru n do
        (
          if(i = j) then (d[i,  j]:A[i,  j])
          else ( d[i,  j]:0)
        )
      ),
      D:genmatrix(d,  n,  n),

      D2:invert(D+L),
      Tgs:D2.(−U),
      cgs:D2.b,

      for i:1 thru n do t[i,  1]:0.0,
      x0:genmatrix(t,  n,  1),

      for i:1 thru iter do
      (
        x1:Tgs.x0+cgs,
        disp(i, "−−−−", transpose(x1)),
        x0:x1
      )
    );
```

(%o1) gauss_seidel $(A,b,iter):=$ block $(n:(\text{matrix\_size}(A))_1,$ for i thru n do for j thru n do if $i>j$ then $l_{i,j}:A_{i,j}$ else $l_{i,j}:0,L:$ genmatrix $(l,n,n),$ for i thru n do for j thru n do if $i<j$ then

Note :

```
disp(i, "----",  x1)
```

can be used for printing as col vectors

```
A1:matrix([5, 1, 2], [−3, 9, 4], [1, 2, −7]);
b1:matrix([10], [−14], [−33]);
gauss_seidel(A1, b1, 10);
```

(%o2) $\begin{pmatrix} 5 & 1 & 2 \\ -3 & 9 & 4 \\ 1 & 2 & -7 \end{pmatrix}$

(%o3) $\begin{pmatrix} 10 \\ -14 \\ -33 \end{pmatrix}$

1
\----
$\begin{pmatrix} 2.0 & -0.8888888888888888 & 4.746031746031746 \end{pmatrix}$
2
\----
$\begin{pmatrix} 0.2793650793650792 & -3.571781305114638 & 3.7336860670194 \end{pmatrix}$
3
\----
$\begin{pmatrix} 1.220881834215167 & -2.808010973936899 & 4.086408555191624 \end{pmatrix}$
4
\----
$\begin{pmatrix} 0.9270387727107303 & -3.062724211403811 & 3.971655764271872 \end{pmatrix}$
5
\----
$\begin{pmatrix} 1.023882536572013 & -2.979441716374605 & 4.0092855862604 \end{pmatrix}$
6
\----
$\begin{pmatrix} 0.992174108770761 & -3.006735557636591 & 3.996957570499654 \end{pmatrix}$
7
\----
$\begin{pmatrix} 1.002564083327456 & -2.997793114668471 & 4.000996836284359 \end{pmatrix}$
8
\----
$\begin{pmatrix} 0.9991598884199506 & -3.000723075541953 & 3.999673391048006 \end{pmatrix}$
9
\----
$\begin{pmatrix} 1.000275258689188 & -2.999763087569384 & 4.000107011935774 \end{pmatrix}$
10
\----
$\begin{pmatrix} 0.9999098127395674 & -3.00007762328048 & 3.999964938025513 \end{pmatrix}$

(%o4) *done*

1.4

check

Figure 5:

k	$\mathbf{x}^{(k)}$		
0	$[\ 0.000000$	0.000000	$0.000000\ ]^T$
1	$[\ 2.000000$	-0.888889	$4.746032\ ]^T$
2	$[\ 0.279365$	-3.571781	$3.733686\ ]^T$
3	$[\ 1.220882$	-2.808011	$4.086409\ ]^T$
4	$[\ 0.927039$	-3.062724	$3.971656\ ]^T$
5	$[\ 1.023883$	-2.979442	$4.009286\ ]^T$
6	$[\ 0.992174$	-3.006736	$3.996958\ ]^T$
7	$[\ 1.002564$	-2.997793	$4.000997\ ]^T$
8	$[\ 0.999160$	-3.000723	$3.999673\ ]^T$
9	$[\ 1.000275$	-2.999763	$4.000107\ ]^T$
10	$[\ 0.999910$	-3.000078	$3.999965\ ]^T$

1.5

function for Gauss-Seidel method

```
(%i5) kill(all);
(%o0) done
```

```
(%i1) gauss_seidel(A,  b,  init,  iter):=block(
        n:matrix_size(A)[1],
        for i:1 thru n do
        (
          for j:1 thru n do
          (
            if(i > j) then (l[i,  j]:A[i,  j])
            else ( l[i,  j]:0)
          )
        ),
        L:genmatrix(l,  n,  n),

        for i:1 thru n do
        (
          for j:1 thru n do
          (
            if(i < j) then (u[i,  j]:A[i,  j])
            else ( u[i,  j]:0)
          )
        ),
        U:genmatrix(u,  n,  n),

        for i:1 thru n do
        (
          for j:1 thru n do
          (
            if(i = j) then (d[i,  j]:A[i,  j])
            else ( d[i,  j]:0)
          )
        ),
        D:genmatrix(d,  n,  n),

        D2:invert(D+L),
        Tgs:D2.(−U),
        cgs:D2.b,


        x0:init,
        for i:1 thru iter do
        (
          x1:Tgs.x0+cgs,
          disp(i, "−−−−", transpose(x1)),
          x0:x1
        )
      );
```

(%o1) gauss_seidel$(A,b,init,iter)$:=block$(n$:(matrix_size$(A))_1$, for i thru n do for j thru n do if $i>j$ then $l_{i,j}:A_{i,j}$ else $l_{i,j}:0$, L :genmatrix(l,n,n), for i thru n do for j thru n do if $i<j$ then $u_{i,j}:A_{i,j}$ else $u_{i,j}:0$, U:genmatrix(u,n,n), for i thru n do for j

Note :

```
disp(i, "----",  x1)
```

can be used for printing as col vectors

(%i5)
```
A1:matrix([5, 1, 2], [-3, 9, 4],[1, 2, -7]);
b1:matrix([10], [-14], [-33]);
start:matrix([0.0], [0.0], [0.0]);
gauss_seidel(A1, b1, start, 10);
```

(%o2) $$\begin{pmatrix} 5 & 1 & 2 \\ -3 & 9 & 4 \\ 1 & 2 & -7 \end{pmatrix}$$

(%o3) $$\begin{pmatrix} 10 \\ -14 \\ -33 \end{pmatrix}$$

(%o4) $$\begin{pmatrix} 0.0 \\ 0.0 \\ 0.0 \end{pmatrix}$$

1

$\begin{pmatrix} 2.0 & -0.8888888888888888 & 4.746031746031746 \end{pmatrix}$
2

$\begin{pmatrix} 0.2793650793650792 & -3.571781305114638 & 3.7336860670194 \end{pmatrix}$
3

$\begin{pmatrix} 1.220881834215167 & -2.808010973936899 & 4.086408555191624 \end{pmatrix}$
4

$\begin{pmatrix} 0.9270387727107303 & -3.062724211403811 & 3.971655764271872 \end{pmatrix}$
5

$\begin{pmatrix} 1.023882536572013 & -2.979441716374605 & 4.0092855862604 \end{pmatrix}$
6

$\begin{pmatrix} 0.992174108770761 & -3.006735557636591 & 3.996957570499654 \end{pmatrix}$
7

$\begin{pmatrix} 1.002564083327456 & -2.997793114668471 & 4.000996836284359 \end{pmatrix}$
8

$\begin{pmatrix} 0.9991598884199506 & -3.000723075541953 & 3.999673391048006 \end{pmatrix}$
9

$\begin{pmatrix} 1.000275258689188 & -2.99976308756938 4 & 4.000107011935774 \end{pmatrix}$
10

2

Exercise

Starting with initial vector x0 = 0, perform 6 iterations of Gauss-Seidel method

Figure 6:

(a) $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 4 & 2 \\ 0 & 2 & 6 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix}$

(b) $\begin{bmatrix} 3 & -1 & 1 \\ 2 & -6 & 3 \\ -9 & 7 & -20 \end{bmatrix}, \quad \begin{bmatrix} 4 \\ -13 \\ 7 \end{bmatrix}$

(c) $\begin{bmatrix} 4 & 2 & -1 \\ 2 & 4 & 1 \\ -1 & 1 & 4 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

(d) $\begin{bmatrix} 4 & -1 & 0 & 0 \\ 2 & 4 & -1 & 0 \\ 0 & -2 & 4 & -1 \\ 0 & 0 & -2 & 4 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 2 \\ -3 \\ 1 \end{bmatrix}$