# Practical 13 : Euler method

## *1*

The Method

Figure 1:

**EULER'S METHOD**

Consider the scalar, first-order initial value problem:

$$\begin{aligned} y'(t) &= f(t, y(t)), \quad a \le t \le b \\ y(a) &= \alpha. \end{aligned} \tag{1}$$

Our objective is to determine a numerical approximation $w \approx y$, where $y(t)$ is the exact solution of (1). As indicated in Section 7.1, we will determine values of $w$ at the discrete set of points

$$a = t_0 < t_1 < t_2 < \cdots < t_{N-1} < t_N = b,$$

only, and we will adopt the notational convention that $w_i$ represents the approximation to $y_i = y(t_i)$. For simplicity, the approximate solution will be sought at

Figure 2:

equally spaced points; that is, for some positive integer $N$, we will define the step size

$$h = (b-a)/N,$$

and then the $t_i$ will be given by

$$t_i = a + ih \quad (i = 0, 1, 2, \ldots, N).$$

Derivation

Figure 3:

**Derivation of Method**

Euler's method is the simplest of the one-step methods for approximating the solution to the initial value problem (1). The derivation of the method begins by assuming that the true solution of (1), $y(t)$, has two continuous derivatives. Expanding this true solution in a Taylor series about the point $t = t_i$ produces

$$y(t) = y_i + (t - t_i)y_i' + \frac{1}{2}(t - t_i)^2 y''(\xi),$$

where $\xi$ is guaranteed to lie between $t$ and $t_i$. Evaluating the above Taylor expansion at $t = t_{i+1}$ and substituting for $y_i'$ from the right-hand side of the differential equation, we obtain

$$y_{i+1} = y_i + hf(t_i, y_i) + \frac{1}{2}h^2 y''(\xi).$$

Euler's method arises by dropping the error term and replacing $y_i$ (exact solution) by $w_i$ (approximate solution):

$$\begin{aligned} w_0 &= \alpha \\ w_{i+1} &= w_i + hf(t_i, w_i) \qquad i = 0, 1, 2, \ldots, N - 1. \end{aligned} \tag{2}$$

## 2

Example

Figure 4:

The initial value problem

$$\frac{dx}{dt} = 1 + \frac{x}{t}, \quad 1 \le t \le 6$$
$$x(1) = 1$$

has as its exact solution $x(t) = t(1 + \ln t)$. For this problem, $f(t, x)$ is given by $f(t, x) = 1 + x/t$, so that the Euler's method difference equation takes the form

$$w_0 = 1$$
$$w_{i+1} = w_i + h\left(1 + \frac{w_i}{t_i}\right).$$

Take step size h = 0.5

```
kill(all);
```

(%o0) *done*

```
w:1.0;
f(t, x):=1+(x/t);
x(t):=t·(1+log(t));
h:0.5;
t:1;
disp("t                              w                        exact solution                        | x(ti)−wi |");
while( t <= 6) do
(
   print(t, "          ",  w, "            ",  x(t), "         ",  abs(x(t)−w)),
   w:w+(h·f(t,  w)),
   t:t+h
);
```

(%o8) 1.0

(%o9) $f(t,x):=1+\frac{x}{t}$

(%o10) $x(t):=t\,(1+\log(t))$

(%o11) 0.5

(%o12) 1

*t* *w* *exact solution* *| x(ti)−wi |*

(%o13) *done*

```
1           1.0           1          0.0
1.5           2.0           2.108197662162246
0.1081976621622464
2.0           3.166666666666666           3.386294361119891
         0.2196276944532242
2.5           4.458333333333333           4.790726829685388
         0.3323934963520552
3.0           5.85           6.295836866004330
0.4458368660043303
3.5           7.324999999999999           7.884670389733788
         0.5596703897337889
4.0           8.87142857142857           9.545177444479564
         0.6737488730509931
4.5           10.48035714285714           11.26834828549323
         0.787991142636093
5.0           12.14484126984126           13.0471895621705
         0.9023482923292345
5.5           13.85932539682539           14.87611450731133
         1.016789110485945
6.0           15.61926406926406           16.75055681536833
         1.131292746104264
```

(%o14) *done*

check

Figure 5:

| $t$ | Approximate Solution | Exact Solution | $\lvert y(t_i) - w_i \rvert$ |
|---|---|---|---|
| 1.0 | 1.00000000 | 1.00000000 | |
| 1.5 | 2.00000000 | 2.10819766 | 0.108198 |
| 2.0 | 3.16666667 | 3.38629436 | 0.219628 |
| 2.5 | 4.45833333 | 4.79072683 | 0.332393 |
| 3.0 | 5.85000000 | 6.29583687 | 0.445837 |
| 3.5 | 7.32500000 | 7.88467039 | 0.559670 |
| 4.0 | 8.87142857 | 9.54517744 | 0.673749 |
| 4.5 | 10.48035714 | 11.26834829 | 0.787991 |
| 5.0 | 12.14484127 | 13.04718956 | 0.902348 |
| 5.5 | 13.85932540 | 14.87611451 | 1.016789 |
| 6.0 | 15.61926407 | 16.75055682 | 1.131293 |

# 3

Example

Figure 6:

The initial value problem

$$\frac{dx}{dt} = \frac{t}{x}, \quad 0 \le t \le 5$$
$$x(0) = 1$$

has as its exact solution $x(t) = \sqrt{t^2+1}$. Here the Euler's method difference equation takes the form

$$w_0 = 1$$
$$w_{i+1} = w_i + h\frac{t_i}{w_i}.$$

Take step size h = 0.5

```
kill(all);
```

(%o0) *done*

```
→   w:1.0;
    f(t, x):=(t/x);
    x(t):=sqrt(t^2+1);
    h:0.5;
    t:0.0;
    disp("t                    w               exact solution              | x(ti)−wi |");
    while( t <= 5) do
    (
      print(t, "        ",  w, "          ",  x(t), "        ",  abs(x(t)−w)),
      w:w+(h·f(t, w)),
      t:t+h
    );
```

(%o1) 1.0

(%o2) $f(t,x):=\frac{t}{x}$

(%o3) $x(t):=\sqrt{t^2+1}$

(%o4) 0.5

(%o5) 0.0

*t* *w* *exact solution* *| x(ti)−wi |*

(%o6) *done*

```
0.0          1.0          1.0          0.0
0.5          1.0          1.118033988749895
0.1180339887498949
1.0          1.25          1.414213562373095
0.1642135623730951
1.5          1.65          1.802775637731994
0.1527756377319946
2.0          2.104545454545454          2.23606797749979
     0.131522529543353
2.5          2.579707441586491          2.692582403567252
     0.1128749619807605
3.0          3.064258511157103          3.162277660168379
     0.09801914901127651
3.5          3.553773346324708          3.640054944640259
     0.08628159831555094
4.0          4.046207677233808          4.123105625617661
     0.07689794838385 3
4.5          4.54049767901822          4.609772228646443
     0.06927454962822388
5.0          5.036038071076412          5.099019513592784
     0.06298144251637261
```

(%o7) *done*

check

Figure 7:

| $t$ | Approximate Solution | Exact Solution | $\lvert y(t_i) - w_i \rvert$ |
|---|---|---|---|
| 0.0 | 1.00000000 | 1.00000000 | |
| 0.5 | 1.00000000 | 1.11803399 | 0.118034 |
| 1.0 | 1.25000000 | 1.41421356 | 0.164214 |
| 1.5 | 1.65000000 | 1.80277564 | 0.152776 |
| 2.0 | 2.10454545 | 2.23606798 | 0.131523 |
| 2.5 | 2.57970744 | 2.69258240 | 0.112875 |
| 3.0 | 3.06425851 | 3.16227766 | 0.098019 |
| 3.5 | 3.55377335 | 3.64005494 | 0.086282 |
| 4.0 | 4.04620768 | 4.12310563 | 0.076898 |
| 4.5 | 4.54049768 | 4.60977223 | 0.069275 |
| 5.0 | 5.03603807 | 5.09901951 | 0.062981 |

# *4*

```
(%i1) kill(all);
(%o0) done
```

Function for Euler Method

```
(%i1) eulerm(g, a, x0, b, N):=block
(
   [t, x, h, t0],
   define(f(t, x), g),
   t0:a,
   h:(b−a)/N,
   x:x0,
   while(t0<=b) do
   (
      print(t0, "−−−− ", x),
      x:x+(h·f(t0, x)),
      t0:t0+h
   )
);
```

(%o1) eulerm $(g,a,x0,b,N):=$ block$([t,x,h,t0],$ define $(f(t,x),g),t0:a,h:\frac{b-a}{N},x:x0,$

```
(%i2) eulerm(1+(x/t), 1.0, 1.0, 6, 10);
1.0 ---- 1.0
1.5 ---- 2.0
2.0 ---- 3.166666666666666
2.5 ---- 4.458333333333333
3.0 ---- 5.85
3.5 ---- 7.324999999999999
4.0 ---- 8.87142857142857
4.5 ---- 10.48035714285714
5.0 ---- 12.14484126984126
5.5 ---- 13.85932539682539
6.0 ---- 15.61926406926406
(%o2) done

(%i3) eulerm((t/x), 0.0, 1.0, 5, 10);
0.0 ---- 1.0
0.5 ---- 1.0
1.0 ---- 1.25
1.5 ---- 1.65
2.0 ---- 2.104545454545454
2.5 ---- 2.579707441586491
3.0 ---- 3.064258511157103
3.5 ---- 3.553773346324708
4.0 ---- 4.046207677233808
4.5 ---- 4.54049767901822
5.0 ---- 5.036038071076412
(%o3) done

(%i4) eulerm(t·x^3−x, 0.0, 1.0, 1, 4);
0.0 ---- 1.0
0.25 ---- 0.75
0.5 ---- 0.5888671875
0.75 ---- 0.4671751748537645
1.0 ---- 0.3694992968508541
(%o4) done
```

(%i5) eulerm(t·y^3−y, 0.0, 1.0, 1, 4);

0.0 ---- 1.0

0.25 ---- $1.0-0.25\,y$

0.5 ---- $0.25\,(0.25\,y^3-y)-0.25\,y+1.0$

0.75 ---- $0.25\,(0.5\,y^3-y)+0.25\,(0.25\,y^3-y)-0.25\,y+1.0$

1.0 ---- $0.25\,(0.75\,y^3-y)+0.25\,(0.5\,y^3-y)+0.25\,(0.25\,y^3-y)-0.25\,y+1.0$

(%o5) *done*

f must be given in t and x only.

## 5

Exercise

Figure 8:

For Exercises 1–6, apply Euler's method to approximate the solution of the given initial value problem over the indicated interval in $t$ using the indicated number of time steps.

1. $x' = tx^3 - x \quad (0 \le t \le 1), \quad x(0) = 1, \quad N = 4$
2. $x' + (4/t)x = t^4 \quad (1 \le t \le 3), \quad x(1) = 1, \quad N = 5$
3. $x' = (\sin x - e^t)/\cos x \quad (0 \le t \le 1), \quad x(0) = 0, \quad N = 3$
4. $x' = (1 + x^2)/t \quad (1 \le t \le 4), \quad x(1) = 0, \quad N = 5$
5. $x' = t^2 - 2x^2 - 1 \quad (0 \le t \le 1), \quad x(0) = 0, \quad N = 4$
6. $x' = 2(1 - x)/(t^2 \sin x) \quad (1 \le t \le 2), \quad x(1) = 2, \quad N = 3$

For Exercises 7–10, apply Euler's method to approximate the solution of the given initial value problem over the indicated interval in $t$ using the indicated number of time steps. Compare the approximate solution with the given exact solution, and compare the actual error with the theoretical error bound. When determining the Lipschitz constant, consider the indicated set $D$.

7. $x' = e^t/x \quad (0 \le t \le 2), \quad x(0) = 1, \quad N = 4, \quad x(t) = \sqrt{2e^t - 1},$
   $D = \{(t, x) |\ 0 \le t \le 2,\ x \ge 1\}$
8. $x' = -t\tan x/(1 + t^2) \quad (0 \le t \le 1), \quad x(0) = \pi/4, \quad N = 4,$
   $x(t) = \sin^{-1}\sqrt{(2 + 2t^2)^{-1}}, \quad D = \{(t, x) |\ 0 \le t \le 1,\ 0 \le x \le \pi/4\}$
9. $x' = t - x \quad (0 \le t \le 4), \quad x(0) = 1, \quad N = 4, \quad x(t) = 2e^{-t} + t - 1,$
   $D = \{(t, x) |\ 0 \le t \le 4,\ x \in R\}$
10. $x' = x - t \quad (0 \le t \le 2), \quad x(0) = 2, \quad N = 4, \quad x(t) = e^t + t + 1,$
    $D = \{(t, x) |\ 0 \le t \le 4,\ x \in R\}$