Practical 5 : LU Decomposition

Method

1

1.1

Doolittle factorization

Figure 1:

integer i, k, n; **real array** $(a_{ij})_{1:n\times 1:n}, (\ell_{ij})_{1:n\times 1:n}, (u_{ij})_{1:n\times 1:n}$
for $k = 1$ **to** n **do**
    $\ell_{kk} \leftarrow 1$
    for $j = k$ **to** n **do**

$$u_{kj} \leftarrow a_{kj} - \sum_{s=1}^{k-1} \ell_{ks} u_{sj}$$

    end do
    for $i = k + 1$ **to** n **do**

$$\ell_{ik} \leftarrow \left(a_{ik} - \sum_{s=1}^{k-1} \ell_{is} u_{sk} \right) \bigg/ u_{kk}$$

    end do
end do

Figure 2:

integer i, n; **real array** $(b_i)_{1:n}, (\ell_{ij})_{1:n\times 1:n}, (z_i)_{1:n}$
$z_1 \leftarrow b_1$
for $i = 2$ **to** n **do**

$$z_i \leftarrow b_i - \sum_{j=1}^{i-1} \ell_{ij} z_j$$

end for

Figure 3:

integer i, n; **real array** $(u_{ij})_{1:n\times 1:n}, (x_i)_{1:n}, (z_i)_{1:n}$
$x_n \leftarrow z_n / u_{nn}$
for $i = n-1$ **to** 1 **step** -1 **do**
$$x_i \leftarrow \left(z_i - \sum_{j=i+1}^{n} u_{ij} x_j \right) \bigg/ u_{ii}$$
end for

1.2

Figure 4:

Solving Linear Systems Using *LU* Factorization

Once the $\boldsymbol{LU}$ factorization of $\boldsymbol{A}$ is available, we can solve the system
$$\boldsymbol{Ax} = \boldsymbol{b}$$
by writing
$$\boldsymbol{LUx} = \boldsymbol{b}$$
Then we solve two triangular systems:
$$\boldsymbol{Lz} = \boldsymbol{b}$$

Figure 5:

for z and
$$\boldsymbol{Ux} = \boldsymbol{z} \qquad (21)$$
for $\boldsymbol{x}$. This is particularly useful for problems that involve the same coefficient matrix $\boldsymbol{A}$ and many different right-hand vectors $\boldsymbol{b}$.

2

returns a matrix

```
kill(all);
```
(%o0) *done*

```
A:matrix([1, 2, 3, 4, 5], [6, 7, 8, 9, 10]);
```
(%o1) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \end{pmatrix}$

```
→ B:transpose(A);
```

(%o2) $\begin{pmatrix} 1 & 6 \\ 2 & 7 \\ 3 & 8 \\ 4 & 9 \\ 5 & 10 \end{pmatrix}$

a row matrix

```
→ C:matrix([1, 0, −1, 0]);
```

(%o3) $\begin{pmatrix} 1 & 0 & -1 & 0 \end{pmatrix}$

a column matrix

```
→ C1:transpose(C);
```

(%o4) $\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$

matrix multiplication

```
→ C.C1;
```

(%o5) 2

3

```
→ kill(all);
```

(%o0) *done*

```
→ a[i, j]:i+j;
```

(%o1) $j+i$

```
→ a[2, 3];
```

(%o2) $a_{2,3}$

```
→ b[i, j]:=i+j;
```

(%o3) $b_{i,j}:=i+j$

```
→ b[4, 7];
```

(%o4) 11

```
--> A:genmatrix(a, 2, 5);
```

(%o5) $$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & a_{2,5} \end{pmatrix}$$

```
--> B:genmatrix(b, 2, 5);
```

(%o6) $$\begin{pmatrix} 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \end{pmatrix}$$

```
--> c[i, j]:=0;
```

(%o7) $c_{i,j}:=0$

```
--> O:genmatrix(c, 2, 5);
```

(%o8) $$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

```
--> O1:transpose(O);
```

(%o9) $$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

4

```
--> kill(all);
```

(%o0) *done*

```
--> A:genmatrix(a, 2, 5, 1, 3);
```

(%o1) $$\begin{pmatrix} a_{1,3} & a_{1,4} & a_{1,5} \\ a_{2,3} & a_{2,4} & a_{2,5} \end{pmatrix}$$

```
--> A:genmatrix(a, 2, 5, 1);
```

(%o2) $$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & a_{2,5} \end{pmatrix}$$

```
--> A:genmatrix(a, 2, 5);
```

(%o3) $$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & a_{2,5} \end{pmatrix}$$

```
-->  A:genmatrix(a, 3, 3);
```

(%o4) $\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix}$

```
-->  kill(all);
```

(%o0) *done*

```
-->  b[i, j]:i+j;
```

(%o1) $j+i$

```
-->  b[1, 3];
```

(%o2) $b_{1,3}$

```
-->  for i:1 thru 3 do
     (
         for J:1 thru 3 do
         (
             a[i, j]:=0
         )
     );
```

(%o3) *done*

```
-->  a[1, 2];
```

(%o4) 0

5

```
-->  kill(all);
```

(%o0) *done*

```
-->  A:zeromatrix(2, 5);
```

(%o1) $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

```
-->  matrix_size(A);
```

(%o2) [2,5]

```
-->  matrix_size(A)[1];
```

(%o3) 2

```
-->  matrix_size(A)[2];
```

(%o4) 5

```
--> B:zeromatrix(1, 5);
```

(%o5) $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

```
--> C:zeromatrix(5, 1);
```

(%o6) $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

6

6.1

```
--> kill(all);
```

(%o0) *done*

```
➔ luFactor(A):=block
(
    [n,  L, U, i,  j,   k, ans],
    n:matrix_size(A)[1],
    L:zeromatrix(n,  n),
    U:zeromatrix(n,  n),
    for k:1 thru n do
    (
        L[k][k]:1,
        for j:k thru n do
        (
            U[k][j]:A[k][j]−(sum(L[k][s]·U[s][j],  s,  1,  k−1))
        ),
        for i:k+1 thru n do
        (
            L[i][k]:(A[i][k]−sum(L[i][s]·U[s][k],   s,  1,  k−1))/U[k][k]
        )
    ),
    ans:['L=L,  'U=U]

);
```

(%o1) luFactor(A):=block$([n,L,U,i,j,k,ans],n:$ $($matrix_size$(A))_1$, $L:$zeromatrix(n,n), $U:$zeromatrix(n,n), for k thru n do $((L_k)_k:1$, for j from k thru n do $(U_k)_j:(A_k)_j-$

$$\sum_{s=1}^{k-1}(L_k)_s(U_s)_j$$

, for i from $k+1$ thru n do $(L_i)_k:$

$$\frac{(A_i)_k-\sum_{s=1}^{k-1}(L_i)_s(U_s)_k}{(U_k)_k}$$

$),ans:['L=L,'U=U])$

6.2

Example

```
A:matrix([1, 2, 3], [3, 5, 7], [4, 6, 9]);
```

(%o2) $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 5 & 7 \\ 4 & 6 & 9 \end{pmatrix}$

```
luFactor(A);
```

(%o3) $\left[L=\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 4 & 2 & 1 \end{pmatrix}, U=\begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{pmatrix}\right]$

```
luFactor(A)[1];
```

(%o4) $L=\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 4 & 2 & 1 \end{pmatrix}$

```
B:matrix([3, 2, 1], [2, 5, 4], [5, 6, 8]);
```

(%o5) $\begin{pmatrix} 3 & 2 & 1 \\ 2 & 5 & 4 \\ 5 & 6 & 8 \end{pmatrix}$

```
luFactor(B);
```

(%o6) $\left[L=\begin{pmatrix} 1 & 0 & 0 \\ \frac{2}{3} & 1 & 0 \\ \frac{5}{3} & \frac{8}{11} & 1 \end{pmatrix}, U=\begin{pmatrix} 3 & 2 & 1 \\ 0 & \frac{11}{3} & \frac{10}{3} \\ 0 & 0 & \frac{43}{11} \end{pmatrix}\right]$

```
rhs(luFactor(B)[1]);
```

(%o7) $\begin{pmatrix} 1 & 0 & 0 \\ \frac{2}{3} & 1 & 0 \\ \frac{5}{3} & \frac{8}{11} & 1 \end{pmatrix}$

```
C:matrix([1, 4, 3], [2, 7, 9], [5, 8, -2]);
```

(%o19) $\begin{pmatrix} 1 & 4 & 3 \\ 2 & 7 & 9 \\ 5 & 8 & -2 \end{pmatrix}$

```
luFactor(C);
```

(%o20) $\left[L=\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 12 & 1 \end{pmatrix}, U=\begin{pmatrix} 1 & 4 & 3 \\ 0 & -1 & 3 \\ 0 & 0 & -53 \end{pmatrix}\right]$

6.3

solve the system Ax = b using LU method

```
➔   solve_by_lu(A,  b):=block
    (
        [n,  i,  j],
        n:matrix_size(A)[1],
        L:rhs(luFactor(A)[1]),
        U:rhs(luFactor(A)[2]),
        z:zeromatrix(n,  1),
        x:zeromatrix(n,  1),

        z[1]:b[1],
        for i:2 thru n do
        (
            z[i]:b[i]−sum(L[i][j]·z[j],  j,  1,  i−1)
        ),

        x[n]:z[n]/U[n][n],
        for i:n−1 thru 1 step −1 do
        (
            x[i]:(z[i]−sum(U[i][j]·x[j],  j,  i+1,  n))/U[i][i]
        ),

        ans1:['z=z, 'x=x ]
    );
```

(%o8) $\mathrm{solve\_by\_lu}(A,b):=\mathrm{block}\left([n,i,j],n:(\mathrm{matrix\_size}(A))_1,L:\mathrm{rhs}((\mathrm{luFactor}(A))_1),U:\mathrm{rhs}((\mathrm{luFactor}(A))_2),z:\mathrm{zeromatrix}(n,1),x:\mathrm{zeromatrix}(n,1),z_1:b_1,\mathrm{for}\ i\ \mathrm{from}\ 2\ \mathrm{thru}\ n\ \mathrm{do}\ z_i:b_i-\sum_{j=1}^{i-1}(L_i)_j z_j,x_n:\frac{z_n}{(U_n)_n},\mathrm{for}\ i\ \mathrm{from}\ n-1\ \mathrm{step}\ -1\ \mathrm{thru}\ 1\ \mathrm{do}\ x_i:\frac{z_i-\sum_{j=i+1}^{n}(U_i)_j x_j}{(U_i)_i},ans1:['z=z,'x=x]\right)$

```
A:matrix([1, 2, 3], [3, 5, 7], [4, 6, 9]);
b:matrix([1], [2], [3]);
```

(%o9) $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 5 & 7 \\ 4 & 6 & 9 \end{pmatrix}$

(%o10) $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

```
solve_by_lu(A, b);
```

(%o11) $[z=\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, x=\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}]$

6.4

Ques1. solve the equations :-
x1+2x2+3x3=2
3x1+5x2+6x3=4
7x1+8x2+9x3=7

```
A:matrix([1, 2, 3], [3, 5, 6], [7, 8, 9]);
b:matrix([2], [4], [7]);
```

(%o12) $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

(%o13) $\begin{pmatrix} 2 \\ 4 \\ 7 \end{pmatrix}$

```
solve_by_lu(A, b);
```

(%o14) $[z=\begin{pmatrix} 2 \\ -2 \\ 5 \end{pmatrix}, x=\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{5}{6} \end{pmatrix}]$

6.5

Ques2. solve the linear system :-
x1+3x2+1x3-2x4=3
2x1+4x2-x3+2x4=7
3x1+x2+x3+5x4=10
4x1+2x2-x3+6x4=11

```
A:matrix([1, 3, 1, −2], [2, 4, −1, 2], [3, 1, 1, 5], [4, 2, −1, 6]);
b:matrix([3], [7], [10], [11]);
```

(%o15) $\begin{pmatrix} 1 & 3 & 1 & -2 \\ 2 & 4 & -1 & 2 \\ 3 & 1 & 1 & 5 \\ 4 & 2 & -1 & 6 \end{pmatrix}$

(%o16) $\begin{pmatrix} 3 \\ 7 \\ 10 \\ 11 \end{pmatrix}$

```
luFactor(A);
```

(%o17) $\left[L=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ 4 & 5 & 1 & 1 \end{pmatrix}, U=\begin{pmatrix} 1 & 3 & 1 & -2 \\ 0 & -2 & -3 & 6 \\ 0 & 0 & 10 & -13 \\ 0 & 0 & 0 & -3 \end{pmatrix}\right]$

```
solve_by_lu(A, b);
```

(%o18) $\left[z=\begin{pmatrix} 3 \\ 1 \\ -3 \\ -3 \end{pmatrix}, x=\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}\right]$

6.6

```
A:matrix([2, 7, 5], [6, 20, 10], [4, 3, 0]);
```

(%o21) $\begin{pmatrix} 2 & 7 & 5 \\ 6 & 20 & 10 \\ 4 & 3 & 0 \end{pmatrix}$

```
luFactor(A);
```

(%o22) $\left[L=\begin{pmatrix}1 & 0 & 0\\ 3 & 1 & 0\\ 2 & 11 & 1\end{pmatrix}, U=\begin{pmatrix}2 & 7 & 5\\ 0 & -1 & -5\\ 0 & 0 & 45\end{pmatrix}\right]$

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Exercise

Solve by LU method

Figure 6:

$$\textbf{11.}\ A=\begin{bmatrix}1 & 2 & 3 & 4\\ -1 & 1 & 2 & 3\\ 1 & -1 & 1 & 2\\ -1 & 1 & -1 & 5\end{bmatrix} \quad \mathbf{b}_1=\begin{bmatrix}10\\ 5\\ 3\\ 4\end{bmatrix},\quad \mathbf{b}_2=\begin{bmatrix}-4\\ -5\\ -3\\ -4\end{bmatrix},\quad \mathbf{b}_3=\begin{bmatrix}-2\\ -3\\ 1\\ -8\end{bmatrix}$$

$$\textbf{12.}\ A=\begin{bmatrix}1 & 0 & 2 & 0\\ -1 & 4 & 3 & 6\\ 0 & -2 & 5 & -3\\ 3 & 1 & 1 & 0\end{bmatrix} \quad \mathbf{b}_1=\begin{bmatrix}3\\ 12\\ 0\\ 5\end{bmatrix},\quad \mathbf{b}_2=\begin{bmatrix}-1\\ -6\\ -4\\ 3\end{bmatrix},\quad \mathbf{b}_3=\begin{bmatrix}3\\ -8\\ 10\\ 2\end{bmatrix}$$

$$\textbf{13.}\ A=\begin{bmatrix}1 & 3 & 1 & -2\\ 2 & 4 & -1 & 2\\ 3 & 1 & 1 & 5\\ 4 & 2 & 6 & -1\end{bmatrix} \quad \mathbf{b}_1=\begin{bmatrix}1\\ -5\\ -2\\ 9\end{bmatrix},\quad \mathbf{b}_2=\begin{bmatrix}-5\\ -3\\ 6\\ -5\end{bmatrix},\quad \mathbf{b}_3=\begin{bmatrix}5\\ 5\\ -2\\ 1\end{bmatrix}$$

$$\textbf{14.}\ A=\begin{bmatrix}2 & 7 & 5\\ 6 & 20 & 10\\ 4 & 3 & 0\end{bmatrix} \quad \mathbf{b}_1=\begin{bmatrix}14\\ 36\\ 7\end{bmatrix},\quad \mathbf{b}_2=\begin{bmatrix}-4\\ -16\\ -7\end{bmatrix},\quad \mathbf{b}_3=\begin{bmatrix}-3\\ -12\\ 6\end{bmatrix}$$

$$\textbf{15.}\ A=\begin{bmatrix}13 & 39 & 2 & 57 & 28\\ -4 & -12 & 0 & -19 & -9\\ 3 & 0 & -9 & 2 & 1\\ 6 & 17 & 9 & 5 & 7\\ 19 & 42 & -17 & 107 & 44\end{bmatrix} \quad \mathbf{b}_1=\begin{bmatrix}-53\\ 18\\ -7\\ 0\\ -103\end{bmatrix},\ \mathbf{b}_2=\begin{bmatrix}57\\ -18\\ -11\\ 18\\ 69\end{bmatrix},$$

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8.1 (14) Solution

```
--> A:matrix([2, 7, 5], [6, 20, 10], [4, 3, 0]);
    b1:matrix([14], [36], [7]);
    b2:matrix([−4], [−16], [−7]);
    b3:matrix([−3], [−12], [6]);
```

(%o27) $\begin{pmatrix} 2 & 7 & 5 \\ 6 & 20 & 10 \\ 4 & 3 & 0 \end{pmatrix}$

(%o28) $\begin{pmatrix} 14 \\ 36 \\ 7 \end{pmatrix}$

(%o29) $\begin{pmatrix} -4 \\ -16 \\ -7 \end{pmatrix}$

(%o30) $\begin{pmatrix} -3 \\ -12 \\ 6 \end{pmatrix}$

```
--> luFactor(A);
```

(%o31) $[L=\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 11 & 1 \end{pmatrix}, U=\begin{pmatrix} 2 & 7 & 5 \\ 0 & -1 & -5 \\ 0 & 0 & 45 \end{pmatrix}]$

```
--> solve_by_lu(A, b1);
```

(%o32) $[z=\begin{pmatrix} 14 \\ -6 \\ 45 \end{pmatrix}, x=\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}]$

```
--> solve_by_lu(A, b2);
```

(%o33) $[z=\begin{pmatrix} -4 \\ -4 \\ 45 \end{pmatrix}, x=\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}]$

```
--> solve_by_lu(A, b3);
```

(%o34) $[z=\begin{pmatrix} -3 \\ -3 \\ 45 \end{pmatrix}, x=\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}]$