Practical 9 : Lagrange Interpolation

1

Lagrange form of the interpolation polynomial

Figure 1:

Definition. The LAGRANGE POLYNOMIAL $L_{n,j}(x)$ has degree n and is associated with the interpolating point x_j in the sense

$$L_{n,j}(x_i) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}.$$

With this family of functions, it is straightforward to demonstrate that

$$P_n(x) = \sum_{i=0}^{n} L_{n,i}(x) f_i$$

interpolates the data (x_j, f_j) for $j = 0, 1, 2, \ldots, n$. For each x_j

$$\begin{aligned} P_n(x_j) &= \sum_{i=0}^{n} \underbrace{L_{n,i}(x_j)}_{\substack{1 \text{ if } i=j, \\ 0 \text{ otherwise}}} f_i \\ &= 0 \cdot f_0 + \cdots + 0 \cdot f_{j-1} + 1 \cdot f_j + 0 \cdot f_{j+1} + \cdots + 0 \cdot f_n \\ &= f_j. \end{aligned}$$

Figure 2:

Since $P_n(x) = \sum_{i=0}^{n} L_{n,i}(x) f_i$ is based on Lagrange polynomials, P_n is referred to as the Lagrange Form of the Interpolating Polynomial.

The final piece needed to construct the Lagrange form of the interpolating polynomial is to obtain explicit formulas for the $L_{n,j}$. Fortunately, these can be determined by directly applying the conditions stated in the definition. Since $L_{n,j}$ is an nth-degree polynomial with n roots located at $x = x_i$ $(i \neq j)$, it follows that $L_{n,j}$ must be of the form

$$c(x - x_0)(x - x_1) \cdots (x - x_{j-1})(x - x_{j+1}) \cdots (x - x_n)$$

Figure 3:

for some constant c. The final condition of the definition, $L_{n,j}(x_j) = 1$, determines the value of c:

$$c = \frac{1}{(x_j - x_0)(x_j - x_1)\cdots(x_j - x_{j-1})(x_j - x_{j+1})\cdots(x_j - x_n)}.$$

Therefore,

$$L_{n,j}(x) = \frac{(x - x_0)(x - x_1)\cdots(x - x_{j-1})(x - x_{j+1})\cdots(x - x_n)}{(x_j - x_0)(x_j - x_1)\cdots(x_j - x_{j-1})(x_j - x_{j+1})\cdots(x_j - x_n)}$$

$$= \prod_{i=0, i\neq j}^{n} \frac{x - x_i}{x_j - x_i}.$$

2

function for Lagrange interpolating polynomial

```
(%i1) kill(all);
(%o0) done

(%i1) P(a,  f,  x):=block
(
   [n,  i,  temp,  s,  j,  k],
   n:length(a),
   L(i,  x):=block
   (
      temp:1,
      for j:1 thru n do ( if(j#i) then temp:temp·(x−a[j])/(a[i]−a[j]) ),
      return(temp)
   ),
   s:0,
   for k:1 thru n do s:s+(L(k,  x)·f[k]),
   return(s)
)
;
```

(%o1) $P(a,f,x):=\mathrm{block}([n,i,temp,s,j,k],n:\mathrm{length}(a),L(i,x):=\mathrm{block}(temp:1,\mathrm{for}\ j\ \mathrm{thru}\ n\ \mathrm{do}\ \mathrm{if}\ j\neq i\ \mathrm{then}\ temp:\frac{temp\,(x-a_j)}{a_i-a_j},\mathrm{return}(temp)),s:0,\mathrm{for}\ k\ \mathrm{thru}\ n\ \mathrm{do}\ s:s+L(k,x)\,f_k,\mathrm{return}(s))$

3

(%i2) P([1/3, 1/4, 1], [2, −1, 7], x);

(%o2) $$14\left(x-\frac{1}{3}\right)\left(x-\frac{1}{4}\right)-36\,(x-1)\left(x-\frac{1}{4}\right)-16\,(x-1)\left(x-\frac{1}{3}\right)$$

(%i3) P([1/3, 1/4, 1], [2, −1, 7], 1);

(%o3) 7

(%i4) P([0, 1, −1, 2, −2], [−5, −3, −15, 39, −9], x);

(%o4) $$\frac{13\,(x-1)\,x\,(x+1)\,(x+2)}{8}+\frac{(x-2)\,x\,(x+1)\,(x+2)}{2}+\frac{5\,(1-x)\,(x-2)\,(x+1)\,(x+2)}{4}+\frac{5\,(x-2)\,(x-1)\,x\,(x+2)}{2}-\frac{3\,(x-2)\,(x-1)\,x\,(x+1)}{8}$$

(%i5) P([0, 1, −1, 2, −2], [−5, −3, −15, 39, −9], 3);

(%o5) 241

(%i6) P([0, 1, −1, 2, −2], [−5, −3, −15, 39, −9], 0);

(%o6) −5

3.1 Q7

Figure 4:

7. Consider the data set

x	-1	0	1	2
y	5	1	1	11

(a) Show that the polynomials $f(x) = x^3 + 2x^2 - 3x + 1$ and $g(x) = \frac{1}{8}x^4 + \frac{3}{4}x^3 + \frac{15}{8}x^2 - \frac{11}{4}x + 1$ both interpolate all of the data.

(%i7) P([−1, 0, 1, 2], [5, 1, 1, 11], x);

(%o7) $$\frac{11\,(x-1)\,x\,(x+1)}{6}-\frac{(x-2)\,x\,(x+1)}{2}+\frac{(x-2)\,(x-1)\,(x+1)}{2}-\frac{5\,(x-2)\,(x-1)\,x}{6}$$

(%i8) ratsimp(P([−1, 0, 1, 2], [5, 1, 1, 11], x));

(%o8) $x^3+2\,x^2-3\,x+1$

3.2 Q8

Figure 5:

8. Consider the data set

x	-3	1	2	5
y	-23	-11	-23	1

(a) Show that the polynomials $f(x) = x^3 - 3x^2 - 10x + 1$ and $g(x) = -23 + 3(x-3) - 3(x+3)(x-1) + (x+3)(x-1)(x-2)$ both interpolate all of the data.

(%i9) P([−3, 1, 2, 5], [−23, −11, −23, 1], x);

(%o9) $$\frac{(x-2)(x-1)(x+3)}{96}+\frac{23(x-5)(x-1)(x+3)}{15}-\frac{11(x-5)(x-2)(x+3)}{16}+\frac{23(x-5)(x-2)(x-1)}{160}$$

(%i10) ratsimp(P([−3, 1, 2, 5], [−23, −11, −23, 1], x));

(%o10) $$x^3-3x^2-10x+1$$

4

Exercise

4.1

Figure 6:

4. Consider the function $f(x) = \ln x$.
   (a) Construct the Lagrange form of the interpolating polynomial for f passing through the points $(1, \ln 1)$, $(2, \ln 2)$, and $(3, \ln 3)$.
   (b) Plot the polynomial obtained in part (a) on the same set of axes as $f(x) = \ln x$. Use an x range of $[1, 3]$. Next, generate a plot of the difference between the polynomial obtained in part (a) and $f(x) = \ln x$.
   (c) Use the polynomial obtained in part (a) to estimate both $\ln(1.5)$ and $\ln(2.4)$. What is the error in each approximation?

4.2

Figure 7:

5. Consider the function $f(x) = \sin x$.
   (a) Construct the Lagrange form of the interpolating polynomial for f passing through the points $(0, \sin 0)$, $(\pi/4, \sin \pi/4)$, and $(\pi/2, \sin \pi/2)$.
   (b) Plot the polynomial obtained in part (a) on the same set of axes as $f(x) = \sin x$. Use an x range of $[0, \pi/2]$. Next, generate a plot of the difference between the polynomial obtained in part (a) and $f(x) = \sin x$.
   (c) Use the polynomial obtained in part (a) to estimate both $\sin(\pi/3)$ and $\sin(\pi/6)$. What is the error in each approximation?

4.3

Figure 8:

6. Consider the function $f(x) = e^x$.

(a) Construct the Lagrange form of the interpolating polynomial for f passing through the points $(-1,\, e^{-1})$, $(0,\, e^0)$, and $(1,\, e^1)$.

(b) Plot the polynomial obtained in part (a) on the same set of axes as $f(x) = e^x$. Use an x range of $[-1, 1]$. Next, generate a plot of the difference between the polynomial obtained in part (a) and $f(x) = e^x$.

(c) Use the polynomial obtained in part (a) to estimate both $\sqrt{e}$ and $e^{-1/3}$. What is the error in each approximation?