Practical 14 : Second Order Runge Kutta Methods

1

Figure 1:

Modified Euler method $(a_1 = 0,\ a_2 = 1,\ \alpha_2 = \delta_2 = h/2)$

$$\tilde{w} = w_i + \tfrac{h}{2} f(t_i, w_i) \qquad \text{Euler's method with step } h/2$$
$$w_{i+1} = w_i + h f\left(t_i + \tfrac{h}{2}, \tilde{w}\right) \qquad \text{Midpoint integration}$$

Figure 2:

Heun method $(a_1 = a_2 = 1/2,\ \alpha_2 = \delta_2 = h)$

$$\tilde{w} = w_i + h f(t_i, w_i) \qquad \text{Euler's method with step } h$$
$$w_{i+1} = w_i + \tfrac{h}{2}\left[f(t_i, w_i) + f(t_i + h, \tilde{w})\right] \qquad \text{Trapezoidal integration}$$

Optimal RK2 method $(a_1 = 1/4,\ a_2 = 3/4,\ \alpha_2 = \delta_2 = 2h/3)$

$$\tilde{w} = w_i + \tfrac{2h}{3} f(t_i, w_i)$$
$$w_{i+1} = w_i + \tfrac{h}{4} f(t_i, w_i) + \tfrac{3h}{4} f\left(t_i + \tfrac{2h}{3}, \tilde{w}\right)$$

2

Modified Euler Method

Figure 3:

initial value problem

$$\frac{dx}{dt} = 1 + \frac{x}{t} \qquad (1 \le t \le 6), \qquad x(1) = 1,$$

2.1

```
kill(all);
```

(%o0) *done*

```
w:1.0;
f(t, x):=1+(x/t);
x(t):=t·(1+log(t));
h:0.5;
t:1;
disp("t                          w                    exact solution                    | x(ti)−wi |");
while( t <= 6) do
(
   print(t, "          ", w, "          ", x(t), "        ", abs(x(t)−w)),
   wt:w+((h/2)·f(t, w)),
   w:w+(h·f(t+(h/2), wt)),
   t:t+h
);
```

(%o3) 1.0

(%o4) $f(t,x):=1+\frac{x}{t}$

(%o5) $x(t):=t\,(1+\log(t))$

(%o6) 0.5

(%o7) 1

t *w* *exact solution* *| x(ti)−wi |*

(%o8) *done*

```
1           1.0             1          0.0
1.5            2.1              2.108197662162246
0.008197662162246378
2.0            3.371428571428571            3.386294361119891
         0.01486578969131935
2.5            4.76984126984127            4.790726829685388
         0.02088555984411844
3.0            6.26926406926407            6.295836866004329
0.02657279674026025
3.5            7.852602952602953            7.884670389733788
         0.03206743713083515
4.0            9.507736707736708            9.545177444479564
         0.03744073674285531
4.5            11.22561556090967            11.26834828549323
         0.04273272458355492
5.0            12.99922196826221            13.0471895621705
         0.04796759390828598
5.5            14.82295368889796            14.87611450731133
         0.05316081841337805
6.0            16.69223406377801            16.75055681536833
         0.05832275159031752
```

(%o9) *done*

2.2

function

```
→ kill(all);
(%o0) done

→ eulermod(g, a, x0, b, N):=block
(
    [t, x, h, t0, w, wt],
    define(f(t, x), g),
    t0:a,
    h:(b−a)/N,
    w:x0,
    while(t0<=b) do
    (
        print(t0, "−−−− ", w),
        wt:w+((h/2)·f(t0, w)),
        w:w+(h·f(t0+(h/2), wt)),
        t0:t0+h
    )
);
```

(%o1) eulermod$(g,a,x0,b,N)$:=block$([t,x,h,t0,w,wt],$ define$(f(t,x),g),t0:a,h:\frac{b-a}{N},w:x0,$while $t0\le b$ do $($ print$(t0,----\ ,w),wt:w+\frac{h}{2}f(t0,w),w:w+h\,f\left(t0+\frac{h}{2},wt\right),t0:t0+h))$

```
→ eulermod(1+(x/t), 1.0, 1.0, 6, 10);
1.0 −−−− 1.0
1.5 −−−− 2.1
2.0 −−−− 3.371428571428571
2.5 −−−− 4.76984126984127
3.0 −−−− 6.26926406926407
3.5 −−−− 7.852602952602953
4.0 −−−− 9.507736707736708
4.5 −−−− 11.22561556090967
5.0 −−−− 12.99922196826221
5.5 −−−− 14.82295368889796
6.0 −−−− 16.69223406377801
(%o2) done
```

3

Heun Method

Figure 4:

initial value problem

$$\frac{dx}{dt} = 1 + \frac{x}{t} \qquad (1 \leq t \leq 6), \qquad x(1) = 1,$$

3.1

```
→ kill(all);
(%o0) done
```

```
→  w:1.0;
   f(t, x):=1+(x/t);
   x(t):=t·(1+log(t));
   h:0.5;
   t:1;
   disp("t                    w                    exact solution                    | x(ti)−wi |");
   while( t <= 6) do
   (
      print(t, "          ",  w, "            ",  x(t), "         ",  abs(x(t)−w)),
      wt:w+(h·f(t,  w)),
      w:w+(h/2)·(f(t,  w)+f(t+h,  wt)),
      t:t+h
   );
```

(%o1) 1.0

(%o2) $f(t,x):=1+\frac{x}{t}$

(%o3) $x(t):=t\,(1+\log(t))$

(%o4) 0.5

(%o5) 1

t *w* *exact solution* *| x(ti)−wi |*

(%o6) *done*

```
1          1.0              1          0.0
1.5           2.083333333333333             2.108197662162246
      0.02486432882891343
2.0           3.340277777777777             3.386294361119891
      0.04601658334211356
2.5           4.725347222222221             4.790726829685388
      0.06537960746316696
3.0           6.212083333333332             6.29583686600433
      0.08375353267099772
3.5           7.78314484126984             7.884670389733788
      0.1015255484639485
4.0           9.426272675736959             9.545177444479564
      0.1189047687426043
4.5           11.13233453798185             11.26834828549323
      0.1360137475113774
5.0           12.89426059775761             13.0471895621705
      0.152928964412883
5.5           14.70641393026065             14.87611450731133
      0.1697005770506852
6.0           16.56419398452677             16.75055681536833
      0.1863628308415563
```

(%o7) *done*

check

Figure 5:

t_i	w_i	$x(t_i)$	$\lvert x(t_i) - w_i \rvert$
1.0	1.000000	1.000000	
1.5	2.083333	2.108198	0.024864
2.0	3.340278	3.386294	0.046017
2.5	4.725347	4.790727	0.065380
3.0	6.212083	6.295837	0.083754
3.5	7.783145	7.884670	0.101526
4.0	9.426273	9.545177	0.118905
4.5	11.132335	11.268348	0.136014
5.0	12.894261	13.047190	0.152929
5.5	14.706414	14.876115	0.169701
6.0	16.564194	16.750557	0.186363

3.2

function

```
-->  kill(all);
(%o0) done
```

```
-->  heun(g, a, x0, b, N):=block
     (
        [t, x, h, t0, w, wt],
        define(f(t, x), g),
        t0:a,
        h:(b−a)/N,
        w:x0,
        while(t0<=b) do
        (
           print(t0, "−−−− ", w),
           wt:w+(h·f(t0, w)),
           w:w+(h/2)·(f(t0, w)+f(t0+h, wt)),
           t0:t0+h
        )
     );
```

(%o1) $\mathrm{heun}(g,a,x0,b,N):=\mathrm{block}\left([t,x,h,t0,w,wt],\ \mathrm{define}(f(t,x),g),t0:a,h:\frac{b-a}{N},w:x0,\mathrm{while}\ t0\leq b\ \mathrm{do}\ \left(\mathrm{print}(t0,----\ ,w),wt:w+h\,f(t0,w),w:w+\frac{h}{2}(f(t0,w)+f(t0+h,wt)),t0:t0+h\right)\right)$

```
→ heun(1+(x/t), 1.0, 1.0, 6, 10);
1.0 ---- 1.0
1.5 ---- 2.083333333333333
2.0 ---- 3.340277777777777
2.5 ---- 4.725347222222221
3.0 ---- 6.212083333333332
3.5 ---- 7.78314484126984
4.0 ---- 9.426272675736959
4.5 ---- 11.13233453798185
5.0 ---- 12.89426059775761
5.5 ---- 14.70641393026065
6.0 ---- 16.56419398452677
(%o2) done
```

4

Optimal RK2 Method

Figure 6:

initial value problem

$$\frac{dx}{dt} = 1 + \frac{x}{t} \qquad (1 \le t \le 6), \qquad x(1) = 1,$$

4.1

```
→ kill(all);
(%o0) done
```

```
-->   w:1.0;
      f(t, x):=1+(x/t);
      x(t):=t·(1+log(t));
      h:0.5;
      t:1;
      disp("t                    w                exact solution                | x(ti)−wi |");
      while( t <= 6) do
      (
        print(t, "       ", w, "        ", x(t), "      ", abs(x(t)−w)),
        wt:w+((2·h/3)·f(t, w)),
        w:w+(h/4)·f(t, w)+(3·h/4)·f(t+(2·h/3), wt),
        t:t+h
      );
```

(%o1) 1.0

(%o2) $f(t,x):=1+\frac{x}{t}$

(%o3) $x(t):=t\,(1+\log(t))$

(%o4) 0.5

(%o5) 1

t *w* *exact solution* *| x(ti)−wi |*

(%o6) *done*

```
1           1.0           1         0.0
1.5           2.09375            2.108197662162246
0.01444766216224646
2.0           3.359848484848484            3.386294361119891
         0.02644587627140637
2.5           4.753382034632034            4.790726829685388
         0.03734479505335386
3.0           6.248176088617265            6.29583686600433
         0.04766077738706542
3.5           7.827038770053475            7.884670389733788
         0.05763161968031305
4.0           9.477795861427574            9.545177444479564
         0.06738158305198994
4.5           11.19136649795217            11.26834828549323
         0.07698178754105989
5.0           12.96071373335682            13.0471895621705
         0.08647582881368088
5.5           14.7802226066925            14.87611450731133
         0.09589190061883457
6.0           16.64530777872948            16.75055681536833
         0.1052490366388454
```

(%o7) *done*

4.2

function

```
kill(all);
```

(%o0) *done*

```
rk2(g, a, x0, b, N):=block
(
  [t, x, h, t0, w, wt],
  define(f(t, x), g),
  t0:a,
  h:(b−a)/N,
  w:x0,
  while(t0<=b) do
  (
    print(t0, "---- ", w),
    wt:w+((2·h/3)·f(t0, w)),
    w:w+(h/4)·f(t0, w)+(3·h/4)·f(t0+(2·h/3), wt),
    t0:t0+h
  )
);
```

(%o1) $\mathrm{rk2}(g,a,x0,b,N):=\mathrm{block}\left([t,x,h,t0,w,wt],\ \mathrm{define}(\mathrm{f}(t,x),g),t0:a,h:\frac{b-a}{N},w:x0,\mathrm{while}\ t0\le b\ \mathrm{do}\ \left(\mathrm{print}(t0,----\ ,w),wt:w+\frac{2\,h}{3}\mathrm{f}(t0,w),w:w+\frac{h}{4}\mathrm{f}(t0,w)+\frac{3\,h}{4}\mathrm{f}\left(t0+\frac{2\,h}{3},wt\right),t0:t0+h\right)\right)$

```
rk2(1+(x/t), 1.0, 1.0, 6, 10);
```

```
1.0 ---- 1.0
1.5 ---- 2.09375
2.0 ---- 3.359848484848484
2.5 ---- 4.753382034632034
3.0 ---- 6.248176088617265
3.5 ---- 7.827038770053475
4.0 ---- 9.477795861427574
4.5 ---- 11.19136649795217
5.0 ---- 12.96071373335682
5.5 ---- 14.7802226066925
6.0 ---- 16.64530777872948
```

(%o2) *done*

5

Exercise

Figure 7:

4. Apply the modified Euler method to approximate the solution of the given initial value problem over the indicated interval in t using the indicated number of time steps.

(a) $x' = tx^3 - x \quad (0 \le t \le 1), \quad x(0) = 1, \quad N = 4$

(b) $x' + (4/t)x = t^4 \quad (1 \le t \le 3), \quad x(1) = 1, \quad N = 5$

(c) $x' = (\sin x - e^t)/\cos x \quad (0 \le t \le 1), \quad x(0) = 0, \quad N = 3$

(d) $x' = (1 + x^2)/t \quad (1 \le t \le 4), \quad x(1) = 0, \quad N = 5$

(e) $x' = t^2 - 2x^2 - 1 \quad (0 \le t \le 1), \quad x(0) = 0, \quad N = 4$

(f) $x' = 2(1 - x)/(t^2 \sin x) \quad (1 \le t \le 2), \quad x(1) = 2, \quad N = 3$

5. Repeat Exercise 4 using the Heun method.

6. Repeat Exercise 4 using the optimal RK2 method.