Practical 6 : Gauss-Jacobi Method

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Gauss_jacobi Method

Figure 1:

The Jacobi Method, Gauss-Seidel Method, and SOR Method

To identify the splittings associated with the Jacobi method, the Gauss-Seidel method, and the SOR method, first express the coefficient matrix A in the form

$$A = D - L - U.$$

Here, D is the diagonal part of A, $-L$ is the strictly lower triangular part of A, and $-U$ is the strictly upper triangular part. It is important to keep in mind that the matrices L and U used here are in no way related to the LU decomposition of the coefficient matrix. As an example, suppose

$$A = \begin{bmatrix} 5 & 1 & 2 \\ -3 & 9 & 4 \\ 1 & 2 & -7 \end{bmatrix}.$$

Then

$$D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & -7 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ -1 & -2 & 0 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 0 & -1 & -2 \\ 0 & 0 & -4 \\ 0 & 0 & 0 \end{bmatrix}.$$

The Jacobi method is based on the splitting $M = D$ and $N = L + U$. In order for M to be nonsingular, it must be the case that, for each i, $d_{i,i} \equiv a_{i,i} \neq 0$. If this relationship does not hold for even a single value of i, then the equations in the system must be reordered before the Jacobi method can be applied. With

Figure 2:

the specific choice of splitting indicated above, the iteration scheme for the Jacobi method is defined by

$$\mathbf{x}^{(k+1)} = T_{jac}\mathbf{x}^{(k)} + \mathbf{c}_{jac}, \tag{2}$$

where

$$T_{jac} = D^{-1}(L+U) \quad \text{and} \quad \mathbf{c}_{jac} = D^{-1}\mathbf{b}.$$

Taking into account the structure of the iteration matrix, T_{jac}, and the vector $\mathbf{c}_{jac}$, the individual components of equation (2) can be written as

$$x_i^{(k+1)} = \frac{1}{a_{i,i}}\left[b_i - \sum_{j=1}^{i-1} a_{i,j}x_j^{(k)} - \sum_{j=i+1}^{n} a_{i,j}x_j^{(k)}\right]. \tag{3}$$

Hence, the Jacobi method is equivalent to solving the i-th equation in the system for the unknown x_i.

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2.1

Figure 3:

Consider the system of equations

$$\begin{array}{rcrcrcr} 5x_1 & + & x_2 & + & 2x_3 & = & 10 \\ -3x_1 & + & 9x_2 & + & 4x_3 & = & -14 \\ x_1 & + & 2x_2 & - & 7x_3 & = & -33\,. \end{array}$$

The Jacobi method, when applied to this system, will produce the sequence of approximations $\{\mathbf{x}^{(k)}\}$ according to the rules

$$\begin{aligned} x_1^{(k+1)} &= \frac{1}{5}\left[10 - x_2^{(k)} - 2x_3^{(k)}\right] \\ x_2^{(k+1)} &= \frac{1}{9}\left[-14 + 3x_1^{(k)} - 4x_3^{(k)}\right] \\ x_3^{(k+1)} &= -\frac{1}{7}\left[-33 - x_1^{(k)} - 2x_2^{(k)}\right]\,. \end{aligned}$$

2.2

function for Gauss-Jacobi method

```
(%i1) kill(all);
(%o0) done
```

(%i1)
```
gauss_jacobi(A,  b,   iter):=block(
    n:matrix_size(A)[1],
    for i:1 thru n do
    (
        for j:1 thru n do
        (
            if(i > j) then (l[i,  j]:A[i,  j])
            else ( l[i,  j]:0)
        )
    ),
    L:genmatrix(l,  n,  n),
    for i:1 thru n do
    (
        for j:1 thru n do
        (
            if(i < j) then (u[i,  j]:A[i,  j])
            else ( u[i,  j]:0)
        )
    ),
    U:genmatrix(u,  n,  n),
    for i:1 thru n do
    (
        for j:1 thru n do
        (
            if(i = j) then (d[i,  j]:A[i,  j])
            else ( d[i,  j]:0)
        )
    ),
    D:genmatrix(d,  n,  n),
    D1:invert(D),
    Tgj:D1.(−L−U),
    cgj:D1.b,
    for i:1 thru n do t[i,  1]:0.0,
    x0:genmatrix(t,  n,  1),
    for i:1 thru iter do
    (
        x1:Tgj.x0+cgj,
        disp(i, "−−−−",  x1),
        x0:x1
    )
);
```

(%o1) gauss_jacobi$(A,b,iter)$:=block$(n$:(matrix_size$(A))_1$,for i thru n do for j thru n do if $i>j$ then $l_{i,j}:A_{i,j}$ else $l_{i,j}:0$,L:genmatrix(l,n,n),for i thru n do for j thru n do if $i<j$ then $u_{i,j}:A_{i,j}$ else $u_{i,j}:0$,U:genmatrix(u,n,n),for i thru n do for j thru n do if $i=j$ then $d_{i,j}:A_{i,j}$ else $d_{i,j}:0$,D:genmatrix(d,n,n), $D1$:invert(D),$Tgj:D1 . (-L-U)$,$cgj:D1 . b$,for i thru n do $t_{i,1}$:0.0,$x0$:genmatrix$(t,n,1)$,for i thru $iter$ do

(%i3) A1:matrix([5, 1, 2], [−3, 9, 4],[1, 2, −7]);
b1:matrix([10], [−14], [−33]);

(%o2) $$\begin{pmatrix} 5 & 1 & 2 \\ -3 & 9 & 4 \\ 1 & 2 & -7 \end{pmatrix}$$

(%o3) $$\begin{pmatrix} 10 \\ -14 \\ -33 \end{pmatrix}$$

(%i4) gauss_jacobi(A1, b1, 14);

1

$$\begin{pmatrix} 2.0 \\ -1.555555555555555 \\ 4.714285714285714 \end{pmatrix}$$
2

$$\begin{pmatrix} 0.4253968253968252 \\ -2.984126984126984 \\ 4.555555555555555 \end{pmatrix}$$
3

$$\begin{pmatrix} 0.7746031746031745 \\ -3.438447971781305 \\ 3.922448979591836 \end{pmatrix}$$
4

$$\begin{pmatrix} 1.118710002519526 \\ -3.040665154950869 \\ 3.842529604434366 \end{pmatrix}$$
5

$$\begin{pmatrix} 1.071121189216427 \\ -2.890443156686543 \\ 4.005339956088256 \end{pmatrix}$$
6

$$\begin{pmatrix} 0.9759526489020063 \\ -2.97866625074486 \\ 4.041462125120478 \end{pmatrix}$$
7

$$\begin{pmatrix} 0.9791484001007809 \\ -3.026443394863988 \\ 4.002660021058898 \end{pmatrix}$$
8

$$\begin{pmatrix} 1.004224670549238 \\ -3.008132764881472 \\ 3.989465944338972 \end{pmatrix}$$

Note :

```
disp(i, "----", transpose(x1))
```

can be used for printing as row vectors

2.3

check

Figure 4:

k	$\mathbf{x}^{(k)}$
0	$[\ 0.000000 \quad 0.000000 \quad 0.000000\ ]^T$
1	$[\ 2.000000 \quad -1.555556 \quad 4.714286\ ]^T$
2	$[\ 0.425397 \quad -2.984127 \quad 4.555556\ ]^T$
3	$[\ 0.774603 \quad -3.438448 \quad 3.922449\ ]^T$
4	$[\ 1.118710 \quad -3.040665 \quad 3.842530\ ]^T$
5	$[\ 1.071121 \quad -2.890443 \quad 4.005340\ ]^T$
6	$[\ 0.975953 \quad -2.978666 \quad 4.041462\ ]^T$
7	$[\ 0.979148 \quad -3.026443 \quad 4.002660\ ]^T$
8	$[\ 1.004225 \quad -3.008133 \quad 3.989466\ ]^T$
9	$[\ 1.005840 \quad -2.993910 \quad 3.998280\ ]^T$
10	$[\ 0.999470 \quad -2.997289 \quad 4.002574\ ]^T$
11	$[\ 0.998428 \quad -3.001321 \quad 4.000699\ ]^T$
12	$[\ 0.999985 \quad -3.000835 \quad 3.999398\ ]^T$
13	$[\ 1.000408 \quad -2.999738 \quad 3.999759\ ]^T$
14	$[\ 1.000044 \quad -2.999757 \quad 4.000133\ ]^T$

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Exercise

Starting with initial vector x0 = 0, perform 6 iterations of Gauss-Jacobi method

Figure 5:

$$\text{(a)} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 4 & 2 \\ 0 & 2 & 6 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix}$$

$$\text{(b)} \begin{bmatrix} 3 & -1 & 1 \\ 2 & -6 & 3 \\ -9 & 7 & -20 \end{bmatrix}, \quad \begin{bmatrix} 4 \\ -13 \\ 7 \end{bmatrix}$$

$$\text{(c)} \begin{bmatrix} 4 & 2 & -1 \\ 2 & 4 & 1 \\ -1 & 1 & 4 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\text{(d)} \begin{bmatrix} 4 & -1 & 0 & 0 \\ 2 & 4 & -1 & 0 \\ 0 & -2 & 4 & -1 \\ 0 & 0 & -2 & 4 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 2 \\ -3 \\ 1 \end{bmatrix}$$