EQUIVALENCE OF THE PERTURBATION THEORIES OF HORI AND DEPRIT

J. A. CAMPBELL

Dept. of Physics, Center for Particle Theory, and Dept. of Computer Science, University of Texas at Austin, Tex., U.S.A.

and

WILLIAM H. JEFFERYS

Dept. of Astronomy, University of Texas at Austin, Tex., U.S.A.

(Received 3 December, 1969).

Abstract. A comparison is made between the perturbation theories of Hori and Deprit which are based on the use of Poisson brackets. General recurrence formulae are presented for Hori's theory which are analogous to those in Deprit's theory. Explicit relations between the determining functions for the two theories are indicated through the sixth order, these results having been obtained by a novel computer program. A general argument for the equivalence of the theories to all orders is given.

1. Introduction

Perturbation theories based on the use of Poisson brackets have several advantages over the usual von Zeipel's method (Hori, 1966; Deprit, 1969). The determining function is not a mixed function of old and new coordinates; the theory is canonically invariant (because this is true of the Poisson brackets), and it is possible to give a direct expression of any function of old variables in terms of the new variables.

Both Hori and Deprit have proposed formulations of such a perturbation theory, which are not obviously equivalent. Deprit's equations, in particular, involve extra terms containing partial derivatives with respect to the small parameter ε, and thus have greater complexity. On the other hand, Deprit (1969) has expressed reservations about the correctness of Hori's formulation, while remarking that a comparative study of the two methods may be informative.

In this paper we present just such a comparative study. First we derive general formulae for Hori's theory analogous to those in Deprit's theory. Next we show by explicit calculation that the two theories are equivalent through the sixth order in ε (the fifth-order and sixth-order calculations having been carried out by a computer program), and point out why they should be equivalent to all orders. Through the sixth order we shall obtain for every determining function W for Deprit's theory, a corresponding function S for Hori's theory which produces the identical canonical transformation.

2. Brief Description of Deprit's Theory

With minor modifications we use the notation of the paper by Deprit (1969). Since his treatment is extensive, we shall give only the principal equations. If $W(y, Y, \varepsilon)$ is an arbitrary function of canonically conjugate vector variables (y, Y), with ε as a small parameter, Deprit defines the operator L_W on functions f by

$$L_W f = \{f, W\} = \sum_j \left(\frac{\partial f}{\partial y_j}\frac{\partial W}{\partial Y_j} - \frac{\partial f}{\partial Y_j}\frac{\partial W}{\partial y_j}\right). \tag{1}$$

Introducing the operator Δ_W by

$$\Delta_W f = L_W f + \frac{\partial f}{\partial \varepsilon}, \tag{2}$$

he shows that if we define the exponential mapping

$$E_W f = \sum_{n \geqslant 0} \frac{\varepsilon^n}{n!}(\Delta_W^n f)_{\varepsilon=0}, \tag{3}$$

where Δ_W^n is the n'th iteration of Δ_W, and $(\Delta_W^n f)_{\varepsilon=0}$ is the result of setting $\varepsilon=0$ in the expansion of $\Delta_W^n f$, then the formulae

$$x_i = E_W y_i, \quad X_i = E_W Y_i \tag{4}$$

define a canonical transformation $(y, Y) \rightarrow (x, X)$. If f is any function of (y, Y), then its expression in the new coordinates is given by $E_W f$. Deprit then defines

$$W = \sum_{n \geqslant 0} \frac{\varepsilon^n}{n!} W_{n+1}(y, Y) \tag{5a}$$

and

$$f = \sum_{n \geqslant 0} \frac{\varepsilon^n}{n!} f_n(y, Y). \tag{5b}$$

Since the Poisson brackets $\{\ ,\ \}$ are bilinear, this leads to

$$L_W = \sum_{n \geqslant 0} \frac{\varepsilon^n}{n!} L_{n+1}, \tag{6}$$

where we have set $L_{n+1} = L_{W_{n+1}}$. With

$$f^k = \Delta_W^k f = \sum_{n \geqslant 0} \frac{\varepsilon^n}{n!} f_n^k, \tag{7}$$

Deprit shows that

$$f_n^{k+1} = f_{n+1}^k + \sum_{m=0}^{n} \binom{n}{m} L_{m+1} f_{n-m}^k, \tag{8}$$

which is his basic recursion relationship. With all the f_n^k known, the transformation is constructed through

$$E_W f = \sum_{n\geqslant 0} \frac{\varepsilon^n}{n!} f_0^n . \tag{9}$$

By using the recursive formula (8) repeatedly and making substitutions into (7), one eventually arrives at the relationship between the quantities f_0^n and f_n. Through the fourth order, these are:

$$f_0^0 = f_0 \tag{10a}$$

$$f_0^1 = L_1 f_0 + f_1 , \tag{10b}$$

$$f_0^2 = (L_1^2 + L_2)\, f_0 + 2L_1 f_1 + f_2 , \tag{10c}$$

$$f_0^3 = (L_1^3 + 2L_1L_2 + L_2L_1 + L_3)\, f_0 + (3L_1^2 + 3L_2)\, f_1 + 3L_1 f_2 + f_3 , \tag{10d}$$

$$\begin{aligned} f_0^4 = {} & (L_1^4 + 3L_1L_3 + 3L_2^2 + L_3L_1 + 3L_1^2L_2 + 2L_1L_2L_1 \\ & + L_2L_1^2 + L_4)\, f_0 + (4L_1^3 + 8L_1L_2 + 4L_2L_1 + 4L_3)\, f_1 \\ & + (6L_1^2 + 6L_2)\, f_2 + 4L_1 f_3 + f_4 . \end{aligned} \tag{10e}$$

Note that $L_iL_j \neq L_jL_i$ in general, because we are working with noncommuting operators. We have derived results for the next two orders by computer, and we shall discuss these results in Section 5.

3. General Recurrence Formulae for Hori's Theory

The basic exponential map, corresponding to the determining function S, is given by Equation (2) of Hori (1966), which we write as

$$F_S f = \sum_{n\geqslant 0} \frac{\varepsilon^n}{n!} D_S^n f , \tag{11}$$

where $D_S f = \{f, S\}$, and (11) is the equation that corresponds to (9) above for Deprit's theory. In (11) there is no setting of $\varepsilon = 0$ in any term, in contrast to what occurs in (3) above, which is a counterpart of (9). Again we expand f and S in power series in ε, this time with the choice

$$S = \sum_{n\geqslant 0} \frac{\varepsilon^n}{(n+1)!} S_{n+1}(y, Y). \tag{12}$$

Equation (12) is not identical in form to (5a). The extra factor of $1/(n+1)$ under the summation makes no change in the interpretation of the method, but it allows an easier comparison between formulae without the occurrence of superfluous fractional coefficients. The expansion for f is unchanged. Then, corresponding to (6), we obtain

$$D_S = \sum_{n \geqslant 0} \frac{\varepsilon^n}{(n+1)!} D_{n+1}, \tag{13}$$

where $D_{n+1} = D_{S_{n+1}}$. When these definitions are put into (11), the right-hand side expanded, and terms in each power of ε collected, one arrives at an equation for $F_S f$ in the form

$$F_S f = \sum_{n \geqslant 0} \frac{\varepsilon^n}{n!} f^n \tag{14}$$

where, with *different* functions f_n^k from those in (7), we have

$$f^n = \sum_{m=0}^{n} \binom{n}{m} f_{n-m}^m, \tag{15}$$

$$f_n^{k+1} = \sum_{m=0}^{n} \binom{n}{m} \frac{D_{m+1}}{m+1} f_{n-m}^k, \tag{16}$$

and

$$D_S^k f = \sum_{n \geqslant 0} \frac{\varepsilon^n}{n!} f_n^k, \tag{17}$$

analogous to Deprit's formulae in Section 2. To the fourth order, the analogues of the Equations (10) are

$$f^0 = f_0, \tag{18a}$$

$$f^1 = D_1 f_0 + f_1, \tag{18b}$$

$$f^2 = (D_1^2 + D_2) f_0 + 2D_1 f_1 + f_2, \tag{18c}$$

$$f^3 = (D_1^3 + \tfrac{3}{2}D_1 D_2 + \tfrac{3}{2}D_2 D_1 + D_3) f_0 + (3D_1^2 + 3D_2) f_1 + 3D_1 f_2 + f_3, \tag{18d}$$

$$f^4 = (D_1^4 + 2D_1 D_3 + 3D_2^2 + 2D_3 D_1 + 2D_1^2 D_2 + 2D_1 D_2 D_1 + 2D_2 D_1^2 + D_4) f_0 + (4D_1^3 + 6D_1 D_2 + 6D_2 D_1 + 4D_3) f_1 + (6D_1^2 + 6D_2) f_2 + 4D_1 f_3 + f_4 \tag{18e}$$

Once again, we have produced the fifth-order and sixth-order terms with a computer program.

4. Comparison of the Two Theories

In order for the two methods to define identical canonical transformations, we must have $E_W f = F_S f$ for every function f. This implies that we must be able to select W

and S such that, in Equations (10) and (18), we have $f_0^k = f^k$ for every choice of f_n. This will imply in turn a set of relations between the operators D_m and L_n.

The zeroth order equations are automatically satisfied. The first order requires D_1 and L_1 to be equal; equivalently, $S_1 = W_1$. The second-order comparison requires that $D_2 = L_2$, i.e., $S_2 = W_2$. All of these results follow trivially from the observation that the coefficients of corresponding terms are identical.

In the third order, it looks at first inspection as if the equations are in conflict. However, if we define

$$D_3 = L_3 + \tfrac{1}{2}[L_1, L_2], \tag{19}$$

where $[L_1, L_2] = L_1L_2 - L_2L_1$, then in view of the previous paragraph we get a correspondence. Because of the Jacobi identity we have

$$[L_{W_i}, L_{W_j}] = -L_{\{W_i, W_j\}}, \tag{20}$$

which implies that

$$S_3 = W_3 - \tfrac{1}{2}\{W_1, W_2\}. \tag{21}$$

Similarly, in the fourth order, the choice

$$D_4 = L_4 + [L_1, L_3] \tag{22}$$

leads to a correspondence, and from that result we deduce the relation

$$S_4 = W_4 - \{W_1, W_3\}. \tag{23}$$

5. Comparison by Computer Program in the Fifth and Sixth Orders

Beyond the fourth order the calculations become unwieldy (the number of terms in the n'th order for each method is 2^n), and automated calculation is highly desirable. Several symbolic programs exist already for the type of formula manipulation which leads to sets of equations like (10) or (18), but in general these programs are not suitable for the final stage of the present calculation, which requires the derivation of equations such as (19) or (22). This is so because conventional symbolic programs usually first expand expressions to be simplified, and then contract the result in size simply by cancellation between terms. A more ingenious design is needed if, for example, a program is intended to generate the result (22) and not stop short at the form $D_4 = L_4 + L_1L_3 - L_3L_1$. One of us (J.A.C.) is now developing a program to deal with simplification of expressions containing noncommuting quantities and to produce results in terms of commutators or anti-commutators. Although the original test example for the program was a problem in elementary-particle physics (Gell-Mann *et al.*, 1968) quite different from the problem examined in this paper, it required only eight days of not very concentrated effort to extend the program to the point where it was also capable of turning out the relations quoted in Equations (19), (22), and (24).

The program has been written in the versatile list-processing language LISP 1.5 (McCarthy *et al.*, 1965). The complete calculation through sixth order occupied 74.1 seconds of central-processor time on a Control Data 6600 computer. It is a good working estimate that, if there exists a general descriptive parameter N for a calculation (e.g., the order N in perturbation theory), then the time needed for a symbolic computation grows at least as fast as an exponential in N. We have found the estimate to be confirmed here, and conclude that a seventh-order calculation, if needed, may take between 6 and 7 minutes of 6600 computing time. Almost certainly, a program written specifically to solve the problem under discussion in this paper will require less time for computation, but the present program is intended to be as general as possible, and to function in several fields other than celestial mechanics.

New test examples to make the program more efficient are still needed, and we shall be pleased to hear of suggestions of such examples.

It does not seem profitable to list all the details of the fifth-order and sixth-order expansions here, as these are best reproduced by use of the recursive schemes described above, rather than by the use of (10) or (18). However, we do give the central results, which are the relationships between the D and L operators:

$$\begin{aligned} D_5 = L_5 &+ \tfrac{3}{2}[L_1, L_4] + [L_2, L_3] + \tfrac{1}{6}[L_1, [L_1, L_3]] \\ &+ \tfrac{1}{2}[[L_1, L_2], L_2] - \tfrac{1}{6}[L_1, [L_1, [L_1, L_2]]], \end{aligned} \tag{24a}$$

$$\begin{aligned} D_6 = L_6 &+ 2[L_1, L_5] + \tfrac{5}{2}[L_2, L_4] - \tfrac{1}{2}[L_1, [L_1, [L_1, L_3]]] \\ &+ 2[L_1, [L_2, L_3]] + \tfrac{5}{2}[[L_1, L_3], L_2] \\ &+ [L_1, [[L_1, L_2], L_2]] + \tfrac{1}{2}[L_1, [L_1, L_4]]. \end{aligned} \tag{24b}$$

These equations are now readily converted into the relations between the quantities S_m and W_n by the use of Equation (20).

6. The Validity of Hori's Formulation

At the beginning of Section 3 of his paper, Deprit makes some remarks concerning the validity of the straightforward expansions in Lie series when W depends on ε. He is here referring to developments of functions $f(y, Y)$ which are equivalent to those used by Hori. Superficially his statement suggests that Hori's method is invalid, but on closer examination of the statement and of Deprit's discussion, it appears that we are actually led to the opposite conclusion.

Equations (1) through (17) of Deprit's paper show the validity of the quoted developments for arbitrary functions which are independent of ε. Now consider a family of functions dependent on ε. For a fixed but arbitrary number $\varepsilon = \varepsilon_0$, these become arbitrary functions independent of ε. Hence equations (1) through (17) of his paper assert the validity of the developments for that fixed value of ε_0. (Here we are using ε_0 merely as a *label* to identify particular functions of the phase variables). But ε_0 is arbitrary, which shows that the expansions are valid for any ε.

The crucial formula is Deprit's Equation (3e), which states that

$$\exp(\varepsilon L_W)\{f, g\} = \{\exp(\varepsilon L_W) f, \exp(\varepsilon L_W) g\}, \tag{25}$$

where

$$\exp(\varepsilon L_W) = \sum_{n\geqslant 0} \frac{\varepsilon^n}{n!} L_W^n. \tag{26}$$

Since the definition of L_W, and hence of $\exp(\varepsilon L_W)$, does not involve the possible dependence of W on ε, (25) holds regardless of any such dependence. Hence by defining

$$x = \exp(\varepsilon L_W)\, y, \quad X = \exp(\varepsilon L_W)\, Y, \tag{27}$$

we find with the use of (25) that the commutation relations for (x, X), from the corresponding ones for (y, Y), are:

$$\begin{aligned} \{x_i, x_j\} &= \{X_i, X_j\} = 0, \\ \{x_i, X_j\} &= 1 \cdot \delta_{ij}, \end{aligned} \tag{28}$$

without regard to the dependence of W on ε. This shows that (x, X) are also canonical variables.

Because the part of the development summarized in Equations (25) through (28) is directly applicable to Hori's theory, it follows that the theory is, in fact, canonical. There is thus no further obstacle to the calculation of relations between the determining functions for the theories of Hori and Deprit in terms of Poisson brackets, as in Equations (20) and (23), to arbitrarily high orders.

Acknowledgements

We wish to thank J. Fred Nurke for computational assistance. The work described in this paper was supported in part by the National Science Foundation, and by the National Aeronautics and Space Administration under NASA Grant No. NGR 44-012-046.

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