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The Hori–Deprit Method for Averaged Motion Equations of the Planetary Problem in Elements of the Second Poincaré System

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Abstract—We consider an algorithm to construct averaged motion equations for four-planetary systems by means of the Hori–Deprit method. We obtain the generating function of the transformation, change-variable functions and right-hand sides of the equations of motion in elements of the second Poincaré system. Analytical computations are implemented by means of the Piranha echeloned Poisson processor. The obtained equations are to be used to investigate the orbital evolution of giant planets of the Solar system and various extrasolar planetary systems.

Keywords: second system of Poincaré elements, Hori–Deprit method, Poisson processor, Poisson brackets, equations of motion, planetary problem

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INTRODUCTION

This is the second work of the series devoted to the dynamical evolution of planetary systems. The first one (Perminov and Kuznetsov, 2015) provides an algorithm to expand the Hamiltonian of an N-planet problem to the Poisson series with respect to elements of the second Poincaré system; for the special case of the four-planet problem, the Hamiltonian expansion is found.

To investigate long-period orbital evolution of planetary systems, one has to apply averaging methods such as the Hori–Deprit method, also known as the Lee transformation method (Kholshchevnikov, 1985). The said method possesses a number of theoretical and practical advantages. All unknown values are expressed via Poisson brackets; this implies their invariance with respect to canonical transformations. Using Poisson brackets, one can pass from osculating elements to mean elements (and vice versa) by means of a simple change of variables. The algorithm of this method can be easily implemented on computers.

The analytical transformations are executed by the Piranha computer algebra system (Biscani, 2015), which is an echeloned Poisson processor implemented by means of the C++ programming language.

We provide an algorithm to construct averaged motion equations for the four-planet problem and the results of the application of that algorithm. Using the Hori–Deprit method, we construct an averaged Hamiltonian of the problem as a series with respect to powers of a small parameter. In averaged elements of the second Poincaré system, we obtain the generating function of the transformation, change-variables equations, and the right-hand side of the equations of motion.

Integrating the equations of motion, one can investigate the orbital evolution of giant planets of the Solar System and various extrasolar planetary systems on cosmogonic time intervals.

HAMILTONIAN OF PLANETARY SYSTEM

We briefly consider the expansion algorithm for the Hamiltonian of the planetary system, provided in (Perminov and Kuznetsov, 2015). The Hamiltonian is expressed by canonical Jacobi coordinates as the sum of the unperturbed Hamiltonian h_0 and the perturbing part μh_1:

$$h = h_0 + \mu h_1. \tag{1}$$

Here μ is a small parameter interpreted as the ratio of the sum of planetary masses over the star mass m_0. For example, for the Solar System, one can assign $\mu = 0.001$. Thus, the mass of the ith planet is equal to $\mu m_0 m_i$, where the values m_i are either less than 1 or of the same order as 1. The unperturbed Hamiltonian of the problem is represented as

$$h_0 = -\sum_{i=1}^{N} \frac{M_i \kappa_i^2}{2a_i}, \tag{2}$$

where a_i is the semimajor axis, $M_i = m_i \overline{m}_{i-1} / \overline{m}_i$ is the reduced mass, $\kappa_i^2 = G m_0 \, \overline{m}_i / \overline{m}_{i-1}$ is the gravitational parameter, $\bar{m}_i = 1 + \mu\sum_{k=1}^{i} m_k$, and the number N of the planets is equal to 4.

The perturbing function h_1 in the Jacobi coordinates is represented as follows:

$$h_1 = Gm_0 \left\{ \sum_{i=2}^{N} \frac{m_i\left(2\mathbf{r}_i\mathbf{R}_i + \mu R_i^2\right)}{r_i\tilde{R}_i\left(r_i + \tilde{R}_i\right)} - \sum_{i=1}^{N}\sum_{j=1}^{i-1} \frac{m_i m_j}{\left|\boldsymbol{\rho}_i - \boldsymbol{\rho}_j\right|} \right\}, \quad (3)$$

where $\mathbf{r}_i$ and $\boldsymbol{\rho}_i$ are the radius vectors of the planet in the Jacobi coordinate system and the barycentric coordinate system, respectively. The values $\mathbf{R}_i$, $\tilde{R}_i$, and $\left|\boldsymbol{\rho}_i - \boldsymbol{\rho}_j\right|$ are defined as follows:

$$\mathbf{R}_i = \sum_{k=1}^{i} \frac{m_k}{\bar{m}_k}\mathbf{r}_k, \quad \tilde{R}_i = \sqrt{r_i^2 + 2\mu\mathbf{r}_i\mathbf{R}_i + \mu^2 R_i^2},$$
$$\left|\boldsymbol{\rho}_i - \boldsymbol{\rho}_j\right| = \mathbf{r}_i - \mathbf{r}_j + \mu\sum_{k=j}^{i-1}\frac{m_k}{\bar{m}_k}\mathbf{r}_k. \quad (4)$$

Taking into account (4), we see that the expansion of terms of the principal part of the perturbing function (3) to the series to the first power of the small parameter is as follows:

$$\frac{1}{\left|\boldsymbol{\rho}_i - \boldsymbol{\rho}_j\right|} = \frac{1}{\Delta_{ij}} - \mu\frac{A_{ij}}{\Delta_{ij}^3} + \ldots . \quad (5)$$

The series expansion of the second part of the perturbing function is as follows:

$$\frac{2\mathbf{r}_i\mathbf{R}_i + \mu R_i^2}{r_i\tilde{R}_i\left(r_i + \tilde{R}_i\right)} = \frac{C_i}{r_i^3} + \mu\left(\frac{1}{2}\frac{D_i}{r_i^3} - \frac{3}{2}\frac{C_i^2}{r_i^5}\right) + \ldots . \quad (6)$$

The values A_{ij}, C_i, and D_i are combinations of scalar products:

$$A_{ij} = \left(\mathbf{r}_i - \mathbf{r}_j\right)\sum_{k=j}^{i-1}\frac{m_k}{\bar{m}_k}\mathbf{r}_k, \quad C_i = \mathbf{r}_i\sum_{k=j}^{i-1}\frac{m_k}{\bar{m}_k}\mathbf{r}_k,$$
$$D_i = \left(\sum_{k=1}^{i-1}\frac{m_k}{\bar{m}_k}\mathbf{r}_k\right)^2. \quad (7)$$

The value $1/\Delta_{ij}$ can be expanded in a series of Legendre polynomials

$$\frac{1}{\Delta_{ij}} = \frac{1}{\left|\mathbf{r}_i - \mathbf{r}_j\right|} = \frac{1}{r_j}\sum_{k=0}^{\infty}\rho_{ij}^k P_k\left(\cos H_{ij}\right), \quad (8)$$

where $\rho_{ij} = r_i/r_j$, H_{ij} is the angle between $\mathbf{r}_i$ and $\mathbf{r}_j$, and P_k is the Legendre polynomial of the degree k.

According to the algorithm provided in (Perminov and Kuznetsov, 2015), scalar products arising in (7) are represented by series of orbital elements. Legendre polynomials are preserved in symbolic form under analytic transformations.

In this work, we use an expansion of the Hamiltonian such that scalar products of radius vectors $\mathbf{r}_i$ and $\mathbf{r}_j$ in relations (7) are expressed via $\cos H_{ij}$, which are the cosines of the angles between the radius vectors $\mathbf{r}_i$ and $\mathbf{r}_j$. In expansions (8), we use known relations to represent Legendre polynomials as polynomials in powers of their independent variable, which is the cosine, while cosines of angles are preserved as symbolic variables for any series. Passing to variables $\cos H_{ij}$, we reduce the number of variables in the expansion of the Hamiltonian, increase the number of its terms, and improve the approximation precision of the series expansion of the Hamiltonian. Properties of the Hamiltonian expansion obtained by means of this algorithm (including the number of terms and estimates of the approximation precision) are provided in (Perminov and Kuznetsov, 2016a, 2016b).

Number coefficients in series terms are preserved in the rational form to exclude rounding errors of analytic transforms.

Functions (5)–(8) are expanded in the Poisson series with respect to elements of the second Poincaré system (Charlier, 1927; Subbotin, 1968) expressed via Kepler elements as follows:

$$L_k = M_k\sqrt{\kappa_k^2 a_k}, \quad \lambda_k = \omega_k + \Omega_k + l_k,$$
$$\xi_{1,k} = \sqrt{2L_k\left(1 - \sqrt{1 - e_k^2}\right)}\cos(\omega_k + \Omega_k),$$
$$\eta_{1,k} = -\sqrt{2L_k\left(1 - \sqrt{1 - e_k^2}\right)}\sin(\omega_k + \Omega_k), \quad (9)$$
$$\xi_{2,k} = \sqrt{2L_k\sqrt{1 - e_k^2}(1 - \cos I_k)}\cos\Omega_k,$$
$$\eta_{2,k} = -\sqrt{2L_k\sqrt{1 - e_k^2}(1 - \cos I_k)}\sin\Omega_k.$$

We say that $\xi_{1,k}$, $\eta_{1,k}$ are eccentric elements, while $\xi_{2,k}$, $\eta_{2,k}$ are oblique ones; in this notation, the second index corresponds to the number of the planet. By a_k, e_k, I_k, Ω_k, ω_k, and l_k we denote the following Kepler elements of the kth planet, respectively: the semimajor axis, the eccentricity, the inclination, the longitude of the ascending node, the pericenter argument, and the mean anomaly of the planet k.

In this element system, L_k and λ_k, $\xi_{1,k}$ and $\eta_{1,k}$, and $\xi_{2,k}$ and $\eta_{2,k}$ are pairwise canonically conjugate variables such that L_k, $\xi_{1,k}$, and $\xi_{2,k}$ play roles of momenta, while λ_k, $\eta_{1,k}$, and $\eta_{2,k}$ play roles of coordinates.

The unperturbed Hamiltonian is expressed via elements of the second Poincaré system as follows:

$$h_0 = -\sum_{k=1}^{N}\frac{M_k^3\kappa_k^4}{2L_k^2}. \quad (10)$$

The Poisson series for the perturbing function h_1 is as follows:

$$h_1 = \sum A_{np} x^p \cos n\lambda, \tag{11}$$

where A_{np} are number coefficients, x^p is the product of the corresponding powers of positional Poincaré elements, and $n\lambda$ is the argument of the angular part of the expansion. In the explicit form, we have

$$\begin{gathered} x^p = L_1^{p_1}\,\xi_{1,1}^{p_2}\,\eta_{1,1}^{p_3}\,\xi_{2,1}^{p_4}\,\eta_{2,1}^{p_5}\,\ldots\,L_N^{p_{5N-4}}\,\xi_{1,N}^{p_{5N-3}}\,\eta_{1,N}^{p_{5N-2}}\,\xi_{2,N}^{p_{5N-1}}\,\eta_{2,N}^{p_{5N}}, \\ n\lambda = n_1\lambda_1 + n_2\lambda_2 + \ldots + n_N\lambda_N. \end{gathered} \tag{12}$$

Two expansions are constructed for the Hamiltonian of the problem: the expansion to the second degree of a small parameter and to the third degree. In this paper, we construct equations of motion of the planetary problem, using the first expansion, i.e., the expansion to the second degree of a small parameter. This expansion approximates the Hamiltonian of the system with the relative precision of 10^{-9} (Perminov and Kuznetsov, 2016a, 2016b); it can be used to construct the equations of motion for extrasolar planetary systems with moderate values of eccentricities and orbit inclinations.

Series for the Hamiltonian of the planetary system are constructed in osculating elements. To investigate the orbital evolution on cosmogonic time intervals, one has to pass to averaged elements. To do that, we use the Hori–Deprit method.

THE HORI–DEPRIT METHOD

The Hori–Deprit method is based on the use of Poisson brackets. All unknowns are invariant with respect to the canonical transformations. The new Hamiltonian is provided in a clear and simple form. Constraint equations (between old and new orbit elements) are expressed explicitly as well (unlike other averaging methods providing recurrent relations).

In our problem, we decompose the set of all orbital elements into two groups according to the velocity of their variation (fast and slow elements). Fast variables are mean longitudes $\lambda = \{\lambda_1, \ldots, \lambda_N\}$ such that their variation period is close to the motion period along the orbit. All other variables are slow. They are denoted by the vector $x = \{x_1, \ldots, x_{5N}\}$ such that its components are defined as follows:

$$\begin{gathered} x_{5k-4} = L_k, \quad x_{5k-3} = \xi_{1,k}, \quad x_{5k-2} = \eta_{1,k}, \\ x_{5k-1} = \xi_{2,k}, \quad x_{5k} = \eta_{2,k}. \end{gathered} \tag{13}$$

The corresponding averaged of the Poincaré elements are denoted by $\Lambda = \{\Lambda_1, \ldots, \Lambda_N\}$ and $X = \{X_1, \ldots, X_{5N}\}$, where the components of the vector X are denoted as follows:

$$\begin{gathered} X_{5k-4} = \mathrm{L}_k, \quad X_{5k-3} = \Xi_{1,k}, \quad X_{5k-2} = \mathrm{H}_{1,k}, \\ X_{5k-1} = \Xi_{2,k}, \quad X_{5k} = \mathrm{H}_{2,k}. \end{gathered} \tag{14}$$

Here the index k is changed from 1 to N.

The Hamiltonian of the problem is averaged with respect to the fast variables, i.e., with respect to the mean longitudes. The averaged Hamiltonian can be represented as the following series in powers of the small parameter

$$H(X) = H_0(X) + \sum_{m=1}^{\infty} \mu^m H_m(X), \tag{15}$$

where H_0 is the unperturbed Hamiltonian in averaged elements. Similarly, (10) depends only on L_k.

To find H_m, we use the main equation of the Hori–Deprit method (Kholshchevnikov, 1985; Kuznetsov and Kholshchevnikov, 2006):

$$H_m = h_m + \sum \frac{1}{r!}\{T_{j_r}, \ldots, T_{j_1}, h_{j0}\}. \tag{16}$$

Here the summation indices satisfy the following conditions: $0 \le j_0 \le m-1$; $j_1, j_2, \ldots, j_r \ge 1$; $\sum_{s=0}^{k} j_s = m$; $1 \le r \le m$. Here the braces denote the multiple Poisson brackets computed from the right to the left as follows:

$$\{f_4, f_3, f_2, f_1\} = \{f_4, \{f_3, \{f_2, f_1\}\}\}. \tag{17}$$

The order of action is important for the computation of such brackets because they are noncommutative and nonassociative.

The generating function of the transformation T is defined by a series in powers of the small parameter μ as well:

$$T(X, \Lambda) = \sum_{m=1}^{\infty} \mu^m T_m(X, \Lambda). \tag{18}$$

For example, the initial terms of (15) to the third degree of the small parameter are as follows:

$$\begin{gathered} H_0 = h_0, \quad H_1 = \{T_1, h_0\} + h_1, \quad H_2 = \{T_2, h_0\} \\ + \{T_1, h_1\} + \frac{1}{2}\{T_1, T_1, h_0\}, \quad H_3 = \{T_3, h_0\} \\ + \{T_2, h_1\} + \frac{1}{2}(\{T_2, T_1, h_0\} + \{T_1, T_1, h_1\}) + \frac{1}{6}\{T_1, T_1, T_1 h_0\}. \end{gathered} \tag{19}$$

Here h_0 and h_1 are the unperturbed Hamiltonian and the perturbing function of the problem, respectively. In our case, we have $h_i = 0$ for $i > 1$.

The main equation of the method is solved as follows. At each computation step, relation (16) can be represented in the form

$$H_m = \{T_m, h_0\} + \Phi_m, \quad (20)$$

where Φ_m is defined at the previous step of the algorithm. Representing the Poisson bracket explicitly in that relation, we see that

$$\Phi_m = H_m + \sum_{k=1}^{N} \omega_k \frac{\partial T_m}{\partial \Lambda_k}, \quad (21)$$

where $\omega_k = \partial h_0 / \partial X_{5k-4}$ is the variation frequency of the fast variable Λ_k (the mean motion). In the general case, the functions Φ_m are echeloned Poisson series, i.e.,

$$\Phi_m(X,\Lambda) = \sum_{k,n} B_{kn} X^k \cos(n\Lambda), \quad (22)$$

where the coefficients of the echeloned Poisson series are defined as follows:

$$B_{kn} = \sum_i \frac{A_{kn}^{(i)}}{\prod_j (n^{(i,j)}\omega)^{q(i,j)}}.$$

Here A_{kn} are number coefficients, the denominator contains the product of various linear combinations of frequencies ω of fast variables, n is the vectors of coefficients of the frequencies, and q are integer degrees of the denominator.

Let the components of the averaged Hamiltonian have the form

$$H_m(X) = \sum_{\substack{k \\ n \in I_0}} B_{kn} X^k, \quad (23)$$

where $I_0 = \{n : n_1 = \ldots = n_N = 0\}$. Then Eq. (21) has the formal solution

$$T_m(X,\Lambda) = \sum_{\substack{k,n \\ n \in I_1}} \frac{B_{kn}}{n\omega} X^k \sin(n\Lambda), \quad (24)$$

where $I_1 = \{n : n_1^2 + \ldots + n_N^2 \neq 0\}$. Thus, if the generating function T is known, then one can obtain the averaged Hamiltonian H of the system.

Equations for the change of variables providing the relation between averaged elements and osculating elements are as follows:

$$x = X + \sum_{m=1}^{\infty} \mu^m u_m(X,\Lambda), \ \lambda = \Lambda + \sum_{m=1}^{\infty} \mu^m v_m(X,\Lambda). (25)$$

To pass from osculating elements to averaged ones, we use the following transformations:

$$X = x + \sum_{m=1}^{\infty} (-1)^m \mu^m u_m(x,\lambda),$$
$$\Lambda = \lambda + \sum_{m=1}^{\infty} (-1)^m \mu^m v_m(x,\lambda). \quad (26)$$

The said transformations can be obtained from (25) if T is changed to $-T$ and (X, Λ) is changed to (x, λ).

To find the expressions for the change of variables functions for the variables u_m and v_m, one has to use relation (16), where h_0 is assigned to be equal to X_i or Λ_i, $h_1 = h_2 = \ldots = 0$, and H_m is assigned to be equal to u_{im} or v_{im}.

$$u_{im} = \sum \frac{1}{r!}\{T_{j_r}, \ldots, T_{j_1}, X_i\}, \quad i = 1, \ldots, 20,$$
$$v_{im} = \sum \frac{1}{r!}\{T_{j_r}, \ldots, T_{j_1}, \Lambda_i\}, \quad i = 1, \ldots, 4. \quad (27)$$

Here the summation region is defined as follows: $j_1, j_2, \ldots, j_r \geq 1; \sum_{s=0}^{k} j_s = m; 1 \leq r \leq m.$

In averaged elements, the equations of motion have the form:

$$\frac{\mathrm{d}X}{\mathrm{d}t} = \{H, X\}, \quad \frac{\mathrm{d}\Lambda}{\mathrm{d}t} = \{H, \Lambda\}. \quad (28)$$

Equations of motion for slow variables X do not depend on fast variables Λ. Thus, we obtain two independent systems of equations describing the evolution of a planetary system on cosmogonic time intervals.

THE PIRANHA COMPUTER ALGEBRA SYSTEM

The Hamiltonian expansion used in this work and the algorithms of the Hori–Deprit method are implemented by means of the Piranha computer algebra system authored by Biscani (Max Planck Institute for Astronomy, Heidelberg, Germany). In the present paper, a new version of that system is used (Biscani, 2015). Its computation speed is higher and it uses the operating storage more efficiently. Also, the new version supports echeloned Poisson series necessary for the implementation of the algorithms of the Hori–Deprit method.

The Piranha computer algebra system created by means of the C++ programming language is cross-platform software. For convenience, all user functions are compiled into a program library set available through the Python language. It is possible to use the system either through a standard Python terminal or via executable scripts.

The following analytic operations are accessible in the framework of the system.

Series adding and multiplying is implemented by means of the standard mathematical operators "+", "–", and "*".

Binomial expansions computations are implemented automatically once an expression of the kind $(1+s)^q$, where s is a series and q is an arbitrary rational number is inputted to the terminal. The user has to indicate the greatest degree of the variable such that expansion terms up to that degree are saved.

Series truncation up to terms of a given order is implemented automatically by means of the invoking of the global function *set_auto_truncate_degree*(*max_-degree*, *names*), where *max_degree* is the greatest degree of the variables indicated in *names*: expansion terms of degrees up to that value are saved. If the degree sum of a term is greater than that value, then the term is excluded from the expansion. To cancel a set truncation rule, one has to use the corresponding global function *unset_auto_truncate_degree*().

To substitute another series (or a value) in a series, one has to invoke the function *subs*(*arg*1, *name*, *arg*2): another series or the value *arg*2 is substituted in the variable *name* of the series *arg*1.

To estimate the series *arg* by a collection of scalar values such that each one corresponds to its variable, one has to invoke the function *evaluate*(*arg*, *eval_dict*), where *eval_dict* is the dictionary consisting of pairs "variable name" – "number". A scalar value is outputted.

To integrate or differentiate the series *arg* with respect to the variable *name*, one has to use the function *integrate*(*arg*, *name*) or *partial*(*arg*, *name*) respectively.

To integrate the angular part s of the Poisson series with respect to time, one has to use the method *s.t_integrate*();

To differentiate an expression with respect to an implicit independent variable *name* included in the expression, one has to invoke the global function *register_custom_derivative*(*name*, *lambda_expr*) defining a superposition derivative and using it for the differentiating with respect to the variable *name* afterwards. Here *lambda_expr* is a λ-expression describing the superposition derivative.

To save the series *arg* to the file *path_to_file* and to download it afterwards, one has to use the functions *save*(*arg*, *path_to_file*) and *load*(*path_to_file*), respectively.

LEE TRANSFORMATIONS

We implement the algorithm to average the Hamilton function and to compute the generating function of the transformation. The unperturbed Hamiltonian of the problem does not depend on mean longitudes and has not to be averaged, i.e., $H_0 = h_0$. Functions Φ_m computed below are functions of averaged Poincaré elements.

For the first approximation of the Hori–Deprit method, we assign $\Phi_1 = h_1$. Terms of the averaged Hamiltonian from H_1 correspond to terms from Φ_1 without the trigonometric part. To find T_1, one has to integrate the trigonometric part of Φ_1 with respect to t. Since the mean longitude is expressed by the relation $\Lambda_k = \omega_k t$, it follows that, integrating the trigonometric part of (21), we obtain the expression for T_1.

Once the expression for T_1 is known, one can find the second Hori–Deprit approximations of H_2 and T_2. To do that, we construct Φ_2 as follows:

$$\Phi_2 = \{T_1, h_1\} + \frac{1}{2}\{T_1, T_1, h_0\}. \tag{29}$$

The Poisson bracket $\{T_1, h_1\}$ is represented as the following sum:

$$\begin{aligned}\{T_1, h_1\} = \sum_{k=1}^{N}\Bigg(&\frac{\partial T_1}{\partial L_k}\frac{\partial h_1}{\partial \Lambda_k} - \frac{\partial T_1}{\partial \Lambda_k}\frac{\partial h_1}{\partial L_k} + \frac{\partial T_1}{\partial \Xi_{1,k}}\frac{\partial h_1}{\partial H_{1,k}} - \\ &- \frac{\partial T_1}{\partial H_{1,k}}\frac{\partial h_1}{\partial \Xi_{1,k}} + \frac{\partial T_1}{\partial \Xi_{2,k}}\frac{\partial h_1}{\partial H_{2,k}} - \frac{\partial T_1}{\partial H_{2,k}}\frac{\partial h_1}{\partial \Xi_{2,k}}\Bigg).\end{aligned} \tag{30}$$

Thus, we have to compute first partial derivatives of T_1 and h_1 with respect to all elements of the second Poincaré system.

For the Poisson bracket $\{T_1, h_0\}$, we have the relation

$$\{T_1, h_0\} = -\sum_{k=1}^{N}\frac{\partial T_1}{\partial \Lambda_k}\frac{\partial h_0}{\partial L_k} = \sum_{k=1}^{N}\frac{M_k^3\kappa_k^4}{L_k^3}\frac{\partial T_1}{\partial \Lambda_k}, \tag{31}$$

which allows us to express the bracket $\{T_1, T_1, h_0\}$,similarly to (30): just h_1 is changed for expression (31). Thus, it is necessary to compute both first and second partial derivatives of T_1 with respect to orbital elements.

Further, similarly to the first approximation of the method, we put terms from Φ_2 without the trigonometric part in correspondence to H_2 and define T_2 as the integral of the trigonometric part of Φ_2 with respect to time. This yields the averaged Hamiltonian of the problem in the form $H = H_0 + \mu H_1 + \mu^2 H_2$ and the generating function of the transformation in the form $T = \mu T_1 + \mu^2 T_2$.

Note that h_1 (and, therefore, T_1) contains terms with μ^1 apart from terms with μ^0. Therefore, H_1 should contain only terms from h_1 such that they do not contain the small parameter. Once we multiply derivatives in (30), terms containing the first and the second degree of the small parameter arise. This can be taken into account at further approximations of the averaging method. In H_2, only terms without the small parameter should be taken into account.

Functions of the change of variables u_1 and v_1 are the Poisson brackets of the function T_1 with

$$u_{k,1} = \{T_1, X_k\}, \quad v_{j,1} = \{T_1, \Lambda_j\}, \tag{32}$$

Table 1. The number of terms in series representing the averaged Hamiltonian of the problem and generating function of the transformation

H_0	H_1	H_2	T_1	T_2
4	12906	26190371	365556	632478386

respectively, which reduces to the computation of the partial derivative of the function T_1 with respect to the canonically conjugate element with the corresponding sign. All such derivatives are computed in (30). The index k is running through the values $k = 1, \ldots, 20$, while $j = 1, \ldots, 4$.

The functions u_2 and v_2 are the sum of the two Poisson brackets:

$$u_{k,2} = \{T_2, X_k\} + \frac{1}{2}\{T_1, T_1, X_k\}, \quad v_{j,2} = \{T_2, \Lambda_j\} + \frac{1}{2}\{T_1, T_1, \Lambda_j\}. \tag{33}$$

The brackets $\{T_1, X_k\}$ and $\{T_1, \Lambda_j\}$ are computed in (31), while the second-order partial derivatives of T_1 are obtained during the computation of the Poisson bracket $\{T_1, T_1, h_0\}$.

Once the brackets in (28) are computed, the second Hori–Deprit approximations of the equations of motion of the kth planet take the form

$$\begin{gathered}\frac{\partial L_k}{\partial t} = 0, \quad \frac{\partial \Lambda_k}{\partial t} = \frac{M_k^3 \kappa_k^4}{L_k^3} + \mu \frac{\partial H_1}{\partial L_k} + \mu^2 \frac{\partial H_2}{\partial L_k}, \\ \frac{\partial \Xi_{1,k}}{\partial t} = -\mu \frac{\partial H_1}{\partial H_{1,k}} - \mu^2 \frac{\partial H_2}{\partial H_{1,k}}, \quad \frac{\partial \eta_{1,k}}{\partial t} = \mu \frac{\partial H_1}{\partial \Xi_{1,k}} + \mu^2 \frac{\partial H_2}{\partial \Xi_{1,k}}, \quad \frac{\partial \Xi_{2,k}}{\partial t} = -\mu \frac{\partial H_1}{\partial H_{2,k}} - \mu^2 \frac{\partial H_2}{\partial H_{2,k}}, \quad \frac{\partial \eta_{2,k}}{\partial t} = \mu \frac{\partial H_1}{\partial \Xi_{2,k}} + \mu^2 \frac{\partial H_2}{\partial \Xi_{2,k}}.\end{gathered} \tag{34}$$

If the series are differentiated by means of the Piranha system, then the derivatives of cosines of angles between radius vectors with respect to orbital elements are preserved in the symbolic form. This allows us to reduce the memory usage and the number of terms in the resulting series.

RESULTS

Analytical transformations are implemented by means of a personal computer with the quad-core processor Core i5 with a clock speed of 3600 MHz and 32 GB of RAM. For resource-intensive computations, up to 60 GB of the virtual memory were used. To implement all transformations, we used echeloned Poisson series with coefficients in the fractional rational form.

Two Hori–Deprit approximations are implemented. We obtain series for the averaged Hamiltonian of the problem (i.e., H_1 and H_2) and the generating function of the transformation between osculating elements and averaged elements (i.e., T_1 and T_2). In Table 1, we provide the number of terms of the series representing the averaged Hamiltonian of the problem $H = H_0 + \mu H_1 + \mu^2 H_2$ and the generating function of the transformation $T = \mu T_1 + \mu^2 T_2$.

For the averaged Hamiltonian H_1 and generating function T_1, the greatest sum of degrees of eccentric and oblique elements is retained at 5 (as for the nonaveraged Hamiltonian h_1). While we compute Φ_2 (this value defines terms of the Hamiltonian and generating function at the second power of the small parameter), the greatest sum of degrees of eccentric and oblique elements is set equal to 2. Other terms are truncated under analytical transformations.

Further, we construct the first Hori–Deprit approximation for the expansions of the right-hand parts of the equations of motion for the four-planet problem in averaged elements and of the change-variable function. Two different variants of those expansions are constructed. In the first case, derivatives of cosines of angles with respect to orbital elements are expressed as series with respect to orbital elements (under analytical transformations). In the second case, derivatives of cosines of angles between radius vectors are preserved in the form of corresponding symbolic variables.

In Tables 2 and 3, for any orbital element, we provide the number of terms contained in the right-hand sides of the equations of motion and change-variable functions, respectively. The column N_1 of each table displays the number of terms for the case where derivatives of cosines of angles are expressed via corresponding expansions; the column N_2 displays the same number for the case where those derivatives are preserved in the symbolic form.

We see that the equations of motion for elements that are analogues of semimajor axes contain no terms. This corresponds to the assumptions of the Lagrange–Laplace theorem on the absence of secular perturbations of the semimajor axes of planetary orbits.

If symbolic variables are used to denote derivatives, then the number of terms of the series representing the right-hand sides of the equations of motion becomes two orders less. The advantage of the symbolic notation for derivatives of cosines of angles in change-variable functions is especially clear for the case of oblique orbit elements. The reason is that oblique elements in the explicit form arise only when cosines of angles are differentiated.

In the second Hori–Deprit approximation, the expansions for right-hand sides of the equations of motion are constructed by means of symbolic variables used to denote first derivatives of cosines of

Table 2. The number of terms in series representing right-hand sides of the equations of motion in averaged elements for the first approximation of the Hori–Deprit method

	N_1	N_2		N_1	N_2		N_1	N_2
L_1	0	0	$\xi_{1,1}$	711771	7110	$\xi_{2,1}$	463150	5235
L_2	0	0	$\xi_{1,2}$	927747	9235	$\xi_{2,2}$	603740	6810
L_3	0	0	$\xi_{1,3}$	927713	9235	$\xi_{2,3}$	603735	6810
L_4	0	0	$\xi_{1,4}$	711669	7110	$\xi_{2,4}$	463135	5235
λ_1	784429	10842	$\eta_{1,1}$	711771	7110	$\eta_{2,1}$	463150	5235
λ_2	1022441	14074	$\eta_{1,2}$	927747	9235	$\eta_{2,2}$	603740	6810
λ_3	1022448	14081	$\eta_{1,3}$	927713	9235	$\eta_{2,3}$	603735	6810
λ_4	784450	10863	$\eta_{1,4}$	711669	7110	$\eta_{2,4}$	463135	5235

Table 3. The number of terms in series representing the first Hori–Deprit approximation of the change-variable functions

	N_1	N_2		N_1	N_2		N_1	N_2
L_1	335263	280380	$\xi_{1,1}$	334373	242679	$\xi_{2,1}$	542249	148152
L_2	470434	364108	$\xi_{1,2}$	454826	315272	$\xi_{2,2}$	708920	192882
L_3	510454	364166	$\xi_{1,3}$	478125	315305	$\xi_{2,3}$	708861	192882
L_4	412372	280554	$\xi_{1,4}$	381776	242778	$\xi_{2,4}$	542062	148152
λ_1	322142	307074	$\eta_{1,1}$	334373	242679	$\eta_{2,1}$	542249	148152
λ_2	446171	398806	$\eta_{1,2}$	454826	315272	$\eta_{2,2}$	708920	192882
λ_3	472703	398948	$\eta_{1,3}$	478125	315305	$\eta_{2,3}$	708861	192882
λ_4	380013	307500	$\eta_{1,4}$	381776	242778	$\eta_{2,4}$	542062	148152

angles with respect to orbital elements. To reduce the number of terms and the time required for analytical computations, second derivatives of cosines of angles with respect to orbital elements are preserved as symbolic variables in change-of-variable functions as well.

In Table 4, we provide an estimate of the number of terms contained in series representing the right-hand sides of the equations of motion for orbital elements. In Table 5, we provide an estimate of the number of terms contained in series for change-variable functions for any orbital element.

Series representing the right-hand sides of the equations of motion and functions changing variables for numeric values are estimated as well. In Table 6, values of elements of the second Poincaré system are provided for orbits of giant planets of the Solar System

Table 4. The number of terms in series representing right-hand sides of the equations of motion in averaged elements for the second approximation of the Hori–Deprit method

	Number of terms		Number of terms		Number of terms
L_1	0	$\xi_{1,1}$	25.2×10^6	$\xi_{2,1}$	21.5×10^6
L_2	0	$\xi_{1,2}$	33.8×10^6	$\xi_{2,2}$	28.8×10^6
L_3	0	$\xi_{1,3}$	33.1×10^6	$\xi_{2,3}$	28.8×10^6
L_4	0	$\xi_{1,4}$	24.6×10^6	$\xi_{2,4}$	21.7×10^6
λ_1	41.5×10^6	$\eta_{1,1}$	25.2×10^6	$\eta_{2,1}$	21.5×10^6
λ_2	52.4×10^6	$\eta_{1,2}$	33.8×10^6	$\eta_{2,2}$	28.8×10^6
λ_3	52.6×10^6	$\eta_{1,3}$	33.1×10^6	$\eta_{2,3}$	28.8×10^6
λ_4	42.8×10^6	$\eta_{1,4}$	24.6×10^6	$\eta_{2,4}$	21.7×10^6

Table 5. The number of terms in series representing the second Hori–Deprit approximation of the change-variable functions

	Number of terms		Number of terms		Number of terms
L_1	108×10^6	$\xi_{1,1}$	116×10^6	$\xi_{2,1}$	53×10^6
L_2	144×10^6	$\xi_{1,2}$	145×10^6	$\xi_{2,2}$	71×10^6
L_3	144×10^6	$\xi_{1,3}$	145×10^6	$\xi_{2,3}$	71×10^6
L_4	108×10^6	$\xi_{1,4}$	116×10^6	$\xi_{2,4}$	53×10^6
λ_1	138×10^6	$\eta_{1,1}$	116×10^6	$\eta_{2,1}$	53×10^6
λ_2	176×10^6	$\eta_{1,2}$	145×10^6	$\eta_{2,2}$	71×10^6
λ_3	176×10^6	$\eta_{1,3}$	145×10^6	$\eta_{2,3}$	71×10^6
λ_4	138×10^6	$\eta_{1,4}$	116×10^6	$\eta_{2,4}$	53×10^6

Table 6. Elements of the second Poincaré system for orbits of the giant planets of the Solar System to the epoch of J2000.0 with respect to the mean ecliptic (to the date Jan. 1, 2000)

	Jupiter	Saturn	Uranus	Neptune
L	0.037445975	0.015184844	0.0032900915	0.0048591503
λ	1.51369753	1.61615531	2.98371499	2.98389800
ξ_1	9.0581952×10^{-3}	$-3.0106342 \times 10^{-4}$	$-2.6776907 \times 10^{-3}$	4.2369519×10^{-4}
η_1	$-2.3811861 \times 10^{-3}$	$-6.6327968 \times 10^{-3}$	$-4.2629530 \times 10^{-4}$	$-4.2317436 \times 10^{-4}$
ξ_2	3.2663278×10^{-4}	4.9853915×10^{-3}	$-9.3372864 \times 10^{-5}$	1.1947330×10^{-4}
η_2	4.3906355×10^{-3}	1.9200592×10^{-3}	$-7.6739691 \times 10^{-4}$	2.1500408×10^{-3}

(such as Jupiter, Saturn, Uranus, and Neptune), computed using relations (9) based on Kepler elements. Here, semimajor axes of orbits are expressed in astronomical units, while angular elements are expressed in radians. The day was selected as the unit of time. Kepler elements of planetary orbits are taken with respect to the mean ecliptic to the epoch of equinox J2000.0 according to http://www.ssd.jpl.nasa.gov. Planetary masses correspond to the system of constants of the DE421 ephemeris. We treat orbital elements provided in Table 6 as osculating ones.

In Table 7, estimates of the right-hand sides of the equations of motion are presented for all orbital elements of two Hori–Deprit approximations. The corresponding columns of the table display estimates of terms at the first and second power of the small parameter in the right-hand sides of the equations of motion. The contribution of the corresponding powers of the small parameter is taken into account in the numerical estimate. The results are provided for mean values of Poincaré elements obtained due to relations (25) by means of change-of-variable functions. To obtain them, we used estimates of change-of-variable functions for the Poincaré elements presented in Table 8 and obtained by means of the first and second Hori–Deprit approximations. To obtain those estimates, we use the numerical values of the orbital elements, provided in Table 6.

For example, we express the series representing the right-hand side of the equation of motion for the element $\Xi_{1,1}$ to within first degree for the eccentric elements

$$
\begin{aligned}
\frac{1}{Gm_0}\frac{\partial \Xi_{1,1}}{\partial t} &= \mu m_1 m_2 \left\{ \sum_{k=2}^{13} \left(a_k \frac{\kappa_2^{4k-2} M_2^{4k-2} L_1^{2k-1}}{\kappa_1^{4k-4} M_1^{4k-4} L_2^{4k-2}} \right) H_{1,1} \right. \\
&\quad \left. + \sum_{k=2}^{13} \left(b_k \frac{\kappa_2^{4k} M_2^{4k} L_1^{\frac{8k-5}{2}}}{\kappa_1^{4k-2} M_1^{4k-2} L_2^{\frac{8k+5}{2}}} \right) H_{1,2} \right\} + \mu m_1 m_3 \\
&\quad \times \left\{ \sum_{k=2}^{11} \left(a_k \frac{\kappa_3^{4k-2} M_3^{4k-2} L_1^{2k-1}}{\kappa_1^{4k-4} M_1^{4k-4} L_3^{4k-2}} \right) H_{1,1} \right. \\
&\quad \left. + \sum_{k=2}^{10} \left(b_k \frac{\kappa_3^{4k} M_3^{4k} L_1^{\frac{8k-5}{2}}}{\kappa_1^{4k-2} M_1^{4k-2} L_3^{\frac{8k+5}{2}}} \right) H_{1,3} \right\} + \mu m_1 m_4 \\
&\quad \times \left\{ \sum_{k=2}^{8} \left(2a_k \frac{\kappa_4^{4k-2} M_4^{4k-2} L_1^{2k-1}}{\kappa_1^{4k-4} M_1^{4k-4} L_4^{4k-2}} \right) H_{1,1} \right. \\
&\quad \left. + \sum_{k=2}^{8} \left(2b_k \frac{\kappa_4^{4k} M_4^{4k} L_1^{\frac{8k-5}{2}}}{\kappa_1^{4k-2} M_1^{4k-2} L_4^{\frac{8k+5}{2}}} \right) H_{1,4} \right\}.
\end{aligned}
\tag{35}
$$

Table 7. Estimates of series representing right-hand sides of the equations of motion by numerical values

	μ	μ^2		μ	μ^2		μ	μ^2
L_1	0	0	$\xi_{1,1}$	-3.28×10^{-9}	2.05×10^{-10}	$\xi_{2,1}$	-3.70×10^{-10}	3.06×10^{-11}
L_2	0	0	$\xi_{1,2}$	-1.95×10^{-8}	6.64×10^{-10}	$\xi_{2,2}$	6.74×10^{-10}	-5.39×10^{-11}
L_3	0	0	$\xi_{1,3}$	1.69×10^{-9}	-6.78×10^{-11}	$\xi_{2,3}$	1.61×10^{-10}	-3.17×10^{-12}
L_4	0	0	$\xi_{1,4}$	3.84×10^{-9}	-4.17×10^{-11}	$\xi_{2,4}$	-2.14×10^{-10}	2.45×10^{-12}
λ_1	-1.21×10^{-6}	7.90×10^{-8}	$\eta_{1,1}$	-1.92×10^{-8}	1.02×10^{-9}	$\eta_{2,1}$	-1.02×10^{-8}	8.45×10^{-10}
λ_2	5.45×10^{-6}	-2.34×10^{-7}	$\eta_{1,2}$	6.27×10^{-9}	-1.35×10^{-9}	$\eta_{2,2}$	1.60×10^{-8}	-1.33×10^{-9}
λ_3	4.21×10^{-7}	-5.20×10^{-10}	$\eta_{1,3}$	9.26×10^{-6}	-2.52×10^{-11}	$\eta_{2,3}$	5.57×10^{-11}	-6.32×10^{-13}
λ_4	3.30×10^{-7}	-1.19×10^{-9}	$\eta_{1,4}$	5.49×10^{-10}	-2.14×10^{-11}	$\eta_{2,4}$	-6.12×10^{-11}	5.38×10^{-13}

Table 8. Estimates of change-variable functions by numerical values

	μ	μ^2		μ	μ^2		μ	μ^2
L_1	4.20×10^{-7}	-1.10×10^{-6}	$\xi_{1,1}$	-6.82×10^{-6}	4.21×10^{-8}	$\xi_{2,1}$	8.84×10^{-8}	-2.11×10^{-8}
L_2	-4.35×10^{-6}	1.13×10^{-6}	$\xi_{1,2}$	-2.01×10^{-5}	-1.05×10^{-6}	$\xi_{2,2}$	-1.55×10^{-7}	-1.39×10^{-7}
L_3	-1.79×10^{-7}	1.15×10^{-8}	$\xi_{1,3}$	-6.22×10^{-5}	1.12×10^{-6}	$\xi_{2,3}$	1.44×10^{-7}	-1.46×10^{-9}
L_4	5.61×10^{-8}	4.70×10^{-9}	$\xi_{1,4}$	-1.10×10^{-4}	-5.89×10^{-7}	$\xi_{2,4}$	-1.22×10^{-7}	7.69×10^{-10}
λ_1	2.84×10^{-4}	-1.03×10^{-5}	$\eta_{1,1}$	1.10×10^{-4}	9.79×10^{-6}	$\eta_{2,1}$	2.42×10^{-6}	-1.23×10^{-7}
λ_2	-6.80×10^{-4}	2.07×10^{-5}	$\eta_{1,2}$	-6.41×10^{-4}	-6.59×10^{-6}	$\eta_{2,2}$	-3.80×10^{-6}	2.27×10^{-7}
λ_3	1.04×10^{-7}	-4.33×10^{-7}	$\eta_{1,3}$	-9.25×10^{-6}	5.05×10^{-7}	$\eta_{2,3}$	2.19×10^{-8}	1.17×10^{-9}
λ_4	-5.23×10^{-6}	-4.72×10^{-8}	$\eta_{1,4}$	-1.84×10^{-5}	-2.71×10^{-6}	$\eta_{2,4}$	-2.12×10^{-8}	-5.53×10^{-9}

Replacing variable $u_{5,1}$ for the element $\Xi_{1,1}$ is represented as follows (up to terms such that $\Sigma|n_s| = 1$):

$$\begin{aligned}
\frac{1}{Gm_0}u_{5,1} &= \mu m_1 m_2 \left\{ \sum_{k=2}^{13} \left(c_k \frac{\kappa_2^{4k-2} M_2^{4k-2} L_1^{\frac{8k-9}{2}}}{\kappa_1^{4k-4} M_1^{4k-4} L_2^{4k-2}} \right) \frac{\cos\lambda_1}{\omega_1} \right. \\
&\left. + \sum_{k=2}^{14} \left(d_k \frac{\kappa_2^{4k-4} M_2^{4k-4} L_1^{\frac{8k-13}{2}}}{\kappa_1^{4k-6} M_1^{4k-6} L_2^{4k-4}} \right) \frac{\cos\lambda_2}{\omega_2} \right\} + \mu m_1 m_3 \\
&\times \left\{ \sum_{k=2}^{11} \left(c_k \frac{\kappa_2^{4k-2} M_2^{4k-2} L_1^{\frac{8k-9}{2}}}{\kappa_1^{4k-4} M_1^{4k-4} L_2^{4k-2}} \right) \frac{\cos\lambda_1}{\omega_1} \right. \\
&\left. + \sum_{k=2}^{11} \left(d_k \frac{\kappa_2^{4k-4} M_2^{4k-4} L_1^{\frac{8k-13}{2}}}{\kappa_1^{4k-6} M_1^{4k-6} L_2^{4k-4}} \right) \frac{\cos\lambda_3}{\omega_3} \right\} + \mu m_1 m_4 \qquad (36) \\
&\times \left\{ \sum_{k=2}^{8} \left(c_k \frac{\kappa_2^{4k-2} M_2^{4k-2} L_1^{\frac{8k-9}{2}}}{\kappa_1^{4k-4} M_1^{4k-4} L_2^{4k-2}} \right) \frac{\cos\lambda_1}{\omega_1} \right. \\
&\left. + \sum_{k=2}^{9} \left(d_k \frac{\kappa_2^{4k-4} M_2^{4k-4} L_1^{\frac{8k-13}{2}}}{\kappa_1^{4k-6} M_1^{4k-6} L_2^{4k-4}} \right) \frac{\cos\lambda_4}{\omega_4} \right\} + \mu \frac{3}{2} \frac{m_1}{\overline{m}_1} \\
&\times \left\{ m_2 \frac{\kappa_2^4 M_2^4 L_1^{\frac{3}{2}}}{\kappa_1^2 M_1^2 L_2^4} + m_3 \frac{\kappa_3^4 M_3^4 L_1^{\frac{3}{2}}}{\kappa_1^2 M_1^2 L_3^4} + m_4 \frac{\kappa_4^4 M_4^4 L_1^{\frac{3}{2}}}{\kappa_1^2 M_1^2 L_4^4} \right\}.
\end{aligned}$$

For convenience, series variables for cosines of angles between the radius vectors of planets are substituted in (35) and (36) accordingly.

To obtain the equation for the element $H_{1,1}$, one can change all eccentric and oblique elements of (35) for canonically conjugate ones and change the sign of each term. To obtain the replacement function $u_{6,1}$ for the element $H_{1,1}$, one can change all cosines of expression (36) for sines and change the sign of each of its terms. In Table 9, the numerical values of the coefficients a_k, b_k, c_k, and d_k are provided.

These expansions are to be used to construct averaged motion equations for the investigation of the long-period evolution of planetary systems. The change-variable functions are to be used to find estimates of short-period perturbations of orbital elements.

CONCLUSIONS

We present an algorithm of the Hori–Deprit method to construct averaged motion equations for the four-planet problem in elements of the second Poincaré system. In the second Hori–Deprit approximation, we construct the Hamiltonian of the problem in averaged elements and the generating function of

Table 9. Coefficients of series representing equations of motion and substitution functions for the elements $\Xi_{1,1}$ and $H_{1,1}$

k	a_k	b_k	c_k	d_k
2	$\frac{3}{4}$	$-\frac{15}{16}$	$-\frac{1}{2}$	$-\frac{3}{2}$
3	$\frac{45}{32}$	$-\frac{105}{64}$	$-\frac{9}{16}$	$-\frac{15}{16}$
4	$\frac{525}{256}$	$-\frac{4725}{2048}$	$-\frac{75}{128}$	$-\frac{105}{128}$
5	$\frac{11025}{4096}$	$-\frac{24255}{8192}$	$-\frac{1225}{2048}$	$-\frac{1575}{2048}$
6	$\frac{218295}{65536}$	$-\frac{945945}{262144}$	$-\frac{19845}{32768}$	$-\frac{24255}{32768}$
7	$\frac{2081079}{524288}$	$-\frac{4459455}{1048576}$	$-\frac{160083}{262144}$	$-\frac{189189}{262144}$
8	$\frac{19324305}{4194304}$	$-\frac{328513185}{67108864}$	$-\frac{1288287}{2097152}$	$-\frac{1486485}{2097152}$
9	$\frac{703956825}{134217728}$	$-\frac{1486131075}{268435456}$	$-\frac{41409225}{67108864}$	$-\frac{46930455}{67108864}$
10	$\frac{25264228275}{4294967296}$	$-\frac{106109758755}{17179869184}$	$-\frac{1329696225}{2147483648}$	$-\frac{1486131075}{2147483648}$
11	$\frac{224009490705}{34359738368}$	$-\frac{468383480565}{68719476736}$	$-\frac{10667118605}{17179869184}$	$-\frac{11789973195}{17179869184}$
12	$\frac{1967210618373}{274877906944}$	$-\frac{16393421819775}{2199023255552}$	$-\frac{85530896451}{137438953472}$	$-\frac{93676696113}{137438953472}$
13	$\frac{34277154714075}{4398046511104}$	$-\frac{71191013636925}{8796093022208}$	$-\frac{1371086188563}{2199023255552}$	$-\frac{1490311074525}{2199023255552}$
14	–	–	–	$-\frac{23730337878975}{35184372088832}$

the transformation from osculating elements to averaged ones.

For the four-planet problem, we construct equations of motion and the change-variable functions, taking into account terms including the second power of the small parameter. The right-hand sides of the equations of motion and change-variable functions are estimated by numerical values corresponding to orbital elements of the giant planets of the Solar System. We provide estimates of numbers of terms of series representing the averaged Hamiltonian, generating function of the transformation, right-hand sides of the equations of motion, and change-variable functions.

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