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Article · November 1983

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(Submitted May 30, 1983; revised July 4, 1983)
Pis'ma Astron. Zh. 9, 750–754 (December 1983)

A technique proposed for eliminating both short- and long-period terms from the Hamiltonian of the artificial-satellite problem by a single canonical change of variables.

PACS numbers: 95.10.Ce, 94.80.Rz

The classical method of perturbation theory, that of von Zeipel, is practically the same as the method which Poincaré expounded in the second volume of his celebrated Les Méthodes Nouvelles de la Mécanique Céleste. Applying this methdd to study asteroid motion, von Zeipel introduced separate reductions for the short- and long-period terms. The reason was that the problem is degenerate (the undisturbed Hamiltonian does not depend on all the action variables), but it also reflects the fact that the various short-period perturbations are of little interest: in evolutionary studies it is the secular long-period perturbations which are of greatest import.

Such a separation of the periodic terms is immaterial, however, in formulating analytic theories of satellite motion, for all the angle variables will drop out of the Hamiltonian of the problem up to some order with respect to the small parameter C_{20}, the coefficient of the second zonal harmonic of the geopotential.

After Hori[1] and Deprit[2] had emphasized the explicit advantages of Lie series over Taylor series in von Zeipel's method, analytic theories of satellite motion began to be constructed chiefly by the route which these authors marked out.

The best examples are the fourth-order theories of Deprit and Rom[3] and Kutuzov,[4] which allow for the second zonal harmonic, and the third-order theory of Kinoshita,[5] in which the effects of the second, third, and fourth zonal harmonics are considered. Let us first describe the basic features of the method employed in these investigations —now the standard approach.

Written in terms of Delaunay elements, the differential equations determining a satellite trajectory are:

$$\frac{d}{dt}(L, G, H)=\frac{\partial F}{\partial (l, g, h)}, \quad \frac{d}{dt}(l, g, h)=-\frac{\partial F}{\partial (L, G, H)}. \quad (1)$$

Here

$$F=F_0+F_1+F_2, \quad F_0=(2L^2)^{-1},$$

with F_1 denoting the part of the Hamiltonian due to the expansion of the second zonal harmonic, while F_2 results from the expansion of the remaining harmonics in the geopotential.

The standard Hori–Deprit method of developing an analytic theory involves making two successive canonical changes of variables:

$$(L, G, H, l, g, h)\rightarrow(L', G', H', l', g', h')\rightarrow(L'', G'', H'', l'', g'', h''),$$

which transform the system (1) first into a system of the form

$$\frac{d}{dt}(L', G', H')=\frac{\partial F^*}{\partial (l', g', h')}, \quad \frac{d}{dt}(l', g', h')=-\frac{\partial F^*}{\partial (L', G', H')},$$

where the secular and long-period terms are retained in the Hamiltonian, so that $F^* = F^*(L', G', H', g')$, and then into the system

$$\frac{d}{dt}(L'', G'', H'')=\frac{\partial F^{**}}{\partial (l'', g'', h'')}, \quad \frac{d}{dt}(l'', g'', h'')=-\frac{\partial F^{**}}{\partial (L'', G'', H'')}, \quad (2)$$

where F^{**} now contains only secular terms, so that the system (2) is integrable in quadratures.

The generating function $S^* = S_1^* + S_2^* + \ldots$, $F^* = F_0^* + F_1^* + \ldots$ of the first canonical transformation (the S_i^*, F_i^* are of order $C_{20}{}^i$) is to be determined iteratively from the chain of partial differential equations

$$\begin{gathered} F_0=F_0^*,\\ \{F_0, S_1^*\}+F_1=F_1^*,\\ \{F_0, S_2^*\}+\frac{1}{2}\{F_1+F_1^*, S_1^*\}+F_2=F_2^*,\\ \{F_0, S_3^*\}+\frac{1}{2}\{F_1+F_1^*, S_2^*\}+\frac{1}{2}\{F_2+F_2^*, S_1^*\}\\ +\frac{1}{12}\{\{F_1-F_1^*, S_1^*\}S_1^*\}=F_3^*, \ldots \end{gathered} \quad (3)$$

The braces here represent Poisson brackets

$$\{A, B\}=\frac{\partial A}{\partial L}\frac{\partial B}{\partial l}+\frac{\partial A}{\partial G}\frac{\partial B}{\partial g}+\frac{\partial A}{\partial H}\frac{\partial B}{\partial h}-\frac{\partial A}{\partial l}\frac{\partial B}{\partial L}-\frac{\partial A}{\partial g}\frac{\partial B}{\partial G}-\frac{\partial A}{\partial h}\frac{\partial B}{\partial H},$$

and in the degenerate case at hand, $\{F_0, S_i^*\} = -\bar{n}\partial S_i^*/\partial l'$, $\bar{n} = 1/(L')^3$.

The generating function $S^{**} = S_1^{**} + S_2^{**} + \ldots$, $F^{**} = F_0^{**} + F_1^{**} + \ldots$ of the second canonical transformation (the S_i^{**}, F_i^{**} are of order $C_{20}{}^i$) is to be determined from the following equations:

$$\begin{gathered} F_0^*=F_0^{**}, \quad F_1^*=F_1^{**},\\ \{F_1^*, S_1^{**}\}+F_2^*=F_2^{**},\\ \{F_1^*, S_2^{**}\}+\frac{1}{2}\{F_2^*+F_2^{**}, S_1^{**}\}+F_3^*=F_3^{**}, \ldots \end{gathered} \quad (4)$$

Here

$$\{F_1^*, S_i^{**}\}=\frac{\partial F_1^*}{\partial G''}\frac{\partial S_i^{**}}{\partial g''},$$

and the S_i^{**} are in fact of order C_{20}^i, Because when they are determined from Eqs. (4), a factor

$$\left(\frac{\partial F_1^*}{\partial G''}\right)^{-1} = -\frac{4}{3}\frac{(L'')^3(G'')^4}{C_{20}(1-5\cos^2 I)}$$

enters. This procedure gives the standard terms that appear in the second-order theory but are small first-order quantities with respect to C_{20}. One excludes from consideration the critical-slope case in which $1 - 5\cos^2 I \approx 0$.

If ε_i'' denotes one of the intermediate elements L'', G'', H'', l'', g'', h'', and ε_i' one of the elements L', G', H', l', g', h', then the corresponding osculating element ε_i can be obtained from the expressions

$$\varepsilon_i' = \varepsilon_i'' + \{\varepsilon_i'', S^{**}\} + \frac{1}{2}\{\{\varepsilon_i'', S^{**}\}, S^{**}\} + \frac{1}{6}\{\{\{\varepsilon_i'', S^{**}\}, S^{**}\}, S^{**}\} + \dots,$$

$$\varepsilon_i = \varepsilon_i' + \{\varepsilon_i', \varepsilon^*\} + \frac{1}{2}\{\{\varepsilon_i', S^*\}, S^*\} + \frac{1}{6}\{\{\{\varepsilon_i', S^*\}, S^*\}, S^*\} + \dots \tag{5}$$

In general, one can express any function $f(\varepsilon)$ of the osculating elements in terms of $f(\varepsilon'')$, $f(\varepsilon')$ by formulas of the same type as Eqs. (5):

$$f(\varepsilon') = f(\varepsilon'') + \{f(\varepsilon''), S^{**}\} + \frac{1}{2}\{\{f(\varepsilon''), S^{**}\}, S^{**}\} + \dots,$$
$$f(\varepsilon) = f(\varepsilon') + \{f(\varepsilon'), S^*\} + \frac{1}{2}\{\{f(\varepsilon'), S^*\}, S^*\} + \dots$$

This is one of the main advantages of the Hori–Deprit method, and it is particularly valuable because in computing the position of a satellite in its orbit one ordinarily uses not the Delaunay elements themselves but suitable combinations of them forming systems of elements that are nonsingular for zero values of the eccentricity and inclination.

The relations (5) between the intermediate and the osculating elements will simplify, of course if the initial system of differential equations (1) can be reduced by a single change of variables to a system of the form

$$\frac{d}{dt}(L'', G'', H'') = 0, \ \frac{d}{dt}(l'', g'', h'') = -\frac{\partial F'(L'', G'', H'')}{\partial(L'', G'', H'')}, \tag{6}$$

which is integrable in quadratures to the accuracy required. In this event the corrections for subsidiary perturbations (other harmonics in the geopotential, lunisolar perturbations, ...) simplify as well. I intend to consider in future papers the procedures for introducing those corrections.

In 1969 Mersman[6] proposed a modified von Zeipel algorithm which enables all the angle variables to be eliminated from the Hamiltonian of the problem simultaneously. I will now show that this change of variables can be carried out by modifying the Hori–Deprit method somewhat, while retaining all its advantages.

Let $f(L, G, H, l, g)$ be a function depending periodically on the angle variables l, g. Introduce the notation $(f)_{sec}$ for the secular part of f, $(f)_g$ for the part of f depending on only one of the angle variable, g; and $(f)_{l,g}$ for the part of f depending on l.

We shall seek a generating function S of the form

$$S = S_1 + S_2 + S_3 + \dots,$$
$$S_i = S_i'(l, g) + S_i''(g), \ S_i', \ S_i'' \sim C_{20}^i \ (i = 1, 2, \dots).$$

In our case

$$\{F_0, S_i\} = \{F_0, S_i'\}$$

and, using Eqs. (3), we arrive at the following equations for determining the generating function and the new Hamiltonian $F' = F_0' + F_1' + \dots$. In the zeroth order $F_0' = F_0$. In the first order

$$F_1' = (F_1)_{sec},$$
$$\{F_0, S_1'\} + (F_1)_{l,g} = 0, \tag{7}$$

since $F_1 = (F_1)_{l,g} + (F_1)_{sec}$. Clearly S_1' will coincide with the S_1^* given by Eqs. (3), and F_1' with the F_1^*. In the second order,

$$\{F_0, S_2'\} + \frac{1}{2}\{F_1 + F_1', S_1''\} + \Phi_2 = F_2',$$
$$\Phi_2 = \frac{1}{2}\{F_1 + F_1', S_1'\} + F_2. \tag{8}$$

Determining S_1'' from the differential equation

$$\{F_1', S_1''\} + (\Phi_2)_g = 0 \tag{9}$$

and setting $F_2' = (\Phi_2)_{sec}$, we can determine S_2' from the equation

$$\{F_0, S_2'\} + \frac{1}{2}\{(F_1)_{l,g}, S_1''\} + (\Phi_2)_{l,g} = 0 \tag{10}$$

Although S_1'', F_2' coincide with the S_1^{**}, F_2^{**} given by Eqs. (4), now S_2' differs from the S_2^* given by Eqs. (3).

For the third order, on solving the equation

$$\{F_0, S_3'\} + \frac{1}{2}\{F_1 + F_1', S_2''\} + \Phi_3 = F_3',$$
$$\Phi_3 = \frac{1}{2}\{F_1 + F_1', S_2'\} + \frac{1}{2}\{F_2 + F_2', S_1' + S_1''\} + \frac{1}{12}\{\{(F_1)_{l,g}, S_1' + S_1''\}, S_1' + S_1''\}, \tag{11}$$

we first find S_2'' from the differential equation

$$\{F_1', S_2''\} + (\Phi_3)_g = 0, \tag{12}$$

and then, setting $F_3' = (\Phi_3)_{sec}$, we determine S_3' from the equation

$$\{F_0, S_3'\} + \frac{1}{2}\{(F_1)_{l,g}, S_2''\} + (\Phi_3)_{l,g} = 0. \tag{13}$$

The quantity S_3'' can be evaluated from the equation for the fourth order, and so on.

The generating functions S_i', S_{i-1}'' and the Hamiltonian F_i' ($i \geq 3$) will differ from S_i^*, S_{i-1}^{**} and F_i^{**}, respectively. To obtain a solution exact up to order N, we evidently have to solve the equations by successive approximations as far as the N-th order and then determine S_N'' from an (N+1)-th-order differential equation.

Thus by a single change of variables we have arrived at a system in terms of the intermediate elements, Eqs. (6).

The expressions relating the intermediate and the osculating elements will now take the form

$$\varepsilon_i = \varepsilon_i'' + \{\varepsilon_i'', S\} + \frac{1}{2}\{\{\varepsilon_i'', S\}, S\} + \frac{1}{6}\{\{\{\varepsilon_i'', S\}, S\}, S\} + \dots \quad (14)$$

and any function $f(\varepsilon)$ can be obtained in terms of $f(\varepsilon'') \equiv f''$ by the formula

$$f(\varepsilon) = f'' + \{f'', S\} + \frac{1}{2}\{\{f'', S\}, S\} + \dots$$

For nonresonance satellites, one can develop an analytic theory that includes the effects of the tesseral harmonics as well. In this event

$$F_0 = \frac{1}{2L^2} + \nu H,$$

F_2 will represent an expansion of the zonal and tesseral harmonics, h will be equal to $\Omega - \theta$, ν will be the earth's angular rotational velocity, θ will denote Greenwich sidereal time, Ω will be the longitude of the ascending node of the satellite orbit, and the Poisson bracket

$$\{F_0, S_i'\} = -\bar{n}\frac{\partial S_i'}{\partial l} + \nu\frac{\partial S_i'}{\partial h},$$

where $S_i'(l, g, h)$ depends on the angle variable l or h.

The equations for the generating function and the new Hamiltonian [Eqs. (7)–(13)] will remain unchanged. This procedure for simultaneously eliminating the short- and long-period terms may be used in conjunction with various types of averaging. If, for instance, one is developing a semianalytic theory of resonance-satellite motion, the Hamiltonian F_1' may be allowed to keep its secular and resonance terms and may be used to integrate the averaged system numerically. The osculating elements can be evaluated from the intermediate elements by changes of variables of the type (14), in which the generating function S characterizes the periodic perturbations.

The algorithm has here been discussed only formally, of course; whether it is convergent remains an open question.

It is worth noting that in principle one can eliminate all the periodic variables from the Hamiltonian of the problem simultaneously if one supplements the degenerate part F_0 of the Hamiltonian by a secular part F_1, setting $\tilde{F}_0 = F_0 + (F_1)_{sec}$, and then applies the formal apparatus of the standard Hori–Deprit method to the system with the Hamiltonian, $F = \tilde{F}_0 + (F_1)_{l,g} + F_2$, so that

$$\{\tilde{F}_0, S_i\} = \frac{\partial \tilde{F}_0}{\partial L}\frac{\partial S_i}{\partial l} + \frac{\partial \tilde{F}_0}{\partial G}\frac{\partial S_i}{\partial g} + \frac{\partial \tilde{F}_0}{\partial H}\frac{\partial S_i}{\partial h}.$$

But while this idea is a natural one, it is not applicable to our problem: the algorithm turns out to be divergent. In fact, although F_1 contains no long-period terms (so that S_1 will indeed be of order C_{20}), terms of order C_{20}^2 as well as terms of order C_{20} will appear in S_2. One can show that as a result, while $F_i' \sim C_{20}^i$ to third order inclusive, F_4' will acquire terms of order C_{20}^3; F_5', terms of order C_{20}^4; F_6', terms of order C_{20}^4, and so on; and it does not seem possible by successive approximations to make the process convergent.

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Érnst A. Dibaĭ: In memoriam

Pis'ma Astron. Zh. 9, 755 (December 1983)

PACS numbers: 01.60. + q

On November 11, 1983, the life of a distinguished Soviet astrophysicist, a talented scientist and teacher, the Executive Secretary of the Astronomicheskii Zhurnal Editorial Board, and a member of the Editorial Board of this journal, Érnst Apushevich Dibai, was prematurely cut short.

Dibai was born in Kazan' in 1931. On graduating from Kazan' University he enrolled as a graduate student in Moscow University, at the Shternberg Astronomical Institute. In 1958 he successfully defended his dissertation and began work as a member of the Shternberg staff. From that time to the close of his life, his activity was closely tied to this institution.

Between 1961 and 1977 Dibai was head of the Crimean station, the Shternberg Institute's principal base for teaching and observing. It was here that his outstanding capacities for organization came into full flower. Under Dibai's direction and with his personal participation the 125-cm reflector was mounted, a number of new instruments were installed, a modern laboratory was built, and promising opportunities opened up for observers and the Station staff alike. Dibai was one of the principal designers of

