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DEMONSTRATION OF THE DIFFERENTIAL EQUATIONS EMPLOYED BY DELAUNAY IN THE LUNAR THEORY.

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THE method of treating the lunar theory adopted by Delaunay is so elegant that it cannot fail to become in the future the classic method of treating all the problems of celestial mechanics. The canonical system of equations employed by Delaunay is not demonstrated by him in his work, but he refers to a memoir of Binet inserted in the *Journal de l' Ecole Polytechnique*, Cahier XXVIII. Among the innumerable sets of canonical elements it does not appear that a better can be selected. These equations can be established in a very elegant manner by using the properties of Lagrange's and Poisson's quantities (a, b) and $[a, b]$. But a demonstration, founded on more direct and elementary considerations, is, on some accounts, to be preferred.

Let a denote the mean distance, e the eccentricity, i the inclination of the orbit to a fixed plane, l the mean anomaly, g the angular distance of the lower apsis from the ascending node, h the longitude of the ascending node measured from a fixed line in the fixed plane, μ the sum of the masses of the bodies whose relative motion is considered, and R the ordinary perturbative function augmented by the term $\frac{\mu^2}{2L^2}$. Then if we put $L=\sqrt{(\mu a)}$, $G=\sqrt{[\mu a(1-e^2)]}$, $H=\sqrt{[\mu a(1-e^2)]}\cos i$, Delaunay's equations are

$$\frac{dL}{dt}=\frac{dR}{dl}, \qquad \frac{dG}{dt}=\frac{dR}{dg}, \qquad \frac{dH}{dt}=\frac{dR}{dh},$$

$$\frac{dl}{dt}=-\frac{dR}{dL}, \qquad \frac{dg}{dt}=-\frac{dR}{dG}, \qquad \frac{dh}{dt}=-\frac{dR}{dH}.$$

In terms of rectangular coordinates

$$R=\frac{\mu^2}{2L^2}+\frac{m'}{[(x'-x)^2+(y'-y)^2+(z'-z)^2]^{\frac{1}{2}}}-\frac{m'(xx'+yy'+zz')}{r'^3}.$$

In this expression for x, y, z ought to be substituted their values deduced from the formulas of elliptic motion, and expressed in terms of L, G, H, l, g, h. It should be noted that the term $\frac{\mu^2}{2L^2} = \frac{\mu}{2a}$, of the zero order with respect to the disturbing forces, has been added to R only to preserve in the equations the canonical form: it is only by amplifying the signification of the word that l can be called an element, as it is not constant in elliptic motion but augments proportionally to the time and $\frac{dl}{dt} = n = \frac{\mu^2}{L^3}$. It is chosen as a variable in preference to the element attached to it by addition simply to prevent t from appearing in the derivatives of R outside of the functional signs sine and cosine.

The equations

$$\frac{d^2x}{dt^2} + \frac{\mu x}{r^3} = \frac{dR}{dx}, \qquad \frac{d^2y}{dt^2} + \frac{\mu y}{r^3} = \frac{dR}{dy}, \qquad \frac{d^2z}{dt^2} + \frac{\mu z}{r^3} = \frac{dR}{dz},$$

are well known; here, however, R does not contain the term $\frac{\mu}{2L^2}$. By multiplying them severally by dx, dy, dz adding and integrating is obtained

$$\frac{dx^2 + dy^2 + dz^2}{2dt^2} - \frac{\mu}{r} + \frac{\mu}{2a} = \int \left(\frac{dR}{dx} dx + \frac{dR}{dy} dy + \frac{dR}{dz} dz \right).$$

When the elements are made variable this gives

$$\frac{d}{dt}\left(\frac{\mu}{2a}\right) = -\left(\frac{dR}{dx}\frac{dx}{dt} + \frac{dR}{dy}\frac{dy}{dt} + \frac{dR}{dz}\frac{dz}{dt}\right).$$

But we have

$$\frac{dx}{dt} = n\frac{dx}{dl}, \qquad \frac{dy}{dt} = n\frac{dy}{dl}, \qquad \frac{dz}{dt} = n\frac{dz}{dl},$$

and hence

$$\frac{d}{dt}\left(\frac{\mu}{2a}\right) = -n\left(\frac{dR}{dx}\frac{dx}{dl} + \frac{dR}{dy}\frac{dy}{dl} + \frac{dR}{dz}\frac{dz}{dl}\right)$$

$$= -n\frac{dR}{dl}.$$

Dividing both members of this equation by $-n = -\sqrt{(\mu a^{-3})}$, the left member is seen to be the differential of $\sqrt{(\mu a)} = L$. Consequently

$$\frac{dL}{dt} = \frac{dR}{dl}.$$

Denoting the true anomaly by v, the orthogonal projection of the radius vector on the line of nodes is $r \cos(v+g)$, on a line perpendicular to it and in the plane of the orbit $r \sin(v+g)$. And the latter, projected on the plane of reference, is $r \sin(v+g) \cos i$, and on a line perpendicular to this plane $r \sin(v+g) \sin i$. If the two projections lying in the plane of reference are again each projected on the axis of x, their sum will be the value of the coordinate x, and the sum of their projections on the axis of y, the value of the coordinate y. Hence

$$x = r\cos(v+g)\cos h - r\sin(v+g)\cos i\sin h,$$
$$y = r\cos(v+g)\sin h + r\sin(v+g)\cos i\cos h,$$
$$z = r\sin(v+g)\sin i,$$

or substituting for i its value in terms of G and H,

$$x = r\cos(v+g)\cos h - \frac{H}{G} r\sin(v+g)\sin h,$$

$$y = r\cos(v+g)\sin h + \frac{H}{G} r\sin(v+g)\cos h,$$

$$z = \frac{\sqrt{(G^2 - H^2)}}{G} r\sin(v+g).$$

As r and v are functions of L, G and l only, the preceding equations show the manner in which H, g and h are involved in R.

H denotes double the areal velocity projected on the plane xy, or

$$\frac{xdy - ydx}{dt} = H.$$

Consequently
$$\frac{dH}{dt} = x\frac{dR}{dy} - y\frac{dR}{dx}.$$

But the foregoing values of x, y, z show that we have

$$\frac{dx}{dh} = -y, \qquad \frac{dy}{dh} = x, \qquad \frac{dz}{dh} = 0;$$

and thus
$$\frac{dH}{dt} = \frac{dR}{dx}\frac{dx}{dh} + \frac{dR}{dy}\frac{dy}{dh} + \frac{dR}{dz}\frac{dz}{dh} = \frac{dR}{dh}.$$

G denotes double the areal velocity, and evidently, if for the moment we suppose x and y to be drawn in the plane of the orbit, the axis of x towards the lower apsis,

$$\frac{dG}{dt} = x\frac{dR}{dy} - y\frac{dR}{dx} = \frac{dR}{dv},$$

where, in the last R, for x, y and z must be substituted their values given above in terms of r, v, G, H, g, h. Now as the only way, in which g is involved in these values, is by addition to v, it follows that

$$\frac{dR}{dv} = \frac{dR}{dg};$$

and this equation is not affected when, for r and v in R, are substituted their values in terms of L, G and l. Consequently

$$\frac{dG}{dt} = \frac{dR}{dg}.$$

In the elliptic theory

$$\frac{xdz - zdx}{dt} = \sqrt{(G^2 - H^2)} \cos h,$$

$$\frac{ydz - zdy}{dt} = \sqrt{(G^2 - H^2)} \sin h.$$

Whence we deduce

$$\frac{d.[\sqrt{(G^2 - H^2)} \cos h]}{dt} = x\frac{dR}{dz} - z\frac{dR}{dx},$$

$$\frac{d.[\sqrt{(G^2 - H^2)} \sin h]}{dt} = y\frac{dR}{dz} - z\frac{dR}{dy}.$$

Eliminating $\frac{d.\sqrt{(G^2 - H^2)}}{dt}$ from these equations, we obtain

$$\frac{dh}{dt} = \frac{z \sin h}{\sqrt{(G^2 - H^2)}}\frac{dR}{dx} - \frac{z \cos h}{\sqrt{(G^2 - H^2)}}\frac{dR}{dy} - \frac{x \sin h - y \cos h}{\sqrt{(G^2 - H^2)}}\frac{dR}{dz}.$$

Comparing the coefficients of the three derivatives of R in the right member of this equation with the values of x, y and z in terms of r, v, G, H, g, h, we recognize that they are severally equivalent to the negative of the partial derivatives of these quantities with respect to H. So that

$$\frac{dh}{dt} = -\left(\frac{dR}{dx}\frac{dx}{dH} + \frac{dR}{dy}\frac{dy}{dH} + \frac{dR}{dz}\frac{dz}{dH}\right) = -\frac{dR}{dH}.$$

It is a well known principle in the theory of varying elements, that if we differentiate any function, which is a function of the coordinates and t only, but expressed in terms of t and the elements, with respect to t only inasmuch as it is explicitly involved, we obtain the correct value. Hence if the differentiation is performed on the supposition that the elements are alone variable, the result should be zero. Applying this to the function r, we get

$$\frac{dr}{dL}\frac{dL}{dt} + \frac{dr}{dG}\frac{dG}{dt} + \frac{dr}{dl}\left(\frac{dl}{dt} - n\right) = 0,$$

or

$$\frac{dr}{dL}\frac{dR}{dl} + \frac{dr}{dG}\frac{dR}{dg} + \frac{dr}{dl}\left(\frac{dl}{dt} - n\right) = 0,$$

or again

$$\frac{dr}{dL}\left(\frac{dR}{dr}\frac{dr}{dl} + \frac{dR}{dv}\frac{dv}{dl}\right) + \frac{dr}{dG}\frac{dR}{dv} + \frac{dr}{dl}\left(\frac{dl}{dt} - n\right) = 0.$$

Whence we derive

$$\frac{dl}{dt} = n - \frac{dr}{dL}\frac{dR}{dr} - \left(\frac{dr}{dl}\right)^{-1}\left[\frac{dr}{dL}\frac{dv}{dl} + \frac{dr}{dG}\right]\frac{dR}{dv}.$$

From the expression for r we can eliminate l and introduce v in its place by means of the expression for v in terms of L, G and l; the result is the well-known equation

$$r=\frac{a(1-e^2)}{1+e\cos v}=\frac{G^2}{\mu\left[1+\frac{\sqrt{(L^2-G^2)}}{L}\cos v\right]}.$$

And we have

$$\frac{dr}{dl}=\frac{dr}{dv}\frac{dv}{dl}, \qquad \frac{dr}{dL}=\left(\frac{dr}{dL}\right)+\frac{dr}{dv}\frac{dv}{dL},$$

the parentheses denoting the derivative with respect to L only insomuch as it enters the preceding equation for r. By making these substitutions, the coefficient of $\frac{dR}{dv}$ in the expression for $\frac{dl}{dt}$ becomes

$$-\frac{dv}{dL}-\left(\frac{dr}{dl}\right)^{-1}\left[\left(\frac{dr}{dL}\right)\frac{dv}{dl}+\frac{dr}{dG}\right].$$

From the preceding equation for r, we derive

$$\left(\frac{dr}{dL}\right)=-\frac{\mu r^2\cos v}{L^3e},$$

also the following is a well known equation in the elliptic theory

$$\frac{dv}{dl}=\frac{G}{nr^2}.$$

For obtaining the value of $\frac{dr}{dG}$, u being the eccentric anomaly, we have the equations $\quad r=a(1-e\cos u), \qquad l=u-e\sin u.$
Their differentials give

$$\frac{dr}{de}=-a\cos u+ae\sin u\frac{du}{de},$$

$$0=(1-e\cos u)du-\sin u.de.$$

Whence $$\frac{dr}{de}=-a\frac{\cos u-e}{1-e\cos u}=-a\cos v.$$

And $$e=\frac{\sqrt{(L^2-G^2)}}{L}, \qquad \frac{de}{dG}=-\frac{G}{L^2e},$$

$$\frac{dr}{dG}=\frac{dr}{de}\frac{de}{dG}=\frac{G\cos v}{\mu e}.$$

By substituting the values, it is found that

$$\left(\frac{dr}{dL}\right)\frac{dv}{dl}+\frac{dr}{dG}=0.$$

In consequence $$\frac{dl}{dt}=n-\frac{dr}{dL}\frac{dR}{dr}-\frac{dv}{dL}\frac{dR}{dv}$$

$$=n-\frac{dR}{dL}.$$

As R is a function of the coordinates and the time only, we can treat it as we have done r. Then

$$\frac{dR}{dL}\frac{dL}{dt}+\frac{dR}{dl}\left(\frac{dl}{dt}-n\right)+\frac{dR}{dG}\frac{dG}{dt}+\frac{dR}{dg}\frac{dg}{dt}+\frac{dR}{dH}\frac{dH}{dt}+\frac{dR}{dh}\frac{dh}{dt}=0.$$

On substituting in this the values of the differentials of the elements which have already been determined, it is seen that all the terms but two mutually cancel each other. And on dividing the result by $\frac{dR}{dg}$, we get

$$\frac{dg}{dt}=-\frac{dR}{dG}.$$

By adding to R the term $\frac{\mu^2}{2L^2}=\frac{\mu}{2a}$, its partial derivative with respect to L is augmented by the term $-\frac{\mu^2}{L^3}=-n$, but all the other derivatives are unchanged. In consequence of this addition the value of the differential of l becomes

$$\frac{dl}{dt}=-\frac{dR}{dL}.$$

An objection may be made against the preceding method of obtaining the differentials of l and g, in that the quantities $\frac{dr}{dl}$ and $\frac{dR}{dg}$, which both periodically vanish, have been employed as divisors. But this objection has force only when it is admitted that the differentials of l and g or the corresponding derivatives of R may be discontinuous. For having proved the truth of the equations for all times, except when the divisors, just mentioned, vanish, it follows, that if both members are continuous, the equations must still hold even for the moments of time when $\frac{dr}{dl}=0$ or $\frac{dR}{dg}=0$.

RECENT RESULTS IN THE STUDY OF LINKAGES.

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(*Continued from page* 46.)

THE so called kite-shaped quadrilateral, that which has two pairs of equal adjacent sides, may be employed as a cell in which the rays are always at right angles. Let $ABCD$ be such a quadrilateral the length of its unequal sides being a and b. Take any point on AB for the fulcrum O and draw OQ and OP parallel to the diagonals. It is evident that however the