NOTE ON THE EVOLUTION OF THE LIE-DEPRIT TRANSFORM

Ahmed A. Kamel

Ford Aerospace & Communications Corporation

Palo Alto, CA 94303

1. INTRODUCTION

Deprit's introduction of the Lie transform early in 1969 [8] stirred great interest in the following three areas:

a. Equivalence of Deprit's formulation to that of Hori [14] and von Zeipel (Schniad [33], Kamel [17], Mersman [27], [28], Campbell and Jeffereys [6], Hori [15], Henrard and Roels [12]).

b. Improvement of the computational aspects of Deprit's algorithm (Kamel [17, 18]) and the development of similar recursive algorithms for Hori's Lie Series (Kamel [17], Mersman [27, 28], Campbell and Jeffereys [6]).

c. Generalization of the Lie-Deprit transform and Lie-Hori series to non-Hamiltonian systems, (Kamel [19, 20], Henrard [10, 11], Hori [15]).

This note will be limited to a summary of the evolution of the Lie-Deprit transform and its equivalence to the well known method of averaging (Bogoliubov and Mitropolsky [3]) and the method of multiple scales (Nayfeh [30], Kevorkian and Cole [24]).

2. DEPRIT'S FUNDAMENTAL ALGORITHM

In the spirit of Lie's ideas [25], Deprit introduced the Lie transform $(x, X) \rightarrow (y, Y)$ using the canonical differential equations

$$\frac{dx}{d\epsilon} = W_X(x, X; \epsilon) \tag{1a}$$

$$\frac{dX}{d\epsilon} = -W_x(x, X; \epsilon) \tag{1b}$$

with

$$x = y,\ X = Y, \text{ when } \epsilon = 0.$$

In Eq. (1) the independent variable is the small parameter ϵ and the function W is the generator of the canonical transformation $(x, X) \rightarrow (y, Y)$. The solution of Eq. (1) can be represented by a Taylor series in ϵ. More generally, for any function F, the Taylor series can be written as

$$F(x, X; \epsilon) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} F_n(x, X) = \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!} F^{(k)}(y, Y), \tag{2}$$

with

$$F_n(x, X) = \left[\frac{\partial^n}{\partial \epsilon^n} F(x, X; \epsilon) \right]_{\epsilon=0}, \; n \geqslant 0, \tag{3a}$$

$$F^{(k)}(y, Y) = \left[\frac{d^k}{d\epsilon^k} F(x, X; \epsilon) \right]_{\epsilon=0}, \; k \geqslant 0. \tag{3b}$$

The total derivative operator $d/d\epsilon$ is related to the partial derivative operator $\partial/\partial\epsilon$ and it has the following properties

$$\begin{aligned} \frac{dF}{d\epsilon} &= \frac{\partial F}{\partial \epsilon} + F_x \frac{dx}{d\epsilon} + F_X \frac{dX}{d\epsilon} \\ &= \frac{\partial F}{\partial \epsilon} + L_W F, \end{aligned} \tag{4a}$$

$$\frac{d^k F}{d\epsilon^k} = \frac{d}{d\epsilon} \frac{d^{k-1} F}{d\epsilon^{k-1}}, \; k \geqslant 2, \tag{4b}$$

where L_W is the Lie-Deprit operator. The use of Eqs. (1) and (4a) leads to the usual Poisson bracket

$$L_W F = (F;\, W) = F_x \cdot W_X - F_X W_x. \tag{5}$$

Deprit selected the generating function W in the form

$$W(x, X; \epsilon) = \sum_{m=0}^{\infty} \frac{\epsilon^m}{m!} W_{m+1}(x, X), \tag{6}$$

and substituted Eqs. (2) and (6) in Eqs. (4a) and (4b) to obtain

$$\frac{d^k}{d\epsilon^k} F = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} F_n^{(k)}(x, X), \; k \geqslant 1, \tag{7}$$

with

$$F_n^{(k)} = F_{n+1}^{(k-1)} + \sum_{m=0}^{n} C_m^n L_{m+1} F_{n-m}^{(k-1)}, \; k \geqslant 1, \, n \geqslant 0, \tag{8}$$

where C_m^n is the usual binomial coefficient

$$C_m^n = \frac{n!}{(n-m)!\, m!} \tag{9}$$

and L_p is the Lie-Deprit operator defined by the Poisson bracket of Eq. (5):

$$L_p F = (F;\ W_p),\ \ p \geqslant 1. \tag{10}$$

Note that in view of Eqs. (2), (3), (7), and (8), the Poisson bracket operations can be performed in terms of a dummy set of variables, e.g., $(z,\ Z)$, which are selected at the end of the calculations to be either $(x,\ X)$ or $(y,\ Y)$ as appropriate to calculate the coefficients of ϵ in Eq. (2). These are given by

$$F_n = F_n^{(0)} \text{ and } F^{(k)} = F_0^{(k)}. \tag{11}$$

To show how Eq. (8) is used to generate $F^{(k)}$ from F_n, Deprit used the forward triangle of Fig. 1 similar to Pascal's triangle. The forward flow from left to right leads to the required $F^{(k)}$.

Now, to obtain the inverse transformation $(y,\ Y) \rightarrow (x,\ X)$, or more generally to obtain F_n from $F^{(k)}$, Deprit proposed to perform this inversion using two additional triangles similar to that of Fig. 1 (see Eqs. (32) to (34) of Deprit [8]). The first triangle is used to calculate the inverse generating function V from W of Eq. (6), and the selected triangle is used to calculate F_n from $F^{(k)}$. This procedure for the calculation of the inverse was in marked contrast to Hori's method in which no new calculations are needed for the inverse (Mersman [27]). However, it was found (Kamel [17], [18]) that Deprit's algorithm of Eq. (8) can be simplified such that, like Hori's method, no new calculations are required for the inverse. This is summarized in the next section.

3. SIMPLIFICATION OF DEPRIT'S ALGORITHM

The simplification of Deprit's algorithm was performed in two steps. In the first step, Deprit's Eq. (8) was rearranged to eliminate the need for the calculation of the inverse generating function V. This was achieved by observing that Deprit's triangle can be constructed with backward flow from the right to the left, as shown in Fig. 2, to obtain F_n from $F^{(k)}$. The general backward algorithm was simply obtained by replacing k by $k + 1$ and $n + 1$ by n in Eq. (8) to get

$$F_n^{(k)} = F_{n-1}^{(k+1)} - \sum_{m=0}^{n-1} C_m^{n-1}\, L_{m+1} F_{n-m-1}^{(k)},\ n \geqslant 1,\ k \geqslant 0. \tag{12}$$

The second step in the simplification of Deprit's algorithm was to avoid any new operations for the inverse. This was achieved by observing that successive substitution of Eq. (12) into itself from $n = 1$ up leads to the form

$$F_n^{(k)} = - \sum_{j=0}^{n} C_j^n G_j F^{(k+n-j)}, \; n \geqslant 1, \; k \geqslant 0, \tag{13}$$

where G_j is a linear operator given by Kamel [17]-[19]:

$$G_0 = -1, \tag{14a}$$

$$G_j = - \sum_{m=1}^{j} C_{m-1}^{j-1} L_m G_{j-m}. \tag{14b}$$

The present author (Kamel [20]) later followed a suggestion by Dr. Henrard (1970) to use an operator $N_j = -G_j$ to eliminate the negative signs in Eqs. (13) and (14a). In a later publication, however, Dr. Henrard [11] offered a new derivation for the algorithm given by Eqs. (13) and (14) with $N_j = -G_j$ and referred to it as the algorithm of the inverse. Now, for $k = 0$ and $k = 1$, Eqs. (8), (11), and (13) yield

$$F_n = F^{(n)} - \sum_{j=1}^{n} C_j^n \, G_j F^{(n-j)}, \; n \geqslant 1, \tag{15a}$$

$$F^{(n)} = F_n + \sum_{j=1}^{n-1} \left[C_j^{n-1} G_j \, F^{(n-j)} + C_{j-1}^{n-1} L_j F_{n-j} \right] + L_n F_0, \; n \geqslant 1. \tag{15b}$$

The use of $F_{j,i} = -G_j F^{(i)}$ in Eqs. (14) and (15) leads to the recursive algorithm

$$F_n = F^{(n)} + \sum_{j=1}^{n} C_j^n F_{j,n-j}, \; n \geqslant 1, \tag{16a}$$

$$F^{(n)} = F_n - \sum_{j=1}^{n-1} \left[C_j^{n-1} F_{j,n-j} - C_{j-1}^{n-1} L_j F_{n-j} \right] + L_n F_0, \; n \geqslant 1, \tag{16b}$$

with

$$F_{0,0} = F^{(0)} = F_0 \, , \; F_{0,i} = F^{(i)}, \tag{16c}$$

$$F_{j,i} = - \sum_{m=1}^{j} C_{m-1}^{j-1} L_m F_{j-m,i}, \tag{16d}$$

which is the simplest algorithm to obtain F_n in terms of $F^{(n)}$ using Eq. (16a) or $F^{(n)}$ in terms of F_n using (16b). Except for the binomial coefficients, the same $F_{j,i}$ are used in both equations. This proves the fact that no new Poisson bracket operations are required to calculate the inverse (see also Eqs. (29) and (30) of Kamel, [18]). Formula (16b) is particularly useful in the calculation of the Hamiltonian function. At each order of perturbation ($n = 1, 2, \ldots$), the generating function W_n is selected such that $L_n F_0$ eliminates the short period terms in Eq. (16b) leaving the new Hamiltonian, $F^{(n)}$, with the long period and secular terms. In some cases, however, some reduction in the number of the Poisson bracket operations may be possible by choosing $F^{(n)}$ to be the old Hamiltonian instead of the new Hamiltonian. In this case $F = 0$ for all values of i as shown by Eq. (16d). Further reduction in the number of Poisson bracket operations can be achieved by observing that the use of the small parameter ϵ is only formal and, therefore, the functions $F^{(n)}$ can be redefined to contain other terms. For example, $F^{(n)} + \epsilon^m F^{(n+m)}$ can be used to replace $F^{(n)}$ in Eq. (16) in a similar manner as used by Howland [13].

Equation (16d) can be visualized using the triangle of Fig. 3. This triangle is simpler than the triangles of Figs. 1 and 2 because of the absence of any cross computations. This is due to the fact that each element $F_{j,i}$ in the ith column is computed in terms of the elements above it in the same column and, as a result, the ith column drops out from the computations when $F^{(i)} = 0$. Similarly, Eq. (16a) can be considered as the sum of certain combinations of the horizontal elements in Fig. 3.

Formula (16b) is essentially the same as formula (49) of Kamel [18], which was implemented by Char using the MACSYMA algebraic program (see McNamara [26]).

4. GENERALIZED LIE-DEPRIT TRANSFORM

The initial effort to extend the Lie-Deprit transform to non-Hamiltonian systems was published by the present author in a SUDAAR 389 report in Oct. 1969 (Kamel [19]). Private communication of this report to Drs. Deprit and Henrard and to Prof. Hori resulted in a more rigorous derivation by Henrard [10] and similar extension of Lie series by Hori [16]. Professor Hori's first presentation of his extension was at the Summer Institute in Orbital Mechanics at the University of Texas at Austin in May 1970. This, apparently, led Prof. Giacaglia [9] to conclude that Prof. Hori's extension was independent of the present author's development published in *Celestial Mechanics* (Kamel [20]) at about the same time that Prof. Hori presented his extension at Texas. However, it is important to point out that Prof. Hori referred to the present author's SUDAAR 389 report in February of 1970 (Hori [15]). Also, Prof. Hori's publication of his extension (Hori [16]) referred to the present author's two publications on the subject and never claimed that his finding was independent.

Now, the extension of the Lie-Deprit transform to non-Hamiltonian systems starts with the same basic idea of the Krylov-Bogoliubov method of averaging that applies a near identity transformation $y \to x$ in the form of power series in the small parameter ϵ,

$$y = x + \sum_{n=1}^{\infty} \frac{\epsilon^n}{n!} x_n(x), \tag{17}$$

to the N-dimensional system of differential equations,

$$\dot{y} = g(y; \epsilon) = \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!} g^{(k)}(y), \tag{18}$$

so that the resulting system of differential equations

$$\dot{x} = f(x; \epsilon) = \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!} f_k(x) \tag{19}$$

has a simple form. To obtain the relationship between f_k and $g^{(k)}$, a generalized Lie-transform of Eq. (1) was introduced (Kamel [19], [20]):

$$\frac{dx}{d\epsilon} = \tilde{W}(x; \epsilon) = \sum_{m=0}^{\infty} \frac{\epsilon^m}{m!} \tilde{W}_{m+1}(x), \tag{20}$$

with $x = y$ when $\epsilon = 0$.

Now, differentiating Eq. (17) with respect to time and using Eqs. (18) and (19) yields

$$g = [A]f, \tag{21}$$

where A is the $N \times N$ Jacobian matrix given by

$$A = \left[\frac{\partial y}{\partial x}\right] = I + \sum_{n=1}^{\infty} \frac{\epsilon^n}{n!} \left[\frac{\partial x_n}{\partial x}\right]. \tag{22}$$

Note that the sequence $g^{(k)}(y)$ of Eq. (18) is related to $g(y; \epsilon)$ by

$$g^{(k)}(y) = \left[\frac{d^k}{d\epsilon^k} g\right]_{\epsilon=0}, \; k \geqslant 0, \tag{23}$$

which can be obtained by successive differentiation of Eq. (21) with respect to ϵ. For $k = 1$

$$\frac{dg}{d\epsilon} = [A] \frac{dt}{d\epsilon} + \left[\frac{dA}{d\epsilon}\right] f, \tag{24}$$

where, using Eq. (20),

$$\frac{df}{d\epsilon} = \frac{\partial f}{\partial \epsilon} + \left[\frac{\partial f}{\partial x}\right] \tilde{W}, \tag{25}$$

and $dA/d\epsilon$ may be obtained in terms of $\tilde{W}$ using the fact that y is the initial condition of Eq. (20) and, therefore, is independent of ϵ. Thus, in view of Eqs. (22) and (25) with $f = y$,

$$\frac{dy}{d\epsilon} = \frac{\partial y}{\partial \epsilon} + [A]\ \tilde{W} = 0. \tag{26}$$

Also, since the right-hand side of Eq. (26) is zero, its partial derivative must equal its total derivative with respect to ϵ. This leads to

$$\frac{dA}{d\epsilon} = -\ [A] \left[\frac{\partial \tilde{W}}{\partial x}\right], \tag{27a}$$

where

$$A = \mathrm{I} \text{ when } \epsilon = 0. \tag{27b}$$

Substitution of Eqs. (25) and (27a) into Eq. (24) lead to

$$\frac{dg}{d\epsilon} = [A]\Delta f, \tag{28}$$

with

$$\Delta f = \frac{\partial f}{\partial \epsilon} + \tilde{L}_W f \tag{29}$$

where $\tilde{L}_W$ is the generalized Lie-Deprit operator similar to L_W of Eq. (5),

$$\tilde{L}_W f = \left[\frac{\partial f}{\partial x}\right] \tilde{W} - \left[\frac{\partial \tilde{W}}{\partial x}\right] f \tag{30}$$

and $[\partial f/\partial x]$, $[\partial W/\partial x]$ are the $N \times N$ Jacobian matrices of f and W.

Now, in view of Eqs. (21) and (28), the successive differentiation of Eq. (28) leads to

$$\frac{d^k g}{d\epsilon^k} = [A]\Delta^k f,\ k \geqslant 0, \tag{31}$$

where

$$\Delta^0 f = f, \tag{32a}$$

$$\Delta^k f = \Delta\Delta^{k-1} f,\ k \geqslant 1. \tag{32b}$$

In view of Eqs. (22), (23), and (31), the coefficients of Eq. (18) are given by

$$g^{(k)}\ (y) = [\Delta^k f(x;\epsilon)]_{\epsilon=0}. \tag{33}$$

Note the similarity between Eqs. (29), (32), (33) and (4a), (4b), (3b), respectively. This shows that recursive algorithms similar to those of Sections 2 and 3 can be obtained. In particular, an algorithm similar to Eq. (16b) is given by

$$g_{0,0} = g^{(0)} = f_0,\ g_{0,i} = g^{(i)}, \tag{34b}$$

$$g_{j,i} = - \sum_{m=1}^{j} C_{m-1}^{j-1}\ \tilde{L}_m\ g_{j-m,i}, \tag{34c}$$

$$\tilde{L}_j f = \left[\frac{\partial f}{\partial x}\right] \tilde{W}_j - \left[\frac{\partial \tilde{W}_j}{\partial x}\right] f, \tag{34d}$$

which is essentially the same as Eqs. (47) to (49) of Kamel [19] and Eq. (37) of Kamel [20]. In these equations, the generator $\tilde{W}_n$ is selected such that $\tilde{L}_n f_0$ eliminates the short-period terms in Eq. (34a). In the Krylov-Bogoliubov method of averaging, however, the x_n of Eq. (17) is used to eliminate the short-period terms. To obtain x_n in terms of $\tilde{W}_n$, substitute Eqs. (17), (20), and (22) in Eq. (26) to get

$$x_n = - \tilde{W}_n - \sum_{m=1}^{n-1} C_{m-1}^{n-1}\ L'_m\ x_{n-m}, \tag{35a}$$

with

$$L'_m\ F = \left[\frac{\partial F}{\partial x}\right] \tilde{W}_m. \tag{35b}$$

Also, for any function F of the form

$$F = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!}\ F_n\ (x) = \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!}\ F^{(k)}(y), \tag{36}$$

the $F^{(k)}$ are related to F_n by a formula similar to (16a) where $F_{j,i}$ is defined by (also, see Eq. 26 of Kamel [20])

$$F_{j,i} = - \sum_{m=1}^{j} C_{m-1}^{j-1}\ L'_m\ F_{j-m,i}, \tag{37}$$

where L'_m is defined by Eq. (35b).

It should be pointed out that the use of $\tilde{W}_n$ instead of x_n to eliminate the short-period terms in Eq. (34a) allowed the development of the compact recursive formulation for the Krylov-Bogoliubov method of averaging. This is in marked contrast to Musen's method (Musen [29]). Also, since the Lie-Hori series is equivalent to the Lie-Deprit transform, which was shown here to be equivalent to the Krylov-Bogoliubov method, it is expected that the Lie-Hori series is also equivalent to the method of averaging as shown by Ahmed and Tapley [1].

Simple examples that demonstrate the use of Eq. (34) can be found in Kamel [20] and Nayfeh [30]. More sophisticated application can be found in Kamel and Hassan [22] and Brown [4].

The generalized Lie-Deprit transform can be shown to be also equivalent to the method of multiple scales whose basic idea is to solve Eq. (18) by using the near identify transformation of Eq. (17) and an expansion of the total derivative operator d/dt in terms of multiple time scales

$$\frac{d}{dt} = \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!} \frac{\partial}{\partial t_k}, \tag{38a}$$

where

$$t_k = \frac{\epsilon^k}{k!} t, \; k \geqslant 0. \tag{38b}$$

At each order of the expansion, the vector $\partial x_n / \partial t_0$ $(n = 1,2, \ldots)$ is selected to eliminate the short-period terms and the vector $\partial x / \partial t_k$ $(k \geqslant 0)$ is selected to eliminate the secular terms. At the end of the process, the total derivative of the vector x is obtained from

$$\frac{dx}{dt} = \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!} \frac{\partial x}{\partial t_k}. \tag{39}$$

To establish the equivalence to the generalized Lie-Deprit transform, compare Eq. (39) with Eq. (19) to get

$$f_k = \frac{\partial x}{\partial t_k} \;, \; k \geqslant 0, \tag{40}$$

and use Eq. (35a) to get the vector x_n from the generating vector $\tilde{W}_n$. Note that, as in the case of the method of averaging, the use of $\tilde{W}_n$ instead of x_n allowed the development of the compact recursive formulation of Eq. (34) for the method of multiple scales.

Based on the above analysis, the generalized Lie-Deprit transform is equivalent to both the method of averaging and the method of multiple scales. However, it should be pointed out that the near identity transformation of Eq. (17) is not the same for both methods. In the method of averaging, x is usually defined in terms of polar coordinates, i.e., radii and phase angles while in the method of multiple scales, x is given by the equation

$$\frac{\partial x}{\partial t_0} = g(0), \tag{41}$$

whose solution may be written as

$$x = x(x_a) \tag{42}$$

where x_a are the polar coordinates used as x in the method of averaging.

The choice of x in the method of multiple scales leads to fewer trigonometric terms than in the method of averaging (Kamel and Duhamel [23]). On the other hand, the integration of the $\tilde{W}_n$ equations become more complicated (Eq. 64, Kamel [20]). Also, some of the partial derivatives $\partial F/\partial x$ become more difficult to perform unless a change of variables from x to x_a is used. In this case, $\partial F/\partial x$ is replaced by

$$\frac{\partial F}{\partial x} = \left[\frac{\partial F}{\partial x_a}\right]\left[\frac{\partial x_a}{\partial x}\right], \tag{43}$$

where $[\partial x_a/\partial x]$ is the Jacobian of the inverse of Eq. (42),

$$\frac{\partial x_a}{\partial x} = \left[\frac{\partial x}{\partial x_a}\right]. \tag{44}$$

Finally, the equivalence between the method of averaging and Kevorkian's two-scale method was given by Sarlet [32].

5. THE HAMILTONIZATION OF NON-HAMILTONIAN SYSTEMS

A completely different approach to develop the recursive formulas of Eqs. (34) to (37), for the methods of averaging and multiple time scales, was given by Kamel [21]. In this approach, the non-Hamiltonian system was expressed in form of a Hamiltonian of twice the order of the original system as shown by Birkhoff (pp. 55-58, Birkhoff [2]).

This Hamiltonization process has been long known to researchers in optimization and control (see for example, Bryson and Ho [5]), Powers and Tapley [31]; Choi and Tapley [7], Kamel and Hassan [22]) and it is achieved by using adjoint vectors Y and X (also known as Lagrange multipliers). In this case, the Hamiltonian

F is defined by

$$F = Y \cdot g(y; \epsilon) = X \cdot f(x; \epsilon), \tag{45}$$

where the linearity in the adjoint variables is preserved by selecting the generating function W to be

$$W = X \cdot \tilde{W}(x). \tag{46}$$

In view of Eqs. (2), (6), (18), (19), and (20),

$$F^{(n)} = Y \cdot g^{(n)}(y), \; F_n = X \cdot f_n(x), \; W_{n+1} = X \cdot \tilde{W}_{n+1}(x), \quad n \geqslant 0. \tag{47}$$

Substitution of Eq. (47) in Eq. (16) leads to Eqs. (34) to (37). This is because the Lie-Deprit operators L_W, $\tilde{L}_W$, and L'_W of Eqs. (5), (30), and (35) are related as follows:

$$L_W[X \cdot F(x)] = X \cdot \tilde{L}_W F(x), \tag{48a}$$

$$L_W[F(x)] = L'_W F(x). \tag{48b}$$

Therefore, the canonical algorithm of Eq. (16) is also equivalent to the methods of averaging and multiple scales. Finally, the solution of the equations of motion of the adjoint vectors (X, Y) are not required. These equations, however, are given by

$$\dot{X} = -\frac{\partial F}{\partial x} = -\left[\frac{\partial f}{\partial x}\right] X, \tag{49a}$$

$$\dot{Y} = -\frac{\partial F}{\partial y} = -\left[\frac{\partial g}{\partial y}\right] Y, \tag{49b}$$

and in view of Eqs. (18) and (19), (X, Y) can be interpreted as the variations ∂x and ∂y propagated backward in time. These variations may be useful in the study of the dynamical stability about a given nominal trajectory (x, y).

6. CONCLUSION

The methods of averaging and multiple scales were shown to be equivalent to the Lie-Deprit transform. General recursive algorithms (Eqs. (34) and (35)) were obtained for these methods using the generating vector defined by Eq. (20). These algorithms are convenient for computerized algebraic manipulation using existing scientific computers or the ever increasing capability of personal computers. Finally, the extension of the Lie-Deprit transform to the perturbation solution of partial differential equations is an open subject for future research.

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REFERENCES

[1] A. H. Ahmed and B. D. Tapley, "Equivalence of generalized Lie-Hori method and the method of averaging," *Celestial Mechanics* **33**, 1-20 (1984).

[2] G. D. Birkhoff, *Dynamical Systems* (Am. Math. Soc. Colloq. Pub., IX, Providence, Rhode Island, 1927).

[3] N. N. Bogoliubov and Y. A. Mitropolsky, *Asymptotic Methods in the Theory of Nonlinear Oscillations* (Gordon and Breach, New York, 1961).

[4] B. C. Brown, "Secular effects in the orbits of the Galilean satellites," *Celestial Mechanics* **23**, 203-221 (1981).

[5] A. E. Bryson and Y. C. Ho, *Applied Optimal Control* (Hemisphere, Washington, D.C., 1981), revised ed.

[6] J. A. Campbell and W. H. Jeffereys, "Equivalence of the perturbation theories of Hori and Deprit," *Celestial Mechanics* **2**, 467-473 (1970).

[7] J. S. Choi and B. D. Tapley, "An extended canonical perturbation method," *Celestial Mechanics* **7**, 77-90 (1973).

[8] A. Deprit, "Canonical transformations depending on a small parameter," *Celestial Mechanics* **1**, 12-30 (1969).

[9] G. E. O. Giacaglia, *Perturbation Methods in Non-Linear Systems*, (Springer, New York, 1972).

[10] J. Henrard, "On a perturbation theory using Lie transforms," *Celestial Mechanics*, **3**, 107-120 (1970).

[11] J. Henrard, "The algorithm of the inverse for Lie transforms," *Recent Advances in Dynamical Astronomy*,

Tapley and Szebehely, Eds. (Reidel, Dordrecht, Holland), pp. 250-259.

[12] J. Henrard and J. Roels, "Equivalence for Lie transforms," *Celestial Mechanics* **10**, 497-512 (1974).

[13] R. A. Howland, "An accelerated elimination technique for the solution of perturbed Hamiltonian systems," *Celestial Mechanics* **15**, 327-352 (1977).

[14] G. Hori, "Theory of general perturbations with unspecified canonical variables," *Publ. Astron. Soc. Japan* **18**, 287-296 (1966).

[15] G. Hori, "Comparison of two perturbation theories based on canonical transformations," *Publ. Astron. Soc. Japan* **22**, 191-198 (1970).

[16] G. Hori, "Theory of general perturbations for noncanonical systems," *Pub. Astron. Soc. Japan*, **23**, 567-587 (1971).

[17] A. A. Kamel, "Perturbation theory based on Lie transforms and its application to the stability of motion near Sun-Perturbed Earth-Moon triangular libration points," PhD Thesis, SUDAAR No. 391, Dec 1969, Dept. of Aeronautics and Astronautics, Stanford University. See also NASA CR-1622, August 1970.

[18] A. A. Kamel, "Expansion formulae in canonical transformation depending on a small parameter," *Celestial Mechanics* **1**, 190-199 (1969).

[19] A. A. Kamel, "Perturbation method in the theory of nonlinear oscillations," SUDAAR No. 389, Oct 1969, Dept. of Aeronautics and Astronautics, Stanford University. See also NASA CR-1554, March 1970.

[20] A. A. Kamel, "Perturbation method of the theory of nonlinear oscillations," *Celestial Mechanics*, **3**, 90-106 (1970).

[21] A. A. Kamel, "Lie transforms and the Hamiltonization of non-Hamiltonian systems," *Celestial Mechanics* **4**, 397-405 (1971).

[22] A. A. Kamel and S. D. Hassan, "A perturbation treatment for optimal slightly nonlinear systems with

linear control and quadratic criteria," *J. Optimization Theory and Applications* **2**, 386-404 (1973).

[23] A. A. Kamel and T. Duhamel, "A second order solution of the main problem of artificial satellites using multiple scales," to be published in *J. Guidance and Control.*

[24] J. Kevorkian and J. D. Cole, *Perturbation Methods in Applied Mathematics* (Springer, New York, 1981).

[25] M. S. Lie, *Theorie der Transformationsgruppen* (Teubner, Leipzig, 1888, Vol. 1, Chap. 1).

[26] B. McNamara, "Super-convergent adiabatic invariants with resonant denominators by Lie Transforms," *J. Math. Phys.* **19**, 2154-2164 (1978).

[27] W. A. Mersman, "A new algorithm for the Lie transformation," *Celestial Mechanics* **3**, 81-89 (1970).

[28] W. A. Mersman, "Explicit recursive algorithm for the construction of equivalent canonical transformation," *Celestial Mechanics* **3**, 384-389 (1971).

[29] P. Musen, "On the high order effects in the methods of Krylov-Bogoliubov and Poincaré," *J. Astr. Sciences* **12**, No. 4, 129-134 (1965).

[30] A. H. Nayfeh, *Perturbation Methods* (Wiley, New York, 1973).

[31] W. F. Powers and B. D. Tapley, "Canonical transformation applications to optimal trajectory analysis," *AIAA Journal* **7**, 394-399 (1969).

[32] W. Sarlet, "On a common derivation of the averaging method and the two-timescale Method," *Celestial Mechanics* **17** 299-312 (1978).

[33] H. Shniad, "The equivalence of von Zeipel mappings and Lie transforms," *Celestial Mechanics* **2**, 114-120 (1969).