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An efficient algorithm for orbital evolution of artificial satellite

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Abstract

Searching for an accurate model to evaluate the orbital position of the operating satellites and space debris is very important at the time being. This is actually to design different maneuvering schemes to avoid catastrophic consequences of collision. In the present paper a second order theory of perturbations (in the sense of the Hori–Lie perturbation method) is developed. The most dominating perturbations, namely geopotential effects, luni-solar perturbations, solar radiation pressure and atmospheric drag are included. Resonance and very long period perturbations are modeled with the use of semi-secular terms for short time span predictions. A comparison of our analytical solution with a numerical integration (Burlich–Stoer) of the equations of motion for chosen artificial satellites at (LEO, MEO, GEO) is presented. The computations are carried out for different satellite with different area-to-mass ratios showing a good accuracy of the theory.

Keywords: Analytical theory; Lie method; Artificial satellite; Orbital motion; Luni-solar; Solar radiation pressure; Air drag

1. Introduction

More than four decades of space exploration has led to accumulation of significant quantities of debris around the Earth. These objects range in size from tiny pieces of junk to large inoperable satellites. Although these objects, that have small size, have high area-to-mass ratios, consequently their orbits are strongly influenced by solar radiation pressure and atmospheric drag. So, the large existing population of space debris in the LEO, MEO and GEO is rapidly growing with time. Serious hazard for the survival of operating spacecrafts, particularly satellites and space astronomical observatories, is frequently faced.

Since the average collision velocity between any spacecraft orbiting in the LEO and debris is about 10 km/s and about 3 km/s in the GEO, space debris may significantly disturb or cause catastrophic damage to a spacecraft or a satellite. Different shielding techniques of spacecraft may be useful in protecting against impacts of space debris with diameters smaller than 1 cm. For larger debris, the only effective method to avoid catastrophic consequences of collision is to change the spacecraft's orbit using maneuver. The necessary condition for maneuvering to avoid collision is to evaluate and predict future positions of the spacecraft and space debris with sufficient highly accuracy.

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Several theories have been developed by many authors to include the effects of the luni-solar perturbations on artificial satellite's orbits. The formulas for secular and periodic perturbations in the orbital elements were obtained using different methods [1–3]. Practical applications of the existing theories are rather limited due to low precision of the formulas for perturbations. For satellites moving in Geostationary ring the luni-solar perturbations and solar radiation pressure become more and more significant [4,5]. On the other hand, atmospheric drag has no effect at such high altitudes. Therefore, the motion of the satellite and debris in HEO can be treated taking into account the asphericity of the Earth, the gravitational attraction of the Moon and the Sun as well as solar radiation pressure. In the LEO the atmospheric drag becomes significant, besides the aforementioned forces.

1.1. The main features of the present work

Existing analytical methods can solve this problem only with low accuracy. Difficulties are caused mainly by the lack of satisfactory analytical solution of the resonance problem for geosynchronous orbit as well as from the lack of efficient analytical theory combining luni-solar perturbation and solar radiation pressure with geopotential attraction.

A variety of numerical integration methods has been developed to solve ordinary differential equations and many of these methods have successfully been applied in the field of celestial mechanics, (e.g. Cowell, Rung–Kutta, Burlich–Stoer, Edgar Everhart).

In spite of the importance of the numerical integration, we still need analytical theories to overcome the accumulation of errors of numerical integration and to investigate the physical and dynamical phenomena affecting the orbits.

The present work aims to meet the following goals:

(1) Constructing an analytical theory up to the second order taking into account geopotential effects, luni-solar perturbations, solar radiation pressure and atmospheric drag. The theory includes the following three parts: (a) the first order perturbations due to J_2 coefficient of the geopotential, Moon, Sun and atmospheric drag, (b) the second order perturbation due to J_2^2, the rest of zonal and tesseral harmonics, Moon, Sun and solar radiation pressure, (c) the second order perturbation due to the different combinations of first order terms.
(2) Analyzing the dynamics of the perturbative effects due to luni-solar effects, solar radiation pressure and atmospheric drag.
(3) Predicting the position of artificial satellite and space debris in GEO in a level of few meters, and in LEO in tens of centimeters level.

2. Formulation of the problem

Applying Hori–Lie technique [6], the Hamiltonian of the problem should be expressed in terms of the Delaunay canonical variables: $L = \sqrt{\mu a}$, $G = L\sqrt{1-e^2}$, $H = G\cos I$, $\ell = M$, $g = \omega$, $h = \Omega$, where a, e, I, ω, Ω and M are the Keplerian elements.

Because of their complex form of the eccentricity and the inclination functions are not expressed here in terms of Delaunay variables and since that the Hamiltonian $\boldsymbol{H}$ depends not only on the Delaunay variables, but on the inclination I and eccentricity e as well. The partial derivatives with respect to L, G, and H are then calculated using the following formulas:

$$\frac{\partial}{\partial L} = \left(\frac{\partial}{\partial L}\right) + \frac{1-e^2}{eL}\frac{\partial}{\partial e},$$
$$\frac{\partial}{\partial G} = -\frac{\sqrt{1-e^2}}{eL}\frac{\partial}{\partial e} + \frac{1}{L\sqrt{1-e^2}}\frac{\sin I}{\cos I}\frac{\partial}{\partial I},$$
$$\frac{\partial}{\partial H} = -\frac{1}{L\sqrt{1-e^2}\sin I}\frac{\partial}{\partial I}.$$

The geopotential including all spherical harmonics coefficients, luni-solar attraction, solar radiation pressure and atmospheric drag are taken into account.

It is well known that for high orbit altitudes the luni-solar perturbations are of the same order as the J_2 oblatness term [7]

$$\mathrm{o}\left(\frac{\mu_k}{a_k}\left(\frac{a}{a_k}\right)^2\right) \approx \mathrm{o}\left(\frac{\mu}{a}\left(\frac{R_\mathrm{e}}{a}\right)^2 J_2\right),$$

where, μ is the gravitational parameter, k refers to the Moon or the Sun, and R_e the mean Earth's equatorial radius. Therefore, the terms are ordered so as to consider J_2, the small parameter of the problem, of order one. The order of perturbations can be changed differently according to the altitude of the object.

The Hamiltonian $\boldsymbol{H}$ can be written in the form,

$$\boldsymbol{H} = \boldsymbol{H}_0 + \boldsymbol{H}_1 + \boldsymbol{H}_2, \tag{1}$$

where,

$$\boldsymbol{H}_0 = \frac{\mu^2}{2L^2} - \dot{\Theta}H, \tag{2}$$

$$\boldsymbol{H}_1 = \boldsymbol{H}_1^\mathrm{G}(L, G, H, \ell, g, -; C_{2,0}) + \boldsymbol{H}_\mathrm{M}(L, G, H, \ell, g, h) + \boldsymbol{H}_\mathrm{S}(L, G, H, \ell, g, h), \tag{3}$$

$$\boldsymbol{H}_2 = \boldsymbol{H}_2^\mathrm{G}(L, G, H, \ell, g, h; C_{nm}, S_{nm}) + \boldsymbol{H}_\mathrm{SRP}(L, G, H, K, \ell, g, h, -), \tag{4}$$

$$\boldsymbol{H}_\mathrm{M} = \sum_{l\geqslant 2}\sum_{m=0}^{l}\sum_{s=0}^{l}\sum_{p=0}^{l}\sum_{q=0}^{l}\sum_{j=-\infty}^{\infty}\sum_{r=-\infty}^{\infty} D_\mathrm{M}\{(-1)^{k_2}U_l^{m,-s}\cos\Psi^+ + (-1)^{k_3}U_l^{m,s}\cos\Psi^-\}, \tag{5}$$

$$\boldsymbol{H}_\mathrm{S} = \sum_{l\geqslant 2}\sum_{m=0}^{l}\sum_{s=0}^{l}\sum_{p=0}^{l}\sum_{q=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} D_\mathrm{S}\cos\Phi, \tag{6}$$

$$\boldsymbol{H}_\mathrm{SRP} = -\boldsymbol{F}_\mathrm{SRP}\cdot\boldsymbol{r}, \tag{7}$$

$$D_\mathrm{S} = \mu_\mathrm{S}\frac{\varepsilon_\mathrm{S}(l-m)!}{a_\mathrm{S}(l+m)!}\left(\frac{a}{a_\mathrm{S}}\right)^l F_{lmp}(I)F_{lms}(I_\mathrm{S})G_{lsj}(e_\mathrm{S})H_{lpq}(e),$$

$$\Phi = (l-2p)\omega + (l-2p+q)\ell - (l-2s)\omega_\mathrm{S} - (l-2s+j)\ell_\mathrm{S} + m(\Omega - \Omega_\mathrm{S}),$$

$$D_\mathrm{M} = (-1)^{k_1}\frac{L^{2l}\mu_\mathrm{M}\varepsilon_m\varepsilon_s(l-s)!}{2\mu^l a_\mathrm{M}^{l+1}(l+m)!}F_{lmp}(I)F_{lsq}(I_\mathrm{M})G_{lqr}(e_\mathrm{M})H_{lpj}(e),$$

$$\varepsilon_m = \begin{cases} 1 & \text{if } m = 0 \\ 2 & \text{if } m \neq 0, \end{cases}$$

$$\varepsilon_s = \begin{cases} 1 & \text{if } s = 0 \\ 2 & \text{if } s \neq 0, \end{cases}$$

$k_1 = \|m/2\|,$ The greatest integer part of $m/2$;

$$k_2 = t(m+s-1)+1;$$

$$k_3 = t(m+s),$$

$$t = \begin{cases} 0 & \text{if } (l-1) \text{ is even}, \\ 1 & \text{if } (l-1) \text{ is odd}, \end{cases}$$

$$y_s = \begin{cases} 0 & \text{if } s \text{ is even}, \\ 1/2 & \text{if } s \text{ is odd}, \end{cases}$$

$$\Psi^+ = \theta_{lmpj} + \theta'_{lsqr} - y_s\pi,$$

$$\Psi^+ = \theta_{lmpj} - \theta'_{lsqr} - y_s\pi,$$

$$\theta_{lmpj} = (l-2p)\omega + (l-2p+j)l + m\Omega,$$

$$\theta'_{lsqr} = (l-2q)\omega_{\mathrm{M}} + (l-2q+r)\ell_{\mathrm{M}} + s(\Omega_{\mathrm{M}} - (\pi/2)),$$
$$U_l^{m,s}(\varepsilon) = (-1)^{m-s} \sum_{r=\max[0,-(m+s)]}^{\min[l-s,l-m]} (-1)^{l-m-r} \binom{l+m}{m+s+r} \binom{l-m}{r} \cos^{m+s+2r}\left(\frac{\varepsilon}{2}\right) \sin^{-m-s+2l-2r}\left(\frac{\varepsilon}{2}\right)$$

with $\varepsilon = 23.4^\circ\ 27'\ 08.26''$, the ecliptic inclination on the equator. The eccentricity function H_{lpj} and G_{lqr} can be related to the Hansen function $X_k^{n,m}$ [8]. The inclination functions F_{lmp} are given in [9], where $\dot{\Theta}$ is the Earth's mean angular velocity, the subscripts M and S refer to the Moon and Sun respectively. The parameters D_{M}, D_{S}, $\Psi^{\pm}$ and Φ are the amplitudes and arguments of lunar and solar perturbations respectively. For more mathematical details about geopotential terms $\boldsymbol{H}_1^{\mathrm{G}}$ and $\boldsymbol{H}_2^{\mathrm{G}}$ (composed of zonal $\boldsymbol{H}_2^{\mathrm{Z}}$ and tesseral $\boldsymbol{H}_2^{\mathrm{T}}$ components), the reader is referred to [10]. The ecliptic reference system is considered for the lunar ephemeris while the equatorial reference frame is considered for both satellite and Sun Keplerian elements (see [2,3,11]). The details of all involved quantities in $\boldsymbol{H}_{\mathrm{M}}$ and $\boldsymbol{H}_{\mathrm{S}}$ functions can be found in [12].

The solar radiation pressure contribution to the Hamiltonian $\boldsymbol{H}_{\mathrm{SRP}}$ is made as shown above to allow the application of the canonical methods. The disturbing function of solar radiation $\mathbf{F}_{\mathrm{SRP}}$, [5] depends explicitly on time, to recover the autonomous form of the equations, we adjoin to the Delaunay set a new pair of conjugate variables (k, K) such that the mean longitude of the Sun k is given by $k = \upsilon t + \mathrm{const}$, υ the Sun's mean motion, υK can represent the kinetic energy of the Sun [6].

The disturbing function of the solar radiation pressure $\boldsymbol{F}_{\mathrm{SRP}}$ and the radius vector $\boldsymbol{r}$ are given by

$$\boldsymbol{F}_{\mathrm{SRP}} = -\left(\frac{r_0}{r_{\mathrm{S}}}\right)^2 \{R_1(\gamma_i)\hat{s} + R_2(\gamma_i)\hat{n}_i\}, \quad \boldsymbol{r} = r[\cos f \hat{P} + \sin f \hat{Q}],$$

where $r_0,\ r_{\mathrm{S}}$ are the Earth–Sun, Satellite–Sun distances respectively, $\hat{s}$ is a unit vector along r_0 and $\hat{n}_i$ is a unit vector along the normal to the ith elemental surface, γ_i is the angle of incidence, f is the true anomaly, $\hat{P}, \hat{Q}$ are the direction cosines of the perifocal coordinate system with respect to the geocentric equatorial coordinate system, and $R_1,\ R_2$ are given by

$$R_1(\gamma_i) = \left(\frac{A}{m}\right)\frac{S_0}{c_l}\{(1-\beta_i)\cos\gamma_i\}, \quad R_2(\gamma_i) = \left(\frac{A}{m}\right)\frac{S_0}{c_l}\left\{\frac{2}{3}\delta_i + 2\beta_i\cos\gamma_i\right\},$$
$$\hat{S} = \cos k\hat{i} + \cos\varepsilon\sin k\hat{j} + \sin\varepsilon\cos k\hat{k}, \quad \hat{n}_i = -\cos\alpha_i\hat{e}_s - \sin\alpha_i\hat{e}_t,$$

where, $\frac{A}{m}$ is the area-to-mass ratio, S_0 is the solar constant, c_l is the velocity of light, $\beta_i,\ \delta_i$ are the specular and diffusive reflectivity coefficients respectively, $\varepsilon,\ \hat{e}_s,\ \hat{e}_t$ are the obliquity, a unit vector along the geocentric radius vector, a unit vector perpendicular to $\hat{e}_s$ in the orbital plane respectively, and α_i is a constant angle between $\hat{e}_s$ and $-\hat{n}_i$. The final form of $\boldsymbol{H}_{\mathrm{SRP}}$ which are used in this work is

$$\begin{aligned}\mathsf{H}_{\mathrm{SRP}} = A_1\frac{L^2}{\phi}\{&(1+c)(1+c_\varepsilon)\cos F_{11}^{1,-1} + (1+c)(1+c_\varepsilon)\cos F_{11}^{1,-1} + (1-c)(1+c_\varepsilon)\cos F_{11}^{-1,1}\\ &+ (1-c)(1-c_\varepsilon)\cos F_{11}^{-1,-1} + 2ss_\varepsilon(\cos F_{11}^{0,-1} - \cos F_{11}^{0,1})\}\\ &+ A_2\frac{L^4}{\phi^2}\{(1+c^2) + (1+c)(1-c_\varepsilon)\cos F_{11}^{00}\},\end{aligned} \tag{8}$$

where A_1, A_2 are constants, $F_{ij}^{mn} = if + j\omega + mh + nk$, and $c = \cos I,\ c_\varepsilon = \cos\varepsilon,\ \phi = \frac{a}{r}$.

In order to include the drag force into the canonical equations of motions, a theorem due to [13] will be utilized

$$\frac{\mathrm{d}L_j}{\mathrm{d}t} = -\frac{\partial \boldsymbol{H}}{\partial \ell_j} + P_j, \quad \frac{\mathrm{d}\ell_j}{\mathrm{d}t} = \frac{\partial \boldsymbol{H}}{\partial L_j} - Q_j,$$
$$P_j = -\frac{1}{2}C_{\mathrm{D}}\frac{A}{m}\rho|\vec{V}|p_j, \quad Q_j = -\frac{1}{2}C_{\mathrm{D}}\frac{A}{m}\rho|\vec{V}|q_j,$$

where C_{D} is the drag dimensionless coefficient, ρ is the atmospheric density, $\vec{V}$ is the satellite velocity, p_j and q_j are the momenta and angular variables.

3. Canonical transformations

The canonical equations of motion of an Earth artificial satellite are given by:

$$\frac{\mathrm{d}}{\mathrm{d}t}(L,G,H)=\frac{\partial \boldsymbol{H}}{\partial(\ell,g,h)},\quad \frac{\mathrm{d}}{\mathrm{d}t}(\ell,g,h)=-\frac{\partial \boldsymbol{H}}{\partial(L,G,H)}. \tag{9}$$

The single canonical transformation:

$$(L,G,H,\ell,g,h)\rightarrow(L',G',H',-,-,-)$$

transform Eq. (9) into

$$\frac{\mathrm{d}}{\mathrm{d}t}(L',G',H')=\frac{\partial \boldsymbol{H}^*}{\partial(\ell',g',h')},\quad \frac{\mathrm{d}}{\mathrm{d}t}(\ell',g',h')=-\frac{\partial \boldsymbol{H}^*}{\partial(L',G',H')}. \tag{9a}$$

The new Hamiltonian $\boldsymbol{H}^*=\boldsymbol{H}^*(L',G',H',-,-,-)$ averaged over (ℓ,g,h) does not contain any of periodic terms and is of the following form:

$$\boldsymbol{H}^*(L',G',H',-,g',h')=\boldsymbol{H}_0^*+\boldsymbol{H}_1^*+\boldsymbol{H}_2^*+\cdots,\quad \boldsymbol{H}_n^*\approx\varepsilon^n. \tag{10}$$

And the generating function $S^*=S^*(L',G',H',\ell',g',h')$ of this transformation is of the form

$$S^*(L',G',H',\ell',g',h')=S_1^*+S_2^*+\cdots,\quad S_n^*\approx\varepsilon^n. \tag{11}$$

Applying Hori–Lie method according to theory presented by [14], the new Hamiltonian and the generating functions have the following forms:

Order 0:

$$\boldsymbol{H}_0^*=\frac{\mu}{2L'^2}-\dot{\Theta}H'. \tag{12}$$

Order 1:

$$\{\boldsymbol{H}_0,S_1^*\}+\boldsymbol{H}_1=\boldsymbol{H}_1^*,$$

i.e.

$$\frac{\partial \boldsymbol{H}_0}{\partial L'}\frac{\partial S_1^*}{\partial \ell'}+\frac{\partial \boldsymbol{H}_0}{\partial H'}\frac{\partial S_1^*}{\partial h'}=\boldsymbol{H}_1^*-\boldsymbol{H}_1, \tag{13}$$

$$\frac{\partial \boldsymbol{H}_0}{\partial L'}=-\frac{\mu^2}{L'^3}=-n',\quad \frac{\partial \boldsymbol{H}_0}{\partial H'}=-\dot{\Theta}. \tag{14}$$

Let us introduce the independent variable t^* by the definition:

$$\frac{\partial \ell'}{\partial t^*}=n',\quad \frac{\partial h'}{\partial t^*}=\dot{\Theta}, \tag{15}$$

where, n' is the mean motion of the satellite, then

$$\frac{\partial \boldsymbol{H}_0}{\partial L'}=\frac{\partial \ell'}{\partial t^*},\quad \frac{\partial \boldsymbol{H}_0}{\partial H'}=\frac{\partial h'}{\partial t^*}. \tag{16}$$

Substituting from (16) into (13) we obtain,

$$\frac{\partial S_1^*}{\partial \ell'}\frac{\partial \ell'}{\partial t^*}+\frac{\partial S_1^*}{\partial h'}\frac{\partial h'}{\partial t^*}=-(\boldsymbol{H}_1^*-\boldsymbol{H}_1), \tag{17}$$

i.e.

$$\frac{\partial S_1^*}{\partial t^*}=-(\boldsymbol{H}_1^*-\boldsymbol{H}_1). \tag{18}$$

Now, let us divide the $\boldsymbol{H}_1$ into two parts: the first part is $\langle \boldsymbol{H}_1 \rangle_{t^*}$ which is independent on ℓ', g', h' and the second one is $(\boldsymbol{H}_1)_{t^*}$ which depend on ℓ', g', h'. Then we have

$$\boldsymbol{H}_1^* = \langle \boldsymbol{H}_1 \rangle_{t^*} = \boldsymbol{H}_{J2}^* + \boldsymbol{H}_{\mathrm{M}}^* + \boldsymbol{H}_{\mathrm{S}}^* = \langle \boldsymbol{H}_{J2} \rangle_{t^*} + \langle \boldsymbol{H}_{\mathrm{M}} \rangle_{t^*} + \langle \boldsymbol{H}_{\mathrm{M}} \rangle_{t^*}, \tag{19}$$

$$S_1^* = \frac{1}{2}\frac{L'^3}{\mu^2}\int (\boldsymbol{H}_1)_{t^*}\,\mathrm{d}t^* = S_{J2}^* + S_{\mathrm{M}}^* + S_{\mathrm{S}}^*, \tag{20}$$

where,

$$\boldsymbol{H}_{J2}^* = \langle \boldsymbol{H}_{J2} \rangle_{t^*}, \quad \boldsymbol{H}_{\mathrm{M}}^* = \langle \boldsymbol{H}_{\mathrm{M}} \rangle_{t^*}, \quad \boldsymbol{H}_{\mathrm{S}}^* = \langle \boldsymbol{H}_{\mathrm{S}} \rangle_{t^*}, \tag{21}$$

$$S_{J2}^* = \int (\boldsymbol{H}_{J2})_{t^*}\,\mathrm{d}t^*, \quad S_{\mathrm{M}}^* = \int (\boldsymbol{H}_{\mathrm{M}})_{t^*}\,\mathrm{d}t^*, \quad S_{\mathrm{S}}^* = \int (\boldsymbol{H}_{\mathrm{S}})_{t^*}\,\mathrm{d}t^* \tag{22}$$

Using (15) we obtain

$$S_{J2}^* = \frac{1}{n'}\int (\boldsymbol{H}_{J2})_{t^*}\,\mathrm{d}\ell', \quad S_{\mathrm{M}}^* = \frac{1}{n'}\int (\boldsymbol{H}_{\mathrm{M}})_{t^*}\,\mathrm{d}\ell', \quad S_{\mathrm{S}}^* = \frac{1}{n'}\int (\boldsymbol{H}_{\mathrm{S}})_{t^*}\,\mathrm{d}\ell'. \tag{23}$$

Order 2:

$$\{\boldsymbol{H}_0, S_2^*\} + \frac{1}{2}\{\boldsymbol{H}_1 + \boldsymbol{H}_1^*, S_1^*\} + \boldsymbol{H}_2 = \boldsymbol{H}_2^*. \tag{24}$$

By introducing the variable t^* in Eq. (15), we get

$$-\frac{\mathrm{d}S_2^*}{\mathrm{d}t^*} + \frac{1}{2}\{\boldsymbol{H}_1 + \boldsymbol{H}_1^*, S_1^*\} + \boldsymbol{H}_2 = \boldsymbol{H}_2^*. \tag{25}$$

Now, let us divide $\frac{1}{2}\{\boldsymbol{H}_1 + \boldsymbol{H}_1^*, S_1^*\} + \boldsymbol{H}_2$ into two parts: the first part is $\langle \frac{1}{2}\{\boldsymbol{H}_1 + \boldsymbol{H}_1^*, S_1^*\} + \boldsymbol{H}_2 \rangle_{t^*}$ which is independent on ℓ', g', h' and the second one is $\left(\frac{1}{2}\{\boldsymbol{H}_1 + \boldsymbol{H}_1^*, S_1^*\} + \boldsymbol{H}_2\right)_{t^*}$ which depend on ℓ', g', h'. Then we have

$$S_2^* = \frac{L'^3}{\mu^2}\int \left(\frac{1}{2}\{\boldsymbol{H}_1 + \boldsymbol{H}_1^*, S_1^*\} + \boldsymbol{H}_2\right)_{t^*}\mathrm{d}t^* = S_2 + S_2^{\mathrm{Z}} + S_2^{\mathrm{T}} + S_2^{\mathrm{SRP}}, \tag{26}$$

where,

$$S_2 = \frac{1}{2}\frac{L'^3}{\mu^2}\int (\{\boldsymbol{H}_1 + \boldsymbol{H}_1^*, S_1^*\})_{t^*}\,\mathrm{d}t^*, \tag{27}$$

$$S_2^{\mathrm{Z}} = \frac{L'^3}{\mu^2}\int (\boldsymbol{H}_2^{\mathrm{Z}})_{t^*}\,\mathrm{d}t^*, \quad S_2^{\mathrm{T}} = \frac{L'^3}{\mu^2}\int (\boldsymbol{H}_2^{\mathrm{T}})_{t^*}\,\mathrm{d}t^*, \quad S_2^{\mathrm{SRP}} = \frac{L'^3}{\mu^2}\int (\boldsymbol{H}_2^{\mathrm{SRP}})_{t^*}\,\mathrm{d}t^*. \tag{28}$$

Using (15), we obtain

$$S_2 = \frac{1}{2n'}\int (\{\boldsymbol{H}_1 + \boldsymbol{H}_1^*, S_1^*\})_{t^*}\,\mathrm{d}\ell', \tag{29}$$

$$S_2^{\mathrm{Z}} = \frac{1}{2n'}\int (\boldsymbol{H}_2^{\mathrm{Z}})_{t^*}\,\mathrm{d}\ell', \quad S_2^{\mathrm{T}} = \frac{1}{2n'}\int (\boldsymbol{H}_2^{\mathrm{T}})_{t^*}\,\mathrm{d}\ell', \quad S_2^{\mathrm{SRP}} = \frac{1}{2n'}\int (\boldsymbol{H}_2^{\mathrm{SRP}})_{t^*}\,\mathrm{d}\ell'. \tag{30}$$

Let $\boldsymbol{H}_1' = \boldsymbol{H}_1 + \boldsymbol{H}_1^*$, and using Eq. (3) we can rewrite (29) as the following:

$$\begin{aligned} S_2 &= \frac{1}{2n'}\int (\{\boldsymbol{H}_{J2}' + \boldsymbol{H}_{\mathrm{M}}' + \boldsymbol{H}_{\mathrm{S}}', S_{J2}^* + S_{\mathrm{M}}^* + S_{\mathrm{S}}^*\})_{t^*}\,\mathrm{d}\ell' \\ &= S_2^{*\mathrm{I}} + S_2^{*\mathrm{II}} + S_2^{*\mathrm{III}} + S_2^{*\mathrm{IV}} + S_2^{*\mathrm{V}} + S_2^{*\mathrm{VI}} + S_2^{*\mathrm{VII}} + S_2^{*\mathrm{VIII}} + S_2^{*\mathrm{X}}, \end{aligned} \tag{31}$$

where,

$$S_2^{*\mathrm{I}} = \frac{1}{2n'}\int (\{\boldsymbol{H}'_{J2}, S^*_{J2}\})_{t^*}\,\mathrm{d}\ell', \quad S_2^{*\mathrm{II}} = \frac{1}{2n'}\int (\{\boldsymbol{H}'_{J2}, S^*_{\mathrm{M}}\})_{t^*}\,\mathrm{d}\ell',$$
$$S_2^{*\mathrm{III}} = \frac{1}{2n'}\int (\{\boldsymbol{H}'_{J2}, S^*_{\mathrm{S}}\})_{t^*}\,\mathrm{d}\ell', \quad S_2^{*\mathrm{IV}} = \frac{1}{2n'}\int (\{\boldsymbol{H}'_{\mathrm{M}}, S^*_{J2}\})_{t^*}\,\mathrm{d}\ell',$$
$$S_2^{*\mathrm{V}} = \frac{1}{2n'}\int (\{\boldsymbol{H}'_{\mathrm{M}}, S^*_{\mathrm{M}}\})_{t^*}\,\mathrm{d}\ell', \quad S_2^{*\mathrm{VI}} = \frac{1}{2n'}\int (\{\boldsymbol{H}'_{\mathrm{M}}, S^*_{\mathrm{S}}\})_{t^*}\,\mathrm{d}\ell',$$
$$S_2^{*\mathrm{VII}} = \frac{1}{2n'}\int (\{\boldsymbol{H}'_{\mathrm{S}}, S^*_{J2}\})_{t^*}\,\mathrm{d}\ell', \quad S_2^{*\mathrm{VIII}} = \frac{1}{2n'}\int (\{\boldsymbol{H}'_{\mathrm{S}}, S^*_{\mathrm{M}}\})_{t^*}\,\mathrm{d}\ell',$$
$$S_2^{*\mathrm{X}} = \frac{1}{2n'}\int (\{\boldsymbol{H}'_{\mathrm{S}}, S^*_{\mathrm{S}}\})_{t^*}\,\mathrm{d}\ell'.$$

4. Secular and periodic perturbations

According to Hori–Lie perturbation method and using S^*_σ as one of the generating functions (S^*_1, S^*_2), the perturbations in any orbital elements or in any function E of these elements, can be in the following expression [14]:

$$\Delta_\sigma E = \{E, S^*_\sigma\} = \frac{\partial E}{\partial L}\frac{\partial S^*_\sigma}{\partial \ell} - \frac{\partial E}{\partial \ell}\frac{\partial S^*_\sigma}{\partial L} + \frac{\partial E}{\partial G}\frac{\partial S^*_\sigma}{\partial g} - \frac{\partial E}{\partial g}\frac{\partial S^*_\sigma}{\partial G} + \frac{\partial E}{\partial H}\frac{\partial S^*_\sigma}{\partial h} - \frac{\partial E}{\partial h}\frac{\partial S^*_\sigma}{\partial H}. \tag{32}$$

4.1. First order periodic perturbations in Keplerian elements

Using Eq. (32), the perturbations $\Delta_{\mathrm{M}}E$ in Keplerian elements due to Moon perturbations can be found from the following formula:

$$\Delta_{\mathrm{M}} E = \{E, S^*_{\mathrm{M}}\} = \frac{\partial E}{\partial L}\frac{\partial S^*_{\mathrm{M}}}{\partial \ell} - \frac{\partial E}{\partial \ell}\frac{\partial S^*_{\mathrm{M}}}{\partial L} + \frac{\partial E}{\partial G}\frac{\partial S^*_{\mathrm{M}}}{\partial g} - \frac{\partial E}{\partial g}\frac{\partial S^*_{\mathrm{M}}}{\partial G} + \frac{\partial E}{\partial H}\frac{\partial S^*_{\mathrm{M}}}{\partial h} - \frac{\partial E}{\partial h}\frac{\partial S^*_{\mathrm{M}}}{\partial H}. \tag{33}$$

From Eqs. (5) and (23), we get

$$S^*_{\mathrm{M}} = \sum^{M} D_{\mathrm{M}}\left\{\frac{(-1)^{k_2} U_l^{m,-s}}{\dot{\Psi}^+}\sin\Psi^+ + \frac{(-1)^{k_3} U_l^{m,s}}{\dot{\Psi}^-}\sin\Psi^-\right\}, \tag{34}$$

where,

$$\begin{aligned}\dot{\Psi}^\pm = \frac{\mathrm{d}\dot{\Psi}^\pm}{\mathrm{d}t} &= \frac{\mathrm{d}g}{\mathrm{d}t}(l-2p)\omega + \frac{\mathrm{d}\ell}{\mathrm{d}t}(l-2p+j) + m\frac{\mathrm{d}h}{\mathrm{d}t}\\ &\pm\left\{(l-2q)\frac{\mathrm{d}g_{\mathrm{M}}}{\mathrm{d}t} + (l-2q+r)\frac{\mathrm{d}\ell_{\mathrm{M}}}{\mathrm{d}t} + s\frac{\mathrm{d}h_{\mathrm{M}}}{\mathrm{d}t}\right\},\end{aligned} \tag{35}$$

$$\begin{aligned}\frac{\mathrm{d}g}{\mathrm{d}t} &= \frac{3}{4}\frac{\mu^4}{L^3G^4}(1-5\cos^2 I)C_{2,0},\\ \frac{\mathrm{d}h}{\mathrm{d}t} &= \frac{3}{2}\frac{\mu^4}{L^3G^4}\cos I C_{2,0},\\ \frac{\mathrm{d}\ell}{\mathrm{d}t} &= n + \frac{3}{4}\frac{\mu^4}{L^3G^4}(1-3\cos^2 I)C_{2,0},\end{aligned} \tag{36}$$

$$\sum^{M} = \sum_{l\geqslant 2}\sum_{m=0}^{l}\sum_{s=0}^{l}\sum_{p=0}^{l}\sum_{q=0}^{l}\sum_{j=-\infty}^{\infty}\sum_{r=-\infty}^{\infty}. \tag{37}$$

Using Eq. (34) and the definition in Eq. (33), we get the periodic perturbations in Keplerian elements due to the Moon effects in first order as the following:

$$\begin{aligned}
\Delta_{\mathrm{M}}a &= \frac{2(-1)^{k_2}}{na}\sum^{M}(l-2p+j)D_{\mathrm{M}}\left\{\frac{(-1)^{k_2}U_l^{m,-s}}{\dot{\Psi}^{+}}\cos\Psi^{+}+\frac{(-1)^{k_3}U_l^{m,s}}{\dot{\Psi}^{-}}\cos\Psi^{-}\right\},\\
\Delta_{\mathrm{M}}e &= \frac{\sqrt{1-e^2}}{na^2e}\sum^{M}\left(\sqrt{1-e^2}(l-2p+j)-(l-2p)\right)D_{\mathrm{M}}\left\{\frac{(-1)^{k_2}U_l^{m,-s}}{\dot{\Psi}^{+}}\cos\Psi^{+}+\frac{(-1)^{k_3}U_l^{m,s}}{\dot{\Psi}^{-}}\cos\Psi^{-}\right\},\\
\Delta_{\mathrm{M}}I &= \frac{1}{na^2\sqrt{1-e^2}\sin I}\sum^{M}((l-2p)\cos I-m)D_{\mathrm{M}}\left\{\frac{(-1)^{k_2}U_l^{m,-s}}{\dot{\Psi}^{+}}\cos\Psi^{+}+\frac{(-1)^{k_3}U_l^{m,s}}{\dot{\Psi}^{-}}\cos\Psi^{-}\right\},\\
\Delta_{\mathrm{M}}\omega &= \frac{1}{na^2\sqrt{1-e^2}}\sum^{M}\left(-\cot I D_{\mathrm{M}}^{I}+\frac{1-e^2}{e}D_{\mathrm{M}}^{e}\right)\left\{\frac{(-1)^{k_2}U_l^{m,-s}}{\dot{\Psi}^{+}}\sin\Psi^{+}+\frac{(-1)^{k_3}U_l^{m,s}}{\dot{\Psi}^{-}}\sin\Psi^{-}\right\},\\
\Delta_{\mathrm{M}}\Omega &= \frac{1}{na^2\sqrt{1-e^2}\sin I}\sum^{M}D_{\mathrm{M}}^{I}\left\{\frac{(-1)^{k_2}U_l^{m,-s}}{\dot{\Psi}^{+}}\sin\Psi^{+}+\frac{(-1)^{k_3}U_l^{m,s}}{\dot{\Psi}^{-}}\sin\Psi^{-}\right\},\\
\Delta_{\mathrm{M}}M &= \frac{1}{na^2}\sum^{M}\left(-\frac{1-e^2}{e}D_{\mathrm{M}}^{e}-2lD_{\mathrm{M}}\right)\left\{\frac{(-1)^{k_2}U_l^{m,-s}}{\dot{\Psi}^{+}}\sin\Psi^{+}+\frac{(-1)^{k_3}U_l^{m,s}}{\dot{\Psi}^{-}}\sin\Psi^{-}\right\}\\
&\quad -\frac{3}{a^2}\sum^{M}(l-2p+j)D_{\mathrm{M}}\left\{\frac{(-1)^{k_2}U_l^{m,-s}}{\dot{\Psi}^{+}}\sin\Psi^{+}+\frac{(-1)^{k_3}U_l^{m,s}}{\dot{\Psi}^{-}}\sin\Psi^{-}\right\}.
\end{aligned}\tag{38}$$

4.2. First order secular perturbations in Keplerian elements

By integrating Eq. (9a) we have:

$$\begin{aligned}
&L' = \text{const.} = L'_0,\quad G' = \text{const.} = G'_0,\quad H' = \text{const.} = H'_0,\\
&\ell' = \frac{\partial \boldsymbol{H}^*}{\partial L'}(t-t_0),\quad g' = \frac{\partial \boldsymbol{H}^*}{\partial G'}(t-t_0)+g'_0,\quad h' = \frac{\partial \boldsymbol{H}^*}{\partial H'}(t-t_0)+h'_0,
\end{aligned}\tag{39}$$

where,

$$L'_0 = L'(t_0),\quad G'_0 = G'(t_0),\quad H' = H'(t_0),\quad \ell'_0 = \ell'(t_0),\quad g'_0 = g'(t_0),\quad h'_0 = h'(t_0).$$

The conditions for secular perturbations are:

$$\left.\begin{aligned}
&m = 0,\\
&s = 0,\\
&2p = l = \text{even} = 2\gamma,\\
&2q = l = \text{even} = 2\gamma,\\
&r = 0.
\end{aligned}\right\}\tag{40}$$

From Eq. (38) and using Eq. (5) with the condition (40), we get the secular perturbations in first order due to the Moon in the following formula:

$$\begin{aligned}
g'_{\mathrm{M}} &= -\left\{\frac{1}{na^2\sqrt{1-e^2}}\sum^{M}\left(-\cot I D_{\mathrm{M}}^{I}+\frac{1-e^2}{e}D_{\mathrm{M}}^{e}\right)\left((-1)^{k_2}U_l^{m,-s}+(-1)^{k_3}U_l^{m,s}\right)\right\}(t-t_0),\\
h'_{\mathrm{M}} &= -\left\{\frac{1}{na^2\sqrt{1-e^2}\sin I}\sum^{M}D_{\mathrm{M}}^{I}\left((-1)^{k_2}U_l^{m,-s}+(-1)^{k_3}U_l^{m,s}\right)\right\}(t-t_0),\\
\ell'_{\mathrm{M}} &= \left\{\frac{1}{na^2}\sum^{M}\left(-\frac{1-e^2}{e}D_{\mathrm{M}}^{e}-2elD_{\mathrm{M}}\right)\left((-1)^{k_2}U_l^{m,-s}+(-1)^{k_3}U_l^{m,s}\right)\right\}(t-t_0),
\end{aligned}\tag{41}$$

where,

$$D_{\mathrm{M}}^{e} = D_{\mathrm{M}}\left\{H_{lpj}(e) \to \frac{\mathrm{d}H_{lpj}(e)}{\mathrm{d}e},\right.$$

$$D_{\mathrm{M}}^{I} = D_{\mathrm{M}}\left\{F_{lmp}(I) \to \frac{\mathrm{d}F_{lmp}(I)}{\mathrm{d}I}.\right.$$

Similarly as in Eq. (34), the rest of generating functions S_{J2}^{*}, S_{S}^{*}, S_2, S_2^{Z}, S_2^{T}, and S_2^{SRP} can be derived. After that similarly, as in Eqs. (38) and (41) the periodic and secular perturbations in Keplerian elements in first and second order due to J_2, Moon, Sun, and the combined terms of these perturbations are obtained. Details of the theory (especially for luni-solar perturbations) have been presented in [12] and the formula of the geopotential perturbations which are used as in [10].

5. Model explanation of the algorithm

The SGP4/SDP4 models (maintains by the US Space Command) are commonly used for predictions of the satellite state vectors when the NORAD elements sets (two-line elements obtained by removing periodic variation as in [15]) are applied. The SGP4 models (for low Earth orbit) includes the first and second order secular perturbations due to second zonal harmonic J_2, the first order secular perturbations due to J_4, the short and long period perturbations due J_2–J_4, and atmospheric drag perturbation. The SDP4 models (for High Earth orbit) includes doubly averaged the luni-solar effects and some resonances effects for 12 and 24 h orbits.

OMOP algorithm [16] includes an arbitrary degree and order geopotential coefficients, luni-solar attraction and atmospheric drag. The geopotential periodic and secular perturbations are taken into account up to second and third order respectively. This algorithm OMOP has been applied for LEO. [16] concluded that OMOP can predict satellite position with a much better accuracy than the SGP4/SDP4 models (maintains by the US Space Command).

Since our work is Extended to the algorithm OMOP which was done by Wnuk [10,16]. Therefore our present algorithm can predict the satellite positions with a much better accuracy than the SGP4/SDP4 models.

The present algorithm includes an arbitrary degree and order of geopotential coefficients, luni-solar attraction, solar radiation pressure and atmospheric drag. The periodic and secular perturbations due those forces are retained up to second and third order respectively. The new terms of the present algorithm coming from second order of solar radian pressure and the coupling terms arising from the interaction of Moon and Sun gravity fields, Moon with J_2, and Sun with J_2. In addition the secular perturbations up to the second order, which induced by the atmospheric drag is taken into account. Resonance and very long period perturbations are modeled with the use of semi-secular terms for short time span predictions.

6. Numerical realization

Our computing algorithm evaluates: (1) the perturbations in the Keplerian orbital elements, (2) the position and velocity of the satellite after these perturbations taking into account the geopotential up to 70×70 and 40×40 for zonal and tesseral harmonics respectively, luni-solar attraction, radiation pressure and the secular perturbations in the perigee and apogee heights coming from the atmospheric drag [17].

One Final note should be made in this section regarding the computation of the algorithm, the expansion involving the Hansen coefficients $X_k^{n,m}$ [8] which is related to the eccentricity functions H_{lpj} and G_{lqr} in Eqs. (5) and (6) are valid for all eccentricities less than one, although the convergence of these infinite series is slow for high eccentricity. The interested reader is encouraged to consult [18] for detailed study of this point. Moreover, it is well known that the sum over j cannot be truncated to just a few values if the eccentricity is high, therefore series over $j,\ r,\ q$ in Eqs. (5) and (6) are truncated in our computation between -10 to 10. Also, the series over l in the same equations are truncated in our calculation from 2 to 5, which can involve literally thousand of terms in any one solution.

The accuracy of our analytical theory has been compared with the accuracy obtained from numerical integration (Burlich–Stoer). Starting from the satellite positions and velocities $\mathsf{r}_0,\ \dot{\mathsf{r}}_0$ or Keplerian elements at the epoch t_0, future satellite position and velocities r, $\dot{\mathrm{r}}$ for the epoch t as well as Keplerian elements were calculated both by numerical integration and using the analytical theory. Numerical integration of equation of motion enables a determination of the satellite positions at the epoch t directly from the initial positions and velocities at the epoch t_0.

Fig. 1 presents the differences in satellite positions determined by the analytical theory and the numerical integration (Burlich–Stoer) on the time span interval 5 days. The differences are calculated by two methods: (a) using the first order only, (b) included the perturbations up to the second order. The figure shows that the accuracy of the theory for predicting geostationary satellite positions using the first order (see dot curve) is in a level of 90 m, on the other hand, after included the second order theory, the accuracy of the theory for predicting geostationary satellite positions became about 40 m. Therefore, including the theory up to second order is very essential to improve the accuracy of the theory.

Figs. 2 and 3 show the perigee/apogee decays for ROHINI and MIMOSA satellites respectively on (y-axis in km) against the time on (x-axis in days). ROHINI lifetime has been subjected to several analyses by the

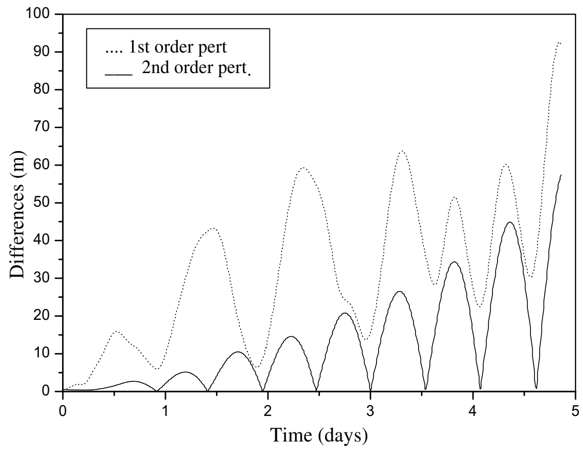

Fig. 1. The accuracy in the satellite positions is visualized as the differences between the analytical theory and numerical integration predictions for GEO, using first order theory only and the theory up to second order.

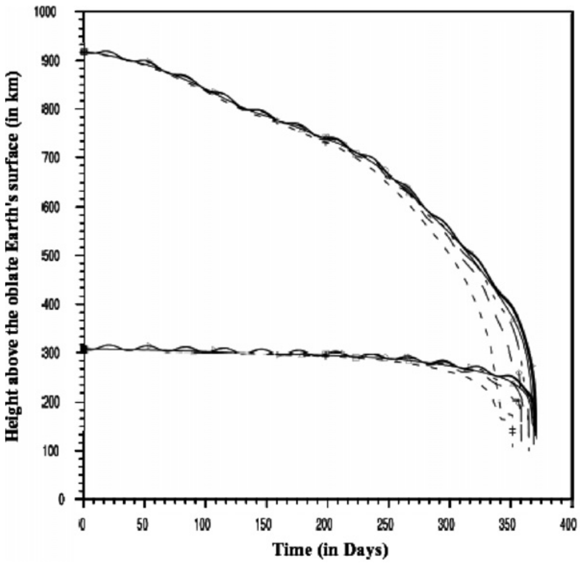

Fig. 2. The perigee/apogee decays for ROHINI satellite.

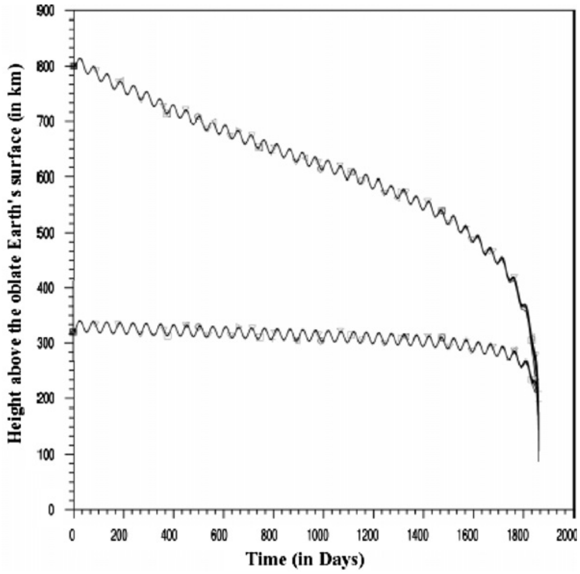

Fig. 3. The perigee/apogee decays for MIMOSA satellite.

Indian Space Agency. The most detailed prediction was made in [19]. They used numerical integration of the equations of motion consisting of about 216000 integration steps yielding the predicted lifetime as 400 days.

Our second order theory yields lifetime to be 371.4 days whereas the observed decay occurred after 371 days, which reflects the high accuracy provided by the presented theory. While the predicted lifetime of the MIMOSA satellite is 1860.7 days. It is worth noting that the MIMOSA is still in its operational orbit. We used TD88 density model because it proves its high level from Figs. 2 and 3. The figures explain the high precession of lifetime of the already decay ROHINI satellite. Therefore it is saving to use this model in any dynamical model involving drag effect. The study of the lifetime of those satellite was not the target of the theory but, to show how much efficient the TD88 density model that we are using for secular perturbation from drag effects.

Fig. 4 presents the perturbation in eccentricity due to solar radiation pressure in long period (6000 days) for a geostationary satellite. Notice that those perturbations change the eccentricity between −4.E−12 to +4.E−12.

Fig. 5 displays the perturbation in eccentricity due to luni-solar perturbations in long period (9000 days) for a geostationary satellite. Notice that those perturbations change the eccentricity almost periodically each 10.5 years.

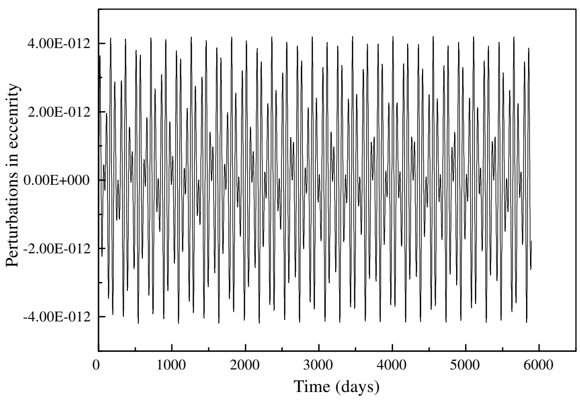

Fig. 4. The solar radiation pressure perturbations in eccentricity for GEO.

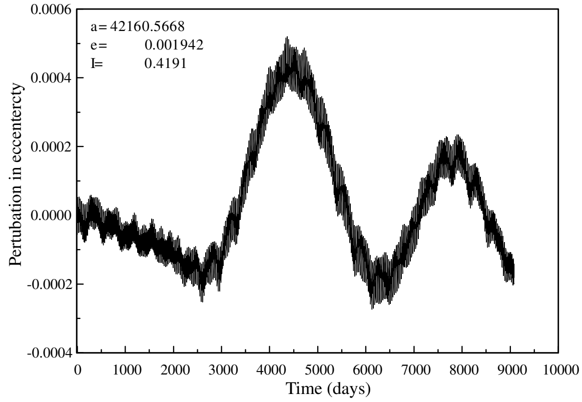

Fig. 5. The luni-solar perturbations in eccentricity for GEO.

Fig. 6 shows that the influence of luni-solar perturbations in means longitude $\lambda = \omega + \Omega + \ell$ for a geostationary satellite, which has a strong secular rate from 1° to −0.6°.

Figs. 7–9 represent the differences between satellite position calculated with the use of the analytical method and with the use of numerical integration (Burlich–Stoer) on the time span interval 5 days.

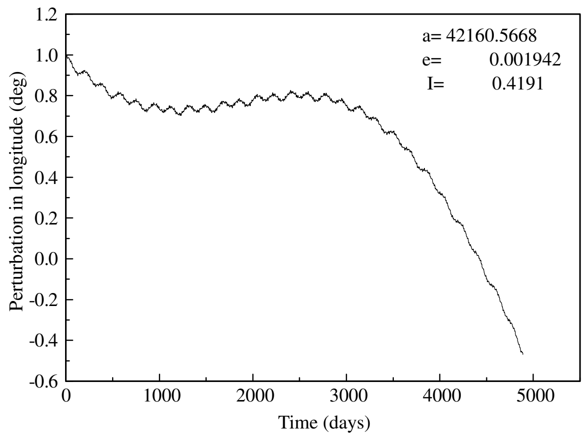

Fig. 6. The luni-solar perturbations in longitude for GEO.

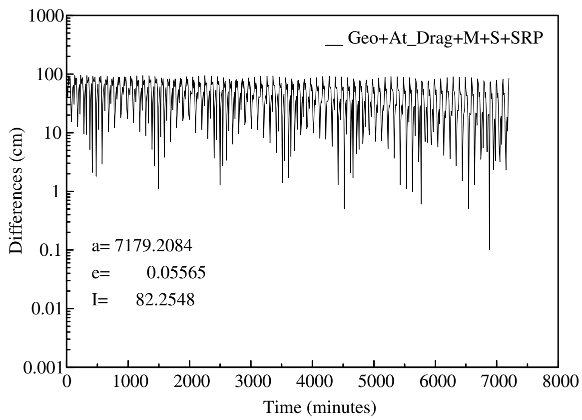

Fig. 7. The accuracy in the satellite positions is visualized as the differences between the analytical theory and numerical integration predictions for LEO.

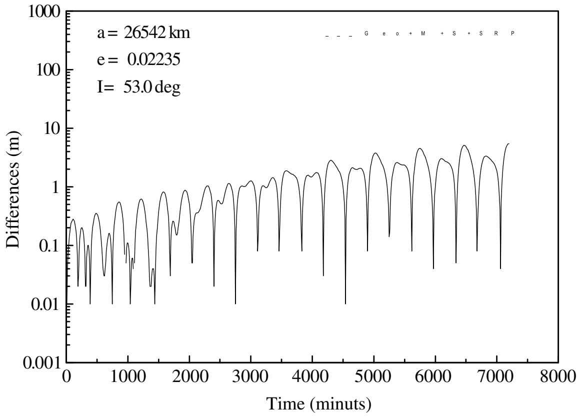

Fig. 8. The accuracy in the satellite positions is visualized as the differences between the analytical theory and numerical integration predictions for GPS satellite.

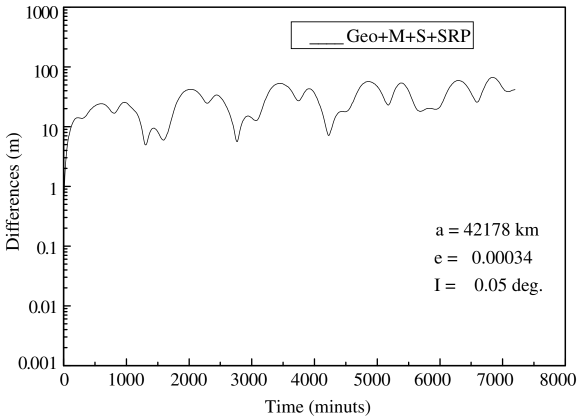

Fig. 9. The accuracy in the satellite positions is visualized as the differences between the analytical theory and numerical integration predictions for GEO.

Fig. 7 shows that the accuracy of the analytical solution, which included all perturbations, geopotential, luni-solar, solar radiation pressure and the atmospheric drag, for a predicted Low Earth satellite is on a level of 95 cm. Fig. 8 shows that the accuracy of the analytical solution for a predicted GPS satellite is on a level of 45 cm after one day and 200 cm after 5 days, in that case all perturbations are included except the atmospheric drag, which has no any effects on the GPS satellite.

Fig. 9 has shown that the accuracy of the analytical solution for a predicted geostationary satellite is on a level of 10 m after one day and 35 m after 5 days, also, in that case all perturbations are included except the atmospheric drag, which has no any effects on the GEO satellite.

These results showed a good accuracy for predicting satellites. It is possible to get a much better accuracy, which is important for avoiding collision between specific satellites with space debris, but using JPL ephemeris must be used instead of usual astronomical ephemeris for solar and Lunar coordinates.

7. Conclusions

The work presents an analytical model (algorithm) of the satellites and space debris orbital evolution. It can be applied both for the short- and the long-term propagation. The algorithm includes all perturbing forces, which have a significant influence on a satellite motion: geopotential effects with arbitrary degree and order spherical harmonic coefficients, luni-solar attractions, solar radiation pressure and Atmospheric Drag. Resonance and very long period perturbations, for a short time span predictions, are modeled using semi-secular terms.

The algorithm was tested for different satellite orbits and compared with the numerical integration. The accuracy of the theory is on the level of: 85 cm for LEO satellite; 1 m for GPS satellites; and 15 m for the geostationary satellite. The work presented an accurate algorithm, which can predict the satellite position with better accuracy than OMOP algorithm particularly for LEO. For more precise prediction of artificial satellite and space debris orbits, high precision coordinates of the Moon and Sun must be included. This is very crucial for the orbital maneuvers to avoid collisions with space debris. Beside the accuracy of initial conditions, i.e. the accuracy of orbital elements calculated from observations, the quality of the applied theory of motion (numerical or analytical) is essential.

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