1 Introduction and Mathematical Tools

1.1 Introduction

Electrodynamics is the study of moving electric charges and their effects. Now the first question that comes to my mind is what is an electric charge? and the answer is I don't know. But interestingly, the whole field is about interaction of electric charges and I don't what it is. Anyway let's put a hand wavy definition for now and move on and I believe it'll work for most part of the course.

> **Definition 1.1** (Electric Charge): Electric Charge is a physical property of the matter that cause it to experience a force when placed in an electromagnetic field.

Not a very impressive definition but it'll do for now. But on the other hand, I don't think there are any better definitions out there. Let me ask you this question, what is a mass? and one would give a same kind of definition for mass as well. So let's not bother ourselves with the definitions and move on. Now the next question why one should study electrodynamics? Well, one of the aspirations of the physics community is to unify all the fundamental forces in the universe and write a lagrangian which explain all phenomena in the universe. Electrodynamics is the success story of this aspiration. First comes Faraday unifying electric and magnetic forces and leading to electromagnetism and then Maxwell enters the picture and unifies electrodyamics with optics and hence reducing three courses,in principle into one. After Maxwell comes Einstein unifying electromagnetism, optics with gravity and completely changes the way we think about the very fundamental ideas of the universe. The story doesn't ends here but still continues leading to the program of string theory in modern times. So in that sense this topic is a part of quite a big legacy.
Now another reason to study this topic is that it is the first field theory that one encounters in the physics curriculum. So it is a good starting point to learn classical field theory but not in a big framework but a very specific one. But one only realise this after a couple of advanced courses in field theory. And definitely one'll learn quite a few advanced mathematical techniques. In rest of the chapter I'll introduce mathematical techniques required to study this course, starting with vector analysis.

1.2 Some Important Definitions and Conventions

Before talking about vectors and tensors let me introduce some basic definitions of vector analysis. So what is a scalar? well we all learnt in high school that a scalar is something that has only magnitude and no direction. For example, mass, temperature, pressure etc. are all scalars.

> **Definition 1.2** (Scalar): A scalar is quantity specified by a single number and remains invariant under rotations.

on the other hand a vector is something that has both direction and magnitude. For example, velocity, force, electric field etc. are all vectors. Let's write a more formal definition of a vector. But before that let me introduce a powerful notation.

1.2.1 Einstein Summation Convention

Consider a 3×3 matrix given as

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \tag{1}$$

Now consider a vector $\boldsymbol{x}$ given as

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{2}$$

Now by matrix multiplication we can write $\tilde{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$ and we can this expression more explicitly as

$$\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{pmatrix} \tag{3}$$

this equation can further be shrinked into the following form

$$\tilde{x}_i = a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 \tag{4}$$

where i is the i^{th} index of the vector and one realise that this can written as sum over a repeated index. So the final form of the equation is

$$\tilde{x}_i = a_{ij}x_j \tag{5}$$

so the ensence of the einstein summation convention is that *whenever there is a repeated index in a term, it is understood that it is summed over all possible values of the index.*
Now let us see the definition of a vector.

Definition 1.3 (Vector): A vector in a n-dimensional vector space is defined as quantity with n components in a given frame that transforms like

$$a_i = l_{ij}b_j \tag{6}$$

Similarly one can also define a tensor as

Definition 1.4 (Tensor of nth order): A tensor of n^{th} order is a quantity that transforms like

$$\tilde{T}_{i_1 i_2 \cdots i_n} = l_{i_1 j_1} l_{i_2 j_2} \ldots l_{i_n j_n} T_{j_1 j_2 \cdots j_n} \tag{7}$$

In simple words, if some quantity have n-indexes then it is a tensor of n^{th} order.

1.3 Vector Algebra and Transformation

As for the rule of algebra are concerned, the following rules are followed
1. Each vector have n components and can be given as a column matrix.
2. All rules of matrix algebra are valid.

Suppose we have a vector $\boldsymbol{A}$ and it's representation in some frame O is given as

$$\vec{r} = a_j \hat{\boldsymbol{e}}_j \tag{8}$$

and say in some other frame O' the representation is

$$\vec{r} = a'_j \hat{\boldsymbol{e}}'_j \tag{9}$$

but how these two representations are related? So we have

$$a'_i = a_j \hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_j \tag{10}$$

Now one can define $l_{ij} \coloneqq \hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_j$ and we arrive at the following equation defining vector transformations

$$a'_j = l_{ij} a_i \tag{11}$$

Now how to determine l_{ij}?

Exercise 1.1: Show that l_{ij} is a rotation matrix.*Hint: Use the fact that* $\hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}'_j = l_{ij}$

Now this rule is clearly valid for every vector in O_1 and O_2 and I would like state the following results
1. If we have $\boldsymbol{c} = \alpha\boldsymbol{a} + \beta\boldsymbol{b}$ then well have $c'_j = l_{ij} c_i$.
2. Say for the nth derivative of $\boldsymbol{r}$ the $\frac{\mathrm{d}^n r'_j}{\mathrm{d}t^n} = l_{ij} \frac{\mathrm{d}^n r_i}{\mathrm{d}t^n}$.

Exercise 1.2: Prove the above the two properties.

Now it is the time to introduce another important concept in vector analysis, the concept of a **scalar product**.

Definition 1.5 (Scalar Product): The scalar product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is defined as

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_i b_i \tag{12}$$

Now one would love to ask this question how dot product change under coordinate transformation and we'll leave it as an exercise for you.

Exercise 1.3: Show that scalar product is invariant under the coordinate transformation.

One more way to define scalar product or dot product is the following,

$$\boldsymbol{a} \cdot \boldsymbol{b} = |a||b|\cos(\theta) \tag{13}$$

Think: How Eq. 12 and Eq. 13 are related?

Now another important tool is **vector product**,

Definition 1.6 (Vector Product): For to vectors $\vec{a}$ and $\vec{b}$ one can define their vector product in the following manner

$$\boldsymbol{a} \times \boldsymbol{b} = \varepsilon_{ijk}\hat{e}_i a_j b_k \tag{14}$$

where ε_{ijk} is Levi-Civita-Tensor. One can also give a geometric definition in the following manner

$$\boldsymbol{a} \times \boldsymbol{b} = |a||b|\sin(\theta)\hat{n} \tag{15}$$

where $\hat{n}$ is perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$.

Think: How Eq. 15 and Eq. 14 are related?

here we are using Levi-Civita tensor ε_{ijk} so I would like to enlist few to its properties

1. Let we have ε_{ijk} for even permutation of i,j,k we have $+1$, for odd permutations of i,j,k we'll have -1 and other we'll have 0.
2. $$\varepsilon_{ijk}\varepsilon_{pqk} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp} \tag{16}$$
3. $$\varepsilon_{ijk}\varepsilon_{pjk} = 2\delta_{ip} \tag{17}$$
4. $$\varepsilon_{ijk}\varepsilon_{ijk} = 6 \tag{18}$$

Exercise 1.4: Show that for the vectors $\vec{a}$,$\vec{b}$ and $\vec{c}$ to be coplanar then

$$\varepsilon_{ijk}a_i b_j c_k = 0 \tag{19}$$

Exercise 1.5: If $\vec{a}$ and $\vec{b}$ are any vectors then,

$$\left(\vec{a} \times \vec{b}\right) \cdot \left(\vec{a} \times \vec{b}\right) + \left(\vec{a} \cdot \vec{b}\right)^2 = |\vec{a}|^2 \left|\vec{b}\right|^2 \tag{20}$$

Another important type of product that one encounters quite often is **triple product**. Two kinds of triple product quite famous are **Vector Triple Product** and **Vector Scalar Product**. So let us start with triple scalar product simply can be given as $\vec{a} \cdot \left(\vec{a} \times \vec{b}\right) = \varepsilon_{ijk}a_i b_j c_k$ and triple vector product

$$\vec{c} \times \left(\vec{b} \times \vec{c}\right) = \varepsilon_{ijk} \hat{\boldsymbol{e}}_{\boldsymbol{i}} c_{j\left(\vec{b} \times \vec{c}\right)_k} \tag{21}$$

I'll leave the rest to reader to work out and get this result

$$\vec{a} \times \left(\vec{b} \times \vec{c}\right) = \vec{b}(\vec{a} \times \vec{c}) - \vec{c}\left(\vec{a} \times \vec{b}\right) \tag{22}$$

and there is one more result I would like to put forward

$$\vec{a} \times \left(\vec{b} \times \vec{c}\right) + \vec{c} \times \left(\vec{a} \times \vec{b}\right) + \vec{b} \times (\vec{c} \times \vec{a}) = 0 \tag{23}$$

Example 1.1: Show $\vec{a} \cdot \left(\vec{b} \times \vec{c}\right)$ vanishes identically if two of three vectors are proportional to each other.

Solution: By using levi-civita tensor we have

$$\vec{a} \cdot \left(\vec{b} \times \vec{c}\right) = \varepsilon_{ijk} a_i b_j c_k \tag{24}$$

without loss of generality, let $a = \lambda b$

$$\vec{a} \cdot \left(\vec{b} \times \vec{c}\right) = \lambda \varepsilon_{ijk} b_i b_j c_k \tag{25}$$

Now the expression $\varepsilon_{ijk} b_i b_j c_k$ is actually the determinant of the following matrix

$$\varepsilon_{ijk} b_i b_j c_k = \begin{vmatrix} b_1 & b_2 & b_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \tag{26}$$

because we have two rows same so by the properties of the determinant we have

$$\varepsilon_{ijk} b_i b_j c_k = 0 \tag{27}$$

hence proved that, *if two of three vectors are proportional to each other.* ■

Exercise 1.6: If $\hat{\boldsymbol{e}}$ is an unit vector and $\hat{\boldsymbol{a}}$ an arbitrary vector then show that

$$\vec{a} = (\vec{a} \cdot \hat{\boldsymbol{e}}) + \hat{\boldsymbol{e}} \times (\vec{a} \times \hat{\boldsymbol{e}}) \tag{28}$$

This shows that $\vec{a}$ can be resolved into component parallel to and one perpendicular to an arbitrary direction $\hat{\boldsymbol{e}}$.

Exercise 1.7: Prove the following identites

1.
$$\left(\vec{a}\times\vec{b}\right)\cdot\left(\vec{c}\times\vec{d}\right)=\begin{vmatrix}\vec{a}\cdot\vec{c} & \vec{b}\cdot\vec{c}\\ \vec{a}\cdot\vec{d} & \vec{b}\cdot\vec{d}\end{vmatrix} \tag{29}$$

2.
$$\left(\vec{a}\times\vec{b}\right)\times\left(\vec{c}\times\vec{d}\right)=\left[\vec{c}\cdot\left(\vec{d}\times\vec{a}\right)\right]\vec{b}-\left[\vec{c}\cdot\left(\vec{d}\times\vec{b}\right)\right]\vec{a} \tag{30}$$

$$\left(\vec{a}\times\vec{b}\right)\times\left(\vec{c}\times\vec{d}\right)=\left[\vec{a}\cdot\left(\vec{b}\times\vec{d}\right)\right]\vec{c}-\left[\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)\right]\vec{d} \tag{31}$$

3.
$$\left[\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)\right]\vec{d}=\left[\vec{d}\cdot\left(\vec{b}\times\vec{c}\right)\right]\vec{a}+\left[\vec{a}\cdot\left(\vec{d}\times\vec{c}\right)\right]\vec{b}+\left[\vec{a}\cdot\left(\vec{b}\times\vec{d}\right)\right]\vec{c} \tag{32}$$

1.4 Differential Calculus

Suppose we have a function $f(x)$ and we want to know how much of the function changes in a certain direction we can take a derivative in that particular directionand talk about the rate of change in that direction.

$$\mathrm{d}f=\frac{\mathrm{d}f}{\mathrm{d}x}\,\mathrm{d}x \tag{33}$$

Now this is the case for a single variable function but now consider a multivalued function as $f(x_1,...x_N)$ how one talks about infinitesimal change in the function itself? One can define the change in same way as one defined for single variable function

$$\mathrm{d}f=\sum_{i=1}^{N}\frac{\partial f}{\partial x_i}\,\mathrm{d}x_i \tag{34}$$

and the quantity $\frac{\partial f}{\partial x_i}$ is called the partial derivative of the function f with respect to x_i keeping other variables constant. Now one can define the gradient of the function as

Definition 1.7 (Gradient of the function): The gradient of the function f is defined as

$$\boldsymbol{\nabla} f(x_1...x_N)=\frac{\partial f}{\partial x_i}\hat{\boldsymbol{e}}_i\forall i \tag{35}$$

Now one can relate the change in the function with the gradient of the function as

$$\mathrm{d}f=\boldsymbol{\nabla} f(x_1...x_N)\cdot\mathrm{d}\vec{x} \tag{36}$$

Lemma 1.4.1: The direction of the gradient of the function is the direction of the maximum change in the function.

Proof: Now consider $\mathrm{d}f$ be the change in the function around some point $\vec{x}_o$ and then for a fixed amount of change $\mathrm{d}\vec{r}$ the change in the function

$$\mathrm{d}f = |\boldsymbol{\nabla} f||\mathrm{d}\vec{r}|\cos(\theta) \tag{37}$$

where θ is the angle between the gradient and the change in the length element. Now the direction of maximum change in the function is that of the gradient of the function and more over the $|\boldsymbol{\nabla} f|$ gives the slope of the function along this maximal direction. ■

Remark: For $\boldsymbol{\nabla} f = 0$ we have a stationary point so in can be either a valley or a top or just a saddle point.

Think: Now as we saw earlier that we took gradient of a scalar function and got something to do with vectors. Is the converse also true that for each vector field one can define a scalar field using a gradeint of a scalar function?

Now knowing or unknowingly we have used the concept of a **differential operator** or **del operator** which is defined as

$$\boldsymbol{\nabla} = \hat{e}_i \frac{\partial}{\partial x_i} \tag{38}$$

Now as for a vector we have three possible multiplicative operations similarly we have three possible ways in which del can act on different functions?
1. On a scalar function: $\boldsymbol{\nabla} f$
2. On a vector function: $\boldsymbol{\nabla} \cdot \vec{v}$,(the divergence)
3. On a vector function: $\boldsymbol{\nabla} \times \vec{v}$,(the curl)

Definition 1.8 (Divergence of a vector field): The divergence of a vector field $\vec{v}$ is defined as

$$\boldsymbol{\nabla} \cdot \vec{v} = \boldsymbol{\nabla} \cdot \vec{v} = \frac{\partial v_i}{\partial x_i} \tag{39}$$

one can also geometric interpertation of the divergence as how much of field is flowing out of a point. But this point becomes more clear when we talk about **The Divergence Theorem** after a couple of sections.
Now similarly one can define a new quantity called **curl** of a vector field.

Definition 1.9 (Curl of a vector field): The curl of a vector field $\vec{v}$ is defined as

$$\boldsymbol{\nabla} \times \vec{v} = \boldsymbol{\nabla} \times \vec{v} = \varepsilon_{ijk} \frac{\partial v_i}{\partial x_j} \hat{\mathbf{e}}_k \tag{40}$$

one can also give a geometric interpretation of the curl as how much of the field is rotating around a point. But this point will also becomes more clear when we talk about **Stokes' Theorem** after a couple of sections.

1.4.1 Some Important Identities

So there are 6 important product rules that are quite useful in vector calculus which I list down here

1. $$\boldsymbol{\nabla} fg = f\boldsymbol{\nabla} g + g\boldsymbol{\nabla} f \tag{41}$$
2. $$\boldsymbol{\nabla}\left(\vec{A} \cdot \vec{B}\right) = \vec{A} \times (\boldsymbol{\nabla} \times B) + \vec{B} \times (\boldsymbol{\nabla} \times A) + \left(\vec{A} \cdot \boldsymbol{\nabla}\right)\vec{B} + \left(\vec{B} \cdot \boldsymbol{\nabla}\right)\vec{A} \tag{42}$$
3. $$\boldsymbol{\nabla} \cdot \left(f\vec{A}\right) = f\boldsymbol{\nabla} \cdot \vec{A} + \vec{A} \cdot \boldsymbol{\nabla} f \tag{43}$$
4. $$\boldsymbol{\nabla} \cdot \left(\vec{A} \times \vec{B}\right) = \vec{B} \cdot \boldsymbol{\nabla} \times \vec{A} - \vec{A} \cdot \boldsymbol{\nabla} \times \vec{B} \tag{44}$$
5. $$\boldsymbol{\nabla} \times \left(f\vec{A}\right) = \boldsymbol{\nabla} f \times \vec{A} + f\boldsymbol{\nabla} \times \vec{A} \tag{45}$$
6. $$\boldsymbol{\nabla} \times \left(\vec{A} \times \vec{B}\right) = \left(\vec{B} \cdot \boldsymbol{\nabla}\right)A - \left(\vec{A} \cdot \boldsymbol{\nabla}\right)B + \vec{A}\left(\boldsymbol{\nabla} \cdot \vec{B}\right) - \vec{B}\left(\boldsymbol{\nabla} \cdot \vec{A}\right) \tag{46}$$