1 Markov Process

Here we'll consider markov process and try to understand the dynamics of the markov process. Markov Process can be picturesquly called as *'process without memory.'*

1.1 Conditional Probability

Definition 1.1 (Conditional Probability): Consider a continous random variable $X(t)$, given at times $\{t_i\}$ the random variable accquires a value $\{x_i\}$ then the probability of having a value x_n at time t_n $(t_n > t_{n-1} > t_{n-1} > ... > t_1)$is given by

$$p_{1|n-1}(x_n, t_n | x_1, t_1; ...; x_{n-1}, t_{n-1}) = \frac{p(x_1, t_1; ...x_n, t_n)}{p(x_1, t_1; ...; x_{n-1}, t_{n-1})} \tag{1}$$

where $p(x_1, t_1; ...; x_n, t_n)$ is the joint probability of having $x_n s$ at times $t_n s$ and $p(x_1, t_1; ...; x_{n-1}, t_{n-1})$ is the probability of having $x_{n-1}s$ at times $t_{n-1}s$.

Now inversly any joint probability distribution funciton p_n can be written as

$$p_n(x_1, t_1; ...; x_n, t_n) = p(x_1, t_1)p(x_2, t_2|x_1, t_1)...p(x_n, t_n|x_1, t_1; ...; x_{n-1}, t_{n-1}) \tag{2}$$

one can motivate oneself to write Eq. 2 as n coin toss and thinking about some final outcome and checking what is the probability of getting that outcome.
Now by elementary notion of joint probability and by using Eq. 1

$$\int p_{1|1}(x_2, t_2|x_1, t_1)p_1(x_1, t_1)dx_1 = p_1(x_2, t_2) \tag{3}$$

this equation allows us to predict the future state of the system given the probability of the present state and the conditional probability of the future state given the present state. The above equation can be generalised as

$$\int p(x_1, t_1)p(x_2, t_2|x_1, t_1)...p(x_n, t_n|x_1, t_1; ...; x_{n-1}, t_{n-1})\Pi_{i=1}^{N-1} dx_i = p_1(x_2, t_2) \tag{4}$$

this equation is nothing but a **path integral** for a system to start a point (x_1, t_1) and ending at (x_n, t_n). Now at this point, in view Eq. 4, we can get any joint probability distribution function from a define two types of process

Definition 1.2 (Completely Random Process): A stochastic process $X(t)$ is said to be completely random, given that, at any time (t_n) the conditional probability of an event (x_n) is independent of all it's earlier events $\{x_{n-1}, t_{n-1}\}$.

$$p_{n|k}(x_{k+1}, t_{k+1}; ...; x_{k+n}, t_{k+n}|x_1, t_1; ...; x_n, t_n) = p_n(x_{k+1}, t_{k+1}; ...; x_{k+n}, t_{k+n}) \tag{5}$$

1.2 Markov Process

Definition 1.3 (Markov Process): A Markov Process is defined as a stochastic process with the property that for any set of n successive times one has

$$P_{1|n-1}(x_n, t_n \mid x_1, t_1; ...; x_{n-1}, t_{n-1}) = P_{1|1}(x_n, t_n \mid x_{n-1}, t_{n-1}) \tag{6}$$

- Any markov process is characterised by $P_1(y_1, t_1)$ and the whole hierarchy can constructed from them.

Example 1.1: A gambler plays head and tails. Let Y_t be the total earning after t throws. Show that Y_t is a discrete-time Markov process and find its transition probability.

Solution: Consider flipping of coin thought experiment and one can arrive at all possible outcomes of the sequential coin toss.

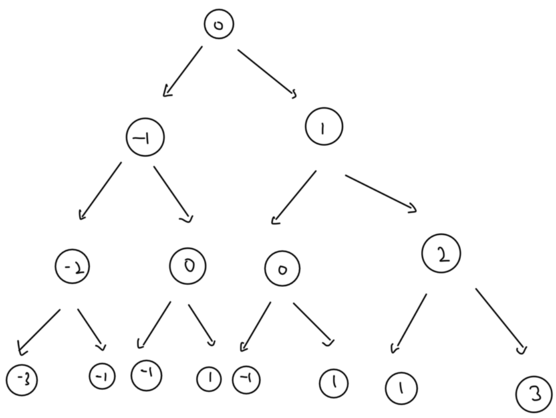

Figure 1: All possible earnings of the gambler after 3 tosses

Now from Figure 1 one can see that for earning at any Y_t is independent of all the earlier earnings except Y_{t-1} so we have

$$P(Y_t) = P(Y_t|Y_{t-1})P(Y_{t-1}) \tag{7}$$

this is the definition of a markov process and each time we have $P(Y_t|Y_{t-1}) = \frac{1}{2}$ ■

Example 1.2: Consider the ordinary differential equation $\frac{\mathrm{d}x}{\mathrm{d}t} = f(x)$. Write the solution with initial value xo at to in the form $x(t) = \varphi(x_o, t - t_o)$. Show that x obeys the definition of the Markov process with transition probability $P(x, t|x_o, t_o) = \delta(x - \varphi(x_o, t - t_o))$.

Solution: The solution of the ordinary differential equation is given by $x(t) = \varphi(x_o, t - t_o)$ now because the system in deterministic and the solution is unique, so we have the transition probability as

$$P(x, t) = P_{1|1}(x, t|x_o, t_o)P(x_o, t_o) \tag{8}$$

because of the uniqueness of the solution, the transition probability is a delta function, given as $P(x, t|x_o, t_o) = \delta(x - \varphi(x_o, t - t_o))$.
The above example can be extended to higher dimension and one can conclude that every deterministic system is a markov process, albeit of a singular type. ■

1.2.1 Physical Examples of Markov Process

1. **Brownian Motion of a particle in a fluid**, consider a heavy mass particle moving in a fluid will undergoing collisons with lighter particles in the fluid, having an average velocity to be (V_e) suppose the velocity of the heavy particle is V and the difference between them $\Delta V = V - V_e$ and we'll have more collisions on front than the back so the change in the velocity, $\Delta V \propto V$, in time Δt but not on anytime earlier. **Thus velocity of a heavy particle is a markov process.**
   But Smoluchowski and Einstein showed that at the time scales we do the experiments we can't see the positions of indiviual particles but we measure the average displacement of the heavy particle. Let $\{X_1, ..., X_k\}$ be the set of positions measured then the average displacement is given by $\Delta X_k(k, k-1) = X_k - X_{k-1}$ is independent of all earlier readings hence the average displacement is also a markov process.

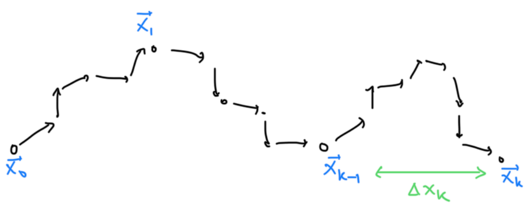

Figure 2: Brownian motion of a heavy particle in a fluid

2. Dissocitation of a diatomic gas to it's monoatomic constiuents, now the probability of finding a type of a molecule at any time $p(t + \mathrm{d}t)$ is only dependent on $p(t)$ but not on its past history hence a markov process.
$$AB \rightarrow A + B \tag{9}$$
3. **Radioactive decay**, the probability of finding a radioactive particle at time t is only dependent on the probability of finding it at time t_0 and not on any earlier time.

   Remark: *For any r-component stochastic process one may ignore a number of components and the remaining s components again constitute a stochastic process. But, if the r-component process is Markovian, the process formed by the $s < r$ components in general is not.*

1.3 Chapman Kolmogrov Equation

Consider Eq. 2, we'll rewrite the joint probability for (y_1, y_2, y_3) at time ordered time (t_1, t_2, t_3) and integrating it over y_2 and dividing it by $p(y_1, t_1)$ we have

$$p(y_3, t_3|y_1, t_1) = \int p(y_3, t_3|y_2, t_2)p(y_2, t_2|y_1, t_1)\,\mathrm{d}y_2 \tag{10}$$

similarly we can extend this result

$$\begin{aligned} p_{1|1}(y_n, t_n|y_1, t_1) = \int p_{1|1}(y_n, t_n|y_{n-1}, t_{n-1})p_{1|1}(y_{n-1}, t_{n-1}|y_{n-2}, t_{n-2})... \\ p_{1|1}(y_2, t_2|y_1, t_1)\Pi_{i=2}^{N-2}\,\mathrm{d}y_i \end{aligned} \tag{11}$$

This equation is kinda a specific case of Eq. 4 but famously known as Chapman Kolmogrov equation.

Example 1.3: Let Y have the range ± 1. Show that

$$P_{1|1}(y,t|y',t') = \frac{1}{2}\left\{1+e^{-2\gamma(t-t')}\right\}\delta_{y,y'} + \frac{1}{2}\left\{1-e^{-2\gamma(t-t')}\right\}\delta_{y,-y'} \tag{12}$$

obeys the Chapman-Kolmogrov equation. Show that this is consistent with $P_1(y,t) = \frac{1}{2}\left(\delta_{y,1}+\delta_{y,-1}\right)$. The Markov process so defined is called a *dichotomic* or *two-valued Markov process*, or also *random telegraph process.*

Solution: To check: $P_{1|1}(y_3,t_3|y_1,t_1) = \int P_{1|1}(y_3,t_3|y_2,t_2)P_{1|1}(y_2,t_2|y_1,t_1)\,\mathrm{d}y_2$ Now LHS is given as

$$P_{1|1}(y_3,t_3|y_1,t_1) = \frac{1}{2}\left\{1+e^{-2\gamma(t_3-t_1)}\right\}\delta_{y_3,y_1} + \frac{1}{2}\left\{1-e^{-2\gamma(t_3-t_1)}\right\}\delta_{y_3,-y_1} \tag{13}$$

and the RHS is given as

$$\int P_{1|1}(y_3,t_3|y_2,t_2)P_{1|1}(y_2,t_2|y_1,t_1)\,\mathrm{d}y_2 = \tag{14}$$
$$\left(\frac{1}{2}\left\{1+e^{-2\gamma(t_3-t_2)}\right\}\delta_{y_3,y_2} + \frac{1}{2}\left\{1-e^{-2\gamma(t_3-t_2)}\right\}\delta_{y_3,-y_2}\right)\left(\frac{1}{2}\left\{1+e^{-2\gamma(t_2-t_1)}\right\}\delta_{y_2,y_1} + \frac{1}{2}\left\{1-e^{-2\gamma(t_2-t_1)}\right\}\delta_{y_2,-y_1}\right)$$

Now this can be further simplified and one can show that LHS=RHS. Similarly using

$$P_1(y_3,t_3) = \int \mathrm{d}y_1\, P_{1|1}(y_3,t_3|y_1,t_1)P_1(y_1,t_1) \tag{15}$$

we can show that $P_1(y_3,t_3) = \frac{1}{2}\left(\delta_{y_3,1}+\delta_{y_3,-1}\right)$ ■

There are two famous examples of Markov process which I'll mention here,

1. It can be easily verified that following expression follows Chapman-Kolmogrov equation Eq. 11 for $-\infty \leq y_1, y_2 \leq \infty$ and $t_1 < t_2$

$$P_{1|1}(y_2,t_2|y_1,t_1) = \frac{1}{\sqrt{2\pi(t_2-t_1)}}e^{-\frac{(y_2-y_1)^2}{2(t_2-t_1)}} \tag{16}$$

   Now for $P_1(y_1,0) = \delta(y_1)$ a non stationary markov process is defined is called the *Weiner Process* or *Wenier-Levy Process* . It is usually valid for only $t > 0$ alone and was orginally invented for position of Brownian particle

2. $Y(t)$ takes only values $n = 0, 1, 2..$ and $t \geq 0$. A Markov process is defined by $(t_2 \geq t_1 \geq 0)$

$$P_{1|1}(n_2,t_2|n_1,t_1) = \frac{(t_2-t_1)^{n_2-n_1}}{(n_2-n_1)!}e^{-(t_2-t_1)}, P_1(n,0) = \delta_{n,0} \tag{17}$$

   it is understood that $P_{1|1} = 0$ for $n_2 < n_1$,. Thus each sample function $y(t)$ is a succession of steps of unit height and at random moments. It is uniquely determined by the time points at which the steps take place. These time points constitute a random set of dots on the time axis. Their number between any two times t_2, t_1 is distributed according to the Poisson distribution. Hence $Y(t)$ is called Poisson process.

1.4 Stationary Markov Process

For stationary markov process, the transition probability is independent of time but only dependent on the interval as $\tau = t_2 - t_1$ so we introduce the special notation that as

$$P_{1|1}(y_2, t_2|y_1, t_1) = T_\tau(y_2|y_1) \tag{18}$$

and hence the Chapman-Kolmogrov equation becomes

$$T_{\tau_1+\tau_2}(y_3|y_1) = \int T_{\tau_1}(y_3|y_2)T_{\tau_2}(y_2|y_1)\,\mathrm{d}y_2 \tag{19}$$

If one reads the integral as a product of two matrices, or integral kernels, this equation may be written $T_{\tau_1+\tau_2} = T_{\tau_1}T_{\tau_2}(\tau_1, \tau_2 > 0)$.

Remark: T_τ is indeed defined for $\tau > 0$ but we can have $T_{-\tau}$ as well so we have an identity which relate the two as

$$T_\tau(y_2|y_1)P_1(y_1) = T_{-\tau}(y_1|y_2)P_1(y_2) \tag{20}$$

because $P_2(y_2, t_2; y_1, t_1)$ is symmetric in nature.

Exercise 1.1: Show that autocorrelation function can be given as

$$k(\tau) = \int \mathrm{d}y_2\,\mathrm{d}y_1\, y_1y_2T_\tau(y_2|y_1)P_1(y_1) \tag{21}$$

Exercise 1.2: Prove the following identities as

1.
$$\int T_\tau(y_2|y_1)\,\mathrm{d}y_2 = 1 \tag{22}$$

2.
$$\int T_\tau(y_2|y_1)P_1(y_1)\,\mathrm{d}y_2 = P_1(y_1) \tag{23}$$

Now this is an eigenvalue equation.

1.5 The Decay Process

Consider a piece of radioactive material containing n_o active nuclei a t = 0. The number $N(t)$ of active nuclei at anytime $t > 0$ is a non-stationary stochastic process. It is clearly Markovian because the probability of $N(t_2)$ at $t_1 > t_2$, conditional on $N(t_1) = n_1$, is independent of the previous history. Now the probability of at any time t is give as

$$P_1(n_1, t) = \binom{n_o}{n_1} w^{n_1}(1-w)^{n_o-n_1} \tag{24}$$

Now the mean number of particle at any time is

$$\langle N(t)\rangle = \sum_n n_1\binom{n_o}{n_1} w^{n_1}v^{n_o-n_1} = w\frac{\partial}{\partial w}(w+v)^{n_o} = wn_o \tag{25}$$

where $v = 1 - w$. Now at any time we have $w(t) = e^{-\gamma t}$ in Eq. 24

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$$P_1(n_1, t) = \binom{n_o}{n_1} e^{-\gamma n_o t} (1 - e^{-\gamma t})^{n_o - n_1} \tag{26}$$

Now suppose we consider particles to be n_1 at time t_1 and n_2 at time t_2 then the transition probability is given as

$$P_{1|1}(n_2, t_2 | n_1, t_1) = \binom{n_1}{n_2} e^{-\gamma n_1 (t_2 - t_1)} \left(1 - e^{-\gamma (t_2 - t_1)}\right)^{n_1 - n_2} \tag{27}$$