

# Theory of Angular Momentum

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## 1 Introduction

## 2 Rotations and Angular Momentum Commutation Relations

In classical mechanics we cause rotation of a vector  $\vec{v}$  by applying a rotation matrix  $R$  such that  $\vec{v}' = R\vec{v}$  where  $\vec{v}'$  is the rotated vector. The rotation matrix is suppose to follow the following properties

- $R^T R = R R^T = I$
- $\sqrt{v_x^2 + v_y^2 + v_z^2} = \sqrt{v_{x'}^2 + v_{y'}^2 + v_{z'}^2}$

Now in 3D we can generate all the rotations possible using 3 matrices around x,y and z axis using  $R_x(\phi)$ ,  $R_y(\phi)$  and  $R_z(\phi)$ . For very small angle  $\phi$ , the above matrices allow the commutation relation as  $[R_x(\epsilon), R_y(\epsilon)] = R_z(\epsilon^2) - I$

**i** Note 1: Exercise

Generalise the above commutation relation.

Now one of the important lessons of quantum mechanics is that we can rotate a physical system in  $\mathbb{R}^3$  then there should be a corresponding operator  $\mathcal{D}(R)$  such that it rotates the wavefunction in the hilbert space  $\mathcal{H}$  such that  $|\alpha\rangle_R = \mathcal{D}(R)|\alpha\rangle$ . And the *physics* in the rotated frame is given by  $|\alpha\rangle_R$ .

### **i** Note 2: Exercise

Show that  $\mathcal{D}(R)$  is an unitary operator.

Now let us consider a unitary operator which generates the rotation and let  $\vec{J} \cdot \hat{n}$  be the generator of the rotation where  $\vec{J} = \{\hat{J}_x, \hat{J}_y, \hat{J}_z\}$  and  $\hat{n}$  is a unit vector in the direction of the rotation axis. So we can write the  $\mathcal{D}(R) = e^{-i\vec{J} \cdot \hat{n}\theta}$ . Now in the small angle approximation we have  $\mathcal{D}(R) = 1 - i\vec{J} \cdot \hat{n}\theta + \mathcal{O}(\theta^2)$

## 3 Finite Rotation in Quantum mechanics

We can re-write the rotation around an axis as compounding of multiple infinitesimal rotation around the axis as

$$\mathcal{D}_z(\phi) = \lim_{N \rightarrow \infty} \left( 1 - i\hat{J}_z \frac{\phi}{N} \right)^N$$

which is

$$\mathcal{D}_z(\phi) = \exp(-i\hat{J}_z \phi) = 1 - \frac{i\hat{J}_z \phi}{\hbar} - \frac{\hat{J}_z^2 \phi^2}{2\hbar^2} + \mathcal{O}(\phi^3)$$

Now let us try to figure-out the angular momentum algebra using the rotation operator. Now we assume that rotation operator have same group properties as the classical rotation matrices.

1. Identity:  $R \cdot I = I \cdot R = R \implies \mathcal{D}(R) \cdot I = I \cdot \mathcal{D}(R) = \mathcal{D}(R)$
2. Closure:  $R_1 R_2 = R_3 \implies \mathcal{D}(R_1) \mathcal{D}(R_2) = \mathcal{D}(R_3)$
3. Inverses:  $RR^{-1} = R^{-1}R = I \implies \mathcal{D}(R) \mathcal{D}(R)^{-1} = \mathcal{D}(R)^{-1} \mathcal{D}(R) = I$
4. Associativity:  $R_1(R_2 R_3) = (R_1 R_2) R_3 \implies \mathcal{D}(R_1) (\mathcal{D}(R_2) \mathcal{D}(R_3)) = (\mathcal{D}(R_1) \mathcal{D}(R_2)) \mathcal{D}(R_3)$

Now using the commutation relation of classical rotation matrix one can workout the commutation relation of the angular momentum operator.

### **i** Note 3: Exercise

Workout the commutation relation of angular momentum operator using rotation operator commutation relation, which follows from the fact that they have same group structure.

So one of the central results of this section is the angular momentum algebra given as

$$[\hat{J}_i, \hat{J}_j] = i\hbar \epsilon_{ijk} \hat{J}_k \quad (1)$$

Equation 1 shows that the angular momentum operators forms an **non-abelian group**, which is different from the translation as the generators  $p_j$  form an **abelian group**. Now in principle I only need Equation 1 to give all the properties of angular momentum operator and hence rotation

operators. But now the question is how the expectation values change with the rotation operator? Consider the expectation value  $\langle \hat{J}_x \rangle$ , it is defined as

$$\langle \hat{J}_x \rangle = \langle \alpha_R | \hat{J}_x | \alpha_R \rangle = \langle \alpha | \mathcal{D}_z^\dagger \hat{J}_x \mathcal{D}_z | \alpha \rangle$$

Now let us see how  $\hat{J}_x$  transform under the rotation around z-axis?

$$\begin{aligned} \mathcal{D}_z^\dagger \hat{J}_x \mathcal{D}_z &= e^{-i\hat{J}_z\phi} \hat{J}_x e^{i\hat{J}_z\phi} = 1 + \left( \frac{i\phi}{\hbar} \right) [\hat{J}_z, \hat{J}_x] + \frac{1}{2!} \left( \frac{i\phi}{\hbar} \right)^2 [\hat{J}_z, [\hat{J}_z, \hat{J}_x]] + \dots \\ &= \hat{J}_x \cos(\phi) - \hat{J}_y \sin(\phi) \end{aligned}$$

Now note that these operator transforms as a vector under rotation.

#### **i** Note 4: Exercise

Show explicitly that the expectation value of angular momentum operators along different axis transform as a vector under rotation.

## 4 Spin- $\frac{1}{2}$ Systems and Finite Rotations

The lowest number,  $N$ , of dimensions in which the angular momentum commutation in Equation 1 are realised is  $N = 2$ . Now the generators of the rotation for  $N = 2$  which satisfy the commutation relation are

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (2)$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad (3)$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (4)$$

and to make sure the dimensions are satisfied we'll add an extra  $\frac{\hbar}{2}$  in front of all of them and they are generator of rotation for spin- $\frac{1}{2}$  particles. Now like showed in the earlier section that the expectation values transform as vector. Let us consider the transformation of a expectation value of spin- $\frac{1}{2}$  matrices around z-axis as,

$$\begin{pmatrix} \cos(\phi) & -\sin(\phi) & 0 \\ \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \langle \hat{S}_x \rangle \\ \langle \hat{S}_y \rangle \\ \langle \hat{S}_z \rangle \end{pmatrix} = \begin{pmatrix} \langle \hat{S}_x \rangle \cos(\phi) - \langle \hat{S}_y \rangle \sin(\phi) \\ \langle \hat{S}_x \rangle \sin(\phi) + \langle \hat{S}_y \rangle \cos(\phi) \\ \langle \hat{S}_z \rangle \end{pmatrix} \quad (5)$$

Now the above can be extended to any angular momentum operator  $\hat{J}_i$  as  $\hat{J}'_j = R_{ji} \hat{J}_i$ . But how a general ket  $|\alpha\rangle$  is transformed under rotation,

$$\begin{aligned}
e^{-i\hat{J}_z\phi/\hbar} |\alpha\rangle &= e^{-i\hat{J}_z\phi/\hbar} (|0\rangle\langle 0|\alpha\rangle + |1\rangle\langle 1|\alpha\rangle) \\
&= (e^{-i\phi/2}|0\rangle\langle 0|\alpha\rangle + e^{i\phi/2}|1\rangle\langle 1|\alpha\rangle)
\end{aligned}$$

Now for  $\phi = 2\pi$ ,  $|\alpha\rangle_{R_z(2\pi)} = -|\alpha\rangle$ . This is the most astonishing result that after a  $2\pi$  rotation one picks up a negative sign and in principle it takes two  $2\pi$  rotations to return to the original state. But in quantum theory we only measure the expectation value of operators and the phase becomes irrelevant so how we detect the sign?

## 4.1 Pauli Two Component Formalism

Introduced by Wolfgang Pauli in 1927, the two component formalism is a way to represent spin- $\frac{1}{2}$  particles using two component spinors, which becomes more apparent when we consider the full relativistic treatment. As we understood in earlier sections that vectors can be represented using column matrix depending upon the dimensions of the Hilbert space. So for spin- $\frac{1}{2}$  particle state can be represented as

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and

$$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and any vector can be written as

$$|\alpha\rangle = |0\rangle\langle 0|\alpha\rangle + |1\rangle\langle 1|\alpha\rangle = \begin{pmatrix} c_0 \\ c_1 \end{pmatrix}$$

and

$$\langle\alpha| = \langle\alpha|0\rangle\langle 0| + \langle\alpha|1\rangle\langle 1| = (c_0^* \ c_1^*)$$

where  $c_0 = \langle 0 | \alpha \rangle$  and  $c_1 = \langle 1 | \alpha \rangle$ . Now, we can represent the Pauli matrices using two component formalism as

$$\sigma_x = |0\rangle\langle 1| + |1\rangle\langle 0| = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_y = -i|0\rangle\langle 1| + i|1\rangle\langle 0| = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1| = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

### 4.1.a Some Properties of Pauli Matrices

1. Hermitian:  $\sigma_i^\dagger = \sigma_i$
2. Unitary:  $\sigma_i^\dagger = \sigma_i$
3. Traceless:  $\text{Tr}(\sigma_i) = 0$

4. Determinant:  $\det(\sigma_i) = -1$
5. Commutation Relation:  $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$
6. Anticommutation Relation:  $\{\sigma_i, \sigma_j\} = 2\delta_{ij}I$
7. Squared:  $\sigma_i^2 = I$

**i** Note 5: Exercise

Prove the above properties of pauli matrices.

Now consider  $\vec{a} \cdot \vec{\sigma}$  where  $\vec{a}$  is a vector and  $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ . We can show that

$$\vec{a} \cdot \vec{\sigma} = a_k \sigma_k = \begin{pmatrix} a_z & a_x - ia_y \\ a_x + ia_y & -a_z \end{pmatrix}$$

Now we'll show a very important identity which will be much useful later,

**i** Note 6: Theorem

$$(\vec{\sigma} \cdot \vec{a})(\vec{\sigma} \cdot \vec{b}) = \vec{a} \cdot \vec{b}I + i\vec{\sigma} \cdot (\vec{a} \times \vec{b})$$

**Proof:** Let us re-write the left hand side, as  $(\vec{\sigma} \cdot \vec{a})(\vec{\sigma} \cdot \vec{b}) = a_i b_j \sigma_i \sigma_j$  and using the commutation and anticommutation relation and

$$\sigma_i \sigma_j = \frac{1}{2}[\sigma_i, \sigma_j] + \frac{1}{2}\{\sigma_i, \sigma_j\} = i\epsilon_{ijk}\sigma_k + \delta_{ij}I \quad (6)$$

hence the left hand side becomes

$$a_i b_j \sigma_i \sigma_j = a_i b_j (i\epsilon_{ijk}\sigma_k + \delta_{ij}I) = i\vec{\sigma} \cdot (\vec{a} \times \vec{b}) + \vec{a} \cdot \vec{b}I$$

Hence proved.

For  $\vec{a} \in \mathbb{R}^3$  we can show that  $(\vec{\sigma} \cdot \vec{a})^2 = |\vec{a}|^2 I$  using Equation 6.

## 4.2 Rotation in Two Component Formalism

As we have shown in the earlier sections that the  $\{\sigma_i\}$  are the generators of the rotation in the two dimensional hilbert space. So we can write the rotation operator as a  $2 \times 2$  matrix as

**i** Note 7: Theorem

$$\mathcal{D}(\hat{n}, \phi) = e^{-i\frac{\phi}{2}\hat{n}\cdot\vec{\sigma}} = \cos\left(\frac{\phi}{2}\right)I - i\sin\left(\frac{\phi}{2}\right)(\hat{n}\cdot\vec{\sigma}) \quad (7)$$

**Proof:** Using the Taylor expansion of exponential operator we can write

$$e^{-i\frac{\phi}{2}\hat{n}\cdot\vec{\sigma}} = \sum_{k=0}^{\infty} \frac{1}{k!} \left( -i\frac{\phi}{2}\hat{n}\cdot\vec{\sigma} \right)^k$$

Now using the property that for  $\hat{n} \in \mathbb{R}^3$ ,  $(\hat{n}\cdot\vec{\sigma})^2 = I$  we can split the above summation into odd and even terms as

$$\begin{aligned} e^{-i\frac{\phi}{2}\hat{n}\cdot\vec{\sigma}} &= \sum_{m=0}^{\infty} \frac{1}{(2m)!} \left( -i\frac{\phi}{2} \right)^{2m} (\hat{n}\cdot\vec{\sigma})^{2m} + \sum_{m=0}^{\infty} \frac{1}{(2m+1)!} \left( -i\frac{\phi}{2} \right)^{2m+1} (\hat{n}\cdot\vec{\sigma})^{2m+1} \\ &= \sum_{m=0}^{\infty} \frac{1}{(2m)!} \left( -i\frac{\phi}{2} \right)^{2m} I + \sum_{m=0}^{\infty} \frac{1}{(2m+1)!} \left( -i\frac{\phi}{2} \right)^{2m+1} (\hat{n}\cdot\vec{\sigma}) \end{aligned}$$

Now using the Taylor expansion of cosine and sine function we get

$$e^{-i\frac{\phi}{2}\hat{n}\cdot\vec{\sigma}} = \cos\left(\frac{\phi}{2}\right)I - i\sin\left(\frac{\phi}{2}\right)(\hat{n}\cdot\vec{\sigma})$$

hence proved.

Now in principle  $\chi$  which is a 2-component spinor and transforms under rotation as  $\chi \rightarrow \mathcal{D}(\phi, \hat{n})\chi$  and as I have asked to show that the expectation value of an angular momentum operator transforms as a vector under rotation, and one can go one step ahead and make claims that the angular momentum operator indeed transforms as vector under rotation as  $\hat{J}'_i = R_{ij}\hat{J}_j$ , but in reality it is  $\chi^\dagger \sigma_k \chi$  that transforms as a vector under rotation

$$\chi^\dagger \sigma_i \chi \rightarrow R_{ij} \chi^\dagger \sigma_j \chi$$

Now as we have seen earlier that under  $2\pi$  rotation the spinor picks up a negative sign but what is the eigenvalue of the rotation operator. So we are interested in finding  $\mathcal{D}(\phi, \hat{n})\chi = \chi$ .

## 5 Eigenvalues and Eigenvectors of Angular Momentum