1 Quantum Kinematics and Dynamics

1.1 Position

Now clearly its the time we should talk about some real stuff. Alright the very first idea one encounters in the real physics is position where you are located? and as we define operators for everything we'll define a position a operator as $\mathcal{X}$ and I define the corresponding eigenkets as $\{|x\rangle\}$, where the $\mathcal{X}$ acts on $|x\rangle$ in the following manner

$$\mathcal{X}|x\rangle = x|x\rangle \tag{1}$$

here x is just a number with appropriate units and tells about where the particle is on the real line. Clearly the size of the space $\{|x\rangle\}$ is infinite as there are inifinte number of possible positions. And since it is an infinite dimensional vector space so the inner product is replaced in the following manner.

$$\langle x''|x'\rangle = \delta(x' - x'') \tag{2}$$

So now one would like to ask how one should write a wavefunction in terms of position eigenkets? Well I would just say expand it in terms of position eigenkets as

$$|\alpha\rangle = \int_x dx|x\rangle\langle x||\alpha\rangle \tag{3}$$

where this quantity $\langle x|\alpha\rangle$ is actually our wavefunction $\psi(x)$ and as we know from finite dimensional case

$$\int_{-\infty}^{\infty} \psi(x)^*\psi(x)dx = 1 \tag{4}$$

which is our normalisation condition and from our studies in wave mechanics we call $\psi(x)$ as our wave function.

1.1.1 Translation

Now that we are done with forming the position space let's talk about a very fundamental operation in the position space that is translation,'movement' is more informal language. So I define an operator called textbf{Translation Operator ($\mathcal{T}(dx)$)} which acts on the eigenkets of $\mathcal{X}$ operator in the following manner

$$\mathcal{T}(dx)|x\rangle = |x + dx\rangle \tag{5}$$

Now after applying $\mathcal{X}$ on the resulting ket we'll have

$$\mathcal{X}|x + dx\rangle = x + dx|x + dx\rangle \tag{6}$$

here, $x + dx$ is just a number.

Example 1.1: Here is an example where I would like to illustrate how to show a small but an important result.Prove that $[\mathcal{X}, \mathcal{T}] \neq 0$

Solution: Here we need to show that $\mathcal{X}\mathcal{T} - \mathcal{T}\mathcal{X} \neq 0$ Now from above just observe that in Eq. 6 we have one part of the result now we'll do the next half.

$$\mathcal{T}\mathcal{X}|x\rangle = x|x+dx\rangle \tag{7}$$

So now we have our commutation relation as

$$[\mathcal{X}, \mathcal{T}] = dx \tag{8}$$

Hence proved. ■

So now I know translation and position don't commute and let us work out further properties we think of should the translation operator should have.

1. $\mathcal{T}$ should be just identity for $dx \to 0$.
2. $\mathcal{T}$ is a unitary operator given as it should conserve probabilites as ket moves in space.$\mathcal{T}\mathcal{T}^\dagger = \mathbb{I}$
3. If we have a translation operator moving a ket by dx the I can similarly define a translation operator which moves a ket by $-dx$, which I should call its inverse. So I have $\mathcal{T}\mathcal{T}^{-1} = \mathbb{I}$ with this I may say $\mathcal{T}^\dagger = \mathcal{T}^{-1}$

So now with the above properties all said I propose the form of translation operator.

$$\mathcal{T} = 1 - iKdx \tag{9}$$

Now I'll leave it as an exercise for you to verify the above properties with this operator.

Exercise 1.1: With the form of translation operator as given in Eq. 9 verify that it satisfy the given three properties discussed. In second property you would realise that K must be some special kind of operator for this the property to satisfy what it is?

So from above exercise textit{K turns out to be a hermitian operator}.

Exercise 1.2: Show that $[x, K] = i$

1.1.2 Generator of Translation

We talked about translation operator shifting our positions in the position space so for some dx translation in position space we can have N number of small translations as

$$|x+dx\rangle = \lim_{N\to\infty}\left[1 - \frac{iKdx}{N}\right]^N |x\rangle \tag{10}$$

Now this repeated action of translation operator I'll write it in a different form

$$\lim_{N\to\infty}\left[1 - \frac{iKdx}{N}\right]^N = e^{-iKdx} \tag{11}$$

Now what it this **K** till now I haven't talked about it till now so now I would take up upon the idea of dimension analysis and classical mechanics. So by dimensional analysis I claim that Kdx should be dimensionless and thus $[K] = [L]^{-1}$. Now I'll take up the idea of translation in classical mechanics which talk about about the form of K. So now with

$$K = \frac{p}{\hbar} \tag{12}$$

> **Exercise 1.3**: Show that the from of K should be above using dimensional analysis.

in denominator we have $\hbar$ and it is no surprise that something of dimensions of action should be there. But the use of $\hbar$ is no surprise as it is the "number of Quantum Mechanics" and comes from famous **De Broglie's Equation**.
Similar arguements can be extended to more than one dimensions and one can study that these translation operators make an **abelian group**. One can go a step ahead and using form of K we can write the commutation relation between x and p as

$$[x,p] = i\hbar \tag{13}$$

and in higher dimensions we have

$$\left[x_i, x_j\right] = 0 \tag{14}$$

$$\left[p_i, p_j\right] = 0 \tag{15}$$

$$\left[x_i, p_j\right] = i\hbar\delta_{ij} \tag{16}$$

But with all this jargon in front of me I don't know how the momentum operator looks like? Consider the translation operator applied on the wavefunction as

$$\begin{aligned}\left(1 - \frac{ipdx'}{\hbar}\right)|\alpha\rangle = \int_x dx'\mathcal{T}(dx')|x'\rangle\langle x'||\alpha\rangle &= \int_x dx'|x' + dx'\rangle\langle x'||\alpha\rangle \\ &= \int_x dx'|x'\rangle\langle x' - dx'||\alpha\rangle\end{aligned} \tag{17}$$

Now by using the taylor expansion we have

$$\int_x dx'|x'\rangle\langle x' - dx'||\alpha\rangle = \int_x dx'\left(\langle x'|\alpha\rangle - dx'\frac{\partial\langle x'|\alpha\rangle}{\partial x'}\right) \tag{18}$$

By comparing the above two Equations,

$$p|\alpha\rangle = \int_x dx'|x'\rangle\left(-i\hbar\frac{\partial\langle x'|\alpha\rangle}{\partial x'}\right) \tag{19}$$

hence the form of momentum operator is

$$p = -i\hbar\frac{\partial}{\partial x} \tag{20}$$

and in higher dimension we have

$$\boldsymbol{p} = -i\hbar\boldsymbol{\nabla} \tag{21}$$

1.2 Momentum

1.3 Time Evolution Operator and Schrödinger's Equations

1.4 Schrödinger's Picture and Hisenberg Picture

1.5 Simple Harmonic Oscillator-Operator Style

Now we'll consider a practical and one of most useful examples in quantum mechanics is about the a particle trapped in harmonic trap and now I'll show how to solve a problem in quantum mechanics, first step is write the hamiltonian,

$$\mathcal{H} = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 \tag{22}$$

now at this point there are two ways, one is easier and of more insight and further use in future and the other one is of more insight in terms of classical understanding. So I'll go with the first one, operator method, Eq. 22 can be factorised but because x and p are operators and they don't commute. Consider these two new operators $a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{ip}{m\omega}\right)$ and $a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{ip}{m\omega}\right)$ and these are **annihilation operator** and **creation operator** respectively.

Exercise 1.4: Show that

$$\left[a, a^\dagger\right] = 1 \tag{23}$$

Now we define the number operator as $N = a^\dagger a$ and one can check that

$$\mathcal{H} = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right) \tag{24}$$

1.6 Schrödinger's Wave Equation

1.6.1 1D Potential Infinite Well

1.6.2 Free Particle

1.6.3 Delta Potential

1.6.4 Finite Well Potential