Theory of Angular Momentum

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In classical mechanics we cause rotation of a vector $\vec{v}$ by applying a rotation matrix R such that $\vec{v'} = R\vec{v}$ where $\vec{v'}$ is the rotated vector. The rotation matrix is suppose to follow the following properties

- $$R^T R = RR^T = I$$
- $$\sqrt{v_x^2 + v_y^2 + v_z^2} = \sqrt{v_{x'}^2 + v_{y'}^2 + v_{z'}^2}$$

Now in 3D we can generate all the rotations possible using 3 matricies around x,y and z axis using $R_x(\phi)$, $R_y(\phi)$ and $R_z(\phi)$. For very small angle ϕ, the above matricies allow the commutation relation as $\left[R_x(\epsilon), R_y(\epsilon)\right] = R_z(\epsilon^2) - I$

> **i Note 1: Exercise**
>
> Generalise the above commuation relation.

Now one of the important lessons of quantum mechanics is that we can rotate a physical system in R^3 then there should be a corresponding operator $\mathcal{D}(R)$ such that it rotates the wavefunction in the hilbert space $\mathcal{H}$ such that $|\alpha\rangle_R = \mathcal{D}(R)|\alpha\rangle$. And the *physics* in the rotated frame is given by $\mid \alpha\rangle_R$.

> **i Note 2: Exercise**
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> Show that $\mathcal{D}(R)$ is an unitary operator.

Now let us consider a unitary operator which generates the rotation and let $\vec{J} \cdot \hat{n}$ be the generator of the rotation where $\vec{J} = \left\{ \hat{J}_x, \hat{J}_y, \hat{J}_z \right\}$ and $\hat{n}$ is a unit vector in the direction of the rotation axis. So we can write the $\mathcal{D}(R) = e^{-i\vec{J}\cdot\hat{n}\theta}$. Now in the small angle approximation we have $\mathcal{D}(R) = 1 - i\vec{J} \cdot \hat{n}\theta + \mathcal{O}(\theta^2)$

3 Finite Rotation in Quantum mechanics

We can re-write the rotation around an axis as compounding of multiple infinitesimal rotation around the axis as

$$\mathcal{D}_z(\phi) = \lim_{N\to\infty} \left(1 - i\hat{J}_z \frac{\phi}{N} \right)^N$$

which is

$$\mathcal{D}_z(\phi) = exp\left(-i\hat{J}_z\phi\right) = 1 - \frac{i\hat{J}_z\phi}{\hbar} - \frac{\hat{J}_z^2\phi^2}{2\hbar^2} + \mathcal{O}(\phi^3)$$

Now let us try to figure-out the angular momentum algebra using the rotation operator. Now we assume that rotation operator have same group properties as the classical rotation matrices.

1. Identity: $R \cdot I = I \cdot R = R \implies \mathcal{D}(R) \cdot I = I \cdot \mathcal{D}(R) = \mathcal{D}(R)$
2. Closure: $R_1 R_2 = R_3 \implies \mathcal{D}(R_1)\mathcal{D}(R_2) = \mathcal{D}(R_3)$
3. Inverses: $RR^{-1} = R^{-1}R = I \implies \mathcal{D}(R)\mathcal{D}(R)^{-1} = \mathcal{D}(R)^{-1}\mathcal{D}(R) = I$
4. Associativity: $R_1(R_2 R_3) = (R_1 R_2)R_3 \implies \mathcal{D}(R_1)(\mathcal{D}(R_2)\mathcal{D}(R_3)) = (\mathcal{D}(R_1)\mathcal{D}(R_2))\mathcal{D}(R_3)$

Now using the commutation relation of classical rotaion matrix one can workout the commutation relation of the angular momentum operator.

> **i Note 3: Exercise**
>
> Workout the commutation relation of angular momentum operator using rotation operator commutation relation, which follows from the fact that they have same group structure.

So one of the central results of this section is the angular momentum algebra given as

$$\left[\hat{J}_i, \hat{J}_j\right] = i\hbar\epsilon_{ijk}\hat{J}_k \tag{1}$$

Equation 1 shows that the angular momentum operators forms an **non-abelian group**, which is different from the translation as the generators p_j form an **abelian group**. Now in principle I only need Equation 1 to give all the properties of angular momentum operator and hence rotation operators. But now the question is how the expectation values change with the rotation operator? Consider the expectation value $\langle \hat{J}_x \rangle$, it is defined as

$$\langle \hat{J}_x \rangle = \langle \alpha_R | \hat{J}_x | \alpha_R \rangle = \langle \alpha | \mathcal{D}_z^\dagger \hat{J}_x \mathcal{D}_z | \alpha \rangle$$

Now the let us see how $\hat{J}_x$ transform under the rotation around z-axis?

$$\mathcal{D}_z^\dagger \hat{J}_x \mathcal{D}_z = e^{-i\hat{J}_z\phi} \hat{J}_x e^{i\hat{J}_z\phi} = 1 + \left(\frac{i\phi}{\hbar}\right)\left[\hat{J}_z, \hat{J}_x\right] + \frac{1}{2!}\left(\frac{i\phi}{\hbar}\right)^2 \left[\hat{J}_z, \left[\hat{J}_z, \hat{J}_x\right]\right] + \dots$$

$$= \hat{J}_x \cos(\phi) - \hat{J}_y \sin(\phi)$$

Now note that these operator transforms as a vector under rotation.

> **ⓘ Note 4: Exercise**
>
> Show explicitly that angular momemtum operators along different axis transform as a vector under rotation.