IC112 Calculus
Assignment 3

1. If $\lim\limits_{x\to a} f(x)$ exists, prove that it must be unique.

2. Using the ε–δ definition of limits:

   (a) Show that $\lim\limits_{x\to 0} \frac{1}{x}\sin\frac{1}{x}$ does not exist.

   (b) Prove that $\lim\limits_{x\to 0} x\sin\frac{1}{x} = 0$.

3. Discuss the continuity at $x = a$ of the function $f(x)$ given by:

   (a) $f(x) = \begin{cases} \dfrac{1}{x-a}\ \mathit{cosec}\ \dfrac{1}{x-a} & \text{if } x \neq a, \\ 0 & \text{if } x = a. \end{cases}$

   (b) $f(x) = \begin{cases} \dfrac{x^2}{a} - a & \text{for } 0 < x < a, \\ 0 & \text{for } x = a, \\ a - \dfrac{a^2}{x^2} & \text{for } x > a. \end{cases}$

4. Discuss the continuity and differentiability of the function $f(x)$ defined as follows:

   (a) $f(x) = \begin{cases} x^2 \sin\left(\dfrac{1}{x}\right) & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$

   (b) $f(x) = \begin{cases} 1 & \text{for } -\infty < x < 0, \\ 1 + \sin x & \text{for } 0 \leq x < \dfrac{\pi}{2}, \\ 2 + \left(x - \dfrac{\pi}{2}\right)^2 & \text{for } \dfrac{\pi}{2} \leq x < \infty. \end{cases}$

5. If $f(x) = \dfrac{xe^{1/x}}{1+e^{1/x}}$ for $x \neq 0$ and $f(0) = 0$, show that $f(x)$ is continuous at $x = 0$, but $f'(0)$ does not exist.

6. Let $f(x) = x\dfrac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}}$ for $x \neq 0$ and $f(0) = 0$. Show that $f(x)$ is continuous but not differentiable at $x = 0$.

7. Using the concept of increasing and decreasing functions, show that

   (i) $\dfrac{x}{1+x} < \log(1+x) < x, \ \forall x > 0.$

   (ii) $\dfrac{x}{1+x^2} < \tan^{-1} x < x, \ \forall x > 0.$

   (iii) $\dfrac{2}{2x+1} < \log\left(1 + \dfrac{1}{x}\right) < \dfrac{1}{\sqrt{x(x+1)}}.$

8. Discuss the applicability of Rolle's theorem to

$$f(x) = \log\left[\frac{x^2 + ab}{(a+b)x}\right]$$

   in the interval $[a, b]$.

9. Verify Rolle's theorem for the functions:

   (a) $x(x+3)e^{-x/2}$ on $[-3, 0]$,

   (b) $e^x(\sin x - \cos x)$ on $\left[\dfrac{\pi}{4}, \dfrac{5\pi}{4}\right]$.

10. Compute the value of θ in Lagrange's Mean Value Theorem:

$$f(x+h) = f(x) + hf'(x + \theta h),$$

    if $f(x) = ax^2 + bx + c$.

11. A function $f(x)$ is continuous on $[0, 1]$ and differentiable on $(0, 1)$. Prove that there exists $x_1 \in (0, 1)$ such that $f'(x_1) = f(1) - f(0)$.

12. In Cauchy's Mean Value Theorem, let $f(x) = e^x$ and $g(x) = e^{-x}$. Show that the point c is the arithmetic mean of a and b.