Rationale
The English-channel/Eurotunnel[1] is an underwater tunnel that was constructed in 1994 to connect the port of Folkestone (England) to the port of Calais (France) by rail. Over 10.6 million people crossed between the 2 countries in the year 2016. As a frequent passenger of these trains during my time in England, I was fascinated by the idea of a train that could travel through the sea. However, as I grew older, and my fascination waned, the time taken to cross the channel began to seem like an eternity.

The Eurotunnel that connects England and France

Introduction
This mathematical exploration will seek to provide a solution to this childhood problem of mine. I shall use the concepts of gravitational potential energy and kinetic energy, as well as the Euler-Lagrange equation to approximate the ideal shape of a tunnel that would minimize the travel time between England and France. The actual travel time is 35 minutes[2].

Aim
To investigate the optimum shape of a hypothetical tunnel, such that travel time is minimized, between England and France.
To compare the time taken to travel through a hypothetical straight tunnel and a hypothetical Brachistochrone tunnel, between England and France.

Assumptions
This exploration is limited by these assumptions: -
a. Absence of coulomb friction.
b. Absence of air resistance.
c. There is no friction between the rails of the tunnel and the train.
d. The train acts as a point object with uniform motion all across its body.
e. All functions in this paper as well as their derivatives are continuous, and differentiable.

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[1] "The History of the Eurostar." The Telegraph, Telegraph Media Group, 13 Oct. 2014, www.telegraph.co.uk/finance/newsbysector/transport/11158665/The-history-of-the-Eurostar.html.
[2] http://www.chunnel.org.uk/

f. Density of Earth is assumed constant throughout its structure/composition.

Note: a'(v) = the derivative of a(v) with respect to x, for any function a, and for any independent variable v.

Exploration

The Euler-Lagrange equation is based on the principle of least action or Hamilton's principle. This means that the equation is used to derive the most energy efficient pathway to move from one point to another.

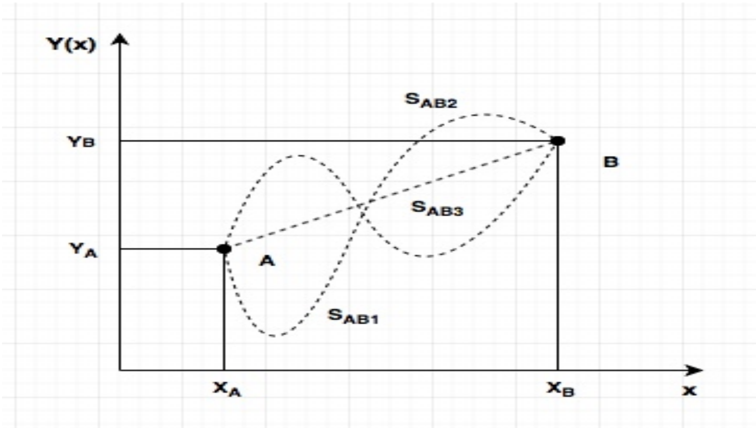


Figure 1.1

Let x be the horizontal distance and y(x) be a function that gives output of vertical position, as a function of horizontal distance.

Let A and B be two points on the graph y(x) against x, with coordinates (x_A, y_A), and (x_B, y_B).

Note that x is the independent variable, and y is the dependent variable in this case.

There can be an infinite number of paths between points A and B. Three of which, S_{AB1}, S_{AB2}, S_{AB3} as shown in the diagram above.

Intuitively, we know that the path of least action (least work, least distance, least time) is given by the straight path between points A and B.

However, this is not the case in all situations, particularly situations where gravity is involved; as this exploration will show.

The Euler Lagrange equation[3] gives us an equation for the path of least action for **any system of motion**. The equation is: -

$$\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right) = 0, \text{when } s = \int_{X_A}^{x_B} [f(x, y(x), y'(x)]\, dx \qquad (1)$$

where $f(x, y(x), y'(x))$ is a functional that gives the work function, i.e. the parameter that we wish to minimize (could be time, or distance).

[3] Ueding, Martin. "Derivation of the Euler-Lagrange-Equation¶." Derivation of the Euler-Lagrange-Equation - Martin Ueding, martin-ueding.de/articles/euler-lagrange-equation-derivation/.

Proof

To find the distance or length of each of the different paths illustrated in figure 1.1, we need to add up the length of each incremental portions of the path. Upon magnification, each incremental portion of the path takes on a shape like the one below: -

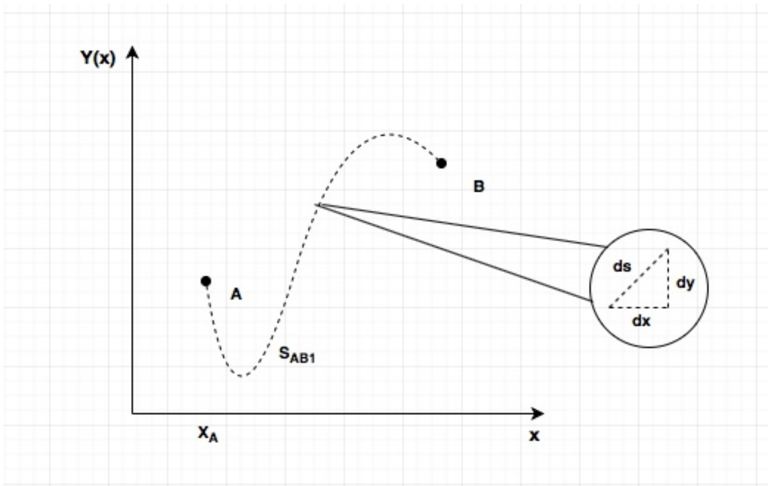


Figure 1.2

Where s is the distance between the two points A and B. To find the path of least action/least work, we will find the equation of the path that gives the least distance, and hence the least work.

The incremental portion ds can be written in terms of the incremental change in x (dx) and the incremental change in y (dy).

$$(ds)^2 = (dx)^2 + (dy)^2 \qquad (2)$$

This equation can be rewritten as: -

$$ds = \sqrt{dx^2 + dy^2} \qquad (3)$$

$$ds = \sqrt{1 + (y'(x))^2}dx \qquad (4)$$

(From figure 1.2) To find the length of the path between these two points, we can integrate both sides and derive an equation for s, the distance between the two points.

$$\int_{x_A}^{x_B} ds = \int_{x_A}^{x_B} \sqrt{1 + (y'(x))^2}dx (5)$$

$$s_{AB} = \int_{x_A}^{x_B} \sqrt{1 + (y'(x))^2}dx \qquad (6)$$

$$s = \int \sqrt{1 + (y'(x))^2}\, dx \qquad (7)$$

From (1), and (6), (7) can be rewritten as a Euler-Lagrange function f.

$$[f(x, y(x), y'(x)] = \sqrt{1 + (y'(x))^2} \qquad (8)$$

Since we are attempting to minimise the distance of the path (i.e. work/action), based on (1), we must represent s, the distance (the length of path), with the function f

$\sqrt{1+(y'(x))^2}$ **is a function of y'(x), not of y(x),** therefore, from (1), for any function f representing length of path s,

$$\frac{\partial f}{\partial y} = 0 \quad (9)$$

From (1) and (8),

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \qquad (10)$$

$$\frac{\partial f}{\partial y'} = \frac{1}{2\sqrt{1+(y'(x))^2}}\left(2(y'(x))\right) = \frac{(y'(x))}{\sqrt{1+(y'(x))^2}} \qquad (11)$$

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 = \frac{d}{dx}\left[\frac{(y'(x))}{\sqrt{1+(y'(x))^2}}\right] \qquad (12)$$

Therefore,

$$\frac{(y'(x))}{\sqrt{1+(y'(x))^2}} = C \qquad (13)$$

$$\frac{\left(\frac{dy}{dx}\right)^2}{1+\left(\frac{dy}{dx}\right)^2} = C^2 = A \qquad (14)$$

where A and C are constants.
Therefore, when we rearrange (14), we get an expression for $(y'(x))$,

$$(y'(x)) = \sqrt{\frac{A}{1-A}} \qquad (15)$$

Let the **constant** $\sqrt{\frac{A}{1-A}}$ be denoted by m.

$$\int (y'(x))dx = \int m\, dx \quad (16)$$

$$y = mx + c \qquad (17)$$

(17) is the equation of a straight line, which is intuitively, the shortest path between the two points.

<u>The path of least action for an object influenced by force of gravity</u>

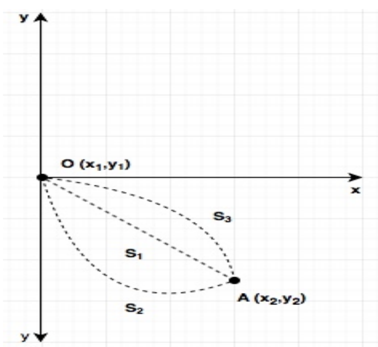


Figure 1.3

The shortest **time** taken to travel from one point to another (where the force of gravity is involved), is not necessarily achieved by taking the **shortest path (straight line)** between the two points. The total time taken to travel from one point to another is given by adding the incremental time taken to travel through incremental distances in the path. Hence,

$$\Delta t = \int \frac{ds}{v} \qquad (18)$$

Mechanism: The object has gravitational potential energy when stationary at O. As it rolls down, it gains velocity as it moves downward, with GPE being converted into kinetic energy (KE). By the time it reaches A, all its GPE (denoted by p) has been converted into KE, and the object has attained maximum possible velocity. During this motion, we will assume that point A lies on the ground, and that all of the potential energy of the object at O, will have been converted to kinetic energy by the time it reaches point A.

$$(p)\ at\ point\ O(0{,}0) = m\vec{g}\vec{y} \qquad (19)$$

where g is acceleration due to gravity, $g = \frac{9.8m}{s^2}\ downward$, m is the mass of the object, and y is the **vertical** height of the object.

The kinetic energy (KE) (k)of the object at any given moment is given by: -

$$k = \frac{1}{2}mv^2 \qquad (20)$$

where v is the velocity of the object at any given moment.

Based on the ***law of conservation of energy***, and assuming that these are the only two forms of energy that the object possesses during its motion, we derive

$$k + p = constant = Total\ energy\ (T) \qquad (21)$$
$$\Delta p = -\Delta k, and\ vice\ versa \qquad (22)$$

Considering the velocity of the object and indeed its horizontal velocity and position, and indeed the displacement of it, is determined by its GPE, and its GPE depends on its **vertical position y, not its horizontal position x**. Hence y is the independent variable, and x is the dependent variable. Considering this, and from (20),

$$ds = \sqrt{1 + (\mathrm{x}'(\mathrm{y})\ )^2}dy \qquad (23)$$

When p is maximum, k is equal to 0, and when p is equal to 0, k is maximum. Based on this, and the ***law of conservation of energy***, we can say that the maximum of p is equal to the maximum of k. Hence,

$$\frac{1}{2}mv^2 = mgy \qquad (24)$$
$$v = \sqrt{2gy} \qquad (25)$$

From (23) and (25), (18) can be rewritten as,

$$\Delta t = \int_{y_1}^{y_2} \frac{\sqrt{1+(x'(y))^2}}{\sqrt{2gy}}\, dy \qquad (26)$$

$$\Delta t = \frac{1}{\sqrt{2g}} \int_{y_1}^{y_2} \frac{\sqrt{1+(x'(y))^2}}{\sqrt{y}}\, dy \qquad (27)$$

Considering **y is the independent variable, and x is the dependent variable in this case,** (1) can be rewritten as: -

$$\left(\frac{\partial f}{\partial x} - \frac{d}{dy}\frac{\partial f}{\partial x'}\right) = 0, \text{when } s = \int_{y_1}^{y_2} [f(y, x(y), x'(y)]\, dy \quad (28)$$

From (26) and (28), $[f\ (y, x(y), x'(y)]$ in this case is the function

$$[f\ (y, x(y), x'(y)] = \frac{\sqrt{1+(x'(y))^2}}{\sqrt{2gy}} \qquad (29)$$

For path of least action, condition in (28) must be met. Therefore,

$$\frac{\partial f}{\partial x} = \frac{d}{dy}\frac{\partial f}{\partial x'} \qquad (30)$$

Since f is a function of y, x'(y), and not x(y),

$$\frac{\partial f}{\partial x} = 0 \qquad (31)$$

$$\frac{d}{dy}\left(\frac{\partial f}{\partial x\prime}\right) = 0 \qquad (32)$$

$$Let\ \left(\frac{\partial f}{\partial x\prime}\right) = c, where\ c\ is\ a\ constant \quad (33)$$

From (28) and (33),

$$\left(\frac{\partial f}{\partial x\prime}\right) = c = \frac{1}{\sqrt{2g}}\left(\frac{1}{2\sqrt{y(1+(x'(y))^2)}}\right)(2(x'(y))) \qquad (34)$$

$$\left(\frac{\partial f}{\partial x\prime}\right) = c = \left(\frac{x'(y)}{\sqrt{y(1+(x'(y))^2)}}\right) \qquad (35)$$

$$\left(\frac{\partial f}{\partial x\prime}\right)^2 = c^2 = \frac{(x'(y))^2}{y(1+(x'(y))^2)} \qquad (36)$$

$$\frac{1}{c^2}(x'(y))^2 = y(1 + (x'(y))^2) \qquad (37)$$

$$\left(\frac{1}{c^2} - y\right)(x'(y))^2 = y \qquad (38)$$

$$x'(y) = \sqrt{\frac{y}{\frac{1}{c^2} - y}} \qquad (39)$$

For the sake of simplicity, instead of c^2, we can take the **constant**, to be some $\frac{1}{2a}$[4]. Therefore,

$$x = \int_{y_1}^{y_2} \sqrt{\frac{y}{2a-y}}\,dy \tag{40}$$

Using substitution, we can write x and y in terms of some parameter θ,

$$Let\ y = a(1-\cos\theta) \tag{41}$$
$$y'(\theta) \ = a\sin\theta \tag{42}$$

Therefore (40) can be rewritten as,

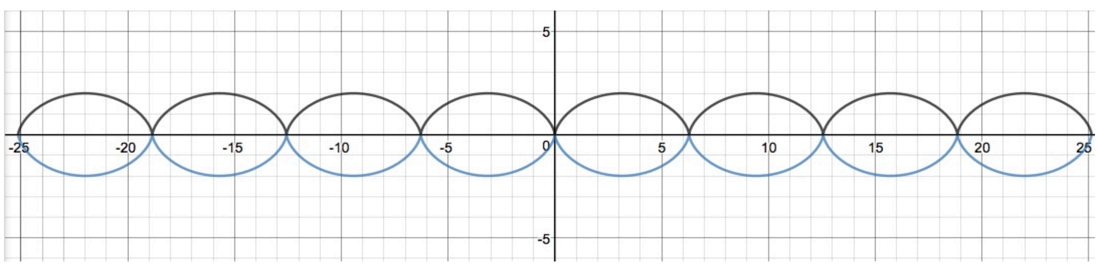
$$x = \int_{y_1}^{y_2} \sqrt{\frac{a(1-\cos\theta)}{a(1+\cos\theta)}}(a\sin\theta)\,d\theta \tag{43}$$
$$\sin\theta = \sqrt{1-\cos^2\theta} \tag{44}$$
$$\sin\theta = \sqrt{(1-\cos\theta)(1+\cos\theta)} \tag{45}$$

Therefore (43) can be rewritten as,

$$x = -a\int_{y_1}^{y_2} \sqrt{\frac{(1-\cos\theta)(1-\cos\theta)(1+\cos\theta)}{(1+\cos\theta)}}\,d\theta \tag{46}$$
$$x = -a\int_{y_1}^{y_2} \sqrt{(1-\cos\theta)^2}\,d\theta \tag{47}$$
$$x = -a\int_{y_1}^{y_2}(1-\cos\theta)\,d\theta \tag{48}$$

$$\boldsymbol{x = a(\theta - \sin\theta)} \tag{49}$$
$$\boldsymbol{y = a(1 - \cos\theta)} \tag{50}$$

(49) and (50) now give a parametric expression for x and y, in terms of the parameter θ. Parametric equations[5] are a set of equations that express a set of quantities as explicit functions of a number of independent variables, known as "parameters."



[4] Adam. "Lagrangian Mechanics - Lesson 3: The Brachistochrone Problem." YouTube, YouTube, 1 June 2016, www.youtube.com/watch?v=UI0FE8NhU7U&t=2814s.

[5] "Parametric Equations." From Wolfram MathWorld, Wolfram Alpha, mathworld.wolfram.com/ParametricEquations.html.

Figure 1.4: Graphing the parametric equations in (49) and (50)

Depending on the value of $\boldsymbol{a}$**,** the graph can take on either the blue shape or the green shape.

$Through\ \boldsymbol{experimentation}\ we\ can\ ascertain\ that\ the\ path\ of\ least\ time\ taken\ is\ given$ by the blue curve. The path of least action taken by an object, under the influence of gravity, is a cycloid on a straight path[6]

Hypocycloid[7]

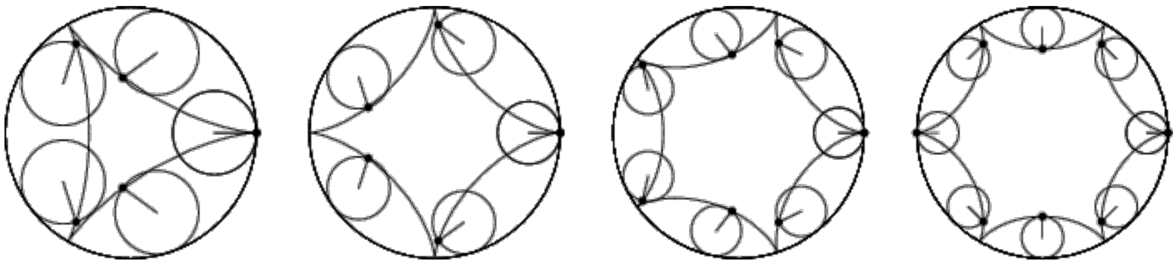


Figure 1.5

The cycloid is defined as a curve (resembling a series of arches) traced by a point on a circle being rolled along a straight line. To find the path of least action ***underground*** i.e. a cycloid path that goes under the surface of the earth must be found. The hypocycloid is an adaptation of the cycloid, and perfectly fits the description of this underground path.

In geometry, a **hypocycloid** is a special plane curve generated by the trace of a fixed point on a small circle that rolls within a larger circle. It is given by a pair of parametric equations: -

$$x = (a-b)\cos\emptyset + b\cos\left(\frac{a-b}{b}\emptyset\right) \qquad (51)$$

$$y = (a-b)\sin\emptyset + b\sin\left(\frac{a-b}{b}\emptyset\right) \qquad (52)$$

where a is the radius of the larger circle, b is the radius of the smaller circle, and ∅ is the angle between the line that connects the center of the larger circle to the fixed point on the smaller circle and the line that connects the center of the larger circle to the center of the smaller circle.

Another form of equation that shows a hypocycloid is: -

$$\theta = \tan^{-1}\left[p\tan\left(\frac{\emptyset}{2}\right)\right] - p\frac{\emptyset}{2} \qquad (53)$$

[6] Utah State University. “Wheeler Classical Mechanics.” 1. The Brachistochrone.

[7] “Hypocycloid.” From Wolfram MathWorld, Wolfram Alpha, mathworld.wolfram.com/Hypocycloid.html.

where $\phi = 0$ when $r = r_o$, and $\phi = \pi$ or $-\pi$ when $r = R$, where r_o is the minimum distance between the point and the center of the larger circle. Where Ø is the angle between the line that joins the arbitrary point and the center of the small circle, and the line that joins the center of the small circle and the center of the large circle.

Mechanism underground (*Straight* tunnel)[8]

A straight tunnel through the earth means that the journey will take **42.2 minutes**[9]; regardless of the distance between the destination and point of departure. This is proven using Newton's laws of gravitation and shell theorem, in the appendix, due to lack of space.

Mechanism underground (*Brachistochrone* tunnel)

Premises: Tunnels are structures that are dug underground. The channel is a tunnel designed as a pathway for a locomotive to travel through from England to France. For the following working it is important to keep **assumption (f)** in mind.

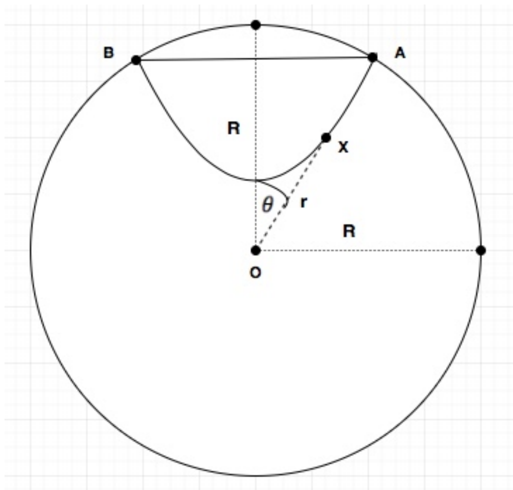


Figure 1.6

The above diagram shows a pathway, or possible tunnel, between A and B, given by AXB. X is the varying position of the locomotive as it passes through the curved tunnel. The value *r*, is given by AX. The gravitational force on an object is given by: -

$$F(r) = m\frac{g}{R}r \qquad (54)$$

$$\textit{Where } R \textit{ is the radius of the earth}, m \textit{ is the mass of the train, and}$$
$$g \textit{ is a constant and is equal to } \frac{GMm}{R^2} = 9.81\, {}^{N}/_{kg}$$

This formula is given by Newton's laws of gravitation, where G is the gravitational constant, M is the mass of the earth, and m is the mass of the train.

The gravitational potential energy (*p*) of an object is given by: -

[8] ETH Zurich. "Brachistochrone." Swiss Federal Institute of Technonology Zurich.
[9] https://thatsmaths.com/2013/08/29/a-hole-through-the-earth/

$$p = \int_0^r F(r)\,dr = m\frac{g}{R}\int_0^r r\,dr \qquad (55)$$

$$p = m\frac{g}{R}\left(\frac{r^2}{2}\right) = \left(\frac{1}{2}\right)mr^2\frac{g}{R} \qquad (56)$$

Recalling (20), (21), and **the law of conservation of energy,**

$$k + p = constant = Total\ energy\ (T) \qquad (57)$$

$$\frac{1}{2}mv^2 + \left(\frac{1}{2}\right)mr^2\frac{g}{R} = T \qquad (58)$$

Assuming that the locomotive starts off with no kinetic energy ($k = 0$) at the surface of the Earth (where $r = R$), and only has potential energy, then T can be found as,

$$\left(\frac{1}{2}\right)mgR = T \qquad (59)$$

Therefore, T is constant for any position of X (**Figure 1.6**), throughout the path. Hence for any position of X, (58) can be rewritten as: -

$$\left(\frac{1}{2}\right)mv^2 + \left(\frac{1}{2}\right)mr^2\frac{g}{R} = \left(\frac{1}{2}\right)mgR \qquad (60)$$

$$v^2 + r^2\frac{g}{R} = gR \qquad (61)$$

Hence, we can derive an equation for velocity, v, as a function of the distance of the locomotive from the center of the Earth, r.

$$v(r) = \sqrt{\frac{g(R-r^2)}{R}} \qquad (62)$$

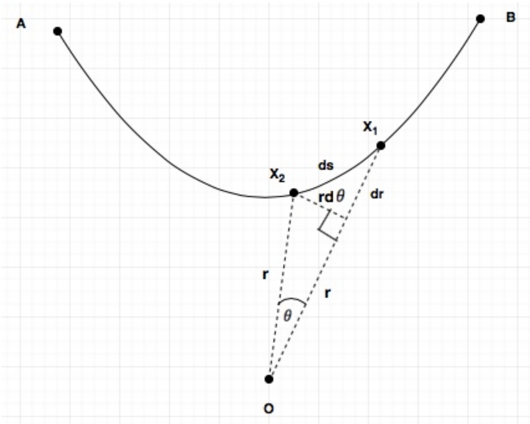


Figure 1.7

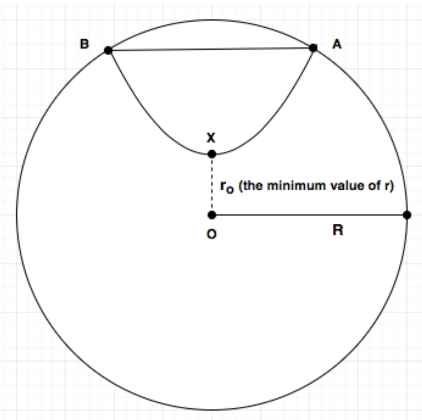


Figure 1.8

The above figure shows a portion of the path, ds, and the components that compose it. It is important to note that these are for small values of θ. Hence, the length of an arc is assumed to be equal to the length of a chord.
From the diagram, ds can be written as: -

$$ds = \sqrt{r^2(d\theta)^2 + (dr)^2} \quad (63)$$

$$ds = \sqrt{r^2 + (r'(\theta))^2}d\theta \quad (64)$$

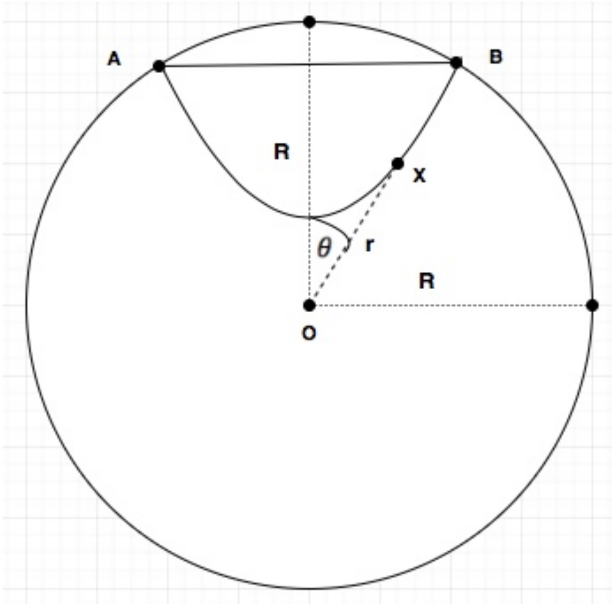


Figure 1.8

Recalling (18), time taken for locomotive to travel through path AB can be written as: -

$$T_{AB} = \int_A^B \frac{ds}{v(r)} \quad (65)$$

From (62) and figure 1.8,

$$T_{AB} = \int_{\theta_A}^{\theta_B} \left| \sqrt{\frac{R}{g}} \left(\sqrt{\frac{r^2 + (\boldsymbol{r'(\theta)})^2}{R^2 - r^2}} \right) d\theta \right| \quad (66)$$

The absolute value of the function is taken as time intervals cannot be negative.

$$Let\ f(r(\theta), \theta, \frac{dr}{d\theta}) = \left| \sqrt{\frac{R}{g}} \left(\sqrt{\frac{r^2 + (\boldsymbol{r'(\theta)})^2}{R^2 - r^2}} \right) \right| \quad (67)$$

Recalling the Euler-Lagrange equation in (1), and the condition required to minimize action (i.e. minimize time taken/minimize T_{AB})

An adaptation of the Euler-Lagrange equation in (1) gives: -

$$\left(\frac{\partial f}{\partial r} - \frac{d}{d\theta} \frac{\partial f}{\partial r'} \right) = 0 \quad (68)$$

where $\boldsymbol{r'} = \frac{\boldsymbol{dr}}{\boldsymbol{d\theta}}$.

Therefore,

$$\frac{dr}{d\theta}\left(\frac{\partial f}{\partial r}-\frac{d}{d\theta}\frac{\partial f}{\partial r'}\right)=0 \tag{69}$$
$$\frac{\partial r}{\partial\theta}\left(\frac{\partial f}{\partial r}\right)-\frac{\partial r}{\partial\theta}\left(\frac{d}{d\theta}\frac{\partial f}{\partial r'}\right)=0 \tag{70}$$
$$\frac{df}{d\theta}=\frac{\partial f}{\partial r}\left(\frac{\partial r}{\partial\theta}\right)+\frac{\partial f}{\partial r\prime}\left(\frac{\partial r'}{\partial\theta}\right)+\frac{\partial f}{\partial\theta} \tag{71}$$
$$\frac{\partial f}{\partial r}\left(\frac{\partial r}{\partial\theta}\right)=\frac{df}{d\theta}-\frac{\partial f}{\partial r'}\left(\frac{\partial r'}{\partial\theta}\right)-\frac{\partial f}{\partial\theta} \tag{72}$$

Substituting (72) into (70) gives the homogenous equation: -

$$\frac{df}{d\theta}-\frac{\partial f}{\partial r'}\left(\frac{\partial r\prime'}{\partial\theta}\right)-\frac{\partial f}{\partial\theta}-\frac{\partial r}{\partial\theta}\left(\frac{d}{d\theta}\frac{\partial f}{\partial r'}\right)=0 \tag{73}$$
$$-\frac{\partial f}{\partial\theta}+\frac{d}{dx}\left(f-r'\left(\frac{df}{dr'}\right)\right)=0 \tag{74}$$

Considering that the function f has no θ in it,

$$\frac{\partial f}{\partial\theta}=0 \tag{75}$$

Therefore,

$$\frac{d}{dx}\left(f-r'\left(\frac{df}{dr'}\right)\right)=0 \tag{76}$$
$$\left(f-r'\left(\frac{df}{dr'}\right)\right)=C \tag{77}$$

where C is a constant.

$$\frac{df}{dr'}=\sqrt{\frac{R}{g}}\left(\frac{2r\prime}{2\sqrt{(r^2+r\prime^2)(R^2-r^2)}}\right) \tag{78}$$
$$\sqrt{\frac{R}{g}}\left(\sqrt{\frac{r^2+(r\prime)^2}{R^2-r^2}}\right)-\sqrt{\frac{R}{g}}r'\left(\frac{r\prime}{\sqrt{(r^2+r\prime^2)(R^2-r^2)}}\right)=C \tag{79}$$
$$\sqrt{\frac{R}{g}}\left(\frac{r^2+(r\prime)^2}{\sqrt{(r^2+r\prime^2)(R^2-r^2)}}\right)-\sqrt{\frac{R}{g}}\left(\frac{(r\prime)^2}{\sqrt{(r^2+r\prime^2)(R^2-r^2)}}\right)=C \tag{80}$$
$$\sqrt{\frac{R}{g}}\left(\frac{r^2}{\sqrt{(r^2+r\prime^2)(R^2-r^2)}}\right)=C \tag{81}$$

To solve for the constant C, we can note and use the fact that, when r reaches a **minimum (stationary point),** i.e. the base of the curve, where $r = r_o$, then $r' = \frac{dr}{d\theta} = 0$. Therefore, by subbing r = r_o, and $r' = 0$, the value of C is given by: -

$$\sqrt{\frac{R}{g}}\left(\frac{{r_o}^2}{\sqrt{({r_o}^2)(R^2-{r_o}^2)}}\right)=C \tag{82}$$
$$\sqrt{\frac{R}{g}}\left(\frac{r_o}{\sqrt{(R^2-{r_o}^2)}}\right)=C \tag{83}$$

Subbing (83) into (81), we get

$$\sqrt{\frac{R}{g}}\left(\frac{r^2}{\sqrt{(r^2+r\prime^2)(R^2-r^2)}}\right)=\sqrt{\frac{R}{g}}\left(\frac{r_o}{\sqrt{(R^2-{r_o}^2)}}\right) \tag{84}$$

$$\left(\frac{r^2}{\sqrt{(r^2+r\prime^2)(R^2-r^2)}}\right) = \left(\frac{r_o}{\sqrt{(R^2-{r_o}^2)}}\right) \tag{85}$$

$$r' = \frac{dr}{d\theta} = \frac{Rr}{r_o}\sqrt{\frac{r^2-{r_o}^2}{R^2-r^2}} \tag{86}$$

$$d\theta = \frac{r_o}{Rr}\sqrt{\frac{R^2-r^2}{r^2-{r_o}^2}}dr \tag{87}$$

Through integration, an expression for θ can be formed.

$$\int d\theta = \int \frac{r_o}{Rr}\sqrt{\frac{R^2-r^2}{r^2-{r_o}^2}}dr \tag{88}$$

We can find the change in θ, as the point X moves from point at which r = r_o, to the point where r = r.

$$\int_{\theta_A}^{\theta_B} d\theta = \frac{r_o}{R}\int_{r_A}^{r_B}\left|\frac{1}{r}\sqrt{\frac{R^2-r^2}{r^2-{r_o}^2}}dr\right| \tag{89}$$

$$\theta = \frac{r_o}{R}\int_{r_A}^{r_B}\left|\frac{1}{r}\sqrt{\frac{R^2-r^2}{r^2-{r_o}^2}}dr\right| \tag{90}$$

$$Let\ u = \sqrt{\frac{r^2-{r_o}^2}{R^2-r^2}} \tag{91}$$

When $r = r_o, u = 0. When\, r = R, u = \infty$. Hence, we can replace the limits in (90) with 0 and ∞.

$$\theta = \frac{r_o}{R}\int_{\theta}^{\theta}\left|\frac{1}{\left(\frac{r_0}{R}\right)^2+u^2} - \frac{1}{1+u^2}du\right| \tag{92}$$

$$\theta = \tan^{-1}\left(\frac{r_o}{R}u\right) - \frac{r_o}{R}\tan^{-1}(u) \tag{93}$$

$$\theta = \tan^{-1}\left(\frac{r_o}{R}\sqrt{\frac{r^2-{r_o}^2}{R^2-r^2}}\right) - \frac{r_o}{R}\tan^{-1}\left(\sqrt{\frac{r^2-{r_o}^2}{R^2-r^2}}\right) \tag{94}$$

From the **hypocycloid equation stated earlier**, we can infer that the tan substitution below is appropriate.

$$Let\ \tan\frac{\phi}{2} = \sqrt{\frac{r^2-{r_o}^2}{R^2-r^2}} \tag{95}$$

where $\phi = 0$ when $r = r_o$, and $\phi = \pi\ or - \pi$ when $r = R$

$$\theta = \tan^{-1}\left(\frac{r_o}{R}\tan\frac{\phi}{2}\right) - \frac{r_o}{R}\tan^{-1}\left(\tan\frac{\phi}{2}\right) \tag{96}$$

$$\theta = \tan^{-1}\left(\frac{r_o}{R}\tan\frac{\phi}{2}\right) - \frac{r_o}{R}\frac{\phi}{2} \tag{97}$$

This resembles the hypocycloid equation mentioned above.

Using the tan substitution in (95), we can form an equation for r,

$$\tan\frac{\phi}{2} = \sqrt{\frac{R^2-r^2}{r^2-{r_o}^2}} \quad (98)$$

$$\tan^2\frac{\phi}{2} = \frac{R^2-r^2}{r^2-{r_o}^2} \quad (99)$$

$$\frac{\sin^2\frac{\phi}{2}}{\cos^2\frac{\phi}{2}} = \frac{R^2-r^2}{r^2-{r_o}^2} \quad (100)$$

$$\sin^2\frac{\phi}{2}(r^2-{r_o}^2) = \cos^2\frac{\phi}{2}(R^2-r^2) \quad (101)$$

$$\frac{1}{2}(1-\cos\phi)(r^2-{r_o}^2) = \frac{1}{2}(1+\cos\phi)(R^2-r^2) \quad (102)$$

$$r^2 - r^2\cos\phi - {r_o}^2 + {r_o}^2\cos\phi = R^2 - r^2 + R^2\cos\phi - r^2\cos\phi \quad (103)$$

$$2r^2 = (R^2+{r_o}^2) + (R^2-{r_o}^2)\cos\phi \quad (104)$$

$$r^2 = \frac{1}{2}(R^2+{r_o}^2) + (R^2-{r_o}^2)\cos\phi \quad (105)$$

This gives us an equation for the value of r at any point on the tunnel as a function of ϕ
Recalling the equation for time taken, and for $r'(\theta)$,

$$T_{AB} = \int_{\vartheta_A}^{\vartheta_B}\left|\sqrt{\frac{R}{g}}\left(\sqrt{\frac{r^2+(r\prime(\theta))^2}{R^2-r^2}}\right)d\theta\right| \quad (106)$$

$$\frac{dr}{d\theta} = \frac{Rr}{r_o}\sqrt{\frac{r^2-{r_o}^2}{R^2-r^2}} \quad (107)$$

$$d\theta = \frac{r_o}{Rr}\sqrt{\frac{R^2-r^2}{r^2-{r_o}^2}}dr \quad (108)$$

By substituting $d\theta$ in the equation for time in (106) with the expression from (108), we get

$$T_{AB} = \sqrt{\frac{R^2-{r_o}^2}{gR}}\int_{r_A}^{r_B}\left|\frac{r}{\sqrt{(R^2-r^2)(r^2-{r_o}^2)}}dr\right| \quad (109)$$

Since this equation was too complicated to solve manually, I used wolfram alpha to do so.
The result was: -

$$T_{AB} = \left[\left(\tan^{-1}\sqrt{\frac{R^2-r^2}{r^2-{r_o}^2}}\right)\sqrt{\frac{R^2-{r_o}^2}{gR}}\right]\Bigg|_A^B \quad (110)$$

Recalling the tan substitution in (95), (110) becomes

$$T_{AB} = \left[\left(\tan^{-1}\tan\frac{\phi}{2}\right)\sqrt{\frac{R^2-{r_o}^2}{gR}}\right]\Bigg|_A^B \quad (111)$$

$$T_{AB} = \left(\frac{\phi}{2}\times\sqrt{\frac{R^2-{r_o}^2}{gR}}\right)\Bigg|_A^B \quad (112)$$

Remembering our condition for ϕ, it goes from
$0\ to\ \pi\ and\ back\ to\ 0\ when\ moving\ from\ A\ to\ B$, thus having a **range of 2π**
Therefore, to travel from A to B, the time is given by: -

$$T_{AB} = \frac{2\pi}{2}\sqrt{\frac{R^2-{r_o}^2}{gR}} \quad (113)$$

$$T_{AB} = \frac{\pi}{\sqrt{gR}}\sqrt{R^2 - {r_o}^2} \tag{114}$$

$$T_{AB} = \frac{\pi}{\sqrt{gR}}R\sqrt{1 - \frac{{r_o}^2}{R^2}} \tag{115}$$

$$T_{AB} = \pi\sqrt{\frac{R}{g}}\sqrt{1 - \frac{{r_o}^2}{R^2}} \tag{116}$$

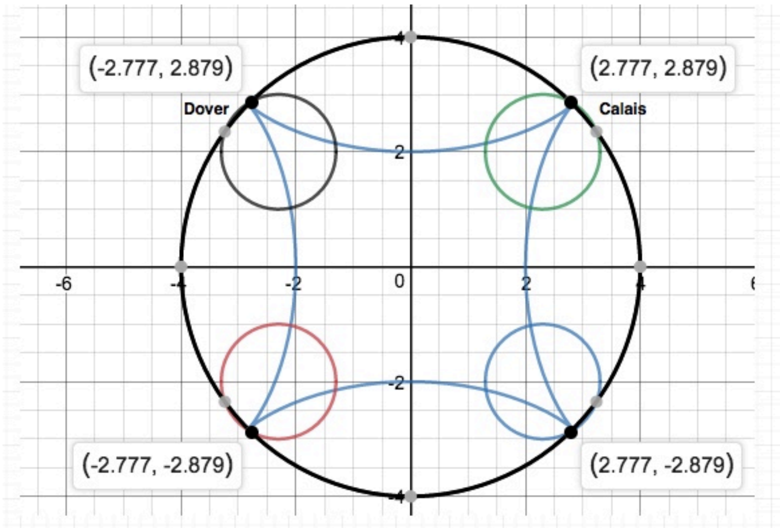


Figure 1.9

The above graph was made using *Desmos* graphing software, and shows the possible (hypocycloid) path of an object between Calais and Folkestone. The distance between Calais and Folkestone on the surface of the Earth is equal to the circumference of the smaller circle. This is because the arbitrary point on the smaller circle that traces the hypocycloid path undergoes one single full rotation to travel from Folkestone to Calais. If the radius of the small circle is a, and the surface distance between Folkestone and Calais is about 50km. The radius of the Earth is 6400km[10].
Then,

$$2\pi(a) = 50 \tag{117}$$

$$a = \frac{50}{2\pi} = 7.95774715459km \tag{118}$$

From figure 1.8, we can gather that the r_o is given by: -

$$r_o = R - 2a \tag{119}$$

$$r_o = 6400 - 15.9154943092 = 6384.08450569km \tag{120}$$

$$T\ (in\ seconds) = \pi\sqrt{\frac{6400}{9.81}}\sqrt{1 - \left(\frac{6384.08450569}{6400}\right)^2} \tag{121}$$

Therefore, the time taken to travel from Folkestone to Calais or vice versa is: -

$$T = 5.65\ minutes$$

[10] Editor, Tim Sharp Reference. “How Big Is Earth?” Space.com, www.space.com/17638-how-big-is-earth.html.

Conclusion

From this paper, we can draw 3 conclusions: -

1. The fastest path to travel between two points is that which is given by a hypocycloid.
2. The time taken to travel from one point on the Earth to another, through a straight tunnel, is 42.2 minutes.
3. The time taken to travel from one point on the Earth to another, through a Brachistochrone tunnel, is 5.65 minutes.

Evaluation

Although the idea of being able to travel from Folkestone to Calais within 6 minutes is very intriguing, it is impossible for a few reasons: -

1. Geographical problems in building tunnels of an adequate depth to suit the Brachistochrone shape. The presence of a water body, and the resultant water pressures make it very difficult to construct a tunnel deep enough.
2. In real life objects face friction. A locomotive moving between the two countries will definitely face friction, which is a resistive force that works against the direction of motion of the train. Hence, the speed of the train will be lesser than this theoretical model assumes, and the time taken will be conversely greater.
3. Tunnels cost a lot of capital to construct. The Increased length of the Brachistochrone tunnel is bad for the commercial value of the tunnel, as it requires more materials to construct the tunnel.
4. Another limitation of this model is that it ignores the difference in elevation of Folkestone and Calais. Folkestone is at a greater height above sea level than Calais, and this makes a difference in the travel time between the two places. However, the elevation difference is very little, and hence for the sake of this exploration, we assume this difference to be negligible.

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Appendix – Straight tunnel mechanism

If we take 2 arbitrary points on the surface of the earth, A and B, as the initial and final points of a journey respectively, then we can draw a diagram like the one below, in figure 1.6. Y is the midpoint between A and B. R is the radius of the Earth. Point X is a moving point that moves from B to A, with a varying distance to the center of the Earth (O). The shortest distance between point X and the center O, assuming X travels in a straight line from A to B, is given by r_0.

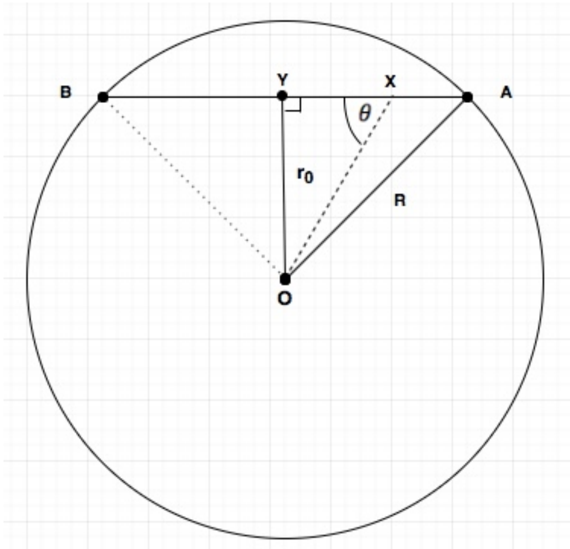

Figure 2.1

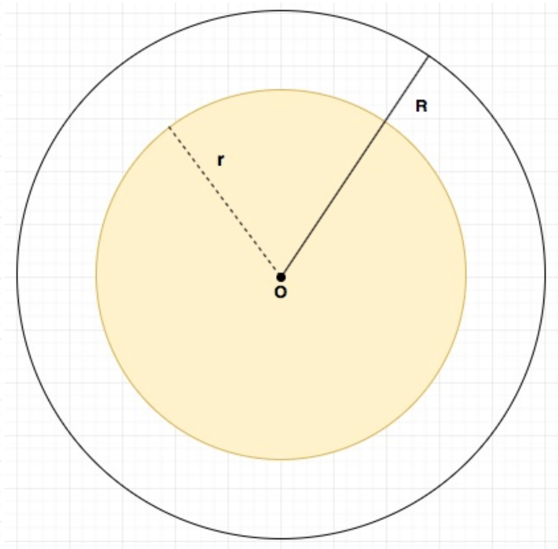

Figure 2.2

According to Newton's law of universal gravitation, the formula for the force between two masses, due to gravity, is given by: -

$$F = \frac{GMm}{R^2} \qquad (122)$$

where M and m are the two masses, and R is the distance between them.
In the case of an object on the surface of the Earth, the force exerted on it by the Earth is given by: -

$$F = \frac{GM_{Earth}m}{{R_{Earth}}^2} \; (123)$$

where G is the gravitational constant, given by 6.67 x 10^{-11} Nm^2/kg^2, M_{Earth} is the mass of the earth, m is the mass of the object (the locomotive in this case), and R is the radius of the Earth.
From Newton's shell theorem[11], it is known that for any object or mass inside the Earth, the gravitational force is provided solely by the mass present within the inner shell whose circumference the mass/object lies on. The inner shell is shown in **Figure 1.7**, with a radius of r. Hence, the gravitational object is a function of radius r and the mass within the shell, M_{shell}, and is given by: -

[11] Sparknotes. "Gravitation: Potential." SparkNotes, SparkNotes, www.sparknotes.com/physics/gravitation/potential/section3/.

$$F(M_{shell}, r) = \frac{GM_{shell}m}{r^2} \qquad (124)$$

We cannot know for certain the value of M_{shell}. The above function is a function of two variables. To simplify this, we can rewrite M_{Earth}, and M_{shell} as: -

$$M_{Earth} = V_{Earth}(\rho) \qquad (125)$$
$$M_{shell} = V_{shell}(\rho) \qquad (126)$$

where V_{Earth} is the volume of the Earth, where V_{shell} is the Volume of the inner shell and ρ is the density of the Earth.

Considering the Earth and its shells approximate to spherical shapes,

$$V_{Earth} = \frac{4}{3}\pi R^3 \qquad (127)$$
$$V_{shell} = \frac{4}{3}\pi r^3 \qquad (128)$$

From (127) and (128),

$$V_{shell} = V_{Earth}\left(\frac{r^3}{R^3}\right) \qquad (129)$$

From (125), (126), and (129),

$$M_{shell} = V_{Earth}\left(\frac{r^3}{R^3}\right)(\rho) \qquad (130)$$
$$M_{shell} = M_{Earth}\left(\frac{r^3}{R^3}\right) \qquad (131)$$

Therefore (124) can be rewritten as: -

$$F(r) = \frac{GM_{Earth}m}{R^3}(r) \qquad (132)$$

Since the values of G, M_{Earth} and R are **constants of known values**,

$$Let\ \frac{GM_{Earth}}{R^2} = g, where\ g = \frac{9.81m}{s^2} \qquad (133)$$
$$F(r) = m\frac{g}{R}r \qquad (134)$$

The function is now a single variable function.
From triangle OAB in **Figure 2.1**, we can isolate triangle OYX and isolate the function, and its driving component (the component of the force that drives the locomotive through the tunnel)

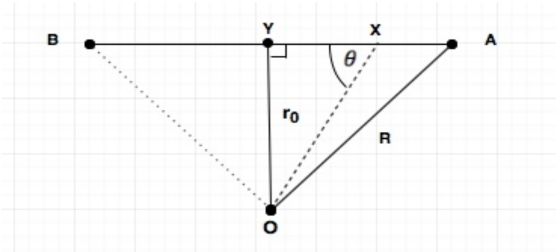

Figure 2.3

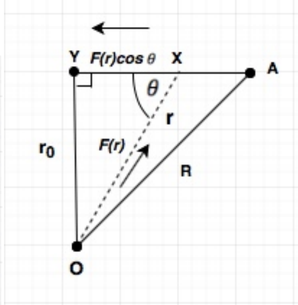

Figure 2.4

Let S be the length of AB (from figure 2.1, it is similar to the x coordinate of B – the x coordinate of A), length of YX = x,

This motion from A to B is half of an oscillation cycle of a **Simple Harmonic Oscillator**, which means that the object's displacement, velocity, and acceleration can be written as functions of cosine/sine, and their time period.

$$x = x_o \cos \omega t \qquad (135)$$

where x_o is the value of the maximum displacement of the object from the center (point Y), where $YX = YA$, t is the time period of one complete oscillation (from A to B, back to A, i.e. to cover a distance of 2S), and ω is the angular speed of the object.

$$a = \frac{d^2x}{dt^2} \qquad (136)$$
$$\frac{dx}{dt} = -\omega x_o \sin \omega t \qquad (137)$$
$$a = \frac{d^2x}{dt^2} = -\omega^2 x_o \cos \omega t \qquad (138)$$
$$a = \frac{d^2x}{dt^2} = -\omega^2 x \qquad (139)$$
$$\omega = \frac{2\pi}{t} \qquad (140)$$

From **Figure 2.4,**

$$F_{driving} = F(r) \cos \theta \qquad (141)$$
$$\cos \theta = \frac{-x}{r} \qquad (142)$$

From (134) and (141), (142) is rewritten as: -

$$F_{driving} = m\left(-\frac{g}{R}\right)x \qquad (143)$$

From Newton's second law of motion, we know that: -

$$F = ma \qquad (144)$$

From (143) and (144),

$$-\frac{g}{R}x = a \qquad (145)$$

From (139) and (145),

$$-\frac{g}{R}x = -\omega^2 x \qquad (146)$$

$$\omega = \sqrt{\frac{g}{R}} \qquad (147)$$

From (140) and (147),

$$\frac{2\pi}{t} = \sqrt{\frac{g}{R}} \qquad (148)$$

$$t = 2\pi\sqrt{\frac{R}{g}} \qquad (149)$$

t is the time taken to complete **one full oscillation (A to B and B to A)**, therefore time taken to travel from A to B, t_{AB}, only is half of time taken for one full oscillation. t_{AB} is given by: -

$$t_{AB} = \pi\sqrt{\frac{R}{g}} \qquad (150)$$

Taking $\boldsymbol{R = 6.38 \times 10^6 m}$***, and*** $\boldsymbol{g = \frac{9.81m}{s}}$***,***

$$t_{AB} = 2537.4959396878\text{s}$$

$$\boldsymbol{t_{AB} = 42.2}\ \textbf{minutes}$$

Time taken to travel through a tunnel from one point to another, with only the force of activity acting on it, is independent of mass and is equal to 42.2 minutes.