On the Moebius Trigonometric Polynomial

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Abstract

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1. A preliminary stroll

In 1937, Harold Davenport[1] was newly appointed at the University of Manchester, where Louis Mordell had also a chair. H.Davenport proved in [*Davenport*, 1937a] and among other things[2] that

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n}\{n\theta\} = -\frac{\sin 2\pi\theta}{\pi} \qquad (a.e.) \tag{1.1}$$

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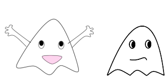
Key words and phrases:

[1]H. Davenport was a student of J.E. Littlewood. He graduated in 1927, but got his thesis from Cambridge in 1937 only.

[2]The Riemann zeta function is defined for $\Re s > 1$ by $\zeta(s) = \sum_{n\ge 1} 1/n^s$. It also verifies $\zeta(s) = \prod_{p\ge 2}\bigl(1-p^{-s}\bigr)^{-1}$ where the product is absolutely convergent in the [strong] sense of Godement and the variable p ranges over the primes. We deduce from that that $\zeta(s)^1 = \sum_{n\ge 1}\mu(n)/n^s$ for a function $\mu(n)$ named after the german mathematician Auguste Moebius who studied it more systematically in 1832 (but this function appears already in Euler in 1748 or in Gauss's Disquisitiones arithmeticae from 1801). We have $\sum_{d|n}\mu(d) = \mathbb{1}_{n=1}$, almost by definition. The fact that $\sum_{n\ge 1}\mu(n)/n$ is known to be "equivalent" to the prime number theorem, this was E. Landau's phd thesis in 1899 in fact, but the term "equivalent" may sound strange between two results that are ... true! Such a statement may be quantified and it is the purpose of [*Kienast*, 1926] and more recently of [*Ramaré*, 2013]. In terms of Dirichlet series, we have $-\zeta'(s)/\zeta(s) = \sum_{p\ge 2}\log(p)/(p^s-1)$.



for almost every θ. Giving a formal proof is easy enough: we use $\{u\}-\frac{1}{2}=-\sum_{h\geq 1}\sin(2\pi hu)/(\pi h)$, valid when h is not an integer, and get

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\{n\theta\}=\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\big(\{n\theta\}-\tfrac{1}{2}\big)=-\sum_{n\geq 1}\frac{\mu(n)}{n}\sum_{h\geq 1}\frac{\sin(2\pi nh\theta)}{\pi h}$$
$$=-\sum_{m\geq 1}\Big(\sum_{nh=m}\mu(n)\Big)\frac{\sin(2\pi m\theta)}{\pi m}=-\frac{\sin(2\pi\theta)}{\pi}.$$

but how can one actually *prove* the identity above? Let us introduce the notation (from Davenport's paper)

$$R_N(\theta)=\sum_{1\leq n\leq N}\frac{\mu(n)}{n}\{n\theta\}+\frac{\sin 2\pi\theta}{\pi}. \tag{1.2}$$

H. Davenport first notices a property that enables one to go from a single value to an average sum[1].

Lemma 1 *We have* $R_N(\theta_2)-R_N(\theta_1)\ll N|\theta_2-\theta_1|+(\log N)^{-h}$, *for arbitrary* N, θ_2, θ_1 *and* h.

We use the usual notation for the Bernoulli periodical functions $\psi_1(x)=\{x\}-\frac{1}{2}$ and $\psi_2(x)=\{x\}^2-\{x\}+\frac{1}{6}$ so that

$$2\int_0^x\psi_1(t)dt=\psi_2(x)-\tfrac{1}{6},\quad \psi_2(x)=-\sum_{k\geq 1}\frac{\cos 2\pi kx}{\pi^2k^2}.$$

We define

$$T_N(\theta)=\sum_{n>N}\frac{\mu(n)}{2n^2}\psi_2(n\theta). \tag{1.3}$$

The next Lemma reduces the problem to estimating $T_N(\theta)$.

Lemma 2

$$\int_{\theta_1}^{\theta_2}R_N(\theta)d\theta=T_N(\theta_1)-T_N(\theta_2)+\sum_{1\leq n\leq N}\frac{\mu(n)(\theta_2-\theta_1)}{2n}.$$

We can now write

$$T_N(\theta)=\sum_{k\geq 1}\frac{1}{2\pi^2k^2}\sum_{n>N}\frac{\mu(n)\cos 2\pi kn\theta}{n^2}$$

[1] A simple but tricky, tricky proof!

which should be $o(1/N)$. We will even show that is uniformly $o(1/N)$, thus showing that (1.1) holds everywhere with a uniform convergence. The proof rests on the trigonometric polynomial

$$U(\alpha) = \sum_{n\le X} \mu(n)e(n\alpha). \tag{1.4}$$

and on the following Theorem.

Theorem 3 *For any constant $A \ge 1$, we have* $\max_{\alpha\in[0,1]} |U(\alpha)| \ll_A X/(\log X)^A$.

[*Davenport*, 1937b] proved that immediately after the breakthrough of [*Vinogradov*, 1937]. His proof shows clearly that he had understood fully the main points in Vinogradov's method. It was however indirect, and we propose now a direct method.

2. Expectations

In [*Hajela & Smith*, 1987], the authors prove among other things that, *under the Generalized Riemann Hypothesis*, we have

$$\max_{\alpha\in[0,1]} |U(\alpha)| \ll_\varepsilon X^{\frac{5}{6}+\varepsilon} \tag{2.1}$$

and, together with M. Huxley, we have most probably proved that

$$\max_{\alpha\in[0,1]} |U(\alpha)| \ll_\varepsilon X^{\frac{3}{4}+\varepsilon}. \tag{2.2}$$

What can be said about lower bounds? What are possible heuristics? The L^2-average immediately gives us[1]

$$\max_{\alpha\in[0,1]} |U(\alpha)| \ge \Big(\frac{\sqrt{6}}{\pi} - \varepsilon\Big)X. \tag{2.3}$$

Already for $\alpha = 0$, we expect that (see [*Ng*, 2004])

$$U(0) = \Omega_\pm\big(\sqrt{X}(\log\log\log X)^{5/4}\big). \tag{2.4}$$

but the best bound proved is due to [*Odlyzko & te Riele*, 1985] and says that $\limsup_{X\to\infty} |U(0)|/\sqrt{X} \ge 1.06$. A proper comparison is surely given by the polynomial $\sum_{n\le N} \varepsilon_n e(n\theta)$ where ε_n are independant random variables taking

[1] Can one improve on this bound, even simply increase the constant for every X large enough?

the values ± 1 with equal probability. [*Halász*, 1973], following [*Salem & Zygmund*, 1954], shows that in such circumstances, the maximum is asymptotic to $\sqrt{X \log X}$. This constitutes ground for the conjecture:

$$\max_{\alpha \in [0,1]} |U(\alpha)| \overset{?}{\ll} \sqrt{X \log X}$$

The litterature on this trigonometric polynomial is not very dense, and we mention [*Codecà & Nair*, 2000, Theorem 3], [*Hajela & Smith*, 1987, Theorem 2.1] and [*Iwaniec & Kowalski*, 2004, Theorem 13.9]. The L^1-mean has also been considered in [*Balog & Ruzsa*, 1999] (the corresponding problem on the primes has received a much more satisfactory answer in [*Vaughan*, 1988]). There has been recently some more interest for this problem [*Bourgain* et al., 2012].

3. Splitting the circle

We partition the circle $\mathbb{R}/\mathbb{Z}$ in neighborhoods of rational points by appealing to the Dirichlet Theorem. Every $\alpha \in \mathbb{R}/\mathbb{Z}$ can be written in the following form:

$$\alpha = \frac{a}{q} + \beta, \quad q \le Q,\ (a,q) = 1,\ |\beta| \le \frac{1}{qQ} \tag{3.1}$$

where $Q = X/(\log X)^{2A+7}$, the constant A being the one from Theorem 3. The result may be uniform, the proof is *not* uniform! There are three main times:

- **The arithmetical time**. By arithmetical here, we mean information on the distribution of the Moebius function when the variable belongs to some arithmetic progression.
- **The Fourier analysis time**. This will extend the information obtained via arithmetic to some neighbourhood.
- **The combinatorial time**. The previous part took take of the *first harmonics*, in a vague sense, and we have to control the *noise*. This is the main novelty.

4. The arithmetical time

The treatment of the $q \le (\log X)^{2A+7}$ uses the zero-free region of the Dirichlet L-functions and a classical analysis shows the following. Let C_1 and C_2 be two positive constants. For every Dirichlet (not necessarily primitive) character χ of modulus $q \le (\log X)^{C_1}$, we have

$$\sum_{n \le X} \mu(n)\chi(n) \ll_{C_1,C_2} X/(\log X)^{C_2}. \tag{4.1}$$

We infer the following.

Lemma 4 *For every positive constants C_1 and C_3, for every b and every $q \leq (\log X)^{C_1}$, we have*

$$\sum_{\substack{n \leq X, \\ (n,q)=1}} \mu(n) e(nb/q) \ll_{C_1,C_3} X/(\log X)^{C_3}. \tag{4.2}$$

We next remove the condition $(n, q) = 1$ that appears in (4.2).

Lemma 5 *We have, for any $C_1, C_4 \geq 1$ and every $q \leq (\log X)^{C_1}$,*

$$\sum_{n \leq X} \mu(n) e(na/q) \ll_{c_1,C_4} X/(\log X)^{C_4}. \tag{4.3}$$

We will select $C_4 = 3A + 7$.

5. The Fourier analysis time

The next step is to extend the estimate of Lemma 5 to nearby values. The manner to do so is classical within the framework of the circle method and one should retain the principle: if we know $\sum_{n \leq Y} h(n)$ not only in $Y = X$, but also for nearby values of Y (here between $X/(\log X)^C$ and X), then we can extend the bound to $\sum_{n \leq Y} h(n) e(\beta n)$ for $\beta = 0$ to small βs.

Lemma 6 *We have, for $A, C \geq 1$ and $q \leq (\log X)^{2A+7}$,*

$$\sum_{n \leq X} \mu(n) e\left(n\left(\frac{a}{q} + \beta\right)\right) \ll X/(\log X)^A. \tag{5.1}$$

provided that $|q\beta| \leq 1/Q$.

6. The combinatorial time

Here is the Theorem we will need.

Theorem 7 [a] *Let α be a point of $\mathbb{R}/\mathbb{Z}$. We assume given two positive real parameters $X \geq Q \geq 2$ and a rational number a/q such that $|q\alpha - a| \leq 1/Q$ and $q \leq Q$. We have*

$$\Big|\sum_{n\leq X} \mu(n)e(n\alpha)\Big| \leq \frac{30\,X}{\sqrt{\min(q, X/q, \sqrt{Q})}}(1+\log X)^{7/2}.$$

[a] It improves on [*Codecà & Nair*, 2000, Theorem 3], [*Hajela & Smith*, 1987, Theorem 2.1] and [*Iwaniec & Kowalski*, 2004, Theorem 13.9].

The identity (‡) used here is a simplified version of the one proven in [*Ramaré*, 2013].

The parameter X from the Theorem is large and given. We select another parameter, z, that we will choose below in choice (6.12). Here the hypothesis we make:

$$z \leq q \leq X/z \quad , \quad z^2 \leq Q/2 \quad \text{and} \quad X/z \leq Q/4. \tag{6.1}$$

We however keep this parameter it unspecified and optimize it at the end. A possible value is $z = X^{1/5}$ since Q will be typically of size $X^{1-\varepsilon}$. We use here the following function:

$$\mu_z(d) = \begin{cases} \mu(d) & \text{when } d \leq z, \\ 0 & \text{else.} \end{cases} \tag{6.2}$$

We also specify a notation for its Dirichlet series:

$$M_z(s) = \sum_{d\leq z} \frac{\mu(d)}{d^s}. \tag{6.3}$$

6.1 An identity

The most conceptual proof runs as follows. We define

$$V(s) = \sum_{n\geq 2}\Big(\sum_{d|n}\mu_z(d)\Big)/n^s = \sum_{n\geq 2} v(n)/n^s. \tag{6.4}$$

Notice that n ranges over the integers that are greater than z. This series almost factors:

$$1 + V(s) = \zeta(s)M(s) \tag{6.5}$$

Here is the formal decomposition of 1 that we will use:

$$1 = -V + \zeta M_z.$$

On using this, we decompose $1/\zeta$ as follow:

$$\frac{1}{\zeta} = -\frac{1}{\zeta}V + M_z.$$

The work is almost over but we still have to refine the decomposition and put $(\zeta^{-1} - M_z)$ in the first factor. This is easily done

$$\frac{1}{\zeta} = -\Big(\frac{1}{\zeta} - M_z\Big)V - M_z V + M_z$$

and finally, by using again $V = M_z\zeta - 1$,

$$\frac{1}{\zeta} = -\Big(\frac{1}{\zeta} - M_z\Big)V - M_z^2\zeta + 2M_z. \tag{6.6}$$

We can translate this identity on the Moebius function by equating the coefficients of the corresponding Dirichlet series and get[1]:

((‡) $$\mu = -(\mu - \mu_z) \star v - \mu_z \star \mu_z \star \mathbb{1} + 2\mu_z.$$

This identity enables us to write

$$\sum_{n\le X} \mu(n)e(n\alpha) = -\sum_{n\le X} \big((\mu - \mu_z) \star v\big)(n)e(n\alpha) + \sum_{n\le X} (\mu_z \star \mu_z \star \mathbb{1})(n)e(n\alpha) + 2\sum_{n\le z} \mu(n)e(n\alpha). \tag{6.7}$$

The combinatorial input ends here and we now study each sum independently.

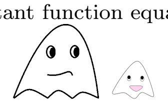
The first sum is said to be *bilinear*, a name due to the way it is studied, or *of type II* by Vinogradov; the second one is said *linear*, or *of type I* by Vinogradov, while the third one is simply a remainder term, here bounded in absolute value by $2z$. The function $n \mapsto (\mu-\mu_z)*v(n)$ is fully unknown, but we are going to use only its shape as a convolution product over localized divisors of not-too-large functions.

6.2 Treating the linear sum

Lemma 8

$$\Big|\sum_{d\le z^2} (\mu_z \star \mu_z)(d) \sum_{m\le X/d} e(dm\alpha)\Big| \le \frac{2X}{\sqrt{q}}(1 + 2\log z)^2 + z^2 q.$$

[1]The reader could have also simply developped $(\mathbb{1}\star\mu_z - e)\star(\mu-\mu_z)$ where e is the neutral element of the arithmetical convolution product: this function takes value 1 at $n = 1$ and 0 everywhere else (its Dirichlet series is simply the constant function equal to 1).



Proof. We start with

$$\sum_{n\le X}(\mu_z\star\mu_z\star\mathbb{1})(n)e(n\alpha)=\sum_{d\le z^2}(\mu_z\star\mu_z)(d)\sum_{m\le X/d}e(dm\alpha).$$

Since $z^2\le Q/2$ by (6.1), we have $|d\alpha-\frac{da}{q}|\le\frac{1}{2q}$. When $q\nmid d$, we simply sum the inner geometric progression and get

$$\Big|\sum_{m\le X/d}e(dm\alpha)\Big|\le 1/|\sin(\pi d\alpha)|\le 1/\|da/q\|\le q \tag{6.8}$$

where $\|\alpha\|$ is the distance to the closest integer and where we have used the classical inequality

$$\frac{1}{\sin(\pi x)}\le\frac{1}{2x}\quad(0<x\le 1/2).$$

We continue with $\sum_{d\le z^2}|(\mu_z\star\mu_z)(d)|\le z^2$. This finally amount to the contribution:

$$\Bigg|\sum_{\substack{d\le z^2,\\ q\nmid d}}(\mu_z\star\mu_z)(d)\sum_{m\le X/d}e(dm\alpha)\Bigg|\le z^2q. \tag{6.9}$$

When $q|d$, we bound trivially $e(dm\alpha)$ by 1, i.e. we use

$$\Big|\sum_{m\le X/d}e(dm\alpha)\Big|\le X/d.$$

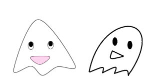
Hence[1]

$$\Bigg|\sum_{\substack{d\le z^2,\\ q|d}}(\mu_z\star\mu_z)(d)\sum_{m\le X/d}e(dm\alpha)\Bigg|\le X\sum_{\substack{d\le z^2,\\ q|d}}\frac{\tau(d)}{d}.$$

We use the simple inequalities $\tau(q\ell)\le\tau(q)\tau(\ell)$ and $\tau(q)\le 2\sqrt{q}$ together with

$$\sum_{\substack{d\le z^2,\\ q|d}}\frac{\tau(d)}{d}\le\frac{\tau(q)}{q}\sum_{\ell\le z^2/q}\frac{\tau(\ell)}{\ell}\le\frac{\tau(q)}{q}(1+2\log z)^2$$
$$\le\frac{2}{\sqrt{q}}(1+2\log z)^2$$

[1] The number $\tau(d)$ is the number of (positive) divisors of d.



since we readily check that

$$\sum_{\ell \le L} \frac{\tau(\ell)}{\ell} \le \left(\sum_{m \le L} \frac{1}{m}\right)^2 \le (1 + \log L)^2. \tag{6.10}$$

Note that we may have $z^2/q < 1$ above, but the bound $\sum_{\ell \le z^2/q} \frac{\tau(\ell)}{\ell} \le (1 + 2\log z)^2$ holds nonetheless. All of that amounts to

$$\left| \sum_{\substack{d \le z^2, \\ q|d}} (\mu_z \star \mu_z)(d) \sum_{m \le X/d} e(dm\alpha) \right| \le \frac{2X}{\sqrt{q}} (1 + 2\log z)^2. \tag{6.11}$$

We combine (6.9) and (6.11) to conclude. □

6.3 Treating the bilinear sum

The previous part do not require anything in particular and could have been done well before Vinogradov's time. What this latter brought is (one) the recognition of the bilinear sums and (two) the way they can be treated. Vinogradov explains that fully in the introduction of his book [*Vinogradov*, 2004]. He will further use the very same technique to study exponential sums with polynomial phases. Said in simplistic terms, the idea is to use the Cauchy inequality, which has the effect of creating a smooth variable over which we can sum a geometric series. To be able to employ the Cauchy inequality with a minimum amount of loss, we first have to have sums of similar lengths, and since this length is ruled by X/m in the left-hand part of (6.13), we need m to be about of constant size. The proof has three steps:

- **A diadic decomposition**.
- **Using Cauchy's inequality**.
- **Treating the diagonal and off-diagonal terms separately**.

Here is the fundamental result we prove.

Lemma 9
$$\sum_{n \le X} \mu(n) e(n\alpha) \le 15 \frac{X}{\sqrt{z}} (1 + \log X)^{7/2}.$$

Theorem 3 follows by chosing

$$z = \min(q, X/q, \sqrt{Q})/4 \tag{6.12}$$

which indeed verifies (6.1).

Proof.

Diadic decomposition

We readily show that[1]

$$
\left|\sum_{\substack{dm\le X,\\ d>z}}\mu(d)v(m)e(dm\alpha)\right|
\le 3(\log X)\max_{\substack{z\le M\le X/z,\\ M\le M'\le\min(2M,X/z)}}\left|\sum_{\substack{dm\le X,\\ M\le m\le M',\\ d>z}}\mu(d)v(m)e(dm\alpha)\right|. \tag{6.13}
$$

Or, when expressed in a shorter way: if we are ready to loose a factor $\log X$, we can localize the variable m. We could also localize the variable d, by starting at the very beginning with the sum $\sum_{X/2<n\le X}\mu(n)e(n\alpha)$. This is not required in the simplistic treatment we propose here.

Using the Cauchy inequality

We first sum on the m-variable and apply the Cauchy inequality, i.e. we write

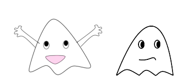
$$
\left|\sum_{\substack{dm\le X,\\ M\le m\le M',\\ d>z}}\mu(d)v(m)e(dm\alpha)\right|^2 \le \sum_{M\le m\le M'}|v(m)|^2
\times\sum_{z<d_1,d_2\le\frac{X}{M}}\mu(d_1)\mu(d_2)\sum_{m\in I(d_1,d_2)}e((d_1-d_2)m\alpha).
$$

where $I(d_1,d_2)$ is the interval $[M,\min(M',X/\max(d_1,d_2))]$. This interval contains at most $M+1$ integer points. The announced phenomenom occurs: in the second sum, the m-variable is now *smooth*, by which we mean that its coefficient does not have any arithmetical part anymore (and does not vary abnormally fast). We are left with a geometric series[2].

[1] Since $1+(\log X)/\log 2\le 3\log X$ when $X\ge 2$.

[2] Here is a parenthesis concerning $\sum_{M\le m\le M'}|v(m)|^2$. We have (on following a path similar to (6.10) since $|v(m)|\le\tau(m)$)

$$
\begin{aligned}
\sum_{M\le m\le M'}|v(m)|^2 &\le M'\sum_{m\le M'}\frac{|v(m)|^2}{m}\le M'\sum_{m\le M'}\frac{\tau(m)^2}{m}\\
&\le M'\sum_{d_1,d_2\le M'}\frac{\tau(d_1d_2)}{d_1d_2}\le M'\sum_{d_1,d_2\le M'}\frac{\tau(d_1)\tau(d_2)}{d_1d_2}\\
&\le M'\left(\sum_{d_1\le M'}\frac{\tau(d_1)}{d_1}\right)^2
\end{aligned}
$$



We again use (compare with (6.8)), when $q \nmid d_1 - d_2$ and $X/z \leq Q/4$ (which is guaranted by (6.1)):

$$\Bigl|\sum_{m\in I(d_1,d_2)} e((d_1-d_2)m\alpha)\Bigr| \leq 1/|\sin(\pi(d_1-d_2)\alpha)| \leq 1/\|(d_1-d_2)a/q\|.$$

On introducing the variable $r = d_1 - d_2$ we infer that

$$\Biggl|\sum_{\substack{dm\leq X,\\ M\leq m\leq M'\\ d>z}} \mu(d)v(m)e(dm\alpha)\Biggr|^2 \leq 2M(1+\log X)^4$$
$$\times\Biggl(\sum_{\substack{d_1,d_2\leq \frac{X}{M},\\ d_1\equiv d_2[q]}}(M+1) + \frac{X}{M}\sum_{\substack{-X/M\leq r\leq X/M,\\ r\not\equiv 0[q]}} 1/\|ra/q\|\Biggr).$$

The first summation is bounded above by

$$(M+1)\sum_{d_1\leq \frac{X}{M}}\sum_{\substack{d_2\leq \frac{X}{M},\\ d_1\equiv d_2[q]}} 1 \leq (M+1)\sum_{d_1\leq\frac{X}{M}}\biggl(\frac{X}{Mq}+1\biggr) \leq 2X\biggl(\frac{X}{Mq}+1\biggr).$$

We handle the second sum by splitting the interval $[1, X/M]$ in intervals J of length at most q. On each of these intervals we see that[1]

$$\sum_{\substack{r\in J,\\ r\not\equiv 0[q]}}\frac{1}{\|ra/q\|} \leq 2q(1+\log(q/2)) \leq 2q\log(2q).$$

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which implies that

$$\sum_{M\leq m\leq M'}|v(m)|^2 \leq M'\biggl(\sum_{m\leq M'}\frac{1}{m}\biggr)^4 \leq M'(1+\log M')^4.$$

A more careful study would show that the left-hand side is in fact $\ll M(\log X)^3$, but the loss a $\log X$ is here of no consequence.

[1]Indeed, a is prime to q, and this implies that the function $r \mapsto (ar + q\mathbb{Z})$ covers at most once each point of $\mathbb{Z}/q\mathbb{Z}$, with the exception of the point 0 which is not reached. When $ar+q\mathbb{Z}$ is equal to some $s + q\mathbb{Z}$ with $1 \leq s \leq q/2$, we use $\|ar/q\| = s/q$. Else, $ar + q\mathbb{Z}$ is equel to some $q - s + q\mathbb{Z}$ where again $1 \leq s \leq q/2$ and $\|ar/q\| = s/q$. The sum of the $1/\|ar/q\|$ for r ranging J but not divisible by q is thus bounded above by $2\sum_{1\leq s\leq q/2} q/s$, hence our claim.

Since there are at most $\frac{X}{Mq}+1$ such intervals J, we have reached

$$
\left|\sum_{\substack{dm\le X,\\ M\le m\le M'\\ d>z}}\mu(d)v(m)e(dm\alpha)\right|^2 \tag{6.14}
$$
$$
\le 2M(1+\log X)^4\left(\frac{2X^2}{Mq}+2X+2\frac{X}{M}\left(\frac{X}{Mq}+1\right)q\log(2q)\right)
$$
$$
\le 4\left(\frac{X^2}{q}+XM+\frac{X^2}{M}+Xq\right)(1+\log X)^4\log(2q)\le 16\frac{X^2}{z}(1+\log X)^5.
$$

On carrying (6.14) into (6.13) and adding the rseult of Lemma 8, we see that we have proved that[1]

□

7. Proof of some Lemmas

Proof. [Proof of Lemma 2] We first notice that

$$
\sum_{1\le n\le N}\frac{\mu(n)}{n}\int_{\theta_1}^{\theta_2}\psi_1(n\theta)d\theta=\sum_{1\le n\le N}\frac{\mu(n)}{2n^2}(\psi_2(n\theta_2)-\psi_2(n\theta_1)).
$$

The second step reads

$$
\sum_{n\ge 1}\frac{\mu(n)}{n^2}\psi_2(n\theta)=\sum_{k,n\ge 1}\frac{\mu(n)}{\pi^2(nk)^2}\cos(2\pi kn\theta)=\frac{\cos 2\pi\theta}{\pi^2}
$$

from which we infer that

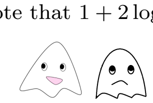
$$
\sum_{n\le N}\frac{\mu(n)}{2n^2}\psi_2(n\theta)-\frac{\cos 2\pi\theta}{2\pi^2}=-\sum_{n>N}\frac{\mu(n)}{2n^2}\psi_2(n\theta)=-T_N(\theta),
$$

say. We now resume the main line of the proof:

$$
\begin{aligned}
\int_{\theta_1}^{\theta_2}R_N(\theta)d\theta &= \sum_{1\le n\le N}\frac{\mu(n)}{2n^2}(\psi_2(n\theta_2)-\psi_2(n\theta_1))+\sum_{1\le n\le N}\frac{\mu(n)(\theta_2-\theta_1)}{2n}\\
&\qquad -\frac{\cos 2\pi\theta_2}{2\pi^2}+\frac{\cos 2\pi\theta_1}{2\pi^2}\\
&= T_N(\theta_1)-T_N(\theta_2)+\sum_{1\le n\le N}\frac{\mu(n)(\theta_2-\theta_1)}{2n}
\end{aligned}
$$

as announced. The Lemma follows readily. □

[1] Note that $1+2\log z\le 1+\log X$.



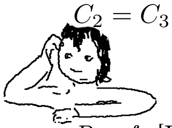
Proof. [Proof of Lemma 4] We can assume $(b, q) = 1$ by removing the common factor. We then note that

$$f_{b/q} : n \mapsto \begin{cases} e(na/q) & \text{when } (n, q) = 1, \\ 0 & \text{else}, \end{cases} \tag{7.1}$$

is periodical of period q and of support the multiplicatif group of $\mathbb{Z}/q\mathbb{Z}$. This function can be written as a linear combination of Dirichlet characters modulo q, and this combination is easy to determine thanks to the hermitian structure. We get

$$\begin{aligned} f_{b/q} &= \sum_{\chi \mod q} \frac{1}{\varphi(q)} \sum_{\substack{n \mod q, \\ (n,q)=1}} e(nb/q)\overline{\chi(n)}\chi \\ &= \sum_{\chi \mod q} \frac{1}{\varphi(q)} \chi(b) \sum_{\substack{n \mod q, \\ (n,q)=1}} e(n/q)\overline{\chi(n)}\chi, \end{aligned}$$

which shows that the coefficients are of absolute value not more than $\sqrt{q}/\phi(q)$ (use the value of the modulus of the Gauss sum). We conclude by selecting $C_2 = C_3 + \frac{1}{2}C_1$ in (4.1). □



Proof. [Proof of Lemma 5] We write

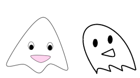
$$\begin{aligned} \sum_{n \le X} \mu(n) e(na/q) &= \sum_{\delta | q} \sum_{\substack{n \le X, \\ (n,q)=\delta}} \mu(n) e(na/q) \\ &= \sum_{\delta | q} \mu(\delta) \sum_{\substack{m \le X/\delta, \\ (m,q)=1}} \mu(m) e(m\delta a/q) \end{aligned}$$

where we can now use (4.2), since we do not assume $(a, q) = 1$ therein, with the parameters C_1 and $C_3 = C_4 + 1$. We get

$$\sum_{n \le X} \mu(n) e(na/q) \ll \sum_{\delta | q} \frac{X}{\delta (\log X)^{C_3}} \ll \frac{X}{(\log X)^{C_4}}$$

as soon as $q \le (\log X)^{C_1}$. □

Proof. [Proof of Lemma 6] The last step is to extend the bound in a/q to the neighbourhood. We use the shortcut $h(n) = \mu(n)e(na/q)$. We use the inequality



$|\beta| \le 1/Q$. We get, by using Step number 4 with $C_4 = 3A + 7$,

$$\begin{aligned}\sum_{n\le X} h(n)e(n\beta) &= \sum_{n\le X} h(n)\Big(e(X\beta) - 2i\pi\beta\int_n^X e(t\beta)dt\Big)\\ &= e(X\beta)\sum_{n\le X} h(n) - 2i\pi\beta\int_1^X e(t\beta)\sum_{n\le t} h(n)dt\\ &\ll \frac{X}{(\log X)^A} + \frac{1}{Q}\int_1^{X/(\log X)^{2A+4}} tdt + \frac{X^2}{Q(\log X)^{3A+7}} \ll \frac{X}{(\log X)^A}.\end{aligned}$$

$\square$

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