

Fibonacci numbers and trigonometric identities

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Abstract

Webb & Parberry proved in 1969 a startling trigonometric identity involving Fibonacci numbers. This identity has remained isolated up to now, despite the amount of work on related polynomials. We provide a wide generalization of this identity together with what we believe (and hope!) to be its proper understanding.

1. Introduction

Fibonacci numbers verify a wealth of identities, see e.g. [5], [6], [7], [11]. By specifying x and y to 1 in Corollary 10 of [4], we get an intriguing one which states that for $n \geq 1$:

$$F_n = \prod_{k=1}^{[(n-1)/2]} \left(1 + 4 \cos^2 \frac{k\pi}{n} \right). \quad (1)$$

Webb & Parberry's paper [14] contains already all the necessary material to write this identity, but they do not state it explicitly. This formula is indeed intriguing: the left hand side satisfies a second order recursion formula while no such recursion arises from the right hand side expression. Indeed, how could we connect $\cos \frac{2k\pi}{n}$ and $\cos \frac{2k\pi}{n+1}$? Taking a number theoretic point of view leads to more dismay: Fibonacci numbers are linked with the arithmetic of $\mathbb{Q}(\sqrt{5})$ and not with that of $\mathbb{Q}(\exp(2i\pi/n))$.

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The mystery gets somewhat lifted by the proof of the above identity: we introduce the second order Chebyshev polynomials U_n by

$$U_0(x) = 1, \quad U_1(x) = x, \quad U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x) \quad (2)$$

so that they verify

$$U_n(\cos \theta) = \frac{\sin(n+1)\theta}{\sin \theta}. \quad (3)$$

This last expression leads to the recursion above as well as to the formula

$$U_n(x) = 2^n \prod_{k=1}^n \left(x - \cos \frac{k\pi}{n+1} \right). \quad (4)$$

Since it is not difficult to discover that $F_n = i^{n-1}U_{n-1}(-i/2)$ we get the result. This factorization has been noted and studied by [14], [4], [1], [15]. The main arguments of this proof are the recurrence relation (2) and the rule of additions of sine. However this sheds no light on how the arithmetic of $\mathbb{Q}(\sqrt{5})$ and the one of $\mathbb{Q}(\exp(2i\pi/n))$ get entangled.

Elaborating on this argument, Webb & Parberry in [14] and Hoggatt & Long in [4] defined a sequence of polynomials by

$$u_0(x, y) = 0, \quad u_1(x, y) = 1, \quad u_{n+2}(x, y) = xu_{n+1}(x, y) + yu_n(x, y). \quad (5)$$

The case $y = 1$, the only one considered in [14], leads to what is sometimes called Fibonacci polynomials. A further generalization led to Morgan-Voyce polynomials in [13], [10] and [12], and more recently to Brahmagupta's polynomials in [8] and [9]. In particular, it is proved that

$$u_n(x, y) = \prod_{k=1}^n \left(x - 2i\sqrt{y} \cos \frac{k\pi}{n} \right) \quad (6)$$

and this leads to amazing identities like:

Corollary 1. *We have for $n \geq 3$*

$$\prod_{1 \leq \ell \leq n} \left(1 + 4 \sin^2 \frac{2\pi\ell}{n} \right) = (1 + F_n - 2F_{n+1} + (-1)^n)^2$$

where F_n is the n -th Fibonacci number, with $F_0 = 0$ and $F_1 = 1$.

This corollary does not appear as such in [4], but can be derived from the material presented therein. We provide a simple proof later. Again the LHS satisfies a linear recursion, which we now compute explicitly. First express

F_n in terms of ϕ and $-1/\phi$ where ϕ , the golden ratio, is the larger root of $x^2 - x - 1$ and expand the square: there comes a linear combination of terms of the form p^n , where all the p 's appearing are roots of $(x^2 - 3x + 1)(x^2 - x - 1)(x + 1)(x - 1) = x^6 - 4x^5 + 2x^4 + 6x^3 - 4x^2 - 2x + 1$ since ϕ^2 and $1/\phi^2$ are the roots of $x^2 - 3x + 1$. If we call $g(n)$ the LHS of the quantity computed in the above corollary, we have

$$g(n+6) = 2g(n+5) + 4g(n+4) - 6g(n+3) - 2g(n+2) + 4g(n+1) - g(n) \quad (7)$$

since each sequence (p^n) satisfies it.

The next question is whether one can obtain such identities with three homogeneous variables x , y and z . One path to such a generalization has been to study the dynamics of the zeros of Fibonacci polynomials in [3], getting interesting by-products but no trigonometric identity.

2. Results and proofs

We prove here such a trigonometric identity. Let us start with a simple case.

Theorem 1. *We have for $n \geq 3$ and $\xi = \exp(2i\pi/n)$*

$$\prod_{1 \leq \ell \leq n} (1 - x\xi^\ell - y\xi^{2\ell}) = 1 - G_n + (-y)^n$$

where G_n is defined by the recursion $G_n = xG_{n-1} + yG_{n-2}$ for $n \geq 2$ with $G_0 = 2$, $G_1 = x$ and $G_2 = x^2 + 2y$.

The polynomials G_n are a special case of Brahmagupta polynomials (take $t = 1$) introduced and studied in [8], [9]. These are linked with Morgan-Voyce polynomials, see [13] and are the Fibonacci polynomials when we further specialize y to be equal to 1.

At this level, our proof was a mystery to us, we did in fact stumble on it while studying a completely different problem. It was from then onwards tempting to prove such trigonometric relations with r terms instead of 3. This led us to a very simple and direct proof of a general relation, via considerations of circulant matrices. Most earlier connections between determinants and Fibonacci numbers, or Chebyshev polynomials involved continuants: determinant of a family of increasing size tridiagonal matrices whose first elements are properly chosen while the later ones are fixed. See [1], [2], see also exercise 24 page 85 of [5].

And while studying this proof, we discovered that the heart of such identities was a hundred years old one liner:

Fundamental Lemma. Let $(x_s)_{0 \leq s \leq r}$ be complex numbers and set $\xi = \exp(2i\pi/n)$. We have

$$\prod_{0 \leq \ell \leq n-1} \sum_{0 \leq s \leq r} x_s \xi^{s\ell} = (-1)^{rn} x_r^n \prod_{\rho/P(\rho)=0} (\rho^n - 1)$$

where $P(Y) = \sum_{s=0}^r x_s Y^s$.

Simply because the quantity to be computed is up to the sign the resultant of $P(Y)$ and $Q_n(Y) = Y^n - 1$ for which we have

$$\text{Res}(P, Q_n) = (-1)^{rn} \prod_{u/Q_n(u)=0} P(u) = x_r^n \prod_{\rho/P(\rho)=0} Q_n(\rho). \quad (8)$$

So, on one side, we have the roots of one polynomial, while on the other one we have the roots of another polynomial. This identity being so fundamental, we recall its one line proof: consider two polynomials A and B with respective leading terms a_m and b_n , degrees m and n and roots $(\alpha_i)_{1 \leq i \leq m}$ and $(\beta_j)_{1 \leq j \leq n}$ repeated with multiplicity. Then, and it can be taken as a definition

$$\text{Res}(A, B) = a_m^n b_n^m \prod_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} (\alpha_i - \beta_j), \quad (9)$$

from which (8) follows trivially. There is an expression of this resultant as a (Sylvester) determinant whose entries are the coefficients of A and B , as well as some 0's, but we will not invoke such an expression.

Let us use our fundamental lemma on an example. Taking $P = Y^2 - Y - 1$, with roots the golden ratio $\phi = (1 + \sqrt{5})/2$ and $-1/\phi$ we get

$$\begin{aligned} \prod_{0 \leq \ell \leq n-1} \left(1 + 2i \sin \frac{2\pi\ell}{n} \right) &= 1 - \phi^n - (-\phi)^{-n} + (-1)^n \\ &= 1 + F_n - 2F_{n+1} + (-1)^n \end{aligned} \quad (10)$$

since $\phi^n = \phi^{-1}F_n + F_{n+1}$ and $(-\phi)^{-n} = -\phi F_n + F_{n+1}$, yielding an illuminating proof of Corollary 1!!

We saw in this example how linear recurrent sequences can be introduced and we now address the general case treated in the Theorem. The adaptation is straightforward. To avoid denominators, we set $P(Y) = Y^2 - xY - y$ with roots ρ_1 and ρ_2 in $\mathbb{C}(x, y)$. By our fundamental lemma, we have

$$\begin{aligned} \prod_{1 \leq \ell \leq n} (\xi^{2\ell} - x\xi^\ell - y) &= (\rho_1\rho_2)^n - \rho_1^n - \rho_2^n + 1 \\ &= (-y)^n - \rho_1^n - \rho_2^n + 1. \end{aligned}$$

We set $G_n(x, y) = \rho_1^n + \rho_2^n$, which yields $G_0 = 2$, $G_1 = x$ and $G_{n+2} = xG_{n+1} + yG_n$. We need only to factor $\xi^{2\ell}$ out and exchange ℓ by $-\ell$ to get our Theorem!

Let us now treat the general case. Let x_0, \dots, x_r be r indeterminates. Let ρ_1, \dots, ρ_r be the r roots of $P(Y) = \sum_{0 \leq s \leq r} x_s Y^s$ in an algebraic closure of $K(x_0, \dots, x_r)$. We set

$$H_n = (-1)^{rn} x_r^n \prod_{1 \leq s \leq r} (\rho_s^n - 1) \quad (11)$$

which in fact belongs to $K[x_0, \dots, x_r]$ and

$$\mathcal{H}(X) = \sum_{n \geq 0} H_n X^n = \sum_{n \geq 0} \sum_{S \subset \{1, \dots, r\}} (-1)^{r-|S|} \prod_{s \in S} \rho_s^n ((-1)^r x_r X)^n \quad (12)$$

$$= \sum_{S \subset \{1, \dots, r\}} \frac{(-1)^{r-|S|}}{1 - (-1)^r \prod_{s \in S} \rho_s x_r X}. \quad (13)$$

This shows that $\mathcal{H}(X)$ is a rational fraction with denominator of degree at most 2^r . As a consequence H_n verifies a linear recursion of degree at most 2^r which we easily establish by writing $\mathcal{H}(X) = \mathcal{A}(X)/\mathcal{B}(X)$ with $\mathcal{A}(X)$ and $\mathcal{B}(X)$ polynomials. When equating coefficients in the equation

$$\mathcal{B}(X) \sum_{n \geq 0} H_n X^n = \mathcal{A}(X)$$

we recover an explicit form of the required recursion. Summarizing, we get

Theorem 2. *Let $(x_s)_{0 \leq s \leq r}$ be complex numbers and set $\xi = \exp(2i\pi/n)$. We have*

$$\prod_{0 \leq \ell \leq n-1} \sum_{0 \leq s \leq r} x_s \xi^{s\ell} = H_n$$

where $P(Y) = \sum_{s=0}^r x_s Y^s$ and H_n defined by (11) satisfies a linear recursion of degree at most 2^r .

Since $\mathcal{B}(X)$ is fairly universal and determine the coefficients of the recursion satisfied by H_n , it would be satisfactory to have a complete description of it solely in terms of the x_s 's. We have not been able to derive such a description. We can of course group together the products $1 - (-1)^r \prod_{s \in S} \rho_s x_r X$ over S with a fixed cardinality. As symmetric expressions of the roots, each can be expressed as a polynomial of $\mathbb{C}[x_0, x_1, \dots, x_r]$. For $|S| = 0$, we get $1 - (-1)^r x_r X$. For $|S| = 1$, we get $(-x_r X)^r P((-1)^r/(x_r X))$.

We end this part with yet another identity:

Corollary 2. *We have for $m \geq 1$*

$$\prod_{k=1}^m \left(5 + 4 \sin^2 \frac{k\pi}{2m+1} \right) = F_{4m+2}.$$

We achieve this by considering the polynomial $P = Y^2 - 3Y + 1$ with roots ϕ^2 and $(-\phi)^{-2}$. We then follow the path described above with $n = 2(2m+1)$:

$$\begin{aligned} \prod_{k=0}^{n-1} (1 - 3\xi^k + \xi^{2k}) &= (-1)^{n-1} \prod_{k=0}^{n-1} \left(-3 + 2 \cos \frac{2\pi k}{n} \right). \\ &= -5 \prod_{k=1}^{2m} \left(-3 + 2 \cos \frac{2\pi k}{n} \right)^2 \end{aligned}$$

since n is even and by using the symmetry $k \mapsto n-k$. We next use the symmetry $k \mapsto (n/2) - k$ which sends $[1 \cdots m]$ over $[m+1 \cdots 2m]$ to finally find that

$$\prod_{k=0}^{n-1} (1 - 3\xi^k + \xi^{2k}) = -5 \prod_{k=1}^m \left(9 - 4 \cos^2 \frac{2\pi k}{n} \right)^2.$$

We proceed as in (10) to find that the above equals $2 + F_{2n} - 2F_{2n+1}$. To take its squareroot, we note that (for instance by using Binet's formula)

$$2F_{2\ell+1} - F_{2\ell} - 2(-1)^\ell = 5F_\ell^2 \quad (14)$$

and the Lemma follows readily.

We can prove (1) along these lines (with m instead of m), starting with the polynomial $P = 1 + iY + Y^2$. This leads after some manipulations to the square of the right-hand side of (1). We simplify the squareroot of the left-hand side by using (14).

3. Extensions and limitations

The above approach would work if we were to replace $Y^n - 1$ by $Q(Y^n)$ for a fixed polynomial Q . It would also extend to the case when the coefficients of Q are polynomials in n . The same remark holds for the coefficients of P . In these cases, the roots do not depend on roots of unity, which means that the rule of addition of sine does not in fact intervene. We can even handle the more difficult case $Q_n = Y^{2n} + Y^{n+1} + 1$.

However, we are not able to treat the case $Y^{n^2} - 1$ or any other non sub-sequence of $Y^n - 1$ where the exponents would not be taken in an arithmetic

progression. No linear recursion in that case exists, for it would mean a linear recursion for the values H_{n^2} , and their growth is too steep to allow such a fact (once we specialize the x_s 's).

Let us end this paper with a related problem. Restricting product to indices ℓ prime to n in our Theorem, one gets the norm of $1 - x\xi - y\xi^2$ in $\mathbb{Q}(\xi)$:

Corollary 3. *We have*

$$\prod_{\substack{1 \leq \ell \leq n \\ (\ell, n)=1}} (1 - x\xi^\ell - y\xi^{2\ell}) = \prod_{d|n} (1 - G_d + (-y)^d)^{\mu(n/d)}$$

where μ denotes the Moebius function.

The question arose to decide whether this norm verifies a linear recursion as above or not. Our approach via a resultant supports a negative answer; We have

$$\prod_{\substack{1 \leq \ell \leq n \\ (\ell, n)=1}} (1 - x\xi^\ell - y\xi^{2\ell}) = (-1)^{(r-1)n} y^n \prod_{\rho/P(\rho)=0} \Phi_n(\rho)$$

where Φ_n is the n -th cyclotomic polynomial. This polynomial does not in general have a finite number of monomes (for instance when n is prime), which ruins the approach we used, but also make us believe no recursion does indeed exist.

We end this paper with thanks to the referee for his/her careful reading.

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