# An explicit density estimate for Dirichlet $L$-series

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## 1 Introduction

Dirichlet $L$-series $L(s,\chi) = \sum_{n\geq 1}\chi(n)n^{-s}$ associated to primitive Dirichlet characters $\chi$ are one of the keys to the distribution of primes. Even the simple case $\chi = 1$ which corresponds to the Riemann zeta-function contains many informations on primes and on the Farey dissection. There have been many generalizations of these notions, and they all have arithmetical properties and/or applications, see [45, 29, 33] for instance. Investigations concerning these functions range over many directions, see [14] or [43]. We note furthermore that Dirichlet characters have been the subject of numerous studies, see [2, 50, 4]; Dirichlet series in themselves are still mysterious, see [3] and [6].

One of the main problem concerns the location of the zeroes of these functions in the strip $0 < \Re s < 1$; the Generalized Riemann Hypothesis asserts that all of those are on the line $\Re s = 1/2$. We concentrate in this paper on estimating

$$N(\sigma, T, \chi) = \sum_{\substack{\rho=\beta+i\gamma, \\ L(\rho,\chi)=0, \\ \sigma\leq\beta, |\gamma|\leq T}} 1.$$

On the generalised Riemann hypothesis, this quantity vanishes when $\sigma > 1/2$ and we want to bound it from above. An upper bound is however often very powerful, one of the more striking uses of such an estimate being surely Hoheisel Theorem. In [31, Theorem 7], the authors already prove an explicit density estimates for $L$-functions, namely

$$\sum_{\chi \mod q} N(\sigma, T, \chi) \leq \left(\frac{254\,231}{\log qT} + 17\,102\right)(q^3 T^4)^{1-\sigma}(\log qT)^{6\sigma} + 16\,541(\log T)^6$$

under some size conditions on $T$ and $q$ we do not reproduce. [10] had in fact proved most of this result, but his bound had the restriction $\chi \neq \chi_0$, the principal character. This result is used in [32] to prove to show that every odd integer $\geq \exp(3\,100)$ is a sum of at most three primes.

As it turns out, I proved long ago in my M. Phil. memoir a bound in case $\chi = \chi_0$ that was better than that. This was never published but several versions circulated, at various stages of improvement. This paper will fix a version. We do so because of a regain of interest in the field (see of course [49], [25] and [26]) and more precisely [28] where these authors manage to use a density estimate from [27] to improve on the numerical bounds for the Tchebyschef-$\psi$ function. After more than fifty years of very limited theoretical progress in this field

(though there has been work on it, see [15], [16], [17], [36]), this is quite a news and announces further improvements. The second main news in this area is due to the doctoral thesis of D. Platt [38] where the Riemann hypothesis has been verified for all Dirichlet characters of conductor $q \leq Q_0 = 400\,000$ aand up to the height $10^8/q$, improving in a drastic fashion on the previous works [44] and [7]. This author has also checked that the Riemann zeta function has no non-real zero off the critical line and of height bounded in absolute value by $3 \cdot 10^{10}$ (see also [53] and [21], though these results have not been the subject of any academic publications).

Here is our main Theorem:

**Theorem 1.1.** *For $T \geq 2\,000$ and $T \geq Q \geq 10$, as well as $\sigma \geq 0.52$, we have*

$$\sum_{q \leq Q} \frac{q}{\varphi(q)} \sum_{\chi \bmod^* q} N(\sigma, T, \chi) \leq 83 \big(55 Q^5 T^3\big)^{1-\sigma} \log^{5-2\sigma}(Q^2 T) + 32\, Q^2 \log^2(Q^2 T)$$

*where $\chi \bmod^* q$ denotes a sum over all primitive Dirichlet character $\chi$ to the modulus $q$. Furthermore, we have*

$$N(\sigma, T, \mathbb{1}) \leq 2T \log\big(1 + 9.8(3T)^{(5-8\sigma)/3} \log^{5-2\sigma}(T)\big) + 103(\log T)^2.$$

Our result is asymptotically better in case $Q = 1$ than Ingham's, from which we borrow most of the proof, by almost two powers of logarithm: we get the exponent $5 - 2\sigma$ instead of the classical 5. See [51, Theorem 9.19].

In case $Q = 1$, the form we have chosen for our density estimate is unusual but numerically efficient. If a simpler form is required, we can degrade the above (via $\log(1 + x) \leq x$) into

$$N(\sigma, T, \mathbb{1}) \leq 9.7(3T)^{8(1-\sigma)/3} \log^{5-2\sigma}(T) + 103(\log T)^2. \tag{1}$$

However the form we have chosen also implies for instance that

$$\frac{N(3/4, T, \mathbb{1})}{(3T)^{2/3}(\log T)^{7/2}} \leq \begin{cases} 1/4 & \text{when } T \leq 4.5 \cdot 10^{10}, \\ 1/2 & \text{when } T \leq 3.3 \cdot 10^{12}, \\ 1 & \text{when } T \leq 2.1 \cdot 10^{14}, \\ 9.9 & \text{when } T \geq 0. \end{cases}$$

while (1) would only prove the last line.

For comparison, Chen/ Liu & Wang's result is useless here because of the exponent of $T$. We should however mention that, when comparing this estimate to the total number of zeroes, see Lemma 9.1, the above bound at $\sigma = 3/4$ is not more than 1/2 this total number (and this is required because of the symetry of the zeroes with respect to $\rho \mapsto 1 - \bar{\rho}$) only when at least $T \geq 10^{16}$. This is a really small bound in such a field. The choice $17/20 = 0.85$ seems interesting. Our result yields

$$N(17/20, T, \mathbb{1}) \leq 9.9\,(3T)^{2/5}(\log T)^{33/10} \tag{2}$$

which is this time always not more than 0.079 times the trivial bound.

Let us compare our result in case $Q = 1$ with [27].

- When $\sigma = 17/20$, [27] yields $\frac{1}{2}N(17/20, T, \mathbb{1}) \leq 0.5561\,T + 0.7586\,\log T - 268\,658$ (the factor $\frac{1}{2}$ is required: in classical notation $N(\sigma, T)$ counts the non-trivial zeros of the Riemann zeta function with abscissa between 0 and $T$ and *not* between $-T$ and $T$). The estimate (2) is nearly twice better.

- However when $\sigma = 4/5$, [27] yields $\frac{1}{2}N(4/5, T, \mathbb{1}) \leq 0.7269\,T + 0.9566\,\log T - 209\,795$ which is better than $9.5(4T)^{2/3}(\log T)^{7/2}$ when $T \leq 5.3 \cdot 10^{20}$. It is even better than the more refined bound we have given when $T \leq 4.2 \cdot 10^{12}$.

- And when $\sigma = 7/10$, [27] yields $\frac{1}{2}N(7/10, T, \mathbb{1}) \leq 1.4934\,T + 1.4609\,\log T - 136\,370$ which is smaller than our better bound at least on the range $T \leq 3.3 \cdot 10^{37}$.

Let us note here that some intermediate results are of independant interest: lemma 4.3 is a complement of [41, Lemma 3.2] for evaluating averages of non-negative multiplicative functions, corollaries 6.1 and 6.2 are sharp explicit versions of [20, Theorem 3]. Lemma 6.2 is more straightforward but is indeed a numerical refinement of [35, Corollary 3]. Lemma 5.4 has been quoted earlier in [5] but this is the first published proof (as far as I know).

## Acknowledgement

## Notation and some definitions

We follow closely Ingham's proof as given in [51], paragraph 9.16 through 9.19. We extend it to cover the case of Dirichlet characters.

We consider a real parameter $X \geq 2000$ and the following kernel that we use to "mollify" $L(s, \chi)$ (see [13] for instance)

$$M_X(s, \chi) = \sum_{n \leq X} \chi(n)/n^s. \tag{3}$$

We consider

$$\begin{cases} f_X(s, \chi) = M_X(s, \chi)L(s, \chi) - 1, \\ h_X(s, \chi) = 1 - f_X(s, \chi)^2 = L(s, \chi)M_X(s, \chi)(1 - L(s, \chi)M_X(s, \chi)), \\ g_X(s, \chi) = h_X(s, \chi)h_X(s, \overline{\chi}). \end{cases} \tag{4}$$

We observe that zeroes of $L(s, \chi)$ are zeroes of $h_X(s, \chi)$. We use here the fact that $M_X(s, \chi)$ is expected to be a partial inverse of $L(s, \chi)$, due to combinatorial properties of the Moebius function.

We use the shorthand

$$\sum_{\substack{q \leq Q, \\ \chi \bmod^* q}}^{\star} h(q, \chi) = \sum_{q \leq Q} \frac{q}{\varphi(q)} \sum_{\substack{\chi \bmod q, \\ \chi \text{ primitive}}} h(q, \chi) \tag{5}$$

for any arbitrary function $h$.

We denote by $N_1(\sigma, T, \chi)$ the zeroes $\rho$ of $h_X(s, \chi)$ in the rectangle

$$\Re\rho \geq \sigma, \quad T \geq |\Im\rho| \tag{6}$$

*to the exception of those with $\Im\rho = 0$.* They are also the zeroes of $g_X(s, \chi)$ with $T \geq \Im\rho \geq 0$ and $\Re s \geq \sigma$, counted according to multiplicities. We define furthermore $N_1(\sigma, T_1, T_2, \chi) = N_1(\sigma, T_2, \chi) - N_1(\sigma, T_1, \chi)$ as well as

$$N_1(\sigma, T_1, T_2, Q) = \sum_{\substack{q \leq Q, \\ \chi \bmod^* q}}^{\star} N_1(\sigma, T_1, T_2, \chi).$$

In the course of the proof, we shall also require

$$F_Q(\sigma, T) = \int_{-T}^{T} \sum_{\substack{q \leq Q, \\ \chi \bmod^* q}}^{\star} |f_X(\sigma + it, \chi)|^2 dt \tag{7}$$

which of course depends on the paremeter $X$ as well. The variable $t$ ranges $[-T, T]$ and we sometimes will have result where the variable $t$ ranges $[0, T]$. In such a case we will use the notation $\frac{1}{2}F_Q(\sigma, T)$, thanks to the symmetry induced by $\overline{f_X(\sigma + it, \chi)} = f_X(\sigma - it, \overline{\chi})$.

The remainder of the notation is standard, but here are some points: the arithmetical functions are the Moebius function $\mu$, the number of prime factors counted without multiplicity $\omega$, the Euler-totient function $\varphi$; the arithmetical convolution product is denoted by $\star$. The letter $\psi$ does *not* represent the Chebyschef-$\psi$ function but the digamma function, though $\vartheta$ *is* the Chebyschef $\vartheta$-function. The letter $p$ represents a prime number in summations. We use $f = \mathcal{O}^*(g)$ to say that $|f| \leq g$.

### Minimal orders of magnitude

The parameters that will decide on size are $Q$ and $T$. Most of the time, we will only require bounds on $X = Q^2 T$. When $Q = 1$, we can assume that $T \geq 3 \cdot 10^{10}$, while in general, we can assume that either $Q > Q_0 = 400\,000$ or $T > 10^8/Q$. Since in that case ($Q \neq 1$), we also assume that $Q \geq 10$, this means that we can in any case assume that $X \geq 10^9$. We also consider only the case $T \geq 2000$, which implies that $X \geq 2000Q^2$ (valid also when $Q = 1$). Note however that a parameter $T$ is often used in Lemmas, and it is *not* always subject to $T \geq 2000$. The parameter $X$ is always linked to the final choices.

## 2   On the size of $L$-functions

**Lemma 2.1.** *Let $\chi$ be a primitive character of conductor $q > 1$. For $-\frac{1}{2} \leq -\eta \leq \sigma \leq 1 + \eta \leq \frac{3}{2}$, we have*

$$|L(s, \chi)| \leq \left(\frac{q|1 + s|}{2\pi}\right)^{\frac{1}{2}(1+\eta-\sigma)} \zeta(1 + \eta)$$

See [40, Theorem 3]. In the same paper, Theorem 4 treats in passing the case $q = 1$, where the above bound for $q = 1$ simply has to be multiplied by $3|\frac{1+s}{1-s}|$. We can treat the term $\zeta(1 + \eta)$ by using the inequality (see also Lemma 5.4 below)

$$\zeta(1 + \eta) \leq \frac{1 + \eta}{\eta} \tag{8}$$

valid for $\eta > 0$. Our main application will be for $\sigma = \Re s = \frac{1}{2}$, for which we can invoke the following result of [11, Corollary to Theorem 3], modified according to [52, Section 5]:

**Lemma 2.2.** *For $0 \leq t \leq e$, we have $|\zeta(\frac{1}{2} + it)| \leq 2.657$. For $t \geq e$, we have $|\zeta(\frac{1}{2} + it)| \leq 2.4\,t^{1/6} \log t$.*

The modification in question leads to the constant 2.4 instead of the initial 3.

**Lemma 2.3.** *Let $\chi$ be a primitive character of conductor $q \geq 1$. We have (for $T \geq 4$)*
$$\max\{|L(s, \chi)|, \Re s \geq 0, |\Im s| \leq T\} \leq 4.42(qT)^{5/8}.$$

*Proof.* We use Lemma 2.1 with $\eta = 1/4$ in case $q > 1$, to get the upper bound

$$\left(\frac{q(1 + \sigma + T)}{2\pi}\right)^{\frac{1}{2}(\frac{5}{4} - \sigma)} \zeta(5/4)$$

In the quotient, worst case is $\sigma = 0$. The quantity $\zeta(5/4) \leq 4.6$ is trivially an upper bound in case $\Re s \geq 5/4$. In case $q = 1$, we multiply this bound by 3.001. $\qquad\square$

**Lemma 2.4.** *For $\sigma \geq 0$ and $|t| \leq T$ where $T \geq 2000$, we have*

$$\log|h_X(\sigma + it, \chi)| \leq 4 \log\big((qT)^{5/8} X\big) + 6.$$

*Proof.* We use the preceding Lemma and get

$$|h_X(\sigma + it, \chi)| \leq \left(\big(4.42(qT)^{5/8} X\big)^2 + 1\right)^2.$$

$\qquad\square$

**Lemma 2.5.** *We have, when $Q \geq 10$ and $T \geq 0$,*

$$\max\{|L(\tfrac{1}{2} + it, \chi)|, \chi \bmod^* q \leq Q, |t| \leq T\}$$
$$\leq 2\,(QT)^{1/4} \log(QT) + 3Q^{1/4} \log Q.$$

*When $Q = 1$, we have*

$$\max\{|\zeta(\tfrac{1}{2} + it)|, |t| \leq T\} \leq 2.4\,T^{1/6} \log(T) + 6.8.$$

*Proof.* We use lemma 2.1 with $\eta = 1/\log(QT)$ in case $q > 1$ and get the upper bound
$$e^{1/2}\left(\frac{q(\frac{3}{2} + T)}{2\pi}\right)^{1/4} (\log(QT) + 1) \leq 2\,(QT)^{1/4} \log(QT)$$

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for $QT \geq 5$. When $QT \leq 5$, then we take $\eta = 1/\log Q$ and numerically check that

$$\left(1 + \frac{1}{\log 2}\right)e^{1/2}\left(\frac{\frac{3}{2}+T}{2\pi}\right)^{\frac{1}{4}} Q^{1/4}\log Q - 2\,(QT)^{1/4}\log Q \leq 1.7 Q^{1/4}\log Q$$

when $T \geq 0$. As for the remaining case $QT \leq 5$ and $T \leq 1$, we add the maximum of $-2T^{1/4}\log T$ divided by $\log 10$ (this is $8/(e\log 10)$) to the coefficient of $Q^{1/4}\log Q$. This readily extends to encompass case $q = 1$ and this concludes the first half of the Lemma.

Let us turn to the estimate concerning solely the Riemann zeta-function. We first check that $\min_{0 \leq t \leq 3}(14.4\,T^{1/6}\log(T^{1/6}) + 7.96) \geq 2.657$ since the minimum is reached when $T^{1/6} = 1/e$. One can in fact be more precise by relying on explicit computations of $\zeta(1/2 + it)$ on the very restricted range $t \in [0,3]$. This hints at the property $|\zeta(\frac{1}{2} + it)| \leq 2.4\,T^{1/6}\log(T) + 6.78$. The RHS is more than 2.657 if $t \geq 0.07$, so the only the range $[0, 0.07]$ needs to be covered. It is then not necessary to give more details. $\qquad\square$

# 3 Some arithmetical lemmas

Here is a lemma from [12]:

**Lemma 3.1.** *We have, for $D \geq 1\,004$*

$$\sum_{d \leq D}\mu^2(d) = \frac{6D}{\pi^2} + \mathcal{O}^*\!\left(0.1333\sqrt{D}\right).$$

*In particular, this is not more than $0.62D$ when $D \geq 1700$.*

We shall require explicit computations that involve sums over primes (we convert products in sums via the logarithm). We shall truncate these sums and here is a handy lemma to control the error term.

**Lemma 3.2.** *Let $f$ be a $C^1$ non-negative, non-increasing function over $[P, \infty[$, where $P \geq 3\,600\,000$ is a real number. We have*

$$\sum_{p \geq P} f(p)\log p \leq (1 + \epsilon)\int_P^\infty f(t)dt + \epsilon f(P) + Pf(P)/(5\log^2 P)$$

*with $\epsilon = 1/36260$. When we can only ensure $P \geq 2$, then a similar inequality holds, simply replacing the last $1/5$ by a 4.*

*Proof.* Indeed, a summation by parts tells us that

$$\sum_{p \geq P} f(p)\log p = -\int_P^\infty f'(t)\vartheta(t)dt - \vartheta(P)f(P)$$

where $\vartheta(x) = \sum_{p \leq x}\log p$. At this level, we recall two results from [17], Proposition 5.1

$$\vartheta(x) - x \leq x/36260 \quad (x > 0)$$

and Theorem 5.2 therein (these results may also be found in [15]):

$$|\vartheta(x) - x| \leq 0.2x/(\log^2 x) \quad (x \geq 3\,600\,000).$$

The Lemma follows readily on applying these estimates. $\qquad\square$

**Lemma 3.3.** *We have*

$$\sum_{d \leq D} \mu^2(d) \frac{\varphi(d)}{d^2} = a \log D + b + \mathcal{O}^*(0.174)$$

*with* $a = \prod_{p \geq 2}(p^3 - 2p + 1)/p^3 = 0.4282 + \mathcal{O}^*(10^{-4})$ *and*

$$b/a = \gamma + \sum_{p \geq 2} \frac{3p - 2}{p^3 - 2p + 1} \log p = 2.046 + \mathcal{O}^*(10^{-4}).$$

*Furthermore the 0.174 can be reduced to 0.0533 when* $D \geq 10$ *and to 0.0194 when* $D \geq 48$.

*Proof.* We appeal to [41, Lemma 3.2]. First note that

$$D(s) = \sum_{d \geq 1} \frac{\mu^2(d) \varphi(d)}{d^{2+s}} = \prod_{p \geq 2} \left( 1 + \frac{p-1}{p^{2+s}} \right)$$

$$= \zeta(s) \prod_{p \geq 2} \left( 1 - \frac{1}{p^{2+s}} - \frac{1}{p^{2+2s}} + \frac{1}{p^{3+2s}} \right) = \zeta(s) H(s)$$

say. We thus get, for $D \geq 1$:

$$\sum_{d \leq D} \mu^2(d) \frac{\varphi(d)}{d^2} = H(0) \log D + H'(0) + \gamma H(0) + \mathcal{O}^*\left( c/D^{1/3} \right)$$

where the constants are given by

$$c = \prod_{p \geq 2} \left( 1 + \frac{1}{p^{5/3}} + \frac{1}{p^{4/3}} + \frac{1}{p^{7/3}} \right) \leq 6,$$

and

$$a = H(0) = \prod_{p \geq 2} \frac{p^3 - 2p + 1}{p^3} = 0.4282 + \mathcal{O}^*(10^{-4}).$$

Furthermore

$$\frac{H'(0)}{H(0)} = \sum_{p \geq 2} \frac{3p - 2}{p^3 - 2p + 1} \log p = 1.4695 + \mathcal{O}^*(10^{-4})$$

We use the following Sage program, see [48], since it implements interval arithmetic from [19]:

```
# File lemma32.sage

R = RealIntervalField(64)

def g(n):
    res = 1
    l = factor(n)
    for p in l:
        if p[1] > 1:
```

```
                return R(0)
            else:
                res *= (p[0]-1)/p[0]^2
    return R(res)


P = 10000
aaa = R(1)
p = 2
while p <= P:
    aaa *= R(1-2/p^2+1/p^3)
    p = next_prime (p)
eps = 1/R(36260)
x = 3*(1+eps)/R(P)/log(R(P))+3*eps/R(P)^2/log(R(P))+3/4/R(P)/log(R(P))^3
x = exp(-x)
aaa = aaa * x.union(R(1))


P = 100000
bbb = R(0)
p = 2
while p <= P:
    bbb += R((3*p-2)/(p^3-2*p+1))*log(R(p))
    p = next_prime (p)
x = (log(R(P))+1)/R(P)
bbb = bbb + x.union(R(0)) + R(euler_gamma)


ccc = R(6)


def model(z):
    return aaa * (log(R(z)) + bbb)


def getbounds (zmin, zmax):
    zmin = max (0, floor (zmin))
    zmax = ceil (zmax)
    res = R(0)
    for n in range (1, zmin + 1):
        res += g(n)
    maxi = abs(res - model (zmin)).upper()
    maxiall = maxi
    for n in xrange (zmin + 1, zmax + 1):
        m = model (n)
        maxi = max (maxi, abs(res - m).upper())
        res += g(n)
        maxi = max (maxi, abs(res - m).upper())
        if n % 100000 == 0:
            print "Upto ", n, " : ", maxi, cputime()
            maxiall = max (maxiall, maxi)
            maxi = R(-1000).upper()
    maxi = max (maxi, abs (res - model (zmax)).upper())
    maxiall = max (maxiall, maxi)
    print "La borne pour z >= ", zmax, " : "
```

```
    bound = ccc/R(zmax)^(1/3)
    print bound.upper()
    return [maxiall, maxi]
```

Assuming this file is called `lemma32.sage`, the command `load ''lemma32.sage''` within Sage indeed loads the included functions. The command `getbounds(1000, 30000000)` brings the output

```
sage: getbounds(1000, 30000000)
...
La borne pour z >=  30000000  :
0.0193097876921125952
[0.00214646012080014072, 0.000202372251756890651]
```

showing that

$$\left| \sum_{d \leq D} \mu^2(d) \frac{\varphi(d)}{d^2} - a \log D - b \right| \leq 0.00215$$

when $1000 \leq D \leq 30\,000\,000$. We then check that we can in fact start at $x = 48$. The conclusion is easy. □

**Lemma 3.4.** *Let $N \geq 1$ be a real number. We have*

$$\frac{6}{\pi^2} \log N + 0.578 \leq \sum_{n \leq N} \mu^2(n)/n \leq \frac{6}{\pi^2} \log N + 1.166.$$

*When $N \geq 1000$, the couple $(0.578, 1.166)$ may be replaced by $(1.040, 1.048)$*

A similar lemma occurs in [46], but with worst constants.

*Proof.* We proceed as above and get

$$\sum_{n \leq N} \mu^2(n)/n = \frac{6}{\pi^2} \left( \log N + 2 \sum_{p \geq 2} \frac{\log p}{p^2 - 1} + \gamma \right) + \mathcal{O}^*(3/N^{1/3}).$$

A similar script as in the previous Lemma yields

$$\left| \sum_{d \leq D} \frac{\mu^2(d)}{d^2} - \frac{6}{\pi^2} \log D - b' \right| \leq 0.00340$$

when $1000 \leq D \leq 30\,000\,000$. We present here an easier GP script, see [37], to extend it. Though such a script is usually enough (by which we mean, its result can in most examples be certified by Sage as in the previous Lemma), only the program using MPFR handles correctly the error term.

```
{g(n) =
   my(res = 1.0, dec = factor(n), P = dec[,1], E = dec[,2]);
   for(i = 1, #P,
      my(p = P[i]);
      if(E[i] != 1, return(0));
      res *= 1/p);
   return(res);}
```

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```
aaa = 6/Pi^2;
bbb = 1.7171176851;
ccc = 3;

{model(z)=aaa*(log(z)+bbb)}

{getsidedbounds(zmin,zmax)=
   my(res = 0.0, m, maxiplus, maximinus, maxiplusall, maximinusall);
   zmin = max( 0, floor(zmin));
   zmax = ceil(zmax);
   for(n=1, zmin, res += g(n));
   m = model(zmin);
   maxiplus = res - m;
   maxiplusall = maxiplus;
   maximinus = res - m;
   maximinusall = maximinus;
   for(n = zmin+1, zmax,
      m = model(n);
      maxiplus = max(maxiplus, res-m);
      maximinus = min(maximinus, res-m);
      res += g(n);
      maxiplus = max(maxiplus, res-m);
      maximinus = min(maximinus, res-m);

      if(n%100000==0,
         print("Upto ",n," : ", maximinus, " / ", maxiplus);
         maxiplusall = max(maxiplusall, maxiplus);
         maximinusall = min(maximinusall, maximinus);
         maxiplus = -1000;
         maximinus = 1000));
   m = model(zmax);
   maxiplus = max(maxiplus, res - m);
   maxiplusall = max(maxiplusall, maxiplus);
   maximinus = min(maximinus, res - m);
   maximinusall = min(maximinusall, maximinus);
   print("La borne pour z >= ", zmax, " : ", ccc/zmax^(1/3));
   return( [maximinusall, maximinus, maxiplusall, maxiplus]);
}
```

The conclusion is easy. □

# 4   On the total weight

In this section we prove an upper bound for $\sum_{\substack{q \leq Q, \\ \chi \bmod^* q}}^{\star} 1$.

**Lemma 4.1.** *The number $\eta(q)$ of primitive characters modulo $q$ is a multiplicative function given for any prime $p$ by*

$$\eta(p) = p - 2, \quad \eta(p^k) = p^k \left(1 - \frac{1}{p}\right)^2 \quad (\forall k \geq 2).$$

This also [47, Theorem 8].

*Proof.* Indeed, there are $\varphi(q)$ characters modulo $q$, which we can split according to their conductor: for each $d|q$, there are $\eta(d)$ characters modulo $q$ of conductor $d$. Hence $\mathbb{1} \star \eta = \varphi$ which is readily solved in $\eta = \mu \star \varphi$. This expression proves the multiplicativity as well as the values we have given. $\qquad\square$

By using a script similar to the one used for Lemma 3.4, we prove that

**Lemma 4.2.** *When $Q \in [10, 100\,000\,000]$, we have*

$$\sum_{q \leq Q} \frac{q}{\varphi(q)} \eta(q) \leq 0.29\, Q^2$$

*where the function $\eta$ is defined in Lemma 4.1.*

Here is a Lemma that will lead to a proof similar to the one of [41, Lemma 3.2].

**Lemma 4.3.** *We have, for any real number $X \geq 0$ and any real number $c \in [1, 2]$,*

$$\sum_{q \leq X} q = \tfrac{1}{2} X^2 + \mathcal{O}^*(\tfrac{1}{2} X^c).$$

*Proof.* When $X < 1$, we check that $\tfrac{1}{2} X^c \geq \tfrac{1}{2} X^2$ for any $c \in [1, 2]$. This proves that

$$\sum_{q \leq X} q = \tfrac{1}{2} X^2 + \mathcal{O}^*(\tfrac{1}{2} X^c) \quad (\forall X \in [X^*, X^* + 1)) \tag{9}$$

for any $c \in [1, 2]$ and $X^* = 0$.

Let $Q$ be a positive integer and $N = \sum_{q \leq Q} q = Q(Q+1)/2$. Note that $Q \leq \sqrt{2N} < Q + 1$. Then

- When $Q \leq X < \sqrt{2N}$, we have $|N - \tfrac{1}{2} X^2| \leq \tfrac{1}{2} X^c$ for every $c \in [1, 2]$. Indeed this is equivalent to $N \leq \tfrac{1}{2} X^2 + \tfrac{1}{2} X^c$ which is implied by $N \leq \tfrac{1}{2} Q^2 + \tfrac{1}{2} Q \leq \tfrac{1}{2} X^2 + \tfrac{1}{2} X^c$.

- When $\sqrt{2N} \leq X < Q+1$, the inequality $|N - \tfrac{1}{2} X^2| \leq \tfrac{1}{2} X^c$ is equivalent to $X^c - X^2 + 2N \geq 0$. The derivative of the involved function is $cX^{c-1} - 2X$ which is non-positive ($c < 2$). So we have to check that $(Q+1)^c - (Q+1)^2 + 2N \geq 0$ i.e. $2N \geq Q^2 + Q$ which is true.

This proves that (9) holds for $X^* = Q$, for all $Q$. Our Lemma follows readily. $\quad\square$

**Lemma 4.4.** *When $Q \geq 1$, we have*

$$\sum_{Q_0 < q \leq Q} \frac{q}{\varphi(q)} \eta(q) \leq 0.215\, Q^2 + 3\, Q^{3/2}$$

*where the function $\eta$ is defined in Lemma 4.1. Hence the sum in question is $\leq 0.29\, Q^2$ when $Q \geq 10$. And, when $Q \geq Q_0 = 400\,000$, we have*

$$\sum_{Q_0 < q \leq Q} \frac{q}{\varphi(q)} \eta(q) \leq 0.215\, \left(Q^2 - Q_0^2\right) + 3\, \left(Q^{3/2} + Q_0^{3/2}\right).$$

*Proof.* We introduce the multiplicative function

$$g(q) = \frac{q}{\varphi(q)}\eta(q).$$

We have $g(q)/q = (\mathbb{1} \star h)(q)$ where $h$ is the multiplicative function defined on powers of each prime $p$ by

$$h(p) = -2/p, \quad h(p^2) = 1/p, \quad h(p^k) = 0 \quad (\forall k \geq 3).$$

This enables us to write

$$g(q) = q \sum_{\substack{ab^2 \mid q, \\ (a,b)=1}} \frac{\mu(a)\mu^2(b)2^{\omega(a)}}{ab}. \tag{10}$$

Hence, on denoting by $S$ the sum to be studied, we use Lemma 4.3 with some parameter $c \in (1,2)$ and get

$$S = \sum_{\substack{a,b \geq 1, \\ (a,b)=1}} \frac{\mu(a)\mu^2(b)2^{\omega(a)}}{ab} \sum_{ab^2 \mid q \leq Q} q$$

$$= \sum_{\substack{a,b \geq 1, \\ (a,b)=1}} \mu(a)\mu^2(b)2^{\omega(a)}b \left( \frac{Q^2}{2a^2b^4} + \mathcal{O}^*\left( \frac{Q^c}{2a^cb^{2c}} \right) \right)$$

$$= \frac{Q^2}{2} \prod_{p \geq 2} \left(1 + \frac{1}{p^3}\right)\left(1 - \frac{2}{p^2\frac{1}{1+p^3}}\right) + \mathcal{O}^*\left( \frac{Q^c}{2} \prod_{p \geq 2}\left(1 + \frac{2}{p^c} + \frac{1}{p^{2c-1}}\right) \right)$$

hence

$$S = \frac{Q^2}{2} \prod_{p \geq 2} \left(1 - \frac{2}{p^2} + \frac{1}{p^3}\right) + \mathcal{O}^*\left( \frac{Q^c}{2} \prod_{p \geq 2}\left(1 + \frac{2}{p^c} + \frac{1}{p^{2c-1}}\right) \right).$$

We choose $c = 3/2$ and compute

$$S \leq 0.215\,Q^2 + 3Q^{3/2}.$$

On appealing to Lemma 4.2, the second part of our Lemma follows readily. The third part is straightforward. $\qquad\square$

# 5 Estimates concerning the Moebius function

Here is a handy Lemma taken from [42, Theorem 1.1], generalizing [23, Lemma 10.2].

**Lemma 5.1.** *We have uniformly for any real numbers $N \geq 1$ and $\varepsilon > 0$, and any integer $d$*

$$\left| \sum_{\substack{n \leq N, \\ (n,d)=1}} \mu(n)/n^{1+\varepsilon} \right| \leq 1 + \varepsilon.$$

**Lemma 5.2.** *When $\sigma \geq 1$ and $q \geq 2$, we have $q^{-\sigma} - q^{-2\sigma} \leq q^{-1} - q^{-2}$.*

*Proof.* We coonsider the auxiliary function $f(\sigma) = e^{-\sigma y} - e^{-2\sigma y}$, whose derivative is $f'(\sigma) = -ye^{-\sigma y}(1 - 2e^{-\sigma y})$. When $y \geq \log 2$ and $\sigma \geq 1$, we have $1 - 2e^{-\sigma y} \geq 0$. The lemma follows readily. $\qquad\square$

**Lemma 5.3.** *When $|s - 1| \leq 1/2$ is real, we have*

$$\zeta(s) = \frac{1}{s-1} + \gamma - \gamma_1(s-1) + \mathcal{O}^*(20|s-1|^2)$$

*where $\gamma = 0.57721\cdots$ and $\gamma_1 = 0.07281\cdots$ are the Laurent-Stieltjes constants.*

See [18] for the latest bounds on the Laurent-Stieltjes constants.

*Proof.* We first note then inequality $|e^{-z} - 1 + z| \leq |z|^2 e^{|z|}$. We then proceed as follows:

$$\begin{aligned}
\zeta(1+z) &= \frac{1+z}{z} - (1+z)\int_1^\infty \{t\}\frac{dt}{t^{2+z}} \\
&= \frac{1+z}{z} - (1+z)\int_1^\infty \{t\}(1 - z\log t)\frac{dt}{t^2} + \mathcal{O}^*\left(|z^2 + z^3|\int_1^\infty \frac{(\log t)^2 dt}{t^{2-|z|}}\right) \\
&= \frac{1}{z} + \gamma - \gamma_1 z + \mathcal{O}^*\left(|z^2 + z^3|\int_1^\infty (\log t + (\log t)^2)\frac{dt}{t^{2-|z|}}\right).
\end{aligned}$$

Note that these lines show that the constant $\gamma$ and $\gamma_1$ exist. Since they are unique, we can identify them with the usual ones. An integration by parts takes care of the remainder term. $\qquad\square$

**Lemma 5.4.** *When $s > 1$ is real, we have $\zeta(s) \leq e^{\gamma(s-1)}/(s-1)$.*

*Proof.* This inequality appears as [5, Lemma 1], reference being given to a private communication of ... me. In the mentionned paper, the authors prove among other thing this inequality with $\log 2$ instead of the optimal $\gamma$. We publish the proof here.

Since $x \mapsto e^{\gamma x}/x$ is increasing when $\gamma x \geq 1$, while $x \mapsto \zeta(1+x)$ is decreasing, and $\zeta(1 + \gamma^{-1}) \leq e\gamma$, the inequality is proved for $x \geq 1/\gamma$.

By splitting the interval $[0, 2]$ in $K+1 = 10001$ subintervals $[k/K, (k+1)/K]$ and trying to check numerically that $\zeta(1 + 2k/K) \leq K\, e^{2\gamma(k+1)/K}/(2(k+1))$ we obtain that is fails when $k \leq 632$ and succeeds otherwise: we have proved the inequality for $s > 1 + 2 \times 633/10000 = 1.1266$. We reiterate this process, but replacing the interval $[0, 2]$ by the interval $[0, 2 \times 633/10000]$ and prove the inequality for $s > 1 + 2 \times 633/10^4 \times 4025/10^4$, and in particular for $s > 1.06$.

We should now prove the inequality in the vicinity of $s = 1$, for which we use Lemma 5.3. We find that

$$\zeta(1 + \varepsilon) \leq \frac{1}{\varepsilon} + \gamma - \gamma_1\varepsilon + 20\varepsilon^2$$

and we readily that this not more than $e^{\gamma\varepsilon}/\varepsilon$ when $\varepsilon \in [0, 1/10]$. The Lemma follows readily. $\qquad\square$

**Lemma 5.5.** *For $\sigma > 1$ and $X \geq 10^9$, we have*

$$\sum_{n \geq 1} \frac{\left( \sum_{\substack{d|n, \\ d \leq X}} \mu(d) \right)^2}{\sigma n^\sigma} \leq 0.529 \frac{e^{\gamma(\sigma-1)}\sigma}{\sigma - 1} \log X.$$

*Proof.* Let $G(\sigma)$ be our sum. On expanding the square, we find that

$$G(\sigma) = \frac{\zeta(\sigma)}{\sigma} \sum_{d_1, d_2 \leq X} \frac{\mu(d_1)\mu(d_2)}{[d_1, d_2]^\sigma}.$$

We define for any $\sigma$ the auxiliary function

$$\varphi_\sigma(d) = d^\sigma \prod_{p|d}(1 - p^{-\sigma}) \tag{11}$$

which verifies $d^\sigma = (\mathbb{1} \star \varphi_\sigma)(d)$; here $\star$ denotes the usual convolution product. On using Lemma 5.1 and Selberg diagonalisation method, we get

$$G(\sigma)/\zeta(\sigma) = \frac{1}{\sigma} \sum_{\delta \leq X} \varphi_{\sigma-1}(\delta) \left( \sum_{\delta|d \leq X} \frac{\mu(d)}{d^\sigma} \right)^2 \leq \sigma \sum_{\delta \leq X} \frac{\mu^2(\delta)\varphi_\sigma(\delta)}{\delta^{2\sigma}}.$$

We readily check that $\varphi_\sigma(\delta)/\delta^{2\sigma} \leq \varphi(\delta)/\delta^2$, since Lemma 5.2 establishes this fact on prime powers. We are now in a position to appeal to Lemma 3.3 and reach

$$G(\sigma) \leq \sigma\zeta(\sigma)\big(0.4283 \log X + 2.047 + 0.0194\big).$$

We conclude by appealing to Lemma 5.4. $\qquad\square$

**Lemma 5.6.** *When $X \geq 10^9$, we have*

$$\sum_{X < n < 5X} \frac{\left( \sum_{\substack{d|n, \\ d \leq X}} \mu(d) \right)^2}{n^2} \leq \frac{0.605}{X}$$

*Proof.* We compute separately the contributions arising from $n \in (X, 2X]$ and from $n \in (2X, 5X)$. When $n \in (X, 2X]$, the coefficient $\sum_{\substack{d|n, \\ d \leq X}} \mu(d)$ equals $-\mu(n)$, which means we have to bound above

$$S_1 = \sum_{X < n \leq 2X} \frac{\mu^2(n)}{n^2}.$$

We proceed by integration by parts, relying on Lemma 3.1:

$$S_1 = \frac{\sum_{X < n \leq 2X} \mu^2(n)}{(2X)^2} + 2\int_X^{2X} \sum_{X < n \leq t} \mu^2(n) \frac{dt}{t^3}$$

$$= \frac{\frac{6}{\pi^2}X + 0.1333(1 + \sqrt{2})\sqrt{X}}{(2X)^2} + 2\int_X^{2X} \left( \frac{6}{\pi^2}(t - X) + 0.1333(\sqrt{X} + \sqrt{t}) \right) \frac{dt}{t^3}$$

$$= \frac{3}{\pi^2 X} + \frac{0.1333}{X^{3/2}} \left( 1 + \sqrt{2} + 1 - \frac{1}{4} + \frac{2}{3/2}\left( 1 - \frac{1}{2^{3/2}} \right) \right) \leq 0.304/X.$$

When $n \in (2X, 5X)$, we readily see by inspecting all the possible cases that the coefficient $\sum_{\substack{d|n, \\ d \leq X}} \mu(d)$ takes values in $\{-1, 0, 1\}$. The only non-trivial cases is when $n$ is divisible by 2 and 3, where the coefficient has value $-\mu(n) - \mu(n/2) - \mu(n/3)$. When $\mu(n) \neq 0$, the conclusion is straighforward, but otherwise we are left with $-\mu(n/2) - \mu(n/3)$. However, if both $\mu(n/2)$ and $\mu(n/3)$ do not vanish, then so does $\mu(n)$. It is thus enough to bound $S_2 = \sum_{2X < n < 5X} 1/n^2$. We write simply

$$S_2 \leq \frac{1}{(2X)^2} + \int_{2X}^{5X} \frac{dt}{t^2} = \left(\frac{1}{2} - \frac{1}{5} + \frac{1}{4X}\right)\frac{1}{X} \leq 0.301/X.$$

$\square$

## 6 Large sieve estimates and the like

Here is the classical large sieve inequality for primitive characters (see [35], [20]) stated with notation (5):

**Lemma 6.1.** *We have*

$$\sum_{\substack{q \leq Q, \\ \chi \bmod^* q}}^{\star} \left| \sum_{1 \leq n \leq N} b_n \chi(n) \right|^2 \leq (N - 1 + Q^2) \sum_n |b_n|^2.$$

**Theorem 6.1.** *Let $(u_n)_{n \geq 1}$ be a sequence of complex numbers. Let $k \geq 1$ be an integer parameter, and $c^* \geq 2/k$ and $T^* > 0$ be two real parameters. For any real numbers $T \geq 0$ we have*

$$\sum_{\substack{q \leq Q, \\ \chi \bmod^* q}}^{\star} \int_{-T}^{T} \left| \sum_n u_n \chi(n) n^{it} \right|^2 dt$$

$$\leq 2\pi \beta_k^2 \left( \frac{\pi/c^*}{\sin \frac{2\pi}{c^* k}} \right)^{2k} \sum_{n \leq N} |u_n|^2 \left( 4\pi \frac{\operatorname{sh} \frac{4\pi}{c^* T^*}}{4\pi/(c^* T^*)} n + c^* Q^2 \max(T^*, T) \right)$$

*where the positive constants $\beta_k$ are defined in (16).*

When $c^* \geq 5.03$ and $T^*$ is large enough, it is best to select $k = 1$. The choice $c = 12.5876$ leads to the following corollary:

**Corollary 6.1.** *We have, for $T \geq 0$:*

$$\sum_{\substack{q \leq Q, \\ \chi \bmod^* q}}^{\star} \int_{-T}^{T} \left| \sum_n a_n \chi(n) n^{it} \right|^2 dt \leq 43 \sum_n |a_n|^2 (n + Q^2 \max(10, T)).$$

It is possible to diminish the constant in front of the $Q^2 T$-term at the cost of a higher one in front of the $n$-term. For instance, on selecting $c = 2.33$ and $k = 6$, we get

**Corollary 6.2.** *We have, for $T \geq 0$:*

$$\sum_{\substack{q \leq Q, \\ \chi \bmod^* q}}^{\star} \int_{-T}^{T} \left| \sum_n a_n \chi(n) n^{it} \right|^2 dt \leq \sum_n |a_n|^2 (141n + 26Q^2 \max(30, T)).$$

We follow the idea of [35, Corollary 3] but rely on [39] to get that

**Lemma 6.2.** *We have*

$$\int_0^T \left| \sum_n u_n n^{it} \right|^2 dt \leq \sum_{n \leq N} |u_n|^2 (2\pi c_0(n+1) + T),$$

*where* $c_0 = \sqrt{1 + \frac{2}{3}\sqrt{\frac{6}{5}}}$. *Moreover, when $u_n$ is real-valued, the constant $2\pi c_0$ may be reduced to $c_0\pi$.*

*Proof.* As noted in [35, Last paragraph], we have $|\log(m/n)| \geq 1/(n+1)$ when $m$ and $n$ are distinct positive integers and it is thus a triviality to give explicit constants in [35, Corollary 3]. When the sequence $(u_n)$ is real-valued, we write

$$\int_0^T \left| \sum_n u_n n^{it} \right|^2 dt = T \sum_n |u_n|^2 + \sum_{m \neq n} \frac{u_m m^{iT} \overline{u_n} n^{-iT}}{\log(m/n)} - \sum_{m \neq n} \frac{u_m \overline{u_n}}{\log(m/n)}.$$

This third summand vanishes identically when $(u_n)$ is real-valued as shown by combining the pairs $(m, n)$ and $(n, m)$. The Lemma follows readily. $\qquad\square$

## 6.1 Proof of Theorem 6.1, I: a generic proof

We follow closely the proof of a Lemma due to Gallagher (this is [20, Lemma 1], as well as [9, Theorem 9]).

We first present a "generic" proof and choose the parameters later. Let $F$ be a function to be chosen later. We assume that $F(t) = 0$ as soon as $|t| \geq \eta$ for some parameter $\eta > 0$. Let $\delta > 0$ be a parameter that we shall also chose later. We define

$$F_\delta(x) = F(x/\delta).$$

Let us start from an arbitrary sequence of complex numbers $(v_n)$ such that $\sum_n |v_n| < \infty$. We readily get

$$\sum_n v_n e^{2i\pi t (\log n)/(2\pi)} \hat{F}_\delta(t) = \sum_n v_n F_\delta\left(\widehat{t - \frac{\log n}{2\pi}}\right).$$

Parseval identity yields

$$\int_{-\infty}^{\infty} \left| \sum_n v_n e^{2i\pi t (\log n)/(2\pi)} \hat{F}_\delta(t) \right|^2 dt = \int_{-\infty}^{\infty} \left| \sum_n v_n F_\delta\left(x - \frac{\log n}{2\pi}\right) \right|^2 dx$$

$$= \int_0^{\infty} \left| \sum_n v_n F_\delta\left(\frac{\log(y/n)}{2\pi}\right) \right|^2 dy/y.$$

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Our hypothesis on the support of $F$ implies that the $y$'s in the relevant range verify $e^{-\eta\pi\delta} \le y/n \le e^{\eta\pi\delta}$. We apply the above for $v_n = u_n n^{it}\chi(n)$ and, as a consequence, we find that

$$\sum_{\substack{q \le Q, \\ \chi \bmod^* q}}^{\star} \int_{-\infty}^{\infty} \Bigl| \sum_n u_n n^{it}\chi(n)\hat{F}_\delta(t) \Bigr|^2 dt$$

$$\le \int_0^\infty \sum_n |u_n|^2 \Bigl| F_\delta\Bigl(\frac{\log(y/n)}{2\pi}\Bigr) \Bigr|^2 \bigl(y(e^{2\eta\pi\delta} - e^{-2\eta\pi\delta}) + Q^2\bigr)dy/y$$

$$\le \sum_n |u_n|^2 \int_0^\infty \Bigl| F_\delta\Bigl(\frac{\log u}{2\pi}\Bigr) \Bigr|^2 \bigl(n(e^{2\eta\pi\delta} - e^{-2\eta\pi\delta}) + Q^2 u^{-1}\bigr)du.$$

We change variable by setting $u = \exp(2\pi\delta w)$ and recall that the kernel function $F$ is assumed to be even. The right-hand side is thus bounded above by

$$2\pi\delta \sum_n |u_n|^2 \int_{-\infty}^\infty |F(w)|^2 \Bigl(n(e^{2\eta\pi\delta} - e^{-2\eta\pi\delta})e^{2\pi\delta w} + Q^2\Bigr)dw$$

$$\le 2\pi\delta \sum_n |u_n|^2 \bigl(n(e^{2\eta\pi\delta} - e^{-2\eta\pi\delta})(e^{2\eta\pi\delta} + e^{-2\eta\pi\delta}) + 2Q^2\bigr) \int_0^\infty |F(w)|^2 dw.$$

Since $\hat{F}_\delta(t) = \delta\hat{F}(\delta t)$, we have finally reached

$$\sum_{\substack{q \le Q, \\ \chi \bmod^* q}}^{\star} \int_{-\infty}^{\infty} \Bigl| \sum_n u_n n^{it}\chi(n) \Bigr|^2 |\hat{F}(\delta t)|^2 dt$$

$$\le 2\pi \sum_n |u_n|^2 \Bigl(n\frac{\operatorname{sh}(4\eta\pi\delta)}{\delta} + Q^2\delta^{-1}\Bigr) \int_{-\infty}^\infty |F(w)|^2 dw. \quad (12)$$

## 6.2 Proof of Theorem 6.1, II: searching for a good kernel

Now that we have this generic proof at our disposal, we simple have to optimise the choice of the function $F$. We want $|\hat{F}(\delta t)| > 0$ when $|t| \le T$ as well as $F(t) = 0$ when $|t| \ge \eta$. The only regularity conditions are that $F$ is even and belongs to $\mathrm{L}^2[-\eta, \eta]$. We obviously have $\hat{F}(y) = \int_{-\eta}^\eta F(t)e(yt)dt$ and we look for

$$m(F, c, \eta) = \min_{|y| \le 1/c} |\hat{F}(y)|$$

with $c = 1/(\delta T)$. We assume further that $\int_{-\eta}^\eta |F(x)|^2 dx = 1$ since $\int_{-\infty}^\infty |\hat{F}(y)|^2 dy = \int_{-\eta}^\eta |F(t)|^2 dt$. This would imply that via (12), provided that $m(F, c, \eta) > 0$,

$$\sum_{\substack{q \le Q, \\ \chi \bmod^* q}}^{\star} \int_{-T}^{T} \Bigl| \sum_n u_n n^{it}\chi(n) \Bigr|^2 dt$$

$$\le \frac{2\pi}{m(F, c, \eta)^2} \sum_n |u_n|^2 \bigl(ncT \operatorname{sh}\frac{4\eta\pi}{cT} + cQ^2 T\bigr). \quad (13)$$

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We define $G(x) = \sqrt{\eta} F(\eta x) \in \mathrm{L}^2[-1,1]$, which verifies $\int_{-1}^1 |G(x)|^2 dx = 1$. Furthermore $\hat{G}(y) = \hat{F}(y/\eta)/\sqrt{\eta}$, and thus the right-hand side of (13) becomes

$$\frac{2\pi}{m(G, c/\eta, 1)^2 \eta} \sum_n |u_n|^2 \big(ncT \,\mathrm{sh}\, \frac{4\eta\pi}{cT} + cQ^2 T\big).$$

We thus set $c^* = c/\eta$ and get

$$\sideset{}{^\star}\sum_{\substack{q \leq Q, \\ \chi \bmod^* q}} \int_{-T}^T \Big| \sum_n u_n n^{it} \chi(n) \Big|^2 dt$$

$$\leq \frac{2\pi}{m(G, c^*, 1)^2} \sum_n |u_n|^2 \big(nc^*T \,\mathrm{sh}\, \frac{4\pi}{c^*T} + c^*Q^2 T\big). \quad (14)$$

By using Cauchy's inequality, we see that the condition $\int_{-1}^1 |G(x)|^2 dx = 1$ implies that $m(G, c^*, 1)^2 \leq 2$. When $c^*$ tends to $\infty$, reducing $m(G, c^*, 1)$ to $\hat{G}(0)$, the bound $m(G, \infty, 1)^2 = 2$ is reached with the choice $G(x) = \mathbb{1}_{|x| \leq 1}$. Let us now select a positive integer $k$ and consider $g_k(x) = \mathbb{1}_{|x| \leq 1/k}$. Its $k$-th convolution power $G_k = g_k^{(\star k)}/\beta_k$ verifies the condition support, is indeed even and we have

$$\hat{G}_k(y) = \left( \frac{\sin \frac{2\pi y}{k}}{\pi y} \right)^k / \beta_k.$$

The constant $\beta_k$ is $\|g_k\|_2$. We use [22, (3.836), part 2] with $m = 0$ to get

$$\int_0^\infty \left( \frac{\sin x}{x} \right)^n dx = \frac{n\pi}{2^n} \sum_{0 \leq \ell \leq n/2} \frac{(-1)^\ell (n - 2\ell)^{n-1}}{\ell!(n - \ell)!}. \quad (15)$$

We infer from this formula that

$$\|g_k\|_2^2 = \int_{-\infty}^\infty \left( \frac{\sin \frac{2\pi y}{k}}{\pi y} \right)^{2k} dy$$

$$= \left( \frac{2}{k} \right)^{2k} \frac{k}{2\pi} \frac{4k\pi}{2^{2k}} \sum_{0 \leq \ell \leq k} \frac{(-1)^\ell (2k - 2\ell)^{2k-1}}{\ell!(2k - \ell)!}$$

$$\beta_k^2 = \frac{2^{2k}}{k^{2k-2}} \sum_{0 \leq \ell \leq k-1} \frac{(-1)^\ell (k - \ell)^{2k-1}}{\ell!(2k - \ell)!}. \quad (16)$$

This gives

$$\beta_1^2 = \|g_1\|_2^2 = 2, \quad \beta_2^2 = \|g_2\|_2^2 = \frac{2}{3}, \quad \beta_3^2 = \|g_3\|_2^2 = \frac{88}{1215}.$$

Numerically, we find that $\beta_k^2 = \|g_k\|_2^2 \leq (2/k)^{2k} \frac{\sqrt{k}}{2}$ and in fact may be asymptotic to this expression. If $c^* \geq 2/k$, which implies that $2\pi/(kc^*) \leq \pi$, we have

$$m(G_k, c^*, 1)^{-2} = \beta_k^2 \left( \frac{\frac{\pi}{c^*}}{\sin \frac{2\pi}{c^*k}} \right)^{2k}. \quad (17)$$

This together with (14) ends the proof of Theorem 6.1: indeed, we first check that it is enough to prove the stated inequality for $T \geq T^*$ and the result then follows readily.

## 6.3 Proof of Corollary 6.1

Let us start by a remark. Numerically, we see that, given $c^*$, the sequence $(m(G_k, c^*, 1)^{-2})_k$ decreases and then increases. The guess $\beta_k^2 \sim (2/k)^{2k} \frac{\sqrt{k}}{2}$ implies that the limit is indeed infinity, and we can only look for the best value of $k$. It is then straighforward to compute numerical values.

Concerning Lemma 6.1, we first took $c = 4\pi$ and rounding the constant in front of the $Q^2 T$-term led to the value 26. Once this value was set, we increased the value of $c$ so as to get a small constant in front of the $n$-term. This process has been carried with $T^* = 1000$, which we then check that it could be reduced to $T^* = 10$.

Concerning Lemma 6.2, we first took $c = 2$ and rounding the constant in front of the $Q^2 T$-term led to the value 26. Once this value was set, we increased the value of $c$ so as to get a small constant in front of the $n$-term. This process has been carried with $T^* = 1000$, which we then check that it could be reduced to $T^* = 30$.

Here is the GP-script we have used

```
{norme(k) = my(res = 0, sgn = 1);
   for(l = 0, k-1,res += sgn*(k-l)^(2*k-1)/l!/(2*k-l)!;
                  sgn = -sgn);
   return(res*2^(2*k)/(k^(2*k-2)));}

{ct(k,c) = my(u = Pi/c);
   return(norme(k)*(u/sin(2*u/k))^(2*k)*2*Pi);}

{show(c, borneinf = 10, bornesup = 100, Tstar = 1000) =
   my(ct0, corr = sinh(4*Pi/c/Tstar)/4/Pi*c*Tstar);
   for(k = max(borneinf,ceil(2/c+0.00000000001)), bornesup,
      ct0 = ct(k,c);
      print(" k = ", k,
              " --> ( ", ct0*4*Pi*corr, ", ", ct0*c, ")"));}
```

The value 26 is the smallest (integer) value we have been able to get in front of the $Q^2 T$-term. The best real value we have been able to reach is $25.89 \cdots$ by taking $k = 18$ and $c = 1.21$.

## 7 Usage of Theorem 6.1

**Lemma 7.1.** *We have, for $X \geq 10^9$, $X \geq 2\,000\,Q^2$, and $Q \geq 10$, $T \geq 0$,*

$$\sideset{}{^\star}\sum_{\substack{q \leq Q, \\ \chi \bmod^* q}} \int_0^T |M_X(\tfrac{1}{2} + it, \chi)|^2 dt \leq (8.57 Q^2 T + 2.24 X) \log X.$$

*We also have*

$$\int_0^T |M_X(\tfrac{1}{2} + it, \mathbb{1})|^2 dt \leq (0.77 T + 0.126\, X) \log X.$$

*Proof.* From Corollary 6.2, Lemma 3.4 and 3.1, we readily get the first quantity to be not more than (note that the integration is between 0 and $T$ and *not* between $-T$ and $T$)

$$\sum_{n \le X} \frac{\mu^2(n)}{n}(70.5n + 13Q^2 \max(T, 30))$$

$$\le 13Q^2 \max(T, 30)\Big(\frac{6}{\pi^2}\log X + 1.048\Big) + 70.5 \times 0.62X.$$

This ensures, on taking into account the bound $X \ge 10^9$, that we have

$$\sideset{}{^\star}\sum_{\substack{q \le Q, \\ \chi \bmod^* q}} \int_0^T |M_X(\tfrac{1}{2} + it, \chi)|^2 dt \le (8.57\,Q^2 \max(T, 30) + 2.11\,X) \log X.$$

Thus, when $T \le 30$, we have

$$\sideset{}{^\star}\sum_{\substack{q \le Q, \\ \chi \bmod^* q}} \int_0^T |M_X(\tfrac{1}{2} + it, \chi)|^2 dt \le (8.57\,Q^2 \times 30 + 2.11\,X) \log X$$

$$\le \Big(\frac{8.57 \times 30}{2000} + 2.11\Big)X \log X.$$

Hence the Lemma.

When considering only the principal character modulo 1, we can rely on Lemma 6.2, which gives us the bound

$$\sum_{n \le X} \frac{\mu^2(n)}{n}(4.2\,n + T + 4.2) \le 0.77(T + 4.2)\log X + 4.2 \times 0.62X.$$

$\square$

**Lemma 7.2.** *We have, for $X \ge 10^9$, $X \ge 2\,000\,Q^2$, $Q \ge 10$ and $T \ge 0$,*

$$\tfrac{1}{2}F_Q(1/2, T) \le 8.79Q^{1/2}(Q^2 T + 0.27X)(2T^{1/4}\log(QT) + 3\log Q)^2 \log X.$$

*When $Q = 1$, we get:*

$$\tfrac{1}{2}F_1(1/2, T) \le 4.45(T + 0.164X)(T^{1/6}\log(T) + 2.83)^2 \log X.$$

Note that it is important that this Lemma should hold for small $T$'s as well. The method developped here is of course very elementary since we want to be able to compute all the involved constants, and has nothing in common with the technology developped for instance in [14].

*Proof.* On using (7) and the Minkowski inequality, we readily see that

$$\sqrt{\tfrac{1}{2}F_Q(1/2, T)} \le \sqrt{A} + \sqrt{0.3\,TQ^2}$$

where

$$A = \sideset{}{^\star}\sum_{\substack{q \le Q, \\ \chi \bmod^* q}} \int_0^T |M_X(\tfrac{1}{2} + it, \chi)|^2 dt \max_{\substack{q \le Q, \\ \Re s = \frac{1}{2}, |\Im s| \le T}} |L(s, \chi)|^2$$

and on appealing to Lemma 4.4. On appealing to Lemma 2.5 and 7.1, we reach the upper bound

$$A \le A_0 = Q^{1/2}(8.57Q^2T + 2.24X)(2T^{1/4}\log(QT) + 3\log Q)^2\log X.$$

We notice that $2T^{1/4}\log(QT) + 3\log Q = 8Q^{-1/4}x\log x + 3\log Q$ with $x = (QT)^{1/4}$. The minimum of this quantity is obtained with $x = 1/e$ and then $Q = 10$ and has value $\ge 5.25$. We check that $A_0 \ge 8.57Q^{5/2}T \cdot 5.25\log X \ge 5788 \cdot 0.3\,Q^2T$ and, as a consequence

$$\tfrac{1}{2}F_Q(1/2, T) \le (1 + \sqrt{1/5788})^2 8.57Q^{1/2} \times$$
$$(Q^2\max(T, 30) + 0.27X)(2T^{1/4}\log(QT) + 3\log Q)^2\log X.$$

When $Q = 1$, we start from $\sqrt{\tfrac{1}{2}F_1(1/2, T)} \le \sqrt{A} + 1$ and use the second part of Lemma 2.5 and 7.1 to get the bound:

$$A \le A_0 = (2.4T^{1/6}\log(T) + 6.8)^2(0.77T + 0.126X)\log X.$$

We check that $T \mapsto 2.4T^{1/6}\log(T) + 6.8$ stays greater than 1.5025, and then that $A_0 \ge 1.5025^2 \cdot 0.126 \cdot 10^9 \cdot \log(10^9) \ge 10^9$ and thus

$$\tfrac{1}{2}F_1(1/2, T) \le (1 + \sqrt{1/10^9})^2 2.403^2(T^{1/6}\log T + \tfrac{6.8}{2.403})^2(0.77T + 0.126X)\log X.$$

We pulled out 2.403 and not 2.4 so that we get in front the nicely rounded coefficient 4.45 and inside the equally nicely rounded coefficient 2.83. $\qquad\square$

**Lemma 7.3.** *We have, when $\delta = 1/\log X$, $X \ge 2000Q^2$, $X \ge 10^9$ and $T \ge 0$,*

$$\tfrac{1}{2}F_Q(1 + \delta, T) \le \left(7.26 + 0.951\frac{Q^2T}{X}\right)(\log X)^2.$$

*When $Q = 1$, we get:*

$$\tfrac{1}{2}F_1(1 + \delta, T) \le \left(1.40 + 0.0442\frac{T}{X}\right)(\log X)^2.$$

*Proof.* We readily get from (7) and Theorem 6.1 the upper bound

$$\frac{43}{2}\sum_{X < n}\left(\sum_{\substack{d\mid n, \\ d \le X}}\mu(d)\right)^2 n^{-2-2\delta}(n + Q^2\max(T, 10)).$$

We note that, by Lemma 5.5, we have

$$\sum_{X < n}\left(\sum_{\substack{d\mid n, \\ d \le X}}\mu(d)\right)^2 n^{-1-2\delta} \le \frac{(1 + 2\delta)^2 e^{2\gamma\delta}}{2\delta}0.529\log X$$

$$\le 0.337(\log X)^2$$

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Concerning the second sum, we appeal to Lemma 5.6 together with a simple version of Rankin's trick[1] to bound above this quantity by

$$Q^2 \max(T, 10) \left( \frac{0.605}{X^{1+2\delta}} + \sum_{n \geq 1} \left( \sum_{\substack{d|n, \\ d \leq X}} \mu(d) \right)^2 n^{-2-2\delta} \left( \frac{n}{5X} \right)^{1+\delta} \right).$$

This is not more by Lemma 5.5 than

$$Q^2 \max(T, 10) \left( \frac{0.605}{X^{1+2\delta}} + \frac{(1+\delta)^2 e^{\gamma\delta}}{5X^{1+\delta}\delta} 0.529 \log X \right)$$

$$\leq 0.0442 \frac{Q^2 \max(T, 10)}{X} (\log X)^2.$$

All of that gives us

$$\frac{43}{2} \left( 0.337 + 0.0442 \frac{Q^2 \max(T, 10)}{X} \right) (\log X)^2$$

When $T \leq 10$, we find

$$\tfrac{1}{2} F_Q(1 + \delta, T) \leq \frac{43}{2} \left( 0.337 + \frac{0.0442 \times 10}{2000} \right) (\log X)^2 \leq 7.26(\log X)^2$$

We include this contribution to our estimate by replacing $\frac{43}{2} \times 0.3371$ by 7.26. The first part of the Lemma follows readily.

When considering only the principal character modulo 1, we can rely on Lemma 6.2 and get the upper bound

$$\sum_{X < n} \left( \sum_{\substack{d|n, \\ d \leq X}} \mu(d) \right)^2 n^{-2-2\delta} (4.14\,n + T + 4.14).$$

We proceed as above via Rankin's trick, after some steps similar to what has been done, we reach the bound

$$4.14 \frac{(1+2\delta)^2 e^{2\gamma\delta}}{2\delta} 0.529 \log X + \frac{T + 4.14}{X} \left( \frac{0.605}{X^{2\delta}} + \frac{(1+\delta)^2 e^{\gamma\delta}}{5X^\delta\delta} 0.529 \log X \right)$$

amounting to

$$\left( 1.393 + 0.0442 \frac{T + 4.14}{X} \right) (\log X)^2.$$

The Lemma follows readily. □

# 8 Computing some values of $\Gamma$ and its derivatives

We shall require values of $\Gamma$ and $\Gamma'$ at special points. The values we require are tabulated in [1, Section 6, pages 253–277] and the values of the $\Gamma$-function

---

[1]The reason we use the exponent $1 + \delta$ is the following: we using $1 + k\delta$ say, we get $Q^2T/X$ divided by $(2 - k)X^{k\delta}$. With the choice $\delta = 1/\log X$, we check that the maximum of this denominator is attained close to the point $k = 1$. Hence our choice.

may also be asked to GP/Pari, but some explanations are called for. We get to $\Gamma'$ via $\Gamma'(s) = \psi(s)\Gamma(s)$, where $\psi$ is the Digamma function which is well known. In particular it verifies $\psi(x+1) = \psi(x) + (1/x)$. There are ways to compute explicitely the values of the $\psi$-function at rational argument (see Gauss formula), but we will simply use the `psi` function of Gp/Pari.

We proceed in a similar fashion for the trigamma function $\psi_1(x) = \psi'(x)$. It verifies $\psi_1(x+1) = \psi_1(x) - (1/x^2)$. Again some values are missing and we recall the following simplistic representation of $\psi_1$ that we used to compute $\psi_1(4/3)$:

$$\psi_1(x) = \sum_{n \geq 0} \frac{1}{(x+n)^2}. \tag{18}$$

This series converges rather slowly but we can use the `sumpos` function of Gp/Pari via `psi1(x)=sumpos(X=0,1/(X+x)^2)` to get excellent results instantly. Here are the values we will need ($\Gamma'' = (\psi^2 + \psi_1)\Gamma$):

$$
\begin{aligned}
&\Gamma(1) = 1 \\
&\Gamma(7/6) = 0.927\cdots \quad \Gamma'(7/6) = -0.308\cdots \\
&\Gamma(4/3) = 0.892\cdots \quad \Gamma'(4/3) = -0.117\cdots \quad \Gamma''(4/3) = 0.993\cdots \\
&\Gamma(3/2) = 0.886\cdots \quad \Gamma'(3/2) = 0.0323\cdots \quad \Gamma''(3/2) = 0.829\cdots \\
&\Gamma(2) = 1 \\
&\Gamma(13/6) = 1.08\cdots \quad \Gamma'(13/6) = 0.568\cdots \\
&\Gamma(9/4) = 1.14\cdots \quad \Gamma'(9/4) = 1.20\cdots \\
&\Gamma(7/3) = 1.19\cdots \quad \Gamma'(7/3) = 0.735\cdots \quad \Gamma''(7/3) = 1.08\cdots \\
&\Gamma(5/2) = 1.32\cdots \quad \Gamma'(5/2) = 0.934\cdots \quad \Gamma''(5/2) = 1.30\cdots
\end{aligned} \tag{19}
$$

# 9   On the total number of zeroes

Here is a lemma we took from [34].

**Lemma 9.1.** *If $\chi$ is a Dirichlet character of conductor $k$, if $T \geq 1$ is a real number, and if $N(T, \chi)$ denotes the number of zeros $\beta + i\gamma$ of $L(s, \chi)$ in the rectangle $0 < \beta < 1$, $|\gamma| \leq T$, then*

$$\left| N(T, \chi) - \frac{T}{\pi} \log\left(\frac{qT}{2\pi e}\right) \right| \leq C_2 \log(qT) + C_3$$

*with $C_2 = 0.9185$ and $C_3 = 5.512$.*

We recall that [38] in his thesis has shown that no Dirichlet $L$-series with conductor $\leq Q_0 = 400\,000$ has no zeros of height $10^8/Q_0$ off the critical line and (his result is more extended than that). In particular $N(\sigma, 6, \chi) = 0$ whenever $\sigma > 0$.

In particular, we have when $Q \geq 10$ and $\sigma > 1/2$, on using the third part of Lemma 4.4

$$\sum_{\substack{q \leq Q, \\ \chi \bmod^* q}}^{\star} N(\sigma, 6, \chi) \leq \left( 0.215 \left( Q^2 - Q_0^{3/2} \right) + 3 \left( Q^{3/2} + Q_0^{3/2} \right) \right) \tag{20}$$

$$\times \left( \frac{6}{2\pi} \log \frac{6Q}{2\pi e} + \tfrac{1}{2}(0.92 \log(6Q) + 5.6) \right)$$

$$\leq 0.246\, Q^2 \log Q. \tag{21}$$

The maximal value being about $0.245127483233716$ reached next to $Q = 4\,585\,014$.

# 10 Bounding $F_Q(T_2, \sigma) - F_Q(T_1, \sigma)$

This part contains the heart of the argument. Here are the results we prove in this section.

**Lemma 10.1.** *Let $T \geq T_1 \geq 2$ and $Q$ be positive real parameters that verify $X \geq 10^9$, $X \geq 2\,000\,Q^2$ and $Q \geq 10$. When $X = Q^2T$, we have, for any $\sigma \in [1/2, 1]$:*

$$F_Q(T, \sigma) - F_Q(T_1, \sigma) \leq \frac{13.4}{0.367}(31Q^5T^3)^{1-\sigma}\log^{4-2\sigma}(Q^2T).$$

And here its counterpart concerning solely the principal character:

**Lemma 10.2.** *Let $T \geq 3 \cdot 10^{10}$ and $\sigma \in [1/2, 1]$. We have*

$$F_1(T, \sigma) - F_1(6, \sigma) \leq \frac{2.91}{0.367}(2.77\,T)^{8(1-\sigma)/3}\log^{4-2\sigma}(T).$$

Both proofs are rather easy in their principle: we majorize $F_Q(T, \sigma) - F_Q(T_1, \sigma)$ by a smoother quantity (replacing the cutoff at $T$ by essentially an exponential smoothing). This is done at subsection 10.5. We evaluate this smoother version by a convexity argument which we develop in subsection 10.1. In order to apply the resulting bound, we need to bound the smoothed version on $\sigma = 1/2$ (this is subsection 10.2) and on $\sigma = 1 + \delta$ for some small $\delta$ (this is subsection 10.3). This last part is where the fact that the coefficients of the Dirichlet series $f_X$ vanish at the beginning will be used.

## 10.1 A convexity argument

To evaluate $\int_{T_1}^{T_2} \sum_{\substack{q \leq Q, \\ \chi \bmod^* q}}^{\star} |f_X(\sigma_0 + it, \chi)|^2 dt$, we use a slight extension convexity argument due to [24]. We first are to evaluate this integral in $\frac{1}{2}$ and in $1 + \delta$. We set

$$\Phi(s) = \frac{s - 1}{s(\cos s)^{1/(2\tau)}} \quad \Re s \in [\tfrac{1}{2}, 1 + \delta] \tag{22}$$

for some parameter $\tau \geq 2\,000$ that we will at the end take to be $T_2$. Here $\delta = 1/\log(Q^2T_2)$. The function $s \mapsto \cos s$ does not vanish in the strip we consider since $|\cos(\sigma + it)|^2 = (\cos \sigma)^2 + (\sinh t)^2$. The factor $s - 1$ is to take care of the pole of $\zeta$ at $s = 1$, and its growth is compensated by the $1/s$. The $(\cos s)^{1/2\tau}$ is here that $\Phi(s)f_X(s, \chi) = o(1)$ uniformly in $\Re s$ and as $|\Im s|$ goes to infinity while giving enough weight to the $s$ with $|\Im s|$ between 0 and $T$. Let us set

$$a = \frac{1 + \delta - \sigma}{1 + \delta - \frac{1}{2}} \quad , \quad b = \frac{\sigma - \frac{1}{2}}{1 + \delta - \frac{1}{2}}. \tag{23}$$

A slight extension of the Hardy-Ingham-Pólya inequality which we prove thereafter reads

$$\mathfrak{M}_Q(\sigma) \leq \mathfrak{M}_Q(1/2)^a \mathfrak{M}_Q(1 + \delta)^b \tag{24}$$

with

$$\mathfrak{M}_Q(\sigma) = \int_{-\infty}^{\infty} \sum_{\substack{q \leq Q, \\ \chi \bmod^* q}}^{\star} |\Phi(\sigma + it)f_X(\sigma + it, \chi)|^2 dt. \tag{25}$$

The extension comes from the fact that we have added a summation over characters instead of considering a single function.

*Proof.* Indeed we follow closely [51, section 7.8] and set

$$\phi(z,\chi) = \frac{1}{2i\pi} \int_{\sigma-i\infty}^{\sigma+i\infty} \Phi(s) f_X(s,\chi) z^{-s} ds \qquad (\sigma \geq 1/2, |\arg z| < \pi/2). \quad (26)$$

Setting $z = ixe^{-i\delta}$ with $0 < \delta < \pi/2$, we readily see that

$$\Phi(\sigma + it) f_X(\sigma + it, \chi) e^{-i(\sigma+it)(\frac{1}{2}\pi-\delta)} \quad \text{and} \quad \phi(ixe^{-i\delta}, \chi)$$

are Mellin transforms. Using Parseval's formula and Hölder inequality, we obtain:

$$\mathfrak{M}_Q(\sigma) = 2\pi \int_0^\infty \sum_{\substack{q \leq Q, \\ \chi \bmod^* q}}^\star |\phi(ixe^{-i\delta}, \chi)|^2 x^{2\sigma-1} dx$$

$$\leq 2\pi \Big( \int_0^\infty \sum_{\substack{q \leq Q, \\ \chi \bmod^* q}}^\star |\phi(ixe^{-i\delta}, \chi)|^2 dx \Big)^a \Big( \int_0^\infty \sum_{\substack{q \leq Q, \\ \chi \bmod^* q}}^\star |\phi(ixe^{-i\delta}, \chi)|^2 x^{1+2\delta} dx \Big)^b$$

$$\leq \mathfrak{M}_Q(1/2)^a \mathfrak{M}_Q(1+\delta)^b.$$

$\square$

**Lemma 10.3.** *We have* $|\cos(\sigma + it)| \geq |\cos\sigma\sin\sigma| e^{|t|}$.

*Proof.* We have $|\cos(\sigma + it)|^2 = (\cos\sigma)^2 + (\operatorname{sh} t)^2$, hence the inequality to be proved is equivalent to $(\cos\sigma)^2 e^{-2|t|} + (e^{-2t}\operatorname{sh}|t|)^2 \geq |\cos\sigma\sin\sigma|^2$, and this later is equivalent to $1 + (4|\cos\sigma|^2 - 2)e^{-2|t|} + e^{-4t} \geq 4|\cos\sigma\sin\sigma|^2$, and also to $(2|\cos\sigma|^2 - 1 + e^{-2|t|})^2 + 1 - (1 - 2|\cos\sigma|^2)^2 \geq 4|\cos\sigma\sin\sigma|^2$. This concludes the proof since $1 - (1 - 2|\cos\sigma|^2)^2 = 4|\cos\sigma\sin\sigma|^2$. $\square$

We now exploit inequality (24), still following [51, section 7.8], on appealing to Lemma 10.3 as well as $|(s-1)/s| \leq 1$ when $1/2 \leq \Re s \leq 1+\delta \leq 3/2$. We bound above the RHS of (25) via

$$\mathfrak{M}_Q(\sigma) \leq \Big( \frac{1}{\cos\sigma\sin\sigma} \Big)^{1/\tau} \int_{-\infty}^\infty e^{-|t|/\tau} \sum_{\substack{q \leq Q, \\ \chi \bmod^* q}}^\star |f_X(\sigma + it, \chi)|^2 dt.$$

On recalling (7), we see that an integration by parts give us

$$\mathfrak{M}_Q(\sigma) \leq \Big( \frac{1}{\cos\sigma\sin\sigma} \Big)^{1/\tau} \int_0^\infty e^{-T/\tau} F_Q(\sigma, T) dT/\tau$$

$$\leq \Big( \frac{2}{\sin 2\sigma} \Big)^{1/\tau} \int_0^\infty e^{-t} F_Q(\sigma, t\tau) dt$$

## 10.2 An upper bound for $\mathfrak{M}_Q(1/2)$

**Lemma 10.4.** *Let $T$ and $Q$ be positive real parameters that verify $Q^2T \geq 10^9$, $X \geq 2\,000\,Q^2$ and $Q \geq 10$. With the choice $X = Q^2T$ and $\tau = T$, we have*

$$\mathfrak{M}_Q(1/2) \leq 122(Q^5T^3)^{1/2}\log^3(Q^2T).$$

*Proof.* We appeal to Lemma 7.2 to infer that $\mathfrak{M}_Q(1/2)/(2 \times 8.80\sqrt{Q}\log X)$ is bounded above by

$$\int_0^\infty \mathfrak{N}_Q(t)e^{-t}dt$$

where $\mathfrak{N}_Q(t)$ is $(Q^2\tau t+0.27X)(2\tau^{1/4}t^{1/4}\log(Q\tau)+2\tau^{1/4}t^{1/4}\log t+3\log Q)^2$ i.e.

$4((\log t)^2 + 2(\log Q\tau)\log t) + (\log Q\tau)^2)Q^2\tau^{3/2}t^{3/2}$
$+12(\log Q)((\log t) + (\log Q\tau))Q^2\tau^{5/4}t^{5/4}$ $\qquad +9(\log Q)^2Q^2\tau t$
$+2 \times 0.27((\log t)^2 + 2(\log Q\tau)(\log t) + (\log Q\tau)^2)X\tau^{1/2}t^{1/2}$ $+9 \times 0.27(\log Q)^2X$

Note that in this proof, we keep $X$ and $\tau$ independant of $T$ and $Q$ until the integration has been done. On using values of $\Gamma$, $\Gamma'$ and $\Gamma''$ (see section 8), we get the bound

$4(1.31 + 2(\log Q\tau)0.935 + 1.33(\log Q\tau)^2)Q^2\tau^{3/2}$
$+12(\log Q)(1.21 + 1.15(\log Q\tau))Q^2\tau^{5/4}$ $\qquad +9(\log Q)^2Q^2\tau$
$+0.54(0.830 + 2 \times 0.0324(\log Q\tau) + (\log Q\tau)^20.887)X\tau^{1/2}$ $+2.43(\log Q)^2X$

We now take $X = Q^2T$ and $\tau = T$. We use the inequalities $\log Q \leq \log T$, $\log Q \leq (1/3)\log(Q^2T)$ and $\log(Q\tau) \leq \log(Q^2T)$. Here is the upper bound we get for $\mathfrak{M}_Q(1/2)/(17.6Q^{5/2}T^{3/2}\log^3(Q^2T))$

$4(\frac{1.31}{\log^2(Q^2T)} + 2\frac{0.935}{\log Q^2T} + 1.33)$
$+\frac{12}{3}(\frac{1.21}{\log Q^2T} + 1.15)T^{-1/4}$ $\qquad +T^{-1/2}$
$+0.54(\frac{0.830}{\log^2(Q^2T)} + 2\frac{0.0324}{\log Q^2T} + 0.887) +\frac{2.43}{9}T^{-1/2}$

which simplifies into the claimed quantity since $\log(Q^2T) \geq 9\log(10)$. $\qquad\square$

And here is the counterpart corresponding to the case $Q = 1$.

**Lemma 10.5.** *Let $T \geq 3 \cdot 10^{10}$. On selecting $X = \tau = T$, we have*

$$\mathfrak{M}_1(1/2) \leq 11.3\, T^{4/3}\log^3(T).$$

*Proof.* We appeal to Lemma 7.2 to infer that $\mathfrak{M}_1(1/2)/(2\times4.45\log X)$ is bounded above by

$$\int_0^\infty \mathfrak{N}_1(t)e^{-t}dt$$

where $\mathfrak{N}_1(t)$ is $(t\tau + 0.164X)((t\tau)^{1/6}\log\tau + (\tau t)^{1/6}\log t + 2.83)^2$, i.e.

$((\log\tau)^2 + 2(\log t)(\log\tau) + (\log t)^2)\tau^{4/3}t^{4/3}$
$+5.66(\log\tau + \log t)\tau^{7/6}t^{7/6}$ $\qquad +8.0089\tau t$
$+0.164(X(\log\tau)^2 + 2X(\log t)(\log\tau) + X(\log t)^2)\tau^{1/3}t^{1/3}$
$+0.92824(X\log\tau + X\log t)\tau^{1/6}t^{1/6}$ $\qquad +1.3134596X$

Note again that in this proof, we keep $X$ and $\tau$ independant of $T$ until the integration has been done. On using values of $\Gamma$, $\Gamma'$ and $\Gamma''$ (see section 8), and substituting $X = \tau = T$. we get the bound

$$(1.20(\log T)^2 + 1.472 \log T + 1.09)T^{4/3}$$
$$+5.66(0.569 \log T + 1.09)T^{7/6} \qquad\qquad +8.0089T$$
$$+0.164(0.893(\log T)^2 - 0.117 \log T + 0.0994)T^{1/3}$$
$$+0.92824(0.927 \log T - 0.308)T^{1/6} \qquad\qquad +1.3134596T$$

Here is the upper bound we get when $T \geq 3 \cdot 10^{10}$

$$\mathfrak{M}_1(1/2) \leq 11.3\, T^{4/3} \log^3(T).$$

$\square$

## 10.3 An upper bound for $\mathfrak{M}_Q(1 + \delta)$

We appeal to Lemma 7.3 (and recall that $\delta = 1/\log X$) to infer that

$$\mathfrak{M}_Q(1 + \delta) \leq 2 \int_0^\infty \left(7.27 + 0.951 Q^2 t\tau X^{-1}\right) e^{-t} dt (\log X)^2$$
$$\leq 2 \left(7.27 + 0.951 Q^2 T X^{-1}\right) (\log X)^2 \leq 16.45 \log^2(Q^2 T).$$

In case $Q = 1$, we get

$$\mathfrak{M}_1(1 + \delta) \leq \int_0^\infty 2 \left(1.41 + 0.0443 \frac{t\tau}{eX}\right) e^{-t} dt (\log X)^2$$
$$\leq 2.909 (\log T)^2.$$

## 10.4 An upper bound for $\mathfrak{M}_Q(\sigma)$

We thus conclude via (24) that (note that $b = 1 - a$)

$$\mathfrak{M}_Q(\sigma) \leq \left(122(Q^5 T^3)^{1/2} \log^3(Q^2 T)\right)^a \left(16.5 \log^2(Q^2 T)\right)^b$$
$$\leq 16.5 \, (122/16.5)^a \, (Q^5 T^3)^{a/2} \log^{2+a}(Q^2 T).$$

We note that

$$(122/16.5)\frac{\sqrt{Q^5 T^3}}{\log(Q^2 T)} = (122/16.5)\sqrt{QT}\frac{Q^2 T}{\log(Q^2 T)} \geq 1.$$

Hence

$$\left(\frac{122 Q^{5/2} T^{3/2} \log^2(Q^2 T)}{16.5}\right)^{1 - \frac{\sigma - \frac{1}{2}}{1 + \delta - \frac{1}{2}}} \leq \left(\frac{122 Q^{5/2} T^{3/2} \log^2(Q^2 T)}{16.5}\right)^{1 - \frac{\sigma - \frac{1}{2}}{1 - \frac{1}{2}}}.$$

Hence the bound:

$$\boxed{\mathfrak{M}_Q(\sigma) \leq 16.5 \left(55 Q^5 T^3\right)^{1 - \sigma} \log^{4 - 2\sigma}(Q^2 T).}$$

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Let us now prove the counterpart of this bound when $Q = 1$.

$$\mathfrak{M}_Q(\sigma) \leq \left(11.3\, T^{4/3} \log^3(T)\right)^a \left(2.909 \log^2(T)\right)^b$$
$$\leq 2.91\, (11.3/2.91)^a\, T^{4a/3} \log^{2+a}(T).$$

Which we proceed as above to simplify into

$$\boxed{\mathfrak{M}_Q(\sigma) \leq 2.91\, (2.77\, T)^{8(1-\sigma)/3} \log^{4-2\sigma}(T)} \qquad (27)$$

## 10.5  End of the proof of Lemma 10.1 and 10.2

We first notice that

$$\left|\frac{s-1}{s(\cos s)^2}\right| = \left|1 - \frac{1}{s}\right| \frac{1}{\cos^2 \sigma + \sinh^2 t} \geq \left(1 - \frac{1}{|t|}\right) \frac{1}{1 + \sinh^2 |t|}.$$

The derivative of the right-hand-side is $(1+\sinh^2 t - 2(t^2 - t)\sinh t \cosh t)/(t(1+\sinh^2 t))^2$ which is negative when $t \geq 2$. The assumption $T_1 \geq 2$ comes here into play. From which we deduce that

$$\min_{T_1 \leq |t| \leq T_2} \frac{|s-1|^2}{|s|^2 |\cos s|^{1/\tau}} \geq \frac{(1 - \frac{1}{T_2})^2}{(1 + \sinh^2 T_2)^4} \frac{1}{\sqrt{1 + \sinh^2 T_2}^{\tau^{-1}-4}}.$$

We continue by (with $T_2 = \tau = T \geq 2000$)

$$\left(1 - \frac{1}{T}\right)^2 \frac{1}{(1 + \sinh^2 T)^{2/T}} \geq \left(1 - \frac{1}{T}\right)^2 \frac{2.00001^{1/T}}{e} \geq 0.367.$$

This leads to

$$\mathfrak{M}_Q(\sigma) \geq 0.367 \int_{T_1 \leq |t| \leq T_2} \sideset{}{^\star}\sum_{\substack{q \leq Q, \\ \chi \bmod^* q}} |f_X(\sigma + it, \chi)|^2 dt. \qquad (28)$$

# 11  The zero detection Lemma and proof of Theorem 1.1

## 11.1  $N_1(\sigma_1, 6, T, \chi)$: from a pointwise version to an averaged one

We use $\sigma_0 = \sigma_1 - 1/(3\log(Q^2 T))$ and write

$$N_1(\sigma_1, 6, T, \chi) \leq \int_{\sigma_0}^{\sigma_1} N_1(\sigma, 6, T, \chi) d\sigma / (\sigma_1 - \sigma_0).$$

We have to note here that the condition $\sigma \geq 0.65$ of Theorem 1.1 ensures that $\sigma_0 > 1/2$. Indeed the parameter $\sigma$ from this Theorem is $\sigma_1$.

## 11.2 From the averaged version to $F_Q(T,\sigma) - F_Q(6,\sigma)$: character per character

In 1914, H. Bohr and E. Landau proved in [8] for the first time that the number of zeroes off the critial line but within the critical strip is negligible when compared to the total number of zeroes. Their argument was qualitative and H.-E. Littlewood made it quantitative in [30]. We follow this approch as reproduced in [51, section 9.9]. For $\sigma_0 \in [\frac{1}{2}, 1]$, we have

$$2\pi \int_{\sigma_0}^{2} N_1(\sigma, T_1, T_2, \chi) d\sigma = \int_{T_1}^{T_2} \big(\log |g_X(\sigma_0 + it, \chi)| - \log |g_X(2 + it, \chi)|\big) dt$$

$$+ \int_{\sigma_0}^{2} \big(\arg g_X(\sigma + iT_2, \chi) - \arg g_X(\sigma + iT_1, \chi)\big) d\sigma \quad (29)$$

where $\arg g_X(s, \chi)$ is taken to be 0 on the line $\Re s = 2$.

There are two ways of studying the first integral. They both start by noticing

$$\log |h_X(s, \chi)| \leq \log(1 + |f_X(s, \chi)|^2). \quad (30)$$

The usual fashion is to continue by the inequality $\log(1 + |f_X(s, \chi)|^2) \leq |f_X(s, \chi)|^2$. We can however also appeal to the Jensen inequality and use instead:

$$\int_{T_1}^{T_2} \log |g_X(\sigma_0 + it, \chi)| dt \leq \int_{T_1 \leq |t| \leq T_2} \sum_{\substack{q \leq Q, \\ \chi \bmod^* q}}^{\star} \log(1 + |f_X(s, \chi)|^2) dt$$

$$\leq W \log \left(1 + \frac{1}{W} \int_{T_1 \leq |t| \leq T_2} \sum_{\substack{q \leq Q, \\ \chi \bmod^* q}}^{\star} |f_X(s, \chi)|^2 dt\right)$$

with $W = 2(T_2 - T_1) \sum\limits_{\substack{q \leq Q, \\ \chi \bmod^* q}}^{\star} 1$. This inequality is increasing in $W$ and we can take for $W$ and upper bound for the stated value. And in fact, when $W$ tends to infinity, we reach the former inequality. We will use this variation when $Q = 1$, with $W = 2T_2$.

Concerning the other summand in (29), we note that

$$-\log |h_X(2 + it, \chi)| \leq -\log(1 - |f_X(2 + it, \chi)|^2) \leq 2|f_X(2 + it, \chi)|^2 \quad (31)$$

provided $|f_X(2 + it, \chi)|^2 \leq 1/2$ which we prove now:

$$|f_X(2 + it, \chi)| \leq \sum_{n \geq X} \frac{|\sum_{d|n} \mu(d)|}{n^2} \leq \sum_{n \geq X} \frac{2^{\omega(n)}}{n^2}$$

$$\leq \sqrt{8/3} \sum_{n \geq X} \frac{1}{n^{3/2}} \leq \frac{2\sqrt{8/3}}{(X-1)^{1/2}} \leq 1/\sqrt{2}$$

since $X \geq 2000$ and $2^{\omega(n)} \leq \sqrt{8/3}\sqrt{n}$ (use multiplicativity).

**Bounding the argument**

Getting an upper bound for the argument is more tricky and relies on the following Lemma from [51, section 9.4]:

**Lemma 11.1.** *Let $0 \leq \alpha < \beta \leq 2$ and $F$ be an analytical function, real for real $s$, holomorphic for $\sigma \geq \alpha$ except maybe at $s = 1$. Let us assume that $|\Re F(2+it)| \geq m > 0$ and that $|F(\sigma'+it')| \leq M$ for $\sigma' \geq \sigma$ and $T \geq t' \geq T_0 - 2$. Then, if $T - 2 \geq T_0$ is not the ordinate of a zero of $F(s)$, we have*

$$|\arg F(\sigma + iT)| \leq \frac{\pi}{\log \frac{2-\alpha}{2-\beta}} \log(M/m) + \frac{3\pi}{2}$$

*valid for $\sigma \geq \beta$.*

The condition concerning the *ordinate* comes from the way we define the logarithm, and hence the argument. It is usually harmless since one can otherwise argue by continuity at the level of the resulting bound.

We use this lemma with $\alpha = 0$, $\beta = 1/2$ and $F = g_X(s, \chi)$ which is indeed real on the real axis. We already showed that

$$|\Re g_X(2+it, \chi)| \geq (1-|f_X(2+it, \chi)^2|)(1-|f_X(2+it, \overline{\chi})^2|) \geq (1-0.214^2)^2 \geq 0.91.$$

Hence, for $j = 1, 2$, on using Lemma 2.3 together with Lemma 3.1 to bound $|M_X(s)|$ by $0.62\,X$, we find that

$$|\arg g_X(\sigma + iT_j, \chi)| \leq 11 \times 2\log(1 + (0.62(qT_j)^{5/8}X)^2) + 17.$$

The use of this lemma asks for $T_1 = 4 + 2$ (the smallest value available). Since we fix this value, we can dispense with the index in $T_2$ and denote it by $T$. We continue as follows, since $X = Q^2T$:

$$|\arg g_X(\sigma + iT, \chi)| + |\arg g_X(\sigma + 6i, \chi)|$$
$$\leq 22\log(1 + (0.62(QT)^{5/8}Q^2T)^2) + 22\log(1 + (0.62(6Q)^{5/8}Q^2T)^2) + 34.$$

We have to compare this quantity with $\log(Q^2T)$, knowing that $Q^2T \geq 10^9$ and $T \geq 2000$. We use

$$\log(1 + (0.62(QT)^{5/8}Q^2T)^2) \leq \frac{13}{4}\log(Q^2T) + \log(0.62^2 + 10^{-9\times 13/4})$$
$$\leq \frac{13}{4}\log(Q^2T) - 0.956$$

and

$$\log(1 + (0.62(6Q)^{5/8}Q^2T)^2) \leq \log(1 + (0.62(\tfrac{6}{2000}TQ)^{5/8}Q^2T)^2)$$
$$\leq \log(1 + (0.62(\tfrac{6}{2000}TQ^2)^{5/8}Q^2T)^2)$$
$$\leq \frac{13}{4}\log(Q^2T) + \log(10^{-9\times 13/4} + (0.62(\tfrac{6}{2000})^{5/8})^2)$$
$$\leq \frac{13}{4}\log(Q^2T) - 8.21.$$

This finally amounts to

$$|\arg g_X(\sigma + iT, \chi)| + |\arg g_X(\sigma + 6i, \chi)| \leq 143\log(Q^2T) - 167.$$

We will multiply this bound by $3/2$ to take care of the integration over $\sigma$ in $[\sigma_0, 2]$ in (29).

**Partial conclusion**

Since $|f_X(2+it)| \le 1/(X-1)$, we get for $\sigma_0 \ge 1/2$

$$2\pi \int_{\sigma_0}^{2} N_1(\sigma, 6, T, \chi)d\sigma \le \int_{6}^{T} \left( |f_X(\sigma_0 + it, \chi)|^2 + |f_X(\sigma_0 + it, \overline{\chi})|^2 \right) dt$$
$$+ \frac{4(T-6)}{X-1} + 215 \log(Q^2 T) - 250. \quad (32)$$

We have been careful not to use the bound $Q \ge 10$ up to now to cover the two cases $Q = 1$ but $T \ge 3 \cdot 10^{10}$, and $Q \ge 10$, $T \ge 2000$ and $Q^2 T \ge 10^9$. We now have to distinguish both cases as the estimate from Lemma 4.4 requires a bound on $Q$.

## 11.3 From the averaged version to $F_Q(T, \sigma) - F_Q(6, \sigma)$: summing over characters

We sum (32) over $q$, use Lemma 4.4, and join the two previous steps. We get

$$\sum_{\substack{q \le Q, \\ \chi \bmod^* q}}^{\star} N_1(\sigma_1, 6, T, Q) \le \int_{6}^{T} \sum_{\substack{q \le Q, \\ \chi \bmod^* q}}^{\star} |f_X(\sigma_0 + it, \chi)|^2 dt \frac{3 \log(Q^2 T)}{\pi}$$
$$+ 31 \, Q^2 \log(Q^2 T).$$

But one should be careful: the variable $t$ ranges positive values only while it ranges a symmetric interval in $F_Q(T, \sigma) - F_Q(6, \sigma)$.

## 11.4 Using Lemma 10.1

We finally use $X = Q^2 T$, $Q \ge 10$ and $T \ge 2\,000$ and Lemma 10.1 to get

$$N_1(\sigma_1, 6, T, Q) \le \frac{16.5 \times 3}{0.367 \times 2\pi} \left( 55 Q^5 T^3 \right)^{1-\sigma_0} \log^{5-2\sigma_0}(Q^2 T)$$
$$+ 31 \, Q^2 \log(Q^2 T).$$

We have to replace $\sigma_0$ by $\sigma_1$. We define $\delta_1 = 1/(3 \log(Q^2 T))$ and note that, with $x = \log(Q^2 T)$, we have

$$\left( 55 Q^5 T^3 \right)^{\delta_1} \log^{2\delta_1}(Q^2 T) \le 5.5^{\delta_1} \exp\left( 3x \frac{1}{3x} + (\log x) \frac{2}{3x} \right) \le 3.84.$$

All of that amounts to

$$N_1(\sigma_1, 6, T, Q) \le 83 \left( 55 Q^5 T^3 \right)^{1-\sigma_1} \log^{5-2\sigma_1}(Q^2 T) + 31 \, Q^2 \log^2(Q^2 T).$$

We simplify and use (21) to get the stated result.

## 11.5 Using Lemma 10.2

Here is the counterpart when $Q = 1$. We combine (32) together with Lemma 10.2 to get

$$N_1(\sigma_1, 6, T, 1) \leq \frac{2.91 \times 3}{0.367 \times 2\pi}(3.77\,T)^{8(1-\sigma_0)/3}\log^{5-2\sigma_0}(T)$$
$$+ 103\,(\log T)^2. \quad (33)$$

Hence

$$N_1(\sigma_1, 6, T, 1) \leq 9.72(3T)^{8(1-\sigma_1)/3}\log^{5-2\sigma_1}(T) + 103(\log T)^2.$$

If we are to use the variation implying the Jensen inequality, we reach

$$N_1(\sigma_1, 6, T, 1) \leq 2T\log\big(1 + 9.72(3T)^{\frac{8}{3}8(1-\sigma_1)-1}\big)\log^{5-2\sigma_1}(T) + 103(\log T)^2.$$

The main Theorem follows readily.

# References

[1] M. Abramowitz and I.A. Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, volume 55 of *National Bureau of Standards Applied Mathematics Series.* For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C., 1964. `http://mintaka.sdsu.edu/faculty/wfw/ABRAMOWITZ-STEGUN`.

[2] R. Bacher. Determinants related to Dirichlet characters modulo 2, 4 and 8 of binomial coefficients and the algebra of recurrence matrices. *Internat. J. Algebra Comput.*, 18(3):535–566, 2008.

[3] R. Balasubramanian, B. Calado, and H. Queffélec. The Bohr inequality for ordinary Dirichlet series. *Studia Math.*, 175(3):285–304, 2006.

[4] P. Barrucand and F. Laubie. Sur les symboles des restes quadratiques des discriminants. *Acta Arith.*, 48(1):81–88, 1987.

[5] G. Bastien and M. Rogalski. Convexité, complète monotonie et inégalités sur les fonctions zêta et gamma, sur les fonctions des opérateurs de Baskakov et sur des fonctions arithmétiques. *Canad. J. Math.*, 54(5):916–944, 2002.

[6] F. Bayart, C. Finet, D. Li, and H. Queffélec. Composition operators on the Wiener-Dirichlet algebra. *J. Operator Theory*, 60(1):45–70, 2008.

[7] M. Bennett. Rational approximation to algebraic numbers of small height: the Diophantine equation $|ax^n - by^n| = 1$. *J. reine angew. Math.*, 535:1–49, 2001.

[8] H. Bohr and E. Landau. Sur les zéros de la fonction $\zeta(s)$ de *Riemann. C. R.*, 158:106–110, 1914.

[9] E. Bombieri. Le grand crible dans la théorie analytique des nombres. *Astérisque*, 18:103pp, 1987/1974.

[10] Jingrun Chen. On zeros of Dirichlet's $L$ functions. *Sci. Sinica Ser. A*, 29(9):897–913, 1986.

[11] Y. Cheng and S.W. Graham. Explicit estimates for the Riemann zeta function. *Rocky Mountain J. Math.*, 34(4):1261–1280, 2004.

[12] H. Cohen and F. Dress. Estimations numériques du reste de la fonction sommatoire relative aux entiers sans facteur carré. *Prépublications mathématiques d'Orsay : Colloque de théorie analytique des nombres, Marseille*, pages 73–76, 1988.

[13] J.B. Conrey. At least two-fifths of the zeros of the Riemann zeta function are on the critical line. *Bull. Amer. Math. Soc. (N.S.)*, 20(1):79–81, 1989.

[14] J.-M. Deshouillers and H. Iwaniec. Power mean-values for Dirichlet's polynomials and the Riemann zeta-function. II. *Acta Arith.*, 43(3):305–312, 1984.

[15] P. Dusart. *Autour de la fonction qui compte le nombre de nombres premiers.* PhD thesis, Limoges, `http\string://www.unilim.fr/laco/theses/1998/T1998_01.pdf`, 1998. 173 pp.

[16] P. Dusart. Inégalités explicites pour $\psi(x)$, $\theta(x)$, $\pi(x)$ et les nombres premiers. *C. R. Math. Acad. Sci., Soc. R. Can.*, 21(2):53–59, 1999.

[17] P. Dusart. Estimates of some functions over primes without R. H. *http://arxiv.org/abs/1002.0442*, 2010.

[18] S. Saad Eddin. Explicit upper bounds for the Stieltjes constants. *J. Number Theory*, 133(3):1027–1044, 2013.

[19] L. Fousse, G. Hanrot, V. Lefèvre, P. Pélissier, and P. Zimmermann. MPFR: a multiple-precision binary floating-point library with correct rounding. *ACM Trans. Math. Software*, 33(2):Art. 13, 15, 2007.

[20] P.X. Gallagher. A large sieve density estimate near $\sigma = 1$. *Invent. Math.*, 11:329–339, 1970.

[21] X. Gourdon. The $10^{13}$ first zeros of the Riemann Zeta Function and zeros computations at very large height. *http://numbers.computation.free.fr/Constants/Miscellaneous/zetazeros1e13-1e24.pdf*, 2004.

[22] I.S. Gradshteyn and I.M. Ryzhik. *Table of integrals, series, and products. Translated from the Russian. Translation edited and with a preface by Alan Jeffrey and Daniel Zwillinger. 6th ed.* San Diego, CA: Academic Press. xlvii, 1163 p., 2000.

[23] A. Granville and O. Ramaré. Explicit bounds on exponential sums and the scarcity of squarefree binomial coefficients. *Mathematika*, 43(1):73–107, 1996.

[24] G.H. Hardy, A.E. Ingham, and G. Pólya. Theorems concerning mean values of analytic functions. *Proceedings Royal Soc. London (A)*, 113:542–569, 1927.

[25] H.A. Helfgott. Minor arcs for Goldbach's problem. *Submitted*, 2012. arXiv:1205.5252.

[26] H.A. Helfgott. Major arcs for Goldbach's theorem. *Submitted*, 2013. arXiv:1305.2897.

[27] H. Kadiri. A zero density result for the Riemann zeta function. page 15pp, 2013.

[28] H. Kadiri and L. Faber. New bounds for $\psi(x)$. *preprint*, page 15pp, 2013.

[29] M. Kolster and Thong Nguyen Quang Do. Syntomic regulators and special values of $p$-adic $L$-functions. *Invent. Math.*, 133(2):417–447, 1998.

[30] J.E. Littlewood. On the zeros of the Riemann Zeta-function. *Cambr. Phil. Soc. Proc.*, 22:295–318, 1924.

[31] Ming-Chit Liu and Tianze Wang. Distribution of zeros of Dirichlet $l$-functions and an explicit formula for $\psi(t, \chi)$. *Acta Arith.*, 102(3):261–293, 2002.

[32] Ming-Chit Liu and Tianze Wang. On the Vinogradov bound in the three primes Goldbach conjecture. *Acta Arith.*, 105(2):133–175, 2002.

[33] F. Martin and E. Royer. Formes modulaires et périodes. In *Formes modulaires et transcendance*, volume 12 of *Sémin. Congr.*, pages 1–117. Soc. Math. France, Paris, 2005.

[34] K.S. McCurley. Explicit estimates for the error term in the prime number theorem for arithmetic progressions. *Math. Comp.*, 42:265–285, 1984.

[35] H.L. Montgomery and R.C. Vaughan. Hilbert's inequality. *J. Lond. Math. Soc., II Ser.*, 8:73–82, 1974.

[36] S. Nazardonyavi and S. Yakubovich. Sharper estimates for Chebyshev's functions $\vartheta$ and $\psi$. *preprint*, 2013. arXiv:1302.7208.

[37] The PARI Group, Bordeaux. *PARI/GP, version* 2.5.2, 2011. `http://pari.math.u-bordeaux.fr/`.

[38] D.J. Platt. *Computing degree 1 L-function rigorously*. PhD thesis, Mathematics, 2011. arXiv:1305.3087.

[39] E. Preissmann. Sur une inégalité de Montgomery et Vaughan. *Enseign. Math.*, 30:95–113, 1984.

[40] H. Rademacher. On the Phragmén-Lindelöf theorem and some applications. *Math. Z.*, 72:192–204, 1959.

[41] O. Ramaré. On Snirel'man's constant. *Ann. Scu. Norm. Pisa*, 21:645–706, 1995. `http://math.univ-lille1.fr/~ramare/Maths/Article.pdf`.

[42] O. Ramaré. Explicit estimates on the summatory functions of the Moebius function with coprimality restrictions. *Submitted*, page 12pp, 2013.

[43] G. Ricotta. Real zeros and size of Rankin-Selberg $L$-functions in the level aspect. *Duke Math. J.*, 131(2):291–350, 2006.

[44] R. Rumely. Numerical Computations Concerning the ERH. *Math. Comp.*, 61:415–440, 1993.

[45] N. Saby. Théorie d'Iwasawa géométrique: un théorème de comparaison. *J. Number Theory*, 59(2):225–247, 1996.

[46] L. Schoenfeld. An improved estimate for the summatory function of the Möbius function. *Acta Arith.*, 15:223–233, 1969.

[47] R. Spira. Calculation of Dirichlet $L$-functions. *Math. Comp.*, 23:489–497, 1969.

[48] W. A. Stein et al. *Sage Mathematics Software (Version 5.9)*. The Sage Development Team, 2013. `http://www.sagemath.org`.

[49] T. Tao. Every odd number greater than 1 is the sum of at most five primes. *Math. Comp.*, 2013.

[50] J.-D. Therond. Les idéaux de l'anneau des entiers d'un corps quadratique. In *Publications du Centre de Recherches en Mathématiques Pures. Sér. II, Tome 13*, volume 13 of *Publ. Centre Rech. Math. Pures Sér. II*, pages 17–24. Univ. Neuchâtel, Neuchâtel, 1995.

[51] E.C. Titchmarsh. *The Theory of Riemann Zeta Function*. Oxford Univ. Press, Oxford 1951, 1951.

[52] T. Trudgian. An improved upper bound for the argument of the Riemann zeta-function on the critical line, ii. *Submitted Math. Comp.*, 2013. `http://arxiv.org/abs/1208.5846`.

[53] S. Wedeniwski. On the Riemann hypothesis. *http://www.zetagrid.net*, 2009.