A PURELY ANALYTICAL LOWER BOUND FOR $L(1,\chi)$

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[1] [2]

Abstract

We give a simple proof of $L(1,\chi)\sqrt{q} \gg 2^{\omega(q)}$ where χ is a quadratic character of conductor q. In particular we do not use the Dirichlet class-number formula

1 Main results

For a Dirichlet quadratic character χ of conductor q, several techniques were devised to get a lower bound for $L(1,\chi)$. One of them consists in estimating

$$S(\alpha) = \sum_{n\geq 1} \big(\sum_{d|n} \chi(d)\big) e^{-\alpha n} \tag{1}$$

in two ways, where $\alpha > 0$ is a parameter to be chosen. We first notice that $(1 \star \chi)(n) \geq 0$ and even ≥ 1 if n is a square, thus obtaining the lower bound $S(\alpha) \gg \alpha^{-1/2}$. On an other side, reversing the inner summation yields

$$S(\alpha) = L(1,\chi)\alpha^{-1} + \sum_{d\geq 1} \chi(d) \left(\frac{1}{e^{\alpha d} - 1} - \frac{1}{\alpha d} \right). \tag{2}$$

Using partial summation and the Polya-Vinogradov inequality, the "remainder term" is $\mathcal{O}(\sqrt{q}\,\mathrm{Log}\, q)$. Taking $\alpha^{-1} = c\sqrt{q}\,\mathrm{Log}^2 q$ for a large enough

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constant c yields $L(1,\chi)\sqrt{q} \gg 1/\operatorname{Log} q$. It is fairly easy to remove this $\operatorname{Log} q$ by noticing that only a smoothed version of the Polya-Vinogradov inequality is required, thus getting $L(1,\chi)\sqrt{q} \geq c$ for some positive constant c.

The next question is to evaluate c. Recalling the Dirichlet class number formula

$$L(1,\chi)\sqrt{q} = \begin{cases} \pi h(-q) & \text{if } \chi(-1) = -1, q \geq 5 \\ \operatorname{Log} \epsilon_q \; h(q) & \text{if } \chi(-1) = +1, \end{cases} \tag{3}$$

we see that $c > \pi$ for q larger than an explicit value would give another solution of the class number 1 problem. As it turns out the previous method can be made to yield the bound $c = \pi - o(1)$. Such a result is a priori surprising since the method being analytical, the smoothing e^{-x} can be modified. We have not been able to solve the resulting extremal problem in order to get the best possible smoothing, but we have tried numerically several of them (including all the smoothings of the form $P(x)e^{-x}$ where P is a polynomial) and found no better one, which suggests that this choice is (sadly) optimal. The general case being cumbersome, we have chosen to present a shorter proof which is well adapted to this particular choice and enables us to give a more complete description.

Using genus theory of quadratic forms, one can deduce from the Dirichlet class number formula that $L(1,\chi) \geq \frac{\pi}{2} 2^{\omega(q)}/\sqrt{q}$ and it turns out that the previous method can be modified to yield a bound of similar strength, without any appeal either to the theory of quadratic forms, or to the Dirichlet class number formula. We can prove

Theorem 1 *When q goes to infinity, we have*

$$\begin{cases} L(1,\chi) \geq (1+o(1))\pi/\sqrt{q} & (\chi(-1) = -1), \\ L(1,\chi) \geq (1+o(1))\sqrt{\pi}/\sqrt{q} & (\chi(-1) = +1). \end{cases}$$

Moreover, uniformly for $q \geq 3$, we have

$$\begin{cases} L(1,\chi) \geq \dfrac{2^{\omega(q)-1}\pi}{10\sqrt{q}} & (\chi(-1) = -1), \\ L(1,\chi) \geq \dfrac{2^{\omega(q)-1}\sqrt{\pi}}{2\sqrt{q}} & (\chi(-1) = +1). \end{cases}$$

To do so we first rewrite the above proof. We define $2k = 3 + \chi(-1)$. The reader may get a clearer understanding of the argument by considering only the case $k = 1$. The function

$$\Phi_k(s) = \left(\frac{2\pi}{kq^{k/2}}\right)^{-s} \Gamma(s)\zeta(ks)L(ks, \chi), \tag{4}$$

verifies the relation $\Phi_k(\frac{1}{k} - s) = \Phi_k(s)$. Consequently and following Hecke's theory on Dirichlet series having a functional equation, we deduce that the function of the variable $\tau = x + iy$ with $y > 0$

$$f_k(\tau) = \frac{\sqrt{q}}{2\pi^{1/k}} L(1, \chi) + \sum_{n\geq 1} (1 \star \chi)(n) e^{\frac{2i\pi n^k \tau}{kq^{k/2}}}, \tag{5}$$

verifies $f_k(-1/\tau) = (\tau/i)^{1/k} f_k(\tau)$. To get these relations, we simply express f_k in term of Φ_k by the Mellin formula and use the functional equation for Φ_k.

The next step consists in elaborating on the argument $(1 \star \chi)(n^2) \geq 1$. To do so let λ be Liouville's function (whose associated Dirichlet series is $\zeta(2s)\zeta(s)^{-1}$). We check that μ^2 is the convolution inverse of λ and that $1 \star \lambda$ is the characteristic function of the squares. Defining $\nu = \mu^2 \star \chi \geq 0$, we get

$$1 \star \chi = 1 \star \lambda \star \mu^2 \star \chi = \nu \star (1 \star \lambda)$$

which yields

$$f_k(iz) = \frac{\sqrt{q}}{2\pi^{1/k}} L(1, \chi) + \sum_{m\geq 1} \nu(m) \theta_k(2\pi m^k z/(kq^{k/2})) \tag{6}$$

where

$$\theta_k(z) = \sum_{n\geq 1} e^{-zn^{2k}}. \tag{7}$$

Using the functional equation of f_k, we infer

$$\frac{1 - x^{1/k}}{2} \frac{\sqrt{q}}{\pi^{1/k}} L(1, \chi) = \sum_{m\geq 1} \nu(m) \mathcal{H}_k(x, 2\pi m^k x/(kq^{k/2})) \tag{8}$$

where

$$\mathcal{H}_k(x, y) = x^{1/k} \theta_k(y) - \theta_k(y/x^2). \tag{9}$$

The point is that $\mathcal{H}_k(x,y) \geq 0$ as soon as x is not too close neither to 1 nor to 0. This property is highly non-obvious since the main terms (as y is close to 0) of both summands of the RHS of (9) cancel each other as may be seen in the proof below. $\mathcal{H}_k(x,y)$ is then about $(1-x^{1/k})/2$ when y is small and then decays to 0.

Lemma 1 *For $k=1$ or $k=2$, and $x^{1/k}=0.43542$, we have $\mathcal{H}_k(x,y)\geq 0$ for $y\geq 0$. Moreover the estimation $\mathcal{H}_k(x,y)=\frac{1-x^{1/k}}{2}+\mathcal{O}(y^{1/k})$ holds. Finally we have*

$$\min_{y\leq 2\pi x}\mathcal{H}_1(x,y)\geq\frac{1-x}{20}\qquad,\qquad \min_{y\leq \pi x}\mathcal{H}_2(x,y)\geq\frac{1-\sqrt{x}}{4}.$$

Proof. We assume throughout that $0<x<1$. The proof is separated in two parts according as to y is small or not. We start with the case of small y's. By Euler-MacLaurin summation formula, we get

$$\theta_k(y)=\int_0^\infty e^{-yt^{2k}}dt-\frac{1}{2}+\frac{(-1)^{h+1}}{h!}\int_0^\infty B_h(t)f^{(h)}(t)dt\qquad (h\geq 2)$$

where B_h is the h-th Bernoulli function and $f(t)=\exp(-yt^{2k})$. This formula is so simple because f is even so that $f^{(r)}(0)B_{r+1}(0)=0$ if $r\geq 1$. We use the above expression with $h=3$. Recalling that $B_3(t)=t^3-3t^2/2+t/2$ for $t\in[0,1]$, we get $|B_3(t)|\leq 1/(12\sqrt{3})$ and thus

$$\theta_k(y)=y^{-1/(2k)}\Gamma(1+\tfrac{1}{2k})-\tfrac{1}{2}+\mathcal{O}^*(C(k)y^{1/k})$$

where $f(t)=\mathcal{O}^*(g(t))$ means $|f(t)|\leq g(t)$ and where

$$C(k)=\frac{1}{72\sqrt{3}}\int_0^\infty\left|\frac{d^3}{dt^3}e^{-t^{2k}}\right|dt=\begin{cases}0.024919+\mathcal{O}^*(10^{-6}) & (k=1)\\ 0.113326+\mathcal{O}^*(10^{-6}) & (k=2).\end{cases}$$

Details of this computation can be found in the appendix. We thus get

$$\mathcal{H}_k(x,y)\geq\frac{1-x^{1/k}}{2}\left(1-2C(k)\,y^{1/k}\frac{x^{1/k}+x^{-2/k}}{1-x^{1/k}}\right).\tag{10}$$

The value of x has been chosen so as to minimize the coefficient of $y^{1/k}$ inside the parentheses. From this expression it follows that $\mathcal{H}_k(x,y)\geq 0$ as soon as

$$y/x^2\leq w_k(x)=\left(\frac{1-x^{1/k}}{2C(k)(1+x^{3/k})}\right)^k.$$

Let us assume now that $y/x^2 \geq w_k(x)$. Using geometrical progressions, we get $\theta_k(y/x^2) \leq e^{-y/x^2}/[1-\exp(-(4^k-1)w_k(x))]$ and thus $\mathcal{H}_k(x,y) \geq 0$ as soon as

$$(1-e^{-(4^k-1)w_k(x)})e^{(x^{-2}-1)y} \geq x^{-1/k}$$

Since we want to cover the range $y \in]0,\infty[$, the condition on x reads

$$(1-e^{-(4^k-1)w_k(x)})e^{(1-x^2)w_k(x)} \geq x^{-1/k} \tag{11}$$

which holds for the chosen value of x. See the Appendix for further details. To get the lower bound of $\mathcal{H}_k(x,y)$ over $y \leq 2\pi x/k$ given in the lemma, we first check that (10) implies it if $y \leq 0.8x$ and $k=1$ or if $y \leq 0.25x$ and $k=2$. We extend both ranges by using

$$\mathcal{H}_k(x,y) \geq x^{1/k}e^{-y} - \frac{e^{-y/x^2}}{1-e^{-(4^k-1)\xi/x}} \quad \text{for} \quad y \geq \xi x. \tag{12}$$

This lower bound is first increasing and then decreasing as a function of y. The lemma follows readily.

We now continue the proof of Theorem 1. Using (8) and the lemma, we get

$$L(1,\chi)\sqrt{q} \geq \pi^{1/k}\big(1+\mathcal{O}(q^{-1/2})\big)$$

by discarding all the terms except the one corresponding to $m=1$. By discarding only the terms corresponding to $m > \sqrt{q}$, we get

Theorem 2 *We have*

$$\begin{cases} L(1,\chi)\sqrt{q} \geq \dfrac{\pi}{10} \displaystyle\sum_{m\leq\sqrt{q}} \nu(m) & (\chi(-1)=-1), \\ L(1,\chi)\sqrt{q} \geq \dfrac{\sqrt{\pi}}{2} \displaystyle\sum_{m\leq\sqrt{q}} \nu(m) & (\chi(-1)=+1). \end{cases}$$

To end the proof of Theorem 1, note that $\nu(p)=1$ if $p|q$. Since half the divisors of q are less than $\sqrt{q}$, we conclude easily.

Theorem 2 is to be compared with what one can get by using the theory of quadratic forms. For instance (cf [2], 1.4), if $\chi(-1)=-1$, Oesterlé proves

$$L(1,\chi)\sqrt{q}/\pi \geq \sum_{m\leq\sqrt{q}/2} \nu(m),$$

and Goldfeld [1] already proves similar inequalities. Modifying this technique we can extend the range of summation to $m \leq \sqrt{q}$ if we divide the lower bound by 3. We do not know of any similar results in case $\chi(-1) = 1$ (though we tend to believe they exist).

2 Numerical Appendix

2.1

With $g(t) = e^{-t^\ell}$, we get

$$\begin{aligned} g''(t) &= (-\ell(\ell-1)t^{\ell-2} + \ell^2 t^{2\ell-2})g(t), \\ g'''(t) &= \ell t^{\ell-3}(-(\ell-1)(\ell-2) + 3\ell(\ell-1)t^\ell - \ell^2 t^{2\ell})g(t). \end{aligned}$$

The function $g'''(t)$ is negative except for $t \in [t_{-1}, t_1]$ where it is non-negative and where

$$t_\varepsilon^\ell = \left(3(\ell-1) + \varepsilon\sqrt{(\ell-1)(5\ell-1)}\right)/(2\ell).$$

The L^1-norm of g''' is $g''(0) + 2(g''(t_1) - g''(t_{-1}))$, which gives the values of $C(1)$ ($\ell = 2$) and $C(2)$ ($\ell = 4$).

2.2

Note that the RHS of (11) is increasing as a function of w_k and that $w_k(x)$ is a decreasing function of x. By splitting the interval $[0, 1]$ in 1000 subintervals $[x_-, x_+]$, and using the fact that (12) is implied by

$$e^{(1-x_+^2)w_k(x_+)} - e^{(2-4^k-x_-^2)w_k(x_+)} - x_-^{-1/k} \geq 0 \qquad (x_- \leq x \leq x_+) \tag{13}$$

we check that (12) is verified in the following two ranges :

$$0.001 \leq x \leq 0.917 \quad (k=1), \qquad \text{and} \qquad 0.001 \leq x \leq 0.621 \quad (k=2).$$

References

[1] D. M. Goldfeld (1976) The class number of quadratic fields and the conjectures of Birch and Swinnerton-Dyer. Bull. Am. Math. Soc. **12**:623–663.

[2] J. Oesterlé (1985) Nombres de classes des corps quadratiques imaginaires. Astérisque **121-122**:309–323.