Extracting informations from the linear entire model of peg solitaire*†

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January 19, 2008

Abstract

We introduce several geometrical notions, on a solitaire board, that can be computed via linear programming. The notions are the *thickness* and *obligation* of a move, the maximal and minimal *tension*, *call* and *load* of a point.

1 Introduction

In (Berlekamp *et al.*, 1982), (Avis *et al.*, 2000) and more explicitly in (Ramaré, 2008b), we showed that, when it was possible to derive a position J from an initial position I in legal moves on a solitaire board $\mathfrak{S}$, then $\mathbb{1}_I - \mathbb{1}_J$ was a linear combination with integral non-negative coefficients of moves $\mathfrak{f} = \check{P} + \check{Q} - \check{R}$, i.e. elements from $\mathscr{D}(\mathfrak{S})$. This leads to a fast test, and our aim here is to extract more information from this conceptualisation. We define $V^+(\mathfrak{S}, \mathbb{Z})$ to be the set of such linear combination with integral non-negative coefficients. We already defined in (Ramaré, 2008b) the *thickness* $\mathcal{E}(A, I, J)$ of a move $\mathfrak{f}$ with respect to the two positions I and J to be the maximum number of times $\mathfrak{f}$ is used in any legal derivation from I to J. We similarly define the *obligation* $\mathcal{O}(A, I, J)$ of this move with respect to the two positions I and J to be the minimal number of times this move is used.

These two notions deal with moves, and we have six corresponding notions concerning points, namely

*keywords: Peg solitaire, Hi-Q, Pagoda function.
†AMS classification: primary 05A99, secondary 91A46, 52B12, 90C08.

1. The maximal, resp. minimal, *load* at a point A with respect to I and J is the maximal, resp. minimal, number $\mathcal{L}_\pm(A, I, J)$ of times this point is used as R-point (i.e. the point where the peg lands) in any legal writing from I to J. We set $\mathcal{L}_+(A, I, J) = \infty$ and $\mathcal{L}_-(A, I, J) = 0$ when no such writing exists.

2. The maximal, resp. minimal, *call* at a point A with respect to I and J is the maximal, resp. minimal, number $\mathcal{C}_\pm(A, I, J)$ of times this point is used as Q-point (i.e. the point whose peg is being eaten) in any legal writing from I to J. We set $\mathcal{C}_+(A, I, J) = \infty$ and $\mathcal{C}_-(A, I, J) = 0$ when no such writing exists.

3. The maximal, resp. minimal, *tension* at a point A with respect to I and J is the maximal, resp. minimal number, $\mathcal{T}_\pm(A, I, J)$ of times this point is used as P-point (i.e. the point from which a peg jumps over the other one) in any legal writing from I to J. We set $\mathcal{T}_+(A, I, J) = \infty$ and $\mathcal{T}_-(A, I, J) = 0$ when no such writing exists.

We now provide lower and upper bounds for these quantities by using the linear model just recalled.

2 Thickness and obligation

Here is our main result:

Theorem 2.1 *Let I and J be two positions of $\mathfrak{S}$, $\mathfrak{f}_0 \in \mathscr{D}(\mathfrak{S})$ and π be a resource count on $\mathfrak{S}$ such that $\langle \pi, \mathfrak{f}_0 \rangle = 1$. Assume that one can derive J from I with legal moves. The move $\mathfrak{f}_0$ can appear at most $\langle \pi, \mathbb{1}_I - \mathbb{1}_J \rangle$ in any writing of $\mathbb{1}_I - \mathbb{1}_J$ as a linear combination of elements of $\mathscr{D}(\mathfrak{S})$ with non-negative integer coefficients.*

This criterium provides us with an upper bound m of the thickness, which we have shown in (Ramaré, 2008b) to be not more than 17. We refine it as follows: if $\mathbb{1}_I - \mathbb{1}_J - m\mathfrak{f}_0 \notin V^+(\mathfrak{S}, \mathbb{Z})$ then we may decrease m by one and try again.

Similarly, to find a lower bound for the obligation at $\mathfrak{f}_0$, we first seek writings with integral non-positive coefficients of $\mathbb{1}_I - \mathbb{1}_J$ that do not use $\mathfrak{f}_0$. If any such writing exists, we keep the lower bound 0. Else we proceed by looking for any such writing of $\mathbb{1}_I - \mathbb{1}_J - \mathfrak{f}_0$, then $\mathbb{1}_I - \mathbb{1}_J - 2\mathfrak{f}_0$ and so on. This process is limited by the thickness.

This provides us with decent upper and lower bounds for the thickness and obligation when the initial and final positions are fixed. But we have seen in (Ramaré, 2008b) that we have the universal bound 17 for the thickness. This bound is most probably too large but I do not know how to get any improvement of it. By using a criterium like the one of Theorem 2.1 but restricted to non-negative resource counts, we get the shown upper bounds for the thickness of each move on an english board. The upper bound 5 is again most probably too high, but I have not been able to improve on it.

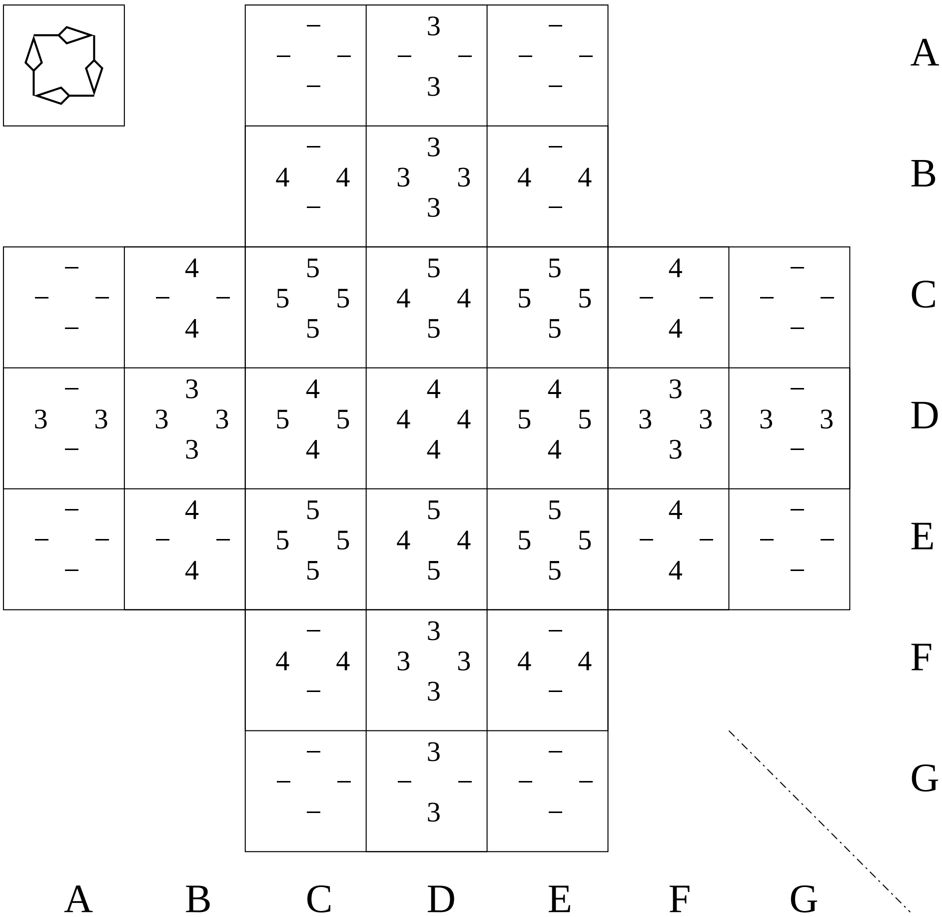

Figure 1: Upper bound for the maximal thickness on an english board

Here is a precise and specialized conjecture:

Conjecture 2.1

The move $\check{AC} + \check{AD} - \check{AE}$ cannot be used more than twice.

3 Load, call and tension at a point

Theorem 3.1 *Let I and J be two positions of $\mathfrak{S}$, $A \in \mathfrak{S}$ and π be a resource count on $\mathfrak{S}$ such that $\langle \pi, \mathfrak{f} \rangle \geq 1$ for all $\mathfrak{f} \in \mathscr{D}(\mathfrak{S})$ such that A is the P-point of $\mathfrak{f}$. Assume that one can derive J from I with legal moves. The point A can appear at the P position in any derivation in legal moves of J from I at most $\langle \pi, \mathbb{1}_I - \mathbb{1}_J \rangle$ times.*

Let us consider the resource count on $\mathbb{Z}^2$ given by $\pi_0((m,n)) = \rho^{|m|+|n|}$ where $\rho = (-1+\sqrt{5})/2$ (note that $\rho^2 + \rho = 1$) and put $\mathfrak{S}$ on $\mathbb{Z}^2$ in such a way that A falls on $(0,0)$. Any $\mathfrak{f}$ having A as P-point verifies $\langle \pi_0, \mathfrak{f} \rangle = 2\rho$ and $\langle \pi_0, \mathbb{1}_{\mathfrak{S}} \rangle = 8\rho + 13$. By using the resource count $\pi = \pi_0/(2\rho)$, we get that A is used at most

$$(13\rho + 21)/2 \leq 14.517\cdots$$

times as P-point. Note that this proof shows that the set of resource counts verifying the hypothesis of the Theorem above is non-empty.

We have just proved that the tension is majorised by 14. The same resource count shows that the call is at most 17. This bound is huge and 12 seems more reasonable but, again, I do not have any proof! One can also modify Theorem 3.1 to get a Theorem applicable to this case or to majorize the load, but getting a universal bound requires a slightly more complicated resource count. We use the following one:

$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\cdots$	ρ^5	ρ^4	ρ^3	ρ^2	ρ^3	ρ^4	ρ^5	$\cdots$
$\cdots$	ρ^4	ρ^3	ρ^2	ρ	ρ^2	ρ^3	ρ^4	$\cdots$
$\cdots$	ρ^3	ρ^2	ρ	1	ρ	ρ^2	ρ^3	$\cdots$
$\cdots$	ρ^2	ρ	1	0	1	ρ	ρ^2	$\cdots$
$\cdots$	ρ^3	ρ^2	ρ	1	ρ	ρ^2	ρ^3	$\cdots$
$\cdots$	ρ^4	ρ^3	ρ^2	ρ	ρ^2	ρ^3	ρ^4	$\cdots$
$\cdots$	ρ^5	ρ^4	ρ^3	ρ^2	ρ^3	ρ^4	ρ^5	$\cdots$
$\vdots$	$\vdots$	$\vdots\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$

Its total weight (namely $\langle \pi, \mathbb{1}_{\mathfrak{S}} \rangle$) is $28 + 17\rho$. A move $\mathfrak{f}$ having the central point as R-point verifies $\langle \pi, \mathfrak{f} \rangle = 1 + \rho$, so that the maximal number of them is $(28+17\rho)/(1+\rho) = 17 + 11\rho = 23.79\ldots$

We could also realize that the peg landing in A can come from at most 4 points and that these can be used only 14 times as P-point, but this leads to the bound $4 \times 14 = 56$.

Given a derivation $\mathfrak{f}_1, \ldots, \mathfrak{f}_k$ in legal moves, assume ℓ of them have A as R-point, while c of them have it as Q-point while t of them have it as P-point. We have

$$-\ell + c + t = \mathbb{1}_I(P) - \mathbb{1}_J(P). \tag{1}$$

Practically, we proceed in a different fashion, namely by looking at the dual problem (which was the initial one in fact; resource count *are* on the dual side). We simply solve the following linear program:

$$\min / \max \left\{ \sum_{\mathfrak{f} \in \mathfrak{Q}} x_{\mathfrak{f}}, \quad / \quad \sum_{\mathfrak{f} \in \mathscr{D}(\mathfrak{S})} x_{\mathfrak{f}} \mathfrak{f} = \mathbb{1}_I - \mathbb{1}_J \right\} \tag{2}$$

where $\mathfrak{Q}$ is chosen according to the problem and the point chosen. For instance, to get an upper bound for the load at A, we take for $\mathfrak{Q}$ the set of all moves $\mathfrak{f}$ that have A as their R-point.

Note finally that this way of getting bounds for loads, calls and tensions from the ones on thicknesses and obligations works on special boards also and we readily conclude that all these quantities are not larger than 20 on an english board, and much better bounds are available for points close to corners.

4 Convex hull, landscape and joint landscape

Given a position I we defined the *convex hull* of this position as being the set of points where one can put a peg via legal moves from I. The points outside this convex hull can simply be discarded in any problem starting from I. We even introduced a more refined notion in (Ramaré, 2008b), namely the *landscape*. It is the function $\mathfrak{L}(A, I)$ on $\mathfrak{S}$ which is equal, in A and when $A \notin I$, to be the minimal number of moves required to put a peg in A when starting from I, a number we have called the *height* of I in A, and, when $A \in I$, to be -1 times the minimal number of moves required to remove the peg from A, what we have called the *depth* of A with respect to I. The height, resp. depth, in A is infinite when it is not possible to put a peg in A, resp. not possible to remove the peg located in A.

This landscape is difficult to compute or even to evaluate. It turns out that a more refined notion is easier to evaluate, and is what we call the *joint landscape*. This is with reference to an initial position I and a final one J. We define $\mathfrak{L}(A, I, J)$ as above but we limit the possible paths. When $A \in I$, the height is the minimal number of moves required to put a peg in A, as above, but we add the condition that J may still be derived from the position reached.

The *joint convex hull* is thus the place where $\mathfrak{L}(A, I, J) \neq \pm\infty$ and can be evaluated by noticing that when $A \notin I$ and the maximal load in A vanishes, then $\mathfrak{L}(A, I, J) = \infty$. Similarly, when $A \in I$ and the maximal tension and call in A both vanish, then $\mathfrak{L}(A, I, J) = -\infty$. This our main result and we extract it as a Theorem to emphasize its importance (and for future reference!).

Theorem 4.1

When $A \notin I$, then $\mathfrak{L}(A, I, J) = \infty$ as soon as $\mathcal{L}_+(A, I, J) = 0$.
When $A \in I$, then $\mathfrak{L}(A, I, J) = -\infty$ as soon as $(\mathcal{C}_+ + \mathcal{T}_+)(A, I, J) = 0$.

Let us consider two examples. First the problem known as the "short bow":

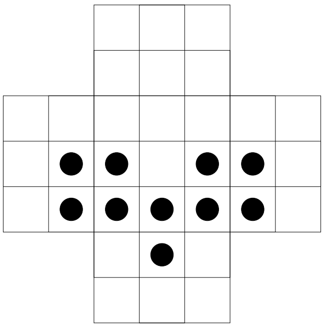

Figure 2: The short bow : start

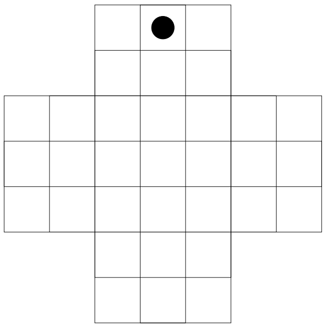

Figure 3: The short bow : ending

We draw in figure 4 and 5 the landscape and the joint landscape. We used 0 instead of $\pm\infty$:

		0	4	0		
		4	3	4		
2	1	1	2	1	1	2
1	−1	−1	1	−1	−1	1
1	−1	−1	−1	−1	−1	1
		1	−1	1		
		3	1	3		

Figure 4: Landscape

		0	4	0		
		0	4	0		
0	1	1	2	1	1	0
0	−1	−1	1	−1	−1	0
0	−1	−1	−1	−1	−1	0
		0	−1	0		
		0	0	0		

Figure 5: Joint landscape

It is thus possible to restrict considerably the board. Note that this is one of the reasons why it is interesting to study this game with a fairly general board.

We see in this example that the joint landscape is much more efficient than the simple one. Here is an example with more points:

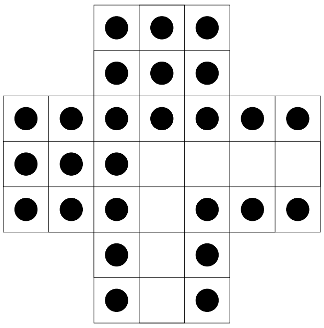

Figure 6: The angle : start

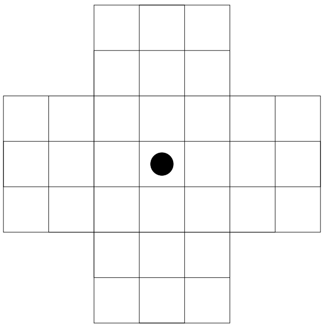

Figure 7: The angle : ending

We represent in figure 8 and 9 the landscape and the joint landscape, where we have again used 0 instead of $\pm\infty$:

		−3	−3	−3		
		−2	−1	−1		
−3	−2	−2	−1	−1	−2	−2
−3	−1	−1	1	1	3	5
−3	−1	−1	1	−1	−1	−2
		−2	3	−1		
		−2	5	−2		

Figure 8: Landscape

		−3	−3	−3		
		−2	−1	−1		
−3	−2	−2	−1	−1	−2	−2
−3	−1	−1	1	1	0	0
−3	−1	−1	1	−1	−1	−2
		−2	0	−1		
		−2	0	−2		

Figure 9: Joint landscape

We continue this practical exploration with yet another example, taken from (Ramaré, 2008b), figures 26, 27 and 28.

In this paper, we had not been able to prove that the top corners where outside the convex hull of the initial position. We do not resolve this question here, but are able to show that they are not within the joint convex hull, and this is enough for our purpose.

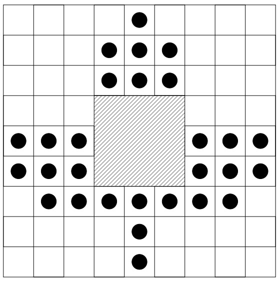

Figure 10: The palm : start

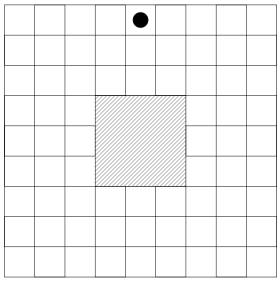

Figure 11: The palm : ending

0	0	2	1	−2	1	2	0	0
0	6	1	−1	−1	−1	1	6	0
3	4	1	−1	−1	−1	1	4	3
1	1	1				1	1	1
−1	−1	−1				−1	−1	−1
−1	−1	−1				−1	−1	−1
1	−1	−1	−1	−1	−1	−1	−1	1
2	1	1	3	−3	3	1	1	2
6	3	3	4	−3	4	3	3	6

Figure 12: Joint landscape

We end this part with a problem on Wiegleb's board that is reputed to be the most difficult complement problem on this board, and the only one to be impossible according to (Beasley, 1985). See also (Bell, 2007). Up to now, we had not been able to show whether the quadratic test was or not satisfied by this position. With the problem being properly reduced, and in particular the *corrections* $\mathscr{C}(A, I, J)$ defined by equation (27) of (Ramaré, 2008b) being computed according to the joint landscape, it is now proved to be impossible via the quadratic test!

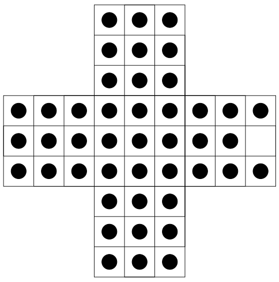

Figure 13:
The difficult Six

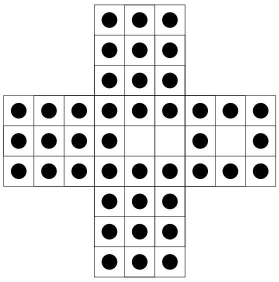

Figure 14:
The difficult Six:
an equivalent start

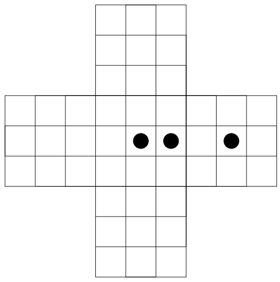

Figure 15:
The difficult Six:
an equivalent ending

5 Joint landscape and external height

We study here the notion of depth more thoroughly and reach to a lower bound of it requiring only integer linear programming.

We define the *external external height* $\mathrm{H}(A, I, J)$ to be the maximum number of moves such that, starting from I, the position reached I' contains A and that there exists a legal way of deriving J from I'. We thus look at paths $\mathfrak{f}_1, \ldots, \mathfrak{f}_k, \cdots \mathfrak{f}_\ell$ such that

$$\begin{cases} \mathbb{1}_I & \text{is legal,} \\ \mathbb{1}_I - \mathfrak{f}_1 & \text{is legal,} \\ \vdots & \\ \mathbb{1}_I - \mathfrak{f}_1 \cdots - \mathfrak{f}_{k-1} & \text{is legal,} \\ \mathbb{1}_I - \mathfrak{f}_1 - \cdots - \mathfrak{f}_k = h & \text{is legal and verifies } h(A) = 0, \\ \mathbb{1}_I - \mathfrak{f}_1 \cdots - \mathfrak{f}_{k+1} & \text{is legal,} \\ \vdots & \\ \mathbb{1}_J = \mathbb{1}_I - \mathfrak{f}_1 \cdots - \mathfrak{f}_\ell & \text{is legal} \end{cases} \tag{3}$$

and look for k minimal in the case of the joint height $\mathrm{h}(A, I, J)$ and for k maximal in the case of the external joint height $\mathrm{H}(A, I, J)$. This joint external height is infinite whenever no path verify (3), so that we have

$$\mathrm{h}(A, I, J) \leq \mathrm{H}(A, I, J). \tag{4}$$

We have developped techniques to evaluate the height, and this function is fairly well understood, see (Ramaré, 2008a), but the depth remains less tractable. We introduced in (Ramaré, 2008b) a mean of getting a non-trivial lower bound, but this does not provide a clear view of the situation. Our main remark here is that the joint depth is more tractable than the depth, for we have:

Theorem 5.1 *We have*

$$\mathrm{Depth}(A, I, J) + \mathrm{H}(A, \mathfrak{S} \setminus J, \mathfrak{S} \setminus I) = |I| - |J|$$

whenever both quantities are finite.

Proof: Indeed, the joint depth corresponds to paths $\mathfrak{f}_1, \ldots, \mathfrak{f}_k, \ldots, \mathfrak{f}_\ell$ such that (3) holds and this is equivalent to

$$\begin{array}{rl}
\mathbb{1}_{\mathfrak{S}\setminus I} = \mathbb{1}_{\mathfrak{S}\setminus J} - \mathfrak{f}_\ell \cdots - \mathfrak{f}_1 & \text{is legal,} \\
\vdots & \vdots \\
\mathbb{1}_{\mathfrak{S}\setminus J} - \mathfrak{f}_\ell \cdots - \mathfrak{f}_k & \text{is legal,} \\
\mathbb{1}_{\mathfrak{S}\setminus J} - \mathfrak{f}_\ell \cdots - \mathfrak{f}_{k+1} = g & \text{is legal and verifies } g(A) = 1, \\
\mathbb{1}_{\mathfrak{S}\setminus J} - \mathfrak{f}_\ell \cdots - \mathfrak{f}_{k+2} & \text{is legal,} \\
\vdots & \vdots \\
\mathbb{1}_{\mathfrak{S}\setminus J} - \mathfrak{f}_\ell & \text{is legal,} \\
\mathbb{1}_{\mathfrak{S}\setminus J} & \text{is legal.}
\end{array}$$

This should be read from bottom to top. We recognize the definition of a path occuring in the definition of $\mathrm{H}(A, \mathfrak{S} \setminus J, \mathfrak{S} \setminus I)$. If $k = \mathrm{Depth}(A, I, J)$, then the above shows that

$$\mathrm{H}(A, \mathfrak{S} \setminus J, \mathfrak{S} \setminus I) \leq \ell - k.$$

The minimal character of k in the proof ensures the maximal character of $\ell - k$. $\diamond\diamond\diamond$

It is maybe better to take an example. In the problem below, the depth of A is clearly 2:

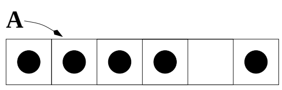

Figure 16: Initial position I

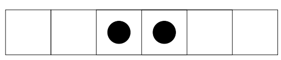

Figure 17: Final position J

We now have to check that the joint external height of A with respect to $\mathfrak{S} \setminus J$ and $\mathfrak{S} \setminus I$ is indeed 1. Here is the problem:

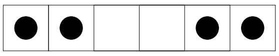

Figure 18: Initial position $\mathfrak{S} \setminus J$

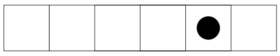

Figure 19: Final position $\mathfrak{S} \setminus I$

The reader will easily prove our claim.

Let us next consider the classical on the english solitaire board: we remove the central peg and we aim at leaving only one peg there. And we want to compute the joint depth of the point A indicated below:

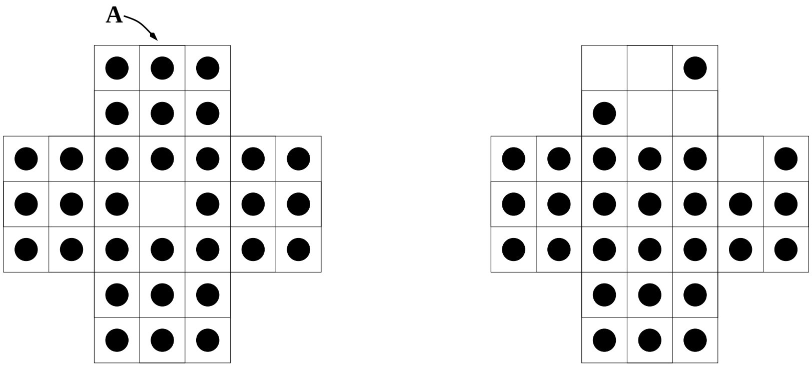

Figure 20: Joint depth of A is 4

We show that $\mathrm{Depth}(A, I, J) = 4$ by using the start given by figure 20 (and we leave it to prove that $\mathrm{Depth}(A, I, J) > 3$, to the reader to attain this position and to continue from it and leave a sole peg in the central hole!!). Computing $\mathrm{H}(A, \mathfrak{S} \setminus J, \mathfrak{S} \setminus I)$ is however even more tricky in this case, so how can we use our Theorem?

We have at our disposal another geometrical interpretation of $|I| - |J| - \mathrm{H}(A, I, J) = \mathrm{Depth}(A, \mathfrak{S} \setminus J, \mathfrak{S} \setminus I)$: it is the minimal number of *reversed* moves from J that leaves a peg in A, provided the position attained can be further continued to reach I, still by reversed legal moves. By reversed moves, we mean $-\mathfrak{f}$ where $\mathfrak{f} \in \mathscr{D}(\mathfrak{S})$. This interpretation is handy in that it is almost the same as the definition of joined height, where we simply change the set of moves! In particular, we have at our disposal the corresponding notion of lesser external height for a part K of $\mathfrak{S}$:

$$\mathrm{lH}(K, J) = \min_{\mathfrak{P}} \Big\{ \langle \Psi(\mathfrak{P}), \mathbb{1}_{\mathfrak{S}} \rangle, \quad \mathbb{1}_{\mathfrak{S}} \geq \mathbb{1}_J + \Psi(\mathfrak{P}) \geq \mathbb{1}_K \Big\} \tag{5}$$

which can be evaluated by linear programming and which verifies

$$\mathrm{lH}(\{A\}, J) \leq \mathrm{Depth}(\{A\}, \mathfrak{S} \setminus J, \mathfrak{S} \setminus I). \tag{6}$$

This leads to

$$\mathrm{lH}(\{A\}, \mathfrak{S} \setminus I) \leq \mathrm{Depth}(\{A\}, I). \tag{7}$$

It is likely that this lower bound refines (26) of (Ramaré, 2008b), but since we do not provide any proof, we do not claim anything formally.

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