MATHEMATICS OF COMPUTATION
Volume 00, Number 0, Pages 000–000
S 0025-5718(XX)0000-0

EXPLICIT ESTIMATES ON SEVERAL SUMMATORY FUNCTIONS INVOLVING THE MOEBIUS FUNCTION

OLIVIER RAMARÉ

ABSTRACT. We prove that $|\sum_{d\le x}\mu(d)/d|\log x\le 1/69$ when $x\ge 96\,955$ and deduce from that: $|\sum_{\substack{d\le x,\\(d,q)=1}}\mu(d)/d|\log(x/q)\le \frac45 q/\varphi(q)$ for every $x>q\ge 1$. We give also better constants when x/q is larger. Furthermore we prove that $|1-\sum_{d\le x}\mu(d)\log(x/d)/d|\le \frac{3}{14}/\log x$ and several similar bounds, from which we also prove corresponding bounds when summing the same quantity, but with the additional condition $(d,q)=1$. We prove similar results for $\sum_{d\le x}\mu(d)\log^2(x/d)/d$, among which we mention the bound $|\sum_{d\le x}\mu(d)\log^2(x/d)/d-2\log x+2\gamma_0|\le\frac{5}{24}/\log x$, where γ_0 is the Euler constant. We complete this collection by bounds like $|\sum_{\substack{d\le x,\\(d,q)=1}}\mu(d)|/x\le \frac{q}{\varphi(q)}/\log(x/q)$. We also provide all these bounds with variations where $1/\log x$ is replaced by $1/(1+\log x)$.

1. Introduction

Explicit estimates in multiplicative number theory have a long history. Concerning prime-related questions, one can distinguish between two main lines of inquiry: estimates on the Chebyshev ψ-function and estimates for the summatory function M of the Moebius function. In the first case the explicit formula for the ψ-function enables us to introduce in the argument the result of heavy computations regarding the zeros of the Riemann ζ-function, see for instance [22] and [23]. This is so because the residues of the Mellin transform of the ψ-function i.e. $-\zeta'(s)/\zeta$, are known: they are simply equal to 1 counted with multiplicity. No such fact happens in the case of the Moebius function, the Mellin transform that appears being $1/\zeta$. No one has yet obtained an explicit error term for the function M from the Mellin transform / Perron formula machinery, though there are no *theoretical* obstructions. The implied constants are however expected to be too large for any decent use.

Once the analytical path is discarded, two distinct paths of inquiries have been used for the summatory function $M(x)=\sum_{d\le x}\mu(d)$. Either follow the idea of Chebyshev, and this is done in [13], [6], [8]. Either follow an idea that I see as coming from Landau: Landau proved that $\psi(x)\sim x$ is equivalent to $M(x)=o(x)$ and we need an quantitative version of it. See also [12] for this kind of questions. This path is followed in [24] and continued in [9]. Recently in [20], I trode a similar path but relied on a more efficient set of identities.

In practice, one needs estimates for families of functions that are derived from the Moebius function, with as small a loss as one can manage. There are three

Key words and phrases: Explicit estimates, Moebius function.
2010 *Mathematics Subject Classification.* Primary 11N37, 11Y35; Secondary 11A25.

main variations, and we deal here with two of them, see [21] for the third family. The first problem is to go from $M(x)$ to $m(x) = \sum_{n\le x}\mu(d)/d$ and the second one is to add a coprimality condition $(d,q) = 1$ for some fixed q. Let us introduce some actors before continuing the description of the present work. We define

$$m_q(x) = \sum_{\substack{n\le x,\\ (n,q)=1}} \mu(n)/n, \quad m(x) = m_1(x) \tag{1.1}$$

Note that [11, Lemma 10.2] already proposed explicit wide ranging estimates. This investigation is extended in [21] to the second family

$$\check{m}_q(x) = \sum_{\substack{n\le x,\\ (n,q)=1}} \mu(n)\log(x/n)/n, \quad \check{m}(x) = \check{m}_1(x). \tag{1.2}$$

And, with some applications in mind, we also cover the family

$$\check{\check{m}}_q(x) = \sum_{\substack{n\le x,\\ (n,q)=1}} \mu(n)\log^2(x/n)/n, \quad \check{\check{m}}(x) = \check{\check{m}}_1(x). \tag{1.3}$$

The general problem consists in getting estimates for $m(x)$ and $\check{m}(x)$ from $M(x) = \sum_{d\le x}\mu(d)$. A summation by parts loses a factor of $\log x$. As kindly pointed out to me by H. Diamond and is clear in [3], a method of A. Axer from [1] already provides an answer to this problem. This method can be found presented in an elementary way in [25, Theorem 2.5], and in [15, Theorem 8.1] and in a more refined setting in [7, Lemma 3.1] (see also [4, Lemma 5.7]).

As an extension of the method of Axer, M. Balazard in [3], furthering work of R.A. MacLeod [14] developed a line of work that led to good identities linking $m(x)$ to $M(x)$ and $\check{m}(x)$ also to $M(x)$. A consequence is the following Theorem.

Theorem 1.1 (Balazard). *We have, when $x \ge 1$:*

$$|m(x)| \le \frac{|M(x)|}{x} + \frac{1}{x^2}\int_1^x |M(t)|dt + \frac{8}{3x}.$$

One of the advantages of the method and identity used in [3] is that it implies a differentiable function instead of the fractional-part-function as in Axer's work and this leads to a much smoother treatment.

We readily deduce from the above that

$$|m(x)| \le \bigl(\tfrac{3}{2} + o(1)\bigr)\exp\Bigl(-\max_{x^{7/8}\le t\le x}\log\frac{2+|M(t)|}{t}\Bigr) + \mathcal{O}(x^{-1/4}). \tag{1.4}$$

This is an excellent quantitative link between the error term of $m(x)$ and the one of $M(x)$, though it is not perfect: if we assume that $M(t) \ll t^{3/4}$ we deduce from the above that $m(x) \ll x^{-7/32}$, while one would like to infer that $m(x) \ll x^{-1/4}$. This kind of identity is used in [20, Corollary 1.2], where I relied on a identity due to El Marraki in [10]. On using the new identity above we improve on this work.

Theorem 1.2. *We have*

$$\Bigl|\sum_{n\le x}\mu(n)/n\Bigr| \le \frac{0.0144\log x - 0.1}{(\log x)^2} \quad (x \ge 463\,421).$$

The following simpler bounds also hold true:

$$(\log x)\Big|\sum_{n\le x}\mu(n)/n\Big| \le \begin{cases} 1/69 & \text{when } x\ge 96\,955, \\ 1/65 & \text{when } x\ge 60\,298, \\ 1/40 & \text{when } x\ge 24\,270, \end{cases} \begin{cases} 1/25 & \text{when } x\ge 3\,470, \\ 1/20 & \text{when } x\ge 1\,426, \\ 1/12 & \text{when } x\ge 687. \end{cases}$$

We can also replace $1/69$ *by* 0.0144.

This result improves by a factor a bit larger than 2 the previous estimate of [20] and by a factor a bit more than 31/2 the estimate of [10].

Corollary 1.3.

$$\Big|\sum_{n\le x}\mu(n)/n\Big| \le \begin{cases} 1 & \text{when } x\ge 1, \\ 1/10 & \text{when } x\ge 7, \\ 1/20 & \text{when } x\ge 41, \\ 1/100 & \text{when } x\ge 694. \end{cases}$$

On studying the proof of Theorem 1.1 and some other identity due to the same author concerning $\check{m}(x)$, I obtained the following result.

Theorem 1.4. *We have, when* $x\ge 1$*:*

$$|\check{m}(x)-1| \le \frac{\frac74-\gamma_0}{x^2}\int_1^x |M(t)|dt + \frac{2}{x}.$$

Here γ_0 *is the Euler constant.*

This second very simple inequality (in particular, the term $|M(x)|/x$ does not appear) leads to a much more surprising result. Section 2 and 3 contain somewhat more precise versions (i.e. identities) and it is in fact these forms that we use (enabling us in practice to reduce the coefficient $\frac74-\gamma_0 = 1.172\cdots$ to 0.321, see section 4).

Theorem 1.5. *We have*

$$|\check{m}(x)-1| \le \frac{0.00257\log x - 0.0077}{(\log x)^2} \quad (x\ge 3\,846).$$

The following simpler bound also holds:

$$|\check{m}(x)-1|\log x \le \begin{cases} 0.213 < 3/14 & \text{when } x\ge 1, \\ 0.0203 < 4/197 & \text{when } x\ge 16, \\ 1/389 & \text{when } x\ge 3\,155. \end{cases}$$

Corollary 1.6.

$$\Big|-1+\sum_{n\le x}\mu(n)\log(x/n)/n\Big| \le \begin{cases} 1 & \text{when } x\ge 1, \\ 1/125 & \text{when } x\ge 7, \\ 1/500 & \text{when } x\ge 44, \\ 1/1250 & \text{when } x\ge 222. \end{cases}$$

We follow a similar path with $\check{\check{m}}$, the relevant identities being much more cumbersome (see Lemma 5.1). We get the following.

Theorem 1.7. *We have, when $x \ge 1$:*

$$|\check{\check{m}}(x) - 2\log x + 2\gamma_0| \le 2(\gamma_0 - \tfrac{1}{2})^2 \frac{|M(x)|}{x} + \frac{3/2}{x^2}\int_1^x |M(t)|dt + \frac{4+2\gamma_0}{x}$$

The good surprise is that the term $|M(x)|/x$ is affected with a very small coefficient. Section 5 contains a more precise identity that led to this inequality. Lacking however a positivity argument we used in the case of $m(x)$ and $\check{m}(x)$, we cannot proceed as with those: the coefficient $3/2$ has to be taken at flat value (well, 1.46 is available).

Theorem 1.8. *We have*

$$|\check{\check{m}}(x) - 2\log x + 2\gamma_0| \le \frac{0.00965\log x - 0.0818}{(\log x)^2} \quad (x \ge 10).$$

The following simpler bounds also hold:

$$|\check{\check{m}}(x) - 2\log x + 2\gamma_0|\log x \le \begin{cases} 0.2062 < 5/24 & \text{when } x > 1,\\ 1/103 & \text{when } x \ge 9.\end{cases}$$

More surprisingly, it numerically seems that the function $\check{\check{m}}(x) - 2\log x + 2\gamma_0$ is non-increasing. We formulate that formally in the form of a conjecture:

Conjecture. *The function $\check{\check{m}}(x) - 2\log x + 2\gamma_0$ is positive decreasing.*

One would need a representation for this difference that exhibits some positivity. The representation we have contains oscillating terms, typically $M(t)$. Similarly, section 13 contains an equally surprising question whose veracity is sustained by numerical results. The function $\check{m}(x) - 2$ does not exhibit such a behaviour: for instance, it changes its sign of variations around $x = 5$.

The coprimality condition is somewhat difficult to handle from a numerical point of view. The classical path (used for instance in [16, near (7)]) consists in determining a function g_q such that $\mathbb{1}_{(n,q)=1}\mu(n) = g_q \star \mu(n)$, where $\star$ denotes the arithmetic convolution product. The drawback of this method is that the support of g_q is not bounded, and indeed, we have

$$\begin{cases} g_q(p^k) = 1 & \text{when } p|q \text{ and } k \ge 0,\\ g_q(p^k) = 0 & \text{when } (p,q) = 1 \text{ and } k \ge 1.\end{cases} \tag{1.5}$$

We proposed in [19] an approach via the Liouville function $\lambda(n)$ (the completely multiplicative function that is 1 on integers that have an even number of prime factors – counted with multiplicity – and -1 otherwise). Such an approach splits the evaluation in three steps: expressing $m_q(x)$ in terms of ℓ_q, where

$$\ell_q(x) = \sum_{\substack{n\le x,\\ (n,q)=1}} \lambda(n)/n, \quad \ell(x) = \ell_1(x); \tag{1.6}$$

then expressing $\ell_q(x)$ in terms of $\ell(y)$; and finally expressing $\ell(x)$ in terms of $m(y)$. It turns out that we can combine the first and the third step in a single one, allowing for some non-trivial savings. This time, the drawback of this new method is that the intermediate computations required are heavier and require much more RAM memory that in the previous method (see section 7). Here is our starting Lemma.

Lemma 1.9. *We have*

$$m_q(x) = \sum_{d|q} \frac{\mu^2(d)}{d} \sum_{\substack{w \le \sqrt{x/d},\\ w|q^\infty}} m\Big(\frac{x}{dw^2}\Big)/w^2$$

where $w|q^\infty$ means that every prime factor of w divides q. The same identity holds with $\check{m}_q$ (resp. $\check{\check{m}}_q$) and $\check{m}$ (resp. $\check{\check{m}}$) instead respectively of m_q and m.

Proof. We use the decomposition $\mathbb{1}_{(n,q)=1}\mu(n) = g_q \star \mu(n)$ where g_q is defined in (1.5) and get

$$m_q(x) = \sum_{a \le x} \frac{g_q(a)}{a} m(x/a).$$

Next we decompose a in the form $a = dw^2$ where d is squarefree. The Lemma follows readily. □

An alternative proof consists in combining [19, Lemma 1.1-1.3] and binding the two squares in a single variable w. This identity has an almost immediate consequence.

Corollary 1.10. *For any real number $x \ge 1$ and any positive integer q, we have*

$$0 \le \check{m}_q(x) = \sum_{\substack{n \le x,\\ (n,q)=1}} \mu(n)\log(x/n)/n \le 1.00303 \cdot q/\varphi(q).$$

The constant 1.00303 can be replaced by the optimal one: $\check{m}(30)$. Indeed, Balazard established this for $\check{m}(x)$ in [2] and Lemma 1.9 enables us to extend it. This kind of wide-ranging estimate is useful. We proceed in the same way with $\check{\check{m}}(x)$. We first establish that $0 \le \check{\check{m}}(x) \le 2\log x$ by combining numerical verifications and Theorem 1.8, from which we infer the following.

Corollary 1.11. *For any real number $x \ge 1$ and any positive integer q, we have*

$$0 \le \check{\check{m}}_q(x) = \sum_{\substack{n \le x,\\ (n,q)=1}} \mu(n)\log^2(x/n)/n \le 2\log x \cdot q/\varphi(q).$$

In both corollaries, q is not bounded with respect to x. The factor $q/\varphi(q)$ is shown to be necessary by selecting q to be the product of every prime not more than x. With some more work, this method leads to the following Theorem:

Theorem 1.12. *We have, when $1 \le q < x$, where q is an integer and x a real number,*

$$\frac{\varphi(q)}{q}\log(x/q)|m_q(x)| \le \begin{cases} 0.78 < 4/5 & \textit{when } x/q > 1,\\ 5/16 & \textit{when } x/q \ge 687,\\ 17/125 & \textit{when } x/q \ge 24\,233. \end{cases}$$

In these estimates, we can replace $\varphi(q)/q$ by $1/\prod_{p|q}(1+p^{-1})$.

We specify for clarity that one cannot, at least from our proofs, replace $\varphi(q)/q$ by $1/\prod_{p|q}(1+p^{-1})$ in Corollaries 1.10 and 1.11. It is sometimes better in usage to have $1 + \log x$ instead of $\log x$.

Theorem 1.13. *We have, when $1 \leq q < x$, where q is an integer and x a real number,*

$$\frac{\varphi(q)}{q}(1+\log(x/q))|m_q(x)| \leq \begin{cases} 1.694 < 17/10 & \textit{when } x/q > 1, \\ 1/2 & \textit{when } x/q \geq 296, \\ 5/38 & \textit{when } x/q \geq 882, \\ 1/7 & \textit{when } x/q \geq 11\,811. \end{cases}$$

In these estimates, we can replace $\varphi(q)/q$ by $1/\prod_{p|q}(1+p^{-1})$.

This improves on the corresponding estimates proved in [19]. Prior to this paper, the sole estimate on $m_q(x)$ seems to be [11, Lemma 10.2] which bounds $|m_q(x)|$ uniformly by 1. Some numerical investigations produced the example

$$|m_2(10)| = \frac{31}{105} \geq 0.260\,\frac{2}{\log(10/2)} \tag{1.7}$$

and I have not been able to get any example with a lower bound larger than $0.260\ldots$ for $|m_q(x)|\frac{\varphi(q)}{q}\log(x/r)$ when $x \geq 3$. It thus seems likely that

$$|m_q(x)| \overset{?}{\leq} \frac{0.261\ q}{\varphi(q)\log(x/q)}, \quad (x \geq 3, x > q \geq 1). \tag{1.8}$$

The inequality (9.1) obtained below shows that (1.8) holds true when $x/q \geq 2\,438$, but this leaves still infinitely many cases to cover.

Theorem 1.14. *We have, when $1 \leq q < x$, where q is an integer and x a real number,*

$$\frac{q}{\varphi(q)}\log(x/q)\big|\check{m}_q(x) - \check{m}_q^\sharp(x)\big| \leq \begin{cases} 1/4 & \textit{when } x/q > 1, \\ 1/5 & \textit{when } x/q \geq 100, \\ 0.0538 < 2/37 & \textit{when } x/q \geq 3\,158. \end{cases}$$

where we use the notation

$$\check{m}_q^\sharp(x) = \sum_{\substack{d|q, w|q^\infty, \\ dw^2 \leq x}} \frac{\mu^2(d)}{dw^2}. \tag{1.9}$$

In these estimates, we can replace $\varphi(q)/q$ by $1/\prod_{p|q}(1+p^{-1})$.

Using the same notations, we get the following variations.

Theorem 1.15. *We have, when $1 \leq q < x$, where q is an integer and x a real number,*

$$\frac{q}{\varphi(q)}(1+\log(x/q))\big|\check{m}_q(x) - \check{m}_q^\sharp(x)\big| \leq \begin{cases} 1 & \textit{when } x/q > 1, \\ 1/5 & \textit{when } x/q \geq 171, \\ 0.066 < 1/15 & \textit{when } x/q \geq 3\,150. \end{cases}$$

At this level of generality, it is not possible to simplify the main term $\check{m}_q^\sharp(x)$. We proceed in a similar fashion for $\check{\check{m}}_q$

Theorem 1.16. *We have, when $1 \le q < x$, where q is an integer and x a real number,*

$$\frac{\varphi(q)}{q}\log(x/q)\left|\check{\check{m}}_q(x)-\check{m}_q^\sharp(x)\right| \le \begin{cases} 4/19 & \text{when } x/q > 1,\\ 1/48 & \text{when } x/q \ge 5.\end{cases}$$

We have used the notation

$$\check{\check{m}}_q^\sharp(x) = 2\sum_{\substack{d|q,w|q^\infty,\\ dw^2\le x}} \frac{\mu^2(d)(\gamma_0-\log(x/(dw^2)))}{dw^2}. \tag{1.10}$$

In these estimates, we can replace $\varphi(q)/q$ by $1/\prod_{p|q}(1+p^{-1})$.

And the variation:

Theorem 1.17. *We have, when $1 \le q < x$, where q is an integer and x a real number,*

$$\frac{\varphi(q)}{q}(1+\log(x/q))\left|\check{\check{m}}_q(x)-\check{m}_q^\sharp(x)\right| \le \begin{cases} 1.155 < 7/6 & \text{when } x/q > 1,\\ 3/17 & \text{when } x/q \ge 2,\\ 1/22 & \text{when } x \ge 6.\end{cases}$$

In these estimates, we can replace $\varphi(q)/q$ by $1/\prod_{p|q}(1+p^{-1})$.

We finally complete this series of results with one concerning $M_q(x)$, where we follow the previous convention:

$$M_q(x) = \sum_{\substack{n\le x,\\ (n,q)=1}} \mu(n), \quad M(x) = M_1(x). \tag{1.11}$$

We adapt Lemma 1.9 in (16.1) and a similar routine leads to the following.

Theorem 1.18. *We have, when $1 \le q < x$, where q is an integer and x a real number,*

$$\frac{\varphi(q)}{q}\log(x/q)\left|M_q(x)\right|/x \le \begin{cases} 0.997 & \text{when } x/q > 1,\\ 0.429 < 9/20 & \text{when } x/q \ge 490,\\ 1/5 & \text{when } x/q \ge 4\,536,\\ 0.0918 < 9/98 & \text{when } x/q \ge 48\,513.\end{cases}$$

In these estimates, we can replace $\varphi(q)/q$ by $1/\prod_{p|q}(1+p^{-1})$.

Theorem 1.19. *We have, when $1 \le q < x$, where q is an integer and x a real number,*

$$\frac{\varphi(q)}{q}(1+\log(x/q))\left|M_q(x)\right|/x \le \begin{cases} 1 & \text{when } x/q > 1,\\ 1/2 & \text{when } x/q \ge 490,\\ 1/5 & \text{when } x/q \ge 7\,100,\\ 39/400 & \text{when } x/q \ge 48\,645.\end{cases}$$

In these estimates, we can replace $\varphi(q)/q$ by $1/\prod_{p|q}(1+p^{-1})$.

Scripts. All the computations used have been achieved via GP/PARI, see [18], often speeded by using `gp2c` as described for instance in [5]. To give a flavor, we have used for instance the command `gp-run -g AsymptoticBoundsFor_M.gp`. The flag "-g" enables automatic memory management and garbage collection. The computations have been run on a 64-bit dual core running at 3.0GHz and having 8 Gbytes of RAM. One also to increase the heap size via `allocatemem(7500000000)`. The amount of RAM available decided on all the ranges of initial computation; we did not rely on any swapping capacity, as this slows dramatically the computations. Practically we increased the range of the bound under inspection until most of the RAM has been used. I am indebted to Bill Alombert for helping me in writing many scripts in a near optimal way. The important scripts are available online at

`http://math.univ-lille1.fr/~ramare/index.html#GP`

Acknowledgement. The referee is to be thanked for his/her careful reading and constructive remarks that have helped improve this paper.

2. An identity for $m(x)$, proof of Theorem 1.2 and of Lemma 1.3

Here is the main identity we use, taken from [2, Proof of proposition 6]:

$$m(x) = \frac{M(x)}{x} + \frac{4(1-x^{-1})^2}{x} - \frac{4(1-x^{-1})^3}{3x^2} + \frac{1}{x}\int_1^x M(x/t)\varepsilon_1'(t)dt \tag{2.1}$$

where

$$\varepsilon_1'(t) = \left(\frac{(2\{t\}-1)t + \{t\} - \{t\}^2}{t^2}\right)^2 \tag{2.2}$$

is the derivative at every non-integer point of

$$\varepsilon_1(t) = \frac{1}{3} - \frac{1}{3t} + \frac{4}{3}\frac{\{t\}^3 - \frac{3}{2}\{t\}^2 + \frac{1}{2}\{t\}}{t^2} - \frac{1}{3}\frac{\{t\}^4 - 2\{t\}^3 + \{t\}^2}{t^4}. \tag{2.3}$$

Moreover $0 \le \varepsilon_1'(t) \le 1/t^2$. Let us recall [20, Theorem 1.1]

Lemma 2.1. *For $D \ge 1\,078\,853$, we have*

$$\left|\sum_{d\le D}\mu(d)\right| \le \frac{0.0130\log D - 0.118}{(\log D)^2}D.$$

We use the following computations.

Lemma 2.2. *We have, with $D_0 = 1\,078\,853$*

$$\int_{\substack{1\le t\le D_0,\\ M(t)\ge 0}} M(t)dt = 58\,909\,800, \quad \int_{\substack{1\le t\le D_0,\\ M(t)\le 0}} M(t)dt = -54\,647\,032.$$

We use the file `CompIntM.gp` whose main function is `getintM`. Since we have this script at hand, we also prove the following version that will of use in section 4.

Lemma 2.3. *We have, with $D_0 = 464\,402$*

$$\int_{\substack{1\le t\le D_0,\\ M(t)\ge 0}} M(t)dt = 15\,512\,101, \quad \int_{\substack{1\le t\le D_0,\\ M(t)\le 0}} M(t)dt = -14\,504\,264.$$

Lemma 2.4. *The function*

$$x \mapsto \frac{\log x}{3}\int_{xD_0/(x+D_0)}^{x/2} \frac{0.0130\log u - 0.118}{(\log u)^2}\frac{udu}{(x-u)^2} + \frac{58\,909\,800\log x}{x^2}$$

is increasing to $\frac{1}{3}0.0130(1-\log 2) = 0.00132\cdots$. *As a conclusion, it is* ≤ 0.00134 *when* $1\,078\,853 = D_0 \le x$.

Proof. Let us denote by $x \mapsto f(x)\log x$ the function to be studied. In practice, it would be enough to plot the function and check it is "numerically" increasing. Here we can show that it is indeed increasing and as a matter of fact, the function $x \mapsto f(x)$ is increasing. We recall that

$$\frac{d}{dx}\int_a^{h(x)} H(x,u)dx = \int_a^{h(x)} \frac{\partial}{\partial x}H(x,u)du + h'(X)H(x,h(x)) \tag{2.4}$$

provided that the integrals $\int_a^{h(x)}\left|\frac{\partial}{\partial x}H(x,u)\right|du$ and $\int_a^{h(x)}\left|H(x,u)\right|du$ are uniformly bounded in a neighborhood of the point where we take the derivative. Here, this formula yields

$$\begin{aligned} f'(x) = \frac{1}{6}&\frac{0.0130\log(x/2) - 0.118}{(\log(x/2))^2}\frac{x}{(x-x/2)^2} \\ &+ \frac{D_0^2}{3(x+D_0)^2}\frac{0.0130\log(xD_0/(x+D_0)) - 0.118}{(\log(xD_0/(x+D_0)))^2}\frac{xD_0/(x+D_0)}{(x-xD_0/(x+D_0))^2} \\ &\qquad - \frac{2}{3}\int_{xD_0/(x+D_0)}^{x/2}\frac{0.0130\log u - 0.118}{(\log u)^2}\frac{udu}{(x-u)^3} - \frac{58\,909\,800}{x^3}. \end{aligned}$$

The sum of the two last terms is larger than

$$-\frac{2}{3}\frac{0.0130\log x - 0.118}{(\log x)^2}\frac{(x/2)^2}{x^3/8} - \frac{58\,909\,800}{x^3} \ge \frac{-0.00682}{x\log x}$$

provided that $x \ge D_0 = 1\,078\,853$. The reader will easily conclude the proof that f is indeed increasing. Regarding the value at infinity, on using equivalents, we deduce it is

$$\lim_{x\to\infty}\frac{\log x}{3}\int_1^{x/2}\frac{0.0130}{\log u}\frac{udu}{(x-u)^2} = \lim_{x\to\infty}\frac{0.0130}{3}\int_1^{x/2}\frac{udu}{(x-u)^2}$$

which is also $\frac{0.0130}{3}\int_0^{1/2}\frac{vdv}{(1-v)^2} = 0.0130(1-\log 2)/3$. □

To use (2.1), we have to handle the last term. We note that, when n is an integer, we have

$$\begin{aligned} \int_n^{n+1}|M(x/t)|\varepsilon_1'(t)dt &= \max_{n\le t\le n+1}|M(x/t)|(\varepsilon_1(n+1) - \varepsilon_1(n)) \\ &= \max_{n\le t\le n+1}|M(x/t)|\left(\frac{1}{3n} - \frac{1}{3(n+1)}\right) = \tfrac{1}{3}\int_n^{n+1}\max_{n\le t\le n+1}|M(x/t)|dt/t^2. \end{aligned}$$

Thus

$$\begin{aligned}\int_1^x |M(x/t)||\varepsilon_1'(t)dt &\le \int_1^{x/D_0} |M(x/t)||\varepsilon_1'(t)dt + \frac{58\,909\,800}{x}\\ &\le \frac{x}{3}\int_1^{x/D_0} \frac{0.0146\log(x/(t+1)) - 0.1098}{(\log(x/(t+1)))^2}\frac{dt}{t^2(t+1)} + \frac{58\,909\,800}{x}\\ &\le \frac{x}{3}\int_{xD_0/(x+D_0)}^{x/2} \frac{0.0146\log u - 0.1098}{(\log u)^2}\frac{udu}{(x-u)^2} + \frac{58\,909\,800}{x}\\ &\le 0.00134x/\log x\end{aligned}$$

by Lemma 2.4. This yields

$$|m(x)| \le \frac{0.0130\log x - 0.118}{(\log x)^2} + \frac{4}{x} + \frac{0.00134}{\log x} \le \frac{0.0144\log x - 0.118}{(\log x)^2}$$

when $x \ge D_0$. We checked it numerically for $619\,000 \le x \le 3\cdot 10^7$. Theorem 1.2 and Corollary 1.3 follow readily. The script used is `AsymptoticBoundsFor_m.gp` whose main function if `getboundsbis`. The function `modelmult` has to be chosen according to the needs.

3. An identity for $\check{m}(x)$ and proof of Theorem 1.4

As a preparation for next section, we prove some more than is strictly required here (only case $\ell = 0$ of what follows is needed in this section). A simple use of the Euler-MacLaurin summation formula gives, when ℓ is a non-negative integer:

$$\begin{aligned}H_\ell(x) &= \sum_{n\le x}\frac{\log^\ell n}{n}\\ &= \frac{(\log x)^{\ell+1}}{\ell+1} + \gamma_\ell - B_1(x)f_\ell(x) + \frac{B_2(x)}{2}f_\ell'(x) + \varepsilon_{6,\ell}(x)\end{aligned}\tag{3.1}$$

with $f_\ell(t) = (\log t)^\ell/t$ and the classical notation $B_1(t) = \{t\} - \frac12$, $b_2(x) = x^2 - x + \frac16$, $B_2(t) = b_2(\{t\})$ as well as

$$\varepsilon_{6,\ell}(x) = \tfrac12\int_x^\infty B_2(t)f_\ell''(t)dt.\tag{3.2}$$

The constant γ_ℓ is the Euler-Stieltjes constant, and γ_0 is the classical Euler constant, while $\gamma_1 = -0.07281584548\cdots$.

Lemma 3.1. *When m is a non-negative integer and $y \ge 1$ a real number we have*

$$\int_1^y t\log^m t dt/m! = \frac{y^2}{2}\sum_{\substack{a+b=m,\\ a,b\ge 0}}\frac{(-1)^b(\log y)^a}{a!2^b} + \frac{(-1)^{m+1}}{2^{m+1}}.$$

Proof. The proof goes by generating series. We consider

$$\begin{aligned}\sum_{n\ge 0}\frac{z^n}{n!}\int_1^y t\log^n t dt &= \int_1^y t^{1+z}dt = \frac{y^2y^z - 1}{2+z}\\ &= \frac{y^2}{2}\sum_{a\ge 0}\frac{z^a(\log y)^a}{a!}\sum_{b\ge 0}(-z/2)^b - \tfrac12\sum_{b\ge 0}(-z/2)^b.\end{aligned}$$

Identification leads to the lemma. □

Furthermore, still following Balazard's notation from [3] and [2] in case $\ell = 0$,

$$\beta_{2,\ell}(x) = \frac{2}{x}\int_0^x \sum_{n\le t}\frac{t}{n}\log^\ell(t/n)dt \tag{3.3}$$

$$= \frac{2}{x}\sum_{n\le x} n\int_1^{x/n} u\log^\ell u du$$

$$= \sum_{n\le x}\biggl(\frac{x}{n}\sum_{\substack{a+b=\ell,\\ a,b\ge 0}}\frac{(-1)^b\log^a(x/n)}{a!2^b} - \frac{(-1)^\ell}{2^\ell}\frac{n}{x}\biggr). \tag{3.4}$$

We specialize to $\ell = 0$ from now on.

$$\beta_{2,0}(x) = \sum_{n\le x}(x/n - n/x) = xH_0(x) - x + \{x\} + \frac{b_2(x) - b_2(\{x\})}{2x}$$
$$= x\log x + x(\gamma_0 - \tfrac12) - \frac{1}{12x} - \frac{\{x\}^2 - \{x\}}{x} + x\varepsilon_{6,0}(x).$$

This equation implies that $\varepsilon_{6,0}(1) - \frac{1}{12} = \frac12 - \gamma_0$. We thus have

$$\sum_{n\le x}\mu(n)\Bigl(\frac{x\log(x/n)}{n} + \frac{x}{n}(\gamma_0 - \tfrac12) - \frac{n}{12x} - \frac{\{x/n\}^2 - \{x/n\}}{x/n} + \frac{x}{n}\varepsilon_{6,0}(x/n)\Bigr) = x - x^{-1}.$$

Recall [2, (8)]:

$$\sum_{n\le x}\mu(n)\Bigl(\frac{x}{n} - 1 - \frac{\{x/n\}^2 - \{x/n\}}{x/n}\Bigr) = 2 - 2x^{-1} \quad (x\ge 1) \tag{3.5}$$

and get

$$\sum_{n\le x}\mu(n)\Bigl(\frac{x\log(x/n)}{n} + \frac{x}{n}(\gamma_0 - \tfrac32) + 1 - \frac{n}{12x} + \frac{x}{n}\varepsilon_{6,0}(x/n)\Bigr) = x - 2 + x^{-1}.$$

Let us continue the main proof. We get

Lemma 3.2. *When $x \ge 1$, we have*

$$\check{m}(x) - 1 = \frac{6 - 8\gamma_0}{3x} - \frac{6 - 4\gamma_0}{x^2} + \frac{6 - 4\gamma_0}{3x^4} - \frac{1}{x}\int_1^x M(x/t)h'(t)dt$$

where $h'(t) = (\frac32 - \gamma_0)\varepsilon_1'(t) + g'(t)$ is continuous and differentiable except at integers where it has left and right derivative. It satisfies

$$0 \le t^2 h'(t) \le \tfrac74 - \gamma_0. \tag{3.6}$$

Theorem 1.4 is a straightforward consequence of this Lemma. Let us specify that h' is typically not continuous at integer points.

Proof. Indeed, we have already reached

$$\check{m}(x) - 1 = (\tfrac32 - \gamma_0)m(x) - (\tfrac32 - \gamma_0)\frac{M(x)}{x} - \frac{2}{x} + \frac{1}{x^2} - \frac{1}{x}\int_1^x M(x/t)g'(t)dt$$

where the continuous and piece-wise differentiable function g is defined by:

$$g(x) = \frac{-1}{12x} + x\varepsilon_{6,0}(x) \quad (g(1) = \tfrac12 - \gamma_0). \tag{3.7}$$

We further recall (2.1):

$$m(x)=\frac{M(x)}{x}+\frac{4(1-x^{-1})^2}{x}-\frac{4(1-x^{-1})^3}{3x}+\frac{1}{x}\int_1^x M(x/t)\varepsilon_1'(t)dt.$$

This leads to

$$\begin{aligned}x(\check{m}(x)-1)=2-\tfrac{8}{3}\gamma_0-\frac{2(3-2\gamma_0)}{x}+\frac{2-\frac{4}{3}\gamma_0}{x^3}\\ -\int_1^x M(x/t)((\tfrac32-\gamma_0)\varepsilon_1'(t)+g'(t))dt.\end{aligned}$$

With the notation

$$h'(t)=(\tfrac32-\gamma_0)\varepsilon_1'(t)+g'(t),$$

we find that (with $u=\{x\}$)

$$\begin{aligned}t^2h'(t)&=-u^2+u-\tfrac12+(\tfrac32-\gamma_0)(2u-1)^2\\ &\quad+t^2\int_t^\infty\frac{B_2(v)dt}{v^3}+(3-2\gamma_0)\frac{(2u-1)(u-u^2)}{t}+(\tfrac32-\gamma_0)\frac{(u-u^2)^2}{t^2}.\end{aligned}$$

The polynomial in u of the first line is positive between 0 and 1, while the second part is $\ll 1/x$. Some straightforward numerical work yields $h'(t)>0$. On the other hand we recalled after (2.3) that $0\le\varepsilon'(t)\le 1/t^2$. Concerning $g'(t)$, we first notice that $|B_2(v)|\le 1/6$, so that

$$|t^2g'(t)|\le\left|\{t\}^2-\{t\}+\tfrac{1}{12}\right|+\frac{1}{6\times 2}\le 1/4.$$

The Lemma readily follows. □

4. Proof of Theorem 1.5 and of Corollary 1.6

Lemma 4.1. *For any positive integer k we have*

$$\int_k^{k+1}B_4(t)dt/t^4\le 0.$$

Proof. We have, on using the shortcut $u=\{t\}$,

$$\begin{aligned}B_4(t)&=u^4-2u^3+u^2-\tfrac{1}{30}\\ &=t^4-(4k+2)t^3+(6k^2+6k+1)t^2-(4k+2)(k^2+k)t+(k^2+k)^2-\tfrac{1}{30}.\end{aligned}$$

As a consequence and on denoting by I_k the integral to be computed, we find that

$$\begin{aligned}I_k&=1-2(2k+1)\log\frac{k+1}{k}+\frac{6k^2+6k+1}{k(k+1)}-\frac{(2k+1)^2}{k(k+1)}\\ &\quad+\frac{(k^4+2k^3+k^2-\frac{1}{30})(3k^2+3k+1)}{3k^3(k+1)^3}\\ &=-\frac{1}{63k^6}-\frac{19}{42k^7}+\frac{49}{90k^8}+\mathcal{O}(1/k^9).\end{aligned}$$

We readily convert this asymptotic expresssion into an explicit inequality. It finally remains to check the inequality for the first few k's, and this is readily done. □

Lemma 3.2 puts us in the same position as when majorising $m(x)$ in section 3. We thus proceed in a similar way. We first notice that, when n is a positive integer,

$$\begin{aligned}\int_n^{n+1} |M(x/t)|h'(t)dt &= \max_{n\le t\le n+1} |M(x/t)|(h(n+1)-h(n))\\ &= \max_{n\le t\le n+1} |M(x/t)|\left(\frac{3-2\gamma_0}{6n} - \frac{3-2\gamma_0}{6(n+1)} + g(n+1)-g(n)\right)\end{aligned}$$

and the coefficient of $\max_{n\le t\le n+1}|M(x/t)|$ also reads

$$\begin{aligned}&\frac{3-\frac12-2\gamma_0}{6n} - \frac{3-\frac12-2\gamma_0}{6(n+1)} - n\int_n^\infty B_2(t)\frac{dt}{t^3} + (n+1)\int_{n+1}^\infty B_2(t)\frac{dt}{t^3}\\ &\quad= \frac{3-\frac12-2\gamma_0}{6n(n+1)} - n\log\frac{n+1}{n} + \frac{2n+1}{2(n+1)} - \frac{2n+1}{12n(n+1)^2} + \int_{n+1}^\infty B_2(t)\frac{dt}{t^3}.\end{aligned}$$

We now notice that

$$\int_{n+1}^\infty B_2(t)\frac{dt}{t^3} = \int_{n+1}^\infty B_3(t)\frac{dt}{t^3} = \frac{1}{30(n+1)^3} + \int_{n+1}^\infty B_4(t)\frac{dt}{t^4} \le \frac{1}{30(n+1)^3}$$

by Lemma 4.1. We finally show that, when $n\ge 60$

$$\begin{aligned}&\frac{\frac52-2\gamma_0}{6n(n+1)} - n\log\frac{n+1}{n} + \frac{2n+1}{2(n+1)} - \frac{2n+1}{12n(n+1)^2}\\ &\qquad\qquad + \frac{1}{30(n+1)^3} \le 0.321\int_n^{n+1}\frac{dt}{t^2}.\end{aligned}$$

(The constant is decreasing in the lower bound for n, provided this lower bound be ≥ 3, the optimal constant being $(5-2\gamma_0)/12 = 0.32046\cdots$).

Lemma 4.2. *The function which to $x\ge D_0 = 464\,402$ associates*

$$0.321\cdot\log x\int_{xD_0/(x+D_0)}^{x/2}\frac{0.0146\log u - 0.1098}{(\log u)^2}\frac{udu}{(x-u)^2} + \frac{18\,192\,350\log x}{x^2}$$

is decreasing between D_0 and $D_1 = 1.60510\cdots 10^6$ and increasing afterwards to $0.321\cdot 0.0146(1-\log 2) = 0.001438\cdots$. As a conclusion, it is ≤ 0.00144 when $x\ge D_0$.

Proof. We simply plot the function via

```
{f(x) = intnum(u = x*D0/(x+D0), x/2,
               (0.0146*log(u)-0.1098)*u/( (log(u))^2*(x-u)^2) )
        *log(x)*0.321 + 18192350*log(x)/x^2}

default(realprecision, 10);
plot(t=10^6, 10^8, f(t));
```

□

Thus, when $x \geq D_0$ and on recalling Lemma 2.3:

$$\begin{aligned}\int_1^x |M(x/t)|h'(t)dt &\leq \int_1^{x/D_0} |M(x/t)|h'(t)dt + (\tfrac{7}{4} - \gamma_0)\frac{15\,512\,101}{x} \\ &\leq 0.321 \cdot x \int_1^{x/D_0} \frac{0.0146\log(x/(t+1)) - 0.1098}{(\log(x/(t+1)))^2}\frac{dt}{t^2(t+1)} + \frac{18\,192\,350}{x} \\ &\leq 0.321 \cdot x \int_{xD_0/(x+D_0)}^{x/2} \frac{0.0146\log u - 0.1098}{(\log u)^2}\frac{udu}{(x-u)^2} + \frac{18\,192\,350}{x} \\ &\leq 0.00144x/\log x\end{aligned}$$

by Lemma 4.2. We then use Lemma 3.2 (notice that $\frac{1}{2} - \gamma_0 < 0$). This yields

$$\begin{aligned}|\check{m}(x) - 1| &\leq (\gamma_0 - \tfrac{1}{2})\frac{0.0146\log x - 0.1098}{(\log x)^2} + \frac{2}{x} + \frac{0.00144}{\log x} \\ &\leq \frac{0.00257\log x - 0.0077}{(\log x)^2}\end{aligned}$$

when $x \geq D_0$. We then appeal to Lemma 10.2 to extend the range to $x \geq 13\,950$ and checked numerically its extension to the range $3\,846 \leq x \leq 10^6$. Theorem 1.5 follows readily. The GP-script is called `AsymptoticBoundsFor_checkm.gp` and the main function is `getboundsbisaux`.

The proof of Corollary 1.6 is immediate.

5. An identity for $\check{\check{m}}(x)$ and proof of Theorem 1.7

We start with (3.4) with the choice $\ell = 1$ and on using (3.1):

$$\begin{aligned}\beta_{2,1}(x) &= \sum_{n\leq x}\Big(\frac{x}{n}(\log(x/n) - \tfrac{1}{2}) + \frac{n}{2x}\Big) \\ &= x\log(x/e)H_0(x) + \frac{x}{2}H_0(x) - xH_1(x) + \frac{x}{4} - \frac{B_1(x)}{2} + \frac{B_2(x)}{4x} - \frac{1}{24x} \\ &= \frac{x\log^2 x}{2} + x(\gamma_0 - \tfrac{1}{2})\log x + x(-\tfrac{1}{2}\gamma_0 - \gamma_1 + \tfrac{1}{4}) \\ &\quad + \frac{B_2(x)}{4x} - \frac{1}{24x} + x(\log x - \tfrac{1}{2})\varepsilon_{6,0}(x) - x\varepsilon_{6,1}(x).\end{aligned}$$

We note here, that by construction, cf (3.3), the error term above is continuous all over, and continuously differentiable except maybe at integer point, where it has left and right derivatives. Furthermore $-\frac{1}{2}\varepsilon_{6,0}(1) - \varepsilon_{6,1}(1) - \frac{1}{2}\gamma_0 - \gamma_1 + \frac{1}{4} = 0$. The above equation implies that

$$\begin{aligned}\sum_{n\leq x}\mu(n)\Big(&\frac{x\log^2(x/n)}{2n} + \frac{x}{n}(\gamma_0 - \tfrac{1}{2})\log\frac{x}{n} - \frac{x}{n}(\tfrac{1}{2}\gamma_0 + \gamma_1 - \tfrac{1}{4}) \\ &+ \frac{\{x/n\}^2 - \{x/n\}}{4x/n} + \frac{x(\log(x/n) - \frac{1}{2})}{n}\varepsilon_{6,0}(x/n) - \frac{x}{n}\varepsilon_{6,1}(x/n)\Big) \\ &= x\log x - \frac{x}{2} + \frac{1}{2x}.\end{aligned}$$

We again use (3.5) in the form

$$\sum_{n\le x}\mu(n)\frac{\{x/n\}^2-\{x/n\}}{x/n}=-2+2x^{-1}\sum_{n\le x}\mu(n)\Big(\frac{x}{n}-1\Big)$$

to get

$$\sum_{n\le x}\mu(n)\Big(\frac{x\log^2(x/n)}{2n}+\frac{x}{n}(\gamma_0-\tfrac12)\log\frac{x}{n}-\frac{x}{n}(\tfrac12\gamma_0+\gamma_1+\tfrac14)$$
$$+\frac12+\frac{x(\log(x/n)-\frac12)}{n}\varepsilon_{6,0}(x/n)-\frac{x}{n}\varepsilon_{6,1}(x/n)\Big)$$
$$=x\log x-\frac{x}{2}+\frac1x-\frac12.$$

Here is our main lemma:

Lemma 5.1. *When $x\ge1$, we have*

$$\check{\check{m}}(x)-2\log x+2\gamma_0=2(\gamma_0-\tfrac12)^2\frac{M(x)}{x}+\frac1x\int_1^x M(x/t)k_2'(t)dt+\frac{K_2(1/x)}{x}$$

where

$$K_2(v)=1+4\gamma_0+(3-2\gamma_0)v-(8\gamma_0^2-12\gamma_0+2\gamma_1+2)\frac{(1-v)(2+v)}{3}\tag{5.1}$$

satisfies $4.5\le K_2(v)\le 4+2\gamma_0$ when $0\le v\le1$ and

$$k_2'(t)=(2\gamma_0^2-3\gamma_0+2\gamma_1+\tfrac12)\varepsilon_1(t)+2(\gamma_0-\log t)t\varepsilon_{6,0}(t)+2t\varepsilon_{6,1}(t)-\frac{2\gamma_0-1}{12t}\tag{5.2}$$

is continuous and differentiable except at integers where it has left and right derivative. It satisfies

$$t^2|k_2'(t)|\le1.46.\tag{5.3}$$

Theorem 1.7 is a straightforward consequence of this Lemma. Let us specify that h' is typically not continuous at integer points.

Proof. Indeed, we have already reached

$$\tfrac12\check{\check{m}}(x)+(\gamma_0-\tfrac12)(\check{m}(x)-1)-(\tfrac12\gamma_0+\gamma_1+\tfrac14)(m(x)-M(x)x^{-1})$$
$$+\frac1x\int_1^x M(x/t)q'(t)dt=\log x-\gamma_0+\frac{1}{x^2}-\frac{1}{2x}.$$

with the notation

$$q(t)=t(\log t-\tfrac12)\varepsilon_{6,0}(t)-t\varepsilon_{6,1}(t)\tag{5.4}$$

($q(1)=-\frac12\gamma_0+\gamma_1$). During the proof of Lemma (3.2), we have noticed that

$$\check{m}(x)-1=(\tfrac32-\gamma_0)m(x)-\frac{M(x)}{x}-\frac2x+\frac{1}{x^2}-\frac1x\int_1^x M(x/t)g'(t)dt$$

where the function g is defined by

$$g(t)=\frac{-1}{12t}+t\varepsilon_{6,0}(t).\tag{5.5}$$

Combining both identities leads to

$$\tag{5.6} \tfrac{1}{2}\check{\check{m}}(x) + (-\gamma_0^2 + \tfrac{3}{2}\gamma_0 - \gamma_1 - \tfrac{1}{4})(m(x) - M(x)x^{-1}) - (\gamma_0 - \tfrac{1}{2})^2\frac{M(x)}{x} + \frac{1}{x}\int_1^x M(x/t)(q'(t) - (\gamma - \tfrac{1}{2})g'(t))dt = \log x - \gamma_0 + \frac{1+4\gamma_0}{2x} + \frac{3-2\gamma_0}{2x^2}$$

We further recall (2.1):

$$m(x) - \frac{M(x)}{x} = \frac{4(1-x^{-1})^2}{x} - \frac{4(1-x^{-1})^3}{3x} + \frac{1}{x}\int_1^x M(x/t)\varepsilon_1'(t)dt$$

and thus

$$\check{\check{m}}(x) - 2\log x + 2\gamma_0 = 2(\gamma_0 - \tfrac{1}{2})^2\frac{M(x)}{x} + \frac{1}{x}\int_1^x M(x/t)k_2'(t)dt + \frac{K_2(1/x)}{x}$$

where

$$k_2'(t) = -2(-\gamma_0^2 + \tfrac{3}{2}\gamma_0 - \gamma_1 - \tfrac{1}{4})\varepsilon_1'(t) - 2q'(t) + (2\gamma_0 - 1)g'(t)$$

and K_2 is defined in (5.1). We have

$$k_2(t) = (2\gamma_0^2 - 3\gamma_0 + 2\gamma_1 + \tfrac{1}{2})\varepsilon_1(t) + 2(\gamma_0 - \log t)t\varepsilon_{6,0}(t) + 2t\varepsilon_{6,1}(t) - \frac{2\gamma_0 - 1}{12t}.$$

Let us recall the definitions of $\varepsilon_{6,0}$ and $\varepsilon_{6,1}$:

$$\varepsilon_{6,0}(x) = \int_x^\infty B_2(t)\frac{dt}{t^3}, \quad \varepsilon_{6,1}(x) = \tfrac{1}{2}\int_x^\infty B_2(t)\frac{(2\log t + 1)dt}{t^3}.$$

We deduce from that that (when x is not an integer)

$$\varepsilon_{6,0}'(x) = -\frac{B_2(x)}{x^3}, \quad \varepsilon_{6,1}'(x) = -\frac{B_2(x)(2\log x + 1)}{2x^3}.$$

This leads to the following expression for $k_2'(t)$:

$$t^2k_2'(t) = (2\gamma_0^2 - 3\gamma_0 + 2\gamma_1 + \tfrac{1}{2})t^2\varepsilon_1'(t) - 2(2\gamma_0 + 1)B_2(t) + \frac{2\gamma_0 - 1}{12} + 2(\gamma_0 - \log t)t^2\int_t^\infty B_2(u)\frac{du}{u^3} + t^2\int_t^\infty B_2(u)\frac{(2\log u + 1)du}{u^3}.$$

An integration by parts using $B_3'(t) = 3B_2(t)$ (where $b_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$ and $B_3(t) = b_3(\{t\})$) gives us

$$t^2k_2'(t) = (2\gamma_0^2 - 3\gamma_0 + 2\gamma_1 + \tfrac{1}{2})t^2\varepsilon_1'(t) - 2(2\gamma_0 + 1)B_2(t) + \frac{2\gamma_0 - 1}{12} - (1 + 2\gamma_0)\frac{B_3(t)}{3t} + \frac{6\gamma_0 + 1}{3}t^2\int_t^\infty B_3(u)\frac{du}{u^4} + 2t^2\int_t^\infty B_3(u)\frac{\log(u/t)}{u^4}du.$$

We find that the part without the last two integrals equals (on using $u = \{t\}$)

$$(2\gamma_0^2 - 3\gamma_0 + 2\gamma_1 + \tfrac{1}{2})\frac{((2u-1)t + u - u^2)^2}{t^2} - (4\gamma_0 + 2)(u^2 - u + \tfrac{1}{6}) + \frac{2\gamma_0 - 1}{12} - (1 + 2\gamma_0)\frac{2u^3 - 3u^2 + u}{6t}$$

with $\gamma_1 = -0.07281584548\cdots$. Some numerical analysis tells us that this function is increasing then decreasing between two consecutive integers. It takes its minimal value there and this value is:

$$2\gamma_0^2 - 3\gamma_0 + 2\gamma_1 + \tfrac{1}{2} - \frac{2\gamma_0+1}{3} + \frac{2\gamma_0-1}{12} = 2\gamma_0^2 - \frac{7}{2}\gamma_0 + 2\gamma_1 + \tfrac{1}{12} = -1.41687\cdots$$

The local maxima are all not more than 0.38. The function B_3 satisfies $B_3(1-t) = -B_3(t)$. It attains its maximum at $(3-\sqrt{3})/6$ and minimum at $(3+\sqrt{3})/6$. The value of this maximum is $\sqrt{3}/36 = 0.04811\cdots$. As a consequence

$$\begin{aligned}\frac{6\gamma_0+1}{3}t^2\int_t^\infty |B_3(u)|\frac{du}{u^4} + 2t^2\int_t^\infty |B_3(u)|\frac{\log(u/t)}{u^4}du \\ \le \frac{6\gamma_0+1}{3}\frac{\sqrt{3}}{36}\frac{1}{3t} + \frac{\sqrt{3}}{36}\frac{2}{t}\int_1^\infty \frac{\log v}{v^4}dv = (2\gamma_0+1)\frac{\sqrt{3}}{108t}.\end{aligned}$$

The part concerning k_2' follows readily. It is easy to study $K_2(u)$ and show the bounds claimed. The maximum is attained at $u=1$. □

6. Proof of Theorem 1.8

On using Lemma 5.1 together with Lemma 2.1, we find that with $D_0 = 1\,078\,853$

$$\begin{aligned}|\check{m}(x) - 2\log x + 2\gamma_0| &\le 2(\gamma_0 - \tfrac{1}{2})^2\frac{|M(x)|}{x} + \frac{1.46}{x^2}\int_1^x |M(t)|dt + \frac{5.2}{x} \\ &\le 2(\gamma_0 - \tfrac{1}{2})^2\frac{0.0130\log x - 0.118}{(\log x)^2} + \frac{1.46}{x^2}\int_1^{D_0}|M(t)|dt \\ &\quad + \frac{1.46}{x^2}\int_{D_0}^x \frac{0.0130\log t - 0.118}{(\log t)^2}tdt + \frac{5.2}{x}.\end{aligned}$$

We have

$$\begin{aligned}\int_{D_0}^x \frac{0.0130\log t - 0.118}{(\log t)^2}tdt &= \frac{0.0130x^2}{2\log x} - \frac{0.0130D_0^2}{2\log D_0} - \int_{D_0}^x \frac{0.118 - \frac{0.0130}{2}}{(\log t)^2}tdt \\ &= \frac{0.0130x^2}{2\log x} - \frac{0.0130D_0^2}{2\log D_0} - \frac{0.1115x^2}{2(\log x)^2} - \frac{0.1115D_0^2}{2(\log D_0)^2} - \int_{D_0}^x \frac{0.1115t}{(\log t)^3}dt \\ &\le \frac{0.0130x^2}{2\log x} - \frac{0.0130D_0^2}{2\log D_0} - \frac{0.1115x^2}{2(\log x)^2} - \frac{0.1115D_0^2}{2(\log D_0)^2}.\end{aligned}$$

Moreover, GP/PARI tells us (see script `CompIntM.gp` of lemma 2.2) that

$$\int_1^{1\,100\,000} |M(t)|dt = 118577036$$

and

$$118577036 - \frac{0.0130D_0^2}{2\log D_0} - \frac{0.1115D_0^2}{2(\log D_0)^2} \le -10^9.$$

We have thus reached:

$$\begin{aligned}|\check{m}(x) - 2\log x + 2\gamma_0| &\le 2(\gamma_0 - \tfrac{1}{2})^2\frac{0.0130\log x - 0.118}{(\log x)^2} + 1.46\frac{0.0130\log x - 0.1115}{2(\log x)^2} + \frac{5.2}{x} \\ &\le \frac{0.00965\log x - 0.0818}{(\log x)^2}.\end{aligned}$$

We check numerically this bound with Gp/Pari for $10 \le x \le 10^7$. The script used is `AsymptoticBoundsFor_checkcheckm.gp`, function `getboundsteraux`.

7. An intermediate function for m

In this part, we produce upper bounds for

$$m^*(y) = \sum_{w \le \sqrt{y+1}} \max_{y/w^2 \le z < (y+1)/w^2} |m(z)|/w^2. \tag{7.1}$$

We have taken the maximum so that we may restrict our attention to integer values of y. Notice that the interval $[y/w^2, (y+1)/w^2)$ contains at most one integer.

Lemma 7.1. *For* integer x*, we have*

$$m^*(x)\sqrt{x}/\sqrt{1+\log x} \le \begin{cases} 0.5 & \textit{when } x \in [171\,454, 33\,000\,000], \\ 0.557 & \textit{when } x \in [100, 33\,000\,000], \\ 0.578 & \textit{when } x \in [1, 33\,000\,000]. \end{cases}$$

We precomputed the values of $\mu(n)$ for n up to $3.3 \cdot 10^7$, then the values of $m(x)$ for x up to $3.3 \cdot 10^7$. The script is called `AsymptoticBoundsFor_mstar.gp`, its main function being `getboundsmstar` and the total run time has been about 25 hours on the material specified in the introduction.

We compared $|m^*(x)|$ with $\sqrt{1+\log x}/\sqrt{x}$ as this seems coherent with the numerical outputs: the successive maxima seemed to be oscillating in the neighbourhood $0.5\sqrt{1+\log x}/\sqrt{x}$ and slightly dipping when x grows. The numerical data is however way too thin to allow any conjecture. It seems clear that the work [17] can be adapted to $m(x)$, but no such straightforward heuristic can be deduced for $m^*(x)$.

Here is an generalization of [19, Lemma 2.2]:

Lemma 7.2. *Let $A > e$ be a given parameter. The function*

$$T(y) : y \mapsto \frac{\log y}{y} \int_A^y \frac{dv}{\log v}$$

is first increasing and then decreasing. It reaches its maximum at $y_0(A)$ where $y_0(A)$ is the unique solution of $y = (\log y - 1)\int_A^y dv/\log v$. Moreover we have $T(y_0(A)) = (\log y_0(A))/(\log y_0(A) - 1)$.

Proof. We compute its derivative and get

$$\frac{y^2}{\log y - 1} T'(y) = \frac{y}{\log y - 1} - \int_A^y \frac{dv}{\log v} = h(y)$$

say. We check that $h'(y) = -1/[\log y\,(\log y - 1)^2]$. We have $h(A) > 0$ and h tends to be negative in the vicinity of infinity. Indeed, after several integration by parts, we reach

$$h(y) = \frac{y}{\log y - 1} - \frac{y}{\log y} - \frac{y}{\log^2 y} - \frac{2y}{\log^3 y} + \frac{A}{\log A} + \frac{A}{\log^2 A} + \frac{2A}{\log^3 A} - 6\int_A^y \frac{dv}{\log^4 v}.$$

□

Lemma 7.3. *With $X_0 = 1\,426$, the function*

$$T(y) : y \mapsto \frac{\log y}{y} \int_{\sqrt{X_0}}^{y} \frac{dv}{\log v}$$

is increasing and then decreasing, reaching a maximum around 144.803 *with value* $1.251\,548 + \mathcal{O}^*(10^{-6})$. *Moreover* $T(\sqrt{3.3 \cdot 10^7}) \le 1.139$.

Lemma 7.4. *For $x \ge 11\,808$, we have $|m^*(x)| \le 0.132/\log x$.*
For $x \ge 687$, we have $|m^*(x)| \le 0.320/\log x$.
For $x > 1$, we have $|m^*(x)| \le 0.779/\log x$.

Proof. The initial step is provided by Theorem 1.2:

$$|m(x)| \le 1/(20 \log x) \quad (x \ge X_0 = 1\,426). \tag{7.2}$$

Thus, for $x/U^2 = X_0$,

$$|m^*(x)| \le \frac{1}{20} \sum_{u \le U} \frac{1}{u^2 \log(x/u^2)} + \frac{\max_{U < u \le U+1} |m(x/u^2)|}{U^2} + \int_U^\infty \frac{|m(x/u^2)| du}{u^2}.$$

We appeal to Corollary 1.3 and get that the sum of the last terms is bounded above by

$$\begin{aligned}
&\frac{1}{100U^2} + \int_U^{\sqrt{x/694}} \frac{du}{100u^2} + \int_{\sqrt{x/694}}^{\sqrt{x/41}} \frac{du}{20u^2} + \int_{\sqrt{x/41}}^{\infty} \frac{du}{u^2} \\
&\quad \le \frac{1}{U}\Big(\frac{1}{100U} + \frac{1}{100} + \frac{U}{20\sqrt{x/694}} + \frac{U}{\sqrt{x/41}}\Big) \\
&\quad \le \frac{1}{U}\Big(\frac{1}{100\sqrt{x/X_0}} + \frac{1}{100} + \frac{1}{20\sqrt{X_0/694}} + \frac{1}{\sqrt{X_0/41}}\Big) \le 0.215/U.
\end{aligned}$$

We continue by using a comparison with an integral

$$\begin{aligned}
|m^*(x)| &\le \frac{1}{20} \sum_{u \le U} \frac{1}{u^2 \log(x/u^2)} + \frac{0.215}{U} \\
&\le \frac{1}{20 \log x} + \frac{1}{20\sqrt{x}} \int_{\sqrt{x/U^2}}^{\sqrt{x}} \frac{dv}{2 \log v} + \frac{0.205}{U} \\
&\le \frac{1}{20 \log x} + \frac{1}{20\sqrt{x}} \int_{\sqrt{X_0}}^{\sqrt{x}} \frac{dv}{2 \log v} + \frac{0.215}{\sqrt{x/X_0}}.
\end{aligned}$$

We employ Lemma 7.3 at this level. Hence, when $x \ge 3.3 \cdot 10^7$,

$$|m^*(x)| \le \frac{1}{20 \log x} + \frac{1.139}{20 \log x} + \frac{0.215}{\sqrt{x/X_0}} \le \frac{0.132}{\log x}. \tag{7.3}$$

We extend it to $x \ge 21\,500$ by using Lemma 7.1. We reduce this bound by direct verification by adapting the script `AsymptoticBoundsFor_mstar.gp`, function `getboundsmstar`. □

8. An intermediate function for m, bis repetita

The previous section was dedicated with getting bounds of the shape $1/\log x$ and we aim here at bounds of the shape $1/(1+\log x)$. There are no difficulties, but the computations need to be put down.

Here is another version of Lemma 8.1.

Lemma 8.1. *Let $A > e$ be a given parameter. The function*

$$T^*(y) : y \mapsto \frac{1+\log y}{y}\int_A^y \frac{dv}{\log v}$$

is first increasing and then decreasing. It has a maximum at $y^*(A)$ where $y^*(A)$ is the solution of $(1+\log y)y = (\log y)^2 \int_A^y dv/\log v$ and we have $T^*(y^*(A)) = (1+\log y^*(A))^2/(\log y^*(A))^2$.

Proof. We compute its derivative and get

$$\frac{y^2}{\log y}T^{*\prime}(y) = \frac{y(1+\log y)}{(\log y)^2} - \int_A^y \frac{dv}{\log v} = h^*(y)$$

say. We check that $h^{*\prime}(y) = -2/(\log y)^3$. We have $h^*(A) > 0$ and h^* tends to be negative in the vicinity of infinity. Indeed, after several integrations by parts, we reach

$$h^*(y) = \frac{A}{\log A} + \frac{A}{\log^2 A} - 2\int_A^y \frac{dv}{\log^3 v}.$$

□

Lemma 8.2. *With $X_0 = 1\,426$, the function*

$$T^*(y) : y \mapsto \frac{1+\log y}{y}\int_{\sqrt{X_0}}^y \frac{dv}{\log v}$$

is increasing and then decreasing, reaching a maximum around 1613.183 *with value* $1.289\,114 + \mathcal{O}^*(10^{-6})$. *Moreover* $T^*(\sqrt{3.3\cdot 10^7}) \le 1.28$.

Lemma 8.3. *For $x \ge 11\,811$, we have $|m^*(x)| \le 0.143/(1+\log x)$.*
For $x \ge 882$, we have $|m^*(x)| \le 0.320/(1+\log x)$.
For $x \ge 296$, we have $|m^*(x)| \le 0.5/(1+\log x)$.
For $x > 1$, we have $|m^*(x)| \le 1.694/(1+\log x)$.

Proof. We follow the proof of Lemma 7.4 until (7.3) which we modify into

$$|m^*(x)| \le \frac{1}{20\log x} + \frac{1.28}{25(1+\log x)} + \frac{0.215}{\sqrt{x/X_0}} \le \frac{0.143}{1+\log x}. \tag{8.1}$$

We extend it to $x \ge 22\,100$ by using Lemma 7.1. We reduce this bound by direct verification by again adapting the script `AsymptoticBoundsFor_mstar.gp`, function `getboundsmstar`. □

9. Proof of Theorem 1.12 and 1.13

We proceed to prove the estimate concerning $m_q(x)$. Lemmas 1.9 and 7.4 give us, for a real parameter U such that $x/q \geq X_1$,

$$|m_q(x)| \leq \sum_{d|q} \frac{\mu^2(d)}{d} m^*(x/d)$$

$$\leq \frac{q}{\varphi(q)\log(x/q)} \begin{cases} 0.132 & \text{when } x/q \geq 11\,808, \\ 0.320 & \text{when } x/q \geq 687, \\ 0.779 & \text{when } x/q > 1. \end{cases} \tag{9.1}$$

Theorem 1.12 follows readily. We proceed in a similar way with Theorem 1.13 but on using Lemma 8.3.

10. An intermediate function for $\check{m}$

In this part, we produce upper bounds for

$$\check{m}^*(y) = \sum_{w \leq \sqrt{y+1}} \max_{y/w^2 \leq z < (y+1)/w^2} |\check{m}(z) - 1|/w^2. \tag{10.1}$$

We have again taken the maximum so that we may restrict our attention to integer values of y. With such a definition, our script will spend most of its time computing logarithms, which are costly. Let us investigate somewhat further what happens. The interval is $[y/w^2, (y+1)/w^2)$. If there are no integer lying in this interval, then let us set

$$A = \sum_{n \leq y/w^2} \mu(n)/n, \quad B = -1 + \sum_{n \leq y/w^2} \mu(n)(\log n)/n.$$

The function $|A \log z - B|$ is maximized at $\log z = \log y - 2\log w$ or at $\log z = \log(y+1) - 2\log w$. If there is an integer, say m lying in this interval, then we have to maximize $|A \log z - B|$ when $z \in [y/w^2, m)$ and a similar expression when $z \in [n, y/(w+1)^2)$. In both cases, we can recover the relevant $\log z$ in terms of the logarithms of integers $\leq y$. It is thus enough to precompute these.

Lemma 10.1. *For* integer x*, we have*

$$\check{m}^*(x)\sqrt{x} \leq \begin{cases} 0.440 & \textit{when } x \in [64, 12\,000\,000], \\ 0.504 & \textit{when } x \in [16, 12\,000\,000], \\ 1 & \textit{when } x \in [1, 12\,000\,000]. \end{cases}$$

We precomputed the values of $\mu(n)$ and $\log n$ for n up to $1.2\cdot 10^7$, then the values of $m(x)$ and of $-1+\sum_{n\leq x}\mu(n)(\log n)/n$ for x up to $1.2\cdot 10^7$. The total run time has been about 8 hours. The constant 0.440 comes from the behaviour of $\check{m}^*(x)$ for $x \in [9\,400\,000, 9\,600\,000]$. The script is called `AsymptoticBoundsFor_checkmstar.gp`, and its main function is called `getboundscheckmstar`

We compared $|\check{m}^*(x)|$ to $1/\sqrt{x}$ as this seems coherent with the numerical outputs; the quantity $\check{m}^*(x)\sqrt{x}$ oscillates between 0.425 and 0.439, this maximum increasing slightly with x.

We used a very similar scheme to show that

Lemma 10.2. *We have, for* real x,

$$|\check{m}(x)-1|\sqrt{x} \le \begin{cases} 0.0192 & \text{when } 340\,000 \le x \le 12\,000\,000,\\ 0.0218 & \text{when } 11 \le x \le 12\,000\,000,\\ 1 & \text{when } 1 \le x \le 12\,000\,000. \end{cases}$$

Another numerical application of Lemma 7.2 yields:

Lemma 10.3. *With* $Y_0 = 16$, *the function*

$$T(y) : y \mapsto \frac{\log y}{y}\int_{\sqrt{Y_0}}^{y}\frac{dv}{\log v}$$

is increasing and then decreasing, reaching a maximum around 139.45 *with value* $1.253951 + \mathcal{O}^*(10^{-6})$.

Lemma 10.4. *We have, for* real x,

$$|\check{m}^*(x)|\log x \le \begin{cases} 0.0538 & \text{when } x \ge 3\,158,\\ 0.200 & \text{when } x \ge 101,\\ 0.250 & \text{when } x > 1. \end{cases}$$

Proof. The initial step is provided by Theorem 1.5:

$$|\check{m}(x)-1| \le 1/(389\log x) \quad (x \ge Y_0 = 16). \tag{10.2}$$

For smaller x, we use $0 \le \check{m}(x) \le 1.004$ which implies that $|\check{m}(x)-1| \le 1$. Thus, for $x/U^2 = Y_0$,

$$|\check{m}^*(x)| \le \sum_{u\le U}\frac{0.0203}{u^2\log(x/u^2)} + \frac{\max_{U<u\le U+1}|\check{m}(x/u^2)-1|}{U^2} + \int_U^\infty \frac{|\check{m}(x/u^2)-1|du}{u^2}.$$

We appeal to Corollary 1.3 and get that the sum of the last terms is bounded above by

$$\begin{aligned}\frac{1}{125U^2} &+ \int_U^{\sqrt{x/7}}\frac{du}{125u^2} + \int_{\sqrt{x/7}}^\infty \frac{du}{u^2}\\ &\le \frac{1}{U}\Big(\frac{1}{125U} + \frac{1}{125} + \frac{U}{\sqrt{x}}\Big)\\ &\le \frac{1}{U}\Big(\frac{1}{125\sqrt{x/Y_0}} + \frac{1}{125} + \frac{1}{\sqrt{Y_0/7}}\Big) \le 0.667/U.\end{aligned}$$

We continue by using a comparison with an integral

$$\begin{aligned}|\check{m}^*(x)| &\le \sum_{u\le U}\frac{0.0203}{u^2\log(x/u^2)} + \frac{0.667}{U}\\ &\le \frac{0.0203}{\log x} + \frac{0.0203}{\sqrt{x}}\int_{\sqrt{x/U^2}}^{\sqrt{x}}\frac{dv}{2\log v} + \frac{0.667}{U}.\\ &\le \frac{0.0203}{\log x} + \frac{0.0203}{\sqrt{x}}\int_{\sqrt{Y_0}}^{\sqrt{x}}\frac{dv}{2\log v} + \frac{0.667}{\sqrt{x/Y_0}}.\end{aligned}$$

We again employ Lemma 10.3 at this level. Hence, when $x \ge 1.2\cdot 10^7$,

$$|\check{m}^*(x)| \le \frac{0.0203}{\log x} + \frac{1.254\times 0.0203}{\log x} + \frac{0.667}{\sqrt{x/Y_0}} \le \frac{0.0584}{\log x}. \tag{10.3}$$

We extend it to $x \geq 3\,900$ by using Lemma 10.1 and to $x \geq 3\,158$ by direct checking done by modifying suitably `AsymptoticBoundsFor_checkmstar.gp`. □

11. An intermediate function for $\check{m}$, bis repetita

The previous section was dedicated with getting bounds of the shape $1/\log x$ and we aim here at bounds of the shape $1/(1+\log x)$. There are no difficulties, but the computations need to be put down.

Another numerical application of Lemma 8.1 yields:

Lemma 11.1. *With $Y_0 = 16$, the function*

$$T^*(y) : y \mapsto \frac{1+\log y}{y} \int_{\sqrt{Y_0}}^{y} \frac{dv}{\log v}$$

is increasing and then decreasing, reaching a maximum around 68.49 *with value* $1.529154 + \mathcal{O}^*(10^{-6})$.

Lemma 11.2. *We have, for* real x,

$$|\check{m}^*(x)|(1+\log x) \leq \begin{cases} 0.0660 & \textit{when } x \geq 3\,150, \\ 0.200 & \textit{when } x \geq 171, \\ 1 & \textit{when } x > 1. \end{cases}$$

Proof. We follow the proof of Lemma 7.4 until (10.3) which we modify into

$$|\check{m}^*(x)| \leq \frac{0.0203}{\log x} + \frac{1.530 \times 0.0203}{1+\log x} + \frac{0.667}{\sqrt{x/Y_0}} \leq \frac{0.0660}{1+\log x}. \tag{11.1}$$

We extend it to $x \geq 3\,800$ by using Lemma 7.1. We reduce this bound by direct verification, again realised by modifying suitably `AsymptoticBoundsFor_checkmstar.gp`. □

12. Proof of Theorem 1.14 and 1.15

We proceed to prove the estimate concerning $\check{m}_q(x)$. Lemma 1.9 gives us,

$$\begin{aligned}\check{m}_q(x) &= \sum_{d|q} \frac{\mu^2(d)}{d} \sum_{\substack{w \leq \sqrt{x/d}, \\ w|q^\infty}} \check{m}(x/(dw^2))/w^2 \\ &= \sum_{d|q} \frac{\mu^2(d)}{d} \sum_{\substack{w \leq \sqrt{x/d}, \\ w|q^\infty}} \frac{\check{m}(x/(dw^2)) - 1}{w^2} + \check{m}_q^\sharp(x)\end{aligned}$$

for a real parameter U such that $x/q \geq X_1$,

$$\begin{aligned}\left|\check{m}_q(x) - \check{m}_q^\sharp(x)\right| &\leq \sum_{d|q} \frac{\mu^2(d)}{d} m^*(x/d) \\ &\leq \frac{q}{\varphi(q)\log(x/q)} \begin{cases} 0.0538 & \text{when } x \geq 3\,158, \\ 0.200 & \text{when } x \geq 101, \\ 0.250 & \text{when } x > 1. \end{cases}\end{aligned}$$

13. An intermediate function for $\check{\check{m}}$

In this part, we produce upper bounds for

$$\check{\check{m}}^*(y) = \sum_{w\le\sqrt{y+1}} \max_{y/w^2\le z<(y+1)/w^2} |\check{\check{m}}(z) - 2\log z + 2\gamma_0|/w^2 \tag{13.1}$$

by following closely section 10. We have again taken the maximum so that we may restrict our attention to integer values of y. With such a definition, our script will spend most of its time computing logarithms, which are costly. Let us investigate somewhat further what happens. The interval is $[y/w^2, (y+1)/w^2)$. If there are no integers lying in this interval, then let us set

$$A = \sum_{n\le y/w^2} \frac{\mu(n)}{n},\ B = \sum_{n\le y/w^2} \frac{\mu(n)\log n}{n},\ C = 2\gamma_0 + \sum_{n\le y/w^2} \frac{\mu(n)(\log n)^2}{n}.$$

The function $|A(\log z)^2 - 2(B+1)\log z + C|$ is maximum at $\log z = \log(y/w^2)$, $\log z = \log((y+1)/w^2)$ or $\log z = (B+1)/A$ if this latter value falls within the proper bounds. If there is an integer, say m lying in this interval, then we have to maximize $|A\log z - B|$ when $z\in[y/w^2, m)$ and a similar expression when $z\in[m, y/(w+1)^2)$. In both cases, we can recover the relevant $\log z$ in terms of the logarithms of integers $\le y$. It is thus enough to precompute these.

Lemma 13.1. *For* integer x, *we have*

$$\check{\check{m}}^*(x)\sqrt{x} \le \begin{cases} 0.333 & \textit{when } x\in[16\,900, 12\,000\,000],\\ 0.474 & \textit{when } x\in[10, 12\,000\,000],\\ 1.16 & \textit{when } x\in[1, 12\,000\,000].\end{cases}$$

We precomputed the values of $\mu(n)$ and $\log n$ for n up to $1.2\cdot 10^7$, then the values of $m(x)$, of $-1+\sum_{n\le x}\mu(n)(\log n)/n$ and of $2\gamma_0 + \sum_{n\le x}\mu(n)(\log n)^2/n$ for x up to $1.2\cdot 10^7$. The total run time has been about 16 hours. The script is called `AsymptoticBoundsFor_checkcheckmstar.gp` and its main function is named `getboundscheckcheckmstar`.

We compared $|\check{\check{m}}^*(x)|$ with $1/\sqrt{x}$ as this seems coherent with the numerical outputs. These outputs varies extremely tamely, as shown by the following sample:

beginning	end	maximal constant
6800000	7000000	0.3318...
7000000	7200000	0.3313...
7200000	7400000	0.3318...
7400000	7600000	0.3318...
7600000	7800000	0.3316...

.

Question. *Does there exist a constant c such that* $\lim_{x\to\infty}\sqrt{x}\,|\check{\check{m}}^*(x)| = c$ *?*

We report also the results of a slight modification of the script used above.

Lemma 13.2. *We have*

$$|\check{\check{m}}(x) - 2\log x + 2\gamma_0|\sqrt{x} \leq \begin{cases} 0.00234 & \textit{when } 3\,393 \leq x \leq 12\,000\,000, \\ 0.00650 & \textit{when } 10 \leq x \leq 12\,000\,000, \\ 0.0717 & \textit{when } 4 \leq x \leq 12\,000\,000, \\ 1.155 & \textit{when } 1 \leq x \leq 12\,000\,000. \end{cases}$$

Lemma 13.3. *We have*

$$|\check{\check{m}}^*(x)|\log x \leq \begin{cases} 0.0408 & \textit{when } x \geq 5, \\ 0.206 & \textit{when } x > 1. \end{cases}$$

Proof. The proof follows closely the one of Lemma 10.4, but relies on Theorem 1.8 instead of Theorem 1.5, namely:

$$|\check{\check{m}}(x) - 2\log x + 2\gamma_0| \leq 1/(103\log x) \quad (x \geq 9). \tag{13.2}$$

We also have $0 \leq \check{\check{m}}(x) - 2\log x + 2\gamma_0 \leq 2\gamma_0$ when $x \leq Y_0 = 16$, in agreement to the conjecture. Thus, for $x/U^2 = Y_0$,

$$|\check{\check{m}}^*(x)| \leq \frac{1}{103}\sum_{u\leq U}\frac{1}{u^2\log(x/u^2)} + 2\gamma_0\frac{1+U^{-1}}{U}.$$

We continue by using a comparison with an integral

$$\begin{aligned}|\check{\check{m}}^*(x)| &\leq \frac{1}{103}\sum_{u\leq U}\frac{1}{u^2\log(x/u^2)} + 2\gamma_0\frac{1+U^{-1}}{U} \\ &\leq \frac{1}{103\log x} + \frac{1}{103\sqrt{x}}\int_{\sqrt{x/U^2}}^{\sqrt{x}}\frac{dv}{2\log v} + 2\gamma_0\frac{1+U^{-1}}{U}. \\ &\leq \frac{1}{103\log x} + \frac{1}{103\sqrt{x}}\int_{\sqrt{Y_0}}^{\sqrt{x}}\frac{dv}{2\log v} + 2\gamma_0\frac{1+\sqrt{Y_0/x}}{\sqrt{x/Y_0}}.\end{aligned}$$

We again employ Lemma 10.3 at this level. Hence, when $x \geq 1.2\cdot 10^7$,

$$|\check{\check{m}}^*(x)| \leq \frac{1}{103\log x} + \frac{1.254}{103\log x} + 2\gamma_0\frac{1+\sqrt{Y_0/x}}{\sqrt{x/Y_0}} \leq \frac{0.0408}{\log x}. \tag{13.3}$$

We extend it to $x \geq 10$ by using Lemma 13.1, and to $x \geq 5$ by direct check, by modifying `AsymptoticBoundsFor_checkcheckmstar.gp`. We also have

$$|\check{\check{m}}^*(x)| \leq 0.207/\log x \quad (x > 1).$$

□

14. An intermediate function for $\check{\check{m}}$, bis repetita

The previous section was dedicated with getting bounds of the shape $1/\log x$ and we aim here at bounds of the shape $1/(1+\log x)$. There are no difficulties, but the computations need to be put down.

Lemma 14.1. *We have, for* real *x,*

$$|\check{\check{m}}^*(x)|(1+\log x) \leq \begin{cases} 0.0452 & \textit{when } x \geq 6, \\ 0.172 & \textit{when } x \geq 2, \\ 1.155 & \textit{when } x > 1. \end{cases}$$

Proof. We follow the proof of Lemma 13.3 until (13.3) which we modify via Lemma 11.1 into

$$|\check{\check{m}}^*(x)| \le \frac{1}{103\log x} + \frac{1.530}{103(1+\log x)} + 2\gamma_0\frac{1+\sqrt{Y_0/x}}{\sqrt{x/Y_0}} \le \frac{0.0452}{1+\log x}. \tag{14.1}$$

We extend it to $x \ge 540$ by using Lemma 7.1. We reduce this bound by direct verification, again by modifying `AsymptoticBoundsFor_checkcheckmstar.gp` . □

15. Proof of Theorem 1.16 and 1.17

We proceed to prove the estimate concerning $\check{\check{m}}_q(x)$. Lemma 1.9 gives us,

$$\begin{aligned}\check{\check{m}}_q(x) &= \sum_{d|q}\frac{\mu^2(d)}{d}\sum_{\substack{w\le\sqrt{x/d},\\ w|q^\infty}}\check{\check{m}}(x/(dw^2))/w^2\\ &= \sum_{d|q}\frac{\mu^2(d)}{d}\sum_{\substack{w\le\sqrt{x/d},\\ w|q^\infty}}\frac{\check{\check{m}}(x/(dw^2)) - 2\log(x/(dw^2)+2\gamma_0}{w^2} + \check{\check{m}}_q^\sharp(x).\end{aligned}$$

Then, for a real parameter U such that $x/q \ge X_1$, we have

$$\left|\check{\check{m}}_q(x) - \check{\check{m}}_q^\sharp(x)\right| \le \sum_{d|q}\frac{\mu^2(d)}{d}m^*(x/d) \le \frac{q/\varphi(q)}{\log(x/q)}\begin{cases}0.0408 & \text{when } x\ge 5,\\ 0.206 & \text{when } x>1.\end{cases}$$

16. Proof of Theorems 1.18 and 1.19

We prove in this section estimates relative to M_q. As in the case of m_q, $\check{m}_q$ and $\check{\check{m}}_q$, we rely on an identity that links this quantity with its counterpart with no coprimality condition. Indeed, we readily modify the proof of Lemma 1.9 to get

$$M_q(x) = \sum_{d|q}\mu^2(d)\sum_{\substack{w\le\sqrt{x/d},\\ w|q^\infty}}M\Big(\frac{x}{dw^2}\Big). \tag{16.1}$$

We recall [20, Theorem 1]:

Lemma 16.1. *For $D \ge 1\,078\,853$, we have*

$$\Big|\sum_{d\le D}\mu(d)\Big| \le \frac{0.0130\log D - 0.118}{(\log D)^2}D.$$

We extend this bound to get the simpler

$$\Big|\sum_{d\le D}\mu(d)\Big|/D \le \frac{0.0130}{\log D} \quad (D\ge D_1 = 97\,067), \tag{16.2}$$

$$\Big|\sum_{d\le D}\mu(d)\Big|/D \le \frac{0.0950}{\log D} \quad (D\ge D_0 = 688). \tag{16.3}$$

$$\Big|\sum_{d\le D}\mu(d)\Big|/D \le \frac{0.644}{\log D} \quad (D > 1). \tag{16.4}$$

The script we used is called `AsymptoticBoundsFor_M.gp` and its main function is `getboundsM`. But as in the previous cases of m_q, $\check{m}_q$ and $\check{\check{m}}_q$, simply plugging these estimates in (16.1) and estimating the tail via Rankin's trick leads to bad numerical results. We again take another path that requires many more computations but is numerically much better. We define

$$M^*(y) = \sum_{w \le \sqrt{y}} \max_{y/w^2 < z \le (y+1)/w^2} |M(z)|. \tag{16.5}$$

Lemma 16.2. *For an n integer satisfying $99\,000 \le n \le 80\,000\,000$, we have*

$$M^*(n) \le 0.574\sqrt{n(1+\log n)}.$$

It is enough to replace the constant 0.574 by 0.595 to extend this result to $x \ge 9\,933$. Furthermore, for n integer satisfying $1 \le n \le 80\,000\,000$, we have

$$M^*(n) \le 2.403\sqrt{n}.$$

As it turns out, the quantity $M^*(n)/\sqrt{n(1+\log n)}$ tends to oscillate slowly and dips somewhat when x increases while the quantity $M^*(n)/\sqrt{n}$ tends to rise slowly, still in an oscillating manner of course. Since we have to precompute only $M(n)$ which happens to be an integer of type `small`, the requirement in memory space is much less acute, when compared with the other computations of this kind we had to carry up to now. This is why we have been able to increase significantly the upper bound for n. The running time is however much larger, as it increases quadratically: on the same machine as before, it took about three days. The script is called `AsymptoticBoundsFor_Mstar.gp` and its main function is `getboundsMstar`.

Yet another numerical application of Lemma 7.2 yields:

Lemma 16.3. *With $D_1 = 97\,067$, the function*

$$T(y) : y \mapsto \frac{\log y}{y}\int_{\sqrt{D_1}}^{y} \frac{dv}{\log v}$$

is increasing and then decreasing, reaching a maximum around $45\,443.09$ with value $1.102836 + \mathcal{O}^*(10^{-7})$.

All this has prepared the ground for estimating $|M^*(x)|\log x$.

Lemma 16.4. *We have*

$$|M^*(x)|\log x \le \begin{cases} 0.0918 & \textit{when } x \ge 48\,513, \\ 0.199 & \textit{when } x \ge 4\,536, \\ 0.429 & \textit{when } x \ge 490, \\ 0.997 & \textit{when } x > 1. \end{cases}$$

Proof. The proof is patterned on the one of Lemma 10.4. We find that, for $x/U_0^2 = D_0 = 688$ and $x/U_1^2 = D_1 = 97\,067$,

$$\begin{aligned}
M^*(x)/x &\le \sum_{w\le U_1}\frac{0.0130}{w^2\log(x/w^2)} + \sum_{U_1<w\le U_0}\frac{0.0950}{w^2\log(x/w^2)} + \frac{1+U_0^{-1}}{U_0}\\
&\le \frac{0.0130}{\log x} + \frac{0.0130}{\sqrt{x}}\int_{\sqrt{D_1}}^{\sqrt{x}}\frac{dv}{2\log v} + \frac{0.0950\sqrt{D_1}}{\sqrt{x}\log D_1}\\
&\qquad + \frac{0.0950}{\sqrt{x}}\int_{\sqrt{D_0}}^{\sqrt{D_1}}\frac{dv}{2\log v} + \frac{1+\sqrt{D_0/x}}{\sqrt{x/D_0}}\\
&\le \frac{0.0130(1+1.102836)}{\log x} + \frac{2.578}{\sqrt{x}} + \frac{2.777}{\sqrt{x}} + \frac{1+\sqrt{D_0/x}}{\sqrt{x/D_0}} \le \frac{0.0918}{\log x}
\end{aligned}$$

when $x \ge 80\,000\,000$. We use Lemma 16.2 to extend this bound to $x \ge 61\,408$ and to $x \ge 48\,513$ by direct inspection; this is done by modifying suitably the script `AsymptoticBoundsFor_Mstar.gp` □

We can thus write

$$\left|M_q(x)\right| \le \sum_{d|q}\mu^2(d)M^*\Bigl(\frac{x}{d}\Bigr)$$

and conclude as before.

Variation. We start with an application of Lemma 8.1:

Lemma 16.5. *With $D_1 = 97\,067$, the function*

$$T^*(y) : y \mapsto \frac{1+\log y}{y}\int_{\sqrt{D_1}}^{y}\frac{dv}{\log v}$$

is increasing and then decreasing, reaching a maximum around $22\,545.85$ *with value* $1.209\,488 + \mathcal{O}^*(10^{-6})$.

Lemma 16.6. *We have*

$$|M^*(x)|(1+\log x) \le \begin{cases} 0.0975 & \text{when } x \ge 48\,645,\\ 0.198 & \text{when } x \ge 7\,100,\\ 0.498 & \text{when } x \ge 490,\\ 1 & \text{when } x > 1.\end{cases}$$

Proof. We adapt the proof of Lemma 16.4 and get

$$\begin{aligned}
M^*(x)/x &\le \frac{0.0130}{\log x} + \frac{1.210\times 0.0130}{1+\log x} + \frac{2.578}{\sqrt{x}} + \frac{2.777}{\sqrt{x}} + \frac{1+\sqrt{D_0/x}}{\sqrt{x/D_0}}\\
&\le \frac{0.0975}{1+\log x}
\end{aligned}$$

when $x \ge 80\,000\,000$. We use Lemma 16.2 to extend this bound to $x \ge 66\,000$ and to $x \ge 48\,645$ by direct inspection, again by modifying suitably the script `AsymptoticBoundsFor_Mstar.gp`. □

References

1. A. Axer, *Über einige Grenzwertsätze*, Wien. Ber. **120** (1911), 1253–1298 (German).
2. M. Balazard, *Remarques élémentaires sur la fonction de Moebius*, private communication (2011).
3. ______, *Elementary remarks on Möbius' function*, Proceedings of the Steklov Intitute of Mathematics **276** (2012).
4. P.T. Bateman and H.G. Diamond, *Asymptotic distribution of Beurling's generalized prime numbers*, Studies in Number Theory, Math. Assoc. Amer. (distributed by Prentice-Hall, Englewood Cliffs, N.J.), 1969, pp. 152–210. MR 0242778 (39 #4105)
5. D. Berkane, O. Bordellès, and O. Ramaré, *Explicit upper bounds for the remainder term in the divisor problem*, Math. of Comp. **81** (2012), no. 278, 1025–1051.
6. N. Costa Pereira, *Elementary estimates for the Chebyshev function $\psi(X)$ and for the Möbius function $M(X)$*, Acta Arith. **52** (1989), 307–337.
7. H.G. Diamond and Wen-Bin Zhang, *A PNT equivalence for Beurling numbers*, Submitted to Functiones et approximatio (2012).
8. F. Dress and M. El Marraki, *Fonction sommatoire de la fonction de Möbius 2. Majorations asymptotiques élémentaires*, Exp. Math. **2** (1993), no. 2.
9. M. El Marraki, *Fonction sommatoire de la fonction μ de Möbius, majorations asymptotiques effectives fortes*, J. Théor. Nombres Bordx. **7** (1995), no. 2.
10. ______, *Majorations de la fonction sommatoire de la fonction $\frac{\mu(n)}{n}$*, Univ. Bordeaux 1 **Prépublication** (1996), no. 96-8.
11. A. Granville and O. Ramaré, *Explicit bounds on exponential sums and the scarcity of squarefree binomial coefficients*, Mathematika **43** (1996), no. 1, 73–107.
12. A. Kienast, *Über die Äquivalenz zweier Ergebnisse der analytischen Zahlentheorie*, Mathematische Annalen **95** (1926), 427–445, 10.1007/BF01206619.
13. R.A. Mac Leod, *A new estimate for the sum $m(x) = \sum_{n\leq x}\mu(n)$*, Acta Arith. **13** (1967), Erratum, ibid. 16 (1969), 99-100.
14. R.A. MacLeod, *A curious identity for the Möbius function*, Utilitas Math. **46** (1994), 91–95. MR 1301298 (95g:11002)
15. H.L. Montgomery and R.C. Vaughan, *Multiplicative Number Theory: I. Classical Theory*, Cambridge Studies in Advanced Mathematics, vol. 97, Cambridge University Press, 2006.
16. Y. Motohashi, *Primes in arithmetic progressions*, Invent. Math. **44** (1978), no. 2, 163–178.
17. Nathan Ng, *The distribution of the summatory function of the Möbius function*, Proc. London Math. Soc. (3) **89** (2004), no. 2, 361–389. MR 2078705 (2005f:11215)
18. The PARI Group, Bordeaux, *PARI/GP, version* `2.5.2`, 2011, `http://pari.math.u-bordeaux.fr/`.
19. O. Ramaré, *Explicit estimates on the summatory functions of the Moebius function with coprimality restrictions*, Submitted (2013), 12pp.
20. ______, *From explicit estimates for the primes to explicit estimates for the Moebius function*, Acta Arith. **157** (2013), no. 4, 365–379.
21. ______, *Some elementary explicit bounds for two mollifications of the Moebius function*, Functiones et Approximatio (2013), 12pp.
22. J.B. Rosser, *Explicit bounds for some functions of prime numbers*, American Journal of Math. **63** (1941), 211–232.
23. J.B. Rosser and L. Schoenfeld, *Approximate formulas for some functions of prime numbers*, Illinois J. Math. **6** (1962), 64–94.
24. L. Schoenfeld, *An improved estimate for the summatory function of the Möbius function*, Acta Arith. **15** (1969), 223–233.
25. W. Schwartz and J. Spilker, *Arithmetical functions, an introduction to elementary and analytic properties of arithmetic functions and to some of their almost-periodic properties*, Lectures Notes Series, vol. 184, London Math. Soc., Cambridge, 1994.

Laboratoire CNRS Paul Painlevé, Université Lille 1, 59 655 Villeneuve d'Ascq, France
E-mail address: `ramare@math.univ-lille1.fr`