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EIGENVALUES IN THE LARGE SIEVE INEQUALITY

O. Ramaré

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Abstract: *D'un spécialiste des nombres moyens à un spécialiste des nombres ... et des moyens.* We provide some evidence that the eigenvalues of the hermitian form $\sum_{a/q} |\sum_{n\leqslant N} \varphi_n e(na/q)|^2$ tend to have a limit distribution when N and Q go simultaneously to infinity in such a way that N/Q^2 tends to a constant. We also present some background material, as well as a large sieve equality, when $N \operatorname{Log}^7 N = o(Q)$, that follows from our results.

Keywords: large sieve inequality, circle method.

1. Introduction

The additive arithmetical form of the large sieve inequality relies on a bound for

$$\sum_{q\leqslant Q} \sum_{a \bmod^* q} \Big|\sum_{n\leqslant N} \varphi_n e(na/q)\Big|^2$$

usually when Q^2 and N are of comparable size – in a vague sense. In this context, the result comes from forgetting the arithmetical nature and bounding the hermitian form

$$\sum_{\substack{\theta\in\Theta(Q),\\ m,n\leqslant N}} \varphi_n \overline{\varphi_m} e((n-m)\theta) \tag{1}$$

where $\Theta(Q) = \{a/q, q \leqslant Q, a \bmod {}^* q\}$ is the beginning of the Farey series. The information we use is that the largest of its eigenvalues is $\leqslant N + Q^2$, and it comes from the sole fact that any two points of $\Theta(Q)$ are at least Q^{-2} apart. See [23]. For such an approach to hold, every point a/q has to appear with an identical weight and deviating from this line costs a high price in understanding, as shown for instance by the weigheted version of [24], or of [27]. The quantity we evaluate can also be thought of as a version of the circle method, and Gallagher in

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[15] indeed derived a good bound for the above inequality by comparing it to the integral $\int_0^1 |\sum_n \varphi_n e(n\theta)|^2 d\theta$. The same path is taken in [12]. However, this comparison is not the key point in many applications, and this line of thought leads to Iwaniec' δ*-symbol* method as it appears for instance in [8]. Examining this generalization/modification, it transpires that the circle method situation corresponds more to the above inequality weighted with a $1/(qQ)$ and that is also what appears in the form of a reversed large sieve inequality that Duke & Iwaniec proved in [9]. Conversely, it is not obvious that the important points in the circle method should be the rationals, and [6] shows that the quadratic Harcos sequence may be equally regular. Our first idea was simply to try to understand the bilinear form in (1) as it stands.

When Q^2 is small with respect to N, most (non-zero) eigenvalues are expected to be close to N. The reason is as follows: the matrix B of the above hermitian form has (m,n)-entry $\sum_\theta e((n-m)\theta)$ and can be written as A^*A where $A = (e(n\theta))$ has lines indexed by the integers between 1 and N and columns by points of Θ. Its non-zero eigenvalues are the same as the ones of AA^*, whose entries are now

$$\sum_{1\leqslant n\leqslant N} e(n(\theta-\theta')).$$

Such an expression divided by N is 1 if $\theta = \theta'$ and small otherwise, explaining our claim. See [10] for more on this aspect.

The situation when Q^2 is of size N is much less satisfactorily understood (in arithmetical situations, Q^2 is most often $\ll N$ but, ideally, of its size) and we can look at the large sieve inequality as a comparison between our hermitian form and $(N+Q^2)\,\mathrm{Id}$. Another comparison pertaining to the same area is due to Franel [13] and consists in measuring the discrepancy between $\Theta(Q)$ and a set of uniformly spaced points, but there does not seem to have been any use of this idea in the large sieve area despite the fairly large amount of work it induces (see e.g. [20], [3], [7], [18], [16], [14]).

The large sieve inequality being so efficient, one can believe it to be nearly an equality, meaning that most eigenvalues are indeed close to the largest one. With such a belief, we started to compute the distribution function of the eigenvalues, namely

$$\mathscr{D}(N,Q,\lambda) = \#\{i/\lambda_i \leqslant \lambda N\}/N \tag{2}$$

where (λ_i) are the eigenvalues associated with N and Q. A particularly interesting case is when $|\Theta(Q)| = N$, and is the one we chose for the computations, but it is clear that we are most generally interested in the situation when $|\Theta(Q)|$ is around N. Recall that (see (16) and the discussion therein)

$$|\Theta(Q)| = \sum_{q\leqslant Q} \phi(q) = \frac{3}{\pi^2}Q^2(1+o(1)). \tag{3}$$

As a consequence, in the plot below, we have $N \sim 3Q^2/\pi^2$; note that the large sieve inequality ensures us that $\mathscr{D}(N,Q,1+Q^2N^{-1}) = 1$. As a further consequence

$\mathscr{D}(N,Q,u)$ equals 1 when u is just larger than $1+\pi^2/3$, where the "just" is a $o(1)$ when N goes to infinity.

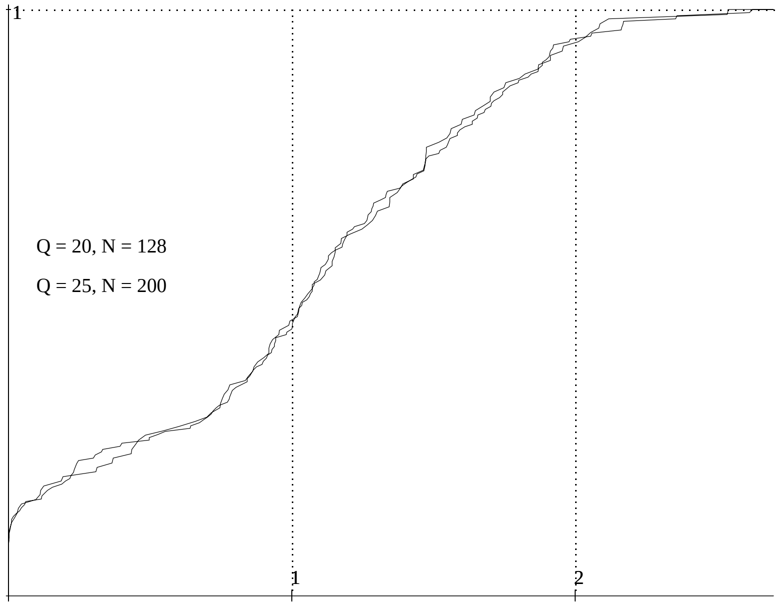

Figure 1. Densities of eigenvalues $/N$ for $Q=20$ and $Q=25$.

We see on this plot that only a fraction of the eigenvalues are indeed close to the maximal value! Furthermore, an extremely strong asymptotic behaviour arises, for which we do not have the slightest proof. In fact we are not able to relate in any way the eigenvalues corresponding to Q to the ones corresponding to another Q', and surely not able to relate them when they are rescaled. We note here that Selberg's proof of the large sieve inequality in the "lectures on sieve" published in [29] provides a refinement of the upper bound $N+Q^2$ when Q^2 is comparable to N (see discussion around (20.22) there).

We are however able to provide some support to the existence of an asymptotical distribution. To do so, we introduce the moments

$$\mathfrak{M}(\ell)=\sum_i \lambda_i^\ell. \tag{4}$$

One easily discovers that $\mathfrak{M}(1)=N|\Theta(Q)|$ and the main result of this paper is an evaluation of $\mathfrak{M}(2)$. Our statement requires the bounded function $\mathscr{G}$ of the real variable u defined by

$$\mathscr{G}(u)=\int_1^\infty \frac{\sin(u\xi)}{\xi^5}\,(1-4\,\mathrm{Log}\,\xi)\,d\xi. \tag{5}$$

With it we form C^2-function $\mathfrak{f}$ by

$$\mathfrak{f}(x)=\frac{1}{9\pi^2\zeta(3)x^2}+\frac{3}{\pi^4x^3}\sum_{n\geqslant 1}\frac{\phi(n)}{n^4}\mathscr{G}(2\pi xn). \tag{6}$$

We prove here

Theorem 1.1. *For $N, Q \geqslant 2$, we have*

$$\mathfrak{M}(2) = N^2|\Theta(Q)| + N^2 Q^2 \mathfrak{f}(N/Q^2) + \mathcal{O}\big(NQ^3 \operatorname{Log}^7 Q\big).$$

It is likely that the $\operatorname{Log}^7 Q$ is too large, but any further improvement on the exponent of Q would be highly valuable. Note that using $\mathfrak{M}(1)$, we get

$$|\Theta(Q)|^{-1} \sum_{1 \leqslant i \leqslant |\Theta(Q)|} (N^{-1}\lambda_i - 1)^2 = Q^2 |\Theta(Q)|^{-1} \mathfrak{f}(N/Q^2) + \mathcal{O}\big(QN^{-1} \operatorname{Log}^7 Q\big) \quad (7)$$

which shows that deviation to the mean is measured by $\mathfrak{f}$. The reader should see that we are able to save a power which is not obvious when noticing that the summatory function of the Moebius function intervenes. Since it is shown that $x^2\mathfrak{f}(x)$ is asymptotic to a positive constant when x goes to infinity, we see that the $N^{-1}\lambda_i$ differs noticeably from 1 at least when $Q^{8/3} \gg N$, and this corresponds to a Q appreciably smaller than $\sqrt{N}$. When x nears 0, we show in the next section that $\mathfrak{f}(x) = \frac{9}{\pi^4 x} + o(1)$.

Theorem 1.1 is robust in three ways: the interval in $[1, N]$ may be replaced by any interval of length N; further the method shows that we can even replace the characteristic function of $[1, N]$ by a smoothed version, say g, provided $\sum_n |g(n) - 1| = o(\sqrt{N})$; and it gives an asymptotic for Q and N in a fairly wide range.

Lemma 10.1 below enables us to compute $\frac{\pi^2}{3}\mathfrak{f}(3/\pi^2) = 0.447\,7\cdots$ which is in excellent agreement with the numerical data we compiled

Q	N	$\mathfrak{M}(2)/N^3$	$\mathfrak{M}(3)/N^4$	$\mathfrak{M}(4)/N^5$	$\mathfrak{M}(5)/N^6$
4	6	1.37037	2.11111	3.46091	5.89712
7	18	1.41118	2.20165	3.63594	6.22759
9	28	1.41308	2.20770	3.68719	6.46904
11	42	1.42782	2.26342	3.84129	6.84514
14	64	1.41824	2.24114	3.82707	6.94221
16	80	1.44517	2.34570	4.13771	7.79376
22	150	1.44206	2.33303	4.10684	7.74225
24	180	1.44144	2.32558	4.07203	7.61456
27	230	1.44469	2.34246	4.12686	7.76518
30	278	1.44488	2.33670	4.10450	7.71174
35	384	1.44388	2.33885	4.12364	7.78983
37	432	1.44459	2.33884	4.11674	7.75043

For $Q \geqslant 30$, the precision used for the computations was beginning to show its weakness and some marginally negative eigenvalues showed up. This means that any further computations will require more precision and a more sturdy algorithm than the simplistic use of `charpoly` followed by a `polroots` in PARI/GP.

Theorem 1.1 is difficult to prove It holds for N and Q varying independently. For $N = |\Theta(Q)|$, we can add the following estimate:

Theorem 1.2. *When* $N = |\Theta(Q)|$ *and for any* $c \geqslant 1$, *we have*

$$N^{-1} \sum_{1\leqslant i\leqslant N} \mathrm{Log}\,(N^{-1}\lambda_i) = -\frac{1}{2}\,\mathrm{Log}\,N - \mathfrak{c} + \mathcal{O}_c((\mathrm{Log}\,Q)^{-c})$$

where

$$\mathfrak{c} = \frac{1}{2} + \frac{6\zeta'(2)}{\pi^2} + \frac{1}{2}\,\mathrm{Log}\,(\pi^2/3) = 0.525\,538\ldots$$

It shows that one eigenvalue at least is not more than $\sqrt{N}$ if N is large enough, but the whole average could be dominated by a single eigenvalue of size $\exp(-\frac{1}{2}(1+o(1))N\,\mathrm{Log}\,N)$, which is the sole lower bound for λ_i that stems from this Theorem (and the large sieve inequality).

This theorem is much less flexible than the previous one but it should be noted that the error term depends on the one in the prime number theorem.

Even in case $|\Theta(Q)| = N$, we do not have an expression of $\mathfrak{f}(N/Q^2)$ in terms of more usual constants. Chasing for an understanding of this apparent asymptotic distribution (in order to phrase a conjecture), we tried different random processi. The only model that looks promising gives that:

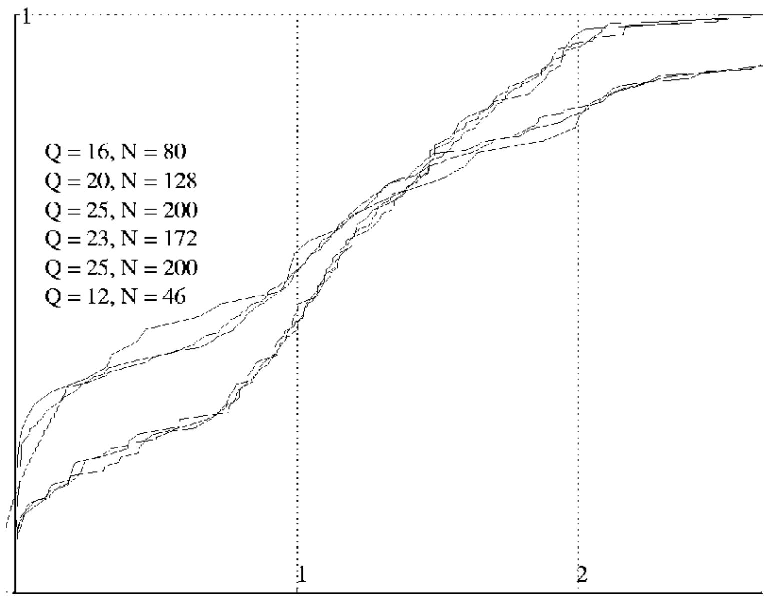

Figure 2. ????

We have plotted the previous distribution as well as the new arising one (which starts above and ends below). We generated $N = |\Theta(Q)|$ random integer points between 1 and $100N$ and divided them by $100N$ to get points θ in $[0,1]$. In this model, the generation imposes the spacement condition: starting from all points being 0, each of them is successively changed randomly so as to be at distance at least $1/Q^2$ of all the others. We then considered the eigenvalues of A^*A with $A = (e(n\theta))$.

2. Change of viewpoint and perspectives

We put in this section material linked with our main matter but slightly off focus. The exposition is pedestrian and this walk will be much less painful by setting

$$\mathfrak{r} = \mathrm{Log}^7 Q. \tag{8}$$

(This is a lower case gothic r). We initiate our journey by noticing that

$$\sum_{\theta,\theta'\in\Theta(Q)} \Big| N^{-1}\sum_n e(n(\theta-\theta')) - \delta_{\theta=\theta'}\Big|^2 = Q^2\mathfrak{f}(N/Q^2) + \mathcal{O}\big(N^{-1}Q^3\mathfrak{r}\big) \tag{9}$$

which is surprising enough in that we do not reach a $o(Q^2)$ when Q^2 is about N. The main term we have removed, namely $\delta_{\theta=\theta'}$, corresponds to the comparison of the initial hermitian form with N Id; And it would be interesting to find a main term to remove that would lead to a smaller right hand side. Note that $\mathfrak{f}(x) \ll 1/x^2$ so that (9) is fairly good when x is large enough.

The statement of Theorem 1.1 is dissymetrical in N and $|\Theta(Q)|$; we exchange their roles by writing

$$N^2Q^2\Big(\frac{3}{\pi^2} + \mathfrak{f}(N/Q^2)\Big) = N^2Q^2\Big(\frac{9}{\pi^4}Q^2N^{-1} + \mathfrak{g}(Q^2/N)\Big) \tag{10}$$

where $\mathfrak{g}(y) + \frac{9}{\pi^4}y = \frac{3}{\pi^2} + \mathfrak{f}(1/y)$. This leads to

$$\mathfrak{M}(2) = |\Theta(Q)|^2N + N^2Q^2\mathfrak{g}(Q^2/N) + \mathcal{O}\big(N^2Q\,\mathrm{Log}\,Q + NQ^3\mathfrak{r}\big) \tag{11}$$

and thus to

$$\begin{aligned}\sum_{m,n}\Big| |\Theta(Q)|^{-1}\sum_{\theta\in\Theta(Q)} e((m-n)\theta) - \delta_{m=n}\Big|^2 \\ = N^2Q^{-2}\mathfrak{g}(Q^2/N) + \mathcal{O}\big(N^2Q^{-3}\,\mathrm{Log}\,Q + NQ^{-1}\mathfrak{r}\big).\end{aligned} \tag{12}$$

This inequality has of course the same weakness when Q^2 is about N, but is also rather good when Q^2 is much larger than N since $\mathfrak{g}(y) = \frac{3}{\pi^2} + o(1)$ as y goes to infinity.

Proof. We start from (50)

$$\mathfrak{g}(y) = -\frac{9y}{\pi^4} + \frac{3}{\pi^2} + \frac{24/\pi^2}{2i\pi}\int_{\frac{9}{8}-i\infty}^{\frac{9}{8}+i\infty} \frac{\zeta(s)}{\zeta(1+s)}\frac{(y/(2\pi))^s\cos(\pi s/2)\Gamma(s)ds}{(1-s)(2-s)(1+s)^2}$$

and we shift the line of integration to the curve $\mathscr{L}$ of equation

$$\sigma = -c(\mathrm{Log}\, t)^{-2/3}(\mathrm{LogLog}\, t)^{-1/3} \tag{13}$$

for a suitable c (see chapter 8 of [19]) chosen so that $\zeta(1+s)$ does not vanish on the right hand side of $\mathscr{L}$. We encounter a single simple pole at $s=1$ with residue $9y/\pi^4$. No further pole arising, and especially none at $s=0$, we see that the term $3/\pi^2$ remains uncancelled! Completing the proof is then a matter of routine. ∎

As a consequence, we note that when $\sum_n |\varphi_n|^2 = 1$ we have

$$\begin{aligned}\sum_{\theta\in\Theta(Q)}\biggl|\sum_n \varphi_n e(n\theta)\biggr|^2 &= \sum_{m,n}\overline{\varphi}_m\varphi_n \sum_{\theta\in\Theta(Q)} e((n-m)\theta) \\ &= |\Theta(Q)| + \sum_{m,n}\overline{\varphi}_m\varphi_n\biggl(\sum_{\theta\in\Theta(Q)} e((n-m)\theta) - |\Theta(Q)|\delta_{m=n}\biggr) \\ &= |\Theta(Q)| + \mathcal{O}\left(NQ\sqrt{\mathfrak{g}(Q^2/N)} + N\sqrt{Q}\,\mathrm{Log}\, Q + \sqrt{NQ^3\mathfrak{r}}\right)\end{aligned}$$

thus getting equality in the large sieve, but only when $N\,\mathrm{Log}^7 N = o(Q)$. This is a drastic condition that is never met in any example I could think of. At best, the above proof could yield equality when $Q = N$ but any further reduction seems hopeless with the material presented here. We state formally:

Theorem 2.1. *We have when $Q \geqslant N$:*

$$\sum_{\theta\in\Theta(Q)}\biggl|\sum_n \varphi_n e(n\theta)\biggr|^2 = |\Theta(Q)|\sum_n |\varphi_n|^2\left(1 + \mathcal{O}\left(\sqrt{NQ^{-1}}\,\mathrm{Log}^{7/2} Q\right)\right).$$

The corresponding "dual" statement we can get is, when $Q \leqslant N^{1/3}$,

$$\sum_n\biggl|\sum_\theta \psi_\theta e(n\theta)\biggr|^2 = N\sum_\theta |\psi_\theta|^2\left(1 + \mathcal{O}\left(\sqrt{Q^3N^{-1}}\,\mathrm{Log}^{7/2} Q\right)\right). \tag{14}$$

We restrained the statements to the cases $Q \geqslant N$ and $Q^3 \leqslant N$ respectively because otherwise, the bound we got is superseded by the one steming from the large sieve inequality. Furthermore Kobayashi in Theorem 2.1 of [21] proves in a general framework that

$$\sum_n\biggl|\sum_\theta \psi_\theta e(n\theta)\biggr|^2 = N\sum_\theta |\psi_\theta|^2\left(1 + \mathcal{O}(N^{-1}Q^2)\right) \tag{15}$$

which is much better. An equality as the one of our Theorem 2.1 for slightly smaller Q's would provide a general approach to the Barban-Davenport-Halbertam Theorem (see [1], [4] and [5]) in the version of [22] and [17] (more recent developments appear in the sequel of papers of Hooley with the same title and also in [31]).

The equalities above tell us that all eigenvalues are at least as large as $|\Theta(Q)| - C_1\sqrt{NQ}\ \mathrm{Log}^{7/2}\ Q$ for some constant C_1 and provided $Q \geqslant N$. Kobayashi's equality yields that these eigenvalues are $\geqslant N - C_2 Q^2$. In both cases, these eigenvalues are as large as possible. But Theorem 1.2 tells us that this is *not* the case when $|\Theta(Q)| = N$.

We use this last paragraph to show where a usual approach fails to work. One of the main problem is to prove the existence of an asymptotic distribution for these eigenvalues. A first idea would be to consider the step function on $[0,1]$ that at t equals the number of points in $\Theta(Q)$ that are not more than t, number that we divide by $|\Theta(Q)|$. By Helly's selection principle (see for instance Theorems 2.2/2.3 of chapter 2 of [2]), this collection admits a subsequence that converges towards a function, say $\mathcal{H}$, and in such a way that, for any continuous function on $[0,1]$, we have

$$|\Theta(Q)|^{-1}\sum_{a,q} f(a/q) \to \int_0^1 f(t)d\mathcal{H}(t)$$

when Q runs through this special sequence. We may thus think that we have found an invariant "at infinity". However the fact that the Farey quotients are well-distributed implies that $\mathcal{H}(t)$ is ... t! This equi-distribution follows from Theorem 2.1 in an interesting manner: approximate uniformly the characteristic function of an interval $[\alpha,\beta]$ by a finite trigonometric polynomial and use our Theorem to deduce that the number of points of $\Theta(Q)$ in this interval is asympotically $(\beta-\alpha)|\Theta(Q)|$. The reader may want to consult [18] for shapenings of this equi-distribution. Several different constructions of an asymptotic limit in various spaces and relying on the Banach-Alaoglu Theorem ended in this very same way. Kargaev & Zhigljavsky got in [20] an extremely interesting result that partially relies on this type of argument and that indeed pertains to the local distribution of the Farey quotients. It should be pointed out that such an understanding arose from Kloostermann's memoir on representation by quaternary diagonal quadratic forms; this is made most clear in Iwaniec's presentation of it, as can be for instance seen in chapter 20 of [19] (see in particular proposition 20.7 therein). Any link with our present work is left for a future paper.

The proof we presented shows exactly where the catch lies: we needed only a trigonometric polynomial of fixed length, while we should consider also polynomials of length up to Q^2.

3. General store

We open this section to store some general comments, facts and other kind of notes. The first of these concerns the quantity $|\Theta(Q)|$ for which we shall often use the estimate

$$|\Theta(Q)| = \frac{3}{\pi^2}Q^2 + \mathcal{O}(Q\ \mathrm{Log}\ Q) \quad (Q \geqslant 2). \tag{16}$$

The reader may use (33) for $s = 0$, but the remainder term there is larger than the above by a Log factor. A direct use of the convolution method using

$$\phi(q)/q = \sum_{d|q} \mu(d)/d \tag{17}$$

would suffice. As a matter of fact, obtaining the proper power of V in (33) is more difficult than it seems, when one imposes the use of the Mellin transform. Using this transform indeed allows for some flexibility than the convolution misses. That is why we devoted an entire section to the problem. We shall also use the estimate

$$|\zeta(s)| \ll (2+|t|)^{\frac{1}{2}-\Re s} \quad (\Re s \leqslant 0, t = \Im s) \tag{18}$$

which is classical. In the critical strip $|\zeta(s)|$ is surely not more than $(2+|t|)^{(1-\Re s)/2}$ Log $(2+|t|)$ which is enough to ensure the various convergences conditions we shall meet.

We need numerous notations. We decided to use many variations of the letters f and g for the functions. The reader will thus meet f, F, $\mathscr{F}$ and $\mathfrak{f}$.

$\{u\}$ denotes the fractionnal part of u and the parenthesis are used with no other meaning. When numerical approximations are involved, we will use the $\mathcal{O}$-like notation defined by $f = \mathcal{O}^*(g)$ if $|f| \leqslant g$.

4. Access to the moments

We now proceed to compute an expression of $\mathfrak{M}(\ell)$ in terms of the coefficients of the hermitian form only. We first note that the matrix of the hermitian form in (1) has (n_1, n_2)-coefficient:

$$\sum_{\theta_1} e((n_1 - n_2)\theta_1). \tag{19}$$

We readily compute that the $(n_1, n_{\ell+1})$-coefficient of its ℓ-th power is

$$\sum_{\substack{\theta_1,\ldots,\theta_\ell,\\ n_2,\ldots,n_\ell}} e((n_1 - n_2)\theta_1 + \ldots + (n_\ell - n_{\ell+1})\theta_\ell). \tag{20}$$

The trace of the ℓ-th power is thus

$$\mathfrak{M}(\ell) = \sum_{\substack{\theta_1,\ldots,\theta_\ell,\\ n_1,n_2,\ldots,n_\ell}} e((n_1 - n_2)\theta_1 + \ldots + (n_\ell - n_1)\theta_\ell). \tag{21}$$

According to (4), its size is at most $N(N+Q^2)^\ell$. Since the length of summation is $N^{2\ell}$, this means tremendous cancellation when ℓ is large. Note the easy

$$\mathfrak{M}(1) = N|\Theta(Q)|. \tag{22}$$

Expression (21) has another interest: it is possible to introduce therein a smoothing on the variables n_i. This smoothing would not alter significantly the main term if it would differ from the characteristic function of the interval $[1, N]$ only on a finite number of intervals of length $o(N^{1/\ell})$. So, the larger the ℓ, the more difficult it would be to smoothen our summation. In case $\ell = 2$, the next paragraph offers an unusual approach that asks for the variables to be *non-smoothed* but let us first provide a sketch of our claim on the smoothing part. We consider functions g_i over $[1, N]$ such that

$$\sum_{n\leqslant N} \bigl|g_i(n) - 1\bigr| \ll L$$

for some $L \geqslant 1$. As a direct consequence, we get

$$\Bigl|\sum_{n\leqslant N} g_i(n)e(n\theta)\Bigr| \ll \min(N, \|\theta\|^{-1}) + L.$$

We consider

$$\begin{aligned}\mathfrak{M}(\ell, (g_i)) &= \sum_{\substack{\theta_1,\ldots,\theta_\ell,\\ n_1,n_2,\ldots,n_\ell}} g_1(n_1)\ldots g_\ell(n_\ell)e((n_1 - n_2)\theta_1 + \ldots + (n_\ell - n_1)\theta_\ell)\\ &= \sum_{\substack{\theta_1,\ldots,\theta_\ell,\\ n_1,n_2,\ldots,n_\ell}} g_1(n_1)\ldots g_\ell(n_\ell)e(n_1(\theta_1 - \theta_\ell) + \ldots + n_\ell(\theta_\ell - \theta_{\ell-1}))\end{aligned}$$

which we are to compare to $\mathfrak{M}(\ell)$. It is enough to compare any two such expressions with same g_2, ... , g_ℓ and different g_1 since the variables n_1, ... , n_ℓ have an identical role. The difference between the corresponding smoothed moments is

$$\begin{aligned}\ll \sum_{n_1} |g_1(n_1) - g_1'(n_1)| \sum_{\theta_1,\ldots,\theta_\ell} \bigl(\min(N, \|\theta_2 - \theta_1\|^{-1}) + L\bigr) \times\\ \bigl(\min(N, \|\theta_3 - \theta_2\|^{-1}) + L\bigr)\ldots\bigl(\min(N, \|\theta_\ell - \theta_{\ell-1}\|^{-1}) + L\bigr)\end{aligned}$$

in which we evaluate the summation in θ_ℓ with $\theta_{\ell-1}$ being fixed. It is not more up to a multiplicative constant than

$$N + Q^2 \operatorname{Log} Q + Q^2 L.$$

Next we proceed similarly with the summation over $\theta_{\ell-1}$ and so on until we reach the variable θ_2. The variable θ_1 yields the contribution $|\Theta(Q)| \ll Q^2$ so that our difference is

$$\mathcal{O}\Bigl(L\bigl(N + Q^2(L + \operatorname{Log} Q)\bigr)^{\ell-1}Q^2\Bigr).$$

Since we are interested in the case when $N \simeq Q^2$, the introduction of a smoothing would not alter the main term if $L^\ell = o(N)$ since our moment is of size $N^{\ell+1}$. In fact the following inequalities hold for $\ell > 1/2$

$$\Bigl(N/\sqrt{N + Q^2}\Bigr)^{2\ell} \leqslant \mathfrak{M}(\ell)/|\Theta(Q)| \leqslant (N + Q^2)^\ell. \tag{23}$$

Proof. We simply apply Hölder inequality:

$$\sum_{1\leqslant i\leqslant|\Theta(Q)|}\lambda_i \leqslant \Biggl(\sum_{1\leqslant i\leqslant|\Theta(Q)|}\sqrt{\lambda_i}^{2\ell}\Biggr)^{1/(2\ell)}\Biggl(\sum_{1\leqslant i\leqslant|\Theta(Q)|}\sqrt{\lambda_i}^{\frac{2\ell}{2\ell-1}}\Biggr)^{(2\ell-1)/(2\ell)}$$

and conclude by using (22) for the first summation and $\lambda_i \leqslant N+Q^2$ in the last one. ∎

Variations of this proof involving $\mathfrak{M}(2)$ instead of $\mathfrak{M}(1)$ would slightly improve the bounds of (23).

5. Exact expression for the dispersion

Lemma 5.1. *For integer $N\geqslant 1$ and $d\geqslant 1$, we have*

$$(1/d)\sum_{\substack{n,m\leqslant N\\ d|n-m}}1=\frac{N^2}{d^2}+\bigl\{N/d\bigr\}-\bigl\{N/d\bigr\}^2.$$

This lemma tells us that we can compute the left-hand side by only using N/d and its fractional part. The most straightforward path yields only

$$\sum_{\substack{n,m\leqslant N\\ d|n-m}}1=\sum_{a \bmod d}\Bigl(\frac{N}{d}+\mathcal{O}(1)\Bigr)^2=\frac{N^2}{d}+\mathcal{O}(d+Nd^{-1})$$

but such an approximation is valid for N any *real* number while the one of the lemma is *not*.

Proof. Case $N<d$ is most readily handled so we assume $N\geqslant d$. We find that

$$S=\sum_{\substack{n,m\leqslant N\\ d|n-m}}1=\sum_{1\leqslant a\leqslant d}\left(\frac{N-a}{d}-\left\{\frac{N-a}{d}\right\}+1\right)^2.$$

Let b be the integer in $[1,d]$ that is congruent to N modulo d. We have

$$\begin{aligned}
S&=\sum_{1\leqslant a\leqslant b}\left(\frac{N-a}{d}-\frac{b-a}{d}+1\right)^2+\sum_{b+1\leqslant a\leqslant d}\left(\frac{N-a}{d}-\frac{b-a}{d}\right)^2\\
&=\sum_{1\leqslant a\leqslant b}\left(\frac{N-b}{d}+1\right)^2+\sum_{b+1\leqslant a\leqslant d}\left(\frac{N-b}{d}\right)^2\\
&=d\left(\frac{N-b}{d}\right)^2+2b\frac{N-b}{d}+b=\frac{N^2}{d}-b^2/d+b.
\end{aligned}$$

If d does not divide N, then $b = d\{N/d\}$, while $b = d$ if $d|N$. In both cases we get

$$\sum_{\substack{n,m\leqslant N\\ d|n-m}} 1 = \frac{N^2}{d} + d\Big(\Big\{\frac{N}{d}\Big\} - \Big\{\frac{N}{d}\Big\}^2\Big). \tag{24}$$

■

6. A Mellin transform

We introduce a Mellin transform that transforms the expression of Lemma 5.1 into something multiplicative in d.

Lemma 6.1. *For any positive real number X of fractionnal part ξ, we have*

$$\frac{1}{2i\pi}\int_{-\frac14-i\infty}^{-\frac14+i\infty}\zeta(s)\frac{X^{s+1}}{s(s+1)}ds = \frac12(\xi-\xi^2).$$

Proof. Let us first note that

$$\frac{1}{2i\pi}\int_{2-i\infty}^{2+i\infty}\frac{t^{s+1}}{s(s+1)}ds = \begin{cases} t-1 & \text{if } t\geqslant 1,\\ 0 & \text{if } 1>t>0,\end{cases} \tag{25}$$

which we prove by shifting the line of integration to the far right if $t<1$ and to the far left otherwise. We thus get for X a positive real number

$$\begin{aligned}\sum_{n\leqslant X} n\left(\frac{X}{n}-1\right) &= \sum_{n\geqslant 1}\frac{n}{2i\pi}\int_{2-i\infty}^{2+i\infty}\frac{(X/n)^{s+1}}{s(s+1)}ds\\ &= \frac{1}{2i\pi}\int_{2-i\infty}^{2+i\infty}\zeta(s)\frac{X^{s+1}}{s(s+1)}ds. \end{aligned}\tag{26}$$

We next modify both sides of this equation. Write $X = N+\xi$, with $N\in\mathbb{N}$ and $\xi\in[0,1[$. We get

$$\sum_{n\leqslant X} n\left(\frac{X}{n}-1\right) = XN - \frac12(N^2+N) = \frac12(N^2-N)+\xi N. \tag{27}$$

We now transform the RHS of (26)

$$\frac{1}{2i\pi}\int_{2-i\infty}^{2+i\infty}\zeta(s)\frac{X^{s+1}}{s(s+1)}ds = \frac12X^2+\zeta(0)X+\frac{1}{2i\pi}\int_{-\frac14-i\infty}^{-\frac14+i\infty}\zeta(s)\frac{X^{s+1}}{s(s+1)}ds$$

and recall that $\zeta(0) = -1/2$. The last integral converges since $\zeta(-\frac14+it) = \mathcal{O}((1+|t|)^{3/4})$. By comparing what we just obtained with (27), we infer

$$\frac{1}{2i\pi}\int_{-\frac14-i\infty}^{-\frac14+i\infty}\zeta(s)\frac{X^{s+1}}{s(s+1)}ds = \frac12(N^2-N)+\xi N-\frac12X^2+\frac12X = \frac12(\xi-\xi^2)$$

as claimed. ■

7. A truncated Perron summation formula

The easiest way to deal with the evaluation of the next paragraph goes through some standard techniques that were already well-known in the beginning of the twentieth century. We present in this preliminary paragraph our own version, which we find more convenient.

Let Y be the function that is 0 on $]0,1[$, then $1/2$ in 1 and 1 afterwards.

Lemma 7.1. *For $\kappa > 0$ and $x > 0$, we have*

$$\left| Y(x) - \frac{1}{2i\pi}\int_{\kappa-iT}^{\kappa+iT} \frac{x^z dz}{z} \right| \leqslant \frac{x^\kappa}{\pi} \min\left(\frac{7}{2}, \frac{1}{T|\operatorname{Log} x|}\right).$$

The proof will show that we could have taken any value for $Y(1)$, provided it lies in $[0,1]$.

Proof. When $x < 1$, we write for $K > \kappa$ going to infinity :

$$\left(\int_{\kappa-iT}^{\kappa+iT} + \int_{\kappa+iT}^{K+iT} + \int_{K+iT}^{K-iT} + \int_{K-iT}^{\kappa-iT}\right) \frac{x^z dz}{z} = 0.$$

The third integral dwindles to zero when K increases. Both integral on the horizontal segments are bounded by $x^\kappa/(T|\operatorname{Log} x|)$. This implies

$$\left| Y(x) - \frac{1}{2i\pi}\int_{\kappa-iT}^{\kappa+iT} \frac{x^z dz}{z} \right| \leqslant \frac{x^\kappa}{\pi T|\operatorname{Log} x|} \quad (0 < x < 1).$$

The same bound holds for $x > 1$: the proof goes as above except that we shift the line of integration towards the left hand side. These bounds are efficients when $T|\operatorname{Log} x|$ is large enough; else we write

$$\int_{\kappa-iT}^{\kappa+iT} \frac{x^z dz}{z} = x^\kappa \int_{\kappa-iT}^{\kappa+iT} \frac{dz}{z} + x^\kappa \int_{-T}^{T} \frac{(x^{it}-1)idt}{\kappa+it}.$$

The first integral is $2\arctan(T/\kappa) \leqslant \pi$ while we deal with the second one by using

$$\left| \frac{x^{it}-1}{it \operatorname{Log} x} \right| = \left| \int_0^1 e^{iut\operatorname{Log} x} du \right| \leqslant 1.$$

This leads to the upper bound $2T|\operatorname{Log} x|$ (even if $x = 1$), and thus

$$\left| \frac{1}{2i\pi}\int_{\kappa-iT}^{\kappa+iT} \frac{x^z dz}{z} \right| \leqslant \frac{x^\kappa}{\pi}\left(\frac{\pi}{2} + T|\operatorname{Log} x|\right).$$

This is enough if $x < 1$. If $x > 1$, we note that

$$1 - \frac{x^\kappa}{2i\pi}\int_{\kappa-iT}^{\kappa+iT}\frac{dz}{z} = 1 - \frac{x^\kappa}{\pi}\arctan(T/\kappa)$$

which is bounded below by $-x^\kappa/2$ and above by $1 \leqslant x^\kappa$. As a consequence, we reach

$$\left|Y(x) - \frac{1}{2i\pi}\int_{\kappa-iT}^{\kappa+iT}\frac{x^z dz}{z}\right| \leqslant \frac{x^\kappa}{\pi}\min\left(\pi + T|\operatorname{Log} x|, \frac{1}{T|\operatorname{Log} x|}\right).$$

We simplify this upper bound by noticing that

$$\min(\pi + u, 1/u) \leqslant \min(\alpha, 1/u)$$

with $\alpha = 1/u_0 = \pi + u_0$. This entails $\alpha \leqslant 7/2$, and the lemma follows readily. ■

This lemma leads to the aforementioned Theorem.

Theorem 7.1 [Truncated Perron's formula]. *Let $F(z) = \sum_n a_n/n^z$ be a Dirichlet series that converges absolutely for $\Re z > \kappa_a$, and let $\kappa > 0$ be strictly larger than κ_a. For $x \geqslant 1$ and $T \geqslant 1$, we have*

$$\sum_{n\leqslant x} a_n = \frac{1}{2i\pi}\int_{\kappa-iT}^{\kappa+iT} F(z)\frac{x^z dz}{z} + \mathcal{O}^*\left(\int_{1/T}^{\infty}\sum_{|\operatorname{Log}(x/n)|\leqslant u}\frac{|a_n|}{n^\kappa}\frac{2x^\kappa du}{Tu^2}\right).$$

In this Theorem, the error term is essentially raw. There appear the sums

$$\sum_{|\operatorname{Log}(x/n)|\leqslant u}|a_n|/n^\kappa$$

where the conditions on n may be rewritten as $e^{-u}x \leqslant n \leqslant e^u x$. When $u \geqslant 1$, the majorant $\sum_{n\geqslant 1}|a_n|/n^\kappa$ is usually enough. When u is smaller, we appeal most of the times to an upper bound of the shape $ux^{\kappa_a}B/x^\kappa$ for some sensible B (a constant times $\operatorname{Log} x$ for instance), which leads to the error term

$$\mathcal{O}\left(\frac{Bx^{\kappa_a}\operatorname{Log} T}{T} + \frac{x^\kappa}{T}\sum_{n\geqslant 1}|a_n|/n^\kappa\right).$$

Note that the shorter sums we are to consider are of length $\simeq x/T$.

Proof. Following Lemma 7.1, we first write

$$\begin{aligned}\sum_{n\leqslant x} a_n &= \sum_{n\geqslant 1} a_n Y(x/n)\\ &= \sum_{n\geqslant 1} a_n \frac{1}{2i\pi}\int_{\kappa-iT}^{\kappa+iT}\frac{(x/n)^z dz}{z} + \mathcal{O}^*\left(\sum_{n\geqslant 1}\frac{|a_n|x^\kappa}{\pi n^\kappa}\min\left(\frac{7}{2}, \frac{1}{T|\operatorname{Log}(x/n)|}\right)\right).\end{aligned}$$

Let us set $\varepsilon = 1/T$. Integers n such that $|\operatorname{Log}(x/n)| \leqslant \varepsilon$ give a contribution to the error term that we keep as it is. Else we write

$$\begin{aligned}
&\sum_{\varepsilon \leqslant |\operatorname{Log}(x/n)|} \frac{|a_n| x^\kappa}{n^\kappa |\operatorname{Log}(x/n)|} \\
&= \sum_{\varepsilon \leqslant |\operatorname{Log}(x/n)|} \frac{|a_n| x^\kappa}{n^\kappa} \int_{|\operatorname{Log}(x/n)|}^{\infty} \frac{du}{u^2} \\
&= \int_{\varepsilon}^{\infty} \sum_{|\operatorname{Log}(x/n)| \leqslant u} \frac{|a_n| x^\kappa}{n^\kappa} \frac{du}{u^2} - \int_{\varepsilon}^{\infty} \sum_{|\operatorname{Log}(x/n)| \leqslant \varepsilon} \frac{|a_n| x^\kappa}{n^\kappa} \frac{du}{u^2}
\end{aligned}$$

which is enough. ■

8. An arithmetical sum

This part is of independant interest. Analytic number theorists know that Theorem does not lead to a proof of $\sum_{n \leqslant X} 1 = X + \mathcal{O}(1)$; indeed the natural error term that arises is at least of size $\mathcal{O}(X^{1/3})$ (by shifting the line of integration to $\Re s = 1/\operatorname{Log} X$). One can reach $\mathcal{O}_\varepsilon(X^\varepsilon)$ for any positive ε by shifting the line of integration far to the left hand side, but not to $\Re s = -\infty$. However we do not have this scheme at our disposal when looking at a simple arithmetical modification of 1, like for summing $\phi(n)/n$ and $\Re s = 0$ seems the limit as to where we can shift $\Re s$. We propose a method that still uses Theorem , relies only on shifting the line of integration essentially to $\Re s = 0$ and that gives $\sum_{n \leqslant X} 1 = X + \mathcal{O}(\operatorname{Log}^3 X)$. The main outcome is the flexibility of the method which we can thus apply to arithmetical modifications of 1, like in the example we treat here.

The leading idea of the method is to use the dependance if $\Im s$ and not to work only on $\Re s$ as usually. The main lemma reads as follows.

Lemma 8.1. *There exists a constant For $0 < a \leqslant 1/2$, $D \geqslant 2$ and any real number b, we have*

$$\left| \int_0^b \zeta(a+it) D^{a+it} dt \right| \leqslant 4D^a (1+|b|) \Big(|\operatorname{LogLog} D| + \operatorname{Log}(1+|b|) + \frac{3}{a} \Big).$$

The term $2/a$ is most probably not required but is harmless in our subsequent proof while allowing for the simplistic proof that follows.

Proof. For $|b| \leqslant 1$, we simply introduce the absolute values inside the integral and are left with bounding $\int_0^1 |\zeta(a+it)| dt$. This is a constant and its value has no impact whatsoever on our result. However, we ran the following simpleminded GP-script (with 28 digits precision):

```
maximum = 0;
auxiliary = 0;
forstep(a = 0, 1/2, 0.01,
      auxiliary = intnum(t=0,1,abs(zeta(a+I*t)));
      if(auxiliary > maximum, print(a," -> ",auxiliary);
           maximum = auxiliary, ));
```

which told us the maximum to be at $a = 0$ with value $0.465\cdots$. This is by no means a rigorous proof for this maximum, and especially not of the fact that it is indeed reached at $a = 0$, but tells us that we will be able to prove this maximum to be not more than 1. This is the value we use. We handle only the case $b > 0$. The expression we use for ζ on the critical line is simple enough:

$$\zeta(s) = 1 + \frac{1}{s-1} - s\int_1^\infty \{u\}\frac{du}{u^{s+1}}. \tag{28}$$

We start with an integration by parts

$$\int_a^{a+ib} s\,(D/u)^s ds = \frac{(a+ib)(D/u)^{a+ib} - a(D/u)^a}{\operatorname{Log}\,(D/u)} - \int_a^{a+ib} (D/u)^s \frac{ds}{\operatorname{Log}\,(D/u)}$$

from which we readily infer the bound for $u > 0$

$$\left|\int_a^{a+ib} s(D/u)^s ds\right| \leqslant \min\left(\frac{2(a+b)}{|\operatorname{Log}\,(D/u)|}, (a+b)b\right)(D/u)^a. \tag{29}$$

This gives us that

$$\begin{aligned}
\left|\int_a^{a+ib} s\int_1^\infty \{u\}\frac{du}{u^{s+1}}\frac{D^s ds}{(a+b)D^a}\right| &\leqslant \int_1^\infty \min\left(\frac{2}{|\operatorname{Log}\,(D/u)|}, b\right)\frac{\{u\}du}{u^{1+a}} \\
&\leqslant \int_{De^{-2/b}}^{De^{2/b}} \frac{b\,du}{D^a e^{-2a/b}u} + \int_1^{De^{-2/b}} \frac{2du}{u\operatorname{Log}\,(D/u)} + \int_{De^{2/b}}^\infty \frac{2du}{u^{1+a}\operatorname{Log}\,(u/D)} \\
&\leqslant 4D^{-a}e^{2a/b} + \int_{e^{2/b}}^D \frac{2dv}{v\operatorname{Log}\,v} + D^{-a}\int_{e^{2/b}}^\infty \frac{2dv}{v^{1+a}\operatorname{Log}\,v} \\
&\leqslant 4D^{-a}e^{2a/b} + 2\operatorname{LogLog}\,D - 2\operatorname{Log}\,(2/b) \\
&\quad + 2D^{-a}\left(\frac{\operatorname{LogLog}\,e - \operatorname{Log}\,(2/b)}{e^{2a/b}} + \frac{1}{ae^a\operatorname{Log}\,e}\right).
\end{aligned}$$

We simplify our bound further by noticing that $2e^{2a/b} \leqslant 12 \leqslant 6/a$. ∎

We are to study

$$W(s,V) = \sum_{v\leqslant V}\prod_{p^k\|v}\left(p^{k(1-s)} - p^{(k-1)(1-s)}\right) \tag{30}$$

for $\Re s < 0$. In our application $\Re s$ will be larger than $-1/\operatorname{Log} V$, which means that the dependance in this parameter will be nominal only. The dependance in $t = \Im s$ will also not be very important but has to be carried out. We proceed via a here properly tuned but otherwise standard method and introduce

$$\mathcal{W}(s,z) = \zeta(z-1+s)\mathscr{M}_{V/2}(z) = \sum_{v\geqslant 1} f^\sharp(v,1-s)v^{-z} \tag{31}$$

with

$$\mathscr{M}_{V/2}(z) = \sum_{\ell\leqslant V/2} \mu(\ell)/\ell^z.$$

The coefficients $f^\sharp(v,1-s)$ are indeed $f(v,1-s)$ when $v \leqslant V/2$. The cut at $V/2$ will be important later on. For v such that $V/2 < v \leqslant V$, the only missing divisor is v and then $f^\sharp(v,1-s) + \mu(v) = f(v,1-s)$. The fact that $\mu(v)$ is bounded in absolute value will be enough to ensure that it does not change our final result.

We invoke Theorem 7.1 to write with $a = 1/\operatorname{Log}(2+V)$ and $\kappa = 2-\Re s+a$ that $W(s,V)$ equals

$$\frac{1}{2i\pi}\int_{\kappa-iT}^{\kappa+iT} \mathcal{W}(s,z)\frac{V^z dz}{z} + \mathcal{O}^*\left(\int_{1/T}^{\infty} \sum_{|\operatorname{Log}(V/v)|\leqslant u} \frac{|f^\sharp(v,1-s)|}{v^\kappa}\frac{2V^\kappa du}{Tu^2}\right). \tag{32}$$

Note that $f^\sharp(v,1-s) = \sum_{w|v, 2w\leqslant V} \mu(w)(v/w)^{1-s}$ so that

$$|f^\sharp(v,1-s)| \leqslant v^{1-\Re s}\sum_{w|v}\mu^2(w)/w.$$

As a consequence, this yields for $0 < u < 1$

$$\begin{aligned}\sum_{|\operatorname{Log}(V/v)|\leqslant u} |f^\sharp(v,1-s)|/v^\kappa &\leqslant (e^uV)^{1-\Re s-\kappa}\sum_{e^{-u}V\leqslant v\leqslant e^uV}\ \sum_{w|v}\mu^2(w)/w \\ &\leqslant (e^uV)^{1-\Re s-\kappa}\sum_{w\leqslant e^uV}\frac{\mu^2(w)}{w}\left(\frac{(e^u-e^{-u})V}{w}+1\right) \\ &\ll (e^uV)^{1-\Re s-\kappa}\left(uV + u + \operatorname{Log} V\right).\end{aligned}$$

By considering separately the cases $1/T < u \leqslant 1/V$, $1/V < u \leqslant 1$ and $u > 1$ of the error term of (32), we find it is $\mathcal{O}$ of

$$V^{2-\Re s}T^{-1}\bigl(\operatorname{Log} T + \operatorname{Log}(2+V)\bigr) + V\operatorname{Log} V.$$

We then shift κ to $\kappa' = 1-\Re s + a$; we encounter a pole at $z = 2-s$, getting

$$\begin{aligned}W(s,V) = \frac{V^{2-s}\mathscr{M}_{V/2}(2-s)}{(2-s)} &+ \frac{1}{2i\pi}\int_{\kappa'-iT}^{\kappa'+iT}\mathscr{M}_{V/2}(z)\zeta(z-1+s)\frac{V^z dz}{z} \\ + \mathcal{O}\biggl(\int_1^{2-\Re s}(|t|+T+1)^{(1-w)/2}V^w\frac{dw}{T}\biggr) &+ \mathcal{O}\bigl(V^{2-\Re s}T^{-1}\operatorname{Log}(T(2+V))\bigr).\end{aligned}$$

This last integral asks for a refined treatment and is the reason why we introduced $\mathscr{M}_{V/2}(z)$ instead of $1/\zeta(z)$. We expand $\mathscr{M}_{V/2}(z)$ and get

$$\sum_{\ell\leqslant V/2}\frac{\mu(\ell)V^{1-s}}{\ell^{1-s}}\frac{1}{2i\pi}\int_{\kappa'-iT}^{\kappa'+iT}\zeta(z-1+s)\frac{(V/\ell)^{z-1+s}dz}{z}.$$

At this level, we recall Lemma . It has been patterned to handle the above integral via an integration by parts while the restriction $\ell\leqslant V/2$ has been introduced to be able to use this lemma. We thus get that the above is not more, up to a multiplicative constant, than

$$\sum_{\ell\leqslant V/2}(V/\ell)^{1-\Re s}(V/\ell)^{a}\times$$
$$(\mathrm{LogLog}\,(V+3)+\,\mathrm{Log}\,(T+|t|+3)+\,\mathrm{Log}\,(2+V))\,\mathrm{Log}\,(2+T)$$

which is $\mathcal{O}(V^{1-\Re s}\,\mathrm{Log}^3\,(2+V+T+|t|))$. We take $T=(2+V)$. We can replace $\mathscr{M}_{V/2}(2-s)$ by $1/\zeta(2-s)$ with a cost of $\mathcal{O}(1/V)$ and conclude that

$$W(s,V)=\frac{V^{2-s}}{\zeta(2-s)(2-s)}+\mathcal{O}\big(V^{1-\Re s}\,\mathrm{Log}^3\,(2+V+|t|)\big). \tag{33}$$

Note as a mean of verification that we recover the classical result on the Euler ϕ-function for $s=0$, upto the power of logarithm.

9. An undergraduate divertimento

We continue our preparation by studying the special function

$$\mathscr{F}(u)=\frac{1}{2i\pi}\int_{\frac{9}{8}-i\infty}^{\frac{9}{8}+i\infty}\frac{u^{-s}\cos(\pi s/2)\Gamma(s)ds}{(1-s)(2-s)(1+s)^2}. \tag{34}$$

that will appear in the subsequent study. This function clearly does not carry any arithmetic anymore and we should be able to grasp its behaviour quite fully. It turns out that achieving such an understanding is more difficult expected and we prefer to spend a full section on this task. The study will have two distinct parts : one for small u's, and one for large ones.

Bounded values

Recall that the Stirling formula tells us that $\Gamma(s)\sim|t|^{\sigma-\frac12}e^{-\pi|t|/2}$ when σ is fixed and $|t|$ goes to infinity. By writing

$$\Gamma(z-m)=\frac{\Gamma(z)}{(z-m)(z-m+1)\dots(z-1)}$$

we see that we can send the line of integration to the far left provided we compute the contribution of the poles. The first three candidates, namely $s=1$, $s=0$ and $s=-1$, are singular because of the denominator. In $s=1$ there is in fact no pole due to the $\cos(\pi s/2)$. In $s=0$ the pole is simple with residue $1/2$.

In $s=-1$, the pole is double, so we have access to the residue via the computation of the derivative at $s=-1$ of

$$\frac{u^{-s}\big(\cos(\pi s/2)/(s+1)\big)\Gamma(s+2)}{s(1-s)(2-s)} = \frac{u^{-s}\big(\sin(\pi(s+1)/2)/(s+1)\big)\Gamma(s+2)}{s(1-s)(2-s)}$$

which is $\frac{\pi}{12}u \operatorname{Log} u + \frac{(6\gamma-11)\pi}{72}u$. Next we are to take care of the contribution of the poles at $-m$ for $m \geqslant 2$. To do so we use the complement formula and write

$$\begin{aligned}\frac{u^{-s}\cos(\pi s/2)\Gamma(s)}{(1-s)(2-s)(1+s)^2} &= \frac{\pi u^{-s}\cos(\pi s/2)}{\sin(\pi s)(1-s)(2-s)(1+s)^2\Gamma(1-s)}\\ &= \frac{\pi u^{-s}}{2\sin(\pi s/2)(1+s)^2\Gamma(3-s)}\end{aligned}$$

which shows we have a pole only if $m=2\ell$ and that its residue is then

$$\frac{(-1)^\ell u^{2\ell}}{(2\ell-1)^2(2\ell+2)!}.$$

Since this gives rise to an entire series of infinite radius of convergence, we conclude:

Lemma 9.1. *For $u>0$, we have*

$$\mathscr{F}(u) = \frac{1}{2} + \frac{\pi}{12}u \operatorname{Log} u + \frac{(6\gamma-11)\pi}{72}u + \sum_{\ell\geqslant 1}\frac{(-1)^\ell u^{2\ell}}{(2\ell-1)^2(2\ell+2)!}.$$

Larges values

Note first that the Stirling formula tells us that $\Gamma(s) \sim |t|^{\sigma-\frac12}e^{-\pi|t|/2}$ when σ is fixed and $|t|$ goes to infinity. By sending the line of integration to the right hand side, one can then easily prove that $|\mathscr{F}(u)-1/(9u^2)| \ll_\varepsilon |u|^{-7/2+\varepsilon}$ for any $\varepsilon>0$. Our aim here is to find expressions that will lead to a better understanding to $\mathscr{F}$, and in particular will enable a fast computation. Here is our Theorem.

Theorem 9.1. *We have*

$$\mathscr{F}(u) = \frac{1}{9u^2} + \int_u^\infty \frac{\sin y}{y^4}\left(\frac{u}{y} + 4\frac{u}{y}\operatorname{Log}\frac{u}{y}\right)dy$$

Furthermore, $|\mathscr{F}(u)-1/(9u^2)| \leqslant 12/u^4$.

Our first lemma runs as follows.

Lemma 9.2. *For any real number c verifying $1 > c > 0$, we have*

$$\frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty}\frac{x^{-s}ds}{(1-s)(2-s)(s+1)^2}=\begin{cases}\frac{5}{36}x-\frac{1}{6}x\,\mathrm{Log}\,x & \textit{if } 0<x\leqslant 1,\\ \frac{1}{4x}-\frac{1}{9x^2} & \textit{if } x\geqslant 1.\end{cases}$$

Proof. When $x > 1$, we send the line of integration to the far right. When $x \leqslant 1$ we send it to the far left. We compute the residues by using

$$\frac{1}{(1-s)(2-s)(1+s)^2}=-\frac{1}{4(s-1)}+\frac{1}{9(s-2)}+\frac{5}{36(s+1)}+\frac{1}{6(s+1)^2}. \qquad \blacksquare$$

We next transform the initial expression for F into a real variable integral, i.e. perform an explicit Mellin inversion. The main tool is the following formula

$$\cos\frac{\pi s}{2}\,\Gamma(s)=\int_0^\infty \cos(y)y^{s-1}dy=\int_0^\infty \cos(y)y^s dy/y \tag{35}$$

for $0 < \Re s < 1$. An integration by parts easily gives the following approximation of the above:

$$u^{-s}\cos\frac{\pi s}{2}\,\Gamma(s)=\int_0^Y \cos(y)(u/y)^{-s}dy/y+\mathcal{O}(u^{-\Re s}Y^{\Re s-1}). \tag{36}$$

We first note that

$$\mathscr{F}(u)=\frac{1}{2i\pi}\int_{\frac12-i\infty}^{\frac12+i\infty}\frac{u^{-s}\cos(\pi s/2)\Gamma(s)ds}{(1-s)(2-s)(1+s)^2} \tag{37}$$

and we use the above representation of the integrand. We get

$$\mathscr{F}(u)=\frac{1}{2i\pi}\int_{\frac12-i\infty}^{\frac12+i\infty}\int_0^Y \cos(y)(u/y)^{-s}\frac{dy}{y}\frac{ds}{(1-s)(2-s)(1+s)^2}+\mathcal{O}(u^{-1/2}Y^{-1/2})$$

and using our lemma, we find that

$$\mathscr{F}(u)=\int_u^\infty \cos y\Big(\frac{5u}{6y}-\frac{u}{y}\,\mathrm{Log}\,(u/y)\Big)\frac{dy}{6y}+\int_0^u \cos y\Big(\frac{1}{4u}-\frac{y}{9u^2}\Big)dy. \tag{38}$$

We integrate by parts the first term to get

$$\mathscr{F}(u)=-\frac{5\sin u}{36u}+\int_u^\infty \sin y\Big(\frac{u}{3y}-\frac{u}{y}\,\mathrm{Log}\,(u/y)\Big)\frac{dy}{3y^2}+\frac{5\sin u}{36u}-\frac{\cos u-1}{9u^2}.$$

We integrate by parts one more time and get

$$\mathscr{F}(u)=\frac{1}{9u^2}+\int_u^\infty \frac{u\cos y}{y^4}\,\mathrm{Log}\,\frac{u}{y}\,dy. \tag{39}$$

And yet another integration yields

$$\mathscr{F}(u) = \frac{1}{9u^2} + \int_u^\infty \frac{\sin y}{y^4}\left(\frac{u}{y} + 4\frac{u}{y}\,\mathrm{Log}\,\frac{u}{y}\right)dy$$

and we carry a last one to reach

$$\mathscr{F}(u) = \frac{1}{9u^2} + \frac{\cos u}{u^4} - \int_u^\infty \frac{\cos y}{y^5}\left(\frac{9u}{y} + 20\frac{u}{y}\,\mathrm{Log}\,\frac{u}{y}\right)dy$$

Noticing that the maximum of $t \mapsto |9t - 20t\,\mathrm{Log}\,t|$ over $[0,1]$ is at $\exp(-11/20)$, we get

$$|\mathscr{F}(u) - 1/(9u^2)| \leqslant 12/u^4.$$

We set

$$\mathscr{G}_X(u) = \int_1^X \frac{\sin(u\xi)}{\xi^5}\,(1 - 4\,\mathrm{Log}\,\xi)\,d\xi \tag{40}$$

and $\mathscr{G}(u) = \mathscr{G}_\infty(u)$, which is (5), so that

$$\mathscr{F}(u) = \frac{1}{9u^2} + \frac{\mathscr{G}_X(u)}{u^3} + \mathcal{O}^*((5 + 4\,\mathrm{Log}\,X)/(u^3X^4)). \tag{41}$$

10. The moment of order 2

From section 4, we infer

$$\mathfrak{M}(2) = \sum_{\substack{\theta,\theta' \in \Theta(Q) \\ n,m \leqslant N}} e((m-n)(\theta - \theta')). \tag{42}$$

Introducing therein the value of the Ramanujan sum, we find

$$\mathfrak{M}(2) = \sum_{1 \leqslant n,m \leqslant N} \sum_{\substack{u,v \\ u,v|m-n}} uvM(Q/u)M(Q/v) \tag{43}$$

with $M(z) = \sum_{d \leqslant z} \mu(d)$. We use Lemma 5.1 with Lemma 6.1 to modify (43). There comes

$$\mathfrak{M}(2) = N^2 \sum_{u,v} (u,v) M(Q/u)M(Q/v)$$
$$+ \sum_{u,v} uvM(Q/u)M(Q/v)\frac{[u,v]}{i\pi} \int_{-\frac14 - i\infty}^{-\frac14 + i\infty} \zeta(s)\frac{N^{s+1}}{[u,v]^{s+1}s(s+1)}ds.$$

We note that (using (45) below with $s=1$)

$$\sum_{u,v\leqslant Q}(u,v)M(Q/u)M(Q/v)=\sum_{q\leqslant Q}\phi(q)=|\Theta(Q)|$$

so that

$$\mathfrak{M}(2)=N^2|\Theta(Q)|+\frac{1}{i\pi}\int_{-\frac14-i\infty}^{-\frac14+i\infty}\zeta(s)\sum_{u,v}\frac{uvM(Q/u)M(Q/v)}{[u,v]^s}\frac{N^{s+1}}{s(s+1)}ds.$$

We shift the line of integration to $-\epsilon=-1/\operatorname{Log}(QN)$ so that $|\zeta(s)|\ll(1+|t|)^{\frac12+\epsilon}\operatorname{Log}(QN)$. Concerning the sum over u and v, we get

$$F(s)=\sum_{u,v}\frac{uvM(Q/u)M(Q/v)}{[u,v]^s}=\sum_{u,v}(u,s)^s\frac{uvM(Q/u)M(Q/v)}{u^sv^s}.$$

We use at this level the so called Selberg diagonalization process which most commonly appears in the study of Selberg sieve. See also [25] and [26]. It consists in introducing the multiplicative function defined by

$$f(p^k,s)=p^{ks}-p^{(k-1)s}\quad(k\geqslant 1)\tag{44}$$

when p is a prime. This function enables us to write

$$(u,v)^s=\sum_{d|(u,v)}f(d,s)=\sum_{\substack{d|u\\ d|v}}f(d,s)\tag{45}$$

with the neat effect of separating the contribution of u and v. This leads us to

$$F(s)=\sum_{d\leqslant Q}f(d,s)\Big(\sum_{u/d|u\leqslant Q}u^{1-s}M(Q/u)\Big)^2.$$

Further recall (30) and note that

$$\begin{aligned}\sum_{u/d|u\leqslant Q}u^{1-s}M(Q/u)&=\sum_{v\leqslant Q}\sum_{u/d|u|v}u^{1-s}\mu(v/u)\\&=d^{1-s}\sum_{v\leqslant Q/d}f(v,1-s)=d^{1-s}W(s,Q/d).\end{aligned}$$

Let us start with a wide range upper bound for the inner sum:

$$\Big|\sum_{u/d|u\leqslant Q}u^{1-s}M(Q/u)\Big|\leqslant Q\sum_{u/d|u\leqslant Q}u^{\epsilon}\ll Q^2/d.$$

This enables us to truncate on $\Re s = -\epsilon$ the series defining F:

$$F(s) = \sum_{d \leqslant D} \frac{f(d,s)}{d^{2s-2}} W(s, Q/d)^2 + \mathcal{O}(Q^4 D^{-1} \operatorname{Log} Q).$$

For $D = 1/2$ this also yields an upper bound for $F(s)$ that will enables us to restrict the integral in height. We get

$$\begin{aligned}\mathfrak{M}(2) = N^2 |\Theta(Q)| + \frac{1}{i\pi} \int_{-\epsilon - iT_0}^{-\epsilon + iT_0} \zeta(s) \sum_{d \leqslant D} \frac{f(d,s)}{d^{2s-2}} W(s, Q/d)^2 \frac{N^{s+1}}{s(s+1)} ds \\ + \mathcal{O}\Big(\frac{NQ^4 \operatorname{Log} Q}{D} \Big) + \mathcal{O}\Big(\frac{NQ^4 \operatorname{Log} Q}{\sqrt{T_0}} \Big).\end{aligned}$$

We use (33) to replace $W(s, Q/d)^2$ by $(Q/d)^{4-2s}/(\zeta(2-s)(2-s))^2$ with cost (since $D \leqslant Q$ and $T_0 \leqslant N^2$)

$$\begin{aligned}\ll \int_0^{T_0} (1+t)^{\frac{1}{2}+\epsilon} \sum_{d \leqslant D} \frac{|f(d,s)|}{d^{2\Re s - 2}} \Big(\frac{(Q/d)^{3-2\Re s}}{1+t} \operatorname{Log}^3 (Q + |t|) \\ + (Q/d)^{2-2\Re s} \operatorname{Log}^6 (Q + |t|) \Big) \frac{N}{(1+t)^2} dt.\end{aligned}$$

The above is

$$\ll \int_0^{T_0} \sum_{d \leqslant D} |f(d,s)| d^2 \Big(\frac{(Q/d)^3}{1+t} + (Q/d)^2 \operatorname{Log}^3 (Q+t) \Big) \frac{N \operatorname{Log}^3 (Q+t)\, dt}{(1+t)^{3/2}}$$

or

$$\ll \Big(Q^3 \operatorname{Log}^2 D + Q^2 D \operatorname{Log} D \operatorname{Log}^3 Q \Big) N \operatorname{Log}^3 Q.$$

We select

$$D = Q, \quad T_0 = N^2. \tag{46}$$

We reach

$$\begin{aligned}\mathfrak{M}(2) = N^2 |\Theta(Q)| + \frac{NQ^4}{i\pi} \int_{-\epsilon - iT_0}^{-\epsilon + iT_0} \sum_{d \leqslant D} \frac{f(d,s)}{d^2} \frac{\zeta(s)(N/Q^2)^s ds}{\zeta(2-s)^2 s(s+1)(2-s)^2} \\ + \mathcal{O}(NQ^3 \operatorname{Log}^7 Q).\end{aligned}$$

We replace the sum over d by a complete sum with loss at most $\mathcal{O}(NQ^4 T_0^{-5/2})$ and note that

$$\sum_{d \geqslant 1} \frac{f(d,s)}{d^2} = \zeta(2-s)/\zeta(2). \tag{47}$$

We then replace T_0 by ∞ and ϵ by $-1/8$. We have thus reached an expression of the shape $\mathfrak{M}(2) = N^2|\Theta(Q)| + N^2Q^2\mathfrak{f}(N/Q^2) +$ *error* but we still have to modify the expression defining $\mathfrak{f}$ to recognize (6).

A different expression for $\mathfrak{f}$

Using the change of variables $s \mapsto 1-s$, we reach

$$\mathfrak{f}(x) = \frac{6/\pi^2}{i\pi}\int_{\frac{9}{8}-i\infty}^{\frac{9}{8}+i\infty} \frac{\zeta(1-s)x^{-s}ds}{\zeta(1+s)(1-s)(2-s)(1+s)^2} \tag{48}$$

The functionnal equation of the Riemann ζ-funtion (see [30] or [19]) may be written as

$$\zeta(1-s) = 2^{1-s}\pi^{-s}\cos(\pi s/2)\Gamma(s)\zeta(s). \tag{49}$$

so that

$$\mathfrak{f}(x) = \frac{24/\pi^2}{2i\pi}\int_{\frac{9}{8}-i\infty}^{\frac{9}{8}+i\infty} \frac{\zeta(s)}{\zeta(1+s)}\frac{(2\pi x)^{-s}\cos(\pi s/2)\Gamma(s)ds}{(1-s)(2-s)(1+s)^2}. \tag{50}$$

Note that

$$\frac{\zeta(s)}{\zeta(s+1)} = \sum_{n\geqslant 1}\frac{\phi(n)}{n^{1+s}}. \tag{51}$$

Recalling (34) we get

$$\mathfrak{f}(x) = \frac{24}{\pi^2}\sum_{n\geqslant 1}\frac{\phi(n)}{n}\mathscr{F}(2\pi x n). \tag{52}$$

We approximate $\mathfrak{f}$ as follows:

$$\mathfrak{f}(x) = \frac{24}{\pi^2}\sum_{n\geqslant 1}\frac{\phi(n)}{n}\left(\mathscr{F}(2\pi x n) - \frac{1}{36\pi^2x^2n^2}\right) + \frac{2}{3\pi^4x^2}\sum_{n\geqslant 1}\frac{\phi(n)}{n^3}$$

and this is Theorem 1.1. We continue by appealing to Theorem 9.1:

$$\mathfrak{f}(x) = \frac{1}{9\pi^2\zeta(3)x^2} + \frac{3}{\pi^5x^3}\sum_{1\leqslant n\leqslant N}\frac{\phi(n)}{n^4}\mathscr{G}(2\pi x n) + \mathcal{O}^*\Big(\frac{18}{\pi^6x^4}\sum_{n\geqslant N}\phi(n)n^{-5}\Big).$$

A more thorough numerical study of the remainder term would use a summation by parts together with Lemma 3.2 of [28], but we contend ourselves with a simple appeal to the inequality $\phi(n)\leqslant n$.

Lemma 10.1. *For $N \geqslant 2$, we have*

$$\mathfrak{f}(x) = \frac{1}{9\pi^2\zeta(3)x^2} + \frac{3}{\pi^5x^3}\sum_{1\leqslant n\leqslant N}\frac{\phi(n)}{n^4}\mathscr{G}(2\pi x n) + \mathcal{O}^*\Big(\frac{6}{\pi^6x^4(N-1)^3}\Big).$$

We are now in a position to compute values of $\mathfrak{f}$: we simply use (40) via (41) to restrict the domain of integration for $\mathscr{G}$. The reader should keep in mind that

$\mathfrak{f}$ is most probably a *non-negative* function, simply because of (7). This is a highly non trivial property when looking at the expression given by the lemma!

11. Additional information

Using the notation of the introduction, we note $B = AA^*$ and remark that A is a Vandermonde matrix. This last statement is true only if A is square, namely if $|\Theta(Q)| = N$. As a consequence, and when $|\Theta(Q)| = N$, the determinant of B is

$$\det B = \prod_{\substack{0<\theta\neq\theta'\leqslant 1,\\ \theta,\theta'\in\Theta(Q)}} |e(\theta) - e(\theta')|.$$

This is valid for any set Θ and we now utilize more specific informations:

$$\begin{aligned}
\operatorname{Log}\det B &= -\sum_{\theta\neq\theta'\in\Theta(Q)} \operatorname{Log}\,(1 - e(\theta-\theta')) = -\sum_{\theta\neq\theta'\in\Theta(Q)}\sum_{k\geqslant 1}\frac{e(k(\theta-\theta'))}{k} \\
&= -\sum_{k\geqslant 1}\frac{\sum_{\theta\neq\theta'\in\Theta(Q)} e(k(\theta-\theta'))}{k} \\
&= -\sum_{k\geqslant 1}\frac{\left|\sum_{\theta\in\Theta(Q)} e(k\theta)\right|^2 - |\Theta(Q)|}{k} \\
&= -\sum_{k\geqslant 1}\frac{\left(\sum_{d|k} dM(Q/d)\right)^2 - |\Theta(Q)|}{k} \\
&= -\lim_{K\to\infty}\sum_{1\leqslant k\leqslant K}\frac{\left(\sum_{d|k} dM(Q/d)\right)^2 - |\Theta(Q)|}{k}.
\end{aligned}$$

The development of $-\operatorname{Log}\,(1-z)$ is valid of course for any $|z|<1$, but also for all $z\neq 1$ on the unit circle, the limit being this time definitely not absolute. The next step consists in expanding the square to find that our quantity is

$$\begin{aligned}
&\sum_{d_1,d_2\leqslant Q}\frac{d_1d_2}{[d_1,d_2]}M(Q/d_1)M(Q/d_2)\Big(\operatorname{Log}\frac{K}{[d_1,d_2]} + \gamma + \mathcal{O}([d_1,d_2]/K)\Big) \\
&\qquad - |\Theta(Q)|\Big(\operatorname{Log} K + \gamma + \mathcal{O}(1/K)\Big)
\end{aligned}$$

which gives us our first expression of $\operatorname{Log}\det B$:

$$\operatorname{Log}\det B = \sum_{d_1,d_2\leqslant Q}\frac{d_1d_2}{[d_1,d_2]}M(Q/d_1)M(Q/d_2)\operatorname{Log}\,[d_1,d_2].$$

Estimation of this sum is easy enough, appealing to $[d_1, d_2] = d_1 d_2/(d_1, d_2)$. We first note that

$$\sum_{d_1, d_2 \leqslant Q} \frac{d_1 d_2}{[d_1, d_2]} M(Q/d_1) M(Q/d_2) \operatorname{Log} d_1$$
$$= \sum_{\delta \leqslant Q} \phi(\delta) \sum_{\delta | d_1 \leqslant Q} M(Q/d_1) \operatorname{Log} d_1 \sum_{\delta | d_2 \leqslant Q} M(Q/d_2).$$

The sum over d_2 is 1, and the one over d_1 is

$$\begin{aligned}\sum_{\delta | d_1 \leqslant Q} M(Q/d_1) \operatorname{Log} d_1 = &\sum_{d_1 \leqslant Q/\delta} M((Q/\delta)/d_1)(\operatorname{Log} \delta + \operatorname{Log} d_1) \\ =&\operatorname{Log} \delta + \sum_{d_1 \leqslant Q/\delta} M((Q/\delta)/d_1) \sum_{\ell | d_1} \Lambda(\ell) \\ =&\operatorname{Log} \delta + \psi(Q/\delta).\end{aligned}$$

We proceed similarly to get

$$\sum_{d_1, d_2 \leqslant Q} (d_1, d_2) \operatorname{Log} (d_1, d_2) M(Q/d_1) M(Q/d_2)$$
$$= \sum_{\ell \leqslant Q} \Lambda(\ell) \sum_{\ell | d_1, d_2 \leqslant Q} (d_1, d_2) M(Q/d_1) M(Q/d_2)$$
$$= \sum_{\ell \leqslant Q} \Lambda(\ell) \ell \sum_{d \leqslant Q/\ell} \phi(d)$$

so that we reach a second and much more tractable expression for $\operatorname{Log} \det B$:

$$\operatorname{Log} \det B = \sum_{\delta \leqslant Q} \phi(\delta) \Big(2 \operatorname{Log} \delta + 2\psi(Q/\delta) - \sum_{\ell \leqslant Q/\delta} \ell \Lambda(\ell) \Big). \tag{53}$$

At this level, it is better to recall two simple results.

Lemma 11.1. *For any $c > 0$, we have*

$$\sum_{\ell \leqslant L} \Lambda(\ell)/\ell = \operatorname{Log} L - \gamma + \mathcal{O}_c((\operatorname{Log} L)^{-c}).$$

We have

$$\sum_{\ell \leqslant L} \frac{\phi(\ell)}{\ell^2} = \frac{6}{\pi^2} \left(\operatorname{Log} L - \frac{6\zeta'(2)}{\pi^2} + \gamma + \mathcal{O}(L^{-1/3}) \right).$$

The first estimate is a form classically equivalent to the prime number Theorem. The error term therein could be $\mathcal{O}(r(L))$ with

$$r(L) = \exp\big(-c (\operatorname{Log} L)^{3/5} (\operatorname{LogLog} L)^{-1/5} \big) \tag{54}$$

for some positive constant c (and $c = 0.2$ is a possible choice thanks to [11]) or even $r(L) = (\mathrm{Log}\ L)/\sqrt{L}$ if the Riemann hypothesis holds. The second estimate is equally classical and can for instance be obtained by applying Lemma 3.2 of [28], the relevant Dirichlet series being the one from (51), though shifted by $s \mapsto s+1$ (in rough details: $g(n) = \phi(n)/n^2$, $H(s) = 1/\zeta(s+2)$ and $k_n = 1/n$).

Evaluating the last sum in (53), say S_2, requires Dirichlet hyperbola principle. We write

$$\begin{aligned}
S_2 &= \sum_{\ell \leqslant L} \Lambda(\ell)\ell \sum_{d \leqslant Q/\ell} \phi(d) + \sum_{d \leqslant Q/L} \phi(d) \sum_{L < \ell \leqslant Q/d} \Lambda(\ell)\ell \\
&= \sum_{\ell \leqslant L} \Lambda(\ell)\ell \left(\frac{3Q^2}{\pi^2 \ell^2} + \mathcal{O}((Q/\ell)\ \mathrm{Log}\ Q) \right) \\
&\quad + \sum_{d \leqslant Q/L} \phi(d) \left(\frac{1}{2} \left((Q/d)^2 - L^2 \right) + \mathcal{O}_c((Q/d)^2 (\mathrm{Log}\ L)^{-c}) \right) \\
&= \frac{3Q^2}{\pi^2} (\mathrm{Log}\ L - \gamma) + \mathcal{O}(QL\ \mathrm{Log}^2\ Q) + \mathcal{O}_c(Q^2 (\mathrm{Log}\ L)^{-c}) \\
&\quad + \frac{Q^2}{2} \sum_{d \leqslant Q/L} \frac{\phi(d)}{d^2} - L^2 \frac{3Q^2}{\pi^2 L^2}
\end{aligned}$$

which ends in

$$S_2 = \frac{3Q^2}{\pi^2} \mathrm{Log}\ Q - \frac{18\zeta'(2)}{\pi^4} Q^2 - \frac{3}{2\pi^2} Q^2 + \mathcal{O}(Q^2 (\mathrm{Log}\ Q)^{-c}) \tag{55}$$

by taking $L = Q^{1/4}$. On another side we have

$$\begin{aligned}
S_{1,1} &= \sum_{\delta \leqslant Q} \phi(\delta)\ \mathrm{Log}\ \delta = \sum_{\delta \leqslant Q} \phi(\delta)\ \mathrm{Log}\ Q - \sum_{\delta \leqslant Q} \phi(\delta) \int_\delta^Q \frac{dt}{t} \\
&= \mathrm{Log}\ Q \left(\frac{3}{\pi^2} Q^2 + \mathcal{O}(Q\ \mathrm{Log}\ Q) \right) - \int_1^Q \left(\frac{3}{\pi^2} t^2 + \mathcal{O}(t\ \mathrm{Log}\ Q) \right) \frac{dt}{t} \\
&= \frac{3}{\pi^2} Q^2\ \mathrm{Log}\ Q - \frac{3}{2\pi^2} Q^2 + \mathcal{O}\big(Q\ \mathrm{Log}^2\ Q\big)
\end{aligned}$$

while another use of Dirichlet hyperbola principle yields

$$\begin{aligned}
S_{1,2} &= \sum_{\delta \leqslant Q} \phi(\delta)\psi(Q/\delta) = \sum_{\delta \leqslant \Delta} \phi(\delta)\psi(Q/\delta) + \sum_{\ell \leqslant Q/\Delta} \Lambda(\ell) \Big(\sum_{\Delta < \delta \leqslant Q/\ell} \phi(\delta) \Big) \\
&= Q \sum_{\delta \leqslant \Delta} \frac{\phi(\delta)}{\delta} + \mathcal{O}(Q^2 (\mathrm{Log}\ (Q/\Delta))^{-c}) + \frac{3Q^2}{\pi^2} \sum_{\ell \leqslant Q/\Delta} \frac{\Lambda(\ell)}{\ell^2} \\
&= -\frac{18\zeta'(2)}{\pi^4} Q^2 + \mathcal{O}(Q^2 (\mathrm{Log}\ Q)^{-c})
\end{aligned}$$

on taking $\Delta = Q^{1/2}$. Finally

$$\begin{aligned}\operatorname{Log}\det B &= 2S_{1,1} + 2S_{1,2} - S_2\\ &= \frac{6}{\pi^2}Q^2 \operatorname{Log} Q - \frac{3}{\pi^2}Q^2 - \frac{36\zeta'(2)}{\pi^4}Q^2\\ &\quad - \frac{3Q^2}{\pi^2}\operatorname{Log} Q + \frac{18\zeta'(2)}{\pi^4}Q^2 + \frac{3}{2\pi^2}Q^2 + \mathcal{O}(Q^2(\operatorname{Log} Q)^{-c})\\ &= \frac{3}{\pi^2}Q^2 \operatorname{Log} Q - \left(\frac{1}{2} + \frac{6\zeta'(2)}{\pi^2}\right)\frac{3}{\pi^2}Q^2 + \mathcal{O}_c(Q^2(\operatorname{Log} Q)^{-c})\end{aligned}$$

Gathering our results yields (remember (16) and that $N = |\Theta(Q)|$)

$$N^{-1}\operatorname{Log}\det B = \frac{1}{2}\operatorname{Log} N - \frac{1}{2} - \frac{6\zeta'(2)}{\pi^2} + \frac{1}{2}\operatorname{Log}(\pi^2/3) + \mathcal{O}_c((\operatorname{Log} Q)^{-c}). \quad (56)$$

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Address: ????
E-mail: `Olivier.Ramare@math.univ-lille1.fr`
Received: ????