On the behaviour of $\gamma \operatorname{Log} p$ modulo 1

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Abstract

We prove non-trivial lower bounds for sums of type $\sum_{p\sim P} g(\gamma \operatorname{Log} p)$, where g is a non-negative 2π-periodical function and γ is a given parameter. As an application we prove that $\zeta(1+it)^{\pm 1} \ll \operatorname{Log}\operatorname{Log}(9+|t|)$ and extend the zero-free region of the Riemann zeta-function.

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1 Introduction

Sums over primes and their evaluation is one of the main subject of multiplicative number theory. We are concerned in this paper with lower bounds for sums of type $\sum_{p\sim P} g(\gamma \operatorname{Log} p)$ for a non-negative 2π-periodical function g and some parameter γ. Such sums appear in the connection of zero-free region problems. The case $g(t) = 1 + \cos t$ shows that we have to handle the case of g being essentially a *multiplicative* character associated to the infinite place; bilinear forms techniques are not available in such a context since the bilinear form reduces here simply to a product of two linear forms. Using Mellin transforms is not possible either since we are asking for too much precision in the location of p. Sieve techniques are not efficient either since we are seeking a lower bound. However Balasubramanian & Ramachandra introduced in in [1] and [2] a technique that transforms a sieve upper bound for a sum over primes (namely the Brun-Titchmarsh inequality) into a lower bound. This is the starting point of this paper. To cut a long story short, let us rapidly say that we use an enveloping sieve as in [21]; when γ is a power of P, we are able to take the z-parameter of this sieve to be as large as $P^{\frac{1}{2}+\delta}$ for some positive δ. When $\gamma \geq P \geq (\operatorname{Log}\gamma)^4$ and γ is large enough, we can even take $z = P/\exp\big((\operatorname{Log}\operatorname{Log} P)^3\big)$. This is surprising as one would expect the case of large values of γ to be more difficult to handle than the case of small values.

To express our result in full generality, we define for positive real number η and integrable function g:

$$V(g,\eta) = \min_{\begin{cases} \mathcal{A} \subset [0,2\pi], \\ \mu(\mathcal{A}) \geq \eta \end{cases}} \int_{\mathcal{A}} |g(u)| du/\eta \tag{1}$$

where $\mathcal{A}$ ranges the set of mesurable subsets within $[0,2\pi]$ of measure at least η. More classically, we use the continuity modulus of g, when g is moreover assumed to be continuous, defined

$$\omega_g(\delta) = \max_{u \leq v \leq u+\delta} |g(u) - g(v)|. \tag{2}$$

Theorem 1.1. *There exists $P_0 \geq 10$ with the following property. Let g be a continuous 2π-periodical non-negative real-valued function bounded in absolute value by 1. Assume the parameters P, $\gamma \geq 2$ and $\xi \in (0,1]$ verify*

$$P \geq \max(P_0, (\operatorname{Log}\gamma)^4),\ \ \xi \geq 1/(\operatorname{Log} P)^2.$$

Then we have

$$\sum_{P < p \leq (1+\xi)\cdot P} g(\gamma \operatorname{Log} p) \geq \big(V(g, 2\pi\kappa_\gamma(P)) + \mathcal{O}(H)\big)\xi P / \operatorname{Log} P$$

where

$$H = \frac{1}{\operatorname{Log} P} + \omega_g\left(\frac{6\pi}{(\operatorname{Log} P)^4}\right)$$

and where $\kappa_\gamma(P)$ is given by the following table:

P		γ		$\gamma^{5/3}$		$\gamma^{5/2}$	
$\kappa_\gamma(P)$	*1*		$(1+\theta)/2$		$4/5$		$(2+3\theta)/4$

with $\theta = (\operatorname{Log}\gamma)/\operatorname{Log} P$.

As already mentionned, we will obtain this result by improving on the method developped in [1] and [2]. In term of the z-parameter in the Selberg sieve, the path we take can be roughly described as follows. Balasubramanian & Ramachandra in [2] were looking at each separate subinterval where $g(t)$ is close to some fixed value. They were limited to $z \simeq \sqrt{P/\gamma}$. On looking at the collection of each of these subintervals, with still a fixed value of $g(t)$, we are already able to select $z \simeq \sqrt{P}$. At this level, we are treating the error term via a Mellin transform with variable s. We can next take special advantage of our selector/smoothing function and gain an additional saving when $\Im s$ is large enough, and this is embodied in inequality (52). A more classical part of this treatment depends on good average bounds for $\zeta(s)M(s)$ on the line $\Re s = 1/2$ for a fairly long Dirichlet polynomial M. This allows us to take z larger than $\sqrt{P}$ when γ is large enough, if we are able to handle the case of variables s with a not too large imaginary part. We treat this problem in two steps. We first remark that the relevant error term depends only weakly on γ so that we can average over this variable. When we revert to a summation over integers, we see that our averaging has the same effect as not specifying the value of $g(t)$ anymore. We still have to compute the full sum, and it is for this purpose that we sieve from the very beginning with Barban & Vehov's weights from [4]. We need here a better result than the one proved in [8] and in [19], and this is our Theorem 10.1.

Under the Riemann Hypothesis, the above line of work leads to the following.

Theorem 1.2. *Let us assume the truth of the Riemann Hypothesis. For every small positive constant $\varepsilon > 0$, there exists $P_0(\varepsilon) \geq 10$ with the following property. Let g be a continuous 2π-periodical non-negative real-valued function bounded in absolute value by 1. Assume the parameters P, $\gamma \geq 2$ and $\xi \in (0,1]$ verify*

$$P \geq \max(P_0, (\operatorname{Log}\gamma)^{1/\varepsilon}),\ \ \xi \geq 1/(\operatorname{Log}P)^2.$$

Then we have

$$\sum_{P<p\leq(1+\xi)\cdot P} g(\gamma \operatorname{Log} p) \geq \big(V(g, 2\pi(1-4\varepsilon)) + \mathcal{O}(H)\big)\xi P/\operatorname{Log}P$$

where H is defined in Theorem 1.1.

In particular, given a fixed small $\varepsilon > 0$, and some ξ in $(0,1]$, we can find $\gamma_0(\varepsilon, \xi)$ such that, when $\gamma \geq \gamma_0$, and *either* $\gamma \geq P \geq (\operatorname{Log}\gamma)^4$ *or the Riemann Hypothesis holds and* $P \geq (\operatorname{Log}\gamma)^{1/\varepsilon}$, we have

$$\left|\frac{\operatorname{Log}P}{\xi P}\sum_{P<p\leq(1+\xi)\cdot P} g(\gamma\operatorname{Log}p) - \frac{1}{2\pi}\int_0^{2\pi} g(u)du\right| \leq \varepsilon.$$

This is because $2\pi V(g, 2\pi) = \int_0^{2\pi} g(u)du$ and we can apply the same inequality with $A-g$ for a large enough constant A. The condition linking P and γ cannot be fully dispensed with since none of the sequences $(\gamma \cdot \operatorname{Log} p)_p$ is equidistributed modulo 1, when γ is fixed. We try to clarify the situation in sections 2 and 3. Our proof in fact works as soon as $P \gg (\operatorname{Log}\gamma)^{3+\varepsilon}$.

Here is a corollary of Theorem 1.1:

Theorem 1.3. *There exists γ_0 such that, for every real valued, non-negative, continuously differentiable and 2π-periodical function g, for every every $\gamma \geq \gamma_0$ and every $\sigma \in]1,2]$, we have*

$$\sum_{p\geq1}\frac{g(\gamma\operatorname{Log}p)\operatorname{Log}p}{p^\sigma} \geq \int_{(\operatorname{Log}\gamma)^4}^{\infty} V(g, 2\pi\kappa_\gamma(t))\frac{dt}{t^\sigma} + \mathcal{O}(-\operatorname{Log}(\sigma-1))$$

On selecting first $g(t) = 1 \pm \cos t$ and then $g(t) = 1 \pm \sin t$ in Theorem 1.1, we immediately reach:

Corollary 1.4. *There exists t_0 such that, for every $t \geq t_0$ and every $\sigma \in]1, 2]$ such that $(\sigma - 1) \operatorname{Log} \operatorname{Log} t$ is small enough in terms of t_0, we have*

$$\left| \Re \frac{\zeta'}{\zeta}(\sigma + it) \right| \leq \frac{h((\sigma - 1) \operatorname{Log} t)}{\sigma - 1} + \mathcal{O}(-\operatorname{Log}(\sigma - 1) + \operatorname{Log} \operatorname{Log} t)$$

where

$$h(x) = x \int_1^{5/3} \frac{2v/\pi}{1+v} \sin\Big(\frac{1+v}{2v/\pi}\Big) e^{-xv} dv + (e^{-5x/3} - e^{-5x/2}) \frac{5}{4\pi} \sin \frac{4\pi}{5} + x \int_{5/2}^{\infty} \frac{4v/\pi}{2v+3} \sin\Big(\frac{2v+3}{4v/\pi}\Big) e^{-xv} dv$$

The function h verifies $h(x) \leq 2e^{-x}/\pi$. The same inequality holds with $\Im$ instead of $\Re$.

The main point is that $2/\pi < 1$. Note that to reach this $2/\pi$, taking the z-parameter of the Selberg sieve to be $\sqrt{P}/(\operatorname{Log} P)^{40}$ would be enough. Here is a table of (upper bounds for) values for h which shows that this function decreases rapidly to 0:

x	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	∞
$h(x)$	0.416	0.314	0.247	0.200	0.165	0.138	0.117	0.100	0.086	0.074	0

Note that the bound of Corollary 1.4 exploits the dependence in σ. We do not know how to improve on this Corollary under the sole Lindelöf Hypothesis. Under the Riemann Hypothesis the same proof but relying also on Theorem 1.2 yields:

Corollary 1.5. *We assume the Riemann Hypothesis. For every $\varepsilon > 0$, there exists a $t_0(\varepsilon)$ such that, for every $t \geq t_0(\varepsilon)$ and every $\sigma \in]1, 2]$ such that $(\sigma - 1) \operatorname{Log} \operatorname{Log} t$ is small enough in terms of t_0, we have*

$$\left| \frac{\zeta'}{\zeta}(\sigma + it) \right| \leq \frac{\varepsilon / t^{\sigma - 1}}{\sigma - 1} + \mathcal{O}(-\operatorname{Log}(\sigma - 1) + \operatorname{Log} \operatorname{Log} t).$$

Corollary 1.4 has the following remarkable consequences.

Corollary 1.6. *When $\sigma \geq 1$ and $t \geq 100$, we have*

$$\zeta(\sigma + it), 1/\zeta(\sigma + it) \ll \operatorname{Log} \operatorname{Log} t.$$

This is proved at the end of section 19. These two bound are optimal up to the implied constant, as shown in [16]. The reader may also have a look at the numerical study from [13]. The effect on a zero-free region of a bound for $|\zeta(1 + it)|$, as opposed to one valid in the vicinity of the line $\Re s = 1$, is not immediate. As a matter of fact, our bound on this line is as good as what would follow from the Riemann Hypothesis (see [24, Theorem 14.8]) though we are not able to infer the Riemann Hypothesis from it. Our bound is however strong enough to lead so some improvement on the existing zero-free region:

Theorem 1.7. *There exists a constant $c > 0$ such that the Riemann zeta-function has no zero in the region*

$$\sigma \geq 1 - \frac{c}{(\operatorname{Log} t)^{2/3}(\operatorname{Log}\operatorname{Log}\operatorname{Log} t)^{1/3}}.$$

Moreover we can choose c such that, in the region above, we have

$$\frac{\zeta'}{\zeta}(\sigma + it) \ll (\operatorname{Log} t)^{2/3}(\operatorname{Log}\operatorname{Log}\operatorname{Log} t)^{1/3}.$$

This is the first time the Vinogradov-Korobov's result [12] from 1958 (see Lemma 19.1 below) is being refined, aside from numerical sharpening, see [6]. The above zero-free region leads mechanically to the following prime number Theorem:

$$\psi(x) - x \ll x \exp\Bigl(-c_3(\operatorname{Log} x)^{3/5}/(\operatorname{Log}\operatorname{Log}\operatorname{Log} x)^{1/5}\Bigr) \tag{3}$$

for some positive constant c_3 and provided $x \geq 100$.

Notation. Some of our expressions need parenthesing and it is better to have the brackets [and] at our disposal. As a consequence, we do not use them to indicate the integer part, for which we use $\lfloor$ and $\rfloor$. We will also use $\lceil x \rceil$ to denote the smallest integer larger than x. Special weights have been introduced in [4]. They are similar in some aspects to the Selberg weights, see [9], but verify the property proved below in Theorem 10.1. Here is their definition:

$$\beta(n) = \Bigl(\sum_{d|n} \lambda_d\Bigr)^2 \tag{4}$$

where λ_d is defined by

$$\lambda_d = \mu(d)\frac{\operatorname{Log}(z/d)}{\operatorname{Log} z}. \tag{5}$$

2 A direct approach when γ is bounded

The truncated Perron summation formula, see e.g. [22, Theorem 7.1], gives us

$$\sum_{n \leq P} \Lambda(n) n^{i\gamma} = \frac{1}{2i\pi}\int_{c-iT}^{c+iT} \frac{-\zeta'}{\zeta}(z - i\gamma)\frac{P^z dz}{z} + \mathcal{O}\Bigl(\frac{P(\operatorname{Log} P)(\operatorname{Log} T)}{T} + \operatorname{Log} P\Bigr).$$

for $c = 1 + (\operatorname{Log} P)^{-1}$. We select $T = (\operatorname{Log} P)^4$ and skip some classical steps. As a conclusion, we find that, for every $\gamma \geq 0$, and P large enough satisfying $\operatorname{Log} P \geq c_2(\operatorname{Log}(9+\gamma))^{2/3}(\operatorname{Log}\operatorname{Log}(9+\gamma))^{1/3}$ for a large enough constant c_2, we have

$$\Bigl|\sum_{p \leq P} p^{i\gamma} - \frac{P^{1+i\gamma}/\operatorname{Log} P}{1 + i\gamma}\Bigr| \leq P/(\operatorname{Log} P)^2. \tag{6}$$

3 Full equidistribution modulo 0

The previous section shows that the sequence $(\mathrm{Log}\, p)$ is not equidistributed modulo 2π, and in fact is not equidistributed modulo m for every $m > 0$. In order to proceed further, we need a definition.

Definition 3.1. A real-valued sequence $(u_n)_n$ is said to be fully equidistributed modulo 0 if for every $\varepsilon > 0$, there exists $m_0(\varepsilon) > 0$ such that, for every $m_1 \geq m_0$, we have

$$\limsup_{N\to\infty} \max_{\substack{m_0 \leq m \leq m_1,\\ m\in\mathbb{R}}} \max_{0\leq a<b<1} \Big| \#\big\{n \leq N, m \cdot u_n \in [a,b] + \mathbb{Z}\big\}/N - (b-a)\Big| \leq \varepsilon.$$

The inner maximum over a and b is the discrepancy of the sequence $(mu_n)_n$ at N. If a sequence is equidistributed modulo $q > 0$ say, then it is equidistibuted modulo hq for every integer $h \geq 1$, and so the discrepancy of (hqu_n) tends to zero. The above definition adds a maximum over m which implies some uniformity and the fact that m is a *real* number and is not restricted to the integers multiple of a given period. A fully equidistibuted modulo 0 sequence is a priori not dense modulo m for some positive m though I do not have any example showing that density is not a consequence of being fully equidistibuted modulo 0. Concerning sets of reals numbers m that are such that $(mu_n)_n$ is equidistributed modulo 1, the reader may look at [17] and [23].

Here is the equivalent of Weyl's criterium in this context:

Theorem 3.1. *A real-valued sequence $(u_n)_n$ is fully equidistributed modulo 0 if and only if for every $\varepsilon > 0$, there exists $\gamma_0(\varepsilon) > 0$ such that, for every $\gamma_1 \geq \gamma_0$, we have*

$$\limsup_{N\to\infty} \max_{\gamma_0\leq\gamma\leq\gamma_1} \Big|\sum_{n\leq N} e(\gamma u_n)\Big|/N \leq \varepsilon.$$

Proof. The Erdös-Turán inequality (cf [25, Theorem 20]) readily shows that a sequence verifying the conditions of our Theorem is fully equidistributed modulo 0. Let us turn our attention to the reverse implication. Let $\varepsilon > 0$ and select $\gamma_0(\varepsilon) = m_0(\varepsilon/3)$. Let $\gamma_1 > \gamma_0(\varepsilon)$ and K the smallest integer larger than $6\pi/\varepsilon$. Let $\gamma \in [\gamma_0, \gamma_1]$. We have that

$$\begin{aligned} e(\gamma x) &= \sum_{0<k\leq K} \mathbb{1}_{k-1\leq\gamma K x<k} \cdot e(\gamma x) \\ &= \sum_{0<k\leq K} \mathbb{1}_{k-1\leq\gamma K x<k} \cdot e(k/K) + \mathcal{O}^*(\varepsilon/3). \end{aligned}$$

Thus

$$\sum_{n\leq N} e(\gamma u_n) = \sum_{0<k\leq K} \mathbb{1}_{k-1\leq\gamma K u_n<k} \cdot e(k/K) + \mathcal{O}^*(N\varepsilon/3).$$

We use our property with $\varepsilon/3$ and $m_1 = K\gamma_1$. For N large enough, we have

$$(1/N)\sum_{n\leq N} e(\gamma u_n) = \frac{1}{K}\sum_{0<k\leq K} e(k/K) + \mathcal{O}^*(\varepsilon/3) + \mathcal{O}^*(\varepsilon/2) = \mathcal{O}^*(\varepsilon).$$

$\square$

Let us first notice that the sequence $(\operatorname{Log} n)$ is not equidistributed modulo m, for any $m > 0$, but is fully equidistibuted modulo 0. It is furthermore dense modulo m for any positive real number m since the difference between two consecutive terms is $o(1)$. Here is a consequence of (6).

Theorem 3.2. *The sequence* $(\operatorname{Log} p)$ *is fully equidistributed modulo 0.*

The sequence $(\operatorname{Log} p)$ is also dense modulo m for every positive real number m by the same reason as above, i.e. the difference between two consecutive terms is $o(1)$. The nature of fully equidistributed modulo 0 sequences is an intriguing problem. Theorems 1.1 and 1.2 suggest that we can expect such a sequence to be not too erratically distributed in intervals.

4 Some properties of $V(g, \eta)$

We gather in this section several general properties of the quantity $V(g, \eta)$ that will lead to smoother treatments later on.

Lemma 4.1. *We have, when g and h are in $L^1([0, 2\pi])$ and $\eta > 0$,*

$$|V(g, \eta) - V(h, \eta)| \leq \|g - h\|_1/\eta.$$

Proof. Let B be a measurable subset of $[0, 2\pi]$ of measure $\leq \eta$. We have

$$\Big|\int_B |g(u)|du - \int_B |h(u)|du\Big| \leq \||g| - |h|\|_1 \leq \|g - h\|_1.$$

As a conclusion

$$\min_{A/\mu(A)\geq\eta} \int_A |g(u)|du \leq \min_{B/\mu(B)\geq\eta} \int_B |h(u)|du + \|g - h\|_1$$

and the reverse inequality follows by exchanging h and g. The Lemma follows readily. $\square$

We continue by the following Lemma which converts our geometrical definition into a functional one.

Lemma 4.2. *We have*

$$V(g, \eta) = \min_{\substack{0\leq h\leq 1,\\ \|h(u)\|_1\geq\eta}} \int_0^{2\pi} h(u)|g(u)|du/\eta$$

where the minimum is taken over all mesurable functions h satisfying the stated conditions.

Proof. Let us momentarily call W the RHS above. We have $V(g,\eta) \le W$ since the set of possible h contains all the characteristic functions of measurable subsets of cardinality at least η. Reciprocally, let h be as in the definition of W, and let A be its support. We have

$$\eta \le \int_0^{2\pi} h(u)du = \int_A h(u)du \le \mu(A)$$

while $\int_0^{2\pi} h(u)|g(u)|du \le \int_A |g(u)|du$. This proves the reverse inequality. □

We continue with a more combinatorial definition.

Lemma 4.3. *We have*

$$\eta V(g,\eta) = \lim_{N\to\infty} \min_{\substack{\mathcal{N}\subset\{0,\cdots,N-1\},\\ |\mathcal{N}|\ge \eta\cdot N/(2\pi)}} \sum_{n\in\mathcal{N}} \int_{2\pi n/N}^{2\pi(n+1)/N} |g(u)|du$$

We can also restrict the above limit over N to any sequence that goes to infinity.

Proof. Let us momentarily call W the RHS above. We have $V(g,\eta) \ge W$ since the set

$$A = \bigcup_{n\in\mathcal{N}} \left[\frac{2\pi n}{N}, \frac{2\pi(n+1)}{N}\right]$$

is of measure $\ge \eta$. Reciproqually, let us select a positive ε. Let A be a measurable subset of measure at least η. Since the Lebesgue measure is regular, we can find a set

$$A_1 = \bigcup_{1\le \ell\le L}]\alpha_\ell, \beta_\ell[\supset A$$

and with $\mu(A_1) \le \mu(A) + \varepsilon$. Given a positive integer N, we can find a subset $\mathcal{N}$ of $\{0,\cdots,N-1\}$ such that

$$A_2 = \bigcup_{n\in\mathcal{N}} \left[\frac{2\pi n}{N}, \frac{2\pi(n+1)}{N}\right] \supset A_1$$

with $\mu(A_2) \le \mu(A_1) + 2L\frac{2\pi}{N}$. The conclusion is straighforward. □

We now prove the following monotonicity property.

Lemma 4.4. *When $0 < \eta' < \eta$, we have $V(g,\eta) \ge V(g,\eta')$.*

Proof. We simply rewrite Lemma 4.2 in the form

$$V(g,\eta) = \min_{\substack{0\le h\le 1/\eta,\\ \|h(u)\|_1\ge 1}} \int_0^{2\pi} h(u)|g(u)|du.$$

The variations as a function of η are clear on this expression. This elegant proof is due to J. Kaczorowski. □

4.0.1 The special case of step-functions

We now consider the special case of non-negative step functions:

$$g(u) = \sum_{0\le n\le N-1} G(n)\mathbb{1}_{n<uN/(2\pi)\le n+1}.$$

We have

$$\int_A g(u)du = \sum_{0\le n\le N-1} G(n)\sigma^*(n)$$

with

$$\sigma^*(n) = \int_{A\cap[2\pi n/N, 2\pi(n+1)/N]} du.$$

We have $\sum_n \sigma^*(n) \ge \eta$ and $\sigma^*(n) \le 2\pi/N$. On elaborating on this we find that $V(g,\eta)$ is the minimum of

$$\sum_{0\le n\le N-1} G(n)x_n$$

under the two conditions $\sum_n x_n \ge \eta$ and $0 \le x_n \le 2\pi/N$

5 A partition of identity

We start with an integer $C \ge 1$ and the sequel of inequalities

$$\begin{array}{rcl}
\mathbb{1}_{|u-t|\le 1/2} = 1 & \iff & u-\frac{1}{2} \le t \le u+\frac{1}{2} \\
\mathbb{1}_{|u-t-1|\le 1/2} = 1 & \iff & u-\frac{3}{2} \le t \le u-\frac{1}{2} \\
\vdots & \vdots & \vdots \\
\mathbb{1}_{|u-t-C|\le 1} = 1 & \iff & u-\frac{1}{2}-C \le t \le u+\frac{1}{2}-C
\end{array}$$

to obtain that

$$\mathbb{1}_{|u-t|\le 1/2}+\mathbb{1}_{|u-t-1|\le 1/2}+\cdots+\mathbb{1}_{|u-t-C|\le 1/2} = \begin{cases} 1 & \text{when } -\frac{1}{2}-C \le t-u \le \frac{1}{2}, \\ 0 & \text{else.} \end{cases}$$

Let f be any element from $\mathrm{L}^1([-1,1])$. We notice that

$$\int_{-1}^{1} \big(\mathbb{1}_{|u-t|\le 1/2}+\mathbb{1}_{|u-t-1|\le 1/2}+\cdots+\mathbb{1}_{|u-t-C|\le 1/2}\big) f(t)dt$$
$$= \begin{cases} \int_{-1}^{1} f(t)dt & \text{when } \frac{1}{2} \le u \le C-\frac{1}{2}, \\ 0 & \text{when } u \le -1/2 \text{ or } u \ge C+\frac{1}{2}. \end{cases}$$

Proof. Indeed the integral we consider is

$$\int_{-1}^{1} \mathbb{1}_{u-\frac{1}{2}-C\le t\le u+\frac{1}{2}} f(t)dt.$$

We can guarantee that this integral equals $\int_{-1}^{1} f(t)dt$ when

$$u - \tfrac{1}{2} - C \leq -1, \quad 1 \leq u + \tfrac{1}{2}.$$

We proceed in a similar fashion to guarantee that the initial integral vanishes. $\square$

Thus, and assuming $\int_{-1}^{1} f(t)dt = 1$, the sequence (in $c \in \{0, \cdots, C\}$) of functions

$$\rho_c(u) = \int_{-1}^{1} \mathbb{1}_{|u-t-c|\leq 1/2} f(t)dt \quad (= \rho_0(u-c)) \tag{7}$$

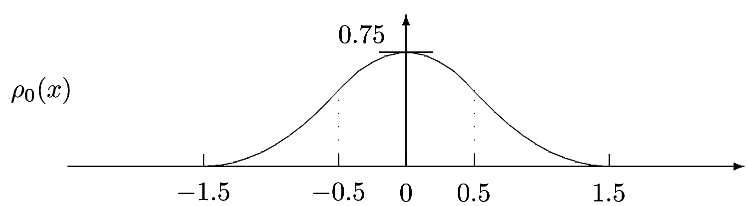
(its support lies within $[c - \frac{3}{2}, c + \frac{3}{2}]$) enables us to represent the characteristic function of $[0, C]$ as a sum of smooth functions having a support of length 3, upto a bounded function with support contained in the union of the intervals $[-3/2, 0]$ and between $[C, C + \frac{3}{2}]$. Furthermore, as it is a convolution product, the Fourier transform of each part is easily computed, once we know the one of $f(t)$. In the sequel we select $f(t) = (1 - |t|)^+$, so that

$$\hat{\rho}_c(v) = e(cv)\left(\frac{\sin \pi v}{\pi v}\right)^3 \tag{8}$$

while we have specifically

$$\rho_0(u) = \begin{cases} 0 & \text{when } |u| \geq 3/2, \\ (3 - 2|u|)^2/8 & \text{when } 1/2 < |u| \leq 3/2, \\ \frac{3}{4} - u^2 & \text{when } |u| \leq 1/2. \end{cases}$$

Here is a plot of this function:



We shall make use of a rescaled version of the family $(\rho_c)_c$, namely (with $\delta_B = 2\pi/B$):

$$w_{c,B}(u) = \rho_c\big(\delta_B^{-1} u\big) \quad \left(\hat{w}_{c,B}(v) = e(c\delta_B v)\delta_B \left(\frac{\sin \pi\delta_B v}{\pi\delta_B v}\right)^3\right) \tag{9}$$

which has its support within $\big[\delta_B(c - \frac{3}{2}), \delta_B(c + \frac{3}{2})\big]$. Let us note that the dependencies in c and B are (rather) well separated in $\hat{w}_{c,B}(v)$.

As a consequence of the above analysis, we get

$$\mathbb{1}_{\gamma \operatorname{Log} P \le x \le \gamma \operatorname{Log}[(1+\xi)P]} \ge \sum_{C_1 \le c \le C_2} w_{c,B}(x) \tag{10}$$

with

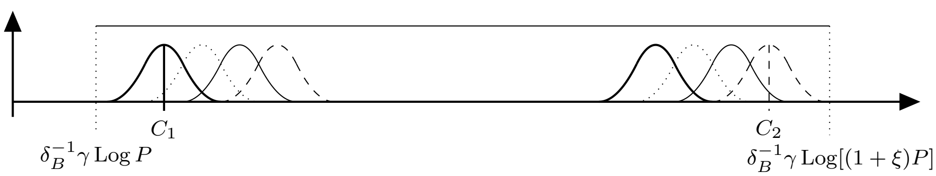
$$C_1 = \left\lceil \delta_B^{-1} \gamma \operatorname{Log} P + \tfrac{3}{2} \right\rceil, \quad C_2 = \left\lfloor \delta_B^{-1} \gamma \operatorname{Log}[(1+\xi)P] - \tfrac{3}{2} \right\rfloor. \tag{11}$$

Our aim is to find a lower bound for

$$S(g;\gamma) = \sum_{P < p \le (1+\xi)\cdot P} g(\gamma \operatorname{Log} p) \tag{12}$$

and we replace this function here by a smoothed version on using (10):

$$S(g;\gamma) \ge S^-(g;\gamma) = \sum_p \sum_{C_1 \le c \le C_2} w_{c,B}(\gamma \operatorname{Log} P) g(\gamma \operatorname{Log} p). \tag{13}$$



(See (62)). The function g is periodical modulo 2π and we want to use incorporate (some of) this fundamental property in our smoothing. Note that

$$w_{c+B,B}(u) = \rho_{c+B}\big(\delta_B^{-1} u\big) = \rho_c\big(\delta_B^{-1}(u - B)\big) = w_{c,B}(u - 2\pi)$$

which leads us to consider, for b ranging $\{0, \cdots, B-1\}$, the family of functions defined by

$$W(x; b, B, J_1(b), J_2(b)) = \sum_{J_1 \le j \le J_2} w_{b+jB,B}(x). \tag{14}$$

The parameter $J_1(b)$ is the smallest integer such that $b + J_1(b)B \ge C_1$, while $J_2(b)$ is the largest integer such that $b + J_2(b)B \le C_2$. The required inequalities on $J_1(b)$ and $J_2(b)$ have been gathered in section 7. We note here that the function $H_{b,B}(y)$ is only a short form defined in (16) for $W(\gamma \operatorname{Log} y; b, B, J_1(b), J_2(b))$ when working with fixed parameters γ, $J_1(b)$ and $J_2(b)$.

$$S^-_{b,B}(g;\gamma) = \sum_{P < p \le (1+\xi)\cdot P} W(\gamma \operatorname{Log} p; b, B, J_1(b), J_2(b))\, g(\gamma \operatorname{Log} p) \tag{15}$$

6 A selector function

We consider here the function $W(x; b, B, J_1, J_2)$ defined in (14) where the integer parameters $J_1 \le J_2$ can be kept arbitrary but, again, read section 7 to see how they are related to the datas of the initial problem. Our object here is

$$H_{b,B}(y) = W(\gamma \operatorname{Log} y; b, B, J_1, J_2). \tag{16}$$

We compute its Mellin transform:

$$\begin{aligned}\mathscr{M}\,H_{b,B}(s) &= \int_0^\infty H_{b,B}(y)y^{s-1}ds = \int_{-\infty}^\infty W(\gamma x; b, B, J_1, J_2)e^{sx}dx \\ &= \sum_{J_1\le j\le J_2}\int_{-\infty}^\infty w_{b+jB,B}(t)e^{ts/\gamma}dt/\gamma.\end{aligned}$$

We introduce Fourier transforms and find that

$$\begin{aligned}\mathscr{M}\,H_{b,B}(s) &= \sum_{J_1\le j\le J_2}\hat{w}_{b+jB,B}(s/(2i\gamma\pi))/\gamma \\ &= \sum_{J_1\le j\le J_2} e\Big(\delta_B\frac{(b+jB)s}{2i\pi\gamma}\Big)\hat{w}_{0,B}(s/(2i\gamma\pi))/\gamma \\ &= \frac{e^{2\pi(J_2+1)s/\gamma}-e^{2\pi J_1 s/\gamma}}{e^{2\pi s/\gamma}-1}e^{2\pi bs/(B\gamma)}\hat{w}_{0,B}(s/(2i\gamma\pi))/\gamma \qquad (17)\end{aligned}$$

Here is a Lemma that enables us to control the size of $\mathscr{M}\,H_{b,B}$.

Lemma 6.1. *When $\sigma = 1/2$, and provided $B, \gamma \ge 10$, we have*

$$\big|\mathscr{M}\,H_{b,B}(s)\big| \ll e^{\pi J_2/\gamma}\begin{cases}\delta_B/\sqrt{1+t^2} & \text{when } |t|\le\gamma/2, \\ \frac{\delta_B^{-2}\gamma^3}{(1+|t|)^3\sqrt{1+\gamma^2\sin^2(\pi t/\gamma)}} & \text{else.}\end{cases}$$

Proof. We start from (9) and get immediately

$$\hat{w}_{b,B}(v) = e(b\delta_B v)\delta_B\left(\frac{\sin\pi\delta_B v}{\pi\delta_B v}\right)^3.$$

When $|t|\le\gamma/2$, we bound this part by $\mathcal{O}(1)$ and use (17) to conclude. When $|t|\ge\gamma/2$, we recall the simple identity

$$|\sin(-i\sigma+\tau)|^2 = \operatorname{sh}^2\sigma + \sin^2\tau \qquad (18)$$

which implies that $\sin\pi\delta_B v$ is bounded when $v = s/(2i\gamma\pi)$ and s is on the line $\Re s = 1/2$. Hence

$$\left|\hat{w}_{0,B}\Big(\frac{s}{2i\gamma\pi}\Big)\right| \ll \frac{\delta_B^{-2}\gamma^3}{(1+t^2)^{3/2}}.$$

Then we write

$$\frac{e^{2\pi(J_2+1)s/\gamma}-e^{2\pi J_1 s/\gamma}}{e^{2\pi s/\gamma}-1} = \frac{e^{\pi(J_2+J_1+1)s/\gamma}}{e^{\pi s/\gamma}}\frac{\operatorname{sh}(\pi(J_2+1-J_1)s/\gamma)}{\operatorname{sh}(\pi s/\gamma)}$$

and, on using again (18) with $\sigma = 1/2$, we obtain

$$\begin{aligned}\left|\frac{e^{2\pi(J_2+1)s/\gamma}-e^{2\pi J_1 s/\gamma}}{e^{2\pi s/\gamma}-1}\right| &= \frac{e^{\frac{\pi(J_2+J_1+1)}{2\gamma}}}{e^{\pi/(2\gamma)}}\sqrt{\frac{\operatorname{sh}^2(\pi\frac{J_2-J_1+1}{2\gamma})+\sin^2(\pi\frac{J_2-J_1+1}{\gamma}t)}{\operatorname{sh}^2(\pi/(2\gamma))+\sin^2(\pi t/\gamma)}} \\ &\le e^{\pi(J_2+\frac12)/\gamma}\sqrt{\frac{2\gamma^2}{\pi^2+4\gamma^2\sin^2(\pi t/\gamma)}}.\end{aligned}$$

The Lemma follows readily. ∎

7 On the choice of J_1 and J_2

We have used integer parameters J_1 and J_2 that do not occur in the initial problem, and we need to relate them to the parameters P and $(1+\xi)P$. Note that they are dependant on the parameter $b \in \{0, \cdots, B-1\}$. We need

$$\delta_B^{-1}\gamma \operatorname{Log} P \le b + J_1 B - \tfrac{3}{2}, \quad b + J_2 B + \tfrac{3}{2} \le \delta_B^{-1}\gamma \operatorname{Log}\big[(1+\xi)P\big] \tag{19}$$

As a matter of fact we will choose J_1 to be the smallest integer verifying the first inequality, and J_2 to be the largest integer verifying the second one. With such a choice, we have

$$e^{2\pi J_2/\gamma} - e^{2\pi J_1/\gamma} \le \xi P e^{2\pi \frac{-b+\frac{3}{2}}{B\gamma}} \tag{20}$$

The bound from below for this difference is $Pe^{2\pi\frac{-b}{B\gamma}}((1+\xi)e^{\frac{-5\pi}{B\gamma}} - e^{\frac{3\pi}{B\gamma}})$ and thus

$$e^{2\pi J_2/\gamma} - e^{2\pi J_1/\gamma} \ge \xi P e^{\frac{-2\pi b}{B\gamma}}\Big(1 + \mathcal{O}\Big(\frac{1}{\xi B\gamma}\Big)\Big). \tag{21}$$

Furthermore

$$e^{2\pi J_2/\gamma} \ge (1+\xi)Pe^{-\frac{4\pi}{\gamma} - \frac{5\pi}{B\gamma}}, \quad Pe^{\frac{5\pi}{\gamma B}} \ge e^{2\pi J_1/\gamma}. \tag{22}$$

We will need a lower bound for the number of primes in $[P_1, P_2]$ where P_1 and P_2 are defined as follows: replace $b+J_1B$ and $b+J_2B$ by c in (19). Say C_1 is the smallest such integer, and C_2 is the largest, and define $P_1 = \exp(2\pi(C_1 + \frac{3}{2})/\gamma)$ and $P_2 = \exp(2\pi(C_2 + \frac{3}{2})/\gamma)$. We use

$$\begin{aligned}
\int_{P_1}^{P_2} \frac{dt}{\operatorname{Log} t} &\ge \frac{\big((1+\xi)e^{\frac{-5\pi}{B\gamma}} - e^{\frac{5\pi}{B\gamma}}\big)P}{\operatorname{Log}((1+\xi)P)} \\
&\ge \Big(1 + \mathcal{O}\Big(\frac{1}{\xi B\gamma}\Big)\Big)\frac{\xi P}{\operatorname{Log}((1+\xi)P)} \\
&\ge \Big(1 + \mathcal{O}\Big(\frac{\xi}{\operatorname{Log} P} + \frac{1}{\xi B\gamma}\Big)\Big)\frac{\xi P}{\operatorname{Log} P}.
\end{aligned}$$

Consequently we have

$$\frac{\pi(P_2) - \pi(P_1)}{\xi P/\operatorname{Log} P} \ge 1 + \mathcal{O}\Big(\frac{\xi}{\operatorname{Log} P} + \frac{1}{\xi B\gamma} + \frac{1}{\xi \exp\sqrt{\operatorname{Log} P}}\Big). \tag{23}$$

Assuming $\xi \gg 1/(\operatorname{Log} P)^2$, and anticipating on the choice $B = \rfloor(\operatorname{Log} P)^4\lfloor$ (the largest integer $\le (\operatorname{Log} P)^4$), this implies that

$$\frac{\pi(P_2) - \pi(P_1)}{\xi P/\operatorname{Log} P} \ge 1 + \mathcal{O}\Big(\frac{\xi}{\operatorname{Log} P} + \frac{1}{\xi(\operatorname{Log} P)^4}\Big). \tag{24}$$

8 More on our selector

Let us assume the parameters J_1 and J_2 are selected as in section 7. We use the more explicit notation

$$\mathscr{M} H_{b,B}(s; P, (1+\xi)P, \gamma) = \mathscr{M} H_{b,B}(s). \tag{25}$$

When $s = \frac{1}{2} + it$ with small enough t in terms of γ, this function can be approximated independently of γ, as shown in the next Lemma.

Lemma 8.1. *When $s = \frac{1}{2} + it$, with $|t| \ll \gamma$, we have*

$$\mathscr{M} H_{b,B}(s; P, (1+\xi)P, \gamma) = \delta_B \frac{[(1+\xi)P]^s - P^s}{2\pi s} + \mathcal{O}(\delta_B \sqrt{P}/\gamma).$$

The constant in the $\mathcal{O}$-symbol depends only on the upper bound for $|t|/\gamma$.

Proof. We start from (17). First, we have

$$J_2 = \frac{\gamma \operatorname{Log}[(1+\xi)P]}{2\pi} + \mathcal{O}(1)$$

and thus

$$2\pi(J_2+1)s/\gamma = s \operatorname{Log}[(1+\xi)P] + \mathcal{O}(|s|/\gamma)$$

and similarly for J_1; this finally gives us

$$e^{2\pi(J_2+1)s/\gamma} - e^{2\pi J_1 s/\gamma} = [(1+\xi)P]^s - P^s + \mathcal{O}\Big(\frac{\sqrt{P}|s|}{\gamma}\Big).$$

Then, provided $|s| \ll \gamma$, we have

$$\frac{1/\gamma}{e^{2\pi s/\gamma} - 1} = \frac{1}{2\pi s} + \mathcal{O}(|s|/\gamma)$$

and

$$e^{2\pi b s/(B\gamma)} = 1 + \mathcal{O}(|s|/\gamma).$$

Finally

$$\delta_B^{-1} \hat{w}_{0,B}(s/(2i\gamma\pi)) = 1 + \mathcal{O}(|s|/\gamma).$$

The Lemma follows readily from these estimates. $\square$

9 Auxiliaries on a function from sieve theory

We define

$$M(s_0, \delta) = \sum_{m \le z/\delta} \frac{\mu(m\delta) \operatorname{Log} \frac{z}{m\delta}}{(\delta m)^{s_0}}. \tag{26}$$

We start by recalling the first part of [8, Lemma 2].

Lemma 9.1. *We have, when $\delta \geq 1$ and for any $A \geq 1$,*

$$M(1,\delta) = \frac{\mu(\delta)}{\phi(\delta)\operatorname{Log} z} + \mathcal{O}_A\bigl(\sigma_{-1/2}(\delta)\operatorname{Log}(2z)^{-A}/\delta\bigr)$$

where, when δ is squarefree,

$$\sigma_{-1/2}(\delta) = \prod_{p|r}(1+p^{-1/2}).$$

Lemma 9.2. *We have, for any $A \geq 1$ and any $z \geq 2$,*

$$\sum_{d_1,d_2} \frac{\lambda_{d_1}\lambda_{d_2}}{[d_1,d_2]} = \frac{1}{\operatorname{Log} z}\Bigl(1 + \frac{c_0}{\operatorname{Log} z} + \mathcal{O}_A\bigl(1/(\operatorname{Log} z)^A\bigr)\Bigr)$$

where λ_d is defined in (5)*, and*

$$c_0 = \gamma + \sum_{p\geq 2} \frac{\operatorname{Log} p}{p(p-1)} = 1.332\,582\ldots \tag{27}$$

Comparing this result with [5], the reader will notice that the factor $\operatorname{Log}(z/d)/\operatorname{Log} z$ by which we multiply $\mu(d)$ to get λ_d has a critical effect. See also [20] and [3]. This fact ought to be emphasized as there is a popular belief that the smoothing is only here to render this quantity amenable to computations. See also [4], [8] for a possible modification of this smoothing factor and its effect on the result.

Proof. Let us denote by S the sum to be evaluated. We have classically (see (30))

$$S\operatorname{Log}^2 z = \sum_{\delta\leq z} \mu^2(\delta)\phi(\delta)M(1,\delta)^2.$$

Routine manipulations give us, on using Lemma 9.1, that

$$S = \sum_{\delta\leq z} \frac{\mu^2(\delta)}{\phi(\delta)\operatorname{Log}^2 z} + \mathcal{O}_A\bigl((\operatorname{Log} z)^{-A}\bigr)$$

for any $A \geq 1$. We combine this with the classical evaluation of the summation over δ which is for instance contained in [21, Lemma 3.4]. The Lemma follows readily. □

Lemma 9.3. *When $x > 0$, we have*

$$\frac{1}{2i\pi}\int_{2-i\infty}^{2+i\infty} \frac{x^{-s}ds}{s(s+1)} = \begin{cases} 1-x, & \text{when } 0 < x \leq 1,\\ 0, & \text{when } 1 \leq x.\end{cases}$$

Proof. Indeed, when $x \geq 1$, shift the line of integration to the far right. When $x < 1$, shift the line to the far left. The polar contribution in this latter case amounts to 1 in 0, and to $-x$ in -1. The Lemma follows readily. □

This gives us, when $\Re s_0 \geq 0$,

$$M(s_0, \delta) = \frac{1}{2i\pi} \int_{2-\infty}^{2+i\infty} \sum_{\substack{d \geq 1, \\ (d,\delta)=1}} \frac{\mu(d)\mu(\delta)}{(d\delta)^{s_0+s}} \frac{z^s ds}{s^2}$$

which easily gets rewritten in the form

$$M(s_0, \delta) = \frac{\mu(\delta)}{2i\pi} \int_{2-\infty}^{2+i\infty} \prod_{p|\delta} (p^{s_0+s} - 1)^{-1} \frac{z^s ds}{s^2 \zeta(s+s_0)}. \tag{28}$$

We now use this formula for $s_0 = \frac{1}{2} + it$.

Lemma 9.4. *We have, when $\delta \leq z$,*

$$M(\tfrac{1}{2} + it, \delta) \ll \frac{\sqrt{z}}{\phi(\delta)}.$$

Proof. Push the line of integration to $\Re s = 1/2$. □

Lemma 9.5. *Under the Riemann Hypothesis, we have, when $\delta \leq z$ and for any $\varepsilon > 0$,*

$$\sqrt{\delta} M(\tfrac{1}{2} + it, \delta) \ll_\varepsilon [(1 + |t|)z]^\varepsilon \delta^{-\varepsilon/2}.$$

Proof. Push the line of integration to $\Re s = \varepsilon > 0$. We use [24, Theorem 14.14 (B)] to bound $1/\zeta(s + \frac{1}{2} + it_0)$ by $[(1 + |t|)(1 + |t_0|)]^\varepsilon$. □

Lemma 9.6. *We define M_2, when s is a complex number, by*

$$M_2(s) = \sum_{d_1, d_2 \leq z} \frac{\lambda_{d_1} \lambda_{d_2}}{[d_1, d_2]^s}. \tag{29}$$

We have $|M_2(1/2 + it)| \ll z/(\operatorname{Log} z)^2$.

Proof. We use the Selberg diagonalisation process to infer

$$M_2(s) = \sum_{d_1, d_2 \leq z} (d_1, d_2)^s \frac{\lambda_{d_1} \lambda_{d_2}}{d_1^s d_2^s} = \sum_{\delta \leq z} \frac{\phi_s(\delta)}{\delta^{2s}} \Bigg(\sum_{\ell \leq z/\delta} \frac{\lambda_{\delta\ell}}{\ell^s} \Bigg)^2 \tag{30}$$

with

$$\phi_s(\delta) = \prod_{p^\alpha \| \delta} (p^{as} - p^{(a-1)s}). \tag{31}$$

The Lemma follows readily from Lemma 9.4. □

Lemma 9.7. *Under the Riemann Hypothesis and on recalling (29), we have, for any $\varepsilon > 0$,*

$$|M_2(1/2 + it)| \ll_\varepsilon [(1 + |t|)z]^\varepsilon \sqrt{z}.$$

Proof. Same proof as the one of Lemma 9.6 but employing Lemma 9.5 instead of Lemma 9.4 (and change 3ε by ε). $\square$

On bounding simply $|\lambda_d|$ by 1, we reach easily:

Lemma 9.8. *We have, when $z \geq 2$*

$$\sum_n \Big(\sum_{[d_1,d_2]=n} \lambda_{d_1}\lambda_{d_2} \Big)^2 \ll z^2 (\operatorname{Log} z)^8$$

and

$$\sum_n \Big(\sum_{[d_1,d_2]=n} \lambda_{d_1}\lambda_{d_2} \Big)^2 / n \leq (\operatorname{Log} z)^9.$$

We recall [18, Corollary 3] :

Lemma 9.9. *When $\sum_{n\geq 1} |a_n| + \sum_{n\geq 1} n|a_n|^2 < \infty$, we have*

$$\int_0^T \Big| \sum_{n\geq 1} a_n n^{-it} \Big|^2 dt = \sum_{n\geq 1} |a_n|^2 \big(T + \mathcal{O}(n)\big)$$

This Lemma yields readily the next one.

Lemma 9.10. *We have, for $z \geq 2$,*

$$\int_0^T |M_2(\tfrac{1}{2} + it)|^2 dt \leq (z^2 + T)(\operatorname{Log} z)^9.$$

10 A refined estimates for the Barban & Vehov weights

The result of this section is the following Theorem:

Theorem 10.1. *We have*

$$\sum_{n\leq N} \Big(\sum_{d|n} \lambda_d \Big)^2 = \frac{N}{\operatorname{Log}^2 z} \bigg(\operatorname{Log} z + c_0 + \mathcal{O}\Big(\exp\big(-\sqrt{\operatorname{Log}\min(z, N/z)}\big)\Big)\bigg)$$

for any $N \geq z \geq 10$ and where c_0 is defined in (27)*.*

With the error term $\mathcal{O}(N/\operatorname{Log}^2 z)$, this is the main result of [8]. Graham's proof relies on a switching to the complementary divisor that is absent from the proof we present below. The remainder of this section is devoted to a proof of this Theorem.

We consider the function (decreasing when $t \geq 0$)

$$\varphi(t) = \frac{1}{100(\operatorname{Log}(|t|+10))^{2/3}(\operatorname{Log}\operatorname{Log}(|t|+10))^{1/3}} \tag{32}$$

so that we have, by [6] (see also Lemma 19.1 below), we have

$$|\zeta(\sigma + it)|^{\pm 1} \ll \operatorname{Log}(2 + |t|) \quad \text{when } \sigma \geq 1 - \varphi(t). \tag{33}$$

Notation:

We denote by $\mathfrak{L}(\delta, c)$ the line of the points $\sigma + it$ that are such that

$$\mathfrak{L}(\delta, c): \quad \sigma = \delta + c\varphi(t). \tag{34}$$

Here is another classical Lemma whose proof goes along the same lines as the proof of Lemma 9.3.

Lemma 10.2. *When $k \geq 1$ is an integer and $x > 0$, we have*

$$\frac{1}{2i\pi}\int_{2-i\infty}^{2+i\infty}\frac{x^{-s}ds}{s^{k+1}} = \begin{cases}(-\operatorname{Log} x)^k/k!, & \textit{when } 0 < x \leq 1,\\ 0, & \textit{when } 1 \leq x.\end{cases}$$

We recall that, with the help of definition (31), equation (28) reads, for squarefree δ,

$$M(s,\delta) = \frac{\mu(\delta)}{2i\pi}\int_{2-i\infty}^{2+i\infty}\frac{z^{s_1}ds_1}{s_1^2\zeta(s+s_1)\phi_{s+s_1}(\delta)}. \tag{35}$$

Recall also that, by (30), we have

$$M_2(s) = \sum_{\delta \leq z}\phi_s(\delta)M(s,\delta)^2/\operatorname{Log}^2 z. \tag{36}$$

Handling the sieve coefficient, step 2

We introduce yet another Dirichlet series, namely

$$D(s, s_1, s_2) = \prod_{p\geq 2}\left(1 - \frac{p^{s_1+s_2} - p^{s_1} - p^{s_2} + 1}{(p^{s+s_1}-1)(p^{s+s_2}-1)p^{s_1+s_2}}\right) \tag{37}$$

which is absolutely convergent as a product in the region defined by

$$\Re(s+s_1), \Re(s+s_2) > 0,\ \Re(2s+s_1+s_2), \Re(2s+2s_1+2s_2) > 1. \tag{$\mathcal{R}$}$$

It and its inverse are moreover bounded in any region defined as above but where, in the last two joinded inequalities, the "> 1" is replaced by "$\geq 1+\delta$" for some positive δ. This function appears via

$$\sum_{\delta\geq 1}\frac{\mu^2(\delta)\phi_s(\delta)}{\phi_{s+s_1}(\delta)\phi_{s+s_2}(\delta)} = \zeta(s+s_1+s_2)D(s,s_1,s_2). \tag{38}$$

10.1 Proof of a smoothed version of Theorem 10.1

We assume, without any loss of generality that $N \geq z \geq 2$ and study

$$S(N;z) = \sum_{n\geq 1}\Big(\sum_{d|n}\lambda_d \operatorname{Log} z\Big)^2\Big(1 - \frac{n}{N}\Big)^+.$$

We use Lemma 9.3 to get the expression:

$$S(N;z) = \frac{1}{2i\pi}\int_{2-i\infty}^{2+i\infty} \zeta(s) M_2(s) \frac{N^s ds}{s(s+1)}. \tag{39}$$

We continue with (36) and (26), recall (35) and (38) to get

$$\begin{aligned} S(N;z) = \frac{1}{(2i\pi)^3}\int_{2-i\infty}^{2+i\infty}\int_{1-i\infty}^{1+i\infty}\int_{1-i\infty}^{1+i\infty} \\ D(s,s_1,s_2)\frac{\zeta(s)\zeta(s+s_1+s_2)}{\zeta(s+s_1)\zeta(s+s_2)}\frac{z^{s_1}ds_1}{s_1^2}\frac{z^{s_2}ds_2}{s_2^2}\frac{N^s ds}{s(s+1)}. \end{aligned} \tag{40}$$

We need some notation to be able to handle such an expression. We first define the function

$$\mathfrak{F}(s,s_1,s_2) = D(s,s_1,s_2)\frac{\zeta(s)\zeta(s+s_1+s_2)}{\zeta(s+s_1)\zeta(s+s_2)}\frac{z^{s_1+s_2}}{s_1^2 s_2^2}\frac{N^s}{s(s+1)}. \tag{41}$$

We then write (40) as

$$S(N;z) = \frac{1}{(2i\pi)^3}\int_{(2)} ds \int_{(1)} ds_1 \int_{(1)} ds_2 \ \mathfrak{F}(s,s_1,s_2). \tag{42}$$

First step

On shifting the lines of integration in s, we meet the first pole at $s=1$, and, checking that we stay within $(\mathcal{R})$, we find that

$$S(N;z) = S_0 + S^{(0)}$$

where S_0 and $S^{(0)}$ are being defined by

$$S^{(0)} = \frac{1}{(2i\pi)^3}\int_{\mathfrak{L}(1,-1)} ds \int_{(1)} ds_1 \int_{(1)} ds_2 \ \mathfrak{F}(s,s_1,s_2) \tag{43}$$

and

$$S_0 = \frac{N}{2}\frac{1}{(2i\pi)^2}\int_{(1)} ds_1 \int_{(1)} ds_2 \ \mathfrak{F}^*(s_1,s_2) \tag{44}$$

with the additional notation

$$\mathfrak{F}^*(s_1,s_2) = D(1,s_1,s_2)\frac{\zeta(1+s_1+s_2)}{\zeta(1+s_1)\zeta(1+s_2)}\frac{z^{s_1+s_2}}{s_1^2 s_2^2}. \tag{45}$$

Treatment of S_0

We shift the line of integration in s_1 and get

$$\begin{aligned} \frac{2}{N}S_0 &= \frac{1}{2i\pi}\int_{(1)} ds_2 D(1,0,s_2)\frac{z^{s_2}}{s_2^2} + \frac{1}{(2i\pi)^2}\int_{\mathfrak{L}(0,-1/3)} ds_1 \int_{(1)} ds_2 \ \mathfrak{F}^*(s_1,s_2) \\ &= \operatorname{Log} z + S^{(2)} \end{aligned}$$

say, since $D(1,0,s_2) = 1$, while the remaining integral is computed in Lemma 10.2. As for $S^{(2)}$, we shift the line of integration in s_2 to $\mathfrak{L}(0,1/6)$, meeting a pole at $s_2 = -s_1$. Concerning the remainder part, it is routine to show it is

$$\ll \exp\bigl(-2\sqrt{\operatorname{Log} z}\bigr)$$

for some positive constant c'. Let us explain rapidly how this is achieved: we select a large positive T_0. The contribution of the s_1 that are such that $|\Im s_1| \geq T_0$ is $\ll (\operatorname{Log} T_0)^2/T_0$. A same contribution comes from the s_2 that are such that $|\Im s_2| \geq T_0$. However, when both imaginary parts are smaller than T_0 in absolute value, we can save $z^{-\varphi(T_0)/6}$ and the remaining contribution is bounded. We choose $\operatorname{Log} T_0 = (\operatorname{Log} z)^{\frac{3}{5}-0.001}$. As a conclusion

$$S_0 = \frac{N}{2}(\operatorname{Log} z + c_0) + \mathcal{O}\Bigl(N \exp\bigl(-2\sqrt{\operatorname{Log} z}\bigr)\Bigr)$$

for some constant c_0. This constant is irrelevant to the main argument, but it is interesting to get its value. To do so, we can assume that z is small with respect to N, and so simply opening the square in the summation to be computed is enough. The computation of the appearing main term is then achieved by appealing to Lemma 9.2.

Second step: splitting $S^{(0)}$

On shifting the lines of integration, we find that

$$S^{(0)} = S_1 + S^{(1)}$$

with

$$S^{(1)} = \frac{1}{(2i\pi)^3}\int_{\mathfrak{L}(1,-1)} ds \int_{\mathfrak{L}(0,1/3)} ds_1 \int_{\mathfrak{L}(0,1/3)} ds_2 \ \mathfrak{F}(s,s_1,s_2) \tag{46}$$

and

$$S_1 = \frac{1}{(2i\pi)^2}\int_{\mathfrak{L}(1,-1)} ds \int_{\mathfrak{L}(0,1/3)} ds_2 \ \mathfrak{F}^{**}(s,s_2), \tag{47}$$

the integrand $\mathfrak{F}^{**}(s,s_2)$ being

$$N\, D(s, 1-s-s_2, s_2)\frac{\zeta(s)}{\zeta(1-s_2)\zeta(1+s_2)}\,\frac{(N/z)^{s-1}}{(1-s-s_2)^2 s_2^2 s(s+1)}. \tag{48}$$

The paths have been chosen so as not to pass through the poles at $s_1 = 0$ or $s_2 = 0$, use the pole at $s + s_1 + s_2 = 1$. It is again routine to show it is

$$\ll \exp\bigl(-2\sqrt{\operatorname{Log}(N/z)}\bigr)$$

with the conclusion that

$$S_1 \ll N \exp\bigl(-2\sqrt{\operatorname{Log}(N/z)}\bigr).$$

Concerning $S^{(1)}$, we simply shift the line of integration in s_1 and s_2 to the lines $\Re s_1 = 0$ and $\Re s_2 = 0$ on avoiding the vicinity of $s_1 = 0$ and $s_2 = 0$. The saving in N is largely enough.

10.2 Proof of Theorem 10.1: removing the smoothing

We remove the smoothing with the simple identity

$$\mathbb{1}_{n\le N} = \frac{(N+L-n)^+ - (N-n)^+}{L} - \mathbb{1}_{N<n\le N+L}\frac{N+L-n}{L}.$$

with

$$L = N/\exp\sqrt{\operatorname{Log}\min(z, N/z)}. \tag{49}$$

The Theorem follows readily.

11 Moment estimates

In this section, we consider bounds for

$$F_0(T,z) = \int_T^{2T} \left|\zeta(\tfrac12+iu)M_2(\tfrac12+iu)\right|du. \tag{50}$$

Lemma 11.1. *We have, for $z\ge 2$ and with $s=\frac12+it$*

$$F_0(T,z) \ll (T+\sqrt{T}z)\sqrt{\operatorname{Log} T\operatorname{Log}^9 z}.$$

Proof. We use the L^2-estimate for M_2 from Lemma 9.10 and the L^2-moment for ζ from [24, Theorem 7.4], via Cauchy's inequality. □

Lemma 11.2. *We have, for $z\ge 2$ and with $s=\frac12+it$*

$$F_0(T,z) \ll (Tz^{5/8}+T^{1/4}z)(\operatorname{Log} z)^{-2/3}\operatorname{Log} T.$$

Proof. We use

$$F_0(T,z) \ll \Big(\int_T^{2T}\left|\zeta(\tfrac12+iu)\right|^4 du\Big)^{1/4}\Big(\int_T^{2T}\left|M_2(\tfrac12+iu)\right|^{4/3}du\Big)^{3/4}.$$

The first integral is $\ll T(\operatorname{Log} T)^4$ on using the bound for the fourth power moment of ζ, see [24, (7.6.2)]. In the second one, we bound $|M_2(\frac12+iu)|^{1/3}$ by $z^{1/3}(\operatorname{Log} z)^{-2/3}$. For the remaining part, we use (30) to write

$$\begin{aligned}\int_T^{2T}\left|M_2(\tfrac12+iu)\right|du &\ll \sum_{\delta\le z}\frac{\prod_{p|\delta}(\sqrt{p}+1)}{\delta}\int_T^{eT}\Bigg|\sum_{\substack{\ell\le z/\delta,\\ (\ell,\delta)=1}}\frac{\lambda_{\ell\delta}}{\ell^{\frac12+iu}}\Bigg|^2 du\\ &\ll \sum_{\delta\le z}\frac{\prod_{p|\delta}(\sqrt{p}+1)}{\delta}(T\operatorname{Log}(z/\delta)+z/\delta)\\ &\ll T\sqrt{z}+z.\end{aligned}$$

□

12 Treatment of the error term: the large ordinates

In this section we consider the integral

$$I(\gamma) = \int_{\gamma/2}^{\infty} \bigl|\zeta(\tfrac{1}{2}+it)M_2(\tfrac{1}{2}+it)\bigr| \frac{dt}{t^3\sqrt{1+\gamma^2\sin^2(\pi t/\gamma)}}. \tag{51}$$

We define

$$F(t) = \int_{\gamma/2}^{t} \bigl|\zeta(\tfrac{1}{2}+iu)M_2(\tfrac{1}{2}+iu)\bigr| du$$

which is bounded above by $(z\sqrt{t}+t)\sqrt{\operatorname{Log} t}(\operatorname{Log} z)^{9/2}$ by Lemma 11.1. An integration by parts yields

$$I(t) \ll \int_{\gamma/2}^{\infty} \frac{F(t)dt}{t^4\sqrt{1+\gamma^2\sin^2(\pi t/\gamma)}} + \gamma \int_{\gamma/2}^{\infty} \frac{F(t)|\sin(2\pi t/\gamma)|dt}{t^3\bigl(1+\gamma^2\sin^2(\pi t/\gamma)\bigr)^{3/2}}$$

by using $2\cos x \sin x = \sin(2x)$. We next note that, when $T \geq \gamma/2$, we have

$$\begin{aligned}
\int_{T}^{2T} \frac{dt}{\sqrt{1+\gamma^2\sin^2(\pi t/\gamma)}} &\ll \gamma \int_{T/\gamma}^{2T/\gamma} \frac{du}{\sqrt{1+\gamma^2\sin^2(\pi u)}} \\
&\ll T \int_{0}^{1/2} \frac{du}{\sqrt{1+\gamma^2\sin^2(\pi u)}} \\
&\ll T \int_{0}^{1/2} \frac{du}{\sqrt{1+\gamma^2u^2}} \ll \frac{T \operatorname{Log} \gamma}{\gamma}.
\end{aligned} \tag{52}$$

We also have, also when $T \geq \gamma/2$, and on following a similar path

$$\begin{aligned}
\int_{T}^{2T} \frac{|\sin(2\pi t/\gamma)|dt}{\bigl(1+\gamma^2\sin^2(\pi t/\gamma)\bigr)^{3/2}} &\ll \gamma \int_{T/\gamma}^{2T/\gamma} \frac{|\sin(2\pi u)|du}{\sqrt{1+\gamma^2\sin^2(\pi u)}} \\
&\ll T \int_{0}^{1/2} \frac{udu}{(1+\gamma^2u^2)^{3/2}} \ll \frac{T}{\gamma^2}.
\end{aligned}$$

A diadic decomposition with Lemma 11.1 then leads to

$$I(\gamma) \ll \bigl(1+z/\sqrt{\gamma}\bigr)(\operatorname{Log}\gamma)^{3/2}(\operatorname{Log} z)^{9/2}/\gamma^3. \tag{53}$$

On using Lemma 11.2 instead, we get

$$I(\gamma) \ll \bigl(z^{5/8}+z/\gamma^{3/4}\bigr)(\operatorname{Log}\gamma)^2(\operatorname{Log} z)^{-2/3}/\gamma^3. \tag{54}$$

13 Averaging over γ

We somehow anticipate on the sequel and define

$$S^+_{b,B}(1;\gamma) = \sum_{P<n\le(1+\xi)P} H_{b,B}(n)\beta(n). \tag{55}$$

We find that, on using (16), (14), (9) and (7),

$$\begin{aligned} S^+_{b,B}(1;\gamma) &= \sum_{P<n\le(1+\xi)P} \sum_{\substack{k\equiv b[B],\\ b+J_1B\le k\le b+J_2B}} w_{k,B}(\gamma \operatorname{Log} n)\beta(n) \\ &= \sum_{P<n\le(1+\xi)P} \sum_{\substack{k\equiv b[B],\\ b+J_1B\le k\le b+J_2B}} \rho_k\Big(\frac{B}{2\pi}\gamma \operatorname{Log} n\Big)\beta(n) \\ &= \sum_{P<n\le(1+\xi)P} \sum_{\substack{k\equiv b[B],\\ b+J_1B\le k\le b+J_2B}} \rho_0\Big(\frac{B}{2\pi}\gamma \operatorname{Log} n - k\Big)\beta(n). \end{aligned}$$

We now show that the size conditions on k are in fact redundant. Indeed, $\rho_0\Big(\frac{B}{2\pi}\gamma \operatorname{Log} n - k\Big)$ vanishes unless

$$k+\tfrac{3}{2} > \frac{B}{2\pi}\gamma \operatorname{Log} n > k-\tfrac{3}{2}.$$

This implies that $k+\frac{3}{2} > \delta_B^{-1}\gamma \operatorname{Log} P$ and in particular $k\ge b+J_1B$ and a similar reasoning applies to the upper bound. Going back to (7), we find that (for an arbitrary integrable function u that is going to be the characteristic function of an interval)

$$\begin{aligned} \int_0^\infty u(\gamma)\rho_0\Big(\frac{B}{2\pi}\gamma \operatorname{Log} n - k\Big)d\gamma &= \int_{-1}^1 f(t)\int_0^\infty u(\gamma)\mathbb{1}_{\left|\frac{B\gamma \operatorname{Log} n}{2\pi}-k-t\right|\le 1/2} d\gamma dt \\ &= \int_{-1}^1 f(t)\int_{\frac{2\pi}{B \operatorname{Log} n}(k+t-1/2)}^{\frac{2\pi}{B \operatorname{Log} n}(k+t+1/2)} u(\gamma)d\gamma dt \end{aligned}$$

and thus, on writing $k=b+B\ell$,

$$\sum_{k\equiv b[B]}\int_{-1}^1 f(t)\int_{\frac{2\pi}{B \operatorname{Log} n}(k+t-1/2)}^{\frac{2\pi}{B \operatorname{Log} n}(k+t+1/2)} u(\gamma)d\gamma dt = \\ \int_{-1}^1 f(t)\sum_{\ell\in\mathbb{Z}}\int_{\frac{2\pi}{\operatorname{Log} n}(\ell+(t+b-1/2)/B)}^{\frac{2\pi}{\operatorname{Log} n}(\ell+(t+b+1/2)/B)} u(\gamma)d\gamma dt.$$

We take for $u(\gamma)$ to be the characteristic function of the interval $[\Gamma, 2\Gamma]$. The number of ℓ's is

$$\frac{\Gamma}{2\pi/\operatorname{Log} n} + \mathcal{O}(1)$$

and we consider a subinterval of length $2\pi/(B\operatorname{Log} n)$ next to each of these, so the total measure is

$$\frac{\Gamma}{B} + \mathcal{O}(1/(B\operatorname{Log} n)).$$

Since $\int_{-1}^{1} f(t)dt = 1$, we have reached

$$\int_{\Gamma}^{2\Gamma} S^{+}_{b,B}(1;\gamma)d\gamma = \frac{\Gamma + \mathcal{O}(1/\operatorname{Log} P)}{B} \sum_{P<n\le(1+\xi)P} \beta(n)$$

on assuming essentially nothing on z. When $\Gamma \ge (\operatorname{Log} P)^4$, we can find a $\gamma^* \in [\Gamma, 2\Gamma]$ such that

$$S^{+}_{b,B}(1;\gamma^*) \le \frac{1 + \mathcal{O}(1/(\operatorname{Log} P)^5)}{B} \sum_{P<n\le(1+\xi)P} \beta(n).$$

The evaluation of the summation over n is a consequence of Theorem 10.1:

$$\sum_{P<n\le(1+\xi)P} \beta(n) = \Big(1 + \mathcal{O}\Big(\frac{1}{\operatorname{Log} z} + \frac{1}{\xi(\operatorname{Log} z)^9}\Big)\Big)\frac{\xi P}{\operatorname{Log} z} \tag{56}$$

provided $z \le P/\exp((\operatorname{Log}\operatorname{Log} P)^3)$. The term $(\operatorname{Log} z)^9$ will be of no interest.

14 A hybrid Brun-Titchmarsh Theorem

We are seeking an upper bound for $S^{-}_{b,B}(1;\gamma)$ defined in (15). On recalling (55), we find that, when $z \le P$,

$$S^{-}_{b,B}(1;\gamma) = \sum_{p\ge1} H_{b,B}(p) \le \sum_{n\ge1} H_{b,B}(n)\beta(n) = S^{+}_{b,B}(1;\gamma).$$

We expand the square in $\beta(n)$ and use the Mellin transform to get

$$\begin{aligned} S^{+}_{b,B}(1;\gamma) &= \frac{1}{2i\pi}\sum_{d_1,d_2} \lambda_{d_1}\lambda_{d_2} \int_{2-i\infty}^{2+i\infty} \mathscr{M}\, H_{b,B}(s) \sum_{m\ge1} \frac{ds}{[d_1,d_2]^s m^s} \\ &= \frac{1}{2i\pi}\int_{2-i\infty}^{2+i\infty} \mathscr{M}\, H_{b,B}(s) \sum_{d_1,d_2} \frac{\lambda_{d_1}\lambda_{d_2}}{[d_1,d_2]^s}\zeta(s)ds. \end{aligned}$$

We select $\Gamma = \gamma$ in section 13 (see particularly (56)). We recover a γ^* in the interval $[\gamma, 2\gamma]$ such that

$$S^{+}_{b,B}(1;\gamma^*) \le \Big(1 + \mathcal{O}\Big(\frac{1}{(\operatorname{Log} P)^5} + \frac{1}{\operatorname{Log} z}\Big)\Big)\frac{\xi P}{B\operatorname{Log} z}$$

provided $z \ge P^{1/4}$. We shift the line of integration to $\Re s = 1/2$, but we do not evaluate the integration on this line by introducing absolute values.

When $|t| \leq \gamma$, we compare the integral to the one corresponding to γ^* by using Lemma 8.1. This introduces an error bounded above up to a multiplicative constant by

$$\delta_B \frac{\sqrt{P}}{\gamma} \int_0^\gamma \bigl|\zeta(\tfrac{1}{2}+it) M_2(\tfrac{1}{2}+it)\bigr| dt. \tag{57}$$

First treatment

We appeal to Lemma 11.1 and a diadic decomposition to get a contribution of the term (57) which is at most (up to a multiplicative constant)

$$\delta_B \sqrt{P} (\operatorname{Log}\gamma + z/\sqrt{\gamma}) \sqrt{\operatorname{Log}\gamma \operatorname{Log}^9 z}.$$

We shift the line of integration to $\Re s = 1/2$. The integral on the segment $[1/2, 1/2 + i\gamma]$ is controlled by the above while the remaining integral is $\ll \sqrt{P} B^2 \gamma^3 I(\gamma)$ by Lemma 6.1. We appeal to (53) and reach the inequality (since $\xi \geq 1/(\operatorname{Log} P)^2$)

$$\begin{aligned} \frac{S^-_{b,B}(1;\gamma) - S^+_{b,B}(1;\gamma^*)}{\delta_B \xi P / \operatorname{Log} P} \ll \sqrt{P}^{-1} &(1 + z/\sqrt{\gamma})(\operatorname{Log}\gamma)^{3/2} (\operatorname{Log} P)^{15/2} \\ &+ \frac{B^3}{\sqrt{P}} \bigl(1 + z/\sqrt{\gamma}\bigr) (\operatorname{Log}\gamma)^{3/2} (\operatorname{Log} P)^{9/2}. \end{aligned}$$

We select $B = \lfloor (\operatorname{Log} P)^4 \rfloor$. We take, provided $P \geq (\operatorname{Log}\gamma)^4$,

$$z = \min\bigl(P, \sqrt{P\gamma}\bigr) / \exp((\operatorname{Log}\operatorname{Log} P)^3). \tag{58}$$

This choice is optimal when $\gamma \geq P$. We finally get

$$S^-_{b,B}(1;\gamma) \leq \Bigl(1 + \mathcal{O}\Bigl(\frac{1}{(\operatorname{Log} P)^5} + \frac{1}{\operatorname{Log} z}\Bigr)\Bigr) \frac{\xi P}{B \operatorname{Log} z}. \tag{59}$$

Second treatment

If we use Lemma 11.2 instead, we get a contribution from (57):

$$\ll \delta_B \frac{\sqrt{P}}{\gamma} (\gamma z^{5/8} + \gamma^{1/4} z)(\operatorname{Log}\gamma)/(\operatorname{Log} z)^{2/3}.$$

Note that we can assume $\gamma \leq P$ since the previous section already contains an optimal result otherwise. We proceed as above by use (54). This allows us to select

$$z = \min\bigl(P^{4/5}, P^{1/2}\gamma^{3/4}\bigr) / \exp((\operatorname{Log}\operatorname{Log} P)^3) \tag{60}$$

and again reach (59).

Summary

We select $B = \lfloor (\operatorname{Log} P)^2 \rfloor$ and get when $\xi \geq 1/(\operatorname{Log} P)^2$ and provided $P \geq (\operatorname{Log} \gamma)^4$

$$S^-_{b,B}(1;\gamma) \leq \Big(1 + \mathcal{O}\Big(\frac{1}{\operatorname{Log} P}\Big)\Big) \frac{\xi P}{B \operatorname{Log} z} \tag{61}$$

where z is chosen as follows:

P		γ		$\gamma^{5/3}$		$\gamma^{5/2}$	
$\exp((\operatorname{Log}\operatorname{Log} P)^3) z$	P		$\sqrt{P\gamma}$		$P^{4/5}$		$P^{1/2}\gamma^{3/4}$

In the above bound we have removed the term $\mathcal{O}(1/(\operatorname{Log} P)^5)$ because it is absorbed in the term $\mathcal{O}(1/\operatorname{Log} z)$, since z is roughly between $\sqrt{P}$ and P, so that $\operatorname{Log} z \asymp \operatorname{Log} P$.

15 Proof of Theorem 1.1

We are seeking a lower bound for $S(g;\gamma)$, and we start with (12). Recalling (15), we consider

$$S^-(g;\gamma) = \sum_{0 \leq b \leq B-1} S^-_{b,B}(g;\gamma) \quad (\leq S(g;\gamma)). \tag{62}$$

On looking at the support of $H_{b,B}$, we infer that

$$S^-_{b,B}(g;\gamma) \geq S^-_{b,B}(1;\gamma) g^*(\delta_B \cdot b)$$

where we have used the definition

$$g^*(u) = \sum_{0 \leq b' \leq B-1} \Bigg(\min_{b' - \frac{3}{2} \leq \theta/\delta_B \leq b' + \frac{3}{2}} g(\theta) \Bigg) \mathbb{1}_{\delta_B(b'-1/2) < u \leq \delta_B(b'+1/2)}. \tag{63}$$

We know also that by the prime number Theorem and the discussion that occured at the end of section 7 that

$$\sum_{0 \leq b \leq B-1} S^-_{b,B}(1;\gamma) \geq \Big(1 + \mathcal{O}\Big(\frac{\xi}{\operatorname{Log} P} + \frac{1}{\xi (\operatorname{Log} P)^4}\Big)\Big) \frac{\xi P}{\operatorname{Log} P}.$$

We define

$$\sigma_b = \frac{B \operatorname{Log} P}{\xi\, P} S_{b,B}(1;\gamma).$$

Our problem is to minimize

$$\sum_{0 \leq b \leq B-1} \sigma_b\, g^*(\delta_B \cdot b) \tag{64}$$

under the conditions (here $P^\kappa = z$), see (61),

$$\kappa^{-1}\Theta \geq \sigma_b, \qquad \sum_{0 \leq b \leq B-1} \sigma_b \geq \Theta^{-1} B,$$

where, to clarify the situation, Θ is the maximum of the $(1+\mathcal{O}(1/\operatorname{Log} P))$ from (61) and one above the $1+\mathcal{O}(\xi/\operatorname{Log} P + 1/(\xi \operatorname{Log}^4 P))$ from (24). We are left with the conditions:

$$2\pi/B \geq 2\pi\kappa\Theta^{-1}\sigma_b/B, \qquad \sum_{0\leq b\leq B-1} 2\pi\Theta\kappa\sigma_b/B \geq 2\pi\Theta\kappa.$$

We use the sub-section 4.0.1 to infer that the minimum of (64) is

$$\geq (2\pi\Theta\kappa)^{-1} 2\pi\Theta\kappa V(g^*, 2\pi\Theta\kappa) \geq \Theta V(g^*, 2\pi\Theta\kappa).$$

Since g, and thus g^*, is bounded, we have

$$V(g^*, 2\pi\Theta\kappa) \geq V(g^*, 2\pi\kappa) + \mathcal{O}(\Theta).$$

We then appeal to Lemma 4.1 to show that

$$V(g^*, 2\pi\kappa) \geq V(g, 2\pi\kappa) + \omega_g(6\pi/B)/\kappa$$

where ω_g is the continuity modulus of g defined in (2).

16 Proof of Theorem 1.3

The proof is a simple application of Theorem 1.1. Since g is here assumed to be continuously differentiable, its modulus of continuity verifies $\omega_g(\delta) \ll \delta$. Let us say that we study the sum starting from $P_0 = (\operatorname{Log}\gamma)^4$. We recursively define, when $k \geq 0$,

$$P_{k+1} = P_k + \xi(P_k)P_k, \qquad \text{with } \xi(x) = 1/\operatorname{Log} x.$$

When P_0 is large enough in terms of ε, we have

$$\begin{aligned}
\sum_{p>P_0} \frac{g(\gamma \operatorname{Log} p)\operatorname{Log} p}{p^\sigma} &= \sum_{k\geq 0}\sum_{P_k<p\leq P_{k+1}} \frac{g(\gamma \operatorname{Log} p)\operatorname{Log} p}{p^\sigma} \\
&\geq \sum_{k\geq 0} \frac{1}{P_{k+1}^\sigma}\big(V(g, 2\pi\kappa_\gamma(P_k)) - \mathcal{O}((\operatorname{Log} P_k)^{-1})\big)\xi(P_k)P_k \\
&\geq \sum_{k\geq 0}\big(V(g, 2\pi\kappa_\gamma(P_k)) - \mathcal{O}((\operatorname{Log} P_k)^{-1})\big)\left(\frac{P_k}{P_{k+1}}\right)^\sigma \int_{P_k}^{P_{k+1}} \frac{dt}{t^\sigma}.
\end{aligned}$$

We first note that

$$V(g, 2\pi\kappa_\gamma(P_k))\left(1 - \left(\frac{P_k}{P_{k+1}}\right)^\sigma\right) \ll \frac{1}{\operatorname{Log} P_k}.$$

Moreover

$$\sum_{k\geq 0} \frac{1}{\operatorname{Log} P_k}\int_{P_k}^{P_{k+1}} \frac{dt}{t^\sigma} \ll \int_{P_0}^{\infty} \frac{dt}{t^\sigma \operatorname{Log} t} \ll \int_0^{P_0^{1-\sigma}} \frac{dv}{-\operatorname{Log} v}$$

with $v = t^{1-\sigma}$. We continue by $u = 1/v$ and isolate the singularity at $u = 1$, getting

$$\sum_{k\geq 0} \frac{1}{\operatorname{Log} P_k} \int_{P_k}^{P_{k+1}} \frac{dt}{t^\sigma} \ll \int_{P_0^{\sigma-1}}^{\infty} \frac{du}{u^2 \operatorname{Log} u} \ll \int_{\min(2, P_0^{\sigma-1})}^{2} \frac{du}{u-1} + \mathcal{O}(1)$$
$$\ll -\operatorname{Log}(\sigma - 1) + \mathcal{O}(1).$$

This is valid provided γ is large enough. We appeal to the monotony property proved in Lemma 4.4 and remark that $\kappa(t)$ is non-increasing to infer that

$$\sum_{p>P_0} \frac{g(\gamma \operatorname{Log} p) \operatorname{Log} p}{p^\sigma} \geq \int_{P_0}^{\infty} V(g, 2\pi\kappa_\gamma(t)) \frac{dt}{t^\sigma} - \mathcal{O}\big(\operatorname{Log}(\sigma - 1)\big).$$

Theorem 1.3 follows readily on extending the summation to encompass $p \leq P_0$ by positivity.

17 Proof of Corollary 1.4

Proving Corollary 1.4 is a simple application of Theorem 1.3 on selecting $g(t) = 1 \pm \cos t$ and $g(t) = 1 \pm \sin t$, but we provide details in the case $g(t) = 1 - \cos t$ so as to render checking easier. Let κ be a real number between 1/2 and 1. We readily check that

$$V(g, 2\pi\kappa) = \frac{2}{2\pi\kappa} \int_0^{\pi\kappa} (1 - \cos t) dt = 1 - \frac{\sin \pi\kappa}{\pi\kappa}.$$

We thus get

$$\frac{-\zeta'}{\zeta}(\sigma) + \Re \frac{-\zeta'}{\zeta}(\sigma + i\gamma) \geq \int_{(\operatorname{Log}\gamma)^4}^{\gamma} \frac{dt}{t^\sigma} + \int_{\gamma}^{\gamma^{5/3}} \left(1 - \frac{\sin \pi \frac{1+\vartheta(t)}{2}}{\pi(1+\vartheta(t))/2}\right) \frac{dt}{t^\sigma}$$
$$+ \int_{\gamma^{5/3}}^{\gamma^{5/2}} \left(1 - \frac{\sin \pi \frac{4}{5}}{\pi 4/5}\right) \frac{dt}{t^\sigma} + \int_{\gamma^{5/2}}^{\infty} \left(1 - \frac{\sin \pi \frac{2+3\vartheta(t)}{4}}{\pi(2+3\vartheta(t))/4}\right) \frac{dt}{t^\sigma} + \mathcal{O}(\operatorname{Log}(\sigma - 1))$$

where $\vartheta(t) = (\operatorname{Log}\gamma)/\operatorname{Log} t$. We use $\frac{-\zeta'}{\zeta}(\sigma) = 1/(\sigma - 1) + \mathcal{O}(1)$ and the change of variable $t = \gamma^v$ to get infer from the above that

$$\Re \frac{\zeta'}{\zeta}(\sigma + i\gamma) \leq \frac{e^{-4(\sigma-1)\operatorname{Log}\operatorname{Log}\gamma} - 1}{\sigma - 1} + \int_1^{5/3} \frac{\sin \pi \frac{1+v^{-1}}{2}}{\pi(1+v^{-1})/2} \frac{dv \operatorname{Log}\gamma}{e^{v(\sigma-1)\operatorname{Log}\gamma}}$$
$$+ \int_{5/3}^{5/2} \frac{\sin \pi \frac{4}{5}}{\pi 4/5} \frac{dv \operatorname{Log}\gamma}{e^{v(\sigma-1)\operatorname{Log}\gamma}} + \int_{5/2}^{\infty} \frac{\sin \pi \frac{2+3v^{-1}}{4}}{\pi(2+3v^{-1})/4} \frac{dv \operatorname{Log}\gamma}{e^{v(\sigma-1)\operatorname{Log}\gamma}} + \mathcal{O}(\operatorname{Log}(\sigma - 1)).$$

The Corollary follows readily.

The bound $h(x) \leq 2e^{-x}/\pi$ comes from the bound

$$V(g, 2\pi\kappa_\gamma(t)) \geq V(g, \pi) = 1 - \frac{2}{\pi}$$

which we use when $t \geq \gamma$.

18 Proof of Theorem 1.2

On using Lemma 9.7, we find that

$$F_0(T,z) = \int_T^{2T} \left|\zeta(\tfrac{1}{2}+it)M_2(\tfrac{1}{2}+it)\right| dt \ll_\varepsilon T\sqrt{z}(Tz)^\varepsilon$$

for every $\varepsilon > 0$. It is a simple matter to modify section 12 in order to obtain

$$I(\gamma) \ll_\varepsilon \sqrt{z}(\operatorname{Log}\gamma)(\gamma z)^\varepsilon/\gamma^3$$

when $\gamma \geq 2$ and where $I(\gamma)$ is being defined in (51). We reach in this fashion:

$$\frac{S^-_{b,B}(1;\gamma) - S^+_{b,B}(1;\gamma^*)}{\delta_B P/\operatorname{Log} P} \ll_\varepsilon B^3 \frac{(Pz)^{\frac{1}{2}+\varepsilon}}{P} \operatorname{Log}\gamma \ll_\varepsilon (Pz)^{\frac{1}{2}+2\varepsilon} P^{-1} \operatorname{Log}\gamma. \quad (65)$$

We thus take $z = P^{1-4\varepsilon}$ and get

$$\frac{S^-_{b,B}(1;\gamma) - S^+_{b,B}(1;\gamma^*)}{\delta_B P/\operatorname{Log} P} \ll_\varepsilon P^{-\varepsilon} \operatorname{Log}\gamma.$$

We conclude the proof as we did the one of Theorem 1.1.

19 On the size of $\zeta(s)^{\pm 1}$ next to the line $\Re s = 1$

Let us recall a classical consequence of the Vinogradov-Korobov bound (68) and of the classical method of Mertens. See [11, Corollary 8.28, Theorem 8.29].

Lemma 19.1. *There exists a constant $c > 0$ such that, in the region*

$$\sigma \geq 1 - \frac{c}{(\operatorname{Log} t)^{2/3}(\operatorname{Log}\operatorname{Log} t)^{1/3}},$$

the Riemann zeta-function has no zero, and we furthermore have

$$\left|\frac{\zeta'}{\zeta}(\sigma+it)\right| + \left|\frac{1}{\zeta(\sigma+it)}\right| \ll (\operatorname{Log} t)^{2/3}(\operatorname{Log}\operatorname{Log} t)^{1/3}$$

as well as

$$\zeta(\sigma+it) \ll (\operatorname{Log} t)^{2/3}, \quad \zeta'(\sigma+it) \ll (\operatorname{Log} t)^{4/3}.$$

Proof of Corollary 1.6

We define $\sigma_1 - 1 = c/\operatorname{Log}\operatorname{Log} t$ for a small enough constant $c > 0$ and readily check that

$$\pm \operatorname{Log}|\zeta(\sigma+it)| \leq \pm \operatorname{Log}|\zeta(\sigma_1+it)| + \int_\sigma^{\sigma_1} \left|\Re\frac{\zeta'}{\zeta}(u+it)\right| du.$$

On appealing to Corollary 1.4, we get

$$\begin{aligned}
\pm \operatorname{Log}|\zeta(\sigma+it)| &\le \int_{\sigma-1}^{\sigma_1-1} \frac{2/\pi}{e^{v \operatorname{Log} t} v} dv + \mathcal{O}(1) \pm \operatorname{Log}|\zeta(\sigma_1+it)| \\
&\le (2/\pi)\int_{(\sigma-1)\operatorname{Log} t}^{\infty} \frac{dw}{e^w w} + \mathcal{O}(1) \pm \operatorname{Log}|\zeta(\sigma_1+it)| \\
&\le \frac{2/\pi}{t^{\sigma-1}(\sigma-1)\operatorname{Log} t}\big(1+\mathcal{O}(\delta)\big) + \mathcal{O}(1) \pm \operatorname{Log}|\zeta(\sigma_1+it)|
\end{aligned}$$

by using an integration by parts and with $\delta^{-1} = (\sigma-1)\operatorname{Log} t$. We first apply this inequality with $\delta = 1$, getting

$$\zeta((\operatorname{Log} t)^{-1} + it) \ll \operatorname{Log}\operatorname{Log} t.$$

Since $\zeta'(\sigma+it) \ll (\operatorname{Log} t)^{4/3}$ for $\sigma \ge 1$, we get, when $\sigma \in [1, (\operatorname{Log} t)^{-1}]$,

$$\zeta(\sigma+it) = \zeta((\operatorname{Log} t)^{-1} + it) + \mathcal{O}\big((\operatorname{Log} t)^{-1}(\operatorname{Log} t)^{4/3}\big) \tag{66}$$

and thus $\zeta(\sigma+it) \ll (\operatorname{Log} t)^{1/3}$. Once this first inequality has been established, we use $\zeta' = \zeta\cdot(\zeta'/\zeta)$ to infer the bound

$$\zeta'(\sigma+it) \ll (\operatorname{Log} t)(\operatorname{Log}\operatorname{Log} t)^{1/3}.$$

We finally use this bound in (66) to reach

$$\zeta(\sigma+it) \ll \operatorname{Log}\operatorname{Log} t, \qquad (\sigma \ge 1).$$

We prove similarly that

$$1/\zeta(\sigma+it) \ll \operatorname{Log}\operatorname{Log} t, \qquad (\sigma \ge 1)$$

by using $\zeta'/\zeta^2(\sigma+it) \ll (\operatorname{Log} t)^{4/3}(\operatorname{Log}\operatorname{Log} t)^{2/3}$. The proof is complete.

20 Extension of an upper bound of $|\zeta(1+it)|$

We define

$$G(t,\Delta) = \max_{|t-t'|\le\Delta} |\zeta(1+it')| \tag{67}$$

and seek upper bounds of $|\zeta(\sigma+it)|$ and $|\zeta'(\sigma+it)|$ for σ close to 1 in terms of $G(t,\Delta)$. We will then appeal to Corollary 1.6 to bound $G(t,\Delta)$ by $\mathcal{O}(\operatorname{Log}\operatorname{Log}(|t|+\Delta))$.

Lemma 20.1. *For $\Delta \le 1$, $\delta \ge 0$ and $|1-\sigma| \le (\operatorname{Log}_2 t)^{2\delta}(\operatorname{Log} t)^{-2/3}$, we have*

$$|\zeta(\sigma+it)| \ll \exp\Big\{C_9(\operatorname{Log}_2 t)^{3\delta}\Big\}\left(G(t,\Delta)\operatorname{Log}\operatorname{Log} t + \frac{1}{\Delta(\operatorname{Log}_2 t)^{\delta}}\right) + 1.$$

This Lemma is proved in subsection 20.2.

20.1 A good representation of $\zeta(\sigma + it)$

We introduce the smoothing function

$$\Omega(u) = \begin{cases} 1 & \text{when } |u| \le U, \\ (U + L - |u|)/L & \text{when } U \le |u| \le U + L, \\ 0 & \text{else} \end{cases}$$

whose Fourier transform is given by

$$\hat{\Omega}(z) = \int_{-\infty}^{\infty} \Omega(u) e(zu) = \frac{2}{\pi L z^2} \sin(Lz/2) \sin((2U + L)z/2).$$

Let us recall Vinogradov-Korobov's estimate:

$$\Bigg| \sum_{N < n \le N'} n^{-it} \Bigg| \ll N \exp\left\{ -\alpha \frac{(\operatorname{Log} N)^3}{(\operatorname{Log} t)^2} \right\} \qquad (N' \le 2N) \tag{68}$$

where α is some positive constant. The best such result can be found in [7]. Equipped with this result, we readily shorten the classical representation $\zeta(\sigma + it) = \sum_{n \le t} n^{-\sigma - it} + \mathcal{O}(1)$ (valid when $\sigma \ge \frac{1}{2}$ and $t \ge 1$) and get

$$\zeta(\sigma + it) = \sum_{n \ge 1} \frac{\Omega(\operatorname{Log} n)}{n^{\sigma + it}} + \mathcal{O}(1). \tag{69}$$

This is valid provided $U + L \le \operatorname{Log} t$ and

$$U \ge C_7(\delta) \max\left(\max(0, (1 - \sigma))^{1/2} \operatorname{Log} t, (\operatorname{Log} t)^{2/3} (\operatorname{Log}_2 t)^{\delta} \right)$$

for any $\delta > 0$ and some constant C_7 depending on δ.

20.2 Expressing $\zeta(\sigma + it)$ in terms of $\zeta(1 + it)$

We rewrite formula (69) as follows:

$$\zeta(\sigma + it) = \sum_{n \ge 1} \frac{\Omega(\operatorname{Log} n) n^{\eta}}{n^{1 + it}} + \mathcal{O}(1) \qquad (\eta = 1 - \sigma)$$

which, on using

$$\Omega(\operatorname{Log} n) e^{\eta \operatorname{Log} n} = \int_{-\infty}^{\infty} \hat{\Omega}(u + \eta/(2i\pi)) e(-u \operatorname{Log} n) du,$$

yields the representation we were looking for, namely

$$\zeta(\sigma + it) = \int_{-\infty}^{\infty} \hat{\Omega}(u + \eta/(2i\pi)) \zeta(1 + it - 2i\pi u) du + \mathcal{O}(1). \tag{70}$$

The conclusion is then just a matter of routine. We first check that

$$|\hat{\Omega}(u+\eta/(2i\pi))| \le e^{(U+L)|\eta|/(2\pi)} \min\Bigl(\frac{2U+L}{2\pi}, \frac{1}{\pi|z|}, \frac{2}{\pi L|z|^2}\Bigr)$$

where $z = u - i\eta/(2\pi)$ by using $|\sin z| \le e^{\sigma}\min(1,|z|)$. We deduce from them that

$$\begin{aligned}|\zeta(\sigma+it)| \ll e^{(U+L)|\eta|}\biggl(\int_{|t-v|\le 1/(U+L)} &(U+L)|\zeta(1+iv)|dv \\ &+ \int_{1/(U+L)\le|t-v|\le\Delta} |\zeta(1+iv)|\frac{dv}{|v-t|} + \frac{(\operatorname{Log} t)^{2/3}}{L\Delta}\biggr) + 1\end{aligned}$$

whence

$$\begin{aligned}|\zeta(\sigma+it)| &\ll e^{(U+L)|\eta|}\biggl(G(t,(U+L)^{-1}) \\ &\qquad + \int_{1/(U+L)\le|t-v|\le\Delta} |\zeta(1+iv)|\frac{dv}{|v-t|} + \frac{(\operatorname{Log} t)^{2/3}}{L\Delta}\biggr)+1, \\ &\ll e^{(U+L)|\eta|}\biggl(G(t,\Delta)\operatorname{Log}((U+L)\Delta) + \frac{(\operatorname{Log} t)^{2/3}}{L\Delta}\biggr) + 1,\end{aligned}$$

for $\Delta \le 1$ (whether $\Delta \le 1/L$ or not). On taking $U = L = C_8(\operatorname{Log} t)^{2/3}(\operatorname{Log}_2 t)^{\delta}$, Lemma 20.1 follows readily.

21 A zero-free region: proof of Theorem 1.7

The proof and the statement of the following Lemma has taken some years to find a proper shape. One can find traces of it in [14], between equations (92) and (93), see the definition of F. It will evolve until [15, Lemma 1] to yield a bound on $\zeta'/\zeta(s)$ next to the line $\Re s = 1$. At the time, Gronwall and Landau were improving each other's bound. See also [24, section 3.9, Lemma α].

Lemma 21.1. *Let M be an upper bound for the holomorphic function F in $|s-s_0| \le R$. Assume we know of a lower bound $m > 0$ for $|F(s_0)|$. Then*

$$\frac{F'(s)}{F(s)} = \sum_{|\rho-s_0|\le R/2} \frac{1}{s-\rho} + \mathcal{O}\Bigl(\frac{\operatorname{Log}(M/m)}{R}\Bigr)$$

for every s such that $|s-s_0| \le R/4$ and where the summation variable ρ ranges the zeros ρ of F in the region $|\rho - s_0| \le R/2$, repeated according to multiplicity.

This Lemma is our main tool in what follows. The reader should look at the very interesting section 3 of [10], and more precisely to [10, Lemma 3.2]. An expression for the real part of $F'(s)/F(s)$ in terms of the possible zeros is obtained there. In fact, the proof therein contains an expression for $F'(s)/F(s)$, but it seems necessary to take the real part to bound it solely in term of M/m (notation as above).

Proof of Theorem 1.7

Let γ_0 be the (large enough) ordinate of a zero $\rho_0 = \beta_0 + i\gamma_0$ of the Riemann zeta-function. It will be easier in this proof to use the notation $\operatorname{Log}_k t$ to denote the k-fold iterate of the logarithm, so that $\operatorname{Log}_2 t = \operatorname{Log}\operatorname{Log} t$.

We apply Lemma 21.1 with parameters $F = \zeta$, $s_0 = 1 + i\gamma_0$, and $R = (\log_3 \gamma_0 / \log \gamma_0)^{2/3}$. The bound M required is provided by Lemma 20.1 where we select $\Delta = 1$, $\delta = (\operatorname{Log}_4 \gamma_0)/\operatorname{Log}_3 \gamma_0$ and we bound $G(\gamma_0, 1)$ by appealing to Corollary 1.6. We also use this Corollary to show that we can take $m \gg 1/\operatorname{Log}\operatorname{Log}\gamma_0$. As a conclusion, we find that

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{|\rho - 1 - i\gamma_0| \le R/2} \frac{1}{s-\rho} + \mathcal{O}\big((\operatorname{Log}\gamma_0)^{2/3}(\operatorname{Log}_3 \gamma_0)^{1/3}\big).$$

We select $s = \sigma + it$ for some $\sigma > 1$ that we choose below. Note that

$$\Re \frac{1}{s-\rho} \ge 0$$

so that we have

$$\Re \frac{\zeta'(s)}{\zeta(s)} \le \Re \frac{1}{s-\rho_0} + \mathcal{O}\big((\operatorname{Log}\gamma_0)^{2/3}(\operatorname{Log}_3 \gamma_0)^{1/3}\big).$$

We differ here from the usual treatment (but this one would also do) to show the strength of Corollary 1.4. On using this Corollary, we find that

$$\frac{2/\pi}{\sigma - 1} + \mathcal{O}(-\operatorname{Log}(\sigma - 1)) \le \Re \frac{\zeta'(s)}{\zeta(s)}.$$

We conclude, on assuming $\sigma - 1 \ge 1/\operatorname{Log} t$, that

$$\frac{2/\pi}{\sigma - 1} \le \frac{1}{\sigma - \beta_0} + \mathcal{O}\big((\operatorname{Log}\gamma_0)^{2/3}\operatorname{Log}\operatorname{Log}\operatorname{Log}\gamma_0\big).$$

We take $\sigma - 1 = 1 - \beta_0$ and find that

$$1 \ll \frac{2}{\pi} - \frac{1}{2} \ll (1 - \beta_0)(\operatorname{Log}\gamma_0)^{2/3}(\operatorname{Log}_3 \gamma_0)^{1/3}.$$

The proof of Theorem 1.7 is now complete.

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