

A REMARK ON THE GEOMETRY OF SPACES OF FUNCTIONS WITH PRIME FREQUENCIES.

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ABSTRACT. For any positive integer r , denote by \mathcal{P}_r the set of all integers $\gamma \in \mathbb{Z}$ having at most r prime divisors. We show $C_{\mathcal{P}_r}(\mathbb{T})$, the space of all continuous functions on the circle \mathbb{T} whose Fourier spectrum lies in \mathcal{P}_r , contains a complemented copy of ℓ^1 . In particular, $C_{\mathcal{P}_r}(\mathbb{T})$ is not isomorphic to $C(\mathbb{T})$, nor to the disc algebra $A(\mathbb{D})$. A similar result holds in the L^1 setting.

For any set of “frequencies” $\Lambda \subset \mathbb{Z}$, let us denote by $C_\Lambda(\mathbb{T})$ the space of all continuous functions f on the circle \mathbb{T} whose Fourier spectrum lies in Λ , i.e. $\hat{f}(\gamma) = 0$ for every $\gamma \in \mathbb{Z} \setminus \Lambda$. A classical topic in harmonic analysis (see e. g. the monographs [8] or [15]) is to try to understand how various “thinness” properties of the set Λ are reflected in geometrical properties of the Banach space $C_\Lambda(\mathbb{T})$. For example, by famous results of Varopoulos and Pisier, Λ is a Sidon set if and only if C_Λ is isomorphic to ℓ^1 , if and only if C_Λ has cotype 2. More generally, one can start with any Banach space X naturally contained in $L^1(\mathbb{T})$ and consider the spaces X_Λ of all functions $f \in X$ whose Fourier spectrum lies in Λ .

We concentrate here on sets $\Lambda \subset \mathbb{Z}$ closely related to the set \mathcal{P} of prime numbers. Considering the primes as a “thin” set is a question of point of view. On the one hand, \mathcal{P} is a small set since it has zero density in \mathbb{Z} . On the other hand, the envelopping sieve philosophy as exposed in [13] and [12] says that the primes can be looked at as a subsequence of density of a linear combination of arithmetic sequences, a fact that is brilliantly illustrated in [3] by showing that \mathcal{P} contains arbitrarily long arithmetic progressions. From the harmonic analysis standpoint, let us just quote two results going in opposite directions: it was shown by F. Lust-Piquard [9] that $C_{\mathcal{P}}(\mathbb{T})$ contains subspaces isomorphic to c_0 , a property shared by $C(\mathbb{T})$ and the disc algebra $A(\mathbb{D}) = C_{\mathbb{Z}_+}(\mathbb{T})$, which indicates that \mathcal{P} is not a very thin set; but on the other hand ([6]) any $f \in L^\infty_{\mathcal{P}}(\mathbb{T})$ is “totally ergodic” (i.e. fg has a unique invariant mean for any $g \in C(\mathbb{T})$), a property shared by very thin sets such as Sidon sets and not satisfied by \mathbb{Z} and \mathbb{Z}_+ .

In this note, we show that the geometry of $C_{\mathcal{P}}(\mathbb{T})$ is quite different from that of $C(\mathbb{T})$ or the disc algebra $A(\mathbb{D}) = C_{\mathbb{Z}_+}(\mathbb{T})$, and that $L^1_{\mathcal{P}}(\mathbb{T})$ is very different from $L^1(\mathbb{T})$. Hence, the primes definitely do not behave like an arithmetic progression. Further, this disruption occurs even with the set of integers having at most r prime factors, for some given r , though the sieve is known to behave rather well for such sequences as soon as $r \geq 2$ ([2], [16]).

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For any positive integer r , let us denote by \mathcal{P}_r the set of all positive integers having at most r prime factors:

$$\mathcal{P}_r = \left\{ \prod_{1 \leq i \leq r} p_i^{\alpha_i} \mid p_i \in \mathcal{P}, \alpha_i \in \mathbb{Z}_+ \right\}.$$

Our main result reads as follows.

Theorem. *Let $\Lambda \subset \mathbb{Z}$ and assume that $\mathcal{P} \subset \Lambda \subset \mathcal{P}_r \cup (-\mathcal{P}_r)$. Then*

- (1) $C_\Lambda(\mathbb{T})$ contains a complemented copy of ℓ^1 ;
- (2) $L_\Lambda^1(\mathbb{T})$ and $L_{\mathbb{Z}-\cup\Lambda}^1(\mathbb{T})$ contain a complemented copy of ℓ^2 .

It is well known that neither $C(\mathbb{T})$ nor to the disc algebra $A(\mathbb{D})$ contains a complemented copy of ℓ^1 , and that $L^1(\mathbb{T})$ does not contain a complemented copy of ℓ^2 (see the remark below). Hence, we immediately get

Corollary. *If $\Lambda \subset \mathbb{Z}$ satisfies $\mathcal{P} \subset \Lambda \subset \mathcal{P}_r \cup (-\mathcal{P}_r)$, then $C_\Lambda(\mathbb{T})$ is isomorphic neither to a complemented subspace of $C(\mathbb{T})$ nor to a complemented subspace of the disc algebra $A(\mathbb{D})$, and neither $L_\Lambda^1(\mathbb{T})$ nor $L_{\mathbb{Z}-\cup\Lambda}^1(\mathbb{T})$ is isomorphic to a complemented subspace of $L^1(\mathbb{T})$.*

Remark. (i) One way of showing that neither $C(\mathbb{T})$ nor to the disc algebra $A(\mathbb{D})$ contains a complemented copy of ℓ^1 is to use Pełczyński's *property (V)*. A Banach space X has property (V) if every non weakly compact (bounded) operator $T : X \rightarrow Y$ into a Banach space Y “fixes a copy of c_0 ”, i.e. there is a (closed) subspace $X_0 \subset X$ isomorphic to c_0 such that $T|_{X_0}$ is an isomorphism from X_0 onto $T(X_0)$. It is well known that $C(\mathbb{T})$ and $A(\mathbb{D})$ have property (V) (see [4]). Moreover, property (V) is clearly inherited by complemented subspaces. Since obviously ℓ^1 does not have property (V), the result follows.

(ii) To show that $L^1(\mathbb{T})$ does not contain a complemented copy of ℓ^2 one can use another well known Banach space property. Let us say that Banach space X *satisfies Grothendieck's theorem* if every operator from X into a Hilbert space is absolutely summing. A typical example is the space L^1 : this is precisely “Grothendieck's theorem” (see e.g. [11]). Moreover, this property is again clearly inherited by complemented subspaces. Since obviously ℓ^2 does not satisfy it, the result follows.

Having introduced property (V) and GT, we can state a more general version of the above corollary:

Corollary'. *If $\Lambda \subset \mathbb{Z}$ satisfies $\mathcal{P} \subset \Lambda \subset \mathcal{P}_r \cup (-\mathcal{P}_r)$, then*

- (i) $C_\Lambda(\mathbb{T})$ does not have property (V);
- (ii) $L_\Lambda^1(\mathbb{T})$ and $L_{\mathbb{Z}-\cup\Lambda}^1(\mathbb{T})$ do not satisfy Grothendieck's theorem.

The proof of the theorem relies on the following “gap” property of \mathcal{P}_r . Denote by $d(\lambda, A)$ denotes the distance of an integer λ to a set $A \subset \mathbb{Z}$,

$$d(\lambda, A) = \inf\{|\lambda - a|; a \in A\}.$$

Lemma. *For any $r \in \mathbb{N}$, one can find an infinite set $E_r \subset \mathcal{P}$ such that*

$$\sup_{\lambda \in E_r} d(\lambda, \mathcal{P}_r \setminus \{\lambda\}) = +\infty.$$

In other words, there are *big symmetric holes* in any set Λ such that $\mathcal{P} \subset \Lambda \subset \mathcal{P}_r$. It is of course well known that there are big holes in \mathcal{P} . However, the standard argument is to consider the set $\{n!+2, \dots, n!+n\}$ with n arbitrarily large (which is obviously disjoint from the set of prime numbers), but this provides only *one-sided* holes in \mathcal{P} . (One may consult [14] and [10] where the quantitative question of the sizes of these holes is studied). Actually, the referee has kindly informed us that the case $\Lambda = \mathcal{P}$ was already known: see [1] and [5]. Nevertheless, our proof is both short and general. It would be interesting to estimate the (optimal) size of these symmetric holes.

Proof of the lemma. For convenience, we put $\mathcal{P}_0 = \{1\}$. We prove by induction on $r \in \mathbb{Z}^+$ the following assertion, denoted by (\mathcal{H}_r) :

For every $a \geq 1$, there exists a prime number p such that

$$\{p-a, \dots, p-1, p+1, \dots, p+a\} \cap \mathcal{P}_r = \emptyset.$$

Property (\mathcal{H}_0) is obvious. Now, assume that (\mathcal{H}_{r-1}) has been proved for some $r \geq 1$, and let us fix $a \geq 1$.

We choose a prime number $A > a+1$ such that

$$\{A-a, \dots, A-1, A+1, \dots, A+a\} \cap \mathcal{P}_{r-1} = \emptyset,$$

and we put

$$Q = \prod_{\substack{1 \leq k \leq 2A \\ k \neq A}} k^2 = \left(\frac{(2A-1)!}{A} \right)^2.$$

Note that A and Q are coprime.

By Dirichlet's theorem on arithmetic progressions, one can find a prime number $p > A$ such that $p \equiv A \pmod{Q}$. We claim that this prime number p satisfies

$$\{p-a, \dots, p-1, p+1, \dots, p+a\} \cap \mathcal{P}_r = \emptyset,$$

which will give (\mathcal{H}_r) .

To prove the claim, let us fix $j \in \{\pm 1, \dots, \pm a\}$ and, towards a contradiction, assume that $p+j \in \mathcal{P}_r$, i.e.

$$p+j = \prod_{1 \leq i \leq r} q_i^{\alpha_i},$$

where $q_i \in \mathcal{P}$ and $\alpha_i \in \mathbb{Z}_+$. Since $A+j$ divides Q and $p+j \equiv A+j \pmod{Q}$, the number $A+j$ divides $p+j$. Hence we can write

$$A+j = \prod_{1 \leq i \leq r} q_i^{\beta_i},$$

where $\beta_i \leq \alpha_i$. If at least one of the β_i vanished, then $A+j$ would belong to \mathcal{P}_{r-1} , but this is false. Hence for every $1 \leq i \leq r$, we have $\beta_i \geq 1$. On the other hand, since $A < p$, there exists some $i_0 \in \{1, \dots, r\}$ such that $\beta_{i_0} < \alpha_{i_0}$. Now, consider $N = q_{i_0}(A+j)$. Since $\beta_{i_0} \geq 1$, we have that q_{i_0} divides $A+j$, hence N divides $(A+j)^2$; but $(A+j)^2$ divides Q so that N divides Q . On the other hand N divides $p+j$ since $\beta_{i_0} + 1 \leq \alpha_{i_0}$. Hence N must divide the gcd of Q and $p+j$. The latter is equal $A+j$ since $A+j$ divides Q and $p+j \equiv A+j \pmod{Q}$. This is the required contradiction.

Thus, (\mathcal{H}_r) is true for all $r \in \mathbb{Z}^+$, and the lemma follows immediately. \square

Proof of the theorem. It would suffice to apply the lemma and [6, Proposition 2]. However, we repeat quickly the argument in our framework.

From the lemma, we get by induction a sequence of prime numbers $H = (p_n)_{n \geq 1}$ (that we identify with the set $\{p_1, p_2, \dots\}$) with the following properties:

- (a) H is a Hadamard sequence with ratio at least 3;
- (b) the “mesh” $[H] = \left\{ \sum_{1 \leq n \leq m} \varepsilon_n p_n \mid \varepsilon_n \in \{-1, 0, 1\}; m \geq 1 \right\}$ intersects the set $\mathcal{P}_r \cup (-\mathcal{P}_r)$ only at the $\pm p_n$ ’s.

Then we consider the Riesz products

$$R_N(x) = 2 \prod_{n=1}^N [1 + \cos(2\pi p_n x)].$$

Identifying R_N with the positive measure $R_N(x)dx$ (where dx is the normalized Lebesgue measure on \mathbb{T}), we have $\|R_N\| = \int_{\mathbb{T}} R_N(x)dx = 2$. If μ is any limit point of the sequence (R_N) in $M(\mathbb{T})$, then μ is a positive measure on \mathbb{T} whose Fourier coefficients are 1 on H and vanish on $\Lambda \setminus [H]$.

Since $H \subset \Lambda \subset \mathcal{P}_r \cup (-\mathcal{P}_r)$ and $[H] \cap (\mathcal{P}_r \cup (-\mathcal{P}_r)) = H \cup (-H)$ by (b), it follows that the convolution operator associated with μ is then a translation-invariant projection from $C_{\Lambda}(\mathbb{T})$ onto $C_E(\mathbb{T})$, where $E = H \cup (\Lambda \cap (-H))$. Since E is a Sidon set by (a), $C_E(\mathbb{T})$ is isomorphic to ℓ^1 , and this proves part (1) of the theorem.

The same convolution operator also defines a projection from $L_{\Lambda}^1(\mathbb{T})$ onto $L_E^1(\mathbb{T})$, which is isomorphic to ℓ^2 because E is a $\Lambda(2)$ set by (a). Hence, $L_{\Lambda}^1(\mathbb{T})$ contains a complemented copy of ℓ^2 . Finally, using [7, Proposition 1.4] this holds for $L_{\mathbb{Z}^- \cup \Lambda}^1(\mathbb{T})$ as well (see [6] for some details). This proves part (2) of the theorem. \square

Remark. If we knew that there are infinitely many primorial numbers (resp. factorial numbers), then the above proposition would not be needed: we could apply directly [6, Proposition 2] to prove the theorem.

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