Joint landscape on a peg solitaire board[*,†]

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Abstract

We investigate and develop further the notions of joint landscape.

1 Introduction

We introduced in (Ramaré, 2008c) the notion of *landscape* for a position I and in (Ramaré, 2008a) (see also the end of (Ramaré, 2008b)) the notion of *joint landscape* for two positions I and J. It is the function $\mathfrak{L}(A, I, J)$ on $\mathfrak{S}$ which is equal in A, when $A \notin I$, to the minimal number of moves required to reach a position from which we may still derive J and which contains A, when starting from I. We have called this number the *joint height* $\mathrm{h}(A, I, J)$ of A with respect to be I and J, When $A \in I$, then $-\mathfrak{L}(A, I, J)$ is the minimal number of moves required to reach a position from which we may still derive J and which does not contain A. We call this number the *joint depth* of A with respect to I and J, an easy specialisation of the notion of depth introduced in section 11 of (Ramaré, 2008c). These heights, resp. depths, in A are infinite when it is not possible to put a peg in A, resp. not possible to remove the peg in A, with the above conditions.

We define the *external external height* $\mathrm{H}(A, I, J)$ to be the maximum number of moves such that, starting from I, the position reached I' contains A and that there exists a legal way of deriving J from I'. We thus look at

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path $\mathfrak{f}_1, \ldots, \mathfrak{f}_k, \cdots \mathfrak{f}_\ell$ such that

$$\begin{cases} \mathbb{1}_I & \text{is legal,} \\ \mathbb{1}_I - \mathfrak{f}_1 & \text{is legal,} \\ \vdots & \\ \mathbb{1}_I - \mathfrak{f}_1 \cdots - \mathfrak{f}_{k-1} & \text{is legal,} \\ \mathbb{1}_I - \mathfrak{f}_1 - \cdots - \mathfrak{f}_k = h & \text{is legal and verifies } h(A) = 0, \\ \mathbb{1}_I - \mathfrak{f}_1 \cdots - \mathfrak{f}_{k+1} & \text{is legal,} \\ \vdots & \\ \mathbb{1}_J = \mathbb{1}_I - \mathfrak{f}_1 \cdots - \mathfrak{f}_\ell & \text{is legal} \end{cases} \tag{1}$$

and look for k minimal in the case of the joint height $\mathrm{h}(A, I, J)$ and for k maximal in the case of the external joint height $\mathrm{H}(A, I, J)$. This joint external height is infinite whenever no path verify (1), so that we have

$$\mathrm{h}(A, I, J) \leq \mathrm{H}(A, I, J). \tag{2}$$

We have developped techniques to evaluate the height, and this function is fairly well understood, see (Ramaré, 2008b), but the depth remains less tractable. We introduced in (Ramaré, 2008c) a mean of getting a non-trivial lower bound, but this does not provide a clear view of the situation. Our main remark here is that the joint depth is more tractable than the depth, for we have:

Theorem 1.1 *We have*

$$\mathrm{Depth}(A, I, J) + \mathrm{H}(A, \mathfrak{S} \setminus J, \mathfrak{S} \setminus I) = |I| - |J|$$

whenever both quantities are finite.

Proof: Indeed, the joint depth corresponds to paths $\mathfrak{f}_1, \ldots, \mathfrak{f}_k, \ldots, \mathfrak{f}_\ell$ such that

$$\begin{array}{rl} \mathbb{1}_I & \text{is legal,} \\ \mathbb{1}_I - \mathfrak{f}_1 & \text{is legal,} \\ \vdots & \vdots \\ \mathbb{1}_I - \mathfrak{f}_1 \cdots - \mathfrak{f}_{k-1} & \text{is legal,} \\ \mathbb{1}_I - \mathfrak{f}_1 - \cdots - \mathfrak{f}_k = h & \text{is legal and verifies } h(A) = 0, \\ \mathbb{1}_I - \mathfrak{f}_1 \cdots - \mathfrak{f}_{k+1} & \text{is legal,} \\ \vdots & \vdots \\ \mathbb{1}_J = \mathbb{1}_I - \mathfrak{f}_1 \cdots - \mathfrak{f}_\ell & \text{is legal} \end{array}$$

and this is equivalent to

$$
\begin{array}{rl}
\mathbb{1}_{\mathfrak{S}\setminus I} = \mathbb{1}_{\mathfrak{S}\setminus J} - \mathfrak{f}_\ell \cdots - \mathfrak{f}_1 & \text{is legal,} \\
\vdots & \vdots \\
\mathbb{1}_{\mathfrak{S}\setminus J} - \mathfrak{f}_\ell \cdots - \mathfrak{f}_k & \text{is legal,} \\
\mathbb{1}_{\mathfrak{S}\setminus J} - \mathfrak{f}_\ell \cdots - \mathfrak{f}_{k+1} = g & \text{is legal and verifies } g(A) = 1, \\
\mathbb{1}_{\mathfrak{S}\setminus J} - \mathfrak{f}_\ell \cdots - \mathfrak{f}_{k+2} & \text{is legal,} \\
\vdots & \vdots \\
\mathbb{1}_{\mathfrak{S}\setminus J} - \mathfrak{f}_\ell & \text{is legal,} \\
\mathbb{1}_{\mathfrak{S}\setminus J} & \text{is legal}
\end{array}
$$

which should be read from bottom to top. We recognize the definition of a path occuring in the definition of $\mathrm{H}(A, \mathfrak{S} \setminus J, \mathfrak{S} \setminus I)$. If $k = \mathrm{Depth}(A, I, J)$, then the above shows that

$$\mathrm{H}(A, \mathfrak{S} \setminus J, \mathfrak{S} \setminus I) \leq \ell - k.$$

The minimal character of k in the proof ensures the maximal character of $\ell - k$. ◇◇◇

It is maybe better to take an example. In the problem below, the depth of A is clearly 2:

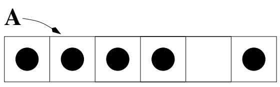

Figure 1: Initial position I

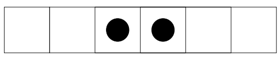

Figure 2: Final position J

We now have to check that the joint external height of A with respect to $\mathfrak{S} \setminus J$ and $\mathfrak{S} \setminus I$ is indeed 1. Here is the problem:

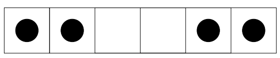

Figure 3: Initial position $\mathfrak{S} \setminus J$

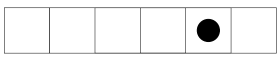

Figure 4: Final position $\mathfrak{S} \setminus I$

The reader will easily prove our claim.

Let us next consider the classical on the english solitaire board: we remove the central peg and we aim at leaving only one peg there. And we want to compute the joint depth of the point A indicated below:

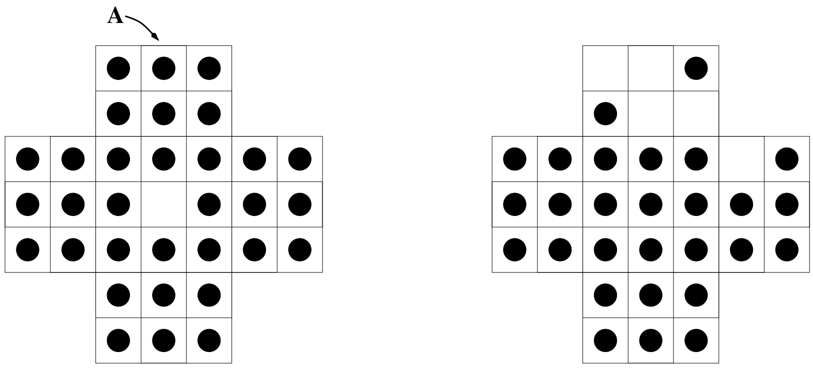

Figure 5: Joint depth of A is 4

We show that $\mathrm{Depth}(A, I, J) = 4$ by using the start given by figure 5 (and we leave it to prove that $\mathrm{Depth}(A, I, J) > 3$, to the reader to attain this position and to continue from it and leave a sole peg in the central hole!!). Computing $\mathrm{H}(A, \mathfrak{S} \setminus J, \mathfrak{S} \setminus I)$ is however even more tricky in this case, so how can we use our Theorem?

We have at our disposal another geometrical interpretation of $|I| - |J| - \mathrm{H}(A, I, J) = \mathrm{Depth}(A, \mathfrak{S} \setminus J, \mathfrak{S} \setminus I)$: it is the minimal number of *reversed* moves from J that leaves a peg in A, provided the position attained can be further continued to reached I, still by reversed legal moves. By reversed moves, we mean $-\mathfrak{f}$ where $\mathfrak{f} \in \mathscr{D}(\mathfrak{S})$. This interpretation is handy in that it is almost the same as the definition of joined height, where we simply change the set of moves! In particular, we have at our disposal the corresponding notion of lesser external height for a part K of $\mathfrak{S}$:

$$\mathrm{lH}(K, J) = \min_{\mathfrak{P}} \Big\{ \langle \Psi(\mathfrak{P}), \mathbb{1}_{\mathfrak{S}} \rangle, \quad \mathbb{1}_{\mathfrak{S}} \geq \mathbb{1}_J + \Psi(\mathfrak{P}) \geq \mathbb{1}_K \Big\} \tag{3}$$

which can be evaluated by linear programming and which verifies

$$\mathrm{lH}(\{A\}, J) \leq \mathrm{Depth}(\{A\}, \mathfrak{S} \setminus J, \mathfrak{S} \setminus I) \tag{4}$$

References

Ramaré, O. 2008a. Extracting informations from the linear entire model of peg solitaire. 1–9.

Ramaré, O. 2008b. Height functions on a peg solitaire board. 1–10.

Ramaré, O. 2008c. A stronger model on peg solitaire, II. 1–27. arXiv:0801.0679v1.