ON TWO PROBLEMS IN ANALYTIC NUMBER THEORY

by

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Thesis

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Abstract

Let χ be a Dirichlet character modulo q, let $L(s,\chi)$ be the attached Dirichlet L-function, and let $L'(s,\chi)$ denotes its derivative with respect to the complex variable s. Let $d(n)$ be the divisor function. In this thesis we prove two main results. The first result is to give an asymptotic formula for the $2k$-th power mean value of $\left|L'/L(1,\chi)\right|$ when χ ranges a primitive Dirichlet character modulo q for q prime. We derive some consequences, in particular a bound for the number of χ such that $\left|L'/L(1,\chi)\right|$ is large. In the second result, we prove inequalities of the shape $d^s(n) \le K(s,\delta,\beta)\sum_{\ell|n,\ell\le n^\delta} d^\beta(\ell)$ with explicit values of $K(s,\delta,\beta)$ and optimal values for β. When $s=1$ and $\delta \in (0,1/2]$, any $\beta \ge 2$ of the form $\beta = \epsilon + \delta^{-1} + (\delta\log\delta + (1-\delta)\log(1-\delta))/(\delta\log 2)$ for some $\epsilon > 0$ is admissible, with a constant $K(s,\delta,\beta)$ equal to $(\delta^{-5}\min(1,\epsilon))^{-3/\delta^3}$. This elaborates on work of Munshi in that the constants are effective (and even explicit) and that the inequality is also valid for non-especially square-free integers. We prove more specific inequalities for squarefree integers.

Keywords: Dirichlet characters, L-functions, Distribution function, divisor function, multiplicative function.

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I dedicate this thesis to my father Mohammed.

Dad. I will never forget your big heart.

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Contents

Notation

Throughout this we will have the following notation:

- Usually $s = \sigma + it$, but for a zero of L-function we use ρ.

- If $T \geq 1$ is a real number, $N(T,\chi,\sigma)$ denotes the number of zeros ρ of the function $L(s,\chi)$ in the region $|\Im\rho| \leq T$, $\Re\rho \geq \sigma$. We define, for a primitive character χ modulo q :

$$N(T,\chi,\sigma) = \#\left\{\rho = \beta + i\gamma; \quad |\gamma| \leq T \ and \ \beta \geq \sigma, L(\rho,\chi) = 0\right\}$$

- The notation $\underline{m}$ and $\underline{n}$ mean that $\prod_{i=1}^{k} m_i$ and $\prod_{i=1}^{k} n_i$ respectively.

- Let $q(\underline{m} \cdot \underline{n})$ be the largest divisor of q that is prime to $\underline{m} \cdot \underline{n}$.

- log refers to the natural logarithm, i.e., the logarithm in base e.

- $\tau(\chi)$ denotes the Gauss sum.

- The notation of Vinogradov $f \ll g$ means that $\left|f(t)\right| \leq Cg(t)$ for some constant C independent of the variable t.

- The notation $f \ll_A g$ means that $f \ll g$ with a constant C that may depend on A.

- We say $f(t) = \mathcal{O}\left(g(t)\right)$ if $\left|f(t)\right| \leq \ Cg(t)$ as $t \to \infty$ for some constant C, where $g(t) \geq 0$. This is usually called the big Oh notation or Landau O notation.

- We say $f(t) = \mathcal{O}^{\star}\left(g(t)\right)$ if $\left|f(t)\right| \leq g(t)$ as $t \to \infty$, where $g(t) \geq 0$.

- The notation $f = \mathcal{O}_A(g(t))$ means that $f(t) = \mathcal{O}(g(t))$ with the constant C that may depend on A.

- We write $f(t) \sim g(t)$ when $\lim f(t)/g(t) = 1$ as t tends to some limit.
- We write $f(t) = o\big(g(t)\big)$ when $\lim f(t)/g(t) = 0$ as t tends to infinity.
- $\Lambda(n)$ denotes the von Mangoldt Λ-function, which is defined by

$$\Lambda(n) = \begin{cases} \log p & \text{when } n = p^k \text{, and } k \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

- $\varphi(n)$ denotes the Euler totient or phi function, which counts the number of positive integers less than or equal to n that are relatively prime to n.

Part I

An asymptotic distribution of $\left|\frac{L'}{L}(1, \chi)\right|$

Chapter 1

Introduction

1.1 History

Let χ be a Dirichlet character modulo q, let $L(s,\chi)$ be the attached Dirichlet L-function, and let $L'(s,\chi)$ denotes its derivative with respect to the complex variable s. Evaluation the power moments of L-functions or of a family of L-functions has been a fundamental problem for may years. It is related to many other problems in analytic number theory, for example, the order estimate for L-functions, non vanishing problem at the central point, distribution of primes in arithmetic progressions.

1.1.1 Work on $L(s,\chi)$ and $L'(s,\chi)$ at $s = 1/2$

In the early 1930s, Paley [24] obtained the asymptotic formula

$$\sum_{\chi \bmod q} \left| L\left(\frac{1}{2},\chi\right)\right|^2 \sim \frac{\varphi(q)\varphi^\star(q)}{q}\log q, \quad q \to \infty,$$

where the sum above is over all primitive Dirichlet characters modulo q, $\varphi(q)$ denotes Euler's totient function, and $\varphi^\star(q)$ denotes the number of primitive character modulo q. In 1981, Heath Brown [8] established an asymptotic formula for the fourth moment of L-functions, in which the main term dominates the error term at least when q does not have so many distinct prime factors.

$$\sum_{\chi \bmod q} \left|L(1/2,\chi)\right|^4 = \frac{\varphi^\star(q)}{2\pi^2}\prod_{p|q}\frac{(1-p^{-1})^3}{(1+p^{-1})}\log^4 q + \mathcal{O}\left(2^{\omega(q)} q \log^3 q\right),$$

where the summation is over primitive characters and $\omega(q)$ is the number of distinct prime factors of q. Later, Soundararajan [33] improved the evaluation of the error terms of Heath- Brown's result and obtained the asymptotic formula. For all large q, we have

$$\sum_{\chi \, mod \, q} \left|L(1/2,\chi)\right|^4 = \frac{\varphi^\star(q)}{2\pi^2}\prod_{p|q}\frac{(1-p^{-1})^3}{(1+p^{-1})}\log^4 q\left(1+\mathcal{O}\left(\frac{\omega(q)}{\log q}\sqrt{\frac{q}{\varphi(q)}}\right)\right)+\mathcal{O}\left(q(\log q)^{7/2}\right).$$

Since $\omega(q) \ll \log q/\log\log q$ and $q/\varphi(q) \ll \log\log q$. Then, we have

$$\frac{\omega(q)}{\log q}\sqrt{\frac{q}{\varphi(q)}} \ll \frac{1}{\sqrt{\log\log q}}.$$

More recently, Young [38] succeeded in establishing good upper bounds for off-diagonal terms of certain sum involving divisor functions by using the Weil bound for Kloosterman sums and proved that the asymptotic formula

$$\sum_{\chi \, mod \, q} \left|L(1/2,\chi)\right|^4 = \varphi^\star(q)\left(P_4(\log q)+\mathcal{O}_\epsilon\left(q^{-5/512+\epsilon}\right)\right), \quad \epsilon>0,$$

holds for primes $q \neq 2$, where P_4 is certain computable polynomial of degree 4.

1.1.2 Work on $L'(s,\chi)$ at $s = 1/2$

Despite of many efforts, no asymptotic formulas for the 2k-th moment of Dirichlet L-functions have been known except for the cases k = 1, 2. However, there are a number of conditional or unconditional results on lower and upper bounds for the moments of Dirichlet L-functions. For example, Rudnick and Soundararajan [27] showed that the lower bound

$$\sum_{\substack{\chi \, mod \, q \\ \chi\neq\chi_0}} \left|L(1/2,\chi)\right|^{2k} \gg_k \varphi(q)\log^{k^2} q$$

holds unconditionally for all integers $k > 1$, when q is prime number. Here the sum above is over all non-principal characters modulo q. In 2010, Heath-Brown [11] proved that

$$\sum_{\substack{\chi \, mod \, q \\ \chi\neq\chi_0}} \left|L(1/2,\chi)\right|^{2k} \ll_k \varphi(q)\log^{k^2} q,$$

holds for all $k \in (0,2)$ under GRH, and that this result holds unconditionally for $k = 1/v$, $v \in \mathbb{N}$. More recently, Soundararajan [34] showed thatthe upper bound

$$\sum_{\chi \, mod \, q} \left|L(1/2,\chi)\right|^{2k} \ll_\epsilon \varphi(q)\log^{k^2+\epsilon} q, \quad \epsilon>0,$$

holds for all $k > 0$, assuming the GRH. By the extending to result of Rundnick and Soundararajan, in 2012, Sono [30] gave lower bound for the moments of $L^{\ell}(s,\chi)$, the ℓ-derivatives of $L(s,\chi)$ for $\ell > 0$, at $s = 1/2$. The author proved that for any fixed $k \in \mathbb{Z}_{\geq 2}$ and $\ell \in \mathbb{N}$, the inequality

$$\sum_{\substack{\chi \bmod q \\ \chi \neq \chi_0}} \left| L^{(\ell)}(1/2,\chi) \right|^{2k} \geq \frac{c_{k,\ell}}{(2k)^{k^2+2k\ell}} (1+o(1))\, \varphi(q) \log^{k^2+2k\ell} q$$

holds unconditionally for all large primes q, where $c_{k,\ell}$ above is a constant dependent only on k and ℓ, explicitly given in [30]. The recent work of Chandee and Li [2] gives lower bounds for the moments above when $\ell = 0$ and $0 < k < 1$ is rational. On the other hand, the first upper bounds for the moments $L^{(\ell)}(1/2,\chi)$ has been given by Sono [31] in 2014, he proved that

$$\sum_{\substack{\chi \bmod q \\ \chi \neq \chi_0}} \left| L^{(\ell)}(1/2,\chi) \right|^{2k} \ll \varphi(q) \log^{k^2+2k\ell} q,$$

holds for $1/2 \leq k < 2$ and $\ell \in \mathbb{Z}_{\geq 0}$, assuming the GRH. The sum above is over all non-principal characters modulo q. This result is a generalization of Heath-Brown's estimate. On the other hand, by extending Soundararajan's method for the moments of Riemann zeta function [34] and using Milinovich's ideas introduced in [20], Sono [31] obtained in the case of $k \geq 2$ the following upper bound

$$\sum_{\chi \bmod q} \left| L^{(\ell)}(1/2,\chi) \right|^{2k} \ll_{\epsilon} \varphi(q) \log^{k^2+2k\ell+\epsilon} q.$$

The sum above is over all primitive Dirichlet characters modulo q.

1.1.3 Work on $L(s,\chi)$ at $s = 1$

Concerning the values of L-functions at $s = 1$. Ankeny and Chowla [1] in 1951 proved that, for $q = p \geq 3$ prime, we have

$$\sum_{\substack{\chi \bmod p \\ \chi \neq \chi_0}} \left| L(1,\chi) \right|^2 = \zeta(2)p + \mathcal{O}\left(\log^2 p\right).$$

This result was improved by Slavutskil [28] [29], who proved that

$$\sum_{\substack{\chi \bmod p \\ \chi \neq \chi_0}} \left| L(1,\chi) \right|^2 = \zeta(2)p - (\log p)^2 + \mathcal{O}\left(\log p\right).$$

In 1990, Zhang improved the error estimate in the result of Slavutskil to $\mathcal{O}\left(\log\log p\right)$ in [41] and then in [39], the author obtained

$$\sum_{\substack{\chi \bmod p \\ \chi\neq\chi_0}} \left|L(1,\chi)\right|^2 =$$

$$\zeta(2)p-(\log p)^2-\left(1+\zeta(2)+2\sum_{n=1}^{\infty}\frac{\log(n+1)}{n(n+1)}-\int_0^1\left(\sum_{n=1}^{\infty}\frac{y}{n(n+y)}\right)^2 dy\right)+\mathcal{O}\left(\frac{1}{\log p}\right).$$

In 1994, Katsurada and Matsumoto [18] showed that, for any integer $N\geq 1$, we have

$$\sum_{\substack{\chi \bmod p \\ \chi\neq\chi_0}} \left|L(1,\chi)\right|^2 = \zeta(2)p-(\log p)^2+(\gamma_0^2-2\gamma_1-3\zeta(2))-(\gamma_0^2-2\gamma_1-2\zeta(2))p^{-1}$$

$$+2(1-p^{-1})\left[\sum_{n=1}^{N-1}(-1)^n\zeta(1-n)\zeta(1+n)p^{-n}+\mathcal{O}\left(p^{-N}\right)\right],$$

where the $\mathcal{O}$-constant depends only on N, and the constants γ_0 and γ_1 are the Laurent expansion coefficients of $\zeta(s)$ at $s=1$. Concerning the fourth power mean value of L-function at $s=1$, Zhang [40] gave in 1990 the following asymptotic formula, for all integer $q\geq 3$, we have

$$\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \left|L(1,\chi)\right|^4 = \frac{5}{72}\pi^4\varphi(q)\prod_{p|q}\frac{(p^2-1)^3}{p^4(p^2+1)}+\mathcal{O}\left(\exp\left(\frac{4\log q}{\log\log q}\right)\right).$$

Zhang conclude his paper by an asymptotic formula for kth mean value of L function at $s=1$, for any $q\geq 3$, let $k\geq 3$ be a fixed integer, we have

$$\sum_{\chi \bmod q} \left|L(1,\chi)\right|^{2k} = \varphi(q)\sum_{n=1}^{\infty}\frac{d_k^2(n)}{n^2}+\mathcal{O}\left(\exp\left(\frac{2k\log q}{\log\log q}\right)\right),$$

where $d_k(n)=\sum_{r_1\cdots r_k=n} 1$ is the k-th divisor function. But is this case, it is very difficult to express the series $\sum_{n=1}^{\infty}\frac{d_k^2(n)}{n^2}$ as a rational fraction of Riemann zera-function.

1.1.4 Work on $L'/L(s,\chi)$ at $s=1$

Less is known about L'/L evaluated also at the point $s=1$, through these values are known to be fundamental in studying the distribution of primes since Dirichlet in 1837.

Recently, Zhang [42] gave an asymptotic formula for L'/L at $s=1$. For a real number $Q>3$. we have

$$\sum_{q\le Q}\frac{q}{\varphi(q)}\sum_{\chi\ne\chi_0}\left|\frac{L'(1,\chi)}{L(1,\chi)}\right|^4=\frac{Q^2}{2}\sum_p\frac{(p^2+1)\log^4 p}{p(p+1)(p^2-1)^2}+2Q^2\left(\sum_p\frac{\log^2 p}{p^2-1}\right)\left(\sum_p\frac{\log^2 p}{p(p+1)}\right)$$
$$-2Q^2\sum_p\frac{(p^2-p+1)\log^4 p}{p^2(p^2-1)^2}+2Q^2\left(\sum_p\frac{\log^2 p}{p(p^2-1)}\right)^2+\mathcal{O}\left(Q\log^5 Q\right),$$

where the summation $\sum_p$ over all primes. Zhang proved his result by using the estimates of the character sums and the Bombieri-Vinogradov's theorem. Under the generalized Riemann hypothesis (and later in [14] unconditionally), Ihara and Matsumoto [13] gave a stronger result related to the value-distributions of $\{L'/L(s,\chi)\}_\chi$ and of $\{\zeta'/\zeta(s+i\tau)\}_\tau$, where χ runs over Dirichlet characters with prime conductors and τ runs over $\mathbb{R}$.
In the case of a quadratic field $\mathbb{Q}\left(\sqrt{D}\right)$ and D run over the odd prime numbers. Ihara, Murty and Shimura [15] in 2009 gave the following asymptotic formula

$$\frac{1}{|X_D|}\sum_{\chi\in X_D}P^{(a,b)}\left(\frac{L'}{L}(1,\chi)\right)=(-1)^{a+b}\mu^{(a,b)}+\mathcal{O}\left(D^{\epsilon-1}\right).$$

Here, we have $\mu^{(a,b)}$ is defined as

$$\mu^{(a,b)}=\sum_{n=1}^{\infty}\frac{\Lambda_a(n)\Lambda_b(n)}{n^2},$$

where the arithmetic function $\Lambda_k(n)$ is defined by $\Lambda_k(n)=\sum_{n=n_1\cdots n_k}\Lambda(n_1)\cdots\Lambda(n_k)$ for $k>0$ and $\Lambda_0(n)=0$ for all n except for $\Lambda_0(1)=1$. The X_D is the collection of all non-principal primitive Dirichlet characters $\chi:\ (\mathbb{Z}/D)^\times\longrightarrow\mathbb{C}^\times$. For each pair (a,b) of non-negative integers, we have $P^{(a,b)}(z)=z^a\overline{z}^b$. The authors proved this result unconditionally and for all $\epsilon>0$. More recently, Mourtada and Murty [22] gave an improvement of the result of Ihara and al. They proved that for $Y>1$, $k\ge 1$ and $\beta=\pm 1$, we have

$$\sum_{0<\beta D\le Y}\left(-\frac{L'}{L}(1,\chi_D)\right)^k=C_kY+\mathcal{O}_k\left(Y^{5/6+\epsilon}\right),$$

where χ_D be a quadratic Dirichlet character of conductor D and

$$C_k=\frac{3}{\pi^2}\sum_{n=1}^{\infty}\frac{\Lambda_k(n^2)}{n^2}\prod_{p|n}\left(1+\frac{1}{p}\right)^{-1},$$

Here the summation is over fundamental discriminants in $0<\beta D\le Y$.

1.2 Statement of results

In this thesis, we show that the values $\left|L'/L(1,\chi)\right|$ behave according to a distribution law. Let us state this result formally.

Theorem 1.1 *There exists a unique probability measure μ such that every continuous function f, we have*

$$\frac{1}{q-2}\sum_{\substack{\chi \bmod q\\ \chi\neq\chi_0}} f\left(\left|\frac{L'}{L}(1,\chi)\right|\right) \underset{q\to+\infty}{\longrightarrow} \int_0^{+\infty} f(t)\,d\mu(t), \tag{1.1}$$

where $\sum_{\chi \bmod q}$ denotes the summation over all the primitive characters $\chi \mod q$. Here the variable q ranges the odd primes.

We deduce the existence of μ by the general solution to the Stieltjes moment problem and the unicity by the criterion of Carleman.

This is an existence (and unicity) result, but getting an actual description of μ is a tantalizing problem. As we noted before, it is likely to have a geometrical or arithmetical interpretation, on which our approach gives no information. Here is a plot of the distribution function

$$D_q(t) = \frac{1}{q-2}\,\#\left\{\chi\neq\chi_0 \bmod q\,;\, \left|\frac{L'}{L}(1,\chi)\right|\leq t\right\}, \tag{1.2}$$

for $q = 59, 101$ and 257.

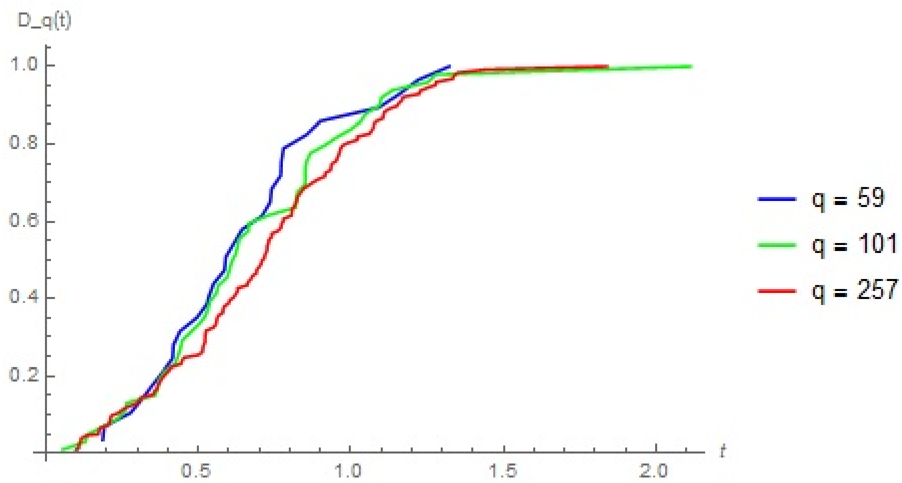

Figure 1.1: The distribution function $D_q(t)$.

The key to this result is to give an asymptotic formula of the $2k$-th power mean

$$\sum_{\substack{\chi \bmod q \\ \chi\neq\chi_0}} \left|\frac{L'}{L}(1,\chi)\right|^{2k} \tag{1.3}$$

Here is our result.

Theorem 1.2 *Let $\epsilon > 0$ and let χ be a primitive Dirichlet character modulo prime q. For k is an arbitrary non-negative integer, we have*

$$\sum_{\substack{\chi \bmod q \\ \chi\neq\chi_0}} \left|\frac{L'}{L}(1,\chi)\right|^{2k} = (q-2)\sum_{m\geq 1}\frac{\left(\sum_{m=m_1\cdot m_2\cdots m_k}\Lambda(m_1)\cdots\Lambda(m_k)\right)^2}{m^2} + \mathcal{O}_\epsilon\left(q^{9/10+\epsilon}\right),$$

where Λ is the Von Mangoldt's function.

The error term is not effective as we use the full force of Siegel's theorem on exceptional zeros.

We use an analytical method that is already used in [25] in contrast with the previous works that used essentially only elementary and combinatorial arguments. We introduce many simplifications in their intricate combinatorial argument, which is why we can handle the case of general k.

This result is strong in two aspects: it is valid for general k and we save a power of q (we did not try to optimize this saving). The method of proof relies in particular on a suitable average density estimate for the zeros of Dirichlet L-functions. Let us note that this approach does not work for $\frac{L'}{L}(1+it,\chi)$ when $t\neq 0$.

We also computed

$$\sum_{m\geq 1}\frac{\left(\sum_{m=m_1\cdot m_2}\Lambda(m_1)\Lambda(m_2)\right)^2}{m^2} = 0.80508\cdots,$$

and

$$\sum_{m\geq 1}\frac{\left(\sum_{m=m_1\cdots m_4}\Lambda(m_1)\cdots\Lambda(m_4)\right)^2}{m^2} = 1.242\cdots.$$

Here are some consequences of our main Theorem.

Corollary 1.3 *There exists $a > 0$ and $c > 0$ such that*

$$\liminf_q \frac{1}{q-2}\#\left\{\chi\neq\chi_0 \bmod q\,;\,\left|\frac{L'}{L}(1,\chi)\right|\leq a\right\}\leq 1-c.$$

Since we can control all the moments, we have an estimation for the size of the tail of our distribution.

Corollary 1.4 *For $t \geq 1$. We have*

$$\liminf_{q} \frac{1}{q-2} \# \left\{ \chi \neq \chi_0 \ mod \ q \ ; \ \left| \frac{L'}{L}(1,\chi) \right| \geq t \right\} \ll e^{-\sqrt{t}/2}.$$

We do not know whether this bound can be improved upon.

Chapter 2

Preliminary lemma

In this section we present some useful tools to prove our results. We finish the chapter by viewing two more important lemmas.

Lemma 2.1 *Let m and n be two positive integers. We have*

$$\sum_{f|q} \sum_{\chi \bmod^\star f} \chi(m)\overline{\chi(n)} = \begin{cases} \varphi(q(mn)) & \text{when } m \equiv n\,[q(mn)] \\ 0 & \text{otherwise,} \end{cases}$$

where χ ranges the primitive characters $\bmod f$ and φ is the Euler's function.

Proof. See [26], section 4.3. ⋄⋄⋄

Lemma 2.2 *Let M be an upper bound for the holomorphic function F in $|s-s_0| \le R$. Assume we know of a lower bound $m > 0$, for $|F(s_0)|$. Then*

$$\frac{F'(s)}{F(s)} = \sum_{|\rho - s_0| \le R/2} \frac{1}{s-\rho} + \mathcal{O}\left(\frac{\mathrm{Log}(M/m)}{R}\right)$$

for every s such that $|s-s_0| \le R/4$ and where the summation variable ρ ranges the zeros ρ of F in the region $|\rho - s_0| \le R/2$, repeated according to multiplicity.

Proof. See[9], Lemma 3.2 and [10]. See also [35], section 3.9 [Lemma a]. ⋄⋄⋄

Lemma 2.3 *There is a constant c such that, for any non-principal character χ modulo q, we have*

$$\frac{L'}{L}(s,\chi) \ll \mathrm{Log}\big(q|t|\big)$$

provided that

$$\Re s \geq 1 - \frac{c}{\operatorname{Log} q}, \quad |t| \leq q$$

except for at most one of them, which we call exceptional*, and for which we have*

$$\frac{L'}{L}(s,\chi) \ll_\varepsilon q^\varepsilon$$

in the above region.

Lemma 2.4 *We have, when $\sigma \geq 4/5$ and for any $\varepsilon > 0$,*

$$\sum_{f|q} \sum_{\chi \bmod^\star f} N(T,\chi,\sigma) \ll_\varepsilon (qT)^{2(1-\sigma)+\varepsilon},$$

where χ ranges the primitive characters $\bmod f$.

Proof. A proof of this Lemma can be found in [12]. ◇◇◇

2.1 Main lemma

Our main theorems 1.1 and 1.2 will follow from the following two lemmas under the hypothesis q is prime, otherwise, the combination will be complicated .

Lemma 2.5 *Let m_i, n_i and k be the positive integers for $i \in \{1,2,\cdots,k\}$ and let q be a prime number. Then, we have*

$$\sum_{\substack{m_1,\cdots,m_k,n_1,\cdots n_k \geq 1 \\ \underline{m} \equiv \underline{n} \, mod\, [q(\underline{m}\cdot\underline{n})]}} \varphi\left(q(\underline{m}\cdot\underline{n})\right) \frac{\prod_{i=1}^{k} \Lambda(m_i)\Lambda(n_i)}{\underline{m}\cdot\underline{n}} e^{-\underline{m}\cdot\underline{n}/X} =$$

$$(q-2) \sum_{m \geq 1} \frac{\left(\sum_{m = m_1 \cdot m_2 \cdots m_k} \Lambda(m_1)\cdots\Lambda(m_k) \right)^2}{m^2} + \mathcal{O}_\varepsilon\left(q e^{-q/X} + \frac{q}{X^{1-\varepsilon}} \right),$$

where φ is the Euler's function.

Proof. We study this summation

$$F_q(X) = \sum_{\substack{m_1,\cdots,m_k,n_1,\cdots n_k \geq 1 \\ \underline{m} \equiv \underline{n} \, mod\, [q(\underline{m}\cdot\underline{n})]}} \varphi\left(q(\underline{m}\cdot\underline{n})\right) \frac{\prod_{i=1}^{k} \Lambda(m_i)\Lambda(n_i)}{\underline{m}\cdot\underline{n}} e^{-\underline{m}\cdot\underline{n}/X}. \tag{2.1}$$

We split the domain defined by the condition $\underline{m} \equiv \underline{n}$ *mod* $\left[q(\underline{m}\cdot\underline{n})\right]$ in fourth parts.

- The first case is when $(q, \underline{m}\cdot\underline{n}) = 1$ and $\underline{m} \neq \underline{n}$. It follows that $q(\underline{m}\cdot\underline{n}) = q$, and thus $\underline{m} \equiv \underline{n}$ *mod* $[q]$. We get the contribution

$$\begin{aligned}
A_q(X) &= \sum_{\substack{m_1,\cdots,m_k,n_1,\cdots n_k\geq 1\\ \underline{m}\equiv\underline{n}\,\mathrm{mod}\,[q]\\ \underline{m}\neq\underline{n}}} \varphi(q(\underline{m}\cdot\underline{n}))\prod_{i=1}^{k}\Lambda(m_i)\Lambda(n_i)\frac{e^{-\underline{m}\cdot\underline{n}/X}}{\underline{m}\cdot\underline{n}}\\
&= \varphi(q)\sum_{\substack{\underline{m},\underline{n}\geq 1\\ \underline{m}\equiv\underline{n}\,\mathrm{mod}\,[q]\\ \underline{m}\neq\underline{n}}}\left(\sum_{\underline{m}=\prod_{i=1}^{k} m_i}\Lambda(m_1)\cdots\Lambda(m_k)\sum_{\underline{n}=\prod_{i=1}^{k} n_i}\Lambda(n_1\cdots\Lambda(n_k))\right)\frac{e^{-\underline{m}\cdot\underline{n}/X}}{\underline{m}\cdot\underline{n}}.
\end{aligned}$$

By using the fact that

$$\sum_{\underline{m}=\prod_{i=1}^{k} m_i}\Lambda(m_1)\cdots\Lambda(m_k)\sum_{\underline{n}=\prod_{i=1}^{k} n_i}\Lambda(n_1)\cdots\Lambda(n_k)\ll_\varepsilon \underline{m}^\varepsilon\underline{n}^\varepsilon,$$

we get

$$\begin{aligned}
A_q(X) &\ll_\varepsilon \varphi(q)\sum_{\substack{\underline{m},\underline{n}\geq 1\\ \underline{m}\equiv\underline{n}\,\mathrm{mod}\,[q]\\ \underline{m}<\underline{n}}}\frac{e^{-\underline{m}\cdot\underline{n}/X}}{\underline{m}^{1-\varepsilon}\underline{n}^{1-\varepsilon}}\\
&\ll_\varepsilon \varphi(q)\sum_{\underline{m}\geq 1}\sum_{\substack{k\geq 1\\ \underline{n}=\underline{m}+kq}}\frac{e^{-\underline{m}\cdot\underline{n}/X}}{\underline{m}^{1-\varepsilon}\underline{n}^{1-\varepsilon}} = \varphi(q)\sum_{\underline{m}\geq 1}\sum_{k\geq 1}\frac{e^{-\underline{m}(\underline{m}+kq)/X}}{\underline{m}^{1-\varepsilon}\left(\underline{m}+kq\right)^{1-\varepsilon}}\\
&\ll_\varepsilon \varphi(q)\sum_{k\geq 1}e^{-kq/X}\ll_\varepsilon \varphi(q)e^{-q/X}.
\end{aligned}$$

Therefore, we have

$$\boxed{A_q(X)\ll_\varepsilon qe^{-q/X}.} \tag{2.2}$$

- The second case is when $(q, \underline{m}\cdot\underline{n}) = 1$ and $\underline{m} = \underline{n}$, it follows that $q(\underline{m}) = q$. We get the contribution

$$B_q(X) = \sum_{\substack{m_1,\cdots,m_k,n_1,\cdots n_k\geq 1\\ (q,\underline{m}\cdot\underline{n})=1,\ \underline{m}=\underline{n}}}\varphi(q(\underline{m}\cdot\underline{n}))\prod_{i=1}^{k}\Lambda(m_i)\Lambda(n_i)\frac{e^{-\underline{m}\cdot\underline{n}/X}}{\underline{m}\cdot\underline{n}}.$$

Let us write the function $B_q(X)$ as follows

$$B_q(X) = B_q^\sharp(X) + B_q^\flat(X),$$

where

$$B_q^{\sharp}(X) = \sum_{\substack{m_1,\cdots,m_k,n_1,\cdots n_k \geq 1 \\ (q,\underline{m}\cdot\underline{n})=1,\ \underline{m}=\underline{n} \\ \underline{m} \leq X^{1/2}}} \varphi\big(q(\underline{m})\big) \prod_{i=1}^{k} \Lambda(m_i)\Lambda(n_i) \frac{e^{-\underline{m}^2/X}}{\underline{m}^2},$$

and

$$B_q^{\flat}(X) = \sum_{\substack{m_1,\cdots,m_k,n_1,\cdots n_k \geq 1 \\ (q,\underline{m}\cdot\underline{n})=1,\ \underline{m}=\underline{n} \\ \underline{m} > X^{1/2}}} \varphi\big(q(\underline{m})\big) \prod_{i=1}^{k} \Lambda(m_i)\Lambda(n_i) \frac{e^{-\underline{m}^2/X}}{\underline{m}^2}.$$

(A) **For the function $B_q^{\flat}(X)$.** Since $\underline{m} > X^{1/2}$, it follows that $e^{-\underline{m}^2/X} \leq 1$. Thus, we get

$$B_q^{\flat}(X) \ll \varphi(q) \sum_{\substack{m_1,\cdots,m_k,n_1,\cdots n_k \geq 1 \\ (q,\underline{m}\cdot\underline{n})=1,\ \underline{m}=\underline{n} \\ \underline{m} > X^{1/2}}} \frac{\prod_{i=1}^{k} \Lambda(m_i)\Lambda(n_i)}{\underline{m}^2}.$$

Using again the fact that $\prod_{i=1}^{k} \Lambda(m_i)\Lambda(n_i) \ll_\varepsilon \underline{m}^\varepsilon \cdot \underline{n}^\varepsilon \ll_\varepsilon \underline{m}^{2\varepsilon}$. We have

$$B_q^{\flat}(X) \ll_\varepsilon \frac{q}{X^{1-\varepsilon}}. \tag{2.3}$$

(B) **For the function $B_q^{\sharp}(X)$.** Since $\underline{m}^2$ is small enough, we can rely on the approximation

$$e^{-\underline{m}^2/X} = 1 + \mathcal{O}\left(\frac{\underline{m}^2}{X}\right),$$

which gives us

$$B_q^{\sharp}(X) = \varphi(q) \sum_{\substack{m_1,\cdots,m_k,n_1,\cdots n_k \geq 1 \\ (q,\underline{m}\cdot\underline{n})=1,\ \underline{m}=\underline{n} \\ \underline{m} \leq X^{1/2}}} \frac{\prod_{i=1}^{k} \Lambda(m_i)\Lambda(n_i)}{\underline{m}^2} + \mathcal{O}\left(\frac{\varphi(q)}{X} \sum_{\substack{m_1,\cdots,m_k,n_1,\cdots n_k \geq 1 \\ (q,\underline{m}\cdot\underline{n})=1,\ \underline{m}=\underline{n} \\ \underline{m} \leq X^{1/2}}} \prod_{i=1}^{k} \Lambda(m_i)\Lambda(n_i) \right).$$

By a similar argument as above, we get

$$B_q^{\sharp}(X) = \varphi(q) \sum_{\substack{m_1,\cdots,m_k,n_1,\cdots n_k \geq 1 \\ (q,\underline{m}\cdot\underline{n})=1,\ \underline{m}=\underline{n} \\ \underline{m} \leq X^{1/2}}} \frac{\prod_{i=1}^{k} \Lambda(m_i)\Lambda(n_i)}{\underline{m}^2} + \mathcal{O}_\varepsilon\left(\frac{\varphi(q)}{X^{1-\varepsilon}}\right).$$

Thus, we have

$$B_q^\sharp(X) = (q-2) \sum_{\substack{m_1,\cdots,m_k,n_1,\cdots n_k \geq 1 \\ (q,\underline{m}\cdot\underline{n})=1,\ \underline{m}=\underline{n}}} \frac{\prod_{i=1}^k \Lambda(m_i)\Lambda(n_i)}{\underline{m}^2} + \mathcal{O}_\varepsilon\left(\frac{q}{X^{1-\varepsilon}}\right). \tag{2.4}$$

From Eq (2.3) and (2.4), we find that

$$\boxed{B_q(X) = (q-2) \sum_{\substack{m_1,\cdots,m_k,n_1,\cdots n_k \geq 1 \\ (q,\underline{m}\cdot\underline{n})=1,\ \underline{m}=\underline{n}}} \frac{\prod_{i=1}^k \Lambda(m_i)\Lambda(n_i)}{\underline{m}^2} + \mathcal{O}_\varepsilon\left(\frac{q}{X^{1-\varepsilon}}\right).} \tag{2.5}$$

- The third case is when $(q,\underline{m}\cdot\underline{n}) \neq 1$ and $\underline{m} \neq \underline{n}$. It follows that $q(\underline{m}\cdot\underline{n}) = 1$ and thus $\varphi\left(q(\underline{m}\cdot\underline{n})\right) = 1$. We get the contribution

$$\begin{aligned} C_q(X) &= \sum_{\substack{m_1,\cdots,m_k,n_1,\cdots n_k \geq 1 \\ q|\underline{m}\cdot\underline{n},\ \underline{m}\neq\underline{n}}} \varphi\left(q(\underline{m}\cdot\underline{n})\right) \frac{\prod_{i=1}^k \Lambda(m_i)\Lambda(n_i)}{\underline{m}\cdot\underline{n}} e^{-\underline{m}\cdot\underline{n}/X} \\ &= \sum_{\substack{\ell\geq 1 \\ q|\ell}} \sum_{\ell=\underline{m}\cdot\underline{n}} \prod_{i=1}^k \Lambda(m_i)\Lambda(n_i) \frac{e^{-\ell/X}}{\ell}. \end{aligned}$$

We notice that

$$\sum_{\ell=\underline{m}\cdot\underline{n}} \prod_{i=1}^k \Lambda(m_i)\Lambda(n_i) \ll_\varepsilon \ell^\varepsilon.$$

Thus, we have

$$C_q(X) \ll_\varepsilon \sum_{\substack{\ell\geq q \\ q|\ell}} \frac{e^{-\ell/X}}{\ell^{1-\varepsilon}}.$$

Since $q|\ell$, we write $\ell = qu$. Then

$$\boxed{C_q(X) \ll_\varepsilon \frac{1}{q^{1-\varepsilon}} \sum_{u\geq 1} \frac{e^{-qu/X}}{u^{1-\varepsilon}} \ll_\varepsilon \frac{e^{-q/X}}{q^{1-\varepsilon}}.} \tag{2.6}$$

- The fourth case is when $(q,\underline{m}\cdot\underline{n}) \neq 1$ and $\underline{m} = \underline{n}$. It follows that $q(\underline{m}) = 1$. We get the contribution

$$\begin{aligned}
D_q(X) &= \sum_{\substack{m_1,\cdots,m_k,n_1,\cdots n_k\geq 1\\ q|\underline{m},\ \underline{m}=\underline{n}}} \varphi\left(q(\underline{m}\cdot\underline{n})\right)\prod_{i=1}^{k}\Lambda(m_i)\Lambda(n_i)\frac{e^{-\underline{m}\cdot\underline{n}/X}}{\underline{m}\cdot\underline{n}}\\
&= \sum_{\substack{m_1,\cdots,m_k,n_1,\cdots n_k\geq 1\\ q|\underline{m},\ \underline{m}=\underline{n}}} \prod_{i=1}^{k}\Lambda(m_i)\Lambda(n_i)\frac{e^{-\underline{m}^2/X}}{\underline{m}^2}.
\end{aligned}$$

It follows that

$$\begin{aligned}
D_q(X) &= \sum_{\substack{\underline{m},\underline{n}\geq 1\\ q|\underline{m},\ \underline{m}=\underline{n}}}\left(\sum_{\underline{m}=\prod_{i=1}^{k} m_i}\Lambda(m_1)\cdots\Lambda(m_k)\sum_{\underline{n}=\prod_{i=1}^{k} n_i}\Lambda(n_1)\cdots\Lambda(n_k)\right)\frac{e^{-\underline{m}^2/X}}{\underline{m}^2}\\
&\ll_\varepsilon \sum_{\substack{\underline{m}\geq 1\\ q|\underline{m}}}\frac{e^{-\underline{m}^2/X}}{\underline{m}^{2(1-\varepsilon)}} = \sum_{u\geq 1}\frac{e^{-(qu)^2/X}}{(qu)^{2(1-\varepsilon)}} \ll_\varepsilon \frac{e^{-q^2/X}}{q^{2(1-\varepsilon)}}.
\end{aligned}$$

Therefore, we have

$$\boxed{D_q(X) \ll_\varepsilon \frac{e^{-q^2/X}}{q^{2(1-\varepsilon)}}.} \tag{2.7}$$

From Eq (2.2), (2.5), (2.6) and (2.7), we conclude that

$$F_q(X) = (q-2)\sum_{\substack{m_1,\cdots,m_k,n_1,\cdots n_k\geq 1\\ (q,\underline{m}\cdot\underline{n})=1,\ \underline{m}=\underline{n}}}\frac{\prod_{i=1}^{k}\Lambda(m_i)\Lambda(n_i)}{\underline{m}^2} + \mathcal{O}_\varepsilon\left(qe^{-q/X} + \frac{q}{X^{1-\varepsilon}}\right).$$

This completes the proof. ◇◇◇

Lemma 2.6 *For any integer number $k \geq 1$, we have*

$$\sum_{m_1\cdot m_2\cdots m_k=m}\Lambda(m_1)\cdots\Lambda(m_k) \leq (\log m)^k \tag{2.8}$$

Proof. We prove this lemma by induction on k. For $k = 1$ is easily. In order to show that Eq (2.8) is valid for $k = 2$, we write

$$\sum_{m_1 m_2=m}\Lambda(m_1)\Lambda(m_2) \leq \log m \sum_{m_1 m_2=m}\Lambda(m_2) \leq (\log m)^2$$

Now, we assume that Eq (2.8) is valid for any fixed and non-negative integer $k-1$. Then we have to prove that it is also valid for k. By induction hypothesis, we have

$$\begin{aligned}\sum_{m_1\cdot m_2\cdots m_k=m}\Lambda(m_1)\cdots\Lambda(m_k) &= \sum_{m_1 n=m}\Lambda(m_1)\sum_{m_2\cdot m_3\cdots m_k=n}\Lambda(m_2)\cdots\Lambda(m_k)\\ &\leq \sum_{m_1 n=m}\Lambda(m_1)\log^{k-1} n \leq (\log m)^k\end{aligned}$$

We conclude from the above that Eq (2.8) is valid for k. Then it is valid for all $k \geq 1$. The lemma is proved. ⋄⋄⋄

Chapter 3

Poofs

In this chapter we prove our results.

3.1 Proof of Theorem 1.2

For q is prime, we consider the function

$$G_q(s) = \sum_{\substack{\chi \bmod q \\ \chi \neq \chi_0}} \left(\frac{L'}{L}(s,\chi)\right)^k \left(\frac{L'}{L}(s,\bar{\chi})\right)^k$$

where χ ranges the non-principle primitive characters modulo q. When $\Re s > 1$, the series converges absolutely. Applying the following definition of L'/L

$$\frac{L'}{L}(s,\chi) = \sum_{n\geq 1} \frac{\chi(n)\Lambda(n)}{n^s},$$

we write the function $G_q(s)$ as

$$G_q(s) = \sum_{\substack{\chi \bmod q \\ \chi \neq \chi_0}} \sum_{\substack{m_1\cdots m_k \geq 1 \\ n_1 \cdots n_k \geq 1}} \frac{\prod_{i=1}^{k} \Lambda(m_i)\chi(m_i) \prod_{i=1}^{k} \Lambda(n_i)\bar{\chi}(n_i)}{\left(\prod_{i=1}^{k} m_i n_i\right)^s}.$$

The proof relies on two distinct evaluations of the quantity:

$$S_q(X) = \frac{1}{2} \int_{2-i\infty}^{2+i\infty} G_q(s) X^{s-1} \Gamma(s-1)\, ds. \tag{3.1}$$

1. **The first evaluation** relies on the formula $e^{-y} = \frac{1}{2i\pi}\int_{2-i\infty}^{2+i\infty} y^{-s}\Gamma(s)\,ds$ (valid for positive y) and on using Lemma 2.1. We readily find that

$$S_q(X) = \sum_{\substack{m_1,\cdots,m_k,n_1,\cdots,n_k\geq 1\\ \underline{m}\equiv\underline{n} \bmod [q(\underline{m}\cdot\underline{n})]}} \varphi\left(q(\underline{m}\cdot\underline{n})\right)\frac{\prod_{i=1}^{k}\Lambda(m_i)\Lambda(n_i)}{\underline{m}\cdot\underline{n}}e^{-\underline{m}\cdot\underline{n}/X}.$$

Thanks to Lemma 2.5, we get

$$\boxed{S_q(X) = (q-2)\sum_{m\geq 1}\frac{\left(\sum_{m=m_1\cdots m_k}\Lambda(m_1)\cdots\Lambda(m_k)\right)^2}{m^2} + \mathcal{O}_\varepsilon\left(qe^{-q/X}+\frac{q}{X^{1-\varepsilon}}\right).} \quad (3.2)$$

2. **The second evaluation:** On selecting $\sigma = 9/10$, $\varepsilon = 1/10$ and $T = q$ in Lemma 2.4, we see that at most $\mathcal{O}(q^{3/5})$ characters modulo a divisor of q have a zero in the region

$$|\Im\rho| \leq q,\ \Re\rho \geq 9/10.$$

We call the characters which are in this set **bad characters** and the other set, the one of **good characters**. Now, we shift the line of integration in Eq (3.1) to

- $\Re s = 9/10$ and $|\Im s| \leq q$ when χ belongs to the good set;
- $\Re s = 1 - c/\log q$ and $|\Im s| \leq q$ when χ belongs to the bad set; Here c is the constant from Lemma 2.3.

Then, we have:

(i) For the bad character, when $\Re s = 1 - c/\log q$ and $|\Im s| \leq q$. Lemma 2.3 gives us that

$$L'/L(s,\chi) \ll \log q.$$

We have

$$\left|X^{s-1}\right| \leq X^{-\frac{c}{\log q}}.$$

Thus

$$\begin{aligned}
\int_{1-\frac{c}{\log q}-iq}^{1-\frac{c}{\log q}+iq} \left|G_q(s)X^{s-1}\Gamma(s-1)\,ds\right| &\leq q^{3/5}(\log q)^{2k}X^{-\frac{c}{\log q}}\int_{1-\frac{c}{\log q}-iq}^{1-\frac{c}{\log q}+iq}|\Gamma(s-1)|\,ds \\
&\leq q^{3/5}(\log q)^{2k}X^{-\frac{c}{\log q}}\int_{-q}^{q}\left|\Gamma\left(-\frac{c}{\log q}+it\right)\right|dt
\end{aligned}$$

Recall that the formula of complex Stirling is given by

$$\Gamma(z+a) = \sqrt{2\pi}\, e^{-z} z^{z+a-1/2} \left(1 + \mathcal{O}\left(1/|z|\right)\right),$$

where a is a complex number fixed, $|\arg z| \leq \pi$ and $|z| \geq 1$. Thanks to this formula, we deduce that

$$\left|\Gamma\left(-\frac{c}{\log q} + it\right)\right| \ll \sqrt{2\pi}\, |t|^{-\frac{c}{\log q} - \frac{1}{2}} e^{-\pi t/2},$$

and that

$$\int_{1-\frac{c}{\log q}-iq}^{1-\frac{c}{\log q}+iq} \left|G_q(s) X^{s-1} \Gamma(s-1)\, ds\right| \ll q^{3/5} (\log q)^{2k} X^{-\frac{c}{\log q}}. \tag{3.3}$$

(ii) Even for the exceptional one, we have:

$$L'/L(s,\chi) \ll_\varepsilon q^\varepsilon,$$

when $\Re s = 1 - c/q^\varepsilon$ and $|\Im s| \leq 1$. Then, we get

$$\begin{aligned} \int_{1-\frac{c}{q^\varepsilon}-i}^{1-\frac{c}{q^\varepsilon}+i} \left|G_q(s) X^{s-1} \Gamma(s-1)\, ds\right| &\leq q^{2k\varepsilon} X^{-\frac{c}{q^\varepsilon}} \int_{-1}^{1} \left|\Gamma\left(-\frac{c}{q^\varepsilon} + it\right)\right| dt \\ &\ll_\varepsilon q^{2k\varepsilon} X^{-\frac{c}{q^\varepsilon}} \log^\varepsilon q \end{aligned} \tag{3.4}$$

(iii) For a good character, when $\Re s = \frac{9}{10}$ and $|\Im s| \leq q$. Lemma 2.2 gives us that

$$L'/L(s,\chi) \ll \log q.$$

We have also

$$\left|X^{s-1}\right| \leq X^{-\frac{1}{10}}.$$

We deduce that

$$\begin{aligned} \int_{\frac{9}{10}-iq}^{\frac{9}{10}+iq} \left|G_q(s) X^{s-1} \Gamma(s-1)\, ds\right| &\leq \varphi(q) X^{-\frac{1}{10}} \log^{2k} q \int_{\frac{9}{10}-iq}^{\frac{9}{10}+iq} |\Gamma(s-1)|\, ds \\ &\leq q X^{-\frac{1}{10}} \log^{2k} q \int_{0}^{1} \left|\Gamma\left(-\frac{1}{10} + it\right)\right| dt \end{aligned}$$

Applying again the formula of Stirling, we get

$$\left|\Gamma\left(-\frac{1}{10}+it\right)\right| \ll \sqrt{2\pi}\,|t|^{-\frac{3}{5}}\, e^{-\pi t/2}.$$

Therefore, we have

$$\int_{\frac{9}{10}-iq}^{\frac{9}{10}+iq} \left|G_q(s)X^{s-1}\Gamma(s-1)\,ds\right| \ll qX^{-\frac{1}{10}}\log^{2k} q \tag{3.5}$$

(vi) For all characters in the region $\Re s = 2$ and $|\Im s| \geq q$, we find that

$$\int_{\substack{\Re s=2\\|\Im s|\geq q}} \left|G_q(s)X^{s-1}\Gamma(s-1)\,ds\right| \ll Xq\log^{2k} q \int_{t\geq q} |\Gamma(1+it)|\,dt \ll Xq\log^{2k} q\, e^{-q\pi/2} \tag{3.6}$$

Since q is a prime number. From Eq (3.3), (3.4), (3.5) and (3.6), we obtain that

$$\boxed{S_q(X) = \sum_{\substack{\chi \bmod q\\ \chi\neq\chi_0}} \left|\frac{L'}{L}(1,\chi)\right|^{2k} + \mathcal{O}\left(q^{3/5}X^{-\frac{c}{\log q}} + qX^{-\frac{1}{10}}\right).} \tag{3.7}$$

The first term on the right-hand side above comes from case $s=1$.
From Eq (3.2) and (3.7), we conclude that

$$\sum_{\substack{\chi \bmod q\\ \chi\neq\chi_0}} \left|\frac{L'}{L}(1,\chi)\right|^{2k} = (q-2)\sum_{m\geq 1}\frac{\left(\sum_{m=m_1\cdots m_k}\Lambda(m_1)\cdots\Lambda(m_k)\right)^2}{m^2}$$
$$+\mathcal{O}_\varepsilon\left(qe^{-q/X} + \frac{q}{X^{1-\varepsilon}} + q^{3/5}X^{-\frac{c}{\log q}} + qX^{-\frac{1}{10}}\right)$$

For $X = q^{1-\varepsilon}$ and $\varepsilon > 0$, we conclude that

$$\sum_{\substack{\chi \bmod q\\ \chi\neq\chi_0}} \left|\frac{L'}{L}(1,\chi)\right|^{2k} = (q-2)\sum_{m\geq 1}\frac{\left(\sum_{m=m_1\cdots m_k}\Lambda(m_1)\cdots\Lambda(m_k)\right)^2}{m^2} + \mathcal{O}_\varepsilon\left(q^{9/10+\varepsilon}\right).$$

This completes the proof.

3.2 Proof of Theorem 1.1

We recall that μ_q is defined by $\mu_q([0,t]) = D_q(t)$, where $D_q(t)$ is given by Eq (1.2). Then, we have μ_q is non-negative and $\mu_q([0,\infty[) = 1$. Setting

$$m_k = m_k(q) = \int_0^\infty t^k \, d\mu_q(t),$$

and

$$\Delta_k(q) = \begin{vmatrix} m_0 & m_1 & m_2 & \cdots & m_k \\ m_1 & m_2 & m_3 & \cdots & m_{k+1} \\ m_2 & m_3 & m_4 & \cdots & m_{k+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ m_k & m_{k+1} & m_{k+2} & \cdots & m_{2k} \end{vmatrix}, \quad \Delta_k^\star(q) = \begin{vmatrix} m_1 & m_2 & m_3 & \cdots & m_{k+1} \\ m_2 & m_3 & m_4 & \cdots & m_{k+2} \\ m_3 & m_4 & m_5 & \cdots & m_{k+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ m_{k+1} & m_{k+2} & m_{k+3} & \cdots & m_{2k+1} \end{vmatrix}$$

For q is fixed, we have $\Delta_k(q)$ and $\Delta_k^\star(q)$ are non-negative. Thus, when q goes to infinity, we get $\Delta_k(\infty)$ and $\Delta_k^\star(\infty)$ are non-negative for all $k \geq 1$. By the solution of the Stieltjes moment problem, we deduce that μ exists.

In order to complete our proof, it remains to show that μ unique. Define

$$M_k = \sum_{m\geq 1} \frac{\left(\sum\limits_{m=m_1\cdot m_2 \cdots m_k} \Lambda(m_1)\cdots\Lambda(m_k) \right)^2}{m^2}. \tag{3.8}$$

Using Lemma 2.6, we get

$$M_k \leq \sum_{m\geq 2^k} \frac{(\log m)^{2k}}{m^2}.$$

Now, we notice that

$$\begin{aligned} \sum_{m\geq 2^k} \frac{(\log m)^{2k}}{m^2} &\leq \int_{2^k}^\infty \frac{(\log t)^{2k}}{t^2} dt + \frac{(\log e^k)^{2k}}{e^{2k}} \\ &\leq \int_{k\log 2}^\infty u^{2k} e^{-u} du + \left(\frac{k}{e}\right)^{2k} \leq \Gamma(2k+1) + \left(\frac{k}{e}\right)^{2k} = (2k)! + \left(\frac{k}{e}\right)^{2k}. \end{aligned}$$

Then, we have

$$M_k \leq (2k)! + \left(\frac{k}{e}\right)^{2k}. \tag{3.9}$$

Therefore, we get

$$\sum_{k\geq 1} \frac{1}{M_k^{\frac{1}{2k}}} \gg \sum_{k\geq 1} \left(\frac{1}{(2k)!}\right)^{\frac{1}{2k}} + \sum_{k\geq 1} \frac{e}{k} = \infty$$

It follows that the condition of Carleman is checked and thus the function μ is unique. This completes the proof.

3.3 Proof of Corollary 1.3

To do so, it is enough to note that $\mu \neq \delta_0$ where δ_0 is the measure of Dirac at 0 (compare the moments). Thus, there exists a compact interval $I = [a,b]$ such that $\mu(I)$ is positive. It follows that

$$\mu([a,b]) = \liminf_q \frac{1}{q-2}\#\left\{\chi \neq \chi_0 \ mod\ q\ ;\ \left|\frac{L'}{L}(1,\chi)\right| \in [a,b]\right\} > 0.$$

This completes the proof.

3.4 Proof of Corollary 1.4

We note that

$$\frac{1}{q-2}\sum_{\substack{\chi \bmod q\\ \chi\neq\chi_0}}\left|\frac{L'}{L}(1,\chi)\right|^{2k} \geq \frac{t^{2k}}{q-2}\#\left\{\chi \neq \chi_0 \ mod\ q\ ;\ \left|\frac{L'}{L}(1,\chi)\right| \geq t\right\}$$

Then, we have

$$\frac{1}{q-2}\#\left\{\chi \neq \chi_0 \ mod\ q\ ;\ \left|\frac{L'}{L}(1,\chi)\right| \geq t\right\} \leq \frac{M_k}{t^{2k}}$$

where M_k is given by Eq (3.8) and $M_k \ll (2k)!$ by the preceding proof. Thanks to the Stirling formula, we write

$$\frac{1}{q-2}\#\left\{\chi \neq \chi_0 \ mod\ q\ ;\ \left|\frac{L'}{L}(1,\chi)\right| \geq t\right\} \ll \left(\frac{4k^2}{e^2\,t}\right)^k \sqrt{4\pi k}.$$

Taking $k = \sqrt{t}$, we get

$$\liminf_q \frac{1}{q-2}\#\left\{\chi \neq \chi_0 \ mod\ q\ ;\ \left|\frac{L'}{L}(1,\chi)\right| \geq t\right\} \ll e^{-\sqrt{t}/2}.$$

This completes the proof.

3.5 Scripts

We present here an easier GP script for computing the values $\left|L'/L(1,\chi)\right|$. In this loop, we use the Pari package " ComputeL" written by Tim Dokchitser to compute values of L- functions and its derivative. This package is available on-line at

`www.maths.bris.ac.uk/~matyd/`

On this base we write the next script. I am indebted to Olivier Ramaré for helping me in writing it. We simply plot Figure (1.1) via

```
 read("computeL"); /* by Tim Dokchitser */
default(realprecision,28);
{run(p=37)=
   local(results, prim, avec);
   prim = znprimroot(p);
   results = vector(p-2, i, 0);
   for(b = 1, p-2,
      avec = vector(p,k,0);
      for (k = 0, p-1, avec[lift(prim^k)+1]=exp(2*b*Pi*I*k/(p-1)));
      conductor = p;
      gammaV    = [1];
      weight    = b%2;
      sgn       = X;
      initLdata("avec[k%p+1]",,"conj(avec[k%p+1])");
      sgneq = Vec(checkfeq());
      sgn   = -sgneq[2]/sgneq[1];
      results[b] = abs(L(1,,1)/L(1));
         \\print(results[b]);
      );
   return(results);
}

{goodrun(borneinf, bornesup)=
   forprime(p = borneinf, bornesup,
            print("------------------------");
            print("p = ",p);
            print(vecsort(run(p))));}
```

Part II

An effective Vander Corput inequality

Chapter 4

Introduction

4.1 Introduction and results

We investigate here inequalities of the shape

$$d^s(n) \le C(s,\delta,\beta) \sum_{\substack{\ell|n,\\ \ell\le n^\delta}} d^\beta(\ell) \tag{4.1}$$

where (s,δ,β) are to be selected; the problems are, given s, to detect whether such an inequality is possible and we define otherwise $C(s,\delta,\beta)=\infty$. Otherwise, we define $C(s,\delta,\beta)$ as being the least possible constant.

Such inequalities stem from the work of Wolke [37] which offers a simpler proof of the van der Corput Theorem [36]. Deshouillers & Dress introduced in [3] inequalities of the shape

$$f(n) \le C \sum_{\substack{\ell|n,\\ \ell\le\sqrt{n}}} g(\ell)$$

where f is a given positive multiplicative function and g is the sought solution. Landreau elaborated on this work in [19] and gave an extremely simple proof of the inequality

$$d^s(n) \le k^{k(k-1)s} \sum_{\substack{\ell|n,\\ \ell\le n^{1/k}}} d^{ks}(\ell)$$

for any positive s and any positive integer k. In this line of work, the proof goes by exhibiting a single divisor ℓ of n for which $d^s(n) \ll d^{ks}(\ell)$. For instance Friedlander & Iwaniec showed in [5, Lemma 1 and 2] that every integer n has a divisor $\ell \le n^{1/k}$ such that

$$d(n) \le (2d(\ell))^{\frac{k\log k}{\log 2}}.$$

We also exhibit a special divisor in Lemma 6.2 when the integer n is very powerfull. When investigating a related problem concerning a non-negative multiplicative function taking small values at primes and a sum over restricted divisors, Alladi, Erdös & Vaaler in [16] and [17] showed that any square-free integer n of the form $n = p_1 p_2 \cdots p_{k\ell}$ for some positive integers k and ℓ has at least $\binom{k\ell}{\ell}/k$ divisors d not more than $n^{1/k}$ and having exactly ℓ prime factors. Mohan gave in [21] a short and self-contained proof of this result and we finally mention that Soundararajan has in [32] the best result on the Alladi-Erdös-Vaaler problem.

Friedlander & Iwaniec in [6] [7, Lemma 6.2-6.3] modified their approach for square-free numbers and also used many divisors of n to get the required inequality. They particularly obtained that, when $k \geq 2$ is an integer and n is square-free:

$$\mu^2(n)d(n) \leq k^{(k\log 2)^2} \sum_{\substack{\ell|n,\\ \ell\leq n^{1/k}}} d^{k-\frac{\log k}{\log 2}}(\ell). \tag{4.2}$$

They further obtained the following two specific inequalities:

$$\mu^2(n)d(n) \leq 9 \sum_{\substack{\ell|n,\\ \ell\leq n^{1/3}}} d(\ell) \tag{4.3}$$

as well as

$$\mu^2(n)d(n) \leq 256 \sum_{\substack{\ell|n,\\ \ell\leq n^{1/4}}} d(\ell)^{2(\log 3)/\log 2}. \tag{4.4}$$

Since $2(\log 3)/\log 2 < 4$, this indeed improves on [19].

This work was pursued by Munshi in [23] for square-free integers and obtained

$$\mu^2(n)d(n) \ll \sum_{\substack{\ell|n,\\ \ell\leq n^{\delta}}} d^{\beta}(\ell) \tag{4.5}$$

for any

$$\beta > \beta(\delta) = \frac{1}{\delta} + \frac{\log H(\delta)}{\delta \log 2}, \quad H(\delta) = \delta^{\delta}(1-\delta)^{1-\delta}. \tag{4.6}$$

We have $H(0) = H(1) = 1$ and the function H is decreasing on $[0, 1/2]$. The constant implied in (4.5) is this time unspecified and the proof cannot be (at least to our eyes!) made to yield a specific value. The restriction to square-free integers seems easier to relax when δ is small but *not* when δ is somewhat large (i.e. about $\delta \geq 1/10$), as explained below. Note that the $\beta(1/3) = 0.245\cdots < 1$ and that $\beta(1/4) = 0.754\cdots < 1$ which improves considerably on [7, Lemma 6.2-6.3].

In this part, we employ a combinatorial method that leads to effective result and recover an effective version of the Theorem of Munshi. We are also just off the optimal exponent.

Theorem 4.1 *Define*

$$\beta_{s,b}(\delta) = \begin{cases} \frac{s\log(b+1)}{b\delta\log 2} + \frac{\log H(b\delta)}{b\delta\log 2} & \text{when } b \le 1/(2\delta), \\ \frac{2s\log(b+1)}{\log 2} - \left(\frac{2b\delta-1}{b-1}\right)\frac{\log b}{\log 2} - 2 & \text{when } b > 1/(2\delta). \end{cases}$$

When $\beta = \beta_{s,b}(\delta) + \epsilon$ for some positive ϵ, we have

$$\mu^2(n) d^s(n^b) \le c_s(\delta, b, \epsilon) \sum_{\substack{\ell | n, \\ \ell \le n^{b\delta}}} d^\beta(\ell)$$

where $c_s(\delta, b, \epsilon)$ is given by

$$c_s(\delta, b, \epsilon) = \begin{cases} (b+1)^{s/(b\delta)} \max\left(1, 3H(b\delta)^{1/(b\delta)} / \sqrt{\min(1, 4\epsilon/3)}\right) & \text{when } b \le 1/(2\delta), \\ (b+1)^{2s} \max\left(1, \frac{3}{4} / \sqrt{\min(1, 4\epsilon/3)}\right) & \text{when } b > 1/(2\delta). \end{cases}$$

When the number $\omega(n)$ of prime factors of n is $> \delta^{-1}$, only the second part of the maximum is necessary.

Though we are not able to reach the best exponent $\beta_{s,b}(\delta)$ but we can be quite close to it when δ tends to 0.

Corollary 4.2 *We have*

$$\mu^2(n) d^s(n) \le 2^{s/\delta} \sum_{\substack{\ell | n, \\ \ell \le n^\delta}} d^\beta(\ell) \quad \text{with} \quad \beta = \beta_{s,1}(\delta) = \frac{s}{\delta} + \frac{\log H(\delta)}{\delta \log 2} + \frac{27}{16}\delta^2,$$

where $H(\delta)$ is defined by Eq (4.6).

We prove also some specific bounds by invoking Theorem 4.1.

Corollary 4.3 *We have*

$$\mu^2(n) d(n) \le 8 \sum_{\substack{\ell | n, \\ \ell \le n^{1/3}}} d(\ell), \quad \mu^2(n) d(n) \le 16 \sum_{\substack{\ell | n, \\ \ell \le n^{1/4}}} d(\ell),$$

and

$$\mu^2(n) d(n) \le 25 \sum_{\substack{\ell | n, \\ \ell \le n^{3/13}}} d(\ell), \quad \mu^2(n) d(n) \le 11 \sum_{\substack{\ell | n, \\ \ell \le n^{3/10}}} d(\ell),$$

as well as

$$\mu^2(n) d^2(n) \le 64 \sum_{\substack{\ell | n, \\ \ell \le n^{1/3}}} d^4(\ell), \quad \mu^2(n) d^2(n) \le 256 \sum_{\substack{\ell | n, \\ \ell \le n^{1/4}}} d^5(\ell),$$

We next prove a theorem that is valid for non especially square-free integers. When δ and $\beta > 1$ be two positive real numbers, we define $A(\delta,\beta)$ as the smallest integer α such that

$$(1+[\alpha\delta])^{\beta} \geq 1+\alpha.$$

We prove in Lemma 6.2 that $A(\delta,\beta) \leq \delta^{-1}(1+\delta^{-1})^{1/(\beta-1)}+1$.

Theorem 4.4 *When $\beta \geq 2$ is of the form*

$$\beta = \beta_{s,1}(\delta)+\epsilon = \frac{s}{\delta}+\frac{\log H(\delta)}{\delta \log 2}+\epsilon,$$

for some positive ϵ, and real valued parameters $\delta > 0$ and $s \geq 1$ that satisfy one of the conditions:

$\delta \leq$	*0.5*	*0.1457*	*0.1007*	*0.0870*	*0.0614*
$s \leq$	*1.455*	*2*	*3*	*4*	∞

.

We have

$$d^{s}(n) \leq c^{\star}(\delta,\beta) \sum_{\substack{\ell \mid n, \\ \ell \leq n^{\delta}}} d^{\beta}(\ell)$$

where $c^{\star}(\delta,\beta)$ is given by

$$c_s^{\star}(\delta,\beta) = A(\delta,\beta)^{s(A(\delta,\beta)-1)/\delta} \prod_{1\leq b\leq 1/(2\delta)} \frac{3(b+1)^{s/(b\delta)}H(b\delta)^{1/(b\delta)}}{\sqrt{\min(1,4(\beta-\beta_{s,b}(\delta))/3)}} \times \prod_{1/(2\delta)<b\leq A(\delta,\beta)-1} \frac{3(b+1)^{2s}}{4\sqrt{\min(1,4(\beta-\beta_{s,b}(\delta))/3)}}.$$

We have $c_s^{\star}(\delta,\beta) \leq (\delta^5 \min(1,\epsilon))^{-3s/\delta^3}$.

The hypotheses have been designed so as to be simple for applications and with the lowest possible β, though always $\beta \geq 2$. We could design differents sets of hypotheses to allow lower values of β.

Chapter 5

A combinatorial lemma

We consider finite set S together with an integer-valued height $h : S \mapsto [1,\infty[$. A subset I of the *multiset* $\langle S, h\rangle$ is an integer valued function such that $0 \le I(s) \le h(s)$ for every s in S. We consider also a non-negative function f on subsets of $\langle S, h\rangle$. For every non-negative real number δ, any non-negative function w on S such that

$$\sum_{s\in S} h(s)\,w(s) = 1, \tag{5.1}$$

we consider

$$T_{S,h}(\delta, w) = \sum_{\substack{I\subset\langle S,h\rangle,\\ \sum_{s\in S} I(s)\,w(s)\le\delta}} f(I) \tag{5.2}$$

as well as

$$T^*_{S,h}(\delta) = \min_w T_{S,h}(\delta, w). \tag{5.3}$$

The main novelty of part of this section is contained in the following lemma.

Lemma 5.1 *The minimum in $T^*_{S,h}(\delta)$ is reached when the probability measure hw is uniform on a subset of S. This means that there exists a subset J of S such that such that $w(s) = 0$ when $s \notin J$ and $h(s)\,w(s) = 1/|J|$ otherwise.*

Proof. We set $r = |S|$ and identify S with $\{1, \cdots, r\}$. We still denote by h the corresponding height function. We consider the smoothed version

$$T_{r,h}(\delta, x_1, \cdots, x_r, F) = \sum_{I\subset\langle S,h\rangle} f(I)\,F\Big(\sum_{s\in S} I(s)x_s/\delta\Big), \quad \sum_{s\in S} h(s)x_s = 1 \tag{5.4}$$

where F is a non-increasing C^1 function over $[0,\infty[$. The partial derivatives being all non-negative, the extrema of this function are reached on extremal points of the considered domain. Those are indexed by subsets J of $\{1, \cdots, r\}$: we have $x_s = 0$ when

$s \notin J$ and $h(s)x_s = 1/|J|$ when $s \in S$ (provided there exists such a s, in which case $|J| > 0$).

Next, we approximate the characteristic function of the interval $[0,1]$ by a sequence of functions F_n as above. For each F_n we thus get a subset J_n, and since the number of possible subsets is finite, one of them appears infinitely many times. We take the limit of F_n along this subsequence and get our lemma. ⋄⋄⋄

Let us prepare the ground for applying this lemma. First we notice that one can write

$$T^*_{S,h}(\delta) = \min_{J\subset S} \sum_{\substack{0\le I\le h,\\ \sum_{s\in J} I(s)/(h(s)|J|)\le\delta}} f(I). \tag{5.5}$$

When $f(I) = \prod_{s\in S} f_0(I(s))$, we find that

$$\begin{aligned} T^*_{S,h}(\delta) &= \min_{J\subset S} \sum_{\substack{0\le I\le h,\\ \sum_{s\in J} I(s)/h(s)\le\delta|J|}} \prod_{s\in S} f_0(I(s)) \\ &= \min_{J\subset S} \prod_{s\in S\setminus J} \bigl(1+f_0(1)+\cdots+f_0(h(s))\bigr) \sum_{\substack{0\le I\le h_{|J},\\ \sum_{s\in J} I(s)/h(s)\le\delta|J|}} \prod_{s\in J} f_0(I(s)) \end{aligned}$$

In our case of application, we select $f_0(t) = x$ for some $x \ge 0$.

Lemma 5.2 *We have, for any $x \ge 0$, any $\delta \in (0,1/2]$ and any $r \ge 1$,*

$$\sum_{\substack{\ell | p_1^{h(1)}\cdots p_r^{h(r)},\\ \ell^{1/\delta}\le p_1^{h(1)}\cdots p_r^{h(r)}}} x^{\omega(\ell)} \ge \min_{J\subset S} \prod_{s\in S\setminus J} (1+h(s)x) \sum_{\substack{K\subset J,\\ |K|\le\delta|J|}} \prod_{s\in K} (h(s)x)$$

Proof. This is obtained with the choice

$$w(s) = \frac{\log p_s}{\log n}.$$

We first get

$$\sum_{\substack{\ell | p_1^{h(1)}\cdots p_r^{h(r)},\\ \ell^{1/\delta}\le p_1^{h(1)}\cdots p_r^{h(r)}}} x^{\omega(\ell)} \ge \min_{J\subset S} \prod_{s\in S\setminus J} (1+h(s)x) \sum_{\substack{0\le I\le h_{|J},\\ \sum_{s\in J} I(s)/h(s)\le\delta|J|}} \prod_{\substack{s\in J,\\ I(s)\neq 0}} x \tag{5.6}$$

and then specialize to $I(s) = h(s)$ or 0 and then sum over all $I' \le I$ but such that $I(s)\neq 0 \implies I'(s)\neq 0$. There are $\prod_{\substack{s\in J,\\ I(s)\neq 0}} h(s)$ such choices for I'. ⋄⋄⋄

The previous lemma has the following consequence.

Lemma 5.3 *We have, for any $x \geq 0$, any $\delta \in (0, 1/2]$ and any $r \geq 1$ and $b \geq 2$,*

$$\sum_{\substack{\ell|(p_1\cdots p_r)^b,\\ \ell\leq(p_1\cdots p_r)^{b\delta}}} x^{\omega(\ell)} \geq \min_{1\leq v\leq r}(1+bx)^{r-v}\sum_{0\leq k\leq\delta v}\binom{v}{k}(bx)^k$$

However, when $b > 1$, it is better to work the proof from anew. This is what we do in the next lemma.

Lemma 5.4 *We have, for any $x \geq 0$, any $\delta \in (0, 1/2]$ and any $r \geq 1$ and $b \geq 2$,*

$$\sum_{\substack{\ell|(p_1\cdots p_r)^b,\\ \ell\leq(p_1\cdots p_r)^{\delta b}}} x^{\omega(\ell)} \geq \min_{0\leq v\leq r}(1+bx)^{r-v}\Big(\sum_{0\leq k\leq\delta v}\binom{v}{k}(bx)^k + \sum_{\delta v<k\leq b\delta v}\binom{v}{k}x^k b^{[\frac{\delta bv-k}{b-1}]}\Big)$$

Proof. We start with Eq (5.6) slightly adapted to our case:

$$\sum_{\substack{\ell|(p_1\cdots p_r)^b,\\ \ell\leq(p_1\cdots p_r)^{\delta b}}} x^{\omega(\ell)} \geq \min_{0\leq v\leq r}(1+bx)^{r-v}\sum_{\substack{0\leq I\leq r,\\ \sum_{s\in J}I(s)\leq\delta b|J|}}\prod_{\substack{s\in J,\\ I(s)\neq 0}} x.$$

We continue in a different fashion: the multiset I can be described as a set K of cardinality k union a multiset L with elements from K and where the repetitions are $\leq b-1$. Hence

$$\sum_{\substack{\ell|(p_1\cdots p_r)^b,\\ \ell\leq(p_1\cdots p_r)^{\delta b}}} x^{\omega(\ell)} \geq \min_{0\leq v\leq r}(1+bx)^{r-v}\sum_{\substack{0\leq k\leq v,\\ k\leq b\delta v}}\binom{r}{k}x^k\sum_{\substack{x_1+\cdots+x_k\leq\delta bv-k,\\ 0\leq x_1,x_2,\cdots,x_k\leq b-1}} 1.$$

When $k \leq \delta v$, all values of $(x_1, x_2, \cdots, x_k)$ are allowed, and this amounts to b^k (this corresponds to the previous proof). When $b = 1$, we cannot do any better, but when $b > 1$, we can let the first $[\frac{\delta vb-k}{b-1}] \leq k$ variables x_i to run freely and set the last one to 0. This gives the lower bound

$$\sum_{\substack{\ell|(p_1\cdots p_r)^b,\\ \ell\leq(p_1\cdots p_r)^{\delta b}}} x^{\omega(\ell)} \geq \min_{0\leq v\leq r}(1+bx)^{r-v}\sum_{\substack{0\leq k\leq v,\\ k\leq b\delta v}}\binom{r}{k}x^k b^{\min(k,[\frac{b\delta v-k}{b-1}])}$$

◇◇◇

One may at first glance believe that the involved function of v, namely

$$F_r(v;x,\delta) = (1+x)^{r-v} \sum_{0\le k\le \delta v} \binom{v}{k} x^k \tag{5.7}$$

is non-increasing in v when $x > 0$ (we have clearly $F_r(0;x,\delta) \ge F_r(v;x,\delta)$), but this is false. In fact:

$$\begin{cases} F_r(v;x,\delta) > F_r(v+1;x,\delta) & \text{when } [\delta v] = [\delta(v+1)], \\ F_r(v;x,\delta) < F_r(v+1;x,\delta) & \text{when } [\delta v] = [\delta(v+1)]-1, \end{cases} \tag{5.8}$$

as is easily checked by noticing the formula

$$(1+x)\sum_{0\le k\le \delta v} \binom{v}{k} x^k = \sum_{0\le k\le \delta v} \binom{v+1}{k} x^k + \sum_{\delta v < k \le \delta v+1} \binom{v}{k-1} x^k. \tag{5.9}$$

Here are some lists of values:

$$(F_{10}(v;2,0.3)) = 59049, 19683, 6561, 2187, 6561, 2673, 1053, 2673, 1161, 489, 1161,$$
$$(F_{10}(v;2,0.2)) = 59049, 19683, 6561, 2187, 729, 2673, 1053, 405, 153, 57, 201,$$
$$(F_{10}(v;2,0.23)) = 59049, 19683, 6561, 2187, 729, 2673, 1053, 405, 153, 489, 201.$$

See also section 6.2. The minimum is indeed reached near the end of the series, but not on exactly at the end. We recall a classical result, see e.g. [4, (2.9)].

Lemma 5.5 *We have* $n! = (2\pi n)^{1/2}(n/e)^n e^{\theta_+/(12n)}$ *for* $n \ge 1$ *and some* $\theta_+ \in]0,1[$

Lemma 5.6 *When* $\nu \ge 1$, $x \ge 0$ *and* $\delta \in [0,1/2]$, *we have*

$$\binom{\nu}{[\delta\nu]} x^{[\delta\nu]} \ge \frac{1}{3x\sqrt{\max(1,\nu\delta)}} \Big(\frac{x^\delta}{H(\delta)}\Big)^\nu.$$

***Proof*.** When $\nu\delta \le 1$, the right-hand side in the lemma is readily seen to be ≤ 1 while the left-hand side equals to 1, hence the lemma in this case.

Let us now assume that $\nu\delta > 1$. We set $c = [\delta\nu] \le \delta\nu \le \nu/2$ and assume that $c \ge 1$. On appealing to Lemma 5.5, we get

$$\begin{aligned} \binom{\nu}{c} x^c &\ge \sqrt{\frac{\nu}{(\nu-c)c}} \frac{\nu^\nu}{(\nu-c)^{\nu-c}c^c} \frac{e^{-\frac{1}{12}(\frac{1}{c}+\frac{1}{\nu-c})}}{\sqrt{2\pi}} x^c \\ &\ge \sqrt{\frac{e^{-1/3}}{2\pi\nu\delta(1-\delta)}} \frac{\nu^\nu}{(\nu-c)^{\nu-c}c^c} x^c \\ &\ge \sqrt{\frac{e^{-1/3}}{2\pi\nu\delta x^2}}\; x^{\delta\nu}/H(\delta)^\nu \end{aligned}$$

since $c \mapsto (\nu-c)^{\nu-c}c^c$ is increasing when $c \le \nu/2$ (compute its derivative) and we may replace c by $\delta\nu$. The lemma follows readily. ◇◇◇

Lemma 5.7 *When $r \geq v \geq 0$, $\eta \in [0,1/2]$ and $y \geq 0$, we have*

$$(1+y)^{r-v}\binom{v}{[\eta v]}y^{[\eta v]} \geq \frac{1}{3y\sqrt{\max(1,r\eta)}}\Big(\frac{y^\eta}{H(\eta)}\Big)^r.$$

Proof. Indeed, it is enough by the previous lemma to check that

$$(1+y)^{r-v}\frac{1}{3y\sqrt{\max(1,v\eta)}}\Big(\frac{y^\eta}{H(\eta)}\Big)^v \geq \frac{1}{3y\sqrt{\max(1,r\eta)}}\Big(\frac{y^\eta}{H(\eta)}\Big)^r$$

and it is readily seen that it is enough to check that

$$1+y \geq \frac{y^\eta}{H(\eta)}.$$

We set $y = zH(\eta)^{1/\eta}$. The derivative of $1+zH(\eta)^{1/\eta} - z^\eta$ with respect to z equals

$$H(\eta)^{1/\eta} - \frac{\eta}{z^{1-\eta}} = \eta\Big((1-\eta)^{\frac{1-\eta}{\eta}} - \frac{1}{z^{1-\eta}}\Big)$$

and it is non-negative if and only if

$$(1-\eta)^{\frac{1}{\eta}} \geq \frac{1}{z}.$$

The value of the initial function when $z = (1-\eta)^{-1/\eta}$ is

$$1+(1-\eta)^{-1/\eta}H(\eta)^{1/\eta} - \frac{1}{1-\eta} = \eta\Big(\frac{-1}{1-\eta} + (1-\eta)^{-1}\Big) = 0.$$

Our lemma is proved. ⋄⋄⋄

Here are the two key results that are implied by the above.

Lemma 5.8 *We have, for any $x \geq 1$, any $\delta \in (0,1/2]$, any $r \geq 1$, and any $b \leq 1/(2\delta)$,*

$$\sum_{\substack{\ell|(p_1\cdots p_r)^b,\\ \ell\leq(p_1\cdots p_r)^{b\delta}}} x^{\omega(\ell)} \geq \frac{1}{3x}\Big(\frac{\max(1,rb\delta)^{\frac{-1}{2r}}x^{b\delta}}{H(b\delta)}\Big)^r.$$

We discovered in between that, when $b=1$ and $\delta = 1/k$, a result of similar strenght is obtainable via the method of Mohan [21]. The reader should however notice the high flexibility of Mohan's Theorem which renders it usable in many situations.

Proof. Use Lemma 5.4. In the lower bound, discard all the terms save $k = [b\delta v]$, minorize $(1+bx)^{r-v}$ by $(1+x)^{r-v}$ and then use Lemma 5.7 with $\eta = b\delta$. ⋄⋄⋄

Lemma 5.9 *We have, for any $x \geq 1$, any $\delta \in (0, 1/2]$, any $r \geq 1$, and any $b > 1/(2\delta)$,*

$$\sum_{\substack{\ell | (p_1 \cdots p_r)^b, \\ \ell \leq (p_1 \cdots p_r)^{b\delta}}} x^{\omega(\ell)} \geq \frac{1}{3x} \Big(\frac{\max(1, r/2)^{\frac{-1}{2r}} x^{1/2} b^{\frac{(2b\delta - 1)}{2(b-1)}}}{H(1/2)} \Big)^r.$$

Proof. Use Lemma 5.4. In the lower bound, discard all the terms save $k = [\nu/2]$, minorize $(1 + bx)^{r-\nu}$ by $(1 + x)^{r-\nu}$, and the term with b to $b^{\frac{(2b\delta-1)r}{2(b-1)}}$, and finally use Lemma 5.7 with $\eta = 1/2$. ⋄⋄⋄

Chapter 6

Proof

In this chapter we prove Theorem 4.1 and 4.4.

6.1 Proof

We consider the function

$$G(n;\delta,\beta)=\sum_{\substack{\ell|n,\\ \ell\le n^{\delta}}} d^{\beta}(\ell)$$

for parameters $\delta>0$, a real number, and $\beta\ge 0$ a real number. Let us start with a trivial lemma.

Lemma 6.1 *When m and n are coprime, we have the super-multiplicativity property: $G(mn;\delta,\beta)\ge G(m;\delta,\beta)G(n;\delta,\beta)$.*

Proof. [Proof of Theorem 4.1] Let $n=p_1p_2\cdots p_r$ be a square-free integer and b be a positive integer. Let $\theta>1$ be a parameter we will choose later.

Let us first consider the case $b\le 1/(2\delta)$.
We select

$$x=\big((b+1)^{s}H(b\delta)\big)^{1/(b\delta)}\theta,$$

for some $\theta\ge 1$. This implies that

$$\frac{1}{\sqrt{rb\delta}}\left(\frac{x}{H(b\delta)^{1/(b\delta)}}\right)^{rb\delta}\ge\frac{\theta^{rb\delta}}{\sqrt{rb\delta}}(b+1)^{sr}.$$

We set

$$c_1=\min_{u\ge 1}\frac{\theta^{u}}{\sqrt{u}}$$

and thus, by Lemma 5.8 and, when $rb\delta \geq 1$, we find that

$$\sum_{\substack{\ell|(p_1\cdots p_r)^b,\\ \ell\leq(p_1\cdots p_r)^{b\delta}}} x^{\omega(\ell)} \geq \frac{1}{3x}c_1\,(b+1)^{sr} = \frac{1}{3x}c_1 d^s((p_1\cdots p_r)^b)$$

On the other hand, when ℓ is square-free, we have

$$x^{\omega(\ell)} = 2^{\omega(\ell)\frac{\log x}{\log 2}} = d(\ell)^{\frac{\log x}{\log 2}}$$

and

$$\beta = \frac{\log x}{\log 2} = \frac{s\log(b+1)}{b\delta\log 2} + \frac{\log H(b\delta)}{b\delta\log 2} + \frac{\log\theta}{\log 2}.$$

When $rb\delta \leq 1$, we simply use $d^s((p_1\cdots p_r)^b) = (b+1)^{sr} \leq (b+1)^{s/(b\delta)}$ and thus

$$\sum_{\substack{\ell|n^b,\\ \ell\leq n^{b\delta}}} x^{\omega(\ell)} \geq \min\Big((b+1)^{-s/(b\delta)}, \frac{1}{3x}c_1\Big) d^s(n^b)\mu^2(n).$$

A quick study of the c_1 as a function of $u = rb\delta$ shows that the minimum is attained at $u = 1/(2\log\theta)$ and has value $\sqrt{2e\log\theta}$. Our minimum reads

$$\min\Big((b+1)^{-s/(b\delta)}, \frac{1/3}{((b+1)^s H(b\delta))^{1/(b\delta)}}\frac{\sqrt{2e\log\theta}}{\theta}\Big).$$

We get

$$d^s(n^b)\mu^2(n) \leq (b+1)^{s/(b\delta)}\max\Big(1, 3H(b\delta)^{1/(b\delta)}\frac{\theta}{\sqrt{2e\log\theta}}\Big)\sum_{\substack{\ell|n^b,\\ \ell\leq n^{b\delta}}} d(\ell)^\beta.$$

Starting from $\log\theta/\log 2 \leq \epsilon$, we find that

$$c_s(\delta,b,\epsilon) = (b+1)^{s/(b\delta)}\begin{cases}\max\big(1, 3H(b\delta)^{1/(b\delta)}\big) & \text{when } 2^\epsilon \geq \sqrt{e},\\ \max\big(1, 3H(b\delta)^{1/(b\delta)}\frac{1}{\sqrt{2\epsilon\log 2}}\big) & \text{when } 2^\epsilon < \sqrt{e}.\end{cases}$$

Since $\epsilon \leq 1/(2\log 2) \leq 3/4$, we put a separate case $\epsilon \geq 3/4$ and $\epsilon < 3/4$. In the second case, we majorize $1/\sqrt{2\epsilon\log 2}$ by $\sqrt{3/(4\epsilon)}$. We can fold both cases in a single expression by using $1/\sqrt{\min(1, 4\epsilon/3)}$ as in the Theorem.

We now consider the case $b > 1/(2\delta)$.
We select

$$x = \big((b+1)^s H(1/2) b^{-\frac{2b\delta-1}{2(b-1)}}\big)^2\theta,$$

for some $\theta \geq 1$, This implies that

$$\frac{1}{\sqrt{r/2}}\Big(\frac{xb^{\frac{2b\delta-1}{(b-1)}}}{H(1/2)^2}\Big)^{r/2} \geq \frac{\theta^{r/2}}{\sqrt{r/2}}(b+1)^{sr}.$$

By Lemma 5.9 and, when $r/2 \geq 1$, we find that

$$\sum_{\substack{m|(p_1\cdots p_r)^b,\\ m\leq (p_1\cdots p_r)^{b\delta}}} x^{\omega(m)} \geq \frac{1}{3x}c_1\,(b+1)^{sr} = \frac{1}{3x}c_1 d^s((p_1\cdots p_r)^b).$$

We again use, when ℓ is square-free, that

$$x^{\omega(\ell)} = 2^{\omega(\ell)\frac{\log x}{\log 2}} = d(\ell)^{\frac{\log x}{\log 2}}$$

and

$$\begin{aligned}\beta = \frac{\log x}{\log 2} &= \frac{2s\log(b+1)}{\log 2} + \frac{2\log H(1/2)}{\log 2} - \left(\frac{2b\delta-1}{b-1}\right)\frac{\log b}{\log 2} + \frac{\log\theta}{\log 2}\\ &= \frac{2s\log(b+1)}{\log 2} - 2 - \left(\frac{2b\delta-1}{b-1}\right)\frac{\log b}{\log 2} + \frac{\log\theta}{\log 2}\end{aligned}$$

When $r/2 \leq 1$, we simply use $d^s((p_1\cdots p_r)^b) = (b+1)^{sr} \leq (b+1)^2$ and thus

$$\sum_{\substack{m|n^b,\\ m\leq n^{b\delta}}} x^{\omega(m)} \geq \min\Big((b+1)^{-2s}, \frac{1}{3x}c_1\Big)d^s(n^b)\mu^2(n).$$

The rest of the proof for $b > 1/(2\delta)$ runs as before in the case $b \leq 1/(2\delta)$. This completes the proof. ⋄⋄⋄

Proof. [Proof of Corollary 4.2] The proof of this result follows directly from Theorem 4.1, it suffices to notice that the function $f(\delta) = (1/\delta - 1)\log(1-\delta)$ has derivative $(-\log(1-\delta)-\delta)/\delta^2$ which is non-negative (since the Taylor development of $-\log(1-\delta)$ is $\sum_{k\geq 1}\delta^k/k$). As a consequence, $H(\delta)^{1/\delta} \leq \delta/2$ when $\delta \in (0,1/2]$ and the rest is easy. ⋄⋄⋄

Proof. [Proof of Corollary 4.3] For (s,b,δ,β) are selected. Thanks to Theorem 4.1, we establish the following table.

⋄⋄⋄

s	b	δ	β	$H(\delta)$	ϵ	$c_s(\delta,b,\epsilon)$
1	1	1/3	1	$2^{2/3}/3$	$0.754\cdots$	8
1	1	1/4	1	$3^{3/4}/4$	$0.245\cdots$	16
1	1	3/13	1	$3^{3/13}\cdot 10^{10/13}/13$	$0.0438\cdots$	$24.071\cdots$
1	1	3/10	1	$3^{3/10}\cdot 7^{7/10}/10$	$0.604\cdots$	$10.079\cdots$
2	1	1/3	4	$2^{2/3}/3$	$0.754\cdots$	64
2	1	1/4	5	$3^{3/4}/4$	$0.245\cdots$	256

6.2 A special study

We investigate the case $\delta = 1/3$ and $s = 1$. Our remark concerns $F_r(v;x,1/3)$. We rewrite formula (5.9) in the form

$$F_r(v;x,\delta) = F_r(v+1;x,\delta) + (1+x)^{r-v-1}\left(\binom{v}{\lceil \delta v\rceil - 1} - 1\!\!1_{\lceil\delta v\rceil \le \delta v+\delta}\binom{v+1}{\lceil \delta v\rceil}\right)x^{\lceil \delta v\rceil}. \quad (6.1)$$

So, when $\delta = 1/3$ and $v \equiv 0[3]$, going from v to $v+1$ means adding $(1+x)^{r-v-1}\binom{v}{\lceil\delta v\rceil-1}x^{\lceil\delta v\rceil}$.

Then going from $v+1$ to $v+2$ means adding $(1+x)^{r-v-2}\binom{v+1}{\lceil\delta v\rceil-1}x^{\lceil\delta v\rceil}$.

And going from $v+2$ to $v+3$ means adding $(1+x)^{r-v-3}(\binom{v+2}{\lceil\delta v\rceil-1}-\binom{v+3}{\lceil\delta v\rceil})x^{\lceil\delta v\rceil}$ which is negative.

Note that $\lceil\delta v\rceil = v/3$ here. We check that the sum of last two terms to be added is positive when $x \ge 1$. Indeed it is at least $(1+x)^{r-v-3}x^{\lceil\delta v\rceil}$ times

$$(1+x)^2\binom{v}{\lceil\delta v\rceil-1} + (1+x)\binom{v+1}{\lceil\delta v\rceil-1} + \binom{v+2}{\lceil\delta v\rceil-1} - \binom{v+3}{\lceil\delta v\rceil}.$$

We write $\lceil\delta v\rceil = a \ge 1$ and have to verify that

$$\frac{4\times(3a)!}{(2a+1)!(a-1)!} + \frac{2\times(3a+1)!}{(2a+2)!(a-1)!} + \frac{(3a+2)!}{(2a+3)!(a-1)!} > \frac{(3a+3)!}{(2a+3)!a!}$$

i.e. that

$$a(4(2a+3)(2a+2) + 2(2a+3)(3a+1) + (3a+1)(3a+2)) > (3a+1)(3a+2)(3a+3).$$

A numerical computation concludes (since a is an integer, we check the initial values) completed by an asymptotic analysis. The conclusion of this study is that the minimum of $F_r(v;x,1/3)$ when $x \ge 1$ is reached when v is the largest integer congruent to 2 modulo 3 below r.

6.3 A divisor for integers with high powers

Lemma 6.2 *Every integer n' that is such that all the exponents α_i in the prime number decomposition $n' = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ are larger than $A(\delta,\beta)$ has a divisor m such that*

$$m \le n'^{\delta} \quad \text{and} \quad d(n') \le d(m)^{\beta}.$$

Moreover $A(\delta,\beta)\delta \ge 1$ and the inequality $A(\delta,\beta) \le ((1+\delta)\delta^{-\beta})^{1/(\beta-1)} + 1 = \delta^{-1}(1+\delta^{-1})^{1/(\beta-1)}$ holds true.

The last inequality is theoretically enough. In practice, it reduces the computation to checking a finite number of cases.

Proof. Most of the proof of this lemma is already contained in its statement. The first slightly non-obvious part is the choice of m; we select

$$m = p_1^{[\delta\alpha_1]} \cdots p_r^{[\delta\alpha_r]}.$$

Concerning the upper bound for $A(\delta,\beta)$ and assuming $\beta > 1$, let α be an integer that verifies

$$\alpha \ge ((1+\delta)\delta^{-\beta})^{1/(\beta-1)}.$$

Then since also $\delta\alpha \ge 1$, we get

$$\frac{(1+[\alpha\delta])^{\beta}}{1+\alpha} \ge \frac{(\alpha\delta)^{\beta}}{1+\alpha} \ge \frac{(\alpha\delta)^{\beta}}{\alpha\delta+\alpha} \ge 1.$$

This shows that indeed $A(\delta,\beta)$ is less than the smallest integer larger than $((1+\delta)\delta^{-\beta})^{1/(\beta-1)}$. The divisor $m = p_1^{[\delta\alpha_1]} \cdots p_r^{[\delta\alpha_r]}$ of n' is less than n'^{δ} and indeed verifies

$$\frac{d(m)^{\beta}}{d(n')} = \prod_{1\le i\le r} \frac{(1+[\delta\alpha_i])^{\beta}}{1+\alpha_i} \ge 1.$$

◇◇◇

6.4 On different auxiliary quantities

Lemma 6.3 *When $b\delta \le 1/2$, b is a positive integer and $s \ge 1$, the function $b \mapsto \beta_{s,b}(\delta)$ is non-increasing.*

Proof. We simply compute the derivative:

$$\begin{aligned}
\frac{d}{db}\big(\beta_{s,b}(\delta)\big) &= \frac{d}{db}\left(\frac{s\log(b+1)}{b\delta\log 2}+\frac{\log H(b\delta)}{b\delta\log 2}\right)\\
&= \frac{s}{b(b+1)\delta\log 2}-\frac{s\log(b+1)}{b^2\delta\log 2}-\frac{\log H(b\delta)}{b^2\delta\log 2}+\frac{\log\frac{b\delta}{1-b\delta}}{\delta\log 2}\\
&\le \frac{1}{b^2\delta\log 2}\left(\frac{b}{b+1}-\log(b+1)-\log H(b\delta)+b^2\log\frac{b\delta}{1-b\delta}\right).
\end{aligned}$$

In the last step we have used the fact that the given quantity is majorized by the value at $s=1$. We set $z=b\delta$ which is assigned to be $\in(0,1/2]$. The computed derivative is non-negative as soon as

$$(b^2-z)\log z+(b^2-(1-z))\log(1-z)\le\log(b+1)-\frac{b}{b+1}$$

which is true since the left-hand side is non-positive while the right-hand side is non-negative. ◇◇◇

Lemma 6.4 *When $s\ge 1$, $b\ge 1$ and $\delta\in(0,1/2]$, the function*

$$b\mapsto\frac{2s\log(b+1)}{\log 2}-\frac{2b\delta-1}{b-1}\frac{\log b}{\log 2}-2$$

is non-decreasing.

Proof. It is enough to prove it in case $s=1$. The quantity to consider, when multiplied by $\delta\log 2$ reads

$$2\log(b+1)-2\delta\log b-\frac{1-2\delta}{b-1}\log b.$$

Its derivative reads

$$\frac{2}{b+1}-\frac{2\delta}{b}+\frac{1-2\delta}{(b-1)^2}\log b-\frac{1-2\delta}{b(b-1)}.$$

We check that, since $\delta\in[0,1/2]$,

$$-\frac{2\delta}{b}+\frac{2\delta}{b(b-1)}\ge\frac{-1}{b}$$

so that it is enough to show that

$$\frac{2}{b+1}-\frac{1}{b}+\frac{1-2\delta}{(b-1)^2}\log b\ge 0$$

which is obviously true. ◇◇◇

Lemma 6.5 *When $\beta \geq 2$, $s \geq 1$, $\delta \in (0, 1/2]$, and $\beta \geq \beta_{s,1}(\delta)$ we have $\beta \geq \max_{1 \leq b \leq A(\delta,\beta)-1} \beta_{s,b}(\delta)$ provided we have*

$\delta \leq$	*0.5*	*0.1457*	*0.1007*	*0.0870*	*0.0614*
$s \leq$	*1.455*	*2*	*3*	*4*	∞

Proof. The case $b\delta \leq 1/2$ is cleared off by Lemma 6.3. When $b \in [1/(2\delta), A(\delta,\beta) - 1] \subset [1/(2\delta), B]$ where $B = \delta^{-1}(1+\delta^{-1})^{1/(\beta_0-1)}$ and $\beta_0 = 2$, we have to check that

$$\beta_{s,1}(\delta) \geq \beta_{s,b}(\delta).$$

We first check that

$$\frac{1}{\delta} - \frac{2\log(b+1)}{\log 2} \geq 0. \tag{6.2}$$

Indeed, we only have to check that

$$e^{\frac{\log 2}{2\delta}} \geq B + 1 = \delta^{-1}(1+\delta^{-1}) + 1$$

This is ok provided $\delta \leq 0.0614$. Equivalently, we need

$$\beta \geq 1 + \frac{\log(1+\delta^{-1})}{\log\delta + \log(e^{(\log 2)/(2\delta)} - 1)}.$$

We can now restrict our attention to the case $s = 1$. This means, we have to show that

$$\log 2 + \log H(\delta) \geq 2\delta\log(B+1) - \delta\frac{2B\delta - 1}{B-1}\log B - 2\delta\log 2.$$

Some numerical analysis is enough to show the wanted inequality.

If we seek to relax the condition on δ at the price on generality on s, here is a reasoning. If inequality (6.2) does not hold, then we can use an upper bound, say $\overline{s}$, for s. We follow the same path as above, but now need to check that

$$\overline{s}\log 2 + \log H(\delta) \geq 2\overline{s}\delta\log(B+1) - \delta\frac{2B\delta - 1}{B-1}\log B - 2\delta\log 2.$$

We then show in the same manner that the wanted result holds true under any one of the following conditions:

$$(s \leq 1.455, \delta \leq 1/2), (s \leq 2, \delta \leq 0.1457), (s \leq 3, \delta \leq 0.1007), (s \leq 4, \delta \leq 0.0870).$$

This completes the proof. $\diamond\diamond\diamond$

6.5 Proof of Theorem 4.4

Proof. [Proof of Theorem 4.4] Let n be the integer to be represented. We use the shorter notation $A = A(\delta,\beta)$. We can write $n = n'n_0$ where $n' = p_1^{\alpha_1}\cdots p_r^{\alpha_r}$ and n_0 are mutually coprime and $\alpha_i \geq A$ for every index i. We further write $n_0 = n_1 n_2^2 \cdots n_{A-1}^{A-1}$ where all the n_b are square-free. When $\omega(n_b) > \delta^{-1}$ we appeal to Theorem 4.1 for the factor n_b^b. This is allowed as (see Lemma 6.5)

$$\beta_{s,b}(\delta) \leq \beta_{s,1}(\delta).$$

The product of the n_b^b for all b such that $\omega(n_b) \leq \delta^{-1}$ is a integer, say ℓ, that has at most $(A-1)/\delta$ prime factors each raised at a power not more than $A-1$. Hence $d^s(\ell) \leq A^{s(A-1)/\delta}$. An appeal to the super-multiplicativity (Lemma 6.1) to conclude. ⋄⋄⋄

Proof. [An easy bound for $c_s^\star(\delta,\beta)$] We simply bound $b+1$ by $A(\delta,\beta)$, then majorize $H(b\delta)$ by 1, use $\min(1, 4(\beta-\beta_{s,b}(\delta)/3 \geq \min(1,\epsilon)$, bound above $A(\delta,\beta)$ by $\delta^{-1}+\delta^{-2}+1+$ as $\beta \geq 2$ and reach

$$\begin{aligned}
c_s^\star(\delta,\beta) &\leq A(\delta,\beta)^{s(A(\delta,\beta)-1)/\delta}\left(\frac{3A(\delta,\beta)}{\min(1,\epsilon)}\right)^{s(A(\delta,\beta)-1)/\delta} \\
&\leq \left(\frac{3A(\delta,\beta)}{\min(1,\epsilon)}\right)^{2s(A(\delta,\beta)-1)/\delta} \leq \left(\frac{3\delta^{-2}+3\delta^{-1}+1}{\min(1,\epsilon)}\right)^{2s(1+\delta^{-1})/\delta^2} \\
&\leq \left(\frac{\delta^{-5}}{\min(1,\epsilon)}\right)^{3s/\delta^3}
\end{aligned}$$

as claimed. ⋄⋄⋄

Bibliography

Bibliography

[1] N. C. Ankeny and S. Chowla, *The class number of the cyclotomic field*, Canadian. Math. III (1951), 486–494.

[2] V. Chandee and X. Li, *Lower bounds for small fractional moments of Dirichlet L-functions*, Int. Math. Res. Not. IMRN (2013), no. 19, 4349–4381.

[3] J.-M Deshouillers and F. Dress, *Sommes de diviseurs et structure multiplicative des entiers*, Acta Arith. **49** (1988), no. 4, 341–375.

[4] H.G. Diamond and J. Steinig, *An Elementary Proof of the Prime Number Theorem with a Remainder Term*, Inventiones math. **11** (1970), 199–258.

[5] J.B. Friedlander and H. Iwaniec, *Asymptotic sieve for primes*, Ann. of Math. (2) **148** (1998), no. 3, 1041–1065. MR MR1670069 (2000c:11150b)

[6] ______, *Divisor weighted sums*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **322** (2005), no. Trudy po Teorii Chisel, 212–219, 255. MR 2138460 (2006b:11116)

[7] ______, *The illusory sieve*, Int. J. Number Theory **1** (2005), no. 4, 459–494. MR 2196790 (2006h:11115)

[8] D.R. Heath-Brown, *Sieve identities and gaps between primes*, Journées Arithmétiques (Metz, 1981), Astérisque, vol. 94, Soc. Math. France, Paris, 1982, pp. 61–65.

[9] ______, *Zero-free regions for Dirichlet L-functions and the least prime in an arithmetic progression*, Proc. London Math. Soc., III Ser. **64** (1992), no. 2, 265–338.

[10] ______, *Zero-free regions of $\zeta(s)$ and $L(s,\chi)$*, Proceedings of the Amalfi conference on analytic number theory (Maiori, Amalfi, Italy, from 25 to 29 September, 1989. Salerno: Universita di Salerno,) (E. (ed.) et al. Bombieri, ed.), 1992, pp. 195–200.

[11] ______, *Fractional moments of Dirichlet L-functions*, Acta Arith **4** (2010), no. 145, 397–409.

[12] M.N. Huxley and M. Jutila, *Large values of Dirichlet polynomials. IV*, Acta Arith. **32** (1977), 297–312.

[13] Y. Ihara and K. Matsumoto, *On certain mean values and the value-distribution of logarithms of Dirichlet L-functions*, Quart. J. Math. (Oxford) (2011), no. 62, 637–677.

[14] ______, *On the value-distribution of logarithms derivatives of Dirichlet L-functions*, Analytic Number Theory, Approximation Theory and Special Functions, in Honor of H. M. Srivastava, G. V. Milovanović and M. Th. Rassias (eds.), Springer (2014), 79–91.

[15] Y. Ihara, V. Kumar Murty, and M. Shimura, *On the logarithmic derivatives of Dirichlet L-functions at* $s = 1$, Acta Arith (2009).

[16] P. Erdős K. Alladi and J.D. Vaaler, *Multiplicative functions and small divisors*, Analytic number theory and Diophantine problems (Stillwater, OK, 1984), Progr. Math., vol. 70, Birkhäuser Boston, Boston, MA, 1987, pp. 1–13. MR 1018367 (90h:11089)

[17] ______, *Multiplicative functions and small divisors. II*, J. Number Theory **31** (1989), no. 2, 183–190. MR 987571 (90h:11090)

[18] M. Katsurada and K. Matsumoto, *The mean values of Dirichlet L-functions at integer points and class numbers of cyclotomic fields*, Nagoya Math. J. **134** (1994), 151–172.

[19] B. Landreau, *A new proof of a theorem of van der Corput*, Bull. Lond. Math. Soc. **21** (1989), no. 4, 366–368.

[20] M.B. Milinovich, *Moments of the Reimann zeta function at its relative extrema on the critical line*, Bull. Lond. Math. Soc. **6** (2011), no. 43, 1119–1129.

[21] P. Surya Mohan, *A note on a class of submultiplicative functions*, Glasg. Math. J. **46** (2004), no. 2, 237–238. MR 2062607 (2005c:11116)

[22] M. Mourtada and V. Kumar Murty, *Omega Theorems of* $\frac{L'}{L}(1,\chi_d)$, International journal of Number Theory (2013), no. 03.

[23] R. Munshi, *Inequalities for divisor functions*, Ramanujan J. **25** (2011), no. 2, 195–201. MR 2800605 (2012d:11192)

[24] R.E.A.C. Paley, *On the k-analogues of some theorems in the theory of the Riemann* ζ*-function*, Proc. Lond. Math. Soc. (1931), no. 32, 273–311.

[25] D. Platt and O. Ramaré, *Explicit estimates: from* $\lambda(n)$ *in arithmetic progressions to* $\lambda(n)/n$, To appear.

[26] O. Ramaré and R. Rumely, *Primes in arithmetic progressions*, Math. Comp. **65** (1996), 397–425.

[27] Z. Rudnick and K. Soundararajan, *Lower bounds for moments of L-functions*, Proc. Natl. Acad. Sci. USA **19** (2005), no. 102, 6837–6838.

[28] I. Sh. Slavutskil, *Mean value of L-functions and the ideal class number of cyclotomic field, in " algebraic systems with one action and relation"*, Leningrad. Gos. Ped. Inst., Leningrad (1985), 122–129.

[29] ______, *Mean value of L-functions and the class number of cyclotomic field, in " algebraic systems with one action and relation"*, Nauchn. Sem. LOMI, Leningrad **154** (1986), 136–143.

[30] K. Sono, *Lower bounds for the moments of the derivatives of the Riemann zeta-function and Dirichlet L-functions*, Lith. Math. J. **4** (2012), no. 52, 420–434.

[31] ______, *Upper bounds for the moments of the derivatives of Dirichlet L-functions*, Cent. Eur. J. Math. **6** (2014), no. 12, 848–860.

[32] K. Soundararajan, *An inequality for multiplicative functions*, J. Number Theory **41** (1992), no. 2, 225–230. MR 1164800 (93j:11064)

[33] ______, *The fourth moment of Dirichlet L-functions*, Analytic Number Theory, Gottingen, June 20-24, 2005, Clay Math. Proc., 7, American Mathematical Society, Providence, (2007), 239–246.

[34] ______, *Moments of the Riemann zeta function*, Ann. of Math. **2** (2009), no. 170, 981–993.

[35] E.C. Titchmarsh, *The Theory of Riemann Zeta Function*, Oxford Univ. Press, Oxford 1951, 1951.

[36] J.G. van der Corput, *Une inégalité relative au nombre des diviseurs*, Nederl. Akad. Wetensch., Proc. **42** (1939), 547–553.

[37] D. Wolke, *A new proof of a theorem of van der Corput*, J. London Math. Soc. (2) **5** (1972), 609–612. MR 0314786 (47 #3336)

[38] M.P. Young, *The fourth moment of Dirichlet L-functions*, Ann. of Math. **1** (2011), no. 173, 1–50.

[39] Wenpeng Zhang, *On an elementary result of L-functions*, Adv. in Math. (china) **19** (1990), 478–487.

[40] ______, *On the fourth power mean of Dirichlet L-functions*, Chinese Science Bulletin (1990), no. 23.

[41] ______, *On the mean value of L-functions*, Math. Res. Exposition **10** (1990), 355–360.

[42] ______, *A new mean value formula of Dirichlet L-functions*, Science in China (series A) (1992), no. 10.