Bulletin of the London Mathematical Society Advance Access published April 18, 2010

Bull. London Math. Soc. Page 1 of 10
doi:10.1112/blms/bdq021

The number of rational numbers determined by large sets of integers

J. Cilleruelo, D. S. Ramana and O. Ramaré

Abstract

When A and B are subsets of the integers in $[1, X]$ and $[1, Y]$, respectively, with $|A| \geqslant \alpha X$ and $|B| \geqslant \beta Y$, we show that the number of rational numbers expressible as a/b with (a, b) in $A \times B$ is $\gg (\alpha\beta)^{1+\epsilon} XY$ for any $\epsilon > 0$, where the implied constant depends on ϵ alone. We then construct examples that show that this bound cannot, in general, be improved to $\gg \alpha\beta XY$. We also resolve the natural generalization of our problem to arbitrary subsets C of the integer points in $[1, X] \times [1, Y]$. Finally, we apply our results to answer a question of Sárközy concerning the differences of consecutive terms of the product sequence of a given integer sequence.

1. *Introduction*

When A and B are subsets of the positive integers let A/B be the set of all rational numbers expressible as a/b with (a, b) in $A \times B$. Suppose now that A and B are intervals in the integers in $[1, X]$ and $[1, Y]$ respectively, satisfying $|A| \geqslant \alpha X$ and $|B| \geqslant \beta Y$, where X, Y real numbers at least 1, α, β are real numbers in $(0, 1]$. A standard application of the Möbius inversion formula then shows that $|A/B| \gg \alpha\beta XY$.

Our purpose is to investigate what might be deduced when in place of *intervals* we consider *arbitrary* subsets A and B of the integers in $[1, X]$ and $[1, Y]$ respectively with $|A| \geqslant \alpha X$ and $|B| \geqslant \beta Y$. In this general situation, the main difficulty arises from the fact that certain integers d may have an abnormally large number of multiples in the sets A and B. Further, these integers d are not determined by the conditions on A and B, which are only in terms of their cardinalities. Nevertheless, since the sets under consideration are large, popular heuristics suggest that a non-trivial conclusion should still be accessible. What is pleasing is that we in fact have the following theorem, which is our principal conclusion.

Theorem 1.1. *Let α and β be real numbers in $(0, 1]$ and let X and Y be real numbers at least 1. When A and B are subsets of the integers in $[1, X]$ and $[1, Y]$, respectively, with $|A| \geqslant \alpha X$ and $|B| \geqslant \beta Y$, we have $|A/B| \gg (\alpha\beta)^{1+\epsilon} XY$ for any $\epsilon > 0$, where the implied constant depends on ϵ alone.*

Deferring the detailed proof of Theorem 1.1 to Section 2, let us summarize our argument for it with the aid of the following notation. For any integer $d \geqslant 1$, A and B subsets of the integers, we write $\mathcal{M}(A, B, d)$ to denote the subset of $A \times B$ consisting of all (a, b) in $A \times B$ with $\gcd(a, b) = d$. We show in Proposition 2.1 that for A and B as in Theorem 1.1, we have $\sup_{d \geqslant 1} |\mathcal{M}(A, B, d)| \geqslant (1/8)(\alpha\beta)^2 XY$. Starting from this initial bound, we then

Received 30 July 2009; revised 27 November 2009.

2000 *Mathematics Subject Classification* 11B05 (primary), 11B30 (secondary).

The first author was supported by Grant CCG08-UAM/ESP-3906 and MTM2008-03880 of the MYCIT, Spain and the third author was supported by the CNRS, France.

obtain $\sup_{d\geqslant 1}|\mathcal{M}(A,B,d)| \gg (\alpha\beta)^{1+\epsilon}XY$ by a bootstrapping argument. Theorem 1.1 follows immediately from this last bound, since, for any integer $d \geqslant 1$, we have $a/b \neq a_1/b_1$ for any two elements (a,b) and (a_1,b_1) of $\mathcal{M}(A,B,d)$, and therefore $|A/B| \geqslant \sup_{d\geqslant 1}|\mathcal{M}(A,B,d)|$.

We supplement Theorem 1.1 with the following result which shows that the bound provided by Theorem 1.1 cannot be replaced with $|A/B| \gg \alpha\beta XY$, which bound, as we have already remarked, holds when A and B are intervals.

Theorem 1.2. *For any $\epsilon > 0$, there exists $\alpha > 0$ such that, for all sufficiently large X, there exists a subset A of the integers in $[1,X]$ satisfying $|A| \geqslant \alpha X$ and $|A/A| < \epsilon\alpha^2X^2$.*

We prove Theorem 1.2 in Section 3. Our method depends on the observation that, for any $\epsilon > 0$ and any set of prime numbers $\mathcal{P}$ with $|\mathcal{P}|$ sufficiently large, we have $|S(\mathcal{P})/S(\mathcal{P})| \leqslant \epsilon|S(\mathcal{P})|^2$, where $S(\mathcal{P})$ is the set of square-free integers formed from the primes in the subsets of $\mathcal{P}$ containing about half the primes in $\mathcal{P}$. By means of this observation we deduce that, for suitable $\mathcal{P}$, the set of multiples of the elements of $S(\mathcal{P})$ in $[1,X]$ meets the conditions of Theorem 1.2.

The questions answered by the above theorems may be viewed as particular cases of a more general problem, namely, for X and Y real numbers at least 1 and γ in $(0,1]$, given a subset C of the integer points in $[1,X]\times[1,Y]$ satisfying $|C| \geqslant \gamma XY$, to determine in terms of γ, X and Y an optimal lower bound for $\mathrm{Frac}(C)$, the number of rational numbers a/b with (a,b) in C. Plainly, the above theorems take up the special case when C is of the form $A\times B$, that is, when C is equal to the product of its projections onto the co-ordinate axes.

It turns out, however, that the aforementioned general problem is somewhat easily resolved. In effect, the method of Proposition 2.1 generalizes without additional effort to give the bound $|\mathrm{Frac}(C)| \geqslant (1/8)\gamma^2XY$ and, interestingly, this bound is in fact optimal up to the constant $\frac{1}{8}$. More precisely, we have the following theorem.

Theorem 1.3. *For any γ in $(0,1]$ and all sufficiently large X and Y, there exists a subset C of the integer points in $[1,X]\times[1,Y]$ satisfying $|C| \geqslant (\gamma/8)XY$ and $|\mathrm{Frac}(C)| \leqslant (\gamma^2/2)XY$.*

We prove Theorem 1.3 at the end of Section 3 by explicitly describing sets C that satisfy the conditions of this theorem. Such sets are, in general, far from being of the form $A\times B$, which is only natural on account of Theorem 1.1. Indeed, our bootstrapping argument for Theorem 1.1 depends crucially on the fact that this theorem is, from the more general viewpoint, about sets C that are of the form $A\times B$.

We conclude this note with Section 4 where we apply Theorem 1.1 to obtain a near-optimal answer to the following question of A. Sárközy. When $\mathcal{A}$ and $\mathcal{B}$ are sequences of integers, let $\mathcal{A}.\mathcal{B}$ be the sequence of integers of the form that are the integers of the form ab, for some $a\in\mathcal{A}$ and $b\in\mathcal{B}$. Then Sárközy (see [**5**, Problem 22]) asks if it is true that, for any $\alpha>0$ and $\mathcal{A}$ such that the lower asymptotic density $\underline{d}(\mathcal{A}) > \alpha$, there is a $c(\alpha)$ such that there are infinitely many pairs of consecutive terms of $\mathcal{A}.\mathcal{A}$ the difference between which is bounded by $c(\alpha)$.

Berczi [**1**] responded to the aforementioned question of Sárközy by showing that $\mathrm{Gap}(\mathcal{A}.\mathcal{A})$, which denotes the minimum of the differences between consecutive terms of $\mathcal{A}.\mathcal{A}$, satisfies $\mathrm{Gap}(\mathcal{A}.\mathcal{A}) \ll \frac{1}{\alpha^4}$, where $\alpha = \underline{d}(\mathcal{A})$. Sandor [**4**] subsequently improved this bound by showing that $\mathrm{Gap}(\mathcal{A}.\mathcal{A}) \ll \frac{1}{\alpha^3}$, with α now the upper asymptotic density $\overline{d}(\mathcal{A})$ of $\mathcal{A}$. Cilleruelo and Le [**3**] obtained the same bound when α is the upper Banach density of $\mathcal{A}$ and showed that this is the best possible bound for this density. The following result improves upon and generalizes Sandor's conclusion.

THEOREM 1.4. *Let α and β be real numbers in $(0,1]$ and let $\epsilon > 0$. When $\mathcal{A}$ and $\mathcal{B}$ are infinite sequences of integers with upper asymptotic densities α and β, respectively, there are infinitely many pairs of consecutive terms of the product sequence $\mathcal{A}.\mathcal{B}$ the difference between which is $\ll 1/(\alpha\beta)^{1+\epsilon}$, where the implied constant depends on ϵ alone.*

When $\mathcal{A}$ and $\mathcal{B}$ are the sequences of multiples of the integers h and k, respectively, the difference between any two consecutive terms of the sequence $\mathcal{A}.\mathcal{B}$ is at least hk. Since we have $\overline{d}(\mathcal{A}) = 1/h$ and $\overline{d}(\mathcal{B}) = 1/k$, we see that the conclusion of Theorem 1.4 is optimal up to a factor $1/(\alpha\beta)^\epsilon$.

We bring this introduction to a close by mentioning a recent result of Bourgain, Konyagin and Shparlinski [**2**] that is closely related to our Theorem 1.1. We are grateful to Professor Shparlinski for drawing our attention to this work. In [**2**] the reader will find a lower bound for the size of the product set $A.B$ of sets of rational numbers A and B together with a number of applications. We state the relevant lower bound from [**2**] as the following theorem.

THEOREM 1.5. *Let X be a real number at least 1 and let A and B be sets of rational numbers r/s with $1 \leqslant r,s \leqslant X$. When X is sufficiently large, we have that*

$$|A.B| > \exp\left(\frac{-9\log X}{\sqrt{\log\log X}}\right)|A||B|.$$

Although stronger than Theorem 1.1 in general, the conclusion of the above theorem, for the reason that it depends on X, is weaker than that of Theorem 1.1 for sets of large cardinality. For the same reason it does not appear to be possible to deduce Theorem 1.1 from the above theorem.

Throughout this note, X and Y shall denote real numbers at least 1, and α, β and γ real numbers in $(0,1]$. Also, the letter p shall denote a prime number. When I and J are subsets of a given set, $I \setminus J$ shall denote the set of elements of I that are not in J. In addition to the notation introduced so far, we shall write A_d to denote the subset of a set of integers A consisting of all multiples of d in A for any integer d. Finally, if $B = \{b\}$ with $b \geqslant 1$, we simply write A/b in place of A/B, by an abuse of notation.

2. *Proof of the bound*

Let A and B be finite subsets of the positive integers. Then the family of subsets $\mathcal{M}(A,B,d)$ of $A \times B$, with d varying over the positive integers, is a partition of $A \times B$ and we have

$$|A \times B| = \sum_{d \geqslant 1} |\mathcal{M}(A,B,d)|. \tag{1}$$

When A and B are contained in $[1,X]$ and $[1,Y]$, respectively, we have $|A_d| \leqslant X/d$ and $|B_d| \leqslant Y/d$, for any $d \geqslant 1$. Since $\mathcal{M}(A,B,d)$ is contained in $A_d \times B_d$, we then obtain $|\mathcal{M}(A,B,d)| \leqslant |A_d||B_d| \leqslant XY/d^2$ for all $d \geqslant 1$.

PROPOSITION 2.1. *When A and B are subsets of the integers in the intervals $[1,X]$ and $[1,Y]$, respectively, with $|A| \geqslant \alpha X$ and $|B| \geqslant \beta Y$, we have $\sup_{d\geqslant 1} |\mathcal{M}(A,B,d)| \geqslant (\alpha\beta)^2 XY/8$.*

Proof. We adapt an argument from [3]. From (1), for any integer $T \geqslant 1$, we have that

$$\begin{aligned} |A \times B| &= \sum_{1 \leqslant d \leqslant T} |\mathcal{M}(A,B,d)| + \sum_{T<d} |\mathcal{M}(A,B,d)| \\ &\leqslant \sum_{1 \leqslant d \leqslant T} |\mathcal{M}(A,B,d)| + \frac{XY}{T}, \end{aligned} \tag{2}$$

where the last inequality follows from $\sum_{T<d} |\mathcal{M}(A,B,d)| \leqslant \sum_{T<d} XY/d^2 \leqslant XY/T$. On noting that $|A \times B| \geqslant \alpha\beta XY$, from (2) we conclude that

$$\sup_{d \geqslant 1} |\mathcal{M}(A,B,d)| \geqslant \frac{1}{T} \sum_{1 \leqslant d \leqslant T} |\mathcal{M}(A,B,d)| \geqslant \left(\frac{\alpha\beta - 1/T}{T}\right) XY \tag{3}$$

for any integer $T \geqslant 1$. Since $2 > \alpha\beta$, the interval $[2/\alpha\beta, 4/\alpha\beta]$ contains an integer $k \geqslant 1$. The proposition now follows on setting $T = k$ in (3). □

Definition 1. We call a real number δ an *admissible exponent* if there exists a real number $C > 0$ such that for any α, β real numbers in $(0,1]$, any X, Y real numbers at least 1 and any subsets A and B of the integers in $[1,X]$ and $[1,Y]$ with $|A| \geqslant \alpha X$ and $|B| \geqslant \beta Y$, we have $\sup_{d \geqslant 1} |\mathcal{M}(A,B,d)| \geqslant C(\alpha\beta)^\delta XY$. We call a C satisfying these conditions a *constant associated* to the admissible exponent δ.

Proposition 2.1 says that $\delta = 2$ is an admissible exponent. Proposition 2.3 below will allow us to conclude that every $\delta > 1$ is an admissible exponent. The following lemma prepares us for an application of Hölder's inequality within the proof of Proposition 2.3.

For any integer $n \geqslant 1$ let $\tau(n)$ denote, as usual, the number of positive integers that divide n. When $D \geqslant 1$ is an integer, we write $\tau_D(n)$ to denote the number of divisors d of n satisfying the condition $p|d \Rightarrow p \leqslant D$ for any prime number p.

Lemma 2.2. *For each integer $q \geqslant 0$ there is a real number $c(q) > 0$ such that, for all real numbers $X \geqslant 1$ and integers $D \geqslant 1$, we have*

$$\sum_{1 \leqslant n \leqslant X} \tau_D(n)^q \leqslant c(q) DX. \tag{4}$$

Proof. In effect, we have

$$\sum_{1 \leqslant n \leqslant X} \tau_D(n)^q \ll X(\log 2D)^{2^q} \ll (2^q!)\, DX, \tag{5}$$

where the implied constants are absolute. Plainly, the second inequality results from the elementary inequality $(\log t)^n \leqslant n!\, t$ for $t \geqslant 1$. We now prove the first inequality in (5). Let us write $\mathcal{D}$ for the set of integers m satisfying the condition $p|m \Rightarrow p \leqslant D$. For any integer $n \geqslant 1$, let $k(n)$ be the largest of the divisors of n lying in $\mathcal{D}$. Then $\tau_D(n)$ is the same as $\tau(k(n))$ for all integers $n \geqslant 1$ and we have that

$$\sum_{1 \leqslant n \leqslant X} \tau_D(n)^q = \sum_{\substack{1 \leqslant m \leqslant X, \\ m \in \mathcal{D}}} \tau(m)^q \sum_{\substack{1 \leqslant n \leqslant X, \\ k(n) = m}} 1 \leqslant X \sum_{m \in \mathcal{D}} \frac{\tau(m)^q}{m}, \tag{6}$$

where we have used the upper bound X/m for the number of integers n in $[1,X]$ with $k(n) = m$. Let us write $S(q)$ for any integer $q \geqslant 0$ to denote the last sum in (6). Since Merten's formula gives $\prod_{1\leqslant p\leqslant D}(1-1/p) \sim e^{-\gamma}/\log D$, with γ here being Euler's constant, we have

$$S(0) = \sum_{m\in D}\frac{1}{m} = \prod_{1\leqslant p\leqslant D}\left(1+\frac{1}{p}+\frac{1}{p^2}+\dots\right) = \prod_{1\leqslant p\leqslant D}\left(1-\frac{1}{p}\right)^{-1} \ll \log 2D, \tag{7}$$

where the implied constant is absolute. On noting that every divisor of an integer in $\mathcal{D}$ is again in $\mathcal{D}$ and using $\tau(dk) \leqslant \tau(d)\tau(k)$, valid for any integers d and $k \geqslant 1$, we obtain

$$\sum_{m\in D}\frac{\tau(m)^q}{m} = \sum_{m\in D}\frac{\tau(m)^{q-1}}{m}\sum_{d|m}1 = \sum_{(d,k)\in D\times D}\frac{\tau(dk)^{q-1}}{dk} \leqslant \left(\sum_{d\in D}\frac{\tau(d)^{q-1}}{d}\right)^2. \tag{8}$$

In other words, $S(q) \leqslant S(q-1)^2$ for any integer $q \geqslant 1$. An induction on q then shows that, for any integer $q \geqslant 0$, we have $S(q) \leqslant S(0)^{2^q} \ll (\log D)^{2^q}$, where the implied constant is absolute. On combining this bound with (6), we obtain the first inequality in (5). □

Proposition 2.3. *If $\delta > 1$ is an admissible exponent, then so is $(3\delta(1+1/q)-2)/(2\delta-1)$ for every integer $q \geqslant 1$.*

Proof. Given an integer $q \geqslant 1$, for the sake of conciseness, we write δ' to denote $(3\delta(1+1/q)-2)/(2\delta-1)$. Since we have $\delta > 1$, we also have $\delta' > 1$.

For a constant C associated to δ let us set C' to be the unique positive real number satisfying

$$\frac{1}{8C'} = \left(\frac{C'}{C}\right)^{1/(2(\delta-1))} 8^{\delta/(\delta-1)}(4c(q))^{\delta/(q(\delta-1))}, \tag{9}$$

where $c(q)$ is the implied constant in (4) of Lemma 2.2. It is easily seen from (9) that by replacing C with a smaller constant associated to δ, if necessary, we may assume that $\frac{1}{4} \geqslant C'$.

We shall show that δ' is an admissible exponent with C' a constant associated to δ'. Thus let α and β be real numbers in $(0,1]$ and let X and Y be real numbers at least 1. Also, let A and B be any subsets of the integers in $[1,X]$ and $[1,Y]$ satisfying $|A| \geqslant \alpha X$ and $|B| \geqslant \beta Y$, respectively. We shall show that

$$\sup_{d\geqslant 1}|\mathcal{M}(A,B,d)| \geqslant C'(\alpha\beta)^{\delta'}XY. \tag{10}$$

On replacing, if necessary, α and β with $\alpha' \geqslant \alpha$ and $\beta' \geqslant \beta$ satisfying $\alpha' X \leqslant |A| \leqslant 2\alpha' X$ and $\beta' Y \leqslant |B| \leqslant 2\beta' Y$, respectively, we reduce to the case when $|A| \leqslant 2\alpha X$ and $|B| \leqslant 2\beta Y$.

Let us first dispose of the possibility that an abnormally large number of the integers in A and B are multiples of a given integer. Thus let $\alpha_d = |A_d|/X$ and $\beta_d = |B_d|/Y$, for any integer $d \geqslant 1$. Suppose that there exists an integer $d \geqslant 1$ such that

$$\alpha_d\beta_d \geqslant \left(\frac{C'}{C}\right)^{1/\delta}(\alpha\beta)^{\delta'/\delta}d^{(2/\delta)-2}. \tag{11}$$

Then A_d and B_d are both non-empty and therefore X and Y are both at least d. Further, the sets A_d/d and B_d/d are subsets of the integers in $[1,X/d]$ and $[1,Y/d]$, respectively. Since δ is an admissible exponent, C is a constant associated to δ and we have $|A_d/d| = (d\alpha_d)|X/d|$ and $|B_d/d| = (d\beta_d)|X/d|$, there exists an integer $d' \geqslant 1$ such that

$$|\mathcal{M}(A_d/d, B_d/d, d')| \geqslant C(d^2\alpha_d\beta_d)^{\delta}\frac{XY}{d^2} \geqslant C'(\alpha\beta)^{\delta'}XY, \tag{12}$$

where the last inequality follows from (11). Since $|\mathcal{M}(A_d/d, B_d/d, d')|$ does not exceed $|\mathcal{M}(A, B, dd')|$, we obtain (10) from (12). We may therefore verify (10) assuming that, for every integer $d \geqslant 1$, we have

$$\alpha_d\beta_d < \left(\frac{C'}{C}\right)^{1/\delta} (\alpha\beta)^{\delta'/\delta} d^{(2/\delta)-2}. \tag{13}$$

With the aid of (13) we shall in fact obtain a more precise conclusion than (10). Let us set $K = (\alpha\beta)^{1-\delta'}/8C'$ and $L = 1 + [K]$. We shall show that

$$\frac{1}{L} \sum_{1\leqslant d\leqslant L} |\mathcal{M}(A, B, d)| \geqslant C'(\alpha\beta)^{\delta'} XY, \tag{14}$$

which of course implies (10). Note that since L is roughly about $(\alpha\beta)^{1-\delta'}/C'$, the inequality (14) is what one might expect from (1).

Let D be an integer in $[2/\alpha\beta, 4/\alpha\beta]$. Thus, in particular $D > 1$. When $L \geqslant D$, we obtain (14) even without assuming (13). In effect, we then have $K \geqslant 1$ and hence that $L \leqslant 2K$ or, what is the same thing, that $L < (\alpha\beta)^{1-\delta'}/4C'$ from which (14) follows on noting that, for any integer $T \geqslant D$, and in particular for $T = L$, from (3) we have that

$$\frac{1}{T} \sum_{1\leqslant d\leqslant T} |\mathcal{M}(A, B, d)| \geqslant \left(\frac{\alpha\beta - 1/T}{T}\right) XY \geqslant \left(\frac{\alpha\beta - 1/D}{T}\right) XY \geqslant \frac{\alpha\beta XY}{2T}. \tag{15}$$

Suppose now that $1 \leqslant L < D$. Let us first verify that, for any integer T such that $1 \leqslant T < D$, we have the following inequality on account of (13):

$$\begin{aligned} &\sum_{T<d\leqslant D} |\mathcal{M}(A, B, d)| \\ &\quad\leqslant \left(\frac{C'}{C}\right)^{1/2\delta} (\alpha\beta)^{\delta'/2\delta} T^{(1/\delta)-1} (XY)^{1/2} \left(\sum_{T<d\leqslant D} |A_d|\right)^{1/2} \left(\sum_{T<d\leqslant D} |B_d|\right)^{1/2}. \end{aligned} \tag{16}$$

Indeed, for any integer d satisfying $T < d \leqslant D$ we have that

$$|A_d||B_d| = (\alpha_d X \beta_d Y)^{1/2} |A_d|^{1/2} |B_d|^{1/2} \leqslant \left(\frac{C'}{C}\right)^{1/2\delta} (\alpha\beta)^{\delta'/2\delta} T^{(1/\delta)-1} (XY)^{1/2} |A_d|^{1/2} |B_d|^{1/2}, \tag{17}$$

where the last inequality follows from (13) on noting that $d^{(1/\delta)-1} \leqslant T^{(1/\delta)-1}$ for $d > T$, since $\delta \geqslant 1$. On combining the bound $|\mathcal{M}(A, B, d)| \leqslant |A_d||B_d|$ with (17) and an application of the Cauchy–Schwarz inequality we obtain (16).

We now estimate the sums on the right-hand side of (16). An application of Hölder's inequality gives

$$\sum_{T<d\leqslant D} |A_d| = \sum_{T<d\leqslant D} \sum_{\substack{n\in A,\\ d|n}} 1 \leqslant \sum_{n\in A} \tau_D(n) \leqslant |A|^{1-1/q} \left(\sum_{1\leqslant n\leqslant X} \tau_D(n)^q\right)^{1/q}. \tag{18}$$

From Lemma 2.2 we have the upper bound $c(q)DX$ for the last sum in (18). Since $|A| \leqslant 2\alpha X$ and $D \leqslant 4/\alpha\beta$, from (18) we deduce that

$$\sum_{T<d\leqslant D} |A_d| \leqslant (2\alpha)^{1-(2/q)} \beta^{-1/q} (4c(q))^{1/q} X. \tag{19}$$

Arguing similarly, we obtain the bound

$$\sum_{T<d\leqslant D} |B_d| \leqslant (2\beta)^{1-(2/q)}\alpha^{-1/q}(4c(q))^{1/q}Y. \tag{20}$$

With these estimates we conclude from (16) that, for any integer T satisfying $1 \leqslant T < D$, we have

$$\sum_{T<d\leqslant D} |\mathcal{M}(A,B,d)| \leqslant 2\left(\frac{C'}{C}\right)^{1/2\delta}(\alpha\beta)^{\delta'/2\delta+1/2-3/2q}T^{(1/\delta)-1}(4c(q))^{1/q}XY. \tag{21}$$

A modest calculation using the expressions defining C' and δ' in terms of C and δ, respectively, now shows that K satisfies the relation

$$2\left(\frac{C'}{C}\right)^{1/2\delta}(\alpha\beta)^{\delta'/2\delta+1/2-3/2q}K^{(1/\delta)-1}(4c(q))^{1/q} = \frac{\alpha\beta}{4}. \tag{22}$$

Consequently, on using (21) with $T = L$ and recalling that $K < L$ and $\delta > 1$, we obtain that

$$\sum_{L<d\leqslant D} |\mathcal{M}(A,B,d)| \leqslant \frac{\alpha\beta}{4}XY. \tag{23}$$

Since (15) applied with $T = D$ gives us $\sum_{1\leqslant d\leqslant D} |\mathcal{M}(A,B,d)| \geqslant (\alpha\beta/2)XY$, we conclude from (23) that, when $1 \leqslant L < D$, we have

$$\frac{1}{L}\sum_{1\leqslant d\leqslant L} |\mathcal{M}(A,B,d)| \geqslant \frac{\alpha\beta}{4L}XY. \tag{24}$$

If $L = 1$ we obtain (14) from (24) on noting that $\alpha\beta/4 \geqslant C'(\alpha\beta)^{\delta'}$, since $1/4 \geqslant C'$ and $1 \leqslant \delta'$. When $1 < L < D$, we have $1 \leqslant K$ and hence $L < (\alpha\beta)^{1-\delta'}/4C'$ so that (14) results from (24) in this final case as well. □

Corollary 2.4. *Every $\delta > 1$ is an admissible exponent.*

Proof. Let q be any integer at least 4 and let $\{\delta_n(q)\}_{n\geqslant 1}$ be the sequence of real numbers determined by the relations $\delta_1(q) = 2$ and

$$\delta_{n+1}(q) = \frac{3\delta_n(q)\,(1 + 1/q) - 2}{2\delta_n(q) - 1} \tag{25}$$

for $n \geqslant 1$. Then each $\delta_n(q)$ is an admissible exponent by Propositions 2.1 and 2.3. It is easily verified that the sequence $\delta_n(q)$ is decreasing and has a limit $\delta(q)$ given by the relation

$$\delta(q) = 1 + \frac{3}{4q} + \frac{1}{2}\sqrt{\frac{6}{q} + \frac{9}{4q^2}}. \tag{26}$$

Plainly, any $\delta > \delta(q)$ is an admissible exponent. The corollary now follows on taking q arbitrarily large in (26). □

Theorem 1.1 follows from the above corollary and the definition of admissible exponents on recalling that $|A/B| \geqslant \sup_{d\geqslant 1} |\mathcal{M}(A,B,d)|$.

3. *Examples*

Let us first prove Theorem 1.2. To this end, given an integer $m \geqslant 1$, let $\mathcal{P}$ denote any set of $2m$ prime numbers and, for any subset I of $\mathcal{P}$, let $d(I) = \prod_{p \in I} p$. If $S(\mathcal{P})$ denotes the set of all $d(I)$, with $|I| = m$, we have the following lemma.

LEMMA 3.1. *For any $\epsilon > 0$ we have $|S(\mathcal{P})/S(\mathcal{P})| \leqslant \epsilon |S(\mathcal{P})|^2$ for all sufficiently large m.*

Proof. Plainly, we have $|S(\mathcal{P})| = \binom{2m}{m}$. Let $\mathcal{Q}$ be the set of ordered pairs of disjoint subsets of $\mathcal{P}$. For any I and J subsets of $\mathcal{P}$, we have

$$\frac{d(I)}{d(J)} = \frac{d(I \setminus J)}{d(J \setminus I)}, \tag{27}$$

and since $I \setminus J$ and $J \setminus I$ are disjoint, it follows that $(I \setminus J, J \setminus I)$ is in $\mathcal{Q}$. Thus $|S(\mathcal{P})/S(\mathcal{P})| \leqslant |\mathcal{Q}|$. Let us associate any (U, V) in $\mathcal{Q}$ to the map from $\mathcal{P}$ to the three-element set $\{1, 2, 3\}$ that takes U to 1, V to 2 and the complement of $U \cup V$ in $\mathcal{P}$ to 3. It is easily seen that this association in fact gives a bijection from $\mathcal{Q}$ onto the set of maps from $\mathcal{P}$ to $\{1, 2, 3\}$ and hence that $|\mathcal{Q}| = 3^{2m}$. In summary, we have that

$$|S(\mathcal{P})/S(\mathcal{P})| \leqslant |\mathcal{Q}| = 3^{2m} = \frac{3^{2m}}{\binom{2m}{m}^2} |S(\mathcal{P})|^2 \leqslant (2m+1)^2 \left(\frac{3}{4}\right)^{2m} |S(\mathcal{P})|^2, \tag{28}$$

where we have used the inequality $\binom{2m}{m} \geqslant 2^{2m}/(2m+1)$. The lemma follows from (28) on noting that $(2m+1)^2\,(3/4)^{2m} \to 0$ as $m \to +\infty$. $\square$

Proof of Theorem 1.2. Given an integer $m \geqslant 1$, it is easily deduced from the prime number theorem that the interval $[T, T + T/m]$ contains at least $2m$ prime numbers when T is sufficiently large. For such a T, let $\mathcal{P}$ be a subset of $2m$ prime numbers in $[T, T + T/m]$. If $\mathcal{A}(\mathcal{P})$ is the sequence of integers at least 1 that are divisible by at least one of the integers $d(I)$ in $S(\mathcal{P})$, then a simple application of the principle of inclusion and exclusion implies that $\mathcal{A}(\mathcal{P})$ has an asymptotic density $\alpha(\mathcal{P})$ that is given by the relation

$$\alpha(\mathcal{P}) = \sum_{1 \leqslant r \leqslant \binom{2m}{m}} (-1)^{r+1} \sum_{1 \leqslant i_1 < i_2 < \ldots < i_r \leqslant \binom{2m}{m}} \frac{1}{d(I_{i_1} \cup I_{i_2} \cup \ldots \cup I_{i_r})}, \tag{29}$$

where $I_1, I_2, \ldots, I_{\binom{2m}{m}}$ are the subsets of cardinality m in $\mathcal{P}$.

For any i we have $T^m \leqslant d(I_i) \leqslant (1 + 1/m)^m T^m < eT^m$. Consequently, for the term $r = 1$ in (29) we obtain

$$\sum_{1 \leqslant i \leqslant \binom{2m}{m}} \frac{1}{d(I_i)} \geqslant \frac{\binom{2m}{m}}{eT^m}. \tag{30}$$

When $r \geqslant 2$, we have that $d(I_{i_1} \cup I_{i_2} \cup \ldots \cup I_{i_r})$, for any distinct indices $i_1, i_2, \ldots, i_r$, has at least $k + 1$ prime factors in $\mathcal{P}$ and hence is at least T^{m+1}. It follows, from (29) and these bounds, that we have

$$\alpha(\mathcal{P}) \geqslant \frac{\binom{2m}{m}}{eT^m} - \frac{2^{\binom{2m}{m}}}{T^{m+1}} \geqslant \frac{\binom{2m}{m}}{3T^m} \tag{31}$$

when T is sufficiently large. In particular, on recalling that $|S(\mathcal{P})| = \binom{2m}{m}$, we obtain that, for any integer $m \geqslant 1$, we have

$$\alpha(\mathcal{P}) \geqslant \frac{|S(\mathcal{P})|}{3T^m} \tag{32}$$

for all sufficiently large T and $\mathcal{P}$ any set of $2m$ prime numbers in $[T, T + T/m]$.

Finally, for $\mathcal{P}$ as above and any $X \geqslant 1$, let us set $A = \mathcal{A}(\mathcal{P}) \cap [1, X]$. Since $\alpha(\mathcal{P})$ is the asymptotic density of $\mathcal{A}(\mathcal{P})$, we have from (32) that $|A| \geqslant (|S(\mathcal{P})|/4T^m)X$, for all large enough X and T. Clearly, each integer in A is of the form $d(I)n$, for some $d(I)$ in $S(\mathcal{P})$ and an integer n, which must necessarily be $\leqslant X/T^m$, since A is in $[1, X]$ and $d(I) \geqslant T^m$. Consequently, we have $|A/A| \leqslant (|S(\mathcal{P})/S(\mathcal{P})|/T^{2m})X^2$, for all large enough X and T. On comparing $|A|$ and $|A/A|$ by means of Lemma 3.1, we see that A meets the conditions of Theorem 1.2 when m, T and X are all sufficiently large. □

Proof of Theorem 1.3. The number of primitive integer points, that is, integer points with coprime co-ordinates, in $[1, \gamma X] \times [1, \gamma Y]$ is $\sim (6/\pi^2)\gamma^2 XY$ as X, $Y \to \infty$. Thus for any γ in $(0, 1]$ and all sufficiently large X and Y, there is a subset S of the primitive integer points in $[1, \gamma X] \times [1, \gamma Y]$ satisfying $(\gamma^2/4)XY \leqslant |S| \leqslant (\gamma^2/2)XY$. Let us take for C the union of the sets $d.S$ with d varying over the integers in the interval $[1, 1/\gamma]$, where each $d.S$ is the set of (da, db) with (a, b) varying over S. Then C is contained in $[1, X] \times [1, Y]$. Moreover, the sets $d.S$ are disjoint but $\mathrm{Frac}(d.S) = \mathrm{Frac}(S)$, for each d, and $|\mathrm{Frac}(S)| = |S|$. We therefore have $|C| = [1/\gamma]|S| \geqslant (\gamma/8)XY$ but $|\mathrm{Frac}(C)| = |\mathrm{Frac}(S)| = |S| \leqslant (\gamma^2/2)XY$. □

4. *Gaps in product sequences*

We now deduce Theorem 1.4 from Theorem 1.1. Let $\mathcal{A}$ and $\mathcal{B}$ be sequences with upper asymptotic densities α and β. Then there exist infinitely many real numbers X and Y at least 1 such that $|\mathcal{A} \cap (X/2, X]| \geqslant \alpha X/4$ and $|\mathcal{B} \cap (Y/2, Y]| \geqslant \beta Y/4$. For such X and Y let us apply Theorem 1.1 to the sets $A = \mathcal{A} \cap (X/2, X]$ and $B = \mathcal{B} \cap (Y/2, Y]$. We then have that $|A/B| \gg (\alpha\beta)^{1+\epsilon} XY$, where the implied constant depends on ϵ alone. Since A/B is a subset of the interval $[X/2Y, 2X/Y]$, which is of length $3X/2Y$, we deduce that there are distinct a/b and a'/b' in A/B such that

$$0 < \left| \frac{a}{b} - \frac{a'}{b'} \right| \ll \frac{X/Y}{(\alpha\beta)^{1+\epsilon} XY} = \frac{1}{(\alpha\beta)^{1+\epsilon} Y^2}. \tag{33}$$

Since $|bb'| \leqslant Y^2$, it follows from (33) that ab' and $a'b$ are distinct terms of the product sequence $\mathcal{A}.\mathcal{B}$ and that $ab' - a'b \ll \frac{1}{(\alpha\beta)^{1+\epsilon}}$. Since there are infinitely many distinct X and Y satisfying the required conditions, there are infinitely many such pairs of terms in the product sequence $\mathcal{A}.\mathcal{B}$.

Acknowledgements. We arrived at Theorem 1.3 in response to a question of Professor Adrian Ubis, whom we gladly thank. We also wish to thank the CRM, Barcelona, Spain and the SYM project, HRI, Allahabad, India for opportunities that supported discussions on the problems addressed here. We are grateful to the referee for a thorough reading of this note and for providing us with a number of useful suggestions.

References

1. G. Berczi, 'On the distribution of products of members of a sequence with positive density', *Period. Math. Hungar.* 44 (2002) 137–145.

2. J. Bourgain, S. Konyagin and I. Shparlinski, 'Product sets of rationals, multiplicative translates of subgroups in residue rings, and fixed points of the discrete logarithm', *Int. Math. Res. Notices* 2008 (2008), doi:10.1093/imrn/rnn90.

3. J. Cilleruelo and T.H. Le, 'Gaps in product sequences', *Israel J. Math.*, to appear.

4. C. Sandor, 'On the minimal gaps between products of members of a sequence with positive density', *Ann. Univ. Sci. Budapest Eötvös Sect. Math.* 28 (2005) 3–7.

5. A. Sárközy, 'Unsolved problems in number theory', *Period. Math. Hungar.* 42 (2001) 17–36, doi:10.1023/A:1015236305093.

J. Cilleruelo
Instituto de Ciencias Matemáticas (CSIC-UAM-UC3M-UCM) and Departamento de Matemáticas
Universidad Autónoma de Madrid
Madrid 28049
Spain

franciscojavier.cilleruelo@uam.es

D. S. Ramana
Harish-Chandra Research Institute
Chhatnag Road
Jhunsi
Allahabad 211 019
India

suri@hri.res.in

O. Ramaré
Laboratoire Paul Painlevé
Université Lille 1
59655 Villeneuve d'Ascq Cedex
France

ramare@math.univ-lille1.fr