Some elementary explicit bounds for two mollifications of the Moebius function

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Abstract

We prove that the sum $\sum_{\begin{cases} d\le x, \\ (d,r)=1\end{cases}} \mu(d)/d^{1+\varepsilon}$ is bounded by $1+\varepsilon$, uniformly in $x\ge 1$, r and $\varepsilon>0$. We prove a similar estimate for the quantity $\sum_{\begin{cases} d\le x, \\ (d,r)=1\end{cases}} \mu(d)\operatorname{Log}(x/d)/d^{1+\varepsilon}$. When $\varepsilon=0$, r varies between 1 and a hundred, and x is below a million, this sum is non-negative and this raises the question as to whether it is non-negative *for every* x.

1 Introduction and results

Our first result is the following:

Theorem 1.1. *When $r\ge 1$ and $\varepsilon\ge 0$, we have*

$$\Bigg|\sum_{\substack{d\le x,\\ (d,r)=1}} \frac{\mu(d)}{d^{1+\varepsilon}}\Bigg| \le 1+\varepsilon.$$

This Lemma generalizes the estimate of [5, Lemme 10.2] which corresponds to the case $\varepsilon=0$. This generalization is *not* straightforward at all and requires a change of proof. The case $\varepsilon=0$ and $r=1$ is classical. The parameter ε that is being introduced induces some flexibility useful when applying Rankin's method (devised in [7]). As it turns out, we can do somewhat better concerning the lower bound, and we prove that

$$-\tfrac{11}{15}(1+\varepsilon)\le \sum_{\substack{d\le x,\\ (d,r)=1}} \frac{\mu(d)}{d^{1+\varepsilon}}.$$

We ran computations covering the range $1\le x\le 10^6$ and $1\le r\le 100$ with $\varepsilon=0$; we found that the lowest lower bound was met at $x=13$ and $r=1$. This raises the following question:

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Question 1. *It is true that*

$$\sum_{\substack{d\le x,\\ (d,r)=1}} \frac{\mu(d)}{d} \ge -2323/30030 \quad ?$$

See section 2 for a very preliminary result in this direction.

We proceed by proving the following more involved form:

Theorem 1.2. *When $r \ge 1$ and $1.38 \ge \varepsilon \ge 0$, we have*

$$\Bigg|\sum_{\substack{d\le x,\\ (d,r)=1}} \frac{\mu(d)}{d^{1+\varepsilon}} \operatorname{Log}\frac{x}{d}\Bigg| \le 1.4 + 4.7\varepsilon + 3.3\varepsilon^2 + (1+\varepsilon)\frac{r^{1+\varepsilon}}{\phi_{1+\varepsilon}(r)} x^{\varepsilon}$$

where

$$\frac{r^{1+\varepsilon}}{\phi_{1+\varepsilon}(r)} = \prod_{p|r} \frac{p^{1+\varepsilon}}{p^{1+\varepsilon}-1}. \tag{1}$$

The dependence in r is optimal as seen by taking for r the product of every primes not more than $\sqrt{x}$. The proof is again unbalanced with respect to the upper and the lower bound, and we prove a somewhat better lower bound:

$$-(1.434 + 4.992\varepsilon + 3.558\varepsilon^2) \le \sum_{\substack{d\le x,\\ (d,r)=1}} \frac{\mu(d)}{d^{1+\varepsilon}} \operatorname{Log}\frac{x}{d}.$$

I expect the factor x^{ε} in the upper bound to be a blemish; however, the (limited) numerical verifications we ran suggest that the factor $r^{1+\varepsilon}/\phi_{1+\varepsilon}(r)$ *cannot* be omitted even if the condition $r \le x$ is added (this condition often appears in practice). It should be added that it is not difficult to prove that

$$\sum_{d\le x} \frac{\mu(d)}{d} \operatorname{Log}\frac{x}{d} \sim 1 \quad (x\to\infty)$$

which means that one cannot expect an arbitary small constant in the right hand side of the inequality given in Theorem 1.2. We have checked that

$$0 \le \sum_{\substack{d\le x,\\ (d,r)=1}} \frac{\mu(d)}{d} \operatorname{Log}\frac{x}{d} \le \frac{r}{\phi(r)} + 0.007 \quad (x \le 10^6, 1 \le r \le 100)$$

(where x is a real number and not especially an integer) and all these maxima were in fact very close to $r/\phi(r)$. These computations raise two questions:

Question 2. *Is it true that*

$$\sum_{\substack{d\le x,\\ (d,r)=1}} \frac{\mu(d)}{d} \operatorname{Log}\frac{x}{d} \ge 0, \quad (x\ge 1,\ r\ge 1) \quad ?$$

Question 3. *Is it true that*

$$\sum_{\substack{d\le x,\\ (d,r)=1}}\frac{\mu(d)}{d}\operatorname{Log}\frac{x}{d}\le\frac{r}{\phi(r)}+1,\quad (x\ge 1,\ r\ge 1)\quad ?$$

In both these questions, x is only assumed to be a positive real number. On recalling what happens in the case of Turán's conjecture on the summatory function of the Liouville function divided by its argument, see [2], we believe that the answer to the first question is no. The sum is however less likely to be very erratical because of the smoothing factor, a factor that is absent in Turán's problem. In direction of these conjecture, we note the following formula

$$\int_1^\infty \sum_{\substack{d\le x,\\ (d,r)=1}}\frac{\mu(d)}{d}\operatorname{Log}\frac{x}{d}\frac{dx}{x^{s+1}}=\frac{r^{1+s}}{\phi_{1+s}(r)}\frac{1}{s^2\zeta(1+s)}$$

from which we easily deduce (on taking $s=\varepsilon>0$ and letting ε go to infinity) that

$$\limsup_x \sum_{\substack{d\le x,\\ (d,r)=1}}\frac{\mu(d)}{d}\operatorname{Log}\frac{x}{d}\ge\frac{r}{\phi(r)}.$$

We discuss some related points in the last section.

Notation

We use here the notation $h=\mathcal{O}^*(k)$ to mean that $|h|\le k$. We denote by $\tau(m)$ the number of (positive) divisors of m, and by (a,b) the gcd of a and b. For $\varepsilon\ge 0$ and $r\ge 1$ any natural squarefree number, we define two functions. The first one is alternatively defined by

$$f_{r,\varepsilon}(n)=\sum_{\substack{\ell|n,\\ (\ell,r)=1}}\frac{\mu(\ell)}{\ell^\varepsilon}\tau(n/\ell) \tag{2}$$

or, in multiplicative form, by:

$$f_{r,\varepsilon}(n)=\prod_{\substack{p^\nu\|n,\\ p\nmid r}}\Big(\nu+1-\frac{\nu}{p^\varepsilon}\Big)\prod_{\substack{p^\nu\|n,\\ p|r}}(\nu+1). \tag{3}$$

We easily determine its Dirichlet series: $\sum_{n\ge1}f_{r,\varepsilon}(n)/n^s=\zeta(s)^2/\zeta(s+\varepsilon)$. We shall further write

$$f_{r,\varepsilon}(n)=\mathbb{1}\star g_{r,\varepsilon}(n) \tag{4}$$

where the function $g_{r,\varepsilon}$ has the essential property of being non-negative and is being defined by:

$$g_{r,\varepsilon}(n) = \sum_{\substack{\ell | n,\\ (\ell,r)=1}} \frac{\mu(\ell)}{\ell^\varepsilon} \geq 0. \tag{5}$$

2 Verifying Theorem 1.1 for small values

We study what happens for small values here. The proof is pedestrian and painful, but I have not seen any way to avoid it, or to present it in a more general frame.

We study the following quantity:

$$m_0(r,x) = \sum_{\substack{d\leq x,\\ (d,r)=1}} \frac{\mu(d)}{d^{1+\varepsilon}}. \tag{6}$$

Lemma 2.1. *When $x < 10$, we have $-1/30 \leq m_0(r,x) \leq 1$.*

Proof. The sum we consider reads

$$1 - \frac{h(2)}{2^{1+\varepsilon}} - \frac{h(3)}{3^{1+\varepsilon}} - \frac{h(4)}{5^{1+\varepsilon}} + \frac{h(6)}{6^{1+\varepsilon}} - \frac{h(7)}{7^{1+\varepsilon}}$$

where h is the characteristic function of the integers $\leq x$ that are coprime with r. The minimum is clearly

$$1 - \frac{1}{2^{1+\varepsilon}} - \frac{1}{3^{1+\varepsilon}}$$

which is minimal when $\varepsilon = 0$. This is the $-1/30$. The maximum contains the summand 1. If the summand $1/6^{1+\varepsilon}$ is present, then so is the summand $-1/2^{1+\varepsilon}$. This concludes the proof. □

3 Auxiliaries

Lemma 3.1. *When $\varepsilon \geq 0$, we have*

$$\sum_{h\leq H} h^\varepsilon = \frac{H^{1+\varepsilon}}{1+\varepsilon} + \mathcal{O}^*\big(H^\varepsilon\big).$$

This is also $\leq H^{1+\varepsilon}$.

Proof. Indeed, a summation by parts gives us directly

$$\begin{aligned}\sum_{h\leq H} h^\varepsilon &= \sum_{h\leq H} \varepsilon \int_0^h dt/t^{1-\varepsilon} = \varepsilon \int_0^H \sum_{t<h\leq H} 1\, dt/t^{1-\varepsilon} \\ &= \varepsilon \int_0^H (H-t)\, dt/t^{1-\varepsilon} + \mathcal{O}^*(H^\varepsilon).\end{aligned}$$

□

Lemma 3.2. *For $L > 1$, we have*

$$\sum_{n \le L} f_{r,\varepsilon}(n) \le L \sum_{\ell \le L} g_{r,\varepsilon}(\ell)/\ell. \tag{7}$$

Proof. We recall (4) and write, since $g_{r,\varepsilon} \ge 0$

$$\sum_{n \le L} f_{r,\varepsilon}(n) = \sum_{km \le L} g_{r,\varepsilon}(m) \le L \sum_{m \le L} g_{r,\varepsilon}(m)/m.$$

The Lemma follows readily. □

Lemma 3.3. *For $x \ge 1$, we have*

$$\frac{1 - e^{-\varepsilon x}}{e^x - 1} \le \varepsilon x / e^x.$$

Proof. We need to prove that

$$g(x) = \varepsilon x - \varepsilon x e^{-x} + e^{-\varepsilon x} - 1 \ge 0$$

We have $g'(x) = (1 - e^{-x} + x e^{-x} - e^{-\varepsilon x})\varepsilon$. This is non negative if $x \ge 1$. □

Lemma 3.4. *We have, when $0 \le \varepsilon \le 1.38$,*

$$\operatorname{Log} \frac{1 - \frac{1}{2^{1+\varepsilon}}}{1 - \frac{1}{2}} \le \frac{\varepsilon \operatorname{Log} 2}{2}.$$

Proof. We need to prove that

$$g(x) = \operatorname{Log} \frac{1 - \frac{1}{2^{1+\varepsilon}}}{1 - \frac{1}{2}} - \frac{\varepsilon \operatorname{Log} 2}{2} \ge 0.$$

We have $g'(x) = \left(\frac{1}{2^{1+\varepsilon} - 1} - \frac{1}{2}\right) \operatorname{Log} 2$. This is non decreasing in ε. The Lemma follows after some numerical computations. □

Lemma 3.5. *We have, when $L \ge 319$,*

$$\sum_{p \le L} \frac{\operatorname{Log} p}{p} \le \operatorname{Log} L - 1.332\,582\,275\,733 + \frac{1}{2 \operatorname{Log} L}.$$

The left-hand side is also $\le \operatorname{Log} L$ for any $L \ge 1$.

This is [8, (3.32),(2.11)].

Lemma 3.6. *For $L > 1$, we have*

$$\sum_{\ell \le L} g_{r,\varepsilon}(\ell)/\ell \le \frac{r^{1+\varepsilon}}{\phi_{1+\varepsilon}(r)} L^{\varepsilon}. \tag{8}$$

Proof. We readily find that the sum in question is not more than

$$T = \prod_{p|r} \frac{1}{1-p^{-1}} \prod_{\substack{p\nmid r,\\ p\le L}} \frac{1-p^{-1-\varepsilon}}{1-p^{-1}} = \frac{r^{1+\varepsilon}}{\phi_{1+\varepsilon}(r)} \prod_{p\le L} \frac{1-p^{-1-\varepsilon}}{1-p^{-1}}$$
$$= \frac{r^{1+\varepsilon}}{\phi_{1+\varepsilon}(r)} \exp \sum_{p\le L} \operatorname{Log}\left(1+\frac{1-p^{-\varepsilon}}{p-1}\right)$$

We apply Lemma 3.3 when $p \geq 3$ and Lemma 3.4 when $p = 2$ to get

$$T \le \frac{r^{1+\varepsilon}}{\phi_{1+\varepsilon}(r)} \exp \varepsilon \sum_{p\le L} \frac{\operatorname{Log} p}{p}.$$

We apply Lemma 3.5 and conclude easily. □

Lemma 3.7.

$$\sum_{m\le M} m^{\varepsilon}\tau(m) = \frac{M^{1+\varepsilon}}{1+\varepsilon}\left(\operatorname{Log} M + 2\gamma - \frac{1}{1+\varepsilon}\right) + \mathcal{O}^*\left(0.961(1+2\varepsilon)M^{\frac{1}{2}+\varepsilon}\right)$$

Proof. We recall part of [1, Theorem 1.1]:

$$\sum_{m\le t} \tau(m) = t\operatorname{Log} t + (2\gamma-1)t + \mathcal{O}^*(0.961\sqrt{t}), \quad (t \ge 1).$$

Since $(t\operatorname{Log} t + (2\gamma-1)t)/\sqrt{t}$ is seen to vary between -0.681 and 0.155 when t varies between 0 and 1, this estimate is also valid for $t > 0$. We use summation by parts and find that

$$\begin{aligned}
\sum_{m\le M} m^{\varepsilon}\tau(m) &= M^{\varepsilon}\sum_{m\le M}\tau(m) - \varepsilon\int_0^M \sum_{m\le t}\tau(m)dt/t^{1-\varepsilon} \\
&= M^{1+\varepsilon}(\operatorname{Log} M + 2\gamma - 1) + \mathcal{O}\Big(0.961 M^{\frac{1}{2}+\varepsilon}\Big) \\
&\quad - \varepsilon\int_0^M (\operatorname{Log} t + 2\gamma - 1)t^{\varepsilon}dt + \mathcal{O}^*\left(0.961\varepsilon\int_0^M t^{\varepsilon-1/2}dt\right) \\
&= \frac{M^{1+\varepsilon}}{1+\varepsilon}\left(\operatorname{Log} M + 2\gamma - \frac{1}{1+\varepsilon}\right) + \mathcal{O}^*\left(0.961(1+2\varepsilon)M^{\frac{1}{2}+\varepsilon}\right).
\end{aligned}$$

□

Lemma 3.8. *We have, when $n \geq 2$,*

$$g_{r,\varepsilon}(n) \le 1 - \frac{\mu^2(n)}{n^{\varepsilon}}.$$

Proof. Indeed, we verify that $(1-a)(1-b) \leq (1-ab)$ when $0 \leq a, b$ and one of them is ≤ 1. The Lemma readily follows by recursion on the number of prime factors of n. □

Lemma 3.9. *We have*

$$\sum_{\substack{n \leq 6, \\ (n,r)=1}} \left(g_{r,\varepsilon}(n) + \frac{\mu^2(n)}{n^\varepsilon} \right) \leq 6.$$

Proof. We simply proceed by considering all possible cases. When r is prime to 30, we have to check that

$$-\frac{1}{2^\varepsilon} - \frac{1}{3^\varepsilon} - \frac{1}{2^\varepsilon} - \frac{1}{5^\varepsilon} + \frac{1}{6^\varepsilon} \leq 0$$

and this is readily done. When $(r, 30) = 2$, we have to check that

$$-\frac{1}{3^\varepsilon} - \frac{1}{5^\varepsilon} \leq 3$$

which clearly holds. When $(r, 30) = 3$, the equation to check is

$$-\frac{1}{2^\varepsilon} - \frac{1}{2^\varepsilon} - \frac{1}{5^\varepsilon} \leq 2.$$

When $(r, 30) = 6$, the equation to check is

$$-\frac{1}{5^\varepsilon} \leq 4.$$

When $(r, 30) = 5$, the equation to check is

$$-\frac{1}{2^\varepsilon} - \frac{1}{3^\varepsilon} - \frac{1}{2^\varepsilon} + \frac{1}{6^\varepsilon} \leq 1.$$

When $(r, 30) = 10$, the equation to check is

$$-\frac{1}{3^\varepsilon} \leq 4.$$

When $(r, 30) = 30$, the equation to check is

$$0 \leq 5.$$

□

4 Some lemmas on squarefree numbers

Here is a Lemma from [4]:

Lemma 4.1. *We have, for $D \geq 1\,664$*

$$\sum_{d \leq D} \mu^2(d) = \frac{6D}{\pi^2} + \mathcal{O}^*\big(0.1333\sqrt{D}\big).$$

In particular, this is not more than $0.62D$ when $D \geq 1700$.

Lemma 4.2. *We have*

$$\sum_{d \leq x} \mu^2(d)/\sqrt{d} \leq 1.33\,\sqrt{x}, \quad (x \geq 1).$$

If we are ready to assume larger, we would not save much since the best constant one can get is $12/\pi^2 = 1.215 + \mathcal{O}^*(0.001)$.

Proof. We use PARI/GP see [10] and the following script:

```
{check(borne) =
   my(res = 0.0, coef = 0);
   for(d = 1, borne,
       res += moebius(d)^2/sqrt(d);
       coef = max(coef, res/sqrt(d)));
   return(coef)}
```

It is then almost immediate to check our result when $x \leq 10^7$, despite the lack of refinement of the script proposed. For larger values, we use a summation by parts together with Lemma 4.1. □

Lemma 4.3. *We have*

$$\sum_{d \leq x} \mu^2(d) \leq \tfrac{11}{15}\, x, \quad (x \geq 9).$$

We note that $11/15 = 0.7333\ldots$ while the asymptotically best constant is rather lower, namely $6/\pi^2 = 0.607927\ldots$. Reaching $73/115 = 0.63478\ldots$ already requires to take $x \geq 75$, and this means we would have to handle the possible divisibility by 21 primes in section 2. This is out of reach of the simple-minded method we have at our disposal.

Proof. We use PARI/GP see [10] and the following script:

```
{check(borneinf, bornesup) =
   my(res = 0.0, coef = 0);
   res = sum(d = 1, borneinf-1, moebius(d)^2);
   for(d = borneinf, bornesup,
```

```
        res += moebius(d)^2;
        coef = max(coef, res/d));
    return(coef)}
```

It is then almost immediate to check our result when $x \leq 10^7$, despite the lack of refinement of the script proposed. For larger values, we use a summation by parts together with Lemma 4.1. $\square$

5 Proof of Theorem 1.1

We start with

$$S_0 = \sum_{n \leq x} n^\varepsilon g_{r,\varepsilon}(n) = \sum_{n \leq x} \sum_{\substack{d|n,\\ (d,r)=1}} \mu(d)(n/d)^\varepsilon.$$

Using the first expression yields $0 \leq S_0 \leq x^{1+\varepsilon} + x^\varepsilon - x^\varepsilon \sum_{\left\{\substack{n \leq 6,\\ (n,r)=1}\right.} \mu^2(n)/n^\varepsilon$ by Lemma 3.8. It is even $\leq x^{1+\varepsilon} - x^\varepsilon \sum_{n \leq x} \mu^2(n)/n^\varepsilon$ when $x \geq 6$ by conjugating Lemma 3.8 and Lemma 3.9. Let us write the second expression for S_0:

$$S_0 = \sum_{\substack{d \leq x,\\ (d,r)=1}} \mu(d) \sum_{m \leq x/d} m^\varepsilon.$$

We employ Lemma 3.1 to reach

$$S_0 \quad = \quad \frac{x^{1+\varepsilon}}{1+\varepsilon} \sum_{\substack{d \leq x,\\ (d,r)=1}} \frac{\mu(d)}{d^{1+\varepsilon}} + \mathcal{O}^*\left(x^\varepsilon \sum_{\substack{d \leq x,\\ (d,r)=1}} \mu^2(d) d^{-\varepsilon} \right).$$

By conjugating both estimates, we get, when $x \geq 6$,

$$\begin{aligned} -x^\varepsilon \sum_{\substack{d \leq x,\\ (d,r)=1}} \mu^2(d) d^{-\varepsilon} \leq \frac{x^{1+\varepsilon}}{1+\varepsilon} & \sum_{\substack{d \leq x,\\ (d,r)=1}} \frac{\mu(d)}{d^{1+\varepsilon}} \\ & \leq x^\varepsilon \sum_{\substack{d \leq x,\\ (d,r)=1}} \mu^2(d) d^{-\varepsilon} + x^{1+\varepsilon} - x^\varepsilon \sum_{n \leq x} \mu^2(n)/n^\varepsilon. \end{aligned}$$

The right hand side is easily handled. We use Lemma 4.3 for the left hand side via, when $x \geq 9$:

$$\sum_{\substack{d \leq x,\\ (d,r)=1}} \mu^2(d) d^{-\varepsilon} \leq \sum_{d \leq x} \mu^2(d) \leq \tfrac{11}{15} x.$$

By conjugating both estimates, we get

$$-\tfrac{11}{15}(1+\varepsilon) \le \sum_{\substack{d\le x,\\ (d,r)=1}} \frac{\mu(d)}{d^{1+\varepsilon}} \le 1+\varepsilon. \quad (x\ge 9) \tag{9}$$

This is extended to any $x \ge 1$ in section 2.

6 Proof of Theorem 1.2

The proof relies on two ways of writing the sum

$$S_1 = \sum_{n\le x} n^{\varepsilon} f_{r,\varepsilon}(n) = \sum_{n\le x}\sum_{\substack{d|n,\\ (d,r)=1}} \mu(d)(n/d)^{\varepsilon}\tau(n/d).$$

The first form shows that $0 \le S_1 \le x^{1+2\varepsilon} r^{1+\varepsilon}/\phi_{1+\varepsilon}(r)$ by combining Lemma 3.2 together with Lemma 3.6. Let us write this sum differently:

$$S_1 = \sum_{\substack{d\le x,\\ (d,r)=1}} \mu(d) \sum_{m\le x/d} m^{\varepsilon}\tau(m)$$

and we use Lemma 3.7 to reach

$$S_1 = \frac{x^{1+\varepsilon}}{1+\varepsilon} \sum_{\substack{d\le x,\\ (d,r)=1}} \frac{\mu(d)}{d^{1+\varepsilon}}\Big(\mathrm{Log}\,\frac{x}{d} + 2\gamma - \frac{1}{1+\varepsilon}\Big) + \mathcal{O}^*\Big(0.961\times 1.33\,(1+2\varepsilon)x^{1+\varepsilon}\Big)$$

since $\sum_{d\le x}\mu^2(d)/\sqrt{d} \le 1.33\sqrt{x}$ by Lemma 4.2. We set

$$\alpha = 2\gamma - \frac{1}{1+\varepsilon} \in [0,1]. \tag{10}$$

All of that amounts to:

$$\begin{aligned} S_1 &= \frac{x^{1+\varepsilon}}{1+\varepsilon} \sum_{\substack{d\le x,\\ (d,r)=1}} \frac{\mu(d)}{d^{1+\varepsilon}}\Big(\mathrm{Log}\,\frac{x}{d} + \alpha\Big) + \mathcal{O}^*\big(1.279(1+2\varepsilon)x^{1+\varepsilon}\big) \\ &= S_1^* + \alpha S_0 + \mathcal{O}^*\big(1.279(1+2\varepsilon)x^{1+\varepsilon}\big) \end{aligned}$$

say. We thus have

$$-1.279(1+2\varepsilon)x^{1+\varepsilon} \le S_1^* + \alpha S_0 \le 1.279(1+2\varepsilon)x^{1+\varepsilon} + x^{1+2\varepsilon}\frac{r^{1+\varepsilon}}{\phi_{1+\varepsilon}(r)}$$

We use (9) and reach

$$-1.279(1+2\varepsilon) - \alpha \le x^{-1-\varepsilon} S_1^* \le 1.279(1+2\varepsilon) + \tfrac{11}{15}\alpha + x^{\varepsilon}\frac{r^{1+\varepsilon}}{\phi_{1+\varepsilon}(r)}.$$

We use $\alpha \leq 2\gamma - 1 + \varepsilon$. This gives

$$-1.434 - 4.992\varepsilon - 3.558\varepsilon^2 \leq \sum_{\substack{d\leq x,\\ (d,r)=1}} \frac{\mu(d)}{d^{1+\varepsilon}} \operatorname{Log}\frac{x}{d}$$

$$\leq 1.393 + 4.684\varepsilon + 3.292\varepsilon^2 + (1+\varepsilon)\frac{x^\varepsilon r^{1+\varepsilon}}{\phi_{1+\varepsilon}(r)}.$$

Since $x^\varepsilon r^{1+\varepsilon}/\phi_{1+\varepsilon}(r) \geq 1$, we check that the right hand side is larger than minus times the left hand side. Theorem 1.2 follows.

7 A generalization and a remark

It is not difficult to get along these lines the following Lemma:

Lemma 7.1. *When $r \geq 1$ and $k \geq 1$, we have*

$$\sum_{\substack{d\leq x,\\ (d,r)=1}} \frac{\mu(d)}{d^{1+\varepsilon}} \operatorname{Log}^k \frac{x}{d} \ll_k \left(\frac{r}{\phi(r)}\right)^k (\operatorname{Log} x)^{k-1}.$$

Such quantities appear for instance in [9] where cases $k = 0$ and $k = 1$ are used, while case $k = 2$ is evaluated (there is a main term), but all with no coprimality conditions (i.e. $r = 1$) and no ε. The reader will find in [3, Chapter 1] the evaluation of case $k = 3$, $r = 1$ and $\varepsilon = 0$. [6] also pertains to these quantities.

Proof. Indeed, we first prove that

$$\sum_{n\leq x} \sum_{\substack{d|n,\\ (d,r)=1}} \mu(d)(n/d)^\varepsilon \tau_{k+1}(n/d) \ll \left(\frac{r}{\phi(r)}\right)^k x(\operatorname{Log} x)^{k-1}.$$

We then continue as in section 6. □

Here is a surprising elementary consequence.

Lemma 7.2. *For any $c > 0$, we have*

$$\sum_{\substack{d\leq x,\\ (d,r)=1}} \frac{\mu(d)}{d} - x^\varepsilon \sum_{\substack{d\leq x,\\ (d,r)=1}} \frac{\mu(d)}{d^{1+\varepsilon}} \ll_c \varepsilon \frac{r}{\phi(r)}$$

provided that $0 \leq \varepsilon \leq c(\operatorname{Log} x)^{-1}$.

Proof. It is enough to consider

$$\int_0^\varepsilon \sum_{\substack{d\leq x,\\ (d,r)=1}} \frac{\mu(d)x^\eta}{d^{1+\eta}} \operatorname{Log}(x/d)d\eta \ll \varepsilon \frac{r}{\phi(r)}.$$

□

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