Height functions on a peg solitaire board[*][†]

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Abstract

We provide a smooth theory for very general height functions on a peg solitaire board and prove several result concerning them. This will lead to a better criterium to show that all the points from a given subset of points cannot simultaneously be reached when starting from a given position.

1 Introduction

In (Ramaré, 2008b), we defined the height $\mathrm{Height}(A, I)$ of a point A with respect to the position I to be the least number of legal moves required to put a peg in A. Let us extend this definition to sets K and define the *geometrical height* $\mathrm{h}_g(K, I)$ of K with respect to I to be the minimal number of legal moves required to put pegs in each points of K, when starting with a board $\mathfrak{S}$ containing pegs in each point of I and only in them. When $K_1 \subset K_2$, we have

$$\mathrm{h}_g(K_1, I) \leq \mathrm{h}_g(K_2, I). \tag{1}$$

Our first Theorem concerning this geometrical height reads:

Theorem 1.1 *Let K, I_1 and I_2 be a three subset of $\mathfrak{S}$. When $I_1 \supset I_2$, we have*

$$\mathrm{h}_g(K, I_1) \leq \mathrm{h}_g(K, I_2). \tag{2}$$

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Proof: Let us assume $\mathrm{h}_g(K, I_2) = d$. The case $d = 0$ is of no interest, so let A be the last point of K reached by a minimal path from I_2 to K, and let B and C be its ancestors. Note that B and C are not in K. We have

$$\mathrm{h}_g(K \cup \{B, C\} \setminus \{A\}, I_2) = d - 1.$$

By induction, this implies $\mathrm{h}_g(K \cup \{B, C\} \setminus \{A\}, I_1) \leq d - 1$. Then, either the path realising this height puts a peg in A and the problem is solved, either it does not and we simply have to play $\check{B} + \check{C} - \check{A}$ to end the proof. $\diamond\diamond\diamond$

With this Theorem at hand, we can define $\mathrm{h}_g(K, I)$ in a more symetrical way: it is the maximum of sets I' containing I of the minimal number of legal moves required to put pegs in each points of K, when starting with a board $\mathfrak{G}$ containing pegs in each point of I' and none outside it. As a consequence, we get

Theorem 1.2

$$\mathrm{h}_g(K, I) \leq \mathrm{h}_g(K, L) + \mathrm{h}_g(L, I). \tag{3}$$

Proof: Indeed, we can assume that there is a path from I to L and one from L to K since otherwise the RHS takes value ∞. The path going from I to L transforms I into a set L_0 that contains L. Since $\mathrm{h}_g(K, L) \geq \mathrm{h}_g(K, L_0)$, there is also a path from L_0 to K and we thus get

$$\mathrm{h}_g(K, I) \leq \mathrm{h}_g(K, L_0) + \mathrm{h}_g(L, I) \leq \mathrm{h}_g(K, L) + \mathrm{h}_g(L, I)$$

as required. $\diamond\diamond\diamond$

Anticipating on Theorem 1.4, it is readily understood that the inequality

$$\mathrm{h}_g(K, I) \overset{?}{\leq} \mathrm{h}_g(L, K) + \mathrm{h}_g(L, I) \tag{4}$$

does not hold in general. Indeed, by going to the complementary sets this would imply

$$\mathrm{h}_g(K, I) \overset{?}{\leq} \mathrm{h}_g(K, L) + \mathrm{h}_g(I, L).$$

Consider now the case where K and L have only one point and $\mathrm{h}_g(K, L) = \mathrm{h}_g(I, L) = 1$: we always have $\mathrm{h}_g(K, I) = \infty$ in this case, provided $K \neq L$, which contradicts the above inequality.

Theorem 1.3 *Assume* $\mathrm{h}_g(K, I) < \infty$. *Then there exists a subset* I' *of* I *of cardinality* $|K| + \mathrm{h}_g(K, I)$ *such that* $\mathrm{h}_g(K, I) = \mathrm{h}_g(K, I')$.

This means that there is a part of I that gets transformed exactly in K, without any residual set.

Proof: We again proceed by induction on $\mathrm{h}_g(K, I)$, when I is fixed. This is obvious when $\mathrm{h}_g(K, I)$ is 1 (or 0 for that matter). The recursion is readily achieved. $\diamond\diamond\diamond$

On combining Leibniz's remark with Theorem 1.3, we infer

Theorem 1.4

$$\mathrm{h}_g(K, I) = \mathrm{h}_g(\mathfrak{S} \setminus I, \mathfrak{S} \setminus K). \tag{5}$$

In particular, they are both simultaneously finite. This Theorem is more tricky than seems since it transfers I on the left-hand side where the monotonity (1) is obvious. It thus contains Theorem 1.1, but this provides no new proof of the latter since it is used in its proof.

Proof: Indeed, there is a subset I' of I that gets exactly transformed in K. By inverting the moves, we get

$$\mathrm{h}_g(K, I) = \mathrm{h}_g(K, I') = \mathrm{h}_g(\mathfrak{S} \setminus I', \mathfrak{S} \setminus K) \geq \mathrm{h}_g(\mathfrak{S} \setminus I, \mathfrak{S} \setminus K).$$

But applying the same reasoning to $\mathfrak{S} \setminus I$ and $\mathfrak{S} \setminus K$, we discover that $\mathrm{h}_g(\mathfrak{S} \setminus I, \mathfrak{S} \setminus K) \geq \mathrm{h}_g(K, I)$ hence the equality. $\diamond\diamond\diamond$

By using the same inductive process, we get

Lemma 1.1 *When $K_1 \cap K_2 = \emptyset$, we have*

$$\mathrm{h}_g(K_1 \cup K_2, I) \geq \mathrm{h}_g(K_1, I) + \mathrm{h}_g(K_2, I).$$

Proof: Indeed, let us look at a minimal path from I to $K_1 \cup K_2$. Let us call A the last point created by this path, and let us assume it belongs to K_1. We further denote its immediate ancestors by B and C. The set $K_1' = K_1 \cup \{B, C\} \setminus \{A\}$ has still empty intersection with K_2. By induction we get

$$\begin{aligned} \mathrm{h}_g(K_1 \cup K_2, I) - 1 = \mathrm{h}_g(K_1' \cup K_2, I) &\geq \mathrm{h}_g(K_1', I) + \mathrm{h}_g(K_2, I) \\ &\geq \mathrm{h}_g(K_1, I) - 1 + \mathrm{h}_g(K_2, I) \end{aligned}$$

which ends the proof. $\diamond\diamond\diamond$

We reach this way the following Theorem:

Theorem 1.5 *We have*

$$\mathrm{h}_g(K_1 \cup K_2, I) + \mathrm{h}_g(K_1 \cap K_2, I) \geq \mathrm{h}_g(K_1, I) + \mathrm{h}_g(K_2, I).$$

Proof: Indeed, this is obvious when $K_1 \subset K_2$ or $K_2 \subset K_1$. Let us write then $K_1 = K_1' \cup K_3'$ and $K_2 = K_2' \cup K_3'$ where the unions are disjoint. Assume a minimal path to $K_1 \cup K_2$ puts the last peg in $A \in K_1'$. Let B and C be its predecessors and set $K_1'' = K_1' \cup \{B, C\} \setminus \{A\}$. By induction, we have

$$\begin{aligned}
\mathrm{h}_g(K_1 \cup K_2, I) - 1 + \mathrm{h}_g(K_3', I) &= \mathrm{h}_g(K_1'' \cup K_2, I) + \mathrm{h}_g(K_3', I) \\
&\geq \mathrm{h}_g(K_1'' \cup K_3', I) + \mathrm{h}_g(K_2, I) \\
&\geq \mathrm{h}_g(K_1, I) - 1 + \mathrm{h}_g(K_2, I)
\end{aligned}$$

which settles this case. The case $A \in K_2'$ is symetrical. Let us thus assume $A \in K_3'$. We then define $K_3'' = K_3' \setminus \{A\}$, $K_2'' = K_2' \cup \{B, C\}$ and $K_1'' = K_1' \cup \{B, C\}$. We get

$$\begin{aligned}
\mathrm{h}_g(K_1 \cup K_2, I) - 1 + \mathrm{h}_g(K_3, I) - 1 &= \mathrm{h}_g(K_1'' \cup K_3'' \cup K_2'', I) + \mathrm{h}_g(K_3'', I) \\
&\geq \mathrm{h}_g(K_1'' \cup K_3'', I) + \mathrm{h}_g(K_2'' \cup K_3'', I) \\
&\geq \mathrm{h}_g(K_1, I) - 1 + \mathrm{h}_g(K_2, I) - 1
\end{aligned}$$

and this ends the proof! $\diamond\diamond\diamond$

We then appeal to Theorem 1.4 to get

Theorem 1.6 *We have*

$$\mathrm{h}_g(K, I_1 \cup I_2) + \mathrm{h}_g(K, I_1 \cap I_2) \geq \mathrm{h}_g(K, I_1) + \mathrm{h}_g(K, I_2).$$

Here are two consequences. We have

$$\mathrm{Height}(A, I) \geq \min\Big\{\mathrm{Height}(B, I) + \mathrm{Height}(C, I) + 1, \check{B} + \check{C} - \check{A} \in \mathscr{D}(\mathfrak{S})\Big\}. \tag{6}$$

We do not have always equality. Furthermore, we have defined $\mathrm{Height}(A, A', I)$ in (Ramaré, 2008b) to be in fact what we denote $\mathrm{h}_g(\{A, A'\}, I)$ here. We have proved that

$$\mathrm{Height}(A, A', I) \geq \mathrm{Height}(A, I) + \mathrm{Height}(A', I) \tag{7}$$

which was conjecture 12.1 in the aforementioned paper.

Determining whether $\mathrm{h}_g(K, I)$ is finite or not is important and we can use Theorem 1.3 to get the following criterium:

Theorem 1.7 *If there exists a potential (resource count, see* (12)*)* π *that is non-negative on* I *and such that* $\langle \mathbb{1}_I, \pi\rangle < \langle \mathbb{1}_K, \pi\rangle$, *then* $\mathrm{h}_g(K, I) = \infty$.

This improves on Lemma 12.2 of (Ramaré, 2008b) in that π is asked to be non-negative only on I.

2 Extensions

Let $\mathscr{P}_\infty(\mathfrak{S})$ be the set of all finite sequences of elements of $\mathscr{D}(\mathfrak{S})$. We call such a sequence a *path*. Given a subset I of $\mathfrak{S}$, we say that such a path $\mathfrak{P} = (\mathfrak{f}_1, \ldots, \mathfrak{f}_k)$ is *legal* with respect to I, or that $(I, \mathfrak{P})$ is legal whenever the moves $\mathfrak{f}_1, \ldots, \mathfrak{f}_k$ can legally be played from I. For such a path $\mathfrak{P}$, we set

$$\Psi(\mathfrak{P}) = \mathfrak{f}_1 + \mathfrak{f}_2 + \cdots + \mathfrak{f}_k. \tag{8}$$

Moreover, we use the natural hermitian product on $\mathscr{F}(\mathfrak{S}, \mathbb{C})$, that is

$$\langle f, g\rangle = \langle f, g\rangle_{\mathfrak{S}} = \sum_{P\in\mathfrak{S}} f(P)\overline{g(P)}. \tag{9}$$

Equipped with these notions, we can now define the height as follows:

$$\mathrm{h}_g(K, I) = \min_{\mathfrak{P}}\Big\{\langle\Psi(\mathfrak{P}), \mathbb{1}_{\mathfrak{S}}\rangle, (I, \mathfrak{P}) \text{ legal and } \mathbb{1}_I - \Psi(\mathfrak{P}) \geq \mathbb{1}_K\Big\} \tag{10}$$

and this leads to the natural extension:

$$\mathrm{h}(K, I, \pi) = \min_{\mathfrak{P}}\Big\{\langle\Psi(\mathfrak{P}), \pi\rangle, (I, \mathfrak{P}) \text{ legal and } \mathbb{1}_I - \Psi(\mathfrak{P}) \geq \mathbb{1}_K\Big\} \tag{11}$$

where π is an integer valued potential (resource count), i.e. a function of $\mathscr{F}(\mathfrak{S}, \mathbb{C})$ such that

$$\forall \mathfrak{f} \in \mathscr{D}(\mathfrak{S}), \quad \langle\pi, \mathfrak{f}\rangle \geq 0. \tag{12}$$

This condition ensures that $\mathrm{h}(K, I, \pi) \geq 0$. Since $\mathbb{1}_{\mathfrak{S}}$ is an integer valued potential, we have $\mathrm{h}_g(K, I) = \mathrm{h}(K, I, \mathbb{1}_{\mathfrak{S}})$ and our notion is indeed a generalization of h_g. Note that all $\mathrm{h}(K, I, \pi)$ are simultaneously finite. Let us inspect which of the properties we have discovered are still valid. We have evidently

$$\mathrm{h}(K_1, I, \pi) \leq \mathrm{h}(K_2, I, \pi), \quad \text{whenever } K_1 \subset K_2. \tag{13}$$

Here is our first Theorem.

Theorem 2.1 *Let K, I_1 and I_2 be a three subset of $\mathfrak{S}$. When $I_1 \supset I_2$, we have*

$$\mathrm{h}(K, I_1, \pi) \leq \mathrm{h}(K, I_2, \pi). \tag{14}$$

Proof: Let us assume $\mathrm{h}(K, I_2, \pi) = d$. The case $d = 0$ is of no interest, so let A be the last point of K reached by a minimal path from I_2 to K, and let B and C be its ancestors. We set $\mathfrak{f} = \check{B} + \check{C} - \check{A}$. Note that B and C are not in K. We have

$$\mathrm{h}\big(K \cup \{B, C\} \setminus \{A\}, I_2, \pi\big) = d - \langle\mathfrak{f}, \pi\rangle.$$

By induction on $(\mathrm{h}_g(K,I_2), |\mathfrak{S}\setminus K|)$, this implies $\mathrm{h}(K\cup\{B,C\}\setminus\{A\}, I_1) \leq d - \langle \mathfrak{f}, \pi\rangle$. Then, either the path realising this height puts a peg in A and the problem is solved, either it does not and we simply have to play $\mathfrak{f}$ to end the proof. $\diamond\diamond\diamond$

This leads to a second definition of $\mathrm{h}(K_1, I, \pi)$, namely

$$\mathrm{h}(K,I,\pi) = \min_{\mathfrak{P}, I'\subset I}\Big\{\langle \Psi(\mathfrak{P}), \pi\rangle, (I',\mathfrak{P}) \text{ legal and } \mathbb{1}_{I'} - \Psi(\mathfrak{P}) \geq \mathbb{1}_K\Big\} \tag{15}$$

and has the same consequence as before:

Theorem 2.2

$$\mathrm{h}(K,I,\pi) \leq \mathrm{h}(K,L,\pi) + \mathrm{h}(L,I,\pi). \tag{16}$$

The proof is the one given for Theorem 1.2. We find next

Theorem 2.3 *Assume $\mathrm{h}_g(K,I) < \infty$. Then for every integer valued potential π, there exists a subset I' of I and a path $\mathfrak{P}_0$ which is legal with respect to I', such that $\mathbb{1}_{I'} - \Psi(\mathfrak{P}_0) = \mathbb{1}_K$ and such that $\mathrm{h}(K,I,\pi) = \mathrm{h}(K,I',\pi) = \langle\Psi(\mathfrak{P}_0),\pi\rangle$.*

By using the path inversion of Leibniz, we get again:

Theorem 2.4

$$\mathrm{h}(K,I,\pi) = \mathrm{h}(\mathfrak{S}\setminus I, \mathfrak{S}\setminus K, \pi). \tag{17}$$

The same induction as for the proof of Theorem 2.1 yields:

Theorem 2.5 *We have*

$$\mathrm{h}(K_1\cup K_2, I, \pi) + \mathrm{h}(K_1\cap K_2, I,\pi) \geq \mathrm{h}(K_1, I,\pi) + \mathrm{h}(K_2, I, \pi).$$

3 A better formalisation

The reader may wonder where and how the condition "$(I,\mathfrak{P})$ legal" has been used, since it always seems fairly obvious in the above proofs. Let us define a *legality space* $\mathscr{L}$ to be a set of integer valued functions on $\mathfrak{S}$. For instance the set $\mathscr{L}_0$ of characteristic functions of sets, or the set $\mathscr{L}_{\geq 0}$ of non-negative integer valued functions, or the set $\mathscr{L}_{\mathrm{all}} = \mathscr{F}(\mathfrak{S},\mathbb{Z})$ of all integer valued functions, or even the set of integer valued that takes only the values -1, 1 and 2. We start from a legal position, that is a member f of $\mathscr{L}$ and say a move $\mathfrak{f}$ is legal whenever $f+\mathfrak{f}$ is again in $\mathscr{L}$. We defined *the complementary space* ${}^c\mathscr{L}$ as the space of all $\mathbb{1}_{\mathfrak{S}} - h$ for all $h\in\mathscr{L}$. All the above theory goes through with such a definition of legality but for Theorems 1.4 and 2.4 which becomes

Theorem 3.1

$$\mathrm{h}(K, I, \mathscr{L}, \pi) = \mathrm{h}(\mathfrak{S} \setminus I, \mathfrak{S} \setminus K, {}^c\mathscr{L}, \pi). \tag{18}$$

On taking $\mathscr{L}$ to be the set $\mathscr{L}_{\text{all}}$, the condition "$(f, \mathfrak{P})$ is $\mathscr{L}_{\text{all}}$-legal" becomes void and we get the analytical height:

$$\mathrm{h}_a(f - g, \pi) = \min_{\mathfrak{P}} \Big\{ \langle \Psi(\mathfrak{P}), \pi \rangle, \quad f - g - \Psi(\mathfrak{P}) \geq 0 \Big\} \tag{19}$$

which is this time computable via simple linear programming. Note that $\mathrm{h}_a(f - g, \pi) = 0$ whenever $f - g \geq 0$ just like $\mathrm{h}(K, I, \pi)$ vanishes whenever K is a subset of I. This height offers a convenient lower bound for $\mathrm{h}(\cdot, \cdot, \pi)$:

$$\mathrm{h}_a(\mathbb{1}_I - \mathbb{1}_K, \pi) \leq \mathrm{h}(K, I, \pi). \tag{20}$$

4 A lower bound

We define the *lesser height* by

$$\mathrm{lh}(K, I, \pi) = \min_{\mathfrak{P}} \Big\{ \langle \Psi(\mathfrak{P}), \pi \rangle, \quad \mathbb{1}_I \geq \mathbb{1}_K + \Psi(\mathfrak{P}) \geq 0 \Big\} \tag{21}$$

and the *analytical lesser height* by $\mathrm{lh}_g(K, I) = \mathrm{lh}(K, I, \mathbb{1}_{\mathfrak{S}})$. They are not heights in the sense given earlier because the conditions deal with the final position only, and this spoils most of the arguments we have used until now. These lesser heights have however the advantage of being computable via linear programming. We have

$$\mathrm{lh}_g(K, I) \leq \mathrm{h}_g(K, I) \tag{22}$$

and this is a marked refinement of Theorem 1.7. Indeed, this Theorem is equivalent to saying that when $\mathrm{h}_g(K, I) < \infty$, then $\mathbb{1}_I - \mathbb{1}_K$ belongs to the cone defined by the potentials used therein. This cone is larger than the one defined by all the potentials: indeed, when $\mathrm{lh}_g(K, I)$ is finite, we can take the scalar product with a potential π that is non-negative on I to show that $\langle \pi, \mathbb{1}_I \rangle \geq \langle \pi, \mathbb{1}_K \rangle$. This cone is not limited to integer points, and indeed, the condition of integrality cannot be detected in this approach, while it is used in the existence of $\Psi(\mathfrak{P})$, in (21).

Let us give an example when $\mathrm{lh}(K, I, \mathbb{1}) = \infty$ while the previous criterium would not imply that $\mathrm{h}_g(K, I, \mathbb{1}) = \infty$. We use the value 0 instead of ∞.

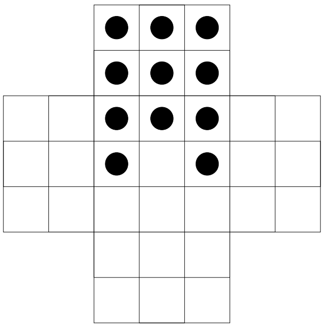

Figure 1: Chimney

		ac	ad	ae		
		bc	bd	be		
ca	cb	cc	cd	ce	cf	cg
da	db	dc	dd	de	df	dg
ea	eb	ec	ed	ee	ef	eg
		fc	fd	fe		
		gc	gd	ge		

Figure 2: Notation

		−2	−2	−2		
		−2	−1	−2		
3	1	−1	−1	−1	1	3
4	2	−1	1	−1	2	4
0	0	1	0	1	0	0
		4	0	4		
		0	0	0		

Figure 3: Height/-Depth

As we can see on the pictures above, the points ea, eb, ed, ef, fd, gc, gd and ge are out of the convex hull of the initial position and can be altogether discarded. Concerning the remaining points, here are the couples that we have detected cannot be simultaneously on the board, we one starts from the "Chimney" position:

$$
\begin{array}{rl}
bc \rightarrow & da \\
bd \rightarrow & da, db, dd, df, dg \\
be \rightarrow & dg \\
ca \rightarrow & cb, da, db, dd, df, dg, fc, \texttt{fe} \\
cb \rightarrow & ca, da, db, dd, df, dg, fc \\
cc \rightarrow & da \\
cd \rightarrow & da, db, dd, df, dg \\
ce \rightarrow & dg \\
cf \rightarrow & cg, da, db, dd, df, dg, fe \\
cg \rightarrow & cf, da, db, dd, df, dg, \texttt{fc}, fe \\
da \rightarrow & bc, bd, ca, cb, cc, cd, cf, cg, db, dc, dd, df, dg, ec, fc, fe \\
db \rightarrow & bd, ca, cb, cd, cf, cg, da, dd, df, dg, ec, fc, fe \\
dc \rightarrow & da, fc \\
dd \rightarrow & bd, ca, cb, cd, cf, cg, da, db, df, dg, fc, fe \\
de \rightarrow & dg, fe \\
df \rightarrow & bd, ca, cb, cd, cf, cg, da, db, dd, dg, ee, fc, fe \\
dg \rightarrow & bd, be, ca, cb, cd, ce, cf, cg, da, db, dd, de, df, ee, fc, fe \\
ec \rightarrow & da, db, fc \\
ee \rightarrow & df, dg, fe \\
fc \rightarrow & ca, cb, \texttt{cg}, da, db, dc, dd, df, dg, ec, fe \\
fe \rightarrow & \texttt{ca}, cf, cg, da, db, dd, de, df, dg, ee, fc
\end{array}
$$

The pairs $\{cg, fc\}$ and $\{ca, fe\}$ where only detected by our new criterium.

We continue this journey with another example:

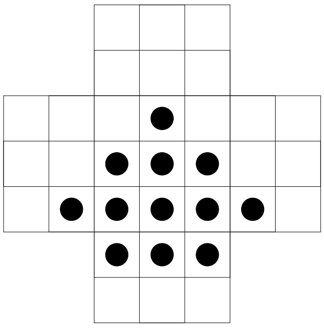

Figure 4: An arrowhead

		0	4	0		
		3	1	3		
4	2	1	−1	1	2	4
3	1	−1	−1	−1	1	3
1	−1	−1	−1	−1	−1	1
		−1	−1	−1		
		1	1	1		

Figure 5: Height/-Depth

The points ac, ae are outside the convex hull. Concerning the remaining points, here are the couples that we have detected cannot be simultaneously on the board, we one starts from the "Arrowhead" position:

$$
\begin{aligned}
ad &\rightarrow\ bc, bd, be, ca, cb, cd, \mathtt{cf}, cg, da, db, dd, df, dg, fd, gd\\
bc &\rightarrow\ ad, be, ca, cc, ce, cg, da, dg\\
bd &\rightarrow\ ad, ca, cg\\
be &\rightarrow\ ad, bc, ca, cc, ce, cg, da, dg\\
ca &\rightarrow\ ad, bc, bd, be, cb, cc, ce, cf, cg, da, \mathtt{df}, dg, ea\\
cb &\rightarrow\ ad, ca, \mathtt{cg}, \mathtt{da}\\
cc &\rightarrow\ bc, be, ca, cg\\
cd &\rightarrow\ ad\\
ce &\rightarrow\ bc, be, ca, cg\\
cf &\rightarrow\ \mathtt{ad}, ca, cg, \mathtt{dg}\\
cg &\rightarrow\ ad, bc, bd, be, ca, \mathtt{cb}, cc, ce, cf, da, db, dg, eg\\
da &\rightarrow\ ad, bc, be, ca, \mathtt{cb}, cg, db\\
db &\rightarrow\ ad, cg, da\\
dd &\rightarrow\ ad\\
df &\rightarrow\ ad, \mathtt{ca}, dg\\
dg &\rightarrow\ ad, bc, be, ca, \mathtt{cf}, cg, df\\
ea &\rightarrow\ ca\\
eg &\rightarrow\ cg\\
fd &\rightarrow\ ad\\
gd &\rightarrow\ ad
\end{aligned}
$$

The pairs $\{ad, cf\}$, $\{ca, df\}$, $\{cb, cg\}$, $\{cb, da\}$ and $\{cf, dg\}$ were only detected by our new criterium.

5 A question

Let us define $n(d, I)$ to be the number of points of height d with respect to I, and $n(d, I, J)$ to be the same quantity but with respect to the joint height. It would be interesting to describe these functions. A first idea, anchored on the fact that two points are required to build a new one, would be to say that these functions should be decreasing, and even rapidly so. This was even what led the author to define $\mathrm{h}(A, I, \pi)$. The two examples given earlier, the Chimney and the Arrowhead, show that $n(d, I)$ is *not* non-increasing in d. The Arrowhead example is even more telling since $n(\cdot, I)$ successively takes the values $10, 2, 4, 3$, showing that this function is not even modal (i.e. convex or concave).

Concerning on one hand the joint height, the function $n(\cdot, I, J)$ for I being the initial position of the Short Bow (Ramaré, 2008a) is again not monotonic. The joint landscape of the Palm, also taken from (Ramaré, 2008a), is again non-monotonic. However, we have not been able to get such examples when the geometry of the board does not intervene, say with $\mathfrak{S} = \mathbb{Z}^2$. Furthermore, equation (6), or (23) below, exhibits some regularity of the height, and it is surprising that this property should not have any consequences on $n(\cdot, I, J)$.

6 A last digression on joint height and joint landscape

In (Ramaré, 2008a), we have defined the notion of *joint landscape*, which relies on the notion of joint height and depth with respect to an initial position I and a final one J. For the *joint height*, we simply need to restrict the legality space to *legality space* $\mathscr{L}(I, J)$ which is the space of all positions reachable from I and from which J can be reached. This definition is so that ${}^{c}\mathscr{L}(I, J) = \mathscr{L}(\mathfrak{S} \setminus J, \mathfrak{S} \setminus I)$. This symetry is pleasant but the dependance of the legality space on I and J spoils part of the general theory. The main recursion loop (i.e. the one of Theorems 1.5 and 2.5) still holds and this yields

Theorem 6.1 *We have*

$$\mathrm{h}(K_1 \cup K_2, I, J, \pi) + \mathrm{h}(K_1 \cap K_2, I, J, \pi) \geq \mathrm{h}(K_1, I, J, \pi) + \mathrm{h}(K_2, I, J, \pi).$$

We have not defined $\mathrm{h}(K, I, J, \pi)$ in so many words, but its meaning should, at this stage, be clear to the reader. This has the same consequence on the joint height, namely:

$$\begin{aligned}&\mathrm{Height}(A,I,J)\\&\geq \min\Big\{\mathrm{Height}(B,I,J)+\mathrm{Height}(C,I,J)+1, \check{B}+\check{C}-\check{A}\in\mathscr{D}(\mathfrak{S})\Big\}.\end{aligned}\tag{23}$$

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