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EXPLICIT ESTIMATES FOR THE SUMMATORY FUNCTION OF $\Lambda(n)/n$ FROM THE ONE OF $\Lambda(n)$

OLIVIER RAMARÉ

ABSTRACT. We prove that the error term $\sum_{n\le x}\Lambda(n)/n - \mathrm{Log}\,x+\gamma$ differs from $(\psi(x)-x)/x$ by a well controlled function. We deduce very precise numerical results from this formula.

1. Introduction

We define classically

$$\psi(x)=\sum_{n\le x}\Lambda(n),\quad \tilde\psi(x)=\sum_{n\le x}\Lambda(n)/n.$$

There has been a good amount of work to find explicit asymptotics for $\psi(x)$, see for instance [12], [13], [14], [15] and [10]. The quantity $\tilde\psi(x)$ has been much less studied though [13, Theorem 6] gives an estimate. The problem here is that one would really want to deduce such an estimate from the ones concerning $\psi(x)$, but such method is missing. The aim of this paper is to provide a fairly simple roundabout, see Theorem 1.1 below.

Let us note that the prime number Theorem in the form $\psi(x)=(1+o(1))x$ is "elementarily" equivalent to

$$\tilde\psi(x)=\mathrm{Log}\,x-\gamma+o(1). \tag{1.1}$$

So in a sense, we are concerned with a quantitative version of this equivalence. A simple integration by parts is *not* enough, as it looses a log-factor. In effect, an estimate of the form $|\psi(x)-x|/x\le 0.01$ for x large enough transfers into something like $|\tilde\psi(x)-\mathrm{Log}\,x+\gamma|\le 0.01\ \mathrm{Log}\,x$ which is of no interest. The Landau equivalence Theorem can however be made explicit, but forbid a saving better than $1/\sqrt{\mathrm{Log}\,x}$ in a rough form ; allowing a saving of any power of $\mathrm{Log}\,x$ is already theoretically not obvious, see [8] for instance. Here is a conjecture.

Conjecture (Strong form of Landau equivalence Theorem, I).
There exist two positive constants c_1 and c_2 such that

$$\left|\tilde\psi(x)-\mathrm{Log}\,x+\gamma\right|\le c_1\max_{x/c_2<y\le c_2x}\frac{|\psi(y)-y|}{y}+c_2x^{-1/4}.$$

Such a conjecture holds (almost trivially) true under the Rieman Hypothesis. The result of [2] indicates that such an inequality does not hold in the case of Beurling generalized integers. Indeed they show that the condition $\psi_{\mathcal P}(x)\sim x$ does not ensure that $\tilde\psi_{\mathcal P}(x)-\mathrm{Log}\,x$ has a limit, with obvious notations.

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Let us end this introduction with a remark: in [6], the authors exhibit, under the Riemann Hypothesis, a pseudo-periodical function that (essentially) takes the value $(\tilde{\psi}(e^{-y})+y)\,e^{y/2}$ when $y>0$ and $(\psi(e^y)-e^y)\,e^{-y/2}$ when $y>0$. This means that the values of ψ and of $\tilde{\psi}$ may share a much more profond link than proposed in the above conjecture.

We are not able to prove our conjecture, but show in Lemma 2.2 that

$$\tilde{\psi}(x)-\operatorname{Log}x+\gamma-\frac{\psi(x)-x}{x}$$

is a well-controlled function. Here are some consequences of this formula.

Theorem 1.1. *For $x\ge 8\,950$, we have*

$$\tilde{\psi}(x)=\operatorname{Log}x-\gamma+\frac{\psi(x)-x}{x}+\mathcal{O}^*\left(\frac{1}{2\sqrt{x}}\right)+\mathcal{O}^*\left(1.75\cdot 10^{-12}\right).$$

Furthermore when $\operatorname{Log}x\ge 9270$, we have (with $R=5.696\,93$)

$$\tilde{\psi}(x)=\operatorname{Log}x-\gamma+\frac{\psi(x)-x}{x}+\mathcal{O}^*\left(\frac{1}{2\sqrt{x}}\right)+\mathcal{O}^*\left(\frac{1+2\sqrt{(\operatorname{Log}x)/R}}{2\pi}\exp\left(-2\sqrt{(\operatorname{Log}x)/R}\right)\right).$$

Corollary. *We have for $x>1$,*

$$\tilde{\psi}(x)=\operatorname{Log}x-\gamma+\mathcal{O}^*\left(3.972/\operatorname{Log}^2x\right).$$

Furthermore, for $1\le x\le 10^{10}$, we have $\tilde{\psi}(x)=\operatorname{Log}x-\gamma+\mathcal{O}^*(1.31/\sqrt{x})$.
For $x\ge 10^{10}$, we have $\tilde{\psi}(x)=\operatorname{Log}x-\gamma+\mathcal{O}^*(0.002\,14)$.
For $x\ge 23$, we have $\tilde{\psi}(x)=\operatorname{Log}x-\gamma+\mathcal{O}^*(0.0067/\operatorname{Log}x)$.

As a comparison, [13, Theorem 6] proposes an inequality similar to the last one above, but with $1/2=0.5$ instead of 0.0067. No error term with a saving of $1/\operatorname{Log}^2x$ is proposed.

Notation. We use the classical counting function

$$N(T)=\sum_{\substack{\rho\\ 0<\gamma\le T}}1, \tag{1.2}$$

where $\rho=\beta+i\gamma$ is a zero of the Riemann zeta-function. Furthermore, by $f(x)=\mathcal{O}^*(g(x))$ we mean $|f(x)|\le g(x)$.

The computations required have been done via Pari/GP, see [16].

2. An explicit formula

We will need [11, Lemma 4]:

Lemma 2.1. *Let g be a continuously differentiable function on $[a,b]$ with $2\le a\le b<+\infty$. We have*

$$\int_a^b\psi(t)g(t)dt=\int_a^b tg(t)dt-\sum_\rho\int_a^b\frac{t^\rho}{\rho}g(t)dt+\int_a^b\bigl(\log 2\pi-\tfrac12\log(1-t^{-2})\bigr)g(t)dt.$$

Here is our main formula.

Lemma 2.2. *We have, for $x \ge 1$:*

$$\tilde{\psi}(x) = \operatorname{Log} x - \gamma + \frac{\psi(x) - x}{x} + \sum_\rho \frac{x^{\rho-1}}{\rho(\rho-1)} + \frac{B(x)}{x}.$$

where the sum is over the zeroes ρ of the Riemann zeta function that lie in the critical strip $0 < \Im s < 1$ (the so-called non trivial zeroes) and $B(x)$ is the bounded function given by

$$B(x) = \tfrac{1}{2} + \operatorname{Log} 2\pi - \frac{x-1}{2} \operatorname{Log}(1 - x^{-1}).$$

The main feature of the Lemma is that the sum over the zeroes is uniformly convergent, a feature not shared by the explicit formulaes for $\psi(x)$ or for $\tilde{\psi}(x)$. In fact, the main difficulty in carried by the term $(\psi(x) - x)/x$.

Proof. We simply proceed by integration by parts:

$$\begin{aligned}\tilde{\psi}(x) &= \int_1^x \psi(t)\frac{dt}{t^2} + \frac{\psi(x)}{x} \\ &= \operatorname{Log} x - \gamma + \int_x^\infty (\psi(t) - t)\frac{dt}{t^2} + \frac{\psi(x) - x}{x}.\end{aligned}$$

Note that the existence of the integral requires a strong enough form of the equivalence between $\psi(t)$ and t. Next we apply the explicit formula given in Lemma 2.2 and get

$$\begin{aligned}\int_x^Y (\psi(t) - t)\frac{dt}{t^2} &= -\sum_\rho \int_x^Y \frac{t^{\rho-2}dt}{\rho} + \int_x^Y \left(\operatorname{Log} 2\pi - \tfrac{1}{2}\operatorname{Log}(1 - t^{-2})\right)\frac{dt}{t^2} \\ &= -\sum_\rho \frac{Y^{\rho-1} - x^{\rho-1}}{\rho(\rho-1)} + \int_x^Y \left(\operatorname{Log} 2\pi - \tfrac{1}{2}\operatorname{Log}(1 - t^{-2})\right)\frac{dt}{t^2}.\end{aligned}$$

Since (1.1) is known to hold, and $\sum_\rho 1/|\rho(\rho-1)|$ is convergent, we can send Y to infinity and get

$$\int_x^Y (\psi(t) - t)\frac{dt}{t^2} = \sum_\rho \frac{x^{\rho-1}}{\rho(\rho-1)} + \int_x^\infty \left(\operatorname{Log} 2\pi - \tfrac{1}{2}\operatorname{Log}(1 - t^{-2})\right)\frac{dt}{t^2}.$$

□

3. Known bounds on $\psi(x)$

In [10], we find that

$$|\psi(x) - x| \le \sqrt{x} \qquad (8 \le x \le 10^{10}). \tag{3.1}$$

If we change this $\sqrt{x}$ by $\sqrt{2x}$, this is valid from $x = 1$ onwards. Furthermore

$$|\psi(x) - x| \le 0.8\sqrt{x} \qquad (1\,500 \le x \le 10^{10}). \tag{3.2}$$

By [3, Théorème 1.3] improving on [15, Theorem 7], we have

$$|\psi(x) - x| \le 0.0065x/\operatorname{Log} x \quad (x \ge \exp(22)) \tag{3.3}$$

We readily extend this estimate to $x \ge 3\,430\,190$ by using (3.1), and then to $x \ge 1\,514\,928$ by direct inspection.

We quote [3, Théorème 1.4] and in [4, Theorem 5.2]

$$|\vartheta(x) - x| \le 3.965x/\operatorname{Log}^2 x \quad (x > 2) \tag{3.4}$$

$$|\vartheta(x) - x| \le 515x/\operatorname{Log}^3 x \quad (x > 2) \tag{3.5}$$

In fact [4, Theorem 5.2] proposes the constant 21 instead of 515 in this inequality, but this preprint has not been published. We will not use this bound but take this opportunity to record this fact.

We go from ϑ to ψ by using [14, Theorem 6]

$$0 \le \psi(x) - \vartheta(x) \le 1.0012\sqrt{x} + 3x^{1/3} \quad (x > 0). \tag{3.6}$$

Lemma 3.1. *For $x \ge 7\,105\,266$, we have*

$$|\psi(x) - x|/x \le 0.000\,213.$$

Proof. We start with the estimate from [14, (4.1)]

$$|\psi(x) - x|/x \le 0.000\,213 \qquad (x \ge 10^{10}). \tag{3.7}$$

We extend it to $x \ge 14\,500\,000$ by using (3.1). We conclude by direct inspection. □

Lemma 3.2. *We have for $x > 1$*

$$|\psi(x) - x| \le 3.971\,x/\operatorname{Log}^2 x, \qquad |\psi(x) - x| \le 516\,x/\operatorname{Log}^3 x.$$

Proof. Indeed, we get

$$|\psi(x) - x| \le |\psi(x) - \vartheta(x)| + |\vartheta(x) - x| \le 1.0012\sqrt{x} + 3x^{1/3} + 3.965x/\operatorname{Log}^2 x$$

which is not more than $3.971x/\operatorname{Log}^2 x$ if $x \ge 10^{10}$. For x lower, we use

$$|\psi(x) - x| \le (\operatorname{Log}^2 x/(3.971\sqrt{x}))3.971x/\operatorname{Log}^2 x$$

which extends our bound till $x \ge 1$. We proceed similarly for the bound with $\operatorname{Log}^3 x$. □

4. Lemmas on the zeroes

We quote from [11]:

Lemma 4.1. *If T is a real number $\ge 10^3$ then*

$$N(T) = \frac{T}{2\pi}\operatorname{Log}\frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + \mathcal{O}^*\big(0.67\operatorname{Log}\frac{T}{2\pi}\big).$$

This is a version of Theorem 19 of [12], relying on [1].

Lemma 4.2. *We have, when $T \ge 10^3$*

$$\sum_{\gamma \ge T_0} 1/\gamma^2 \le \frac{\operatorname{Log}(T/(2\pi))}{2\pi T} + 0.67\frac{2\operatorname{Log}(T/(2\pi)) + (1/2)}{T^2}.$$

Proof. We call S the sum to evaluated and we simply use integration by parts:

$$\begin{aligned} S &= 2\int_T^\infty \frac{N(t)-N(T)}{t^3}dt \\ &\le \frac{2}{(2\pi)^2}\int_{T/(2\pi)}^\infty \frac{u\operatorname{Log} u - u + \frac{7}{8} + 0.67\operatorname{Log} u}{u^3}du \\ &\quad - \frac{\frac{T}{2\pi}\operatorname{Log}\frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} - 0.67\operatorname{Log}\frac{T}{2\pi}}{T^2} \\ &\le \frac{\operatorname{Log}(T/(2\pi))}{2\pi T} + 0.67\frac{2\operatorname{Log}(T/(2\pi)) + (1/2)}{T^2}. \end{aligned}$$

The Lemma follows readily. □

Lemma 4.3. *We have $\sum_\rho 1/|\rho(\rho-1)| \le 0.047$, where ρ ranges over all non trivial zeroes of ζ.*

In particular, we do not impose $\Im\rho > 0$. We prove this Lemma by using the file of the first 10^5 zeroes provided by Odlyzko [9].

We in fact used zeroes only up to height 10 000 and ran the computations using 28 digits precision on GP/Pari. Note that when $\Im\rho = 1/2$, we have $\rho(\rho-1) = -|\rho|^2$. Truncation of the imaginary parts only increases the sum, while the high enough precision takes care of the machine error. The restricted sum is about 0.023 02 (with condition $\Im\rho > 0$). We next use Lemma 4.2 to handle the tail of the series. We finally double the value to remove the condition $\Im\rho > 0$, and round the value up.

We also know, thanks to [5], that the zeroes ρ in the critical strip and verifying $|\Im\rho| \le 2.44\cdot 10^{12} = T_0$ are all on the line $\Re\rho = 1/2$. We handle zeros with large imaginary part with the following Theorem from [7]

Lemma 4.4. *Every zero $\rho = \beta + i\gamma$ of ζ in the strip $0 < \beta < 1$ and $\gamma \ge 10$ verifies*

$$\beta \le 1 - \varphi(\gamma) = 1 - 1/(R\operatorname{Log}\gamma), \quad R = 5.696\,93.$$

5. Proof of Theorem 1.1

We start with Lemma 2.2. Let us set

$$J(x) = \sum_\rho \frac{x^{\rho-1}}{\rho(\rho-1)}. \tag{5.1}$$

By considering the symmetry $\rho \mapsto 1-\rho$, we get (remember that no zero of ζ lies on the segment $[0,1]$)

$$J(x) = \sum_{\substack{\rho,\\ \Im\rho>0}} \frac{x^{\rho-1} + x^{-\rho}}{\rho(\rho-1)}.$$

We are ready to majorize $J(x)$:

$$\begin{aligned} J(x) &\le \sum_{|\gamma|\le T_0} \frac{x^{-1/2}}{|\rho|^2} + 2\sum_{\gamma>T_0}\left(\frac{x^{-1/2}}{2|\rho(\rho-1)|} + \frac{x^{-\varphi(\gamma)}}{2|\rho(\rho-1)|}\right) \\ &\le \frac{0.47}{\sqrt{x}} + \sum_{\gamma>T_0}\frac{x^{-\varphi(\gamma)}}{\gamma^2}. \end{aligned}$$

We first bound $x^{-\varphi(\gamma)}$ by 1 and get, by Lemma 4.2

$$J(x) \le \frac{0.47}{\sqrt{x}} + \frac{\operatorname{Log}(T_0/(2\pi))}{2\pi T_0}\left(1 + 1.36\frac{2\pi}{T_0}\right) \le \frac{0.47}{\sqrt{x}} + 1.75\cdot 10^{-12}.$$

This proves the first part of Theorem 1.1. For large x, we can take advantage of the zero free region. We set $\varphi_2(\gamma) = x^{-\varphi(\gamma)}/\gamma^2$ and get

$$\begin{aligned}
J(x) &\le \frac{0.47}{\sqrt{x}} - \int_{T_0}^{\infty}(N(t)-N(T_0))\varphi_2'(t)dt \\
&\le \frac{0.47}{\sqrt{x}} - \int_{T_0}^{\infty}(N^*(t)-N(T_0))\varphi_2'(t)dt - \int_{T_0}^{\infty}(N(t)-N^*(t))\varphi_2'(t)dt \\
&\le \frac{0.47}{\sqrt{x}} + (N^*(T_0)-N(T_0))\varphi_2(T_0) + \int_{T_0}^{\infty}N^*(t)'\varphi_2(t)dt - \int_{T_0}^{\infty}(N(t)-N^*(t))\varphi_2'(t)dt \\
&\le \frac{0.47}{\sqrt{x}} + 3\cdot 10^{-24}x^{-\varphi(T_0)} + \int_{T_0}^{\infty}\frac{x^{-\varphi(t)}\operatorname{Log}(t/(2\pi))dt}{2\pi t^2} \\
&\qquad + \int_{T_0}^{\infty}\left|\frac{\operatorname{Log} x}{2R} - \operatorname{Log}^2 t\right|\frac{2x^{-\varphi(t)}\operatorname{Log}(t/(2\pi))dt}{t^3\operatorname{Log}^2 t}.
\end{aligned}$$

We now assume $\operatorname{Log} x \ge 2R\operatorname{Log}^2 T_0$ and infer the bound

$$\begin{aligned}
J(x) &\le \frac{0.47}{\sqrt{x}} + 6\cdot 10^{-24}x^{-\varphi(T_0)} + \int_{T_0}^{\infty}\frac{x^{-\varphi(t)}\operatorname{Log}(t/(2\pi))dt}{2\pi t^2} + 0.67\int_{T_0}^{\infty}\frac{x^{-\varphi(t)}dt}{t^3} \\
&\le \frac{0.47}{\sqrt{x}} + 6\cdot 10^{-24}x^{-\varphi(T_0)} + \int_{T_0}^{\infty}\frac{x^{-\varphi(t)}\operatorname{Log}(t/6.25)dt}{2\pi t^2} \\
&\le \frac{0.47}{\sqrt{x}} + \int_{T_0}^{\infty}\frac{x^{-\varphi(t)}\operatorname{Log} t\,dt}{2\pi t^2}.
\end{aligned}$$

$$I = \int_{T_0}^{\infty}\exp\left(-\frac{\operatorname{Log} x}{R\operatorname{Log} t} - \operatorname{Log} t\right)\frac{\operatorname{Log} t\,dt}{2\pi t} = \int_{\operatorname{Log} T_0}^{\infty}\exp\left(-\frac{\operatorname{Log} x}{Ru} - u\right)\frac{u\,du}{2\pi}.$$

We set

$$\frac{\operatorname{Log} x}{Ru} + u = v$$

which gets solved in $(u^2 - uv + (\operatorname{Log} x)/R = 0)$

$$2u = v \pm \sqrt{v^2 - 4(\operatorname{Log} x)/R}.$$

We further get

$$\begin{aligned}
4u\,du &= \left(v \pm \sqrt{v^2-4(\operatorname{Log} x)/R}\right)\left(1 \pm \frac{v}{\sqrt{v^2-4(\operatorname{Log} x)/R}}\right)dv \\
&= \left(v \pm \sqrt{v^2-4(\operatorname{Log} x)/R} \pm \frac{v^2}{\sqrt{v^2-4(\operatorname{Log} x)/R}} + v\right)dv \\
&= \left(2v \pm \frac{2v^2-4(\operatorname{Log} x)/R}{\sqrt{v^2-4(\operatorname{Log} x)/R}}\right)dv
\end{aligned}$$

so that I gets rewritten as

$$I = \int_{2\sqrt{(\operatorname{Log} x)/R}}^{\infty} e^{-v}\left(v + \frac{v^2 - 2(\operatorname{Log} x)/R}{\sqrt{v^2 - 4(\operatorname{Log} x)/R}}\right)\frac{dv}{4\pi}$$
$$+ \int_{2\sqrt{(\operatorname{Log} x)/R}}^{\frac{\operatorname{Log} x}{R \operatorname{Log} T_0} + \operatorname{Log} T_0} e^{-v}\left(v - \frac{v^2 - 2(\operatorname{Log} x)/R}{\sqrt{v^2 - 4(\operatorname{Log} x)/R}}\right)\frac{dv}{4\pi}$$

which yields

$$I \le \int_{2\sqrt{(\operatorname{Log} x)/R}}^{\infty} v e^{-v}\frac{dv}{2\pi} = \frac{1 + 2\sqrt{(\operatorname{Log} x)/R}}{2\pi}\exp\left(-2\sqrt{(\operatorname{Log} x)/R}\right).$$

It is then immediate to conclude the proof of Theorem 1.1.

6. Proof of the Corollary

When $\operatorname{Log} x \le 2R \operatorname{Log}^2 T_0$, but $x \ge 10^{10}$, we use Lemma 3.2 and get

$$\left|\tilde{\psi}(x) - \operatorname{Log} x + \gamma\right| \operatorname{Log}^2 x \le 3.971 + \frac{\operatorname{Log}^2 x}{2\sqrt{x}} + 1.68 \cdot 10^{-12} \operatorname{Log}^2 x \le 3.972.$$

When $8\,950 \le x \le 10^{10}$, we have

$$\left|\tilde{\psi}(x) - \operatorname{Log} x + \gamma\right| \operatorname{Log}^2 x \le \frac{1.3 \operatorname{Log}^2 x}{\sqrt{x}} + 1.68 \cdot 10^{-12} \operatorname{Log}^2 x \le 1.14.$$

When $\operatorname{Log} x \ge 2R \operatorname{Log}^2 T_0$, the bound becomes

$$3.971 + \frac{1 + 2\sqrt{(\operatorname{Log} x)/R}}{2\pi}\exp\left(-2\sqrt{(\operatorname{Log} x)/R}\right)\operatorname{Log}^2 x \le 3.972.$$

We complete the proof by direct inspection. For the limited range bound, we write

$$\left|\tilde{\psi}(x) - \operatorname{Log} x + \gamma\right| \sqrt{x} \le 1.3 + 1.68 \cdot 10^{-12}\sqrt{x} \le 1.31$$

when $x \ge 8\,950$. We again conclude by direct inspection.

When $\operatorname{Log} x \le 2R \operatorname{Log}^2 T_0$, but $x \ge 10^{10}$, we have

$$\left|\tilde{\psi}(x) - \operatorname{Log} x + \gamma\right| \operatorname{Log} x \le 0.0065 + \frac{\operatorname{Log} x}{2\sqrt{x}} + 1.68 \cdot 10^{-12} \operatorname{Log} x \le 0.0067.$$

When $8\,950 \le x \le 10^{10}$, we have

$$\left|\tilde{\psi}(x) - \operatorname{Log} x + \gamma\right| \operatorname{Log} x \le \frac{1.3 \operatorname{Log} x}{\sqrt{x}} + 1.68 \cdot 10^{-12} \operatorname{Log} x \le 0.0003.$$

When $\operatorname{Log} x \ge 2R \operatorname{Log}^2 T_0$, the bound becomes

$$0.0065 + \frac{1 + 2\sqrt{(\operatorname{Log} x)/R}}{2\pi}\exp\left(-2\sqrt{(\operatorname{Log} x)/R}\right)\operatorname{Log}^2 x \le 0.0066.$$

We complete the proof by direct inspection.

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CNRS / Laboratoire Paul Painlevé, Université Lille 1, 59 655 Villeneuce d'Ascq
E-mail address: `ramare@math.univ-lille1.fr`