A sharp bilinear form decomposition for primes and Moebius function

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January 1, 2013

Abstract

We prove via a bilinear decomposition of the van Mangoldt function the optimal $\sum_{\ell \le X} \Lambda(\ell)\, e(\ell\beta)e(\ell a/q)/\ell^{it} \ll \sqrt{q}\, X/\phi(q)$ for q as large as $q \le X^\eta$, for any $\eta < 2/17$ and $|\beta|X + |t| \le X^{1/6-\varepsilon}/q^{11/12}$. The technique is flexible and we also prove that $\sum_{\ell \le X} \mu(\ell)\, e(\ell\beta)e(\ell a/q) \ll \frac{X}{\sqrt{q}} \prod_{p|q}\bigl(1 + p^{-1/2}\bigr)$ under the same conditions. We prove the short interval version $\sum_{X < \ell \le X + X^\theta} \Lambda(\ell)\, e(\ell a/q) \ll (\log q)\sqrt{q}\, X^\theta/\phi(q)$ when $q \le X^\eta$, for any $\eta < 1/6$ and $\theta \in (\theta_0, 1]$ where $X^{\theta_0} = X^{13/15}q^{4/5}$. The technique is simple enough that it leads to explicit estimates and we prove for instance that $|\sum_{X < \ell \le 2X} \Lambda(\ell)\, e(\ell a/q)| \le 13\,000\sqrt{q}\, X/\phi(q)$ for $20 \le q \le X^{1/48}$.

Keywords Exponential sums, Bilinear decomposition, Primes in progressions

MR(2000) Subject Classification 11L07, 11N13 (11L20)

Short title A sharp bilinear decomposition for primes.

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1 Introduction

After Vinogradov's masterpiece [34], the investigation of trigonometric polynomials with arithmetical coefficients received acute attention. The initial treatment of Vinogradov was involved; the book [35] gives an excellent account of it. There has been many developments since, and we concentrate here on the trigonometric polynomial either over the primes, or with coefficients given by the Moebius function. We further focus on sharp estimates valid when the argument α s close to a rational with a rather small denominator. Theorem 2b of chapter IX of [35] gives a first answer in that case (on selecting there $\varepsilon = 2(\log q)/\log\log X$). A simplified version reads:

$$\sum_{n\le X}\Lambda(n)e(na/q) \ll X(\log q)^{10}/\sqrt{q}, \quad q \le \exp\sqrt{0.1\log\log X}. \tag{1}$$

The main feature is that the right-hand side is X times a function of q that goes to 0. The technique is to represent the characteristic function of the primes into a linear combination of linear and bilinear forms in modern terminology (or in type I and type II sums if we are to follow Vinogradov's initial choice of words); this technique is flexible and can for instance be applied when the coefficients are given by the Moebius function. The question is of course to determine the better bilinear forms, once a definite problem has been selected. Let us mention here that the results obtained via such a technique are effective (i.e. all implied constants can be explicitely determined); the possible Siegel zero thus limits our expectations as to what kind of results we may reach (it cannot however be properly termed an obstruction! See section 16). The simplification I gave long ago of Vinogradov's method (see for instance page 31 of [21] and Section 17.2 of [12]) does not lead to any improvement here. In [6] the author obtained an estimate similar to (1) but better when q does not have too many divisors. A first best possible result was reached in [24]: the function of q is $\sqrt{q}/\phi(q)$. The variable q is still restricted to being not more than $\exp(0.02(\log X)^{1/3})$ and one of the purpose of the method we present is to increase this range to $q \le X^\eta$ for any $\eta < 2/17$. The estimate we obtain is also valid not only at a/q but also close to it as often required by applications (a summation by parts enables one to extend the result to $|\alpha - a/q| \ll 1/X$; going further than that is the difficulty). If one is willing to use different techniques, namely dealing with zeroes of L-series, and in fact Gallagher's prime number Theorem from [8], one can reach a result of similar strength, though with an unspecified constant instead of $2/17$ and at the point a/q and not in its neighbourhood, as we explain in section 16 below. We dispense with any reference to zeros density estimates or even to zeros, and in this aspect this work is a sequel of Motohashi's work, see [16] and [17].

Using bilinears forms has several advantages, and one of them is the flexibility of the method. Our initial theorem reads as follows.

Theorem 1. *Let $\eta < 2/17$ and $\varepsilon > 0$ be given. When $q \le X^\eta$, a is prime to q and $|\beta|X + |t| \le X^{1/6-\varepsilon}/q^{11/12}$, we have*

$$\sum_{\ell \le X} \Lambda(\ell)\, e(\ell\beta) e(\ell a/q)/\ell^{it} \ll \sqrt{q}\, X/\phi(q).$$

Even when $t = 0$, this result is a thorough improvement on previous estimates by the range covered. Here and thereafter, we do not try to get the best exponents (like 2/7, 1/6 or 1/12) ; our aim is to describe precisely the method.

While trying to get the method to its limit, one readily discovers that an L^1-estimate on a long range is readily accessible.

Theorem 2. *When $q \le X^{1/6}$, $T \le X^{2/15}/q^{4/5}$, and a is prime to q, we have*

$$\int_{-T}^{T} \Bigl|\sum_{\ell \le X} \Lambda(\ell)\, e(\ell a/q)/\ell^{it}\Bigr| dt \ll (\log \min(q, 2+T))\sqrt{q}\, X/\phi(q).$$

This result is strong but is not optimal due to two blemishes: (1) the factor $\log q$ that I would have prefered to avoid, and (2) this estimate in valid at the precise point a/q and not in its neighborhood (except the immediate one as explained above). This L^1-estimate readily leads to an estimate of the trigonometric polynomial over primes in a short interval.

Theorem 3. *Let $\eta < 1/6$ be given. When $q \le X^\eta$, a is prime to q and θ_0 is defined by $X^{\theta_0} = X^{13/15} q^{4/5}$, , we have, for any $\theta \in (\theta_0, 1]$*

$$\sum_{X < \ell \le X + X^\theta} \Lambda(\ell)\, e(\ell a/q) \ll (\log q)\sqrt{q}\, X^\theta/\phi(q).$$

The factor $\log q$ can be replaced by $\log \min(q, 2 + X^{1-\theta})$, thus offering a smooth transition from the case $\theta = 1$ to $\theta > 0$. Concerning better bounds for the trigonometric polynomial over primes in a short(er) interval, see [19], [28] and [14].

As already announced, we can use the very same method to handle the trigonometric polynomial when the coefficients are given by the Moebius function:

Theorem 4. *When $q \le X^\eta$ for some $\eta < 2/17$, and a is prime to q, we have*

$$\sum_{\substack{\ell \le X,\\ (\ell,q)=1}} \mu(\ell)\, e(\ell\beta) e(\ell a/q) \ll X/\sqrt{q}.$$

There has been a renewal of interest in this problem since and we mention in particular [4] and [10]. Removing the coprimality condition is not *that* obvious, and I do not know how to keep the same level of precision:

Corollary 5. *When $q \le X^\eta$ for some $\eta < 2/17$, and a is prime to q, we have*

$$\sum_{\ell\le X}\mu(\ell)\,e(\ell\beta)e(\ell a/q) \ll \frac{X}{\sqrt{q}}\prod_{p|q}\bigl(1+p^{-1/2}\bigr).$$

We do not state them formally, but L^1-estimates and short interval versions are also accessible in this case: just change the factor $\sqrt{q}/\phi(q)$ by $1/\sqrt{q}$ in Theorem 2 and 3 if the condition $(n,q)=1$ is maintained, and further multiply the bounds by $\prod_{p|q}(1+p^{-1/2})$ is this latter condition is omitted.

The methods we use are fully explicit and even lead to possible numerical bounds. We prove the following.

Theorem 6. *When $20 \le q \le X^{1/48}$ and a is prime to q, we have*

$$\Bigl|\sum_{X<\ell\le 2X}\Lambda(\ell)\,e(\ell a/q)\Bigr| \le 13\,000\sqrt{q}\,X/\phi(q).$$

It is worthwhile noting that the constant 13 000 is maybe large but explicit while the work [24] relies on a Brun sieve-based preliminary sieving process that would make such a computation very hard (it would also most probably result in a much higher constant). We made the effort to get an explicit constant, but there are many places where this work can numerically be improved upon.

We detect the primes via a bilinear decomposition that is the main novelty of this paper. We in fact produce a family of decompositions, in a process akin to Iwaniec's amplification process. Our implementation is rather similar to one that has already been introduced by Selberg around 1973 (read [3] and [17]), and that has shown exceptional power.

This family of decompositions is shown to be "orthogonal" is some sense via a large sieve extension of the classical large sieve inequality for the Farey points, that encompasses *generalized characters* in Selberg's or Motohashi's terminology, also called (up to some rescalling) *local models* in [23]. See Theorem 8 below where the implied constant is improved with respect to the classical one.

While studying an optimisation problem close to the one that classically found's the Selberg sieve for primes, Barban & Vehov* introduced the weights

$$\lambda_d^{(1)} = \begin{cases}\mu(d) & \text{when } d\le z,\\ \mu(d)\frac{\log(z^2/d)}{\log z} & \text{when } z<d\le z^2,\\ 0 & \text{when } z^2<d.\end{cases} \tag{2}$$

*For a reason unknown to me but which stems almost surely from older transliteration rules, Vehov is spelled Vekhov in Zentralblatt.

(They consider in fact slightly more general weights with a y instead of the z^2 that we use here). Motohashi (see section 1.3 of [18]) and [9]. They will be used by Selberg in 1973, and [17] to provide a direct proof of a Hoheisel (named after [11]) type of theorem. Their particular property is that

$$\sum_{n\le B}\Bigl(\sum_{d|n}\lambda_d^{(1)}\Bigr)^2 \ll B/\log z$$

whether $B \ge z^4$ or not. We will follow an idea of Motohashi that the weaker property

$$\sum_{n\ge 1}\Bigl(\sum_{d|n}\lambda_d^{(1)}\Bigr)^2/n^{1+\varepsilon} \ll z^\varepsilon/(\varepsilon\log z)^2, \quad (\varepsilon\in(0,1])$$

can often be enough (this is our case) via Rankin's trick (see [27]) and is easier to establish. The required material is contained in Lemma 13. From an explicit viewpoint, this implies dealing with sums of type $\sum_{d\le D}\mu(d)/d^{1+\varepsilon}$ or $\sum_{d\le D}\mu(d)\log(D/d)/d^{1+\varepsilon}$ with some coprimality conditions added, and such sums are really more difficult to handle than the ones with $\varepsilon = 0$. In this latter case, identities can be used to effect that do not have any counterpart (as far as I can see) in case $\varepsilon > 0$.

Notation

Such a paper requires quite a lot of partial definitions to make the writing easier. It is then easy to forget the meaning of a quantity and we try to recall the most important ones here. The letters θ and H are used in several acceptions, but we have fixed $\varpi = -4\pi/\log 2$.

By $\ell \sim L$, we mean $L < \ell \le 2L$. The main actor is

$$S(a/q,t,\beta) = \sum_{\substack{\ell\sim X,\\ (\ell,q)=1}} \frac{\Lambda(\ell)}{m^{it}} e(\beta\ell) e\Bigl(\frac{\ell a}{q}\Bigr). \tag{3}$$

which we will split into a linear combination of sums of the linear type $L_r^{(1)}(a,t,\beta)$ defined in (36) and $L_r^{(2)}(a,t,\beta)$ defined in (37), and of sums of bilinear type:

$$S_r(a/q,t,\beta,M,N) = \sum_{\substack{mn\sim X,\\ (mn,q)=1,\\ m\sim M, n\sim N}} \frac{\Lambda(m)v_r(n)}{(mn)^{it}} e(\beta mn) e\Bigl(\frac{mna}{q}\Bigr). \tag{4}$$

A parameter $\varpi = -4\pi/\log 2$ will appear. Let us recall the definition of the Ramanujan sum as well as its evaluation in terms of the Moebius function:

$$c_r(m) = \sum_{a \bmod^* r} e(am/r) = \sum_{\substack{u|r,\\ u|n}} u\mu(r/u). \tag{5}$$

Acknowledgement

The gist of this paper emerged while I was giving a course on Local Models and the Hoheisel Theorem at IMSc Chennai in 2010. I thank this institution for the pleasant working conditions it offered to me.

Contents

2 A hybrid large sieve inequality: Theorem 8

This section is devoted to proving Theorem 8, its easy consequence Theorem 9. This material is our main ingredient to control the bilinear form arising from our decomposition of the Λ-function. We quote some rawer material that'll come in handy later.

We quote the following Theorem of Selberg from [3, Théorème 7A].

Theorem 7. *Let N_0 be a given real number. Let $(u_n)_{N_0<n\le N_0+N}$ be a sequence of complex numbers. We have,*

$$\sum_{\substack{\mathfrak{f}r\le R,\\ (\mathfrak{f},r)=1}} \frac{\mathfrak{f}}{\phi(\mathfrak{f}r)} \sum_{\chi\; mod^*\mathfrak{f}} \Big| \sum_{N_0<n\le N_0+N} u_n\chi(n)c_r(n)\Big|^2 \le \sum_{N_0<n\le N_0+N} |u_n|^2(N+R^2)$$

where $c_r(m)$ is the Ramanujan sum modulo r.

The summation is over both coprime variables $\mathfrak{f}$ and r, subject to $\mathfrak{f}r\le R$. The parameter N_0 is required since the left-hand side is *not* (a priori) invariant under translation. We now prove the following hybrid version. Such a result can also be found in [3, Théorème 8A] and has its origin in [8], our input here is a refined constant 71 instead of the 500 that comes out when following Gallagher's proof.

Theorem 8. *Let M be a given real number. Let $(u_n)_{N_0<n\le N_0+N}$ be a sequence of complex numbers. We have, for $T\ge 2\,000$,*

$$\sum_{\substack{\mathfrak{f}r\le R,\\ (\mathfrak{f},r)=1}} \frac{\mathfrak{f}}{\phi(\mathfrak{f}r)} \sum_{\chi\; mod^*\mathfrak{f}} \int_{-T}^{T}\Big| \sum_{N_0<n\le N_0+N} u_n\chi(n)c_r(n)n^{it}\Big|^2 dt$$
$$\le 71 \sum_{N_0<n\le N_0+N} |u_n|^2(N+R^2T)$$

where $c_r(m)$ is the Ramanujan sum modulo r.

2.1 Proof of Theorem 8

We first present a "generic" proof and choose the parameters later. Our argument is very close to the proof of [8, Lemma 1,Theorem 1], see also [3, Théorème 9].

2.1.1 A generic proof

Let F be a function to be chosen later. We assume that $F(t) = 0$ as soon as $|t| \geq 1/2$. The necessary regularity conditon is very mild: we only need it to be in $L^2(\mathbb{R})$ for the Fourier transform to be legit. Let $\delta > 0$ be a parameter that we shall also chose later. Gallagher's choice is $F(x) = \mathbb{1}_{|x|\leq 1/2}$ and we differ from it only for numerical reasons. We define

$$F_\delta(x) = F(x/\delta).$$

We thus get

$$\sum_n u_n e^{2i\pi t(\log n)/(2\pi)} \hat{F}_\delta(t) = \sum_n u_n \widehat{F_\delta\Big(t - \frac{\log n}{2\pi}\Big)}.$$

Parseval identity yields

$$\begin{aligned}\int_{-\infty}^{\infty} \Big|\sum_n u_n e^{2i\pi t(\log n)/(2\pi)} \hat{F}_\delta(t)\Big|^2 dt &= \int_{-\infty}^{\infty} \Big|\sum_n u_n F_\delta\Big(x - \frac{\log n}{2\pi}\Big)\Big|^2 dx \\ &= \int_0^{\infty} \Big|\sum_n u_n F_\delta\Big(\frac{\log(y/n)}{2\pi}\Big)\Big|^2 dy/y.\end{aligned}$$

Our hypothesis on F implies that the y's in the relevant range verify $e^{-\pi\delta} \leq y/n \leq e^{\pi\delta}$. As a consequence

$$\begin{aligned}&\sum_{r\leq R}\sum_{a \bmod^* r} \int_{-\infty}^{\infty} \Big|\sum_n u_n n^{it} e(na/r) \hat{F}_\delta(t)\Big|^2 dt \\ &\leq \int_0^{\infty} \sum_n |u_n|^2 \Big|F_\delta\Big(\frac{\log(y/n)}{2\pi}\Big)\Big|^2 \big(y(e^{\pi\delta} - e^{-\pi\delta}) + R^2\big) dy/y \\ &\leq \sum_n |u_n|^2 \int_0^{\infty} \Big|F_\delta\Big(\frac{\log(y/n)}{2\pi}\Big)\Big|^2 \big(e^{\pi\delta} - e^{-\pi\delta} + R^2 y^{-1}\big) dy \\ &\leq \sum_n |u_n|^2 \int_0^{\infty} \Big|F_\delta\Big(\frac{\log u}{2\pi}\Big)\Big|^2 \big(n(e^{\pi\delta} - e^{-\pi\delta}) + R^2 u^{-1}\big) du.\end{aligned}$$

We change variable by setting $u = \exp(2\pi\delta w)$. We get

$$\begin{aligned}&\sum_{r\leq R}\sum_{a \bmod^* r} \int_{-\infty}^{\infty} \Big|\sum_n u_n n^{it} e(na/r) \hat{F}_\delta(t)\Big|^2 dt \\ &\leq 2\pi\delta \sum_n |u_n|^2 n(e^{\pi\delta} - e^{-\pi\delta}) \int_{-\infty}^{\infty} |F(w)|^2 e^{2\pi\delta w} dw + 2\pi\delta R^2 \sum_n |u_n|^2 \int_{-\infty}^{\infty} |F(w)|^2 dw \\ &\leq 2\pi\delta \sum_n |u_n|^2 \big(n(e^{\pi\delta} - e^{-\pi\delta})e^{\pi\delta} + R^2\big) \int_{-\infty}^{\infty} |F(w)|^2 dw.\end{aligned}$$

Since $\hat{F}_\delta(t) = \delta\hat{F}(\delta t)$, we have finally reached

$$\sum_{r\leq R}\sum_{a \bmod^* r}\int_{-\infty}^{\infty}\Bigl|\sum_n u_n n^{it} e(na/r)\Bigr|^2 |\hat{F}(\delta t)|^2 dt$$
$$\leq 2\pi\sum_n |u_n|^2\bigl(n\frac{e^{2\pi\delta}-1}{\delta} + R^2\delta^{-1}\bigr)\int_{-\infty}^{\infty}|F(w)|^2 dw.$$

Now that we have this generic proof at our disposal, we simple have to optimise the choice of the function F. We want $\hat{F}(\delta t) \geq 1$ when $|t| \leq T$ as well as $F(x) = 0$ when $|x| \geq 1/2$. It is more usual to set $G = \hat{F}$ and to ask that $G(t) \geq 1$ when $|t| \leq \delta T$, and $\hat{G}(x) = 0$ when $|x| \geq 1/2$ (we have to minimize $\int_{-\infty}^{\infty}|G(t)|^2 dt$).

2.1.2 Some material taken from Vaaler

We consider the two functions

$$H(z) = \Bigl(\frac{\sin\pi z}{z}\Bigr)^2\Bigl(\sum_{\substack{m\in\mathbb{Z},\\ m\neq 0}}\frac{\operatorname{sgn} m}{(z-m)^2} + \frac{2}{z}\Bigr) \tag{6}$$

and

$$K(z) = \Bigl(\frac{\sin\pi z}{z}\Bigr)^2. \tag{7}$$

We further select two positive real parameters x_0 and θ, and build from them the following function

$$G(z) = \tfrac{1}{2}\bigl(H(\theta(z+x_0)) + K(\theta(z+x_0)) - H(\theta(z-x_0)) + K(\theta(z-x_0))\bigr). \tag{8}$$

We can also write

$$\begin{aligned}2G(z) = H(\theta(z+x_0)) - \operatorname{sgn}(\theta(z+x_0)) + K(\theta(z+x_0))\\ - H(\theta(z-x_0)) + \operatorname{sgn}(\theta(z-x_0)) + K(\theta(z-x_0)) + 2\mathbb{1}_{|z|\leq x_0}\end{aligned}$$

which shows that $G(z) \geq \mathbb{1}_{|z|\leq x_0}$ thanks to [32, Lemma 5]. This same paper helps us in computing its Fourier transform (see [32, (2.29)] and [32, Corollary 7]) :

$$\begin{aligned}2\hat{G}(t) &= \theta^{-1}\frac{\hat{J}(t/\theta)-1}{i\pi t/\theta}e(tx_0) + \theta^{-1}(1-|t/\theta|)^+ e(tx_0)\\ &- \theta^{-1}\frac{\hat{J}(t/\theta)-1}{i\pi t/\theta}e(-tx_0) + \theta^{-1}(1-|t/\theta|)^+ e(-tx_0) + 2\frac{e(x_0t)-e(-x_0t)}{2i\pi t}\\ &= \frac{\hat{J}(t/\theta)}{\pi t}2\sin(2\pi t x_0) + 2\theta^{-1}(1-|t/\theta|)^+\cos(2\pi t x_0)\end{aligned}$$

where, see [32, (2.32)],

$$\hat{J}(t) = \begin{cases} \pi t(1-|t|)\cot \pi t + |t| & \text{when } |t| \le 1, \\ 0 & \text{when } |t| \ge 1. \end{cases} \tag{9}$$

To accomodate our problem, we select $\theta = 1/2$ and $x_0 = \delta T$. We have reached

$$\hat{G}(t) = \frac{\hat{J}(2t)}{\pi t}\sin(2\pi t x_0) + 2(1-|2t|)^+\cos(2\pi t x_0).$$

We now have to compute its L^2-norm. We have

$$\begin{aligned}\int_{-\infty}^{\infty}|\hat{G}(t)|^2 dt &= 2\int_0^{1/2}\Big|\frac{\hat{J}(2t)}{\pi t}\sin(2\pi t x_0) + 2(1-2t)\cos(2\pi t x_0)\Big|^2 dt \\ &= 2\int_0^{1/2}\Big|\frac{2\pi t(1-2t)\cot(2\pi t)+2t}{\pi t}\sin(2\pi t x_0) + 2(1-2t)\cos(2\pi t x_0)\Big|^2 dt \\ &= \int_0^1\Big|\frac{2\pi(1-t)\cot(\pi t)+2}{\pi}\sin(\pi t x_0) + 2(1-t)\cos(\pi t x_0)\Big|^2 dt = \rho(x_0).\end{aligned}$$

Here is the inequality we have proven so far

$$\sum_{r\le R}\sum_{a \bmod^* r}\int_{-T}^{T}\Big|\sum_n u_n n^{it} e(na/r)\Big|^2 dt \le 2\pi\sum_n |u_n|^2\Big(n\frac{e^{2\pi\delta}-1}{\delta} + R^2\delta^{-1}\Big)\rho(\delta T).$$

We take $\delta = 2\pi/T$ and compute that $\rho(1/(2\pi)) = 1.7762 + \mathcal{O}^*(10^{-4})$, which yields Theorem 8. The best coefficient with respect to $\sum_n |u_n|^2 n$ seems to be $16\pi^2/3 = 52.6378 + \mathcal{O}^*(10^{-4})$, got when $x_0 = 0$.

2.2 From additive to multiplicative characters

We have, when c and q are coprime,

$$\begin{aligned}\phi(q)\mathbb{1}_{(n,q)=1} e(cn/q) &= \sum_{\chi \bmod q}\sum_m \mathbb{1}_{(m,q)=1} e(cm/q)\overline{\chi(m)}\chi(n) \\ &= \sum_{\chi \bmod q}\chi(c)\tau_q(\chi)\chi(n)\end{aligned}$$

and thus, for arbitrary coefficients $(\theta_q(c))_{c \bmod^* q}$,

$$\begin{aligned}\phi(q)^2\sum_{b \bmod^* q}\Big|\sum_{c \bmod^* q}\theta_q(c)e(cb/q)\Big|^2 &= \sum_{b \bmod^* q}\Big|\sum_{c \bmod^* q}\theta_q(c)\sum_{\chi \bmod q}\chi(c)\tau_q(\chi)\chi(b)\Big|^2 \\ &= \sum_{b \bmod^* q}\Big|\sum_{\chi \bmod q}\sum_{c \bmod^* q}\theta_q(c)\chi(c)\tau_q(\chi)\chi(b)\Big|^2 \\ &= \phi(q)\sum_{\chi \bmod q}\Big|\sum_{c \bmod^* q}\theta_q(c)\chi(c)\Big|^2|\tau_q(\chi)|^2\end{aligned}$$

If χ is of conductor $\mathfrak{f}$, we have $q = \mathfrak{f}q'$. Moreover $(q', \mathfrak{f}) = 1$ and $\mu(q') \neq 0$ when $\tau_q(\chi) \neq 0$. Hence

$$\sum_{b \bmod^* q} \left| \sum_{c \bmod^* q} \theta_q(c) e(cb/q) \right|^2 = \sum_{\substack{\mathfrak{f}|q, \\ (q/\mathfrak{f},\mathfrak{f})=1}} \frac{\mu^2(q/\mathfrak{f})\mathfrak{f}}{\phi(q/\mathfrak{f})\phi(\mathfrak{f})} \sum_{\chi \bmod^* \mathfrak{f}} \left| \sum_{c \bmod^* q} \theta_q(c)\chi(c) \right|^2.$$

When combined with Theorem 8, we get

Theorem 9. *Let q be some fixed modulus and $N_0 \in \mathbb{R}$. Let $(u_n)_{N_0 < n \le N_0+N}$ be a sequence of complex numbers such that $u_n = 0$ as soon as $(n, q) \neq 1$. We have, for any $T \geq 2\,000$,*

$$\sum_{\substack{r \le R/q, \\ (q,r)=1}} \frac{1}{\phi(r)} \sum_{a \bmod^* q} \int_{-T}^{T} \left| \sum_{N_0 < n \le N_0+N} u_n c_r(n) n^{it} e(na/q) \right|^2 dt$$

$$\le 71 \sum_{N_0 < n \le N_0+N} |u_n|^2 (N + R^2 T)$$

where $c_r(m)$ is the Ramanujan sum modulo r.

Corollary 10. *Let q be some fixed modulus. Let $(u_n)_{n \le N}$ be a sequence of complex numbers such that $u_n = 0$ as soon as $(n, q) \neq 1$. We have, for any $M \in \mathbb{R}$ and $T \geq 2$,*

$$\sum_{\substack{r \le R/q, \\ (q,r)=1}} \frac{1}{\phi(r)} \sum_{a \bmod^* q} \int_{-T}^{T} \left| \sum_{M < n \le M+N} u_n c_r(n) n^{it} e(na/q) \right|^2 \frac{dt}{1 + |t|}$$

$$\ll \sum_{n \le N} |u_n|^2 (N + R^2 \log T)$$

Proof. Simply use integration by parts and Theorem 9. □

3 General explicit estimates

3.1 Preparing for Rankin's trick

We quote from [26]:

Lemma 11. *When $r \geq 1$ and $1.38 \geq \varepsilon \geq 0$, we have*

$$-(1.44+5\varepsilon+3.6\varepsilon^2) \le \sum_{\substack{d \le x, \\ (d,r)=1}} \frac{\mu(d)}{d^{1+\varepsilon}} \log \frac{x}{d} \le 1.4+4.7\varepsilon+3.3\varepsilon^2+(1+\varepsilon)\frac{r^{1+\varepsilon}}{\phi_{1+\varepsilon}(r)} x^\varepsilon$$

where

$$\frac{r^{1+\varepsilon}}{\phi_{1+\varepsilon}(r)} = \prod_{p|r} \frac{p^{1+\varepsilon}}{p^{1+\varepsilon} - 1}. \tag{10}$$

Next we recall [1, Proposition 3], i.e. that when $s > 1$ is real, we have:

$$\zeta(s) \le e^{\gamma(s-1)}/(s-1). \tag{11}$$

A consequence of Lemma 11 reads:

$$\left|\sum_{\substack{(r/\ell)|d\le z^2,\\ (d,q\ell)=1}} \frac{\lambda_d^{(1)}}{d^{1+\varepsilon}}\right| \le \frac{(\ell/r)^{1+\varepsilon}}{\log z}\left(2.84+9.7\varepsilon+6.9\varepsilon^2+(1+\varepsilon)\frac{(qr)^{1+\varepsilon}}{\phi_{1+\varepsilon}(qr)}z^{2\varepsilon}\right) \tag{12}$$

or, since when $\varepsilon \in [0, 0.168]$, we have $(2.84 + 9.7\varepsilon + 6.9\varepsilon^2)/(1 + \varepsilon) \le 4$, the simpler form holds

$$\left|\sum_{\substack{(r/\ell)|d\le z^2,\\ (d,q\ell)=1}} \frac{\lambda_d^{(1)}}{d^{1+\varepsilon}}\right| \le \frac{5(\ell/r)^{1+\varepsilon}}{\log z}(1+\varepsilon)\frac{(qr)^{1+\varepsilon}}{\phi_{1+\varepsilon}(qr)}z^{2\varepsilon}. \tag{13}$$

Lemma 12. *Let δ be a given integer and $\varepsilon \in (0, 0.16]$. We have*

$$\left|\sum_{\substack{d_1,d_2\le z,\\ \delta|[d_1,d_2]}} \frac{\lambda_{d_1}^{(1)}\lambda_{d_2}^{(1)}}{[d_1,d_2]^{1+\varepsilon}}\right| \le \frac{25(1+\varepsilon)^2(e^\gamma z^4)^\varepsilon}{\varepsilon\log^2 z}\prod_{p|\delta}\frac{3p^{1+\varepsilon}-2}{(p^{1+\varepsilon}-1)p^{1+\varepsilon}}.$$

Proof. Let us call S the sum to be studied. We have

$$\begin{aligned} S &= \sum_{\delta_1\delta_2\delta_3=\delta}\sum_{\substack{\delta_1\delta_3|d_1\le z^2,\\ \delta_2\delta_3|d_2\le z^2,\\ (d_1,\delta_2)=1,\\ (d_2,\delta_1)=1,}} \frac{\lambda_{d_1}^{(1)}\lambda_{d_2}^{(1)}}{[d_1,d_2]^{1+\varepsilon}} \\ &= \sum_{\delta_1\delta_2\delta_3=\delta}\delta_3^{1+\varepsilon}\sum_{\substack{\ell\le z^2,\\ (\ell,\delta)=1}}\phi_{1+\varepsilon}(\ell)\sum_{\substack{\ell\delta_1\delta_3|d_1\le z^2,\\ \ell\delta_2\delta_3|d_2\le z^2,\\ (d_1,\delta_2)=1,\\ (d_2,\delta_1)=1,}}\frac{\lambda_{d_1}^{(1)}\lambda_{d_2}^{(1)}}{(d_1d_2)^{1+\varepsilon}} \end{aligned}$$

and thus, by (13),

$$\begin{aligned} |S| &\le \frac{5^2(1+\varepsilon)^2z^{4\varepsilon}}{\delta^{1+\varepsilon}\log^2 z}\sum_{\delta_1\delta_2\delta_3=\delta}\sum_{\substack{\ell\le z^2,\\ (\ell,\delta)=1}}\frac{\mu^2(\ell)}{\phi_{1+\varepsilon}(\ell)}\frac{(\delta_1\delta_3)^{1+\varepsilon}}{\phi_{1+\varepsilon}(\delta_1\delta_3)}\frac{(\delta_2\delta_3)^{1+\varepsilon}}{\phi_{1+\varepsilon}(\delta_2\delta_3)} \\ &\le \frac{5^2(1+\varepsilon)^2z^{4\varepsilon}}{\phi_{1+\varepsilon}(\delta)\log^2 z}\sum_{\delta_1\delta_2\delta_3=\delta}\frac{\delta_3^{1+\varepsilon}}{\phi_{1+\varepsilon}(\delta_3)}\sum_{\substack{\ell\le z^2,\\ (\ell,\delta)=1}}\frac{\mu^2(\ell)}{\phi_{1+\varepsilon}(\ell)}. \end{aligned}$$

We note that

$$\sum_{\substack{\ell\geq 1,\\ (\ell,\delta)=1}} \frac{\mu^2(\ell)}{\phi_{1+\varepsilon}(\ell)} = \prod_{p\nmid\delta}\Big(1+\frac{1}{p^{1+\varepsilon}-1}\Big)$$
$$= \prod_{p|\delta}\frac{p^{1+\varepsilon}-1}{p^{1+\varepsilon}}\zeta(1+\varepsilon) = \frac{\phi_{1+\varepsilon}(\delta)}{\delta^{1+\varepsilon}}\zeta(1+\varepsilon).$$

Hence we find that

$$|S| \leq \frac{25(1+\varepsilon)^5 z^{4\varepsilon}\zeta(1+\varepsilon)}{\delta^{1+\varepsilon}\log^2 z} 2^{\omega(d)} \sum_{\delta_3|\delta} \frac{\delta_3^{1+\varepsilon}}{2^{\omega(d_3)}\phi_{1+\varepsilon}(\delta_3)}$$
$$\leq \frac{25(1+\varepsilon)^2 z^{4\varepsilon}\zeta(1+\varepsilon)}{\log^2 z}\prod_{p|\delta}\frac{3p^{1+\varepsilon}-2}{(p^{1+\varepsilon}-1)p^{1+\varepsilon}}.$$

□

Lemma 13. *For some parameter $\varepsilon \in (0, 0.16]$, we have*

$$\sum_{n\leq B} \frac{\Big(\sum_{d|n}\lambda_d^{(1)}\Big)^2}{n} \leq \frac{25(1+\varepsilon)^2(e^\gamma B z^4)^\varepsilon}{\varepsilon^2\log^2 z}.$$

Proof. We call S the sum to be bounded. We have for some parameter $\varepsilon \in (0, 0.16]$ and on using Lemma 13 with $\delta = 1$:

$$S \leq B^\varepsilon \sum_{n\geq 1}\frac{\Big(\sum_{d|n}\lambda_d^{(1)}\Big)^2}{n^{1+\varepsilon}} \leq B^\varepsilon\zeta(1+\varepsilon)\sum_{d_1,d_2\leq z^2}\frac{\lambda_{d_1}^{(1)}\lambda_{d_2}^{(1)}}{[d_1,d_2]^{1+\varepsilon}}$$
$$\leq \frac{25(1+\varepsilon)^2\zeta(1+\varepsilon)^2(Bz^4)^\varepsilon}{\log^2 z}.$$

□

3.2 Prime number estimates

We recall some classical results taken from [30].

Lemma 14. *When $M \geq 101$, we have*

$$\sum_{m\sim M}\Lambda(m) \leq \tfrac{6}{5}M.$$

Proof. Indeed, [30, Theorem 12] gives

$$\psi(x) \le 1.03x, \quad (x \ge 0). \tag{14}$$

And [30, Theorem 10] gives

$$\vartheta(x) = \sum_{p \le x} \log p \ge 0.84x, \quad (x \ge 101). \tag{15}$$

□

Finally, from [30, (3.24)]:

$$\sum_{m \le M_0} \frac{\Lambda(m)}{m} \le \log M_0, \quad (M_0 \ge 1). \tag{16}$$

Lemma 15. *When $A \ge B \ge 1$, we have*

$$\sum_{A < m \le A+B} \Lambda(m) \le \Big(\tfrac{37}{18}\frac{B}{\log B} + \log^+ \frac{\log \max(A, e)}{\log B}\Big) \log(A+B).$$

Note that $37/18 = 2.055555\cdots$ so we do not loose much with respect to the constant 2 of the Brun-Titchmarsh inequality. The additional term $\log^+ \frac{\log \max(A,e)}{\log B}$ is usually of not impact. It is an upper bound for the contribution of high powers of small primes.

Proof. Let us first see what happens when B is small. When $1 \le B \le 2$, at most two integers can be inside $(A, A+B]$. Each of them can be a prime power in at most one way, and thus

$$\sum_{A < m \le A+2} \Lambda(m) \le 2B \frac{\log(A+B)}{\log B}, \quad (B \le 2).$$

When $2 < B \le 3$, then the same reasoning applies yielding the bound

$$\sum_{A < m \le A+3} \Lambda(m) \le 3 \log(A+B) \le 2B \frac{\log(A+B)}{\log B}, \quad (2 < B \le 3).$$

In fact, this works for $B \le 7.38$, since below this boud we have $B \le 2B/\log B$. Let us now assume that $B > 7.38$. The interval $(A, A+B]$ contains at most one power of 2, and thus the number of prime powers is not more than $1 + \lceil B/2 \rceil$. And $2 + b/2 < 2b/\log b$ when $b \le 36.93$. Let us now assume that $B > 46.25$ and let us consider powers of 2 and of 3. The number of powers of primes in $(A, A+B]$ is at most $2 + 2\lceil B/6 \rceil$ since any interval of length 6 contains two integers prime to 2 and 3. We check that $4 + b/3 \le 2b/\log b$ when $b \le 326$. We continue with 2, 3 and 5: the number of powers of primes in $(A, A+B]$ is at most $3 + 8\lceil B/30 \rceil$, and $11 + 4b/15 \le 2b/\log b$ when $b \le 1474$. And we proceed to a last step with 2, 3, 5 and 7, but this time by taking the factor $\log\log A \ge \log\log B$ into account: the number of powers of primes in $(A, A+B]$ is at most $4 + \phi(210)\lceil B/210\rceil$, and $52 + 8b/35 \le 2b/\log b + \log\log b$ when $b \le 4022$.

Let us complete the proof. We recall that, when $0 \le \alpha \le 1$, we have $(A+B)^\alpha - A^\alpha \le \alpha B^\alpha$. Indeed

$$(A+B)^\alpha - A^\alpha = \alpha \int_0^B (A+u)^{\alpha-1} du \le \alpha B A^{\alpha-1} \le \alpha B^\alpha.$$

When the integer k is at least $1 + 2\log B$, then $B^{1/k}/k \le \exp(\frac{\log B}{1+\frac12\log B} - \log(1+\frac12\log B))$ and this last quantity is not more than 1 when $B \ge 4000$. As a conclusion, there is at most one prime to the power k within $(A, A+B]$. We have, by the Brun-Titchmarsh Theorem of [15] and with $K_0 = 1+\frac12\log B$ and $K_1 = \log(A+B)/\log 2$

$$\begin{aligned}\sum_{A<m\le A+B} \frac{\Lambda(m)}{\log(A+B)} &\le 2 \sum_{1\le k\le K_1} B^{1/k}\frac{1/k}{\log(B^{1/k})} + \sum_{K_0<k\le K_1}\frac{1}{k}\\ &\le \frac{2B}{\log B}\sum_{1\le k\le K_0} B^{\frac1k -1} + \int_{K_0-1}^{K_1} dt/t\\ &\le \frac{2B}{\log B}\Bigl(1+\frac{1}{\sqrt{B}} + \frac{1}{B}\int_2^{K_0} e^{\frac{\log B}{k}}dk\Bigr) + \log\frac{\log(A+B)}{2\log 2\log B}\\ &\le \frac{2B}{\log B}\Bigl(1+\frac{1}{\sqrt{B}} + \frac{\log B}{B}\int_{\frac{2}{1+\frac{1}{2\log B}}}^{(\log B)/2} e^u\frac{du}{u^2}\Bigr) + \log\frac{\log A+\log 2}{2\log 2\log B}.\end{aligned}$$

We readily check that the function

$$x \mapsto e^{-x/2} + \frac{x}{e^x}\int_{2/(1+1/\log(4000))}^{x/2} e^u\frac{du}{u^2}$$

is ≤ 0.0276 when $\log 4000 \le x$. And finally, we notice that the function

$$y \mapsto \frac{y+\log 2}{2\log 2} - y$$

is decreasing and negative at $y = \log 4000$. The Lemma follows readily. □

Lemma 16. *For any modulus $q \ge 1$, any real number $M \ge \max(121, q^2)$, we have*

$$\sum_{\substack{m\sim M,\\ m\equiv a[q]}} \Lambda(m) \le \tfrac{28}{5} M/\phi(q).$$

Proof. We have, by the Brun-Titchmarsh Theorem of [15]

$$\sum_{\substack{m\sim M,\\ m\equiv a[q]}} \Lambda(m) \le 2\frac{M\log(2M)}{\phi(q)\log(M/q)} + (\psi-\vartheta)(2M) - (\psi-\vartheta)(M)$$

and, by [31, Theorem 6, (5.4)] and [30, Theorem 13],

$$0.99868\sqrt{x} < (\psi-\vartheta)(x) \le 1.42620\sqrt{x}. \tag{17}$$

the lower bound being valid only when $x \ge 121$. We thus get

$$\sum_{\substack{m\sim M,\\ m\equiv a[q]}} \Lambda(m) \le \frac{M}{\phi(q)}\left(2\frac{\log M}{\frac{1}{2}\log M}\left(1+\frac{\log 2}{\log M}\right) + 1.42620\sqrt{2} - 0.99868\right).$$

A numerical application ends the proof. □

Lemma 17. *For any modulus $q \ge 1$, any real number $M \ge \max(121, q^2)$, we have*

$$\sum_{b \bmod^* q}\Bigg|\sum_{\substack{m\equiv b[q],\\ m\sim M}} \Lambda(m)\Bigg|^2 \le \tfrac{27}{4}M^2/\phi(q).$$

Proof. We collect Lemma 16 and 14 getting that the left-hand side above is not more than

$$\frac{28}{5}\frac{M}{\phi(q)}\frac{6}{5}M \le \frac{27}{4}\frac{M^2}{\phi(q)}.$$

□

3.3 On the G-functions

We recall the classical definition

$$G_q(D) = \sum_{\substack{d\le D,\\ (d,q)=1}} \frac{\mu^2(d)}{\phi(d)}, \quad G(D) = G_1(D). \tag{18}$$

We quote from [33]:

$$G(D) \le \frac{q}{\phi(q)}G_q(D) \le G(qD). \tag{19}$$

We quote from [20, Lemma 3.5] (see also [29])

$$G(D) \le \log D + 1.4709, \quad (D \ge 1) \tag{20}$$

and, concerning a lower bound,

$$\log D + 1.06 \le G(D), \quad (D \ge 6). \tag{21}$$

3.4 Other averages of non-negative multiplicative functions

We appeal to [20, Lemma 3.2] that shall be partially recalled during the proof. The input is the [2]

Lemma 18. *We have, for all $t > 0$,*

$$\sum_{n \le t} \frac{\tau(n)}{n} = \tfrac{1}{2}\log^2 t + 2\gamma \log t + \gamma^2 - \gamma_1 + O^*(1.16/t^{1/3})$$

where γ_1 is the second Laurent-Stieljes constant – see for instance [13] and [5]. In particular, we have

$$\gamma_1 \qquad = \qquad -0.072\,815\,845 \qquad + \qquad O^*(10^{-9}). \tag{22}$$

Lemma 19. *For $D > 0$, we have*

$$\sum_{\substack{\ell \le D,\\ (\ell,2)=1}} \mu^2(\ell) 2^{\omega(\ell)}/\ell = a\big(\tfrac{1}{2}\log^2 D + c_1 \log D + c_2\big) + \mathcal{O}^*(30/D^{1/3})$$

where $a = 0.143373 + \mathcal{O}^*(10^{-6})$, $c_1 = 4.71 + \mathcal{O}^*(0.01)$, and $c_2 = 15.81 + \mathcal{O}^*(0.01)$. Also

$$\sum_{\substack{\ell \le D,\\ (\ell,2)=1}} \mu^2(\ell) 2^{\omega(\ell)} = a(\log D + c_1 - 1)D + \mathcal{O}^*(76\, D^{2/3}).$$

Finally we also have, when $D \ge 1$,

$$\sum_{\substack{\ell \le D,\\ (\ell,2)=1}} \mu^2(\ell) 2^{\omega(\ell)} \le a(\log D + 33)D.$$

as well as, when $D \ge 8\,000$,

$$\sum_{\substack{d \le D,\\ (d,2)=1}} \mu^2(d) 2^{\omega(d)}/d \le \tfrac{3}{20}\log^2 D.$$

Proof. We appeal to [20, Lemma 3.2]. First note that

$$\begin{aligned} D(s) &= \sum_{\substack{d \ge 1,\\ (d,2)=1}} \frac{\mu^2(d) 2^{\omega(d)}}{d^{1+s}} = \prod_{p \ge 3}\Big(1 + \frac{2}{p^{1+s}}\Big) \\ &= \zeta(1+s)^2 (1 - 2^{-1-s})^2 \prod_{p \ge 3} \frac{p^{3+3s} - 3p^{1+s} + 2}{p^{3+3s}} = \zeta(1+s) H(s) \end{aligned}$$

say. We thus get, for $D > 0$:

$$\sum_{\substack{d\le D,\\(d,2)=1}} \frac{\mu^2(d)2^{\omega(d)}}{d} = H(0)\tfrac12 \log^2 D + (H'(0) + 2\gamma H(0)) \log D$$
$$+ \tfrac12 H''(0) + 2\gamma H'(0) + (\gamma^2 - \gamma_1)H(0) + \mathcal{O}^*\big(c_4/D^{1/3}\big)$$

with

$$c_4 = 1.16(1 + 2^{-2/3})^2 \prod_{p\ge3} \frac{p^2 + 3p^{2/3} + 2}{p^2} \le 30.$$

Furthermore

$$a = H(0) = \tfrac14 \prod_{p\ge3} \frac{p^3 - 3p + 2}{p^3} = 0.4690862 + \mathcal{O}^*(10^{-6}),$$

and

$$\frac{H'(0)}{H(0)} = 2\log 2 + \sum_{p\ge3} \frac{6\log p}{p^2 - 2} = 3.5621 + \mathcal{O}^*(10^{-4})$$

as well as (see [20, (3.4)])

$$\frac{H''(0)}{H(0)} = \Big(\frac{H'(0)}{H(0)}\Big)^2 - 4\log^2 2 + 6\sum_{p\ge3} \frac{3p^3 - 2p^2 - 12p + 10}{(p-1)(p^2-2)^2} \log^2 p$$
$$= 22.59 + \mathcal{O}^*(0.01).$$

The same Lemma gives us

$$\sum_{\substack{d\le D,\\(d,2)=1}} \mu^2(d)2^{\omega(d)} = a\big(D\log D + (c_1 - 1)D + c_1 - c_2 - 1\big) + \mathcal{O}^*\big(75\, D^{2/3}\big)$$

We prove by GP/Pari (upto 10^6) (noting that $D\log D + (c_1 - 1)D$ is a non-decreasing function) that

$$\sum_{\substack{d\le D,\\(d,2)=1}} \mu^2(d)2^{\omega(d)} = a(\log D + c_1 - 1)D + \mathcal{O}^*(5\, D^{2/3}), \quad (1 \le D \le 500\,000).$$

We also check that

$$\sum_{\substack{d\le D,\\(d,2)=1}} \mu^2(d)2^{\omega(d)} = a(\log D + c_1 - 1)D + \mathcal{O}^*(0.47\, D), \quad (1 \le D \le 500\,000).$$

Also

$$\sum_{\substack{d\le D,\\(d,2)=1}} \mu^2(d)2^{\omega(d)}/d \le \tfrac{3}{20} \log^2 D, \quad (8\,000 \le D \le 5\,000\,000).$$

□

Lemma 20. *When $X \geq 8$, we have*

$$\sum_{\substack{\ell \leq X,\\ (\ell,2)=1}} \mu^2(\ell) 4^{\omega(\ell)} \leq \tfrac{1}{92} X(\log X + 33)\log^2 X.$$

Proof. We will only evaluate the summation S defined by

$$\begin{aligned} S = \sum_{\substack{mn \leq X,\\ (mn,2)=1}} \mu^2(mn) 2^{\omega(m)} 2^{\omega(n)} &\leq 2 \sum_{\substack{m \leq \sqrt{X},\\ (m,2)=1}} \mu^2(m) 2^{\omega(m)} \sum_{\substack{n \leq X/m,\\ (n,2)=1}} \mu^2(n) 2^{\omega(n)} \\ &\leq 2aX \sum_{\substack{m \leq \sqrt{X},\\ (m,2)=1}} \frac{\mu^2(m) 2^{\omega(m)}}{m} \Bigl(\log \frac{X}{m} + 33\Bigr) \\ &\leq a \tfrac{3}{40} X(\log X + 33)\log^2 X \\ &\leq \tfrac{1}{92} X(\log X + 33)\log^2 X \end{aligned}$$

for $X \geq 8000^2$. By using Gp/Pari, we check that

$$\sum_{\substack{\ell \leq X,\\ (\ell,2)=1}} \mu^2(\ell) 4^{\omega(\ell)} \leq 0.0035\, X(\log X + 33)\log^2 X, \quad (1000 \leq X \leq 10^7).$$

□

4 Separating the variables in the bilinear form

In the bilinear form, we will have to handle conditions like

$$a \sim A, \quad b \sim B, \quad \text{and} \quad ab \sim N$$

where $x \sim X$ means that $X < x \leq 2X$. The last inequality is annoying because it links both variables together. This link is mild and we remove it by the process we describe here. The same process is used for instance in [7, Lemma 6]. Our form is more precise in two aspects: we regulate the length of the integration, and we avoid a loss of a log-factor by introducing (or more precisely: preparing the introduction of) some information of the local behaviour of $(|a_n|)$.

Let $\delta \in]0, 1/2]$ be a real parameter. We first note that

$$\int_{-\infty}^{\infty} (1 - |u|)^+ e(uv) du = \Bigl(\frac{\sin \pi v}{\pi v}\Bigr)^2.$$

We then consider the trapezoidal function

$$\begin{aligned} h_0(\delta; u) = \frac{(1-|u|)^+ - (1-\delta-|u|)^+}{\delta} &= \frac{(1-|u|)^+ - (1-\delta)(1-|u|/(1-\delta))^+}{\delta} \\ &= \begin{cases} 1 & \text{when } |u| \leq 1-\delta, \\ \frac{1-|u|}{\delta} & \text{when } 1-\delta \leq |u| \leq 1, \\ 0 & \text{when } 1 \leq |u|. \end{cases} \end{aligned}$$

This function verifies

$$\hat{h}_0(\delta; v) = \int_{-\infty}^{\infty} h_0(\delta; u) e(uv) du = \frac{(\sin \pi v)^2 - (\sin \pi(1-\delta)v)^2}{\pi^2 \delta v^2}$$
$$= \frac{\sin(\pi\delta v)\sin(\pi(2-\delta)v)}{\pi^2\delta v^2}.$$

We further select two additionnal real parameters κ and λ and build:

$$H(\delta, \lambda, \kappa; u) = h_0(\delta; u) e\big(\lambda e^{\kappa u}\big). \tag{23}$$

We have

Lemma 21. *We have the following pointwise bounds, except when $u \in \{\pm 1, \pm(1-\delta)\}$,*

$$|H(\delta, \lambda, \kappa; u)| \le 1, \quad |H'(\delta, \lambda, \kappa; u)| \le \delta^{-1} + 2\pi|\lambda\kappa| e^{|\kappa|}.$$

The following L^1-bounds also hold

$$\int_{-1}^{1} |H'(\delta, \lambda, \kappa; u)| du \le 2 + 4\pi|\lambda\kappa| e^{|\kappa|},$$

and

$$\int_{-1}^{1} |H''(\delta, \lambda, \kappa; u)| du \le 4\pi|\lambda\kappa| e^{|\kappa|}\big(2 + |\kappa| + |\pi\lambda\kappa| e^{|\kappa|}\big).$$

Proof. We have, when $u \notin \{\pm 1, \pm(1-\delta)\}$,

$$H'(\delta, \lambda, \kappa; u) = h_0'(\delta; u) e\big(\lambda e^{\kappa u}\big) + 2i\pi\lambda\kappa e^{\kappa u} H(\delta, \lambda, \kappa; u)$$

and

$$H''(\delta, \lambda, \kappa; u) = h_0''(\delta; u) e\big(\lambda e^{\kappa u}\big) + 4i\pi\lambda\kappa h_0'(\delta; u) e^{\kappa u} e\big(\lambda e^{\kappa u}\big)$$
$$+ 2i\pi\lambda\kappa^2 e^{\kappa u} H(\delta, \lambda, \kappa; u) + (2i\pi\lambda\kappa e^{\kappa u})^2 H(\delta, \lambda, \kappa; u)$$

(and our choice function h_0 verifies $h_0'' = 0$). The reader will easily derive the Lemma from both these expressions. □

The Fourier transform being defined by $\hat{f}(v) = \int_{\mathbb{R}} f(u) e(uv) du$, we deduce from the above Lemma and at most two integrations by parts that

$$|\hat{H}(\delta, \lambda, \kappa; v)| \le \min\Big(1, \frac{1 + 2\pi|\lambda\kappa|}{\pi|v|}, \frac{\delta^{-1}}{\pi^2|v|^2} + \frac{4 + |\kappa| + |\pi\lambda\kappa| e^{|\kappa|}}{\pi|v|^2} |\lambda\kappa| e^{|\kappa|}\Big) \tag{24}$$

We separate the variable by using the next Lemma. At the same time, we handle a variation $e(\beta\ell)$:

Lemma 22. *Let $b > 1$, β be a two given real number. We set*

$$\lambda = 2\pi\beta L, \quad \kappa = -(\log b)/2. \tag{25}$$

There exist three sequences $(a^{(1)})$, $(a^{(2)})$ et $(a^{(3)})$ with $|a^{(1)}|, |a^{(2)}|, |a^{(3)}| \le 1$ depending on b, β and L such that the following holds. Let $(u_\ell)_{b^{-1}L\le\ell\le b^2L}$ be complex numbers. We have

$$\begin{aligned}\sum_{L<\ell\le bL} u_\ell\, e(\beta\ell) &= \int_{-\Delta}^{\Delta} \hat{H}(\delta,\lambda,\kappa;v) \sum_{L/b<\ell\le b^2L} u_\ell \Big(\frac{L}{\ell}\Big)^{\frac{-2i\pi v}{\log b}} dv \\ &\quad + \mathcal{O}^*\Big(\Big|\sum_{L/b<\ell\le b^{\delta/2}L/b} u_\ell a_\ell^{(1)}\Big| + \Big|\sum_{b^{1-\delta/2}L<\ell\le bL} u_\ell a_\ell^{(2)}\Big| + 2\delta\Big|\sum_\ell u_\ell a_\ell^{(3)}\Big|\Big)\end{aligned}$$

for any parameter $\delta \in (0, 1/2)$ and with

$$\Delta = \frac{1}{\pi^2\delta^2} + \frac{2 + \log b + \pi^2(\log b)\beta L}{\delta}(b\log b)\beta L. \tag{26}$$

In usual applications, we simply bound $\sum_\ell u_\ell a_\ell^{(i)}$ by $\sum_\ell |u_\ell|$, but it may be interesting to use a dependence in a parameter in u_ℓ, as when $u_\ell = U_\ell \ell^{it}$ and we further average over t.

Proof. The first step is to introduce $H(\delta,\lambda,\kappa;u)$:

$$\begin{aligned}\sum_{L<\ell\le bL} u_\ell\, e(\beta\ell) &= \sum_{b^{-1}L<\ell\le b^2L} u_\ell H\Big(\delta,\lambda,\kappa; 2\frac{\log(L/\ell)}{\log b} - 1\Big) \\ &\quad + \mathcal{O}^*\Big(\Big|\sum_{0\le\frac{\log(L/\ell)}{\log b}\le\delta} u_\ell a_\ell^{(1)}\Big| + \Big|\sum_{1-\delta\le\frac{\log(L/\ell)}{\log b}\le 1} u_\ell a_\ell^{(2)}\Big|\Big).\end{aligned}$$

Next, we write

$$\begin{aligned}H\Big(\delta,\lambda,\kappa; 2\frac{\log(L/\ell)}{\log b} - 1\Big) &= \int_{-\infty}^{\infty} \hat{H}(\delta,\lambda,\kappa;v)\Big(\frac{L}{\ell}\Big)^{\frac{-4i\pi v}{\log b}} dv \\ &= \int_{-\Delta}^{\Delta} \hat{H}(\delta,\lambda,\kappa;v)\Big(\frac{L}{\ell}\Big)^{\frac{-4i\pi v}{\log b}} dv + \mathcal{O}^*(2\delta).\end{aligned}$$

The Lemma follows readily. □

To handle the error term that arises, we will need the following two Lemmas.

Lemma 23. *When $0 \le \delta \le 1$, we have $2^{\delta/2} - 1 \le \frac{3}{5}\,\delta$.*

Proof. We use the inequality $e^x - 1 \le xe^x$, valid for any $x \ge 0$. □

Technical note

Assume we select a different smoothing function and choose instead of h_0:

$$h_1(\delta;u)=\begin{cases}1 & \text{when } |u|\le 1-\delta,\\ f\big((1-|u|)/\delta\big) & \text{when } 1-\delta\le |u|\le 1,\\ 0 & \text{when } 1\le |u|.\end{cases}$$

where f is some $\mathrm{C}^k[0,1]$ function such that

- $f(0)=1$, and $f(1)=1$,
- $f^{(\ell)}(0)=f^{(\ell)}(1)=0$ for $1\le \ell\le k-1$.

The choice h corresponds to the case $k=1$ and $f(t)=t$. Iterated integrations by parts show that

$$\hat{h}(\delta;v)\ll_{k,f}|v|/(\delta|v|)^{k+1}$$

and the main consequence is that the range of integration $[-\delta^{-2},\delta^{-2}]$ reduces to $[-\delta^{-(k+1)/k},\delta^{-(k+1)/k}]$. This leads to better constants and a larger range for q in terms of X.

5 A decomposition of the van Mangoldt function

5.1 The general theory

We consider the Dirichlet series:

$$V_r(s)=\sum_{n\ge 2}c_r(n)\Big(\sum_{d|n}\lambda_d^{(1)}\Big)/n^s=\sum_{n\ge 2}v_r(n)/n^s. \tag{27}$$

This series has the good idea to (almost) factor. Note that the summation can be restricted to integers $n>z$.

Theorem 24.

$$1+V_r(s)=\zeta(s)M_r(s,\lambda_d^{(1)}) \tag{28}$$

where

$$M_r(s,\lambda_d^{(1)})=\sum_{\substack{u|r,\\ d\le z^2}}\frac{u\mu(r/u)\lambda_d^{(1)}}{[u,d]^s}=\sum_{1\le n\le rz^2}h_r(n)/n^s, \tag{29}$$

where we quote explicitly:

$$h_r(n)=\sum_{\substack{u|r,d\le z^2,\\ [u,d]=n}}u\mu(r/u)\lambda_d^{(1)}. \tag{30}$$

Here is the formal identity that gives us a decomposition of 1:

$$1 = -V_r + (1 + V_r).$$

There follows a decomposition of $-\zeta'/\zeta$ which we modify with the help of (28), and we reach

$$\boxed{-\frac{\zeta'}{\zeta} = \frac{\zeta'}{\zeta} V_r - \zeta' M_r.} \tag{31}$$

This translates in the following point-wise identity:

$$\boxed{\Lambda = -\Lambda \star v_r + \log \star h_r.} \tag{32}$$

The corresponding identity for the Moebius function is even more stricking:

$$\boxed{\mu = -\mu \star v_r + h_r.} \tag{33}$$

Identities (32) and (33) are the core of our approach. It however still needs to be slightly refined, as the variable carried by Λ in the factor $\Lambda \star v_r$ can be small. This is readily taken care of by a simple truncation, see (35). Let us finally mention that we will average over this family of decompositions.

5.2 Starting the proof of Theorem 6

Let us select a squarefree integer $r \le R$. We assume that

$$z^2 R \le X. \tag{34}$$

We further select a (large) parameter M_0 and write (recall (3))

$$\begin{aligned} S(a/q, t, \beta) = L_r^{(1)}(a, t, \beta) + L_r^{(2)}(a, t, \beta) \\ + \sum_{\substack{mn \sim X, \\ (mn,q)=1, \\ m > M_0}} e(\beta mn) \frac{\Lambda(m) v_r(n)}{(mn)^{it}} e\Big(\frac{mna}{q}\Big), \end{aligned} \tag{35}$$

where the first linear form is defined by

$$L_r^{(1)}(a, t, \beta) = \sum_{\substack{mn \sim X, \\ (mn,q)=1}} e(\beta nm) \frac{h_r(m) \log n}{(nm)^{it}} e(nma/q) \tag{36}$$

while the second one is defined by

$$L_r^{(2)}(a, t, \beta) = \sum_{\substack{mn \sim X, \\ (mn,q)=1, \\ m \le M_0}} e(\beta mn) \frac{\Lambda(m) v_r(n)}{(nm)^{it}} e\Big(\frac{mna}{q}\Big). \tag{37}$$

The quantity $S(a/q,t,\beta)$ has a condition $(\ell,q)=1$ while our final estimate does not, and we treat this (trivial) matter in section 6.

We now examine the last sum. We first have to localize the variables m and n. Notice that $n>z$. So we start at $N=z$, go until $2z$, etc until $2^k z \le 2X/M_0 < 2^{k+1}z$, i.e. $0\le k\le \log(X/(M_0 z))/\log 2$. Concerning M, we have $N<n\le N'\le 2N$, and thus $\frac12(X/N)\le X/n<m\le 2X/N$. So for each N, we have two values of M, namely $M_1=\frac12(X/N)$ and $M_2=X/N$. We then use the following inequalities, where $A(m,n)$ is a general summand:

$$\begin{aligned}
\Bigl|\sum_{\substack{mn\sim X,\\(mn,q)=1}} A(m,n)\Bigr|^2 &= \Bigl|\sum_{M,N}\sum_{\substack{mn\sim X,\\(mn,q)=1,\\m\sim M,n\sim N}} A(m,n)\Bigr|^2\\
&\le \sum_{M,N}1\sum_{M,N}\Bigl|\sum_{\substack{mn\sim X,\\(mn,q)=1,\\m\sim M,n\sim N}} A(m,n)\Bigr|^2\\
&\le \frac{2\log\frac{2X}{M_0 z}}{\log 2}\sum_{M,N}\Bigl|\sum_{\substack{mn\sim X,\\(mn,q)=1,\\m\sim M,n\sim N}} A(m,n)\Bigr|^2.
\end{aligned}$$

We relax the condition $mn\sim X$ *and* remove the coefficient $e(\beta mn)$ in this last summation by appealing to Lemma 22, case $b=2$, and $L=X$ and some $\delta=\delta(M,N)\in(0,1)$. With the notation (4), we find that

$$\begin{aligned}
&S_r(a/q,t,\beta,M,N)=\\
&\int_{-\Delta}^{\Delta}\hat H(\delta,\lambda,\kappa;v)\sum_{\substack{(mn,q)=1,\\m\sim M,n\sim N}}\frac{\Lambda(m)}{m^{\frac{-4i\pi v}{\log 2}+it}}\frac{v_r(n)}{n^{\frac{-4i\pi v}{\log 2}+it}}e(mna/q)\frac{dv}{X^{\frac{4i\pi v}{\log 2}}}\\
&\qquad\qquad+\mathcal{O}^*\bigl(E_1(\delta,r)+E_2(\delta,r)+2\delta E_3(r)\bigr)
\end{aligned}$$

with

$$E_1(\delta,r)=\sum_{\substack{X/2<mn\le 2^{\delta/2}X/2,\\(mn,q)=1,\\m\sim M,n\sim N}}\Lambda(m)|v_r(n)|,\quad E_2(\delta,r)=\sum_{\substack{2X/2^{\delta/2}<mn\le 2X,\\(mn,q)=1,\\m\sim M,n\sim N}}\Lambda(m)|v_r(n)| \tag{38}$$

and

$$E_3(r)=\sum_{\substack{(mn,q)=1,\\m\sim M,n\sim N}}\Lambda(m)|v_r(n)|. \tag{39}$$

We recall that the parameters λ and κ are defined in (25), while Δ is defined in (26).

6 Reduction to powers of primes prime to q

This is a trivial matter, but it needs to be recorded. We write

$$\Bigg|\sum_{\substack{\ell\sim X,\\ (\ell,q)\neq 1}} \frac{\Lambda(\ell)}{m^{it}} e(\beta\ell) e\Big(\frac{\ell a}{q}\Big)\Bigg| \le \sum_{p|q} \frac{\log(2X)}{\log p}\log p = \omega(q)\log(2X)$$

which is easily absorbed, even numerically on using $\omega(q) \le (\log q)/\log 2$, by all our error terms.

7 Smooth sums

The study of the linear parts relies on the exact evaluation of smooth sums; we gather this material here.

Lemma 25. *When M and $N > 0$ are real numbers, and a is an integer prime to q, we have*

$$\sum_{\substack{M<n\le M+N,\\ (n,q)=1}} e(na/q) = \frac{\mu(q)N}{q} + \mathcal{O}^*\Big(\frac{\phi(q)+3}{2}\Big).$$

For the proof, split in intervals. By integration by parts, we get

$$\sum_{\substack{n\le N,\\ (n,q)=1}} \log n\, e(na/q) = \sum_{\substack{n\le N,\\ (n,q)=1}} e(na/q)\log N - \int_1^N \sum_{\substack{n\le t,\\ (n,q)=1}} e(na/q)\frac{dt}{t}.$$

We use this formula for $2N$ and N and get By integration by parts, we get

$$\begin{aligned}\sum_{\substack{N<n\le 2N,\\ (n,q)=1}} \log n\, e(na/q) = &\sum_{\substack{n\le N,\\ (n,q)=1}} e(na/q)\log 2\\ &+\sum_{\substack{N<n\le 2N,\\ (n,q)=1}} e(na/q)\log(2N) - \int_N^{2N} \sum_{\substack{n\le t,\\ (n,q)=1}} e(na/q)\frac{dt}{t}\end{aligned}$$

hence

Lemma 26. *When N is a real number, and a is an integer prime to q, we have*

$$\sum_{\substack{N<n\le 2N,\\ (n,q)=1}} \log n\, e(na/q) = \frac{\mu(q)N\log(4N/e)}{q} + \mathcal{O}^*\Big(\frac{\phi(q)+3}{2}\log(8N)\Big).$$

Lemma 27. *When M and $N > 0$ are real numbers, and a is an integer prime to q, we have*

$$\sum_{\substack{M<n\leq M+N,\\ n\equiv b[q]}} \frac{e(\beta n)}{n^{it}} = \frac{1}{q}\int_M^{M+N} \frac{e(\beta v)dv}{v^{it}} + \mathcal{O}\bigl((|t| + |\beta|(M+N)+1)\log(2(M+N))\bigr)$$

and

$$\sum_{\substack{M<n\leq M+N,\\ n\equiv b[q]}} \frac{e(\beta n)\log n}{n^{it}} = \frac{1}{q}\int_M^{M+N} \frac{e(\beta v)\log v}{v^{it}}dv + \mathcal{O}\bigl((|t| + |\beta|(M+N)+1)\log^2(2(M+N))\bigr).$$

Proof. We define $f_1(\alpha,\ell) = e(\beta q\ell)/(\alpha+\ell)^{it}$ and $f_2(\alpha,\ell) = \log(\alpha+\ell)f_1(\alpha,\ell)$ for $\alpha = b/q$ with $1\leq b\leq q-1$ and first study, for $f = f_1$ or $f = f_2$,

$$S(L;f) = \sum_{1\leq \ell\leq L} f(\ell). \tag{40}$$

We have

$$\begin{aligned} S(L;f) &= -\int_1^L [u]f'(u)du + [L]f(L) \\ &= f(1) + \int_1^L f(u)du + \mathcal{O}\Bigl(\int_1^L |f'(u)|du + |f(L)|\Bigr) \\ &= \int_1^L f(u)du + \mathcal{O}((|t| + \beta qL)\log^2(2L)). \end{aligned}$$

We consider

$$\bigl(S((M+N-b)/q;f_1) - S((M-b)/q;f_1)\bigr)\frac{\log q}{q^{it}} + \bigl(S((M+N-b)/q;f_2) - S((M-b)/q;f_2)\bigr)\frac{1}{q^{it}}$$

and, as a consequence, we find that

$$\text{Main Term of} \sum_{\substack{M<n\leq M+N,\\ n\equiv b[q]}} \frac{e(\beta n)\log n}{n^{it}} = \int_{(M-b)/q}^{(M+N-b)/q} \frac{e(\beta(b+uq))\log(b+uq)}{(b+uq)^{it}}du = \frac{1}{q}\int_M^{M+N}\frac{e(\beta v)\log v}{v^{it}}dv.$$

The Lemma follows readily. $\square$

Lemma 28. *When M and $N > 0$ are real numbers, and a is an integer prime to q, we have*

$$\sum_{M<n\le M+N} e(n\beta)\frac{e(an/q)}{n^{it}} = \frac{\mu(q)}{q}\int_M^{M+N}\frac{e(\beta v)dv}{v^{it}} \\ + \mathcal{O}\bigl(q(|t|+|\beta|(M+N)+1)\log(2(M+N))\bigr)$$

and

$$\sum_{M<n\le M+N} e(\beta n)\frac{\log n}{n^{it}}e(an/q) = \frac{\mu(q)}{q}\int_M^{M+N}\frac{e(\beta v)\log v}{v^{it}}dv \\ + \mathcal{O}\bigl(q(|t|+|\beta|(M+N)+1)\log^2(2(M+N))\bigr).$$

Proof. This a simple exercise from the previous Lemma. □

8 The first linear sum

We study here the first linear form defined in (36) and his section is devoted to proving Lemma 29 and 33.

8.1 When $t = \beta = 0$

Lemma 29. *When $Rz^2/q \le N$, we have*

$$\sum_{\substack{r\le R/q,\\ (r,q)=1}} \frac{\mu^2(r)|L_r^{(1)}(a,0,0)|}{\phi(r)} \le \frac{\mu^2(q)X}{\phi(q)}G_q(R/q)\Bigl(2.008\frac{\log\frac{4XR}{eq}}{\log z}+5\Bigr) \\ + 3.3\frac{\phi(q)+3}{2}\frac{R}{q}\frac{z^2+z}{\log z}\log(8X).$$

We start from (36) and sum over n first by using Lemma 26; we find that

$$L_r^{(1)}(a,0,0) = \frac{\mu(q)}{q}X\sum_{\substack{m\le 2X,\\ (m,q)=1}}\frac{h_r(m)\log(4X/(me))}{m} \\ + \mathcal{O}^*\Bigl(\frac{\phi(q)+3}{2}\log(8X)\sum_{\substack{m\le 2X,\\ (m,q)=1}}|h_r(m)|\Bigr).$$

The bound $m\le 2X$ can be replaced in both summations by $m\le rz^2$ since $h_r(m)$ vanishes otherwise. Note also that $rz^2\le 2X$. We set

$$A_r = \sum_{\substack{m\le rz^2,\\ (m,q)=1}}\frac{h_r(m)\log(4X/(me))}{m} = \log\Bigl(\frac{4X}{e}\Bigr)A_r' - A_r'', \tag{41}$$

say. We readily find that

Lemma 30.

$$\frac{A'_r}{\phi(r)} = \sum_{\substack{r|d\le z^2,\\ (d,q)=1}} \frac{\lambda_d^{(1)}}{d}, \quad \frac{A''_r}{\phi(r)} = \sum_{\substack{r|d\le z^2,\\ (d,q)=1}} \log d \frac{\lambda_d^{(1)}}{d} + \sum_{\substack{\ell|r,(r/\ell)|d\le z^2,\\ (d,q\ell)=1}} \frac{\Lambda(\ell)}{\ell-1}\frac{\lambda_d^{(1)}}{d}.$$

Proof.

$$\begin{aligned} A'_r &= \sum_{\substack{u|r,d\le z^2,\\ (ud,q)=1}} \frac{u\mu(r/u)\lambda_d^{(1)}}{[u,d]} \\ &= \sum_{\delta|r}\phi(\delta) \sum_{\substack{\delta|u|r,\delta|d\le z^2,\\ (ud,q)=1}} \frac{\mu(r/u)\lambda_d^{(1)}}{d} = \phi(r) \sum_{\substack{r|d\le z^2,\\ (d,q)=1}} \frac{\lambda_d^{(1)}}{d} \end{aligned}$$

and this settles the case of A'_r. Concerning A''_r, we find that

$$\begin{aligned} A''_r = \sum_{\substack{u|r,d\le z^2,\\ (ud,q)=1}} \frac{u\mu(r/u)\log([u,d])\lambda_d^{(1)}}{[u,d]} &= \sum_{\substack{u|r,d\le z^2,\\ (ud,q)=1}} \frac{u\mu(r/u)\log[u,d]\lambda_d^{(1)}}{[u,d]} \\ = \sum_{\substack{u|r,d\le z^2,\\ (ud,q)=1}} \frac{\mu(r/u)\log(ud)(u,d)\lambda_d^{(1)}}{d} \\ - \sum_{\substack{u|r,d\le z^2,\\ (ud,q)=1}} \frac{\mu(r/u)\log(u,d)(u,d)\lambda_d^{(1)}}{d}. \end{aligned}$$

This calls for the study of two partial quantities, B and C:

$$\begin{aligned} B &= \sum_{\substack{u|r,d\le z^2,\\ (ud,q)=1}} \frac{\mu(r/u)\log(ud)(u,d)\lambda_d^{(1)}}{d} = \sum_{\substack{\delta|u|r,\delta|d\le z^2,\\ (ud,q)=1}} \phi(\delta)\frac{\mu(r/u)\log(ud)\lambda_d^{(1)}}{d} \\ &= \phi(r) \sum_{\substack{r|d\le z^2,\\ (d,q)=1}} \frac{\log d\,\lambda_d^{(1)}}{d} + \sum_{\substack{\delta|r,\delta|d\le z^2,\\ (d,q)=1}} \phi(\delta)\frac{\Lambda(r/\delta)\lambda_d^{(1)}}{d} + \phi(r)\log r \sum_{\substack{r|d\le z^2,\\ (d,q)=1}} \frac{\lambda_d^{(1)}}{d} \end{aligned}$$

i.e. since r and d are squarefree

$$B = \phi(r) \sum_{\substack{r|d\le z^2,\\ (d,q)=1}} \frac{\log d\,\lambda_d^{(1)}}{d} + \phi(r) \sum_{\substack{\ell|r,(r/\ell)|d\le z^2,\\ (d,q)=1}} \frac{\Lambda(\ell)}{\ell-1}\frac{\lambda_d^{(1)}}{d} + \phi(r)\log r \sum_{\substack{r|d\le z^2,\\ (d,q)=1}} \frac{\lambda_d^{(1)}}{d}.$$

The partial quantity C is

$$C = \sum_{\substack{u|r, d\leq z^2,\\ (ud,q)=1}} \frac{\mu(r/u)\log(u,d)(u,d)\lambda_d^{(1)}}{d} = \sum_{\substack{\ell|u|r, \ell|d\leq z^2,\\ (ud,q)=1}} \Lambda(\ell)\frac{\mu(r/u)(u,d)\lambda_d^{(1)}}{d}$$
$$= \sum_{[\delta,\ell]=r} \Lambda(\ell)\phi(\delta) \sum_{r|d\leq z^2} \frac{\lambda_d^{(1)}}{d}.$$

When ℓ is fixed and since r is squarefree, we can have $\delta = r/\ell$ or $\delta = r$, giving the contribution

$$\phi(r/\ell) + \phi(r) = \phi(r)\Big(\frac{1}{\ell-1} + 1\Big).$$

We decompose $\log r = \sum_{\ell|r} \Lambda(\ell)$, so part of the contribution of B cancels out with part of the contribution of C. Hence

$$\begin{aligned} A''_r/\phi(r) &= B/\phi(r) - C/\phi(r) \\ &= \sum_{\substack{r|d\leq z^2,\\ (d,q)=1}} \log d \frac{\lambda_d^{(1)}}{d} + \sum_{\substack{\ell|r, (r/\ell)|d\leq z^2,\\ (d,q)=1}} \frac{\Lambda(\ell)}{\ell-1}\frac{\lambda_d^{(1)}}{d} - \sum_{\ell|r}\frac{\Lambda(\ell)}{\ell-1}\sum_{\ell|d\leq z^2}\frac{\lambda_d^{(1)}}{d} \\ &= \sum_{\substack{r|d\leq z^2,\\ (d,q)=1}} \log d \frac{\lambda_d^{(1)}}{d} + \sum_{\substack{\ell|r, (r/\ell)|d\leq z^2,\\ (d,q\ell)=1}} \frac{\Lambda(\ell)}{\ell-1}\frac{\lambda_d^{(1)}}{d}. \end{aligned}$$

□

We recall [25, Corollary 1.6, Corollary 1.9]:

Lemma 31. *For any real number x and any positive integer r, we have*

$$0 \leq \sum_{\substack{n\leq x,\\ (n,r)=1}} \mu(n)\frac{\log(x/n)}{n} \leq 1.00303 r/\phi(r),$$

and

$$0 \leq \sum_{\substack{n\leq x,\\ (n,r)=1}} \mu(n)\frac{\log^2(x/n)}{n} \leq 2\log x \cdot r/\phi(r).$$

As a consequence, we deduce the following.

Lemma 32. *We have*

$$\Bigg|\sum_{\substack{r'|d\leq z^2,\\ (d,q')=1}} \frac{\lambda_d^{(1)}}{d}\Bigg| \leq \frac{1.004}{r'\log z}\frac{r'q'}{\phi(r'q')},$$

and

$$\Bigg|\sum_{\substack{r|d\le z^2,\\(d,q)=1}}\log\frac{4X}{ed}\frac{\lambda_d^{(1)}}{d}\Bigg|\le\Big(2.008\frac{\log\frac{4X}{e}}{\log z}+5\Big)\frac{q}{\phi(rq)}.$$

Proof. The first estimate is a straightforward. Concerning the second one, we find that

$$\begin{aligned}\sum_{\substack{r|d\le z^2,\\(d,q)=1}}\log\frac{4X}{ed}\frac{\lambda_d^{(1)}}{d}&=\sum_{\substack{r|d\le z^2,\\(d,q)=1}}\log\frac{4X}{ed}\frac{\mu(d)\log\frac{z^2}{d}}{d\log z}-\sum_{\substack{r|d\le z,\\(d,q)=1}}\log\frac{4X}{ed}\frac{\mu(d)\log\frac{z}{d}}{d\log z}\\&=\sum_{\substack{r|d\le z^2,\\(d,q)=1}}\log\frac{4X}{ez^2}\frac{\mu(d)\log\frac{z^2}{d}}{d\log z}+\sum_{\substack{r|d\le z^2,\\(d,q)=1}}\frac{\mu(d)\log^2\frac{z^2}{d}}{d\log z}\\&\quad-\sum_{\substack{r|d\le z,\\(d,q)=1}}\log\frac{4X}{ez}\frac{\mu(d)\log\frac{z}{d}}{d\log z}-\sum_{\substack{r|d\le z,\\(d,q)=1}}\frac{\mu(d)\log^2\frac{z}{d}}{d\log z},\end{aligned}$$

we get

$$\begin{aligned}\Bigg|\sum_{\substack{r|d\le z^2,\\(d,q)=1}}\log\frac{4X}{ed}\frac{\lambda_d^{(1)}}{d}\Bigg|&\le\Big(1.004\frac{\log\frac{4X}{ez^2}}{\log z}+4+1.004\frac{\log\frac{4X}{ez}}{\log z}+4\Big)\frac{q}{\phi(rq)}\\&\le\Big(2.008\frac{\log\frac{4X}{e}}{\log z}+5\Big)\frac{q}{\phi(rq)}.\end{aligned}$$

□

We combine (41) with Lemma 30 to infer that

$$\frac{A_r}{\phi(r)}=\sum_{\substack{r|d\le z^2,\\(d,q)=1}}\log\frac{4X}{ed}\log d\frac{\lambda_d^{(1)}}{d}-\sum_{\substack{\ell|r,(r/\ell)|d\le z^2,\\(d,q\ell)=1}}\frac{\Lambda(\ell)}{\ell-1}\frac{\lambda_d^{(1)}}{d}$$

In this form, Lemma 32 (with $r'=r/\ell$ and $q'=q\ell$) applies directly to yield the bound:

$$|A_r|/\phi(r)\le\Big(2.008\frac{\log\frac{4X}{e}}{\log z}+5\Big)\frac{q}{\phi(rq)}+\sum_{\ell|r}\frac{\Lambda(\ell)}{\ell-1}\frac{1.004}{\log z}\frac{\ell}{r}\frac{rq}{\phi(rq)}.$$

We next sum over r and find that

$$\sum_{\substack{r\le R/q,\\(r,q)=1}}\frac{\mu^2(r)|A_r|}{\phi(r)}\le\frac{q}{\phi(q)}G_q(R/q)\Big(2.008\frac{\log\frac{4X}{e}}{\log z}+5+\frac{2.008\log(R/q)}{\log z}\Big).$$

Regarding the error term, we first notice that

$$\sum_{\substack{m\leq 2X,\\(m,q)=1}} |h_r(m)| = \sum_{u|r, d\leq z^2} |u\mu(r/u)\lambda_d^{(1)}| \leq \frac{z^2+z}{\log z}\prod_{p|r}(p+1)$$

since $\sum_{d\leq y}\log(y/d)\leq y$. And thus

$$\begin{aligned}\sum_{\substack{r\leq R/q,\\(r,q)=1}} \frac{\mu^2(r)\sum_{\substack{m\leq 2X,\\(m,q)=1}}|h_r(m)|}{\phi(r)} &\leq \frac{z^2+z}{\log z}\sum_{\substack{r\leq R/q,\\(r,q)=1}}\mu^2(r)\sum_{\delta|r}\frac{2^{\omega(\delta)}}{\phi(\delta)} \\ &\leq \frac{R}{q}\frac{z^2+z}{\log z}\prod_{p\geq 2}\Big(1+\frac{2}{p(p-1)}\Big) \leq 3.3\,\frac{R}{q}\frac{z^2+z}{\log z}.\end{aligned}$$

The proof of Lemma 29 is complete.

8.2 The general case

Lemma 33. *When $Rz^2/q\leq N$, we have*

$$\begin{aligned}\sum_{\substack{r\leq R/q,\\(r,q)=1}} \frac{\mu^2(r)|L_r^{(1)}(a,t,\beta)|}{\phi(r)} \leq \frac{\mu^2(q)X}{\phi(q)}G_q(R/q)\Big(2.008\frac{\log\frac{4XR}{eq}}{\log z}+5\Big) \\ +\mathcal{O}\Big((|t|+|\beta|X+1)\log^2 X\Big)\frac{Rz^2}{\log z}.\end{aligned}$$

Proof. We start from (36) and sum over n first by using Lemma 28; we find that

$$\begin{aligned}L_r^{(1)}(a,t,\beta) = \frac{\mu(q)}{q}\sum_{\substack{m\leq 2X,\\(m,q)=1}}\frac{h_r(m)}{m^{it}}\int_{X/m}^{2X/m}\frac{e(\beta v)\log v}{v^{it}}dv \\ +\mathcal{O}^*\Big(q(|t|+|\beta|X+1)\log^2 X\sum_{\substack{m\leq 2X,\\(m,q)=1}}|h_r(m)|\Big).\end{aligned}$$

The change of variable $w=vm$ yields:

$$\begin{aligned}L_r^{(1)}(a,t,\beta) = \frac{\mu(q)}{q}\sum_{\substack{m\leq 2X,\\(m,q)=1}}\frac{h_r(m)}{m}\Big(\int_X^{2X}\frac{e(\beta w)\log w}{w^{it}}dw - \int_X^{2X}\frac{e(\beta w)dw}{w^{it}}\log m\Big) \\ +\mathcal{O}\Big(q(|t|+|\beta|X+1)\log^2 X\sum_{\substack{m\leq 2X,\\(m,q)=1}}|h_r(m)|\Big)\end{aligned}$$

so that, with the notation A'_r and A''_r from (41),

$$L_r^{(1)}(a,t,\beta) = \frac{\mu(q)}{q}\int_X^{2X}\frac{e(\beta w)\log w}{w^{it}}dw\,A'_r - \frac{\mu(q)}{q}\int_X^{2X}\frac{e(\beta w)dw}{w^{it}}A''_r$$
$$+\mathcal{O}\Bigl(q(|t|+|\beta|X+1)\log^2 X\sum_{\substack{m\le 2X,\\ (m,q)=1}}|h_r(m)|\Bigr).$$

From then onwards, the treatment of $L_r^{(1)}(a,t,\beta)$ can be mimicked on the one of $L_r^{(1)}(a,0,0)$. We leave the details to the reader. □

9 Study of the second linear form

We study here the seond linear form defined in (37).

Lemma 34.

$$\bigl|L_r^{(2)}(a,0,0)\bigr| \le 1.004\frac{\mu^2(q)qX\log M_0}{\phi(q)^2\log z} + 0.57(\phi(q)+3)\phi_+(r)M_0z^2.$$

Proof. We readily find that

$$L_r^{(2)}(a,t,\beta) = \sum_{\substack{m\le M_0,\\ (m,q)=1}}\frac{\Lambda(m)}{m^{it}}\sum_{\substack{d\le z^2,\\ (d,q)=1}}\lambda_d^{(1)}\sum_{\substack{n\sim X/m,\\ (n,q)=1,\\ d|n}}e(\beta mn)\frac{c_r(n)}{n^{it}}e(mna/q)$$
$$= \sum_{\substack{m\le M_0,\\ (m,q)=1}}\frac{\Lambda(m)}{m^{it}}\sum_{\substack{d\le z^2,\\ (d,q)=1}}\lambda_d^{(1)}\sum_{\ell|r}\ell\mu(r/\ell)\sum_{\substack{n\sim X/m,\\ (n,q)=1,\\ [d,\ell]|n}}e(\beta mn)\frac{e(mna/q)}{n^{it}}.$$

Now we specialize to $t=\beta=0$, getting

$$L_r^{(2)}(a,0,0) = \sum_{\substack{m\le M_0,\\ (m,q)=1}}\Lambda(m)\sum_{\substack{d\le z^2,\\ (d,q)=1}}\lambda_d^{(1)}\sum_{\ell|r}\ell\mu\bigl(\frac{r}{\ell}\bigr)\Bigl(\frac{\mu(q)X}{m[d,\ell]q}+\mathcal{O}^*\bigl(\frac{\phi(q)+3}{2}\bigr)\Bigr)$$
$$= \frac{\mu(q)X\phi(r)}{q}\sum_{\substack{m\le M_0,\\ (m,q)=1}}\frac{\Lambda(m)}{m}\sum_{\substack{d\le z^2,\\ (d,q)=1,\\ r|d}}\frac{\lambda_d^{(1)}}{d} + \mathcal{O}^*\Bigl(0.57(\phi(q)+3)\phi_+(r)M_0z^2\Bigr).$$

We appeal to the first estimate of Lemma 32 (with $r'=r$ and $q'=q$) to get

$$|L_r^{(2)}(a,0,0)| \le \frac{\mu^2(q)1.004\,X\log M_0}{\phi(q)\log z} + 0.57\phi_+(r)(\phi(q)+3)M_0z^2$$

on using (16). The Lemma follows readily. □

Adapting this proof to the general case is not difficult. We get:

Lemma 35.

$$\left|L_r^{(2)}(a,t,\beta)\right| \leq 1.004\frac{\mu^2(q)qX\log M_0}{\phi(q)^2\log z}+\mathcal{O}\big(q(|t|+|\beta|X+1)\phi_+(r)M_0z^2\log X\big).$$

Lemma 36. *We have*

$$\sum_{r\leq R/q}\frac{\mu^2(r)\phi_+(r)}{\phi(r)} \leq 3.28R/q.$$

Proof. The proof is straighforward:

$$\begin{aligned}\sum_{r\leq R/q}\frac{\mu^2(r)\phi_+(r)}{\phi(r)} &\leq \sum_{\ell\leq R/q}\mu^2(\ell)\prod_{p|\ell}\frac{2}{p-1}\sum_{\ell|r\leq R/q}1\\ &\leq \frac{R}{q}\prod_{p\geq 2}\Big(1+\frac{2}{p(p-1)}\Big) \leq 3.28R/q.\end{aligned}$$

□

10 The error term due to the separation of variables

Lemma 37. *When $\varepsilon\in[0,0.16]$, and $R/q\geq 8$, we have*

$$\sum_{\substack{rq\leq R,\\ (r,q)=1}}\frac{\mu^2(r)}{\phi(r)}\sum_{\substack{n\leq B,\\ (n,q)=1}}|v_r(n)|^2/n \leq 0.37\,G_q(R/q)\frac{(e^{2\gamma}Bz^4)^\varepsilon R}{\varepsilon^2 q\log^2 z}\Big(\log\frac{R}{q}+33\Big)\log^2\frac{R}{q}.$$

Proof. We use

$$|c_r(n)|^2 \leq \phi((n,r))^2 = \sum_{\substack{\delta|n,\\ \delta|r}}\delta\phi_2(\delta).$$

Thus, on using (19),

$$\begin{aligned}\sum_{\substack{r\leq R/q,\\ (r,q)=1}}\frac{\mu^2(r)|c_r(n)|^2}{\phi(r)} &= \sum_{\delta|n}\delta\phi_2(\delta)\sum_{\substack{\delta|r\leq R/q,\\ (r,q)=1}}\frac{\mu^2(r)}{\phi(r)}\\ &\leq \sum_{\substack{\delta|n,\\ \delta\leq R/q}}\mu^2(\delta)\phi_2(\delta)G_q(R/q).\end{aligned}$$

On denoting by S the sum to be bounded above, we get

$$\begin{aligned}
S &\leq \sum_{\substack{rq\leq R,\\ (r,q)=1}} \frac{\mu^2(r)}{\phi(r)} \sum_{n\leq B} \Bigl(\sum_{d|n}\lambda_d^{(1)}\Bigr)^2 |c_r(n)|^2/n \\
&\leq G_q(R/q) \sum_{\delta\leq R/q} \mu^2(\delta)\phi_2(\delta) \sum_{\delta|n\leq B} \Bigl(\sum_{d|n}\lambda_d^{(1)}\Bigr)^2/n \\
&\leq G_q(R/q)B^\varepsilon \sum_{\delta\leq R/q} \mu^2(\delta)\phi_2(\delta) \sum_{\delta|n} \Bigl(\sum_{d|n}\lambda_d^{(1)}\Bigr)^2/n^{1+\varepsilon} \\
&\leq G_q(R/q)B^\varepsilon\zeta(1+\varepsilon) \sum_{\delta\leq R/q} \mu^2(\delta)\phi_2(\delta) \sum_{d_1,d_2\leq z^2} \frac{\lambda_{d_1}^{(1)}\lambda_{d_2}^{(1)}}{[\delta,d_1,d_2]^{1+\varepsilon}}.
\end{aligned}$$

We use Selberg's diagonalization process as usual and appeal to Lemma 12 and (11):

$$\begin{aligned}
\frac{S}{G_q(R/q)} &\leq \frac{25(1+\varepsilon)^2(e^{2\gamma}Bz^4)^\varepsilon}{\varepsilon^2\log^2 z} \sum_{\delta\leq R/q} \mu^2(\delta)\frac{\phi_2(\delta)}{\delta^{1+\varepsilon}} \sum_{\ell|\delta}\prod_{p|\ell} \frac{3p^{1+\varepsilon}-2}{p^{1+\varepsilon}-1}. \\
&\leq \frac{25(1+\varepsilon)^2(e^{2\gamma}Bz^4)^\varepsilon}{\varepsilon^2\log^2 z} \sum_{\delta\leq R/q} \mu^2(\delta)\frac{\phi_2(\delta)}{\delta^{1+\varepsilon}} \prod_{p|\delta} \frac{4p^{1+\varepsilon}-3}{p^{1+\varepsilon}-1}.
\end{aligned}$$

Each p-factor is ≤ 4 and decreasing as a function of ε. A use of Lemma 20 gives us

$$S \leq G_q(R/q)\frac{25(1+\varepsilon)^2(e^{2\gamma}Bz^4)^\varepsilon R}{92q\varepsilon^2\log^2 z}\Bigl(\log\frac{R}{q}+33\Bigr)\log^2\frac{R}{q}.$$

The Lemma follows readily. $\square$

11 Proof of the explicit Theorem 6

We continue the argument started in section 5.2.

11.1 Preparation of each $\bigl|S_r(a/q,t,\beta,M,N)\bigr|$

By Lemma 15 first with $A=\max(M,X/(2n))$ and $B=\frac{6}{5}\delta M$, then with $A=\max(M,2^{\delta/2}X/(2n))$ and $B=\frac{6}{5}\delta M$ for some $\delta\geq M^{1/4}$, we find that

$$\begin{aligned}
E_1(\delta,r)+E_2(\delta,r) &\leq 2\Bigl(\frac{\frac{37}{18}\frac{3}{5}\delta M}{\log(3\delta M/5)}+\log\frac{\log M}{\log(3\delta M/5)}\Bigr)\log(2M)\sum_{n\sim N}|v_r(n)| \\
&\leq \tfrac{9}{2}\delta M\sum_{n\sim N}|v_r(n)| \qquad (42)
\end{aligned}$$

when $M \geq 121$. Concerning $E_3(\delta, r)$, we use Lemma 14, getting

$$E_3(\delta, r) \leq \tfrac{6}{5} M \sum_{n \sim N} |v_r(n)|. \tag{43}$$

Note that $\frac{9}{2} + 2 \times \frac{6}{5} \leq 7$. Hence, with $\varpi = -4\pi/\log 2$ and on using the classical $|a+b|^2 \leq 2(|a|^2 + |b|^2)$,

$$\begin{aligned}\bigl|S_r(a/q, t, \beta, M, N)\bigr|^2 \leq 2 \int_{-\Delta}^{\Delta} \sum_{b \bmod^* q} \Biggl| \sum_{\substack{m \equiv b[q],\\ m \sim M}} \frac{\Lambda(m)}{m^{i\varpi v + it}} \Biggr|^2 |\hat{H}(\delta, \lambda, \kappa; v)|^2 dv \\ \times \int_{-\Delta}^{\Delta} \sum_{b \bmod^* q} \Biggl| \sum_{\substack{(n,q)=1,\\ n \sim N}} \frac{v_r(n)}{n^{i\varpi v + it}} e(mna/q) \Biggr|^2 dv + 2 \cdot \Bigl(7\delta M \sum_{n \sim N} |v_r(n)|\Bigr)^2. \end{aligned} \tag{44}$$

By Parseval, we have $\int_{\mathbb{R}} |\hat{H}(\delta, \lambda, \kappa; v)|^2 dv = \int_{\mathbb{R}} |H(\delta, \lambda, \kappa; u)|^2 du = 2 - 2\delta$. We use Lemma 17 provided that $M \geq 121$ and $M \geq q^2$:

$$\begin{aligned}\bigl|S_r(a/q, t, \beta, M, N)\bigr|^2 \leq 2 \times 2\tfrac{27}{4} \frac{M^2}{\phi(q)} \\ \times \int_{-\Delta}^{\Delta} \sum_{b \bmod^* q} \Biggl| \sum_{\substack{(n,q)=1,\\ n \sim N}} \frac{v_r(n)}{n^{i\varpi v + it}} e(nb/q) \Biggr|^2 dv + 98\delta^2 M^2 (N+1) \sum_{n \sim N} |v_r(n)|^2, \end{aligned}$$

i.e.

$$\begin{aligned}\bigl|S_r(a/q, t, \beta, M, N)\bigr|^2 \leq 27 \frac{M^2}{\phi(q)} \int_{-\Delta}^{\Delta} \sum_{b \bmod^* q} \Biggl| \sum_{\substack{(n,q)=1,\\ n \sim N}} \frac{v_r(n)}{n^{i\varpi v + it}} e(nb/q) \Biggr|^2 dv \\ + 196\delta^2 (MN)^2 (1 + N^{-1}) \sum_{n \sim N} \frac{|v_r(n)|^2}{n}. \end{aligned} \tag{45}$$

11.2 Average estimate (over r) of $\bigl|S_r(a/q, t, \beta, M, N)\bigr|$

Let us use the shorter notation

$$\Sigma = \sum_{\substack{rq \leq R,\\ (r,q)=1}} \frac{\mu^2(r)}{\phi(r)} \bigl|S_r(a/q, t, \beta, M, N)\bigr|^2. \tag{46}$$

We sum (45) over r and use the relevant large sieve inequality from Theorem 9 as well as Lemma 37 to get (when $1/\delta \geq 2000$ and since $MN \leq X$

and $1 \le 2N/n$):

$$\Sigma \le 1007 \frac{M^2}{\phi(q)\varpi} 2N \sum_{\substack{(n,q)=1,\\ n\sim N}} \frac{\left(\sum_{d|n} \lambda_d^{(1)}\right)^2}{n} (N + \varpi R^2 \Delta)$$

$$+ 196\delta^2 X^2 (1 + N^{-1}) 0.37\, G_q(R/q) \frac{(e^{2\gamma} 2N z^4)^\varepsilon R}{\varepsilon^2 q \log^2 z} \left(\log \frac{R}{q} + 33\right) \log^2 \frac{R}{q}$$

for any $\varepsilon \in (0, 0.16]$. We take $R = z^{1/4} = \delta^{-1} > q^2$ and $M_0 = z \ge q^8$, with $z \ge 10^{14}$. We assume that

$$|\beta X| \le \delta^{-1}/10. \tag{47}$$

As a conclusion (see (26))

$$\delta^2 \Delta \le \frac{1}{\pi^2} + ((2 + \log 2) 10\delta + \pi^2 (\log 2)/10) \frac{2\log 2}{10} \le 1/5. \tag{48}$$

We select $z = X^{1/4}$ and $\varepsilon = 2/\log X$. We thus have $M/q \ge M^{7/8}$ and $R/q \ge \sqrt{R} = z^{1/8}$. We also forget condition $(n, q) = 1$, getting:

$$\Sigma \le 514 \frac{X^2}{\phi(q)} \sum_{n\sim N} \left(\sum_{d|n} \lambda_d^{(1)}\right)^2 / n + 153\delta \frac{X^2}{q} G_q(R/q) \log^3 X.$$

We sum over M and N and appeal to Lemma 13, getting

$$\Sigma \le 750 \frac{X^2 \log X}{\phi(q)} \frac{25(1+\varepsilon)^2 (e^\gamma X z^4)^\varepsilon}{\varepsilon^2 \log^2 z} + 224\delta \frac{X^2}{q} G_q(R/q) \log^4 X$$

again for any $\varepsilon \in (0, 0.16]$. We select $\varepsilon = 2/\log X$, getting

$$\Sigma \le 3.01 \cdot 10^6 \frac{X^2 \log X}{\phi(q)} + 2 \cdot 10^7 \frac{X^2}{q} G_q(R/q) \le 1.7 \cdot 10^8 \frac{qX^2}{\phi(q)^2} G_q(R/q)$$

since $R/q \ge z^{1/12} = X^{1/48}$, and $G_q(R/q) \ge \frac{\phi(q)/q}{\log}(R/q)$.

11.3 Conclusion of the proof of Theorem 6

We select $z = X^{1/4} \ge q^6$ and assume that $q \ge 10$. We have

$$G_q(R/q) |S(a/q, t, \beta)| \le \sum_{\substack{r \le R/q,\\ (r,q)=1}} \frac{\mu^2(r)}{\phi(r)} (|L_r^{(1)}(a, \beta, t)| + |L_r^{(2)}(a, \beta, t)|)$$

$$+ \sqrt{G_q(R/q)} \left(1.7 \cdot 10^8 \frac{qX^2}{\phi(q)^2} G_q(R/q)\right)^{1/2}$$

This bound is valid for β and t satisfying the hypothesis above. We continue numerically by specializing $t = \beta = 0$. Concerning $L_r^{(1)}$, we use Lemma 29. This means also that we use the hypothesis $Rz^2/q \le X$, but it is implied by our earlier hypothesis (see (34)) $Rz^2 \le X$. We thus get (assuming $q \ge 10$)

$$\begin{aligned}
&\sum_{\substack{r\le R/q,\\(r,q)=1}} \frac{\mu^2(r)}{\phi(r)}|L_r^{(1)}(a,0,0)| \\
&\quad\le \frac{X}{\phi(q)}G_q(R/q)\Big(2.008\frac{\log\frac{4XR}{eq}}{\log z}+5\Big) + 3.3\frac{\phi(q)+3}{2}\frac{R}{q}\frac{z^2+z}{\log z}\log(8X) \\
&\quad\le \frac{X}{\phi(q)}G_q(R/q)\left(2.008\frac{\log\frac{2X^{17/16}}{5e}}{\frac14\log X}+5+\frac{1.7X^{1/24}}{\frac{1}{24}\log X}\frac{X^{9/16}}{\frac14 X\log X}\log(8X)\right) \\
&\qquad\qquad\le \frac{X}{\phi(q)}G_q(R/q)\big(1.05+10^{-20}\big) \le 1.1\frac{X}{\phi(q)}G_q(R/q).
\end{aligned}$$

Concerning $L_r^{(2)}$, we use Lemma 34 and Lemma 36, getting, On another hand, we find that

$$\begin{aligned}
&\sum_{\substack{r\le R/q,\\(r,q)=1}} \frac{\mu^2(r)}{\phi(r)}|L_r^{(2)}(a,0,0)| \\
&\qquad\le G_q(R/q)1.004\frac{\mu^2(q)X\log M_0}{\phi(q)\log z} + 0.57(\phi(q)+3)M_0z^2 3.28R/q \\
&\qquad\quad\le G_q(R/q)\frac{X}{\phi(q)}\left(1.004+1.87\frac{X^{-1/14}}{\frac{1}{32}\log X}\right) \le 1.01\,G_q(R/q)\frac{X}{\phi(q)}.
\end{aligned}$$

This finally amount to:

$$G_q(R/q)|S(a/q,0,0)| \le 3\frac{XG_q(R/q)}{\phi(q)} + \frac{XG_q(R/q)\sqrt{q}}{\phi(q)}\big(1.7\cdot 10^8\big)^{1/2}.$$

We finally get

$$|S(a/q,0,0)| \le 13\,000\frac{\sqrt{q}X}{\phi(q)}$$

provided $10 \le q \le X^{1/48}$.

12 A general estimate: proof of Theorem 1

Proving Theorem 1 is a simple modification of the proof of Theorem 6. The only difficulty being to collect the divers conditions on the parameters R, δ, M_0, z, q and X. We list them below (we simplify them: we neglect the difference from $\phi(q)$ to q, we also impose that $R^2/(z\delta^2) \le 1$ while an upper bound of any constant would also do, and so on):

1. $R \geq 8q$,

2. $M_0, R, z \geq X^\epsilon$,

3. $\delta M_0 \geq M_0^\epsilon$ for the separation of variables,

4. $R^2/(z\delta^2) \leq 1$ for the large sieve argument on the bilinear form,

5. $\delta^2 R \log^4 R \leq 1$ for the large sieve argument on the remainder term coming form the separation of variables,

6. $\delta^{-1} qRz^2 \leq X\sqrt{q}$ for the remainder term linked to $L_r^{(1)}(a, t, \beta)$,

7. $\delta^{-1} qRz^2 M_0 \leq X\sqrt{q}$ for the remainder term linked to $L_r^{(2)}(a, t, \beta)$,

for some $\epsilon > 0$. Condition 6 is a consequence of condition 7. We choose $R = qX^{\epsilon_2}$, $M_0 = \delta^{-1}X^{\epsilon_3}$.

$$q^2 z^{-1} \leq \delta^2 \leq q^2 X^{2\epsilon_2 - \epsilon_1}, \quad \delta^2 q X^{2\epsilon_2} \leq 1, \quad \delta^{-2} z^2 \leq X^{1-\epsilon_1} q^{-3/2}.$$

We choose $z = X^{1/3-\epsilon_4} q^{1/6}$.

$$X^{-1/6} q^{11/12} \leq \delta \leq 1/\sqrt{q}.$$

13 An L^1-estimate: proof of Theorem 2

We rely here on the bound $|\hat{H}(\delta, \lambda, \kappa; v)| \ll (1 + X|\beta|)/(1 + |v|)$ with $\beta = 0$, combined with Corollary 10. This requires a modification at the level of equation (44). here is what we use:

$$\begin{aligned}
\left| \int_{-T}^{T} S_r(a/q, t, \beta, M, N) dt \right|^2 &\ll \delta T M \sum_{n \sim N} |v_r(n)| \\
&+ \int_{-\Delta}^{\Delta} \int_{-T}^{T} \sum_{b \bmod^* q} \left| \sum_{\substack{m \equiv b[q], \\ m \sim M}} \frac{\Lambda(m)}{m^{i\varpi v + it}} \right|^2 |\hat{H}(\delta, \lambda, \kappa; v)| dt dv \\
\times \int_{-\Delta}^{\Delta} \int_{-T}^{T} &\sum_{b \bmod^* q} \left| \sum_{\substack{(n,q)=1, \\ n \sim N}} \frac{v_r(n)}{n^{i\varpi v + it}} e(mna/q) \right|^2 dt |\hat{H}(\delta, \lambda, \kappa; v)| dv. \quad (49)
\end{aligned}$$

We use the variable $w = \varpi v + t$ and the estimate

$$\int_{-\Delta}^{\Delta} \int_{-T}^{T}_{\varpi v + t = w} |\hat{H}(\delta, \lambda, \kappa; v)| dv \ll \log \min(2 + T, \Delta)$$

and from this point onwards, the treatment of the factor containing $v_r(n)$ is as previously. As for the factor containing the Λ-part, we first reduce to prime variables and then use multiplicative characters:

$$\sum_{b \bmod^* q} \Big| \sum_{\substack{m \equiv b[q], \\ m \sim M}} \frac{\Lambda(m)}{m^{iw}} \Big|^2 \le \frac{2}{\phi(q)} \sum_{\chi \quad \bmod q} \Big| \sum_{\substack{(p,q)=1, \\ p \sim M}} \frac{\chi(p) \log p}{p^{iw}} \Big|^2 + \mathcal{O}\Big(\frac{(\log M)^2 M}{q} \Big).$$

The reduction to primitive characters is immediate:

$$\sum_{\chi \quad \bmod q} \Big| \sum_{\substack{(p,q)=1, \\ p \sim M}} \frac{\chi(p) \log p}{p^{iw}} \Big|^2 = \sum_{\mathfrak{f}|q} \sum_{\chi \quad \bmod {}^* \mathfrak{f}} \Big| \sum_{\substack{(m,q)=1, \\ m \sim M}} \frac{\chi(p) \log p}{p^{iw}} \Big|^2.$$

The next step is a way to introduce the Barban-Davenport-Halberstam Theorem via Theorem 7. Here $\mathfrak{f}|q$, and we note that $c_r(p) = \mu(p) = -1$:

$$\sum_{\substack{r \le M/\mathfrak{f}, \\ (r,\mathfrak{f})=1}} \frac{\mu^2(r)}{\phi(r)} \Big| \sum_{\substack{(m,q)=1, \\ m \sim M}} \frac{\chi(p) c_r(p) \log p}{p^{iw}} \Big|^2 = G_{\mathfrak{f}}(M/\mathfrak{f}) \Big| \sum_{\substack{(m,q)=1, \\ m \sim M}} \frac{\chi(p) \log p}{p^{iw}} \Big|^2.$$

We recall (19): $\frac{\mathfrak{f}}{\varphi(\mathfrak{f})} G_{\mathfrak{f}}(M/\mathfrak{f}) \ge G(M/\mathfrak{f})$. We can then appeal to Theorem 8 instead of Lemma 17. Let us summarize the diverse conditions:

1. $R \ge 8q$,
2. $M_0, R, z \ge X^\epsilon$,
3. $\delta M_0 \ge M_0^\epsilon$ for the separation of variables,
4. $TR^2/(\delta^2 z) \le 1$ for the large sieve argument on the bilinear form,
5. $\delta^2 TR \log^4 R \le 1$ for the large sieve argument on the remainder term coming form the separation of variables,
6. $qTRz^2 \le X\sqrt{q}$ for the remainder term linked to $L_r^{(1)}(a,t,\beta)$,
7. $qTRz^2 M_0 \le X\sqrt{q}$ for the remainder term linked to $L_r^{(2)}(a,t,\beta)$,

for some $\epsilon > 0$. Condition 6 is a consequence of condition 7. We choose $R = qX^{\epsilon_2}$, $M_0 = \delta^{-1} X^{\epsilon_3}$, $z = q^2 T^3 X^{\epsilon_4}$.

$$T^{-3} X^{2\epsilon_2 - \epsilon_4} \le \delta^2, \quad \delta^2 q T X^{2\epsilon_2} \le 1, \quad \delta^{-1} \le X^{1-\epsilon_1} T^{-6} q^{-11/2}.$$

i.e.

$$\max\Big(T^{-3/2} q^{-1/2}, X^{-1} T^6 q^{11/2} \Big) \le \delta \le 1/\sqrt{qT}.$$

14 The short interval estimate: proof of Theorem 3

The L^1-estimate opens a clear path to the short interval. Let us start with a methological comment. One can try to compute the Mellin transform of the characteristic function of the interval $[X, X + X^\theta]$ but the lack of continuity results in a transform which is of order $1/s$ when the s goes to $i\infty$. As an implication the integral of the absolute value of this transform over a vertical line is not convergent and this raises complications. One can get a more precise version for the characteristic function of $[1, X]$ that does not rely on the absolute value of the Mellin transform but on more informations on the sequence we consider; this is the truncated Perron's formula. Alternatively, one can consider a smooth sum of the initial charactreristic function, and removing the smoothing depends on short interval estimates, and this is exactly the same information that is required in the truncated Perron's formula.

This means that we get results of similar strength by using the difference of two truncated Perron's formulas. We quote Theorem 7.1 from [22].

Theorem 38 (Truncated Perron's formula)**.** *Let $F(z) = \sum_n u_n/n^z$ be a Dirichlet series that converges absolutely for $\Re z > \kappa_a$, and let $\kappa > 0$ be strictly larger than κ_a. For $x \geq 1$ and $T \geq 1$, we have*

$$\sum_{n\leq x} u_n = \frac{1}{2i\pi}\int_{\kappa-iT}^{\kappa+iT} F(z)\frac{x^z dz}{z} + \mathcal{O}^*\left(\int_{1/T}^{\infty}\sum_{|\operatorname{Log}(x/n)|\leq v}\frac{|u_n|}{n^\kappa}\frac{2x^\kappa dv}{Tv^2}\right).$$

This readily leads to, with $\kappa = 1 + 1/\log X$ and $\omega = X^{\theta-1}$,

$$\begin{aligned}\sum_{X<\ell\leq X+X^\theta}&\Lambda(\ell)\, e(\ell a/q) = \\ &\frac{1}{2i\pi}\int_{1-iT}^{1+iT}\sum_{X<\ell\leq 2X}\frac{\Lambda(\ell)\, e(\ell a/q)}{\ell^z}\frac{X^z((1+\omega)^z-1)dz}{z} \\ &\qquad\qquad + \mathcal{O}\left(\frac{x(\log(xT))^2}{T}\right).\end{aligned}$$

We use the classical $|(1+\omega)^z - 1| \leq z\omega(1+\omega)$ obtained from the representation

$$(1+\omega)^z - 1 = z\int_0^\omega (1+w)^{z-1}dw.$$

The conclusion is straightforward.

15 The trigonometric polynomial of the Moebius function

Adapting the previous argument to the polynomial

$$\tilde{S}(a/q) = \sum_{\ell \sim X} \mu(\ell) e(\ell a/q) \tag{50}$$

and we take the same notations as for the van Mangoldt function, but we add a $\tilde{}$ when they concern the Moebius function. The treatment of the two linear forms carries over with few changes and same result. The treatment of the bilinear form is simpler in one aspect, since we do not need any Brun-Titchmarsh Theorem, but requires more care on one aspect: the condition (ℓ, q). Let us first consider

$$\tilde{\tilde{S}}(a/q) = \sum_{\substack{\ell \sim X, \\ (\ell,q)=1}} \mu(\ell) e(\ell a/q). \tag{51}$$

We have to bound in (44)

$$\sum_{b \bmod^* q} \Big| \sum_{\substack{m \equiv b[q], \\ m \sim M}} \mu(m) \Big|^2. \tag{52}$$

This quantity is $\ll M\phi(q)/q^2$ while we used $\ll M/\phi(q)$ when dealing with primes. As a final output, we multiply the final bound for $\tilde{\tilde{S}}(a/q)$ by $\phi(q)/q$.

Let us now remove this coprimality condition, but it is easier to change the notation and consider the point A/Q instead of a/q. The first process is to write

$$\begin{aligned} \tilde{S}_r(A/Q) &= \sum_{q|Q} \sum_{\substack{\ell \sim X, \\ (\ell,Q)=Q/q}} \mu(\ell) e(\ell A/Q) \\ &= \sum_{q|Q} \mu(Q/q) \sum_{\substack{\ell' \sim Xq/Q, \\ (\ell',Q)=1}} \mu(\ell') e(\ell' A/q). \end{aligned}$$

This means adding a coprimality condition $(\ell', Q) = 1$. The first change is again in (44) which we replace by

$$\sum_{b \bmod^* q} \Big| \sum_{\substack{m \equiv b[q], \\ m \sim M, \\ (m,Q)=1}} \mu(m) \Big|^2 \ll \phi(q) \frac{\phi(Q/q)^2}{(Q/q)^2} \frac{M}{q^2} \ll \frac{\phi(Q)^2}{Q^2} \frac{M^2}{\phi(q)}.$$

This leads to

$$|\tilde{S}_r(A/Q)| \ll \sum_{q|Q} \mu^2(Q/q) \frac{\phi(Q)}{Q} \frac{\sqrt{q} X q/Q}{\phi(q)} \ll \frac{X}{\sqrt{Q}} \prod_{p|Q} \big(1 + p^{-1/2}\big).$$

16 What Gallagher's prime number Theorem yields

Here is a result (essentially due to [8] but which we quote in the formulation given in [3]) and which relies on the Deuring-Heilbronn phenomenom: if an exceptional zero indeed exists, the other L-functions are certified not to vanish in a larger region. This region becomes larger the closer this zero gets to 1. If $L(\beta, \chi_1) = 0$ for some β that satisfies $1 - \beta = o(1/\log q)$, we set $\delta = 1 - \beta$. In that case we have, when b is prime to q,

$$\psi(X; q, b) = \frac{X}{\phi(q)}\left(1 - \chi_1(b)\frac{X^{-\delta}}{\beta}\right) \\ + \mathcal{O}\left(\frac{X\delta \log T}{\phi(q)}\left(\frac{e^{-c_1 \frac{\log X}{\log T}}}{\log X} + \frac{q \log X}{\sqrt{T}} + \frac{T^{5.5}}{\sqrt{X}}\right)\right) \tag{53}$$

for $X \geq T^{c_2} \geq T \geq q$ where $c_1, c_2 > 0$ are two effective constants. The constant implied in the $\mathcal{O}$ is also effective. When an exceptional zero does not exist, the above formula is still valid on putting $\beta = \frac{1}{2}$ in the main term, and $\delta \log T = 1$ in the remainder term. As a consequence, we get

$$S(a/q) = \frac{\mu(q)X}{\phi(q)} - \frac{\chi_1(a)\tau_q(\chi_1)}{\phi(q)}\frac{X^{1-\delta}}{\beta} \\ + \mathcal{O}\left(X\delta \log T\left(\frac{e^{-c_1 \frac{\log X}{\log T}}}{\log X} + \frac{q \log X}{\sqrt{T}} + \frac{T^{5.5}}{\sqrt{X}}\right)\right). \tag{54}$$

We have added an index q to the Gauss sum as χ_1 is not necessarily primitive modulo q. As a consequence, there exist c_3 and c_4 such that when $T = q^{c_4} \log^4 X$ and $X \geq q^{c_3}$, we have

$$S(a/q) = \frac{\mu(q)X}{\phi(q)} - \frac{\chi_1(a)\tau_q(\chi_1)}{\phi(q)}\frac{X^{1-\delta}}{\beta} + \mathcal{O}\big(X/q^{20}\big). \tag{55}$$

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