# Explicit estimates on the summatory functions of the Moebius function with coprimality restrictions

O. Ramaré

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**Abstract**

We prove that $|\sum_{\substack{d\le x,\\ (d,q)=1}}\mu(d)/d|\le 2.02/\operatorname{Log}(x/q)$ for every $x>q\ge 1$ and similar estimates for the Liouville functions. We give also better constants when $x/q$ is larger.

## 1 Introduction

The aim of this note is twofold. We first show how to get explicit estimates for the family of functions

$$m_q(x)=\sum_{\substack{n\le x,\\ (n,q)=1}}\mu(n)/n,\quad m(x)=m_1(x) \tag{1}$$

from explicit estimates concerning solely $m(x)$. And secondly we apply this scheme to prove strong estimates for the sum above with a large range of uniformity and a saving of $1/\operatorname{Log}(x/q)^2$. We proved in [4, Lemma 10.2] and more recently in [9] explicit estimates with a large range of uniformity for sums of the shape

$$\sum_{\substack{d\le x,\\ (d,r)=1}}\frac{\mu(d)}{d^{1+\varepsilon}}$$

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and for a similar sum but with the summand $\mu(d)\operatorname{Log}(x/d)/d^{1+\varepsilon}$. The path we followed there is essentially elementary and the saving is less.

In this problem handling the coprimality condition by Moebius function, i.e. writing

$$\mathbb{1}_{(n,q)=1} = \sum_{\substack{d|q,\\ d|n}} \mu(d),$$

does not work. The classical workaround (used for instance in [7, near (7)]) runs as follows: we determine a function $g$ such that $\mathbb{1}_{(n,q)=1}\mu(n) = g \star \mu(n)$, where $\star$ denotes the arithmetic convolution product. The drawback of this method is that the support of $g$ is not bounded. We propose here a different approach. The Liouville function $\lambda(n)$ (this the completely multiplicative function that is 1 on integers that have an even number of prime factors – counted with multiplicity – and $-1$ otherwise) verifies

$$\sum_{n\ge1} \frac{\lambda(n)}{n^s} = \frac{\zeta(2s)}{\zeta(s)}. \tag{2}$$

We first need to derive estimates for

$$\ell_q(x) = \sum_{\substack{n\le x,\\ (n,q)=1}} \lambda(n)/n, \quad \ell(x) = \ell_1(x). \tag{3}$$

Our process runs as follows: we derive bounds for $\ell(x)$ from bounds on $\mu(x)$ and some computations, derive bounds on $\ell_q(x)$ from bounds on $\ell(x)$, and finally derive bounds on $\mu_q(x)$ from bounds on $\ell_q(x)$. The theoretical steps are contained in the following three lemmas:

**Lemma 1.1** *We have*

$$\ell_q(x) = \sum_{\substack{u^2\le x,\\ (u,q)=1}} m_q\big(x/u^2\big)/u^2. \tag{4}$$

We shall use it only when $q = 1$, but it is equally easy to state it in general.

**Lemma 1.2** *We have*

$$\ell_q(x) = \sum_{d|q} \frac{\mu^2(d)}{d}\ell\big(x/d\big). \tag{5}$$

**Lemma 1.3** *We have*

$$\mu_q(x) = \sum_{\substack{u^2 \le x,\\ (u,q)=1}} \frac{\mu(u)}{u^2} \ell_q\bigl(x/u^2\bigr). \tag{6}$$

On using the initial step provided by [8]

$$|m(x)| \le 0.03/\operatorname{Log} x \quad (x \ge X_0 = 11\,815), \tag{7}$$

this method leads to the following Theorem:

**Theorem 1.1** *We have, when $1 \le q < x$, where $q$ is an integer and $x$ a real number,*

$$\begin{cases} |\ell_q(x)| \le 0.55\, \dfrac{q}{\phi(q)} \Big/ \operatorname{Log}(x/q), \\ |m_q(x)| \le 2.02\, \dfrac{q}{\phi(q)} \Big/ \operatorname{Log}(x/q). \end{cases}$$

*Moreover* $\operatorname{Log}(x/q)|\ell_q(x)| \le 0.155\, \frac{q}{\phi(q)}$ *and* $\operatorname{Log}(x/q)|m_q(x)| \le 1.5 \frac{q}{\phi(q)}$ *when* $x/q \ge 221$. *We also have* $\operatorname{Log}(x/q)|m_q(x)| \le 0.78 \frac{q}{\phi(q)}$ *when* $x/q \ge 663$

The sole previous estimate on $m_q(x)$ seems to be [4, Lemma 10.2] which bounds $|m_q(x)|$ unformly by 1.

I thank Harald Helfgott for interesting discussions that pushed me into pulling this note out of its drawer.

# 2 From the Moebius function to the Liouville function

**Lemma 2.1** *For $2 \le x \le 906\,000\,000$, we have $|\ell(x)| \le 1.347/\sqrt{x}$.*
*For $2 \le x \le 10^{10}$, we have $|\ell(x)| \le 1.41/\sqrt{x}$.*
*For $1 \le x \le 10^{10}$, we have $|\ell(x)| \le \sqrt{2/x}$.*

The computations have been run with PARI/GP (see [10]), speeded by using gp2c as described for instance in [1]. We mention here that [3] proposes an algorithm to compute isolated values of $M(x)$. This can most probably be adapted to compute isolated values of $\ell(x)$, but does not seem to offer any improvement for bounding $|\ell(x)|$ on a large range. In [2], the authors show that

$$\ell(x) \ge 0, \quad (x \le 72\,185\,376\,951\,205)$$

and that

$$\ell(x) \geq -2.0757640 \times 10^{-9}, \quad (x \leq 75\,000\,000\,000\,000)$$

This takes care of the lower bound for $\ell(x)$. The computations we ran are much less demanding in time and algorithm, but however rely on a large enough sieve-kind of table to compute values of $\lambda(n)$ on some very large range. Harald Helfgott (indirectly) pointed out to me that the RAM-memory can be very large nowadays, allowing to precompute large quantities to which one has an almost immediate access.

We compared $|\ell(x)|$ with $1/\sqrt{x}$, and this seems correct for small values, but the works [6] and [5] suggest that the maximal order is larger than that.

**Lemma 2.2** *The function*

$$T(y) : y \mapsto \frac{\operatorname{Log} y}{y} \int_{\sqrt{X_0}}^{y} \frac{dv}{\operatorname{Log} v}$$

*is increasing and then decreasing, reaching a maximum around* 12478.8 *with value* $1.118\,598\,824\,257\,5 + \mathcal{O}^*(10^{-12})$. *Moreover* $T(10^{10}) \leq 1.05$.

*Proof.* This is only the consequence of numerical computations. ◇◇◇

**Lemma 2.3** *For* $x > 1$, *we have* $|\ell(x)| \leq 0.55/\operatorname{Log} x$.
*For* $x \geq 121$, *we have* $|\ell(x)| \leq 0.155/\operatorname{Log} x$.

*Proof.* We appeal to Lemma 1.1 and equation (7) to write, so that for $x/U^2 \geq X_0$

$$|\ell(x)| \leq 0.03 \sum_{u \leq U} \frac{1}{u^2 \operatorname{Log}(x/u^2)} + \frac{1+U^{-1}}{U}$$

We continue by using a comparison with an integral

$$\begin{aligned}
|\ell(x)| &\leq 0.03 \sum_{u \leq U} \frac{1}{u^2 \operatorname{Log}(x/u^2)} + \frac{1+U^{-1}}{U} \\
&\leq \frac{0.03(\pi^2/6)}{\operatorname{Log} x} + \frac{0.03(\pi^2/6)}{\sqrt{x}} \int_{\sqrt{x/U^2}}^{\sqrt{x}} \frac{dv}{2 \operatorname{Log} v} + \frac{1+U^{-1}}{U}. \\
&\leq \frac{0.03(\pi^2/6)}{\operatorname{Log} x} + \frac{0.03(\pi^2/6)}{\sqrt{x}} \int_{\sqrt{X_0}}^{\sqrt{x}} \frac{dv}{2 \operatorname{Log} v} + \frac{1+\sqrt{X_0/x}}{\sqrt{x/X_0}}.
\end{aligned}$$

We employ Lemma 2.2 at this level. Hence, when $x \geq 10^{10}$,

$$\begin{aligned} |\ell(x)| &\leq \frac{0.03(\pi^2/6)}{\operatorname{Log} x} + \frac{0.03(\pi^2/6)\cdot 2\cdot 1.05}{\operatorname{Log} x} + \frac{1+\sqrt{X_0/x}}{\sqrt{x/X_0}} \\ &\leq \frac{0.155}{\operatorname{Log} x}. \end{aligned}$$

We extend it to $x \geq 6500$ via Lemma 2.1, and to $x \geq 221$ by direct inspection. This inequality extends to $x \geq 1$ by weakening the constant 0.155 to 0.55 (indeed $|\ell(x)| \leq 0.55/\operatorname{Log} x$ for $1 \leq x \leq 10^{10}$). ⋄ ⋄ ⋄

### Adding coprimality conditions

The first part of Theorem 1.1 follows immediately by combining Lemma 1.2 together with Lemma 2.3.

## 3 Back to the Moebius function with coprimality coditions

Let us start with a wide ranging estimate:

**Lemma 3.1** *We have, for every integer $q \geq 1$ and every real number $x \geq 1$, $|\ell_q(x)| \leq \pi^2/6$.*

This a direct consequence of Lemma 1.1 and [4, Lemma 10.2].[1]

We proceed to prove the estimate concerning $m_q(x)$. We get, for a real parameter $U$ such that $x > U^2 q$,

$$\begin{aligned} |m_q(x)| &\leq \sum_{u^2\leq x} \frac{1}{u^2}\big|\ell_q\big(x/u^2\big)\big| \\ &\leq \sum_{u\leq U} \frac{q}{\phi(q)} \frac{0.55}{u^2 \operatorname{Log}(x/(u^2 q))} + \frac{\pi^2}{6}\frac{1+U^{-1}}{U}. \end{aligned}$$

### Small values of $x^* = x/q$

We define

$$\rho(U,y) = 0.55 \sum_{u\leq U} \frac{\mu^2(u)}{u^2(1-\frac{2\operatorname{Log} u}{y})} + \frac{\pi^2}{6}\sum_{n>U}\frac{\mu^2(u)}{u^2} y. \tag{8}$$

[1] If we were to adapt the proof presented in [4] to the case of $\lambda$ instead of $\mu$, we would reach the bound 2 and not $\pi^2/6$.

Note that $\rho(U,y)=\rho([U],y)$ where $[U]$ is the integer part of $U$. We want to determine an upper bound for

$$\max_{y>0}\min_{1\le U<\exp(y/2)}\rho(U,y).$$

Here is the GP/Pari (see [10]) script that we have used:

```
{rho(U, y) =
   local(res = 0.0);
   U = floor(U);
   res += 0.55*sum(n=1, U, moebius(n)^2/n^2/(1-2*log(n)/y));
   res += Pi^2/6*y*sum(n=U+1,1000, moebius(n)^2/n^2);
   return(res);}

{rhominloc(U, y) =
   local(res = 10000.0);
   for(n = 1, U, res = min(res, rho(n,y)));
   return(res);}

{rhomin(y) = return(rhominloc(exp(y/2)-0.01,y));}
```

We use this part for $y=\operatorname{Log} x\le 8$. We get a maximum around $y=1.72$ with value $\le 2.0196$. When $x^*\ge 221$, we can single out the term $n=1$ and modify the coefficient 0.55 to 0.155. When $x^*\ge 3\times 221$, we single out the terms of index 1,2, and 3 similarly.

## Large values of $x^*=x/q$

Note that $u\mapsto 1/(u^2\operatorname{Log}(x^*/u^2))$ is non-increasing when $x^*/u^2\ge e$. On assuming $x=eU^2q$, we thus get (with $x^*=x/q$)

$$\begin{aligned}|m_q(x)|&\le\frac{q}{\phi(q)}\frac{0.55}{\operatorname{Log}x^*}+0.55\frac{q}{\phi(q)}\int_1^{\sqrt{x^*/e}}\frac{du}{u^2\operatorname{Log}(x^*/u^2)}+\frac{\pi^2\sqrt{e}}{6}\frac{1+\sqrt{e}x^{*-1/2}}{\sqrt{x^*}}\\&\le\frac{q}{\phi(q)}\frac{0.55}{\operatorname{Log}x^*}+0.55\frac{q}{\phi(q)\sqrt{x^*}}\int_e^{\sqrt{x^*}}\frac{dv}{2\operatorname{Log}v}+\frac{\pi^2\sqrt{e}}{6}\frac{1+\sqrt{e}x^{*-1/2}}{\sqrt{x^*}}\\&\le c(x^*)\frac{q}{\phi(q)\operatorname{Log}x^*}\end{aligned}$$

with

$$c(x^*)=0.55+0.55\frac{\operatorname{Log}x^*}{\sqrt{x^*}}\int_e^{\sqrt{x^*}}\frac{dv}{2\operatorname{Log}v}+\frac{\pi^2\sqrt{e}}{6}\frac{1+\sqrt{e}x^{*-1/2}}{\sqrt{x^*}}\operatorname{Log}x^*.$$

Some numerical work says that the quantity in parentheses is $\leq 1.71$ when $x^* \geq 2\,500$. The modifications required to cover the cases $x^* \geq 221$ and $x^* \geq 3 \times 221$ are immediate.

The proof of Theorem 1.1 is complete.

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