

# An asymptotic seven cubes Theorem <sup>\*†</sup>

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## Abstract

We prove that every integer  $\geq \exp(524)$  is a sum of seven non negative cubes.

## 1 History and statements

In his 1770's "Meditationes Algebraicae", E.Waring asserted that every positive integer is a sum of nine non-negative cubes. A proof was missing, as was fairly usual at the time, the very notion of proof being not so well understood. Notice that here and thereafter, we shall use *cubes* to denote cubes of non-negative integers. Consequently, the integers we want to write as sums of cubes are assumed to be non-negative.

Maillet in [15] proves that twenty-one cubes are enough to represent every (non-negative) integer and later, Wieferich in [30] provides a proof to Waring's statement (though his proof had a leak, mended in [12]). The Göttingen school was in full bloom and Landau [13] shows that eight cubes suffice to represent every *large enough* integer. Dickson [7] improves on this statement by establishing that its only exceptions are 23 and 239. The reader will find in the chapter XXV of [8] a full history of the subject.

Finally, Linnik in [14] shows that every large enough integer is a sum of seven cubes. No further improvements in terms of the number of cubes required are known as of today. Notice that the circle methods readily proves that almost all integers are sums of a most four cubes.

From an experimental and heuristical viewpoint, computations are arguments developed in [2],[27], [16], [1], [6] tend to show that every integer  $\geq 10^{14}$  is a sum of four cubes. [6] provide even a strong argument that leads

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to believe that 7 373 170 279 850 is the last integer that is a sum of five cubes but not a sum of four. When it is required to exhibit a large example of an integer that is a sum of five cubes, but not of four, the simplest example I know is  $10^9 + 4$ .

Similarly, it is believed that 454 is the largest integer that is a sum of eight cubes but no less, that 8 042 is the largest integer that is a sum of seven cubes but of no less and that 1 290 740 plays this role with respect to sums of six cubes.

By density considerations (there are less than  $X^{1/3}$  cubes less than  $X$ ), we see that every integer cannot be a sum of three cubes. The same would follow by studying sums of three cubes in  $\mathbb{Z}/9\mathbb{Z}$ . But nothing prevents sums of three cubes to have positive lower density, a hypothesis believed to be true by Hooley in [11]. There the author proves that this density is strictly less than  $\Gamma(4/3)^3/6$ , this value coming from size considerations only. Hooley even improves on it by taking advantage of some disparities in the distribution of cubes in some arithmetical progressions. The probabilistic models of [5] of sums of 3 cubes supports this conjecture, while giving a density that tends to zero for sums of two squares (a result known to be true by a theorem of Landau), as the model takes more and more local obstructions into account. This is in contrast with a previous probabilistic model (exposed in [10]) of Erdős for such sums that predicted a positive density for sums of three cubes but also for sums of two squares. And of course gives stronger ground to the initial conjecture. The fact is that  $x^3 + y^3 + z^3$  is *not* a norm and in turn, lacks the ensuing multiplicativity.

In [6], it is shown that every integer between 1 290 741 and  $10^{16}$  is a sum of five cubes from which one readily deduces (see [23]) that every integer  $n$  verifying  $455 \leq n \leq \exp(78.7)$  is a sum of seven non-negative cubes. On another side, note that it is proved in [1] that every integer in some special arithmetic progressions, and larger than 455 is indeed a sum of seven cubes.

Our concern in this paper is to show that every integer larger than a given explicit bound is indeed a sum of seven cubes. We prove that

**Theorem 1.1** *Every integer  $n \geq \exp(524)$  is a sum of seven cubes.*

Watson [29] had already simplified greatly Linnik's original proof, though it remained ineffective, and independently, McCurley [16] and Cook [4] corrected this defect. McCurley [16] did even prove a theorem similar to the above though with 1 077 334 instead of 524. The method we use has been partially set in [23] where we had a similar statement but with 205 000 instead of 524. In both approaches, the crux of the method was to show that some arithmetic progression did indeed contain a fairly small prime and the battle lied on the link between the size of the modulus  $q$  involved and the

size of the wanted prime. If the later was of size  $X$ , McCurley's proof needed a modulus of size about  $(\text{Log } X)^{12}$  (barring exceptional moduli) while we required only a modulus of size about  $(\text{Log } X)^6$ . Here, we replace the prime by a product of two well localized primes. We produce moduli  $q$  for which we are sure that more than half of the reduced residue classes contain primes. From which we deduce that every reduced residue class contains a product of two primes. The gain stems from the method used: our main tool is a large sieve extension of the Brun-Titchmarsh inequality and this avoidance of the prime number Theorem in arithmetic progressions explains most of our improvement. This would lead to a lower bound of size about  $\exp(780)$ . An additional gain comes from using the special structure of the moduli we are interested in, namely products of three terms, but we need a digression before pursuing this explanation.

A first remark about the order of magnitude. When working on the lower range, near  $\exp(780)$  say, the product of two primes we are to build should be of size about  $\exp(780/3)/780^4 \simeq \exp(233)$ , and since both primes are going to be of the same size, this latter should be about  $X = \exp(116)$ . On the other side the modulus will be of size  $780^6 \simeq \exp(40)$  which behaves more like  $X^{1/3}$  than like  $\text{Log}^6 X$ . This implies in turn that the prime number theorem in arithmetic progressions is fully unsuitable for such a journey.

Next, the distribution of primes in arithmetic progressions modulo  $q$  often stumbles on the possible existence of so called Siegel zeros, that would have the effect that only about half the residue classes would contain primes. When abording the problem of this distribution through zeros of  $L$ -functions, this effect is well controlled and is avoided by a simple fact: two moduli  $q_1$  and  $q_2$  coprime and not too far apart in size cannot have simultaneously a Siegel zero. The remedy (used in [16] and [4]) is thus to create two moduli, and one of them will be good. The condition of coprimality is not minor in any sense: if  $q$  has a Siegel zero, then the distribution of primes modulo  $3q$  for instance is still going to be unbalanced.

From a sieve point of view, zeros do not appear as such, but a similar role is played by the fact that we can only prove that the number of primes in a given arithmetic progression is about twice what it should be. Indeed, this implies that then primes cannot accumulate on a subset of  $(\mathbb{Z}/q\mathbb{Z})^*$  that would contains less than  $\phi(q)/2$  elements. This can be made accurate if  $q$  is a very small power of a  $X$  (see [9] and [18]). This is only rough philosophy when  $q$  becomes a power of  $X$ . A link between this factor 2 and a possible Siegel zero is also detailed in [19] and in [21].

The fact is that we have a similar effect, even if we are not able to indeed produce a corresponding zero. And, indeed, by using a large sieve extension of the Brun-Titchmarsh inequality, we see that primes cannot accumulate in

some small sets modulo two coprime moduli of similar size. And that the density of the set attained can even be shown to be rather large to one if we are ready to choose one moduli among say  $T$  candidates. Exactly how large depends on the size of the moduli, say  $q$  and of  $T$ , but we can roughly show that more than  $(1 + 2 \log X / (T \log(X/q^2)))^{-1} \phi(q)$  classes are reached and this indeed will be larger than a half if  $T$  is large enough depending on  $\log(X/q^2)$ .

At this level, we reach the mentioned  $\exp(780)$  and there are still some ground to cover. The next idea is to say that the moduli we are interested in are of shape  $u^2 v^2 w^2$ , which  $u$ ,  $v$  and  $w$  of the same size  $B$ . So we can hope for a large sieve inequality for an average on such moduli *once*  $u$  and  $v$  are being fixed. This would replace the  $\log(X/q^2) = \log(X/B^{12})$  by a  $\log(X/B^8)$ . Such an inequality is readily proved via suitable generalization provided section 5, but its use as to give a lower for the number of classes reached finds a hurdle on its path: the possible existence of a Siegel zero modulo  $u^2 v^2$ , or its analogue, a possible Siegel zero effect. To discard this case, we can however apply the same process when fixing  $u$  in  $u^2 v^2$ , meeting a possible Siegel zero effect modulo  $u^2$  and finally avoiding it by applying yet again this process to the moduli  $u^2$ . And this amounts for our final result, when one adds to these main lines some numerical care.

## 2 A modified form of G.L.Watson's lemma

We state and prove a lemma similar to the one used by Watson in [29]. The core identity is however different and is due to E.Bombieri. We still add summands of type  $(a+x)^3 + (a-x)^3$  with a fixed  $a$  to shift the problem from representations by sums of cubes to representations by sums of squares. G.L.Watson's lemma as well as ours relies on the fact that every integer congruent to 3 modulo 8 is a sum of 3 squares, while Yu Linnik introduced coefficients in the resulting ternary quadratic form to encompass all possible residue class. The introduction of the factor  $\gamma$  is the only novelty when compared to the similar lemma proved in [23].

**Lemma 2.1** *Let  $n$ ,  $s$ ,  $u$ ,  $v$  and  $w$  be positive integers,  $t$  a non-negative integer and  $\gamma$  a positive real number. Let us assume that*

- (1)  $1 \leq u < v < w$ , prime.
- (2)  $\gcd(uvw, 6n) = 1$  and  $s$  is odd, (4)  $n - t^3 \equiv 1[2]$ ,
- (3)  $u$ ,  $v$ ,  $w$  and  $s$  are pairwise co- (5)  $n - t^3 \equiv 0[3s]$ ,

$$(6) \begin{cases} 4(n - t^3) \equiv v^6 w^6 s^3 [u^2], \\ 4(n - t^3) \equiv u^6 w^6 s^3 [v^2], \\ 4(n - t^3) \equiv u^6 v^6 s^3 [w^2], \end{cases} \quad (7) \quad \gamma \leq \min(3, 2(w/v)^6, (w/u)^6).$$

Set  $\delta = (1 + (w/u)^6 + (w/v)^6)/4$ . If

$$0 \leq \frac{uv}{6w} \left( \frac{n}{u^6 v^6 s^3} - \delta - \frac{3\gamma}{4} \right)^{1/3} \leq \frac{t}{6uvw s} \leq \frac{uv}{6w} \left( \frac{n}{u^6 v^6 s^3} - \delta \right)^{1/3}$$

then  $n$  is a sum of seven non-negative cubes.

PROOF : Set  $N = 8(n - t^3)$ . Our hypotheses give us the writing

$$N = 2(u^6 v^6 + v^6 w^6 + w^6 u^6) s^3 + 6su^2 v^2 w^2 c$$

where  $c \equiv 3[8]$ . We can then write  $c$  as a sum of 3 non-negative squares:  $c = x^2 + y^2 + z^2$ . We choose  $x, y$  and  $z$  so as to have  $0 \leq x \leq y \leq z$ . This implies that  $x^2 \leq c/3$ , and that  $y^2 \leq c/2$ . Our size condition on  $t$  is also equivalent to

$$0 \leq c \leq \gamma(u^2 v^2 s/w)^2$$

so that, using  $\gamma \leq 3$  and  $x^2 \leq c/3$ , we get

$$0 \leq x \leq u^2 v^2 s/w.$$

Similarly with  $\gamma \leq 2(w/v)^6$  and  $y^2 \leq c/2$ , there comes

$$0 \leq y \leq u^2 w^2 s/v$$

and finally  $\gamma \leq (w/u)^6$  and  $z^2 \leq c$  leads to

$$0 \leq z \leq v^2 w^2 s/u.$$

Next notice than

$$(1) \begin{aligned} & (u^2 v^2 s + wx)^3 + (u^2 v^2 s - wx)^3 \\ & + (u^2 w^2 s + vy)^3 + (u^2 w^2 s - vy)^3 \\ & + (v^2 w^2 s + uz)^3 + (v^2 w^2 s - uz)^3 \\ & = 2(u^6 v^6 + u^6 w^6 + v^6 w^6) s^3 + 6su^2 v^2 w^2 (x^2 + y^2 + z^2) \end{aligned}$$

where the involved cubes are non-negatives due to the upper bound on  $c$ . This gives a writing of  $N$  as 6 non-negatives cubes, all of them even. The lemma follows readily.

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### 3 Using lemma 2.1

Let  $u, v$  and  $w$  be prime numbers  $\equiv 5[6]$  and prime to  $n$ . Let  $\ell$  be a residue class modulo  $u^2v^2w^2$  such that  $\ell^3$  is congruent to  $4n/(v^6w^6)$  modulo  $u^2$ , to  $4n/(u^6w^6)$  modulo  $v^2$ , and to  $4n/(u^6v^6)$  modulo  $w^2$ . This is possible because  $u, v$  and  $w$  being primes  $\equiv 5[6]$ , every invertible residue class modulo  $u^2v^2w^2$  is indeed a cube. Select  $s$  an integer having all its prime factors  $\equiv 5[6]$  and  $\equiv \ell[u^2v^2w^2]$ . Finally select  $t \equiv 0[uvw]$  and so that  $t^3 \equiv n-1[2]$  and  $t^3 \equiv n[6s]$  which can again be achieved because of the condition imposed on the prime factors of  $s$ . It is possible to choose  $t$  in the wanted interval if it contains more than  $6suvw$  integers, which is certainly true if its length is larger than  $6suvw + 1$ . This means

$$(2) \quad \left( \frac{n}{u^6v^6s^3} - \delta \right)^{1/3} - \left( \frac{n}{u^6v^6s^3} - \delta - \frac{3\gamma}{4} \right)^{1/3} \geq \frac{6w}{uv}(1 + \rho)$$

with  $\rho = 1/(6suvw)$ . Before continuing, let us mention that we shall seek  $u, v$  and  $w$  to be as small as possible, and since they are to be coprime with  $n$ , the best we can do is to take them of size  $\text{Log } n$ . This means that  $\delta$  will be about constant in size and  $\rho$  will be very small. Since

$$(3) \quad x^{1/3} - (x - \theta)^{1/3} \geq \theta/(3x^{2/3}) \quad \text{for } x \geq \theta \geq 0$$

it is enough to require  $u^6v^6s^3 \leq n/(\delta + \frac{3\gamma}{4})$  and

$$(4) \quad \gamma \geq 24(1 + \rho) \frac{w}{uv} \left( \frac{n}{u^6v^6s^3} - \delta \right)^{2/3},$$

which reads

$$(5) \quad n^{1/3} / \left( \frac{3\gamma}{4} + \delta \right)^{1/3} \geq su^2v^2 \geq n^{1/3} / \left( \left( \frac{uv\gamma}{24(1 + \rho)w} \right)^{3/2} + \delta \right)^{1/3}.$$

The lower bound being much smaller than the upper bound, the problem is really to find a prime  $s$  in the proper arithmetical progression and of size about and less than  $n^{1/3} / \left( \frac{3\gamma}{4} + \delta \right)^{1/3}$ . Note that in (5), we can replace  $\rho$  by any upper bound (see (4)).

### 4 Creating enough primes $\equiv 5[6]$

**Lemma 4.1** *Let  $B \geq A \geq 1$  be two real numbers. There are more than  $M \geq 1$  prime numbers prime to the integer  $n$  and congruent to  $b$  modulo  $q$  if*

$$\vartheta(B; q, b) - \vartheta(A; q, b) \geq \text{Log } n + M \text{Log } B.$$

PROOF : The product  $P$  of primes  $p \equiv b[q]$  in  $]A, B]$  and dividing  $n$  verifies

$$\text{Log } P \leq \text{Log } n.$$

The condition thus ensures the existence of at least  $M$  other primes in the concerned interval.  $\diamond \diamond \diamond$

We shall only need case  $q = 6$ ,  $b = 5$  and  $y = x$  of the following lemma due to the current author and R.Rumely in [25], but it is no more effort to state it in general.

**Lemma 4.2** *For  $1 \leq x \leq 10^{10}$ , any integer  $q \leq 72$  and any  $b$  prime to  $q$ , we have*

$$\max_{1 \leq y \leq x} \left| \vartheta(y; q, b) - \frac{y}{\phi(q)} \right| \leq 2.072\sqrt{x}.$$

**Lemma 4.3** *There are more than  $M$  prime numbers prime to  $n$  and congruent to 5 modulo 6 lying in the interval  $[\beta \text{Log } n, (\beta + \xi(M, \beta, n_0)) \text{Log } n]$  if  $\text{Log } n$  is larger than  $\text{Log } n_0$ , where  $\xi(M, \beta, n_0)$  is the smallest positive solution of*

$$\frac{\xi - 2}{2} = \frac{2.072(\sqrt{\beta} + \sqrt{\beta + \xi})}{\sqrt{\text{Log } n_0}} + M \frac{\text{Log } \text{Log } n_0 + \text{Log}(\beta + \xi)}{\text{Log } n_0}$$

*provided  $\text{Log } n \leq 10^{10}$ .*

We have  $\xi \geq 2$ .

PROOF : We are to verify the hypothesis of lemma 4.1 with  $q = 6$  and  $b = 5$ . We are to check that

$$\frac{\xi - 2}{2} \geq \frac{2.072(\sqrt{\beta} + \sqrt{\beta + \xi})}{\sqrt{\text{Log } n_0}} + M \frac{\text{Log } \text{Log } n_0 + \text{Log}(\beta + \xi)}{\text{Log } n_0}$$

which is readily done.  $\diamond \diamond \diamond$

When we seek an interval containing primes coprime with  $n$ , we are forced to consider the worst case when all prime factors of  $n$  are indeed in this interval. But in such a case, we could consider a shifted interval: it would this time contain no divisors of  $n$  and would be smaller. Introducing such a dichotomy is however numerically too heavy: we should typically consider whether  $n$  is divisible by the first 200 primes congruent to 5 modulo 6, amounting to  $2^{200}$  cases ... The following lemma is a simple minded way of putting such an idea to practice. We will use it with  $M = 2T$ .

**Lemma 4.4** *Let  $T|M$  be integers and  $\beta > 0$  be a real number. Let  $\alpha = ((2 + \beta + \xi(M, \beta, n_0))/\beta)^{T/M}$ . There exists an interval  $[A, \alpha A]$  with  $A$  in  $[\beta \log n, (\beta + \xi(M, \beta, n_0)) \log n]$  which contains more than  $T$  primes coprime to  $n$  and congruent to 5 modulo 6 if  $\log n$  is larger than  $\log n_0$ .*

PROOF : Set  $A_0 = \beta \log n$ . Among the  $M/T$  intervals  $[\alpha^j A_0, \alpha^{j+1} A_0]$  with  $j \in \{0, 1, \dots, M/T - 1\}$ , one of them contains more than  $T$  primes in the proper congruent class by lemma 4.3. The lemma follows readily.  $\diamond \diamond \diamond$

## 5 General characters in a sieve context

We work here in a general context, which does not cost much more, but enables us to uncover the general lines and to prepare for future uses.

Let  $K$  be a fixed positive integer. Let  $\mathcal{Q}$  be a set of moduli  $K$ -closed under division, by which we mean the following conditions:

1. Every  $q \in \mathcal{Q}$  is divisible by  $K$ .
2. For all  $q \in \mathcal{Q}$  and every positive integer  $\ell$  divisible by  $K$ , if  $\ell|q$  then  $\ell$  belongs to  $\mathcal{Q}$ .

Let also  $\mathcal{K} = (\mathcal{K}_q)_{q \in \mathcal{Q}}$  be a compact set, following here the meaning we have given to this terminology in [26], namely;

1.  $\forall q \in \mathcal{Q}, \mathcal{K}_q \subset \mathbb{Z}/q\mathbb{Z}$ ,
2.  $\forall \ell, q \in \mathcal{Q}$ , with  $\ell|q$ ,  $\mathcal{K}_\ell = \mathcal{K}_q/\ell\mathbb{Z}$ .

We assume further that the Johnsen-Gallagher condition is satisfied, that is to say

$$(JG) \quad \forall \ell, q \in \mathcal{Q}/\ell|q, \quad \text{the number } \sum_{\substack{b \in \mathcal{K}_q \\ b \equiv a[\ell]}} 1 \text{ does not depend on } a \in \mathcal{K}_\ell.$$

Let us note here that similar material is also developed in [24]. In the application we have in mind, we shall take  $\mathcal{K}_q = (\mathbb{Z}/q\mathbb{Z})^*$  the subgroup of invertible elements of  $\mathbb{Z}/q\mathbb{Z}$ . There, all moduli  $q$  will be divisible by a fixed  $K$ ; Since we shall restrict our attention to  $q$ 's such that  $q/K$  and  $K$  are coprime, we could make do by twisting properly usual objects.

We consider next  $\mathcal{F}_q$  the vector space of functions from  $\mathbb{Z}/q\mathbb{Z}$  to  $\mathbb{C}$  that vanish out of  $\mathcal{K}_q$ , which we endow with the hermitian product

$$(6) \quad [f|g]_{\mathcal{K}_q} = \frac{1}{|\mathcal{K}_q|} \sum_{a \in \mathcal{K}_q} f(a) \overline{g(a)}.$$



A definition is required here to clarify our subsequent journey.

**Definition 5.1** A sequence  $(\mathcal{K}_q)_{q \leq Q}$  is said to be an orthonormal system on  $\mathcal{K}$  if

- a. For all  $q \in \mathcal{Q}$ ,  $\mathcal{K}_q \subset \mathcal{F}_q$ .
- b. Let  $\ell$  and  $q$  be both in  $\mathcal{Q}$  with  $\ell|q$  and let  $\chi$  be an element of  $\mathcal{K}_\ell$ . Then  $\tilde{\chi}$  defined by  $\tilde{\chi}(x) = \chi(x + \ell\mathbb{Z})$  if  $x \in \mathcal{K}_q$  and  $\tilde{\chi}(x) = 0$  otherwise, is in  $\mathcal{K}_q$ .
- c.  $\forall (\chi_1, \chi_2) \in \mathcal{K}_q^2$ , we have

$$(7) \quad [\chi_1 | \chi_2]_{\mathcal{K}_q} = \begin{cases} 0 & \text{if } \chi_1 \neq \chi_2, \\ 1 & \text{if } \chi_1 = \chi_2 \end{cases}$$

- d.  $|\mathcal{K}_q| = |\mathcal{K}_\ell|$ .
- e. If  $\chi$  comes (according to (c)) from  $\mathcal{K}_{\ell_1}$  and from  $\mathcal{K}_{\ell_2}$ , then  $\chi$  comes from  $\mathcal{K}_{(\ell_1, \ell_2)}$ , where  $(\ell_1, \ell_2)$  is the gcd of  $\ell_1$  and  $\ell_2$ .

We shall call *characters* the elements of  $\mathcal{K}_q$ , though they are usually linked with no group structure. The notion of *induced* character is natural from (3), while the one of *K-conductor* is simply established from (e). Let  $\mathcal{K}_q^*(K)$  be the set of characters of conductor  $q$ . We shall explain later why it is safe to make the dependence in  $K$  be explicit in our notation.

Condition (e) is more restrictive than it seems and let us give an example where it is not met, while all other conditions will be satisfied. Take  $\mathcal{K}_2(1) = \{1, 2\}$ ,  $\mathcal{K}_3(1) = \{1, 2\}$ , and  $\mathcal{K}_6(1) = \{1, 2\}$ ; In  $\mathcal{K}_2$ ,  $\mathcal{K}_3$  et  $\mathcal{K}_6$ , we put the constant function equal to 1, and of conductor 1, and the function that is 1 on 1 and  $-1$  on 2. This last function is induced by a function modulo 2 and by a function modulo 6 but by no function modulo 1.

Existence of such a system is a problem to which we give a partial answer in a lemma below. When  $\mathcal{K}_q = \mathbb{Z}/q\mathbb{Z}$ , then the additive characters modulo  $q$  ( $n \mapsto e(na/q)$ ) build such a system while if  $\mathcal{K}_q$  is the set of invertible elements, we can take the set of multiplicative characters modulo  $q$  as  $\mathcal{K}$ . In this last example and when  $K = 1$ , the 1-conductor above is the usual conductor. But note that if  $K$  is larger, the  $K$ -conductor  $\mathfrak{f}_K$  is linked with the usual one  $\mathfrak{f}$  by

$$(8) \quad \mathfrak{f}_K = K\mathfrak{f}/(\mathfrak{f}, K).$$

A similar remark holds also for the multiplicative characters. We shall write  $a \bmod_K^* q$  to say that  $a$  ranges the points of  $\mathbb{Z}/q\mathbb{Z}$  such that, when writing

$a/q = b/t$ , with  $(b, t) = 1$ , then  $\text{lcm}(K, t) = [K, t] = q$ . It is clear from these examples at least that the  $K$ -conductor depends strongly on  $K$ , but the reader should be even more wary of this dependence because of *induced system*: usually, we start with a compact set  $\mathcal{K}$  defined by  $\mathcal{K} = (\mathcal{K}_q)_q$  where  $q$  ranges *all* positive integers. We then restrict this system by limiting  $q$  to be in special set  $\mathcal{Q}$  and in particular to be divisible by some  $K$ ; it would then be natural to confuse  $\mathcal{K}_q^*(1)$  and  $\mathcal{K}_q^*(K)$ , a situation our notation prevents.

By (c) and (d),  $\mathcal{K}_q$  is an orthonormal basis of  $\mathcal{F}_q$ . In particular, when  $q \in \mathcal{Q}$  and  $a$  is an integer, the function  $e_{\mathcal{K}_q}(\cdot a/q)$  defined by  $x \mapsto e(xa/q)$  if  $x \in \mathcal{K}_q$  and by  $x \mapsto 0$  otherwise can be written as

$$(9) \quad e_{\mathcal{K}_q}(xa/q) = \sum_{\chi \in \mathcal{K}_q} \left( \frac{1}{|\mathcal{K}_q|} \sum_{k \in \mathcal{K}_q} e(ka/q) \overline{\chi(k)} \right) \chi(x).$$

All this construction has been designed for the following application. Let  $(\varphi_n)_{n \leq N}$  be a sequence of complex numbers such that

$$(10) \quad \forall n \leq N, [a_n \neq 0 \implies \forall q \in \mathcal{Q}, n \in \mathcal{K}_q].$$

We define

$$(11) \quad S(\alpha) = \sum_{n \leq N} \varphi_n e(n\alpha) \quad , \quad (\alpha \in \mathbb{R}/\mathbb{Z})$$

and

$$(12) \quad S(\chi) = \sum_{n \leq N} \varphi_n \chi(n) \quad , \quad (\chi \in \mathcal{K}_q, q \in \mathcal{Q}),$$

distinction between (11) and (12) being made by the context. Let us note that (10) ensures the fundamental equality  $S(\chi) = S(\chi')$  whenever  $\chi$  and  $\chi'$  are induced by a same character.

By (9), we check that

$$(13) \quad \sum_{a \bmod_K^* q} |S(a/q)|^2 = \frac{q}{|\mathcal{K}_q|} \sum_{\chi \in \mathcal{K}_q} |S(\chi)|^2$$

from which we infer

$$\sum_{K|d|q} \sum_{a \bmod_K^* d} |S(a/d)|^2 = \frac{q}{|\mathcal{K}_q|} \sum_{K|f|q} \sum_{\chi \in \mathcal{K}_f^*(K)} |S(\chi)|^2,$$

from which Moebius inversion formula yields

$$(14) \quad \sum_{a \bmod_K^* q} |S(a/q)|^2 = \sum_{K|f|q} \left( \sum_{d|q/f} \mu\left(\frac{q}{df}\right) \frac{df}{|\mathcal{K}_{df}|} \right) \sum_{\chi \in \mathcal{K}_f^*(K)} |S(\chi)|^2.$$

This equation was our main objective here. It turns out that it is one of the main ingredient of Bombieri & Davenport's [3] proof of the Brun-Titchmarsh inequality. Note that we dispense here with the value of the Gauss sums, the reason why we can reach such a generality. Note that usually, the factor of  $\sum_{\chi \in \mathcal{K}_f^*(K)} |S(\chi)|^2$  is non-negative

All that is of no use if no orthonormal system for  $\mathcal{K}$  exist. We now provide a usually satisfied sufficient condition. If  $K = 1$ , our condition is the split multiplicativity introduced in [26], a condition that is inherited by induced system, as defined above, and this would be enough for the application we have in mind. We propose here an intrinsic definition. Let  $\sigma_{q \rightarrow \ell}$  be the canonical surjection from  $\mathbb{Z}/q\mathbb{Z}$  to  $\mathbb{Z}/\ell\mathbb{Z}$  when  $\ell|q$ . We consider the fibered product

$$(15) \quad \mathbb{Z}/q_1\mathbb{Z} \times_K \mathbb{Z}/q_2\mathbb{Z} = \left\{ (x_1, x_2) / \sigma_{q_1 \rightarrow K}(x_1) = \sigma_{q_2 \rightarrow K}(x_2) \right\}$$

whenever  $K$  divides  $q_1$  and  $q_2$  and the somewhat generalized Chinese remainder map

$$(16) \quad \begin{aligned} \rho : \mathbb{Z}/(q_1 q_2 / K)\mathbb{Z} &\rightarrow \mathbb{Z}/q_1\mathbb{Z} \times_K \mathbb{Z}/q_2\mathbb{Z} \\ x &\mapsto (\sigma_{q_1 q_2 / K \rightarrow q_1}(x), \sigma_{q_1 q_2 / K \rightarrow q_2}(x)) \end{aligned}$$

which a ring isomorphism whenever  $(q_1, q_2) = K$  (it is trivially injective and a cardinality argument concludes). Once this is set, we say that  $\mathcal{K}$  is *K-multiplicatively split* if  $\mathcal{K}_{q_1 q_2 / K}$  is isomorphic to  $\mathcal{K}_{q_1} \times_K \mathcal{K}_{q_2}$  via the Chinese remainder map, whenever  $(q_1, q_2) = K$ . In case  $K = 1$ , this only means that  $\mathcal{K}_q$  is isomorphic to the product of the  $\mathcal{K}_{p^\nu}$  for all  $p^\nu \| q$ .

**Theorem 5.1** *If  $\mathcal{K}$  is K-multiplicatively split and verifies the Johnsen-Gallagher condition (JG) then there exists an orthonormal system for  $\mathcal{K}$ . Partially conversely, if an orthonormal system exist then the Johnsen-Gallagher condition is verified.*

PROOF : Let us start with the sufficiency. By  $K$ -split multiplicativity, it is enough to build such a system on  $\mathcal{K}_{p^\nu K}$ . We first select an orthonormal system  $\mathcal{K}_K$  in  $\mathcal{F}_K$ . Let us note that

$$(17) \quad |\mathcal{K}_{p^\nu K}| = \frac{|\mathcal{K}_{pK}|}{|\mathcal{K}_K|} \frac{|\mathcal{K}_{p^2 K}|}{|\mathcal{K}_{pK}|} \cdots \frac{|\mathcal{K}_{p^\nu K}|}{|\mathcal{K}_{p^{\nu-1} K}|}$$

Let us proceed by induction on  $\nu \geq 0$ . Pour  $\nu = 0$ ,  $\mathcal{K}_K$  has already been build (and is thus the same for every  $p$ ). Let us assume that  $\mathcal{K}_{p^{\nu-1}M}$  has been built. We first consider  $\mathcal{K}'$  the set of all pull-backs from  $\mathcal{K}_{p^{\nu-1}M}$  over  $\mathbb{Z}/p^\nu M\mathbb{Z}$ . We have

$$(18) \quad \forall (\chi_1, \chi_2) \in \mathcal{K}', \quad \sum_{a \bmod p^\nu M} \chi_1(a) \overline{\chi_2(a)} = \sum_{b \in \mathcal{K}_{p^{\nu-1}M}} \chi_1(b) \overline{\chi_2(b)} \sum_{\substack{a \in \mathcal{K}_{p^\nu M} \\ a \equiv b \bmod [p^{\nu-1}M]}} 1$$

and  $(JG)$  together with (17) tell us that property (c) is verified on  $\mathcal{K}'$ . It is then enough to complete  $\mathcal{K}'$  in an orthonormal base of  $\mathcal{F}p^\nu M$ .

Conversely, let  $W_d^q(x)$  be the number of points of  $\mathcal{K}_q$  that are congruent to  $x$  modulo  $d$ , when  $d|q$ . Let  $\chi_1$  and  $\chi_2$  be two characters modulo  $d$ . By writing their scalar product modulo  $d$ , we get

$$\sum_{x \in \mathcal{K}_d} W_d^q(x) \chi_1(x) \overline{\chi_2(x)} = \delta_{\chi_1=\chi_2} |\mathcal{K}_q|.$$

However the orthogonality of these characters also gives

$$\sum_{x \in \mathcal{K}_d} \chi_1(x) \overline{\chi_2(x)} = \delta_{\chi_1=\chi_2}$$

from which it is not difficult to conclude that  $W_d^q(x)/|\mathcal{K}_q|$  should be constant and thus equal to  $1/|\mathcal{K}_d|$ . ◇ ◇ ◇

## 6 Products of primes covering every reduced residue class

**Lemma 6.1** *Moreover, for  $X \geq 4$*

$$\sum_{\substack{d \leq X \\ (d,6)=1}} \frac{\mu^2(d)}{\phi(d)} \geq \frac{1}{3} \log X + 0.6.$$

(and  $\frac{1}{3} \log X + 0.463$  if one wants  $X \geq 1$ ).

PROOF : Lemma 3.4 of [22] yields for  $X > 0$

$$\sum_{\substack{d \leq X \\ (d,6)=1}} \frac{\mu^2(d)}{\phi(d)} = \frac{1}{3} \log X + c + \mathcal{O}(4.58X^{-1/3})$$

with

$$c = \frac{1}{3} \left( c_0 + \frac{\text{Log } 2}{2} + \frac{\text{Log } 3}{3} \right) = 0.68178665394 + \mathcal{O}(10^{-10}).$$

This proves our estimate if  $X \geq 10\,000$ . For smaller  $X$  we use the following GP script

```
{res=0.0; cmin=1.0;
for(d=1,1000,
  if (issquarefree(d)&&(d%2==1)&&(d%3!=0),
    res+=1/eulerphi(d),);
  if(d>4,cmin=min(cmin,res-log(d+1)/3),));
cmin}
```

◇ ◇ ◇

**Lemma 6.2** *For  $X > 0$  and  $3.32A \geq X$ , we have*

$$\sum_{\substack{d \leq X \\ (d,6)=1}} \frac{A\mu^2(d)}{(A+d)\phi(d)} \geq \frac{1}{3} \text{Log } X$$

PROOF : This is obviously true if  $X \leq 4$  so we are now only considering case  $X \geq 4$ . Call  $G(X)$  the function of lemma 6.1, or the one we study here but with  $A = \infty$ . With  $c = 0.6$ , we find that

$$\begin{aligned} \sum_{\substack{d \leq X \\ (d,6)=1}} \frac{A\mu^2(d)}{(A+d)\phi(d)} &= \frac{AG(X)}{X+A} + A \int_1^X \frac{G(t)dt}{(t+A)^2} \\ &\geq \frac{Ac}{A+1} + \frac{1}{3} \text{Log} \frac{X(1+A)}{X+A} + A \int_1^4 \frac{(G(t) - \frac{1}{3} \text{Log } t - c)dt}{(t+A)^2}. \end{aligned}$$

When  $t$  varies from 1 to 4,  $G(t)$  is 1, and

$$A \int_1^4 \frac{(1 - \frac{1}{3} \text{Log } t - c)dt}{(t+A)^2} = \frac{3A(1-c)}{(4+A)(1+A)}.$$

The function of  $X$

$$\frac{1}{3} \text{Log} \frac{1+A}{X+A} + \frac{Ac}{A+1} + \frac{3A(1-c)}{(4+A)(1+A)}$$

decreases as  $X$  increases and some numerical analysis tells us it is positive if  $A \geq 10$ . For smaller  $A$  and  $X$ , in fact for  $X \leq 10\,000$ , we use the following script to show our difference to be  $\geq 0.02$

```

{Getc(Amin)=
  exp(log(1+1/Amin)+3*Amin*0.6/(Amin+1)
    +3*Amin*intnum(t=1,4,(0.4-log(t)/3)/(t+Amin)))-1}

{f(X)=
  local(res = 0.0, d);
  for(d = 1, X, if((moebius(d) != 0)&&(gcd(d,6) == 1),
    res += 1/eulerphi(d)/(1+3.32*d/X),));
  return(res)}

{Check(borneX)=
  local(X, res = 1000);
  forstep(X = borneX, 1, -1,
    res = min(res, f(X)-log(X+1)/3);
    if(res > 0, print1("."), print(X, " gives troubles")));
  return(res)}

```

◇ ◇ ◇

**Theorem 6.1** *Let  $(k_i)_{i \in I}$  be a set of moduli, all of them divisible by  $K$ , prime to 6. We further assume that the  $k_i/K$  are two by two coprime. Let  $\mathcal{P}$  be a set of prime numbers all not more than  $X'$ , contained in an interval of length  $X > K$ , and containing only primes congruent to 5 modulo 6 and prime to all  $k_i$ 's. For every modulus  $k$ , we set*

$$\Delta(k, K) = \phi(k) \sum_{a \bmod *k} \left| \sum_{\substack{p \equiv a[k], \\ p \in \mathcal{P}}} 1 \right|^2 - \phi(K) \sum_{a \bmod *K} \left| \sum_{\substack{p \equiv a[K], \\ p \in \mathcal{P}}} 1 \right|^2$$

and  $\Theta = \sum_{p \in \mathcal{P}} \text{Log } p$ ; Then

$$\sum_i \left( 1 - \frac{2 \text{Log}(k_i/K)}{\text{Log}(X/K) - 1} \right) \Delta(k_i, K) / \Theta^2 \leq \frac{(X+6) \text{Log } X'}{\Theta(\text{Log}(X/K) - 1)} - 1.$$

Note that in the proof we shall require condition  $k_i \leq \frac{7}{11} \sqrt{XK}$  but that it is superfluous in the statement, for otherwise the coefficient of  $\Delta(k_i, K)$  is  $\leq 0$ .

PROOF : For typographical simplicity, let us introduce the function  $f$  on primes that is  $\text{Log } p$  if  $p \in \mathcal{P}$  and 0 otherwise. We set

$$V(q) = \sum_{a \bmod *q} \left| \sum_p f(p) e(ap/q) \right|^2.$$

The weighted large sieve inequality of Montgomery & Vaughan in [17] as improved by [20] gives us

$$\sum_{\substack{q \leq Q \\ (q,6)=1}} \frac{V(q)}{N + \rho K q Q} \leq \sum_{p \in \mathcal{P}} \text{Log}^2 p$$

where  $N$  is a real number that majorizes the number of integers  $\equiv 5[6]$  in the interval that contains  $\mathcal{P}$ ; and  $\rho = \sqrt{1 + \frac{2}{3}\sqrt{6/5}}$ . Let us further set

$$W_K(q) = \sum_{\chi \bmod_K^* Kq} \left| \sum_p f(p) \chi(p) \right|^2$$

and (14) gives us

$$V(q) = \sum_{\substack{d|q, \\ (q/d, Kd)=1}} \frac{\mu^2(q/d)}{\phi(q/d)} \frac{d}{\phi(d)} W_K(d)$$

so that

$$\sum_{\substack{d \leq Q \\ (d,6)=1}} \tilde{G}_K(d, Q) W(d) \leq \Theta \text{Log } X'$$

with

$$\tilde{G}_K(d, Q) = \frac{Kd}{\phi(Kd)} \sum_{\substack{q \leq Q/d \\ (q, 6Kd)=1}} \frac{\mu^2(q)}{(N + \rho K Q d q) \phi(q)},$$

We follow the classical treatment of [28] to write

$$\sum_{\delta|dK} \sum_{\substack{q \leq Q/d \\ (q, 6dK)=\delta}} \frac{\mu^2(q)}{(N + \rho K Q d q) \phi(q)} \leq \sum_{\delta|dK} \frac{\mu^2(\delta)}{\phi(\delta)} \sum_{\substack{q \leq Q/d \\ (q, 6dK)=1}} \frac{\mu^2(q)}{(N + \rho K Q d q) \phi(q)},$$

so that

$$\begin{aligned} \tilde{G}_K(d, Q) &\geq \sum_{\substack{q \leq Q/d \\ (q,6)=1}} \frac{\mu^2(q)}{(N + \rho K Q d q) \phi(q)} = \frac{1}{N} \sum_{\substack{q \leq Q/d \\ (q,6)=1}} \frac{\mu^2(q) N / (\rho K d Q)}{(N / (\rho K d Q) + q) \phi(q)} \\ &\geq \frac{\text{Log}(Q/d)}{3N} \end{aligned}$$

provided  $KQ^2 \leq 3.32N/\rho$  et  $d \leq Q$ . Hence

$$\sum_{\substack{d \leq Q \\ (d,6)=1}} \text{Log}(Q/d)W_K(d) \leq 3N\Theta \text{Log } X'.$$

At last note that for any  $k$

$$\sum_{d|k/K} W_K(d) = \phi(k) \sum_{a \bmod *k} \left| \sum_{p \equiv a[k]} f(p) \right|^2 \geq \Theta^2.$$

We now are to get rid of  $N$  and  $Q$ . We can take  $N = (X + 6)/6$  and  $Q = \sqrt{3.32N/(\rho K)} \geq \frac{7}{11}\sqrt{X/K}$ , which yields the result.  $\diamond \diamond \diamond$

We are to observe that a minimization argument readily yields

$$(19) \quad \Delta(k, 1) \geq \Theta^2 \left( \frac{\phi(k)}{|\mathcal{A}(k)|} - 1 \right).$$

We take  $K = 1$  and deduce that for one  $i$ , we have

$$(20) \quad |I| \left( 1 - \frac{2 \text{Log } k_i}{\text{Log } X - 1} \right) \left( \frac{\phi(k_i)}{|\mathcal{A}(k_i)|} - 1 \right) \leq \frac{X + 6}{\sum_p \text{Log } p} \frac{\text{Log } X'}{\text{Log } X - 1} - 1.$$

## 7 A first simple approach for large $n$ 's

We are first to note that we will take  $w$  as upper bounds for the  $u$  and the  $v$  that appear on the denominator of  $Y$ . Which means that we should more write it under the form

$$Y = \frac{n^{1/3}}{w^4 \left( \frac{3u^6 v^6 \gamma}{4w^{12}} + \frac{u^6 v^6}{w^{12}} \delta \right)^{1/3}}.$$

Since  $3u^6 v^6 \gamma / 4w^{12}$  and

$$(21) \quad \frac{u^6 v^6 \delta}{w^{12}} = \frac{1}{4} \left( \frac{u^6 v^6}{w^{12}} + \frac{u^6}{w^6} + \frac{v^6}{w^6} \right)$$

are both largest when  $u/w$  and  $v/w$  are smallest, we have

$$(22) \quad Y \geq \frac{n^{1/3}}{(3(\gamma + 1)/4)^{1/3} B_1^4} = Y'.$$



To derive a lower bound, we write the quantity  $Y/\kappa$  under the form

$$\begin{aligned} & \frac{n^{1/3}}{u^4 \left( \left( \frac{v^5 \gamma}{24(1+\rho')u^3 w} \right)^{3/2} + ((v/u)^6 + (vw/u^2)^6 + (w/u)^6)/4 \right)^{1/3}}, \\ & \leq \frac{n^{1/3}}{u^4 \left( \left( \frac{u\gamma}{24(1+\rho')} \right)^{3/2} + \frac{3}{4} \right)^{1/3}}. \end{aligned}$$

We set

$$\kappa' = \alpha^{-4}((C+1)/(\gamma+1))^{1/3}, \quad C = (2^{1/3}A^-(3(1+\rho'))^{3/2}$$

and are to find our product of two primes in  $[Y'/\kappa', Y']$ . We anticipate over the final result: we have not been able to produce any noticeable improvement by acting on the parameter  $\gamma$ . The trivial value  $\gamma = 1$  yields results that are only slightly worse, but simplifies scripts quite a lot. Thus, from now on, we shall only consider this case. A final note: one could try to select special sets of  $u, v$  and  $w$  were this parameter could be taken appreciably larger than 1; in the present case,  $n$  is still too large and the sets considered too big.

## 8 Base final argument

We write  $s = \hat{s}\check{s}$  with

$$\hat{s}, \check{s} \in [\sqrt{Y'/\kappa'}, \sqrt{Y'}].$$

We know that

$$(23) \quad \vartheta(X; 6, 5) = \frac{X}{2}(1 + \mathcal{O}^*(0.0023)) \quad (X \geq 612\,477)$$

from [25]: directly from the table there for  $X \geq 10^{10}$  and then by using the bounds for the restricted range ( $\leq 1.8/\sqrt{X_0}$ ).

Remember we are sure to have  $T$  primes congruent to 5 modulo 6 and prime to  $n$  in an interval  $[A, \alpha A]$ . Once this is set, the values of  $Y'$  and  $Y'/\kappa'$  are defined, and we are simply to select proper  $u, v$  and  $w$ . We assume  $T$  is divisible by 3 and group them 3 by 3. This gives us  $T/3$  moduli  $k = u^2 v^2 w^2$ , all of them prime to the other. The number of classes reached is at least for one of the  $k$ 's

$$(24) \quad \frac{|\mathcal{A}|}{\phi(k)} \geq \left( 1 + \frac{\frac{L+6}{\Theta} \frac{\frac{1}{2} \log Y'}{\log L-1} - 1}{T/3 \left( 1 - \frac{12 \log B}{\log L-1} \right)} \right)^{-1}$$

where  $L = \sqrt{Y'}(1 - \sqrt{1/\kappa'}) + 1$  and  $\Theta = \sqrt{Y'}(\frac{1}{2}(1 - \sqrt{1/\kappa'}) - \varepsilon(1 + \sqrt{1/\kappa'})) - \log n$  with  $\varepsilon = 0.0023$ . We need this density to be at least  $1/2$ , which translates into

$$(25) \quad \frac{L + 6}{\Theta} \frac{\frac{1}{2} \log Y'}{\log L - 1} - 1 < \frac{T}{3} \left( 1 - \frac{12 \log B}{\log L - 1} \right)$$

We show this way that integers larger than  $\exp(780)$  are sums of seven cubes. Parameters:  $\beta = 1.9$ ,  $T = 51$  and  $M = 2T$ .

## 9 Refinement number I

We now assume  $\log n$  to be not more than 750. Using  $\beta \log n = A^-$  as a lower bound, we find  $B$  such that the interval  $[A^-, B]$  contains more than  $M$  primes congruent to 5 modulo 6 and prime to  $n$ , and find a subinterval that contains  $T$  such primes. This step only differs from the previous one by the fact that we compute exactly  $B$ . Then we can build  $Y'$  and  $Y'/\kappa'$  as before but this time select  $u$ ,  $v$  and  $w$  more shrewdly. Select first  $t_1$  in this set such that

$$(26) \quad \Delta(t_1^2, 1) \leq \frac{D - \log X + 1}{T(\log X - 1 - 4 \log B)}.$$

with  $D = (X + 6) \log(X)/\Theta$ . Having this  $t_1$ , we select  $t_2$  again in this set but distinct from  $t_1$  and such that

$$\Delta(t_1^2 t_2^2, 1) \leq \frac{D - \log X + 2 \log B + 1}{(T - 1)(\log X - 1 - 6 \log B)} + \frac{D - \log X + 1}{T(\log X - 1 - 4 \log B)}$$

And finally, we select  $t_3$  again in this set and distinct from  $t_1$  and  $t_2$  and such that

$$(27) \quad \begin{aligned} \Delta(t_1^2 t_2^2 t_3^2, 1) &\leq \frac{D - \log X + 4 \log B + 1}{(T - 2)(\log X - 1 - 8 \log B)} \\ &\quad + \frac{D - \log X + 2 \log B + 1}{(T - 1)(\log X - 1 - 6 \log B)} \\ &\quad + \frac{D - \log X + 1}{T(\log X - 1 - 4 \log B)} \end{aligned}$$

We simply reorder  $t_1$ ,  $t_2$  and  $t_3$  to get  $u$ ,  $v$  and  $w$  and use again (19) to get a lower bound for the number of classes attained.

We show this way that integers larger than  $\exp(536)$  are sums of seven cubes by selecting the parameters as follows:  $\beta = 2.2$ ,  $T = 33$  and  $M = 2T$ .

## 10 Refinement number II

For small values of  $n$ , we compute  $\xi$  exactly and we reach  $\text{Log } n \geq 524$  by selecting the parameters as follows:  $\beta = 1.79$ ,  $T = 34$ ,  $M = 2T$ .

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