

SOME EULER PRODUCTS WITH HIGH ACCURACY

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ABSTRACT. (File `SomeEulerProducts-01.tex`)

We proposed in [1] an algorithm to compute some Euler products with high precision. Here are some examples for $s = 2$ and small q 's. We decided to produce 100 decimal digits each time. Each computation took at most five seconds and we selected uniformly $P = 100$.

Modulo 3.

$$\begin{aligned} \prod_{p \equiv 1[3]} (1 - p^{-2})^{-1} &= 1.03401\,48754\,14341\,88053\,90306\,44413\,04762\,85789\,65428\,48909 \\ &\quad 98864\,16825\,03842\,12222\,45871\,09635\,80496\,21707\,98262\,05962 \dots \\ \prod_{p \equiv 2[3]} (1 - p^{-2})^{-1} &= 1.41406\,43908\,92147\,63756\,55018\,19079\,82937\,99076\,95069\,39316 \\ &\quad 21750\,39924\,96242\,39281\,06992\,08849\,94537\,54858\,50247\,51141 \dots \end{aligned}$$

Modulo 4.

$$\begin{aligned} \prod_{p \equiv 1[4]} (1 - p^{-2})^{-1} &= 1.05618\,21217\,26816\,14173\,79307\,65316\,21989\,05875\,80425\,46070 \\ &\quad 80120\,04306\,19830\,27928\,16062\,22693\,04895\,12958\,37291\,59718 \dots \\ \prod_{p \equiv 3[4]} (1 - p^{-2})^{-1} &= 1.16807\,55854\,10514\,28866\,96967\,37064\,04040\,13646\,79021\,45554 \\ &\quad 79928\,40563\,68111\,38106\,59377\,71094\,66904\,07472\,79588\,48702 \dots \end{aligned}$$

Modulo 5.

$$\begin{aligned} \prod_{p \equiv 1[5]} (1 - p^{-2})^{-1} &= 1.01091\,51606\,01019\,52260\,49565\,84289\,51492\,09845\,38627\,58173 \\ &\quad 85237\,32024\,20089\,25161\,37424\,56726\,37093\,96197\,69455\,89218 \dots \\ \prod_{p \equiv 2,3[5]} (1 - p^{-2})^{-1} &= 1.55437\,60727\,20889\,22081\,75902\,82565\,55177\,56056\,30147\,34257 \\ &\quad 40072\,50077\,94457\,39239\,00871\,38641\,44091\,80733\,87878\,70683 \dots \\ \prod_{p \equiv 4[5]} (1 - p^{-2})^{-1} &= 1.00496\,03239\,22297\,55899\,37496\,24810\,25218\,47955\,10294\,18802 \\ &\quad 28801\,99528\,37852\,15071\,27700\,70076\,98854\,32491\,36118\,00619 \dots \end{aligned}$$

Modulo 7.

$$\begin{aligned} \prod_{p \equiv 1[7]} (1 - p^{-2})^{-1} &= 1.00222\,95338\,19740\,42627\,18641\,59138\,22019\,24486\,37565\,40128 \\ &\quad 87922\,82973\,79678\,21741\,90308\,08041\,42707\,36575\,28295\,76151 \dots \\ \prod_{p \equiv 2,4[7]} (1 - p^{-2})^{-1} &= 1.34984\,62543\,65273\,20787\,74772\,44978\,62277\,76508\,69021\,24860 \\ &\quad 12031\,69999\,35719\,21654\,93824\,75777\,02051\,36300\,53459\,76601 \dots \\ \prod_{p \equiv 3,5[7]} (1 - p^{-2})^{-1} &= 1.18274\,26007\,67364\,09208\,00286\,83933\,15918\,51718\,05360\,46335 \\ &\quad 82633\,06344\,66854\,90324\,90537\,21799\,81486\,90001\,86365\,91391 \dots \\ \prod_{p \equiv 6[7]} (1 - p^{-2})^{-1} &= 1.00705\,20326\,03074\,04805\,67193\,52428\,88870\,69289\,36714\,73687 \\ &\quad 58335\,65893\,11634\,74829\,60947\,12069\,41243\,26265\,99553\,53536 \dots \end{aligned}$$

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Modulo 8.

$$\begin{aligned}
\prod_{p \equiv 1[8]} (1 - p^{-2})^{-1} &= 1.00483\,50650\,34191\,18711\,83598\,31169\,10411\,95979\,07317\,54340 \\
&\quad 88789\,55156\,06711\,74639\,62051\,31056\,35207\,32105\,88068\,58783 \dots \\
\prod_{p \equiv 3[8]} (1 - p^{-2})^{-1} &= 1.13941\,87771\,08211\,51502\,70589\,30773\,34020\,88725\,59961\,09629 \\
&\quad 48302\,25821\,27411\,02101\,65577\,60742\,91446\,59374\,91512\,33349 \dots \\
\prod_{p \equiv 5[8]} (1 - p^{-2})^{-1} &= 1.05109\,99849\,42183\,30793\,68775\,56006\,33505\,68012\,01018\,45817 \\
&\quad 85080\,59912\,94207\,39729\,30485\,58783\,38889\,50479\,59255\,34495 \dots \\
\prod_{p \equiv 7[8]} (1 - p^{-2})^{-1} &= 1.02515\,03739\,25759\,17991\,61954\,35560\,94158\,79433\,11002\,76024 \\
&\quad 41530\,69566\,94982\,17644\,97960\,41007\,90076\,26943\,14236\,43529 \dots
\end{aligned}$$

Modulo 9.

$$\begin{aligned}
\prod_{p \equiv 1[9]} (1 - p^{-2})^{-1} &= 1.00403\,38350\,51288\,79798\,24781\,19924\,74748\,94825\,22895\,79877 \\
&\quad 28822\,86701\,42359\,63409\,37977\,93839\,33608\,94316\,94860\,37141 \dots \\
\prod_{p \equiv 2,5[9]} (1 - p^{-2})^{-1} &= 1.40783\,70719\,96538\,05093\,52684\,03433\,79823\,18382\,56159\,80878 \\
&\quad 18858\,21039\,93308\,74959\,08486\,21687\,68292\,75777\,90984\,34896 \dots \\
\prod_{p \equiv 4,7[9]} (1 - p^{-2})^{-1} &= 1.02986\,05876\,77826\,18491\,88642\,35135\,21663\,16312\,01666\,87293 \\
&\quad 15881\,63094\,56123\,55333\,65628\,89969\,28513\,96515\,60005\,36245 \dots \\
\prod_{p \equiv 8[9]} (1 - p^{-2})^{-1} &= 1.00442\,33235\,64550\,15978\,66082\,58390\,58205\,39661\,19672\,30788 \\
&\quad 17744\,79626\,23017\,18753\,96410\,76663\,34579\,95134\,16501\,66760 \dots
\end{aligned}$$

Modulo 11.

$$\begin{aligned}
\prod_{p \equiv 1[11]} (1 - p^{-2})^{-1} &= 1.00232\,82408\,97736\,52733\,78057\,92469\,42582\,04345\,78064\,14879 \\
&\quad 23124\,99895\,44150\,38255\,72926\,07516\,98484\,87460\,03110\,08712 \dots \\
\prod_{p \equiv 2,6,7,8[11]} (1 - p^{-2})^{-1} &= 1.38240\,11448\,05788\,71773\,39824\,35954\,70441\,91351\,16435\,84157 \\
&\quad 13863\,06101\,70250\,01900\,59181\,34321\,25138\,72741\,06748\,64687 \dots \\
\prod_{p \equiv 3,4,5,9[11]} (1 - p^{-2})^{-1} &= 1.17640\,19224\,41514\,71776\,56838\,81699\,54785\,03151\,42210\,45715 \\
&\quad 72819\,38133\,44304\,81040\,93008\,74341\,67383\,61950\,21979\,26318 \dots \\
\prod_{p \equiv 10[11]} (1 - p^{-2})^{-1} &= 1.00079\,37707\,14740\,00680\,22327\,79981\,38075\,30993\,79972\,81556 \\
&\quad 86828\,01966\,59824\,89326\,65924\,56171\,20791\,11742\,28212\,98769 \dots
\end{aligned}$$

Modulo 12.

$$\begin{aligned}
\prod_{p \equiv 1[12]} (1 - p^{-2})^{-1} &= 1.00761\,32452\,14144\,96616\,93493\,12247\,73229\,37895\,47142\,90433 \\
&\quad 17666\,43368\,44819\,49208\,97861\,01855\,78530\,60579\,11129\,80649 \dots \\
\prod_{p \equiv 5[12]} (1 - p^{-2})^{-1} &= 1.04820\,19036\,00769\,93683\,49374\,34895\,79267\,34804\,13674\,49481 \\
&\quad 52581\,07376\,14495\,24161\,71571\,43788\,23594\,04990\,88566\,94968 \dots \\
\prod_{p \equiv 7[12]} (1 - p^{-2})^{-1} &= 1.02620\,21468\,31233\,70070\,72018\,66966\,36157\,23611\,09321\,31334 \\
&\quad 95148\,10400\,66496\,54603\,29393\,86454\,19299\,91782\,63867\,91609 \dots \\
\prod_{p \equiv 11[12]} (1 - p^{-2})^{-1} &= 1.01177\,86368\,50332\,58370\,51194\,10267\,33127\,80584\,01230\,89520 \\
&\quad 87028\,35959\,40756\,15016\,41704\,56300\,54442\,19591\,32980\,62727 \dots
\end{aligned}$$

Modulo 13.

$$\prod_{p \equiv 1[13]} (1 - p^{-2})^{-1} = 1.00065\ 68661\ 98289\ 66605\ 74722\ 84730\ 77197\ 91777\ 00717\ 07399 \\ 33554\ 44837\ 12988\ 36602\ 52536\ 84343\ 79642\ 73590\ 88077\ 31673 \dots$$

$$\prod_{p \equiv 2,6,7,11[5]} (1 - p^{-2})^{-1} = 1.38005\ 21671\ 19142\ 93623\ 73358\ 95833\ 59312\ 88490\ 63922\ 76216 \\ 00813\ 27801\ 96170\ 83570\ 07037\ 00666\ 02382\ 19997\ 07055\ 85939 \dots$$

$$\prod_{p \equiv 3,9[13]} (1 - p^{-2})^{-1} = 1.12706\ 12738\ 77030\ 37596\ 05291\ 90459\ 70008\ 03562\ 53668\ 12081 \\ 48604\ 51380\ 13290\ 89754\ 69987\ 12664\ 24897\ 64722\ 52303\ 29593 \dots$$

$$\prod_{p \equiv 4,10[13]} (1 - p^{-2})^{-1} = 1.00628\ 51383\ 85264\ 35654\ 79220\ 78630\ 88874\ 03212\ 24553\ 50607 \\ 59162\ 40959\ 77321\ 01204\ 89381\ 53735\ 74182\ 12805\ 59112\ 51752 \dots$$

$$\prod_{p \equiv 5,8[13]} (1 - p^{-2})^{-1} = 1.04384\ 79529\ 58163\ 48325\ 64453\ 12135\ 62867\ 13038\ 05109\ 49630 \\ 56435\ 71738\ 46465\ 77456\ 29690\ 71263\ 29350\ 03766\ 17988\ 29979 \dots$$

$$\prod_{p \equiv 12[13]} (1 - p^{-2})^{-1} = 1.00019\ 47228\ 43353\ 09720\ 12251\ 29852\ 70839\ 19867\ 65951\ 93000 \\ 49665\ 62593\ 02690\ 92410\ 34974\ 82067\ 06364\ 88262\ 34074\ 53639 \dots$$

Modulo 15.

$$\prod_{p \equiv 1[15]} (1 - p^{-2})^{-1} = 1.00148\ 97422\ 73492\ 93695\ 62022\ 82152\ 29804\ 06202\ 71822\ 24183 \\ 85046\ 92061\ 06460\ 33370\ 47461\ 16170\ 34094\ 66709\ 13158\ 03303 \dots$$

$$\prod_{p \equiv 2,8[15]} (1 - p^{-2})^{-1} = 1.34246\ 04551\ 54995\ 30799\ 30100\ 63345\ 72665\ 24298\ 78723\ 72380 \\ 96524\ 03928\ 73058\ 62457\ 83670\ 07480\ 09151\ 10334\ 06933\ 31380 \dots$$

$$\prod_{p \equiv 4[15]} (1 - p^{-2})^{-1} = 1.00317\ 84700\ 07976\ 58539\ 76886\ 54009\ 35749\ 55893\ 69169\ 67588 \\ 37351\ 26980\ 45622\ 46578\ 84368\ 96080\ 28447\ 94669\ 19055\ 69351 \dots$$

$$\prod_{p \equiv 7,13[15]} (1 - p^{-2})^{-1} = 1.02920\ 54524\ 88970\ 30487\ 46169\ 68199\ 34620\ 53972\ 85734\ 20801 \\ 87576\ 81344\ 73863\ 39397\ 51683\ 30560\ 76995\ 20714\ 09590\ 99521 \dots$$

$$\prod_{p \equiv 11[15]} (1 - p^{-2})^{-1} = 1.00941\ 13977\ 70415\ 34074\ 11140\ 07967\ 71715\ 31828\ 38502\ 83487 \\ 41065\ 68439\ 10926\ 98429\ 51008\ 47969\ 06005\ 15885\ 02338\ 55701 \dots$$

$$\prod_{p \equiv 14[15]} (1 - p^{-2})^{-1} = 1.00177\ 62082\ 89544\ 73626\ 10915\ 43079\ 96283\ 15610\ 57061\ 98467 \\ 19519\ 14691\ 39870\ 02036\ 75682\ 26376\ 90944\ 75824\ 69831\ 96091 \dots$$

Modulo 16.

$$\prod_{p \equiv 1[16]} (1 - p^{-2})^{-1} = 1.00378\ 12963\ 11174\ 37714\ 94711\ 72280\ 61816\ 45658\ 26785\ 28441 \\ 57268\ 63521\ 48911\ 54134\ 99502\ 87194\ 19254\ 71100\ 10645\ 46873 \dots$$

$$\prod_{p \equiv 3,11[16]} (1 - p^{-2})^{-1} = 1.13941\ 87771\ 08211\ 51502\ 70589\ 30773\ 34020\ 88725\ 59961\ 09629 \\ 48302\ 25821\ 27411\ 02101\ 65577\ 60742\ 91446\ 59374\ 91512\ 33349 \dots$$

$$\prod_{p \equiv 5,13[16]} (1 - p^{-2})^{-1} = 1.05109\ 99849\ 42183\ 30793\ 68775\ 56006\ 33505\ 68012\ 01018\ 45817 \\ 85080\ 59912\ 94207\ 39729\ 30485\ 58783\ 38889\ 50479\ 59255\ 34495 \dots$$

$$\prod_{p \equiv 7[16]} (1 - p^{-2})^{-1} = 1.02325\ 48781\ 97407\ 08067\ 95776\ 68614\ 06977\ 00372\ 89157\ 54600 \\ 19844\ 97929\ 83355\ 91253\ 99909\ 55714\ 70317\ 40567\ 85934\ 05044 \dots$$

$$\prod_{p \equiv 9[16]} (1 - p^{-2})^{-1} = 1.00104\ 97991\ 21471\ 31637\ 83963\ 95210\ 10070\ 68052\ 00181\ 57035 \\ 98663\ 81304\ 47589\ 89310\ 55217\ 86340\ 51978\ 44383\ 63621\ 58656 \dots$$

$$\prod_{p \equiv 15[16]} (1 - p^{-2})^{-1} = 1.00185\ 24179\ 73996\ 13159\ 93578\ 02219\ 51678\ 26622\ 68517\ 41444 \\ 99996\ 30754\ 09303\ 19958\ 16127\ 21985\ 97936\ 04820\ 77136\ 34947 \dots$$

REFERENCES

- [1] S. Ettahri, O. Ramaré, and L. Surel. “Fast multi-precision computation of some Euler products”. In: *Submitted* (2019). arxiv.org/pdf/1908.06808.pdf, 23p (cit. on p. 1).

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