Journal of Number Theory 243 (2023) 13–37

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

General Section

Counting ideals in ray classes

Sanoli Gun [a,*], Olivier Ramaré [b], Jyothsnaa Sivaraman [c]

[a] *Institute of Mathematical Sciences, A CI of Homi Bhabha National Institute, CIT Campus, Taramani, Chennai 600 113, India*
[b] *CNRS/Institut de Mathématiques de Marseille, Aix Marseille Université, U.M.R. 7373, Site Sud, Campus de Luminy, Case 907, 13288 Marseille Cedex 9, France*
[c] *Chennai Mathematical Institute, H1, SIPCOT IT Park, Siruseri, Kelambakkam, 603103, India*

ARTICLE INFO

Article history:
Received 13 April 2022
Received in revised form 24 July 2022
Accepted 27 August 2022
Available online 27 September 2022
Communicated by F. Pellarin

MSC:
primary 11R44, 11R45
secondary 11R42

Keywords:
Ray class group
Korkin-Zolotarev basis
Counting ideals

ABSTRACT

Let $\mathbf{K}$ be a number field and $\mathfrak{q}$ an integral ideal in $\mathcal{O}_{\mathbf{K}}$. A result of Tatuzawa [11] from 1973, computes the asymptotic (with an error term) for the number of ideals with norm at most x in a class of the narrow ray class group of $\mathbf{K}$ modulo $\mathfrak{q}$. This result bounds the error term with a constant whose dependence on $\mathfrak{q}$ is explicit but dependence on $\mathbf{K}$ is not explicit. The aim of this paper is to prove this asymptotic with a fully explicit bound for the error term.

Contents

* Corresponding author.
E-mail addresses: sanoli@imsc.res.in (S. Gun), olivier.ramare@univ-amu.fr (O. Ramaré), jyothsnaas@cmi.ac.in (J. Sivaraman).

https://doi.org/10.1016/j.jnt.2022.08.005
0022-314X/

1. Introduction and statement of the theorem

Given a number field $\mathbf{K}$, the problem of counting the number of ideals in a given class of the narrow ray class group $H_{\mathfrak{q}}(\mathbf{K})$ attached to the ideal $\mathfrak{q}$ is classical and goes back, if not to Landau, at least to Hecke. Our query in this paper is the dependence of the error term on the field $\mathbf{K}$ which we describe fully, and even in a completely explicit manner. Let us recall our notation in brief, $n_{\mathbf{K}}$, $h_{\mathbf{K},1}$, $R_{\mathbf{K}}$, $\mu_{\mathbf{K}}$ and $d_{\mathbf{K}}$ are respectively the degree, the narrow class number, the regulator, the group of units of finite order in $\mathbf{K}$ and the discriminant of $\mathbf{K}$ while $\alpha_{\mathbf{K}}$ is the residue of its Dedekind zeta-function at 1. The ring of integers is denoted by $\mathcal{O}_{\mathbf{K}}$ and $h_{\mathbf{K}}$ denotes the class number of $\mathbf{K}$.

On the technical side, notation $f(x) = \mathrm{O}^*(g(x))$ means that $|f(x)| \leqslant g(x)$. In this set-up, we have the following theorem;

Theorem 1. *Let $\mathfrak{q}$ be an integral ideal of $\mathbf{K}$ and $[\mathfrak{b}]$ be an element of $H_{\mathfrak{q}}(\mathbf{K})$. For any real number $x \geqslant 1$, we have*

$$\sum_{\substack{\mathfrak{a}\subset\mathcal{O}_{\mathbf{K}},\\ [\mathfrak{a}]=[\mathfrak{b}],\\ \mathfrak{N}\mathfrak{a}\leqslant x}} 1 = \frac{\alpha_{\mathbf{K}}\varphi(\mathfrak{q})}{|H_{\mathfrak{q}}(\mathbf{K})|}\frac{x}{\mathfrak{N}\mathfrak{q}}$$

$$+\ \mathrm{O}^*\left(E(\mathbf{K})F(\mathfrak{q})^{\frac{1}{n_{\mathbf{K}}}}\log(3F(\mathfrak{q}))^{n_{\mathbf{K}}}\left(\frac{x}{\mathfrak{N}\mathfrak{q}}\right)^{1-\frac{1}{n_{\mathbf{K}}}} + n_{\mathbf{K}}^{8n_{\mathbf{K}}}\frac{R_{\mathbf{K}}}{|\mu_{\mathbf{K}}|}F(\mathfrak{q})\right),$$

where $F(\mathfrak{q}) = 2^{r_1}\varphi(\mathfrak{q})h_{\mathbf{K}}/h_{\mathbf{K},\mathfrak{q}}$ and $E(\mathbf{K}) = 1000 n_{\mathbf{K}}^{12n_{\mathbf{K}}^2}(R_{\mathbf{K}}/|\mu_{\mathbf{K}}|)^{\frac{1}{n_{\mathbf{K}}}}\left[\log\left((2n_{\mathbf{K}})^{4n_{\mathbf{K}}}R_{\mathbf{K}}/|\mu_{\mathbf{K}}|\right)\right]^{n_{\mathbf{K}}}$.

Notice that $F(\mathfrak{q}) \geqslant 1$. Let us briefly recall the definition of the (narrow) ray class group $H_{\mathfrak{q}}(\mathbf{K})$. Let $I(\mathfrak{q})$ be the group of fractional ideals of $\mathbf{K}$ which are co-prime to $\mathfrak{q}$ and $P_{\mathfrak{q}}$ be the subgroup of $I(\mathfrak{q})$ consisting of principal ideals (α) satisfying $v_{\mathfrak{p}}(\alpha - 1) \geqslant v_{\mathfrak{p}}(\mathfrak{q})$ for all prime ideals $\mathfrak{p}$ dividing $\mathfrak{q}$ and $\sigma(\alpha) > 0$ for all embeddings σ of $\mathbf{K}$ in $\mathbb{R}$. We set

$H_{\mathfrak{q}}(\mathbf{K}) = I(\mathfrak{q})/P_{\mathfrak{q}}$. When $\mathfrak{q} = \mathcal{O}_{\mathbf{K}}$, the group $H_{\mathfrak{q}}(\mathbf{K})$ is the usual class group in the narrow sense.

The problem of counting ideals in a class of ray class group can be decomposed in two parts: building a fundamental domain, which turns out to be made of lattice points in some region, and counting such points. Our main effort concerns the building of the fundamental domain. Two hurdles prevent us from directly counting integral ideals of $\mathbf{K}$: the fact that the narrow ray class group $H_{\mathfrak{q}}(\mathbf{K})$ is non-trivial, and the existence of units. To treat both of these, we follow the approach developed by K. Debaene in [1], as it provides us with a very tame dependence in the field (notice that no discriminant appears in our error term). This is combined, as in [1], with two general results: the first one shows that a 'short' enough basis exists for the lattice to be considered, while the second one counts the lattice points in a given domain.

An overall different approach has been followed in [11] by T. Tatuzawa, but his results lack the control of the dependency in $\mathbf{K}$. There also exists an earlier completely explicit result on this subject, and with a better error term as far as the dependence in x is concerned. It is due to J. Sunley in her PhD memoir [8] and is recalled as Theorem 1.1 of [9] (see also [6]). We have few indications as to the proof of this result, as it has not been published in any journal, but, knowing that it originates from the method of Landau, we may surmise that most of the work goes on the dependency of $\mathbf{K}$, relying on a more classical fundamental domain. We finally mention that the present work relies on several highly non-trivial results, like the bound for the regulator given by E. Friedman in [3], and the lower bound for the height of an algebraic number provided by E. Dobrowolski in [2].

We have several applications of our result which we leave for future works. The paper is organized as follows. In section 2, we deal with some notations and preliminaries. In section 3, we prove Theorem 1.

2. Notation and preliminaries

2.1. Notation

Let $\mathbf{K} \neq \mathbb{Q}$ be a number field with discriminant $|d_{\mathbf{K}}| \geqslant 3$ (by Minkowski's bound). Also let $n_{\mathbf{K}} = [\mathbf{K} : \mathbb{Q}] \geqslant 2$ and $\mathfrak{q}$ be an (integral) ideal of $\mathbf{K}$. The number of real embedding of $\mathbf{K}$ is denoted by r_1 whereas the number of complex ones are denoted by $2r_2$. The ring of integers of $\mathbf{K}$ is denoted by $\mathcal{O}_{\mathbf{K}}$, the narrow ray class group modulo $\mathfrak{q}$ is denoted by $H_{\mathfrak{q}}(\mathbf{K})$, its cardinality by $h_{\mathbf{K},\mathfrak{q}}$ and the (absolute) norm is denoted by $\mathfrak{N}$. We shorten $h_{\mathbf{K},\mathfrak{q}}$ by $h_{\mathbf{K},1}$ when $\mathfrak{q} = \mathcal{O}_{\mathbf{K}}$. Whenever required, we shall replace the ideal $\mathfrak{q}$ by the *modulus* $\mathfrak{q}_1 = \mathfrak{q}\mathfrak{q}_\infty$, considered as a set of places, where $\mathfrak{q}_\infty$ is the set of all Archimedean places of $\mathbf{K}$. But, as we do not consider subsets S of Archimedean places, we may safely rely only on $\mathfrak{q}$ and recall regularly that we count *narrow* classes. Still to follow tradition, we denote by $R_{\mathbf{K},\mathfrak{q}_1}$ the $\mathfrak{q}_1$-regulator, by $U_{\mathfrak{q}_1}$ the corresponding group of units and by $\mu_{\mathfrak{q}_1}$ the number of units of finite order in $U_{\mathfrak{q}_1}$, i.e. also, the cardinality of $\mu_{\mathbf{K}} \cap U_{\mathfrak{q}_1}$. Throughout the article $\mathfrak{p}$ will denote a prime ideal in $\mathcal{O}_{\mathbf{K}}$ and p will denote a rational prime number. Further an element of $H_{\mathfrak{q}}(\mathbf{K})$ containing an integral ideal $\mathfrak{a}$ will be denoted by $[\mathfrak{a}]$.

2.2. The Dedekind zeta-function

For $\Re s = \sigma > 1$, the Dedekind zeta-function is defined by

$$\zeta_{\mathbf{K}}(s) = \sum_{\substack{\mathfrak{a} \subseteq \mathcal{O}_{\mathbf{K}} \\ \mathfrak{a} \neq 0}} \frac{1}{\mathfrak{N}(\mathfrak{a})^s},$$

where $\mathfrak{a}$ ranges over the integral ideals of $\mathcal{O}_{\mathbf{K}}$. It has only a simple pole at $s = 1$ of residue $\alpha_{\mathbf{K}}$, say. We know from the analytic class number formula that

$$\alpha_{\mathbf{K}} = \frac{2^{r_1}(2\pi)^{r_2} h_{\mathbf{K}} R_{\mathbf{K}}}{|\mu_{\mathbf{K}}|\sqrt{|d_{\mathbf{K}}|}}, \tag{1}$$

where $h_{\mathbf{K}}, R_{\mathbf{K}}, d_{\mathbf{K}}$ and $\mu_{\mathbf{K}}$ are as before.

2.3. The narrow ray-class group

By narrow ray class group $H_{\mathfrak{q}}(\mathbf{K})$, we consider that ray class group where the integral ideal $\mathfrak{q}$ is completed with all real Archimedean places. We have

$$|H_1(\mathbf{K})| \leqslant |H_{\mathfrak{q}}(\mathbf{K})| \leqslant \varphi(\mathfrak{q})|H_1(\mathbf{K})|, \tag{2}$$

where

$$\varphi(\mathfrak{q}) = \mathfrak{N}(\mathfrak{q}) \prod_{\mathfrak{p}|\mathfrak{q}} \left(1 - \frac{1}{\mathfrak{N}(\mathfrak{p})}\right) \tag{3}$$

and $H_1(\mathbf{K})$ denotes the narrow ray class group corresponding to $\mathcal{O}_{\mathbf{K}}$. A good reference for this are the notes [10] by A. Sutherland.

2.4. Orthogonality defect and successive minima

In this subsection we define some notions and state results about lattices in $\mathbb{R}^n$ that will be required in due course of the proof.

Definition 1. Given a lattice Λ_n of rank n, the orthogonality defect Ω of the lattice Λ_n is given by

$$\Omega = \inf_{(\vec{v_1}, \cdots, \vec{v_n})} \frac{||\vec{v_1}|| \cdots ||\vec{v_n}||}{\text{Vol } (\Lambda_n)}$$

where $\{\vec{v_1}, \cdots, \vec{v_n}\}$ runs over the bases of Λ_n.

Definition 2. Given a basis $V = \{\vec{v_1}, \cdots, \vec{v_n}\}$ of a lattice Λ_n of rank n, let $V^\dagger = \{\vec{v_1}^\dagger, \cdots, \vec{v_n}^\dagger\}$ be the Gram-Schmidt orthogonalization of V. Let

$$\alpha_{i,j} = \frac{\vec{v_i} \cdot \vec{v_j}^\dagger}{||\vec{v_j}^\dagger||^2} \qquad \text{for } i, j \in \{1, \cdots, n\}.$$

When $n = 1$, any basis element of Λ_1 in $\mathbb{R}$ is defined to be a reduced Korkin-Zolotarev basis. When $n > 1$, the basis V of Λ_n is called a reduced Korkin-Zolotarev basis if it satisfies the following properties;

(1) The vector $\vec{v_1}$ is of minimum length among the vectors $\vec{v_i}$ for $1 \leqslant i \leqslant n$ (with respect to the Euclidean norm);
(2) The coefficients $|\alpha_{i,1}| \leqslant \frac{1}{2}$ for $2 \leqslant i \leqslant n$;
(3) If Λ_{n-1} is the orthogonal projection of Λ_n on the orthogonal complement $(\mathbb{R}\vec{v_1})^\perp$, then the vectors $\{\vec{v_2} - \alpha_{2,1}\vec{v_1}, \cdots, \vec{v_n} - \alpha_{n,1}\vec{v_1}\}$ also form a reduced Korkin-Zolotarev basis of Λ_{n-1}.

It is easy to see that reduced Korkin-Zolotarev bases exist for a lattice Λ_n of rank n.

Definition 3. For a lattice Λ_n of rank $n \geqslant 1$ and for $1 \leqslant i \leqslant n$, the i-th successive minimum of Λ_n is defined by

$$\delta_i(\Lambda_n) = \inf\{\lambda \in \mathbb{R} \mid B(0, \lambda) \cap \Lambda_n \text{ contains } i \text{ linearly independent vectors}\}.$$

Here $B(0, \lambda)$ denotes a ball of radius λ around origin in $\mathbb{R}^n$. For $i = 0$, we define $\delta_0(\Lambda_n) = 1$. Further the constant

$$\gamma_n = \sup \left\{ \left(\frac{\delta_1(\Lambda_n)^n}{\mathrm{Vol}(\Lambda_n)} \right)^{2/n} \mid \Lambda_n \text{ is a lattice of rank } n \right\}$$

is called the Hermite's constant.

In this set-up, we have the following theorem.

Theorem 2 *(Lagarias, Lenstra and Schnorr [4]). If $\{\vec{v_1} \ldots \vec{v_n}\}$ is a reduced Korkin-Zolotarev basis for a lattice Λ_n of rank n, then*

$$\prod_{j=1}^{n} ||\vec{v_j}||^2 \leqslant \left(\prod_{j=1}^{n} \frac{j+3}{4} \right) \gamma_n^n \ Vol(\Lambda_n)^2,$$

where γ_n is the Hermite's constant. Further, an upper bound for the Hermite's constant is given by

$$\gamma_n \leqslant n \qquad \textit{for } n \geqslant 1.$$

We now define the notion of the Lipschitz class of a subset of $\mathbb{R}^n$.

Definition 4. Let S be a subset of $\mathbb{R}^n$ with $n \geqslant 2$. We say that S is of Lipschitz class $\mathcal{L}(n, M, L)$ if there are M maps $\phi_1, \ldots \phi_M : [0,1]^{n-1} \to \mathbb{R}^n$ such that S is contained in the union of images of ϕ_i for $i \in \{1, \ldots M\}$ and

$$||\phi_i(\overline{x}) - \phi_i(\overline{y})|| \leqslant L\ ||\overline{x} - \overline{y}||,$$

where $\overline{x}, \overline{y} \in [0,1]^{n-1}$.

We conclude by stating a theorem of Widmer [12] which will play an important role in the proof of Theorem 11.

Theorem 3 *(Widmer [12]). Let Λ_n be a lattice in $\mathbb{R}^n$ with successive minima $\delta_0(\Lambda_n), \cdots, \delta_n(\Lambda_n)$. Let S be a bounded set in $\mathbb{R}^n$ such that its boundary is of Lipschitz class $\mathcal{L}(n, M, L)$ for some natural number M and positive constant L. Then S is measurable and*

$$\left| |S \cap \Lambda_n| - \frac{Vol(S)}{Vol(\Lambda_n)} \right| \leqslant M n^{3n^2/2} \max_{0 \leqslant i < n} \frac{L^i}{\delta_0(\Lambda_n) \cdots \delta_i(\Lambda_n)}.$$

2.5. Lower bounds for algebraic conjugates

In this subsection we recall a theorem of Dobrowolski which gives a lower bound on the absolute value of all the conjugates of an algebraic integer which is not zero or a root of unity.

Theorem 4 *(Dobrowolski [2]). Let α be a non-zero algebraic integer of degree $n > 1$ and let $\tilde{\alpha}$ be the maximum of the absolute values of all conjugates of α. If α is not a root of unity, then*

$$\tilde{\alpha} \geqslant 1 + \frac{\log n}{6n^2}.$$

3. Counting integral ideals in classes of the ray class group

Let $\{\sigma_1, \cdots \sigma_{n_{\mathbf{K}}}\}$ be the set of all embeddings of $\mathbf{K}$ into $\mathbb{C}$. The first r_1 embeddings are all real embeddings and the embeddings $\{\sigma_{r_1+i},\ \sigma_{r_1+r_2+i}\}$ for $1 \leqslant i \leqslant r_2$ are complex conjugates. Consider the first $r_1 + r_2$ embeddings from this set. We will use r to denote $r_1 + r_2 - 1$ and as before $\mathfrak{q}_1 = \mathfrak{q}\mathfrak{q}_\infty$ to denote a modulus, where $\mathfrak{q} \subseteq \mathcal{O}_{\mathbf{K}}$ be an ideal and $\mathfrak{q}_\infty$ contains all real places of $\mathbf{K}$.

3.1. Fundamental domain

Let $\mathcal{O}_{\mathbf{K}}^*$ be the group of units of $\mathcal{O}_{\mathbf{K}}$ and $U_{\mathfrak{q}}$ (respectively $U_{\mathfrak{q}_1}$) be the subgroup of $\mathcal{O}_{\mathbf{K}}^*$ consisting of units which are 1 mod $\mathfrak{q}$ (respectively 1 mod $^*\mathfrak{q}_1$). Both of these subgroups $U_{\mathfrak{q}}$ and $U_{\mathfrak{q}_1}$ are of finite index in $\mathcal{O}_{\mathbf{K}}^*$. Let ϕ denote the embedding

$$\begin{aligned} \phi : \mathbf{K} &\to \mathbb{R}^{r_1} \times \mathbb{C}^{r_2} \\ x &\to (\sigma_i(x))_{i=1}^{r+1} \end{aligned}$$

Further let f denote the map

$$\begin{aligned} f : \mathbb{R}^{r_1} \times \mathbb{C}^{r_2} &\to \mathbb{R}^{r+1} \\ (x_i)_i &\to (\log|x_i|)_{i=1}^{r+1} \end{aligned}$$

Since $[\mathcal{O}_{\mathbf{K}}^* : U_{\mathfrak{q}_1}]$ is finite, the image under the map $f \circ \phi$ of $U_{\mathfrak{q}_1}$ is also a lattice of rank r. Let $\{\eta_1, \dots \eta_r\}$ be a set of multiplicatively independent generators for the group $U_{\mathfrak{q}_1}$ modulo roots of unity and the vectors

$$\vec{v_1} = \left(\frac{1}{n_{\mathbf{K}}}, \cdots, \frac{1}{n_{\mathbf{K}}}\right), \qquad \vec{v_j} = (\log|\sigma_i(\eta_{j-1})|)_{i=1}^{r+1} \quad \text{for } 2 \leqslant j \leqslant r+1$$

form a basis for $\mathbb{R}^{r+1}$. The vectors $\vec{v_2}, \cdots \vec{v_{r+1}}$ form a basis for a lattice of rank r and $\vec{v_1}, \vec{v_2}, \cdots \vec{v_{r+1}}$ are $\mathbb{R}$ linearly independent. We can now write the vector $(\log|x_i|)_{i=1}^{r+1}$ as

$$(\log|x_i|)_{i=1}^{r+1} = \alpha_1(x)\vec{v_1} + \cdots + \alpha_{r+1}(x)\vec{v_{r+1}},$$

and therefore

$$|x_i| = e^{\alpha_1(x)/n_{\mathbf{K}}} \prod_{j=2}^{r+1} |\sigma_i(\eta_{j-1})|^{\alpha_j(x)} = \alpha(x)^{1/n_{\mathbf{K}}} \prod_{j=2}^{r+1} |\sigma_i(\eta_{j-1})|^{\alpha_j(x)}, \tag{4}$$

where $x = (x_1, \cdots, x_{r+1})$ and $\alpha(x) = e^{\alpha_1(x)}$. If $x \in \mathbf{K}$, then $x_i = \sigma_i(x)$ for $1 \leqslant i \leqslant r+1$ and so taking product over all i in (4), we identify $\alpha(x)$:

$$\prod_{i=1}^{r+1} |\sigma_i(x)|^{e_i} = |\mathfrak{N}(x)| = \alpha(x), \tag{5}$$

where $e_i = 1$ when $i \leqslant r_1$ and $e_i = 2$ otherwise. We now define the map

$$\begin{aligned} g : \mathbb{R}^{r_1} \times \mathbb{C}^{r_2} &\to \mathbb{R}^{r+1} \\ x = (x_i)_i &\to (\alpha(x), \alpha_2(x), \cdots \alpha_{r+1}(x)). \end{aligned}$$

We now define $\mathbf{F} = g^{-1}(\mathbb{R}_+ \times [0,1)^r)$. This corresponds to the set $(f \circ \phi)^{-1}(S')$, where S' is given by the points corresponding to the vectors

$$\{\alpha_1 \vec{v_1} + \ldots + \alpha_{r+1} \vec{v_{r+1}} \ \mid \ \alpha_1 \in \mathbb{R},\ \alpha_i \in [0,1) \text{ for } i > 1\}.$$

Since the vectors give rise to a lattice of full rank, given an $x \in \mathbf{K}$, there is an $\eta \in U_{\mathfrak{q}_1}$ such that $\phi(x/\eta) \in \mathbf{F}$. Conversely, for an $x \in \mathbf{K}$, with $\phi(x) \in \mathbf{F}$ and any $\eta \in U_{\mathfrak{q}_1}$, we note that $\phi(\eta x) \in \mathbf{F}$ if and only if η is a root of unity.

3.2. Notation

Throughout the rest of the section, we will use the following notations.

(1) $\mathbf{F}(a_1, b_1, \ldots, a_r, b_r, X) = g^{-1}\left((0, X] \times \prod_{j=1}^r [a_j, b_j)\right)$.
(2) $\mathbf{F}_{\frac{1}{2}}(a_1, b_1, \ldots, a_r, b_r, X) = g^{-1}\left(\left(\frac{X}{2}, X\right] \times \prod_{j=1}^r [a_j, b_j)\right)$.
(3) For $\overline{\gamma} = (\gamma_i)_{i=1}^{r_1} \in \{\pm 1\}^{r_1}$, denote by $\mathbb{R}_{\overline{\gamma}}^{r_1} = \{(x_1, \cdots, x_{r_1}) \mid \mathrm{sign}(x_i) = \gamma_i\}$. Then $\mathbf{F}_{\overline{\gamma}}(a_1, b_1, \ldots, a_r, b_r, X) = \mathbf{F}(a_1, b_1, \ldots, a_r, b_r, X) \cap (\mathbb{R}_{\overline{\gamma}}^{r_1} \times \mathbb{C}^{r_2})$.
Further, $\mathbf{F}_{\frac{1}{2}, \overline{\gamma}}(a_1, b_1, \ldots, a_r, b_r, X) = \mathbf{F}_{\frac{1}{2}}(a_1, b_1, \cdots, a_r, b_r, X) \cap (\mathbb{R}_{\overline{\gamma}}^{r_1} \times \mathbb{C}^{r_2})$.
(4) $\mathbf{F}_{\frac{1}{2}, \overline{\gamma}}(X) = \mathbf{F}_{\frac{1}{2}, \overline{\gamma}}(0, 1, \cdots 0, 1, X)$.
(5) For $j \in \{1, \ldots, r\}$, $m_j = \max_i \lceil \log |\sigma_i(\eta_j)| \rceil$, where $\{\eta_1, \ldots \eta_r\}$ is a set of multiplicatively independent generators for the group $U_{\mathfrak{q}_1}$ modulo roots of unity.

3.3. Computing the Lipschitz class of the boundary

Definition 5. Let $\mathfrak{q}_1$ be a modulus of $\mathbf{K}$ and $U_{\mathfrak{q}_1}$ be the subgroup of units which are 1 mod $^*\mathfrak{q}_1$. Further let $\{\sigma_1, \cdots, \sigma_{r+1}\}$ be a set of $r+1$ embeddings of $\mathbf{K}$ into $\mathbb{C}$ where no two embeddings are conjugates of each other. If $\{\eta_1, \cdots, \eta_r\}$ are multiplicatively independent units generating $U_{\mathfrak{q}_1}$ modulo the roots of unity, then the $\mathfrak{q}_1$ regulator $R_{\mathbf{K}, \mathfrak{q}_1}$ is defined by

$$R_{\mathbf{K}, \mathfrak{q}_1} = \left| \det \left(e_i \log |\sigma_i(\eta_j)| \right)_{i,j} \right|$$

where $i, j \in \{1, 2 \ldots r\}$ and $e_i = 1$ or 2 if the corresponding embedding is real or complex respectively.

Since $\sum_{i=1}^{r+1} e_i \log |\sigma_i(\eta_j)| = 0$ for $1 \leqslant j \leqslant r$, we see that $R_{\mathbf{K}, \mathfrak{q}_1}$ is independent of the generating set of units and the choice of r embeddings.

Lemma 5. *Let Γ be a set of points in $\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$, $\vec{k} = (k_j)_{j=1}^r \in \prod_{j=1}^r ([0, m_j) \cap \mathbb{Z})$ and*

$$\beta_{\vec{k}} = \left(\prod_{j=1}^r |\sigma_i(\eta_j)|^{-k_j/m_j} \right)_{i=1}^{r+1}.$$

Then for any $\overline{\gamma} = (\gamma_i)_{i=1}^{r_1} \in \{\pm 1\}^{r_1}$ and any positive real X, we have

$$|\Gamma \cap \mathbf{F}_{\frac{1}{2},\overline{\gamma}}(X)| = \sum_{\vec{k}} \left|(\Gamma \cdot \beta_{\vec{k}}) \cap \mathbf{F}_{\frac{1}{2},\overline{\gamma}}\left(0, \frac{1}{m_1}, \cdots 0, \frac{1}{m_r}, X\right)\right|,$$

where the sum runs over $\vec{k} \in \prod_{j=1}^{r}([0, m_j) \cap \mathbb{Z})$.

Proof. We have

$$|\Gamma \cap \mathbf{F}_{\frac{1}{2},\overline{\gamma}}(X)| = \sum_{\substack{(k_1,\cdots,k_r)\in\mathbb{Z}^r,\\ 0\leqslant k_i\leqslant m_i-1}} \left|\Gamma \cap \mathbf{F}_{\frac{1}{2},\overline{\gamma}}\left(\frac{k_1}{m_1}, \frac{k_1+1}{m_1}, \cdots \frac{k_r}{m_r}, \frac{k_r+1}{m_r}, X\right)\right|.$$

If we multiply an element of $\mathbf{F}_{\frac{1}{2},\overline{\gamma}}\left(\frac{k_1}{m_1}, \frac{k_1+1}{m_1}, \cdots \frac{k_r}{m_r}, \frac{k_r+1}{m_r}, X\right)$ by $\beta_{\vec{k}=(k_1,\ldots k_r)}$, using (4), we see that we get an element in $\mathbf{F}_{\frac{1}{2},\overline{\gamma}}\left(0, \frac{1}{m_1}, \cdots 0, \frac{1}{m_r}, X\right)$. Therefore

$$|\Gamma \cap \mathbf{F}_{\frac{1}{2},\overline{\gamma}}(X)| = \sum_{\substack{(k_1,\ldots,k_r)\in\mathbb{Z}^r,\\ 0\leqslant k_i\leqslant m_i-1}} \left|(\Gamma \cdot \beta_{\vec{k}}) \cap \mathbf{F}_{\frac{1}{2},\overline{\gamma}}\left(0, \frac{1}{m_1}, \ldots 0, \frac{1}{m_r}, X\right)\right|. \quad \square$$

Let $h' : \mathbb{R}^{r_1} \times \mathbb{C}^{r_2} \to \mathbb{R}^{n_{\mathbf{K}}}$ be a map defined by $h'(y_1, \cdots, y_{r_1+r_2}) = (z_1, \cdots, z_{n_{\mathbf{K}}})$, where for $1 \leqslant i \leqslant r_1$, $z_i = y_i$ and for $r_1 + 1 \leqslant i \leqslant r_1 + r_2$,

$$z_i = |y_i| \cos(\arg(y_i)) \quad \text{and} \quad z_{r_2+i} = |y_i| \sin(\arg(y_i)),$$

where argument for y_i's are inside $[0, 2\pi)$.

Lemma 6. *Let X be any positive real number and*

$$x = (x_1, \cdots x_{r_1+r_2}) \in \mathbf{F}\left(0, \frac{1}{m_1}, \cdots, 0, \frac{1}{m_r}, X\right).$$

Then $||h'(x)|| \leqslant (\sqrt{r+1})\ e^r\ X^{1/n_{\mathbf{K}}}$, where $||\cdot||$ is the Euclidean norm on $\mathbb{R}^{n_{\mathbf{K}}}$.

Proof. From the definition of the norm, we have

$$||h'(x)|| = \sqrt{\sum_{i=1}^{r_1} |x_i|^2 + \sum_{i=r_1+1}^{r_1+r_2} (|\Re(x_i)|^2 + |\Im(x_i)|^2)}.$$

From (4) and the definition of $\mathbf{F}(0, \frac{1}{m_1}, \cdots, 0, \frac{1}{m_r}, X)$, we see that

$$|x_i| = |\alpha(x)|^{1/n_{\mathbf{K}}} \prod_{k=2}^{r+1} |\sigma_i(\eta_{k-1})|^{\alpha_k(x)}$$

where $0 < \alpha(x) \leqslant X$ and $0 \leqslant \alpha_k(x) < 1/m_k$ for $2 \leqslant k \leqslant r+1$. Hence for $1 \leqslant i \leqslant r+1$, we have

$$||h'(x)|| \;\leqslant\; (\sqrt{r+1}\;)\; e^r\; X^{1/n_{\mathbf{K}}}.$$

This completes the proof of the lemma. $\square$

The next lemma can be found in [1, Lemma 6].

Lemma 7. *Let $f : \mathbb{R}^m \to \mathbb{R}$ be a function such that $f(y_1, \dots y_m) = c\prod_{j=1}^m g_j(y_j)$, where $g_j : \mathbb{R} \to \mathbb{R}$ are functions that satisfy $|g_j(y_j) - g_j(y_j')| \leqslant K_j|y_j - y_j'|$ and $|g_j(y_j)| \leqslant M_j$. Then we have*

$$|f(\overline{y}) - f(\overline{y}')| \leqslant \left(c\sum_{j=1}^{m} K_j \prod_{k\neq j} M_k\right)\; ||\overline{y} - \overline{y}'||,$$

where $\overline{y}, \overline{y}' \in \mathbb{R}^m$ and $||.||$ is the Euclidean norm on $\mathbb{R}^m$. Let $h : [0,1]^{m-1} \to \mathbb{R}^m$, $h = (h_1, \dots h_m)$ where $h_i : [0,1]^{m-1} \to \mathbb{R}$ are functions that satisfy $|h_i(\overline{y}) - h_i(\overline{y}')| \leqslant L_i\; ||\overline{y} - \overline{y}'||$. Then we have

$$|h(\overline{y}) - h(\overline{y}')| \leqslant \sqrt{m}\;(max_i L_i)\; ||\overline{y} - \overline{y}'||,$$

where $\overline{y}, \overline{y}' \in [0,1]^{m-1}$.

We now compute the Lipschitz class of the boundary of the fundamental domain $\mathbf{F}$. From now on, we will denote the boundary of a set $\mathbf{F}$ by $\partial\mathbf{F}$.

Lemma 8. *For a number field $\mathbf{K} \neq \mathbb{Q}$, $\overline{\gamma} \in \{\pm 1\}^{r_1}$ and positive real number t, the set*

$$\partial\mathbf{F}_{\frac{1}{2},\overline{\gamma}}\left(0, \frac{1}{m_1}, \cdots, 0, \frac{1}{m_r}, t^{r+1}\right)$$

is of Lipschitz class $\mathcal{L}(n_{\mathbf{K}}, 2r+2, Lt)$, where $L = \sqrt{n_{\mathbf{K}}}(2\pi + r)e^r$.

Proof. Let $x = (x_i)_{i=1}^{r+1} \in \mathbf{F}_{\frac{1}{2}}(0, \frac{1}{m_1}, \cdots, 0, \frac{1}{m_r}, 1)$. Using (4), we see that $|x_i|$ for all i is uniformly bounded away from 0 by a constant that depends only on $\mathbf{K}$ and not on i. Hence

$$\mathbf{F}_{\frac{1}{2}}\left(0, \frac{1}{m_1}, \cdots, 0, \frac{1}{m_r}, 1\right) \;\subseteq\; \bigcup_{\overline{\gamma}} (\mathbb{R}^{r_1}_{\overline{\gamma}} \times \mathbb{C}^{r_2}),$$

where $\overline{\gamma}$ varies over elements of $\{\pm 1\}^{r_1}$. Since $(\mathbb{R}^{r_1}_{\overline{\gamma}} \times \mathbb{C}^{r_2})$ are disjoint for distinct $\overline{\gamma}$, we have

$$\mathbf{F}_{\frac{1}{2}}\left(0, \frac{1}{m_1}, \cdots, 0, \frac{1}{m_r}, 1\right) = \bigcup_{\overline{\gamma}} \mathbf{F}_{\frac{1}{2},\overline{\gamma}}\left(0, \frac{1}{m_1}, \cdots, 0, \frac{1}{m_r}, 1\right).$$

Since $\overline{\mathbf{F}_{\frac{1}{2}}\left(0, \frac{1}{m_1}, \cdots, 0, \frac{1}{m_r}, 1\right)}$ does not contain any point $x = (x_i)_i$ with $x_i = 0$ for some i, we note that $\overline{\mathbf{F}_{\frac{1}{2},\overline{\gamma}}\left(0, \frac{1}{m_1}, \cdots, 0, \frac{1}{m_r}, 1\right)}$ are disjoint for distinct $\overline{\gamma}$. This implies that

$$\partial\mathbf{F}_{\frac{1}{2}}\left(0, \frac{1}{m_1}, \cdots, 0, \frac{1}{m_r}, 1\right) = \bigcup_{\overline{\gamma}} \partial\mathbf{F}_{\frac{1}{2},\overline{\gamma}}\left(0, \frac{1}{m_1}, \cdots, 0, \frac{1}{m_r}, 1\right). \tag{6}$$

More precisely

$$\partial\mathbf{F}_{\frac{1}{2},\overline{\gamma}}\left(0, \frac{1}{m_1}, \cdots, 0, \frac{1}{m_r}, 1\right) = \partial\mathbf{F}_{\frac{1}{2}}\left(0, \frac{1}{m_1}, \cdots, 0, \frac{1}{m_r}, 1\right) \cap (\mathbb{R}^{r_1}_{\overline{\gamma}} \times \mathbb{C}^{r_2}).$$

We want to compute the Lipschitz constant for the set $\partial\mathbf{F}_{\frac{1}{2},\overline{\gamma}}(0, \frac{1}{m_1}, \cdots, 0, \frac{1}{m_r}, 1)$. For any element $\overline{\gamma} = (\gamma_i)_{i=1}^{r_1} \in \{\pm 1\}^{r_1}$, consider the map

$$\begin{aligned} \tilde{g}_{\overline{\gamma}} &: \mathbb{R}^{r_1}_{\overline{\gamma}} \times \mathbb{C}^{r_2} \to \mathbb{R}^{n_{\mathbf{K}}} \\ (y_1, \cdots, y_{r_1+r_2}) &\to (g(y_1, \cdots y_{r_1+r_2}), \arg(y_{r_1+1}), \cdots, \arg(y_{r_1+r_2})) \end{aligned}$$

where the argument is chosen from $[0, 2\pi)$. Then

$$\begin{aligned} \partial\mathbf{F}_{\frac{1}{2}}\left(0, \frac{1}{m_1}, \cdots, 0, \frac{1}{m_r}, 1\right) \cap (\mathbb{R}^{r_1}_{\overline{\gamma}} \times \mathbb{C}^{r_2}) &= \partial g^{-1}\left([\frac{1}{2}, 1) \times \prod_{i=1}^{r} [0, \frac{1}{m_i})\right) \cap (\mathbb{R}^{r_1}_{\overline{\gamma}} \times \mathbb{C}^{r_2}) \\ &= \partial\tilde{g}^{-1}_{\overline{\gamma}}\left([\frac{1}{2}, 1) \times \prod_{i=1}^{r} [0, \frac{1}{m_i}) \times [0, 2\pi)^{r_2}\right). \end{aligned}$$

The map

$$\begin{aligned} \tilde{g}_{\overline{\gamma}} &: \mathbb{R}^{r_1}_{\overline{\gamma}} \times \mathbb{C}^{r_2} \to \mathbb{R}^{r+1} \times [0, 2\pi)^{r_2} \\ (y_1, \cdots, y_{r_1+r_2}) &\to (g(y_1, \cdots, y_{r_1+r_2}), \arg(y_{r_1+1}), \cdots, \arg(y_{r_1+r_2})) \end{aligned}$$

is a bijection. Indeed if $\tilde{g}_{\overline{\gamma}}(y_1, \ldots, y_{r_1+r_2}) = \tilde{g}_{\overline{\gamma}}(y'_1, \ldots, y'_{r_1+r_2})$, then $|y_i| = |y'_i|$ for $1 \leqslant i \leqslant r_1 + r_2$. Further the first r_1 real numbers have the same sign and the last r_2 complex numbers have the same arguments. Hence $(y_1, \ldots, y_{r_1+r_2}) = (y'_1, \ldots, y'_{r_1+r_2})$. When $(x_1, \ldots x_{n_{\mathbf{K}}}) \in R^{n_{\mathbf{K}}}$, then for $1 \leqslant i \leqslant r_1$, define

$$y_i = \gamma_i x_1^{1/n_{\mathbf{K}}} \prod_{k=2}^{r+1} |\sigma_i(\eta_{k-1})|^{x_k} \tag{7}$$

and for $r_1 + 1 \leqslant i \leqslant r_1 + r_2$, define

$$y_i = \left(x_1^{1/n_{\mathbf{K}}} \prod_{k=2}^{r+1} |\sigma_i(\eta_{k-1})|^{x_k} \right) e^{2i\pi x_{r_2+i}}. \tag{8}$$

Clearly $(y_1, \ldots, y_{r_1+r_2}) \in \mathbb{R}_{\overline{\gamma}}^{r_1} \times \mathbb{C}^{r_2}$ and $\tilde{g}_{\overline{\gamma}}(y_1, \ldots, y_{r_1+r_2}) = (x_1, \ldots x_{n_{\mathbf{K}}})$. We now observe that $\tilde{g}_{\overline{\gamma}}^{-1}$ is continuous on $\mathbb{R}^{r+1} \times [0, 2\pi)^{r_2}$. In fact, $[\frac{1}{2}, 1] \times \prod_{i=1}^{r} [0, \frac{1}{m_i}] \times (0, 2\pi)^{r_2}$ is homeomorphic to its image in $\mathbb{R}_{\overline{\gamma}}^{r_1} \times \mathbb{C}^{r_2}$ under the map $\tilde{g}_{\overline{\gamma}}^{-1}$. Therefore

$$\partial \tilde{g}_{\overline{\gamma}}^{-1} \left([\frac{1}{2}, 1] \times \prod_{i=1}^{r} [0, \frac{1}{m_i}] \times (0, 2\pi)^{r_2} \right) = \tilde{g}_{\overline{\gamma}}^{-1} \partial \left([\frac{1}{2}, 1] \times \prod_{i=1}^{r} [0, \frac{1}{m_i}] \times (0, 2\pi)^{r_2} \right).$$

It is easy to see that

$$\partial \tilde{g}_{\overline{\gamma}}^{-1} \left([\frac{1}{2}, 1] \times \prod_{i=1}^{r} [0, \frac{1}{m_i}] \times [0, 2\pi)^{r_2} \right) \subseteq \tilde{g}_{\overline{\gamma}}^{-1} \partial \left([\frac{1}{2}, 1] \times \prod_{i=1}^{r} [0, \frac{1}{m_i}] \times [0, 2\pi)^{r_2} \right).$$

We now define $2(r+1)$ sets as follows; $I_{1,1} = \{\frac{1}{2}\} \times \prod_{i=1}^{r} [0, \frac{1}{m_i}]$, $I_{1,2} = \{1\} \times \prod_{i=1}^{r} [0, \frac{1}{m_i}]$ and for $2 \leqslant j \leqslant r+1$

$$I_{j,l} = \begin{cases} [\frac{1}{2}, 1] \times \prod_{k=1}^{j-2} [0, \frac{1}{m_i}] \times \{0\} \times \prod_{k=j}^{r} [0, \frac{1}{m_i}] & \text{if } l = 1 \\ [\frac{1}{2}, 1] \times \prod_{k=1}^{j-2} [0, \frac{1}{m_i}] \times \{\frac{1}{m_{j-1}}\} \times \prod_{k=j}^{r} [0, \frac{1}{m_i}] & \text{if } l = 2. \end{cases}$$

For $1 \leqslant j \leqslant r+1$, define $\psi_{j,1}, \psi_{j,2} : [0,1]^{n_{\mathbf{K}}-1} \to I_{j,1} \times [0, 2\pi]^{r_2}$ as follows;

$$\begin{aligned} &\psi_{j,1}(t_1, \cdots, t_{n_{\mathbf{K}}-1}) \\ &= \begin{cases} (\frac{1}{2}, \frac{t_1}{m_1}, \cdots \frac{t_r}{m_r}, 2\pi t_{r+1} \cdots, 2\pi t_{n_{\mathbf{K}}-1}) & \text{if } j = 1, \\ (\frac{1+t_1}{2}, \frac{t_2}{m_1}, \cdots \frac{t_{j-1}}{m_{j-2}}, 0, \frac{t_j}{m_j} \cdots \frac{t_r}{m_r}, 2\pi t_{r+1} \cdots, 2\pi t_{n_{\mathbf{K}}-1}) & \text{if } j > 1, \end{cases} \end{aligned}$$

$$\begin{aligned} &\text{and } \psi_{j,2}(t_1, \cdots, t_{n_{\mathbf{K}}-1}) \\ &= \begin{cases} (1, \frac{t_1}{m_1}, \cdots \frac{t_r}{m_r}, 2\pi t_{r+1} \cdots, 2\pi t_{n_{\mathbf{K}}-1}) & \text{if } j = 1, \\ (\frac{1+t_1}{2}, \frac{t_2}{m_1}, \cdots \frac{t_{j-1}}{m_{j-2}}, \frac{1}{m_{j-1}}, \frac{t_j}{m_j} \cdots \frac{t_r}{m_r}, 2\pi t_{r+1} \cdots, 2\pi t_{n_{\mathbf{K}}-1}) & \text{if } j > 1. \end{cases} \end{aligned}$$

Further, for $\overline{\gamma} \in \{\pm 1\}^{r_1}$, we define $h_{\overline{\gamma}} : I_{j,1} \times [0, 2\pi]^{r_2} \to \mathbb{R}_{\overline{\gamma}}^{r_1} \times \mathbb{C}^{r_2}$ and $h'_{\overline{\gamma}} : \mathbb{R}_{\overline{\gamma}}^{r_1} \times \mathbb{C}^{r_2} \to \mathbb{R}^{n_{\mathbf{K}}}$ as follows; $h_{\overline{\gamma}}(x_1, \cdots, x_{n_{\mathbf{K}}}) = (y_1, \cdots, y_{r_1+r_2})$, where y_i's for $1 \leqslant i \leqslant r_1$ are defined by (7) and y_i's for $r_1 + 1 \leqslant i \leqslant r_1 + r_2$ are defined by (8) and $h'_{\overline{\gamma}}(y_1, \cdots, y_{r_1+r_2}) = (z_1, \cdots, z_{n_{\mathbf{K}}})$, where for $1 \leqslant i \leqslant r_1$, $z_i = y_i$ and for $r_1 + 1 \leqslant i \leqslant r_1 + r_2$,

$$z_i = |y_i| \cos(\arg(y_i)) \quad \text{and} \quad z_{r_2+i} = |y_i| \sin(\arg(y_i)),$$

where argument for y_i's is inside $[0, 2\pi)$. We now define $2(r+1)$ maps $\phi_{j,l} : [0,1]^{n_{\mathbf{K}}-1} \to \mathbb{R}^{n_{\mathbf{K}}}$ for $1 \leqslant j \leqslant r+1$ and $l = 1, 2$ as follows; $\phi_{j,l} = h'_{\overline{\gamma}} \circ h_{\overline{\gamma}} \circ \psi_{j,l}$. These $\phi_{j,l}$ cover the boundary of $(h'_{\overline{\gamma}} \circ \tilde{g}_{\overline{\gamma}}^{-1})\left([\frac{1}{2},1] \times \prod_{i=1}^{r}[0,\frac{1}{m_i}] \times [0,2\pi)^{r_2}\right)$ as $h'_{\overline{\gamma}}$ is a homeomorphism from $\mathbb{R}_{\overline{\gamma}}^{r_1} \times \mathbb{C}^{r_2}$ onto its image inside $\mathbb{R}^{n_{\mathbf{K}}}$ and hence

$$\partial\left(h'_{\overline{\gamma}}\left(\tilde{g}_{\overline{\gamma}}^{-1}\left([\frac{1}{2},1] \times \prod_{i=1}^{r}[0,\frac{1}{m_i}] \times [0,2\pi)^{r_2}\right)\right)\right)$$
$$= h'_{\overline{\gamma}}\left(\partial\tilde{g}_{\overline{\gamma}}^{-1}\left([\frac{1}{2},1] \times \prod_{i=1}^{r}[0,\frac{1}{m_i}] \times [0,2\pi)^{r_2}\right)\right).$$

We now compute the constants in Lemma 7 for each of the maps $\phi_{j,l}$. Since we know that $m_{k-1} = \max_i \lceil \log|\sigma_i(\eta_{k-1})| \rceil$ and $0 \leqslant t_{k-1} \leqslant 1$, we have

$$|\sigma_i(\eta_{k-1})|^{\frac{t_{k-1}}{m_{k-1}}} \leqslant e$$

for $1 \leqslant i \leqslant r_1$ and $2 \leqslant k \leqslant r+1$. Further for any $t, t' \in [0,1]$, using mean value theorem, we get

$$\left| |\sigma_i(\eta_{k-1})|^{\frac{t}{m_{k-1}}} - |\sigma_i(\eta_{k-1})|^{\frac{t'}{m_{k-1}}} \right| \leqslant e\,|t-t'|.$$

From now on, we denote the i-th projection map of $\phi_{j,l}$ by $\phi_{j,l}^i$. Applying Lemma 7 and the above observations, for any $\overline{t}, \overline{t}' \in [0,1]^{n_{\mathbf{K}}-1}$ and $1 \leqslant i \leqslant r_1$, we have

$$\left|\phi_{1,1}^i(\overline{t}) - \phi_{1,1}^i(\overline{t}')\right| \leqslant \left(\frac{1}{2}\right)^{\frac{1}{n_{\mathbf{K}}}} re^r\,||\overline{t}-\overline{t}'||.$$

Since for any $t, t' \in [0,1]$

$$|\cos(2\pi t) - \cos(2\pi t')| \leqslant 2\pi\,|t-t'| \quad \text{and} \quad |\sin(2\pi t) - \sin(2\pi t')| \leqslant 2\pi\,|t-t'|, \tag{9}$$

therefore for any $r_1 + 1 \leqslant i \leqslant n_{\mathbf{K}}$ and for any $\overline{t}, \overline{t}' \in [0,1]^{n_{\mathbf{K}}-1}$, as before using Lemma 7, we have

$$\left|\phi_{1,1}^i(\overline{t}) - \phi_{1,1}^i(\overline{t}')\right| \leqslant \left(\frac{1}{2}\right)^{\frac{1}{n_{\mathbf{K}}}} (2\pi + r)\,e^r\,||\overline{t}-\overline{t}'||.$$

Now applying second part of Lemma 7, for any $\overline{t}, \overline{t}' \in [0,1]^{n_{\mathbf{K}}-1}$, we have

$$\left|\phi_{1,1}(\overline{t}) - \phi_{1,1}(\overline{t}')\right| \leqslant \left(\frac{1}{2}\right)^{\frac{1}{n_{\mathbf{K}}}} \sqrt{n_{\mathbf{K}}}\,(2\pi + r)\,e^r\,||\overline{t}-\overline{t}'||. \tag{10}$$

Proceeding similarly, for any $\overline{t}, \overline{t}' \in [0,1]^{n_{\mathbf{K}}-1}$, we get

$$\left|\phi_{1,2}(\overline{t}) - \phi_{1,2}(\overline{t}')\right| \leqslant \sqrt{n_{\mathbf{K}}}\,(2\pi + r)\,e^r\,||\overline{t}-\overline{t}'||. \tag{11}$$

Since for any $t, t' \in [0,1]$

$$\left| \left(\frac{1+t}{2}\right)^{\frac{1}{n_{\mathbf{K}}}} - \left(\frac{1+t'}{2}\right)^{\frac{1}{n_{\mathbf{K}}}} \right| \leqslant \frac{2^{\frac{-1}{n_{\mathbf{K}}}}}{n_{\mathbf{K}}} \, |t-t'|, \tag{12}$$

we have for any $2 \leqslant j \leqslant 2(r+1)$, $1 \leqslant l \leqslant 2$, $\overline{t}, \overline{t}' \in [0,1]^{n_{\mathbf{K}}-1}$ and $1 \leqslant i \leqslant r_1$

$$\left|\phi^i_{j,l}(\overline{t}) - \phi^i_{j,l}(\overline{t}')\right| \leqslant \left(\frac{2^{\frac{-1}{n_{\mathbf{K}}}}}{n_{\mathbf{K}}} + r - 1\right) e^r \, ||\overline{t} - \overline{t}'||.$$

Using (9) and (12) and proceeding as before, we get for any $2 \leqslant j \leqslant 2(r+1)$, $1 \leqslant l \leqslant 2$, $\overline{t}, \overline{t}' \in [0,1]^{n_{\mathbf{K}}-1}$ and $r_1 + 1 \leqslant i \leqslant n_{\mathbf{K}}$

$$\left|\phi^i_{j,l}(\overline{t}) - \phi^i_{j,l}(\overline{t}')\right| \leqslant \left(2\pi + \frac{2^{\frac{-1}{n_{\mathbf{K}}}}}{n_{\mathbf{K}}} + r - 1\right) e^r ||\overline{t} - \overline{t}'|| \leqslant (2\pi + r)e^r \, ||\overline{t} - \overline{t}'||.$$

Combining, for any $2 \leqslant j \leqslant 2(r+1)$, $1 \leqslant l \leqslant 2$ and $\overline{t}, \overline{t}' \in [0,1]^{n_{\mathbf{K}}-1}$, we have

$$\left|\phi_{j,l}(\overline{t}) - \phi_{j,l}(\overline{t}')\right| \leqslant \sqrt{n_{\mathbf{K}}} \, (2\pi + r)e^r \, ||\overline{t} - \overline{t}'||. \tag{13}$$

If the interval $[\frac{1}{2}, 1)$ is replaced by $[\frac{t^{n_{\mathbf{K}}}}{2}, t^{n_{\mathbf{K}}})$ for some positive $t \in \mathbb{R}$, we deduce in a similar way that the bound L in the Definition 4 of Lipschitz class is less than $t\sqrt{n_{\mathbf{K}}}(2\pi + r)e^r$. Hence $\partial \mathbf{F}_{1/2,\overline{\gamma}}(0, \frac{1}{m_1}, \cdots, 0\frac{1}{m_r}, t^{n_{\mathbf{K}}})$ is of Lipschitz class $\mathcal{L}(n_{\mathbf{K}}, 2r+2, Lt)$, where $L = \sqrt{n_{\mathbf{K}}}(2\pi + r)e^r$. □

3.4. Preliminary lemmas

Using Theorem 2 and Theorem 4, we derive the following bounds on the product $\prod_{j=1}^r m_j$, where $m_j = \max_i \lceil \log|\sigma_i(\eta_j)| \rceil$.

Lemma 9. *Let $\mathfrak{q}_1$ be a modulus of $\mathbf{K}$. There exist r units $\eta_1, \cdots \eta_r$ modulo roots of unity which generate $U_{\mathfrak{q}_1}$ such that*

$$\frac{R_{\mathbf{K},\mathfrak{q}_1}}{2^r(r+1)^{\frac{r-1}{2}}} \leqslant \prod_{j=1}^r m_j \leqslant 7^r(r+1)^{r+1/2} n_{\mathbf{K}}^{2r} \, R_{\mathbf{K},\mathfrak{q}_1},$$

where as before $r = r_1 + r_2 - 1$ and $R_{\mathbf{K},\mathfrak{q}_1}$ is the $\mathfrak{q}_1$ regulator of the field $\mathbf{K}$.

Proof. If there are no fundamental units, the regulator $R_{\mathbf{K},\mathfrak{q}_1}$ is defined to be 1 and hence the above inequalities are trivially satisfied. Hence from now on, we assume that $r \geqslant 1$. Let $\{\eta_1', \cdots \eta_r'\}$ be any set of generators of $U_{\mathfrak{q}_1}$ modulo roots of unity. Let l denote the map

$$\begin{aligned} l : \mathbb{R}^{r_1} \times \mathbb{C}^{r_2} &\to \mathbb{R}^{r+1} \\ (x_i)_i &\to (e_i \log |x_i|)_{i=1}^{r+1} \end{aligned}$$

Here e_i is 2 if $r_1 + 1 \leqslant i \leqslant r + 1$ and 1 otherwise. Set $\Lambda_r = l(\phi(U_{\mathfrak{q}_1}))$, where ϕ is as in subsection 3.1. Note that Λ_r is a lattice of rank r. The vectors

$$\vec{v_1} = \left(\frac{1}{\sqrt{r+1}}, \cdots, \frac{1}{\sqrt{r+1}}\right), \quad \vec{v_j} = \left(e_i \log |\sigma_i(\eta'_{j-1})|\right)_{i=1}^{r+1}$$

for $2 \leqslant j \leqslant r+1$ form a basis for $\mathbb{R}^{r+1}$. Since $||\vec{v_1}|| = 1$ and v_1 is orthogonal to the $\mathbb{R}$ vector space generated by $\{v_2, \cdots v_{r+1}\}$, the volume of the lattice generated by the vectors $\{v_1, \cdots v_{r+1}\}$ in $\mathbb{R}^{r+1}$ is the same as the volume of Λ_r in $\mathbb{R}^r$. Hence the volume of Λ_r is $\sqrt{r+1}\ R_{\mathbf{K},\mathfrak{q}_1}$. Note that the volume of Λ_r is independent of the choice of the generators $\{\eta'_1, \cdots \eta'_r\}$ of $U_{\mathfrak{q}_1}$.

Let $\{\vec{w_1} \ldots \vec{w_r}\}$ be a reduced Korkin Zolotarev basis for Λ_r. Choose a set of generators $\eta_1, \cdots \eta_r$ of $U_{\mathfrak{q}_1}$ modulo roots of unity such that $f(\phi(\eta_i)) = \vec{w_i}$. Then for $1 \leqslant j \leqslant r$

$$m_j \ \geqslant\ \frac{1}{2}\max_i(e_i \log(|\sigma_i(\eta_j)|)) \ \geqslant\ \frac{1}{2\sqrt{r+1}}||\vec{w_j}||.$$

Since $\prod_{j=1}^r ||\vec{w_j}||$ is greater than or equal to the volume, we have

$$\prod_{j=1}^r m_j \ \geqslant\ \left(\frac{1}{2\sqrt{r+1}}\right)^r \text{Vol}\ (\Lambda_r) \ \geqslant\ \frac{R_{\mathbf{K},\mathfrak{q}_1}}{2^r(r+1)^{\frac{r-1}{2}}}.$$

We now compute the upper bound. Since there is at least one fundamental unit, we know that $n_{\mathbf{K}} > 1$. If $n_{\mathbf{K}} = 2$, then $R_{\mathbf{K},\mathfrak{q}_1} = |\log|\eta|\ |$, where η generates $U_{\mathfrak{q}_1}$ modulo roots of unity. Since $R_{\mathbf{K},\mathfrak{q}_1} \geqslant R_{\mathbf{K}} \geqslant 1/5$ (see [3]), we get the required upper bound in this case. If $n_{\mathbf{K}} \geqslant 3$, then

$$1 + \frac{\log n_{\mathbf{K}}}{6n_{\mathbf{K}}^2} \ \geqslant\ 1 + \frac{1}{7n_{\mathbf{K}}^2 - 1} \ \geqslant\ e^{\frac{1}{7n_{\mathbf{K}}^2}}.$$

Now by Dobrowolski's theorem, we know that $\max_i(e_i \log|\sigma_i(\eta_j)|) \geqslant \frac{1}{7n_{\mathbf{K}}^2}$. This implies that $m_j \ \leqslant\ 7n_{\mathbf{K}}^2 \max_i(e_i \log|\sigma_i(\eta_j)|) \ \leqslant\ 7n_{\mathbf{K}}^2\ ||\vec{w_j}||$. Using Theorem 2, we now get

$$\begin{aligned} \prod_{j=1}^r m_j \ &\leqslant\ (7n_{\mathbf{K}}^2)^r \prod_{j=1}^r ||\vec{w_j}|| \ \leqslant\ (7n_{\mathbf{K}}^2)^r \left(\frac{3+r}{4}\right)^{r/2} r^{r/2}\ \sqrt{r+1}\ R_{\mathbf{K},\mathfrak{q}_1} \\ &\leqslant\ 7^r(r+1)^{r+1/2} n_{\mathbf{K}}^{2r}\ R_{\mathbf{K},\mathfrak{q}_1}. \quad \square \end{aligned}$$

Lemma 10. *Let $\mathfrak{q}_1$ be a modulus of $\mathbf{K}$ with $n_{\mathbf{K}} > 1$, $\{\sigma_1, \cdots, \sigma_r\}$ be a set of embeddings of $\mathbf{K}$ into $\mathbb{C}$ such that no two embeddings are conjugate to each other. For η_j and m_j as defined earlier and $\alpha \in \mathbf{K}$, the integral*

$$\int_{\mathbb{R}^r} \frac{dx_1 \cdots dx_r}{max_{1\leqslant i\leqslant r+1}(\prod_{j=1}^r |\sigma_i(\eta_j)|^{x_j/m_j}|\sigma_i(\alpha)|)^{n_{\mathbf{K}}-1}} = \frac{m_1\cdots m_r}{R_{\mathbf{K},\mathfrak{q}_1}|\mathfrak{N}(\alpha)|^{\frac{n_{\mathbf{K}}-1}{n_{\mathbf{K}}}}}\left(\frac{n_{\mathbf{K}}}{n_{\mathbf{K}}-1}\right)^r. \tag{14}$$

Proof. Using the transformation $x_j \to m_j x_j$, we see that the left hand side of (14) is equal to

$$m_1\cdots m_r \int_{\mathbb{R}^r} \frac{dx_1\cdots dx_r}{\max_{1\leqslant i\leqslant r+1}(\prod_{j=1}^r |\sigma_i(\eta_j)|^{x_j}|\sigma_i(\alpha)|)^{n_{\mathbf{K}}-1}}. \tag{15}$$

We now make the substitution $y_i = \sum_{j=1}^r x_j \log|\sigma_i(\eta_j)|$, for $1 \leqslant i \leqslant r$. Hence (15) is equal to

$$m_1\cdots m_r \frac{\prod_{i=1}^r e_i}{R_{\mathbf{K},\mathfrak{q}_1}} \int_{\mathbb{R}^r} \frac{dy_1\cdots dy_r}{\max(e^{y_1}|\sigma_1(\alpha)|,\cdots e^{y_r}|\sigma_r(\alpha)|, e^{-Y}|\sigma_{r+1}(\alpha)|)^{n_{\mathbf{K}}-1}}$$

where e_i for $1 \leqslant i \leqslant r$ is 1 or 2 depending on σ_i is real or complex embedding and

$$\begin{aligned} Y = -\sum_{j=1}^r x_j \log|\sigma_{r+1}(\eta_j)| &= \frac{1}{e_{r+1}}\sum_{j=1}^r x_j(-e_{r+1}\log|\sigma_{r+1}(\eta_j)|) \\ &= \frac{1}{e_{r+1}}\sum_{j=1}^r x_j \sum_{k=1}^r e_k \log|\sigma_k(\eta_j)| = \frac{1}{e_{r+1}}\sum_{k=1}^r e_k y_k. \end{aligned}$$

The integral is now identical to the one computed in the proof of Lemma 10 of [1]. □

3.5. Counting points in the fundamental domain

Applying Theorem 3, we will now derive the following counting theorem.

Theorem 11. *Let $\mathfrak{a}, \mathfrak{q}$ be co-prime ideals of $\mathcal{O}_{\mathbf{K}}$, $\overline{\gamma} \in \{\pm 1\}^{r_1}$ and $\mathfrak{C}$ be the ideal class of $\mathfrak{a}\mathfrak{q}$ in the class group of $\mathcal{O}_{\mathbf{K}}$. For $\vec{k} \in \prod_{j=1}^r([0, m_j) \cap \mathbb{Z})$, let $\beta_{\vec{k}} = \left(\prod_{j=1}^r |\sigma_i(\eta_j)|^{-k_j/m_j}\right)_{i=1}^{r+1}$ and $\Lambda_{n_{\mathbf{K}}}(\vec{k})$ be the lattice $h'\left(\phi(\mathfrak{a}\mathfrak{q}) \cdot \beta_{\vec{k}}\right)$ in $\mathbb{R}^{n_{\mathbf{K}}}$, where ϕ is as in subsection 3.1 and h' is as in subsection 3.3. Also let*

$$S\left(\mathfrak{a}, \mathfrak{q}, \mathbf{F}_{\frac{1}{2},\overline{\gamma}}(t^{n_{\mathbf{K}}})\right) = \{\alpha \in \mathfrak{a} \mid \phi(\alpha) \in \mathbf{F}_{\frac{1}{2},\overline{\gamma}}(t^{n_{\mathbf{K}}}),\ \alpha \equiv b \bmod \mathfrak{q}\},$$

where $b \in \mathcal{O}_{\mathbf{K}}$. Then for any real number $t \geqslant 1$, we have

$$\begin{aligned} \left|S\left(\mathfrak{a}, \mathfrak{q}, \mathbf{F}_{\frac{1}{2},\overline{\gamma}}(t^{n_{\mathbf{K}}})\right)\right| = &\frac{(2\pi)^{r_2} R_{\mathbf{K},\mathfrak{q}_1} t^{n_{\mathbf{K}}}}{\sqrt{4|d_{\mathbf{K}}|}\ \mathfrak{N}(\mathfrak{a}\mathfrak{q})} \\ &+\ \mathrm{O}^*\left(\frac{e^{n_{\mathbf{K}}^2+8n_{\mathbf{K}}} n_{\mathbf{K}}^{\frac{3}{2}n_{\mathbf{K}}^2+\frac{11}{2}n_{\mathbf{K}}-\frac{1}{2}} t^{n_{\mathbf{K}}-1}}{\mathfrak{N}(\mathfrak{C}^{-1})^{-1}|\mathfrak{N}(\mathfrak{a}\mathfrak{q})|^{\frac{n_{\mathbf{K}}-1}{n_{\mathbf{K}}}}} + m_1\cdots m_r\right), \end{aligned}$$

where $\mathfrak{q}_1 = \mathfrak{q}\mathfrak{q}_\infty$ *with* $\mathfrak{q}_\infty$ *containing all the infinite places of* $\mathbf{K}$ *and*

$$\mathfrak{N}(\mathfrak{C}^{-1}) = max_{\mathfrak{b}_i \in \mathfrak{C}^{-1}} \sum_{i=1}^{m_1 \cdots m_r} \frac{1}{|\mathfrak{N}(\mathfrak{b}_i)|^{\frac{n_{\mathbf{K}}-1}{n_{\mathbf{K}}}}}.$$

The term $m_1 \cdots m_r$ *may be omitted when* $t \geqslant \frac{\max_{\vec{k}} \ \delta_1\left(\Lambda_{n_{\mathbf{K}}}(\vec{k})\right)}{\sqrt{n_{\mathbf{K}}}(2\pi+r)e^r}$ *or* $\mathfrak{q} = \mathcal{O}_{\mathbf{K}}$*.*

Proof. Since we want to count $\alpha \in \mathfrak{a}$ such that $\alpha \equiv b \bmod \mathfrak{q}$, where $\mathfrak{a}, \mathfrak{q}$ are co-prime, we need to count α in only one residue class, say a modulo $\mathfrak{a}\mathfrak{q}$. Recall that $\mathbf{K} \neq \mathbb{Q}$. We need to count

$$\left|(\phi(\mathfrak{a}\mathfrak{q}) + \phi(a)) \bigcap \mathbf{F}_{\frac{1}{2},\overline{\gamma}}(t^{n_{\mathbf{K}}})\right|.$$

By Lemma 5, we have

$$\begin{aligned} &|(\phi(\mathfrak{a}\mathfrak{q}) + \phi(a)) \bigcap \mathbf{F}_{\frac{1}{2},\overline{\gamma}}(t^{n_{\mathbf{K}}})| \\ &= \sum_{\vec{k}} | \left((\phi(\mathfrak{a}\mathfrak{q}) + \phi(a)) \cdot \beta_{\vec{k}}\right) \bigcap \mathbf{F}_{\frac{1}{2},\overline{\gamma}}(0, \frac{1}{m_1}, \cdots, 0, \frac{1}{m_r}, t^{n_{\mathbf{K}}})|, \end{aligned} \tag{16}$$

where $\vec{k} \in \prod_{j=1}^{r}([0, m_j) \cap \mathbb{Z})$. Since

$$\mathrm{Vol}(h'\left((\phi(\mathfrak{a}\mathfrak{q}) + \phi(a)) \cdot \beta_{\vec{k}}\right)) \ = \ \mathrm{Vol}(h'(\phi(\mathfrak{a}\mathfrak{q}) \cdot \beta_{\vec{k}})),$$

applying Widmer's estimate (Theorem 3), the main term for each $\vec{k}$ is

$$\frac{\mathrm{Vol}\left(h'\left(\mathbf{F}_{\frac{1}{2},\overline{\gamma}}(0, \frac{1}{m_1}, \cdots, 0, \frac{1}{m_r}, t^{n_{\mathbf{K}}})\right)\right)}{\mathrm{Vol}\left(h'\left(\phi(\mathfrak{a}\mathfrak{q}) \cdot \beta_{\vec{k}}\right)\right)}.$$

The volume of $h'(\phi(\mathfrak{a}\mathfrak{q}))$ is the determinant of the matrix whose column vectors form a basis for this lattice $h'(\phi(\mathfrak{a}\mathfrak{q}))$. If $\beta_{\vec{k}} = (\beta_i)_{i=1}^{r+1}$, multiplying the i-the entry of each basis vector with β_i for $1 \leqslant i \leqslant r_1 + r_2$ and for $r_1 + r_2 \leqslant i \leqslant r_1 + 2r_2$ with β_{i-r_2}, we get that

$$\mathrm{Vol}(h'(\phi(\mathfrak{a}\mathfrak{q}) \cdot \beta_{\vec{k}})) = \mathrm{Vol}(h'(\phi(\mathfrak{a}\mathfrak{q}))).$$

This implies that the main term for each $\vec{k}$ is independent of $\vec{k}$ and equal to

$$\frac{\mathrm{Vol}(h'(\mathbf{F}_{\frac{1}{2},\overline{\gamma}}(0, \frac{1}{m_1}, \cdots, 0, \frac{1}{m_r}, t^{n_{\mathbf{K}}})))}{\mathrm{Vol}(h'(\phi(\mathfrak{a}\mathfrak{q})))}. \tag{17}$$

We know that $\mathrm{Vol}(h'(\phi(\mathfrak{a}\mathfrak{q}))) = 2^{-r_2}\sqrt{|d_{\mathbf{K}}|}\mathfrak{N}(\mathfrak{a}\mathfrak{q})$ (see page 31 of [7]). To compute the volume $\mathrm{Vol}(h'(\mathbf{F}_{\frac{1}{2}}(0, \frac{1}{m_1}, \cdots, 0, \frac{1}{m_r}, t^{n_{\mathbf{K}}})))$, we note that

$$\begin{aligned}
&\mathrm{Vol}\left(h'\left(\mathbf{F}_{\frac{1}{2}}(0, \frac{1}{m_1}, \cdots, 0, \frac{1}{m_r},\ t^{n_{\mathbf{K}}})\right)\right) \\
&\quad = \int_{h'(\mathbf{F}_{\frac{1}{2}}(0,\frac{1}{m_1},\cdots,0,\frac{1}{m_r},\ t^{n_{\mathbf{K}}}))} dx_1 \cdots dx_{r_1} dx_{r_1+1} \cdots dx_{n_{\mathbf{K}}},
\end{aligned}$$

where the variables x_{r_1+1} to $x_{n_{\mathbf{K}}}$ are complex. By definition

$$\mathbf{F}_{\frac{1}{2}}\left(0, \frac{1}{m_1}, \cdots, 0, \frac{1}{m_r}, t^{n_{\mathbf{K}}}\right) = g^{-1}\left((\frac{t^{n_{\mathbf{K}}}}{2}, t^{n_{\mathbf{K}}}] \times \prod_{j=1}^{r}[0, \frac{1}{m_j})\right),$$

the argument for each complex coordinate in the pre-image covers the entire interval $[0, 2\pi)$. Replacing the complex variables with polar co-ordinates and integrating over the arguments,

$$\begin{aligned}
&\mathrm{Vol}\left(h'\left(\mathbf{F}_{\frac{1}{2}}(0, \frac{1}{m_1}, \cdots, \frac{1}{m_r}, t^{n_{\mathbf{K}}})\right)\right) \\
&= 2^{r_1}(2\pi)^{r_2}\int_{|x_1|} d|x_1| \cdots \int_{|x_{r_1}|} d|x_{r_1}| \int_{|x_{r_1+1}|} |x_{r_1+1}|\ \ d|x_{r_1+1}| \cdots \int_{|x_{r+1}|} |x_{r+1}|\ \ d|x_{r+1}|.
\end{aligned}$$

The ranges in the above integral are clear from the formulae (4) with $\alpha(x) \in (\frac{t^{n_{\mathbf{K}}}}{2}, t^{n_{\mathbf{K}}}]$ and $\alpha_j(x) \in [0, \frac{1}{m_j})$. We now make a change of variable $x = (x_1, x_2, \cdots x_{r+1})$ by $(\alpha(x), \alpha_1(x), \cdots \alpha_r(x))$. To compute the Jacobian, we note that for $1 \leqslant i \leqslant r+1$ and $1 \leqslant j \leqslant r$, we have

$$\frac{\partial |x_i|}{\partial \alpha(x)} = \frac{|x_i|}{n_{\mathbf{K}}\alpha(x)} \quad \text{and} \quad \frac{\partial |x_i|}{\partial \alpha_j(x)} = |x_i| \log|\sigma_i(\eta_j)|.$$

We also note that $\alpha(x) = \prod_{i=1}^{r+1} |x_i|^{e_i}$ where $e_i = 2$ for complex coordinate and $e_i = 1$ otherwise. Now rewriting the integral, we have

$$\begin{aligned}
&\mathrm{Vol}\left(\mathbf{F}_{\frac{1}{2}}(0, \frac{1}{m_1}, \cdots, 0, \frac{1}{m_r}, t^{n_{\mathbf{K}}})\right) \\
&= 2^{r_1}(2\pi)^{r_2}2^{-r_2}R_{\mathbf{K},\mathfrak{q}_1}\int_0^{1/m_1} d\alpha_1(x) \cdots \int_0^{1/m_r} d\alpha_r(x) \int_{\frac{t^{n_{\mathbf{K}}}}{2}}^{t^{n_{\mathbf{K}}}} d\alpha(x) \\
&= \frac{2^{r_1-1}\pi^{r_2}R_{\mathbf{K},\mathfrak{q}_1}t^{n_{\mathbf{K}}}}{m_1\cdots m_r}.
\end{aligned}$$

Therefore for each $\vec{k}$, the main term (17) is equal to

$$\frac{1}{2^{r_1}} \cdot \frac{2^{r_1-1}\pi^{r_2}R_{\mathbf{K},\mathfrak{q}_1}t^{n_{\mathbf{K}}}}{m_1\cdots m_r} \cdot \frac{1}{2^{-r_2}\sqrt{|d_{\mathbf{K}}|}\mathfrak{N}(\mathfrak{a}\mathfrak{q})} = \frac{(2\pi)^{r_2}R_{\mathbf{K},\mathfrak{q}_1}t^{n_{\mathbf{K}}}}{(m_1\cdots m_r)\sqrt{4|d_{\mathbf{K}}|}\mathfrak{N}(\mathfrak{a}\mathfrak{q})}.$$

Hence the main term after summing over $\vec{k} \in \prod_{j=1}^{r}[0, m_j) \cap \mathbb{Z}$ is equal to

$$\frac{(2\pi)^{r_2} R_{\mathbf{K},\mathfrak{q}_1} t^{n_{\mathbf{K}}}}{\sqrt{4|d_{\mathbf{K}}|\mathfrak{N}(\mathfrak{aq})}}. \tag{18}$$

Applying Theorem 3 and Lemma 8, the error term for each $\vec{k}$ is bounded by

$$\begin{aligned}&(2r+2)n_{\mathbf{K}}^{3n_{\mathbf{K}}^2/2}\max_{0\leqslant i<n_{\mathbf{K}}}\frac{(\sqrt{n_{\mathbf{K}}}(2\pi+r)e^r t)^i}{\prod_{j=1}^i \delta_j(\Lambda_{n_{\mathbf{K}}}(\vec{k}))}\\ &\leqslant (2r+2)n_{\mathbf{K}}^{3n_{\mathbf{K}}^2/2}\max_{0\leqslant i<n_{\mathbf{K}}}\left(\frac{\sqrt{n_{\mathbf{K}}}(2\pi+r)e^r t}{\delta_1(\Lambda_{n_{\mathbf{K}}}(\vec{k}))}\right)^i. \end{aligned} \tag{19}$$

If we have $\sqrt{n_{\mathbf{K}}}(2\pi+r)e^r t \geqslant \delta_1(\Lambda_{n_{\mathbf{K}}}(\vec{k}))$, then to get an upper bound of (19), we can replace i by $n_{\mathbf{K}}-1$. To deduce the asymptotic for all $t \geqslant 1$, we first consider the case where $\delta_1(\Lambda_{n_{\mathbf{K}}}(\vec{k})) > \sqrt{n_{\mathbf{K}}}(2\pi+r)e^r t$ for some vector $\vec{k}$. In this case, we claim that

$$\left|\left((\phi(\mathfrak{aq})+\phi(a))\cdot\beta_{\vec{k}}\right)\bigcap \mathbf{F}_{\frac{1}{2},\overline{\gamma}}\left(0,\frac{1}{m_1},\cdots,0,\frac{1}{m_r},t^{n_{\mathbf{K}}}\right)\right| \leqslant 1.$$

Suppose not and let $x, y \in \left((\phi(\mathfrak{aq})+\phi(a))\cdot\beta_{\vec{k}}\right)\bigcap \mathbf{F}_{\frac{1}{2},\overline{\gamma}}\left(0,\frac{1}{m_1},\cdots,0,\frac{1}{m_r},t^{n_{\mathbf{K}}}\right)$ be distinct. Then $x-y \in \phi(\mathfrak{aq})\cdot\beta_{\vec{k}}$. By definition, this implies that $\delta_1(\Lambda_{n_{\mathbf{K}}}(\vec{k})) \leqslant ||h'(x-y)||$. Applying Lemma 6, we get

$$\sqrt{n_{\mathbf{K}}}(2\pi+r)\ e^r t \ < \ \delta_1(\Lambda_{n_{\mathbf{K}}}(\vec{k})) \ \leqslant \ ||h'(x-y)|| \ \leqslant \ 2(\sqrt{r+1}\ )\ e^r t,$$

a contradiction. If $\mathfrak{q} = \mathcal{O}_{\mathbf{K}}$, the same argument applies to x in place of $x-y$ and this implies that there are no exceptional points. The total number of $\vec{k}$ for which $\delta_1(\Lambda_{n_{\mathbf{K}}}(\vec{k})) > \sqrt{n_{\mathbf{K}}}(2\pi+r)e^r t$ is at most $m_1\cdots m_r$. Hence

$$\begin{aligned}&\sum_{\substack{\vec{k}\\ \sqrt{n_{\mathbf{K}}}(2\pi+r)e^r t\ <\ \delta_1(\Lambda_{n_{\mathbf{K}}}(\vec{k}))}} \left|\left((\phi(\mathfrak{aq})+\phi(a))\cdot\beta_{\vec{k}}\right)\bigcap \mathbf{F}_{\frac{1}{2},\overline{\gamma}}\left(0,\frac{1}{m_1},\cdots,0,\frac{1}{m_r},t^{n_{\mathbf{K}}}\right)\right|\\ &\leqslant\ m_1\cdots m_r.\end{aligned}$$

For each $\vec{k}$ for which $\sqrt{n_{\mathbf{K}}}(2\pi+r)e^r t < \delta_1(\Lambda_{n_{\mathbf{K}}}(\vec{k}))$, we claim that

$$\frac{(2\pi)^{r_2} R_{\mathbf{K},\mathfrak{q}_1} t^{n_{\mathbf{K}}}}{(m_1\cdots m_r)\sqrt{4|d_{\mathbf{K}}|\mathfrak{N}(\mathfrak{aq})}} \ \leqslant\ (2r+2)n_{\mathbf{K}}^{3n_{\mathbf{K}}^2/2}\left(\frac{\sqrt{n_{\mathbf{K}}}(2\pi+r)e^r t}{\delta_1(\Lambda_{n_{\mathbf{K}}}(\vec{k}))}\right)^{n_{\mathbf{K}}-1}.$$

Note that $2^{r-1}\pi^{r_2}(r+1)^{\frac{r-1}{2}} n_{\mathbf{K}}^{n_{\mathbf{K}}} \ \leqslant\ (2r+2)n_{\mathbf{K}}^{\frac{n_{\mathbf{K}}}{2}(3n_{\mathbf{K}}+1)}(2\pi+r)^{n_{\mathbf{K}}}$ when $n_{\mathbf{K}} \geqslant 2$ and hence

$$\frac{(2\pi)^{r_2} 2^r (r+1)^{\frac{r-1}{2}} t^{n_{\mathbf{K}}}}{2\sqrt{|d_{\mathbf{K}}|\mathfrak{N}(\mathfrak{aq})}} \ \leqslant\ \frac{(2r+2)n_{\mathbf{K}}^{\frac{3n_{\mathbf{K}}^2}{2}}}{\mathrm{Vol}(h'(\phi(\mathfrak{aq})\cdot\beta_{\vec{k}}))}\left(\frac{\sqrt{n_{\mathbf{K}}}(2\pi+r)t}{n_{\mathbf{K}}}\right)^{n_{\mathbf{K}}}.$$

By Theorem 2, we have $\delta_1(\Lambda_{n_{\mathbf{K}}}(\vec{k}))^{n_{\mathbf{K}}} \leqslant n_{\mathbf{K}}^{n_{\mathbf{K}}}\mathrm{Vol}(h'(\phi(\mathfrak{a}\mathfrak{q})\cdot\beta_{\vec{k}}))$ and therefore

$$\frac{(2\pi)^{r_2}2^r(r+1)^{\frac{r-1}{2}}t^{n_{\mathbf{K}}}}{2\sqrt{|d_{\mathbf{K}}|}\mathfrak{N}(\mathfrak{a}\mathfrak{q})} \leqslant (2r+2)n_{\mathbf{K}}^{\frac{3n_{\mathbf{K}}^2}{2}}\left(\frac{\sqrt{n_{\mathbf{K}}}(2\pi+r)e^r t}{\delta_1(\Lambda_{n_{\mathbf{K}}}(\vec{k}))}\right)^{n_{\mathbf{K}}}.$$

Finally applying Lemma 9 and $\sqrt{n_{\mathbf{K}}}(2\pi+r)e^r t < \delta_1(\Lambda_{n_{\mathbf{K}}}(\vec{k}))$, we get

$$\frac{(2\pi)^{r_2}R_{\mathbf{K},\mathfrak{q}_1}t^{n_{\mathbf{K}}}}{m_1\cdots m_r\sqrt{4|d_{\mathbf{K}}|}\mathfrak{N}(\mathfrak{a}\mathfrak{q})} \leqslant (2r+2)n_{\mathbf{K}}^{\frac{3n_{\mathbf{K}}^2}{2}}\left(\frac{\sqrt{n_{\mathbf{K}}}(2\pi+r)e^r t}{\delta_1(\Lambda_{n_{\mathbf{K}}}(\vec{k}))}\right)^{n_{\mathbf{K}}-1},$$

as claimed. Therefore for $t \geqslant 1$ the error is bounded by

$$(2r+2)n_{\mathbf{K}}^{\frac{3n_{\mathbf{K}}^2}{2}}(\sqrt{n_{\mathbf{K}}}(2\pi+r)e^r t)^{n_{\mathbf{K}}-1}\sum_{\vec{k}}\frac{1}{\delta_1(\Lambda_{n_{\mathbf{K}}}(\vec{k}))^{n_{\mathbf{K}}-1}} + m_1\cdots m_r. \tag{20}$$

If there are no $\vec{k}$ such that $\sqrt{n_{\mathbf{K}}}(2\pi+r)e^r t < \max_{\vec{k}}\delta_1\left(\Lambda_{n_{\mathbf{K}}}(\vec{k})\right)$, then the error term is bounded by

$$(2r+2)n_{\mathbf{K}}^{3n_{\mathbf{K}}^2/2}(\sqrt{n_{\mathbf{K}}}(2\pi+r)e^r t)^{n_{\mathbf{K}}-1}\sum_{\vec{k}}\frac{1}{\delta_1(\Lambda_{n_{\mathbf{K}}}(\vec{k}))^{n_{\mathbf{K}}-1}}. \tag{21}$$

Let

$$\mu(\mathfrak{a}\mathfrak{q},\vec{k}) = \min_{\alpha\in\mathfrak{a}\mathfrak{q}}\max_{1\leqslant i\leqslant r+1}\left(|\sigma_i(\alpha)|\prod_{j=1}^{r}|\sigma_i(\eta_j)|^{-k_j/m_j}\right).$$

From the definition of successive minima $\delta_1(\Lambda_{n_{\mathbf{K}}}(\vec{k})) \geqslant \mu(\mathfrak{a}\mathfrak{q},\vec{k})$. For any $\alpha \in \mathfrak{a}\mathfrak{q}$, let K_α be the set of all $\vec{k}$ for which the minimum $\mu(\mathfrak{a}\mathfrak{q},\vec{k})$ is attained at α, i.e.

$$\mathrm{K}_\alpha = \left\{\vec{k}\ \middle|\ \mu(\mathfrak{a}\mathfrak{q},\vec{k}) = \max_{1\leqslant i\leqslant r+1}(|\sigma_i(\alpha)|\prod_{j=1}^{r}|\sigma_i(\eta_j)|^{-k_j/m_j})\right\}.$$

We also set $Y_1 = \{\alpha\in\mathfrak{a}\mathfrak{q} \mid \mathrm{K}_\alpha \neq \varnothing\}$. Substituting in (21) (analogously in (20)), we get

$$\begin{aligned}\sum_{\vec{k}}\frac{1}{\delta_1(\Lambda_{n_{\mathbf{K}}}(\vec{k}))^{n_{\mathbf{K}}-1}} &\leqslant \sum_{\vec{k}}\frac{1}{\mu(\mathfrak{a}\mathfrak{q},\vec{k})^{n_{\mathbf{K}}-1}} = \sum_{\alpha\in Y_1}\sum_{\vec{k}\in\mathrm{K}_\alpha}\frac{1}{\mu(\mathfrak{a}\mathfrak{q},\vec{k})^{n_{\mathbf{K}}-1}}\\ &\leqslant \sideset{}{'}\sum_{\alpha\in Y_1}\sum_{u\in U_{\mathfrak{q}_1}}\sum_{\vec{k}\in\mathrm{K}_{u\alpha}}\frac{1}{\max_{1\leqslant i\leqslant r+1}\left(|\sigma_i(u\alpha)|\prod_{j=1}^{r}|\sigma_i(\eta_j)|^{-k_j/m_j}\right)^{n_{\mathbf{K}}-1}}\\ &\leqslant \sideset{}{'}\sum_{\alpha\in Y_1}\sum_{\vec{k}\in\mathbb{Z}^r}\frac{1}{\max_{1\leqslant i\leqslant r+1}(|\sigma_i(\alpha)|\prod_{j=1}^{r}|\sigma_i(\eta_j)|^{-k_j/m_j})^{n_{\mathbf{K}}-1}},\end{aligned}$$

where $'$ on the inner sum indicates that the sum is over non-associate elements of $\mathfrak{aq}$ with respect to the unit group $U_{\mathfrak{q}_1}$. Note that there are at most $m_1 \cdots m_r$ elements in the outermost sum. Applying Lemma 10, we now bound the inner sum to get

$$\sum_{\vec{k}\in\mathbb{Z}^r} \frac{1}{\max_{1\leqslant i\leqslant r+1}(|\sigma_i(\alpha)|\prod_{j=1}^r |\sigma_i(\eta_j)|^{-k_j/m_j})^{n_{\mathbf{K}}-1}} \leqslant \frac{2^r m_1\cdots m_r}{R_{\mathbf{K},\mathfrak{q}_1}|\mathfrak{N}(\alpha)|^{\frac{n_{\mathbf{K}}-1}{n_{\mathbf{K}}}}}.$$

Now the term

$$\begin{aligned}\sideset{}{'}\sum_{\alpha\in Y_1} \frac{1}{|\mathfrak{N}(\alpha)|^{\frac{n_{\mathbf{K}}-1}{n_{\mathbf{K}}}}} &\leqslant \max_{\alpha_i\in\mathfrak{aq}} \sum_{i=1}^{m_1\cdots m_r} \frac{1}{|\mathfrak{N}(\alpha_i)|^{\frac{n_{\mathbf{K}}-1}{n_{\mathbf{K}}}}} \\ &\leqslant \left(\frac{1}{\mathfrak{N}(\mathfrak{aq})}\right)^{\frac{n_{\mathbf{K}}-1}{n_{\mathbf{K}}}} \max_{\mathfrak{b}_i\in\mathfrak{C}^{-1}} \sum_{i=1}^{m_1\cdots m_r} \frac{1}{|\mathfrak{N}(\mathfrak{b}_i)|^{\frac{n_{\mathbf{K}}-1}{n_{\mathbf{K}}}}}.\end{aligned}$$

The inner sum is a constant that depends only on the inverse of the class of $\mathfrak{aq}$ in the class group. We shall denote this by $\mathfrak{N}(\mathfrak{C}^{-1})$. Combining all these estimates and applying Lemma 9, we get that the error term (21) is bounded by

$$\begin{aligned}&2^{r+1}(r+1)^{2r+1}(10n_{\mathbf{K}})^{2r} n_{\mathbf{K}}^{\frac{3n_{\mathbf{K}}^2}{2}} (\sqrt{n_{\mathbf{K}}}(2\pi+r)e^r)^{n_{\mathbf{K}}-1} \frac{\mathfrak{N}(\mathfrak{C}^{-1})t^{n_{\mathbf{K}}-1}}{|\mathfrak{N}(\mathfrak{aq})|^{\frac{n_{\mathbf{K}}-1}{n_{\mathbf{K}}}}} \\ &\leqslant e^{n_{\mathbf{K}}^2+8n_{\mathbf{K}}} n_{\mathbf{K}}^{\frac{3}{2}n_{\mathbf{K}}^2+\frac{11}{2}n_{\mathbf{K}}-\frac{1}{2}} \frac{\mathfrak{N}(\mathfrak{C}^{-1})t^{n_{\mathbf{K}}-1}}{\mathfrak{N}(\mathfrak{aq})^{\frac{n_{\mathbf{K}}-1}{n_{\mathbf{K}}}}}. \quad \square\end{aligned}$$

Remark 3.1. One can replace the condition $t \geqslant \frac{\max_{\vec{k}} \delta_1\left(\Lambda_{n_{\mathbf{K}}}(\vec{k})\right)}{\sqrt{n_{\mathbf{K}}}(2\pi+r)e^r}$ in Theorem 11 by

$$t \geqslant \frac{\left(\left(\frac{2}{\pi}\right)^{r_2}\sqrt{|d_{\mathbf{K}}|}\ \mathfrak{N}(\mathfrak{aq})\right)^{\frac{1}{n_{\mathbf{K}}}} e^{\sum_{j=1}^r (m_j-1)}}{\sqrt{n_{\mathbf{K}}(r+1)}}.$$

Indeed, by Minkowski's lattice point theorem, there exists a point $a \in \mathfrak{aq}$ such that

$$\phi(a)\cdot\beta_{\vec{k}} \in \phi(\mathfrak{aq})\cdot\beta_{\vec{k}} \quad \text{and} \quad \mathfrak{N}(a) < \left(\frac{2}{\pi}\right)^{r_2}\sqrt{|d_{\mathbf{K}}|}\ \mathfrak{N}(\mathfrak{aq}).$$

By (4), we have

$$|\sigma_i(a)| = \mathfrak{N}(a)^{\frac{1}{n_{\mathbf{K}}}} \prod_{j=2}^{r+1} |\sigma_i(\eta_{j-1})|^{\alpha_j(a)}.$$

Let $b_j = \lfloor \alpha_j(a) - \frac{k_{j-1}}{m_{j-1}} \rfloor$ and $\tilde{b_j} = \alpha_j(a) - \frac{k_{j-1}}{m_{j-1}} - b_j$. We observe that $0 \leqslant \tilde{b_j} < 1$. Consider the unit $u_a = \prod_{j=2}^{r+1} \eta_{j-1}^{b_j}$ in $\mathcal{O}_{\mathbf{K}}^*$. We now have

$$
\begin{aligned}
||\phi(a \cdot u_a^{-1}) \cdot \beta_{\vec{k}}|| &= \sqrt{\sum_{i=1}^{r+1} |\sigma_i(a \cdot u_a^{-1})|^2 \prod_{j=2}^{r+1} |\sigma_i(\eta_{j-1})|^{-\frac{2k_{j-1}}{m_{j-1}}}} \\
&= \mathfrak{N}(a)^{\frac{1}{n_{\mathbf{K}}}} \sqrt{\sum_{i=1}^{r+1} \prod_{j=2}^{r+1} |\sigma_i(\eta_{j-1})|^{2\alpha_j(a) - 2b_j - \frac{2k_{j-1}}{m_{j-1}}}} \\
&= \mathfrak{N}(a)^{\frac{1}{n_{\mathbf{K}}}} \sqrt{\sum_{i=1}^{r+1} \prod_{j=2}^{r+1} |\sigma_i(\eta_{j-1})|^{2\tilde{b_j}}} \\
&\leqslant \mathfrak{N}(a)^{\frac{1}{n_{\mathbf{K}}}} \sqrt{r+1}\ e^{\sum_{j=1}^{r} m_j}.
\end{aligned}
$$

3.6. Counting ideals in ray classes

In this subsection, we complete the proof of Theorem 1. We use notations from the previous sections. We start by simplifying the main term in Theorem 11.

Lemma 12. (Debaene [1], Lemma 12) *Let $\mathfrak{b}_1, \mathfrak{b}_2, \cdots$ be integral ideals in $\mathcal{O}_{\mathbf{K}}$, ordered such that $\mathfrak{N}(\mathfrak{b}_1) \leqslant \mathfrak{N}(\mathfrak{b}_2) \cdots$. Then for any real number $y \geqslant 2$*

$$
\sum_{i=1}^{y} \mathfrak{N}(\mathfrak{b}_i)^{\frac{1}{n_{\mathbf{K}}} - 1} \leqslant 6 n_{\mathbf{K}} y^{\frac{1}{n_{\mathbf{K}}}} (\log y)^{\frac{(n_{\mathbf{K}}-1)^2}{n_{\mathbf{K}}}}.
$$

Lemma 13 *(Lang [5], page 127). Let $\mathfrak{q}_1 = \mathfrak{q}\mathfrak{q}_\infty$ be a modulus of $\mathbf{K}$, $r_1, U_{\mathfrak{q}_1}, h_{\mathbf{K}}$ be as defined earlier and $h_{\mathbf{K},\mathfrak{q}}$ be the cardinality of the narrow ray class group of $\mathbf{K}$ modulo $\mathfrak{q}$. Then*

$$
h_{\mathbf{K},\mathfrak{q}} = \frac{2^{r_1} \varphi(\mathfrak{q}) h_{\mathbf{K}}}{[\mathcal{O}_{\mathbf{K}}^* : U_{\mathfrak{q}_1}]}.
$$

Lemma 14. *Let $r_1, r_2, h_{\mathbf{K}}, d_{\mathbf{K}}, R_{\mathbf{K}}, \mathfrak{q}_1, U_{\mathfrak{q}_1}, h_{\mathbf{K},\mathfrak{q}}, R_{\mathbf{K},\mathfrak{q}_1}$ be as in the previous subsections. Also let $\alpha_{\mathbf{K}}$ be the residue of the Dedekind zeta function at $s = 1$, $\mu_{\mathbf{K}}$ be the group of roots of unity in $\mathbf{K}$ and $\mu_{\mathfrak{q}_1}$ be the cardinality of $U_{\mathfrak{q}_1} \cap \mu_{\mathbf{K}}$. We have*

$$
\frac{(2\pi)^{r_2} R_{\mathbf{K},\mathfrak{q}_1}}{\mu_{\mathfrak{q}_1} \sqrt{|d_{\mathbf{K}}|}} = \frac{\alpha_{\mathbf{K}} \varphi(\mathfrak{q})}{h_{\mathbf{K},\mathfrak{q}}}, \qquad \frac{R_{\mathbf{K},\mathfrak{q}_1}}{R_{\mathbf{K}}} = \frac{\mu_{\mathfrak{q}_1}}{|\mu_{\mathbf{K}}|} \frac{2^{r_1} \varphi(\mathfrak{q}) h_{\mathbf{K}}}{h_{\mathbf{K},\mathfrak{q}}}.
$$

Proof. Applying the analytic class number formula (1), we see that

$$
\frac{(2\pi)^{r_2} R_{\mathbf{K},\mathfrak{q}_1}}{\mu_{\mathfrak{q}_1} \sqrt{|d_{\mathbf{K}}|} \mathfrak{N}(\mathfrak{q})} = \frac{2^{r_1} (2\pi)^{r_2} h_{\mathbf{K}} R_{\mathbf{K}}}{\mu_{\mathfrak{q}_1} \sqrt{|d_{\mathbf{K}}|} \mathfrak{N}(\mathfrak{q})} \times \frac{R_{\mathbf{K},\mathfrak{q}_1}}{2^{r_1} h_{\mathbf{K}} R_{\mathbf{K}}} = \frac{\alpha_{\mathbf{K}}}{\mathfrak{N}(\mathfrak{q})} \times \frac{|\mu_{\mathbf{K}}| R_{\mathbf{K},\mathfrak{q}_1}}{2^{r_1} \mu_{\mathfrak{q}_1} h_{\mathbf{K}} R_{\mathbf{K}}}.
$$

As shown in Lemma 9, consider the lattice $l(\phi(U_{\mathfrak{q}_1}))$ and note that $\sqrt{r+1}\ R_{\mathbf{K},\mathfrak{q}_1}$ is the volume of its fundamental domain. Also note that $\sqrt{r+1}\ R_{\mathbf{K}}$ is the volume of the fundamental domain of $l(\phi(\mathcal{O}_{\mathbf{K}}^{*}))$. We know that $l(\phi(U_{\mathfrak{q}_1}))$ and $l(\phi(\mathcal{O}_{\mathbf{K}}^{*}))$ are finitely generated free modules over $\mathbb{Z}$ of the same rank. Therefore by the structure theorem for finitely generated free modules over principal ideal domains, we have

$$[l(\phi(\mathcal{O}_{\mathbf{K}}^{*})) : l(\phi(U_{\mathfrak{q}_1}))] = \frac{\mathrm{Vol}(l(\phi(U_{\mathfrak{q}_1}))}{\mathrm{Vol}(l(\phi(\mathcal{O}_{\mathbf{K}}^{*})))} = \frac{R_{\mathbf{K},\mathfrak{q}_1}}{R_{\mathbf{K}}}.$$

Along with the torsion part coming from the roots of unity, we have

$$[\mathcal{O}_{\mathbf{K}}^{*} : U_{\mathfrak{q}_1}] = [l(\phi(\mathcal{O}_{\mathbf{K}}^{*})) : l(\phi(U_{\mathfrak{q}_1}))]\frac{|\mu_{\mathbf{K}}|}{\mu_{\mathfrak{q}_1}}.$$

Applying the above identities, we have

$$\frac{\alpha_{\mathbf{K}}}{\mathfrak{N}(\mathfrak{q})} \times \frac{|\mu_{\mathbf{K}}|R_{\mathbf{K},\mathfrak{q}_1}}{2^{r_1}\mu_{\mathfrak{q}_1}h_{\mathbf{K}}R_{\mathbf{K}}} = \frac{\alpha_{\mathbf{K}}}{\mathfrak{N}(\mathfrak{q})} \times \frac{[\mathcal{O}_{\mathbf{K}}^{*} : U_{\mathfrak{q}_1}]}{2^{r_1}h_{\mathbf{K}}}.$$

Now using Lemma 13, we get the first formula. The second one follows along similar lines;

$$\frac{R_{\mathbf{K},\mathfrak{q}_1}}{R_{\mathbf{K}}} = \frac{\mu_{\mathfrak{q}_1}}{|\mu_{\mathbf{K}}|}[\mathcal{O}_{\mathbf{K}}^{*} : U_{\mathfrak{q}_1}] = \frac{\mu_{\mathfrak{q}_1}}{|\mu_{\mathbf{K}}|}\frac{2^{r_1}\varphi(\mathfrak{q})h_{\mathbf{K}}}{h_{\mathbf{K},\mathfrak{q}}}.$$

This completes the proof of this lemma. □

We can now proceed to the proof of Theorem 1.

Proof of Theorem 1. Let us fix an ideal $\mathfrak{c} \in [\mathfrak{b}]^{-1}$. Since $\mathfrak{cb} = (\alpha)$ for some $\alpha \equiv 1 \bmod {}^*\mathfrak{q}$, in order to count the number of integral ideals in $[\mathfrak{b}]$ with norm at most x, it is sufficient to count $(\alpha), \alpha \in \mathfrak{c}$ such that $\alpha \equiv 1 \bmod {}^*\mathfrak{q}$ of norm at most $x\mathfrak{N}\mathfrak{c}$. This implies that

$$\sum_{\substack{\mathfrak{a}\subset\mathcal{O}_{\mathbf{K}},\\ [\mathfrak{a}]=[\mathfrak{b}]\\ \mathfrak{N}\mathfrak{a}\leqslant x}} 1 = \frac{1}{\mu_{\mathfrak{q}_1}}\left|\{\alpha \in \mathfrak{c} \mid \phi(\alpha) \in \mathbf{F}_{\overline{\eta}}(0,1,\cdots,0,1,x\mathfrak{N}\mathfrak{c}),\ \alpha \equiv 1 \bmod \mathfrak{q}\}\right|,$$

where $\overline{\eta} = (1,\cdots,1)$. Let $\Lambda_{n_{\mathbf{K}}}(\vec{k})$ be the lattice $h'\left(\phi(\mathfrak{c}\mathfrak{q})\cdot\beta_{\vec{k}}\right)$ in $\mathbb{R}^{n_{\mathbf{K}}}$. Using Theorem 11, we know that if $x \geqslant 1$, we get

$$\sum_{\substack{\mathfrak{a}\subset\mathcal{O}_{\mathbf{K}},\\ [\mathfrak{a}]=[\mathfrak{b}]\\ \mathfrak{N}\mathfrak{a}\leqslant x}} 1 = \frac{(2\pi)^{r_2}R_{\mathbf{K},\mathfrak{q}_1}}{\mu_{\mathfrak{q}_1}\sqrt{|d_{\mathbf{K}}|}\,\mathfrak{N}\mathfrak{q}}x + \mathrm{O}^*\left(\frac{e^{n_{\mathbf{K}}^2+8n_{\mathbf{K}}}}{\mu_{\mathfrak{q}_1}}n_{\mathbf{K}}^{\frac{3}{2}n_{\mathbf{K}}^2+\frac{11}{2}n_{\mathbf{K}}-\frac{1}{2}}\mathfrak{N}(\mathfrak{C}^{-1})\left(\frac{x}{\mathfrak{N}\mathfrak{q}}\right)^{1-\frac{1}{n_{\mathbf{K}}}} + \frac{m_1\cdots m_r}{\mu_{\mathfrak{q}_1}}\right). \quad (22)$$

We rewrite the main term by appealing to the first part of Lemma 14. We then bound above $m_1 \cdots m_r/\mu_{\mathfrak{q}_1}$ first by the quantity $7^r n_{\mathbf{K}}^{4n_{\mathbf{K}}} R_{\mathbf{K},\mathfrak{q}_1}/\mu_{\mathfrak{q}_1}$ thanks to Lemma 9. We bound it further by $2^{3n_{\mathbf{K}}} n_{\mathbf{K}}^{4n_{\mathbf{K}}} 2^{r_1} \varphi(\mathfrak{q}) \frac{R_{\mathbf{K}}}{|\mu_{\mathbf{K}}|} \frac{h_{\mathbf{K}}}{h_{\mathbf{K},\mathfrak{q}}}$ by invoking the second part of Lemma 14. The paper [3] by E. Friedman ensures us that $R_{\mathbf{K}}/|\mu_{\mathbf{K}}| \geqslant 0.2$, and so, this upper bound is at least equal to 2. Lemma 12 may thus be applied to majorize $\mathfrak{N}(\mathfrak{C}^{-1})$. On shortening $F(\mathfrak{q})$ by F, then $n_{\mathbf{K}}$ by n and $R_{\mathbf{K}}/|\mu_{\mathbf{K}}|$ by $\tilde{R}$, this leads to the error term

$$e^{n^2+8n} n^{\frac{3}{2}n^2+\frac{11}{2}n-\frac{1}{2}} 6n\big((2n)^{4n} F\tilde{R}\big)^{1/n} \log\big((2n)^{4n} F\tilde{R}\big)^{\frac{(n-1)^2}{n}} \left(\frac{x}{\mathfrak{N}\mathfrak{q}}\right)^{1-\frac{1}{n}} + (2n)^{4n} F\tilde{R}.$$

We separate the contribution of $\mathfrak{q}$ and of $\mathbf{K}$ by using

$$\log\big((2n)^{4n} F\tilde{R}\big) \leqslant 2\log\big((2n)^{4n}\tilde{R}\big)\log(3F)$$

which is a consequence of the inequality $a+b \leqslant 2ab$ valid when $a, b \geqslant 1$. But we first notice that $\log((2n)^{4n} F\tilde{R}) \geqslant 1$ so that we may simply replace the exponent $(n-1)^2/n$ by n. We further check that

$$e^{n^2+8n} n^{\frac{3}{2}n^2+\frac{11}{2}n-\frac{1}{2}} 6n(2n)^4 \leqslant 500\, n^{12n^2}.$$

Hence the theorem follows. $\square$

Data availability

No data was used for the research described in the article.

Acknowledgments

Research of this article was partially supported by Indo-French Program in Mathematics (IFPM). All authors would like to thank IFPM for financial support. The first and the third authors would also like to acknowledge MTR/2018/000201, SPARC project 445 and DAE number theory plan project for partial financial support.

We also thank Ethan Lee from Canberra for pointing out to us the work [9] of J. Sunley.

References

[1] K. Debaene, Explicit counting of ideals and a Brun-Titchmarsh inequality for the Chebotarev density theorem, Int. J. Number Theory 15 (5) (2019) 883–905.
[2] E. Dobrowolski, On a question of Lehmer and the number of irreducible factors of a polynomial, Acta Arith. 34 (4) (1979) 391–401.
[3] E. Friedman, Analytic formulas for the regulator of a number field, Invent. Math. 98 (3) (1989) 599–622.
[4] J.C. Lagarias, H.W. Lenstra Jr., C.-P. Schnorr, Korkin-Zolotarev bases and successive minima of a lattice and its reciprocal lattice, Combinatorica 10 (4) (1990) 333–348.
[5] S. Lang, Algebraic Number Theory, second edition, Graduate Texts in Mathematics, vol. 110, Springer-Verlag, New York, 1994.

[6] E.S. Lee, On the number of integral ideals in a number field, J. Math. Anal. Appl. 517 (1) (2023) 126585.

[7] J. Neukirch, Algebraic Number Theory, Grundlehren der Mathematischen Wissenschaften (Fundamental Principles of Mathematical Sciences), vol. 322, Springer-Verlag, Berlin, 1999, translated from the 1992 German original and with a note by Norbert Schappacher, with a foreword by G. Harder.

[8] J.E.S. Sunley, On the class numbers of totally imaginary quadratic extensions of totally real fields, Thesis (Ph.D.)–University of Maryland, College Park, ProQuest LLC, Ann Arbor, MI, 1971.

[9] J.E.S. Sunley, Class numbers of totally imaginary quadratic extensions of totally real fields, Trans. Am. Math. Soc. 175 (1973) 209–232.

[10] A. Sutherland, Class field theory, ray class groups and ray class fields, in: 18.785, Number Theory I, Lecture #20, MIT Mathematics, 2015.

[11] T. Tatuzawa, On the number of integral ideals in algebraic number fields, whose norms not exceeding x, Sci. Pap. Coll. Gen. Ed. Univ. Tokyo 23 (1973) 73–86.

[12] M. Widmer, Counting primitive points of bounded height, Trans. Am. Math. Soc. 362 (9) (2010) 4793–4829.