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Modular Ternary Additive Problems with Irregular or Prime Numbers

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To I. M. Vinogradov's pioneering work on prime numbers

Abstract—Our initial problem is to represent classes m modulo q by a sum of three terms, two being taken from rather small sets $\mathcal{A}$ and $\mathcal{B}$ and the third one having an odd number of prime factors (the so-called *irregular numbers* in S. Ramanujan's terminology) and lying in an interval $[q^{20r}, q^{20r}+q^{16r}]$ for some given $r\geq 1$. We show that it is always possible to do so provided that $|\mathcal{A}|\cdot|\mathcal{B}|\geq q(\log q)^2$. The proof leads us to study the trigonometric polynomials over irregular numbers in a short interval and to seek very sharp bounds for them. We prove in particular that $\sum_{q^{20r}\leq s\leq q^{20r}+q^{16r}} e(sa/q) \ll q^{16r}(\log q)/\sqrt{\varphi(q)}$ uniformly in r, where s ranges over the irregular numbers. We develop a technique initiated by Selberg and Motohashi to do so. In short, we express the characteristic function of the irregular numbers via a family of bilinear decompositions akin to Iwaniec's amplification process, and that uses pseudo-characters or local models. The technique applies to the Liouville function, to the Möbius function and also to the von Mangoldt function, in which case it is slightly more difficult. It is however simple enough to warrant explicit estimates, and we prove, for instance, that $\left|\sum_{X<\ell\leq 2X}\Lambda(\ell)\,e(\ell a/q)\right|\leq 1300\sqrt{q}\,X/\varphi(q)$ for $250\leq q\leq X^{1/24}$ and any a prime to q. Several other results are also proved.

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1. INTRODUCTION

We investigate the additive behavior of *irregular numbers* in $\mathbb{Z}/q\mathbb{Z}$ when q is large. We borrow the terminology *irregular numbers* from Ramanujan in [25]: they are the integers having an odd number of prime factors (see sequence A028260 in [20]). The importance of such numbers increased sizeably when Selberg enunciated his *parity principle* around 1949 (for instance, in [37]), as such numbers behaved very similarly to usual integers from a sieve viewpoint. Their characteristic function is easily expressed in terms of more common players: it is $(1-\lambda(n))/2$, where $\lambda(n)=(-1)^{\Omega(n)}$ is the Liouville function, $\Omega(n)$ denoting as usual in such surroundings the number of prime factors of n counted with multiplicity.

Here is our main result.

Theorem 1. *Let $q\geq 3$ be a prime number and $r\geq 1$ be a given number. Let $\mathcal{S}$ be the set of irregular numbers in $[q^{20r}, q^{20r}+q^{16r}]$. Let $\mathcal{A}$ and $\mathcal{B}$ be two arbitrary sets in $\mathbb{Z}/q\mathbb{Z}$ such that*

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$|\mathcal{A}|\cdot|\mathcal{B}|\ge q(\log q)^2$. *We have*

$$\sum_{\substack{a+b+s\equiv m[q]\\ a\in\mathcal{A},\, b\in\mathcal{B},\, s\in\mathcal{S}}} 1 \sim \frac{|\mathcal{A}|\cdot|\mathcal{B}|\cdot|\mathcal{S}|}{q}$$

(*as* q *goes to infinity*) *valid for every* $m\in\mathbb{Z}/q\mathbb{Z}$ *and uniformly in* r.

The difficulty here lies in handling the exponential sum over irregular numbers, or equivalently, the exponential sum $\sum_n \lambda(n)e(na/q)$ when q can be as large as a power of X and n runs over the short interval $[X, X+X^\theta]$. This exponential sum is very similar (and can be reduced, by using the identity $\lambda=\mu\star\mathbb{1}_{X^2}$, where $\star$ denotes the arithmetic convolution product: $(f\star g)(n)=\sum_{d|n}f(d)g(n/d)$) to the sum $\sum_n \mu(n)e(na/q)$.

This latter sum is often compared to $\sum_p e(pa/q)$ where p ranges over primes, but the short interval condition leads to difficulties. However, the method we develop is valid equally for $\lambda(n)$, $\mu(n)$ or the characteristic function of the primes. The proofs are even somewhat easier in the cases of $\lambda(n)$ and $\mu(n)$.

Historically, Davenport saw immediately the strength of Vinogradov's masterpiece [38] and used it for the trigonometric polynomial with Möbius coefficients in [6]. His method is indirect, as he uses Vinogradov's estimate on the primes. The initial treatment of Vinogradov was involved; the book [39] gives an excellent account of it. Since then, the theory of bilinear decompositions for the primes has evolved with the systematic use of combinatorial identities, though mostly over the primes. It is folklore knowledge that such methods adapt mutatis mutandis to the case of the Möbius function, though finding references is not so easy. We found [4, Theorem 3], [11, Theorem 2.1] and [13, Theorem 13.9] as well as [40, Ch. II, § 6]. There has been a renewal of interest in this problem since then, and we mention in particular [3] and [10].

Theorem 1 will be a consequence of the next estimate.

Theorem 2. *When* $q\le X^\eta$ *for some* $\eta<1/8$, $T\le (X^\eta/q)^{16/13}$ *and* a *is prime to* q, *we have*

$$\int_{-T}^{T}\left|\sum_{\ell\le X}\frac{\mu(\ell)}{\ell^{it}}\,e\Big(\frac{\ell a}{q}\Big)\right|dt \ll \frac{X\log\min(q,2+T)}{\sqrt{\varphi(q)}}.$$

Variants of the theorem are also available:

(V2) One can replace $\mu(\ell)$ by $\lambda(\ell)$ and get the same bound.

(V3) One can add the coprimality condition $(\ell,q)=1$ and replace $1/\sqrt{\varphi(q)}$ by $1/\sqrt{q}$.

(V4) One can replace the Möbius function $\mu(\ell)$ by the von Mangoldt function $\Lambda(\ell)$ and change $1/\sqrt{\varphi(q)}$ to $\sqrt{q}/\varphi(q)$.

Variant (V1) is the one of the theorem. All the results below will have the same variants. This L^1-estimate readily leads to an estimate of the trigonometric polynomial over a short interval.

Theorem 3. *Let* $\eta<1/8$ *be given. When* $q\le X^\eta$, a *is prime to* q *and* θ_0 *is defined by* $X^{1-\theta_0}\sqrt{q}=(X^\eta/q)^{16/13}$, *we have*

$$\sum_{X<\ell\le X+X^\theta}\mu(\ell)\,e\Big(\frac{\ell a}{q}\Big)\ll\frac{X^\theta\log q}{\sqrt{\varphi(q)}}\qquad \textit{for any}\quad \theta\in(\theta_0,1].$$

Comparing our bound to the one obtained by Zhan in [41, Theorem 2], we can say that the author has access to smaller intervals, but our bound has no power of $\log X$. This may not be obvious to compare the two results, but a closer scrutiny shows that the constant B in [41, Theorem 2] is smaller than $c_2/2-1$.

The estimates above require that we stay at the precise point a/q and do not shift (except for some trivial amount) from it. When $\theta=1$, we can however relax this condition, and this may be important for the circle method (a summation by parts enables one to extend the result to $|\alpha - a/q| \ll 1/X$; going further than that is the difficulty). Here is what we get.

Theorem 4. *When $q \le X^\eta$ for some $\eta < 2/13$, a is prime to q, $|\beta|X + |t| \le (X^\eta/q)^{13/2}$ and $|\beta|X \le X^{(2-13\eta)/44}$, we have*

$$\sum_{\ell \le X} \frac{\mu(\ell)}{\ell^{it}}\, e(\ell\beta)\, e\Big(\frac{\ell a}{q}\Big) \ll \frac{X}{\sqrt{\varphi(q)}}.$$

Here and thereafter, we do not try to get the best exponents, our aim being to describe precisely the method. These upper bounds have two features: after division by X^θ (the trivial estimate), they tend to 0 with q (no power of $\log X$ comes in to weaken the result); and they are valid for q up to some power of X. The technique developed is to represent the Möbius function by a *family* of linear combinations of linear and bilinear sums in modern terminology (or of type I and type II sums if we are to follow Vinogradov's initial choice of words). Let us specify here that the results obtained are all effective (i.e., all implied constants can be explicitly determined); the possible Siegel zero thus limits our expectation as to what kind of results we may reach (it cannot however be properly termed an obstruction!).

A similar question over the primes has received attention. Theorem 2b of [39, Ch. IX] gives a first answer in that case (on selecting there $\varepsilon = 2(\log q)/\log\log X$). A simplified version reads

$$\sum_{n \le X} \Lambda(n)\, e\Big(\frac{na}{q}\Big) \ll \frac{X(\log q)^{10}}{\sqrt{q}}, \qquad q \le \exp\bigg(\sqrt{\frac{\log\log X}{10}}\bigg). \tag{1.1}$$

In [5] the author obtained an estimate analogous to (1.1) but better when q does not have too many divisors. Both these results rely on bilinear decompositions and are thus extendable to the Möbius function. A first best possible result in the case of primes was reached in [29]: the function of q is $\sqrt{q}/\varphi(q)$. The variable q is still restricted to being not more than $\exp((\log X)^{1/3}/50)$, and, though flexible, the method developed also relies on positivity and is thus not adaptable to the case of the Möbius function. A. Karatsuba in [14, Ch. 10, Sect. 4, Lemma 7] gives the same coefficient $\sqrt{q}/\varphi(q)$ for q up to $\exp(c(\log X)^{1/2})$ for some positive constant $c > 0$, but this time by using analytical methods that are not transposable (more on this issue later). In fact, on using Gallagher's prime number theorem, i.e., [8, Eq. (5)], one can reach a result of similar strength: the same coefficient and for q up to some power of X. Our present method gives an analogous result without using log-free zero density theorem or even mentioning the zeros, and, in this aspect, this project is a sequel of Motohashi's work (see [17, 18]).

Indeed, Motohashi in [18] produced a proof of Gallagher's prime number theorem without using the zeros of the relevant Dirichlet L-series. Such a proof can be adapted to the case of the Möbius function by using the material we develop here. Let us record the result.

Theorem 5. *There exist constants $c_1, c_2 > 0$ such that*

$$\sum_{q \le Q} \sideset{}{^*}\sum_{\chi \bmod^* q} \bigg| \sum_{n \le X} \mu(n)\chi(n) \bigg| \ll X \exp\Big(-c_1 \frac{\log X}{\log Q}\Big)$$

provided that $\exp(\sqrt{\log X}) \le Q \le X^{c_2}$. The constant c_1 is chosen so that at most one Dirichlet L-function has a zero in the region $\sigma \ge 1 - c_1/\log(Q(2+|t|))$. Such a zero, if it exists, is simple and attached to a real character. The symbol $\sum^*$ means that, in case such an exceptional zero exists, say β attached to the character χ^*, we replace $\sum_{n \le X} \mu(n)\chi^*(n)$ by

$$\sum_{n \le X} \mu(n)\chi^*(n) - \frac{X^\beta}{\beta L'(\beta, \chi^*)}. \tag{1.2}$$

In case $L'(\beta,\chi^*)$ is small, this contradicts Theorem 4. Indeed, on choosing X to be a large power of $q=Q$, we deduce from Theorem 5 that

$$\sum_{\ell\le X,\ (\ell,q)=1}\mu(\ell)\,e(\ell\beta)\,e\Big(\frac{\ell a}{q}\Big)\asymp\frac{X\sqrt{q}}{\varphi(q)\,L'(\beta,\chi^*)}.$$

Let us record this result with the proper quantifiers.

Corollary 6. *There exists a constant $c>0$ such that if $L(s,\chi^*)$ admits a real zero $\beta>1-c/\log q$, then $L'(\beta,\chi^*)\gg q/\varphi(q)$.*

This completes the result of Pintz [21], namely that, under the above assumptions, one has $L'(1,\chi^*)\gg q/\varphi(q)$. It is possible to go from one to the other when $1-\beta=o((\log q)^{-2})$ by using simple analysis.

The methods we use are fully explicit and even lead to possible numerical bounds. Because the most difficult case is the one of the primes, we consider this case and prove the following.

Theorem 7. *When $250\le q\le X^{1/24}$ and a is prime to q, we have*

$$\Bigg|\sum_{X<\ell\le 2X}\Lambda(\ell)\ e\Big(\frac{\ell a}{q}\Big)\Bigg|\le 1300\,\frac{\sqrt{q}\,X}{\varphi(q)}.$$

It is worthwhile noting that the constant 1300 is maybe large but explicit, while the work [29] relies on a Brun sieve-based preliminary sieving process that would make such a computation very hard (it would also most probably result in a much higher constant). We made the effort to get an explicit constant, but there are many places where this work can numerically be improved upon. Notice that Platt in [22] has computed the zeroes of all the Dirichlet L-series of conductor $q\le 4\cdot 10^5$ whose imaginary parts are not more than $10^8/q$, so the distribution of primes up to such levels is better handled by using this.

Concerning the method, as already mentioned, we detect the primes via a family of bilinear decompositions, in a mechanism akin to Iwaniec's amplification process. Our implementation is inherited from a technique introduced by Selberg around 1973 (as mentioned by Bombieri in [2] and by Motohashi in [18]), which has shown exceptional power.

This family of decompositions is shown to be "orthogonal," in some sense, via a large sieve extension of the classical large sieve inequality for the Farey points, that encompasses *generalized characters* in Selberg's or Motohashi's terminology, also called (up to some rescaling) *local models* in [28]. See Theorem 10 below, where the implied constant is improved with respect to the classical one.

The building of our decompositions involves Barban–Vehov[1] weights (see [1]). While studying an optimization problem close to the one that classically found the Selberg sieve for primes, Barban and Vehov introduced the weights

$$\lambda_d^{(1)}=\begin{cases}\mu(d) & \text{when}\ \ d\le z,\\ \mu(d)\dfrac{\log(z^2/d)}{\log z} & \text{when}\ \ z<d\le z^2,\\ 0 & \text{when}\ \ z^2<d.\end{cases}\tag{1.3}$$

[1] The surname Vehov is also sometimes spelled Vekhov, in accordance with the traditional transliteration rules for the Cyrillic alphabet.

They consider in fact slightly more general weights with a y instead of z^2 (see Motohashi [19, Sect. 1.3] and Graham [9]). Their particular property is that

$$\sum_{n\le B}\Bigl(\sum_{d|n}\lambda_d^{(1)}\Bigr)^2 \ll \frac{B}{\log z}$$

whether $B\ge z^4$ or not. We will follow an idea of Motohashi that the weaker property

$$\sum_{n\ge 1}\frac{1}{n^{1+\varepsilon}}\Bigl(\sum_{d|n}\lambda_d^{(1)}\Bigr)^2 \ll \frac{z^\varepsilon}{(\varepsilon\log z)^2}, \qquad \varepsilon\in(0,1],$$

can often be enough (this is our case) via Rankin's trick (see [34]) and is easier to establish. The required material is contained in Lemma 27 and comes from [12]. From an explicit viewpoint, this implies dealing with sums of type $\sum_{d\le D}\mu(d)/d^{1+\varepsilon}$ or $\sum_{d\le D}\mu(d)\log(D/d)/d^{1+\varepsilon}$ with some coprimality conditions added, and such sums are really more difficult to handle than the ones with $\varepsilon=0$. In this latter case, identities can be used to effect that do not have any counterpart (as far as we can see) in the case $\varepsilon>0$.

Notation. Our notation is standard except maybe for $f(x)=\mathcal{O}^*(g(x))$, which means that $|f(x)|\le g(x)$. Furthermore, the arithmetic convolution is denoted by $\star$; it is defined by $(f\star g)(n)=\sum_{d|n}f(d)g(n/d)$ under obvious conditions.

We however require quite a lot of partial definitions to make the writing easier. It is then easy to forget the meaning of a quantity, and we try to recall the most important ones here. The letters θ and H are used in several acceptions.

Let us recall the definition of the Ramanujan sum as well as its evaluation in terms of the Möbius function:

$$c_r(m)=\sum_{a \bmod^* r} e\Bigl(\frac{am}{r}\Bigr)=\sum_{u|r,\ u|m} u\mu\Bigl(\frac{r}{u}\Bigr). \tag{1.4}$$

A companion function of these Ramanujan sums c_r are the functions v_r defined by

$$v_r(m)=c_r(m)\sum_{d|m}\lambda_d^{(1)}. \tag{1.5}$$

By $\ell\sim L$, we mean $L<\ell\le 2L$. The main actor is

$$S\Bigl(\frac{a}{q},t,\beta\Bigr)=\sum_{\ell\sim X,\ (\ell,q)=1}\frac{\Lambda(\ell)}{\ell^{it}}\,e(\beta\ell)\,e\Bigl(\frac{\ell a}{q}\Bigr), \tag{1.6}$$

which we will split into a linear combination of sums of the linear type $L_r^{(1)}(a,t,\beta)$ defined in (2.11) and $L_r^{(2)}(a,t,\beta)$ defined in (2.12), and of sums of bilinear type:

$$S_r\Bigl(\frac{a}{q},t,\beta,M,N\Bigr)=\sum_{\substack{mn\sim X,\ (mn,q)=1\\ m\sim M,\ n\sim N}}\frac{\Lambda(m)v_r(n)}{(mn)^{it}}\,e(\beta mn)\,e\Bigl(\frac{mna}{q}\Bigr). \tag{1.7}$$

2. THE GENERAL SETTING OF THE PROOF

We consider the Dirichlet series

$$V_r(s)=\sum_{n\ge 2}\frac{c_r(n)}{n^s}\sum_{d|n}\lambda_d^{(1)}=\sum_{n\ge 2}\frac{v_r(n)}{n^s}. \tag{2.1}$$

This series has the good property to (almost) factor. Note that the summation can be restricted to integers $n>z$.

Theorem 8. *We have*

$$1 + V_r(s) = \zeta(s) M_r\big(s, \lambda_d^{(1)}\big) \tag{2.2}$$

where

$$M_r\big(s, \lambda_d^{(1)}\big) = \sum_{u|r,\, d\le z^2} \frac{u\mu(r/u)\lambda_d^{(1)}}{[u,d]^s} = \sum_{1\le n\le rz^2} \frac{h_r(n)}{n^s} \tag{2.3}$$

with

$$h_r(n) = \sum_{\substack{u|r,\, d\le z^2\\ [u,d]=n}} u\mu\Big(\frac{r}{u}\Big)\lambda_d^{(1)}. \tag{2.4}$$

One can find a similar decomposition in [18, Lemma 4] with $f(n) = 1$. Under this condition, we have $\Psi_r(n) = c_r(n)$ and $g(r) = \mu^2(r)/\varphi(r)$; they are defined between equations (9) and (10) there. The factor $\mu^2(r)$ is absent in Motohashi's definition, but the function g will be used only with square-free argument. In this manner, the large sieve inequality given by [18, Lemma 2] reduces to [2, théorème 7A], which is attributed to A. Selberg by E. Bombieri.

Here is the formal identity that gives us a decomposition of 1:

$$1 = -V_r + (1 + V_r).$$

There follows a decomposition of $-\zeta'/\zeta$, which we modify with the help of (2.2), and we reach

$$-\frac{\zeta'}{\zeta} = \frac{\zeta'}{\zeta} V_r - \zeta' M_r. \tag{2.5}$$

This translates into the following pointwise identity:

$$\Lambda = -\Lambda \star v_r + \log \star h_r. \tag{2.6}$$

The corresponding identity for the Möbius function is even more striking:

$$\mu = -\mu \star v_r + h_r. \tag{2.7}$$

We finally quote the one for the Liouville function:

$$\lambda = -\lambda \star v_r + \mathbb{1}_{X^2} \star h_r. \tag{2.8}$$

Identities (2.6), (2.7) and (2.8) are the core of our approach. They however still need to be slightly refined, as the variable carried by Λ, μ and λ in the factors $\Lambda \star v_r$, $\mu \star v_r$ and $\lambda \star v_r$, respectively, can be small. This is readily taken care of by a simple truncation (see (2.10)). Let us finally mention that we will average over this family of decompositions.

Let us initiate the proof to fix the notation.

The coprimality of our variables with the modulus q will come into play, and we start this section by this point. In situation (V3), the question does not arise. In situation (V4), we write

$$\Bigg| \sum_{\ell\sim X,\, (\ell,q)\ne 1} \frac{\Lambda(\ell)}{\ell^{it}} e(\beta\ell)\, e\Big(\frac{\ell a}{q}\Big) \Bigg| \le \sum_{p|q} \frac{\log(2X)}{\log p} \log p = \omega(q)\log(2X).$$

This leads to an error term that is easily absorbed, even numerically on using $\omega(q) \le (\log q)/\log 2$, by all our subsequent error terms. Section 11 explains how to dispense with the coprimality condition in situations (V1) and (V2).

Let us select a square-free integer $r \le R$. We assume that

$$z^2 R \le X. \tag{2.9}$$

We further select a (large) parameter M_0 and write (recall (1.6))

$$S\Big(\frac{a}{q},t,\beta\Big) = L_r^{(1)}(a,t,\beta) - L_r^{(2)}(a,t,\beta) - \sum_{\substack{mn\sim X,\ (mn,q)=1\\ m>M_0}} e(\beta mn)\frac{\Lambda(m)v_r(n)}{(mn)^{it}}\, e\Big(\frac{mna}{q}\Big), \tag{2.10}$$

where the first linear form is defined by

$$L_r^{(1)}(a,t,\beta) = \sum_{mn\sim X,\ (mn,q)=1} e(\beta nm)\frac{h_r(m)\log n}{(nm)^{it}}\, e\Big(\frac{nma}{q}\Big) \tag{2.11}$$

while the second one is defined by

$$L_r^{(2)}(a,t,\beta) = \sum_{\substack{mn\sim X,\ (mn,q)=1\\ m\le M_0}} e(\beta mn)\frac{\Lambda(m)v_r(n)}{(nm)^{it}}\, e\Big(\frac{mna}{q}\Big). \tag{2.12}$$

We now examine the last sum and localize the variables m and n. Notice that $n > z$. So we start at $N = z$, go until $2z$, etc., until $2^k z \le 2X/M_0 < 2^{k+1}z$, i.e., $0 \le k \le \log(X/(M_0 z))/\log 2$. Concerning M, we have $N < n \le N' \le 2N$, and thus $X/(2N) \le X/n < m \le 2X/N$. So for each N we have two values of M, namely, $M_1 = X/(2N)$ and $M_2 = X/N$. We then use the following inequalities, where $A(m,n) = e(\beta mn)\Lambda(m)v_r(n)(nm)^{-it}e(mna/q)\,\mathbb{1}_{(mn,q)=1}$:

$$\begin{aligned}\Big|\sum_{mn\sim X} A(m,n)\Big|^2 &= \Big|\sum_{M,N}\sum_{\substack{mn\sim X,\\ m\sim M,\ n\sim N}} A(m,n)\Big|^2 \le \sum_{M,N} 1 \sum_{M,N}\Big|\sum_{\substack{mn\sim X,\\ m\sim M,\ n\sim N}} A(m,n)\Big|^2\\ &\le \frac{2\log(2X/(M_0 z))}{\log 2}\sum_{M,N}\Big|\sum_{\substack{mn\sim X,\\ m\sim M,\ n\sim N}} A(m,n)\Big|^2. \end{aligned}\tag{2.13}$$

Let us point out that this last summation over m, n has been denoted by $S_r(a/q,t,\beta,M,N)$ in (1.7). We relax the condition $mn \sim X$ and remove the coefficient $e(\beta mn)$ in this sum by appealing to Lemma 45 (see the Appendix) with $b = 2$, $R = X$ and some $\delta = \delta(M,N) \in (0,1/2)$. We find that

$$\begin{aligned} S_r\Big(\frac{a}{q},t,\beta,M,N\Big) &= \int_{-\Delta}^{\Delta} \sum_{\substack{(mn,q)=1\\ m\sim M,\ n\sim N}} \frac{\Lambda(m)}{m^{i(v+t)}}\,\frac{v_r(n)}{n^{i(v+t)}}\, e\Big(\frac{mna}{q}\Big) X^{iv}\mathscr{H}(v)\,dv\\ &\quad + \mathcal{O}^*\big(E_1(\delta,r) + E_2(\delta,r) + 2\delta E_3(r)\big)\end{aligned}\tag{2.14}$$

where $\mathscr{H}(v)$ is provided by Lemma 45 and

$$E_1(\delta,r) = \sum_{\substack{X<mn\le 2^\delta X\\ (mn,q)=1\\ m\sim M,\ n\sim N}} \Lambda(m)|v_r(n)|, \qquad E_2(\delta,r) = \sum_{\substack{2X/2^\delta<mn\le 2X\\ (mn,q)=1\\ m\sim M,\ n\sim N}} \Lambda(m)|v_r(n)| \tag{2.15}$$

and

$$E_3(r) = \sum_{\substack{(mn,q)=1\\ m\sim M,\ n\sim N}} \Lambda(m)|v_r(n)|. \tag{2.16}$$

The parameter Δ is defined in (A.6) (with $b = 2$).

The proof continues in Section 7.

3. PREPARATIONS

3.1. A hybrid large sieve inequality. This material is our main ingredient to control the bilinear form arising from our decomposition of the Λ-function.

We quote the following theorem of Selberg from [2, théorème 7A].

Theorem 9. *Let N_0 be a given real number. Let $(u_n)_{N_0<n\le N_0+N}$ be a sequence of complex numbers. We have*

$$\sum_{\mathfrak{f}r\le R,\ (\mathfrak{f},r)=1}\frac{\mathfrak{f}}{\phi(\mathfrak{f}r)}\sum_{\chi \bmod^* \mathfrak{f}}\biggl|\sum_{N_0<n\le N_0+N} u_n\chi(n)c_r(n)\biggr|^2 \le \sum_{N_0<n\le N_0+N}|u_n|^2(N+R^2),$$

where $c_r(m)$ is the Ramanujan sum modulo r.

The summation is over both coprime variables $\mathfrak{f}$ and r, subject to $\mathfrak{f}r\le R$. The parameter N_0 is required since the left-hand side is *not* (a priori) invariant under translation. We now prove the following hybrid version. Such a result can also be found in [2, théorème 10] and has its origin in [8]; our contribution here is a refined constant 7. Looking more closely at [32, Corollary 6.4], we see that it proves the following.

Theorem 10. *Let $(u_n)_n$ be a sequence of complex numbers such that $\sum_n(|u_n|+n|u_n|^2)<\infty$. We have*

$$\sum_{d\le D}\sum_{a \bmod^* d}\int_{-T}^{T}\biggl|\sum_n u_n n^{it}\, e\Bigl(\frac{na}{d}\Bigr)\biggr|^2 dt \le 7\sum_n |u_n|^2\bigl(n+D^2\max(T,10)\bigr).$$

Theorem 11. *Let q be some fixed modulus and N_0 be some real number. Let $(u_n)_n$ be a sequence of complex numbers such that $\sum_n(|u_n|+n|u_n|^2)<\infty$. We have, for any $T\ge 0$,*

$$\sum_{r\le R/q,\ (q,r)=1}\frac{1}{\varphi(r)}\sum_{a \bmod q}\int_{-T}^{T}\biggl|\sum_n u_n c_r(n+N_0) n^{it}\, e\Bigl(\frac{na}{q}\Bigr)\biggr|^2 dt \le 7\sum_n |u_n|^2\bigl(n+R^2\max(T,10)\bigr).$$

Proof. We use

$$c_r(n+N_0)=\sum_{a' \bmod^* r} e\biggl(\frac{(n+N_0)a'}{r}\biggr).$$

Applying Cauchy's inequality brings us to a position where we can use Theorem 10, since the set $\{(a/q)+(a'/r)\}$ is a subset of the set of points $\{b/d\}$ with $d\le R$. $\square$

Corollary 12. *Let q be some fixed modulus. Let $(u_n)_n$ be a sequence of complex numbers such that $\sum_n(|u_n|+n|u_n|^2)<\infty$. We have, for any $T\ge 0$,*

$$\sum_{r\le R/q,\ (q,r)=1}\frac{1}{\varphi(r)}\sum_{a \bmod q}\int_{-T}^{T}\biggl|\sum_n u_n c_r(n) n^{it}\, e\Bigl(\frac{na}{q}\Bigr)\biggr|^2 \frac{dt}{1+|t|} \ll \sum_n |u_n|^2\bigl(n+R^2\log(T+2)\bigr).$$

Proof. Simply use integration by parts and Theorem 11. $\square$

3.2. Prime number estimates. We recall some classical results taken from [36].

Lemma 13. *When $M\ge 101$, we have*

$$\sum_{m\sim M}\Lambda(m)\le \frac{5}{4}M.$$

Proof. Indeed, [36, Theorem 12] gives us

$$\psi(x) \leq 1.04x, \qquad x \geq 0, \tag{3.1}$$

while [36, Theorem 10] gives us

$$\vartheta(x) = \sum_{p \leq x} \log p \geq 0.84x, \qquad x \geq 101. \tag{3.2}$$

The lemma follows readily. $\square$

We also infer, from [23, corollary to Theorem 1.1],

$$\sum_{m \leq M_0} \frac{\Lambda(m)}{m} \leq \log M_0, \qquad M_0 \geq 1. \tag{3.3}$$

We next need the following extension of the celebrated version of the Brun–Titchmarsh inequality due to H. L. Montgomery and R. C. Vaughan in [16, Theorem 2].

Lemma 14. *Let d be some integer modulus and a be a reduced residue class modulo d. When $A \geq B > d \geq 1$, we have*

$$\sum_{\substack{A < m \leq A+B \\ m \equiv a[d]}} \frac{\Lambda(m)}{\log m} \leq \frac{2B}{\varphi(d)\log(B/d)}.$$

We will require the cases $d = 1$ and $d = q$. The hypothesis $A \geq B$ is absent in [16, Theorem 2], where only primes are counted. We however need this hypothesis here.

Proof. The hypothesis $A \geq B$ ensures that if a prime power p^k belongs to the interval $(A, A+B]$, then no other powers of this prime belong to this interval. We may thus bound above our quantity by the number of integers within $(A, A+B]$ that have no prime factors below some parameter z, to which we add the number of primes below z. This upper bound is the one used at the beginning of the proof of [16, Theorem 2] in equation (3.3) therein (when B is large) and in Lemma 10 (when B is small). In each occurrence, the summand $\pi(z)$ accounts for the additional primes (or prime powers) to be included. The proof of [16, Theorem 2] thus applies. $\square$

Lemma 15. *For any modulus $q \geq 1$ and any real number $M \geq \max(121, q^3)$, we have*

$$\sum_{m \sim M,\, m \equiv a[q]} \Lambda(m) \leq \frac{9}{2}\frac{M}{\varphi(q)}.$$

Proof. By Lemma 14, we find that

$$\sum_{m \sim M,\, m \equiv a[q]} \Lambda(m) \leq 2\frac{M\log(2M)}{\varphi(q)\log(M/q)}.$$

A numerical application ends the proof. $\square$

Lemma 16. *For any modulus $q \geq 1$ and any real number $M \geq \max(121, q^3)$, we have*

$$\sum_{b \bmod^* q} \Bigg| \sum_{m \equiv b[q],\, m \sim M} \Lambda(m) \Bigg|^2 \leq \frac{45}{8}\frac{M^2}{\varphi(q)}.$$

Proof. We collect Lemmas 15 and 13 getting that the left-hand side above is not more than

$$\frac{9}{2}\frac{M}{\varphi(q)}\frac{5}{4}M \leq \frac{45}{8}\frac{M^2}{\varphi(q)}. \qquad \square$$

Lemma 17. *For any modulus $q \geq 1$, any $\varepsilon > 0$ and any real number $M \geq q^{1+\varepsilon}$, we have*

$$\sum_{b \bmod^* q} \Big| \sum_{m \equiv b[q],\, m \sim M} \Lambda(m) \Big|^2 \ll \frac{M^2}{\varphi(q)}.$$

Proof. Lemma 14 implies that

$$\sum_{m \sim M,\, m \equiv b[q]} \Lambda(m) \ll_\varepsilon \frac{M}{\varphi(q)}$$

when $M \geq q^{1+\varepsilon}$, and the proof is completed by noticing that

$$\sum_{b \bmod^* q} \sum_{m \equiv b[q],\, m \sim M} \Lambda(m) \ll M. \qquad \square$$

3.3. Möbius function estimates. The following lemma is quoted from [30].

Lemma 18. *When $r \geq 1$ and $1.38 \geq \varepsilon \geq 0$, we have*

$$-(1.44 + 5\varepsilon + 3.6\varepsilon^2) \leq \sum_{d \leq x,\, (d,r)=1} \frac{\mu(d)}{d^{1+\varepsilon}} \log \frac{x}{d} \leq 1.4 + 4.7\varepsilon + 3.3\varepsilon^2 + (1+\varepsilon) \frac{r^{1+\varepsilon}}{\varphi_{1+\varepsilon}(r)} x^\varepsilon,$$

where

$$\frac{r^{1+\varepsilon}}{\varphi_{1+\varepsilon}(r)} = \prod_{p|r} \frac{p^{1+\varepsilon}}{p^{1+\varepsilon} - 1}. \tag{3.4}$$

In the case $\varepsilon = 0$, we have access to better and simpler estimates, and we quote from [31, Corollaries 1.10 and 1.11].

Lemma 19. *For any real number $x \geq 1$ and any positive integer r, we have*

$$0 \leq \sum_{n \leq x,\, (n,r)=1} \mu(n) \frac{\log(x/n)}{n} \leq 1.00303 \frac{r}{\varphi(r)}$$

and

$$0 \leq \sum_{n \leq x,\, (n,r)=1} \mu(n) \frac{\log^2(x/n)}{n} \leq 2 \log x \cdot \frac{r}{\varphi(r)}.$$

3.4. Explicit averages of some non-negative multiplicative functions. Let us start by recalling estimates on the G-functions. We recall the classical definition

$$G_q(D) = \sum_{d \leq D,\, (d,q)=1} \frac{\mu^2(d)}{\varphi(d)}, \qquad G(D) = G_1(D). \tag{3.5}$$

We quote from [15]: for any coprime positive integers r and s,

$$G_r(D) \leq \frac{s}{\varphi(s)} G_{rs}(D) \leq G_r(sD), \tag{3.6}$$

and in particular, when $r = 1$ and $s = q$,

$$G(D) \leq \frac{q}{\varphi(q)} G_q(D) \leq G(qD). \tag{3.7}$$

According to [26, Lemma 3.5] (see also [35]),

$$G(D) \le \log D + 1.4709, \qquad D \ge 1, \tag{3.8}$$

and, concerning a lower bound,

$$\log D + 1.06 \le G(D), \qquad D \ge 6. \tag{3.9}$$

We next turn our attention to some of the less studied functions. We appeal to [26, Lemma 3.2], which will be partially recalled during the proof.

Lemma 20. *When $X \ge 6$, we have*

$$\sum_{n\le X,\ (n,6)=1} \frac{\mu^2(n)}{n} \prod_{p|n}\left(1 - \frac{4}{p} + \frac{8}{3p^2}\right) \le 0.225(\log X + 3.1).$$

A bound valid for $X \ge 15$ would be enough.

Proof. We define the multiplicative function g by

$$g(2) = \frac{-1}{2}, \qquad g(3) = \frac{-1}{3}, \qquad g(2^k) = g(3^k) = 0 \quad \forall k \ge 2,$$

$$g(p) = -\frac{4}{p^2} + \frac{8}{3p^3}, \qquad g(p^2) = -\left(1 - \frac{4}{p} + \frac{8}{3p^2}\right)\frac{1}{p^2}, \qquad g(p^k) = 0 \quad \forall k \ge 3,$$

so that we have (simply compare the Dirichlet series)

$$\mathbb{1}_{(n,6)=1} \frac{\mu^2(n)}{n} \prod_{p|n}\left(1 - \frac{4}{p} + \frac{8}{3p^2}\right) = \sum_{\ell m = n} g(\ell)\frac{1}{m}.$$

Hence the sum S that we need satisfies

$$S = \sum_{n\le X} \sum_{\ell m=n} g(\ell)\frac{1}{m} = \sum_{\ell\ge 1} g(\ell) \sum_{m \le X/\ell} \frac{1}{m}.$$

We recall the first half of [26, Lemma 3.3], namely,

$$\sum_{m\le t} \frac{1}{m} = \log t + \gamma + \mathcal{O}^*\big(0.9105\, t^{-1/3}\big)$$

valid for any $t > 0$. This explains why we can dispense with the condition $\ell \le X$ above. We thus get

$$S = \sum_{\ell\ge 1} g(\ell)\left(\log\frac{X}{\ell} + \gamma\right) + \mathcal{O}^*\left(0.9105\, X^{-1/3} \sum_{\ell \ge 1} |g(\ell)|\ell^{1/3}\right).$$

We readily check that

$$\sum_{\ell\ge 1} |g(\ell)|\ell^{1/3} = \left(1 + \frac{1}{2^{2/3}}\right)\left(1 + \frac{1}{3^{2/3}}\right) \prod_{p\ge 5}\left(1 + \frac{4}{p^{5/3}} - \frac{8}{3p^{8/3}} + \frac{1}{p^{4/3}} - \frac{4}{p^{7/3}} + \frac{8}{p^{10/3}}\right) \le 11.$$

Furthermore, with

$$G(s) = \prod_{p\ge 2}\left(1 + \sum_{k\ge 1} \frac{g(p^k)}{p^{ks}}\right),$$

we can rewrite the above in the form

$$S = G(0)\left(\log X + \gamma + \frac{G'(0)}{G(0)}\right) + \mathcal{O}^*\left(\frac{10}{X^{1/3}}\right).$$

We readily find that

$$G(0) = \frac{1}{3}\prod_{p\geq 5}\Bigl(1 - \frac{4}{p^2} + \frac{8}{3p^3} - \frac{1}{p^2} + \frac{4}{p^3} - \frac{8}{3p^4}\Bigr) \leq 0.225.$$

On the other hand,

$$\frac{G'(0)}{G(0)} = \log 2 + \frac{\log 3}{2} - \sum_{p\geq 5}\frac{(g(p)+2g(p^2))\log p}{1+g(p)+g(p^2)} \leq 2.42.$$

Finally, $S \leq 0.225(\log X + 3 + 45/X^{1/3})$. We used a GP-script to show that

$$S \leq 0.225(\log X + 3) \qquad \forall X \in [8, 10^9]. \tag{3.10}$$

As a conclusion,

$$S \leq 0.225(\log X + 3.1) \qquad \forall X \geq 8, \tag{3.11}$$

which can be extended to $X \geq 6$ by direct inspection. $\square$

Lemma 21. *When $X \geq 15$, we have*

$$\sum_{\ell\leq X,\,(\ell,2)=1}\mu^2(\ell)\prod_{p|\ell}\frac{4p^2-12p+8}{p^2} \leq 0.0114\,X(\log X + 3.1)^3.$$

A bound valid for $X \geq 15$ would be enough. This lemma is not optimal, and more work would yield better constants.

Proof. Let us denote by S the sum to be evaluated. We define the multiplicative functions f_1 and f_2 on primes $p \geq 5$ by $f_1(p) = 4 - 12p^{-1} + 8p^{-2} = f_2(p) + 1$ and $f_1(p^k) = f_2(p^k) = 0$ as soon as $k \geq 2$. At $p = 3$, this would give a negative value for $f_2(3)$, so we majorize $f_1(3)$ by 1 to keep the non-negativity of f_2. We thus set $f_2(3^k) = 0$ for every $k \geq 1$. We have $f_1(n) \leq (f_2 \star \mathbb{1})(n)$. Furthermore, we check that $f_2(m) \leq (f_3 \star f_3 \star f_3)(m)$ where

$$f_3(m) = \mu^2(m)\,\mathbb{1}_{(m,6)=1}\prod_{p|m}\Bigl(1 - \frac{4}{p} + \frac{8}{3p^2}\Bigr). \tag{3.12}$$

More precisely, we have the equality $f_2(m) = (f_3 \star f_3 \star f_3)(m)$ when m is square-free, while, when m is not square-free, we have $f_2(m) = 0$ and $(f_3 \star f_3 \star f_3)(m) \geq 0$. Therefore,

$$S \leq \sum_{m\leq X,\,(m,2)=1} f_2(m)\sum_{\ell\leq X/m} 1 \leq X\Biggl(\sum_{m\leq X,\,(m,3)=1}\frac{f_3(m)}{m}\Biggr)^3 \leq 0.0114\,X(\log X + 3.1)^3,$$

where we have used Lemma 20 for the last inequality. $\square$

We define $\varphi_+(r) = \prod_{p|r}(1+p)$.

Lemma 22. *We have*

$$\sum_{r\leq X}\frac{\mu^2(r)\varphi_+(r)}{\varphi(r)} \leq 3.28\,X.$$

This lemma is not optimal and more work would yield better constants.

Proof. The proof is straightforward:

$$\sum_{r\leq X}\frac{\mu^2(r)\varphi_+(r)}{\varphi(r)} \leq \sum_{\ell\leq X}\mu^2(\ell)\prod_{p|\ell}\frac{2}{p-1}\sum_{\ell|r\leq X} 1 \leq X\prod_{p\geq 2}\Bigl(1 + \frac{2}{p(p-1)}\Bigr) \leq 3.28\,X. \qquad \square$$

3.5. Estimates on the Barban–Vehov weights. Let us start with a rough preliminary estimate.

Lemma 23. *We have* $\sum_{d\le z^2}\bigl|\lambda_d^{(1)}\bigr| \le z^2/\log z$.

Proof. We first note that

$$\bigl|\lambda_d^{(1)}\bigr|\log z = \mu^2(d)\biggl(\log^+\frac{z^2}{d} - \log^+\frac{z}{d}\biggr), \tag{3.13}$$

where $\log^+ x = \max(0, \log x)$. We verify this identity by checking it holds for $d \le z$, for $d \in [z, z^2]$ and for larger d. We next note that $\sum_{d\le y}\log(y/d) \le y$, and the lemma follows readily. □

Next we recall [32, Lemma 5.4]; namely, when $s > 1$ is real, we have

$$\zeta(s) \le \frac{e^{\gamma(s-1)}}{s-1}. \tag{3.14}$$

Lemma 24. *Let R and Q be two positive coprime integers. When $\varepsilon \in [0, 0.168]$, we have*

$$\biggl|\sum_{R|d\le z^2,\ (d,Q)=1}\frac{\lambda_d^{(1)}}{d^{1+\varepsilon}}\biggr| \le \frac{5(1+\varepsilon)}{\log z}\,\frac{Q^{1+\varepsilon}z^{2\varepsilon}}{R^{\varepsilon}\varphi_{1+\varepsilon}(QR)}.$$

We also have

$$\biggl|\sum_{R|d\le z^2,\ (d,Q)=1}\frac{\lambda_d^{(1)}}{d}\biggr| \le \frac{1.004}{\log z}\,\frac{Q}{\varphi(RQ)}.$$

Proof. We first note that (cf. (3.13))

$$\lambda_d^{(1)}\log z = \mu(d)\log^+\frac{z^2}{d} - \mu(d)\log^+\frac{z}{d}. \tag{3.15}$$

We prove this identity again by checking it holds for $d \le z$, for $d \in [z, z^2]$ and for larger d. On using the decomposition (3.15), we find that

$$\begin{aligned}\sum_{R|d\le z^2,\ (d,Q)=1}\frac{\lambda_d^{(1)}\log z}{d^{1+\varepsilon}} &= \sum_{R|d\le z^2,\ (d,Q)=1}\frac{\mu(d)\log(z^2/d)}{d^{1+\varepsilon}} - \sum_{R|d\le z,\ (d,Q)=1}\frac{\mu(d)\log(z/d)}{d^{1+\varepsilon}}\\ &= \frac{\mu(R)}{R^{1+\varepsilon}}\sum_{\substack{\ell\le z^2/R\\ (\ell,QR)=1}}\frac{\mu(\ell)\log((z^2/R)/\ell)}{\ell^{1+\varepsilon}} - \frac{\mu(R)}{R^{1+\varepsilon}}\sum_{\substack{\ell\le z/R\\ (\ell,QR)=1}}\frac{\mu(\ell)\log((z/R)/\ell)}{\ell^{1+\varepsilon}}.\end{aligned}$$

We use Lemma 18 to show that the absolute value of the left-hand side is not more than

$$\frac{\mu^2(R)}{R^{1+\varepsilon}}\biggl(1.4 + 4.7\varepsilon + 3.3\varepsilon^2 + (1+\varepsilon)\frac{(RQ)^{1+\varepsilon}}{\varphi_{1+\varepsilon}(RQ)}\,\frac{z^{2\varepsilon}}{R^{\varepsilon}} + 1.44 + 5\varepsilon + 3.6\varepsilon^2\biggr).$$

Since $(2.84 + 9.7\varepsilon + 6.9\varepsilon^2)/(1+\varepsilon) \le 4$ when $\varepsilon \in [0, 0.168]$, the first inequality of the lemma follows. The second inequality follows by using the same decomposition (3.15) and invoking Lemma 19. □

Lemma 25. *Let δ be a given integer and $\varepsilon \in (0, 0.16]$. We have*

$$\biggl|\sum_{d_1,d_2\le z^2,\ \delta|[d_1,d_2]}\frac{\lambda_{d_1}^{(1)}\lambda_{d_2}^{(1)}}{[d_1,d_2]^{1+\varepsilon}}\biggr| \le \frac{25(1+\varepsilon)^2 z^{4\varepsilon}\zeta(1+\varepsilon)}{\delta^{\varepsilon}\log^2 z}\prod_{p|\delta}\frac{3}{p-1}.$$

Proof. Let us denote by S the sum to be studied. We find that

$$S=\sum_{\delta_1\delta_2\delta_3=\delta}\sum_{\substack{\delta_1\delta_3|d_1\le z^2\\ \delta_2\delta_3|d_2\le z^2\\ (d_1,\delta_2)=(d_2,\delta_1)=1}}\frac{\lambda^{(1)}_{d_1}\lambda^{(1)}_{d_2}}{[d_1,d_2]^{1+\epsilon}}=\sum_{\delta_1\delta_2\delta_3=\delta}\delta_3^{1+\epsilon}\sum_{\substack{\ell\le z^2\\(\ell,\delta)=1}}\varphi_{1+\epsilon}(\ell)\sum_{\substack{\ell\delta_1\delta_3|d_1\le z^2\\ \ell\delta_2\delta_3|d_2\le z^2\\ (d_1,\delta_2)=(d_2,\delta_1)=1}}\frac{\lambda^{(1)}_{d_1}\lambda^{(1)}_{d_2}}{(d_1d_2)^{1+\epsilon}}$$

obtained by using

$$\frac{1}{[d_1,d_2]^{1+\epsilon}}=\frac{(d_1,d_2)^{1+\epsilon}}{(d_1d_2)^{1+\epsilon}}=\frac{\delta_3^{1+\varepsilon}}{(d_1d_2)^{1+\epsilon}}\sum_{\ell|d_1/\delta_3,\ \ell|d_2/\delta_3}\varphi_{1+\epsilon}(\ell)$$

and by noticing that ℓ is prime to δ; this implies that, in fact, $\ell\mid d_1/(\delta_1\delta_3)$ and $\ell\mid d_2/(\delta_2\delta_3)$. We apply Lemma 24 twice (once to the sum over d_1 and once with the sum over d_2) to get

$$|S|\le\frac{25(1+\epsilon)^2z^{4\epsilon}\delta^{1+\epsilon}}{(\log z)^2\,\delta^{\epsilon}\varphi^2_{1+\epsilon}(\delta)}\sum_{\delta_1\delta_2\delta_3=\delta}\frac{1}{\delta_3^{\epsilon}}\sum_{\ell\le z^2,\ (\ell,\delta)=1}\frac{\mu^2(\ell)}{\ell^{2\epsilon}\varphi_{1+\epsilon}(\ell)}.$$

But we have

$$\sum_{\ell\ge1,\ (\ell,\delta)=1}\frac{\mu^2(\ell)}{\varphi_{1+\epsilon}(\ell)}=\prod_{(p,\delta)=1}\left(1+\frac{1}{p^{1+\epsilon}-1}\right)=\frac{\varphi_{1+\epsilon}(\delta)}{\delta^{1+\epsilon}}\,\zeta(1+\epsilon).$$

With this we get

$$|S|\le\frac{25(1+\epsilon)^2\,z^{4\epsilon}\zeta(1+\epsilon)}{(\log z)^2\,\delta^{\epsilon}\,\varphi_{1+\epsilon}(\delta)}\sum_{\delta_1\delta_2\delta_3=\delta}\frac{1}{\delta_3^{\epsilon}}.$$

Finally we can see that

$$\frac{1}{\varphi_{1+\epsilon}(\delta)}\sum_{\delta_1\delta_2\delta_3=\delta}\frac{1}{\delta_3^{\epsilon}}=\frac{1}{\varphi_{1+\epsilon}(\delta)}\sum_{\delta_3|\delta}\frac{2^{\omega(\delta/\delta_3)}}{\delta_3^{\epsilon}}=\frac{1}{\varphi_{1+\epsilon}(\delta)}\prod_{p|\delta}2\left(1+\frac{1}{2p^{\epsilon}}\right)\le\prod_{p|\delta}\frac{3}{p-1},$$

from which the proof follows. □

Lemma 26. *Let δ be a given integer and $\varepsilon\in(0,0.16]$. We have*

$$\left|\sum_{d_1,d_2\le z^2}\frac{\lambda^{(1)}_{d_1}\lambda^{(1)}_{d_2}}{[\delta,d_1,d_2]^{1+\varepsilon}}\right|\le\frac{25(1+\varepsilon)^2\,z^{4\varepsilon}\zeta(1+\varepsilon)}{\log^2 z}\prod_{p|\delta}\frac{4}{p}.$$

Proof. We use Selberg's diagonalization process as usual and appeal to Lemma 25 to get

$$\sum_{d_1,d_2\le z^2}\frac{\lambda^{(1)}_{d_1}\lambda^{(1)}_{d_2}}{[\delta,d_1,d_2]^{1+\epsilon}}\le\sum_{t|\delta}\frac{\varphi_{1+\epsilon}(t)}{\delta^{1+\epsilon}}\sum_{\substack{d_1,d_2\le z^2\\ t|[d_1,d_2]}}\frac{\lambda^{(1)}_{d_1}\lambda^{(1)}_{d_2}}{[d_1,d_2]^{1+\epsilon}}\le\frac{25(1+\epsilon)^2z^{4\epsilon}\zeta(1+\epsilon)}{\delta^{1+\epsilon}\log^2 z}\sum_{t|\delta}\frac{\varphi_{1+\epsilon}(t)}{t^{\epsilon}}\prod_{p|t}\frac{3}{p-1}.$$

But we have

$$\frac{1}{\delta^{1+\epsilon}}\sum_{t|\delta}\frac{\varphi_{1+\epsilon}(t)}{t^{\epsilon}}\prod_{p|t}\frac{3}{p-1}\le\frac{1}{\delta}\sum_{t|\delta}\frac{\varphi_{1+\epsilon}(t)}{t^{2\epsilon}}\prod_{p|t}\frac{3}{p-1}\le\prod_{p|\delta}\frac{4}{p},$$

where we have used the following to get the second inequality:

$$\frac{\varphi_{1+\epsilon}(t)}{t^{2\epsilon}}\le\varphi(t).\qquad\square$$

The next lemma is contained in the main theorem of [12].

Lemma 27. *When $B \ge z \ge 100$, we have*

$$\sum_{n \le B} \frac{\left(\sum_{d|n} \lambda_d^{(1)}\right)^2}{n} \le 166 \frac{\log B}{\log z}.$$

In the next two lemmas, we deal with

$$A'_r = \sum_{\substack{u|r,\ d \le z^2 \\ (ud,q)=1}} \frac{u\mu(r/u)\lambda_d^{(1)}}{[u,d]} \qquad \text{and} \qquad A''_r = \sum_{\substack{u|r,\ d \le z^2 \\ (ud,q)=1}} \frac{u\mu(r/u)\log([u,d])\lambda_d^{(1)}}{[u,d]}. \tag{3.16}$$

We first get simpler expressions. The parameter r will always be square-free and prime to q, but we prefer to repeat it when necessary.

Lemma 28. *When r is square-free and prime to q, we have*

$$\frac{A'_r}{\varphi(r)} = \sum_{r|d \le z^2,\ (d,q)=1} \frac{\lambda_d^{(1)}}{d}$$

as well as

$$\frac{A''_r}{\varphi(r)} = \sum_{r|d \le z^2,\ (d,q)=1} \log(rd) \frac{\lambda_d^{(1)}}{d} + \sum_{\ell | r} \frac{\Lambda(\ell)}{\ell - 1} \Bigg(\sum_{(r/\ell)/d \le z^2,\ (d,q)=1} \frac{\lambda_d^{(1)}}{d} - \ell \sum_{r|d \le z^2,\ (d,q)=1} \frac{\lambda_d^{(1)}}{d} \Bigg).$$

Proof. For this proof, it will save on typographical work to define f_d to be $\lambda_d^{(1)}/d$ when d is prime to q and 0 otherwise. The case of A'_r is easily dealt with. We simply write

$$A'_r = \sum_{u|r,\ d \le z^2} \frac{u\mu(r/u) d f_d}{[u,d]} = \sum_{\delta | r} \varphi(\delta) \sum_{\delta | u | r,\ \delta | d \le z^2} \mu\Big(\frac{r}{u}\Big) f_d = \varphi(r) \sum_{r | d \le z^2} f_d,$$

and this ends the proof. Concerning A''_r, we use $[u,d] = ud/(u,d)$ to get

$$A''_r = \sum_{\substack{u|r,\ d \le z^2 \\ (ud,q)=1}} \frac{u\mu(r/u)\log([u,d]) d f_d}{[u,d]} = \sum_{\substack{u|r,\ d \le z^2 \\ (ud,q)=1}} \mu\Big(\frac{r}{u}\Big)(u,d) f_d \big(\log(ud) - \log((u,d))\big) = B - C$$

say. This calls for the study of two partial quantities, B and C:

$$\begin{aligned} B &= \sum_{\delta|u|r,\ \delta|d \le z^2} \varphi(\delta)\mu\Big(\frac{r}{u}\Big) f_d \Big(\log d + \log\frac{u}{\delta} + \log\delta\Big) \\ &= \varphi(r) \sum_{r|d \le z^2} \log d \cdot f_d + \sum_{\delta|r,\ \delta|d \le z^2} \varphi(\delta)\Lambda\Big(\frac{r}{\delta}\Big) f_d + \varphi(r)\log r \sum_{r|d \le z^2} f_d, \end{aligned}$$

since $\mu \star \log = \Lambda$. Moreover, r is square-free, and this implies $\varphi(\delta) = \varphi(r)/\varphi(r/\delta)$. On using $\ell = r/\delta$, we have

$$B = \varphi(r) \sum_{r|d \le z^2} \log d \cdot f_d + \varphi(r) \sum_{\ell|r,\ (r/\ell)|d \le z^2} \frac{\Lambda(\ell)}{\ell - 1} f_d + \varphi(r) \log r \sum_{r|d \le z^2} f_d.$$

The partial quantity C is (use $\log((u,d)) = \sum_{\ell|u,\,\ell|d}\Lambda(\ell)$)

$$C = \sum_{u|r,\, d\le z^2} \mu\Big(\frac{r}{u}\Big)\log((u,d))(u,d)f_d = \sum_{\ell|u|r,\, \ell|d\le z^2}\Lambda(\ell)\mu\Big(\frac{r}{u}\Big)(u,d)f_d = \varphi(r)\sum_{\ell|r}\frac{\Lambda(\ell)}{\ell-1}\sum_{r|d\le z^2} f_d.$$

The last equality asks for some details: ℓ being fixed dividing r and d, we have

$$\sum_{\ell|u|r}\mu\Big(\frac{r}{u}\Big)(u,d) = \ell\sum_{v|r'}\mu\Big(\frac{r'}{v}\Big)(v,d'),$$

where $r' = r/\ell$ and $d' = d/\ell$. This last sum vanishes if $r' \nmid d'$ and has value $\varphi(r') = \varphi(r)/\varphi(\ell)$ otherwise. Hence

$$\frac{A''_r}{\varphi(r)} = \frac{B}{\varphi(r)} - \frac{C}{\varphi(r)} = \sum_{r|d\le z^2,\,(d,q)=1}\log(rd)\,f_d - \sum_{\ell|r}\frac{\Lambda(\ell)}{\ell-1}\Bigg(\sum_{(r/\ell)|d\le z^2} f_d - \ell\sum_{r|d\le z^2} f_d\Bigg).$$

The lemma follows readily. $\square$

Lemma 29. *When r is square-free and prime to q, we have*

$$\Bigg|\sum_{r|d\le z^2,\,(d,q)=1}\log\frac{4X}{erd}\,\frac{\lambda_d^{(1)}}{d}\Bigg| \le \bigg(1.004\frac{\log(4X/(er))}{\log z}+2\bigg)\frac{q}{\varphi(rq)}.$$

Proof. We appeal to the decomposition given by (3.15) and find that

$$\begin{aligned}\sum_{r|d\le z^2,\,(d,q)=1}\log\frac{4X}{erd}\,\frac{\lambda_d^{(1)}}{d} &= \sum_{r|d\le z^2,\,(d,q)=1}\log\frac{4X}{erd}\,\frac{\mu(d)\log(z^2/d)}{d\log z} - \sum_{r|d\le z,\,(d,q)=1}\log\frac{4X}{erd}\,\frac{\mu(d)\log(z/d)}{d\log z}\\ &= \sum_{r|d\le z^2,\,(d,q)=1}\log\frac{4X}{erz^2}\,\frac{\mu(d)\log(z^2/d)}{d\log z} + \sum_{r|d\le z^2,\,(d,q)=1}\frac{\mu(d)\log^2(z^2/d)}{d\log z}\\ &\quad - \sum_{r|d\le z,\,(d,q)=1}\log\frac{4X}{erz}\,\frac{\mu(d)\log(z/d)}{d\log z} - \sum_{r|d\le z,\,(d,q)=1}\frac{\mu(d)\log^2(z/d)}{d\log z}.\end{aligned}$$

Lemma 19 tells us that each sum is signed and is bounded. For instance, when $\mu(r) = 1$, the first two terms are non-negative while the next two are non-positive, and similarly when $\mu(r) = -1$. It is thus enough to bound each term. We further note, with $\alpha = 1.003003$, that

$$\alpha\log\frac{4X}{erz^2} + 2\log\frac{z^2}{r} \le \alpha\log\frac{4X}{erz^2} + 4\log z = \alpha\log\frac{4X}{er} + (4-2\alpha)\log z.$$

The lemma follows readily. $\square$

3.6. Handling some smooth sums. The study of the linear parts relies on the exact evaluation of smooth sums: we gather this material here.

Lemma 30. *When M and $N > 0$ are real numbers such that $M + N \ge 1$ and a is an integer prime to q, we have*

$$\sum_{\substack{M<n\le M+N\\(n,q)=1}} e\Big(\frac{na}{q}\Big) = \frac{\mu(q)N}{q} + \mathcal{O}^*(\varphi(q)).$$

Proof. We split the interval $(M, M+N]$ into $N/q + \mathcal{O}^*(1)$ intervals containing q consecutive integers and a final interval containing, say, h integers prime to q. Since $h \le \varphi(q) - 1$, the lemma is proved. $\square$

By integration by parts, we get

$$\sum_{n\le N,\ (n,q)=1} e\Big(\frac{na}{q}\Big)\log n = \sum_{n\le N,\ (n,q)=1} e\Big(\frac{na}{q}\Big)\log N - \int_1^N \sum_{n\le t,\ (n,q)=1} e\Big(\frac{na}{q}\Big)\frac{dt}{t}.$$

We use this formula for $2N$ and N and get

$$\sum_{\substack{N<n\le 2N\\(n,q)=1}} e\Big(\frac{na}{q}\Big)\log n = \sum_{\substack{n\le N\\(n,q)=1}} e\Big(\frac{na}{q}\Big)\log 2 + \sum_{\substack{N<n\le 2N\\(n,q)=1}} e\Big(\frac{na}{q}\Big)\log(2N) - \int_N^{2N} \sum_{\substack{n\le t\\(n,q)=1}} e\Big(\frac{na}{q}\Big)\frac{dt}{t}.$$

Hence we get the following lemma.

Lemma 31. *When N is a real number and a is an integer prime to q, we have*

$$\sum_{N<n\le 2N,\ (n,q)=1} e\Big(\frac{na}{q}\Big)\log n = \frac{\mu(q)N\log(4N/e)}{q} + \mathcal{O}^*\big(\varphi(q)\log(8N)\big).$$

Lemma 32. *When M and $N>0$ are real numbers and b is an integer, we have*

$$\sum_{\substack{M<n\le M+N\\ n\equiv b[q]}} \frac{e(\beta n)}{n^{it}} = \frac{1}{q}\int_M^{M+N} \frac{e(\beta v)\,dv}{v^{it}} + \mathcal{O}\big((|t|+|\beta|(M+N)+1)\log(2(M+N))\big)$$

and

$$\sum_{\substack{M<n\le M+N\\ n\equiv b[q]}} \frac{e(\beta n)\log n}{n^{it}} = \frac{1}{q}\int_M^{M+N} \frac{e(\beta v)\log v}{v^{it}}\,dv + \mathcal{O}\big((|t|+|\beta|(M+N)+1)\log^2(2(M+N))\big).$$

Proof. We define $f_1(\alpha,\ell)=e(\beta q\ell)/(\alpha+\ell)^{it}$ and $f_2(\alpha,\ell)=\log(\alpha+\ell)f_1(\alpha,\ell)$ for $\alpha=b/q$ with $1\le b\le q-1$ and first study, for $f=f_1$ or $f=f_2$,

$$S(L;f)=\sum_{1\le\ell\le L} f(\ell). \tag{3.17}$$

We have

$$\begin{aligned} S(L;f) &= -\int_1^L [u]f'(u)\,du + [L]f(L) = f(1) + \int_1^L f(u)\,du + \mathcal{O}\left(\int_1^L |f'(u)|\,du + |f(L)|\right)\\ &= \int_1^L f(u)\,du + \mathcal{O}\big((|t|+\beta qL)\log^2(2L)\big). \end{aligned}$$

We consider

$$\left(S\Big(\frac{M+N-b}{q};f_1\Big) - S\Big(\frac{M-b}{q};f_1\Big)\right)\frac{\log q}{q^{it}} + \left(S\Big(\frac{M+N-b}{q};f_2\Big) - S\Big(\frac{M-b}{q};f_2\Big)\right)\frac{1}{q^{it}}$$

and, as a consequence, we find that

$$\mathrm{MT}\left(\sum_{\substack{M<n\le M+N\\ n\equiv b[q]}} \frac{e(\beta n)\log n}{n^{it}}\right) = \int_{(M-b)/q}^{(M+N-b)/q} \frac{e(\beta(b+uq))\log(b+uq)}{(b+uq)^{it}}\,du = \frac{1}{q}\int_M^{M+N} \frac{e(\beta v)\log v}{v^{it}}\,dv,$$

where $\mathrm{MT}(\cdot)$ means the main term of the parenthesized quantity. The lemma follows readily. □

Lemma 33. *When M and $N>0$ are real numbers and a is an integer prime to q, we have*

$$\sum_{\substack{M<n\le M+N\\(n,q)=1}} e(\beta n)\frac{e(an/q)}{n^{it}} = \frac{\mu(q)}{q}\int_M^{M+N}\frac{e(\beta v)\,dv}{v^{it}} + \mathcal{O}\bigl(q(|t|+|\beta|(M+N)+1)\log(2(M+N))\bigr)$$

and

$$\sum_{\substack{M<n\le M+N\\(n,q)=1}} e(\beta n)\frac{\log n}{n^{it}}\,e\Bigl(\frac{an}{q}\Bigr) = \frac{\mu(q)}{q}\int_M^{M+N}\frac{e(\beta v)\log v}{v^{it}}\,dv$$
$$+\,\mathcal{O}\bigl(q(|t|+|\beta|(M+N)+1)\log^2(2(M+N))\bigr).$$

Proof. This is a simple exercise with the previous lemma. □

4. THE FIRST LINEAR FORM

We study here the first linear form defined in (2.11), and this section is devoted to proving Lemma 34 and 35.

4.1. The case $t=\beta=0$.

Lemma 34. *When $Rz^2/q\le X$ and $(a,q)=1$, we have*

$$\sum_{r\le R/q,\ (r,q)=1}\frac{\mu^2(r)\bigl|L_r^{(1)}(a,0,0)\bigr|}{\varphi(r)} \le \frac{X}{q}G(R)\biggl(3.012\frac{\log(4X/e)}{\log z}+2\biggr)+3.3\varphi(q)\frac{R}{q}\frac{z^2}{\log z}\log(8X).$$

Proof. We start from (2.11) and sum over n first by using Lemma 31; we find that

$$L_r^{(1)}(a,0,0)=\frac{\mu(q)}{q}X\sum_{\substack{m\le 2X\\(m,q)=1}}\frac{h_r(m)\log(4X/(me))}{m}+\mathcal{O}^*\Biggl(\varphi(q)\log(8X)\sum_{\substack{m\le 2X\\(m,q)=1}}|h_r(m)|\Biggr). \tag{4.1}$$

The bound $m\le 2X$ can be replaced in both summations by $m\le rz^2$ since $h_r(m)$ vanishes otherwise. Note also that $rz^2\le 2X$. We set

$$A_r=\sum_{m\le rz^2,\ (m,q)=1}\frac{h_r(m)\log(4X/(me))}{m}=A_r'\log\frac{4X}{e}-A_r'', \tag{4.2}$$

on recalling (3.16).

Bounding the main term in (4.1). As a consequence, we deduce the following. We combine (3.16) with Lemma 28 to infer that

$$\frac{A_r}{\varphi(r)}=\log\frac{4X}{e}\sum_{\substack{r|d\le z^2\\(d,q)=1}}\frac{\lambda_d^{(1)}}{d}-\sum_{\substack{r|d\le z^2\\(d,q)=1}}\log(rd)\frac{\lambda_d^{(1)}}{d}-\sum_{\ell|r}\frac{\Lambda(\ell)}{\ell-1}\Biggl(\sum_{\substack{(r/\ell)|d\le z^2\\(d,q)=1}}\frac{\lambda_d^{(1)}}{d}-\ell\sum_{\substack{r|d\le z^2\\(d,q)=1}}\frac{\lambda_d^{(1)}}{d}\Biggr)$$
$$=\sum_{\substack{r|d\le z^2\\(d,q)=1}}\log\frac{4X}{erd}\frac{\lambda_d^{(1)}}{d}-\sum_{\ell|r}\frac{\Lambda(\ell)}{\ell-1}\Biggl(\sum_{\substack{(r/\ell)|d\le z^2\\(d,q)=1}}\frac{\lambda_d^{(1)}}{d}-\ell\sum_{\substack{r|d\le z^2\\(d,q)=1}}\frac{\lambda_d^{(1)}}{d}\Biggr).$$

In this form, Lemma 29 and the second part of Lemma 24 (with $R = r/\ell$ and $Q = \ell q$) apply directly to yield the bound

$$\frac{|A_r|}{\varphi(r)} \le \left(1.004\frac{\log(4X/(er))}{\log z} + 2\right)\frac{q}{\varphi(rq)} + \sum_{\ell|r}\frac{\Lambda(\ell)}{\ell-1}\,\frac{1.004}{\log z}\,\frac{q}{\varphi(rq/\ell)} + \sum_{\ell|r}\frac{\Lambda(\ell)\ell}{\ell-1}\,\frac{1.004}{\log z}\,\frac{q}{\varphi(qr)}.$$

Note that $\sum_{\ell|r}(\varphi(\ell)\Lambda(\ell)/(\ell-1)) \le \log r$ and $\sum_{\ell|r}(\ell\Lambda(\ell)/(\ell-1)) \le 2\log r$. Thus

$$\frac{|A_r|}{\varphi(r)} \le \left(1.004\frac{\log(4X/(er))}{\log z} + 2\right)\frac{q}{\varphi(rq)} + \log r\,\frac{1.004}{\log z}\,\frac{q}{\varphi(rq)} + 2\log r\,\frac{1.004}{\log z}\,\frac{q}{\varphi(qr)}.$$

We simplify this to

$$\frac{|A_r|}{\varphi(r)} \le \left(1.004\frac{\log(4Xr^2/e)}{\log z} + 2\right)\frac{q}{\varphi(rq)}.$$

We first notice that $r \le 4X/e$ and sum over r. Recalling that (3.7) implies $(q/\varphi(q))G_q(R/q) \le G(R)$, we finally get

$$\sum_{r\le R/q,\ (r,q)=1}\frac{\mu^2(r)|A_r|}{\varphi(r)} \le G(R)\left(3.012\frac{\log(4X/e)}{\log z} + 2\right). \tag{4.3}$$

Bounding the error term in (4.1). Regarding the error term, we first notice that

$$\sum_{m\le 2X,\ (m,q)=1}|h_r(m)| = \sum_{u|r,\ d\le z^2}\left|u\mu\Big(\frac{r}{u}\Big)\lambda_d^{(1)}\right| \le \frac{z^2}{\log z}\prod_{p|r}(p+1)$$

by Lemma 23. And thus

$$\begin{aligned}\sum_{r\le R/q,\ (r,q)=1}\frac{\mu^2(r)}{\varphi(r)}\sum_{m\le 2X,\ (m,q)=1}|h_r(m)| &\le \frac{z^2}{\log z}\sum_{r\le R/q,\ (r,q)=1}\mu^2(r)\sum_{\delta|r}\frac{2^{\omega(\delta)}}{\varphi(\delta)}\\ &\le \frac{R}{q}\,\frac{z^2}{\log z}\prod_{p\ge 2}\left(1+\frac{2}{p(p-1)}\right) \le 3.3\,\frac{R}{q}\,\frac{z^2}{\log z}.\end{aligned}$$

The proof of Lemma 34 is complete. □

4.2. The general case.

Lemma 35. *When $Rz^2/q \le X$, we have*

$$\begin{aligned}\sum_{r\le R/q,\ (r,q)=1}\frac{\mu^2(r)\big|L_r^{(1)}(a,t,\beta)\big|}{\varphi(r)} \le{}& \frac{X}{q}G(R)\left(2.008\frac{\log(4XR/(eq))}{\log z} + 3\right)\\ &+ \mathcal{O}\left(\big((|t| + |\beta|X + 1)\log^2 X\big)\frac{Rz^2}{\log z}\right).\end{aligned}$$

Proof. We start from (2.11) and sum over n first by using Lemma 33; we find that

$$\begin{aligned}L_r^{(1)}(a,t,\beta) ={}& \frac{\mu(q)}{q}\sum_{m\le 2X,\ (m,q)=1}\frac{h_r(m)}{m^{it}}\int_{X/m}^{2X/m}\frac{e(\beta m v)\log v}{v^{it}}\,dv\\ &+ \mathcal{O}\left(q(|t| + |\beta|X + 1)\log^2 X\sum_{m\le 2X,\ (m,q)=1}|h_r(m)|\right).\end{aligned}$$

The change of variable $w = vm$ yields

$$L_r^{(1)}(a,t,\beta) = \frac{\mu(q)}{q}\sum_{m\le 2X,\ (m,q)=1}\frac{h_r(m)}{m}\left(\int_X^{2X}\frac{e(\beta w)\log w}{w^{it}}\,dw - \int_X^{2X}\frac{e(\beta w)\,dw}{w^{it}}\log m\right)$$
$$+\,\mathcal{O}\Biggl(q(|t|+|\beta|X+1)\log^2 X\sum_{m\le 2X,\ (m,q)=1}|h_r(m)|\Biggr)$$

so that, with the notation A_r' and A_r'' from (3.16),

$$L_r^{(1)}(a,t,\beta) = \frac{\mu(q)}{q}\int_X^{2X}\frac{e(\beta w)\log w}{w^{it}}\,dw\,A_r' - \frac{\mu(q)}{q}\int_X^{2X}\frac{e(\beta w)\,dw}{w^{it}}A_r''$$
$$+\,\mathcal{O}\Biggl(q(|t|+|\beta|X+1)\log^2 X\sum_{m\le 2X,\ (m,q)=1}|h_r(m)|\Biggr).$$

From then onwards, the treatment of $L_r^{(1)}(a,t,\beta)$ can be mimicked with the one of $L_r^{(1)}(a,0,0)$. We leave the details to the reader. $\square$

5. THE SECOND LINEAR FORM

We study here the second linear form defined in (2.12).

Lemma 36. *When a and r are such that $(a,q)=(r,q)=1$, we have*

$$\bigl|L_r^{(2)}(a,0,0)\bigr| \le 1.004\frac{\mu^2(q)X\log M_0}{\varphi(q)\log z} + 1.04\varphi(q)\varphi_+(r)M_0\frac{z^2}{\log z}$$

with $\varphi_+(r) = \sum_{\ell|r}\ell = \prod_{p|r}(1+p)$.

Proof. We readily find that

$$L_r^{(2)}(a,t,\beta) = \sum_{\substack{m\le M_0\\(m,q)=1}}\frac{\Lambda(m)}{m^{it}}\sum_{\substack{d\le z^2\\(d,q)=1}}\lambda_d^{(1)}\sum_{\substack{n\sim X/m\\(n,q)=1,\ d|n}}e(\beta mn)\frac{c_r(n)}{n^{it}}\,e\Bigl(\frac{mna}{q}\Bigr)$$
$$= \sum_{\substack{m\le M_0\\(m,q)=1}}\frac{\Lambda(m)}{m^{it}}\sum_{\substack{d\le z^2\\(d,q)=1}}\lambda_d^{(1)}\sum_{\ell|r}\ell\mu\Bigl(\frac{r}{\ell}\Bigr)\sum_{\substack{n\sim X/m\\(n,q)=1,\ [d,\ell]|n}}e(\beta mn)\frac{e(mna/q)}{n^{it}}. \tag{5.1}$$

Now we specialize to $t=\beta=0$, apply Lemma 30 and get

$$L_r^{(2)}(a,0,0) = \sum_{\substack{m\le M_0\\(m,q)=1}}\Lambda(m)\sum_{\substack{d\le z^2\\(d,q)=1}}\lambda_d^{(1)}\sum_{\ell|r}\ell\mu\Bigl(\frac{r}{\ell}\Bigr)\Bigl(\frac{\mu(q)X}{m[d,\ell]q}+\mathcal{O}^*(\varphi(q))\Bigr)$$
$$= \frac{\mu(q)X\varphi(r)}{q}\sum_{\substack{m\le M_0\\(m,q)=1}}\frac{\Lambda(m)}{m}\sum_{\substack{d\le z^2\\(d,q)=1,\ r|d}}\frac{\lambda_d^{(1)}}{d} + \mathcal{O}^*\Bigl(1.04\varphi(q)\varphi_+(r)M_0\frac{z^2}{\log z}\Bigr),$$

since the error in the above equation is

$$\varphi(q)\sum_{m\le M_0}\Lambda(m)\sum_{d\le z^2}\bigl|\lambda_d^{(1)}\bigr|\sum_{\ell|r}\ell \le 1.04\varphi(q)\,M_0\frac{z^2}{\log z}\varphi_+(r),$$

where the sum over m is treated via (3.1) and the sum over d is treated via Lemma 23. We appeal to the second estimate of Lemma 24 (with $R = r$ and $Q = q$) to get

$$\left|L_r^{(2)}(a,0,0)\right| \le \frac{1.004\mu^2(q)\,X\log M_0}{\varphi(q)\log z} + 1.04\varphi_+(r)\varphi(q)\,M_0\frac{z^2}{\log z},$$

where the sum over m is this time treated via (3.3). The lemma follows readily. □

Adapting this proof to the general case from (5.1) is not difficult. We get

Lemma 37. *When a and r are such that $(a,q) = (r,q) = 1$, we have*

$$\left|L_r^{(2)}(a,t,\beta)\right| \ll \frac{\mu^2(q)X\log M_0}{(1+|t|)\varphi(q)\log z} + \mathcal{O}\biggl(q(|t| + |\beta|X + 1)\varphi_+(r)M_0\frac{z^2}{\log z}\log X\biggr).$$

We quote the next direct consequence of this lemma.

Lemma 38. *For any $\varepsilon > 0$, we have*

$$\int_{-T}^{T}\sum_{r\le R/q,\ (r,q)=1}\frac{\mu^2(r)}{\varphi(r)}\left|L_r^{(2)}(a,t,0)\right|dt \ll_\varepsilon \frac{X(\log T)(\log R)\log M_0}{\varphi(q)\log z} + \frac{M_0\,z^2T^2R^{1+\varepsilon}\log X}{\log z}.$$

6. THE ERROR TERM DUE TO THE SEPARATION OF VARIABLES

Lemma 39. *When $\varepsilon \in (0, 0.154]$ and $R/q \ge 15$, we have*

$$\sum_{\substack{rq\le R\\(r,q)=1}}\frac{\mu^2(r)}{\varphi(r)}\sum_{\substack{n\le B\\(n,q)=1}}\frac{|v_r(n)|^2}{n} \le 0.285\,G_q\Bigl(\frac{R}{q}\Bigr)\frac{(1+\varepsilon)^2B^\varepsilon z^{4\varepsilon}\zeta^2(1+\varepsilon)R}{q\log^2 z}\Bigl(\log\frac{R}{q} + 3.1\Bigr)^3.$$

Proof. We use

$$|c_r(n)|^2 \le \varphi((n,r))^2 = \sum_{\delta|n,\ \delta|r}\delta f_2(\delta),$$

where $f_2(\delta) = \prod_{p|\delta}(p-2)$. Thus, on using (3.6),

$$\sum_{\substack{r\le R/q\\(r,q)=1}}\frac{\mu^2(r)|c_r(n)|^2}{\varphi(r)} = \sum_{\delta|n}\delta f_2(\delta)\sum_{\substack{\delta|r\le R/q\\(r,q)=1}}\frac{\mu^2(r)}{\varphi(r)} = \sum_{\delta|n}f_2(\delta)G_{\delta q}\Bigl(\frac{R}{\delta q}\Bigr) \le \sum_{\substack{\delta|n\\\delta\le R/q}}f_3(\delta)G_q\Bigl(\frac{R}{q}\Bigr),$$

where $f_3(\delta) = \mu^2(\delta)\prod_{p|\delta}(p-1)(p-2)/p$. On denoting by S the sum to be bounded above, we get

$$\begin{aligned}
S &\le \sum_{\substack{rq\le R\\(r,q)=1}}\frac{\mu^2(r)}{\varphi(r)}\sum_{n\le B}\Bigl(\sum_{d|n}\lambda_d^{(1)}\Bigr)^2\frac{|c_r(n)|^2}{n} \le G_q\Bigl(\frac{R}{q}\Bigr)\sum_{\delta\le R/q}f_3(\delta)\sum_{\delta|n\le B}\frac{1}{n}\Bigl(\sum_{d|n}\lambda_d^{(1)}\Bigr)^2\\
&\le G_q\Bigl(\frac{R}{q}\Bigr)B^\varepsilon\sum_{\delta\le R/q}f_3(\delta)\sum_{\delta|n}\frac{1}{n^{1+\varepsilon}}\Bigl(\sum_{d|n}\lambda_d^{(1)}\Bigr)^2\\
&\le G_q\Bigl(\frac{R}{q}\Bigr)B^\varepsilon\zeta(1+\varepsilon)\sum_{\delta\le R/q}f_3(\delta)\sum_{d_1,d_2\le z^2}\frac{\lambda_{d_1}^{(1)}\lambda_{d_2}^{(1)}}{[\delta,d_1,d_2]^{1+\varepsilon}}.
\end{aligned}$$

We appeal to Lemma 26 and (3.14):

$$\frac{S}{G_q(R/q)} \le \frac{25(1+\varepsilon)^2 B^\varepsilon z^{4\varepsilon}\zeta^2(1+\varepsilon)}{\log^2 z} \sum_{\delta \le R/q} f_3(\delta) \prod_{p|\delta} \frac{4}{p}.$$

A use of Lemma 21 gives us

$$S \le G_q\Big(\frac{R}{q}\Big) \frac{25(1+\varepsilon)^2 B^\varepsilon z^{4\varepsilon}\zeta^2(1+\varepsilon)R}{q\log^2 z} \cdot 0.0114\Big(\log\frac{R}{q} + 3.1\Big)^3.$$

The lemma follows readily. □

7. PROOF OF THE EXPLICIT THEOREM 7

We continue the argument started in Section 2.

7.1. Preparation of each $|S_r(a/q, t, \beta, M, N)|$. Recall the decomposition (2.14) of S_r. We first have the next explicit bound.

Lemma 40. *When $M \ge 10^{14}$ and $\delta \ge M^{-1/4}$, we have*

$$E_1(\delta, r) + E_2(\delta, r) + 2\delta E_3(r) \le 10.5\,\delta M \sum_{n\sim N,\ (n,q)=1} |v_r(n)|.$$

When ignoring the explicit aspect, here is what we get.

Lemma 41. *Let $\epsilon > 0$. When $\delta M \gg M^\epsilon$, we have*

$$E_1(\delta, r) + E_2(\delta, r) + 2\delta E_3(r) \ll_\epsilon \delta M \sum_{n\sim N,\ (n,q)=1} |v_r(n)|.$$

Proof. We first notice that, when $N < n \le 2N$, $M < m \le 2M$ and $X < mn \le 2^\delta X$, we check that, with $A = \max(M, X/n)$,

$$\Big[\max\Big(M, \frac{X}{n}\Big), \min\Big(2M, \frac{2^\delta X}{n}\Big)\Big] \subset \big[A, A + (2^\delta - 1)M\big] \subset \big[A, A + (2\delta\log 2)M\big],$$

since $2^\delta - 1 = \int_0^{\delta\log 2} e^u\,du \le 2\delta\log 2$. Similarly, concerning the conditions $N < n \le 2N$, $M < m \le 2M$ and $2^{1-\delta}X < mn \le 2X$, we check that, with $A = \max(M, 2^{1-\delta}X/n)$,

$$\Big[\max\Big(M, \frac{2^{1-\delta}X}{n}\Big), \min\Big(2M, \frac{2X}{n}\Big)\Big] \subset \big[A, A + 2(1 - 2^{-\delta})M\big] \subset \big[A, A + (2\delta\log 2)M\big],$$

since this time $1 - 2^{-\delta} = \int_{-\delta\log 2}^0 e^u\,du \le \delta\log 2$. We note that $2\log 2 \le 1.4$. By Lemma 14 applied twice, we find that

$$E_1(\delta, r) + E_2(\delta, r) \le 2\frac{2\cdot 1.4\delta M}{\log(1.4\delta M)} \log(2M) \sum_{n\sim N} |v_r(n)|.$$

Note that, since $\delta \ge M^{-1/4}$, we have $\delta M \ge M^{3/4}$ and

$$\frac{2\cdot 1.4\log(2M)}{\log(1.4M^{3/4})} \le 3.8,$$

and thus

$$E_1(\delta, r) + E_2(\delta, r) \le 3.8\,\delta M \sum_{n\sim N} |v_r(n)|. \tag{7.1}$$

Concerning $E_3(r)$, we use Lemma 13, getting

$$E_3(r) \le \frac{5}{4} M \sum_{n\sim N} |v_r(n)|. \tag{7.2}$$

Note that $2\cdot 3.8 + 2\cdot 5/4 = 10.1 \le 10.5$. $\square$

Hence, on using the classical inequality $|a+b|^2 \le 2(|a|^2+|b|^2)$,

$$\begin{aligned}\Bigl|S_r\Bigl(\frac{a}{q},t,\beta,M,N\Bigr)\Bigr|^2 &\le 2\int_{-\Delta}^{\Delta} \sum_{b \bmod^* q} \Biggl|\sum_{m\equiv b[q],\, m\sim M} \frac{\Lambda(m)}{m^{i(v+t)}}\Biggr|^2 |\mathscr{H}(v)|^2\,dv \\ &\quad\times \int_{-\Delta}^{\Delta} \sum_{b \bmod^* q} \Biggl|\sum_{(n,q)=1,\, n\sim N} \frac{v_r(n)}{n^{i(v+t)}}\, e\Bigl(\frac{nba}{q}\Bigr)\Biggr|^2 dv + 2\Biggl(10.5\,\delta M \sum_{n\sim N}|v_r(n)|\Biggr)^2. \end{aligned}\tag{7.3}$$

By Lemma 45 (see the Appendix), we have $\int_{\mathbb{R}} |\widehat{H}(\delta,\lambda,\kappa;v)|^2\,dv = \int_{\mathbb{R}} |H(\delta,\lambda,\kappa;u)|^2\,du = (2-2\delta)\times(\log 2)^2/(4\pi)^2$. We use Lemma 16 provided that $M \ge \max(121,q^3)$:

$$\begin{aligned}\Bigl|S_r\Bigl(\frac{a}{q},t,\beta,M,N\Bigr)\Bigr|^2 &\le 2\cdot 2\frac{(\log 2)^2}{(4\pi)^2}\,\frac{45}{8}\,\frac{M^2}{\varphi(q)} \int_{-\Delta}^{\Delta} \sum_{b \bmod^* q} \Biggl|\sum_{(n,q)=1,\, n\sim N} \frac{v_r(n)}{n^{i(v+t)}}\, e\Bigl(\frac{nb}{q}\Bigr)\Biggr|^2 dv \\ &\quad + 221\,\delta^2 M^2 (N+1) \sum_{n\sim N} |v_r(n)|^2;\end{aligned}$$

i.e., since $MN \le X$ and $N \ge z \ge 121$,

$$\begin{aligned}\Bigl|S_r\Bigl(\frac{a}{q},t,\beta,M,N\Bigr)\Bigr|^2 &\le \frac{45}{2}\,\frac{(\log 2)^2}{(4\pi)^2}\,\frac{M^2}{\varphi(q)} \int_{-\Delta}^{\Delta} \sum_{b \bmod^* q} \Biggl|\sum_{(n,q)=1,\, n\sim N} \frac{v_r(n)}{n^{i(v+t)}}\, e\Bigl(\frac{nb}{q}\Bigr)\Biggr|^2 dv \\ &\quad + 446\,\delta^2 X^2 \sum_{n\sim N} \frac{|v_r(n)|^2}{n}.\end{aligned}\tag{7.4}$$

7.2. Average estimate (over r and (M,N)) of $|S_r(a/q,t,\beta,M,N)|^2$. Let us use the shorter notation

$$\Sigma(M,N) = \sum_{rq\le R,\, (r,q)=1} \frac{\mu^2(r)}{\varphi(r)} \Bigl|S_r\Bigl(\frac{a}{q},t,\beta,M,N\Bigr)\Bigr|^2. \tag{7.5}$$

We sum (7.4) over r and use the large sieve inequality from Theorem 11 to get (since $MN \le X$)

$$\Sigma(M,N) \le 158\frac{M^2}{\varphi(q)}\,\frac{(\log 2)^2}{(4\pi)^2} \sum_{\substack{(n,q)=1\\ n\sim N}} \Biggl(\sum_{d|n} \lambda_d^{(1)}\Biggr)^2 (n+R^2\Delta) + 446\,\delta^2 X^2 \sum_{\substack{rq\le R\\ (r,q)=1}} \frac{\mu^2(r)}{\varphi(r)} \sum_{n\sim N} \frac{|v_r(n)|^2}{n}.$$

We take $R = z^{1/4} = 2\delta^{-1} > 2q^2$ and $M_0 = z \ge q^8$, with $z \ge 10^{14}$. We assume that $\beta = 0$, and thus (see (A.6))

$$\delta^2\Delta = \frac{2}{\pi\log 2}.$$

Thus

$$n + R^2\Delta \le 2N + \frac{2\delta^{-2}}{\pi\log 2}\cdot 4\delta^{-2} \le 2N + 4\delta^{-4} \le 2N + \frac{z}{4}.$$

This is since $R = 2\delta^{-1}$.

Furthermore, for any $n \in (N, 2N]$, we have $n(2N + z/4) \le 4.5N^2$ since $N \ge z$. Hence

$$\Sigma(M,N) \le 4.4 \frac{X^2}{\varphi(q)} \sum_{(n,q)=1,\, n\sim N} \frac{\left(\sum_{d|n} \lambda_d^{(1)}\right)^2}{n} + 446\,\delta^2 X^2 \sum_{rq\le R,\, (r,q)=1} \frac{\mu^2(r)}{\varphi(r)} \sum_{n\sim N} \frac{|v_r(n)|^2}{n}.$$

It is time to sum over relevant (M, N). We set

$$\Sigma = \sum_{M,N} \Sigma(M,N). \tag{7.6}$$

Summation over M is readily dealt with: there are at most two M's for each N. Thus

$$\Sigma \le 8.8 \frac{X^2}{\varphi(q)} \sum_{(n,q)=1,\, n\le 2X/z} \frac{\left(\sum_{d|n} \lambda_d^{(1)}\right)^2}{n} + 892\,\delta^2 X^2 \sum_{rq\le R,\, (r,q)=1} \frac{\mu^2(r)}{\varphi(r)} \sum_{n\le 2X/z} \frac{|v_r(n)|^2}{n}. \tag{7.7}$$

We forget the condition $(n, q) = 1$ in the first summation and appeal to Lemma 27, while we use Lemma 39 for the second term. All of that yields

$$\Sigma \le 8.8 \frac{X^2}{\varphi(q)} \cdot 166 \frac{\log(2X/z)}{\log z} + 892\delta^2 X^2 G_q\left(\frac{R}{q}\right) \frac{0.285(1+\varepsilon)^2 (2e^{2\gamma} X z^3)^\varepsilon R}{\varepsilon^2 q \log^2 z} \left(\log \frac{R}{q} + 3.1\right)^3.$$

Hence (since $\delta = 2R^{-1}$, $\varphi(q) \le q$, $G_q(R/q) \le G(R/q)$ and $R/q \ge R^{1/3} = X^{1/48} \ge 15$)

$$\frac{\varphi(q)}{X^2} \Sigma \le 1461 \frac{\log(2X/z)}{\log z} + 1016 \frac{(1+\varepsilon)^2 (2e^{2\gamma} X z^3)^\varepsilon}{R\varepsilon^2 \log^2 z} \left(\log \frac{R}{q} + 3.1\right)^4.$$

We used equation (3.8). Now we directly choose $z = X^{1/4}$ and $\varepsilon = 8/(7 \log X) \le 0.009$ (since $X \ge 250^{24}$). We thus get

$$\frac{\varphi(q)}{X^2} \Sigma \le 4414 + 95\,200 \frac{(\log R - \log q + 3.1)^4}{R}.$$

Next, with $R = z^{1/4} \ge q^{3/2} \ge 250^{3/2}$,

$$\Sigma \le 32\,830 \frac{X^2}{\varphi(q)}. \tag{7.8}$$

7.3. Conclusion of the proof of Theorem 7. Recall that $z = X^{1/4} \ge q^6$ and assume that $q \ge 250$. We have

$$G_q\left(\frac{R}{q}\right) \left|S\left(\frac{a}{q}, t, \beta\right)\right| \le \sum_{r\le R/q,\, (r,q)=1} \frac{\mu^2(r)}{\varphi(r)} \left(|L_r^{(1)}(a,\beta,t)| + |L_r^{(2)}(a,\beta,t)|\right) + \sqrt{G_q\left(\frac{R}{q}\right)} \left(32\,830 \frac{X^2}{\varphi(q)} \frac{2\log(2X/z^2)}{\log 2}\right)^{1/2},$$

where the factor $2(\log(2X/z^2))/\log 2$ comes from (2.13) since $M_0 = z$. This bound is valid for $\beta = 0$ (because we specialized Δ) and arbitrary t. We continue numerically by specializing further $t = 0$.

We select $R^4 = z = M_0 = X^{1/4} \ge q^6$ and $q \ge 250$.

Concerning $L_r^{(1)}$, we use Lemma 34: the hypothesis $Rz^2/q \le X$ is met. We thus get

$$\sum_{r\le R/q,\, (r,q)=1} \frac{\mu^2(r)}{\varphi(r)} \left|L_r^{(1)}(a,0,0)\right| \le \frac{X}{q} G(R) \left(3.012 \frac{\log(4X/e)}{\log z} + 2\right) + 3.3\varphi(q) \frac{R}{q} \frac{z^2}{\log z} \log(8X)$$
$$\le \frac{X}{q} G(R) \left(3.012 \frac{\log(4X/e)}{(1/4)\log X} + 2 + \frac{3.3\, X^{-19/48}}{(1/16)\log X + 1.06} \frac{\log(8X)}{(1/4)\log X}\right) \le 16 \frac{X}{q} G(R).$$

We use (3.9) on $G(R)$, which is allowed since $R = X^{1/16} \geq 6$. Concerning $L_r^{(2)}$, we use Lemmas 36 and 22, getting (we check that $\varphi(q)M_0Rz^2 \leq X^{107/192}$)

$$\sum_{r\leq R/q,\, (r,q)=1} \frac{\mu^2(r)}{\varphi(r)}\bigl|L_r^{(2)}(a,0,0)\bigr| \leq 1.004\, G_q\Bigl(\frac{R}{q}\Bigr)\frac{\mu^2(q)X\log M_0}{\varphi(q)\log z} + 1.04\varphi(q)M_0\frac{z^2}{\log z}\cdot 3.28\frac{R}{q}$$
$$\leq G(R)\frac{X}{q}\biggl(1.004 + 3.4112\frac{X^{107/192-1}}{((1/16)\log X + 1.06)(1/4)\log X}\biggr) \leq 1.005\, G(R)\frac{X}{q}.$$

This finally amounts to

$$18\, G(R)\frac{X}{q} + \sqrt{G_q\Bigl(\frac{R}{q}\Bigr)}\biggl(47\,860\,\frac{X^2}{\varphi(q)}\log X\biggr)^{1/2}.$$

We finally get ($R/q \geq R^{1/3} = X^{1/48}$ and $G_q(R/q) \geq (\varphi(q)/q)G(R/q)$)

$$\Bigl|S\Bigl(\frac{a}{q},0,0\Bigr)\Bigr| \leq 1292\frac{\sqrt{q}\,X}{\varphi(q)}$$

provided $250 \leq q \leq X^{1/24}$.

8. PROOF OF THEOREM 4 IN VERSION (V4)

Proving Theorem 4 with primes rather than with the Möbius function is a simple modification of the proof of Theorem 7. Let us recall (2.13) from Section 2:

$$\Bigl|S\Bigl(\frac{a}{q},t,\beta\Bigr) - L_r^{(1)}(a,t,\beta) + L_r^{(2)}(a,t,\beta)\Bigr|^2 \leq \frac{2\log(2X/(M_0z))}{\log 2}$$
$$\times\sum_{M,N}\Biggl(\int_{-\Delta}^{\Delta}\Biggl|\sum_{\substack{(mn,q)=1\\ m\sim M,\, n\sim N}}\frac{\Lambda(m)}{m^{i(v+t)}}\frac{v_r(n)}{n^{i(v+t)}}\,e\Bigl(\frac{mna}{q}\Bigr)\Biggr||\mathscr{H}(v)|\,dv + E_1(\delta,r) + E_2(\delta,r) + 2\delta E_3(r)\Biggr)^2. \quad (8.1)$$

This is so because $\sum_{mn\sim X} A(m,n)$ is nothing other than $S(a/q,t,\beta) - L_r^{(1)}(a,t,\beta) + L_r^{(2)}(a,t,\beta)$. The localized version of this quantity, i.e., with $m\sim M$ and $n\sim N$, is $S_r(a/q,t,\beta,M,N)$. However, in this latter sum, the variables m and n are still constrained by the condition $mn\sim X$. We then only have to introduce (2.14) in (2.13) to get the above.

The only remaining difficulty is to collect the diverse conditions on the parameters R, δ, M_0, z, q and X. We list them below (we simplify them: we neglect the difference between $\varphi(q)$ and q, we also require that $R^2/(z\delta^2) \leq 1$ while an upper bound of any constant would also do, and so on). For some $\epsilon > 0$, we need

(1) $R \geq 8q$;
(2) $M_0, R, z \geq X^\epsilon$;
(3) $\delta M_0 \geq M_0^\epsilon$ for the separation of variables by Lemma 41;
(4) $\delta^2 R\log^6 X \leq 1$ for the large sieve argument on the remainder term coming from the separation of variables: indeed, by Lemmas 39 (with $\varepsilon = 1/\log z \asymp 1/\log N$) and 41, we find that

$$\sum_{r\leq R/q}\frac{\mu^2(r)}{\varphi(r)}\bigl|E_1(\delta,r) + E_2(\delta,r) + 2\delta E_3(r)\bigr|^2 \ll \delta^2(MN)^2(\log R)^4\frac{R}{q}.$$

After summation over N as a power of 2 and over M (at most two values), this bound is multiplied by $\log X$. The factor $\mathcal{O}(\log(X/M_0z))$ in front introduces another logarithm;

(5) $(\beta X)^2 \le \delta^{-1}$, and this ensures that Δ, which is given by $\Delta = 100(\delta^{-1} + (\beta X)^2)/\delta$, satisfies $\Delta \ll \delta^{-2}$;

(6) $R^2/(z\delta^2) \le 1$ and $M_0 \ge q^{1+\epsilon}$ for the large sieve argument on the bilinear form: indeed, we proceed as in (7.3) and apply Corollary 12. The final L^2-norm is handled by Lemma 27. We need $R^2\Delta \ll z$, and that is granted by $R^2\delta^{-2} \ll z$ since N starts at z. We employ Lemma 17 rather than Lemma 16 for its less stringent condition $M \ge q^{1+\varepsilon}$. Since M can be as small as M_0, this explains the second condition above;

(7) $(|t| + 1 + |\beta|X)\sqrt{q}Rz^2 \le X$ for the term coming from $L_r^{(1)}(a,t,\beta)$ by Lemma 35;

(8) $(|t| + 1 + |\beta|X)\sqrt{q}Rz^2M_0 \le X$ for the term coming from $L_r^{(2)}(a,t,\beta)$ by Lemma 37.

Condition (7) is a consequence of condition (8). It is better to choose M_0 as small as possible, so let us select $M_0 = \min(q, \delta^{-1})X^{2\theta}$ for some positive θ. We want the ranges in β and t to be as large as possible, so we want δ and z to be small. However, condition (6) is imposed to reach a balance between these two aims. It is also better to choose R as small as we may, so we choose $R = qX^{2\theta}$ and $z = q^2X^{4\theta}/\delta^2$.

Let us choose $M_0 = qX^{4\theta}$ and $\delta = q^{-1}X^{-2\theta}$. We also assume that

$$|\beta X| \le \sqrt{q}X^{\theta} \qquad \text{and} \qquad (1 + |t| + |\beta|X)q^{13/2}X^{22\theta} \le X.$$

We set $\eta = (2 - 44\theta)/13$ to reach the statement of the theorem.

9. PROOF OF THEOREM 2 IN VERSION (V4)

The proof of the L^1-Theorem 2 is similar to the proof of Theorem 7 with one notch of difficulty added. The required modifications are however rather simple; we explain them in this section. The case of the primes is slightly more difficult as it entails using the Barban–Davenport–Halberstam theorem, while the bounds $|\lambda(m)|, |\mu(m)| \le 1$ will be enough in the other cases. We will rely on the bound $|\mathscr{H}(v)| \ll (1 + X|\beta|)/(1 + |v|)$ with $\beta = 0$, combined with Corollary 12. We start with

$$G_q\Big(\frac{R}{q}\Big)\int_{-T}^{T}\Bigg|\sum_{\ell\sim X,\,(\ell,q)=1}\frac{\Lambda(\ell)}{\ell^{it}}\,e\Big(\frac{\ell a}{q}\Big)\Bigg|\,dt \ll \int_{-T}^{T}\sum_{r\le R/q,\,(r,q)=1}\frac{\mu^2(r)}{\varphi(r)}\big(\big|L_r^{(1)}(a,t,0)\big| + \big|L_r^{(2)}(a,t,0)\big|\big)\,dt$$
$$+\int_{-T}^{T}\sum_{r\le R/q,\,(r,q)=1}\frac{\mu^2(r)}{\varphi(r)}\Bigg|\sum_{\substack{mn\sim X\\(mn,q)=1,\,m>M_0}}\frac{\Lambda(m)v_r(n)}{(mn)^{it}}\,e\Big(\frac{mna}{q}\Big)\Bigg|\,dt.$$

The term with $L_r^{(1)}$ on the right-hand side is estimated as in Lemma 35. The term with $L_r^{(2)}$ is estimated as in Lemma 38. The estimation of the third term on the right-hand side requires a modification at the level of equation (7.3). We first split the summation in (m,n) according to their size (M,N) and then apply Cauchy's inequality on the resulting sums, getting

$$\int_{-T}^{T}\sum_{r\le R/q,\,(r,q)=1}\frac{\mu^2(r)}{\varphi(r)}\Bigg|\sum_{\substack{mn\sim X\\(mn,q)=1,\,m>M_0}}\frac{\Lambda(m)v_r(n)}{(mn)^{it}}\,e\Big(\frac{mna}{q}\Big)\Bigg|\,dt$$
$$\ll \sum_{M,N}\Bigg(G_q\Big(\frac{R}{q}\Big)\sum_{r\le R/q}\frac{\mu^2(r)}{\varphi(r)}\Bigg(\int_{-T}^{T}\Big|S_r\Big(\frac{a}{q},t,\beta,M,N\Big)\Big|\,dt\Bigg)^2\Bigg)^{1/2}.$$

Now we release the condition $mn \sim X$:

$$\begin{aligned}\biggl(\int_{-T}^{T}\Bigl|S_r\Bigl(\frac{a}{q},t,\beta,M,N\Bigr)\Bigr|\,dt\biggr)^2 &\ll \Bigl(\delta TM\sum_{n\sim N}|v_r(n)|\Bigr)^2\\ &\quad+\int_{-\Delta}^{\Delta}\int_{-T}^{T}\sum_{b \bmod^* q}\biggl|\sum_{m\equiv b[q],\, m\sim M}\frac{\Lambda(m)}{m^{i(v+t)}}\biggr|^2|\mathscr{H}(v)|\,dt\,dv\\ &\quad\times\int_{-\Delta}^{\Delta}\int_{-T}^{T}\sum_{b \bmod^* q}\biggl|\sum_{(n,q)=1,\, n\sim N}\frac{v_r(n)}{n^{i(v+t)}}\,e\Bigl(\frac{mna}{q}\Bigr)\biggr|^2 dt\,|\mathscr{H}(v)|\,dv. \qquad (9.1)\end{aligned}$$

We use the estimate

$$\int_{-\Delta}^{\Delta}\int_{-T}^{T}\limits_{v+t=w}|\mathscr{H}(v)|\,dv \ll \log\min(2+T,\Delta),$$

and, from this point onwards, the treatment of the factor containing $v_r(n)$ is as previously. As for the factor containing the Λ-part, we first reduce it to prime variables and then use multiplicative characters. The first step reads

$$\begin{aligned}\sum_{b \bmod^* q}\biggl|\sum_{m\equiv b[q],\, m\sim M}\frac{\Lambda(m)}{m^{iw}}\biggr|^2 &= \sum_{b \bmod^* q}\biggl|\sum_{p\equiv b[q],\, p\sim M}\frac{\log p}{p^{iw}}+\sum_{\substack{m\equiv b[q],\, m\sim M\\ m=p^k,\, k\ge 2}}\frac{\Lambda(m)}{m^{iw}}\biggr|^2\\ &\le 2\sum_{b \bmod^* q}\biggl|\sum_{p\equiv b[q],\, p\sim M}\frac{\log p}{p^{iw}}\biggr|^2+2\sum_{b \bmod^* q}\biggl|\sum_{\substack{m\equiv b[q],\, m\sim M\\ m=p^k,\, k\ge 2}}\frac{\Lambda(m)}{m^{iw}}\biggr|^2.\end{aligned}$$

We then use

$$\begin{aligned}\sum_{b \bmod^* q}\biggl|\sum_{\substack{m\equiv b[q],\, m\sim M\\ m=p^k,\, k\ge 2}}\frac{\Lambda(m)}{m^{iw}}\biggr|^2 &\le \max_{b \bmod^* q}\biggl(\sum_{\substack{m\equiv b[q],\, m\sim M\\ m=p^k,\, k\ge 2}}\Lambda(m)\biggr)\sum_{\substack{m\sim M\\ m=p^k,\, k\ge 2}}\Lambda(m)\\ &\le\biggl(\sum_{\substack{m\le M\\ m=p^k,\, k\ge 2}}\Lambda(m)\biggr)^2 \ll M.\end{aligned}$$

On detecting the congruence condition in $m\equiv b[q]$ by multiplicative characters, we thus find that

$$\sum_{b \bmod^* q}\biggl|\sum_{m\equiv b[q],\, m\sim M}\frac{\Lambda(m)}{m^{iw}}\biggr|^2 \le \frac{2}{\varphi(q)}\sum_{\chi \bmod q}\biggl|\sum_{(p,q)=1,\, p\sim M}\frac{\chi(p)\log p}{p^{iw}}\biggr|^2+\mathcal{O}(M).$$

The reduction to primitive characters is immediate:

$$\sum_{\chi \bmod q}\biggl|\sum_{(p,q)=1,\, p\sim M}\frac{\chi(p)\log p}{p^{iw}}\biggr|^2 = \sum_{\mathfrak{f}|q}\sum_{\chi \bmod^* \mathfrak{f}}\biggl|\sum_{(p,q)=1,\, p\sim M}\frac{\chi(p)\log p}{p^{iw}}\biggr|^2.$$

The next step is a way to introduce the Barban–Davenport–Halberstam theorem via Theorem 9. Here $\mathfrak{f} \mid q$, and we note that $c_r(p) = \mu(p) = -1$:

$$\sum_{r \le M/\mathfrak{f},\,(r,\mathfrak{f})=1} \frac{\mu^2(r)}{\varphi(r)} \Bigg| \sum_{(p,q)=1,\,p\sim M} \frac{\chi(p)c_r(p)\log p}{p^{iw}} \Bigg|^2 = G_{\mathfrak{f}}\Big(\frac{M}{\mathfrak{f}}\Big) \Bigg| \sum_{(p,q)=1,\,p\sim M} \frac{\chi(p)\log p}{p^{iw}} \Bigg|^2 .$$

We recall (3.7): $(\mathfrak{f}/\varphi(\mathfrak{f}))G_{\mathfrak{f}}(M/\mathfrak{f}) \ge G(M/\mathfrak{f})$. We can then appeal to Theorem 9 instead of Lemma 16. Let us summarize (for uniformity with the last section) the diverse conditions: for some $\epsilon > 0$,

(1) $R \ge 8q$;
(2) $M_0, R, z \ge X^\epsilon$;
(3) $\delta M_0 \ge M_0^\epsilon$ for the separation of variables;
(4) $\delta^2 TR \log^6 R \le 1$ for the large sieve argument on the remainder term coming from the separation of variables, since we integrate trivially in t the corresponding error term from Section 8;
(5) $\Delta = 100/\delta^2$ since $\beta = 0$, and this makes condition (5) from Section 8 useless here;
(6) $TR^2/(\delta^2 z) \le 1$ and $M_0 \ge q^{1+\epsilon}$ for the large sieve argument on the bilinear form;
(7) $qT^2Rz^2 \le X\sqrt{q}$ for the remainder term linked to $L_r^{(1)}(a,t,0)$;
(8) $qT^2Rz^2M_0 \le X\sqrt{q}$ for the remainder term linked to $L_r^{(2)}(a,t,0)$.

Condition (7) is yet again a consequence of condition (8). Here are the parameters we choose, for some $\theta > 0$:

$$R = qX^{2\theta}, \qquad \delta^{-1} = \sqrt{qT}X^{2\theta}, \qquad M_0 = \sqrt{qT}X^{3\theta}, \qquad z = q^3T^2X^{4\theta}, \tag{9.2}$$

and assume that $T^{13/2}q^8X^{13\theta} \le X$. We set $\eta = (1-13\theta)/8$ to get the claimed estimate.

10. THE SHORT INTERVAL ESTIMATE: PROOF OF THEOREM 3

The L^1-estimate opens a clear path to the short interval result. Let us start with a methodological comment. One can try to compute the Mellin transform of the characteristic function of the interval $[X, X+X^\theta]$, but the lack of continuity results in a transform which is of order $1/s$ when s goes to $i\infty$. As an implication, the integral of the absolute value of this transform over a vertical line is not convergent, and this raises complications. One can get a more precise version for the characteristic function of $[1,X]$ that does not rely on the absolute value of the Mellin transform but relies on more information on the sequence we consider; this is the truncated Perron's formula. Alternatively, one can consider a smooth sum of the initial characteristic functions; then removing the smoothing depends on short interval estimates, and this is exactly the same information that is required in the truncated Perron's formula.

This means that we get results of similar strength by using the difference of two truncated Perron's formulas. A usual form of this truncated Perron's formula like [27, Theorem 2.1] is enough. We note on the methodological side that the more powerful version [33, Theorem 1.2] by the first author would lead to a refined error term.

Theorem 42 (truncated Perron's formula). *Let $F(z) = \sum_n u_n/n^z$ be a Dirichlet series that converges absolutely for $\operatorname{Re} z > \kappa_{\mathrm{a}}$, and let $\kappa > 0$ be strictly larger than κ_{a}. For $X \ge 1$ and $T \ge 1$, we have*

$$\sum_{n \le X} u_n = \frac{1}{2i\pi} \int_{\kappa - iT}^{\kappa + iT} F(z) \frac{X^z\,dz}{z} + \mathcal{O}^*\Bigg(\int_{1/T}^{\infty} \sum_{|\log(X/n)| \le v} \frac{|u_n|}{n^\kappa} \frac{2X^\kappa\,dv}{Tv^2} \Bigg).$$

We take $F(z) = \sum_{X<\ell\le 2X} \Lambda(\ell)e(\ell\beta)/\ell^z$. We select $\kappa = 1$ and set $\omega = X^{\theta-1}$. To handle the error term, we split the integral at $v = 1$. When $v \ge 1$, we majorize $\sum_{|\log(X/n)|\le v} |u_n|/n$ by $\sum_{n\sim X} \Lambda(n)/n \ll 1$. When $v < 1$, we majorize $1/n$ by $X^{-1}e^{1/T} \ll 1/X$. We assume that $T \le \sqrt{X}$. This implies that, when $1/T \le v \le 1$, we have

$$\sum_{|\log(X/n)|\le v} \frac{|u_n|}{n} \ll X^{-1} \sum_{e^{-v}X\le n\le e^v X} \Lambda(n) \ll X^{-1}\frac{vX}{\log(3vX)}\log X \ll v$$

by using Lemma 14 with $q = 1$, $A = e^{-v}X$ and $B = 3vX \ge 2\sinh(v)X \ge 2X(e^v - e^{-v})/2$. This readily leads to

$$\sum_{X<\ell\le X+X^\theta} \Lambda(\ell)\, e\Big(\frac{\ell a}{q}\Big) = \frac{1}{2i\pi}\int_{1-iT}^{1+iT} \sum_{X<\ell\le 2X} \frac{\Lambda(\ell)\, e(\ell a/q)}{\ell^z}\,\frac{X^z((1+\omega)^z-1)\,dz}{z} + \mathcal{O}\Big(\frac{X\log X}{T}\Big). \quad (10.1)$$

A usage of [33, Theorem 1.2] we mentioned would remove the $\log X$ in the error term, the only slight bump in the proof being that we need the same T^* for the formula up to X and the formula up to $X+X^\theta$: going back to [33, Corollary 5.1] would do the trick.

We end this methodological remark here and proceed by using the classical inequality $|(1+\omega)^z - 1| \le |z\omega(1+\omega)|$ obtained from the representation

$$(1+\omega)^z - 1 = z\int_0^\omega (1+y)^{z-1}\,dy.$$

We choose $T = X^{1-\theta}\sqrt{q}\log X$.

We shift the line of integration in (10.1) to $\operatorname{Re} z = 0$. To bound the contribution of the two horizontal segments, we notice that, when $\operatorname{Im} z = \pm T$ and $0 \le \operatorname{Re} z = \sigma \le 1$, we have $|(1+\omega)^z - 1| \ll 1$, $|1/z| \ll 1/T$ and $\sum_{\ell\sim X} \Lambda(\ell)/|\ell^z| \ll X/X^\sigma$. This gives the error term $\mathcal{O}(X/T)$, which is admissible.

On the line $\operatorname{Re} z = 0$, we use the estimate $|(1+\omega)^z - 1|/|z| \ll X^{\theta-1}$, getting

$$\sum_{X<\ell\le X+X^\theta} \Lambda(\ell)\, e\Big(\frac{\ell a}{q}\Big) \ll X^{\theta-1}\int_{-T}^{T}\Bigg|\sum_{X<\ell\le 2X}\frac{\Lambda(\ell)\, e(\ell a/q)}{\ell^{it}}\Bigg|\,dt + \frac{X^\theta}{\sqrt{q}}. \quad (10.2)$$

We are ready to prove Theorem 3. To meet its hypotheses, we require

$$X^{1-\theta}\sqrt{q}\log X \le \Big(\frac{X^\eta}{q}\Big)^{16/13}. \quad (10.3)$$

The hypotheses $\theta > \theta_0$ and

$$X^{1-\theta_0}\sqrt{q} \le \Big(\frac{X^\eta}{q}\Big)^{16/13} \quad (10.4)$$

are enough to ensure (10.3), when X is large enough.

11. THE TRIGONOMETRIC POLYNOMIAL OF THE MÖBIUS FUNCTION

Adapting the previous argument to the polynomial

$$S^\flat\Big(\frac{a}{q}\Big) = \sum_{\ell\sim X} \mu(\ell)\, e\Big(\frac{\ell a}{q}\Big) \quad (11.1)$$

is easy enough. We take the same notation as for the von Mangoldt function, but we add the superscript $\flat$ when it concerns the Möbius function. The treatment of the two linear forms carries over with few changes and same result. The treatment of the bilinear form is simpler in one aspect, since we do not need any Brun–Titchmarsh theorem and use instead of Lemma 17 the estimate

$$\sum_{b \bmod q}\Bigl|\sum_{m\equiv b[q],\, m\sim M}\mu(m)\Bigr|^2 \ll \frac{M^2}{q} \tag{11.2}$$

valid when $M\geq q^{1+\epsilon}$. We also consider the companion

$$S^\sharp\Bigl(\frac{a}{q}\Bigr)=\sum_{\ell\sim X,\,(\ell,q)=1}\mu(\ell)\,e\Bigl(\frac{\ell a}{q}\Bigr). \tag{11.3}$$

We have to bound in (7.3)

$$\sum_{b \bmod^* q}\Bigl|\sum_{m\equiv b[q],\, m\sim M}\mu(m)\Bigr|^2. \tag{11.4}$$

This quantity is at most $M^2\varphi(q)/q^2$, while we used $M^2/\varphi(q)$ when dealing with primes. We thus multiply this bound by $\varphi(q)^2/q^2$, and we have to take the square root of this factor when modifying the final bound.

12. PROOF OF THEOREM 1

We are now in a position to prove Theorem 1. We consider only the case of irregular numbers. Here is a more general statement.

Theorem 43. *Let $X\geq 3$ and $\theta\in[0.79,1]$ be two parameters. Let $\mathcal{S}$ be the set of irregular numbers in $[X,X+X^\theta]$. Let $q\leq X^{1/20}$ be a prime, and let $\mathcal{A}$ and $\mathcal{B}$ be two arbitrary sets in $\mathbb{Z}/q\mathbb{Z}$ such that $|\mathcal{A}|\cdot|\mathcal{B}|\geq q(\log q)^2$. We have*

$$\sum_{\substack{a+b+s\equiv m[q]\\ a\in\mathcal{A},\, b\in\mathcal{B},\, s\in\mathcal{S}}} 1 \sim \frac{|\mathcal{A}|\cdot|\mathcal{B}|\cdot|\mathcal{S}|}{q}$$

(as q goes to infinity) valid for every $m\in\mathbb{Z}/q\mathbb{Z}$.

Proof. Let us define two trigonometric polynomials

$$T\Bigl(\mathcal{A},\frac{c}{q}\Bigr)=\sum_{a\in\mathcal{A}}e\Bigl(\frac{ac}{q}\Bigr),\qquad T\Bigl(\mathcal{B},\frac{c}{q}\Bigr)=\sum_{b\in\mathcal{B}}e\Bigl(\frac{bc}{q}\Bigr), \tag{12.1}$$

then one over the irregular numbers

$$U(\alpha)=\sum_{X\leq s\leq X+X^\theta}\frac{1-\lambda(s)}{2}e(s\alpha) \tag{12.2}$$

and lastly the number of representations

$$r(\mathcal{A},\mathcal{B},m)=\sum_{\substack{a+b+s\equiv m[q]\\ a\in\mathcal{A},\, b\in\mathcal{B},\, s\in\mathcal{S}}} 1. \tag{12.3}$$

We get classically

$$r(\mathcal{A},\mathcal{B},m)=\frac{1}{q}\sum_{c \bmod^* q}T\Bigl(\mathcal{A},\frac{c}{q}\Bigr)T\Bigl(\mathcal{B},\frac{c}{q}\Bigr)U\Bigl(\frac{c}{q}\Bigr)e\Bigl(-\frac{mc}{q}\Bigr). \tag{12.4}$$

By Theorem 3 with $\eta = 1/12$ and λ instead of μ (i.e., in version (V2)), together with the fact that q is prime, we infer that, when $c \neq 0$, we have

$$U\Big(\frac{c}{q}\Big) \ll X^\theta \frac{\log q}{\sqrt{\varphi(q)}}.$$

Furthermore, $U(0) \sim X^\theta/2$ by the result [24] of K. Ramachandra (since $\theta > 7/12$). Hence, by applying Parseval's equality and since $\varphi(q) = q-1$, we obtain

$$r(\mathcal{A},\mathcal{B},m) = (1+o(1))\frac{X^\theta|\mathcal{A}|\cdot|\mathcal{B}|}{2q} + \mathcal{O}\bigg(X^\theta\sqrt{\frac{|\mathcal{A}|\cdot|\mathcal{B}|\log q}{q}}\bigg).$$

This proves our claim. $\square$

Final note. We have decided to select specific parameters in Theorem 1 to illustrate the relative strength of our result; a larger domain for the parameters $(\theta, \log q/\log X)$ is available.

Proof of Theorem 1. We simply need to select $X = q^{20r}$ and $X^\theta = q^{16r}$ in Theorem 43. $\square$

Appendix. SEPARATING THE VARIABLES AND REMOVING A PHASE

In the bilinear form, we will have to handle conditions like

$$a \sim A, \qquad b \sim B \qquad \text{and} \qquad ab \sim X,$$

where $y \sim Y$ means that $Y < y \leq 2Y$. The last condition is annoying because it links both variables together. This link is mild, and we remove it by the process we describe here. At the same time we remove a phase $e(\beta r)$. Our object here is

$$\sum_{X<r\leq bX} \varphi_r\, e(\beta r) \tag{A.1}$$

for some general (φ_ℓ) and a parameter $b > 1$ that is usually equal to 2. The same process is used, for instance, in [7, Lemma 6]. Our form is more precise in two aspects: we regulate the length of the integration, and we avoid a loss of a log-factor by introducing (or, more precisely, preparing the introduction of) some information on the local behavior of $(|\varphi_r|)$. We rewrite the above as

$$\sum_{r\geq 1} \varphi_r\, \mathbb{1}_{[-1,1]}\bigg(2\frac{\log(r/X)}{\log b} - 1\bigg) e\big(\lambda e^{\kappa(2\log(r/X)/\log b - 1)}\big) \tag{A.2}$$

with $\lambda = \beta X\sqrt{b}$ and $\kappa = (\log b)/2$ and we first seek an approximation for $\mathbb{1}_{[-1,1]}$. Let $\delta \in (0, 1/2]$ be a real parameter. We first note that

$$\int_{-\infty}^{\infty} (1-|u|)^+ e(uv)\,du = \Big(\frac{\sin \pi v}{\pi v}\Big)^2.$$

We then consider the trapezoid function

$$h_0(\delta; u) = \frac{(1-|u|)^+ - (1-\delta-|u|)^+}{\delta} = \begin{cases} 1 & \text{when } |u| \leq 1-\delta, \\ \dfrac{1-|u|}{\delta} & \text{when } 1-\delta \leq |u| \leq 1, \\ 0 & \text{when } 1 \leq |u|. \end{cases}$$

This function satisfies

$$\widehat{h}_0(\delta; v) = \int_{-\infty}^{\infty} h_0(\delta; u) e(uv)\,du = \frac{(\sin \pi v)^2 - (\sin \pi(1-\delta)v)^2}{\pi^2\delta v^2} = \frac{\sin(\pi\delta v)\sin(\pi(2-\delta)v)}{\pi^2\delta v^2}.$$

We further select two additional real parameters κ and λ and build

$$H(\delta,\lambda,\kappa;u)=h_0(\delta;u)\,e(\lambda e^{\kappa u}). \tag{A.3}$$

Lemma 44. *Except when $u\in\{\pm1,\pm(1-\delta)\}$, we have*

$$|H(\delta,\lambda,\kappa;u)|\le 1,\qquad |H'(\delta,\lambda,\kappa;u)|\le \delta^{-1}+2\pi|\lambda\kappa|e^{|\kappa|}.$$

The following L^1-bounds also hold:

$$\int_{-1}^{1}|H'(\delta,\lambda,\kappa;u)|\,du\le 2+4\pi|\lambda\kappa|e^{|\kappa|},\qquad \int_{-1}^{1}|H''(\delta,\lambda,\kappa;u)|\,du\le 4\pi|\lambda\kappa|e^{|\kappa|}\bigl(2+|\kappa|+|2\pi\lambda\kappa|e^{|\kappa|}\bigr).$$

Proof. When $u\notin\{\pm1,\pm(1-\delta)\}$, we have

$$H'(\delta,\lambda,\kappa;u)=h_0'(\delta;u)\,e(\lambda e^{\kappa u})+2i\pi\lambda\kappa e^{\kappa u}H(\delta,\lambda,\kappa;u)$$

and

$$\frac{H''(\delta,\lambda,\kappa;u)}{2i\pi\lambda\kappa e^{\kappa u}}=2h_0'(\delta;u)e(\lambda e^{\kappa u})+\bigl(1+2i\pi\lambda\kappa e^{\kappa u}\bigr)H(\delta,\lambda,\kappa;u).$$

The reader will easily derive the lemma from these two expressions. □

The Fourier transform being defined by $\widehat{f}(v)=\int_{\mathbb{R}}f(u)e(uv)\,du$, we deduce from the above lemma and one and then two integrations by parts that

$$|\widehat{H}(\delta,\lambda,\kappa;v)|\le\frac{1+2\pi|\lambda\kappa|e^{|\kappa|}}{\pi|u|} \tag{A.4}$$

and

$$|\widehat{H}(\delta,\lambda,\kappa;v)|\le\frac{\delta^{-1}}{2\pi^2|v|^2}+\frac{3+|\kappa|+|2\pi\lambda\kappa|e^{|\kappa|}}{\pi|v|^2}|\lambda\kappa|e^{|\kappa|}. \tag{A.5}$$

The separation of variables relies on the next lemma.

Lemma 45. *Let $b>1$, $\delta\in(0,1/2)$, β and $X\ge1$ be four real parameters. There exists a C^1-function $\mathscr{H}$ such that, for any sequence (φ_r) of complex numbers, we have*

$$\sum_{X<r\le bX}\varphi_r\,e(\beta r)=\int_{-\Delta}^{\Delta}\sum_{r\ge1}\frac{\varphi_r}{r^{iu}}\mathscr{H}(u)X^{iu}\,du+\mathcal{O}^*\Biggl(\sum_{X<r\le b^{\delta/2}X}|\varphi_r|+\sum_{b^{1-\delta/2}X<r\le bX}|\varphi_r|+2\delta\sum_{r\ge1}|\varphi_r|\Biggr)$$

where Δ is given by

$$\Delta=\frac{2}{\pi(\log b)\delta^2}+\frac{12b+2b\log b+4\pi b^2(\log b)\,|\beta|X}{\log b}\,\frac{|\beta|X}{\delta}. \tag{A.6}$$

Any choice of Δ larger than this one would also do. We have furthermore

$$|\mathscr{H}(u)|\le\frac{(\log b)(1+2\pi b|\beta|X)}{4\pi^2|u|}\qquad\textit{and}\qquad\int_{-\infty}^{\infty}|\mathscr{H}(u)|^2\,du=\frac{(\log b)^2(2-2\delta)}{(4\pi)^2}.$$

The function $\mathscr{H}$ depends on X except when $\beta=0$, so we could dispense with the factor X^{iu}, but it is more natural to keep it. We have in fact

$$\mathscr{H}(u)=\frac{\log b}{4\pi}\,\widehat{H}\Bigl(\delta,\lambda,\kappa;\frac{u\log b}{4\pi}\Bigr)b^{iu/2}\qquad\text{with}\quad \lambda=\beta X\sqrt{b}\quad\text{and}\quad \kappa=\frac{\log b}{2}. \tag{A.7}$$

Proof. The first step is to introduce $H(\delta,\lambda,\kappa;u)$ (with the parameters we have just specified):

$$\sum_{X<r\le bX}\varphi_r\,e(\beta r)=\sum_{r\ge1}\varphi_r H\biggl(\delta,\lambda,\kappa;2\frac{\log(r/X)}{\log b}-1\biggr) + \mathcal{O}^*\biggl(\sum_{0\le\log(r/X)/\log b\le\delta/2}|\varphi_r|+\sum_{1-\delta/2\le\log(r/X)/\log b\le1}|\varphi_r|\biggr).$$

Next, we write (A.5) in the form $|\widehat{H}(\delta,\lambda,\kappa;v)|\le(\delta\Delta_1)/|v|^2$ with

$$\Delta_1=\frac{1}{2\pi^2\delta^2}+\frac{6b+b\log b+2\pi b^2(\log b)\,|\beta|X}{2\pi}\,\frac{|\beta|X}{\delta}.$$

We truncate the integral and infer that

$$H\biggl(\delta,\lambda,\kappa;2\frac{\log(\ell/L)}{\log b}-1\biggr)=\int_{-\infty}^{\infty}\widehat{H}(\delta,\lambda,\kappa;v)\Bigl(\frac{L}{\ell}\Bigr)^{4i\pi v/\log b}e(v)\,dv = \int_{-\Delta_1}^{\Delta_1}\widehat{H}(\delta,\lambda,\kappa;v)\Bigl(\frac{L}{\ell}\Bigr)^{4i\pi v/\log b}e(v)\,dv+\mathcal{O}^*(2\delta).$$

The change of variable $u=4\pi v/\log b$ concludes. The value of $\int_{-\infty}^{\infty}|\mathscr{H}(u)|^2\,du$ is obtained by appealing to Parseval's equality. □

On selecting $b=2$, using $2^{\delta/2}-1\le\delta$ and $1-2^{-\delta/2}\le\delta/2$, simplifying the constant and using $\delta^{-1}+40|\beta|X+60(\beta X)^2\le100(\delta^{-1}+(\beta X)^2)$, we obtain the following result.

Lemma 46. *Let $\delta\in(0,1/2)$, β and $X\ge1$ be three real parameters. There exists a C^1-function $\mathscr{H}$ such that, for any sequence (φ_r) of complex numbers, we have*

$$\sum_{X<r\le2X}\varphi_r\,e(\beta r)=\int_{-\Delta}^{\Delta}\sum_{r\ge1}\frac{\varphi_r}{r^{iu}}\mathscr{H}(u)X^{iu}\,du+\mathcal{O}^*\Biggl(\sum_{\substack{X<r\le(1+\delta)X\\ \text{or } (2-\delta)X<r\le2X}}|\varphi_r|+2\delta\sum_{r\ge1}|\varphi_r|\Biggr)$$

where $\Delta=100(\delta^{-1}+(\beta X)^2)/\delta$. We have furthermore

$$|\mathscr{H}(u)|\le\frac{25}{73}\,\frac{1+|\beta|X}{1+|u|}\qquad\text{and}\qquad|\mathscr{H}(u)|\le\frac{\delta^{-1}+(\beta X)^2}{1+|u|^2}.$$

Concerning the last bound, we prove in fact that

$$|\mathscr{H}(u)|\le\min\biggl(\frac{\log2}{2\pi},\frac{\log2}{4\pi^2}\,\frac{1+13|\beta|X}{|u|}\biggr),$$

from which we infer the bound stated (we use the first value when $|u|\le2$).

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