CUSPS OF PRIMES IN DENSE SUBSEQUENCES – BYPASSING THE W-TRICK

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Abstract. Let the A-cusps of a dense subset $\mathcal{P}^* \in [\sqrt{N}, N]$ of primes be points $\alpha \in \mathbb{R}/\mathbb{Z}$ that are such that $|\sum_{p\in\mathcal{P}^*} e(\alpha p)| \geq |\mathcal{P}^*|/A$. We establish that any $(1/N)$-well spaced subset of A-cusps contains at most $20A^2K\log(2A)$ points, where $K = N/(|\mathcal{P}^*|\log N)$. We further show that any B-cusps ξ is accompanied, when $B \leq \sqrt{A}$, by a large proportion of A-cusps of the shape $\xi + (a/q)$. We conclude this study by showing that, given $A \geq 2$, the characteristic function $1_{\mathcal{P}^*}$ may be decomposed in the form $1_{\mathcal{P}^*} = (V(z_0)\log N)^{-1} f^\flat + f^\sharp$ where the trigonometric polynomial of $f^\sharp$ takes only values $\leq |\mathcal{P}^*|/A$, and $f^\flat$ is a bounded non-negative function supported on the integers prime to M; the parameters z_0 and M are given in terms of A, while $V(z_0) = \prod_{p<z_0}(1-1/p)$. The function $f^\flat$ satisfies more regularity properties. In particular, its density with respect to the integers $\leq N$ and coprime to M is again K. This transfers questions on $\mathcal{P}^*$ to problems on integers coprime to the modulus M.

1. Results in the large

During their investigations on the primes, B. Green in [13, Proof of Lemma 6.1] and B. Green & T. Tao in [10] were led to consider the large values of the Fourier polynomial built on some dense subset of the primes. Let us introduce a notion for clarity.

Definition 1.1. *Let $\mathcal{P}^* \subset [1, N]$ be a subset of the primes, and let $A \geq 1$ be given. We define the set of A-cusps by*

$$\mathscr{C}(\mathcal{P}^*, A) = \left\{\alpha \in \mathbb{R}/\mathbb{Z} : \left|\sum_{p\in\mathcal{P}^*} e(\alpha p)\right| \geq \sum_{p\in\mathcal{P}^*} 1/A\right\}. \tag{1}$$

As the involved trigonometric polynomial is continuous, the set $\mathscr{C}(\mathcal{P}^*, A)$ is closed, hence compact, and is more precisely a finite union of arcs. The above set may sometimes be called *spectrum*, but, first this word is overloaded and, second the right-hand side is often N/A rather than the one we employ above. The similarity with the same word in automorphic theory is also welcome, see the structure theorem below.

Cusps are scarce. Given a subset $\mathcal{P}^* \subset [\sqrt{N}, N]$ of the primes, of cardinality $N/(K\log N)$, B. Green & T. Tao proved in [10] that the cardinality of a $1/N$-well spaced subset of $\mathscr{C}(\mathcal{P}^*, A)$ is $\mathcal{O}_\varepsilon((A^2K)^{1+\varepsilon})$, for every positive ε. We sharpened this result in [24] to $\mathcal{O}(A^2K\log A)$. We are somewhat more precise in the next result.

Theorem 1.2. *Notation being as in Definition 1.1, the set $\mathscr{C}(\mathcal{P}^*, A)$ is a finite union of arcs. Any $1/N$-well spaced subset of $\mathscr{C}(\mathcal{P}^*, A)$ contains at most* $(4e^\gamma +$

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$o(1))A^2K\log(2A)$ points, where $K = N/(|\mathcal{P}^*|\log N)$. When $N \geq 10^4$, such a $1/N$-well spaced set contains at most $19A^2K\log(2A)$ points.

Let us recall that a set $\mathcal{X} \subset \mathbb{R}/\mathbb{Z}$ is said to be δ-well spaced when $\min_{x\neq x'\in\mathcal{X}} |x - x'|_{\mathbb{Z}} \geq \delta$, where $|y|_{\mathbb{Z}} = \min_{k\in\mathbb{Z}} |y-k|$ denotes in a rather unusual manner the distance to the nearest integer.

In particular, the measure of $\mathscr{C}(\mathcal{P}^*, A)$ is $\mathcal{O}(A^2(\log A)/N)$, as N grows.

Conditional optimality.

Theorem 1.3. *If Theorem 1.2 holds with the bound $\mathcal{O}(A^2Kf(\log A))$ for some non-decreasing positive function f, then*

$$\int_0^1 \biggl|\sum_{p\in\mathcal{P}^*} e(p\alpha)\biggr| d\alpha \ll \sqrt{\frac{|\mathcal{P}^*|\, f(\log N)}{\log N}}.$$

A lower bound of size $|\mathcal{P}^*|/\sqrt{N}$ can be infered from the work [33] of R. C. Vaughan. It is reproved in a more general fashion by E. Eckels, S. Steven, A. Leodan & R. Tobin in [5]. When restricting to the full sequence of primes, the best known upper bound is $(\frac{\sqrt{2}}{2} + o(1))\sqrt{N/\log N}$, due to D. A. Goldston in [8]. Both Vaughan and Goldston tend to believe that the good order of magnitude is $\sqrt{N/\log N}$, in which case Theorem 1.2 would be optimal. However, when A is small, we have only been able to build examples with $\mathcal{O}(A^2)$ cusps (see below).

The case of the full sequence of primes. We may approximate the trigonometric polynomial T on the primes via a local model, i.e. write

$$T(\alpha) = \sum_{p\leq N} e(p\alpha) = \frac{1}{V(z_0)\log N} \sum_{\substack{n\leq N\\ (n,P(z_0))=1}} e(n\alpha) + \mathcal{O}\biggl(\frac{N}{z_0\log N}\biggr) \tag{2}$$

where $z_0 \leq \log\log N$ is a parameter at our disposal and

$$P(z_0) = \prod_{p<z_0} p, \quad V(z_0) = \prod_{p<z_0}\biggl(1 - \frac{1}{p}\biggr). \tag{3}$$

Eq. (2) is proved in Subsection 2.1. As a consequence and when limiting our study to A small, say $A = o(\log\log N)$, we find that the set of A-cusps of T is a union of arcs around points from $\{a/q : (a,q) = 1, q|P(z_0)\}$, for some z_0 (chosen for the error term in (2) to become $< N/(A\log N)$). Around a/q and when $q|P(z_0)$, say in $\alpha = \frac{a}{q} + \beta$, we readily find that $|T(\alpha)| \leq \min(\frac{\mu^2(q)N}{\varphi(q)\log N}, P(z_0)/\|P(z_0)\beta\|) + \mathcal{O}(N/(z_0\log N))$. Furthermore, on adapting the proof of P. Bateman from [1][1], we readily find that

$$\sum_{\varphi(q)\geq A} \mu^2(q)\varphi(q) \sim \frac{A^2}{2}.$$

Gathering our results, we find that $\mathscr{C}(\mathcal{P}\cap[1,N], A)$ is a union of about $(A^2/2)$ arcs of length $\mathcal{O}(1/N)$. The figure we display has been again doctored a bit, but exhibit the presence of very sharp cusps. We produced it by using Sage, cf [29].

[1]The result of Bateman does not include the square-free condition on q.

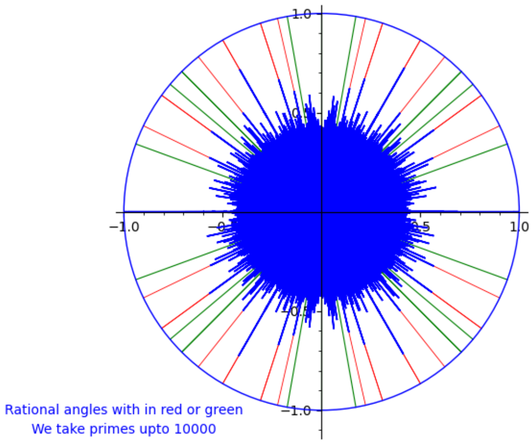

FIGURE 1. The modulus $|T(\alpha)|/T(0)$. In red (resp. green), the first lines from the origin to rational points with square-free (resp. non square-free) denominators.

A subsequence exhibiting non-rational cusps. A part of the Farey sequence, namely the points with square-free denominators, appears as cusps for the full sequence of primes. We found the presence of this sequence in many numerical examples[2]. Let us give an example were non-rational cusps appear. We select

$$\mathcal{P}_0^* = \{p : \{p\sqrt{2}\} \le 1/2\} \tag{4}$$

where $\{x\}$ denotes the fractional part of the real number x. By detecting the condition $\{p\sqrt{2}\} \le 1/2$ through Fourier analysis, we readily find that

$$\begin{aligned} T_0^*(\alpha) = \sum_{\substack{p\le N\\ \{p\sqrt{2}\}\le 1/2}} e(p\alpha) &= \frac{(1/2)}{V(z_0)\log N} \sum_{\substack{2\le n\le N\\ (n,P(z_0))=1}} e(n\alpha) \\ &+ \sum_{\substack{|h|\le H\\ h \text{ odd}}} \frac{1}{i\pi h}\frac{1}{V(z_0)\log N} \sum_{\substack{2\le n\le N\\ (n,P(z_0))=1}} e\big(n(h\sqrt{2}+\alpha)\big) + \mathcal{O}\biggl(\Bigl(\frac{1}{H}+\frac{1}{z_0}\Bigr)\frac{N}{\log N}\biggr). \end{aligned} \tag{5}$$

where $P(z_0) = \prod_{p<z_0} p$ as above. This is proved in Subsection 2.2. The figure we produce has been again doctored; it also exhibits the presence of very sharp cusps.

Cusps and Farey points. In the two previous examples, the Farey sequence plays a major role which seems to be related to the coprimality condition $(n, P(z_0)) = 1$ in (2) and (5). The next theorem shows that this is the general situation.

Theorem 1.4. *Notation being as in Definition 1.1. Let $N \ge 10^4$, $A \in [2, \sqrt{N}]$, $B \in [1, A]$ and $\xi \in \mathscr{C}(\mathcal{P}^*, B)$ be given. Define $t^*(\alpha) = |T^*(\alpha)|/T^*(0)$. The set*

$$\mathcal{F} = \bigl\{\xi + (a/q) : a \text{ mod}^* q, q \le A/B\bigr\} \cap \mathscr{C}(\mathcal{P}^*, A)$$

contains more than $A^2/(6800B^4Z^2K\log A)$ elements, where $Z \in [1/B, 1]$ is the maximum value of $t^*(\xi + (a/q))$ for $\xi + (a/q) \in \mathcal{F}$.

Lemma 9.1 also exhibits some easy properties on $\mathscr{C}(\mathcal{P}^*, A)$.

In particular, the cusps at any of 0, 1/3, 1/2 and 2/3 generate $\asymp A^2/(K\log A)$ A-cusps by simply adding a/q with $q \le A$ to them.

[2]We tried random subsequences of primes of relative density 1/2, then Ramanujan primes and, finally we selected successively one prime out of two.

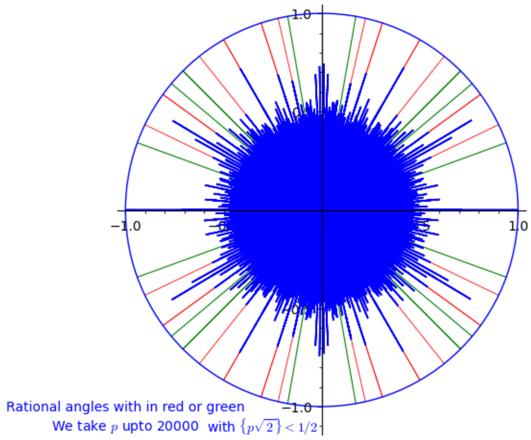

FIGURE 2. The modulus $|T_0^*(\alpha)|/T_0^*(0)$. In red (resp. green), the first lines from the origin to rational points with square-free (resp. non square-free) denominators: some new cusps arise.

A structure theorem without W-trick. The ubiquity of the Farey points with square-free denominators was rather a surprise. We already encountered in [24] the major influence of the parameter z_0 above. Since we prove a numerically better version in Theorem 6.2 below, the reader may see the effect of the "unsieving" parameter z_0. Musing on this effect, we prove the next structure theorem, which may be seen as a general form of (2) and (5). We use the generic notation

$$S(g,\alpha) = \sum_{n\le N} g(n)e(n\alpha). \tag{6}$$

Theorem 1.5. *Let $\epsilon \in (0, 1/3]$ and $A, K \ge 1$ be given. Let*

$$z_0 = \exp\bigl(25000\, A^3 (\log 2A)^2 K\bigr)/\epsilon. \tag{7}$$

Let M be an integer divisible by $P(z_0)$ and all whose prime factors are $< z_0$. Let $N \ge (Mz_0)^{1+\epsilon^{-1}}$. Let $\mathcal{P}^*$ be subset of the primes inside $[\sqrt{N}, N]$ of cardinality (at least) $N/(K\log N)$ and let $T^*(\alpha) = \sum_{p\in\mathcal{P}^*} e(p\alpha)$. We may decompose the characteristic function $1_{\mathcal{P}^*}$ of $\mathcal{P}^*$ as

$$1_{\mathcal{P}^*} = \frac{f^\flat}{V(z_0)\log N} + f^\sharp \tag{8}$$

in such a way that the following properties hold:

- *For every $\alpha \in \mathbb{R}/\mathbb{Z}$, $|S(f^\sharp,\alpha)| \le |T^*(\alpha)|$ and $|S(f^\flat,\alpha)| \le |T^*(\alpha)|V(z_0)\log N$.*
- *For every $\alpha \in \mathbb{R}/\mathbb{Z}$, we have $|S(f^\sharp,\alpha)| < T^*(0)/A$.*
- *$0 \le f^\flat(\ell) \le 2(1+\epsilon)^2$ and $f^\flat(\ell) = 0$ when $(\ell, M) \ne 1$.*
- *For any integer a, we have $S(f^\flat, a/M) = T^*(a/M)V(z_0)\log N$.*

This may be compared with several results of B. Green & T. Tao (see also [6] by J. Fox & Y. Zhao and [4] by D.Conlon, J. Fox & Y. Zhao): the two differences are (a) that we approximate $1_{\mathcal{P}^*}$ and not only a W-tricked version of it and, (b) that this approximation is valid on the full unit circle. This second modification is of lesser importance and essentially improves on the readability. The decomposition (8) may be compared on the Fourier side with the ones given in (2) and (5). In short, the above decomposition now takes care of all the cusps. It is surprising that, in spirit, taking care only of the rational cusps leads to a full description. The decomposition we propose *transfers* the problem on primes from $\mathcal{P}^*$ to a problem on $P(z_0)$-sifted integers (i.e. integers prime to $P(z_0)$) for a weighted sequence $f^\flat$.

The reader should notice that we used another normalization than the one of [10]: the function $f^\flat$ on the $P(z_0)$-sifted integers has the same density, K, with respect to the $P(z_0)$-sifted integers, than $\mathcal{P}^*$ with respect of the primes. Now the local upper bound is around 2, reflecting the loss in the method, while in [10], it is around 1, the loss in the method being reflected by a lower density. This is only a cosmetic change, but is to be noted when comparing distinct sources.

Scope and limitations. Theorem 1.5 has been thought with A small, but still allows this parameter to grow with N. As of now, if $\log M \ll z_0$, we require

$$A \ll \frac{(\log\log N)^{1/3}}{(\log\log\log N)^{2/3}}.$$

This goes to infinity but remains very small. The exponent $1/3$ can be reduced to $1/2$ when asking only for a decomposition over $\mathbb{Z}/\mathcal{O}(1)N\mathbb{Z}$, but this is all the present method offers.

When A is close to 1 and $\mathscr{C}(\mathcal{P}^*, A)$ contains a point away from 1 and $1/2$, we may consider using the methods from G. Freĭman in [7] or from V. Lev in [15].

We concentrate in this paper on the case of the initial segment of the primes, but, as the methods are essentially combinatorial, the results extend to any sieve setting. How so is not exactly clear as far as the dependence in the various parameters is concerned. We delay such generalizations to later papers, but as it is not more work, we prepare the ground by introducing a parameter τ in the enveloping sieve.

Inverse question: an open problem. Theorem 1.5 creates, from a non-negative function on the primes $1_{\mathcal{P}^*}$, a non-negative function $f^\flat$ on the integers sifted by $P(z_0)$. The main question is to know whether we encounter only a special class of functions $f^\flat$, or whether essentially any such function could occur. Notice that the values of $f^\flat$ at each integer is maybe not fully determined.

Remarks on the explicit aspect. Following for instance B. Green & T. Tao in [12], we specify the absolute constants. This is of no consequence and simplifies the understanding of the statements. The reader may forget their value in a first reading, especially so since we did not try to optimize these. Notice however that, if these constants were less, one could test the results against numerical trials with more efficiency.

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2. Local model expansions

2.1. **Local model for the primes. Proof of Eq.** (2)**.** In order to prove Eq. (2), we show that the left and right-hand sides coincide for every $\alpha \in \mathbb{R}/\mathbb{Z}$.

Lemma 2.1. *When $N \ge 2$, $\alpha = (a/q)+\beta$ with a prime to q and $|q\beta| \le N/(\log N)^A$, we have*

$$T(\alpha) = \sum_{p\le N} e(p\alpha) = \frac{\mu(q)}{\varphi(q)\log N}\sum_{n\le N} e(n\beta) + \mathcal{O}\biggl(\frac{N}{(\log N)^2}\biggr).$$

Proof. This is a classical result, obtained for instance by combining [34, Theorem 3.1] of the book of R.C. Vaughan together with an obvious adaptation of [34, Lemma 3.1] in the same book. Since we do not weight the primes by $\log p$, we have to limit our saving to $\mathcal{O}(N/(\log N)^2)$. $\square$

Lemma 2.2. *When $\alpha = (a/q) + \beta$ with a prime to q and $|q\beta| \le 1/(2P(z_0))$ have*

$$\sum_{\substack{n\le N\\ (n,P(z_0))=1}} e(n\alpha) = \frac{\mu(q)V(z_0)}{\varphi(q)}\sum_{n\le N} e(n\beta) + \mathcal{O}\biggl(qP(z_0) + \frac{N|\beta|P(z_0)^2}{\varphi(q)}\biggr).$$

Proof. Let us use the shortcut $M_0 = P(z_0)$. By using the Moebius function to handle the coprimality condition, we readily find that

$$\begin{aligned} L(\alpha) = \sum_{\substack{n\le N\\ (n,P(z_0))=1}} e(n\alpha) &= \sum_{d|M_0}\mu(d)\sum_{m\le N/d} e(dm\alpha) \\ &= \sum_{\substack{d|M_0\\ q|d}}\mu(d)\sum_{m\le N/d} e(dm\alpha) + \mathcal{O}\biggl(\sum_{\substack{d|M_0\\ q\nmid d}} 1/\|d\alpha\|\biggr). \end{aligned}$$

In the error term, we have $1/\|d\alpha\| \ll q$. In the main term (which may be non-zero only when $q|M_0$), we may replace α by β. This quantity is therefore independant on a on which we may sum, getting

$$L(\alpha) = \frac{\mu(q)}{\varphi(q)} L(\beta) + \mathcal{O}(qM_0).$$

To proceed, we use the above with $q = 1$ and notice that $e(n\beta) - e((m+k)\beta) \ll k|\beta|$. As a consequence, we find that

$$d\sum_{\substack{n\le N\\ d|n}} e(n\beta) = \sum_{n\le N} e(n\beta) + \mathcal{O}(dN|\beta|)$$

from which we infer that $L(\beta) = V(z_0)\sum_{n\le N} e(n\beta) + \mathcal{O}(N|\beta|M_0^2)$, ending the proof.

$\square$

Proof of Eq. (2). We compare the expressions of Lemmas 2.1 and 2.2 to get the result. $\square$

2.2. Another local model. Proof of Eq. (5).

Lemma 2.3. *Let $I \subset \mathbb{R}/\mathbb{Z}$ be an interval and χ_I be its indicator function. For each positive integer H there exist coefficients $a_H(h)$ and C_h for $-H \le h \le H$ with $|a_H(h)| \le \min(|I|, 1/(|h|\pi))$ and $|C_h| \le 1$ such that the trigonometric polynomial*

$$\chi^*_{I,H}(t) = |I| + \sum_{0<|h|\le H} a_H(h)e(ht) \tag{9}$$

satisfies, for every $t \in \mathbb{R}/\mathbb{Z}$,

$$|\chi_I(t) - \chi^*_{I,H}(t)| \le \frac{1}{H+1}\sum_{|h|\le H} C_h\biggl(1 - \frac{|h|}{H+1}\biggr)e(ht). \tag{10}$$

We also have $|a_H(h)| \le \min(|I|, 1/(|h|+1))$.

This is, up to a trivial change of notation, a specialisation of [31, Theorem 19] by J. Vaaler, reproduced in [16, Lemma 6.2] by M. Madritsch & R. Tichy.

Proof of Eq. (5). We detect the condition $\{p\sqrt{2}\} \le 1/2$ in $T_0^*(\alpha)$ by using Lemma 2.3. We then employ the local model expansion (2) of $T(\alpha)$ to infer the expression we claim. □

2.3. **General structure and remarks on these expansions.** The gist of (2) and (5) is, on employing the vocabularity of algebraic number theory, to separate the behaviour at the infinite place (size conditions) to the one at the finite places (congruence conditions). However, and since we are handling sequences and not points, the contribution at the finite places do not split prime per prime. Though there are surely limitations to this kind of expansions (see for instance [9] by A. Granville & K. Soundararajan), Theorem 1.5 has a wide range of application.

Together with G.K. Viswanadham in [25], we devised the same kind of expansion for integers that are sums of two squares.

Part 1. The initial large sieve inequality

Most of the work in this part is devoted to proving the next theorem, though the enveloping sieve we build will also be used to prove the structure theorem 1.5.

Theorem 2.4. *Let $\mathcal{X}$ be a δ-well spaced subset of $\mathbb{R}/\mathbb{Z}$ and $N \ge 1$. Let $(u_p)_{p\le N}$ be a sequence of complex numbers. We have*

$$\sum_{x\in\mathcal{X}}\biggl|\sum_{p\le N} u_p e(xp)\biggr|^2 \le (4e^\gamma + o(1))\frac{N+\delta^{-1}}{\log N}\log(2|\mathcal{X}|)\sum_{p\le N}|u_p|^2,$$

where $o(1)$ is a function that goes to 0 when N and $|\mathcal{X}|$ go to infinity. This constant is at least as large as $\frac{1}{2}+o(1)$, and, when $N \ge 10^4$, it is at most 19.

In most applications, δ^{-1} is $\mathcal{O}(N)$. A version of this bound was proved in [24] with $8e^\gamma$ (resp. 280) rather than $4e^\gamma$ (resp. 19). The gain stems from a finer study of the coefficients $w_q(z;z_0)$ resulting in Lemma 5.4. When comparing to the similar [24, Lemma 4.2], one sees that we save $\sqrt{q}$, resulting in the optimal bound $(1+o(1))/z_0$.

3. The enveloping sieve with an unsifted part

We fix two real parameters $z_0 \le z$. It is easy to reproduce the analysis of [21, Section 3] as far as exact formulae are concerned, but one gets easily sidetracked towards slightly different formulae. The reader may for instance compare [22, Lemma 4.2] and [21, (4.1.14)]. Similar material is also the topic of [20, Chapter 12]. So we present a path leading to [21, (4.1.14)] in our special case that we extend a bit: we add a parameter τ and consider integers that prime to $\prod_{p<z, p\nmid\tau} p$. The case $\tau = 1$ is the one we shall require here, but such a parameter τ naturally appears when considering primes in some fixed arithmetic progression. We may also assume that τ is prime to $P(z_0)$.

Main players. We define

$$G_d(y) = \sum_{\substack{\ell\le y,\\ (\ell,d)=1}}\frac{\mu^2(\ell)}{\varphi(\ell)},\quad G(y)=G_1(y). \tag{11}$$

We generalize the definition (11) by

$$G_d(y;z_0) = \sum_{\substack{\ell\le y,\\ (\ell,dP(z_0))=1}}\frac{\mu^2(\ell)}{\varphi(\ell)},\quad G(y;z_0)=G_1(y;z_0). \tag{12}$$

We further set

$$\beta_{z_0,z}(n;\tau)=\Bigl(\sum_{d|n}\lambda_d\Bigr)^2,\qquad \lambda_d=1_{(d,\tau P(z_0))=1}\frac{\mu(d)dG_{d\tau}(z/d;z_0)}{\varphi(d)G_\tau(z;z_0)}. \tag{13}$$

So the reader can see that the parameter τ is just an extension of the $P(z_0)$. Let us recall [24, Theorem 2.1]. Section 3 of [21] corresponds to the case $z_0=1$.

Lemma 3.1. *The coefficients $\beta_{z_0,z}(n)$ admit the expansion*

$$\beta_{z_0,z}(n,\tau)=\sum_{\substack{q\le z^2,\\ q|P(z)/P(z_0)\\ (q,\tau)=1}} w_q(z;z_0,\tau)c_q(n)$$

where $c_q(n)$ is the Ramanujan sum and where

$$w_q(z;z_0,\tau)=\frac{\mu(q)}{\varphi(q)G_\tau(z;z_0)}\frac{G_{[q]}(z;z_0,\tau)}{G_\tau(z;z_0)}$$

with the definitions

$$G_{[q]}(z;z_0,\tau)=\sum_{\substack{\ell\le z/\sqrt{q},\\ (\ell,q\tau P(z_0))=1}}\frac{\mu^2(\ell)}{\varphi(\ell)}\xi_q(z/\ell), \tag{14}$$

then

$$\xi_q(y)=\sum_{\substack{q_1q_2q_3=q,\\ q_1q_3\le y,\\ q_2q_3\le y}}\frac{\mu(q_3)\varphi_2(q_3)}{\varphi(q_3)}\quad \textit{and}\quad \varphi_2(q_3)=\prod_{p|q}(p-2).$$

Proof. We develop the square above and get

$$\begin{aligned}\beta_{z_0,z}(n,\tau)&=\sum_{d_1,d_2}\lambda_{d_1}\lambda_{d_2}1_{[d_1,d_2]|n}=\sum_{d_1,d_2}\frac{\lambda_{d_1}\lambda_{d_2}}{[d_1,d_2]}\sum_{q|[d_1,d_2}\sum_{a \bmod^* q}e(na/q)\\ &=\sum_{\substack{q\le z^2,\\ (q,P(z_0))=1}}w_q(z;z_0)c_q(n)\end{aligned}$$

where

$$w_q(z;z_0,\tau)=\sum_{q|[d_1,d_2]}\frac{\lambda_{d_1}\lambda_{d_2}}{[d_1,d_2]}. \tag{15}$$

We introduce the definition the λ_d's, see (13), and obtain

$$G_\tau(z;z_0)^2w_q(z;z_0,\tau)=\sum_{\substack{\ell_1,\ell_2\le z,\\ (\ell_1\ell_2,\tau P(z_0))=1}}\frac{\mu^2(\ell_1)}{\varphi(\ell_1)}\frac{\mu^2(\ell_2)}{\varphi(\ell_2)}\sum_{\substack{q|[d_1,d_2],\\ d_1|\ell_1,d_2|\ell_2}}\frac{d_1\mu(d_1)d_2\mu(d_2)}{[d_1,d_2]}.$$

The inner sum vanishes is ℓ_1 has a prime factor prime to $q\ell_2$, and similarly for ℓ_2. Furthermore, we need to have $q|[\ell_1,\ell_2]$ for the inner sum not be empty. Whence we may write $\ell_1=q_1q_3\ell$ and $\ell_2=q_2q_3\ell$ where $(\ell,q)=1$ and $q=q_1q_2q_3$. The part of the inner sum corresponding to ℓ has value $\prod_{p|\ell}(p-2+1)=\varphi(\ell)$. We have reached

$$G_\tau(z;z_0)^2w_q(z;z_0,\tau)=\sum_{\substack{\ell\le z,\\ (\ell,q\tau P(z_0))=1}}\frac{\mu^2(\ell)}{\varphi(\ell)}\sum_{\substack{q_1q_2q_3=q,\\ q_1q_3\ell\le z,\\ q_2q_3\ell\le z}}\frac{1}{\varphi(q)\varphi(q_3)}\sum_{\substack{q|[d_1,d_2],\\ d_1|q_1q_3,\\ d_2|q_2q_3}}\frac{d_1\mu(d_1)d_2\mu(d_2)}{[d_1,d_2]}.$$

In this last inner sum, we have necessarily $d_1 = q_1 d_1'$ and $d_2 = q_2 d_2'$, so $q_3 = [d_1', d_2']$. Here is the expression we have obtained

$$G_\tau(z;z_0)^2 w_q(z;z_0,\tau) = \sum_{\substack{\ell\le z,\\ (\ell,q\tau P(z_0))=1}} \frac{\mu^2(\ell)}{\varphi(\ell)} \sum_{\substack{q_1q_2q_3=q,\\ q_1q_3\ell\le z,\\ q_2q_3\ell\le z}} \frac{\mu(q)\mu(q_3)}{\varphi(q)\varphi(q_3)} \sum_{q_3=[d_1',d_2']} \frac{d_1'\mu(d_1')d_2'\mu(d_2')}{[d_1',d_2']}.$$

This last inner sum has value $\varphi_2(q_3)$, whence

$$G_\tau(z;z_0)^2 w_q(z;z_0,\tau) = \frac{\mu(q)}{\varphi(q)} \sum_{\substack{\ell\le z,\\ (\ell,q\tau P(z_0))=1}} \frac{\mu^2(\ell)}{\varphi(\ell)} \sum_{\substack{q_1q_2q_3=q,\\ q_1q_3\ell\le z,\\ q_2q_3\ell\le z}} \frac{\mu(q_3)\varphi_2(q_3)}{\varphi(q_3)}$$

as announced. The size conditions are readily seen to imply that $\ell \le z/\sqrt{q}$. □

Remark I. We have

$$\begin{cases} \xi_q(y) = \dfrac{q}{\varphi(q)} & \text{when } y \ge q,\\ |\xi_q(y)| \le \displaystyle\prod_{p|q} \frac{3p-4}{p-1} & \text{for every } y > 0. \end{cases} \tag{16}$$

Remark II. When developing the theory of *local models* in [20, Chapters 8-11], we show the much neater expression (see Eq. (11.30) in [20]):

$$\alpha(n;\tau) = \sum_{d|n} \lambda_d = \frac{1}{G_\tau(z;z_0)} \sum_{\substack{q\le z,\\ (q,\tau P(z_0))=1}} \frac{\mu(q)}{\varphi(q)} c_q(n).$$

In the theory of local models, we realize $\alpha(n)$ as being (close to) the best *approximation* of the characteristic function of the primes, while in the theory of the Selberg sieve, we also ask for a pointwise *upper bound*.
Remark III. In [14], G.H. Hardy proved in particular that

$$\forall n > 1, \quad \Lambda(n) = \frac{n}{\varphi(n)} \sum_{q\ge 1} \frac{\mu(q)}{\varphi(q)} c_q(n). \tag{17}$$

A proof of this result may be found in the paper [19] by R. Murty. We stress that this expression is *not* valid at $n = 1$. Similarly, if we were to consider the expression

$$\sum_{\substack{q\ge 1\\ (q,P(z_0))=1}} \frac{\mu(q)}{\varphi(q)} c_q(n),$$

we should restrict our attention to integers n that have at least a prime factor larger than z_0.
Remark IV. As can be guessed from Lemma 3.1 and up to some renormalisation, the coefficients w_d are simply the "Fourier" coefficients of the (weighted) sequence $\beta_{z_0,z}(n)$, as shown in the next easily established expression (valid for any a coprime with d):

$$G(z;z_0,\tau) w_d(z;z_0,\tau) = \lim_{N\to\infty} \frac{G(z;z_0,\tau)}{N} \sum_{n\le N} \beta_{z_0,z}(n,\tau) e(na/d).$$

4. On the G-functions

In this section, we investigate explicit estimates for $G_\tau(z;z_0)$. The G-functions have been thoroughly studied when $z_0 = 2$ (this means, no condition on small primes is asked), for instance in [22, Lemmas 3.4] and more precisely when $\tau = 1$ in [18], in [2] by Jan Büthe and in [23, Theorem 3.1].

Lemma 4.1. *When q is prime to $\tau P(z_0)$, we have*

$$G_\tau(z/q;z_0) \le \frac{q}{\varphi(q)} G_{q\tau}(z/q;z_0) \le G_\tau(z;z_0).$$

This comes from [32, Eq. (1.3)] by J. van Lint and H.E. Richert. Let us recall [24, Lemma 2.6].

Lemma 4.2. *When $2 \le z_0 \le z$, we have $G(z;z_0) \ge e^{-\gamma}\dfrac{\log z}{\log 2z_0}$.*

We also need an upper estimate in this work. [18, Theorem 1.1] with $q = P(z_0)$ gives a precise answer.

Lemma 4.3. *When $P(z_0) \le z$, $\tau \le z^4$, $(\tau, P(z_0)) = 1$ and $z_0 \ge 35$, we have*

$$G_\tau(z^2;z_0) \le 3.1 \prod_{p<z_0} \frac{p-1}{p} \frac{\varphi(\tau)}{\tau} \log z \le 2\frac{\varphi(\tau)}{\tau}\frac{\log z}{\log z_0}.$$

Proof. [18, Theorem 1.1] with $q = \tau P(z_0)$ gives us the bound

$$G_q(z^2) = \frac{\varphi(q)}{q}\left(2\log z + \sum_{p|q}\frac{\log p}{p-1} + c_0\right) + \mathcal{O}^*(j(q)/z)$$

where

$$j(q) = \prod_{\substack{p|q\\ p\neq 2}} \frac{p^{3/2}+p-\sqrt{p}-1}{p^{3/2}-\sqrt{p}+1} \prod_{\substack{p|q\\ p=2}} \frac{21}{25} \tag{18}$$

and c_0 is given in Lemma 15.5. Let us note that

$$\forall p \ge 37, \quad \frac{p^{3/2}+p-\sqrt{p}-1}{p^{3/2}-\sqrt{p}+1}\frac{p}{p-1}\frac{1}{p^{1/20}} \le 1.$$

This implies, after a short computation, that $j(\tau P(z_0)) \le 15(\tau P(z_0))^{1/20}$. We next notice that

$$\sum_{p|q}\frac{\log p}{p-1} \le \sum_{p|q}\frac{\log p}{p} + c_0 - \gamma \le \log z_0 + c_0 - \gamma$$

by a classical estimate of Rosser & Schoenfeld in [27, (3.24)]. We have thus reached

$$\begin{aligned} G_\tau(z^2;z_0) = G_q(z^2) &\le \prod_{p<z_0}\frac{p-1}{p}\frac{\varphi(\tau)}{\tau}\left(2\log z + \log z + 2c_0 - \gamma + 15\tau^{1/20}/z^{19/20}\right) \\ &\le \prod_{p<z_0}\frac{p-1}{p}\frac{\varphi(\tau)}{\tau}\left(2\log z + \log z + 2c_0 - \gamma + 15/z^{3/4}\right) \\ &\le 3.1\prod_{p<z_0}\frac{p-1}{p}\frac{\varphi(\tau)}{\tau}\log z \end{aligned}$$

since $z \ge P(35) \ge 10^{11}$. To handle the contribution of the Euler product over up to z_0, we rely on [27, (3.26)] and thus

$$\prod_{p<z_0}\frac{p-1}{p} \le \frac{e^{-\gamma}}{\log z_0}\left(1 + \frac{1}{2\log^2 z_0}\right) \le \frac{0.585}{\log z_0}.$$

The proof is complete. □

Lemma 4.4. *When $P(z_0) \le z$, $\tau \le z^4$, $(\tau, P(z_0)) = 1$ and $z_0 \ge 35$, we have $\frac{\tau}{\varphi(\tau)}G_\tau(z;z_0) \ge V(z_0)\log(zz_0) \ge 0.530\frac{\log z}{\log z_0}$.*

Proof. We use the material of Lemma 4.3 to infer that

$$G_\tau(z;z_0) = V(z_0)\frac{\varphi(\tau)}{\tau}\left(\log z + \sum_{p<z_0}\frac{\log p}{p-1} + \sum_{p|\tau}\frac{\log p}{p-1} + c_0\right) + \mathcal{O}(15(\tau z)^{1/20}/\sqrt{z}).$$

Therefore, and using Lemma 15.3, we get

$$G_\tau(z;z_0) \ge V(z_0)\frac{\varphi(\tau)}{\tau}\left(\log(zz_0) - 0.6 + c_0 - \frac{15}{V(z_0)z^{1/4}}\right).$$

By Lemma 15.2, we may degrade this inequality in

$$\frac{\tau}{\varphi(\tau)}G_\tau(z;z_0)/V(z_0) \ge \log(zz_0) + c_0 - 0.6 - \frac{15e^\gamma \log(1.13z_0)}{z^{1/4}}.$$

Notice that Lemma 15.4 ensures that $\log z \ge \frac{4}{5}z_0$, so that we only need to check that

$$c_0 - 0.6 - \frac{15e^\gamma\log(1.13z_0)}{e^{z_0/5}} \ge 0.$$

This is readily done (this quantity is even ≥ 0.7), therefore closing the proof of the first inequality. On using Lemma 15.2, we further reach the inequality

$$\begin{aligned}\frac{\tau}{\varphi(\tau)}G_\tau(z;z_0) \ge e^{-\gamma}\frac{\log(zz_0)}{\log(1.23z_0)} &\ge e^{-\gamma}\frac{\log z}{\log z_0}\left(\frac{\log z_0}{\log(1.23z_0)} + \frac{\log^2 z_0}{\log(1.23z_0)\log z}\right)\\ &\ge 0.530\frac{\log z}{\log z_0}\end{aligned}$$

as required. □

Lemma 4.5. *When $P(z_0) \le z$, $\tau \le z^4$, $(\tau, P(z_0)) = 1$ and $z_0 \ge 35$, we have $G_\tau(z^2;z_0)/G_\tau(z;z_0) \le 3.8$.*

Proof. We use Lemmas 4.3 and 4.4 to infer that

$$\frac{G_\tau(z^2;z_0)}{G_\tau(z;z_0)} \le \frac{2}{0.530} \le 3.78.$$

The lemma is proved. □

Lemma 4.6. *When $P(z_0) \le z$, $\tau \le z^4$, $(\tau, P(z_0)) = 1$ and $z_0 \ge 35$, we have $G(z;z_0)/G_\tau(z;z_0) \le 1 + 20/z_0$.*

Proof. By Lemma 4.1 (with $q = \tau$ and $\tau = 1$), we have $G(z;z_0)/G_\tau(z;z_0) \le \varphi(\tau)/\tau$ □

5. On the w_q-functions

5.1. Pointwise estimates.

We follow [22, Lemma 4.4].

Lemma 5.1. *When p is prime, we have $-1 \le w_p(z;z_0,\tau)G_\tau(z;z_0)(p-1) \le 0$.*

Proof. When p is prime and prime to τ, we have $\xi_p(y) = 0$ when $y < p$ and $\xi_p(y) = p/(p-1)$ otherwise. Hence $G_{[p]}(z;z_0) = \frac{p}{p-1}G_{p\tau}(z/p;z_0)$ and the latter quantity is at most $G_\tau(z;z_0)$ by Lemma 4.1. The lemma follows readily. □

Lemma 5.2. *When p_1 and p_2 are distinct primes, we have*

$$\max\left(2, \frac{p_1p_2}{\varphi(p_1p_2)}\right) \ge w_{p_1p_2}(z;z_0,\tau)G_\tau(z;z_0)\varphi(p_1p_2) \ge 0.$$

Proof. The lemma is trivial if $(p_1p_2,\tau) \ne 1$, so let us now assume that p_1 and p_2 are prime to τ. Let us assume that $p_1 < p_2$. In (14), when $\ell > z/p_2$, there is no solution in (q_1,q_2,q_3). When $z/p_2 \ge \ell > z/(p_1p_2)$, the only solutions are $(q_1,q_2,q_3) = (p_1,p_2,1)$ and $(q_1,q_2,q_3) = (p_2,p_1,1)$ so that $\xi_{p_1p_2}(z/\ell) = 2$ in that range. When $\ell \le z/(p_1p_2)$, we have $\xi_{p_1p_2}(z/\ell) = \frac{p_1p_2}{\varphi(p_1p_2)}$. □

Lemma 5.3. *When $z_0 \ge 24$, we have $\left|G_\tau(z;z_0)w_q(z;z_0,\tau)\right| \le 1/q^{2/3}$.*
When $z_0 \ge 35$, we have $\left|G_\tau(z;z_0)w_q(z;z_0,\tau)\right| \le 1.04/q^{7/10}$.

Proof. We start from
$$|G_\tau(z;z_0)\varphi(q)w_q(z;z_0,\tau)| \le \prod_{p|q}\frac{3p-4}{p-1}\frac{G_{q\tau}(z/\sqrt{q};z_0)}{G_\tau(z;z_0)} \le \prod_{p|q}\frac{3p-4}{p-1}.$$
We then check that $(3p-4)p^{2/3}/(p-1)^2 \le 1$ when $p > 23$, hence the first result. The second one is proved in much the same manner, except that the inequality $f(p) = (3p-4)p^{7/10}/(p-1)^2 \le 1$ holds true only when $p \ge 43$. We thus have to multiply $1/q^{7/10}$ by $f(37)\cdot f(41)$. This proves the second inequality. □

Lemma 5.4. *When $z \ge 34$, we have*
$$z_0 \max_{q\ge z_0}\left|G_\tau(z;z_0)w_q(z;z_0,\tau)\right| \le 1 + 2.2/z_0.$$

Proof. Any q prime to $P(z_0)$ in the interval $[z_0, z_0^2)$ is a prime. By Lemma 5.1, we then have $|G_\tau(z;z_0)w_q(z;z_0,\tau)| \le 1/(q-1) \le 1/(z_0-1)$. When q has two prime factors, we use Lemma 5.2 and get
$$G_\tau(z;z_0)w_q(z;z_0,\tau) \le \frac{1}{q}\frac{2q}{\varphi(q)} \le \frac{2.11}{q} \le \frac{2.11}{z_0^2} \quad \textit{(provided that } \omega(q)=2\textit{)}.$$
Otherwise, $q \ge z_0^3$ and Lemma 5.3 tells us that $|z_0G_\tau(z;z_0)w_q(z;z_0,\tau)| \le 1$. The lemma follows readily from these three observations. □

We may also imitate the proof of [22, Lemma 4.5] to infer the next lemma.

Lemma 5.5. *We have*
$$|w_q(z;z_0,\tau)G_\tau(z;z_0)| \le \frac{G_\tau(z^2;z_0)}{G_\tau(z;z_0)}\sum_{\substack{q_1q_2q_3=q\\ q_1q_3,q_2q_3\le z}}\frac{1}{q_1q_2q_3}.$$

Proof. We easily infer from Lemma 3.1 that, when q is prime to $\tau P(z_0)$, we have
$$|w_q(z;z_0,\tau)G_\tau(z;z_0)| \le \frac{G_{q\tau}(z/\sqrt{q};z_0)}{\varphi(q)G_\tau(z;z_0)}\sum_{\substack{q_1q_2q_3=q\\ q_1q_3,q_2q_3\le z}}1.$$
We then complete this inequality with $G_{q\tau}(z/\sqrt{q};z_0)/\varphi(q) \le G_\tau(z\sqrt{q};z_0)/q$. □

5.2. **Some variants of the large sieve inequality.**

Lemma 5.6. *When $1 \le Q_1 \le Q_2$ and $\Delta \ge 1$ is a positive integer, we have*
$$\sum_{Q_1\le q\le Q_2}\frac{1}{q}\sum_{a \bmod^* q\Delta}\left|\sum_{n\le N}u_ne\left(\frac{na}{q\Delta}\right)\right|^2 \le (NQ_1^{-1}+2\Delta Q_2)\sum_n|u_n|^2$$
for arbitrary complex coefficients (u_n).

[22, Lemma 8.2] already uses a similar mechanism.

Proof. Let us first notice that the set of points $(\frac{a}{\Delta q})$ for $a \bmod^* q\Delta$ and $q \le t$ is Δt^2-well spaced. Let us use the shortcut

$$W(q) = \sum_{a \bmod^* q}\left|\sum_{n\le N}u_ne(na/q)\right|^2 \tag{19}$$

and assume, by homogeneity, that $\sum_n |u_n|^2 = 1$. By partial summation, we thus infer that

$$\begin{aligned}\sum_{Q_1\le q\le Q_2}\frac{W(\Delta q)}{q} &\le \int_{Q_1}^{Q_2}\sum_{q\le t}W(\Delta q)\frac{dt}{t^2}+\frac{\sum_{Q_1\le q\le Q_2}W(\Delta q)}{Q_2}\\ &\le \int_{Q_1}^{Q_2}(N+\Delta t^2)\frac{dt}{t^2}+\frac{N+\Delta Q_2^2}{Q_2}\le\frac{N}{Q_1}+2Q_2,\end{aligned}$$

by using the large sieve inequality on the second line. This ends the proof. □

5.3. **A w_q-weighted large sieve inequality.** This subsection is devoted to the proof of the next result.

Theorem 5.7. *When $P(z_0)\le z$, $\tau\le z^4$, $(\tau, P(z_0)) = 1$, $z_0\ge 35$ and $z_1\in[z_0,\sqrt{z}/\log z]$, we have*

$$\begin{aligned}\sum_{z_1\le q\le z^2}|G_\tau(z;z_0)w_q(z;z_0,\tau)|\sum_{a\,mod^*q}\Bigl|\sum_{n\le N}u_n e(na/q)\Bigr|^2\\ \le 13(Nz_1^{-1}+z^2(\log 3z_0)^{-1})\sum_n|u_n|^2\end{aligned}$$

for arbitrary complex coefficients (u_n).

This is the extension of [22, Theorem 8.1], though we did not try here to optimize the constants; we restricted our attention to eliminating the log-powers that intervene with a more casual treatment.

Proof of Theorem 5.7. Let us use the notation given by (19) and assume, by homogeneity, that $\sum_n |u_n|^2 = 1$. We readily check, via Lemma 15.4, that the condition $P(z_0)\le z$ implies that $z_0\le z^{1/2}/\log z$. When $q\le z$, we may rely on Lemma 5.4 and write (recall $W(q)$ is defined in (19))

$$\begin{aligned}\Sigma(z_0,z)=\sum_{z_0\le q\le z}|G_\tau(z;z_0)w_q(z;z_0,\tau)|W(q) &\le \frac{1+2.2/z_0}{z_0}\sum_{z_0\le q\le z}W(q)\\ &\le\frac{1+2.2/z_0}{z_0}(N+z^2)\end{aligned}\tag{20}$$

by the large sieve inequality. When $q\le z^{3/2}$, we may use Lemma 5.3 and write

$$\begin{aligned}\Sigma(z,z^{3/2})=\sum_{z\le q\le z^{3/2}}|G_\tau(z;z_0)w_q(z;z_0,\tau)|W(q)&\le 1.04\sum_{z\le q\le z^{3/2}}W(q)/q^{7/10}\\ &\le 1.04\int_z^{z^{3/2}}\sum_{q\le t}W(q)\frac{(7/10)dt}{t^{17/10}}+1.04\frac{\sum_{q\le z^{3/2}}W(q)}{z^{7/5}}\\ &\le 1.04\int_z^{z^{3/2}}(N+t^2)\frac{(7/10)dt}{t^{17/10}}+1.04\frac{N+z^3}{z^{7/5}}\end{aligned}$$

by the large sieve inequality. This simplifies into

$$\Sigma(z,z^{3/2})/1.04\le\frac{N}{z^{7/10}}+\left(\frac{7}{10}\frac{10}{13}+1\right)z^{\frac32\frac{13}{10}}=\frac{N}{z^{7/10}}+\frac{20}{13}z^{2-\frac{1}{20}}.\tag{21}$$

Let us now handle the more difficult part. We employ Lemma 5.5 together with Lemma 4.5 to write:

$$\Sigma(z^{3/2},z^2) = \sum_{z^{3/2}\le q\le z^2} |G_\tau(z;z_0)w_q(z;z_0,\tau)|W(q) \le 3.8 \sum_{\substack{z^{3/2}\le q_1q_2q_3\le z^2\\ q_1q_3,q_2q_3\le z\\ (q_1q_2q_3,P(z_0))=1}} \frac{W(q_1q_2q_3)}{q_1q_2q_3}$$

$$\le 3.8 \sum_{\substack{q_3\le z\\ (q_3,P(z_0))=1}} \frac{\mu^2(q_3)}{q_3} \sum_{\substack{q_1\le z/q_3\\ (q_1,q_3P(z_0))=1}} \frac{\mu^2(q_1)}{q_1} \sum_{\substack{z^{3/2}/(q_1q_3)\le q_2\le z/q_3\\ (q_2,q_1q_3P(z_0))=1}} \frac{\mu^2(q_2)W(q_1q_2q_3)}{q_2}.$$

Notice that, for the last summation to be non-empty, we should require that $q_1 \ge z^{1/2}$ and thus $q_3 \le z^{1/2}$. We get

$$\begin{aligned}
\frac{\Sigma(z^{3/2},z^2)}{3.8} &\le \sum_{\substack{q_3\le z^{1/2}\\ (q_3,P(z_0))=1}} \frac{\mu^2(q_3)}{q_3} \sum_{\substack{z^{1/2}\le q_1\le z/q_3\\ (q_1,q_3P(z_0))=1}} \frac{\mu^2(q_1)}{q_1}\biggl(\frac{q_1q_3N}{z^{3/2}} + 2q_1q_3(z/q_3)\biggr)\\
&\le \frac{N}{z^{3/2}} \sum_{\substack{q_3\le z^{1/2}\\ (q_3,P(z_0))=1}} \sum_{\substack{q_1\le z/q_3\\ (q_1,P(z_0))=1}} 1 + 2z \sum_{\substack{q_3\le z^{1/2}\\ (q_3,P(z_0))=1}} \frac{\mu^2(q_3)}{q_3} \sum_{\substack{q_1\le z/q_3\\ (q_1,P(z_0))=1}} 1\\
&\le 1.1\frac{N}{z^{1/2}}\frac{\log z}{\log 3z_0} + \frac{3z^2}{\log(3z_0)}
\end{aligned}$$

by using Lemma 5.6 on the first line and Lemma 15.8 together with the easily established inequalities

$$\sum_{\substack{q_3\le z^{1/2}\\ (q_3,P(z_0))=1}} \frac{\mu^2(q_3)}{q_3} \le \log z, \qquad \prod_{p\ge z_0}\Bigl(1+\frac{1}{p^2}\Bigr) \le \frac{3}{2\cdot 1.1}.$$

on the third line. On collecting our estimates, we finally find that

$$\begin{aligned}
\Sigma(z_1,z^2) &\le \Sigma(z_1,z) + \Sigma(z,z^{3/2}) + \Sigma(z^{3/2},z^2)\\
&\le \frac{N}{z_1}\biggl(1 + \frac{2.2}{35} + 1.04\frac{z_1}{z^{7/10}} + 3.8\times 1.1\frac{z_1\log z}{z^{1/2}\log(3z_0)}\biggr)\\
&\qquad + \frac{z^2}{\log 3z_0}\biggl(\frac{1+\frac{2.2}{35}}{z_0}\log(3z_0) + 1.04\frac{7\log(3z_0)}{13z^{1/20}} + 3.8\times 3\biggr).
\end{aligned}$$

We use $z \ge \exp(4z_0/5)$ given by Lemma 15.4 to show that

$$\begin{cases}
1 + \frac{2.2}{35} + 1.04\frac{z^{1/2}}{z^{7/10}\log z} + \frac{3.8\times 1.1}{\log(3z_0)} \le 1.97\\
\frac{1+\frac{2.2}{35}}{z_0}\log(3z_0) + 1.04\frac{20\log(3z_0)}{13z^{1/20}} + 3.8\times 3 \le 12.2.
\end{cases}$$

The theorem is proved. □

6. A large sieve inequality with primes and few points

Lemma 6.1. *Let $M \in \mathbb{R}$, and N and δ be positive real number. There exists a smooth function ψ on $\mathbb{R}$ such that*

- *The function ψ is non-negative.*
- *When $t \in [M, M+N]$, we have $\psi(t) \ge 1$.*
- *$\psi(0) = N + \delta^{-1}$.*
- *When $|\alpha| > \delta$, we have $\hat{\psi}(\alpha) = 0$.*
- *We have $\psi(t) = \mathcal{O}_{M,N,\delta}(1/(1+|t|^2))$.*

This is [24, Lemma 3.3] and is a retelling of a result due to A. Selberg, see [28, Section 20].

Theorem 6.2. *Let $N \geq 10^4$. Let $\mathcal{B}$ be a δ-well spaced subset of $\mathbb{R}/\mathbb{Z}$. For any function f on $\mathcal{B}$, we have*

$$\sum_{p\leq N}\left|\sum_{b\in\mathcal{B}} f(b)e(bp)\right|^2 \leq 19(N+\delta^{-1})\|f\|_2^2\frac{\log(2\|f\|_1^2/\|f\|_2^2)}{\log N}.$$

where $\|f\|_q^q = \sum_{b\in\mathcal{B}}|f(b)|^q$ for any positive q.

The best constant that comes out of the proof we propose is $(4e^\gamma + o(1))$, provided $\|f\|_1^2/\|f\|_2^2$ goes to infinity.

Proof. Let us first notice that $\|f\|_1^2 \geq \|f\|_2^2$.

Small $\|f\|_1^2/\|f\|_2^2$. When $y = \|f\|_1^2/\|f\|_2^2$ is small, we simply use

$$\sum_{p\leq N}\left|\sum_{b\in\mathcal{B}} f(b)e(bp)\right|^2 \leq \pi(N)\|f\|_1^2 \leq \frac{5N}{4\log N}\|f\|_2^2\frac{y}{\log(2y)}\log(2\|f\|_1^2/\|f\|_2^2)$$

by Lemma 15.1 and provided that $N \geq 114$. When $y \leq 21$, we obtain

$$\sum_{p\leq N}\left|\sum_{b\in\mathcal{B}} f(b)e(bp)\right|^2 \leq 4e^\gamma\|f\|_2^2\frac{\log(2\|f\|_1^2/\|f\|_2^2)}{\log N}.$$

Large $\|f\|_1^2/\|f\|_2^2$. When $\|f\|_1^2/\|f\|_2^2 \geq N^{e^{-\gamma}/4}$, we use the dual of the usual large sieve inequality (see [17] by H.L. Montgomery) to infer that

$$\begin{aligned}\sum_{p\leq N}\left|\sum_{b\in\mathcal{B}} f(b)e(bp)\right|^2 &\leq (N+\delta^{-1})\|f\|_2^2 \leq (N+\delta^{-1})\|f\|_2^2\frac{\log(2\|f\|_1^2/\|f\|_2^2)}{\log(N^{e^{-\gamma}/4})}\\ &\leq 4e^\gamma(N+\delta^{-1})\|f\|_2^2\frac{\log(2\|f\|_1^2/\|f\|_2^2)}{\log N}.\end{aligned}$$

This establishes our inequality in this case.

Small primes. Let $z = N^{1/4}/10$ and

$$z_0 = \frac{2\|f\|_1^2}{1.23\|f\|_2^2} > 34.$$

We assume that $21 < \|f\|_1^2/\|f\|_2^2 \leq N^{e^{-\gamma}/4}$ and that $N \geq 10^4$. We discard the small primes trivially:

$$\begin{aligned}\sum_{p\leq z}\left|\sum_{b\in\mathcal{B}} f(b)e(bp)\right|^2 &\leq z\|f\|_1^2 \leq N^{(1+e^{-\gamma})/4}\|f\|_2^2/10\\ &\leq N\frac{\|f\|_2^2\log(2\|f\|_1^2/\|f\|_2^2)}{\log N}\frac{\log N}{10N^{(3-e^{-\gamma})/4}\log 42}\\ &\leq \frac{N}{1113}\frac{\|f\|_2^2\log(2\|f\|_1^2/\|f\|_2^2)}{\log N}.\end{aligned}$$

Main proof. Let us define

$$W = \sum_{z<p\le N} \Big|\sum_{b\in\mathcal{B}} f(b)e(bp)\Big|^2. \tag{22}$$

We bound above the characteristic function of the primes from $(z,N]$ by our enveloping sieve and further majorize the characteristic function of the interval $[1,N]$ by a function ψ (see Lemma 6.1) of Fourier transform supported by $[-\delta_1,\delta_1]$ where $\delta_1=\min(\delta,1/(2z^4))$, and which is such that $\hat\psi(0)=N+\delta_1^{-1}$. This leads to

$$W \le \sum_{\substack{q\le z^2,\\ (q,P(z_0))=1}} w_q(z;z_0)\sum_{a \bmod^* q}\sum_{b_1,b_2} f(b_1)\overline{f(b_2)}\sum_{n\in\mathbb{Z}} e((b_1-b_2)n)e(an/q)\psi(n).$$

We have shortened $w_q(z;z_0,1)$ in $w_q(z;z_0)$. We split this quantity according to whether $q<z_0$ or not:

$$W = W(q<z_0)+W(q\ge z_0).$$

When $q\ge z_0$, Poisson summation formula tells us that the inner sum reads as $\sum_{m\in\mathbb{Z}}\hat\psi(b_1-b_2-(a/q)+m)$. The sum over b_1, b_2 and n is thus

$$\le (N+\delta_1^{-1})\sum_{b_1,b_2}|f(b_1)||f(b_2)|\#\big\{a/q : \|b_1-b_2+a/q\|<\delta_1\big\}.$$

Given (b_1,b_2) and since $1/z^4>2\delta_1$, at most one a/q may work. On bounding above $|w_q(z;z_0)|$ by Lemma 5.4, we see that

$$G(z;z_0)W(q\ge z_0)\le (1+2.2\,z_0^{-1})(N+\delta_1^{-1})\frac{\|f\|_1^2}{z_0}. \tag{23}$$

When $w_q(z;z_0)\neq 0$, we have $q|P(z)/P(z_0)$; on adding the condition $q<z_0$, only $q=1$ remains. Since $\mathcal{B}$ is δ-well-spaced and $w_1(z;z_0)=1/G(z;z_0)$, we infer that

$$G(z;z_0)W(q<z_0)\le (N+\delta_1^{-1})\|f\|_2^2.$$

On recalling Lemma 4.2, we thus get

$$W \le (N+\delta_1^{-1})\bigg(\|f\|_2^2+(1+2.2\,z_0^{-1})\frac{\|f\|_1^2}{z_0}\bigg)\frac{e^\gamma\log(1.23\,z_0)}{\log z}. \tag{24}$$

Final estimate. We check that $(N+\delta_1^{-1})\le \frac{N+4N10^{-4}}{N}(N+\delta^{-1})$. We finally get

$$\sum_{p\le N}\Big|\sum_{b\in\mathcal{B}} f(b)e(bn)\Big|^2 \le \bigg(\frac{1}{1113}+4e^\gamma(1+4/10^4)\Big(1+(1+2.2/34)\frac{1.23}{2}\Big)\frac{\log N}{\log N-4\log 4}\bigg) \times (N+\delta^{-1})\|f\|_2^2\frac{\log(2\|f\|_1^2/\|f\|_2^2)}{\log N}.$$

The proof of the theorem follows readily. □

7. Proof of Theorem 2.4

Proof of Theorem 2.4. We follow the usual duality argument, starting from Theorem 6.2. We write

$$\sum_{x\in\mathcal{X}}\Big|\sum_{p\le N} u_p e(xp)\Big|^2 = \sum_{p\le N} u_p\sum_{x\in\mathcal{X}}\overline{S(x)}e(xp)$$

with $S(x) = \sum_{p\le N} u_p e(xp)$. We apply Cauchy's inequality to the resulting expression, then Theorem 6.2 and the inequality

$$\Big(\sum_{x\in\mathcal{X}} |S(x)|\Big)^2 \Big/ \sum_{x\in\mathcal{X}} |S(x)|^2 \le |\mathcal{X}|$$

to simplify the factor $\log(2\|f\|_1^2/\|f\|_2^2)$ that appears in Theorem 6.2. The inequality of the theorem then follows swiftly. On taking $u_p = 1$ and $\mathcal{X} = \{a/q : q \le Q, a(q) = 1\}$ where we have $S(a/q) = \sum_{p\le N} e(ap/q) = \mu(q)(1+o(1))N/(\varphi(q)\log N)$ when Q is at most a power of $\log N$, we obtain that the quantity $\sum_{x\in\mathcal{X}} |S(a/q)|^2$ is asymptotic to $G(Q)\pi(N)^2$. The last part of the theorem then follows on noticing that $\log|\mathcal{X}| \sim 2\log Q \sim 2G(Q)$. □

Part 2. Cusps

8. Bounding the number of cusps

We define, as in Eq. (14) in [24],

$$V = \Big(\frac{N+\delta^{-1}}{\log N} \sum_{p\le N} |u_p|^2\Big)^{1/2}. \tag{25}$$

Lemma 8.1. *Let $N \ge 10^4$ and $\mathcal{X}$ be a δ-well spaced subset of $\mathbb{R}/\mathbb{Z}$. Let $(u_p)_{p\le N}$ be a sequence of complex numbers. We have, for $A \ge 1$,*

$$\#\Big\{x\in\mathcal{X} : \Big|\sum_{p\le N} u_p e(xp)\Big| \ge V/A\Big\} \le 19A^2\log(2A),$$

where V is defined in (25). The constant 19 may be replaced by $4e^\gamma + o(1) = 7.12\cdots$ when $N\to\infty$ and $A\to\infty$.

We used the notation $\#S$ to denote the cardinality of the set S.

Proof of Lemma 8.1. This is a trivial applications of Markov's inequality. □

Proof of Theorem 1.3. We readily getting

$$\begin{aligned}\int_0^1 |T^*(\alpha)|d\alpha &\ll \sum_{0\le r\le R-1} \frac{e^{2r}K}{N} f(r)\frac{T^*(0)}{e^r} + \frac{T^*(0)}{e^R}\\ &\ll T^*(0)\Big(\frac{e^R K}{N} f(R) + e^{-R}\Big) \ll \frac{\sqrt{Nf(\log N)}}{\sqrt{K}\log N}\end{aligned}$$

by choosing $R = [(1/2)\log\frac{N}{Kf(\log N)}]$. The proof is complete. □

Proof of Theorem 1.2. Lemma 8.1 tells us, after a change of variable $(A \mapsto A\sqrt{K(1+(N\delta)^{-1})})$, that

$$\#\Big\{x\in\mathcal{X} : \Big|\sum_{\substack{p\le N\\ p\in\mathcal{P}^*}} e(xp)\Big| \ge \frac{N}{AK\log N}\Big\} \le 19A^2K(1+(N\delta)^{-1}). \tag{26}$$

The above discussion leads to Theorem 1.2. □

9. Finding rational points as cusps

On the structure of the set of cusps. Lemma 1.2 gives an upper bound for the number of cusps. Let us now investigate the structure of this set. The examples we provided tell us it is possible that this set has a structure close to $\mathcal{F}+\big\{\frac{a}{q}, q\le 100A\big\}$ for a small enough set $\mathcal{F}$. This is also a rationale leading to the proof of Theorem 2.4 and why we thought interesting not to sieve the small moduli. Here is a first lemma.

Lemma 9.1. *Notation being as in Definition 1.1, and assuming $N\ge 10$, the following holds:*

- *When α lies in $\mathscr{C}(\mathcal{P}^*, A)$, so do $-\alpha$ and $\frac12+\alpha$.*
- *When $\{\frac13,\frac12,\frac23,1\}\subset\mathscr{C}(\mathcal{P}^*,1/2)$*
- *For every $\xi\in\mathbb{R}/\mathbb{Z}$ with $T^*(\xi)\neq 0$ and every square-free positive integer $q<\sqrt{N}$ such that $\varphi(q)\le AT^*(0)/|T^*(\xi)|$, there exists a, coprime with q, such that $\xi+(a/q)$ lies in $\mathscr{C}(\mathcal{P}^*,A)$.*

Proof. Let us recall that $\mathcal{P}^*\in[\sqrt{N},N]$, so that the elements of $\mathcal{P}^*$ are prime to the moduli 2, 3 and q appearing in the three claimed properties. The first item is a consequence of the facts that (1) the characteristic function $1_{\mathcal{P}^*}$ of $\mathcal{P}^*$ is real valued and (2) that every member of $\mathcal{P}^*$ is odd. Concerning the third item, it is enough to notice the inequality

$$\sum_{a \bmod^* q}\Big|\sum_{p\in\mathcal{P}^*} e(p(\xi+(a/q))\Big|\ge\Big|\sum_{a \bmod^* q}\sum_{p\in\mathcal{P}^*} e(p(\xi+(a/q))\Big|=\mu^2(q)|T^*(\xi)| \tag{27}$$

with $\xi=0$, since $c_q(p)=\mu(q)$ where $c_q(n)$ denotes the value of the Ramanujan sum at n. The second item follows from this same inequality applied to $q=3$, on noticing that the absolute values of the involved Fourier polynomial at $1/3$ and $2/3$ are the same. □

The proof of Theorem 1.4 relies on extracting information from the stream of inequalities (27), for varying q's.

Proof of Theorem 1.4. We may assume that $B\le A/10$, for otherwise the result is obvious: $\mathcal{F}$ contains the element ξ. Let us set $Q=A/(2B)\ge 1$ and

$$\mathcal{Q}=\big\{q: q\le Q, \mu^2(q)=1\big\}. \tag{28}$$

The non-negative real variables $t_\xi(a/q)=t^*(\xi+(a/q))$ are bounded above by $Z\le 1$ and satisfy (see (27))

$$\forall q\in\mathcal{Q},\qquad \sum_{a \bmod^* q} t_\xi(a/q)\ge 1/B$$

as well as, by Lemma 1.2 and since the points in $\mathcal{F}$ are $1/N$-well spaced,

$$\forall C\ge 1,\quad \#\big\{a/q: q\in\mathcal{Q}, t_\xi(a/q)\ge 1/C\big\}\le 19KC^2\log 2C.$$

Let us further introduce the variables

$$x(q,C)=\#\{a \bmod^* q: bZ/C> t_\xi(a/q)\ge bZ/(C+1)\} \tag{29}$$

for some $b\in(1,2)$. We put bZ/C, rather than Z/C, so as to include the case $t_\xi(a/q)=Z$ in the same setting. We thus get

$$\forall q\in\mathcal{Q},\quad \sum_{C\ge 1}\frac{bZ}{C}x(q,C)\ge 1/B.$$

Let us assume momentarily that $AZ = A'$ is a positive integer and degrade the above inequality into

$$\sum_{1\le C\le A'-1}\frac{b}{C}x(q,C)+\Big(\varphi(q)-\sum_{1\le C\le A'-1}x(q,C)\Big)\frac{b}{A'}\ge 1/(BZ),$$

i.e.

$$\sum_{1\le C\le A'-1}\Big(\frac{b}{C}-\frac{b}{A'}\Big)x(q,C)\ge\frac{1}{BZ}-\frac{b\varphi(q)}{A'}\ge\frac{1-b/2}{BZ}.$$

We now sum over $q\in\mathcal{Q}$, getting, by Lemma 15.7,

$$\sum_{1\le C\le A'-1}\Big(\frac{b}{C}-\frac{b}{A'}\Big)\sum_{q\in\mathcal{Q}}x(q,C)\ge\frac{(1-b/2)A'}{4B^2Z^2}.$$

Let us set

$$X(C)=\sum_{D\le C}\sum_{q\in\mathcal{Q}}x(q,D),\quad (X(0)=0)$$

which satisfies

$$\sum_{1\le C\le A'-1}\Big(\frac{b}{C}-\frac{b}{A'}\Big)(X(C)-X(C-1))\ge\frac{(1-b/2)A'}{4B^2Z^2}.$$

On reshuffling the left-hand side, we obtain:

$$\sum_{1\le C\le A'-1}X(C)\Big(\frac{b}{C}-\frac{b}{C+1}\Big)+X(A'-1)\Big(\frac{b}{A'-1}-\frac{b}{A'}\Big)\ge\frac{(1-b/2)A'}{4B^2Z^2}.$$

The non-decreasing function $C\mapsto X(C)$ is therefore constrainted by the two inequalities:

$$\sum_{1\le C\le A'-1}\frac{X(C)}{C(C+1)}\ge\frac{(1-b/2)A'}{4B^2Z^2},\quad X(C)\le 19K(C+1)^2\log(2C+2)/(bZ)^2.$$

Let us split the first sum at $C=\theta A'$. We deduce from the above that

$$19K(bZ)^{-2}\log(\theta A'+1)\sum_{1\le C<\theta A'}\frac{C+1}{C}+X(A'-1)\Big(\frac{1}{[\theta A']}-\frac{1}{A'}\Big)\ge\frac{(1-b/2)A'}{4B^2Z^2}$$

and thus

$$38K(bZ)^{-2}\theta A'\log(2\theta A'+2)+X(A'-1)\frac{4-\theta+A'^{-1}}{4\theta A'}\ge\frac{(1-b/2)A'}{4B^2Z^2}.$$

This calls for $\theta\log A'=b^2/(8\times 38\times 2B^2K)$, so that

$$\tfrac{9}{8}X(A'-1)\ge(3-2b)\frac{A'^2b^2}{4864K(BZ)^2B^2\log 2A'}.$$

On letting b go to 1, this finally amounts to $X(A'-1)\ge\frac{A^2}{5472Z^2B^4K\log 2A'}$. This is proved when AZ is an integer. Otherwise, we use the same bound but for $A" = [AZ]/Z\ge A-1/Z\ge(1-1/(ZA))A\ge 9A/10$ to conclude the proof. □

Part 3. The structure theorem

10. Preliminaries

It is better for clarity to present the parameters first and the object to study later. So let $\epsilon\in(0,1/2]$ and a parameter $A\ge 1$ be given. We define the parameter z_0 by:

$$z_0=\exp\big(25000A^3K(\log 2A)^2\big)/\epsilon. \tag{30}$$

(Notice for numerical purposes that $z_0 \geq 35$) The reader may want to keep this parameter as unknown until Theorem 14.1, so as to understand the above choice. We further introduce a positive integer parameter M that satisfies

$$\text{(H1)} \qquad \text{Every prime } p|M \text{ is } < z_0 \text{ and } P(z_0)|M.$$

At first, the readers may select $M = P(z_0)$, but an application we have in mind will require $M = P(z_0)^3$.

Let then $\mathcal{P}^*$ be a subset of the primes within $[1, N]$, of cardinality $N/(K \log N)$, all larger than $z = \sqrt{N/(Mz_0)}$. We assume that $P(z_0) \leq z$. By [27, Theorem 9] of J.-B. Rosser and L. Schoenfeld, this is implied by $1.02z_0 \leq \log z$.

$$\text{(H2)} \qquad P(z_0) \leq z \quad \text{and} \quad z \geq N^{1/2(1+\epsilon)}.$$

The last inequality is equivalent to $Mz_0 \geq N^{\frac{\varepsilon}{1+\varepsilon}}$. We define

$$(31) \qquad T^*(\alpha) = \sum_{p \in \mathcal{P}^*} e(p\alpha), \quad N' = 240NA, \quad \varepsilon = \frac{1}{240A}.$$

We denote by f the characteristic function of $\mathcal{P}^*$.

11. A finite cover of $\mathscr{C}(\mathcal{P}^*, A)$

Let us cover the set $\mathscr{C}(\mathcal{P}^*, A)$ of A-cusps by a finite set Ξ of points so that every points of $\mathscr{C}(\mathcal{P}^*, A)$ is at distance $\leq 1/(240A)$ of a point of Ξ. To do so, we take, if possible, a point y in each interval $[\frac{a-1}{N'}, \frac{a}{N'})$ with $a \leq N' = 240AN$ and even (resp. odd) such that

$$(32) \qquad |T^*(y)| \geq T^*(0)/A.$$

By Eq. (26) with $\delta = 1/N'$, each set (with a odd and with a even) has at most $5000A^3K\log(2A)$ points. The set Ξ is the union of both. Every point of $\mathscr{C}(\mathcal{P}^*, A)$ is at a distance $\leq \varepsilon/N$ of a point in Ξ. We also consider $\Xi_M = \{My : y \in \Xi\} \subset \mathbb{R}/\mathbb{Z}$, which may be much smaller that Ξ (see Theorem 1.4), it may even be reduced to one point.

12. The associated Bohr set

We consider

$$(33) \qquad \mathcal{B}_M(\varepsilon) = \big\{n \leq N : M|n \quad \& \quad \forall y \in \Xi, \|yn\| \leq \varepsilon\big\},$$

as well as

$$(34) \qquad S_M(\alpha; \Xi, \varepsilon) = \sum_{b \in \mathcal{B}_M(\varepsilon)} e(b\alpha).$$

See [11, Proposition 4.2] by B. Green & I. Ruzsa or [12, Lemma 10.4].

The parameter N is not recalled in our notation. By [30, Lemma 4.20][3] of the book of T. Tao & V. Vu, applied to n/M and Ξ_M, we have

$$(35) \qquad |\mathcal{B}_M| \geq \varepsilon^{|\Xi_M|}\lfloor N/M\rfloor \geq \tfrac{1}{2}\varepsilon^{|\Xi_M|}N/M.$$

Lemma 12.1. *When $u \in \mathbb{R}/\mathbb{Z}$, we have $|e(u) - 1| \leq 2\pi\|u\|$, where $\|u\|$ is the distance to the nearest integer.*

Proof. First replace u by $w = \pm u + k$ for some $k \in \mathbb{Z}$, in such a way that $w = \{u\}$. The result then comes from the mean value theorem or from $e(w) - 1 = \int_0^w 2i\pi e(v)dv$. □

[3] A more usual argument based on the pigeonhole principle yields the marginally weaker bound $\varepsilon^{|\Xi|}N/(1+\varepsilon)^{|\Xi|}$. Asymptotically, when N goes to infinity, ε remains fixed and the set Ξ is made of $\mathbb{Q}$-linear independant points, we have $|\mathcal{B}| \sim (2\varepsilon)^{|\Xi|}N$.

Lemma 12.2. *If, given $\alpha \in \mathbb{R}/\mathbb{Z}$, there exists $y \in \Xi$ such that $|\alpha - y| \le \varepsilon/N$, then $S_M(\alpha; \Xi, \varepsilon) = (1 + \mathcal{O}^*(7\varepsilon))|\mathcal{P}_M(\varepsilon)|$.*

Proof. For any $b \in \mathcal{B}_M(\varepsilon)$, we have $\|b\alpha\| \le 2\varepsilon$. Therefore, on calling to Lemma 12.1, we get

$$S_M(\alpha; \Xi, \varepsilon) = \sum_{b \in \mathcal{P}_M(\varepsilon)} \bigl(1 + e(b\alpha) - 1\bigr) = (1 + \mathcal{O}^*(7\varepsilon))|\mathcal{B}_M(\varepsilon)|.$$

Once this is noticed, the lemma follows readily. □

Furthermore and following B. Green in [13, Eq. (6.5)], we then consider

$$\rho_{M,\varepsilon}(m) = \frac{1}{|\mathcal{B}_M(\varepsilon)|^2} \sum_{\substack{b_1 - b_2 = m \\ b_1, b_2 \in \mathcal{B}_M(\varepsilon)}} 1 \quad \ge 0. \tag{36}$$

13. The approximant

We consider

$$\begin{array}{rcl} f^* : \mathbb{Z} & \to & \mathbb{C} \\ \ell & \mapsto & G(z; z_0)(f \star \rho_{M,\varepsilon})(\ell) = G(z; z_0) \displaystyle\sum_{n+m=\ell} f(n)\rho_{M,\varepsilon}(m) \end{array} \tag{37}$$

This definition is taylored for the next two lemmas.

Lemma 13.1. *We have $f^*(\ell) = 0$ when $(\ell, M) \neq 1$ and $0 \le f^*(\ell) \le 1 + \dfrac{8}{\varepsilon^{|\Xi|} z_0}$.*

Proof. Let us notice that any decomposition $\ell = n + m$ with $\rho_{M,\varepsilon}(m) \neq 0$ has $(n, M) = (\ell, M)$, hence the first property.

By using Section 3 and since $\rho_{M,\varepsilon}(m) \ge 0$, we may write

$$f^*(\ell) \le G(z; z_0) \sum_{n+m=\ell} \beta_{z_0,z}(n)\rho_{M,\varepsilon}(m).$$

We continue with Lemma 3.1:

$$\begin{aligned} \sum_{n+m=\ell} \beta_{z_0,z}(n)\rho_{M,\varepsilon}(m) &= \sum_{q \le z^2} w_q(z; z_0) \sum_{a \bmod^* q} \sum_m \rho_{M,\varepsilon}(m) e((\ell - m)a/q) \\ &= \sum_{q \le z^2} \frac{w_q(z; z_0)}{|\mathcal{B}_M(\varepsilon)|^2} \sum_{a \bmod^* q} e(\ell a/q) \Bigl| \sum_{b \in \mathcal{B}_M(\varepsilon)} e(ba/q) \Bigr|^2 \\ &= \frac{1}{G(z; z_0)} + \sum_{z_0 \le q \le z^2} \frac{w_q(z; z_0)}{|\mathcal{B}_M(\varepsilon)|^2} \sum_{a \bmod^* q} e(\ell a/q) \Bigl| \sum_{b \in \mathcal{B}_M(\varepsilon)} e(ba/q) \Bigr|^2. \end{aligned}$$

Theorem 5.7 applies and gives us the bound

$$f^*(\ell) \le 1 + 4\frac{N(Mz_0)^{-1} + z^2 \log(3z_0)^{-1}}{|\mathcal{B}_M(\varepsilon)|} \le 1 + \frac{6}{\varepsilon^{|\Xi|} z_0} + \frac{6Mz^2}{N\varepsilon^{|\Xi|}\log(3z_0)}.$$

The lemma is proved. □

Lemma 13.2. *We have $S(f^*, \alpha) = G(z; z_0) T^*(\alpha) \bigl| S_M(\alpha, \Xi, \varepsilon)/|\mathcal{B}_M(\varepsilon)| \bigr|^2$.*

If, given $\alpha \in \mathbb{R}/\mathbb{Z}$, there exists $y \in \Xi$ such that $|\alpha - y| \le \varepsilon/N$, then $S(f^*, \alpha) = G(z; z_0) T^*(\alpha)(1 + \mathcal{O}^*(120\varepsilon))$.

For any integer a, we have $S(f^*, a/M) = G(z; z_0) T^*(a/M)$.

Proof. By definition, we have

$$\begin{aligned}S(f^*,\alpha) &= G(z;z_0)\sum_{n,m} f(n)\rho_{M,\varepsilon}(m)e((n+m)\alpha)\\ &= G(z;z_0)T^*(\alpha)\left|\frac{S_M(\alpha,\Xi,\varepsilon)}{|\mathcal{B}_M(\varepsilon)|}\right|^2\end{aligned}$$

as claimed. On using Lemma 12.2, the reader will conclude easily, recalling that $\varepsilon \le 1/2$. □

14. Decomposition

We have reached the main technical point.

Theorem 14.1. *We have $f = f^* G(z;z_0)^{-1} + f^\sharp$ where*

- *For every $\alpha \in \mathbb{R}/\mathbb{Z}$, $|S(f^\sharp,\alpha)| \le |T^*(\alpha)|$ and $|S(f^*,\alpha)| \le |T^*(\alpha)|G(z;z_0)$.*
- *For every $\alpha \in \mathbb{R}/\mathbb{Z}$, we have $|S(f^\sharp,\alpha)| < T^*(0)/A$.*
- $0 \le f^*(\ell) \le 1+\epsilon$ *and* $f^*(\ell) = 0$ *when* $(\ell, M) \ne 1$.
- *For any integer a, we have $S(f^*, a/M) = G(z;z_0)T^*(a/M)$.*

Proof. We have $\varepsilon^{-|\Xi|} \le \epsilon z_0/8$. By construction of Ξ, every point of $\mathscr{C}(\mathcal{P}^*, A)$ is at distance $\le \varepsilon/N$ of a point of Ξ. By Lemma 13.2, this implies that $S(f^\sharp,\alpha) = T^*(\alpha) - S(f^*,\alpha)G(z;z_0)^{-1}$ satisfies

$$\forall \alpha \in \mathscr{C}(\mathcal{P}^*, A), \quad |S(f^\sharp,\alpha)| \le T^*(0)/(2A).$$

When $\alpha \notin \mathscr{C}(\mathcal{P}^*, A)$, the inequality $|S(f^\sharp,\alpha)| \le |T^*(\alpha)|$ implies that $|S(f^\sharp,\alpha)| < T^*(0)/A$ as required. □

15. Beautification and proof of Theorem 1.5

The factor $G(z;z_0)$ in Theorem 14.1 may look awkward to the many. Furthermore, the pointwise bound $f^* \le 1+\epsilon$ does not reflect the fact that the sieve has lost a factor $(\log N)/G(z;z_0)$. We thus propose to replace f^* by

$$f^\flat = f^* G(z;z_0)^{-1} V(z_0)\log N. \tag{38}$$

Only the local upper bound needs to be refreshed. By Lemma 4.4, we find that

$$f^\flat \le (1+\epsilon)\frac{\log N}{\log(zz_0)} \le 2(1+\epsilon)^2.$$

Proof of Theorem 1.5. We have already proved almost everything except that the condition $1.02z_0 < \log z$, where z is defined above, holds true. This is equivalent to $\exp(2.04z_0)Mz_0 < N$. We already know that $N \ge (Mz_0)^{1+\epsilon^{-1}} \ge (Mz_0)^4$ which is why we have imposed $\epsilon \le 1/3$. It is thus enough to prove that $M^3 \ge \exp(2.04z_0)$, which is implied by $P(z_0)^3 \ge \exp(2.04z_0)$. The proof of Lemma 15.4 gives us that it is enough to show that $\exp(3\frac{4}{5}z_0) \ge \exp(2.04z_0)$ which is obvious. □

Part 4. Auxiliaries for explicit computations

On primes. Let us start with some estimates due to J.B. Rosser & L. Schoenfeld in [27, Theorem 1, Corollary 2, Theorem 6-8, Theorem 23].

Lemma 15.1. *We have*

$$\prod_{p\le x}\frac{p}{p-1} \le e^\gamma(\log x)\left(1+\frac{1}{2\log^2 x}\right) \quad \textit{when } x \ge 286,$$

$$e^\gamma(\log x) < \prod_{p\le x}\frac{p}{p-1} \le e^\gamma(\log x) + \frac{2e^\gamma}{\sqrt{x}} \quad \textit{when } x \le 10^8.$$

Futhermore $\pi(x) = \sum_{p\le x} 1 \le \frac{x}{\log x}(1 + \frac{3}{2\log x})$ *and* $\pi(x) \le \frac{5x}{4\log x}$*, both valid when* $x \ge 114$*. Finally,* $\pi(x) \ge x/(\log x)$ *when* $x \ge 17$*.*

Lemma 15.2. *When* $z_0 \ge 2$*, we have* $\prod_{p<z_0} \frac{p-1}{p} \ge \frac{e^{-\gamma}}{\log(9z_0/5)}$*. When* $z_0 > 31$*, we have* $\prod_{p<z_0} \frac{p-1}{p} \ge \frac{e^{-\gamma}}{\log(1.23\, z_0)}$*.*

The constant $9/5$ is somewhat forced on us by $z_0 = 3$.

Proof. This follows from direct inspection for $z_0 \le 100\,000$ and for $z_0 \le 10^8$ by Lemma 15.1. Again on using this lemma, we find that

$$e^{\gamma}\log(1.23z_0)\prod_{p<z_0}\frac{p-1}{p} \ge \left(1 + \frac{\log(1.23)}{\log z_0}\right)\left(1 + \frac{1}{2\log^2 z_0}\right)^{-1}.$$

The right-hand side is readily seen to be > 1 when $y = 1/\log z_0 \le 0.05$. The lemma follows readily. □

Lemma 15.3. *When* $z_0 \ge 3$*, we have* $\sum_{p<z_0} \frac{\log p}{p-1} \ge \log z_0 - 0.6$*.*

Proof. □

Lemma 15.4. *When* $z_0 \ge 35$ *and* $P(z_0) \le z$*, we have* $z_0 \le \frac{5}{4}\log z$*.*

Proof. We have $\vartheta(x) \ge 4x/5$ when $x \ge 30$ (checked up to 10^5). □

On the first G-function.

Lemma 15.5. *When* $z \ge 1$*, we have* $G(z) = \log z + c_0 + \mathcal{O}^*((61/25)/\sqrt{z})$ *where*

$$c_0 = \gamma + \sum_{p\ge 2}\frac{\log p}{p(p-1)} = 1.332\,582\,275\,733\ldots$$

This is part of [23, Theorem 3.1].

Lemma 15.6. *When* $z \ge 2$*, we have* $G(z^2) \le 2G(z)$*. Also when* $z \ge 10$*, we have* $1.4709 \ge G(z) - \log z \ge 1.2$*.*

Proof. Indeed, the inequality

$$G(z^2) - 2G(z) \le -c_0 + \frac{61}{25}\left(\frac{1}{\sqrt{z}} + \frac{1}{z}\right)$$

proves that $G(z^2) - 2G(z) \le 0$ when $z \ge 6.5$. When $z \in [n, n+1)$ and n is an integer, the inequality $G(z^2) \le 2G(z)$ is equivalent to by $G(z^2) \le 2G(n)$ and, for this one to hold throughout the interval $[n, n+1)$, we need $G(n^2 + 2n) \le 2G(n)$. This is readily checked for $n \ge 2$. The next upper bound for $G(z)$ is [22, Lemma 3.5, (1)]. The lower bound follows from Lemma 15.5 when $z \ge 700$. We complete the proof by a direct inspection. The constant 1.2 is forced by the case $z = 29$. □

On some arithmetic functions.

Lemma 15.7. *When real number* $Q \ge 1$*, we have* $\sum_{q\le Q} \mu^2(q) \ge Q/2$*.*

Be cautious: the lower bound of K. Rogers given in [26] is only valid for *integer* values of Q, though this is not specified.

Proof. When $Q \geq 1664$, this is a consequence of [3, Théorème 3] which asserts that $\sum_{q \leq Q} \mu^2(q) = 6\pi^{-2}Q + \mathcal{O}^*(0.1333\sqrt{Q})$. A straightforward numerical check concludes the proof. □

Lemma 15.8. *When $z_0 \geq 35$ and $P(z_0) \leq Q^2$, we have*

$$\sum_{\substack{q \leq Q \\ (q,P(z_0))=1}} 1 \leq 1.1 \frac{Q}{\log(3z_0)}.$$

Proof. The arithmetical form of the large sieve in its simplest instance gives us

$$\sum_{\substack{q \leq Q \\ (q,P(z_0))=1}} 1 \leq \frac{Q + z_0^2}{G(z_0)}.$$

We readily check, via Lemma 15.4, that the condition $P(z_0) \leq Q^2$ implies that $z_0 \leq Q/\log Q$. Lemma 15.6 provides the lower bound $G(z_0) \geq \log(3z_0)$ □

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