NOTES ON RESTRICTION THEORY IN THE PRIMES

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Abstract. We study the mean $\sum_{x\in\mathcal{X}}\bigl|\sum_{p\le N}u_p e(xp)\bigr|^\ell$ when ℓ covers the full range $[2,\infty)$ and $\mathcal{X}\subset\mathbb{R}/\mathbb{Z}$ is a well-spaced set, providing a smooth transition from the case $\ell=2$ to the case $\ell>2$ and improving on the results of J. Bourgain and of B. Green and T. Tao. A uniform Hardy-Littlewood property for the set of primes is established as well as a sharp upper bound for $\sum_{x\in\mathcal{X}}\bigl|\sum_{p\le N}u_p e(xp)\bigr|^\ell$ when $\mathcal{X}$ is small. These results are extended to primes in *any* interval in a last section, provided the primes are numerous enough therein.

1. Introduction and some results

During the proof of Theorem 3 of [1] and by using specific properties of the primes, J. Bourgain established (in Equation (4.39) therein) the estimate

$$(1)\quad \forall \ell>2,\qquad \biggl(\int_0^1\Bigl|\sum_{p\le N}u_p e(p\alpha)\Bigr|^\ell d\alpha\biggr)^{1/\ell}\ll_\ell N^{-1/\ell}\biggl(\frac{N}{\log N}\sum_{p\le N}|u_p|^2\biggr)^{1/2}.$$

This proof was improved by B. Green in [5, Theorem 1.5]. A striking feature of (1) is that it is *not* valid when $\ell=2$ as Parseval formula easily shows. Understanding the transition became then an open question, an answer to which is provided in Corollary 1.2 below. In the paper [3], B. Green & T. Tao reduced the proof to using only sieve properties. They in fact considered a more general setting, that could be encompassed in the framework of *sufficiently sifted sequences* of [11], and the same holds for our estimates. We prefer however to restrict our attention to the better known case of the prime numbers but will nonetheless present some obvious generalization to the case of primes in some interval in the last section. The methods used (essentially the large sieve and envelopping sequences) remain general enough to warrant an extension to a much more general case. The somewhat reverse situation of smooth numbers has been the subject of the work [6] by A.J. Harper, to which we borrow an idea (see around Equation 17 below).

Explicit values for the constants are provided for three reasons: it avoids us saying that these are independant of the involved parameters; it puts forward that our argument is elementary enough; and finally, it shows that some work is still required to improve on them and to determine what are the optimal ones. We did not work overmuch on these constants.

Here is our first result.

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Theorem 1.1. *Let $\mathcal{X}$ be a δ-well spaced subset of $\mathbb{R}/\mathbb{Z}$. Assume $N \ge 10^{11}$ and let $h > 0$. We have*

$$\sum_{x\in\mathcal{X}}\biggl|\sum_{p\le N} u_p e(xp)\biggr|^{2+h} \le 10^7\bigl((1+\tfrac{3}{2\log N})^h + 1/h\bigr)\biggl(\frac{N+\delta^{-1}}{\log N}\sum_{p\le N}|u_p|^2\biggr)^{1+h/2}.$$

Let us recall that a set $\mathcal{X} \subset \mathbb{R}/\mathbb{Z}$ is said to be δ-well spaced when $\min_{x\neq x'\in\mathcal{X}}|x-x'|_{\mathbb{Z}} \ge \delta$, where $|y|_{\mathbb{Z}} = \min_{k\in\mathbb{Z}}|y-k|$ denotes in a rather unusual manner the distance to the nearest integer. On taking $\mathcal{X} = \{\beta + k/N, 0 \le k \le N-1\}$ and integrating over β in $[0, 1/N]$, we get the corollary we advertised above.

Corollary 1.2. *Assume $N \ge 10^{11}$ and let $h > 0$. We have*

$$\int_0^1\biggl|\sum_{p\le N} u_p e(p\alpha)\biggr|^{2+h} d\alpha \le 10^7\frac{(1+\frac{3}{2\log N})^h + 1/h}{N}\biggl(\frac{2N}{\log N}\sum_{p\le N}|u_p|^2\biggr)^{1+h/2}.$$

This result offers an optimal (save for the implied constants) transition to the case $h = 0$. Indeed, on selecting $h = 1/\log N$, this corollary implies that, when $|u_p| \le 1$, we have te optimal

$$\int_0^1\biggl|\sum_{p\le N} u_p e(p\alpha)\biggr|^{2+h} d\alpha \ll \sum_{p\le N}|u_p|^2.$$

In the same line, but maybe more strickingly, our result implies a *uniform* Hardy-Littlewood majorant property, in the sense for instance of the paper [4] of B. Green & I. Ruzsa.

Theorem 1.3. *Assume $N \ge 10^{11}$ and let $\ell \ge 2$. We have*

$$\biggl(\int_0^1\biggl|\sum_{p\le N} u_p e(p\alpha)\biggr|^{\ell} d\alpha\biggr)^{1/\ell} \le 10^7\biggl(\int_0^1\biggl|\sum_{p\le N} e(p\alpha)\biggr|^{\ell} d\alpha\biggr)^{1/\ell}$$

as soon as $\sum_{p\le N}|u_p|^2 \le \sum_{p\le N} 1$.

In other words, the constant $C(\ell)$ in Theorem 1.5 of [5] is *uniformly* bounded, and in fact by 10^7. Guessing and getting the optimal constant is open, whether under the L^∞-condition $|u_p| \le 1$ or under the L^2-condition we use.

At the heart of B. Green & T. Tao's result lies an estimate close to our next theorem. The main difference (aside from the fact that we state it in dual format, Theorem 4.1 below being its true analogue) is that the dependence in $|\mathcal{X}|$ is not $|\mathcal{X}|^\varepsilon$, but $\log(2|\mathcal{X}|)$. This positive ε came from some average of restricted divisors. A more precise proof led to $\exp\frac{c\log|\mathcal{X}|}{\log\log|\mathcal{X}|}$ for some constant $c > 0$, but to nothing better.

Theorem 1.4. *Let $\mathcal{X}$ be a δ-well spaced subset of $\mathbb{R}/\mathbb{Z}$ and $N \ge 10^{11}$. Let $(u_p)_{p\le N}$ be a sequence of complex numbers. We have*

$$\sum_{x\in\mathcal{X}}\biggl|\sum_{1\le p\le N} u_p e(xp)\biggr|^2 \le 10^6\frac{N+\delta^{-1}}{\log N}\sum_{p\le N}|u_p|^2\log(2|\mathcal{X}|).$$

This result should be compared with Theorem 5.3 of [10] where the summation is over the points a/q for $q \le Q_0 \le \sqrt{N}$:

$$\sum_{q \le Q_0} \sum_{a \bmod^* q} \left| \sum_n u_n e(na/q) \right|^2 \le 7 \frac{N \log Q_0}{\log N} \sum_n |u_n|^2 \tag{2}$$

provided u_n vanishes as soon as n has a prime factor less than $\sqrt{N}$. A way to compare both results is to state the maximal estimate we can now get.

Corollary 1.5. *Let $N \ge 10^{11}$ and $Q_0 \in [2, \sqrt{N}]$. Let $(u_p)_{p \le N}$ be a sequence of complex numbers. We have*

$$\sum_{q \le Q_0} \sum_{a \bmod^* q} \max_{|\alpha - \frac{a}{q}| \le \frac{1}{qQ_0}} \left| \sum_{1 \le p \le N} u_p e(\alpha p) \right|^2 \le 10^7 \frac{N \log Q_0}{\log N} \sum_{p \le N} |u_p|^2.$$

From a methological viewpoint, our main innovation lies in the introduction of a preliminary sieving in the envelopping sieve. This completely eliminates any restricted divisors estimate. It has also the effect of reducing the proof sizeably.

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2. Some averages of multiplicative functions

We define $P(z_0) = \prod_{p < z_0} p$ and

$$G_d(y; z_0) = \sum_{\substack{\ell \le y, \\ (\ell, dP(z_0)) = 1}} \frac{\mu^2(\ell)}{\varphi(\ell)}, \quad G(y; z_0) = G_1(y; z_0). \tag{3}$$

We bound it from below in an effective manner, via some steps based of Rankin's trick. We start with some estimates due to J.B. Rosser & L. Schoenfeld in [12, Theorem 1, Corollary 2, Theorem 6-8].

Lemma 2.1. *We have*

$$\sum_{p \le x} \frac{\log p}{p} < \log x + E + \frac{1}{2 \log x} \quad \textit{when } x \ge 319,$$

$$\sum_{p \le x} \frac{\log p}{p} > \log x + E - \frac{1}{2 \log x} \quad \textit{when } x > 1,$$

where $E = -1.33258\,22757\,33220\,87 \cdots$. Also

$$\prod_{p \le x} \frac{p}{p-1} \ge e^{\gamma} (\log x) \left(1 - \frac{1}{2 \log^2 x} \right) \quad \textit{when } x > 1,$$

$$\prod_{p \le x} \frac{p-1}{p} > \frac{e^{-\gamma}}{\log x} \left(1 - \frac{1}{2 \log^2 x} \right) \quad \textit{when } x \ge 285,$$

and

$$\prod_{p\le x}\frac{p}{p-1}\le e^{\gamma}(\log x)\biggl(1+\frac{1}{2\log^2 x}\biggr)\quad \textit{when } x\ge 286,$$
$$\prod_{p\le x}\frac{p-1}{p}\le \frac{e^{-\gamma}}{\log x}\biggl(1-\frac{1}{2\log^2 x}\biggr)\quad \textit{when } x> 1.$$

Futhermore $\pi(x)=\sum_{p\le x}1\le\frac{x}{\log x}(1+\frac{3}{2\log x})$ *and* $\pi(x)\le\frac{5x}{4\log x}$*, both valid when* $x\ge 114$*. Finally,* $\pi(x)\ge x/(\log x)$ *when* $x\ge 17$*.*

We next recall part of [2, Lemma 4] by H. Daboussi & J. Rivat.

Lemma 2.2. *Let* $z\ge 2$ *and* f *be a multiplicative function. Set*

$$S=\sum_{p<z}\frac{f(p)}{1+f(p)}\log p.$$

We assume that $\log y\ge S>0$*. Then*

$$\sum_{\substack{n\le y,\\ k(n)|P(z)}}\mu^2(n)f(n)\ge\prod_{p\le z}(1+f(p))\biggl(1-\exp-\frac{\log z}{\log y}K\biggl(\frac{\log y}{S}\biggr)\biggr)$$

where $k(n)=\prod_{p|n}p$ *and* $K(t)=\log t-1+1/t$*.*

Lemma 2.3. *We have, when* $z\ge 1$

$$\sum_{d\le z}\frac{\mu^2(d)}{\varphi(d)}=\log z+c_0+\mathcal{O}^*(3.95/\sqrt{z})$$

where $c_0=\gamma+\sum_{p\ge 2}\frac{\log p}{p(p-1)}$*. Also* $\sum_{d\le z}\frac{\mu^2(d)}{\varphi(d)}\le\log z+1.4709$*.*

Proof. The first estimate is taken from [8, Theorem 1.2] while the second one is [9, Lemma 3.5, (1)] □

Lemma 2.4. *When* $320\le z_0\le z$*, we have* $G(z;z_0)\ge\dfrac{3}{40}\dfrac{\log z}{\log z_0}$*.*

When $2\le z_0\le z$ *and* $z\ge 320$*, we have* $G(z;z_0)\ge\dfrac{1}{111}\dfrac{\log z}{\log z_0}$*.*

Proof. We define the auxiliary function

$$\tilde G(y,z;z_0)=\sum_{\substack{d\le y,\\ k(n)|P(z)/P(z_0)}}\frac{\mu^2(d)}{\varphi(d)}.$$

We readily find that

$$\tilde G(\infty,z;z_0)=\prod_{z_0\le p<z}\biggl(1-\frac{1}{p}\biggr)^{-1}.$$

By Lemma 2.1, we obtain a lower estimate for this product:

$$\tilde G(\infty,z;z_0)\ge\frac{\log z}{\log z_0}\biggl(1-\frac{1}{2(\log z_0)^2}\biggr)^2\ge 0.97\frac{\log z}{\log z_0}.$$

In Lemma 2.2 with $f(n) = \mu^2(n)\mathbb{1}_{k(n)\ge z_0}/\varphi(n)$, we find that

$$S = \sum_{z_0 \le p < z} \frac{\log p}{p} = \log \frac{z}{z_0} + \mathcal{O}^*\left(\frac{2}{\log z_0}\right)$$

where we appealed to Lemma 2.1 for the approximate estimate. Notice that $S \le \log z$ since $-\log z_0 + 2/\log z_0 \le 0$. We thus get

$$\tilde{G}(y, z; z_0) \ge \tilde{G}(\infty, z; z_0)\left(1 - \exp\left(-\frac{\log z}{\log y} K(3)\right)\right)$$

as soon as $\log y \ge 3 \log z$ since then, $K((\log y)/S) \ge K(2) = 0.4319\cdots$, and thus, in particular,

$$\tilde{G}(y, z; z_0) \ge \tilde{G}(\infty, z; z_0)(1 - e^{-0.4319}) \ge 0.34 \frac{\log z}{\log z_0}.$$

We complete this estimate with

$$\tilde{G}(z^3, z; z_0) - G(z, z; z_0) = \sum_{\substack{z < d \le z^2,\\ z_0 \le k(n) < z}} \frac{\mu^2(d)}{\varphi(d)} \le \sum_{z < d \le z^2} \frac{\mu^2(d)}{\varphi(d)} \le 3 + \frac{8}{\sqrt{z}} \le 3.45$$

by Lemma 2.3. Whence

$$G(z, z; z_0) \ge 0.34 \frac{\log z}{\log z_0} - 3.45.$$

Since also $G(z, z; z_0) \ge 1$ (by considering the contribution of $d = 1$), we find that

$$G(z, z; z_0) \ge \min_{c \ge 1} \max\left(0.34 - \frac{3.45}{c}, \frac{1}{c}\right) \frac{\log z}{\log z_0} \ge \frac{3}{40} \frac{\log z}{\log z_0}.$$

This yields the first estimate of the lemma. The second one is simply infered from this through

$$G(z; z_0) \ge G(z; 320) \ge \frac{3 \log 2}{40 \log 320} \frac{\log z}{\log z}.$$

□

Lemma 2.5. *Let $h > 0$. We have*

$$\sum_{d \le D} \frac{\mu^2(d)}{\varphi(d)^{1+h}} \ge \frac{1 - D^{-h}}{h}.$$

Proof. We first notice that

$$\begin{aligned}\sum_{d \le D} \frac{\mu^2(d)}{\varphi(d)^{1+h}} &\ge \sum_{d \le D} \frac{\mu^2(d)}{d^{1+h}} \prod_{p|d} \left(\sum_{k \ge 0} \frac{1}{p^k}\right)^{1+h} \\ &\ge \sum_{d \le D} \frac{\mu^2(d)}{d^{1+h}} \prod_{p|d} \left(\sum_{k \ge 0} \frac{1}{p^{k(1+h)}}\right) = \sum_{\substack{q \ge 1,\\ k(q) \le D}} \frac{1}{q^{1+h}} \ge \sum_{q \le D} \frac{1}{q^{1+h}}.\end{aligned}$$

Concerning this last quantity, we write

$$\begin{aligned}\sum_{q\le D}\frac{1}{q^{1+h}} &= \int_1^D \sum_{q\le t} 1\frac{(1+h)dt}{t^{2+h}} + \frac{[D]}{D^{1+h}}\\ &\ge \int_1^D (t-1)\frac{(1+h)dt}{t^{2+h}} + \frac{D-1}{D^{1+h}} = \frac{h+1}{h}\left(1-\frac{1}{D^h}\right) + \frac{D-1}{D^{1+h}}\\ &\ge \frac{1-D^{-h}}{h} + 1 - \frac{1}{D^{1+h}} \ge \frac{1-D^{-h}}{h}\end{aligned}$$

as required. □

3. An enveloping sieve

We fix two real parameters $z_0 \le z$ and first consider the sole case of prime numbers. It is easy to reproduce the analysis of [11, Section 3] as far as exact formulae are concerned, but one gets easily sidetracked towards slightly different formulae. The reader may for instance compare [9, Lemma 4.2] and [11, (4.1.14)]. Similar material is also the topic of [10, Chapter 12]. So we present a path leading to [11, (4.1.14)] in our special case.

Following the notation of Section 2, we set

$$\beta_{z_0,z}(n) = \Big(\sum_{d|n}\lambda_d\Big)^2, \qquad \lambda_d = \mathbb{1}_{(d,P(z_0))=1}\frac{\mu(d)dG_d(z/d;z_0)}{\varphi(d)G(z;z_0)}. \tag{4}$$

Section 3 of [11] corresponds to $z_0 = 1$. We introduce

$$\lambda^\sharp_\ell = \mu^2(\ell)\mathbb{1}_{(\ell,P(z_0))=1}\frac{\ell}{\varphi(\ell)}\frac{M_{z_0}(z/\ell)}{G(z;z_0)}, \quad M_{z_0}(y) = \sum_{\substack{m\le y,\\ (m,P(z_0))=1}}\mu(m). \tag{5}$$

We readily check that

$$\lambda_d = \mu(d)\sum_{d|\ell}\lambda^\sharp_\ell, \quad \lambda^\sharp_\ell = \mu(\ell)\sum_{\ell|d}\lambda_d, \tag{6}$$

and a further simple verification leads to

$$\forall n\ge 2,\quad \sum_{d|n}\lambda_d = \sum_{\ell\,/\,(\ell,n)=1}\lambda^\sharp_\ell. \tag{7}$$

We explained in [10, Chapter 11] why working with $(\lambda^\sharp_\ell)$ leads to more regular formulae than working with $(\lambda_d)_d$. We write

$$\beta_{z_0,z}(n) = \Big(\sum_{d|n}\lambda_d\Big)^2 = \Big(\sum_{(n,\ell)=1}\lambda^\sharp_\ell\Big)^2 = \sum_{\ell_1,\ell_2}\lambda^\sharp_{\ell_1}\lambda^\sharp_{\ell_2}\mathbb{1}_{(n,[\ell_1,\ell_2])=1}.$$

Let us now notice the exact formula

$$\forall n\ge 2,\quad \mathbb{1}_{(n,r)=1} = \frac{\varphi(r)}{r}\sum_{q|r}\frac{\mu(q)c_q(n)}{\varphi(q)}. \tag{8}$$

On using this expression, we reach

$$\beta_{z_0,z}(n) = \sum_{\ell_1,\ell_2} \lambda^\sharp_{\ell_1}\lambda^\sharp_{\ell_2} \frac{\varphi([\ell_1,\ell_2])}{[\ell_1,\ell_2]} \sum_{q|[\ell_1,\ell_2]} \frac{\mu(q)c_q(n)}{\varphi(q)}$$

$$= \sum_q \sum_{q|[\ell_1,\ell_2]} \lambda^\sharp_{\ell_1}\lambda^\sharp_{\ell_2} \frac{\varphi([\ell_1,\ell_2])}{[\ell_1,\ell_2]} \frac{\mu(q)c_q(n)}{\varphi(q)} = \sum_{\substack{q\le z^2,\\ (q,P(z_0))=1}} \frac{\mu(q)}{\varphi(q)} w^\sharp_q c_q(n). \tag{9}$$

The formula for $w^\sharp_q$ that emerges from the above reads:

$$w^\sharp_q = \sum_{q|[\ell_1,\ell_2]} \lambda^\sharp_{\ell_1}\lambda^\sharp_{\ell_2} \frac{\varphi([\ell_1,\ell_2])}{[\ell_1,\ell_2]}. \tag{10}$$

We plug (5) in this to get

$$G(z;z_0)^2 w^\sharp_q = \sum_{q|[\ell_1,\ell_2]} \frac{\ell_1}{\varphi(\ell_1)} \frac{\ell_2}{\varphi(\ell_2)} \frac{\varphi([\ell_1,\ell_2])}{[\ell_1,\ell_2]} M_{z_0}(z/\ell_1) M_{z_0}(z/\ell_2)$$
$$= \sum_{q|[\ell_1,\ell_2]} \frac{(\ell_1,\ell_2)}{\varphi((\ell_1,\ell_2))} M_{z_0}(z/\ell_1) M_{z_0}(z/\ell_2).$$

We notice that $\frac{r}{\varphi(r)} = \sum_{\delta|r} \frac{1}{\varphi(\delta)}$, whence

$$G(z;z_0)^2 w^\sharp_q = \sum_{\substack{\delta\le z,\\ (\delta,P(z_0))=1}} \frac{\mu^2(\delta)}{\varphi(\delta)} \sum_{\substack{\delta|\ell_1,\delta|\ell_2,\\ q|[\ell_1,\ell_2]}} M_{z_0}(z/\ell_1) M_{z_0}(z/\ell_2) \tag{11}$$

where we are now just a moment's away from the expression

$$G(z;z_0)^2 w^\sharp_q = \sum_{\substack{\delta\le z,\\ (\delta,P(z_0))=1}} \frac{\mu^2(\delta)}{\varphi(\delta)} \rho_{z_0,z}(q,\delta) \tag{12}$$

with

$$\rho_{z_0,z}(q,\delta) = \sum_{\substack{q/(\delta,q)=q_1q_2q_3,\\ (q_1,q_2)=(q_2,q_3)=(q_1,q_3)=1,\\ \max(q_1q_3\delta,q_2q_3\delta)\le z,\\ (\delta,P(z_0))=1}} (-1)^{\omega(q_3)}. \tag{13}$$

Reaching this expression for $\rho_{z_0,z}(q,\delta)$ is done exactly as in [10, Lemma 12.1] or as next to [11, (4.1.14)].

Lemma 3.1. *When* $2 \le z_0 \le z$ *and* $z \ge 320$*, we have*

$$\left| \frac{w^\sharp_q}{\varphi(q)} \right| \le 11500\, \frac{\log z_0}{\sqrt{q}\log z}.$$

Proof. We deduce from (13) the estimate $|\rho_{z_0,z}(q,\delta)| \le 3^{\omega(q)}$, and thus

$$|G(z;z_0) w^\sharp_q| \le 3^{\omega(q)}. \tag{14}$$

We use Lemma 2.4 to get

$$\left| \frac{w^\sharp_q}{\varphi(q)} \right| \le \prod_p \max\left(\frac{3\sqrt{p}}{p-1}, 1 \right) \frac{1}{G(z;z_0)\sqrt{q}} \le \frac{11500 \log z_0}{\sqrt{q}\log z}.$$

□

4. The fundamental estimate

Theorem 4.1. *Let $N \geq 2 \cdot 10^{10}$. Let B be a δ-well spaced subset of $\mathbb{R}/\mathbb{Z}$. For any function f on B, we have*

$$\sum_{1 \leq p \leq N} \Big| \sum_{b \in B} f(b) e(bp) \Big|^2 \leq 330\,000 (N + \delta^{-1}) \|f\|_2^2 \frac{\log(2\|f\|_1^2/\|f\|_2^2)}{\log N}.$$

Proof. Let $z = N^{1/4} \geq 320$ and

$$z_0 = \left(2 \frac{\|f\|_1^2}{\|f\|_2^2} \right)^2 \geq 2.$$

We have $z_0 \leq z$ when $\|f\|_1^2/\|f\|_2^2 \leq N^{1/8}/2$. When this condition is not met, we use the dual of the usual large sieve inequality to infer that

$$\begin{aligned} \sum_{1 \leq p \leq N} \Big| \sum_{b \in B} f(b) e(bp) \Big|^2 &\leq (N + \delta^{-1}) \|f\|_2^2 \\ &\leq (N + \delta^{-1}) \|f\|_2^2 \frac{\log(2\|f\|_1^2/\|f\|_2^2)}{\log(N^{1/8}/2)}. \end{aligned}$$

Some numerical analysis shows that this establishes our inequality in this case. Henceforth, we assume that $z_0 \leq z$. We first notice that

$$\begin{aligned} \sum_{1 \leq p \leq z} \Big| \sum_{b \in B} f(b) e(bp) \Big|^2 &\leq z \|f\|_1^2 \leq N^{3/8} \|f\|_2^2/2 \\ &\leq N \frac{\|f\|_2^2 \log(2\|f\|_1^2/\|f\|_2^2)}{\log N} \frac{\log N}{N^{5/8} \log 2} \\ &\leq \frac{1}{53000} N \frac{\|f\|_2^2 \log(2\|f\|_1^2/\|f\|_2^2)}{\log N}. \end{aligned}$$

Let us now call W the quantity to be studied with $z < p \leq N$. We bound above the characteristic function of those primes by our enveloping sieve and further majorize the characteristic function of the interval $[1, N]$ by a function ψ of non-negative Fourier transform supported by $[-\delta_1, \delta_1]$ where $\delta_1 = \min(\delta, 1/(2z^4))$. This leads to

$$W \leq \sum_{\substack{q \leq z^2, \\ (q, P(z_0)) = 1}} \frac{w_q^\sharp}{\varphi(q)} \sum_{a \,mod^*\, q} \sum_{b_1, b_2} f(b_1) \overline{f(b_2)} \sum_{n \in \mathbb{Z}} e((b_1 - b_2)n) e(an/q) \psi(n).$$

We split this quantity according to whether $q < z_0$ or not:

$$W = W(q < z_0) + W(q \geq z_0).$$

When $q \geq z_0$, Poisson summation formula tells us that the inner sum is also $\sum_{m \in \mathbb{Z}} \hat{\psi}(b_1 - b_2 - (a/q) + m)$. The sum over b_1, b_2 and n is thus

$$\leq (N + \delta_1^{-1}) \sum_{b_1, b_2} f(b_1) \overline{f(b_2)} \# \big\{ (a/q) / \|b_1 - b_2 + a/q\| < \delta_1 \big\}.$$

Given (b_1, b_2), at most one a/q may work, since $1/z^4 > 2\delta_1$. By bounding above $w_q^\sharp$ by Lemma 3.1, we see that

$$\begin{aligned} W(q \geq z_0) &\leq 11500(N+\delta_1^{-1})\frac{\|f\|_1^2 \log z_0}{\sqrt{z_0}\log z} \\ &\leq \frac{11500}{\sqrt{2}}(N+\delta_1^{-1})\frac{\|f\|_2^2 \log z_0}{\log z}. \end{aligned}$$

When $q < z_0$, only $q = 1$ remains. This yields

$$W(q < z_0) \leq (N+\delta_1^{-1})\frac{111\|f\|_2^2 \log z_0}{\log z}.$$

We check that $(N+\delta_1^{-1}) \leq \frac{N+4N}{N}(N+\delta^{-1})$. We finally get

$$\begin{aligned} \sum_{1\leq p\leq N}\left|\sum_{b\in B} f(b)e(bn)\right|^2 \leq \left(\frac{1}{53000} + 5\times 2\times 4\times\left(\frac{11500}{\sqrt{2}}+111\right)\right) \\ \times (N+\delta^{-1})\|f\|_2^2\frac{\log(2\|f\|_1^2/\|f\|_2^2)}{\log N}. \end{aligned}$$

The proof of the Theorem follows readily. □

5. On moments. Proof of Theorem 1.1

Lemma 5.1. *Assume $y/\log y \leq t$ with $y \geq 2$ and $t \geq 10^7$. Then $y \leq 2t\log t$.*

Proof. Our property is trivial when $y \leq 10^7$. Notice that the function $f : y \mapsto y/\log y$ is non-increasing when $y \geq e$. We find that $f(2t\log t) \geq t \geq f(y)$, whence $2t\log t \geq y$ as sought. □

Proof of Theorem 1.1. For typographical simplification, we define

$$B = \left(\frac{N+\delta^{-1}}{\log N}\sum_{p\leq N}|u_p|^2\right)^{1/2}. \tag{15}$$

We also set $\ell = 2 + h$. For any $\xi > 0$, we examine the set

$$\mathcal{X}_\xi = \left\{x \in \mathcal{X} / \left|\sum_{p\leq N} u_p e(xp)\right| \geq \xi B\right\}. \tag{16}$$

By Lemma 2.1, we see that $\xi \leq c_1 = \min(5/4, 1 + \frac{3}{2\log N})$ or else, the set $\mathcal{X}_\xi$ is empty. We consider (as in [6], bottom of page 1141)

$$\Gamma(\xi) = \sum_{x\in\mathcal{X}_\xi}\left|\sum_{p\leq N} u_p e(xp)\right|. \tag{17}$$

We write is as $\Gamma(\xi) = \sum_{x\in\mathcal{X}_\xi} c(x)\sum_{p\leq N} u_p e(xp)$ for some $c(x)$ of modulus 1 and develop it in

$$\Gamma(\xi) = \sum_{p\leq N} u_p \sum_{x\in\mathcal{X}_\xi} c(x)e(xp).$$

We apply Cauchy's inequality to this expression to get

$$\begin{aligned}\Gamma(\xi)^2 &\le \sum_{p\le N}|u_p|^2\sum_{p\le N}\biggl|\sum_{x\in\mathcal{X}_\xi}c(x)e(xp)\biggr|^2\\ &\le 330\,000\,B^2|\mathcal{X}_\xi|\log(2|\mathcal{X}_\xi|)\end{aligned}$$

by Theorem 4.1. It follows from this upper bound that

$$\xi^2|\mathcal{X}_\xi|^2B^2\le\Gamma(\xi)^2\le 330\,000\,B^2|\mathcal{X}_\xi|\log(2|\mathcal{X}_\xi|)$$

whence

$$2|\mathcal{X}_\xi|/\log(2|\mathcal{X}_\xi|)\le 660\,000\,/\xi^2.$$

which we convert with Lemma 5.1 in $2|\mathcal{X}_\xi|\le 10^6\xi^{-2}\log(10^6/\xi^2)$.

We can now turn towards the proof of the stated inequality and select $\xi_j=c_1/c^j$ for some $c>1$ that we will select later. We get

$$\begin{aligned}\sum_{x\in\mathcal{X}}\biggl|\sum_{p\le N}u_pe(xp)\biggr|^\ell/B^\ell &\le \sum_{j\ge0}\frac{c_1^\ell}{c^{\ell j}}(|\mathcal{X}_{\xi_j}|-|\mathcal{X}_{\xi_{j+1}}|)\\ &\le\frac{10^6}{2}\sum_{j\ge0}\frac{c_1^{\ell-2}(\log(10^6)-2\log c_1+2j\log c)}{c^{(\ell-2)j}}.\\ &\le\frac{10^6}{2}\sum_{j\ge0}\frac{c_1^{\ell-2}(14+2j\log c)}{c^{(\ell-2)j}}.\end{aligned}$$

We note that

$$\frac{10^6}{2}\sum_{j\ge0}\frac{c_1^{\ell-2}\times14}{c^{(\ell-2)j}}=\frac{10^6\times14\times c_1^{\ell-2}}{1-c^{2-\ell}}$$

and

$$\begin{aligned}\frac{10^6}{2}\sum_{j\ge1}\frac{c_1^{\ell-2}j\,2\log c}{c^{(\ell-2)j}} &\le 10^6\frac{(\log c)}{c^{\ell-2}}c_1^{\ell-2}\sum_{j\ge1}\frac{j}{c^{(\ell-2)(j-1)}}\\ &\le\frac{10^6\times(c_1/c)^{\ell-2}\log c}{(1-c^{2-\ell})^2}.\end{aligned}$$

When $\ell\ge3$, we select $c=2$, getting

$$\sum_{x\in\mathcal{X}}\biggl|\sum_{p\le N}u_pe(xp)\biggr|^{2+h}\le(14\cdot10^6(1+\tfrac{3}{2\log N})^h+10^6)\biggl(\frac{N+\delta^{-1}}{\log N}\sum_{p\le N}|u_p|^2\biggr)^{1+h/2}.$$

When $\ell\in(2,3)$, we select $c=\exp(1/h)$, getting

$$\sum_{x\in\mathcal{X}}\biggl|\sum_{p\le N}u_pe(xp)\biggr|^{2+h}\le\biggl(10^7(1+\tfrac{3}{2\log N})^h+\frac{10^6}{h}\biggr)\biggl(\frac{N+\delta^{-1}}{\log N}\sum_{p\le N}|u_p|^2\biggr)^{1+h/2}.$$

Our theorem follows readily. □

6. Large sieve bound on small sets. Proof of Theorem 1.4

Proof of Theorem 1.4. We write

$$W = \sum_{x\in\mathcal{X}}\Big|\sum_{1\le p\le N} u_p e(xp)\Big|^2 = \sum_{x\in\mathcal{X}}\sum_{p\le N} u_p\overline{S(x)}$$

where $S(x) = \sum_{1\le p\le N} u_p e(xp)$. On using the Cauchy-Schwarz inequality, we get

$$W^2 \le \sum_{p\le N}|u_p|^2\sum_{p\le N}\Big|\sum_{x\in\mathcal{X}}\overline{S(x)}e(xp)\Big|^2.$$

We invoque Theorem 4.1 and notice that

$$\Big(\sum_{x\in\mathcal{X}}|\overline{S(x)}|\Big)^2 \le |\mathcal{X}|\sum_{x\in\mathcal{X}}|\overline{S(x)}|^2.$$

This leads to

$$W^2 \le 330\,000\,\frac{N+\delta^{-1}}{\log N}\sum_{p\le N}|u_p|^2\sum_{x\in\mathcal{X}}|S(x)|^2\log(2|\mathcal{X}|).$$

On simplifying by $\sum_{x\in\mathcal{X}}|S(x)|^2$ (after discussing whether it vanishes or not), we get our estimate. □

7. Optimality and uniform boundedness. Proof of Theorem 1.3

We assume that $N\ge 10^{11}$ and set

$$S(\alpha) = \sum_{p\le N} e(p\alpha). \tag{18}$$

The argument employed at the bottom of page 1626 of [5] by B. Green is not enough for us. Instead, we got our inspiration from the argument developped by R.C. Vaughan in [13]. It runs as follows. We first notice that

$$\Big|\sum_{a\,mod^*q} S\Big(\frac{a}{q}+\beta\Big)\Big| \le \Big(\sum_{a\,mod^*q}\Big|S\Big(\frac{a}{q}+\beta\Big)\Big|^\ell\Big)^{1/\ell}\Big(\sum_{a\,mod^*q}1\Big)^{(\ell-1)/\ell}.$$

A direct inspection shows that

$$\sum_{a\,mod^*q} S\Big(\frac{a}{q}+\beta\Big) = \mu(q)S(\beta) + T(q,\beta)$$

where $T(q,\beta) = \sum_{p|q} e(p\beta)(c_q(p)-\mu(q))$. The bound $|c_q(n)|\le\varphi((n,q))$ for the Ramanujan sum $c_q(n)$ (use for instance the von Sterneck expression for $c_q(n)$) gives us

$$|T(q,\beta)| \le \sum_{p|q}(p-1+1) \le q. \tag{19}$$

The last inequality follows from the trivial property that a sum of positive integers is certainly not more than its product. We next get a lower bound for $S(\beta)$ by writing

$$1 - e(\beta p) = 2i\pi\int_0^{\beta p} e(t)dt$$

whence

$$|S(\beta)| \geq S(0) - 2\pi\beta N S(0) \geq (1-2\pi\beta N)S(0) \geq (1-2\pi\beta N)\frac{N}{\log N} \tag{20}$$

by Lemma 2.1. When $|\beta| \leq 1/(7N)$, this leads to $|S(\beta)| \geq c_2 N/\log N$ with $c_2 = 1 - 2\pi/7$, and, when q is squarefree and not more than $\sqrt{N}$, to

$$\bigl|\mu(q)S(\beta) + T(q,\beta)\bigr| \geq c_2\frac{N}{\log N} - \sqrt{N} \geq \frac{N}{10\log N}. \tag{21}$$

We thus get, when $|\beta| \leq 1/(7N)$,

$$\sum_{a\, mod^* q}\Bigl|S\Bigl(\frac{a}{q}+\beta\Bigr)\Bigr|^\ell \geq \frac{\mu^2(q)}{\varphi(q)^{\ell-1}}|S(\beta)+T(q,\beta)|^\ell \geq \frac{\mu^2(q)}{\varphi(q)^{\ell-1}}\Bigl(\frac{N}{10\log N}\Bigr)^\ell.$$

Thus

$$\begin{aligned}\int_0^1 |S(\alpha)|^\ell d\alpha &\geq \sum_{q\leq\sqrt{N}}\sum_{a\, mod^* q}\mu^2(q)\int_{\frac{a}{q}-\frac{1}{7N}}^{\frac{a}{q}+\frac{1}{7N}}\Bigl|S\Bigl(\frac{a}{q}+\beta\Bigr)\Bigr|^\ell d\beta\\ &\geq \frac{2}{7N}\sum_{q\leq\sqrt{N}}\frac{\mu^2(q)}{\varphi(q)^{\ell-1}}\Bigl(\frac{N}{10\log N}\Bigr)^\ell.\end{aligned}$$

By Lemma 2.5, we conclude that

$$\int_0^1 |S(\alpha)|^\ell d\alpha \geq \frac{1-\sqrt{N}^{2-\ell}}{\ell-2}\frac{2}{7N}\Bigl(\frac{N}{10\log N}\Bigr)^\ell.$$

We thus find that

$$\int_0^1\Bigl|\sum_{p\leq N}u_p e(p\alpha)\Bigr|^\ell d\alpha \leq K(\ell)\Bigl(\frac{\log N}{N}\sum_{p\leq N}|u_p|^2\Bigr)^{\ell/2}\int_0^1\Bigl|\sum_{p\leq N}e(p\alpha)\Bigr|^\ell d\alpha \tag{22}$$

where $\ell = 2+h$ and

$$\begin{aligned}K(2+h) &= 10^7\frac{(1+\frac{3}{2\log N})^h + 1/h}{N}\Bigl(\frac{2N^2}{(\log N)^2}\Bigr)^{\ell/2}\frac{h}{1-\sqrt{N}^{-h}}\frac{7N}{2}\Bigl(\frac{N}{10\log N}\Bigr)^{-\ell}\\ &= 10^7\Bigl((1+\tfrac{3}{2\log N})^h + 1/h\Bigr)2^{\ell/2}\frac{h}{1-\sqrt{N}^{-h}}\frac{7}{2}10^\ell.\end{aligned}$$

When $h \geq 1$, this is readily bounded above by $10^8\cdot 20^\ell$. Else, it is bounded above by

$$\frac{3\cdot 10^{12}}{h}\frac{h}{1-\sqrt{N}^{-h}} = \frac{3\cdot 10^{12}}{1-\sqrt{N}^{-h}}.$$

This is bounded above by 10^{13} when $h \geq 1/\log N$. When $0 \leq h \leq 1/\log N$, we use

$$\begin{aligned}\int_0^1\Bigl|\sum_{p\leq N}u_p e(p\alpha)\Bigr|^\ell d\alpha &\leq \Bigl(\pi(N)\sum_{p\leq N}|u_p|^2\Bigr)^{h/2}\int_0^1\Bigl|\sum_{p\leq N}u_p e(p\alpha)\Bigr|^2 d\alpha\\ &\leq \sqrt{5/4}\Bigl(\frac{\log N}{N}\sum_{p\leq N}|u_p|^2\Bigr)^{\ell/2}\Bigl(\frac{N}{\log N}\Bigr)^\ell N^{-1}\log N\end{aligned}$$

which leads to (22) with

$$\begin{aligned}K(2+h) &= \frac{\sqrt{5/4}}{N}\left(\frac{N}{\log N}\right)^{\ell}\frac{h\log N}{1-\sqrt{N}^{-h}}\frac{7N}{2}\left(\frac{N}{10\log N}\right)^{-\ell}\\ &\le \frac{7\sqrt{5/4}}{2(1-\exp(-1/2))} \le 10^{12}.\end{aligned}$$

Theorem 1.3 follows readily.

8. Small sets large sieve estimates. Proof of Theorem 1.4 and of Corollary 1.5

Proof of Theorem 1.4. Under the hypotheses of Theorem 1.4, we simply write

$$\sum_{x\in\mathcal{X}}\Big|\sum_{1\le p\le N} u_p e(xp)\Big|^2 = \sum_{x\in\mathcal{X}}\sum_{1\le p\le N} u_p\overline{S(x)}e(xp)$$

with $S(x)=\sum_{1\le p\le N}u_pe(xp)$. We apply Cauchy's inequality and infer that

$$\begin{aligned}\Big(\sum_{x\in\mathcal{X}}|S(x)|^2)^2 &\le \sum_{p\le N}|u_p|^2\sum_{p\le N}\Big|\sum_{x\in\mathcal{X}}\overline{S(x)}e(xp)\Big|^2\\ &\le 330\,000(N+\delta^{-1})\|S\|_2^2\frac{\log(2\|S\|_1^2/\|S\|_2^2)}{\log N}\sum_{p\le N}|u_p|^2\end{aligned}$$

by Theorem 4.1. We use $\|S\|_1^2\le|\mathcal{X}|\|S\|_2^2$ and simplify by $\sum_{x\in\mathcal{X}}|S(x)|^2$ on both side (after discussing whether this quantity vanishes or not). The theorem is proved. □

Proof of Corollary 1.5. The split the Farey sequence

$$\begin{aligned}F(Q_0) &= \Big\{\frac{a}{q}, 1\le a\le q\le Q_0, (a,q)=1\Big\}\\ &= \big\{0<x_1<x_2<\ldots<x_K=1\big\}\end{aligned}\tag{23}$$

in $F_1(Q_0)=\{x_{2i},1\le i\le K/2\}$ union $F_2(Q_0)=\{x_{2i+1},1\le 0\le(K-1)/2\}$. We recall that the distance between two consecutive points a/q and a'/q' in $F(Q_0)$ is $1/(qq')$; this is at least as large as $\frac{1}{qQ_0}+\frac{1}{q'Q_0}$ by the known property $q+q'\ge Q_0$. Hence two intervals $[\frac{a_1}{q_1}-\frac{1}{q_1Q_0},\frac{a_1}{q_1}+\frac{1}{q_1Q_0}]$ and $[\frac{a_2}{q_2}-\frac{1}{q_2Q_0},\frac{a_2}{q_2}+\frac{1}{q_2Q_0}]$ with $\frac{a_1}{q_1},\frac{a_2}{q_2}\in F_1(Q_0)$ are separated by at least $1/Q_0^2$. We check this is also true when seen on the unit circle: the largest point of $F(Q_0)$ is 1 and its smallest is $\frac{1}{[Q_0]}$. The same applies to $F_2(Q_0)$. We finally notice that $|F(Q_0)|\le Q_0(Q_0+1)/2\le Q_0^2$.

To prove our corollary, for every $x_{2i}\in F_1(Q_0)$, we select a point $\tilde{x}_{2i}$ such that

$$\Big|\sum_{p\le N}u_pe(p\tilde{x_{2i}})\Big| = \max_{|x-x_{2i}|\le\frac{1}{qQ_0}}\Big|\sum_{p\le N}u_pe(px)\Big|\tag{24}$$

and apply Theorem 1.4 to the set $\tilde{X}_1=\{\tilde{x}_{2i}\}$. We proceed similarly with $F_2(Q_0)$. The last details are left to the readers. □

9. Extension to primes in intervals

We discuss here how our results extend from the case of primes in the initial interval to primes in $[M+1, M+N]$ for some non-negative M. During the proof of Theorem 4.1, we used the property that our sequence has at most $N^{1/4}$ elements below $N^{1/4}$, and that the remaining ones are prime to any integer below $N^{1/4}$. This is certainly still true when looking at intervals.

Theorem 9.1. *Let $N \geq 2 \cdot 10^{10}$. Let B be a δ-well spaced subset of $\mathbb{R}/\mathbb{Z}$. For any function f on B, we have*

$$\sum_{M+1 \leq p \leq M+N} \left| \sum_{b \in B} f(b) e(bp) \right|^2 \leq 330\,000 (N + \delta^{-1}) \|f\|_2^2 \frac{\log(2\|f\|_1^2 / \|f\|_2^2)}{\log N}.$$

When defining c_1 in the proof of Theorem 1.1, we used an upper bound for the number of elements in our set. The version of the Brun-Titchmarsh inequality proved by H. Montgomery & R.C. Vaughan in [7] enables us to use $c_1 = 2$. After some trivial modifications, we reach the following.

Theorem 9.2. *Let $\mathcal{X}$ be a δ-well spaced subset of $\mathbb{R}/\mathbb{Z}$. Assume $N \geq 10^{11}$ and let $h > 0$. We have*

$$\sum_{x \in \mathcal{X}} \left| \sum_{M+1 \leq p \leq M+N} u_p e(xp) \right|^{2+h} \leq 10^8 \big(2^h + 1/h\big) \left(\frac{N + \delta^{-1}}{\log N} \sum_{M+1 \leq p \leq M+N} |u_p|^2 \right)^{1+h/2}.$$

Corollary 9.3. *Assume $N \geq 10^{11}$ and let $h > 0$. We have*

$$\int_0^1 \left| \sum_{M+1 \leq p \leq M+N} u_p e(p\alpha) \right|^{2+h} d\alpha \leq 10^8 \frac{2^h + 1/h}{N} \left(\frac{2N}{\log N} \sum_{M+1 \leq p \leq M+N} |u_p|^2 \right)^{1+h/2}.$$

Modifying the proof of Theorem 1.3 is more delicate as it requires bounding the trigonometric polynomial S from below in (20) to discard the contribution of $T(q, \beta)$. A simple solution is to assume that all elements of our sequence are further larger than $\sqrt{N}$, which is readily granted by assuming that $M \geq \sqrt{N}$.

Theorem 9.4. *There exists a constant $C > 0$ such that the following holds. Assume $N \geq 10^{11}$, $M \geq \sqrt{N}$ and let $\ell \geq 2$. We have*

$$\left(\int_0^1 \left| \sum_{M+1 \leq p \leq M+N} u_p e(p\alpha) \right|^\ell d\alpha \right)^{1/\ell} \leq C \sqrt{\frac{N/\log N}{1+R}} \left(\int_0^1 \left| \sum_{M+1 \leq p \leq M+N} e(p\alpha) \right|^\ell d\alpha \right)^{1/\ell}$$

as soon as $\sum_{M+1 \leq p \leq M+N} |u_p|^2 \leq \sum_{M+1 \leq p \leq M+N} 1 = R$.

So the uniform Hardy-Littlewood majorant property holds for primes in the interval $[M+1, M+N]$ provided the number of such primes is $\gg N/\log N$.

Proof of Theorem 9.4. We set

$$S(\alpha) = \sum_{M+1 \leq p \leq M+N} e(p\alpha). \tag{25}$$

We can assume that $S(0)\ge 1$. On following the proof of Theorem 1.3, we readily reach, when $|\beta|\le 1/(7N)$,

$$\sum_{a\,mod^* q}\Big|S\Big(\frac{a}{q}+\beta\Big)\Big|^\ell \ge \frac{\mu^2(q)}{\varphi(q)^{\ell-1}}S(0)^\ell. \tag{26}$$

This again leads to

$$\int_0^1 |S(\alpha)|^\ell d\alpha \ge \frac{1-\sqrt{N}^{2-\ell}}{\ell-2}\frac{2}{7N}S(0)^\ell.$$

Corollary 9.3 gives us with $\ell = 2+h$

$$\begin{aligned}\int_0^1\Big|\sum_{M+1\le p\le M+N}u_p e(p\alpha)\Big|^\ell d\alpha &\le 10^8\frac{2^h+1/h}{N}\Big(\frac{2N}{\log N}S(0)\Big)^{\ell/2}\\ &\le 10^9\frac{h2^h+1}{1-\sqrt{N}^{-h}}\Big(\frac{2N}{S(0)\log N}\Big)^{\ell/2}\int_0^1|S(\alpha)|^\ell d\alpha.\end{aligned}$$

The factor $(\frac{h2^h+1}{1-\sqrt{N}^{-h}})^{1/\ell}$ is bounded when $h\in[1/\log N,\infty)$. We treat separately the case $h\in[0,1/\log N]$. □

However, Theorem 1.4 and Corollary 1.5 go through with no modifications, and are in this manner closer to (2).

Theorem 9.5. *Let $\mathcal{X}$ be a δ-well spaced subset of $\mathbb{R}/\mathbb{Z}$ and $N\ge 10^{11}$. Let $(u_p)_{M+1\le p\le M+N}$ be a sequence of complex numbers. We have*

$$\sum_{x\in\mathcal{X}}\Big|\sum_{M+1\le p\le M+N}u_p e(xp)\Big|^2 \le 10^6\frac{N+\delta^{-1}}{\log N}\sum_{M+1\le p\le M+N}|u_p|^2\log(2|\mathcal{X}|).$$

Corollary 9.6. *Let $N\ge 10^{11}$ and $Q_0\in[2,\sqrt{N}]$. Let $(u_p)_{M+1\le p\le M+N}$ be a sequence of complex numbers. We have*

$$\sum_{q\le Q_0}\sum_{a\,mod^* q}\max_{|\alpha-\frac{a}{q}|\le\frac{1}{qQ_0}}\Big|\sum_{M+1\le p\le M+N}u_p e(\alpha p)\Big|^2\le 10^7\frac{N\log Q_0}{\log N}\sum_{M+1\le p\le M+N}|u_p|^2.$$

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