ARITHMETICAL TRIGONOMETRIC POLYNOMIALS

Olivier Ramaré

October 11, 2023


Flat ground

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- $2, 3, 5, 7, 11, 13, 17, 19, \cdots$.
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- Not always easy: try with 2023 or with 14351.

RSA (1991–2007) and Credit Cards

A key point to information exchanges :

Factorising is difficult.

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But this is not what I'm doing:)

Sums of primes

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Hittmeir and AKS have robbed banks with cardboard guns!

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The main tool for sums of primes

Before 1936: generating series

- Look at
$$r(N) = \#\{(a,b,c) \in \mathbb{N}^3, N = a + 2b + 3c\}.$$
- N-th coefficient of $\sum_{a\geqslant 0} z^a \sum_{b\geqslant 0} z^{2b} \sum_{c\geqslant 0} z^{3c}$.
- $\sum_{a\geqslant 0} z^a \sum_{b\geqslant 0} z^{2b} \sum_{c\geqslant 0} z^{3c} = \dfrac{1}{(1-z)(1-z^2)(1-z^3)} = G(z)$
- Poles at $z = \pm 1$ and at $z = \exp(\pm 2i\pi/3)$.
- Get N-th coefficient by Cauchy's formula:
  integrate on a circle of radius $r < 1$,
  but *close to $r = 1$ to use the poles of $G(z)$.*

This was the leading idea of (Hardy & Ramanujan, 1916), (Hardy & Ramanujan, 2000), (Hardy & Ramanujan, 1917), (Hardy & Ramanujan, 1918).

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$$\sum_{0\leqslant a\leqslant N} z^a \sum_{0\leqslant b\leqslant N} z^{2b} \sum_{0\leqslant c\leqslant N} z^{3c}.$$

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p prime

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This is what we use to prove the three prime theorem.

Initiation

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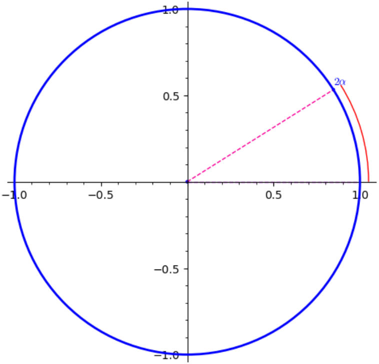
$\alpha = 2\pi/\sqrt{500} = 0.2809\cdots$ for elegant drawings!



The sequence $(p\alpha)$ for $p = 2$

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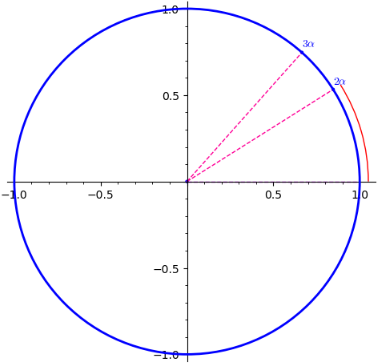

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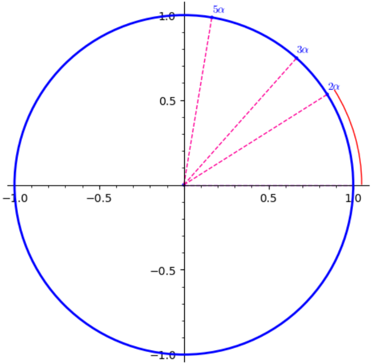

The sequence $(p\alpha)$ for $p = 2, 3, 5$

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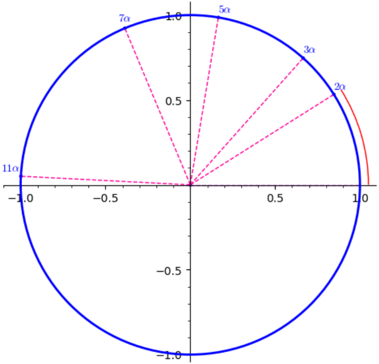

The sequence $(p\alpha)$ for $p = 2, 3, \ldots, 11$

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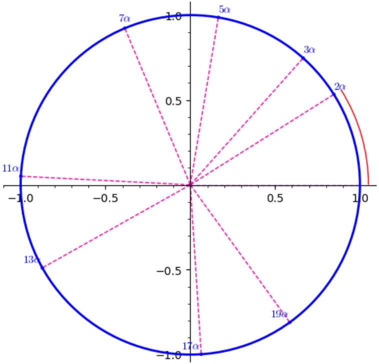

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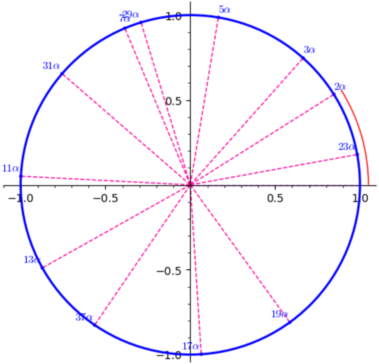

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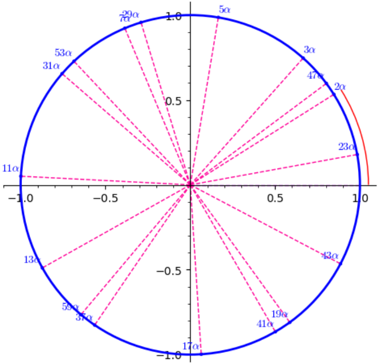

The sequence $(p\alpha)$ for $p = 2, 3, 5, \ldots, 59$

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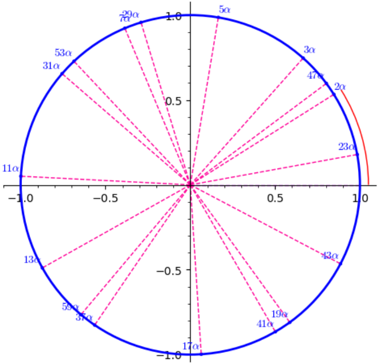

The set $(p\alpha)$ is dense on the unit cercle.

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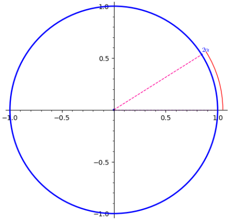
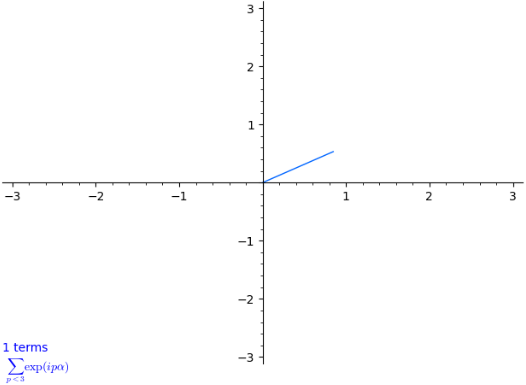
What about the average value?





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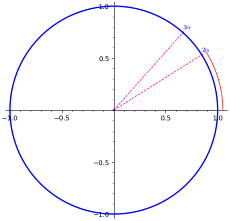

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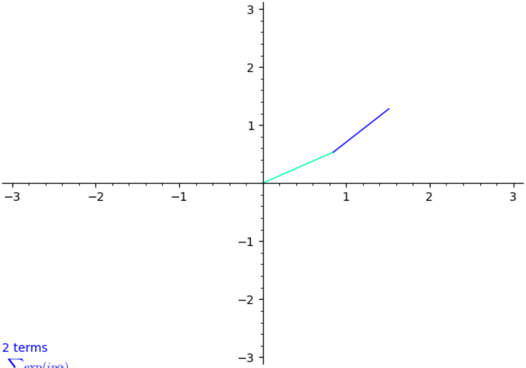

2 terms

$\sum_{p<4} \exp(ip\alpha)$

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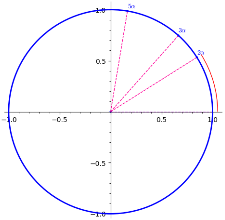

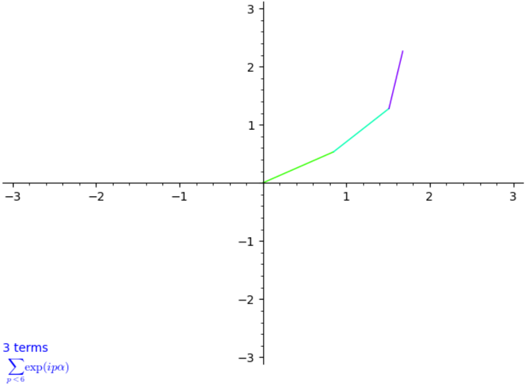

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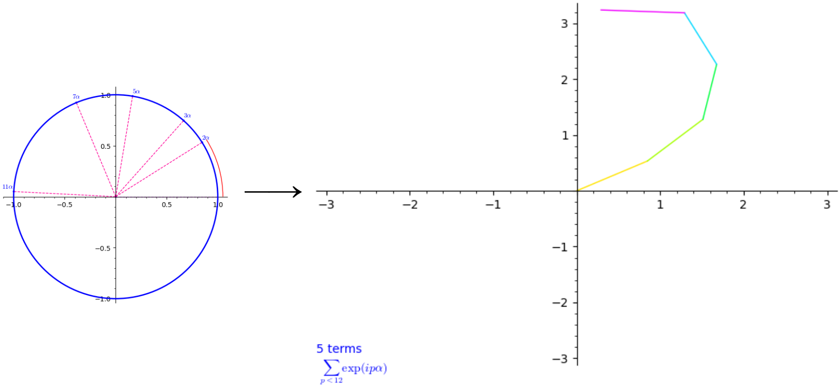

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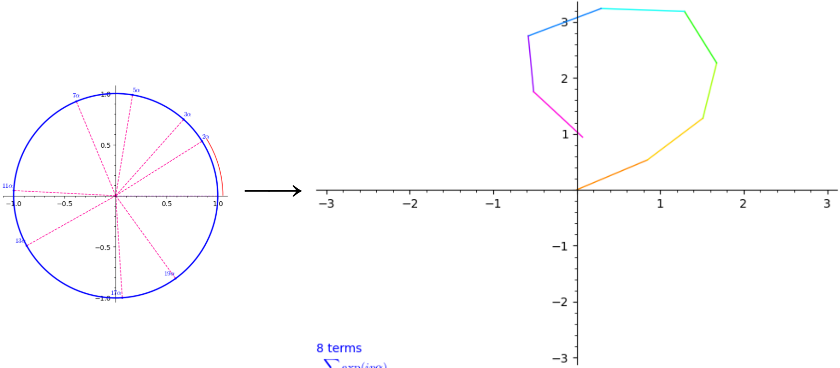

$$\sum_{p<20} \exp(ip\alpha)$$

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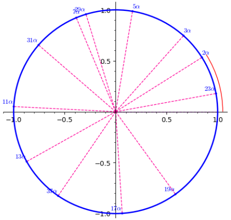

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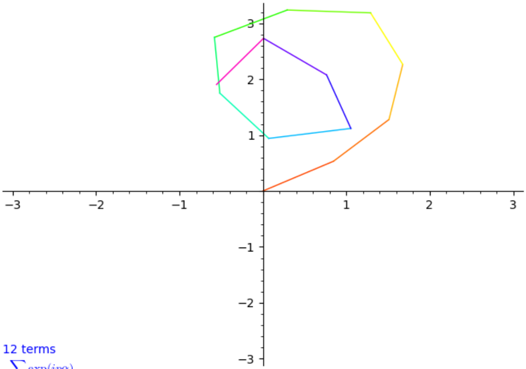
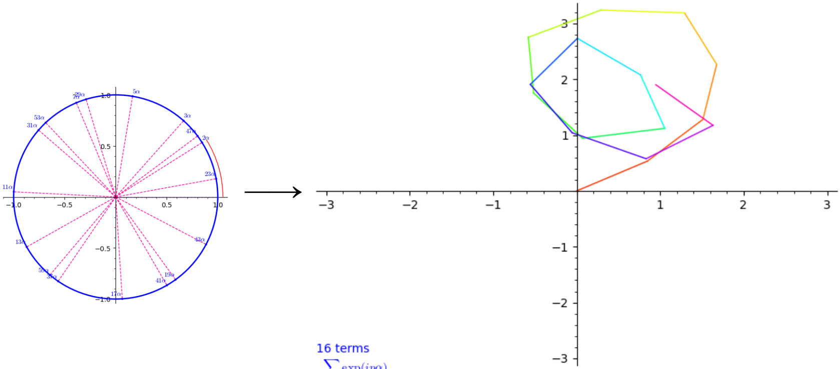


12 terms

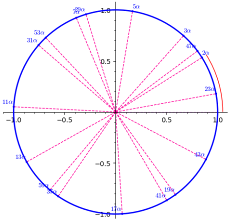
$\sum_{p<38} \exp(ip\alpha)$

What about the average value?

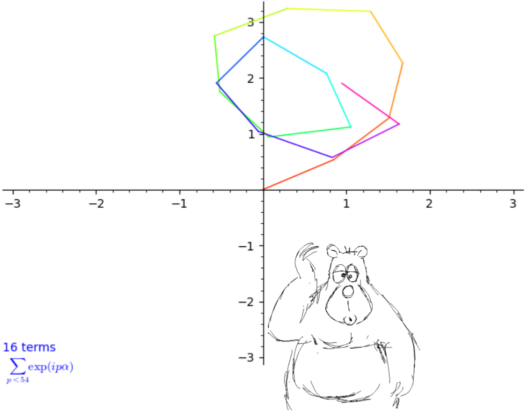


16 terms
$\sum_{p<54} \exp(ip\alpha)$

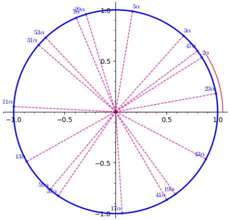
What about the average value?

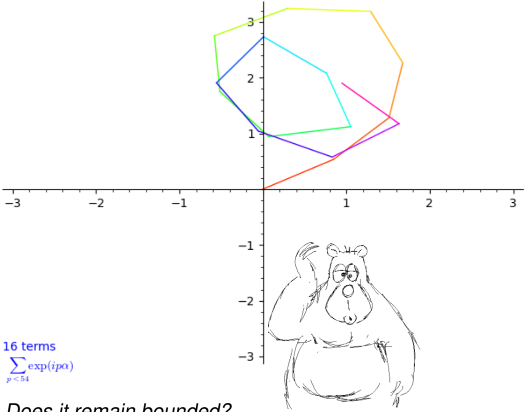


$\longrightarrow$



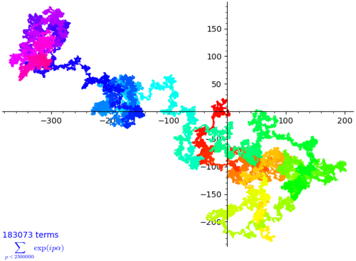
What about the average value?





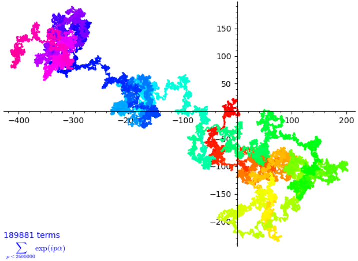
(1) Does it remain bounded?
(2) Does it stay more in some quadrant?
(3) Are there holes?

Seen from farther away



The sequence $\sum_{p \leqslant N} \exp(ip\alpha)$ for $N = 1, 2, \ldots, 2\,500\,000$

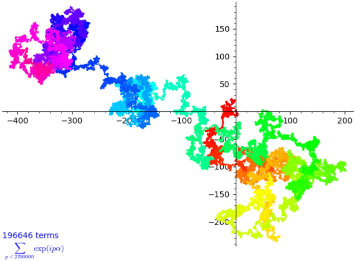
Seen from farther away



The sequence $\sum_{p \leqslant N} \exp(ip\alpha)$ for

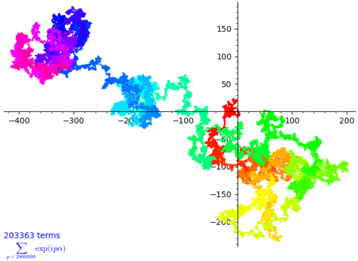
$N = 1, 2, \ldots, 2\,600\,000$

Seen from farther away



The sequence $\sum_{p \leqslant N} \exp(ip\alpha)$ for $N = 1, 2, \ldots, 2\,700\,000$

Seen from farther away



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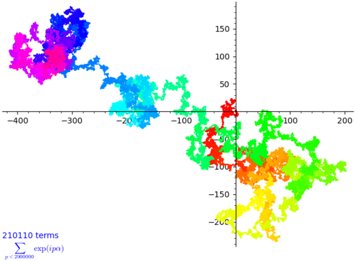

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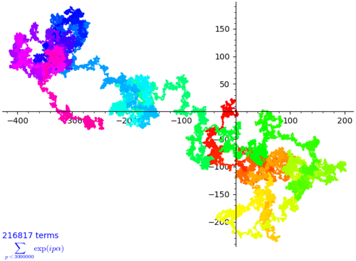
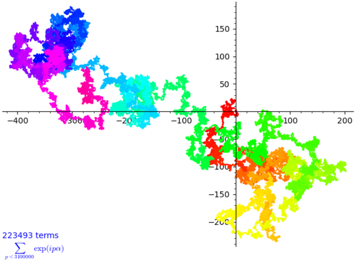


The sequence $\sum_{p \leqslant N} \exp(ip\alpha)$ for $N = 1, 2, \ldots, 3\,000\,000$

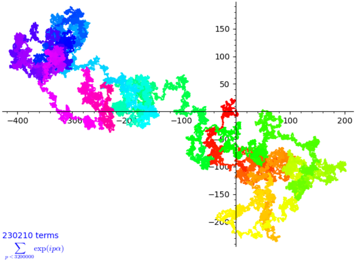
Seen from farther away



The sequence $\sum_{p\leqslant N} \exp(ip\alpha)$ for

$N = 1, 2, \ldots, 3\,100\,000$

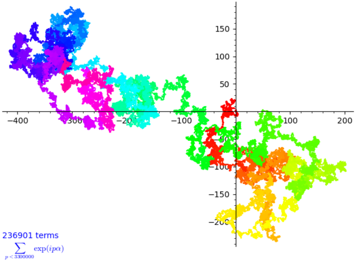
Seen from farther away



The sequence $\sum_{p \leqslant N} \exp(ip\alpha)$ for

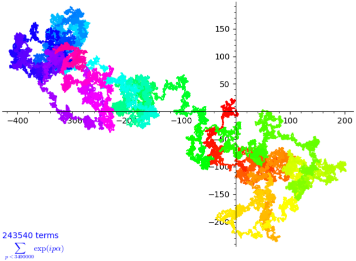
$N = 1, 2, \ldots, 3\,200\,000$

Seen from farther away



The sequence $\sum_{p \leqslant N} \exp(ip\alpha)$ for $N = 1, 2, \ldots, 3\,300\,000$

Seen from farther away



The sequence $\sum_{p \leqslant N} \exp(ip\alpha)$ for $N = 1, 2, \ldots, 3\,400\,000$

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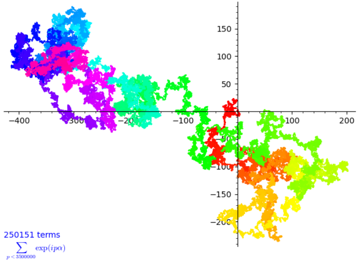
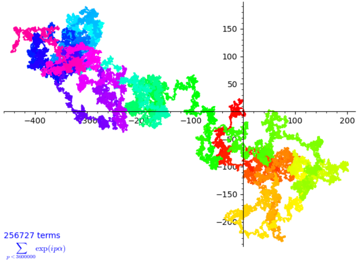


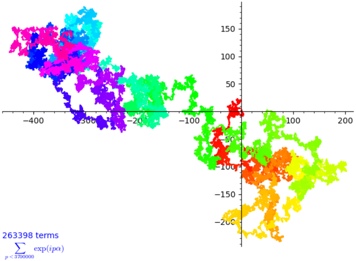
The sequence $\sum_{p \leqslant N} \exp(ip\alpha)$ for $N = 1, 2, \ldots, 3\,500\,000$

Seen from farther away



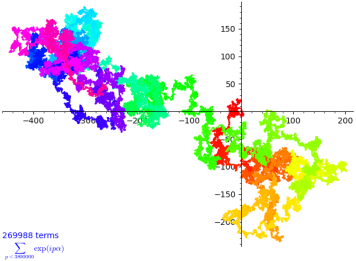
The sequence $\sum_{p \leqslant N} \exp(ip\alpha)$ for $N = 1, 2, \ldots, 3\,600\,000$

Seen from farther away



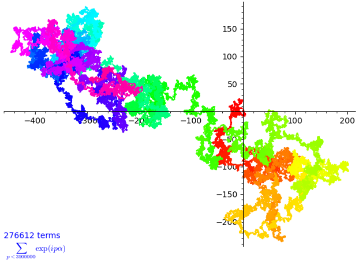
The sequence $\sum_{p \leqslant N} \exp(ip\alpha)$ for $N = 1, 2, \ldots, 3\,700\,000$

Seen from farther away



The sequence $\sum_{p \leqslant N} \exp(ip\alpha)$ for $N = 1, 2, \ldots, 3\,800\,000$

Seen from farther away



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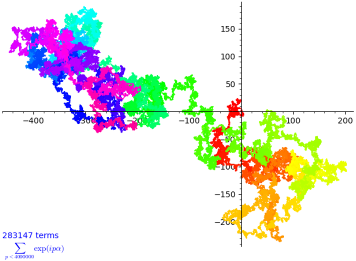
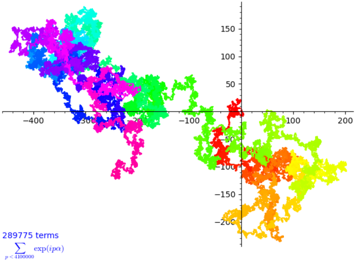


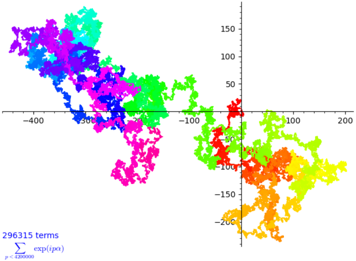
The sequence $\sum_{p \leqslant N} \exp(ip\alpha)$ for $N = 1, 2, \ldots, 4\,000\,000$

Seen from farther away



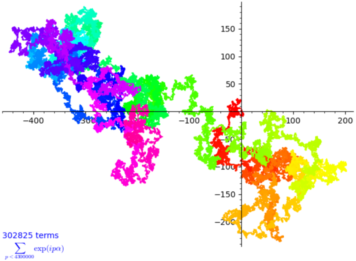
The sequence $\sum_{p \leqslant N} \exp(ip\alpha)$ for $N = 1, 2, \ldots, 4\,100\,000$

Seen from farther away



The sequence $\sum_{p \leqslant N} \exp(ip\alpha)$ for $N = 1, 2, \ldots, 4\,200\,000$

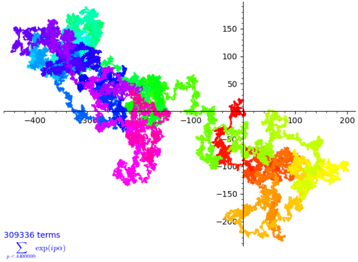
Seen from farther away



The sequence $\sum_{p \leqslant N} \exp(ip\alpha)$ for

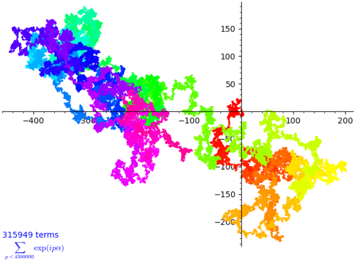
$N = 1, 2, \ldots, 4\,300\,000$

Seen from farther away



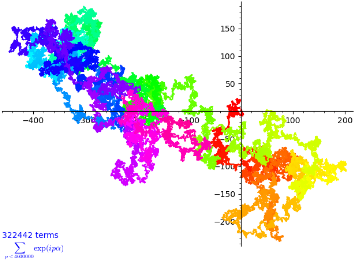
The sequence $\sum_{p \leqslant N} \exp(ip\alpha)$ for $N = 1, 2, \ldots, 4\,400\,000$

Seen from farther away



The sequence $\sum_{p \leqslant N} \exp(ip\alpha)$ for $N = 1, 2, \ldots, 4\,500\,000$

Seen from farther away



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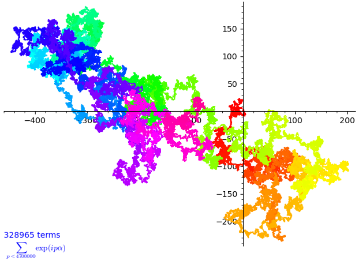
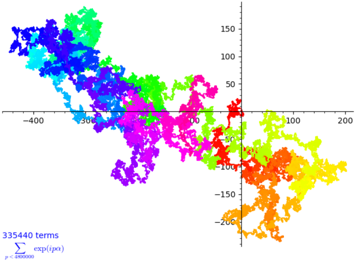
Seen from farther away



The sequence $\sum_{p \leqslant N} \exp(ip\alpha)$ for

$N = 1, 2, \ldots, 4\,700\,000$

Seen from farther away



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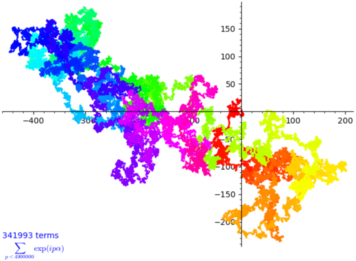
Seen from farther away



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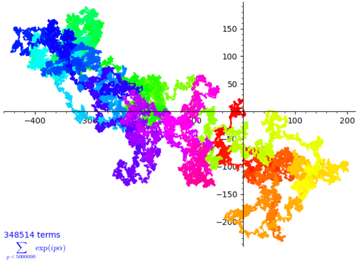
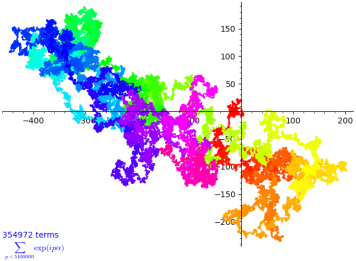


The sequence $\sum_{p \leqslant N} \exp(ip\alpha)$ for $N = 1, 2, \ldots, 5\,000\,000$

Seen from farther away



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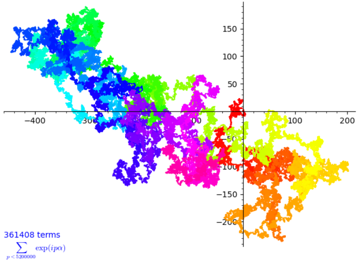
Seen from farther away



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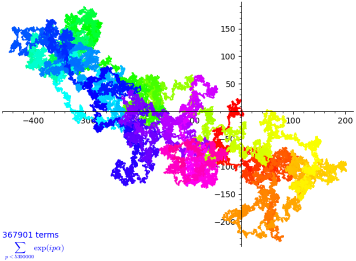
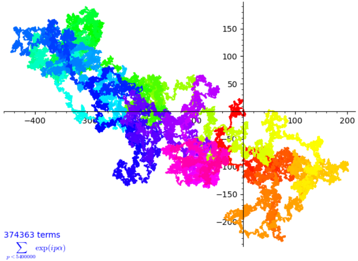


The sequence $\sum_{p \leqslant N} \exp(ip\alpha)$ for $N = 1, 2, \ldots, 5\,300\,000$

Seen from farther away



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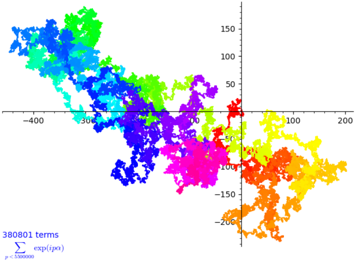
Seen from farther away



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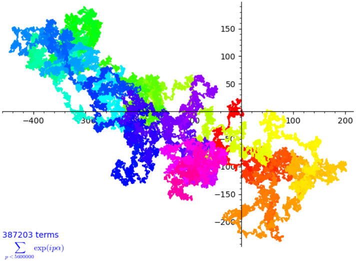

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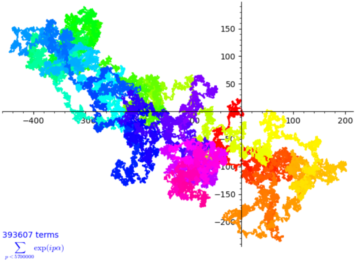

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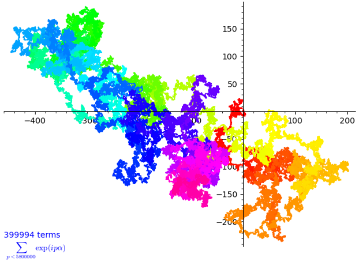

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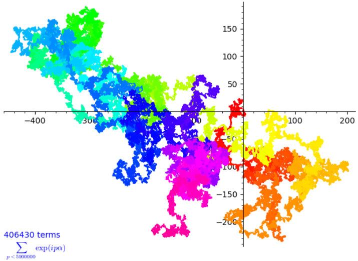

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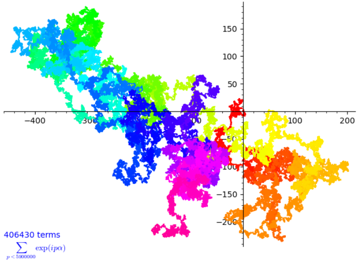

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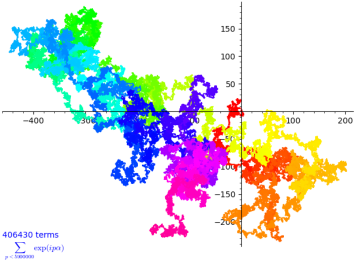

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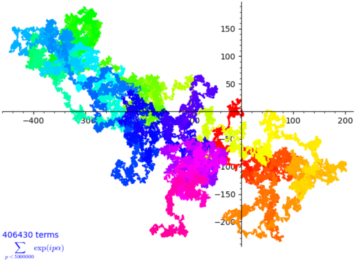

The sequence $\sum_{p \leqslant N} \exp(ip\alpha)$ for

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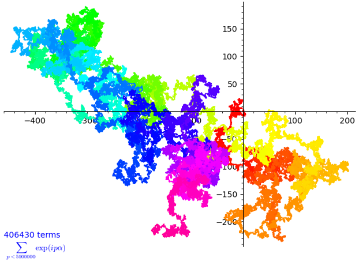
(1) This sum is very, very small!
(2) Where are the loops coming from?
(3) Does this sum “fill” the space?

Seen from farther away



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All what is known: (Vaughan, 1975): $\sum_{p \leqslant N} \exp(ip\alpha) \ll N^{4/5} (\log N)^3$

This is my speciality!

A dynamical viewpoint

Dynamics studies the transform:
$$\begin{aligned} T : [0,2\pi) &\to [0,2\pi) \\ x &\mapsto x+\alpha \mod 2\pi \end{aligned}$$

and the "trajectories" (iterates): $T(x)$, $T^2(x)$, $T^3(x)$, ...

(Starting from $x = 0$, we get $\alpha, 2\alpha, 3\alpha, \ldots$ on the circle).

Take a function $f : [0,2\pi) \to \mathbb{R}$

and study the (Birkhoff) average $\dfrac{1}{N} \displaystyle\sum_{n \leqslant N} f(T^n(x))$.

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There are infinitely many solutions to $p_4 - p_3 = p_3 - p_2 = p_2 - p_1$.

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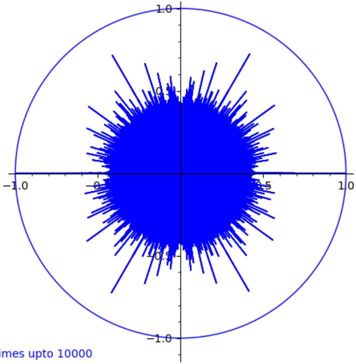
A large body of work emerged on "nilsequences", and on "Sarnak's conjecture".

Varying α / Charted land

Let have α vary

We looked at $S(\alpha) = \sum_{p \leqslant N} \exp(ip\alpha)$ for α fixed and N growing.

Let us look now at $|S(\theta)|$ with θ variable:



We take primes upto 10000

The normalised module in blue,

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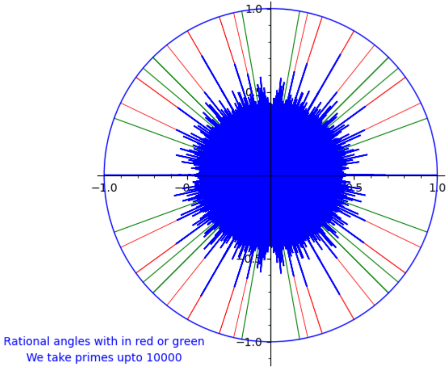

The normalised module in blue, the angles $2\pi a/q$ are in green when... and in red when...

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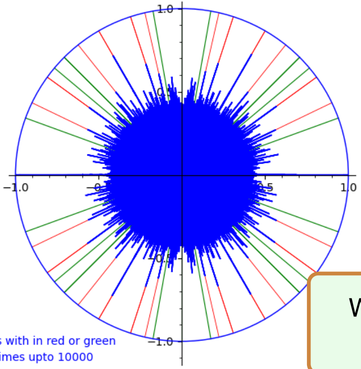


Rational angles with in red or green
We take primes upto 10000

The normalised module in blue, the angles $2\pi a/q$ are in green when... and in red when...

We know how to explain the “cusps”
$\rightarrow$ the Three Primes Theorem

Full sequence: the charted territory

- The value of $\exp(2i\pi pa/q)$ depends only on $p \mod q$.
- Hence PNTAP (Prime Number Theorem in Arithmetic Progressions) controls $S(2\pi a/q)$ when q is “not too large”.

That’s where we see that q squarefree is important.

- Good error term in PNTAP $\rightarrow$ the cusps are sharp.

These are the “major arcs” in Hardy-Littlewood-Ramanujan terminology.

- Away from these? Technique “à la Vinogradov”. Best is $N^{3/4}$.

These are the “minor arcs” in Hardy-Littlewood-Ramanujan terminology.

Cf “sums of type B” in Vinogradov.

- Better bounds? Open territory.
- Angular distribution? Open territory.

Opening the lens

Lack of precision versus generality

We lack of precision,
so maybe we can gain in generality!

(OR+Ruzsa, 98): subsequences $\mathcal{P}^*$ (of the primes) *of density*

for instance, when $N \geqslant 2$, $\#\{p \in \mathcal{P}^* : p \leqslant N\} \geqslant \dfrac{N}{100 \log N}$.

..... Follows lots of work !!! ....

$\bigoplus$ *ergodic aspect (linked to Szemedéri's theorem)*

The leading idea is that primes may be "described" by arithmetic progressions.
Arithmetic progressions share additive properties with the set of every integers.
$\longrightarrow$ "So" primes may share the same properties!
Green set to follow this program on Szemedéri's Theorem.

(Green & Tao, 2008): There are infinitely many solutions to

$$p_4 - p_3 = p_3 - p_2 = p_2 - p_1.$$

And (Tao & Ziegler, 2008), (Green & Tao, 2012), (Tao & Ziegler, 2015), ...

Let us go back to the (theoretical but not historical) beginning!

On the cusps

The number of cusps is finite!

(Green & Tao, 2006): Set $S^*(\alpha) = \sum_{\mathcal{P}^* \ni p \leqslant N} \exp(2i\pi\alpha p)$.

Then

$$\#\left\{\alpha \in \{\tfrac{1}{N}, \tfrac{2}{N}, \cdots, \tfrac{N}{N}\} : |S^*(\alpha)| \geqslant \xi S^*(0)\right\} \ll (1/\xi)^{2+\varepsilon}.$$

Maximum size: $|S^*(\alpha)| \leqslant S^*(0)$.

Number of small Farey points:
$$\#\left\{\tfrac{a}{q} \in [0,1) : q \leqslant 1/\xi\right\} \ll (1/\xi)^2.$$

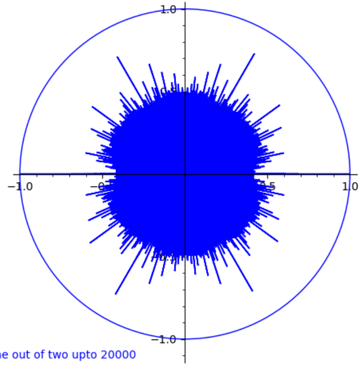
(OR, 2022, 2023):

$$\#\left\{\alpha \in \{\tfrac{1}{N}, \tfrac{2}{N}, \cdots, \tfrac{N}{N}\} : |S^*(\alpha)| \geqslant \xi S^*(0)\right\} \ll (1/\xi)^2.$$

Are all the cusps at Farey points??

A trial

Let us take a prime out of two:



We take one prime out of two upto 20000

The normalised module in blue,

A trial

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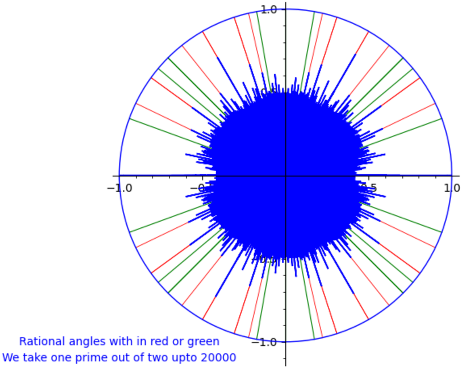

The normalised module in blue, the angles $2\pi a/q$ are in green when... and in red when...

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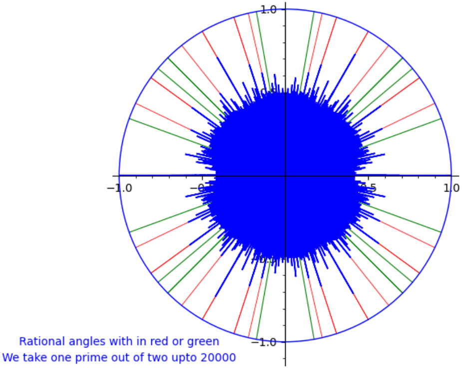
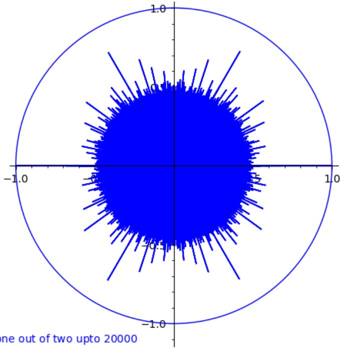


The normalised module in blue, the angles $2\pi a/q$ are in green when... and in red when...

(1) Again the "urchin" structure,
(2) Again cusps at the a/q!!

What about randomly?

Let us take randomly one prime out of two:



We take random p one out of two upto 20000

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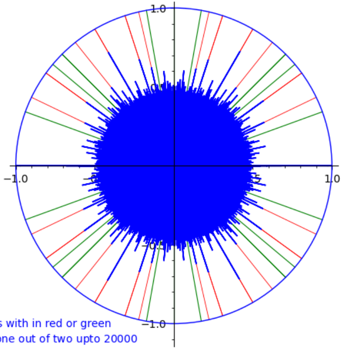

Rational angles with in red or green
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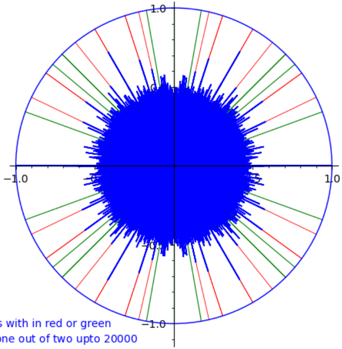

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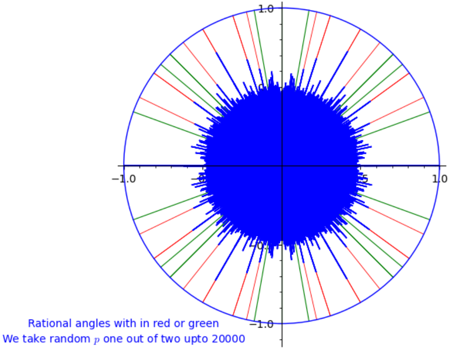

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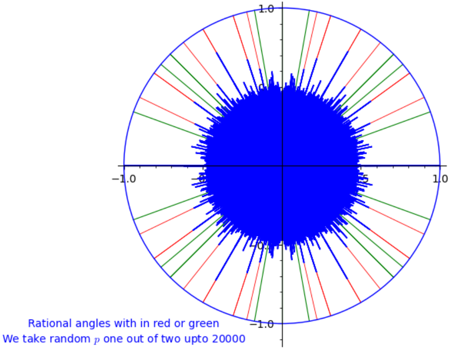

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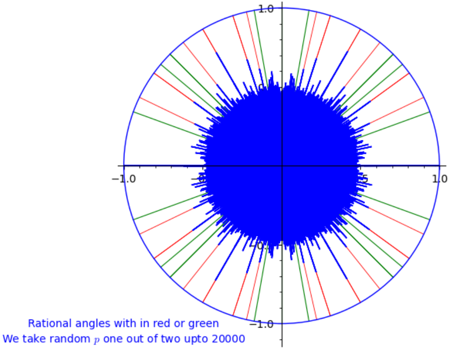
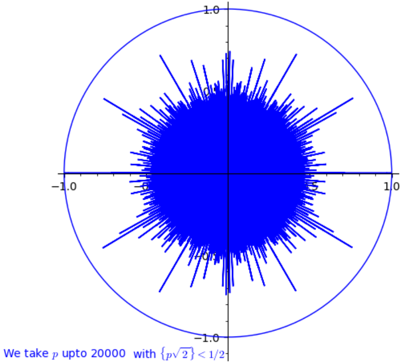


The normalised module in blue, the angles $2\pi a/q$ are in green when... and in red when...

(1) Again the "urchin" structure,
(2) Again cusps at the a/q!!
(3) It can be proved!

May we get a different behaviour???

Select the primes p by $\{p\sqrt{2}\} < 1/2$:



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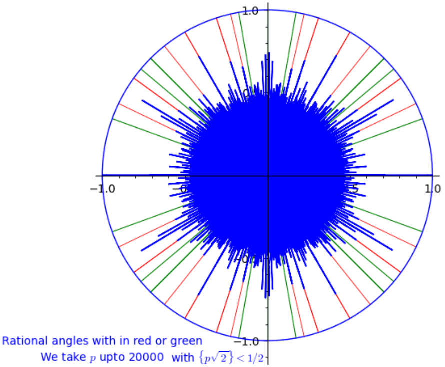

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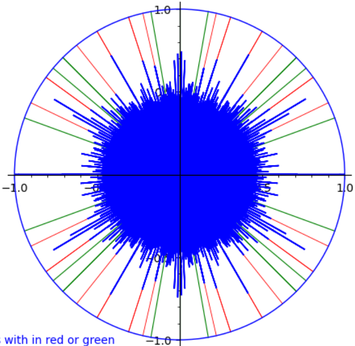

Rational angles with in red or green
We take p upto 20000 with $\{p\sqrt{2}\} < 1/2$

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Cusps are few:

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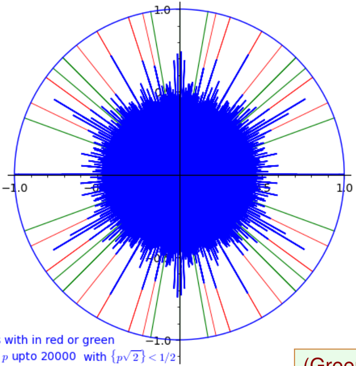

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Cusps are few: (Green & Tao, 2006): For every subsequence of primes of density, the number of "cusps" is finite.

Very few: (OR, 2023): This number of "cusps" is comparable to the number of rationals of small height.

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