Planik and Square Dance, Two Planar Permutation Games*

Solving a Simpler Rubik's Cube With Group Theory

Olivier Ramaré

The Planik is a two-dimensional version of a game close to the Rubik's cube. We present this game and the group theoretical background required to solve it. We adopt a low-profile style for students to be able to follow and add comments aimed at a more experienced audience. We conclude with notes concerning a second family of games that shares similarities with the Planik.

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1. An Introduction for the Players

The focus rests on a game we call 'Planik'. We present later some generalizations as well as a second family of games generically called 'Square Dance'. Planik is a single-player permutation game on the 4×4 square:

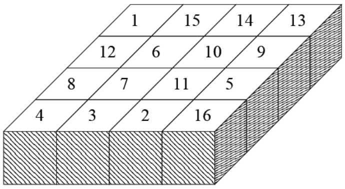

Figure 1. A problem.

The aim is to sort the inscribed numbers using only some given moves. By *sorting*, we mean that the final Planik should have $1-4$ in the first row, $5-8$ on the second one, and so on. The moves allowed can be described swiftly: a player can invert any

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Elementary move, permutation, orbit, signature, signature $\varepsilon(\sigma) = (-1)^{\text{nb of crossings}}$.

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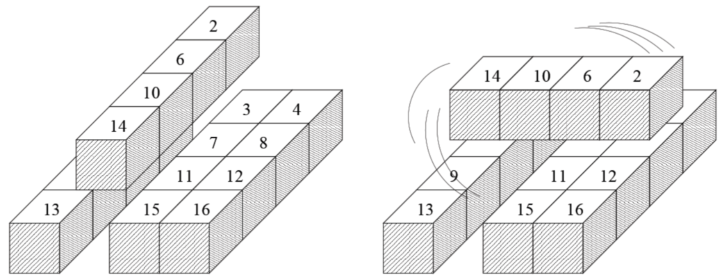
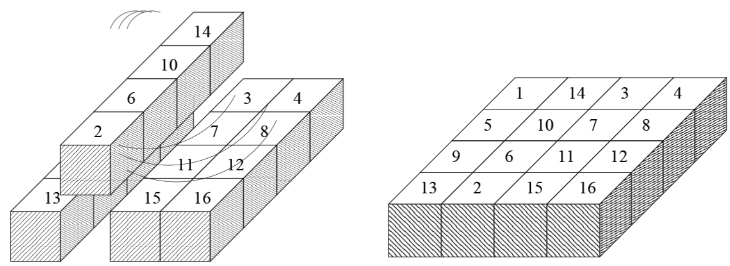
column and any row. In more detail, here is a move on a column:

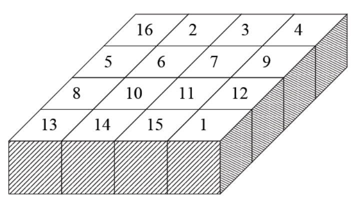


An elementary move

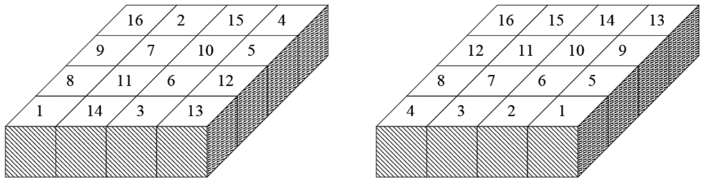


There are only eight elementary moves: four on the rows and four on the columns.

There are only eight elementary moves: four on the rows and four on the columns. In general, we start from a given distribution of the little cubes and seek to sort the Planik out using a sequence of elementary moves. Here is a further problem:



After some effort, the readers will understand that the new challenge is to prove that sorting this Planik out is *not* possible. Here are two more Planiks to sort:



Two additional problems

As the readers would have now understood, we use the word *Planik* to denote two distinct objects: the game in itself or the position of the cubes on this board. The moves we showed are termed *elementary moves*. They are used in sequences, and such a sequence is generically called a *move*. When we need to recall a move we made, we use a, b, c, d for the elementary moves on the rows and A, B, C, and D for the ones on the columns.

2. An Introduction for the Teachers

Group theoretical courses can be very abstract, and this game may be used to introduce essential notions. I would advise to present the game at first and set as a goal the description of all the reachable positions, as well as a clear answer to the question:

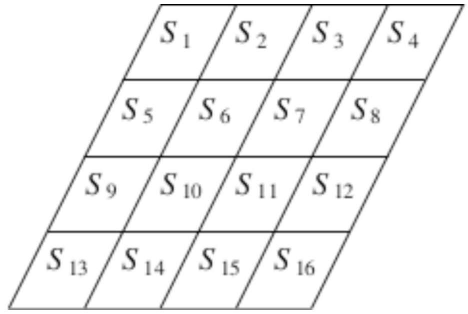
Fundamental Question. *Given any two Planiks, is it possible to go from one to the other by elementary moves?*

The game just described is a *permutation* game, in which some pieces are shifted around, none are destroyed or created, and the board on which to play remains fixed.

Given any two Planiks, is it possible to go from one to the other by elementary moves?

Modelising the Game

In this game, one should distinguish between the physical positions and the cubes on them. Here is our reference *physical* board:



A position is thus a one-to-one map, say σ, from this set of physical positions to the set $\{1, 2, \cdots, 16\}$, and for instance, the position in *Figure* 1 has the square marked 1 on the square S_1, the cube marked 15 on the square marked 2, and so on, so that $\sigma(S_1) = 1$, $\sigma(S_2) = 15, \ldots$ Now, let us apply a move, say B that exchanges the cubes on S_2 and on S_{14} as well as the ones on S_6 and S_{10}, and let us call η the new distribution of the cubes. What is the link between σ, B and η? Obviously, we have $\eta(S_i) = \sigma(S_i)$ when $i \notin \{2, 6, 10, 14\}$ and this clears this part out. Further $\eta(S_2) = \sigma(S_{14})$. As $S_{14} = B(S_1)$, we find that $\eta(S_2) = (\sigma \circ B)(S_2)$. The reader will readily check that this equation holds not only for S_2, but also for S_6, S_{10} and S_{14}, as well as for the other S_i's, as they are left unchanged by B. What we have established finally is that $\eta = \sigma \circ B$. Therefore, we apply a succession of moves, say A, b, c, D, a, A, and the resulting position is $\eta = \sigma \circ A \circ b \circ c \circ D \circ a \circ A$. We aim to reach $\eta(C_i) = C_i$ for all i. Notice that $\gamma = A \circ b \circ c \circ D \circ a \circ A$ is yet another permutation.

Summary

A move is a permutation, say γ, on the physical board. When we start from a distribution σ of the cubes on the physical board and we apply the permutation γ, the resulting distribution is $\sigma \circ \gamma$. To save time, we write $\sigma(i)$ instead of $\sigma(S_i)$, though this may add to confusion! But this has the added consequence that σ is now a permutation on $\{1, 2, \cdots, 16\}$.

Executing two moves is the same as composing two permutations.

Executing two moves is the same as *composing two permutations*, so the collection of positions accessible from $\sigma = Id$ is the set of permutations generated by the elementary moves. We shall show that this set, say G, is a group that we now describe. The subsets

$$\{S_1, S_4, S_{13}, S_{16}\}, \quad \{S_2, S_3, S_{14}, S_{15}\}, \quad \{S_5, S_8, S_9, S_{12}\} \quad \text{and} \quad \{S_6, S_7, S_{10}, S_{11}\},$$

are globally invariant under the possible moves. Our aim is to prove that G is isomorphic to the subgroup of the product of the four permutation groups on these orbits restricted to the permutations for which the product of the *signatures* is 1.

We now need to define all the words we have employed and then proceed to the proof. As the readers will see, the full description of the possible Planiks requires only little background. Attention and energy are all that will be asked on the students. I have done this study with 17-year-old kids, leaving them lots of room for experiments and private investigations.

On the Writing Style

We present in the sequel an answer to the fundamental question above by sticking to rudimentary vocabulary and notions. Comments that may sound obscure to beginners are sometimes added, but on the whole, this article should be accessible to both an advanced or a less specialized audience. As a consequence, our notation is more explicit than usual. For instance, the composition of two permutations, say σ_1 and σ_2, is denoted by $\sigma_1 \circ \sigma_2$ rather than by the short form $\sigma_1\sigma_2$. We will, however, often mix the settings and talk of the 'product' of two permutations and not of their 'composition'. The transposition of a and b is denoted by $\tau_{a,b}$ and not by (a, b); the same holds for cycles.

3. More on Moves

We use the language of *permutations* to describe our game. We identify the board with the set $\mathscr{E} = \{1, 2, \cdots, 15, 16\}$, rather than $\{S_1, S_2, \cdots, S_{16}\}$. A move can be described as a permutation of this set, and for instance, the elementary move A, i.e., the inversion of the first column is also the permutation defined by:

$$\begin{cases} A(1) = 13, \\ A(5) = 9, \\ A(9) = 5, \\ A(13) = 1, \end{cases} \begin{cases} A(2) = 2, \\ A(6) = 6, \\ A(10) = 10, \\ A(14) = 14, \end{cases} \begin{cases} A(3) = 3, \\ A(7) = 7, \\ A(11) = 11, \\ A(10) = 10, \end{cases} \begin{cases} A(4) = 4, \\ A(8) = 8, \\ A(12) = 12, \\ A(16) = 16. \end{cases} \tag{1}$$

The long description given in (1) may be shortened in

$$\left(\begin{array}{cccc|cccc|cccc|cccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 13 & 2 & 3 & 4 & 9 & 6 & 7 & 8 & 5 & 10 & 11 & 12 & 1 & 14 & 15 & 16 \end{array}\right)$$

As noted before, the fundamental property is that when we apply successively two moves, say A followed by a, the resulting permutation is the composition $A \circ a$. We often write this combined move as Aa. The set of all the permutations of $\mathscr{E}$ is denoted by $\mathfrak{S}(\mathscr{E})$ (the letter $\mathfrak{S}$ is the uppercase gothic letter S) which is a *group* under composition. As the symbols used to describe the elements of set $\mathscr{E}$ are irrelevant, we shorten $\mathfrak{S}(\mathscr{E})$ in $\mathfrak{S}(16)$.

When composing an arbitrary number of elementary moves, we build a subset G of $\mathfrak{S}(\mathscr{E})$. This subset also happens to be a group: it contains the identity as, for instance, $A \circ A = \mathrm{Id}$, and every element has an inverse. This last property always holds in such a setting for those who know enough of group theory[1], but it may also be easily established in our case. Take, for instance, the move $AaBCab$. The move $baCBaA$ inverts it, as the readers can readily check and put the Planik back in the initial shape.

[1] Let x be an element of G. The collection $x, x^2, x^3, \cdots$ being finite, there exist $a < b$ for which $x^a = x^b$. Both belong to the larger group $\mathfrak{S}(\mathscr{E})$, so we may simplify by x^a, getting $x^{b-a} = \mathrm{Id}$. Hence x^{b-a-1} in the inverse of x. As it indeed belongs to G, this concludes the proof.

4. Orbits

Our group G acts on the Planik $\mathscr{E}$, and this action has some geometrical properties. One of them is that the cubes of the corners $\mathscr{C} = \{1, 4, 13, 16\}$ remain in the corners, whatever move we make. The same holds for the four middle cubes from $\mathscr{M} = \{6, 7, 10, 11\}$, for the set of the four cubes on the inner horizontal sides, namely $\mathscr{H} = \{2, 3, 14, 15\}$ and for the set of the four cubes on the inner vertical sides $\mathscr{V} = \{5, 8, 9, 12\}$. The classical vocabulary says that these subsets are *globally invariant*. The readers can check that these are the smallest subsets of $\mathscr{E}$ globally invariant under the action of G. They are called the *orbits*.

The invariance of these subsets reduces enormously the number of Planiks one may reach from the initial one. Indeed, the full group of permutations $\mathfrak{S}(\mathscr{E})$ contains $16! = 20\,922\,789\,888\,000$ elements, while there are only $4! = 24$ possible permutations of the corners, and also 24 for each of the central block $\mathscr{M}$, the inner horizontal sides block $\mathscr{H}$ and the innert vertical sides one $\mathscr{V}$. This amounts in all to at most $24^4 = 331\,776$ permutations. This is already good, but the fundamental question is still not answered. We formalize this argument by considering the *structural map*:

$$\begin{aligned} \mathfrak{S}(\mathscr{E}) &\to \mathfrak{S}(\mathscr{C}) \times \mathfrak{S}(\mathscr{M}) \times \mathfrak{S}(\mathscr{H}) \times \mathfrak{S}(\mathscr{V}) \\ \sigma &\mapsto (\sigma_{\mathscr{C}}, \sigma_{\mathscr{M}}, \sigma_{\mathscr{H}}, \sigma_{\mathscr{V}}), \end{aligned} \tag{2}$$

which associates to a move σ on $\mathscr{E}$ the four induced permutations on the orbits. To go further, we need the notion of *signature* of a permutation.

5. Signature on $\mathfrak{S}(4)$

The signature of a permutation is an important concept that is not so easily defined. A key property is that the signature of a product is the product of the signatures. We present four definitions on $\mathfrak{S}(4)$ to adapt to the reader's background. Three of them are generalizable to larger sets. An interesting exercise for the reader is to show that all our definitions indeed amount to the same thing!

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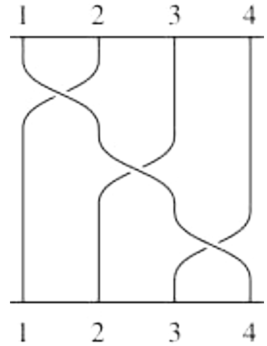
A Popular Definition

Let us start with a popular manner of defining the signature. We select the cyclic permutation $\sigma = \begin{pmatrix} 1\,2\,3\,4 \\ 4\,3\,4\,1 \end{pmatrix}$ to which we attach the following diagram:



There are three crossings of the threads between the top and bottom. We define the signature to be $(-1)^3$. Generally, the signature is -1 when the number of crossings is odd and 1 when it is even. On denoting this signature by ε, we have here $\varepsilon(\sigma) = -1$. This definition makes computing the signature easy in practical situations, but a difficulty appears: proving the fundamental property of the signature, i.e., the signature of a product $\sigma_1 \circ \sigma_2$ is the product of the individual signatures of σ_1 and σ_2. Let us propose definitions that will offer a natural path to this key property.

A Definition Through Spatial Geometry

The four vertices on a regular tetrahedron[2] $\{A, B, C, D\}$ may be permuted by isometries that preserve the orientation. Such isometries that keep our tetrahedron globally invariant induce a group of permutations that is a subgroup of $\mathfrak{S}(4)$: that much is clear. The reader can readily show that all of them may be described in this manner[3].

Our next task is to describe 'orientation preserving'. The definition being that the determinant is positive; we somehow need this notion on 3×3 matrices. The matrix of each of our isometries in the basis $(\vec{i}, \vec{j}, \vec{k})$ has an (integer) determinant. One of them has a negative determinant, for instance, when we fix $\vec{i}$ and exchange $\vec{j}$ and $\vec{k}$, and so half of them have a positive determinant. The sign of this determinant is what we call the *signature* $\varepsilon(\sigma)$ of the permutation σ. It borrows from determinants the property that $\varepsilon(\sigma_1 \circ \sigma_2) = \varepsilon(\sigma_1)\varepsilon(\sigma_2)$.

[2] A *regular* tetrahedron is a tetrahedron whose edges are all of the same length.

[3] To build *all* the permutations of these vertices induced by isometries, we set $\vec{i} = \overrightarrow{AB}$, $\vec{j} = \overrightarrow{AC}$ and $\vec{k} = \overrightarrow{AD}$. Any of the isometries we seek preserves globally[4] the set of edges $\{\pm\vec{i}, \pm\vec{j}, \pm\vec{k}, \pm(\vec{j}-\vec{i}), \pm(\vec{k}-\vec{j}), \pm(\vec{i}-\vec{k})\}$. This a finite though rather large set, but of course, the choice of the image of $\vec{i}$, $\vec{j}$ and $\vec{k}$ determines our permutation. The readers can check that fixing the image of $\vec{i}$ as any one element in this set restricts the images of $(\vec{j}, \vec{k})$ to only two choices. For instance, if $\vec{i}$ is transformed in $-\vec{k}$, then A goes to D and B goes to A, leaving only two choices for the images of C and D. Therefore we generate 24 permutations, i.e., *every* permutation of $\mathfrak{S}(4)$ in this manner.

It is enough to consider the sign of the determinant, but in fact, the signature *is* this determinant. Indeed, the isometries we employ are of finite order, meaning that when we iterate them so many times, we reach the identity. Hence, their determinant is a root of unity that is also a real number: its value must belong to $\{\pm 1\}$.

[4]The readers should notice that we confuse the affine map on *points* and the induced linear map on *vectors*. It may be better, in the first go, to forget this nicety when talking to students.

Definition Through Linear Algebra

If the reader is comfortable with linear algebra, we may define the signature through permutation matrices. Here is how it goes: consider the vector space $\mathbb{R}^4$ and the basis made of $e_1 = (1, 0, 0, 0)$, $e_2 = (0, 1, 0, 0)$, $e_3 = (0, 0, 1, 0)$ and $e_4 = (0, 0, 0, 1)$. Given a permutation σ of $\{1, 2, 3, 4\}$, we may consider the linear map φ_σ that swaps the e_i's accordingly, i.e., such that $\varphi_\sigma(e_i) = e_{\sigma(i)}$, for $i \in \{1, 2, 3, 4\}$. Composing the permutations or composing these maps is the same thing, i.e., $\varphi_{\sigma_1 \circ \sigma_2} = \varphi_{\sigma_1} \circ \varphi_{\sigma_2}$. The matrix of each φ_σ in the basis (e_1, e_2, e_3, e_4) has a determinant that belongs forcibly to $\{\pm 1\}$, thanks to the same argument as above. This we call the signature.

Definition Through Combinatorics

Here is a third way to present the signature, which requires less geometry but more abstract strength. Given a permutation σ on $\{1, 2, 3, 4\}$, both non-zero integers

$$\prod_{1 \le u \ne v \le 4} (\sigma(v) - \sigma(u)) = (-1)^6 \prod_{1 \le u < v \le 4} (\sigma(v) - \sigma(u))^2$$

and

$$\prod_{1 \le u \ne v \le 4} (v - u) = (-1)^6 \prod_{1 \le u < v \le 4} (v - u)^2$$

are equal. Hence the number

$$\varepsilon(\sigma) = \prod_{1 \le u < v \le 4} \frac{\sigma(v) - \sigma(u)}{v - u} \tag{3}$$

is either 1 or -1. This is another definition of the signature. We now have to prove that this is a group morphism, i.e., $\varepsilon(\sigma_1 \circ \sigma_2) = \varepsilon(\sigma_1)\varepsilon(\sigma_2)$. To do so, we rewrite the definition above. Let $\mathcal{P}_4$ the set of (non-ordered) *pairs* of distinct integers from $\{1, 2, 3, 4\}$. A moment's thought is enough to see that (3) may be written in the form

$$\varepsilon(\sigma) = \prod_{\{u,v\}\in\mathcal{P}_4} \frac{\sigma(v) - \sigma(u)}{v - u},$$

simply because the quotient $(\sigma(v) - \sigma(u))/(v - u)$ takes the same value on (u, v) and on (v, u). Once this crucial formula is acquired, we may write

$$\begin{aligned}\varepsilon(\sigma_1 \circ \sigma_2) &= \prod_{\{u,v\}\in\mathcal{P}_4} \frac{\sigma_1(\sigma_2(v)) - \sigma_1(\sigma_2(u))}{\sigma_2(v) - \sigma_2(u)} \prod_{\{u,v\}\in\mathcal{P}_4} \frac{\sigma_2(v) - \sigma_2(u)}{v - u} \\ &= \prod_{\{u,v\}\in\mathcal{P}_4} \frac{\sigma_1(\sigma_2(v)) - \sigma_1(\sigma_2(u))}{\sigma_2(v) - \sigma_2(u)} \varepsilon(\sigma_2).\end{aligned}$$

But since σ_2 is one-to-one on $\{1, 2, 3, 4\}$, we may write the elements of $\mathcal{P}_4$ in the form $\{\sigma_2(u), \sigma_2(v)\}$, from which we conclude that indeed $\varepsilon(\sigma_1 \circ \sigma_2) = \varepsilon(\sigma_1)\varepsilon(\sigma_2)$.

6. Using the Signature

Once the notion of signature on $\mathfrak{S}(4)$ is acquired, we may proceed. The first thing to notice is that the signature of any transposition is -1. In the geometrical context, we simply compute the required determinant, while in the combinatorial one, we preliminarily establish the (very useful) formula

$$\sigma \circ \tau_{a,b} \circ \sigma^{-1} = \tau_{\sigma(a),\sigma(b)}, \tag{4}$$

where we have denoted the transposition of a and b by $\tau_{a,b}$[5].

[5]This formula is readily proved: on denoting $u = \sigma(a)$ and $v = \sigma(b)$, then u (resp. v) is sent to a (resp. b), then swapped and send to v (resp. u). The other points are first moved and then put back on their initial place.

This implies that the signature of every transposition is the same. Since every permutation is a product of transpositions, if we had $\varepsilon(\tau_{a,b}) = 1$, then the signature would be constant on $\mathfrak{S}(4)$. This is readily disproved. We may, of course, prove that the signature of $\tau_{1,2}$ is -1 by investigating the sign of the formula (3) in this case. Additionally, the fact that every permutation is a product of transpositions implies that formula (4) holds not only for transpositions, but also for any cycle, i.e.,

$$\sigma \circ \tau_{a_1,a_2,\ldots,a_r} \circ \sigma^{-1} = \tau_{\sigma(a_1),\sigma(a_2),\ldots,\sigma(a_r)},$$

where we have denoted by $\tau_{a_1,a_2,\ldots,a_r}$ the cycle on $a_1, a_2, \ldots, a_r$. We also check that $\tau_{a_1,a_2}\tau_{a_1,a_2,\ldots,a_r} = \tau_{a_2,\ldots,a_r}$, from which we deduce that

$$\tau_{a_1,a_2,\ldots,a_r} = \tau_{a_1,a_2}\tau_{a_2,a_3}\cdots\tau_{a_{r-1},a_r}.$$

This expression implies that a cycle of length r has signature $(-1)^{r-1}$.

Let us go back to the structural map defined in (2). The fundamental remark is that

$$\forall\sigma \in G, \quad \varepsilon(\sigma_{\mathscr{C}})\varepsilon(\sigma_{\mathscr{M}})\varepsilon(\sigma_{\mathscr{H}})\varepsilon(\sigma_{\mathscr{V}}) = 1. \tag{5}$$

Indeed, this relation is verified on the elementary moves, and thus holds for every permutation of G.

7. Understanding G

The next result determines G fully.

Theorem 7..1. *A permutation $\sigma \in \mathfrak{S}(16)$ belongs to G if and only if the next two conditions are met:*

- *The permutation σ globally preserves the corners, the middle block, and both the inner vertical sides and the horizontal ones.*
- *The permutation σ satisfies* (5).

We owe the proof that follows to Joseph Oesterlé.

Proof. We have shown that these two conditions are necessary, and our task is to show that they are indeed sufficient. Let σ be a permutation that verifies these conditions. We need to represent it as a product of elementary moves, and we shall do so by finding such a product, say π, such that $\pi \circ \sigma = \mathrm{Id}$. We separate the proof in two steps. From a player's viewpoint, we start from a shuffled Planik and need to reorder it.

First Step

Let us first investigate the positions on the corners. The elementary move a transposes the cubes (initially marked) 1 and 4, D transposes 4 and 16 and d transposes 13 and 16. These three transpositions generate $\mathfrak{S}(4)$, so we can use a product of them to put the corners back in the initial position. Similarly, we may properly order the central block using B, C and c without disturbing the distribution on the corners, as these moves do not change the ordering on the corners.

Second Step

We may thus assume that the corners and the central cubes are properly set, i.e., $\sigma_{\mathscr{C}} = \mathrm{Id}$ and $\sigma_{\mathscr{M}} = \mathrm{Id}$. Our information is that $\varepsilon(\sigma_{\mathscr{H}})\varepsilon(\sigma_{\mathscr{V}}) = 1$, i.e., $\varepsilon(\sigma_{\mathscr{H}}) = \varepsilon(\sigma_{\mathscr{V}})$.

The sequel of the proof is based on three observations:

- The succession of moves $aBaB$ (that you get by successively applying the moves a, B, a and B) permutes circularly the cubes on S_2, S_3 and S_{14} and fixes all the other cubes. We generate similarly any other circular permutation on three symbols from the inner vertical sides.

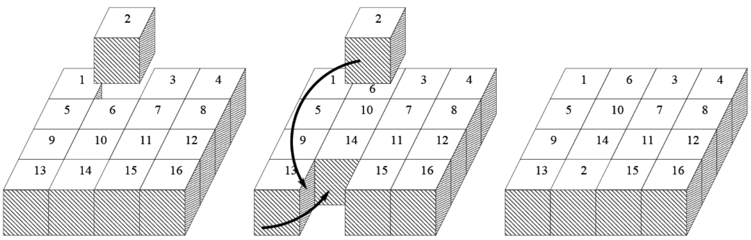
- Similarly, the succession of moves $AbAb$ permutes circularly the cubes on S_5, S_8 and S_9 and fixes all the other cubes, and yet again, we may generate any other circular permutation on three symbols from the inner horizontal sides.

- The succession $aDaDaD$ operates as a product of two transpositions: it transposes the inner side horizontal cubes on S_2 and S_3 and the inner side vertical ones on S_8 and S_{12}.

When $\varepsilon(\sigma_{\mathscr{H}}) = \varepsilon(\sigma_{\mathscr{V}}) = -1$, we first apply the third move above and reach a position where $\varepsilon(\sigma_{\mathscr{H}}) = \varepsilon(\sigma_{\mathscr{V}}) = 1$. The readers will then readily check that any permutation of signature 1 on $\mathfrak{S}(4)$ may be written as a product of circular permutations of length 3: we use that to write both $\sigma_{\mathscr{H}}$ and $\sigma_{\mathscr{V}}$ as product of elements of G. This ends the proof. $\square$

8. The Square Dance Game

Let us complete the Planik (and its extension to larger boards) by another family of planar games. These are played on similar boards, but the moves are distinct, leading to different solutions. The moves are also row per row or column per column, but instead of inverting one of them, we apply a circular permutation. So, for instance, on a 4×4 board, the elementary move a is the circular permutation $\tau_{1,2,3,4}$.



Square dance: An elementary move

We only mention some facts about these games.

The 3×3 *Case*

This game has a single orbit. However, a circular permutation of three symbols has a signature equal to 1, meaning that, on iterating the elementary moves, we may only generate an even permutation. Of course, we need the notion of signature to read this sentence so that this game is only suitable for higher-level students. As it turns out, every even permutation on $\{1, 2, \cdots, 9\}$ is reachable by such moves. One way to prove it is to first sort out the lowest and rightmost 2×2 square. Using the first column as a stack, we may also sort out the upper row. The signature property implies that the last two cubes will indeed be properly placed.

The 4×4 *Case*

We also encounter only one orbit, but the elementary moves now have signature -1. Again, using the first row as a stack, the readers can sort out the lowest rightmost 3×3 square, as well as the upper row. We now show that we may build transpositions by elementary moves. Consider the sequel $aAaaaAaAAaaaA$ (notice that $aaa = a^{-1}$). Here is what happens:

$$
\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 & & & \\ \hline 9 & & & \\ \hline 13 & & & \\ \hline \end{array}
\quad a \to \quad
\begin{array}{|c|c|c|c|}\hline 4 & 1 & 2 & 3 \\ \hline 5 & & & \\ \hline 9 & & & \\ \hline 13 & & & \\ \hline \end{array}
\quad A \to \quad
\begin{array}{|c|c|c|c|}\hline 13 & 1 & 2 & 3 \\ \hline 4 & & & \\ \hline 5 & & & \\ \hline 9 & & & \\ \hline \end{array}
$$

$$
aaa \to \quad
\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 13 \\ \hline 4 & & & \\ \hline 5 & & & \\ \hline 9 & & & \\ \hline \end{array}
\quad A \to \quad
\begin{array}{|c|c|c|c|}\hline 9 & 2 & 3 & 13 \\ \hline 1 & & & \\ \hline 4 & & & \\ \hline 5 & & & \\ \hline \end{array}
\quad a \to \quad
\begin{array}{|c|c|c|c|}\hline 13 & 9 & 2 & 3 \\ \hline 1 & & & \\ \hline 4 & & & \\ \hline 5 & & & \\ \hline \end{array}
$$

$$
AA \to \quad
\begin{array}{|c|c|c|c|}\hline 4 & 9 & 2 & 3 \\ \hline 5 & & & \\ \hline 13 & & & \\ \hline 1 & & & \\ \hline \end{array}
\quad aaa \to \quad
\begin{array}{|c|c|c|c|}\hline 9 & 2 & 3 & 4 \\ \hline 5 & & & \\ \hline 13 & & & \\ \hline 1 & & & \\ \hline \end{array}
\quad A \to \quad
\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 9 & & & \\ \hline 5 & & & \\ \hline 13 & & & \\ \hline \end{array}
$$

From this construction, which we owe to Julien Cassaigne, and after some routine work, we deduce the group of permutations generated is $\mathfrak{S}(16)$.

The 5×5 *Case*

It is again trivial to show that there is only one orbit, but the elementary moves now all have signature 1. We leave the determination of the group of permutations to the interested readers!

Suggested Reading

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