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One Uniform Random Variable Suffices

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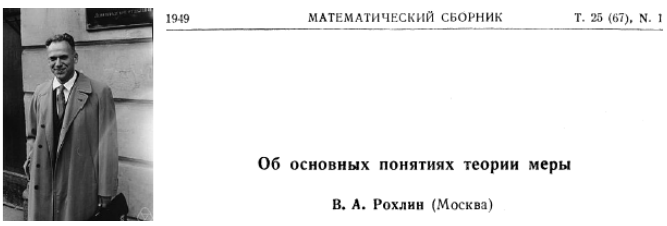
1. Introduction

In this article, we will explore how many different random processes can be simulated with a *single* uniform random variable. What we will discuss is well-known, coming from the idea of a *Standard Lebesgue Space*, introduced by V. A. Rokhlin [1]. We will give a series of probabilistic examples of how a single Uniform [0, 1] random variable can be used to simulate various random sequences and processes.

1.1 V. A. Rokhlin

Rokhlin was a Soviet mathematician who made important contributions in many subjects, including topology, ergodic theory, and measure theory. You can see the title page of his 1949 paper in the original Russian, and a picture of Rokhlin, below.


1949 МАТЕМАТИЧЕСКИЙ СБОРНИК Т. 25 (67), N. 1

Об основных понятиях теории меры

В. А. Рохлин (Москва)

Figure 1. Rokhlin and his famous paper

To quote from the excellent Wikipedia article [2] on **Standard Lebesgue Spaces**

Rokhlin showed that the unit interval endowed with the Lebesgue measure has important advantages over general probability spaces, yet can be effectively substituted for many of these in probability theory. The dimension of the unit interval is not an obstacle, as was clear already to Norbert Wiener. He constructed the Wiener process (also called Brownian motion) in the form of a measurable map from the unit interval to the space of continuous functions.

1.2 Measuring randomness

Let (Ω_1, B_1, P_1) and (Ω_2, B_2, P_2) be probability spaces. We say that (Ω_2, B_2, P_2) is *less random* than (Ω_1, B_1, P_1) if there is a deterministic (B_1, B_2)-measurable map

$$F : \Omega_1 \to \Omega_2$$

where for every $A \in B_2$,

$$P_1(F^{-1}(A)) = P_2(A).$$

In this case, we will say that a random variable with distribution P_1 can be used to *simulate* a random variable with distribution P_2. We will show how (I, B, M) where $I = [0, 1]$, B is the Borel σ-algebra of $[0, 1]$, and M Lebesgue measure, can be used to simulate many random processes.

1.3 Our examples

We will show that one uniform [0, 1] random variable can generate:

- A real-valued random variable with a prescribed cumulative distribution function (CDF).
- Sequences of independent, identically distributed uniform random variables.
- Any sequence of real-valued random variables with prescribed marginal distributions.
- Standard Brownian Motion.

2. Generating a CDF

2.1 Cumulative distribution functions

We say a function $F : \mathbb{R} \to [0, 1]$ is a cumulative distribution function (CDF) if F is:

Right-continuous That is, for all $x_0 \in \mathbb{R}$,

$$\lim_{x \downarrow x_0} F(x) = F(x_0).$$

Non-decreasing That is, for $x_1 \le x_2$,

$$F(x_1) \le F(x_2).$$

Goes from 0 to 1 $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to +\infty} F(x) = 1$.

Given a CDF F, how can we generate a random variable X (that is, a measurable function $X : \mathcal{S} \to \mathbb{R}$ from a probability space $(\mathcal{S}, \mathcal{B}, P)$ to $\mathbb{R}$) which has F as its CDF (that is, $P(X \le x) = F(x)$) using one uniform [0, 1] random variable U? Given a CDF F, and $u \in (0, 1)$, define the deterministic functions F_1^{-1} and F_2^{-1} as follows:

$$F_1^{-1}(u) \equiv \sup\{x \in \mathbb{R} : F(x) < u\},$$

and

$$F_2^{-1}(u) \equiv \inf\{x \in \mathbb{R} : F(x) \ge u\}.$$

We claim that if U is a uniform [0, 1] random variable, *both*

$$X_1 = F_1^{-1}(U) \text{ and } X_2 = F_2^{-1}(U)$$

are random variables with CDF F, since for any $t_0 \in \mathbb{R}$, and $u_0 \in (0, 1)$,

$$F_1^{-1}(u_0) \le t_0 \text{ if and only if } F(t_0) \ge u_0$$

and

$$F_2^{-1}(u_0) > t_0 \text{ if and only if } F(t_0) < u_0.$$

3. Sequences of IID uniform random variables

3.1 From one uniform to countably many

We will show how to use one uniform [0, 1] random variable U to generate a countable sequence $\{U_n\}_{n=1}^{\infty}$ of independent, identically distributed (IID) uniform [0, 1] random variables, via *digit expansions* of U.

3.1.1 Binary digits

Given a uniform [0, 1] random variable U, write it in its *binary expansion*,

$$U = \sum_{i=1}^{\infty} \eta_i(U) 2^{-i}, \eta_i(U) \in \{0, 1\}.$$

We claim that the η_i are *independent* Bernoulli random variables, with

$$P(\eta_1 = 0) = P(\eta_1 = 1) = 1/2.$$

To prove this, note that

$$\eta_1(U) = 0 \text{ if } U < 1/2 \quad \text{and} \quad \eta_1(U) = 1 \text{ if } U \ge 1/2.$$

Next, note that

$$\eta_2(U) = \eta_1(2U - \lfloor 2U \rfloor),$$

and more generally,

$$\eta_i(U) = \eta_1(2^{i-1}U - \lfloor 2^{i-1}U \rfloor).$$

It is not difficult to check that for any $\omega_1, \omega_2, \ldots, \omega_k \in \{0, 1\}$ and $i_1, i_2, \ldots i_k$

$$P(\eta_{i_1} = \omega_1, \eta_{i_2} = \omega_2, \ldots \eta_{i_k} = \omega_k) = 2^{-k}.$$

For example,

$$P(\eta_1(U) = 1, \eta_2(U) = 0) = M([1/2, 3/4)) = 1/4.$$

In general, the event

$$\{\eta_{i_1} = \omega_1, \eta_{i_2} = \omega_2, \ldots \eta_{i_k} = \omega_k\}$$

can be written as a union of dyadic intervals (that is, intervals of the form $(j2^{-n}, (j+1)2^{-n})$, and the probability is the sum of the Lebesgue measure of these intervals. Vice-versa, given IID Bernoulli$(1/2)$ random variables $\{\eta_i\}_{i=1}^{\infty}$, let

$$U = \sum_{i=1}^{\infty} \eta_i 2^{-i}.$$

Then U is a uniform [0, 1] random variable. To prove this, note that any interval (a, b) can be approximated by dyadic intervals, and dyadic intervals can be described by specifying finitely many values of η_i.

3.2 Prime power subsequences

Now let U be a uniform $[0, 1]$ random variable, and consider the binary expansion

$$U = \sum_{i=1}^{\infty} \eta_i(U) 2^{-i}, \eta_i(U) \in \{0, 1\}.$$

Let $p_1, p_2, \ldots p_n, \ldots$ be a sequence of distinct prime integers. Note that the power sequences $\{p_i^k\}_{k=1}^{\infty}$ are disjoint countable sets of integers. For $n \geq 1$, let

$$U_n = \sum_{k=1}^{\infty} \eta_{p_n^k}(U) 2^{-k}.$$

Then

- For any n, the sequence $\{\eta_{p_n^k}\}_{k=1}^{\infty}$ consists of IID Bernoulli random variables.
- For $n \neq m$, the sequences $\{p_n^k\}_{k=1}^{\infty}$ and $\{p_m^k\}_{k=1}^{\infty}$ are disjoint.

Thus $\{U_n\}_{n=1}^{\infty}$ is a sequence of IID uniform $[0, 1]$ random variables. So from one uniform random variable, we have produced a countable sequence of IID uniform random variables.

3.3 d-ary digits

There was nothing special about the binary expansion here. We could have done this with the d-ary expansion of U for any integer $d > 1$. The d-ary digits of a uniform $[0, 1]$ random variable U will be IID random variables valued in $\{0, 1, \ldots, d-1\}$ with equal probabilities. Again taking prime power subsequences, we can build a countable sequence of IID uniform random variables exactly as above. Also, we could take any countable collection of disjoint countable sequences of integers in place of prime power subsequences.

3.4 Connection to dynamical systems

The constructions of countable sequences of uniform random variables using binary (and more generally) d-ary expansions ($d > 1$ an integer), come from the fact that the maps $T_d : [0, 1) \to [0, 1)$ given by

$$T_d(x) = dx - \lfloor dx \rfloor$$

preserve Lebesgue measure M. To see this, note that for any subinterval $(a, b) \subset [0, 1)$,

$$T_d^{-1}(a, b) = \bigcup_{i=0}^{d-1} \left(\frac{i+a}{d}, \frac{i+b}{d} \right),$$

which consists of d disjoint intervals of measure $\frac{b-a}{d}$, so

$$M(T_d^{-1}(a, b)) = d \times \frac{b-a}{d} = b - a = M(a, b).$$

3.5 Luroth expansions

Another nice example of a sequence of random variables one can generate using a dynamical system are *Luroth digits*. For $x \in (0, 1)$, we write

$$N(x) = \left\lfloor \frac{1}{x} \right\rfloor,$$

and define the *Luroth map* T_L by

$$T_L(x) = N(x)((N(x)+1)x - 1).$$

It is an excellent exercise to show that T_L preserves Lebesgue measure M and that if U is a uniform $[0, 1]$ random variable, the sequence $\{X_n\}_{n=0}^{\infty}$ given by

$$X_n(U) = N(T_L^n(U)$$

is a sequence of IID positive integer valued random variables with

$$P(X_1 = k) = \frac{1}{k(k+1)}, k \geq 1.$$

4. General sequences of random variables

4.1 Independent sequences

Combining our two observations:

Generating a CDF We can use a single uniform $[0, 1]$ random variable U to generate any CDF F.

Generating IID sequences We can use a single uniform $[0, 1]$ random variable U to generate a countable sequence of IID uniform random variables $\{U_i\}$.

If we want to simulate an sequence of *independent* random variables X_i with prescribed CDFs F_i, we can start with one uniform random variable U, generate a countable sequence of IID uniform random variables U_i, and use U_i to generate X_i.

4.2 General sequences

Similarly, if we want to simulate an arbitrary sequence of random variables X_i, let G_i be the CDF of the *marginal distribution* of X_i conditioned on $X_1, \ldots, X_{i-1}$. Then we can start with one uniform random variable U, generate a countable sequence of IID uniform random variables U_i, and then use U_i to generate the CDF G_i.

5. Brownian motion

5.1 Standard Brownian motion

Standard Brownian Motion (SBM) on $[0, 1]$ is a random process $\{X(t)\}_{t\in[0,1]}$ satisfying:

- The position $X(t)$ at any time $t \geq 0$ is a mean 0 normal random variable.
- If we sample at finite set of points $t_1, t_2, \ldots t_k$, the vector

$$\mathbf{X} = (X(t_1), \ldots, X(t_k))$$

  is normally distributed in $\mathbb{R}^k$.
- With mean $\mathbf{0} \in \mathbb{R}^k$ and covariance matrix $\mathbf{\Sigma}$ given by

$$\mathbf{\Sigma}_{i,j} = \mathrm{Cov}(X(t_i), X(t_j)) = \min(t_i, t_j).$$

- That is, the density of $\mathbf{X}$ is

$$\phi_{t_1,\ldots,t_k}(\mathbf{x}) = (2\pi)^{-k/2}\|\mathbf{\Sigma}\|^{-1/2} \exp\left(-\frac{1}{2}\mathbf{x}\mathbf{\Sigma}^{-1}\mathbf{x}^{\mathbf{T}}\right).$$

- $\mathbf{\Sigma} = (\min(t_i, t_j))_{1\leq i,j\leq k}$.

We've seen how from one uniform random variable, we can generate countable sequences of random variables. There are many ways to use a countable sequence of random variables to build SBM on $[0, 1]$.

5.2 Rational points

We can use a single uniform $[0, 1]$ random variable U to generate a countable sequence of random variables $\{X(t)\}_{t\in\mathbb{Q}\cap[0,1]}$ indexed by $\mathbb{Q} \cap [0, 1]$, such that

- The position $X(t)$ at any time $t \in [0, 1] \cap \mathbb{Q}$ is a mean 0 normal random variable.
- If we sample at finite set of points $t_1, t_2, \ldots t_k$, the vector

$$\mathbf{X} = (X(t_1), \ldots, X(t_k))$$

  is normally distributed in $\mathbb{R}^k$ with mean $\mathbf{0} \in \mathbb{R}^k$ and covariance matrix $\mathbf{\Sigma}$ given by

$$\mathbf{\Sigma}_{i,j} = \mathrm{Cov}(X(t_i), X(t_j)) = \min(t_i, t_j).$$

By construction of the process, it is not hard to show that if $\{t_n\}$ is a sequence of rational numbers converging to a real number t, the sequence $\{X(t_n)\}$ is a *Cauchy sequence*, so we can define

$$X(t) = \lim_{n\to\infty} X(t_n).$$

Thus, we have used one uniform $[0, 1]$ random variable U to build an SBM!

5.3 Wiener's construction

We recall Wiener's construction of SBM.

- Let

$$\{\eta_i\}_{i\geq 1}$$

  be a sequence of IID $N(0, 1)$ random variables.
- Let $\{\psi_j(\cdot)\}_{j\geq 1}$ be a complete orthonormal basis of continuous functions for $L^2([0, 1], dx)$ where dx is standard Lebesgue measure, for example, Legendre polynomials.
- For a positive integer N, define

$$B_N(t, \omega) \equiv \sum_{j=1}^{N} \eta_j(\omega) \int_0^t \psi_j(u)du.$$

Since we can use a single uniform random $[0, 1]$ variable U to generate the sequence $\{\eta_i\}_{i\geq 1}$ of IID standard normal ($N(0, 1)$) random variables, we can use it to build an SBM!

5.4 Donsker's construction

We now describe a construction of Brownian motion due to Donsker. Let $C_0[0, 1]$ denote the set of continuous functions $f : [0, 1] \to \mathbb{R}$ with $f(0) = 0$.

- X_i be i.i.d. Bernoulli random variables,

$$P(X_1 = 1) = P(X_1 = -1) = 1/2.$$

- Let

$$Y_n\left(\frac{j}{n}\right) = \frac{1}{\sqrt{n}}\sum_{i=1}^{j} X_i$$

for $0 \le j \le n$.
- Define $Y_n(t)$ by linear interpolation, between these values, for $0 \le t \le 1$.
- Let μ_n be the probability measure on $C_0[0, 1]$ supported on the set of realizations of Y_n, i.e., for a subset $A \subset C_0[0, 1]$,

$$\mu_n(A) = P(Y_n(\cdot) \in A).$$

That is, for $A \subset C_0[0, 1]$,

$$\mu_n(A) = P(Y_n(\cdot) \in A).$$

Theorem (Donsker). *There is a probability measure μ on $C_0[0, 1]$ so that for any continuous functional $F : C_0[0, 1] \to \mathbb{R}$,*

$$\lim_{n\to\infty} \int_{C_0[0,1]} F d\mu_n = \int_{C_0[0,1]} F d\mu.$$

That is, $\mu_n \to \mu$ in the weak- topology on the set of probability measures on $C_0[0, 1]$.*

Remarks.

- μ is the same as the Wiener measure.
- Donsker's construction works for any IID sequence $\{X_i\}$ with mean 0 and variance 1.

Since we can use a single uniform [0, 1] random variable U to generate an IID sequence $\{X_i\}$ with mean 0 and variance 1, we can use U to generate SBM!

5.5 Concluding remarks

Finally, we note our constructions are in fact more general.

5.5.1 SBM on $[0, \infty)$

Since SBM on $[0, \infty)$ can be generated by a countable collection of SBM's on $[0, 1)$, we can also generate SBM on $[0, \infty)$ using a single uniform [0, 1] random variable U.

5.5.2 Any continuous random variable

If Y is a real valued random variable such that

$$F(y) \equiv P(Y \le y)$$

is continuous for all $y \in \mathbb{R}$, then

$$U = F(Y)$$

is a uniform random variable. So we can use Y in place of U in all of the above constructions.

5.5.3 Random processes indexed by separable topological spaces

Suppose

$$\{X(t) : t \in T\}$$

is a (real-valued) stochastic process indexed by a topological space T such that

$$t \mapsto X(t)$$

is continuous in distribution. If there is a countable dense set $S \subset T$, then we can simulate $\{X(s) : s \in S\}$ using one uniform [0, 1] random variable U, and by continuity, we can simulate

$$\{X(t) : t \in T\}$$

as above.

Acknowledgments

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[2] https://en.wikipedia.org/wiki/Standard_probability_space.

A Stronger Model for Peg Solitaire

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Abstract. The main problem addressed here is to decide whether it is or not possible to go from a given position on a peg-solitaire board to another one. No non-trivial sufficient conditions are known, but tests have been devised to show it is not possible. We expose the way these tests work in a unified formalism and provide a new one which is strictly stronger than all the previous ones.

Keywords. Peg solitaire, Hi-Q, Pagoda function.

AMS classification. Primary 05A99; Secondary 91A46, 52B12, 90C08.

1. Introduction

Peg solitaire (also called Hi-Q) is a very simple board game that appeared in Europe most probably at the end of the 17th century. Its prior origin is unknown. The first evidence is a painting by Claude-Auguste Berey of Anne Chabot de Rohan (1663–1709) playing it. It seems to have then become popular in some royal courts. The mathematical study of the game started in 1710 when Leibniz writes a memoir on the subject [1]. We refer the reader to the excellent historical account presented in J. Beasley's book [2]. Let us introduce rapidly how this game is being played. The first data is a board $\mathfrak{S}$ which in first approximation may be thought of as a subset of $\mathbb{Z}^2$. The classical ones are the english board and the french one drawn below, and we present a third one introduced by J.C. Wiegleb in 1779 (see [2]).

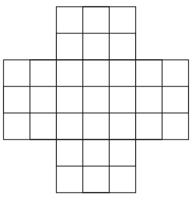

Figure 1. English board

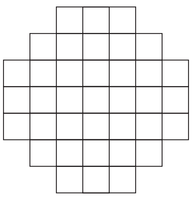

Figure 2. French board

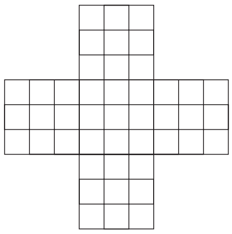

Figure 3. Wiegleb board

Each square of this board can hold at most one peg, and a *problem* as we define it here is to go from a given distribution of these pegs (say I) to another one (say J), via a succession of *legal moves* that we now define. Given three consecutive squares P, Q and R in a row or a column (but *not* on a diagonal), of which two consecutive (say P and Q) contain a peg while the third one (R) does not, a *legal move* consists in removing the two pegs in P and Q and putting one on the empty square R. We classically say that the closest peg in P jumps over the middle one in Q and lands in R, while destroying the peg in Q. As a trivial consequence, the number of pegs on the board decreases when the game proceeds further. For most authors, a problem consists in reducing the initial distribution of pegs, what we call thereafter the *initial position*, to a single peg via legal moves. They qualify the position as *solvable* if this is possible. We shall say that the problem in our sense is *feasible* if one can go from the initial position to the final one by using legal moves. Note that the number of such moves is known and equals the difference between the number of pegs in the initial position and the number of pegs in the final one (that is: $|I| - |J|$).

Given a problem, we can try all possible legal moves and repeat this action until the required number ($|I| - |J|$)

of moves is reached or no further move is possible. This process usually gets stuck because of the combinatorial explosion. For instance E. Harang [3] computed that there are 577 116 156 815 309 849 672 paths on the english board from the initial position consisting of the full board on which we leave the central square empty. Of which 40 861 647 040 079 968 lead to the final peg being on the central square. See also [4]. In fact, numerous setting tend to show that the problem is NP-complete. For this sentence to have a sense, we are to choose a way of extending the board to infinity, and there is no canonical fashion to achieve that. The case of an $n \times n$ board is studied in [5] while the $k \times n$ board with k fixed is shown to be linear in [6]. Of course, one may wonder whether the english board as a subset of a 7×7 board is tractable or not and the answer is still no, at least not without huge resources. The number of paths being enormous, we look for tests that will ensure us that it is not possible to solve a given problem. We would welcome any test that would guarantee the feasibility, but none are yet known.

The first of this test is attributed to M. Reiss in 1857 in [7] though J. Beasley traces it back to A. Suremain de Missery, a former officer of the French artillery, around 1842. We again refer the reader to [2] for more historical details. It is also described in Lucas book [8], which contains also more material and in the dedicated chapter of [9]. A seemingly more algebraic approach is proposed in [10], but it turns out to be only a different setting for the same test. This test is very often reduced by modern authors to the rule-of-three test (see below).

We shall first present these tests in a formalism that will help us clarify the situation; this formalism will also be adequate to present the advances realised on the subject in 1961/1962 at Cambridge university by a group of students (among which were J. Beasley) led by J. H. Conway.

We shall finally present a different test, which we term quadratic, and which is stronger than all previous ones. It however relies on solving a larger integer linear program and can sometime be resource demanding. We provide however numerous examples that we have discovered by exploring thousands of problems, and this in itself shows the practicality of the approach. The theory of this test in its purest form is complete, but we provide in the two last sections several improvements of it, on which we are still working. All examples have been computed via an intensive use of the lp_solve library [12], a GTK interface and a C-program both due to the author.

Let us end this introduction by mentioning that J. Beasley also introduced a very geometrical tool (the *in and out Theorems*), but it does not fit well in our framework and has not been worked out for an arbitrary problem (to the best of my knowledge at least), even if one remains on an english board. We shall not discuss it here. In more recent time, there has been attempts at working out a model of this game via string rewriting as in [6]. This approach remains however fundamentally one dimensional as are string rewriting rules. It has had applications though in describing the complexity of the game.

2. Main formalism of the linear board

Given a board $\mathfrak{S}$, we consider the $\mathbb{Z}$-module $\mathscr{F}(\mathfrak{S}, \mathbb{Z})$ of all rational integer valued functions over this board, and define similarly $\mathscr{F}(\mathfrak{S}, \mathbb{F}_2)$ and $\mathscr{F}(\mathfrak{S}, \mathbb{Q})$. This is one of the main step of the formalization: a position in the game is given by a subset $I \subset \mathfrak{S}$ (the set of squares containing a peg), which we model by its characteristic function $\mathbb{1}_I$. If $P \in \mathfrak{S}$, we note $\check{P}$ the function that is 1 in P and 0 everywhere else. A move is thus the function $\mathfrak{f} = \check{P} + \check{Q} - \check{R}$ and $\mathbb{1}_I - \mathfrak{f}$ should become another characteristic function; we have of course assumed that P, Q and R where three consecutive points in this order either in a row or in a column of $\mathfrak{S}$. We denote the set of these moves by $\mathscr{D}(\mathfrak{S})$. In the case of the english board, $\mathscr{D}(\mathfrak{S})$ has cardinality 76, while $\mathfrak{S}$ has cardinality 33.

Here comes the main remark. Assume we can go from I to J by the succession of legal moves $\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_k$. Then we have

$$\mathbb{1}_I - \mathbb{1}_J = \sum_{1 \le i \le k} \mathfrak{f}_i. \tag{1}$$

There are three ways to exploit this writing. We can say that

- $\mathbb{1}_I - \mathbb{1}_J$ is a rational integer linear combination of members of $\mathscr{D}(\mathfrak{S})$. This leads to the classical Reiss's theory, or to the lattice criterion of [11].
- $\mathbb{1}_I - \mathbb{1}_J$ is a linear combination with non-negative rational coefficients of members of $\mathscr{D}(\mathfrak{S})$. This leads to the main part of Conway's group theory.

- $\mathbb{1}_I - \mathbb{1}_J$ is a linear combination with non-negative integer coefficients of members of $\mathscr{D}(\mathfrak{S})$. This leads to what we call the *full linear test*, or also the non-negative integer test.

We introduce some notations

$$V(\mathfrak{S}, \mathbb{Z}) = \sum_{\mathfrak{f} \in \mathscr{D}(\mathfrak{S})} \mathbb{Z} \cdot \mathfrak{f} \tag{2}$$

and

$$V^+(\mathfrak{S}, \mathbb{Q}) = \sum_{\mathfrak{f} \in \mathscr{D}(\mathfrak{S})} \mathbb{Q}^+ \cdot \mathfrak{f} \quad , V^+(\mathfrak{S}, \mathbb{Z}) = \sum_{\mathfrak{f} \in \mathscr{D}(\mathfrak{S})} \mathbb{Z}^+ \cdot \mathfrak{f}. \tag{3}$$

3. Reiss theory and the rule-of-three test

Let us first expose rapidly and in modern notations the classical material. Since characteristic functions have values 0 or 1, it is tempting to look at $\mathbb{1}_I$ as taking its values in the field with two elements $\mathbb{F}_2$. To avoid confusion, we denote $\tilde{\mathbb{1}}_I$ this characteristic function as an element of $\mathscr{F}(\mathfrak{S}, \mathbb{F}_2)$. If one can go from the initial position I to the final one J by the succession of legal moves $\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_k$, one still has

$$\tilde{\mathbb{1}}_I - \tilde{\mathbb{1}}_J = \sum_{1 \le i \le k} \tilde{\mathfrak{f}}_i$$

where $\tilde{\mathfrak{f}}_i$ are of course the moves seen with values in $\mathbb{F}_2$. If $\mathfrak{f} = \check{P} + \check{Q} - \check{R}$, then $\tilde{\mathfrak{f}}$ is the function over $\mathfrak{S}$ that takes the value $1 \in \mathbb{F}_2$ at all the three points P, Q and R, and vanishes otherwise. However, $\mathbb{F}_2$ is now a field and $V(\mathfrak{S}, \mathbb{F}_2)$ is simply a vector space! Deciding whether $\tilde{\mathbb{1}}_I - \tilde{\mathbb{1}}_J$ belongs to it is a simple matter requiring only linear algebra.

Let us investigate this problem further. One way to characterize $V(\mathfrak{S}, \mathbb{F}_2)$ as a subspace of $\mathscr{F}(\mathfrak{S}, \mathbb{F}_2)$ is to compute equations of it. By using the canonical scalar product, this reduces to computing $V(\mathfrak{S}, \mathbb{F}_2)^\perp$ which means the elements $\chi \in \mathscr{F}(\mathfrak{S}, \mathbb{F}_2)$ such that

$$\forall \mathfrak{f} = \check{P} + \check{Q} - \check{R} \in \mathscr{D}(\mathfrak{S}), \quad \chi(P) + \chi(Q) = \chi(R) \tag{4}$$

since any such χ verifies

$$\forall g \in V(\mathfrak{S}, \mathbb{F}_2), \quad \sum_{A \in \mathfrak{S}} \chi(A) g(A) = 0. \tag{5}$$

We need a name for such elements of $V(\mathfrak{S}, \mathbb{F}_2)^\perp$, and we propose the name *witness*. Let us start to do so on the english board. Let us determine a function χ_0. We first fix four values on a square, for instance

		1	0	
		1	1	

Figure 4. Starting values

By using (4), we can readily extend these values:

		1	0	1
		1	1	0
		0	1	a

Figure 5. Extension

As it turns out, there are two ways to compute a: either by adding the two values on the column above its square or the two on the line containing it. The result is here the same $a = 1$. We can use this process to compute the values of χ_0 on the full board.

What is the dimension of $V(\mathfrak{S}, \mathbb{F}_2)$ in this case? The values on the initial square determine the values everywhere as we have just now remarked, and there is thus 16 witnesses. But these values are not linearly independant and there are linearly generated by the four

0	1
0	0

1	0
0	0

0	0
1	0

0	0
0	1

We can even use this process to extend the values to $\mathbb{Z}^2$. This yields

1	1	0	1	1	0	1	1	0	1	1	0	1
1	1	0	1	1	0	1	1	0	1	1	0	1
0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	0	1	1	0	1	1	0	1	1	0	1
1	1	0	1	1	0	1	1	0	1	1	0	1
0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	0	1	1	0	1	1	0	1	1	0	1
1	1	0	1	1	0	1	1	0	1	1	0	1
0	0	0	0	0	0	0	0	0	0	0	0	0

Figure 6. Over $\mathbb{Z}^2$

Now that the reader sees the regularity of this tiling, he will be convinced that they can be extended to $\mathbb{Z}^2$. The way one drops the english board on it yields for instance this witness:

1	1	0	1	1	0	1	1	0	1	1	0	1
1	1	0	1	1	0	1	1	0	1	1	0	1
0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	0	1	1	0	1	1	0	1	1	0	1
1	1	0	1	1	0	1	1	0	1	1	0	1
0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	0	1	1	0	1	1	0	1	1	0	1
1	1	0	1	1	0	1	1	0	1	1	0	1
0	0	0	0	0	0	0	0	0	0	0	0	0

And once, the witnesses are determined, equations defining $V(\mathfrak{S}, \mathbb{F}_2)$ are obtained by taking the scalar product with (a basis of) them. A classical problem is to determine whether it is possible to start with the french board filled with pegs, except for the central square that is left empty and to end with only one peg. This can be shown to be impossible by using the theory above, but we leave this pleasure to the reader.

This theory of witnesses is essentially what is called Reiss's theory [7], though it is expressed with other words, and is present in Lucas's book [8]. We say "essentially" because they do not use any linear algebra and that their way to reach this result is by using direct move together with reversed ones (to undo a move). They obtain what they call characteristic positions, which is equivalent to the equations defining $V(\mathfrak{S}, \mathbb{F}_2)$. This is however what is presented [10]. There are still a distinction to be made:

1. One can start from witnesses of $\mathbb{Z}^2$, restrict them to $\mathfrak{S}$ and get witnesses for this board. This is called the rule-of-three. Of course, we get only a four independant equations that may not define $V(\mathfrak{S}, \mathbb{F}_2)$ fully. If the board is thick enough, for instance when there exists a defining square from which all the other values of the witnesses can be deduced, this is enough.
2. One can start from $V(\mathfrak{S}, \mathbb{F}_2)$ and directly compute a basis of witnesses. This is required when the board is weakly connected (or even not connected!) and $V(\mathfrak{S}, \mathbb{F}_2)^{\perp}$ has dimension larger than 4. Several examples like that are given in [11].

4. The integer linear test and the lattice criterion

Thinking back in terms of $V(\mathfrak{S}, \mathbb{Z})$, the lattice criterion of [11] is to say that $\mathbb{1}_I - \mathbb{1}_J$ should belong to $V(\mathfrak{S}, \mathbb{Z})$. How is this test connected with the previous one? Or, alternatively: we decided to reduce the problem modulo 2; Why not try to do so modulo 3? Let us first note that we may identify $\mathscr{F}(\mathfrak{S}, \mathbb{F}_2)$ with $\mathscr{F}(\mathfrak{S}, \mathbb{Z})/2 \cdot \mathscr{F}(\mathfrak{S}, \mathbb{Z})$ via

$$\begin{array}{rcl} \tilde{\ }: \mathscr{F}(\mathfrak{S}, \mathbb{Z}) & \to & \mathscr{F}(\mathfrak{S}, \mathbb{F}_2) \\ g & \mapsto & \tilde{g} : \begin{array}{rcl} \mathfrak{S} & \to & \mathbb{F}_2 \\ P & \mapsto & g(P) \mod 2 \end{array} \end{array} . \tag{6}$$

During this process, $V(\mathfrak{S}, \mathbb{Z})$ is of course sent on $V(\mathfrak{S}, \mathbb{F}_2)$. Let us state formally two questions we want to answer:

1. Is $V(\mathfrak{S}, \mathbb{Z})$ a lattice of full rank in $\mathscr{F}(\mathfrak{S}, \mathbb{Z})$?
2. How to compute $\mathscr{F}(\mathfrak{S}, \mathbb{Z})/V(\mathfrak{S}, \mathbb{Z})$?

In the sequel, we introduce a hypothesis on the geometry of the board $\mathfrak{S}$ that will enables us to answer fully these questions. It will turn out that this will also exhibit the very tight link between the integer linear test and the theory of witnesses, as exposed in the previous section.

If the two points P and R of $\mathfrak{S}$ are extremities of a member of $\mathscr{D}(\mathfrak{S})$, we say that P and R are *neighbors* and we note $P \multimap R$. The reflexive and transitive closure of this relation is an equivalence relation, and if two points A and B are equivalent according to it, we note $A \equiv B$. We can now state an important definition:

Definition 4.1. *A board $\mathfrak{S}$ is said to be* with no isolated point *if for every point P of $\mathfrak{S}$, there exists a point $Q \equiv P$ and which is the middle point of a move.*

Most boards will verify this hypothesis. It means that each $\equiv$-equivalence class contains a middle point. However the number of such classes may vary. For a sufficiently thick board, there will be exactly 4 classes, but there may be more, if the board is not connected for instance, or contains thick chambers very weakly connected by only one square. The reader will easily construct examples of boards with no isolated point but where the number of classes is larger than 4. The following Theorem is central in our discussion:

Theorem 4.1. *If $\mathfrak{S}$ is with no isolated point, then $2\mathscr{F}(\mathfrak{S}, \mathbb{Z}) \subset V(\mathfrak{S}, \mathbb{Z})$.*

A final notation before sketching the proof: if $\mathfrak{f} = \check{P} + \check{Q} - \check{R} \in \mathscr{D}(\mathfrak{S})$, we note $\mathfrak{f}' = -\check{P} + \check{Q} + \check{R}$ the reversed move (with equal middle point).

Proof. We show that for every $P \in \mathfrak{S}$, we have $2\check{P} \in V(\mathfrak{S}, \mathbb{Z})$. If P is a middle point, say of the move $\mathfrak{f}$, then $2\check{P} = \mathfrak{f} + \mathfrak{f}'$ belongs to $V(\mathfrak{S}, \mathbb{Z})$. Otherwise, there exists a chain $P = P_0 \mathrel{-\!\circ\!-} P_1 \mathrel{-\!\circ\!-} \dots \mathrel{-\!\circ\!-} P_n$ where P_n is a middle point. Furthermore, by definition, there exists $\mathfrak{f}_i \in \mathscr{D}(\mathfrak{S})$ such that $2\check{P}_i - 2\check{P}_{i+1} = \mathfrak{f}_i - \mathfrak{f}_i'$ for every $i = 0, \dots, n-1$. Finally, we can also write $2\check{P}_n = \mathfrak{f}_n + \mathfrak{f}_n'$ for some $\mathfrak{f}_n \in \mathscr{D}(\mathfrak{S})$. Summing up all these equations, we reach

$$2\check{P}_0 = \mathfrak{f}_0 - \mathfrak{f}_0' + \mathfrak{f}_1 - \mathfrak{f}_1' + \cdots + \mathfrak{f}_{n-1} - \mathfrak{f}_{n-1}' + \mathfrak{f}_n + \mathfrak{f}_n' \in V(\mathfrak{S}, \mathbb{Z}),$$

which is the required conclusion since $P = P_0$. □

This Theorem has several consequences. First of all, on such boards, the $\mathbb{Q}$-vector spanned by the $\mathfrak{f}$'s (that would be $V(\mathfrak{S}, \mathbb{Q})$) is the whole space: $V(\mathfrak{S}, \mathbb{Z})$ is a sublattice of $\mathscr{F}(\mathfrak{S}, \mathbb{Z})$ of full rank. Let us note the following Lemma that will be required later:

Lemma 4.1. *If $\mathfrak{S}$ has no isolated points, we have $|\mathfrak{S}| \leq |\mathscr{D}(\mathfrak{S})| \leq 4|\mathfrak{S}| - 8$.*

Proof. The lower bound comes from the fact that $\mathscr{D}(\mathfrak{S})$ generates $\mathscr{F}(\mathfrak{S}, \mathbb{Q})$. For the upper bound, count horizontal and vertical moves separately. For the horizontal (resp. vertical) ones, count the moves according to their left-hand side (resp. lower) point. The lemma follows readily. □

As a main consequence, we have the following Theorem.

Theorem 4.2. *Assume $\mathfrak{S}$ to be with no isolated point and let $g \in \mathscr{F}(\mathfrak{S}, \mathbb{Z})$. Then*

$$g \in V(\mathfrak{S}, \mathbb{Z}) \iff \tilde{g} \in V(\mathfrak{S}, \mathbb{F}_2).$$

(See (6) *for the definition of $\tilde{g}$).*

Proof. Indeed, the direct implication is obvious, while the reversed one follows from Theorem 4.1: we know that $g \in V(\mathfrak{S}, \mathbb{Z}) + 2 \cdot \mathscr{F}(\mathfrak{S}, \mathbb{Z})$ but this last space is nothing but $V(\mathfrak{S}, \mathbb{Z})$. □

This Theorem tells us that the lattice criterion is *not* stronger than Reiss's theory, when properly understood, and provided we restrict our attention to non-pathological boards.

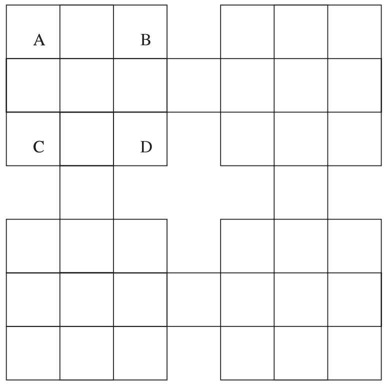

Figure 7. A pathological board

In fact [11] do not even give a single example when reduction modulo 2 does not solve the problem. Here is one:

The total number of pegs on the squares A, B, C and D remains constant. It is not difficult to see that this example is in fact general and we have:

Theorem 4.3. *A board $\mathfrak{S}$ is with no isolated point if and only if $V(\mathfrak{S}, \mathbb{Z})$ has maximal rank in $\mathscr{F}(\mathfrak{S}, \mathbb{Z})$.*

On boards with no isolated points, reducing the situation modulo any odd integer is not going to give any information; indeed Theorem 4.2 implies (after some work) that

$$V(\mathfrak{S}, \mathbb{Z}/m\mathbb{Z}) = \mathscr{F}(\mathfrak{S}, \mathbb{Z}/m\mathbb{Z}) \quad \text{(whenever } m \text{ is odd).}$$

Notice finally that $\mathscr{F}(\mathfrak{S}, \mathbb{Z})/V(\mathfrak{S}, \mathbb{Z})$ is simply a product of copies of $\mathbb{Z}/2\mathbb{Z}$ in this case. It is not difficult to tackle the case with isolated points by generalising the reasoning used for the board drawn Fig. 7, and get that $\mathscr{F}(\mathfrak{S}, \mathbb{Z})/V(\mathfrak{S}, \mathbb{Z})$ is always a product of copies of $\mathbb{Z}/2\mathbb{Z}$ with copies of $\mathbb{Z}$. These results have no influence on what we develop hereafter, so we do not provide any formal proof.

5. Resource counts, pagoda functions and the linear test in non-negative rationals

The next main step takes place in 1961/1962 at Cambridge university when J. H. Conway led a group of students that studied this game. They came out with another and different test, also clearly explained in [9] and that we now describe.

This test exploits the fact that (1) has non-negative coefficients, i.e. the test consists in writing that, if we can go from I to J with legal moves, then

$$\mathbb{1}_I - \mathbb{1}_J \in V^+(\mathfrak{S}, \mathbb{Q}). \tag{7}$$

As it turns out, $V^+(\mathfrak{S},\mathbb{Q})$ is a cone in a vector space, and determining whether a point belongs to it or not is fast. We know generators of this cone (the elements of $\mathscr{D}(\mathfrak{S})$; they can be shown to be generator of its extreme half-lines), and it would be interesting to determine equations for its facets. The paper [14] gives properties of these facets. In [9] as well as in [2], so called *resource counts* or *pagoda functions* are introduced. These are functions π on $\mathfrak{S}$ such that

$$\forall \mathfrak{f} = \check{P} + \check{Q} - \check{R} \in \mathscr{D}(\mathfrak{S}), \quad \pi(P) + \pi(Q) \geq \pi(R). \tag{8}$$

As a consequence, for any such function and if g belongs to $V^+(\mathfrak{S},\mathbb{Q})$, one has

$$\langle \pi, g \rangle = \sum_{A \in \mathfrak{S}} \pi(A) g(A) \geq 0. \tag{9}$$

In particular, if one can derive J from I with legal moves, then $\langle \pi, \mathbb{1}_I \rangle$ is not less than $\langle \pi, \mathbb{1}_J \rangle$. Here are some examples

		-1	0	-1		
		1	0	1		
0	0	0	0	0	0	0
1	0	1	0	1	0	1
0	0	0	0	0	0	0
		1	0	1		
		-1	0	-1		

Figure 8. A resource count

		0	8	0		
		0	5	0		
-3	3	0	3	0	3	-3
2	2	0	2	0	2	2
-1	1	0	1	0	1	-1
		0	1	0		
		0	0	0		

Figure 9. Another resource count

Determining which of these corresponds to equations of facets would be very valuable, but their structure seems too intricate to classify them in a small number of regular families. For instance , a direct computation in case of the english board stumbles on the fact that there are an enormous

							-5
	1	0	1	1	2	3	5
	0	0	0	0	0	0	0
	1	0	1	1	2	3	5
	1	0	1	1	2	3	5
	2	0	2	2	4	6	10
	3	0	3	3	6	9	15
-5	5	0	5	5	10	15	20

Figure 10. A third resource count

quantity of such facets for a human eye to be able to look at them and derive some patterns. It is not sure that this path is blocked, though I tend to believe it is.

We do not dwell any further in this part of the theory since it is extremely well exposed and detailed in [9], [2] and on a number of web pages. The reader will most probably better unterstand the strength of this theory by looking at section 8 of this paper.

We should stress out here that the approach of this Cambridge group is commonly reduced to the use of real-valued "pagoda" functions as above. This is an extremely minimal understanding of their work and for instance does not account for the GNP balance sheet, what J. Beasley in [2] calls Conway's balance sheet in his chapter 6; this one is however one of the main tool of [9]. It mixes *integer valued* pagoda functions together with such functions with values in $\mathbb{F}_2$. Beasley's use of pagoda functions which he calls ressource counts (see chapter 5 of [2]) relies already on the integer character of the values taken: that is how he builds his "move map".

The GNP diagram, or GNP balance sheet, is somewhat off our framework, and is in fact superseded by the next test.

6. The linear test in non-negative integers

The third test consists in combining both preceding ideas and write that if we can go from I to J with legal moves, then

$$\mathbb{1}_I - \mathbb{1}_J \in V^+(\mathfrak{S},\mathbb{Z}). \tag{10}$$

This time, deciding that an element belongs to the integer points of a cone is NP-hard, but in practice, it takes only some fraction of a second on an english board (this was not the case in 1962!). We have of course

$$V^+(\mathfrak{S},\mathbb{Z}) \subset V(\mathfrak{S},\mathbb{Z}) \bigcap V^+(\mathfrak{S},\mathbb{Q}) \tag{11}$$

and this inclusion is strict, even when one restricts our attention to differences of characteristic functions. For instance this test shows that one cannot go from the position of Fig. 11 to only the central peg while the rational and integer linear tests are passed. This example is interesting in showing the impact of the board, for it is feasible in legal moves if we add to the english board the grey square on the upper right side.

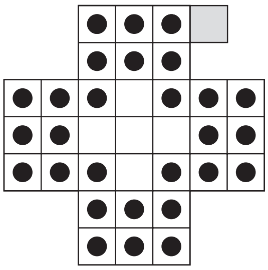

Figure 11. Impossible

We present a smaller counterexample in Figs. 12 and 13 that enables easier direct computations.

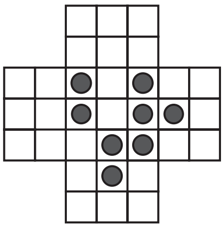

Figure 12. Starting position

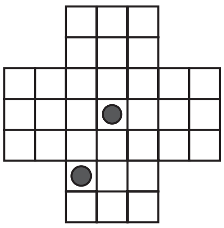

Figure 13. Ending position

When g belongs to this intersection (i.e. the right-hand side of (11)) the denominators in a non-negative writing do not seem to be any worse than $1/2$. Here is the conjecture we make:

Conjecture 6.1. *If $\mathfrak{S}$ has no isolated points, then*

$$V(\mathfrak{S}, \mathbb{Z}) \bigcap V^+(\mathfrak{S}, \mathbb{Q}) \subset \tfrac{1}{2} V^+(\mathfrak{S}, \mathbb{Z}).$$

Here is another related conjecture that may be easier to handle (and maybe easier to disprove!).

Conjecture 6.2. *Let $\mathcal{B} \subset \mathscr{D}(\mathfrak{S})$ be a basis of $V(\mathfrak{S}, \mathbb{Q})$. If $\mathfrak{S}$ has no isolated points, then*

$$2\mathscr{F}(\mathfrak{S}, \mathbb{Z}) \subset V(\mathcal{B}) = \sum_{\mathfrak{f} \in \mathcal{B}} \mathbb{Z} \cdot \mathfrak{f}.$$

The condition on $\mathfrak{S}$ cannot be removed since it is equivalent to $V(\mathfrak{S}, \mathbb{Z})$ being of full rank.

At this point, we have described the situation and we hope the reader is now able to understand properly what is what. The theory so far has two drawbacks: it draws only on properties of $\mathbb{1}_I - \mathbb{1}_J$, and it does not use the order in which the moves are played. Our next criteria, the simple quadratic test, will not go beyond this abelian nature, but will break the first hurdle. It is better to investigate the game a bit further before exposing it.

7. How integer linear programming is used

The cone $V^+(\mathfrak{S}, \mathbb{Z})$ is determined by the set $\mathscr{D}(\mathfrak{S})$ of generators. Let us introduce the notation $\check{\mathfrak{f}}$ for the function over $\mathscr{D}(\mathfrak{S})$ that is 1 in $\mathfrak{f}$ and 0 everywhere else. We consider the map

$$\begin{array}{rccc} \Psi: & \mathscr{F}(\mathscr{D}(\mathfrak{S}), \mathbb{Z}) & \to & V(\mathfrak{S}, \mathbb{Z}) \\ & F = \sum_{\mathfrak{f} \in \mathscr{D}(\mathfrak{S})} x(\mathfrak{f})\, \check{\mathfrak{f}} & \mapsto & \sum_{\mathfrak{f} \in \mathscr{D}(\mathfrak{S})} x(\mathfrak{f})\, \mathfrak{f}. \end{array} \tag{12}$$

The integer linear program we write is simply to minimize any linear form of the $(x(\mathfrak{f}))_{\mathfrak{f}}$ subject to the constraints

$$\forall \mathfrak{f} \in \mathscr{D}(\mathfrak{S}),\ x(\mathfrak{f}) \geq 0, \quad \text{and} \quad \Psi(F) = \mathbb{1}_I - \mathbb{1}_J.$$

The linear form we choose is usually $\sum_{\mathfrak{f}} x(\mathfrak{f})$ since we know what should be its value if a solution exists.

8. Thickness of a move

Given a problem, say from I to J, we define the *thickness* of the move $\mathfrak{f}$ to be the maximum number of times this move can be used, whatever sequence of legal moves $\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_k$ we choose. This thickness is zero allover if the problem is not feasible. In general, given $h \in V^+(\mathfrak{S}, \mathbb{Z})$, we shall speak of the *thickness of* $\mathfrak{f}$ *at* h. Computing this quantity is naturally difficult, but we can bound it from above and even provide a uniform bound for it. The main Theorem reads as follows

Theorem 8.1. *Let $h \in V^{+}(\mathfrak{S}, \mathbb{Z})$, $\mathfrak{f}_0 \in \mathscr{D}(\mathfrak{S})$ and π be a resource count on $\mathfrak{S}$ such that $\langle \pi, \mathfrak{f}_0 \rangle = 1$. The move $\mathfrak{f}_0$ can appear at most $\langle \pi, h \rangle$ in any writing of h as a linear combination of elements of $\mathscr{D}(\mathfrak{S})$ with non-negative integer coefficients.*

The scalar product $\langle \pi, h \rangle$ is defined in (9). We can derive absolute bounds from this Theorem by using a variant of a resource count already used by Conway. First note that we are interested only in the case $h = \mathbb{1}_I - \mathbb{1}_J$ which implies that $|h(A)| \leq 1$ for all $A \in \mathfrak{S}$. Now let $\rho = (\sqrt{5} - 1)/2$ be a solution of $x^2 + x = 1$. To each point $(a, b) \in \mathbb{Z}^2$, we associate the weight $\pi(a, b) = \rho^{|a|+|b|}$. Next, we drop our board $\mathfrak{S}$ on $\mathbb{Z}^2$ in such a way that the middle point of $\mathfrak{f}_0$ be the $(0, 0)$ element. The reader will check that the restriction of π to $\mathfrak{S}$ is a resource count on $\mathfrak{S}$ which we denote again by π. We have $\langle \pi, \mathfrak{f}_0 \rangle = 1$, while

$$|\langle \pi, h \rangle| \leq \langle \pi, \mathbb{1}_{\mathfrak{S}} \rangle \leq 8\rho + 13 = 17.944 \cdots .$$

This short argument show that the thickness of any move on any board is bounded above by 17. This is most probably a way too large majorant (reaching a thickness of 4 is already extremely difficult, and it can be shown on using better resource counts that the maximal thickness on the english board is at most 5), but it is *universal*, i.e. independant of the board we choose.

A similar argument is also the main ingredient of [6] (see Theorem 3.1 therein, with most probably a wrong computation at the end. The 26 of this result is to be replaced by a 34 but this leaves the rest of the argument intact), and is the basis on which rely the low complexity results.

Given a problem, we can refine this upper bound by selecting a more appropriate resource count. Furthermore, once a majorant is given, say m, we can check whether $\mathbb{1}_I - \mathbb{1}_J - m\mathfrak{f}_0$ is feasible or not (this means, whether it passes whichever test we select). If not, we decrement m and repeat the process.

9. A simple quadratic test

Let us consider the two following problems: we are to go from the left hand side position with only the black pegs (or with the grey peg added) to the right hand side one with a sole black peg (or with the grey peg added). Both problems pass the positive integer test. The reader will easily check that the larger problem (with the grey peg) is in fact doable in *legal moves*, which implies that *no* test relying only on $\mathbb{1}_I - \mathbb{1}_J$ would be able to show the first problem to be impossible. The quadratic test we propose now is however able to show this impossibility.

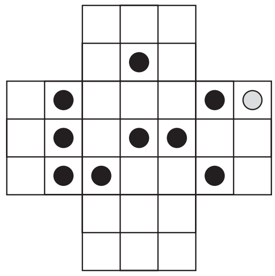

Figure 14. Starting position

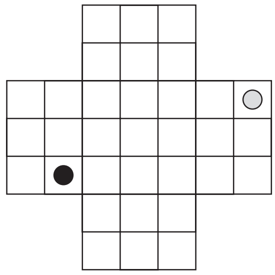

Figure 15. Ending position

Let us start our description of the quadratic test.

9.1 The geometrical support

To each couple $(A, B) \in \mathfrak{S} \times \mathfrak{S}$, we associate a symbol $A \boxtimes B$, to which we add the property

$$A \boxtimes B = B \boxtimes A. \tag{13}$$

We set

$$\mathfrak{S} \boxtimes \mathfrak{S} = \{A \boxtimes B, A, B \in \mathfrak{S}\}. \tag{14}$$

We next consider functions on $\mathfrak{S} \boxtimes \mathfrak{S}$. We denote by $A \check{\boxtimes} B$ the function that is 1 on $A \boxtimes B$ and 0 everywhere else. Note that $A \check{\boxtimes} B = B \check{\boxtimes} A$. We go from $\mathscr{F}(\mathfrak{S}, \mathbb{Q})^2$ to $\mathscr{F}(\mathfrak{S} \boxtimes \mathfrak{S}, \mathbb{Q})$ by

$$\begin{aligned} \boxtimes \quad : \quad \mathscr{F}(\mathfrak{S}, \mathbb{Q}) \times \mathscr{F}(\mathfrak{S}, \mathbb{Q}) &\to \mathscr{F}(\mathfrak{S} \boxtimes \mathfrak{S}, \mathbb{Q}) \\ (g_1, g_2) &\mapsto g_1 \boxtimes g_2 \\ &= \sum_{A, B \in \mathfrak{S}} g_1(A) g_2(B) A \check{\boxtimes} B. \end{aligned}$$

Notice that the value of $g_1 \boxtimes g_2$ on $A \check{\boxtimes} B$ is $g_1(A)g_2(B) + g_1(B)g_2(A)$ if $A \neq B$ and $g_1(A)g_2(A)$ if $A = B$.

9.2 The effect of legal moves

Assume now that we can go from I to J by the legal move $\mathfrak{f} \in \mathscr{D}(\mathfrak{S})$. We have

$$\begin{aligned}\mathbb{1}_I \boxtimes \mathbb{1}_I &= (\mathbb{1}_J + \mathfrak{f}) \boxtimes (\mathbb{1}_J + \mathfrak{f}) \\ &= \mathbb{1}_J \boxtimes \mathbb{1}_J + \mathfrak{f} \boxtimes \mathbb{1}_J + \mathbb{1}_J \boxtimes \mathfrak{f} + \mathfrak{f} \boxtimes \mathfrak{f}.\end{aligned}$$

On using the identity $\mathfrak{f} \boxtimes \mathbb{1}_J = \mathbb{1}_J \boxtimes \mathfrak{f}$, we reach

$$\mathbb{1}_I \boxtimes \mathbb{1}_I = \mathbb{1}_J \boxtimes \mathbb{1}_J + (2\mathbb{1}_J + \mathfrak{f}) \boxtimes \mathfrak{f}.$$

We note that

$$2\mathbb{1}_J + \mathfrak{f} = \sum_{\substack{A \in J \\ \mathfrak{f}(A)=0}} 2\check{A} + |\mathfrak{f}|,$$

from which we infer

$$\mathbb{1}_I \boxtimes \mathbb{1}_I = \mathbb{1}_J \boxtimes \mathbb{1}_J + |\mathfrak{f}| \boxtimes \mathfrak{f} + \sum_{\substack{A \in J \\ \mathfrak{f}(A)=0}} 2\check{A} \boxtimes \mathfrak{f}. \quad (15)$$

This is the equation we want to exploit; we do so in pretty much the same way we exploited (1). We set

$$\begin{aligned}\mathscr{D}(\mathfrak{S} \boxtimes \mathfrak{S}) = \{2\check{A} \boxtimes \mathfrak{f}, \quad A \in \mathfrak{S}, \mathfrak{f} \in \mathscr{D}(\mathfrak{S})/\mathfrak{f}(A) = 0\} \\ \times \bigcup \{|\mathfrak{f}| \boxtimes \mathfrak{f}, \quad \mathfrak{f} \in \mathscr{D}(\mathfrak{S})\}. \quad (16)\end{aligned}$$

Note that if $\mathfrak{f} = \check{P} + \check{Q} - \check{R}$ then

$$|\mathfrak{f}| \boxtimes \mathfrak{f} = P \check{\boxtimes} P + 2P \check{\boxtimes} Q + Q \check{\boxtimes} Q - R \check{\boxtimes} R. \quad (17)$$

We define our cone by

$$V^+(\mathfrak{S} \boxtimes \mathfrak{S}, \mathbb{Z}) = \sum_{\mathfrak{c} \in \mathscr{D}(\mathfrak{S} \boxtimes \mathfrak{S})} \mathbb{Z}^+ \cdot \mathfrak{c}. \quad (18)$$

A problem being given by an initial position I and a final one J, the *simple quadratic test* consists is saying that $\mathbb{1}_I \boxtimes \mathbb{1}_I - \mathbb{1}_J \boxtimes \mathbb{1}_J \in V^+(\mathfrak{S} \boxtimes \mathfrak{S}, \mathbb{Z})$, which can again be solved with integer linear programming. However the spaces are much larger, and the resolution becomes more troublesome. Note the following Lemma:

Lemma 9.1.

$$\begin{aligned}|\mathfrak{S} \boxtimes \mathfrak{S}| &= |\mathfrak{S}|(|\mathfrak{S}| + 1)/2, \\ |\mathscr{D}(\mathfrak{S} \boxtimes \mathfrak{S})| &= (|\mathfrak{S}| - 2)|\mathscr{D}(\mathfrak{S})|.\end{aligned}$$

Indeed, there are $|\mathscr{D}(\mathfrak{S})|$ moves of type $|\mathfrak{f}| \boxtimes \mathfrak{f}$, and, for each $\mathfrak{f} \in \mathscr{D}(\mathfrak{S})$, there are $|\mathfrak{S}| - 3$ moves of type $2\check{A} \boxtimes \mathfrak{f}$ with $\mathfrak{f}(A) = 0$. For the english board, the cardinality of $|\mathscr{D}(\mathfrak{S} \boxtimes \mathfrak{S})|$ is thus 2 356 for a board of 561 squares.

We have already given an example showing that this test is sometimes better than the linear test with non-negative integer coefficients but we show now that this is always the case. To do so, let us define

$$\begin{cases}\mathscr{F}_0(\mathfrak{S} \boxtimes \mathfrak{S}, \mathbb{Z}) = \sum_{A \in \mathfrak{S}} \mathbb{Z} \cdot A \check{\boxtimes} A \\ \qquad\qquad + \sum_{A \neq B \in \mathfrak{S}} \mathbb{Z} \cdot 2A \check{\boxtimes} B \\ W(\mathfrak{S} \boxtimes \mathfrak{S}, \mathbb{Z}) = \sum_{A \neq B \in \mathfrak{S}} \mathbb{Z} \cdot 2A \check{\boxtimes} B.\end{cases}$$

Then we can easily identify $\mathscr{F}_0(\mathfrak{S} \boxtimes \mathfrak{S}, \mathbb{Z})/W(\mathfrak{S} \boxtimes \mathfrak{S}, \mathbb{Z})$ with the space of integer valued functions on $\{A \check{\boxtimes} A, A \in \mathfrak{S}\}$, which we can in turn identify with $\mathfrak{S}$. By these identifications, we start with a function $h \in \mathscr{F}(\mathfrak{S}, \mathbb{Z})$, build $h \boxtimes h \in \mathscr{F}_0(\mathfrak{S} \boxtimes \mathfrak{S}, \mathbb{Z})$ and is next send to h. In particular, we get

$$\begin{aligned}\mathbb{1}_I \boxtimes \mathbb{1}_I - \mathbb{1}_J \boxtimes \mathbb{1}_J &\in V^+(\mathfrak{S} \boxtimes \mathfrak{S}, \mathbb{Z}) \\ \implies \mathbb{1}_I - \mathbb{1}_J &\in V^+(\mathfrak{S}, \mathbb{Z}). \quad (19)\end{aligned}$$

The fact that this test is in fact strictly superior on some boards in shown by the problem described by Figs. 14 and 15.

10. A quadratic test, with flatness constraints

If the simple quadratic test is stronger than the linear one with positive integers, it turns out when used to be lacking in efficiency. The last term in (15) can be written as $2\mathbb{1}_K \boxtimes \mathfrak{f}$ where $K \subset \mathfrak{S}$ avoids the support of $\mathfrak{f}$. This is much better than saying that it is a linear combination of $2\check{A} \boxtimes \mathfrak{f}$, but it leads to $2^{|\mathfrak{S}|-3}|\mathscr{D}(\mathfrak{S})| + |\mathscr{D}(\mathfrak{S})|$ generators! This is of course way too much and makes this new set of generators impractical. However, if $\mathcal{F}$ is a succession of legal moves from I to J, we can write

$$\mathbb{1}_I \boxtimes \mathbb{1}_I - \mathbb{1}_J \boxtimes \mathbb{1}_J = \sum_{\mathfrak{f}} x(\mathfrak{f})|\mathfrak{f}| \boxtimes \mathfrak{f} + \sum_{\mathfrak{f}} \sum_{A} y_{\mathfrak{f}}(A) 2\check{A} \boxtimes \mathfrak{f}. \quad (20)$$

And we readily see that on this writing that the following inequalities are satisfied

$$0 \leq y_{\mathfrak{f}}(A) \leq x(\mathfrak{f}). \quad (21)$$

We call them the *flatness constraints*. Despite their number, these constraints renders the quadratic test much more efficient. In fact, The $x(\mathfrak{f})$ are related to the usual linear moves by

$$\mathbb{1}_I - \mathbb{1}_J = \sum_{\mathfrak{f}} x(\mathfrak{f})\mathfrak{f} \tag{22}$$

(see the process that enabled us to prove (19)) and as such can be controlled in size by the thickness of $\mathfrak{f}$ at $\mathbb{1}_I - \mathbb{1}_J$, as defined in section 8.

On an english board, the $x(\mathfrak{f})$'s are seldomly larger than 4, and on arbitrary board they are anyway bounded.

Notice that if $\mathbb{1}_I \boxtimes \mathbb{1}_I - \mathbb{1}_J \boxtimes \mathbb{1}_J$ passes this test, then actually, it can be written as a linear combination with non-negative integer coefficients of $2\check{A} \boxtimes \mathfrak{f}$ with $\mathfrak{f}(A) = 0$ and diagonal moves $|\mathfrak{f}| \boxtimes \mathfrak{f}$. To realize such a writing, given $\mathfrak{f}$, simply collect together all A's for which $y_{\mathfrak{f}}(A)$ has a given value into a set $\mathcal{A}$. Note that these sets $\mathcal{A}$ are *not* the same as the sets K we used at the very beginning of this section, but are of same use.

The problem described by Figs. 16 and 17 goes through the quadratic test with no flatness constraints, but is shown impossible as soon as we add these constraints:

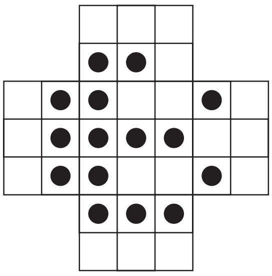

Figure 16. Starting position

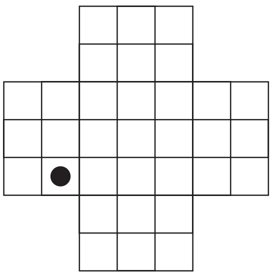

Figure 17. Ending position

This new test is the main novelty of this paper and is extremely efficient in practice, though it requires a processor to carry out the required computations.

We end this part with three further examples of problems shown to be impossible via the quadratic test with flatness constraints. Here are two problems, with a same starting position but different ending positions. None of them go through the quadratic test with flatness constraints:

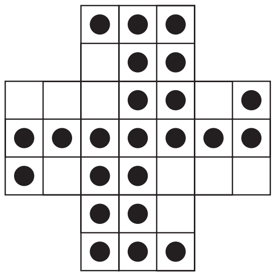

Figure 18. Starting position

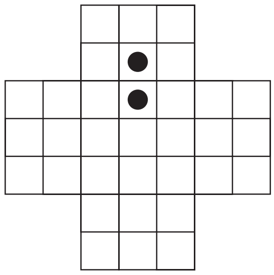

Figure 19. First ending position

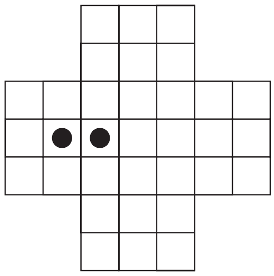

Figure 20. Second ending position

The third example is to go from the initial position to the intermediate ending position. This is shown to be impossible via the quadratic test with flatness constraints, though it again passes the simple quadratic test. Moreover, the problem to go from the initial position to the final ending position is feasible in legal moves.

11. Additional constraints, a first draft

Now that we have seen that the quadratic test with flatness constraints is very efficient, it is tempting to try to add some further constraints. This is the topic of these two last sections, but this part is still very much in progress. The reader may get the impression that it is not so much in progress than more bluntly unfinished. After some months of efforts, I have not been able to derive a unifying setup for what look like protrusions of a hidden structure, which is why I deliver them in this state.

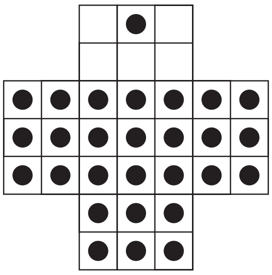

Figure 21. Starting position

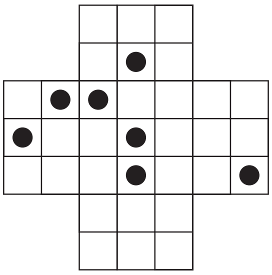

Figure 22. Intermediate ending position

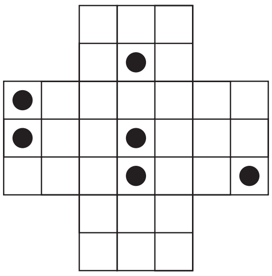

Figure 23. Final ending position

The idea we follow is to add geometrical information to control as much as possible these new variables $y_{\mathfrak{f}}(A)$ in (20).

Let us start with a fundamental inequality.

Proposition 11.1. *Assume we can go from I to J in legal moves. Then there exists a writing of $\mathbb{1}_I \boxtimes \mathbb{1}_I - \mathbb{1}_J \boxtimes \mathbb{1}_J$ (as in* (20)*) such that for every $A \in \mathfrak{S}$ we have*

$$0 \leq \sum_{\mathfrak{f}} y_{\mathfrak{f}}(A) + \sum_{\mathfrak{f}/\mathfrak{f}(A)\neq 0} x(\mathfrak{f}) \leq |I| - |J|. \tag{23}$$

See (28) and (29) for refinements. Let $\mathcal{F}$ be a succession of legal moves from I to J. We set

$$\mathfrak{p}(A, \mathcal{F}) = \sum_{\mathfrak{f}} y_{\mathfrak{f}}(A) + \sum_{\mathfrak{f}/\mathfrak{f}(A)\neq 0} x(\mathfrak{f}) \tag{24}$$

where the $y_{\mathfrak{f}}(A)$'s and the $x(\mathfrak{f})$'s come from (20).

Proof. Given a move $\mathfrak{f}$, let us look at the situation of the board before using this move. There are four possibilities for A:

- $\mathfrak{f}(A) = 1$, which means that A is on the board and participates to the move. It is counted in $x(\mathfrak{f})$ and nowhere else.
- A is not on the board but is created by the move. It is counted in $x(\mathfrak{f})$ and nowhere else.
- A is on the board but does not participate to the move. It is counted in $y_{\mathfrak{f}}(A)$ and nowhere else.
- A is not on the board and not created by the move. It is not counted anywhere.

The proof follows by using this remark and an induction on $|I| - |J|$. We have equality if and only if the last case above never occurs, which means that A is never absent from the position for two consecutive moves. □

We have seen that we can have equality in (23), but we can even show that the right hand side is on average of the correct order of magnitude. Indeed we have

$$\begin{aligned}\sum_{A\in\mathfrak{S}} \mathfrak{p}(A, \mathcal{F}) &= |I| - 2 + |I| - 3 + \cdots + |J| - 1 \\ &\quad + 3(|I| - |J|) \\ &= (|I| - |J|)\frac{|I| + |J| + 3}{2}\end{aligned}$$

since there are $|I| - 2$ points on the first move that are on the board but do not participate to the move, then $|I| - 3$, and so on. As a consequence

$$\frac{1}{|\mathfrak{S}|}\sum_{A\in\mathfrak{S}} \mathfrak{p}(A, \mathcal{F}) = \big(|I| - |J|\big)\frac{|I| + |J| + 3}{2|\mathfrak{S}|}.$$

This shows that (23) prevents too wide deviations from the mean, at least if $|I| + |J|$ and $|\mathfrak{S}|$ are of comparable size. We propose to improve on this double inequality in three ways.

11.1 Using the speed at which a peg gets inside J

We define the *depth* of the point A with respect to the position S containing it to be the minimum number Depth(A, S) of legal moves required to remove the peg in A. If A is not in S, we set $\mathrm{Depth}(A, S) = 0$. Let us recall a classical Lemma.

Lemma 11.1 (Leibniz). *If the sequence of legal moves $\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_k$ goes from I to J, then the sequence of legal moves $\mathfrak{f}_k, \ldots, \mathfrak{f}_2, \mathfrak{f}_1$ goes from $\mathfrak{S}\backslash J$ to $\mathfrak{S}\backslash I$.*

It is enough to verify this property when $k = 1$ where it is obvious. Leibniz expressed this idea in a different manner: he started from the final position J and tried to recover the initial one by playing in reverse; he discovered it was the same game, provided one considered the empty squares as having a peg, and the ones with a peg as being empty. This is exactly what we shall consider. Indeed, given a point A out of our final position J, there is a minimal number a moves that will "bring" its peg inside J, or kill it, namely $\mathrm{Depth}(A, \mathfrak{S}\backslash J)$.

Let us select a minimal path from J to A. Its last move puts a peg in A, i.e. has A as point R since we could otherwise shorten this path. Moreover it does not use A anymore as point P or Q since we could again shorten the path. Consequently, for any $A \notin J$

$$\mathfrak{p}(A, \mathcal{F}) \leq \max(0, |I| - |J| - \mathrm{Depth}(A, \mathfrak{S}\backslash J) + 1). \quad (25)$$

If A is in J, we have $\mathrm{Depth}(A, \mathfrak{S}\backslash J) = 0$ so that (23) is stronger.

Proof. Indeed A not in J implies $\mathrm{Depth}(A, \mathfrak{S}\backslash J) \geq 1$. The $\mathrm{Depth}(A, \mathfrak{S}\backslash J) - 1$ last moves cannot use A in any part of a move, hence we can use (23) with $|J| + \mathrm{Depth}(A, \mathfrak{S}\backslash J) - 1$ points as a final position instead of J if A is at some point of time on the board. Else, it is never here and the upper bound 0 is fine. □

We do not know of any precise mean of computing this depth, but we provide now a fast way to get an excellent lower bound. Let us consider the oriented graph $\mathfrak{G}$ built on the set $\mathfrak{S}$ and where we put an edge from A to B if there exists $\mathfrak{f} \in \mathscr{D}(\mathfrak{S})$ such that $\mathfrak{f}(A) = 1$ and $\mathfrak{f}(B) = -1$. A minimal path that realizes $\mathrm{Depth}(A, \mathfrak{S}\backslash J)$ is readily transformed in a path from A to J on $\mathfrak{G}$. Reciproquely from such a path from A to J on this graph, we deduce a position K by adding the required points P and Q necessary for the $\mathfrak{f}$'s. The only problem is that this process may require to put several pegs on a same square (we do not have any example of such a situation). Denoting by $\delta_{\mathfrak{G}}(A, J)$ the distance on this graph, we have established that

$$\delta_{\mathfrak{G}}(A, J) \leq \mathrm{Depth}(\mathfrak{S}\backslash J) \quad (26)$$

Note that a final position L in case of $\delta_{\mathfrak{G}}$ is reduced to a single point. The distance $\delta_{\mathfrak{G}}(A, J)$ is now readily computed, by using the Dijkstra's algorithm for instance.

Practically, to find a minorant of this depth, we proceed in two steps (with $S = \mathfrak{S}\backslash J$):

- We try every succession of 5 legal moves from S.
- Concerning the remaining ones, we first build the set S_5 of points with $\mathrm{Depth}(A, S) \leq 5$. If $A \in S_5$, we find the minimum of $\delta_{\mathfrak{G}}(A, B) + \mathrm{Depth}(B, S)$ for every $B \in S_5$; this a first lower bound for $\mathrm{Depth}(A, S)$, but sometimes the lower bound 6 is simply better.

11.2 Using the speed at which a point is reached by I

Let us now examine the somewhat reciproqual situation, and try to get the minimum of legal moves from the set I that puts a peg in A. We need two pegs to create one, which means that the distance $\delta_{\mathfrak{G}}(A, I)$ is not a good lower bound anymore. We define the *height* $\mathrm{Height}(A, I)$ of A with respect to I to be his minimal number, and set $\mathrm{Height}(A, I) = \infty$ if A can never be reached. Computing this $\mathrm{Height}(A, I)$ is very difficult.

Lemma 11.2. *Let I be a subset of $\mathfrak{S}$ and A be such that* $\mathrm{Height}(A, I) < \infty$. *For any non-negative resource count π, we have $\langle \mathbb{1}_I, \pi \rangle \geq \pi(A)$.*

Proof. Indeed there is a set J which contains A and that is reachable from I. We thus have $\langle \mathbb{1}_I, \pi \rangle \geq \langle \mathbb{1}_J, \pi \rangle$ which in turn is non less than $\pi(P)$ by the non-negativity assumption on π. □

Using Lemma 11.2 and some direct computations, we get the following height-diagram for the left-hand side position.

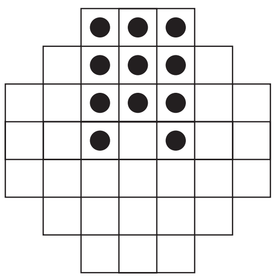

Figure 24. Starting position

		−2	−2	−2		
	1	−1	−1	−1	1	
3	1	−1	−1	−1	1	3
4	2	−1	1	−1	2	4
∞	∞	1	∞	1	∞	∞
	∞	4	∞	4	∞	
		∞	∞	∞		

Figure 25. Height/-Depth

We next provide an example on which Lemma 11.2 is not strong enough to decide whether some points have finite heights or not. This problem passes the linear integer test. We provide the height of each square (we simply computed all position attainable in 5 moves!). The two squares on the left-hand side (and the symmetric ones on the right-hand side) are rather clearly not reachable, but the test deduced from Lemma 11.2 fails to prove that. Even worse, we found for each of this square a position got from the first one in 5 moves and for which this square is not shown to be unreachable by this test.

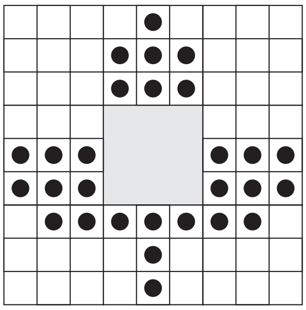

Figure 26. Starting position

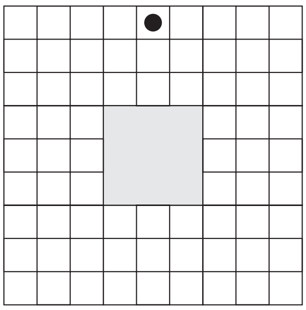

Figure 27. Ending position

Practically, to find a minorant of this height, we proceed in three steps:

?	6	2	1	−2	1	2	6	?
?	4	1	−1	−1	−1	1	4	?
3	4	1	−1	−1	−1	1	4	3
1	1	1				1	1	1
−1	−1	−1				−1	−1	−1
−1	−1	−1				−1	−1	−1
1	−1	−1	−2	−2	−2	−1	−1	−1
2	1	1	3	−3	3	1	1	2
5	3	3	4	−3	4	3	3	5

Figure 28. Height/-Depth

- We try every succession of 5 legal moves.
- We use the Lemma 11.2 to determine those points that are guaranteed to have infinite height. (We apply this test to all of the derived positions).
- Concerning the remaining ones, we first build the set I_5 of points with $\mathrm{Height}(A, I) \leq 5$. If $A \in \mathfrak{S}\backslash I_5$, we find the minimum of $\delta_{\mathfrak{S}}(A, B) + \mathrm{Height}(B, I)$ for every $B \in I_5$; this a first lower bound for $\mathrm{Height}(A, I)$, but sometimes the lower bound 6 is simply better.

Set

$$\mathscr{C}(A, I, J) = \min\big(|I| - |J|, \max\big(\mathrm{Depth}(A, \mathfrak{S}\backslash J) - 1, 0\big) + \max\big(\mathrm{Height}(A, I) - 1, 0\big)\big). \tag{27}$$

We have

$$\mathfrak{p}(A, \mathcal{F}) \leq |I| - |J| - \mathscr{C}(A, I, J). \tag{28}$$

11.3 Using the speed at which a peg comes out of I

We finally improve on the lower bound in (23). The fact is that some points are so much within the starting position I that the peg on them cannot be eliminated before so many moves, and this is precisely how we defined $\mathrm{Depth}(A, I)$. We have then

$$\mathrm{Depth}(A, I) \leq \mathfrak{p}(A, \mathcal{F}). \tag{29}$$

11.4 Final discussion

We end this section with two remarks. First, both notions of depth and height use only one of the two positions of the problem, and this is a loss. For instance concerning height, if we manage to put a peg in a very far away square that is also far from our final position, it is probable that we shall not be able to bring it back to it; for instance, if the starting position is given by Fig. 26, it is likely that we cannot put a point in the lower left corner and finish as in Fig. 27. Secondly, constraints (28) and (29) only avoid extremal cases, as we noted earlier, and there are only $2|\mathfrak{S}|$ of them for a problem with about $|\mathfrak{S}|^2$ variables; in fact, if $\mathfrak{S}$ has no isolated point, Lemma 4.1 yields

$$|\mathfrak{S}|(|\mathfrak{S}|-2) \leq |\mathscr{D}(\mathfrak{S} \boxtimes \mathfrak{S})| \leq 4|\mathfrak{S}|(|\mathfrak{S}|-2).$$

This explains why these constraints are somewhat weak.

12. Additional constraints

Having in mind the counting argument displayed at the end of last section, we see that finding conditions on couples (A, A') of points would not increase too much the size of the problem but may yield more stringent constraints.

As of now, we have only found one such type of constraint, which applies to initial positions I such that $\mathfrak{S}\backslash I$ is large enough.

Let us start with some general considerations. Let $\text{Height}(A, A', I)$ be the minimum number of legal moves necessary to put a peg in each of A and A', starting from a board with pegs on all the points of I. We assign it value ∞ if no such succession exists. Note that the height-function does not behave like a distance, since we can have $\text{Height}(A, A', I) > \text{Height}(A, I) + \text{Height}(A', I)$. We formulate a conjecture:

Conjecture 12.1. $\text{Height}(A, A', I) \geq \text{Height}(A, I) + \text{Height}(A', I)$.

A proof or disproof of this conjecture has sofar escaped the author.

Lemma 12.1. *Consider two points A and A' such that* $\text{Height}(A, A', I) = \infty$. *Then*

$$\mathfrak{p}(A, \mathcal{F}) + \mathfrak{p}(A', \mathcal{F}) - \sum_{\mathfrak{f}/\mathfrak{f}(A)\mathfrak{f}(A')\neq 0} x(\mathfrak{f}) \leq |I| - |J|. \quad (30)$$

Proof. Given a move $\mathfrak{f}$, let us look at the situation of the board before using this move. There are several cases:

- A is on the board and is not moved by $\mathfrak{f}$. Then A' is not on the board, and may not be created by $\mathfrak{f}$. This move is counted in $y_{\mathfrak{f}}(A)$.
- A' is on the board and is not moved by $\mathfrak{f}$. Then A is not on the board, and may not be created by $\mathfrak{f}$. This move is counted in $y_{\mathfrak{f}}(A')$.
- A is on the board and is moved by $\mathfrak{f}$. Then A' is not on the board and may be created. This move is counted in $x(\mathfrak{f})$.
- A' is on the board and is moved by $\mathfrak{f}$. Then A is not on the board and may be created. This move is counted in $x(\mathfrak{f})$. □

The question arises as to whether this Lemma leads or not to improvements, and we provide an example below showing that it indeed does. The geometrical fact that we have used is that a square can either contain a peg, or be empty, a fairly trivial information that was until now absent from our discussion.

Before exposing our example, let us address rapidly the problem of computing couples (A, A') with $\text{Height}(A, A', I) = \infty$.

Lemma 12.2. *Let I be a subset of $\mathfrak{S}$ and A and A' be two points of $\mathfrak{S}$. If there exists a non-negative resource count π, such that $\langle \mathbb{1}_I, \pi \rangle < \pi(A) + \pi(A')$, then* $\text{Height}(A, A', I) = \infty$.

We can improve on this criteria: simply form all positions derived from I by one (or any fixed number) legal move, and apply this criteria to each of them.

Here is a problem that is shown impossible by using this criteria, though it passes the quadratic integer test with flatness constraints:

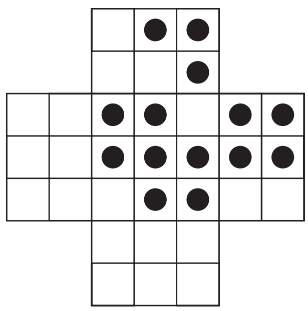

Figure 29. Starting position

This example is also interesting because of the square with an interrogation dot: it is "clearly" of infinite height, but our automatic process is not able to conclude. Here is the list of couples with $\text{Height}(A, A', I) = \infty$ that we have found:

A	A'
(3, 1)	(1, 4), (2, 3), (3, 2), (3, 3)
(4, 1)	(1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 1), (3, 3), (3, 4), (3, 6), (3, 7), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), (4, 7), (5, 1)(1, 4), (2, 3), (3, 1), (3, 2), (3, 3), (4, 1), (4, 2), (5, 2), (5, 3)
(5, 2)	(3, 1), (4, 1)
(5, 3)	(3, 1)
(5, 4)	(4, 1)
(6, 3)	(3, 1), (4, 1), (5, 1)
(6, 4)	(4, 1), (5, 1)
(6, 5)	(4, 1)
(7, 3)	(1, 4), (2, 3), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3), (4, 6), (5, 1), (5, 2), (5, 3), (6, 3), (6, 4), (6, 5), (7, 4), (7, 4)(1, 4), (2, 4), (3, 1), (4, 1), (4, 2), (4, 4), (4, 6), (4, 7), (7, 5)(5, 1), (5, 2), (5, 4), (5, 6), (6, 3), (6, 4), (7, 5)(7, 5)(4, 1), (5, 1), (6, 3), (6, 5)

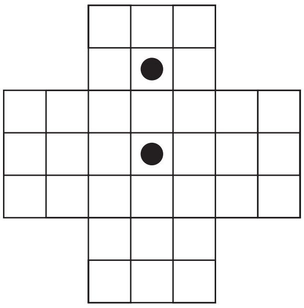

Figure 30. Ending position

		1	−1	−1		
		1	1	−1		
4	1	−1	−1	1	−1	−1
?	1	−1	−1	−1	−1	−1
5	2	1	−1	−1	1	−1
		2	1	1		
		5	4	4		

Figure 31. Height/-Depth

Lemma 12.1 is of course of fairly limited use: we need the starting position to leave free enough squares on the board. However, it shows how more geometrical arguments may be used to get improvements! Our journey ends here.

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Kuttaka Method for Polynomials[1]

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Abstract. In Ganitapada, Aryabhata gave the Kuttaka method for solving integral linear equations in 2 variables. In this article, we shall describe the polynomial version of the Kuttaka method of Aryabhata and illustrate it by examples. We shall also show its applications to Chinese Remainder Theorem and Interpolation, for polynomials in one variable.

0. Introduction

Aryabhata (476-550 AD) is the first known Indian mathematician and astronomer whose writings are available. In 499 AD, Aryabhata wrote his Aryabhatiya in which the second part is known as Ganitapada. This part has 33 verses in Sanskrit and they are written in 'Arya' meter. In Ganitapada, Aryabhata deals with various concepts, methods and formulae that we study in High School Mathematics; for example, decimal place value names, finding square roots and cube roots, Pythagoras Theorem, the sum, sum of squares and sum of cubes of first n natural numbers, heights and distances, values of the sine function, problems whose solutions are based on solutions of quadratic equations, etc.

In the last 2 verses of Ganitapada, Aryabhata deals with solving diophantine equations of the form $ax + b = cy + d$, where a, b, c, d are integers. We can arrange that a, b, c, d are positive integers and $b > d$. Aryabhata gives a method called Kuttaka method or pulveriser, in which successively the problem of getting integer solutions is reduced to simpler problems of the same type. In the Kuttaka algorithm, the numbers involved become successively smaller and smaller and hence the name 'Kuttaka' meaning pulveriser which breaks into pieces. The meaning of kutta or kuttaka is also 'something to be found out' as in a puzzle. Aryabhata's method was explained by Bhaskara-I (6th Century AD) by many illustrative examples especially in astronomy, and later mathematicians such as Brahmagupta (7th Century), Mahaveera (9th Century), Sripati (11th Century), Bhaskara-II (12th Century) discussed it and gave variations of the method. Readers may also refer to the interesting article by Amartya K. Dutta [2] for additional information about the Kuttaka method. André Weil (1906–1999), one of the greatest mathematicians of the 20th century, compared the Kuttaka method with Fermat's method of infinite descent which Fermat (17th Century) used to solve many problems in number theory.

We illustrate the Kuttaka method of Aryabhata for integers by one example.

Example. Find an integer which leaves remainder 11 when divided by 55 and remainder 5 when divided by 21.

Solution. Here the problem can be written in the form

$$N = 55x + 11 = 21y + 5.$$

We are to find x (called kutta or kuttaka), and then N.

The remainders 11 and 5 are called 'agra'. Here $11 > 5$. Their difference $11 - 5 = 6$ is called 'agrantara'.

Divide the coefficient of x viz. 55 by the coefficient of y viz. 21.

$55 = 2 \times 21 + 13$. This division is not counted.

Now divide 21 by 13 to get $21 = 1 \times 13 + 8$. Note the quotient $q_1 = 1$.

Divide 13 by 8. Thus $13 = 1 \times 8 + 5$. Note the quotient $q_2 = 1$.

We have done 2 divisions, i.e. an even number of divisions as required by Aryabhata in his method.

Now consider the simplified problem in which we find an unknown called as 'mati' by Aryabhata such that *mati* $\times$ *last remainder* $+$ *agrantara* is divisible by the last divisor,

i.e. mati $\times 5 + 6$ is divisible by 8.

By trial we get mati $= 2$. Note the quotient q which is 2.

Now form the 'valli' (sequence) $q_1, q_2, mati, q$ which is $1, 1, 2, 2$. Take last 3 numbers $1, 2, 2$ and call them A, B, C.

[1]Expanded version of Talk at International Conference on History of Mathematics, Department of Mathematics, Ramjas College, Delhi, Indian Society for History of Mathematics (December 16–18, 2021).

We have $AB + C$ rule here. Replace A by $AB + C$, keep B as it is, and remove C. Thus we get last 2 numbers as $4, 2$. Using the initial 1 in the valli we get 3 numbers $1, 4, 2$. By $AB + C$ rule we get $6, 4$. Thus we are left with only 2 numbers. Here the first number is the value of x. Thus $x = 6$. So $N = 341$. This is the answer. Check that $341 = 55 \times 6 + 11 = 21 \times 16 + 5$.

We shall now consider a polynomial version of this Kuttaka method of Aryabhata. This is usually not discussed in the literature, but it has applications in Chinese Remainder Theorem for polynomials and interpolation as we shall see.

1. The problem for polynomials over a field

Find a polynomial $P(x)$ which leaves remainder $r(x)$ when divided by $f(x)$ and remainder $s(x)$ when divided by $g(x)$.

Thus we want a polynomial $P(x)$ of the form

$$P(x) = f(x) \cdot X + r(x) = g(x) \cdot Y + s(x), \qquad (1)$$

where $r(x)$ and $s(x)$ have degrees less than those of $f(x)$ and $g(x)$ respectively. The polynomials are over rational numbers. More generally, they can have coefficients over any field.

Here X, Y are unknown polynomials, $r(x), s(x)$ are called remainders (agra) and $r(x)-s(x)$ is called difference between the remainders (agrantara).

For the problem to have a solution, it is necessary and sufficient that the difference $r(x)-s(x)$ is divisible by the gcd of $f(x)$ and $g(x)$.

The problem may be written using polynomial congruences as

$$P(x) \equiv r(x) \pmod{f(x)},$$
$$P(x) \equiv s(x) \pmod{g(x)}. \qquad (2)$$

We seek a common solution $P(x)$ to this system of the two congruences.

In Equation (1), we may take $s(x)$ to the LHS and replacing $r(x) - s(x)$ by new $r(x)$, we may consider the equation in the form

$$P(x) = f(x)X + r(x) = g(x)Y. \qquad (3)$$

If we find any one of X, Y and $P(x)$, we can get the rest.

For (3), the condition for the existence of a solution is that

$$\gcd(f(x), g(x)) \text{ divides } r(x). \qquad (4)$$

If this condition is satisfied, let $\gcd(f(x), g(x)) = d(x)$. Here $d(x)$ is uniquely determined upto a nonzero scalar multiple. If $f(x)$ and $g(x)$ are monic, then their gcd is taken to be the unique monic polynomial.

Write $f(x) = f_1(x)d(x), g(x) = g_1(x)d(x), r(x) = r_1(x)d(x)$ and $P(x) = P_1(x)d(x)$.

Thus (3) becomes

$$P_1(x)d(x) = f_1(x)d(x)X + r_1(x)d(x) = g_1(x)d(x)Y.$$

Cancelling $d(x)$, we get the equivalent system

$$P_1(x) = f_1(x)X + r_1(x) = g_1(x)Y.$$

Here $f_1(x)$ and $g_1(x)$ have gcd 1, i.e. they are relatively prime, i.e. they have no nonconstant polynomial as a common factor.

We thus see that to solve (1) or (2) or (3), it is sufficient to assume that $f(x)$ and $g(x)$ are relatively prime polynomials, as we can reduce the problem to the situation

$$f(x)X + r(x) = g(x)Y,$$

where $f(x)$ and $g(x)$ are relatively prime polynomials.

If we prefer, we may as well assume that $f(x)$ and $g(x)$ are monic polynomials, as the leading coefficients of $f(x)$ and $g(x)$ can be absorbed in X and Y, without changing $r(x)$. However it is not necessary to assume so.

Note that a solution X is unique $\pmod{g(x)}$ and Y unique $\pmod{f(x)}$. Knowing any solution polynomial X, the polynomial Y can be uniquely determined and vice versa.

2. Kuttaka method of Aryabhata in the context of polynomials

We thus solve

$$f(x)X + r(x) = g(x)Y,$$

where $f(x)$ and $g(x)$ are relatively prime polynomials.

Here X is a polynomial to be found and it is called kutta or kuttaka. The method to find X is called the Kuttaka method.

The method to solve (5) for polynomial solutions, analogous to the method of Aryabhata for solving an integral linear equation in 2 variables, is as follows:

Step 1. Divide $f(x)$ by $g(x)$ and get the remainder.

If $\deg f(x) \geq \deg g(x)$, we get a remainder which is of degree $< \deg g(x)$.

Otherwise $f(x)$ itself is the remainder.

In this step, the quotient is not important, but the remainder is.

The division in this step is NOT counted. After this we shall be doing an EVEN number of divisions.

Step 2. We shall now perform divisions, where the divisor is divided by the remainder and we get a new quotient and a new remainder. These computations are similar to the ones we perform in Euclid's gcd algorithm.

In the Kuttaka method, we have to note down the quotient too. We proceed by dividing the divisor (viz. the previous remainder) by the new remainder, and note the quotient. In this way we do an even number divisions. Everytime the divisors and remainders are becoming smaller and smaller in DEGREE. We can stop dividing after an even number of divisions, when we think that the problem is within our reach.

In the method we want to find a polynomial X (mati of Aryabhata) such that $X\times$ (last remainder) + "agrantara = difference between remainders" is divisible by the "last divisor". This simpler problem is

$$f_0(x) \cdot X_0 + r(x) = g_0(x) \cdot Y_0, \tag{5'}$$

where $f_0(x)$ is the last remainder, X_0 is a polynomial to be found, $r(x)$ is the so-called agrantara in (5), $g_0(x)$ is the last divisor.

It may be noted that (5) *is solvable if and only if* (5′) *is.*

If after an even number of steps we get such a solvable equation (5′) then we stop the division process, otherwise we go to the next two divisions so as to get still smaller divisor and remainder, and a solvable equation.

Step 3. We have already noted all the even number of quotients. We make a column of these quotients. Below that we write X_0 (mati).

Now mati $\times$ last remainder + agrantara (remainder difference), i.e.,

$$X_0 \times f_0(x) + r(x)$$

is divisible by the last divisor $g_0(x)$.

Find the quotient Y_0 when $X_0 f_0(x) + r(x)$ is divided by $g_0(x)$.

In our column, we write Y_0 below X_0. We thus have a column of even number of polynomials which were quotients in the divisions, followed by X_0 (mati) and Y_0.

Step 4. In the sequence of polynomials, we have obtained, take the last 3 polynomials and call them A, B, C. Replace A by $AB + C$ and remove C. The polynomial before A, and the polynomials $AB + C$ and B become the last 3 polynomials in the sequence, now. For these last 3 polynomials use $AB + C$ rule and proceed till you are left with only 2 polynomials. The top polynomial is the polynomial X. Then Y may be obtained using (5).

3. Examples

Example 1. Solve $(x + 1) \cdot X + 2 = (x^2 + 1) \cdot Y - x + 1$.

Solution. Here $(x + 1)X + (x + 1) = (x^2 + 1)Y$.
agrantara = remainder difference = $x + 1$.

$x + 1 = 0 \times (x^2 + 1) + x + 1$. "*Omit*" this division for counting. Here since $\deg(x + 1) < \deg(x^2 + 1)$, this division is a formality, and it need not be carried out.

$$x^2 + 1 = (x + 1)(x - 1) + 2, \quad q_1 = x - 1.$$

$$x + 1 = 2\left[\frac{1}{2}(x + 1)\right] + 0, \quad q_2 = \frac{x + 1}{2}.$$

To find mati = X_0 such that mati $\times$ last remainder + "agrantar i.e. remainder difference" is divisible by the last divisor. Remainder difference $= x + 1$.

$X_0 \times 0 + x + 1$ should be divisible by 2. Any X_0 would work, because in polynomials over rational numbers $x + 1$ is divisible by 2.

Take $X_0 = 0$. $Y_0 = q =$ quotient $= \frac{x+1}{2}$.

Sequence (Valli) : q_1, q_2, mati $= X_0, q = Y_0$.

The 3rd and 4th columns below are obtained using $AB+C$ Rule.

q_1	$x - 1$	$x - 1$	$(x - 1)\frac{x+1}{2}$
q_2	$\frac{x+1}{2}$	$\frac{x+1}{2}$	$\frac{x+1}{2}$
mati $= X_0$	0	0	
$q = Y_0$	$\frac{x+1}{2}$		

The top entry in the last column is X. So $X = \frac{x^2-1}{2}$.

Observe that $(x+1)X+(x+1)=(x+1)(X+1)$

$$=(x+1)\left(\frac{x^2+1}{2}\right)=(x^2+1)Y, \text{ so } Y=\frac{x+1}{2}.$$

Thus it works.

Example 2. Solve $(x^2+1)X-x-1=(x+1)Y$.

Solution. Here $x^2+1=(x+1)(x-1)+2$. (Omit for counting.)

0 is an "EVEN no." So we can find mati here.

mati × remainder + agrantar = "remainder difference" is divisible by $x+1$.

$0\times 2+(-x-1)$, is divisible by the last divisor $=x+1$, and the quotient $q=-1$, and mati is 0.

The sequence is $\{\text{mati}, q\}=\{0,-1\}$. So $X=\text{mati}=0$, and so $Y=-1$.

Example 3. Solve $(x^2+1)X+(x+1)=(x^3+2)Y$.

$$x^2+1=0\times(x^3+2)+x^2+1. \qquad \text{(Omit for counting.)}$$

$$\text{First Division: } x^3+2=(x^2+1)(x)-x+2, q_1=x.$$

$$\text{Second Division: } x^2+1=(-x+2)(-x-2)+5,$$

$$q_2=-x-2.$$

Even number of divisions:

$$X_0\times 5+(x+1) \text{ is divisible by } -x+2.$$

$$X_0=-\frac{3}{5}, q=-1.$$

Sequence: $\{q_1,q_2,X_0,q\}=\{x,-x-2,-\frac{3}{5},-1\}$.

Use $AB+C$ rule.

$$\to\left\{x,\frac{3x+1}{5},-\frac{3}{5}\right\}$$

$$\to\left\{\frac{3x^2+x-3}{5},\frac{3x+1}{5}\right\}.$$

So $X=\frac{3x^2+x-3}{5}$.

Verification: $(x^2+1)\left[\frac{3x^2+x-3}{5}\right]+(x+1)=\frac{3x^4+x^3+6x+2}{5}$ has x^3+2 as a factor.

4. Inverse of a polynomial modulo another polynomial

Example 4. Solve $(x^2+1)X=1+(x^3-x+2)Y$.

Here the solution X is an inverse of x^2+1 modulo x^3-x+2 because $(x^2+1)X$ is 1 modulo x^3-x+2. The inverse is unique modulo x^3-x+2.

To find X, consider

$$(x^2+1)X-1=(x^3-x+2)Y.$$

$$\text{First Division: } x^3-x+2=(x^2+1)(x)-2x+2, q_1=x.$$

$$\text{Second Division: } x^2+1=(-2x+2)\left(-\frac{1}{2}x-\frac{1}{2}\right)+2,$$

$$q_2=\frac{-x-1}{2}.$$

Even number of divisions done.

$X_0\times 2+(-1)$ is divisible by $-2x+2$.

By trial, $X_0=\frac{x}{2}, q=-\frac{1}{2}$.

$$\text{Sequence}=\{q_1,q_2,X_0,q\}$$
$$=\left\{x,\frac{-x-1}{2},\frac{x}{2},-\frac{1}{2}\right\}$$
$$\to\left\{x,\frac{-x^2-x-2}{4},\frac{x}{2}\right\}$$
$$\to\left\{\frac{-x^3-x^2}{4},\frac{x}{2}\right\}$$

Thus the inverse is $-\frac{1}{4}(x^3+x^2)$ OR $-\frac{1}{4}(x^2+x-2)$, as the inverse is modulo x^3-x+2.

Verification: $(x^2+1)X-1=(x^2+1)\left(\frac{-1}{4}\right)(x^2+x-2)-1$

$$=-\frac{1}{4}[(x^4+x^3-2x^2)+(x^2+x-2)+4]$$
$$\equiv-\frac{1}{4}[(x^2-2x)+(x-2)-2x^2)$$
$$+(x^2+x-2)+4] \pmod{x^3-x+2}$$
$$\equiv 0 \pmod{x^3-x+2}.$$

Examples. $(x-1)x+1=(x^2+1)y$ is not solvable over $\mathbb{F}_2$ as $\gcd(x-1,x^2+1)=x-1$ over $\mathbb{F}_2$ and $x-1$ does not divided 1.

5. General Chinese remainder Theorem for numbers and polynomials by Kuttaka method

Kuttaka method gives a solution to a system of 2 congruence equations for integers or polynomials.

If there are n congruence equations $x \equiv x_i \pmod{m_i}$ where $i = 1, 2, \ldots, n$ and $m_1, m_2, \ldots, m_n$ are relatively prime in pairs, then the common solution is given by Chinese Remainder Formula as

$$\equiv \sum_{i=1}^{n} x_i M_i \overline{M_i} \pmod{m},$$

where $m = m_1, \ldots, m_n$, $M_i = \frac{m}{m_i}$ and $\overline{M_i}$ is an inverse of $M_i \pmod{m_i}$ i.e. a solution to the congruence $M_i X \equiv 1 \pmod{m_i}$.

This congruence can be solved using Kuttaka method using divisions and '$AB + C$ Rule' as mentioned by Aryabhata.

For polynomials in one variable too, the system

$$f(x) \equiv A_i(x) \mod m_i(x), \quad 1 \le i \le n,$$

has a solution

$$f(x) \equiv \sum_{i=1}^{n} A_i(x) M_i(x) \overline{M_i}(x),$$

when $m_1(x), \ldots, m_n(x)$ are pair-wise relatively prime poly nomials,

$$M_i(x) = \frac{m(x)}{m_i(x)}, m(x) = m_1(x) \ldots m_n(x).$$

Again $\overline{M_i}(x)$, which are inverses of $M_i(x)$ modulo $m_i(x)$, can be obtained using Kuttaka method.

6. Application to interpolation

Find an interpolating curve $y = f(x) =$ a polynomial in x over $\mathbb{R}$ (or other fields of characteristic zero), satisfying

1. $f(-1) = 1$ (i.e. passing through $(-1, 1)$),
2. $f(0) = -1$, $f'(0) = -3$, $f''(0) = 0$,
3. $f(1) = -3$, $f'(1) = 2$.

Solution. By Taylor's formula, for any $n \ge 0$,

$$f(x) = f(a) + (x-a)f'(a) + \frac{1}{2!}(x-a)^2 f''(a) + \cdots$$
$$+ \frac{1}{n!}(x-a)^n f^{(n)}(a) + (x-a)^{n+1} g(x),$$

where $g(x)$ is a polynomial in x.

Using this, the conditions 1, 2, 3 may be written as

$$f(x) \equiv 1 \pmod{(x+1)},$$
$$f(x) \equiv -1 - 3x \pmod{x^3},$$
$$f(x) \equiv -3 + 2(x-1) \pmod{(x-1)^2}.$$

These can be understood as $f(x) \equiv A_i(x) \pmod{m_i(x)}$, $i = 1, 2, 3$.

Let $m(x) = m_1(x)m_2(x)m_3(x)$ and $M_i(x) = \frac{m(x)}{m_i(x)}$.

Then by Chinese Remainder Formula,

$$f(x) \equiv \sum_{i=1}^{3} A_i(x) M_i(x) \overline{M_i}(x) \pmod{m(x)},$$

where $\overline{M_i}(x)$ are inverses of $M_i(x) \pmod{m_i(x)}$.

Here $m_1(x) = (x+1)$, $m_2(x) = x^3$, $m_3(x) = (x-1)^2$.

$$m(x) = (x+1)x^3(x-1)^2 = (x^2-1)(x^4-x^3)$$
$$= x^6 - x^5 - x^4 + x^3.$$

$M_1(x) = x^3(x-1)^2$, $M_2(x) = (x+1)(x-1)^2$, $M_3(x) = (x+1)x^3$.

The inverses $\overline{M_i}(x) \mod m_i(x)$ are to be found by the Kuttaka method.

1. $m_1(x) = x + 1$, $M_1(x) = x^3(x-1)^2$.

$$M_1(x)X \equiv 1 \mod (x+1),$$
$$\text{i.e. } M_1(x)X - 1 = (x+1)Y.$$

   Divide $M_1(x)$ by $x + 1$. By Remainder Theorem, the remainder is $M_1(-1) = -4$. This division is not counted.
   Zero number of divisions, an even number.
   mati × last remainder + agrantar is divisible by last divisor.
   $X_0 \times (-4) + (-1)$ is divisible by $x + 1$.
   $X_0 = -\frac{1}{4}$ works. $q = 0$.
   The sequence is $\{-\frac{1}{4}, 0\}$, so $X = -\frac{1}{4}$.
   i.e. $\overline{M_1}(x) = -\frac{1}{4}$.
2. $m_2(x) = x^3$, $M_2(x) = (x+1)(x-1)^2 = (x^2-1)(x-1) = x^3 - x^2 - x + 1$.
   To find $\overline{M_2}(x)$, we solve

$$M_2(x)X = 1 + x^3 Y.$$

   The 0-th division is: Divide $M_2(x) = x^3 - x^2 - x + 1$ by x^3.

$$x^3 - x^2 - x + 1 = 1 \times x^3 - x^2 - x + 1.$$

   The remainder is $-x^2 - x + 1$. (Quotient is not taken into account.) This step is just 'going modulo x^3'.
   First Division:

$$x^3 = (-x+1)(-x^2-x+1) + 2x - 1, q_1 = -x + 1.$$

   Second division:

$$-x^2 - x + 1 = (2x-1)\left(-\frac{1}{2}x\right) - \frac{3}{2}x + 1, q_2 - \frac{1}{2}x.$$

mati $= X_0$, $X_0 \times \left(-\frac{3}{2}x+1\right)+(-1)$ is divisible by $2x-1$. By trial, $X_0 = 4, q = -3$.

$$\text{Sequence} = \left\{-x+1, -\frac{1}{2}x, 4, -3\right\}$$

$$\to \{-x+1, -2x-3, 4\} \text{ (by } AB+C \text{ Rule)}$$

$$\to \{2x^2+x+1, -2x-3\} \text{ (by } AB+C \text{ Rule)}.$$

So $X = 2x^2+x+1$. Check:

$$M_2(x)X = (x^3-x^2-x+1)(2x^2+x+1)$$
$$= \text{“terms divisible by } x^3\text{”} + 1.$$

So we are done.

3. $M_3(x) = (x+1)x^3, m_3(x) = (x-1)^2$.
To solve: $M_3(x)X = 1+(x-1)^2Y$.
0th Division: Divide $M_3(x)$ by $m_3(x)$.
$x^4+x^3 = (x^2-2x+1)(x^2+3x+5)+7x-5$.

$$\text{1st Division: } x^2-2x+1 = (7x-5)\left(\frac{1}{7}x-\frac{9}{49}\right) + \frac{4}{49},$$

$$q_1 = \frac{1}{7}x - \frac{9}{49}.$$

$$\text{2nd Division: } 7x-5 = \frac{4}{49}\left(\frac{49}{4}(7x-5)\right),$$

$$q_2 = \frac{49}{4}(7x-5).$$

$X_0 \times 0 + (-1)$ is divisible by the last divisor $\frac{4}{49}$.
$X_0 = 0$, quotient $= q = -\frac{49}{4}$.

$$\text{Sequence} = \{q_1, q_2, X_0, q\}$$
$$= \left\{\frac{1}{7}x-\frac{9}{49}, \frac{49}{4}(7x-5), 0, -\frac{49}{4}\right\}$$
$$\to \left\{\frac{1}{7}x-\frac{9}{49}, -\frac{49}{4}, 0\right\} \text{ (by } AB+C \text{ Rule)}$$
$$\to \left\{-\frac{1}{4}(7x-9), -\frac{49}{4}\right\} \text{ (by } AB+C \text{ Rule)}.$$

So $X = \overline{M_3}(x) = -\frac{1}{4}(7x-9)$. We are done.

Finally, $f(x) \equiv \sum_{i=1}^{3} A_i(x)M_i(x)\overline{M_i}(x) \equiv x^5-3x-1 \pmod{m(x)}$.

Here we have used $x^6 \equiv x^5+x^4-x^3 \pmod{m(x)}$.

Thus $y = f(x) = x^5-3x-1$ is the interpolation polynomial of least degree satisfying the given requirements.

7. Euclidean domains

Integers, and polynomials over a field, are examples of Euclidean domains. In these examples we have a division algorithm which helps in solving a the linear diophantine equation in 2 variables using the Kuttaka method. In general Euclidean domains, a ‘division’ exists by definition, however it is not always easy to carry out this division for want of a good division algorithm.

There are conjecturally plenty of Euclidean Domains coming from algebraic number theory. Weinberger [3] has proved that under the assumption of Generalised Riemann Hypothesis, the ring of algebraic integers in an algebraic number field is a Euclidean domain if it is a PID and has infinitely many units. Thus for number fields other than imaginary quadratic fields, under GRH, the ring of integers is a PID if and only if it is a Euclidean Domain. Recently Ram Murty and others have proved such results for many rings of integers without using GRH. Whenever there is an easy division algorithm to carry out in a Euclidean Domain, the Kuttaka method can also be carried out there. The readers may try kuttaka method for simpler Euclidean domains coming from number theory such as $\mathbb{Z}[i]$ and $\mathbb{Z}[\omega]$, where $\omega = \exp(2\pi i/3)$.

Acknowledgement

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On Coherent Edge-Labelling of a Polyhedron in Three Dimensions

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Abstract. In this article we consider the problem of coherently labelling edges of a convex polyhedron in three dimensions. We exhibit all the forty eight possible coherent labellings of the edges of a tetrahedron. We also exhibit that some simplicial polyhedra like bipyramids, Kleetopes, gyroelongated bipyramids are coherently edge-labellable. Also we prove that pyramids and antiprisms over n-gons for $n \geq 4$, which are not simplicial polyhedra, are coherently edge-labellable. We prove that among platonic solids, the cube and the dodecahedron are not coherently edge-labellable, even though, the tetrahedron, the octahedron and the icosahedron are coherently labellable. Unlike the case of a tetrahedron, in general for a polyhedron, we show that a coherent labelling need not induce a coherent labelling of edges incident at a vertex. We also give an obstruction criterion for a polyhedron not to be coherently edge-labellable and consequentially show that any polyhedron obtained from a pyramid with its apex chopped off is not coherently edge-labellable. Finally with the suggestion of the affirmative results we prove the main theorem that any simplicial polyhedron is coherently edge-labellable.

Keywords. Three Dimensional Polyhedron, Tetrahedron, Pyramid, Bipyramid, Cuboid, Kleetope, Dodecahedron, Antiprism, Gyroelongated Bipyramid, Simplicial Polyhedron, Coherent Labelling.

2010 Mathematics Subject Classification. Primary: 52B10.

1. Introduction

Any convex n-gon can be edge-labelled with integers $1, 2, \ldots, n$ such that the labels can be read in a strictly increasing order from a particular edge in an anticlockwise manner. This has an interesting generalization to a coherent edge-labelling of some classes of convex polyhedra in three dimensions. Before we state the main problem we give a definition.

Definition 1.1 (Coherent Labelling). *Let $\mathcal{P}$ be a convex polyhedron in the three-dimensional space. Choosing an outward normal for $\mathcal{P}$, we orient each polygonal face and hence its edges in an anticlockwise manner. We say a labelling of all the edges of $\mathcal{P}$ with integers is coherent if, for each polygonal face, the labels can be read in a strictly increasing order from a particular edge of the face conforming to the orientation of the face. If such a labelling exists then the polyhedron $\mathcal{P}$ is said to be coherently labellable. Otherwise it is said to be not coherently labellable.*

Now we state the problem in its broad sense.

Problem 1.2 (Coherent Labelling Problem). *Classify those convex polyhedra in three dimensions which are coherently labellable.*

In this article, by a polyhedron we mean, a convex polyhedron in three dimensions. Here we make some progress regarding Problem 1.2. We first prove in Theorem 2.1 that there are forty eight coherent essentially distinct labellings of a tetrahedron. We exhibit a coherent labelling for all pyramids in Section 3. In Section 4 we prove Theorem 4.3 which says that any Kleetope (Definition 4.1) is coherently edge-labellable. We also prove Theorem 4.5, an extension theorem for coherently labelling a polyhedron, that is, if a polyhedron is coherently labellable then a coherent labelling exists for the polyhedron which is obtained by attaching a pyramid to one or several faces of the given polyhedron such

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that the apriori edges and vertices of the face do not vanish, but the attached face vanishes. As a consequence bipyramids are coherently edge-labellable.

Not all polyhedra are coherently edge-labellable. For example cubes or cuboids (Theorem 5.1) and triangular prisms (Theorem 5.9) are not coherenly edge-labellable. Actually we prove in Theorem 5.9 that a certain class of polyhedra is not coherently labellable by a giving an obstruction criterion.

By proving that the dodecahedron is not coherently labellable, and the icosahedron or more generally any gyroelongated polyhedron is coherently labellable, we complete all the cases of five regular polyhedra in Theorem 6.1.

The general Problem 1.2 is still open. These results about tetrahedron, bipyramids, Kleetopes and gyroelongated polyhedra, lead us to main Theorem 7 that, any simiplicial polyhedron is coherently labellable.

2. Tetrahedron and its forty eight coherent Edge-Labelling

In this section we list all possible coherent labellings for a tetrahedron.

Theorem 2.1. *Let T be a tetrahedron with three pair-wise edge labelling symbols*

$$(x_1x_4), (x_2x_5), (x_3x_6)$$

as in Fig. 1 where we have clubbed opposite edges. Also $x_1, x_2, \ldots, x_6$ *is a permutation of* $1, 2, \ldots, 6$. *Then there exist forty eight coherent labellings of the tetrahedron given by the cycle elements as*

$$\begin{aligned}(x_1x_2, \ldots, x_6) \in \big\{&(124635), (134625), (125634),\\ &(135624), (135246), (136245),\\ &(145236), (146235)\big\}.\end{aligned}$$

Proof. There are six edges of a tetrahedron T. Hence there are totally 6! labellings. A cyclic change of labels given by

$$x_1 \longrightarrow x_2 \longrightarrow x_3 \longrightarrow x_4 \longrightarrow x_5 \longrightarrow x_6 \longrightarrow x_1$$

of a coherent labelling also gives rise to a coherent labelling. So we fix the least label for an edge. Let $x_1 = 1$. Now the

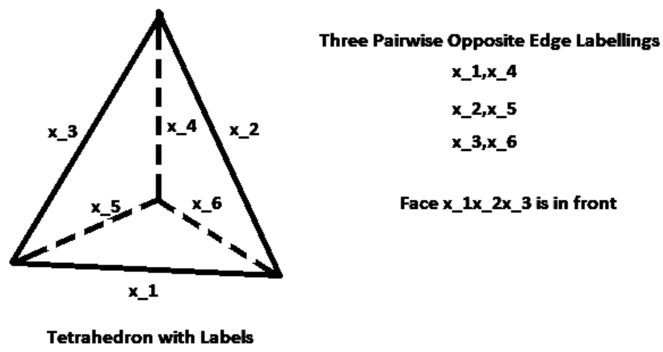

Figure 1. Tetrahedron T with Labelling Symbols

oriented faces corresponding to an outward pointing normal of T from Fig. 1 are given by the cycles

$$(x_1x_2x_3), (x_2x_6x_4), (x_3x_4x_5), (x_1x_5x_6)$$

Since x_1 has the least value, we have

$$x_1 < x_2 < x_3,\ x_1 < x_5 < x_6.$$

Now we have nine possibilities for the remaining as given by

$$x_2 < x_6 < x_4,\ x_6 < x_4 < x_2,\ x_4 < x_2 < x_6$$

$$x_3 < x_4 < x_5,\ x_4 < x_5 < x_3,\ x_5 < x_3 < x_4$$

We will compute the possible consistent labellings.

(1) $x_2 < x_6 < x_4,\ x_3 < x_4 < x_5 \Rightarrow x_6 < x_4 < x_5 < x_6$, a contradiction.

(2) $x_2 < x_6 < x_4,\ x_4 < x_5 < x_3 \Rightarrow x_6 < x_4 < x_5 < x_6$, a contradiction.

(3) $x_2 < x_6 < x_4,\ x_5 < x_3 < x_4 \Rightarrow \{1 = x_1\} < \{x_2, x_5\} < \{x_3, x_6\} < \{x_4\}$, where a set A is less than a set B if for every $a \in A, b \in B$ we have $a < b$. We have four possibilities.

$$\begin{aligned}&x_1 = 1 < x_2 = 2 < x_5 = 3 < x_3 = 4 < x_6 = 5\\ &\quad < x_4 = 6 \Rightarrow (124635)\\ &x_1 = 1 < x_5 = 2 < x_2 = 3 < x_3 = 4 < x_6 = 5\\ &\quad < x_4 = 6 \Rightarrow (134625)\\ &x_1 = 1 < x_2 = 2 < x_5 = 3 < x_6 = 4 < x_3 = 5\\ &\quad < x_4 = 6 \Rightarrow (125634)\\ &x_1 = 1 < x_5 = 2 < x_2 = 3 < x_6 = 4 < x_3 = 5\\ &\quad < x_4 = 6 \Rightarrow (135624)\end{aligned}$$

(4) $x_6 < x_4 < x_2,\ x_3 < x_4 < x_5 \Rightarrow x_6 < x_4 < x_5 < x_6$, a contradiction.

(5) $x_6 < x_4 < x_2,\ x_4 < x_5 < x_3 \Rightarrow x_6 < x_4 < x_5 < x_6$, a contradiction.

(6) $x_6 < x_4 < x_2,\ x_5 < x_3 < x_4 \Rightarrow x_4 < x_2 < x_3 < x_4$, a contradiction.

(7) $x_4 < x_2 < x_6,\ x_3 < x_4 < x_5 \Rightarrow x_4 < x_2 < x_3 < x_4$, a contradiction.

(8) $x_4 < x_2 < x_6,\ x_4 < x_5 < x_3 \Rightarrow \{x_1 = 1\} < \{x_4\} < \{x_2, x_5\} < \{x_3, x_6\}$. We have four possibilities.

$$x_1 = 1 < x_4 = 2 < x_2 = 3 < x_5 = 4 < x_3 = 5 < x_6 = 6 \Rightarrow (135246)$$

$$x_1 = 1 < x_4 = 2 < x_2 = 3 < x_5 = 4 < x_6 = 5 < x_3 = 6 \Rightarrow (136245)$$

$$x_1 = 1 < x_4 = 2 < x_5 = 3 < x_2 = 4 < x_3 = 5 < x_6 = 6 \Rightarrow (145236)$$

$$x_1 = 1 < x_4 = 2 < x_5 = 3 < x_2 = 4 < x_6 = 5 < x_3 = 6 \Rightarrow (146235)$$

(9) $x_4 < x_2 < x_6, x_5 < x_3 < x_4 \Rightarrow x_3 < x_4 < x_2 < x_3$, a contradiction.

So the possible permutations are the following eight permutations with $x_1 = 1$.

$$\big\{(124635), (134625), (125634), (135624), (135246), (136245), (145236), (146235)\big\}.$$

This proves the theorem. ■

2.1 Elementary cyclic group $(\mathbb{Z}/2\mathbb{Z})^3$ action on labellings

Consider the set of 48 coherent labels of the tetrahedron defined as follows.

$$\mathcal{L}_T = \big\{(x_1, x_2, x_3, x_4, x_5, x_6) \in \{1, 2, 3, 4, 5, 6\}^6 \mid (x_1x_2x_3x_4x_5x_6) \in \big\{(124635), (134625), (125634), (135624), (135246), (136245), (145236), (146235)\big\}\big\}.$$

Then we observe that we can interchange the labels in each of the pairs

$$x_1 \longleftrightarrow x_4, x_2 \longleftrightarrow x_5, x_3 \longleftrightarrow x_6$$

independently for a coherent labelling to get another coherent labelling. So we have an action of $(\mathbb{Z}/2\mathbb{Z})^3$ on the set $\mathcal{L}_T$ via these interchanges.

3. Coherent Edge-Labelling of Pyramids

In this section we give a way to coherently label pyramids.

Theorem 3.1. *The pyramid over an n-gon for $n \geq 3$ is coherently labellable.*

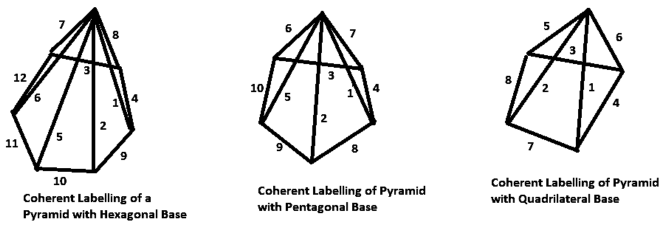

Figure 2. Pyramid Labelling

Proof. We already have a coherent labelling of a tetrahedron. A coherent edge-labelling on pyramids is obtained as follows. A coherent labelling for a pyramid with base either a quadrilateral or a pentagon or a hexagon is given in Fig. 2. In general a coherent labelling is obtained by extending this method of labelling as follows. The base has labels

$$3 < 4 < (n+3) < (n+4) < \cdots < (2n)$$

which gives an outward pointing normal for the pyramid. The faces with apex have the following labels, where all of the them give outward pointing normal as an orientation. Four of the faces with apex have the labels.

$$3 < (n+1) < (n+2), 1 < 4 < (n+2),$$
$$1 < 2 < (n+3), 2 < 5 < (n+4)$$

and the remaining faces have the labels for $n \geq 5$.

$$5 < 6 < (n+5), 6 < 7 < (n+6), \ldots, (n-1) < n < (2n-1), n < (n+1) < 2n.$$

This gives a coherent labelling of any pyramid. ■

4. Coherent Edge-Labelling of Kleetopes

Here in this section we describe one more class of polyhedra which can be coherently labelled. Actually here we prove something more. We begin with a definition.

Definition 4.1. *Let $\mathcal{P}$ be any polyhedron in three dimensional Euclidean Space. For any face F of $\mathcal{P}$ with n-edges we attach a pyramid with n-gon as a base to extend the polyhedron by keeping all the edges of the faces F intact. A Kleetope over $\mathcal{P}$ is a polyhedron $\mathcal{Q}$ obtained by attaching suitable pyramids to all the faces of the polyhedron $\mathcal{P}$ with this type of attachment for each face.*

Example 4.2. An example of a Kleetope polyhedron is as shown in Fig. 3. This is obtained from a dodecahedron and it is called a stellated dodecahedron or a pentakis dodecahedron. For more about polyhedral models refer to M. J. Wenninger [1], [2].

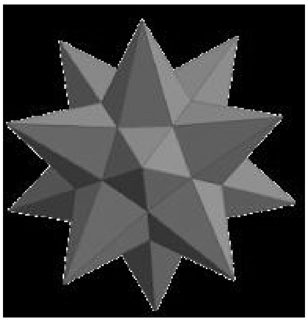

Figure 3. A Stellated Dodecahedron, A Kleetope

Now we prove the following theorem.

Theorem 4.3. *Let $\mathcal{P}$ be any polyhedron with an arbitrary labelling of all the edges. Then there exists a coherent labelling of the Kleetope $\mathcal{Q}$ of $\mathcal{P}$ extending the given labelling of the edges of $\mathcal{P}$.*

Proof. We prove this theorem by extending labelling to the pyramid $\mathcal{P}_F$ in $\mathcal{Q}$ for each face F of the polyhedron $\mathcal{P}$. Consider the arbitrary labelling of the face F given by integers

$$a_1 \longrightarrow a_2 \longrightarrow \cdots \longrightarrow a_n \longrightarrow a_1$$

with a_1 as the least label in this anticlockwise order. Let P_F be the apex of the pyramid for this face in the polyhedron $\mathcal{Q}$. Let

$$b_{12}, b_{23}, \ldots, b_{n1}$$

are the labels of the edges joining a vertex of the face F and the apex P_F. Now after the attachment, the face F vanishes and the new triangular faces with labels are given by

$$\{b_{n1}, a_1, b_{12}\}, \{b_{12}, a_2, b_{23}\}, \ldots, \{b_{(n-2)(n-1)}, a_{n-1}, b_{(n-1)n}\},$$
$$\{b_{(n-1)n}, a_n, b_{n1}\}.$$

We need to prove that there exists a coherent choice for the labels b_{ij} which initially may be fractions and need not be integers.

This is an inductive procedure as follows. In the first step we have $a_1 < a_2$ as a_1 is the least label. Hence choose a rational b_{12} such that $a_1 < b_{12} < a_1 + 1 \leq a_2$. Now in the second step there are two possibilities. If $a_2 < a_3$ then choose b_{23} such that $a_2 < b_{23} < a_2 + 1 \leq a_3$. If $a_2 > a_3$ then choose b_{23} such that $a_1 < b_{23} < b_{12} < a_1 + 1 \leq a_3 < a_2$. Inductively we proceed with the labelling. Suppose $b_{(i-1)i}$ has a coherent value and $b_{(i-1)i} < a_i$ then we need to choose a value for $b_{i(i+1)}$ and $b_{i(i+1)} < a_{i+1}$. Here we have two choices. Suppose $a_i < a_{i+1}$. Then we choose $b_{i(i+1)}$ such that $a_i < b_{i(i+1)} < a_i + 1 \leq a_{i+1}$. Hence we have

$$b_{(i-1)i} < a_i < b_{i(i+1)}.$$

Suppose $a_i > a_{i+1}$. Then choose $b_{i(i+1)}$ such that

$$a_1 < b_{i(i+1)} < \min\{b_{12}, b_{23}, \ldots, b_{(i-1)i}, a_1 + 1\} \leq a_1 + 1.$$

Then we have by choice $b_{i(i+1)} < b_{(i-1)i}$ and by induction $b_{(i-1)i} < a_i$ and hence

$$b_{i(i+1)} < b_{(i-1)i} < a_i.$$

Again here we observe that $b_{i(i+1)} < a_1 + 1 \leq a_{i+1}$ and the induction step is completed. Now we arrive at the last step of choosing a coherent label for b_{n1}. Choose b_{n1} such that $b_{n1} > \max\{a_n, b_{12}\}$ then we have

$$b_{(n-1)n} < a_n < b_{n1}, \; a_1 < b_{12} < b_{n1}.$$

This completes the proof of this theorem. ■

4.1 Labelling of a partial Pyramid

In this section we prove that a partial pyramid (defined in Theorem 4.4 and Remark 4.6) is coherently labellable. This is useful in the proof of main Theorem 7.

Theorem 4.4. *Let $k < n$ be two positive integers. Let $a_1, a_2, \ldots, a_k$ be some labels of k juxtaposed edges of an n-gon in the anticlockwise manner $a_1 \longrightarrow a_2 \longrightarrow \cdots \longrightarrow a_k$. Let P be a vertex not in the plane of the n-gon and is joined by edges with the vertices of the n-gon. Then there exists labels $b_{n1}, b_{12}, b_{23}, \ldots, b_{k(k+1)}$ such that the triangular faces given whose labels are given by*

$$\{b_{n1}, a_1, b_{12}\}, \{b_{12}, a_2, b_{23}\}, \ldots, \{b_{(k-1)k}, a_k, b_{k(k+1)}\}$$

have an anticlockwise orientation of increasing labels.

The method used in the proof of Theorem 4.3 also applies to Theorem 4.4. The same method works for the following theorem as well.

Theorem 4.5. *Let $\mathcal{P}$ be any polyhedron with a labelling of all the edges such that except possibly for one face F with n_F edges, all the other faces have been coherently labelled. Let $\mathcal{Q}$ be the polyhedron obtained by adding a pyramid over an n_F-gon whose base is attached to F such that the face F vanishes in $\mathcal{Q}$ but none of the edges of F vanish in $\mathcal{Q}$. Then there exists a coherent labelling of the polyhedron $\mathcal{Q}$ extending the given labelling of the edges of $\mathcal{P}$.*

Remark 4.6. In Theorem 4.4 if we have arbitrary labelled edges say $a_1, a_2, \ldots, a_k$ instead of juxtaposed edges of an n-gon then we can apply Theorem 4.4 to each subcollection of juxtaposed edges to obtain a labelling of those new triangular faces of the partial pyramid created by these juxtaposed edges for every such subcollection. This remark will be used later in main Theorem 7.

4.2 Coherent Edge-Labelling of Bipyramids

Theorem 4.7. *Let $\mathcal{P}$ be a polyhedron which is a bipyramid over an n-gon. Then the edges of this polyhedron can be labelled coherently.*

Proof. This follows by the same method as given in the proof of Theorem 4.3. ■

Example 4.8 (Coherent Labelling of an Octahedron).
A coherent labelling of an octahedron is given in Fig. 4.

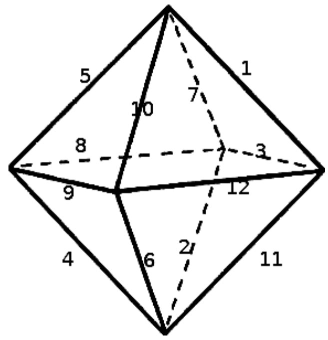

Figure 4. A Labelling of an Octahedron

5. Class of polyhedra which are not Coherently Edge-Labellable

We begin with a couple of counter examples and then prove Theorem 5.9 where a certain class of polyhedra is not coherently labellable.

5.1 The case of a cube or cuboid

Theorem 5.1. *There does not exist a coherent labelling for a cube or a cuboid.*

Proof. By referring to Fig. 5 let us label the edges as $x_1, \ldots, x_{12}$. The faces with possible inequalities are given by

$$\begin{aligned}
F_1 &: x_1 < x_2 < x_3 < x_4 \mid x_2 < x_3 < x_4 < x_1 \\
&\mid x_3 < x_4 < x_1 < x_2 \mid x_4 < x_1 < x_2 < x_3 \\
F_2 &: x_5 < x_6 < x_7 < x_2 \mid x_6 < x_7 < x_2 < x_5 \\
&\mid x_7 < x_2 < x_5 < x_6 \mid x_2 < x_5 < x_6 < x_7 \\
F_3 &: x_{11} < x_{10} < x_8 < x_6 \mid x_{10} < x_8 < x_6 < x_{11} \\
&\mid x_8 < x_6 < x_{11} < x_{10} \mid x_6 < x_{11} < x_{10} < x_8 \\
F_4 &: x_{12} < x_4 < x_9 < x_{10} \mid x_4 < x_9 < x_{10} < x_{12} \\
&\mid x_9 < x_{10} < x_{12} < x_4 \mid x_{10} < x_{12} < x_4 < x_9 \\
F_5 &: x_3 < x_7 < x_8 < x_9 \mid x_7 < x_8 < x_9 < x_3 \\
&\mid x_8 < x_9 < x_3 < x_7 \mid x_9 < x_3 < x_7 < x_8 \\
F_6 &: x_1 < x_{12} < x_{11} < x_5 \mid x_{12} < x_{11} < x_5 < x_1 \\
&\mid x_{11} < x_5 < x_1 < x_{12} \mid x_5 < x_1 < x_{12} < x_{11}.
\end{aligned}$$

Now we prove that there is no choice of inequalities for the labels $x_1, \ldots, x_{12}$ which extends six strings of inequalities with one possibility for each of the six faces.

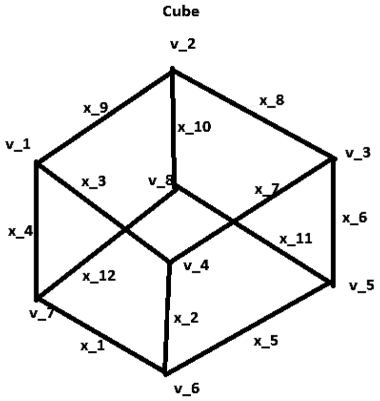

Figure 5. A Cube or a Cuboid

Assume for instance we choose the first possibility for the face F_1 i.e. $F_1 : x_1 < x_2 < x_3 < x_4$. Now we dispose off each possibility as impossible for the face F_2.

(1) Consider $F_2 : x_5 < x_6 < x_7 < x_2$. Then we have $x_7 < x_3$. So only $F_5 : x_7 < x_8 < x_9 < x_3$ occurs and the remaining possibilities for the face F_5 do not occur. Now we have $x_6 < x_8$. So only $F_3 : x_6 < x_{11} < x_{10} < x_8$ occurs and the remaining possibilities for the face F_3 do not occur. Now $x_5 < x_{11}$. So only $F_6 : x_5 < x_1 < x_{12} < x_{11}$ occurs and the remaining possibilities do not occur. Now we conclude that $x_{10} < x_9 < x_4$. This rules out all the possibilities for the face F_4.

(2) Consider $F_2 : x_6 < x_7 < x_2 < x_5$. Then we have $x_7 < x_3$. So only $F_5 : x_7 < x_8 < x_9 < x_3$ occurs and the remaining possibilities for the face F_5 do not occur. Now we have $x_6 < x_8$. So only $F_3 : x_6 < x_{11} < x_{10} < x_8$ occurs and the remaining possibilities for the face F_3 do not occur. Now $x_1 < x_5$. So only $F_6 : x_1 < x_{12} < x_{11} < x_5$ occurs and the remaining possibilities for the face F_6 do not occur. Similarly $x_9 < x_4$. So only $F_4 : x_9 < x_{10} < x_{12} < x_4$ occurs and the remaining possibilities for the face F_4 do not occur. Now we conclude $x_{12} < x_{11} < x_{10} < x_{12}$ which is a contradiction.

(3) Consider $F_2 : x_7 < x_2 < x_5 < x_6$. Then we have $x_7 < x_3$. So only $F_5 : x_7 < x_8 < x_9 < x_3$ occurs and the remaining possibilities for the face F_5 do not occur. Now we have $x_9 < x_4$. So only $F_4 : x_9 < x_{10} < x_{12} < x_4$ occurs and the remaining possibilities for the face F_4 do not occur. Now we have $x_1 < x_5$. So only $F_6 : x_1 < x_{12} < x_{11} < x_5$ occurs and the remaining possibilities for the face F_6 do not occur. Similarly we have $x_{11} < x_6$. So only $F_3 : x_{11} < x_{10} < x_8 < x_6$ occurs and the remaining possibilities for the face F_3 do not occur. Now we conclude $x_{12} < x_{11} < x_{10} < x_{12}$ which is a contradiction.

(4) Consider $F_2 : x_2 < x_5 < x_6 < x_7$. Then we have $x_1 < x_5$. So only $F_6 : x_1 < x_{12} < x_{11} < x_5$ occurs and the remaining possibilities for the face F_6 do not occur. Now we have $x_{11} < x_6$. So only $F_3 : x_{11} < x_{10} < x_8 < x_6$ occurs and the remaining possibilities for the face F_3 do not occur. We have $x_8 < x_7$. So only $F_5 : x_8 < x_9 < x_3 < x_7$ occurs and the remaining possibilities for the face F_5 do not occur. Now we conclude that $x_{12} < x_{10} < x_9$. This rules out all the possibilities for the face F_4.

This proves the theorem. ■

5.2 The case of a triangular Prism

Now we prove that a triangular prism is not coherently labellable. We first mention a definition.

Definition 5.2. *We say a labelling of the edges of a polyhedr on $\mathcal{P}$ is anticlockwise coherent or clockwise coherent at a vertex v of $\mathcal{P}$, if the labelling of the edges of $\mathcal{P}$ with integers, is such that, at the vertex v, we have an anticlockwise traversal or a clockwise traversal respectively, of edges emanating at the vertex v, with an increasing choice of labels, except the first edge and the last edge. (Observer's eye is outside the polyhedron).*

Now we prove a lemma.

Lemma 5.3. *For any coherent labelling of the tetrahedron, the labelling of the edges of the tetrahedron is clockwise coherent at any of its vertices.*

Proof. This follows from Theorem 2.1. ■

Remark 5.4. It is not true that for any coherent labelling of an octahedron, the labelling of the edges of the octahedron is clockwise coherent at a vertex. In fact, at a vertex, the labelling need not be either clockwise coherent or anticlockwise coherent. For example there is no coherency at the topmost and the leftmost vertices in the coherently labelled octahedron in Fig. 4.

Now we make another important observation about the tetrahedron in the following lemma.

Lemma 5.5. *For any coherent labelling of the tetrahedron, if we construct a triangular prism by chopping a vertex of a tetrahedron with new edges labelled a, b, c such that the remaining faces are coherent, then the orientation that this labelling gives rise to, for the new face, is always clockwise.*

Proof. From Lemma 5.3 and Theorem 2.1, we can conclude this lemma. ■

Now we prove that a triangular prism cannot be labelled coherently.

Lemma 5.6. *A triangular prism cannot be labelled coherently.*

Proof. Any coherent labelling of a triangular prism is obtained by a labelling of a tetrahedron and then chopping a vertex and labelling the new edges. The reason is as follows. With a coherently labelled triangular prism, we obtain a coherently labelled tetrahedron by collapsing any one of the triangular faces of the triangular prism to a point or by attaching a tetrahedron to a triangular face and making the three edges and the three vertices vanish. Hence by reversing the process, that is, by chopping a suitable vertex we obtain back the triangular prism and has the given coherent labelling with which we started. This contradicts Lemma 5.5. Hence a triangular prism cannot be coherently labelled. ■

5.3 Obstruction criterion

The proof of the above theorem leads to the following.

Theorem 5.7. *Let $\mathcal{P}$ be a convex polyhedron. Let $\mathcal{Q}$ be a convex polyhedron which is obtained by attaching the base of a pyramid having apex v to a face F of $\mathcal{P}$ so that all the edges and vertices of the face F vanish. We assume the following.*

(1) *$\mathcal{Q}$ is coherently labellable.*
(2) *The obstruction criterion at the vertex v: In all the coherent labellings of $\mathcal{Q}$, the labellings are clockwise coherent at the vertex v of $\mathcal{Q}$.*

Then the polyhedron $\mathcal{P}$ which is obtained by chopping vertex v of $\mathcal{Q}$ is not coherently labellable.

Proof. Suppose the polyhedron $\mathcal{P}$ has a coherent labelling. Then by a pyramid attachment to the face F of $\mathcal{P}$ we obtain a coherent labelling on $\mathcal{Q}$. Hence by the hypothesis, the labelling of edges is clockwise coherent at the vertex v. Now if we chop the vertex v then we can revert back to the initial labels of $\mathcal{P}$ which is coherent to begin with. Now we arrive at a contradiction by proving the following claim.

Claim 5.8. For any coherent labelling of $\mathcal{Q}$, which satisfies the obstruction criterion (2) at the vertex v, if we chop the vertex v of $\mathcal{Q}$ and label the edges of the face F of $\mathcal{P}$ such that we have a coherent labelling on all the faces of $\mathcal{P}$ except possibly F, then the increasing order of any such labelling of edges of F induces a clockwise orientation for the face F.

Hence, using the claim, we immediately arrive at the conclusion that the polyhedron $\mathcal{P}$ is not coherently labellable.

Proof of Claim. Let $a_1 < a_2 < \cdots < a_n$ be the labels of the edges incident to the vertex v of $\mathcal{Q}$ in the clockwise order. Let $b_1, b_2, \ldots, b_n$ be labels of the edges of the face F such that we have the following n-triangular faces for the pyramid attached to $\mathcal{P}$

$$\{a_1, a_2, b_1\}, \{a_2, a_3, b_2\}, \ldots, \{a_{n-1}, a_n, b_{n-1}\}, \{a_n, a_1, b_n\}.$$

Assume without loss of generality that a_1 is the least label among all the edges of $\mathcal{P}$ because the coherency property does not change by cyclically relabelling all the edges of $\mathcal{P}$.

Since the we have a coherent labelling (anticlockwise) on all the faces of $\mathcal{P}$ except possibly the face F. We conclude the following.

- $a_1 < a_n < b_n,\ a_1 < b_1 < a_2$ since a_1 is the least label.
- Either $a_2 < b_2 < a_3$ or $a_3 < a_2 < b_2$. But $a_2 < a_3$ hence we must have $a_2 < b_2 < a_3$.
- Similarly we have $a_i < b_i < a_{i+1}$ for $1 \leq i \leq n-1$.
- Hence we conclude that $b_1 < b_2 < \cdots < b_{n-1} < b_n$. This induces a clockwise orientation on the face F of the polyhedron $\mathcal{P}$.

This proves the claim. ■

Hence we have completed the proof of Theorem 5.7. ■

As an application of Theorem 5.7 we prove the following theorem.

Theorem 5.9. *Any pyramid with its apex chopped off, is not coherently labellable. In particular, the triangular prism and the cube (or the cuboid) are not coherently labellable.*

Proof. It is enough to prove that any coherent labelling of the pyramid induces a labelling of the edges which is clockwise coherent at the apex. This is definitely true for a tetrahedron using Theorem 2.1. Now we prove by induction on the number of edges of the base of the pyramid.

Let $a_1, a_2, \ldots, a_n$ be the labels of the edges incident at the apex in the anticlockwise order with a_1 being the least label among all the labels of the edges of the pyramid. Let $b_1, b_2, \ldots, b_n$ be the labels of the edges of the base such the triangular faces are given by

$$\{a_1, a_2, b_1\}, \{a_2, a_3, b_2\}, \ldots, \{a_{n-1}, a_n, b_{n-1}\}, \{a_n, a_1, b_n\}.$$

Then we first have $a_1 < a_n < b_n$, $a_1 < b_1 < a_2$ since a_1 is the least label. Now we have one of the following three cases.

$$\text{I} : a_{n-1} < b_{n-1} < a_n; \quad \text{II} : a_n < a_{n-1} < b_{n-1};$$
$$\text{III} : b_{n-1} < a_n < a_{n-1}.$$

In the cases I, II we have $a_1 < a_{n-1} < b_{n-1}$. So we delete a_n and b_n in these two cases to obtain a coherent labelling on a pyramid with a base an $(n-1)$-gon with labels $a_1, \ldots, a_{n-1}$. Hence by induction we have

$$a_1 < a_{n-1} < a_{n-2} < \cdots < a_3 < a_2$$

a labelling which is clockwise coherent at the apex of the pyramid with a base an $(n-1)$-gon.

In the cases I, III we have $b_{n-1} < b_n$. Since the labelling on the pyramid is coherent we must have

$$b_{n-1} < b_{n-2} < \cdots < b_2 < b_1 < b_n.$$

Moreover in case III we have $a_1 < a_{n-1} < L$ where L is any positive integer larger than all the labels $a_i, b_i, 1 \leq i \leq n$. Now deleting a_n, b_n and relabelling b_{n-1} by L we obtain a coherent labelling on a pyramid with a base an $(n-1)$-gon with labels $a_i, 1 \leq i \leq n-1, b_j, 1 \leq j \leq n-2$ and L. Here the faces with label L are given by $a_1 < a_{n-1} < L$ and

$$b_{n-2} < b_{n-3} < \cdots < b_2 < b_1 < L.$$

In this case III also, now, by induction we have

$$a_1 < a_{n-1} < a_{n-2} < \cdots < a_3 < a_2$$

a labelling which is clockwise coherent at the apex of the pyramid with a base an $(n-1)$-gon.

Combining with the inequality $a_1 < a_n < a_{n-1}$ in cases II and III we have

$$a_1 < a_n < a_{n-1} < a_{n-2} < \cdots < a_3 < a_2$$

a labelling which is clockwise coherent at the apex of the pyramid with a base an n-gon.

We consider case I. Here we have $a_1 < a_n < b_n$; $a_1 < b_1 < a_2$; $a_{n-1} < b_{n-1} < a_n$ and $b_{n-1} < b_{n-2} < \cdots < b_2 < b_1 < b_n$.

Now either $a_3 < a_2 < b_2$ or $b_2 < a_3 < a_2$. Here $a_3 < a_2 < b_2 \Rightarrow b_1 < a_2 < b_2$ which is a contradiction. So $b_2 < a_3 < a_2$. For $2 \leq i \leq n-3$ if we have if $b_i < a_{i+1} < a_i$ then $a_{i+2} < a_{i+1} < b_{i+1} \Rightarrow b_i < a_{i+1} < b_{i+1}$ which is a contradiction for $1 \leq i \leq n-2$. Hence we must have $b_{i+1} < a_{i+2} < a_{i+1}$. So inductively we obtain $b_i < a_{i+1} < a_i$ for $2 \leq i \leq n-2$. For $i = n-2$ we have $b_{n-2} < a_{n-1} < a_{n-2}$. Now using $a_{n-1} < b_{n-1} < a_n$ we obtain $b_{n-2} < a_{n-1} < b_{n-1}$ which is a contradiction. Hence case I does not arise at all.

So we have proved that if we have a coherent labelling of the pyramid then labels induce clockwise coherency at the apex.

Hence we have completed the proof of Theorem 5.9. ■

6. The Dodecahedron and the Icosahedron

In this section we prove that the dodecahedron is not coherently labellable (Lemma 6.2) and the icosahedron is coherently labellable (Lemma 6.3). As a consequence we have the following theorem.

Theorem 6.1. *Among the five platonic solids or regular polyhedra, the simplicial polyhedra namely the tetrahedron, the octahedron and the icosahedron are coherently labellable and the other two namely the dodecahedron and the cube are not coherently labellable.*

Now we mention the following lemma.

Lemma 6.2. *The dodecahedron is not coherently labellable.*

Proof. Consider the dodecahedron in Fig. 6 with labels $a_1, \ldots, a_{30}$ for the edges of the faces $F_1, \ldots, F_{12}$. Now we prove that the dodecahedron is not coherently labellable in a similar way as given in the proof of Theorem 5.1 for the cube. The possible choices of strings of inequalities for the faces $F_1, \ldots, F_{12}$ are

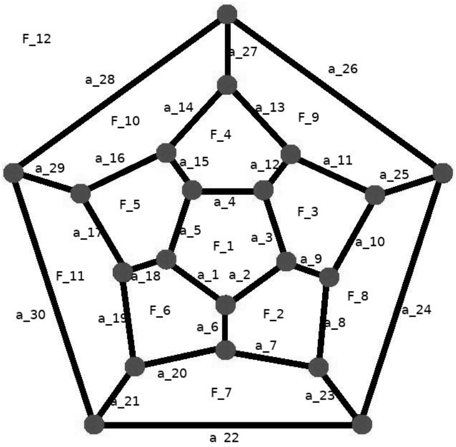

Figure 6. A Dodecahedron

(1) For the face F_1

(A) $a_1 < a_2 < a_3 < a_4 < a_5$.
(B) $a_2 < a_3 < a_4 < a_5 < a_1$.
(C) $a_3 < a_4 < a_5 < a_1 < a_2$.
(D) $a_4 < a_5 < a_1 < a_2 < a_3$.
(E) $a_5 < a_1 < a_2 < a_3 < a_4$.

(2) For the face F_2

(A) $a_2 < a_6 < a_7 < a_8 < a_9$.
(B) $a_6 < a_7 < a_8 < a_9 < a_2$.
(C) $a_7 < a_8 < a_9 < a_2 < a_6$.
(D) $a_8 < a_9 < a_2 < a_6 < a_7$.
(E) $a_9 < a_2 < a_6 < a_7 < a_8$.

(3) For the face F_3

(A) $a_3 < a_9 < a_{10} < a_{11} < a_{12}$.
(B) $a_9 < a_{10} < a_{11} < a_{12} < a_3$.
(C) $a_{10} < a_{11} < a_{12} < a_3 < a_9$.
(D) $a_{11} < a_{12} < a_3 < a_9 < a_{10}$.
(E) $a_{12} < a_3 < a_9 < a_{10} < a_{11}$.

(4) For the face F_4

(A) $a_4 < a_{12} < a_{13} < a_{14} < a_{15}$.
(B) $a_{12} < a_{13} < a_{14} < a_{15} < a_4$.
(C) $a_{13} < a_{14} < a_{15} < a_4 < a_{12}$.
(D) $a_{14} < a_{15} < a_4 < a_{12} < a_{13}$.
(E) $a_{15} < a_4 < a_{12} < a_{13} < a_{14}$.

(5) For the face F_5

(A) $a_5 < a_{15} < a_{16} < a_{17} < a_{18}$.
(B) $a_{15} < a_{16} < a_{17} < a_{18} < a_5$.
(C) $a_{16} < a_{17} < a_{18} < a_5 < a_{15}$.
(D) $a_{17} < a_{18} < a_5 < a_{15} < a_{16}$.
(E) $a_{18} < a_5 < a_{15} < a_{16} < a_{17}$.

(6) For the face F_6

(A) $a_1 < a_{18} < a_{19} < a_{20} < a_6$.
(B) $a_{18} < a_{19} < a_{20} < a_6 < a_1$.
(C) $a_{19} < a_{20} < a_6 < a_1 < a_{18}$.
(D) $a_{20} < a_6 < a_1 < a_{18} < a_{19}$.
(E) $a_6 < a_1 < a_{18} < a_{19} < a_{20}$.

(7) For the face F_7

(A) $a_7 < a_{20} < a_{21} < a_{22} < a_{23}$.
(B) $a_{20} < a_{21} < a_{22} < a_{23} < a_7$.
(C) $a_{21} < a_{22} < a_{23} < a_7 < a_{20}$.
(D) $a_{22} < a_{23} < a_7 < a_{20} < a_{21}$.
(E) $a_{23} < a_7 < a_{20} < a_{21} < a_{22}$.

(8) For the face F_8

(A) $a_8 < a_{23} < a_{24} < a_{25} < a_{10}$.
(B) $a_{23} < a_{24} < a_{25} < a_{10} < a_8$.
(C) $a_{24} < a_{25} < a_{10} < a_8 < a_{23}$.
(D) $a_{25} < a_{10} < a_8 < a_{23} < a_{24}$.
(E) $a_{10} < a_8 < a_{23} < a_{24} < a_{25}$.

(9) For the face F_9

(A) $a_{11} < a_{25} < a_{26} < a_{27} < a_{13}$.
(B) $a_{25} < a_{26} < a_{27} < a_{13} < a_{11}$.
(C) $a_{26} < a_{27} < a_{13} < a_{11} < a_{25}$.
(D) $a_{27} < a_{13} < a_{11} < a_{25} < a_{26}$.
(E) $a_{13} < a_{11} < a_{25} < a_{26} < a_{27}$.

(10) For the face F_{10}

(A) $a_{14} < a_{27} < a_{28} < a_{29} < a_{16}$.
(B) $a_{27} < a_{28} < a_{29} < a_{16} < a_{14}$.
(C) $a_{28} < a_{29} < a_{16} < a_{14} < a_{27}$.
(D) $a_{29} < a_{16} < a_{14} < a_{27} < a_{28}$.
(E) $a_{16} < a_{14} < a_{27} < a_{28} < a_{29}$.

(11) For the face F_{11}

(A) $a_{17} < a_{29} < a_{30} < a_{21} < a_{19}$.
(B) $a_{29} < a_{30} < a_{21} < a_{19} < a_{17}$.
(C) $a_{30} < a_{21} < a_{19} < a_{17} < a_{29}$.
(D) $a_{21} < a_{19} < a_{17} < a_{29} < a_{30}$.
(E) $a_{19} < a_{17} < a_{29} < a_{30} < a_{21}$.

(12) For the face F_{12}

(A) $a_{22} < a_{30} < a_{28} < a_{26} < a_{24}$.
(B) $a_{30} < a_{28} < a_{26} < a_{24} < a_{22}$.
(C) $a_{28} < a_{26} < a_{24} < a_{22} < a_{30}$.
(D) $a_{26} < a_{24} < a_{22} < a_{30} < a_{28}$.
(E) $a_{24} < a_{22} < a_{30} < a_{28} < a_{26}$.

Assume without loss of generality that $(1):(A)$ holds. Now we will exhaust all cases $(2):(A)-(E)$. Suppose $(2):(A)$ holds. Then we have

$$((1):(A) \quad \text{and} \quad (2):(A)) \Rightarrow (6):(A) \Rightarrow (7):(B)$$
$$\Rightarrow (8):(B) \Rightarrow (3):(C) \Rightarrow (9):(B) \Rightarrow (4):(C)$$
$$\Rightarrow (10):(B) \Rightarrow (5):(C) \Rightarrow (11):(B).$$

Now we conclude that $a_{17} \underset{(5):(C)}{<} a_{18} \underset{(6):(A)}{<} a_{19} \underset{(11):(B)}{<} a_{17}$ which is a contradiction.

Suppose $(2):(B)$ holds. Then we have

$$((1):(A) \quad \text{and} \quad (2):(B)) \Rightarrow (3):(B) \Rightarrow (4):(B)$$
$$\Rightarrow (5):(B) \Rightarrow (8):(A) \Rightarrow (9):(A) \Rightarrow (10):(A)$$
$$\Rightarrow (12):(E) \Rightarrow (11):(C) \Rightarrow (7):(D).$$

Now we conclude that $a_8 \underset{(8):(A)}{<} a_{23} \underset{(7):(D)}{<} a_7 \underset{(2):(B)}{<} a_8$ which is a contradiction.

Suppose $(2):(C)$ holds. Then we have

$$((1):(A) \quad \text{and} \quad (2):(C)) \Rightarrow (3):(B) \Rightarrow (4):(B)$$
$$\Rightarrow (5):(B) \Rightarrow (6):(A) \Rightarrow (8):(A) \Rightarrow (9):(A)$$
$$\Rightarrow (10):(A) \Rightarrow (11):(A) \Rightarrow (12):(C).$$

Now we conclude that $a_{26} \underset{(9):(A)}{<} a_{27} \underset{(10):(A)}{<} a_{28} \underset{(12):(C)}{<} a_{26}$ which is a contradiction.

Suppose $(2):(D)$ holds. Then we have

$$((1):(A) \quad \text{and} \quad (2):(D)) \Rightarrow (3):(B) \Rightarrow (4):(B)$$
$$\Rightarrow (5):(B) \Rightarrow (6):(A) \Rightarrow (7):(B) \Rightarrow (8):(A).$$

Now we conclude that $a_{10} \underset{(3):(B)}{<} a_{12} \underset{(4):(B)}{<} a_{15} \underset{(5):(B)}{<} a_{18} \underset{(6):(A)}{<} a_{20} \underset{(7):(B)}{<} a_{23} \underset{(8):(A)}{<} a_{10}$ which is a contradiction.

Suppose $(2):(E)$ holds. Then we have

$$((1):(A) \quad \text{and} \quad (2):(E)) \Rightarrow (3):(B) \Rightarrow (4):(B)$$
$$\Rightarrow (5):(B) \Rightarrow (6):(A) \Rightarrow (7):(B) \Rightarrow (8):(B).$$

Now we conclude that $a_{10} \underset{(3):(B)}{<} a_{12} \underset{(4):(B)}{<} a_{15} \underset{(5):(B)}{<} a_{18} \underset{(6):(A)}{<} a_{20} \underset{(7):(B)}{<} a_{23} \underset{(8):(B)}{<} a_{10}$ which is a contradiction.

This proves that the dodecahedron is not coherently labellable. ■

Lemma 6.3. *The icosahedron is coherently labellable.*

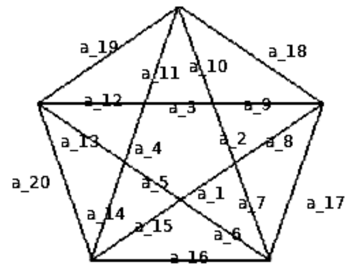

Figure 7. An icosahedron with two opposite vertices removed giving rise to two pentagonal faces for the new polyhedron

Proof. It is enough to prove that following polyhedron in Fig. 7 is coherently labellable. If we attach two pentagonal based pyramids to the two opposite pentagonal faces of the polyhedron in Fig. 7, we get an icosahedron. Hence the icosahedron is coherently labellable using Theorem 4.5. Now assume the edges in the Fig. 7 are labelled $a_1, \ldots, a_{20}$. Then we can have the following strings of inequalities satisfied simultaneously.

- $a_1 < a_2 < a_3 < a_4 < a_5$.
- $a_{16} < a_{20} < a_{19} < a_{18} < a_{17}$.
- $a_1 < a_6 < a_7$; $a_2 < a_8 < a_9$; $a_3 < a_{10} < a_{11}$; $a_4 < a_{12} < a_{13}$; $a_5 < a_{14} < a_{15}$.
- $a_{16} < a_6 < a_{15}$; $a_{17} < a_8 < a_7$; $a_{18} < a_{10} < a_9$; $a_{19} < a_{12} < a_{11}$; $a_{20} < a_{14} < a_{13}$.

This can be reasoned out as follows. Let $A = \{a_1, a_2, a_3, a_4, a_5\}$, $B = \{a_6, a_8, a_{10}, a_{12}, a_{14}\}$, $C = \{a_7, a_9, a_{11}, a_{13}, a_{15}\}$, $D = \{a_{16}, a_{17}, a_{18}, a_{19}, a_{20}\}$. Choose labels such that $A < B < C$ and $D < B < C$, that is, for any $a \in A, b \in B, c \in C, d \in D$ we have $a < b < c, d < b < c$ in addition to the strings of inequalities $a_1 < a_2 < a_3 < a_4 < a_5$ and $a_{16} < a_{20} < a_{19} < a_{18} < a_{17}$. Hence the lemma follows. ■

Now we mention a definition and a theorem.

Definition 6.4. *An n-sided antiprism is a polyhedron composed of two parallel copies of some particular n-sided* *polygon, connected by an alternating band of triangles. A gyroelongated bipyramid is a polyhedron obtained by elongating an n-gonal bipyramid by inserting an n-gonal antiprism between its congruent halves. For example the icosahedron is a gyroelongated bipyramid obtained from pentagonal bipyramid.*

Theorem 6.5. *Any gyroelongated bipyramid is coherently labellable.*

Proof. The proof is similar that of Lemma 6.3. ■

7. Coherent Edge-Labelling of Simplicial Polyhedra

We have observed that apart from the pyramids and antiprisms, the following polyhedra are coherently labellable. The tetrahedron, the bipyramids which include the octahedron, the Kleetopes, the gyroelongated bipyramids which include the icosahedron. This leads us to the following theorem which we prove in this section. The converse of this theorem is not true since pyramids on any base polygon are also coherently labellable.

Theorem Ω. Any three dimensional simplicial polyhedron, that is, a polyhedron with all its faces having exactly three sides, is coherently labellable.

Proof. In a simplicial polyhedron $\mathcal{P}$ we first remove a vertex v along with interior of the edges and faces incident on the vertex. If the remaining figure is coherently labellable then we can coherently label $\mathcal{P}$ using Theorem 4.5 since this is similar to attaching a pyramid. So we continue this process of removing a vertex and all its incident edges and faces till such vertices exist, and keep track of the removed vertices with all their incident edges and faces. In order to label coherently the remaining figure we remove a vertex w (if necessary) along with interior of only those edges and faces that are present and incident at w and continue this process till the remaining figure becomes coherently labellable. Then we reverse the process of putting back vertices of type w and extend the coherent labelling using Theorem 4.4 and Remark 4.6 since this is similar to attaching a partial pyramid. Then we put back vertices of type v and use Theorem 4.5 to extend the labelling since this is similar to attaching a pyramid. ■

References

[1] M. J. Wenninger, Polyhedron models, Cambridge University Press, London-New York (1971), xii+208 pp. ISBN: 978-0-521-09859-5, https://doi.org/10.1017/CBO9780511569746 , MR0467493.

[2] M. J. Wenninger, Dual models, Cambridge University Press, Cambridge (1983), xii+156 pp. ISBN: 978-0-521-54325-5, https://doi.org/10.1017/CBO9780511569371, MR0730208.

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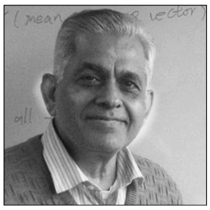

Ravindra Shripad Kulkarni (born 1942) is a Distinguished Professor at the Bhaskaracharya Pratishthana, Pune, specializing in differential geometry. He is known for the Kulkarni–Nomizu product. After doing B.Sc. (Mathematics) from S. P. College, Pune, India, Prof. Ravi Kulkarni received his Ph.D. in 1968 from Harvard University under Shlomo Sternberg with thesis Curvature and Metric. For the academic year 1980–1981 he was a Guggenheim Fellow.

After a research and teaching career spanning over 40 years in the US at Johns Hopkins University, Columbia University, Indiana University, and City University of New York, he returned to India as Distinguished Professor and Director of Harish-Chandra Research Institute, Prayagraj. Later he was a Visiting Professor for 7 years at the Indian Institute of Technology, Bombay. He is interested in the Philosophy of Mathematics and Science.

He has served as the President of the Ramanujan Mathematical Society during 2013–16.

He initiated The Indian Mathematics Consortium in 2014 and was the Founder President of the Consortium, through which he organized a conference in Association with American Mathematical Society at BHU. He is the Chief Editor of the TMC Bulletin. He has guided 10 PhD students and has influenced many younger mathematicians over the years.

Professor Ravindra Kulkarni is turning 80 on 22nd May 2022. In his honour, Bhaskaracharya Pratishthana is organizing an International Conference in Mathematics in a hybrid mode during 21st–25th May, 2022.

Professor Kulkarni's pioneer work from 1968 until 1998 falls in the broad area of differential geometry. From 2000 onwards, he has influenced a few curious directions in group theory whose ideas came from geometry. Professor Kulkarni has also been interested in general mathematics, mathematical philosophy and Indian knowledge system. In the conference, we shall give flavors from all these areas that intrigued him over the years. Accordingly, the conference has been divided (broadly) into three parts: *Differential geometry* (May 21 & 24), *Symmetries from a geometric viewpoint* (May 23), and *Mathematics, Knowledge and Philosophy* (May 22 & 25).

Geometry, Groups and Mathematical Philosophy — An International Conference in Honor of Prof. Ravi S. Kulkarni's 80th Birthday May 21–25, 2022

Confirmed speakers are:

Differential geometry: • Ara Basmajian (CUNY), • Sagun Chanillo (Rutgers University, New Brunswick), • S. G. Dani (CBS Mumbai), • Allan Edmonds (Indiana), • William Goldman (Maryland), • Yunping Jiang (CUNY), • Alexander Mednykh (Novosibirsk) • Ulrich Pinkall (TU Berlin), • Jose Seade (UNAM Mexico), • Dragomir Saric (CUNY), • S.-T. Yau (Harvard, Fields Medalist 1982).

Symmetries: • Gurmeet K. Bakshi (Punjab University, Chandigarh), • Anupam K. Singh (IISER Pune), • Siddhartha Sarkar (IISER Bhopal), • Soham Pradhan (Haifa University, Israel), • Jagmohan Tanti (BBAU Lucknow), • Anthony Weaver (CUNY), • Devendra Tiwari (BP, Pune).

Mathematics, Knowledge and Philosophy: • Satya Deo (HRI), • Jugal Verma (IIT Bombay), • Rohit Parikh (CUNY), • M. A. Sofi (Kashmir University), • Sanjay Pant (Delhi U.), • S. Ponnusamy (IIT Madras).

Conveners: S. A. Katre (SPPU, Pune), Krishnendu Gongopadhyay (IISER Mohali)

More details and the Time Table will be available at `www.bprim.org`.
Details of the Zoom Meeting:
Meeting ID: 865 2547 0673
Passcode: 609430

Details of Workshop/Conferences in India

For details regarding Mathematics Training and Talent Search Programme 2021
Visit: `https://mtts.org.in/programme/mtts2021/`

Name: Combinatorial Methods in Graph Theory
Date: July 14, 2022 – July 16, 2022
Venue: Amrita Vishwa Vidyappeetham, India
Visit: `https://amrita.edu/events/combinatorial-methods-in-graph-theory/`

Name: 34th Annual International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2022)
Date: July 18, 2022 – July 22, 2022
Venue: Indian Institute of Science Bangalore, India
Visit: `math.iisc.ac.in/fpsac2022/`

Name: Elliptic curves and the special values of L-functions
Date: August 08, 2022 – August 19, 2022

Venue: International Centre for Theoretical Sciences (ICTS) of the Tata Institute of Fundamental Research, Bangalore, India

Visit: `https://www.icts.res.in/program/ecl2022`

Details of Workshop/Conferences Abroad

For details regarding SIAM conferences

Visit `http://www.siam.org/meetings/`

Name: 6th Strathmore International Mathematics Conference (SIMC 2021)
Date: June 2, 2021 – July 28, 2021
Venue: Strathmore University, Nairobi, Kenya
Visit: `www.strathmore.edu/events/strathmore-international-mathematics-conference/`

Name: "Stochastic and Deterministic Global Optimization" stream at EURO-2022
Date: July 3, 2022 – July 6, 2022
Venue: Espoo, Finland
Visit: `euro2022espoo.com/`

Name: III. International Conference on Mathematics and its Applications in Science and Engineering
Date: July 4, 2022 – July 7, 2022
Venue: Bucharest, Romania
Visit: `www.icmase.com/`

Name: The 2022 International Conference of Applied and Engineering Mathematics (ICAEM'22)
Date: July 6, 2022 – July 8, 2022
Venue: Imperial College London, London, UK
Visit: `www.iaeng.org/WCE2022/ICAEM2022.html`

Name: Symmetries in Graphs, Maps, and Polytopes Workshop 2022
Date: July 10, 2022 – July 16, 2022
Venue: University of Alaska Fairbanks, Alaska, United States
Visit: `www.alaska.edu/sigmap`

Name: Hybrid: SIAM Conference on the Life Sciences (LS22)
Date: July 11, 2022 – July 14, 2022
Venue: David Lawrence Convention Center, Pittsburgh, Pennsylvania, USA
Visit: `www.siam.org/conferences/cm/conference/ls22`

Name: Hybrid: SIAM Conference on Applied Mathematics Education (ED22)
Date: July 11, 2022 – July 15, 2022
Venue: David Lawrence Convention Center, Pittsburgh, Pennsylvania, USA
Visit: `www.siam.org/conferences/cm/conference/ed22`

Name: 15th Viennese Conference on Optimal Control and Dynamic Games
Date: July 12, 2022 – July 15, 2022
Venue: Vienna University of Technology, Austria
Visit: `orcos.tuwien.ac.at/events/vc2022/`

Name: Hybrid: SIAM Conference on Mathematics of Planet Earth (MPE22)
Date: July 13, 2022 – July 15, 2022
Venue: David Lawrence Convention Center, Pittsburgh, Pennsylvania, USA
Visit: `www.siam.org/conferences/cm/conference/mpe22`

Name: Partial Differential Equations and Related Topics (PDERT'22)
Date: July 15, 2022 – July 19, 2022
Venue: Belgorod State National Research University, Belgorod, Russia
Visit: `agora.guru.ru/display.php?conf=pdert-2022`

Name: Nonlinear Analysis and Extremal Problems
Date: July 15, 2022 – July 22, 2022
Venue: Matrosov Institute for System Dynamics and Control Theory, Irkutsk, Russia
Visit: `conference.icc.ru/e/nla2022`

Name: Modern Trends in Representation Theory - Altai 2022
Date: July 18, 2022 – July 22, 2022
Venue: Altai Republic, Russia
Visit: `mca.nsu.ru/altai2022/`

Name: 34th International Colloquium on Group Theoretical Methods in Physics
Date: July 18, 2022 – July 22, 2022
Venue: University of Strasbourg, Strasbourg, France
Visit: `indico.in2p3.fr/event/23498/`

Name: Transformation Groups 2022
Date: July 18, 2022 – July 23, 2022
Venue: Lomonosov Moscow State University, Moscow, Russia
Visit: `tg2022.mathcenter.ru`

Name: Sixth International Conference Of Mathematical Sciences (Icms 2022)
Date: July 20, 2022 – July 24, 2022
Venue: Maltepe University, Istanbul, Turkey
Visit: `www.maltepe.edu.tr/icms22`

Name: AIM Workshop: Geometry and physics of ALX metrics in gauge theory
Date: July 25, 2022 – July 29, 2022
Venue: American Institute of Mathematics, San Jose, CA, USA
Visit: `aimath.org/workshops/upcoming/geomphysalx`

Name: VBAC 2022: Moduli Spaces and Vector Bundles – New Trends
Date: July 25, 2022 – July 29, 2022
Venue: University of Warwick, UK
Visit: `vbac.wikidot.com/vbac2022`

Name: Texas A&M Workshop in Analysis and Probability: Graduate Student Concentration Week on Metric Geometry
Date: July 25, 2022 – July 29, 2022
Venue: Texas A&M University, College Station, TX, USA
Visit: `sites.google.com/tamu.edu/cw-mg/home`

Name: Topology Students Workshop
Date: July 25, 2022 – July 29, 2022
Venue: Georgia Institute of Technology, Atlanta, GA, USA
Visit: `tsw.gatech.edu`

Name: 19th International Symposium on Dynamic Games and Applications
Date: July 25, 2022 – July 29, 2022
Venue: Faculty of Economics, University of Porto, Portugal
Visit: `www.gerad.ca/colloques/isdg2022/index.html`

Name: Advances in Mathematical Physics, A Conference in Honor of Elliott H. Lieb on his 90th Birthday
Date: July 31, 2022 – August 1, 2022
Venue: Harvard University, Cambridge, MA, USA
Visit: `www.math.harvard.edu/event/conference-mathematics-of-statistical-mechanics-and-quantum-physics/`

Name: Graduate Summer School "Frontiers in Geometry and Topology"
Date: August 1, 2022 – August 5, 2022
Venue: ICTP, Trieste, Italy
Visit: `indico.ictp.it/event/9705/`

Name: Building Bridges 5th EU/US Summer School and Workshop on Automorphic Forms and Related Topics (BB5)
Date: August 1, 2022 – August 13, 2022
Venue: University of Sarajevo, Sarajevo, Bosnia-Herzegovina
Visit: `bb5.pmf.unsa.ba/`

Name: Symmetry, Invariants, and Their Applications: A Celebration of Peter Olver's 70th Birthday
Date: August 3, 2022 – August 5, 2022
Venue: Dalhousie University, Halifax, Nova Scotia, Canada
Visit: `www.math.mun.ca/~movingframes2022/`

Name: Research conference "Frontiers in Geometry and Topology"
Date: August 8, 2022 – August 12, 2022
Venue: ICTP, Trieste (Italy)
Visit: `indico.ictp.it/event/9706/`

Name: Texas A&M Workshop in Analysis and Probability: Geometry and Analysis on Non-smooth Spaces
Date: August 8, 2022 – August 12, 2022
Venue: Texas A&M University, College Station, TX, USA
Visit: sites.google.com/tamu.edu/cw-gans/home

Name: Topological Data Analysis and Persistence Theory - Peter Bubenik, University of Florida, lecturer
Date: August 8, 2022 – August 12, 2022
Venue: Valdosta State University, Valdosta GA, USA
Visit: blog.valdosta.edu/vsu-cbms-conference/

Name: SUMMER SCHOOL ON FINITE GEOMETRY
Date: August 8, 2022 – August 12, 2022
Venue: Sabanci University, Karakoy, Istanbul, Turkey
Visit: scale.gtu.edu.tr/finite_geometry.html

Name: 16th International Conference of The Mathematics Education for the Future Project: Building on the Past to Prepare for the Future
Date: August 8, 2022 – August 13, 2022
Venue: King's College, Cambridge University, UK
Visit: directorymathsed.net/kings-conference/

Name: AIM Workshop: Effective methods in measure and dimension
Date: August 15, 2022 – August 19, 2022
Venue: American Institute of Mathematics, San Jose, CA, USA
Visit: aimath.org/workshops/upcoming/algorandom

Name: Summer Program in Partial Differential Equations 2022
Date: August 15, 2022 – August 21, 2022
Venue: The University of Texas at Austin, Austin TX, USA
Visit: analysispde.ma.utexas.edu/summer-program-in-partial-differential-equations-2022/

Name: Floer Homotopy Theory (FHT)
Date: August 15, 2022 – December 16, 2022
Venue: Mathematical Sciences Research Institute, Berkeley, California, USA
Visit: www.msri.org/programs/335

Name: Analytic and Geometric Aspects of Gauge Theory (GT)
Date: August 22, 2022 – December 21, 2022
Venue: Mathematical Sciences Research Institute, Berkeley, California, USA
Visit: www.msri.org/programs/340

Name: Geometric Structures and Supersymmetry
Date: August 23, 2022 – August 26, 2022
Venue: Tromsø, Norway
Visit: super-tromso.puremath.no/

Name: The 8th International Conference on Control and Optimization with Industrial Applications
Date: August 24, 2022 – August 26, 2022
Venue: Baku, Azerbaijan
Visit: `www.coia-conf.org/en/`

Name: International Conference on Recent Developments in Mathematics
Date: August 24, 2022 – August 26, 2022
Venue: Canadian University Dubai, Dubai, UAE
Visit: `icrdm.org/index.php/icrdm/index`

Name: Interdisciplinary Summer School on Surfaces
Date: August 28, 2022 – September 2, 2022
Venue: Frauenchiemsee, Bavaria, Germany
Visit: `www.tu-chemnitz.de/surfaces22/`

Name: Math of the Brain
Date: August 29, 2022 – September 2, 2022
Venue: Institute for Pure and Applied Mathematics (IPAM), Los Angeles, CA, USA
Visit: `www.ipam.ucla.edu/mb2022`

Name: AIM Workshop: Partial differential equations and conformal geometry
Date: August 29, 2022 – September 2, 2022
Venue: American Institute of Mathematics, San Jose, CA, USA
Visit: `aimath.org/workshops/upcoming/pdeconformal`

Name: XII International Conference of the Georgian Mathematical Union
Date: August 29, 2022 – September 3, 2022
Venue: Shota Rustaveli State University, Georgia, USA
Visit: `gmu.gtu.ge/Batumi2022/`

Name: SIAM Conference on Nonlinear Waves and Coherent Structures (NWCS22)
Date: August 30, 2022 – September 2, 2022
Venue: University of Bremen, Bremen, Germany
Visit: `www.siam.org/conferences/cm/conference/nwcs22`

Name: Geometric applications of microlocal analysis
Date: September 2, 2022 – September 5, 2022
Venue: Stanford University, Stanford, CA, USA

Name: International Conference on Enumerative Combinatorics and Applications ICECA 2022
Date: September 6, 2022 – September 7, 2022
Venue: University of Haifa – Virtual
Visit: `ecajournal.haifa.ac.il/Conf/ICECA2022.html#about`

Name: 13th Conference on Security and Cryptography for Networks
Date: September 12, 2022 – September 14, 2022

Venue: Amalfi (SA), Italy

Visit: `scn.unisa.it`

Name: Real algebraic geometry and singularities. Conference in honor of Wojciech Kucharz's 70th birthday

Date: September 12, 2022 – September 17, 2022

Venue: Jagiellonian University, Kraków, Poland

Visit: `realgeoms2022.c.matinf.uj.edu.pl/`

Name: 4th IMA Conference on the Mathematical Challenges of Big Data

Date: September 19, 2022 – September 20, 2022

Venue: University of Oxford – Hybrid

Visit: `ima.org.uk/17625/4th-ima-conferenceon-the-mathematical-challenges-of-bi g-data/`

Name: Hybrid: SIAM Conference on Mathematics of Data Science (MDS22)

Date: September 26, 2022 – September 30, 2022

Venue: Town and Country, San Diego, California, USA

Visit: `www.siam.org/conferences/cm/conference/mds22`

Name: Horizons in non-linear PDEs

Date: September 26, 2022 – September 30, 2022

Venue: Ulm University, Ulm, Germany

Visit: `www.uni-ulm.de/horizonsPDEs`

Name: AIM Workshop: Higher categories and topological order

Date: September 26, 2022 – September 30, 2022

Venue: American Institute of Mathematics, San Jose, CA, USA

Visit: `aimath.org/workshops/upcoming/highercattopord`

The Mathematics Newsletter may be downloaded from the RMS website at
`www.ramanujanmathsociety.org`