A probability-free proof of an explicit Croot-Łaba-Sisask Lemma

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Abstract

`File Croot-Laba-Sisask-Lemma-02.tex` This note proposes a proof free of probabilistic language of the famous Croot-Łaba-Sisask Lemma. In between, we do the same for the Khintchine and Marcinkiewicz-Zygmund inequalities and explicitate the implied constants.

1 Introduction

We propose a proof free of probabilistic language of (a variant of) [2, Lemma 3.2] by E. Croot, I. Łaba and O. Sisask, which we now state. We follow their notation and in particular, when z is a complex number, z° is defined to be $z/|z|$ when $z \neq 0$, and to be 0 when $z = 0$.

{CLS}

Theorem 1.1. *Let (X, μ) be a probability space and $p \geqslant 2$. Given a function f in the form*

$$f = \sum_{k \leqslant K} \lambda_k g_k$$

where $(g_k)_k$ is a collection of measurable functions on X of $L^p(\mu)$-norm at most 1. Let $\varepsilon > 0$. There exists an L-tuple $(k_1, \cdots, k_\ell) \in \{1, \cdots, K\}^L$ of length $L \leqslant 20p/\varepsilon^2$ such that

$$\int_X \left| \frac{f(x)}{\|\lambda\|_1} - \frac{1}{L} \sum_{\ell \leqslant L} \lambda_{k_\ell}^\circ g_{k_\ell}(x) \right|^p d\mu \leqslant \varepsilon^p,$$

where $\|\lambda\|_1 = \sum_{k \leqslant K} |\lambda_k|$.

This theorem has its origin in the paper [3] by E. Croot and O. Sisask. We refer to the paper [6] by L. Pierce for much deeper background on the Khintchine and Marcinkiewicz-Zygmund inequalities.

[2] E. Croot, I. Łaba, and O. Sisask, 2013, "Arithmetic progressions in sumsets and L^p-almost-periodicity".
[3] E. Croot and O. Sisask, 2010, "A probabilistic technique for finding almost-periods of convolutions".
[6] L. B. Pierce, 2021, "On superorthogonality".

2 An upper explicit Khintchine Inequality

{KI}

Theorem 2.1. *We have, when $p \geqslant 1$,*

$$(1/2^N) \sum_{(\varepsilon_n)\in\{\pm 1\}^N} \left| \sum_{n\leqslant N} c_n \varepsilon_n \right|^p \leqslant p^{p/2} \biggl(\sum_{n\leqslant N} |c_n|^2 \biggr)^{p/2}.$$

This is only half of the Khintchine Inequality and in a special context, but this will be enough for us. We followed [1, Chapter 10, Theorem 1, page 354] by Y.S. Show and H. Teicher.

Proof. Let us start with $p = 2k \geqslant 2$, so that we may open the inner sum and get

$$\begin{aligned} 2^N S(2k) &= \sum_{(\varepsilon_n)\in\{\pm 1\}^N} \left| \sum_{n\leqslant N} c_n \varepsilon_n \right|^p \\ &= \sum_{\substack{s_1+s_2+\cdots+s_N=2k,\\ s_n\geqslant 0}} \binom{2k}{s_1, s_2, \cdots, s_N} \prod_{1\leqslant n\leqslant N} c_n^{s_n} \sum_{(\varepsilon_n)\in\{\pm 1\}^N} \prod_{n\leqslant N} \varepsilon_n^{s_n} \end{aligned}$$

by the multinomial theorem. The inner summand vanishes as soon as some s_n is odd, whence, by letting $2t_n = s_n$, we get

$$\begin{aligned} 2^N S(2k) &= \sum_{\substack{t_1+t_2+\cdots+t_N=k,\\ t_n\geqslant 0}} \binom{2k}{2t_1, 2t_2, \cdots, 2t_N} \prod_{n\leqslant N} (c_n^2)^{t_n} \\ &\leqslant C \sum_{\substack{t_1+t_2+\cdots+t_N=k,\\ t_n\geqslant 0}} \binom{k}{t_1, t_2, \cdots, t_N} \prod_{n\leqslant N} (c_n^2)^{t_n} = C \biggl(\sum_{n\leqslant N} c_n^2 \biggr)^k \end{aligned}$$

where

$$\begin{aligned} C &= \max \binom{2k}{2t_1, 2t_2, \cdots, 2t_N} \binom{k}{t_1, t_2, \cdots, t_N}^{-1} \\ &\leqslant \max \frac{2k(2k-1)\cdots(k+1)}{\prod_j 2t_j(2t_j-1)\cdots(t_j+1)} \\ &\leqslant \max \frac{k^k}{2^{t_1+\cdots+t_k}} \leqslant (k/2)^k. \end{aligned}$$

As $S(p)$ is increasing, we simply choose $k = \lceil p/2 \rceil$. This gives us $2k \geqslant p+2$ and thus

$$(k/2)^k \leqslant (p+2)^{1+p/2} \leqslant (30\,p)^{p/2}.$$

This concludes the main part of the proof, except for the constant 30. We will not continue the proof but simply refer to the paper [5] by U. Haagerup who shows that best constant is (be careful: the abstract of this paper misses a closing parenthesis for the value of B_p, but the value of B_p displayed in the

[1] Y. S. Chow and H. Teicher, 1997, *Probability theory.*
[5] U. Haagerup, 1981, "The best constants in the Khintchine inequality".

middle of page 232 misses a squareroot-sign around the π, as an inspection of the proof at the end the paper rapidly reveals)

$$\begin{cases} 1 & \text{when } 0 < p \leqslant 2, \\ \sqrt{2}\left(\frac{\Gamma((p+1)/2)}{\sqrt{\pi}}\right)^{1/p} & \text{when } 2 < p. \end{cases}$$

We readily check that this implies that the constant 1 rather than 30 is admissible. □

3 An upper explicit Marcinkiewicz-Zygmund Inequality

{MZI}

Theorem 3.1. *Let (X,μ) be a probability space. When $p \geqslant 1$, let $(f_n)_{n\leqslant N}$ be a system of functions such that $\int_X f_n(x)d\mu = 0$. We have*

$$\int_{(x_n)\in X^N} \left|\sum_{1\leqslant n\leqslant N} f_n(x_n)\right|^p d(x_n) \leqslant (4p)^{p/2} \int_{(x_n)\in X^N} \left(\sum_{1\leqslant n\leqslant N} |f_n(x_n)|^2\right)^{p/2} d(x_n).$$

The power of this inequality is that the implied constant do not depend on N, the effect of some orthogonality. Again, this is only half of the Marcinkiewicz-Zygmund Inequality and in a special context, but this will be enough for us. We followed [1, Chapter 10, Theorem 2, page 356] by Y.S. Show and H. Teicher. The relevant constant is the subject of [7] by Y.-F. R and H.-Y. Liang (their value is slightly worse than ours) and [4] by D. Ferger, where the best constant is determined provided the f_n's are "symmetric".

Proof. We first notice that, since $\int_0^1 f_n(x)dx = 0$, we may introduce a symmetrization through

$$\sum_{n\leqslant N} f_n(x_{2n-1}) = -\int_{(x_{2n})\in X^N} \sum_{n\leqslant 2N} (-1)^n f_{\lceil n/2\rceil}(x_n)d(x_{2n}).$$

Jensen's inequality gives us that

$$\int_{(x_{2n-1})\in X^N} \left|\int_{(x_{2n})\in X^N} \sum_{n\leqslant 2N} (-1)^n f_{\lceil n/2\rceil}(x_n)d(x_{2n})\right|^p d(x_{2n-1}) \leqslant \int_{(x_{2n-1})\in X^N} \int_{(x_{2n})\in X^N} \left|\sum_{n\leqslant 2N} (-1)^n f_{\lceil n/2\rceil}(x_n)\right|^p d(x_{2n})d(x_{2n-1})$$

from which we deduce that the L^p-norm of the symmetrization controls the one of the initial sum:

$$\int_{(x_{2n-1})\in X^N} \left|\sum_{n\leqslant N} f_n(x_{2n-1})\right|^p d(x_{2n-1}) \leqslant \int_{(x_n)\in X^{2N}} \left|\sum_{n\leqslant 2N} (-1)^n f_{\lceil n/2\rceil}(x_n)\right|^p d(x_n).$$

[7] Y.-F. Ren and H.-Y. Liang, 2001, "On the best constant in Marcinkiewicz-Zygmund inequality".
[4] D. Ferger, 2014, "Optimal constants in the Marcinkiewicz-Zygmund inequalities".

We next introduce Rademacher's system by noticing that, by successively exchanging x_{2n-1} and x_{2n}, have

$$(1/2^N) \sum_{(\varepsilon_n)\in\{\pm 1\}^N} \int_{(x_n)\in X^{2N}} \left| \sum_{n\leqslant 2N} \varepsilon_{\lceil n/2\rceil}(-1)^n f_{\lceil n/2\rceil}(x_n)\right|^p d(x_n)$$
$$= \int_{(x_n)\in X^{2N}} \left| \sum_{n\leqslant 2N} (-1)^n f_{\lceil n/2\rceil}(x_n)\right|^p d(x_n).$$

We may now remove the symmetrization since:

$$\begin{aligned}\int_{(x_n)\in X^{2N}} &\left| \sum_{n\leqslant 2N} \varepsilon_{\lceil n/2\rceil}(-1)^n f_{\lceil n/2\rceil}(x_n)\right|^p d(x_n)\\ &\leqslant \int_{(x_n)\in X^{2N}} 2^{p-1}\left(\left|\sum_{n\leqslant N}\varepsilon_n f_n(x_{2n})\right|^p + \left|\sum_{n\leqslant N}\varepsilon_n f_n(x_{2n-1})\right|^p\right) d(x_n)\\ &\leqslant 2^p \int_{(x_{2n})\in X^N} \left|\sum_{n\leqslant N}\varepsilon_n f_n(x_{2n})\right|^p d(x_{2n}).\end{aligned}$$

The Khintchine Inequality from Theorem 2.1 now gives us that

$$(1/2^N) \sum_{(\varepsilon_n)\in\{\pm 1\}^N} \left|\sum_{n\leqslant N}\varepsilon_n f_n(x_{2n})\right|^p \leqslant p^{p/2}\left(\sum_{n\leqslant N}|f_n(x_{2n})|^2\right)^{p/2}.$$

The proof is then complete. Concerning the constant 4 in the Theorem, the paper [7] by Y.-F. Ren and H.-Y. Liang gives the upper bound 9/2, which is worse than the above one. □

4 Proof of Theorem 1.1

Proof. We define $\Omega = \{1, \cdots, K\}$ which we equip with the probability measure defined by $\nu(\{k\}) = |\lambda_k|/\|\lambda\|_1$. Given a positive integer L, we consider the family of functions φ_ℓ, for $\ell \leqslant L$ given by

$$\begin{array}{rcl}\varphi_\ell : \Omega^L \times X & \to & \mathbb{C}\\ ((u_h)_{h\leqslant L}, x) & \mapsto & \lambda^\circ_{u_\ell} g_{u_\ell}(x)\end{array}$$

so that

$$\int_{\Omega^L} \varphi_\ell((u_h)_{h\leqslant L}, x) d\nu = \sum_{k\in\Omega} \frac{|\lambda_k|}{\|\lambda\|_1}\lambda^\circ_k g_k(x) = \frac{f}{\|\lambda\|_1} = f_0$$

say. We aim at showing that $(1/L)\sum_{\ell\leqslant L}\varphi_\ell((u_h)_{h\leqslant L}, x)$ closely approximates f_0 for most values of $(u_h)_{h\leqslant L}$. Selecting one such value gives qualitatively our result. To do so, we write

$$\int_{\Omega^L}\int_X \left|\frac{1}{L}\sum_{\ell\leqslant L}\varphi_\ell((u_h)_{h\leqslant L}, x) - f_0\right|^p d((u_h)_{h\leqslant L})dx$$
$$= \frac{1}{L^p}\int_{\Omega^L}\int_X \left|\sum_{\ell\leqslant L}\left(\varphi_\ell((u_h)_{h\leqslant L}, x) - f_0\right)\right|^p d((u_h)_{h\leqslant L})dx.$$

[7] Y.-F. Ren and H.-Y. Liang, 2001, "On the best constant in Marcinkiewicz-Zygmund inequality".

We apply the Marcinkiewicz-Zygmung Inequality, i.e. Theorem 3.1, to this latter expression, getting

$$\begin{aligned}\int_{\Omega^L}\int_X\left|\frac{1}{L}\sum_{\ell\leqslant L}\varphi_\ell((u_h)_{h\leqslant L},x)-f_0\right|^p d((u_h)_{h\leqslant L})dx\\ \leqslant\frac{(4p)^{p/2}}{L^{p/2}}\int_{\Omega^L}\int_X\left|\frac{1}{L}\sum_{\ell\leqslant L}\left|\varphi_\ell((u_h)_{h\leqslant L},x)-f_0\right|^2\right|^{p/2}d((u_h)_{h\leqslant L})dx\\ \leqslant\frac{(4p)^{p/2}}{L^{p/2}}\int_X\int_{\Omega^L}\left|\frac{1}{L}\sum_{\ell\leqslant L}\left|\varphi_1((u_h)_{h\leqslant L},x)-f_0\right|^2\right|^{p/2}d((u_h)_{h\leqslant L})dx.\end{aligned}$$

The end is straightforward:

$$\begin{aligned}\int_{\Omega^L}\int_X\left|\frac{1}{L}\sum_{\ell\leqslant L}\varphi_\ell((u_h)_{h\leqslant L},x)-f_0\right|^p d((u_h)_{h\leqslant L})dx\\ \leqslant\frac{(4p)^{p/2}}{L^{p/2}}\int_X\int_{\Omega^L}\left|\varphi_1((u_h)_{h\leqslant L},x)-f_0\right|^p d((u_h)_{h\leqslant L})dx.\end{aligned}$$

Concerning the relevant p-norms, we make the following observations:

$$\begin{aligned}\int_X\int_{\Omega^L}\left|\varphi_1((u_h)_{h\leqslant L},x)\right|^p d((u_h)_{h\leqslant L})dx\\ =\int_{\Omega^L}\left(\int_X\left|\varphi_1((u_h)_{h\leqslant L},x)\right|^p dx\right)d((u_h)_{h\leqslant L})\leqslant 1,\end{aligned}$$

on the one side while on the other side, by the triangle inequality, we have

$$\|f_0\|_p\leqslant\sum_{k\leqslant K}\frac{|\lambda_k|}{\|\lambda\|_1}\|g_k\|_p\leqslant 1.$$

Therefore

$$\begin{aligned}\left(\int_{\Omega^L}\int_X\left|\frac{1}{L}\sum_{\ell\leqslant L}\varphi_\ell((u_h)_{h\leqslant L},x)-f_0\right|^p d((u_h)_{h\leqslant L})dx\right)^{1/p}\\ \leqslant\frac{(4p)^{1/2}}{L^{1/2}}(1+1)=\sqrt{16p/L}.\end{aligned}$$

We deduce from this inequality that the set of $(u_h)_{h\leqslant L}$ for which

$$\int_X\left|\frac{1}{L}\sum_{\ell\leqslant L}\varphi_\ell((u_h)_{h\leqslant L},x)-f_0\right|^p dx>\varepsilon^p$$

has measure at most $\sqrt{16p/(\varepsilon^2L)}$ which is strictly less than 1 by our assumption on L. The theorem follows readily. □

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[6] L. B. Pierce. "On superorthogonality". In: *J. Geom. Anal.* 31.7 (2021), pp. 7096–7183. ISSN: 1050-6926. DOI: 10.1007/s12220-021-00606-3 (cit. on p. 1).

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