EXPLICIT BOUNDS FOR PRODUCTS OF PRIMES IN AP

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Abstract. For all $q \geq 2$ and for all invertible residue classes a modulo q, there exists a natural number that is congruent to a modulo q and that is the product of exactly three primes, all of which are below $(10^{15}q)^{5/2}$.

Contents

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1. Introduction and results

In this paper we investigate the representation of reduced residue classes modulo q by a product of exactly three small primes. We improve on [15, 16] by introducing an explicit *vertical* Brun–Titchmarsh inequality in Theorem 1.4. Its proof takes most of our efforts and Theorem 6.1 has no predecessor. Please note that this result is new even if one omits the explicit aspect.

Theorem 1.1. *For any $q \geq 2$ and any invertible residue class a modulo q, there exists a natural number that is congruent to a modulo q and that is the product of exactly three primes, all of which are below $(10^{15}q)^{5/2}$.*

We thus go beyond the bound $O(q^3)$ that was a barrier in [15]. Varying the constant 10^{15} is the subject of Theorem 13.2 where it is for instance proved that, $q \geq 10^{88}$, the primes may be assumed to be below $(30q)^{5/2}$. We present in Section 12 some computations for small values of the parameter q. Here is one result that is established there.

Theorem 1.2. *For every $q \in [5208, 2\cdot 10^7]$, every invertible class modulo q contains a product of three primes, each of which being of size at most $q(\log q)^2$.*

To prove Theorem 1.1, we first prove an explicit version of a *vertical* Brun–Titchmarsh Theorem, i.e. an almost-everywhere, in a-aspect, individual upper bound in the Brun–Titchmarsh style. The result in itself is essentially due to Hooley in [8, Theorem 5], but since our phrasing is more precise, we prefer to state it explicitly.

Theorem 1.3 (Hooley [8])**.** *Let $q \geq 1$ be an integer. Let $X \geq q^{3/2}(\log q)^{12}$ be some real number. For every invertible class a modulo q, we have*

$$\sum_{\substack{p\leq X,\\ p\equiv a[q]}} 1 \leq \frac{2(1+\epsilon(a))\cdot X}{\varphi(q)\log(X/\sqrt{q})} \quad \textit{where} \quad \sum_{a \bmod^* q} \epsilon(a)^2 \ll \varphi(q)/\log q$$

where '$p \equiv a[q]$' means that p is congruent to a modulo q while 'a mod* q' means that a ranges through a complete set of reduced residue classes modulo q.

Our proof differs from the one of Hooley and is closer to the work [11] by Motohashi. However, a noticeable distinction is that we avoid using the fourth power moment upper bound for L-functions. For efficiency on the explicit side, we prove a smoothed version of Theorem 1.3. From now on, the symbols η^* and κ shall be kept for the functions defined by

$$\eta^*(t) = \begin{cases} 3t^2, & 0 \leq t \leq 1/3,\\ -6t^2+6t-1, & 1/3 < t \leq 2/3,\\ 3(1-t)^2, & 2/3 < t \leq 1,\\ 0, & \text{otherwise,}\end{cases} \qquad \kappa(t) = \begin{cases} -3t^2, & 0 \leq t \leq 1/3,\\ 6t^2-1, & 1/3 < t \leq 2/3,\\ 3(1-t^2), & 2/3 < t \leq 1,\\ 0, & \text{otherwise.}\end{cases} \tag{1}$$

This choice is explained in Section 2. We have $\|\eta^*\|_1 = 2/9$. A large part of this paper is dedicated to the proof of the next theorem.

Theorem 1.4. *Let $q \geq 2 \cdot 10^7$ be an integer. Let $X \geq (10^{13}q)^{5/2}$ be some real number. For every invertible class a modulo q, we have*

$$\sum_{p \equiv a[q]} \eta^*(p/X) \leq \frac{2(1.004 + \epsilon(a)) \cdot 2X/9}{\varphi(q) \log \frac{X}{10^9\sqrt{q}}} \quad \textit{where} \quad \sum_{a \bmod^* q} \epsilon(a)^2 \leq \varphi(q)/4250.$$

The proof uses directly the Parseval identity for Mellin transforms rather than the more usual result by Gallagher, namely [7, Theorem 1]. A major input is Theorem 6.1 that gives a sharp upper bound for $S(z) = \sum_n (\sum_{[d_1,d_2]=n} |\lambda_{d_1}\lambda_{d_1}|)^2$ where the λ_d's are closely related to some Selberg parameters. We modify these though and this modification is instrumental in our proof. Let us mention that the trivial bound for $S(z)$ is of size $z^2(\log z)^8$ while the bound we prove is of order $z^2/(\log z)^3$. Computations described in Section 7 support the fact that $S(z)(\log z)^3/z^2$ is indeed asymptotic to some positive constant. Theorem 13.1 presents variants of Theorem 1.4.

We recall the definition of the Mellin transform:

$$\check{\eta}(s) = \int_0^\infty \eta(x) x^{s-1} dx. \tag{2}$$

2. Smoothings

Let us quote from Rényi's book [18] the formula:

$$1_{[-1,1]}^{(*m)}(t) = \begin{cases} \displaystyle\sum_{j=0}^{\lfloor (m+|t|)/2 \rfloor} \frac{(-1)^j}{(m-1)!} \binom{m}{j} (m + |t| - 2j)^{m-1} & \text{when } 0 \leq |t| \leq m, \\ 0 & \text{when } m < |t|. \end{cases}$$

Guessing this expression is not obvious, but checking it by recursion is only a matter of routine. The Fourier transform of $1_{[-1,1]}$ is $\sin(2\pi u)/(\pi u)$, so the transform of $1_{[-1,1]}^{(*m)}$ is $\sin(2\pi u)^m/(\pi u)^m$. In the previous paper [15], we used the above with $m = 2$, but we need a smoother function here. By specifying $m = 3$, we get

$$1_{[-1,1]}^{(*3)}(t) = \begin{cases} 3 - t^2 & \text{when } 0 \leq |t| < 1, \\ (3 - |t|)^2/2 & \text{when } 1 \leq |t| < 3, \\ 0 & \text{when } 3 \leq |t|. \end{cases}$$

As indicated in the introduction, the symbol η^* shall be kept for the function defined by

$$\eta^*(t) = 1_{[-1,1]}^{(*3)}(6t - 3)/6. \tag{3}$$

In this section, we use the *Bernoulli polynomials* $(B_m)_{m \geq 0}$ that are defined by

$$\frac{te^{xt}}{e^t - 1} = \sum_{m \geq 0} B_m(x) \frac{t^n}{n!}.$$

It is straightforward to compute the first of these:

$$B_0(x) = 1, B_1(x) = x - \tfrac{1}{2}, B_2(x) = x^2 - x + \tfrac{1}{6}, B_3(x) = x^3 - \tfrac{3}{2}x^2 + \tfrac{1}{2}x.$$

The *Bernoulli functions* are then defined by $b_m(x) = B_m(\{x\})$ where $\{x\}$ is the fractional part of x. These functions are periodic modulo 1, and their Fourier series expansion is well-known.

Lemma 2.1. *We have, when $m \geq 1$,*

$$\sum_{n\geq 1} 1^{(*m)}\Big(\Big(\frac{2n}{X}-1\Big)m\Big) = \frac{2^{m-1}X}{m} - \frac{(2m)^{m-1}}{X^{m-1}m!}\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}b_m\big(2kX/m\big).$$

Proof. Indeed the Fourier transform of $g(u) = 1^{(*m)}((2(u/X)-1)m)$ is given by

$$\begin{aligned}\hat{g}(v) &= \int_{-\infty}^{\infty} 1^{(*m)}((2(u/X)-1)m)e(uv)du \\ &= \frac{X}{2m}\int_{-\infty}^{\infty} 1^{(*m)}(w)e\Big(\frac{Xvw}{2m}+\frac{Xv}{2}\Big)dw = \frac{X}{2m}e(Xv/2)\Big(\frac{2m\sin\frac{\pi Xv}{m}}{\pi Xv}\Big)^m.\end{aligned}$$

By the Poisson summation formula, we get

$$S = \sum_{n\geq 1} 1^{(*m)}\Big(\Big(\frac{2n}{X}-1\Big)m\Big) = \frac{X}{2m}\frac{(2m)^m}{X^m}\sum_{\ell\in\mathbb{Z}} e(X\ell/2)\Big(\frac{\sin\frac{\pi X\ell}{m}}{\pi\ell}\Big)^m.$$

Next notice that, with $x = X\ell/(2m)$,

$$\begin{aligned}(2i)^m e(mx)(\sin 2\pi x)^m &= e(mx)\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}e((2k-m)x) \\ &= \sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}e(2kx).\end{aligned}$$

Hence with $y = X/m$

$$\begin{aligned}S &= \frac{2^{m-1}X}{m} + \frac{X}{2m}\frac{(2m)^m}{X^m}\frac{1}{(2i\pi)^m}\sum_{\substack{\ell\in\mathbb{Z},\\ \ell\neq 0}}\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}\frac{e(2ky\ell)}{\ell^m} \\ &= \frac{2^{m-1}X}{m} - \frac{X}{2m}\frac{(2m)^m}{X^m m!}\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}b_m\big(2kX/m\big).\end{aligned}$$

The proof ends here. □

This lemma may be compared with [20, Lemma 3.1] which corresponds to the case $m = 2$ while the above treats the case $m = 3$.

Lemma 2.2. *With $\rho_3(y) = \sum_{0\leq k\leq 3}(-1)^k\binom{3}{k}b_3(ky)$, we have*

$$\rho_3(y) = \begin{cases} -6\{y\}^3 & \text{when } \{y\}\leq 1/3, \\ 3(1-2\{y\})(\{y\}^2-4\{y\}+1) & \text{when } 1/3<\{y\}\leq 1/2,\end{cases}$$

and $\rho_3(1-y) = -\rho_3(y)$. Here $\{y\}$ is the fractional part of y. In particular this quantity lies in $[3(11-5\sqrt{5})/2, 3(5\sqrt{5}-11)/20]$.

Proof. The property of the Bernoulli polynomials $B_m(1-X) = (-1)^m B_m(X)$ implies that $b_m(1-y) = (-1)^m b_m(y)$, so we may restrict our attention to the case $0 \leq y \leq 1/2$. When $y \in [0, 1/3)$, we find that

$$2\rho_3(y) = -3(2y^3-3y^2+y) + 3(16y^3-12y^2+2y) - (54y^3-27y^2+3y) = -12y^3.$$

We continue with the case $y \in [1/3, 1/2)$ where $\{2y\} = 2y$ but $\{3y\} = 3y - 1$. Therefore

$$\begin{aligned} 2\rho_3(y) &= -3(2y^3 - 3y^2 + y) + 3(16y^3 - 12y^2 + 2y) \\ &\qquad\qquad - (2(3y-1)^3 - 3(3y-1)^2 + 3y - 1) \\ &= 6(1-2y)(y^2 - 4y + 1). \end{aligned}$$

The derivative of $\rho_3(y)$ with respect to y in this interval is $18(-y^2 + 3y - 1)$. The minimum is at $y_0 = (3 - \sqrt{5})/2$ with value $3(11 - 5\sqrt{5})/2$, and this concludes the proof of this lemma. $\square$

Lemma 2.3. *We have*

$$\frac{2X}{9} - \frac{\theta}{X^2} \le \sum_{m \ge 1} \eta^*(m/X) \le \frac{2X}{9} + \frac{\theta}{X^2}$$

where $\theta = 3(5\sqrt{5} - 11)/2 \le 0.28$.

Lemma 2.4. *When* $X \ge 3 \cdot 10^7$ *and* $X \ge q$, *we have*

$$\sum_{(p,q)=1} \eta^*(p/X) \ge (1 - 0.003)\,\frac{2X/9}{\log X}.$$

Proof. Let us call S the sum on the left side. We first find that, as η^* has its support within $[0, 1]$, we have

$$\sum_{(p,q)=1} \eta^*(p/X) \log X = \sum_{(p,q)=1} \eta^*(p/X)\Big(\log p + \log \frac{X}{p}\Big) \ge \sum_{(p,q)=1} \eta^*(p/X) \log p.$$

We next notice that

$$\sum_{(p,q)=1} \eta^*(p/X) \log p \ge \sum_{p} \eta^*(p/X) \log p - \|\eta^*\|_\infty \sum_{p|q} \log p$$

and this last summand is at most $(\log X)/2$. Hence, with the notation $\vartheta(t) = \sum_{p \le t} \log p$, we find that

$$\begin{aligned} (S + \tfrac{1}{2}) \log X &\ge \sum_{p \ge 1} \log p \Big(-\int_{p/X}^{1} {\eta^*}'(t) dt\Big) \\ &\ge -\int_0^1 \vartheta(tX) {\eta^*}'(t) dt = -\int_0^X \vartheta(u) {\eta^*}'(u/X) du/X. \end{aligned}$$

We note that $-{\eta^*}'(t)$ is non-positive when $t \le 1/2$ and non-negative afterwards, hence we need an upper bound for $\vartheta(u)$ when $u \le X/2$ and a lower bound when $u \ge X/2$. Concerning the upper bound, we recall [19, equation (5.1)]:

$$\vartheta(u) = \sum_{p \le u} \log p \le 1.001102u \quad (u > 0). \tag{4}$$

and concerning the lower bound, we use [19, equation (5.2)]:

$$\vartheta(u) \ge (1 - 0.0013156)u \quad (u \ge 1\,319\,007). \tag{5}$$

We mention here that a better lower bound may be found in [2]. We set $\epsilon^* = 0.0013156$ and , on assuming $X/2 \geq 1\,319\,007$, we get

$$\begin{aligned}(S+\tfrac{1}{2})\log X &\geq -\int_0^X u{\eta^*}'(u/X)\frac{du}{X} - \epsilon^*\int_0^X |u{\eta^*}'(u/X)|\frac{du}{X}\\ &\geq -X\int_0^1 t{\eta^*}'(t)dt - \epsilon^* X\int_0^1 |{\eta^*}'(t)|dt = X\int_0^1 \eta^*(t)dt - \epsilon^*\eta^*(1/2)X\\ &= \left(\frac{2}{9} - \frac{\epsilon^*}{2}\right)X.\end{aligned}$$

This gives us a lower estimates for S. □

2.1. **Mellin transforms.**

Lemma 2.5. *We have* $\check{\kappa}(s) = (s+1)\check{\eta^*}(s)$.

Proof. Indeed, by summation by parts, we find that

$$\check{\eta^*}(s) = \int_0^1 (\eta^*(t)/t)t^s dt = -\int_0^1 (\eta^*(t)/t)' t^{s+1}\frac{dt}{s+1}$$

and we swiftly check that $\kappa(t) = -t^2(\eta^*(t)/t)'$. □

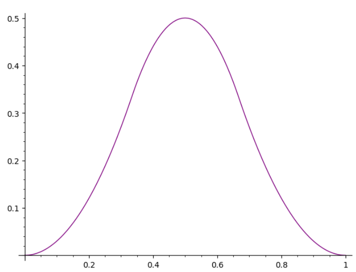

FIGURE 1. $\eta^*(t)$

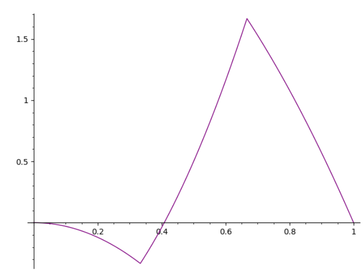

FIGURE 2. $\kappa(t)$

2.2. **Fourier transforms.**

We need also the Fourier transform of κ. We set

$$\hat{\kappa}(\alpha) = \int_0^1 \kappa(t)e(\alpha t)dt. \tag{6}$$

Lemma 2.6. *We have*

$$\hat{\kappa}(\alpha) = 6e(2\alpha/3)\left(\frac{\sin\pi\alpha/3}{\pi\alpha}\right)^2 - 6e(\alpha/2)\left(\frac{\sin\pi\alpha/3}{\pi\alpha}\right)^3.$$

As a sanity check, the above expression ensures that $\hat{\kappa}(0) = 4/9$ while we also have $\check{\kappa}(1) = 2\check{\eta^*}(1) = 4/9$.

Proof. We readily compute that (with $\beta = 2i\pi\alpha$)

$$\begin{aligned}\frac{-\beta}{6}\hat{\kappa}(\alpha) = \tfrac{1}{6}\int_0^1 \kappa'(t)e(\alpha t)dt &= -\int_0^{1/3} te(\alpha t)dt + 2\int_{1/3}^{2/3} te(\alpha t)dt - \int_{2/3}^1 te(\alpha t)dt\\ &= -\int_0^1 te(\alpha t)dt + 3\int_{1/3}^{2/3} te(\alpha t)dt.\end{aligned}$$

Since
$$\int te(\alpha t)dt = \frac{te(\alpha t)}{\beta} - \frac{e(\alpha t)}{\beta^2},$$
we find that
$$\begin{aligned}\frac{-\beta^3}{6}\hat{\kappa}(\alpha) &= -e(\alpha)\beta + e(\alpha) - 1 + 3\big(\tfrac{2}{3}e(2\alpha/3)\beta - e(2\alpha/3) - \tfrac{1}{3}e(\alpha/3)\beta + e(\alpha/3)\big)\\ &= -\beta(e(\alpha) - 2e(2\alpha/3) + e(\alpha/3)) + e(\alpha) - 3e(2\alpha/3) + 3e(\alpha/3) - 1\\ &= -\beta e(\alpha/3)(e(\alpha/3)-1)^2 + (e(\alpha/3)-1)^3.\end{aligned}$$
The lemma follows readily. □

3. Some auxiliary lemmas

3.1. Auxiliaries.

Lemma 3.1. *Let f be an absolutely continuous function on $[0,1]$ such that $f(0) = 0$. For $y, c \geq 0$, we have*
$$\sum_{n\leq cy} f(n/y) = y\int_0^c f(t)dt + (\{cy\} - \tfrac{1}{2})f(c) + \int_0^c (\{ty\} - \tfrac{1}{2})f'(t)dt$$
where $\{u\}$ denotes the integer part of u.

Proof. Since f is absolutely continuous, we have $f(x) = f(c) - \int_x^c f'(t)dt$. The next steps are routine:
$$\sum_{n\leq cy} f(n/y) = -\int_0^c \sum_{n\leq ty} f'(t)dt + [y]f(c) = -\int_0^c \big(ty - \tfrac{1}{2} - \{ty\} + \tfrac{1}{2}\big)f'(t)dt + [y]f(c).$$
The lemma follows readily. □

Lemma 3.2. *For $v > 0$, we have* $\displaystyle\sum_{n\leq 1/v} |\kappa(nv)| \leq \frac{2\sqrt{6}}{9v} + 4.$

Proof. We use Lemma 3.1 with $c = 1$ and $f(t) = |\kappa(t)|$. We readily check that $\int_0^1 |\kappa(t)|dt = 2\sqrt{6}/9$. Furthermore
$$|\kappa|'(t) = \begin{cases} 6t, & 0 \leq t \leq 1/3,\\ -12t, & 1/3 < t \leq 1/\sqrt{6},\\ 12t, & 1/\sqrt{6} < t \leq 2/3,\\ -6t, & 2/3 \leq t \leq 1,\\ 0, & \text{otherwise.}\end{cases}$$
and, after a short calculation, this readily implies that $\int_0^1 ||\kappa|'(t)|dt = 8$. □

3.2. Constant recognition.

Here is an abstraction of an idea contained in [5, Lemma 2.9].

Lemma 3.3. *Let $D(s) = \sum_{n\geq 1} u_n/n^s$ be a Dirichlet series that is absolutely convergent for $\Re s > 1$, has a simple pole of residue ρ at $s = 1$ and can be analytically continuable to $\Re s > c$ for some $c \in (0,1)$. We assume that $\sum_{n\leq X} u_n = \rho X + O(X^c)$. Then for $\Re s = \sigma > c$, we have*
$$\sum_{n\leq X} \frac{u_n}{n^s} = \frac{\rho}{(1-s)X^{s-1}} + D(s) + O(X^{c-\sigma}).$$

Proof. We first notice that $D(s)$ equals

$$\sum_{n\le X}\frac{u_n}{n^s}+s\int_X^\infty\Bigl(\sum_{X<n\le t}u_n-\rho t\Bigr)\frac{s}{t^{s+1}}dt+\frac{s\rho}{(s-1)X^{s-1}}.$$

We infer from this expression that

$$\begin{aligned}D(s)&=\sum_{n\le X}\frac{u_n}{n^s}-\frac{\sum_{n\le X}u_n}{X^s}+s\int_X^\infty\Bigl(\sum_{n\le t}u_n-\rho t\Bigr)\frac{dt}{t^{s+1}}+\frac{s\rho}{(s-1)X^{s-1}}\\&=\sum_{n\le X}\frac{u_n}{n^s}+\frac{\rho}{(s-1)X^{s-1}}-\frac{\sum_{n\le X}u_n-\rho X}{X^s}+s\int_X^\infty\Bigl(\sum_{n\le t}u_n-\rho t\Bigr)\frac{dt}{t^{s+1}}.\end{aligned}$$

This is valid at first for $\Re s>1$, but by analytic continuation and since $\sum_{n\le t}u_n-\rho t\ll t^c$, also when $\Re s>c$. In particular, when $c<\Re s<1$, we have

$$\sum_{n\le X}\frac{u_n}{n^s}=\frac{\rho}{(1-s)X^{s-1}}+D(s)+O(X^{c-\sigma})$$

as required. □

3.3. **On squarefree numbers.** The next lemma is proved, for the first part in [10] and for the second one in [4].

Lemma 3.4. *When $x\ge1$, we have $\sum_{n\le x}\mu^2(n)=\frac{6}{\pi^2}x+\mathcal{O}^*(\sqrt{x})$ while, when $x\ge 438653$, we have $\sum_{n\le x}\mu^2(n)=\frac{6}{\pi^2}x+\mathcal{O}^*(0.02767\sqrt{x})$.*

Lemma 3.5. *When $x\ge1$, we have* $\displaystyle\sum_{n\le x}\frac{\mu^2(n)}{n}\le\frac{6}{\pi^2}\log x+1.166.$

See [14, Lemma 3.4].

Lemma 3.6. *When $y\ge1$, we have* $\displaystyle\sum_{n\le y}\frac{\mu^2(n)}{n^{2/3}}\le\frac{6}{\pi^2}\cdot 3y^{1/3}.$

Proof. We readily check that

$$\begin{aligned}S=\sum_{n\le y}\frac{\mu^2(n)}{n^{2/3}}&=\int_1^y\sum_{n\le t}\mu^2(n)\frac{2dt/3}{t^{5/3}}+\frac{\sum_{n\le y}\mu^2(n)}{y^{2/3}}\\&=\frac{6}{\pi^2}3y^{1/3}-2+\int_1^\infty r(t)\frac{2dt/3}{t^{5/3}}-\int_y^\infty r(t)\frac{2dt/3}{t^{5/3}}+\frac{r(y)}{y^{2/3}}\end{aligned}$$

where $\sum_{n\le t}\mu^2(n)=\frac{6}{\pi^2}t+r(t)$. We apply Lemma 3.3. In our case, this means that the constant we seek is $\zeta(2/3)/\zeta(4/3)$, which is negative! We have thus proved that

$$S=\frac{6}{\pi^2}3y^{1/3}+\frac{\zeta(2/3)}{\zeta(4/3)}+O^*(5/y^{1/6})$$

by using the simple bound of Lemma 3.4. Pari-GP gives us that $\zeta(2/3)/\zeta(4/3)=-0.679\cdots$. This ensures us that our error term is negative when $y\ge160000$. This last range is easily covered by direct computations. □

Lemma 3.7. *When* $1 \le a \le 10$ *and* $x \ge 1$, *we have* $\sum_{n \le x} \mu^2(n)\Big(\log \frac{x}{n}\Big)^a \le a! c_a\, x$

where

c_1	*0.6715*	c_6	*0.6108*
c_2	*0.6365*	c_7	*0.6097*
c_3	*0.6227*	c_8	*0.6101*
c_4	*0.6161*	c_9	*0.6127*
c_5	*0.6127*	c_{10}	*0.6175*

.

Table 1

When $a \ge 8$, the initial computation that we ran are not enough to make the 'error term' negligible. This explains why our values increase from this point onwards. There is little doubt that heavier computations would cure this defect.

Proof. We numerically checked that, when $1 \le x \le 10^8$,

$$\frac{1}{a!\,x}\sum_{n \le x} \mu^2(n)\Big(\log \frac{x}{n}\Big)^a \le \begin{cases} 0.6715 \text{ when } a = 1, \\ 0.6365 \text{ when } a = 2, \\ 0.6227 \text{ when } a = 3, \\ 0.6161 \text{ when } a = 4, \\ 0.6127 \text{ when } a = 5, \end{cases} \begin{cases} 0.6108 \text{ when } a = 6, \\ 0.6097 \text{ when } a = 7, \\ 0.6091 \text{ when } a = 8, \\ 0.6087 \text{ when } a = 9, \\ 0.6077 \text{ when } a = 10. \end{cases}$$

Let us comment somewhat on this script. We enumerate the squarefree integers together with their prime factorisation through the `forsquarefree`-loop of Pari/GP. At each of them, we compute the vector $(\sum_{\ell \le n} \mu^2(\ell)(\log \ell)^a)_{a \le 10}$. We then study the extrema of the functions $\sum_{\ell \le n} \mu^2(\ell)(\log(x/\ell))^a/x$ between two squarefree integers, say n and its successor n^+, a problem which is readily seen as computing the zeros of a polynomial of degree a. As a good amount of time may be spent computing logarithms, we store them and compute them when needed: after an initial precomputation till $n = 10^5$, we compute and store the $\log p$ when new $n = p$ and otherwise get $\log n$ through $\sum_{p|n} \log p$.

For larger values of x, we employ summation by parts in the form

$$\begin{aligned} \sum_{n \le x} \mu^2(n)\Big(\log \frac{x}{n}\Big)^a &= \int_1^x \sum_{n \le x/t} \mu^2(n) \frac{a \log^{a-1} t}{t} dt \\ &\le \frac{6x}{\pi^2} \int_1^x \frac{a \log^{a-1}(x/t)}{t^2} dt + \sqrt{x} \int_1^x \frac{a \log^{a-1}(x/t)}{t^{3/2}} dt \\ &\le \frac{6x}{\pi^2} \cdot a! \Big(1 + \frac{2^a \pi^2}{6\sqrt{x}}\Big). \end{aligned}$$

This is enough for $a \leq 5$. Otherwise, we have to resort to a somewhat more sophisticated bound (with $x_0 = 450000$ and $\varepsilon = 0.02767$):

$$\begin{aligned}\sum_{n\le x}\mu^2(n)\Bigl(\log\frac{x}{n}\Bigr)^a &= \int_1^{x/x_0}\sum_{n\le x/t}\mu^2(n)\frac{a\log^{a-1}t}{t}dt + \int_{x/x_0}^{x}\sum_{n\le x/t}\mu^2(n)\frac{a\log^{a-1}t}{t}dt\\ &\le \frac{6x\cdot a!}{\pi^2} + \varepsilon\sqrt{x}\int_1^x\frac{a\log^{a-1}t}{t^{3/2}}dt + (1-\varepsilon)\sqrt{x}\int_{x/x_0}^{x}\frac{a\log^{a-1}t}{t^{3/2}}dt\\ &\le \frac{6x\cdot a!}{\pi^2} + \varepsilon\sqrt{x}2^a a! + (1-\varepsilon)\int_1^{x_0}\Bigl(\log\frac{x}{u}\Bigr)^{a-1}\frac{adu}{u^{3/2}}\\ &\le \frac{6x\cdot a!}{\pi^2} + \sqrt{x}a!\Bigl(\varepsilon\cdot 2^a + \frac{1-\varepsilon}{2}\frac{(\log x)^{a-1}}{(a-1)!\sqrt{x}}\Bigr).\end{aligned}$$

This suffices to establish that the maximum of our function is attained below 10^8 when $a \in \{6, 7\}$ and to prove the announced values otherwise. □

3.4. **On quadratic subgroups.** Here is a consequence of Axer's method and the Polya-Vinogradov inequality that we take from [15, Lemma 3.3].

Lemma 3.8. *Let $q \geq 3$ and χ be a nontrivial quadratic character modulo q. Then, there is a prime $p \leq 25\,q^2$, such that $\chi(p) = 1$.*

This is far from being optimal even from an explicit viewpoint but thus result is already more than enough for us as an bound $q^{5/2}$ would do. We refer to [12] by P. Pollack for asymptotic bounds on this question.

4. On the λ_d's

We modify slightly the Selberg coefficients λ_d to allow a better control of the error term that arises.

$$\lambda_d = \begin{cases}\mu(d)\dfrac{\log\frac{z}{d}}{\log z}, & d \le z,\\ 0, & \text{otherwise.}\end{cases} \tag{7}$$

Please note that, as in [15], this choice is independent of q. Part of the quality of our explicit evaluations comes from this feature. We define

$$S = S(z) = \sum_{n\le z^2}\Biggl(\sum_{[d_1,d_2]=n}|\lambda_{d_1}||\lambda_{d_2}|\Biggr)^2 \tag{8}$$

then

$$\tilde{S} = \tilde{S}(z) = \sum_{n\le z^2}\Biggl(\sum_{[d_1,d_2]=n}|\lambda_{d_1}||\lambda_{d_2}|\Biggr)^2\Big/n \tag{9}$$

and finally

$$1/G(z) = \sum_{d_1,d_2\le z}\frac{\lambda_{d_1}\lambda_{d_2}}{[d_1,d_2]}. \tag{10}$$

In the setting of the Selberg sieve, this quantity is readily evaluated through using the general theory of averages of non-negative multiplicative functions. No such simplification occurs with our choice but the theory of explicit estimates for the Moebius function is nowadays sufficiently developed to allow our choice of λ_d.

Lemma 4.1. *Let $z > 1$ be a real number. We have $1/G(z) \le \frac{1.00303}{\log z}$.*

Proof. Some classical manipulations give us that

$$1/G(z) = \sum_{\delta \le z} \frac{\mu^2(\delta)\varphi(\delta)}{\delta^2 (\log z)^2} \Bigg(\sum_{\substack{\ell \le z/\delta, \\ (\ell,\delta)=1}} \frac{\mu(\ell) \log \frac{z/\delta}{\ell}}{\ell} \Bigg)^2$$

By [13, Corollary 1.10], the inner sum is non-negative and at most $1.00303\delta/\varphi(\delta)$, hence

$$1/G(z) \le \frac{1.00303}{(\log z)^2} \sum_{m \le z} \frac{\mu^2(m) \log(z/m)}{m} \sum_{\ell\delta = m} \mu(\ell) = \frac{1.00303}{\log z}$$

as announced. □

5. A flexible setting

Sums with $F(\log \frac{x}{n})$. Let us start with a lemma that sets the scene of this section.

Lemma 5.1. *Let $F : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ be differentiable with $F' \ge 0$ and let $x \ge y \ge 1$. Let (u_n) be a sequence of non-negative real numbers for which we have, for any integer $d \ge 1$, any $t \ge 0$ and some $\alpha \in [0, 1)$,*

$$\sum_{m \le t} \frac{u_{dm}}{m^\alpha} \le \rho t^{1-\alpha}/(1-\alpha)$$

for some non-negative real parameter ρ. Then we have

$$\sum_{\substack{n \le y \\ n \equiv 0 (mod\ d)}} \frac{u_n}{n^\alpha} F\left(\log \frac{x}{n}\right) \le \rho \frac{y^{1-\alpha}}{d} (T_{1-\alpha} F)\left(\log \frac{x}{y}\right),$$

where the operator T_β is defined by

$$(T_\beta G)(X) = \int_0^\infty e^{-\beta t} G(X+t)\, dt.$$

We also set $T_1 = T$.

The operator T_β can also be defined on polynomials by

$$T_\beta(X^j) = \sum_{0 \le \ell \le j} \binom{j}{\ell} \ell! \beta^{-\ell-1} X^{j-\ell}.$$

One of the difficulties, when we apply this lemma, is that the assumption on u_n should hold as soon as $t \ge 1$, and this often leads to larger values of ρ than we would like. In our present case of study, the sequence u_n is the characteristic function of the squarefree numbers. We would like to take $\rho = 6/\pi^2$, and this is for instance possible when $\alpha = 2/3$ by Lemma 3.6 but we are often forced to select $\rho = 1$. We present in Lemma 5.2 a way to recover from this loss.

Proof. We see that

$$\sum_{\substack{n \le y \\ n \equiv 0 (\text{mod}\ d)}} \frac{u_n}{n^\alpha} F\left(\log \frac{x}{n}\right) \le \sum_{m \le y/d} \frac{u_{dm}}{(dm)^\alpha} F\left(\log \frac{x/d}{m}\right)$$

which reduces to the case $d = 1$. By considering $F_1(t) = F(\log(x/y)+t)$, we further reduce the question to the case $y = x$. In this special case, and with $\beta = 1-\alpha$, we find that

$$\begin{aligned}\sum_{n\le x}\frac{u_n}{n^\alpha}F\left(\log\frac{x}{n}\right) &\le \sum_{n\le x}\frac{u_n}{n^\alpha}\int_0^{\log(x/n)}F'(t)dt + F(0)\sum_{n\le x}\frac{u_n}{n^\alpha}\\ &\le \int_0^{\log x}\sum_{n\le xe^{-t}}\frac{u_n}{n^\alpha}F'(t)dt + \frac{\rho x^\beta}{\beta}F(0)\\ &\le \int_0^{\log x}\frac{\rho}{\beta}(xe^{-t})^\beta F'(t)dt + \frac{\rho x^\beta}{\beta}F(0)\\ &\le \frac{\rho}{\beta}F(\log x) + \rho x^\beta\int_0^{\log x}e^{-\beta t}F(t)dt.\end{aligned}$$

Since F is non-decreasing, we have $x^\beta\int_{\log x}^\infty e^{-\beta t}F(t)dt \ge xF(\log x)/\beta$, hence

$$\sum_{n\le x}u_nF\left(\log\frac{x}{n}\right) \le \rho x\int_0^\infty e^{-t}F(t)dt.$$

□

Lemma 5.2. *Let $T^\flat$ be the linear operator defined on polynomials of degree at most 10 by*

$$T^\flat(X^a) = \sum_{0\le r\le a}\binom{a}{r}r!c_r\,X^{a-r}$$

where the structure constants c_r are defined in Lemma 3.7. For any real polynomial F of degree at most 10 and with non-negative coefficients, for any $x \ge y > 0$ and any $d \ge 1$, we have

$$\sum_{\substack{n\le y,\\ n\equiv 0[d]}}\mu^2(n)F\Big(\log\frac{x}{n}\Big) \le \frac{y}{d}T^\flat F\Big(\log\frac{x}{y}\Big).$$

Proof. We see that

$$\sum_{\substack{n\le y,\\ n\equiv 0[d]}}\mu^2(n)F\Big(\log\frac{x}{n}\Big) \le \sum_{m\le y/d}\mu^2(m)F\Big(\log\frac{x/d}{m}\Big)$$

(we have forgotten the coprimality condition $(m,d) = 1$), which reduces our problem to the case $d = 1$. By linearity and the non-negativity of the coefficients of F, it suffices to prove the result for $F(y) = y^a$, for $a \le 10$. To do so, we simply note that

$$\begin{aligned}\sum_{n\le y}\mu^2(n)\Big(\log\frac{x}{n}\Big)^a &= \sum_{0\le r\le a}\binom{a}{r}\Big(\log\frac{x}{y}\Big)^{a-r}\sum_{n\le y}\mu^2(n)\Big(\log\frac{y}{n}\Big)^r\\ &\le \sum_{0\le r\le a}\binom{a}{r}\Big(\log\frac{x}{y}\Big)^{a-r}r!c_r\cdot y\end{aligned}$$

by Lemma 3.7. The lemma follows swiftly. □

Lemma 5.3. *Let $A(X)$ be a real polynomial and $B(X) = A(X/3)$. The linear operator $\mathscr{V} : A \mapsto T_{1/3}T_{1/3}T_{1/3}B(0)$ maps X^j to $\frac{27}{2}(j+2)!$.*

Proof. We rapidly check that
$$\mathscr{V}(X^j)=\int_0^\infty\int_0^\infty\int_0^\infty e^{-\frac{u_1+u_2+u_3}{3}}\Big(\frac{u_1+u_2+u_3}{3}\Big)^j du_1du_2du_3.$$
Put $u_i=3v_i$ for $i\in\{1,2,3\}$ and $v=v_1+v_2+v_3$. Then
$$\mathscr{V}(X^j)=27\int_0^\infty e^{-v}v^j\int_0^\infty\int_0^\infty_{u_1+u_2\le v} du_1du_2dv=\frac{27}{2}\int_0^\infty e^{-v}v^{j+2}dv=\frac{27}{2}(j+2)!$$
as announced. □

Lemma 5.4. *When $F\ge 0$ is C^1 and non-decreasing, and $y\ge 1$, we have*
$$\sum_{n\le y}\frac{\mu^2(n)\tau(n)}{n^{2/3}}F\big(\log\frac{y}{n}\big)\le\frac{6}{\pi^2}y^{1/3}\bigg(\frac{6}{\pi^2}\log y+1.166\bigg)T_{1/3}F(0).$$

Proof. We use the non-negativity of F to write
$$\begin{aligned}\sum_{n\le y}\frac{\mu^2(n)\tau(n)}{n^{2/3}}F\big(\log\frac{y}{n}\big)&\le\sum_{\ell\le y}\frac{\mu^2(\ell)}{\ell^{2/3}}\sum_{m\le y/\ell}\frac{\mu^2(m)}{m^{2/3}}F\big(\log\frac{y/\ell}{m}\big)\\&\le\frac{6}{\pi^2}y^{1/3}\sum_{\ell\le y}\frac{\mu^2(\ell)}{\ell}T_{1/3}F(0)\end{aligned}$$
and we conclude by bounding above the sum over ℓ through Lemma 3.5. □

Sums over squarefree n's of $F(\log n)/n^\alpha$.

Lemma 5.5. *For any $x\ge 1$, $d\ge 1$ and for any polynomial F with non-negative coefficients and degree at most 10, we have*
$$\sum_{\substack{n\ge x\\ n\equiv 0(mod\ d)}}\mu^2(n)\frac{F(\log n)}{n^{3/2}}\le 1.088\frac{T_{1/2}F(\log x)}{d\sqrt{x}}.$$

Proof. It is enough to establish this inequality when $d=1$ and on monomials $F(u)=u^a$ for $u\in\{0,1,\cdots,10\}$.

We then notice that
$$\begin{aligned}\int_x^\infty\frac{(\log t)^a dt}{t^{3/2}}&=\frac{1}{\sqrt{x}}\sum_{0\le r\le a}\binom{a}{r}(\log x)^{a-r}\int_1^\infty\frac{(\log u)^r du}{u^{3/2}}\\&=\frac{1}{\sqrt{x}}\sum_{0\le r\le a}\binom{a}{r}(\log x)^{a-r}\int_0^\infty t^r e^{-t/2}dt\\&=\frac{1}{\sqrt{x}}\sum_{0\le r\le a}\binom{a}{r}2^{r+1}(\log x)^{a-r}r!=\frac{T_{1/2}X^a(\log x)}{\sqrt{x}}.\end{aligned}$$
This expression also proves that $x\mapsto T_{1/2}X^a(\log x)/\sqrt{x}$ is increasing in x. When $x\ge A=\exp(2a/3)+1$, the usual comparison sum/integral tells us that
$$\begin{aligned}\Sigma(x)=\sum_{n\ge x}\mu^2(n)\frac{(\log n)^a}{n^{3/2}}&\le\int_{x-1}^\infty\frac{(\log t)^a dt}{t^{3/2}}=\sqrt{\frac{x}{x-1}}\int_N^\infty\frac{(\log(\frac{N-1}{N}u))^a du}{u^{3/2}}\\&\le\sqrt{\frac{x}{x-1}}\int_x^\infty\frac{(\log u)^a du}{u^{3/2}}.\end{aligned}$$

This is enough for large values of x, i.e. when $x \geq 1/(1-(1.088)^{-2})$. For smaller values of x, we proceed as follows. Say $x \in (N-1, N]$, where N is an integer. We need to compare $\Sigma(N)$ to $\displaystyle\int_N^\infty \frac{(\log u)^a du}{u^{3/2}}$. Pari/GP numerical integration for such functions is reliable. We majorize $\Sigma(x)$ by using direct summation when $n \leq 10^5$ and by majorizing the tail on ignoring the squarefree condition and using some Pari/GP inbuilt acceleration of convergence. Here is our Pari/GP script.

```
{Check( a = 4, expo = 3/2) =
   my(bornesup = ceil(2*exp(a/expo) + 1000), intsup, sommesup,
      coeff = sqrt(bornesup/(bornesup-1)), wheremax, hardmax = 10^5);
   intsup = intnum(t = bornesup, oo, (log(t))^a/t^expo);
   sommesup = sum(n = bornesup, hardmax,
                  if(issquarefree(n), log(n)^a/n^expo,0))
              + sumpos(n = hardmax + 1, log(n)^a/n^expo,0);
   coeff = max(coeff, sommesup/intsup);
   forstep(n = bornesup-1, 1, -1,
      intsup += intnum(t = n, n+1, (log(t))^a/t^expo);
      sommesup +=  if(issquarefree(n), (log(n))^a/n^expo, 0);
      if(coeff < sommesup/intsup,
         coeff = sommesup/intsup; wheremax = n,));
   print("*** sum_{n >= N} mu^2(n)(log n)^", a, "/n^(", expo, ") <= ");
   print(coeff," x int(t = N, oo, (log(t))^", a, "/t^(", expo,"))");
   print(" -- reached around N = ", wheremax);
}
```

□

Lemma 5.6. *For any $x \geq 1$, $d \geq 1$ and for any polynomial F with non-negative coefficients and degree at most 10, we have*

$$\sum_{\substack{n \geq x \\ n \equiv 0 (mod\ d)}} \mu^2(n) \frac{F(\log n)}{n^2} \leq 1.520 \frac{TF(\log x)}{d\, x}.$$

Proof. It is again enough to establish this inequality when $d = 1$ and on monomials $F(u) = u^a$ for $u \in \{0, 1, \cdots, 10\}$. When $x \geq A = \exp(2a/3) + 1$, we note that the usual comparison sum/integral tells us that we have

$$\begin{aligned}\sum_{n \geq x} \mu^2(n) \frac{(\log n)^a}{n^2} &\leq \int_{x-1}^\infty \frac{(\log t)^a dt}{t^2} = \frac{x}{x-1} \int_N^\infty \frac{(\log(\frac{x-1}{x} u))^a du}{u^2} \\ &\leq \frac{x}{x-1} \int_x^\infty \frac{(\log u)^a du}{u^2}.\end{aligned}$$

We then proceed as in Lemma 5.5, though with the choice `expo = 2` in the associated Pari/GP script. □

6. Estimation of the sum $S(z)$

Recall that we have defined

$$S = S(z) = \sum_{n \leq z^2} \bigg(\sum_{[d_1, d_2] = n} |\lambda_{d_1}| |\lambda_{d_2}| \bigg)^2.$$

Theorem 6.1. *When $z > 1$, we have*

$$S \le \frac{278\,000\, z^2 \left(\frac{6}{\pi^2}\log z + 1.166\right)}{(\log z)^4}.$$

The computations leading to Lemma 7.1 tend to accredit the idea that S is indeed of order $z^2/(\log z)^3$, though the constant in front should be much smaller than our $\frac{6}{\pi^2}278000$.

Proof of Theorem 6.1. Let

$$F_0(x) = x^4.$$

Define functions F_1, F_2, F_3, F_4 and F_5 by

$$F_{j+1} = T^\flat F_j, \quad 0 \le j \le 5 \tag{11}$$

where $T^\flat$ is defined in Lemma 5.2. Though it is not required for the following, we directly check that

$$\begin{cases} F_1(x) = x^4 + 2.686\, x^3 + 7.638\, x^2 + 14.9448\, x + 14.7864, \\ F_2(x) = x^4 + 5.372\, x^3 + 20.686\cdots x^2 + 50.405\cdots x + 59.366\cdots, \\ F_3(x) = x^4 + 8.058\, x^3 + 39.146\cdots x^2 + 113.648\cdots x + 154.405\cdots, \\ F_4(x) = x^4 + 10.744\, x^3 + 63.017\cdots x^2 + 211.940\cdots x + 325.447\cdots, \\ F_5(x) = x^4 + 13.43\, x^3 + 92.299\cdots x^2 + 352.549\cdots x + 602.915\cdots. \end{cases}$$

Since $[d_1, d_2] = n$, we replace d_i by gd_i, where $g = \gcd(d_1, d_2)$ so that $gd_1d_2 = n$. The proof starts by noting the following chain of inequalities:

$$\begin{aligned}\sum_{[d_1,d_2]=n} |\lambda_{d_1}||\lambda_{d_2}| = \sum_{gd_1d_2=n} \frac{\log\frac{z}{gd_1}}{\log z}\cdot\frac{\log\frac{z}{gd_2}}{\log z} &\le \frac{1}{4(\log z)^2}\sum_{gd_1d_2=n}\left(\log\frac{z^2}{g^2d_1d_2}\right)^2 \\ &\le \frac{1}{4}\left(\frac{\log\frac{z^2}{n}}{\log z}\right)^2 \sum_{d_1d_2|n} 1,\end{aligned}$$

where we used the inequality $\log A\cdot\log B \le (\log AB)^2/4$. Notice this inequality is sharp when $A = B$, or, in our case of application, when $d_1 = d_2$. Moreover, since $n = gd_1d_2$ and both gd_1 and gd_2 are $\le z$, it follows that

$$n \le zd_i, \quad i = 1, 2. \tag{12}$$

The sum over all n, d_i, g and g_i run over squarefree positive integers.

Therefore

$$\begin{aligned} S &\le \frac{1}{16(\log z)^4}\sum_{n\le z^2}\left(\log\frac{z^2}{n}\right)^4 \sum_{\substack{d_1d_2|n\\ d_3d_4|n}} 1 = \frac{1}{16(\log z)^4}\sum_{d_1,\ldots,d_4\le z}\ \sum_{\substack{n\le zd_i\\ n\equiv 0 (\mathrm{mod}\ D)}}\left(\log\frac{z^2}{n}\right)^4 \\ &\le \frac{1}{16(\log z)^4}\sum_{d_1,\ldots,d_4\le z}\ \sum_{n\le zd_i/D}\left(\log\frac{z^2}{nD}\right)^4 \end{aligned}$$

where $D = [d_1d_2, d_3d_4]$.

Without loss of generality (and an extra factor 4), we assume d_4 is the smallest. Then we can take $y = zd_4/D$ in Lemma 5.1 and apply it with $F = F_0$, to get

$$S \leq \frac{1}{4(\log z)^4} \sum_{d_4 \leq d_i \leq z} \frac{zd_4}{D} F_1\left(\log \frac{z}{d_4}\right),$$

We suppose that $g = (d_1d_2, d_3d_4)$, so that $D = d_1d_2d_3d_4/g$. Further, we assume that $(d_1, d_3d_4) = g_1$ and $(d_2, d_3d_4) = g_2$, so that $g_1g_2 = g$. Similarly define $g_3 = (d_3, d_1d_2)$ and $g_4 = (d_4, d_1d_2)$, so that one has $g_3g_4 = g$. Now, owing to the symmetry of d_1, d_2, d_3, we can assume $d_3 \leq d_2 \leq d_1 \leq z$ with an extra 3!. We can then rewrite the sum as

$$S \leq \frac{3z}{2(\log z)^4} \sum_{g \leq z^2} g \sum_{\substack{g_1g_2 = g \\ g_3g_4 = g}} \sum_{\substack{d_4 \leq d_3 \leq d_2 \leq d_1 \leq z \\ g_i | d_i}} \frac{1}{d_1d_2d_3} F_1\left(\log \frac{z}{d_4}\right). \tag{13}$$

For a fixed $g \leq z^2$ and $g_1g_2 = g_3g_4 = g$ with $g_1, g_2, g_3, g_4 \leq z$, the inside sum over the d_j's is at most

$$S' = \sum_{\substack{d_3 \leq d_2 \leq d_1 \leq z \\ g_i | d_i}} \frac{1}{d_1d_2d_3} \sum_{\substack{d_4 \leq d_3 \\ g_4 | d_4}} F_1\left(\log \frac{z}{d_4}\right) \leq \frac{1}{g_4} \sum_{\substack{d_3 \leq d_2 \leq d_1 \leq z \\ g_i | d_i}} \frac{1}{d_1d_2} F_2\left(\log \frac{z}{d_3}\right).$$

Note that we have $[d_1d_2, d_3d_4] = D \leq zd_4$, hence $d_1d_2d_3d_4/g \leq zd_4$ which simplifies into $d_1d_2d_3 \leq zg$. Taking this condition into consideration, we find that

$$S' \leq \frac{1}{g_4} \sum_{\substack{d_2 \leq d_1 \leq z \\ g_i | d_i}} \frac{1}{d_1d_2} \sum_{\substack{d_3 \leq \min\{d_2, \frac{zg}{d_1d_2}\} \\ g_3 | d_3}} F_2\left(\log \frac{z}{d_3}\right).$$

The next is to split the resulting upper bound in three parts as follows:

$$\begin{aligned}
S' \leq \frac{1}{g_4} & \sum_{\substack{d_1 \leq (zg)^{1/3} \\ g_1 | d_1}} \sum_{\substack{d_2 \leq d_1 \\ g_2 | d_2}} \frac{1}{d_1d_2} \sum_{\substack{d_3 \leq d_2 \\ g_3 | d_3}} F_2\left(\log \frac{z}{d_3}\right) \\
&+ \frac{1}{g_4} \sum_{\substack{(zg)^{1/3} < d_1 \leq z \\ g_1 | d_1}} \sum_{\substack{d_2 \leq \left(\frac{zg}{d_1}\right)^{1/2} \\ g_2 | d_2}} \frac{1}{d_1d_2} \sum_{\substack{d_3 \leq d_2 \\ g_3 | d_3}} F_2\left(\log \frac{z}{d_3}\right) \\
&+ \frac{1}{g_4} \sum_{\substack{(zg)^{1/3} < d_1 \leq z \\ g_1 | d_1}} \sum_{\substack{\left(\frac{zg}{d_1}\right)^{1/2} < d_2 \leq d_1 \\ g_2 | d_2}} \frac{1}{d_1d_2} \sum_{\substack{d_3 \leq \frac{zg}{d_1d_2} \\ g_3 | d_3}} F_2\left(\log \frac{z}{d_3}\right) = S'_1 + S'_2 + S'_3.
\end{aligned}$$

The sum S'_1 may treated in a rather straightforward manner:

$$\begin{aligned}
S'_1 &\leq \frac{1}{g} \sum_{\substack{d_1 \leq (zg)^{1/3} \\ g_1 | d_1}} \frac{1}{d_1} \sum_{\substack{d_2 \leq d_1 \\ g_2 | d_2}} F_3\left(\log \frac{z}{d_2}\right) \leq \frac{1}{gg_2} \sum_{\substack{d_1 \leq (zg)^{1/3} \\ g_1 | d_1}} F_4\left(\log \frac{z}{d_1}\right) \\
&\leq \frac{(zg)^{1/3}}{g^2} F_5\left(\log \frac{z}{(zg)^{1/3}}\right) \leq \frac{(zg)^{1/3}}{g^2} F_5\left(\frac{1}{3} \log \frac{z^2}{g}\right).
\end{aligned}$$

Next, the treatment of S_2' starts similarly as the one of S_1' but differs from the second line onwards:

$$\begin{aligned} S_2' &\leq \frac{1}{g} \sum_{\substack{(zg)^{1/3} < d_1 \leq z \\ g_1 | d_1}} \frac{1}{d_1} \sum_{\substack{d_2 \leq \left(\frac{zg}{d_1}\right)^{1/2} \\ g_2 | d_2}} F_3 \left(\log \frac{z}{d_2} \right) \\ &\leq \frac{(zg)^{1/2}}{gg_2} \sum_{\substack{(zg)^{1/3} < d_1 \leq z \\ g_1 | d_1}} \frac{1}{d_1^{3/2}} F_4 \left(\log \frac{z d_1^{1/2}}{(zg)^{1/2}} \right) = \frac{(zg)^{1/2}}{gg_2} \sum_{\substack{(zg)^{1/3} < d_1 \leq z \\ g_1 | d_1}} \frac{1}{d_1^{3/2}} F_4 \left(\frac{1}{2} \log \frac{z d_1}{g} \right) \\ &\leq 1.088 \frac{(zg)^{1/3}}{g^2} (\tilde{T}_{1/2} F_4) \left(\frac{1}{3} \log \frac{z^2}{g} \right) \end{aligned}$$

on denoting by $\tilde{T}_{1/2}$ the operator that, to F, associates $T_{1/2}(G)$, where $G(X) = F(X/2)$.

The last inequality comes from invoking Lemma 5.5 with $F(u) = F_4(\frac{1}{2}u + \frac{1}{2}\log(z/g))$. We continue with a bound for S_3':

$$\begin{aligned} S_3' &= \frac{1}{g_4} \sum_{\substack{(zg)^{1/3} < d_1 \leq z \\ g_1 | d_1}} \sum_{\substack{\left(\frac{zg}{d_1}\right)^{1/2} < d_2 \leq d_1 \\ g_2 | d_2}} \frac{1}{d_1 d_2} \sum_{\substack{d_3 \leq \frac{zg}{d_1 d_2} \\ g_3 | d_3}} F_2 \left(\log \frac{z}{d_3} \right) \\ &\leq \frac{1}{g} \sum_{\substack{(zg)^{1/3} < d_1 \leq z \\ g_1 | d_1}} \sum_{\substack{\left(\frac{zg}{d_1}\right)^{1/2} < d_2 \leq d_1 \\ g_2 | d_2}} \frac{1}{d_1 d_2} \frac{zg}{d_1 d_2} F_3 \left(\log \frac{d_1 d_2}{g} \right) \\ &\leq 1.52 \frac{zg}{gg_2} \sum_{\substack{(zg)^{1/3} < d_1 \leq z \\ g_1 | d_1}} \frac{1}{d_1^2} \frac{d_1^{1/2}}{(zg)^{1/2}} T F_3 \left(\frac{1}{2} \log \frac{z d_1}{g} \right) \end{aligned}$$

by Lemma 5.6. We shuffle some terms and continue:

$$\begin{aligned} S_3' &\leq 1.52 \frac{(zg)^{1/2}}{gg_2} \sum_{\substack{(zg)^{1/3} < d_1 \leq z \\ g_1 | d_1}} \frac{1}{d_1^{3/2}} T F_3 \left(\frac{1}{2} \log \frac{z d_1}{g} \right) \\ &\leq 1.52 \times 1.088 \frac{(zg)^{1/3}}{g^2} (\tilde{T}_{1/2} T F_3) \left(\frac{1}{3} \log \frac{z^2}{g} \right). \end{aligned}$$

Therefore, from the above and (13), we have

$$S \leq \frac{3 z^{4/3}}{2 (\log z)^4} \sum_{g \leq z^2} \frac{\tau^*(g)^2}{g^{2/3}} H \left(\log \frac{z^2}{g} \right), \tag{14}$$

where $H(x) = F_5(x/3) + 1.088\,(\tilde{T}_{1/2} F_4)(x/3) + 1.52 \times 1.088\,(\tilde{T}_{1/2} T F_3)(x/3)$ and

$$\tau^*(g) = \sum_{\substack{g_1 g_2 = g \\ g_i \leq z}} 1.$$

During the proof of [3, Lemma 2.1], the authors proved that $\tau^*(g)^2$ is also the number of solutions of $g = abcd$ with $ab, cd, ac, bd \leq z$. Since this number in invariant under the changes $a \leftrightarrow d$ and $b \leftrightarrow c$, we may assume that $c \le b$ (with an extra factor 2). Thus

$$S \le \frac{3z^{4/3}}{(\log z)^4} \sum_{a\le z} \sum_{b \le z/a} \sum_{c\le b} \sum_{d\le z/b} \frac{1}{(abcd)^{2/3}} H\left(\log \frac{z^2}{abcd}\right).$$

Please notice the '$d \le z/b$'. We do the d-sum, then the c-sum, getting

$$\begin{aligned} S &\le \frac{3z^{5/3}}{(\log z)^4} \sum_{a\le z} \sum_{b\le z/a} \frac{1}{(ab)^{2/3}} T_{1/3}T_{1/3} H\left(\log\frac{z^2}{ab}\right) \\ &= \frac{3\rho^2 z^{5/3}}{(\log z)^4} \sum_{n\le z} \frac{\tau(n)}{n^{2/3}} (T_{1/3}T_{1/3}H)\left(\log \frac{z}{n}\right) \\ &\le \frac{3\rho^3 z^2}{(\log z)^4}\left(\frac{6}{\pi^2}\log z + 1.166\right) T_{1/3}T_{1/3}T_{1/3}H(0) \end{aligned}$$

by Lemma 5.4 with $\rho = 6/\pi^2$ for the last inequality. We also have

$$\begin{aligned} T_{1/3}T_{1/3}T_{1/3}H(0) &= \mathscr{V}(F_5 + 1.088\,(T_{1/2}F_4) + 1.52 \times 1.088\,(T_{1/2}TF_3)) \\ &= 412082.980\cdots. \end{aligned}$$

This gives (the ρ^3 has been taken into account)

$$S \le \frac{277\,760\, z^2\left(\frac{6}{\pi^2}\log z + 1.166\right)}{(\log z)^4}. \tag{15}$$

The proof of our theorem is complete. □

7. Computing $S(z)$

This part is not required a priori. We produce it to support our belief that $S(z)$ is indeed of size $z^2/(\log z)^3$, a fact that is far from obvious from its expression.

Lemma 7.1. *When $z \le z_0 = 15000$, we have $S(z) \le 3.3\, z^2/(\log z)^3$ while $S(z_0) \ge 3.28\, z_0^2/(\log z_0)^3$.*

We numerically observed that the function $S(z)(\log z)^3/z^2$ is non-decreasing. Here is a plot with sample values taken every 50:
To run the required computations, we need precise and sparse expressions. We use the notation $\delta_j = \log d_j$ and restrict the variables d_j to be squarefree. We expand $S(z)$ into

$$\begin{aligned} S(z) = \sum_{\substack{[d_1,d_2]=[d_3,d_4],\\ d_1,d_2,d_3,d_4\le z}} \Biggl(1 - \frac{\delta_1+\delta_2+\delta_3+\delta_4}{\log z} + \frac{\delta_1\delta_2 + \delta_3\delta_4 + (\delta_1+\delta_2)(\delta_3+\delta_4)}{(\log z)^2} \\ - \frac{\delta_1\delta_2(\delta_3+\delta_4) + (\delta_1+\delta_2)\delta_3\delta_4)}{(\log z)^3} + \frac{\delta_1\delta_2\delta_3\delta_4}{(\log z)^4}\Biggr) \end{aligned}$$

i.e. after shuffling the terms,

$$\begin{aligned} S(z) &= \sum_{\substack{[d_1,d_2]=[d_3,d_4],\\ d_1,d_2,d_3,d_4\le z}} \left(1 - \frac{4\delta_1}{\log z} + \frac{2\delta_1\delta_2 + 4\delta_1\delta_3}{(\log z)^2} - \frac{4\delta_1\delta_2\delta_3}{(\log z)^3} + \frac{\delta_1\delta_2\delta_3\delta_4}{(\log z)^4}\right) \\ &= S_0(z) - \frac{S_1(z)}{\log z} + \frac{S_2(z)}{(\log z)^2} - \frac{S_3(z)}{(\log z)^3} + \frac{S_4(z)}{(\log z)^4}. \end{aligned}$$

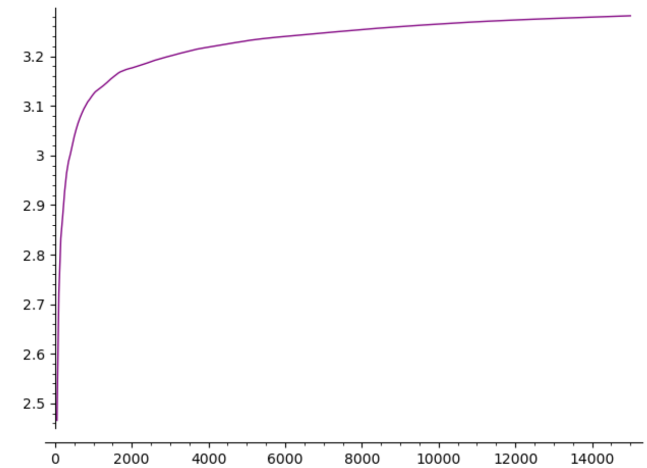

FIGURE 3. Plot of $\max_{x \le z} S(x)(\log x)^3/x^2$ for $10 \le z \le 15000$

Let us assume that z is an integer. When going from z to $z+1$, the sum $S(z)$ changes only when one of the d_i can take the value $z+1$, hence $z+1$ has to be squarefree. We proceed as follows:

- Set $d_1 = z+1$. For each $d_2 \le z+1$, build all the couples (d_3, d_4), with each being $\le z+1$, and add what has to be added to the S_i.
- Assume henceforth that $d_1 \le z$, set $d_2 = z+1$, and again all the couples (d_3, d_4), with each being $\le z+1$, and add what has to be added.
- Assume then that d_1 and d_2 are bounded above by z and set $d_3 = z+1$. For each $d_4 \le z+1$, build all the couples (d_1, d_2) and add what is to be added.
- Finally, do the same with $d_4 = z+1$ and d_1, d_2 and d_3 being bounded above by z.

Local variations. We have to find the maximal absolute value of

$$\left(S_0(z) - \frac{S_1(z)}{\log z} + \frac{S_2(z)}{(\log z)^2} - \frac{S_3(z)}{(\log z)^3} + \frac{S_4(z)}{(\log z)^4}\right)/z^2$$

when z varies in an interval $[z_0, z_0+1)$ for some integer z_0. The derivative times z^3 is given by, with $w = 1/\log z$,

$$\begin{aligned} &-2S_0(z) + 2S_1(z)w + (S_1(z) - 2S_2(z))w^2 \\ &\qquad + (-2S_2(z) + 2S_3(z))w^3 - (-3S_3(z) + 2S_4(z))w^4 - 4S_4(z)w^5 \end{aligned}$$

Final remarks. The previous notes are enough to build an algorithm. Let us specify that we precomputed all the logarithms of the natural numbers up to the final bound z_0 and that we used the `forfactored`-loop of Pari/GP to speed up computations. The script is available on request.

8. Estimation of the sum $\tilde{S}(z)$

Recall that we have have defined

$$\tilde{S} = \tilde{S}(z) = \sum_{n \le z^2} \biggl(\sum_{[d_1, d_2] = n} |\lambda_{d_1}||\lambda_{d_2}| \biggr)^2 / n.$$

Lemma 8.1. *We define the linear operator $\tilde{T}$ on polynomials by $\tilde{T}[Y^j] = Y^j/j!$. For every integer $a \ge 1$ and every real number $x \ge 1$, we have*

$$\sum_{n \le x} \frac{\mu^2(n) a^{\omega(n)}}{n} \le \tilde{T}[(\tfrac{6}{\pi^2} Y + 1.166)^a](\log x).$$

To be precise, we expand the polynomial $(\frac{6}{\pi^2} Y + 1.166)^a$, apply the operator $\tilde{T}$ to this expansion and finally evaluate the result at $Y = \log x$.

Proof. We proceed by recursion on a, the case $a = 1$ being recorded in Lemma 3.5. Let us prove the recursion step. We employ the identity $(a+1)^{\omega(n)} = \sum_{k\ell = n} a^{\omega(k)}$ valid when n is squarefree to infer that

$$\begin{aligned}\sum_{n \le x} \frac{\mu^2(n)(a+1)^{\omega(n)}}{n} &\le \sum_{k\ell \le x} \frac{\mu^2(k)\mu^2(\ell) a^{\omega(k)}}{k\ell} \\ &\le \sum_{k \le x} \frac{\mu^2(k) a^{\omega(k)}}{k} \biggl(\frac{6}{\pi^2} \log \frac{x}{k} + 1.166 \biggr) = S_1 + S_2\end{aligned}$$

where

$$S_1 = \frac{6}{\pi^2} \sum_{k \le x} \frac{\mu^2(k) a^{\omega(k)}}{k} \int_k^x \frac{dt}{t} \le \frac{6}{\pi^2} \int_1^x \tilde{T}[(\tfrac{6}{\pi^2} Y + 1.166)^a](\log t) \frac{dt}{t}$$

by the recursion hypothesis. We now notice that $\int_1^x \tilde{T}[Y^j](\log t) \frac{dt}{t} = \tilde{T}[Y^{j+1}](\log x)$, which implies that, for any polynomial P, we have

$$\int_1^x \tilde{T}[P(Y)](\log t) dt/t = \tilde{T}[P(Y)Y](\log x).$$

This leads to $S_1 \le \tilde{T}[(\frac{6}{\pi^2} Y + 1.166)^a \frac{6}{\pi^2} Y](\log x)$, while by the recursion hypothesis, we have $S_2 \le 1.166\, \tilde{T}[(\frac{6}{\pi^2} Y + 1.166)^a](\log x)$. This gives the required upper bound for $S_1 + S_2$ and concludes the proof. □

Theorem 8.2. *When $z \ge 1000$, we have $\tilde{S} \le 7 \cdot 10^{-7} (\log z)^9 + \frac{6}{\pi^2}$.*

Proof. Note that $\sqrt{ab} \le \frac{a+b}{2}$. With $a = \log(z/d_1)$ and $b = \log(z/d_2)$, this gives us $\sqrt{\log(z/d_1)\log(z/d_2)} \le \frac{1}{2} \log \frac{z^2}{d_1 d_2}$. Notice that this inequality is optimal when $d_1 = d_2$. This implies that, for squarefree n, we have

$$\begin{aligned}\sum_{[d_1, d_2] = n} |\lambda_{d_1}||\lambda_{d_2}| &\le \frac{1}{4(\log z)^2} \sum_{[d_1, d_2] = n} \Bigl(\log \frac{z^2}{d_1 d_2} \Bigr)^2 \\ &\le \frac{1}{4(\log z)^2} \sum_{[d_1, d_2] = n} \Bigl(\log \frac{z^2}{n} \Bigr)^2 = \frac{1}{4(\log z)^2} \Bigl(\log \frac{z^2}{n} \Bigr)^2 3^{\omega(n)}.\end{aligned}$$

This implies that

$$\tilde{S} \le \frac{1}{16(\log z)^4} \sum_{n \le z^2} \frac{\mu^2(n) 9^{\omega(n)}}{n} \Bigl(\log \frac{z^2}{n}\Bigr)^4.$$

We set $y = z^2$ and notice that

$$\Bigl(\log \frac{y}{n}\Bigr)^4 = 4 \int_n^y \Bigl(\log \frac{y}{t}\Bigr)^3 \frac{dt}{t}.$$

On using this and Lemma 8.1 and its notation with the shortcut $\rho = 6/\pi^2$, we get

$$\begin{aligned} \tilde{S} &\le \frac{1}{4(\log z)^4} \int_1^y \sum_{n \le t} \frac{\mu^2(n) 9^{\omega(n)}}{n} \Bigl(\log \frac{y}{t}\Bigr)^3 \frac{dt}{t} \\ &\le \frac{4}{(\log y)^4} \int_1^y \tilde{T}[(\rho Y + 1.166)^9](\log t) \Bigl(\log \frac{y}{t}\Bigr)^3 \frac{dt}{t}. \end{aligned}$$

We recall the classical formula for the Euler beta-function:

$$\int_1^y (\log t)^a \Bigl(\log \frac{y}{t}\Bigr)^b \frac{dt}{t} = \frac{a! b!}{(a+b+1)!} (\log y)^{a+b+1}.$$

We expand $\tilde{T}[(\rho Y + 1.166)^9]$, and get

$$\begin{aligned} \tilde{S} &\le \frac{4}{(\log y)^4} \sum_{0 \le k \le 9} \binom{9}{k} \rho^k 1.166^{9-k} \frac{1}{k!} \frac{6 \cdot k!}{(k+4)!} (\log y)^{k+4} \\ &\le 24 \sum_{0 \le k \le 9} \binom{9}{k} \rho^k 1.166^{9-k} \frac{(\log y)^k}{(k+4)!}. \end{aligned}$$

This gives us

$$\tilde{S} \le 7 \cdot 10^{-6} (\log z)^9 + \frac{6}{\pi^2}$$

when $z \ge 1000$. □

9. Proof of Theorem 1.4

Lemma 9.1. *We have*

$$\Bigl| \sum_{n \le y} \kappa(n/y) \chi(n) \Bigr| \le \min\Bigl(\frac{2\sqrt{6} y}{9} + 4,\ \sqrt{\max(r, 10^7)} \Bigr).$$

Proof. Let us denote our sum by S. The first inequality comes from Lemma 3.2. We then use the identity $\tau(\overline{\chi})\chi(m) = \sum_{1 \le a < r} \overline{\chi}(a) e(am/r)$ where $\tau(\chi)$ is here the Gauss sum attached to χ, and get

$$\begin{aligned} \tau(\overline{\chi}) S &= \sum_{1 \le a < r} \overline{\chi}(a) \sum_{m \in \mathbb{Z}} e(am/r) \kappa(m/y) \\ &= y \sum_{1 \le a < r} \overline{\chi}(a) \sum_{n \in \mathbb{Z}} \hat{\kappa}((n - a/r)y) = y \sum_{1 \le a < r} \overline{\chi}(a) \sum_{\ell \equiv -a[r]} \hat{\kappa}\Bigl(\frac{\ell y}{r}\Bigr). \end{aligned}$$

i.e.

$$\tau(\overline{\chi}) S / y = \sum_{\ell \in \mathbb{Z}} \overline{\chi}(-\ell) \hat{\kappa}\Bigl(\frac{\ell y}{r}\Bigr).$$

Let us now introduce the explicit expression of $\hat{\kappa}$ given in Lemma 2.6, and recall that $\chi(0)=0$. We pair ℓ and $-\ell$ and note that

$$\begin{cases} \hat{\kappa}(x/\pi)+\hat{\kappa}(-x/\pi) = 12\biggl(\Bigl(\dfrac{\sin x/3}{x}\Bigr)^2\cos\dfrac{2x}{3} - \Bigl(\dfrac{\sin x/3}{x}\Bigr)^3\cos\dfrac{x}{2}\biggr), \\ \hat{\kappa}(x/\pi)-\hat{\kappa}(-x/\pi) = 12\biggl(-i\Bigl(\dfrac{\sin x/3}{x}\Bigr)^2\sin\dfrac{2x}{3} + i\Bigl(\dfrac{\sin x/3}{x}\Bigr)^3\sin\dfrac{x}{2}\biggr). \end{cases}$$

The main feature of these expressions lies in the fact that they vanish when $x=0$. We use Lemma 3.1 with $c=\infty$ and $f_\varepsilon(x) = |\hat{\kappa}(x/\pi)+\varepsilon\hat{\kappa}(-x/\pi)|$ for $\varepsilon\in\{\pm1\}$. We find that

$$|\sqrt{r}S/y| \le \frac{r}{\pi y}\int_0^\infty |f_\varepsilon(x)|dx + \frac12\int_0^\infty |f'_\varepsilon(x)|dx.$$

Numerical integration gives us

$$\frac1\pi\int_0^\infty |f_\varepsilon(x)|dx \le \begin{cases} 0.9918 & \text{when } \varepsilon=1, \\ 0.9872 & \text{when } \varepsilon=-1, \end{cases} \quad \int_0^\infty |f'_\pm(x)|dx \le \begin{cases} 2.4 & \text{when } \varepsilon=1, \\ 3.6 & \text{when } \varepsilon=-1. \end{cases}$$

For this last computation, we need explicit expressions for $f'_\pm$, and here they are:

$$\begin{aligned} f'_+(x)/12 = &-\frac23\sin\frac{2x}{3}\Bigl(\frac{\sin x/3}{x}\Bigr)^2 + \frac12\sin\frac{x}{2}\Bigl(\frac{\sin x/3}{x}\Bigr)^3 \\ &+\frac23\cos\frac{x}{3}\cos\frac{2x}{3}\frac{\sin x/3}{x^2} - \frac1x\cos\frac{x}{3}\cos\frac{x}{2}\Bigl(\frac{\sin x/3}{x}\Bigr)^2 \\ &-\frac2x\cos\frac{2x}{3}\Bigl(\frac{\sin x/3}{x}\Bigr)^2 + \frac3x\cos\frac{x}{2}\Bigl(\frac{\sin x/3}{x}\Bigr)^3, \end{aligned}$$

and

$$\begin{aligned} if'_-(x)/12 = &\frac23\cos\frac{2x}{3}\Bigl(\frac{\sin x/3}{x}\Bigr)^2 - \frac12\cos\frac{x}{2}\Bigl(\frac{\sin x/3}{x}\Bigr)^3 \\ &+\frac23\sin\frac{x}{3}\cos\frac{2x}{3}\frac{\sin x/3}{x^2} - \frac1x\sin\frac{x}{3}\cos\frac{x}{2}\Bigl(\frac{\sin x/3}{x}\Bigr)^2 \\ &-\frac2x\sin\frac{2x}{3}\Bigl(\frac{\sin x/3}{x}\Bigr)^2 + \frac3x\sin\frac{x}{2}\Bigl(\frac{\sin x/3}{x}\Bigr)^3. \end{aligned}$$

□

Lemma 9.2. *When χ is a primitive Dirichlet character of conductor $r>1$, we have*

$$\frac{1}{2\pi}\int_{\frac12-i\infty}^{\frac12+i\infty} \bigl|(s+1)\check{\eta}^*(s)L(s,\chi)\bigr|^2|ds| \le 1.1\sqrt{\max(r,10^7)}.$$

Proof. We have

$$(s+1)\check{\eta}^*(s)L(s,\chi) = \check{\kappa}(s)L(s,\chi) = \int_0^\infty \sum_{n\le 1/v}\chi(n)\kappa(vn)v^s\frac{dv}{v} \tag{16}$$

which identifies $(s+1)\check{\eta^*}(s)$ as a Mellin transform. By the Parseval identity for Mellin transforms, we find that

$$\frac{1}{2\pi}\int_{\frac12-i\infty}^{\frac12+i\infty}\left|(s+1)\check{\eta^*}(s)L(s,\chi)\right|^2|ds| = \int_0^\infty\Big|\sum_{n\le 1/v}\kappa(vn)\chi(n)\Big|^2 dv$$
$$= \int_0^\infty\Big|\sum_{n\le t}\kappa(n/t)\chi(n)\Big|^2\frac{dt}{t^2}.$$

We are now in a position to apply Lemma 9.1, and this leads us to

$$\frac{1}{2\pi}\int_{\frac12-i\infty}^{\frac12+i\infty}\left|(s+1)\check{\eta^*}(s)L(s,\chi)\right|^2|ds| \le \int_1^T\Big(\frac{2\sqrt{6}t}{9}+4\Big)^2\frac{dt}{t^2}+\int_T^\infty\sqrt{\max(r,10^7)}^2\frac{dt}{t^2}$$
$$\le \frac{24T}{81}+\frac{16\sqrt{6}}{9}\log T+16+\frac{\max(r,10^7)}{T}.$$

We take $T=\frac{9}{2\sqrt{6}}\sqrt{\max(r,10^7)}$ and get the bound

$$\frac{1}{2\pi}\int_{\frac12-i\infty}^{\frac12+i\infty}\left|(s+1)\check{\eta^*}(s)L(s,\chi)\right|^2|ds| \le 1.089\sqrt{r}+4.4\log(1.84\sqrt{r})+16$$
$$\le 1.1\sqrt{\max(r,10^7)}.$$

This ends the proof. $\square$

Proof of Theorem 1.4. In this proof, we shall use the Dirichlet polynomial

$$K(s,\chi)=\sum_{d_1,d_2}\frac{\lambda_{d_1}\lambda_{d_2}\chi([d_1,d_2])}{[d_1,d_2]^s},\quad u_n=\sum_{[d_1,d_2]=n}\lambda_{d_1}\lambda_{d_2} \tag{17}$$

where the (λ_d)'s come from (7).

We proceed as in the proof of Theorem 1.5 and 1.6 in [15, Section 6], with $H=\{1\}$. This results in having only primitive characters below. We thus reach

$$\sum_{\substack{p\ge1,\\ p\equiv a[q]}}\eta^*(p/X)\le\sum_{\substack{n\ge1,\\ n\equiv a[q]}}\eta^*(n/X)\Big(\sum_{d|n}\lambda_d\Big)^2+\sum_{\substack{p<z,\\ p\equiv a[q]}}\eta^*(p/X)$$
$$\le\sum_{d_1,d_2}\lambda_{d_1}\lambda_{d_2}\frac{1}{\varphi(q)}\sum_{r|q}\sum_{\chi \bmod^* r}\overline{\chi(a)}\sum_{\substack{n\ge1,\\ [d_1,d_2]|n}}\chi(n)\eta^*(n/X)+\frac{zq^{-1}+1}{2}$$
$$\le\frac{1}{\varphi(q)}\sum_{d_1,d_2}\lambda_{d_1}\lambda_{d_2}\sum_{m\ge1}\eta^*([d_1,d_2]m/X)+E(a)+\frac{zq^{-1}+1}{2}$$

where

$$E(a)=\frac{1}{\varphi(q)}\sum_{1<r|q}\sum_{\chi \bmod^* r}\overline{\chi(a)}\frac{1}{2i\pi}\int_{2-i\infty}^{2+i\infty}L(s,\chi)K(s,\chi)X^s\check{\eta^*}(s)ds. \tag{18}$$

Contribution of the principal character. The main term above is easily dealt with. Let

$$\mathscr{M}(q,\eta^*,X)=\frac{1}{\varphi(q)}\sum_{d_1,d_2}\lambda_{d_1}\lambda_{d_2}\sum_{m\geq 1}\eta^*([d_1,d_2]m/X). \tag{19}$$

We find that

$$\begin{aligned}\biggl|\mathscr{M}(q,\eta^*,X)-\frac{2X}{9\varphi(q)}\sum_{d_1,d_2}\frac{\lambda_{d_1}\lambda_{d_2}}{[d_1,d_2]}\biggr| &\leq \frac{\theta}{X^2\varphi(q)}\sum_{d_1,d_2}|\lambda_{d_1}\lambda_{d_2}|[d_1,d_2]^2\\ &\leq \frac{\theta}{X^2\varphi(q)}\biggl|\sum_d d^2|\lambda_d|\biggr|^2\leq\frac{\theta z^6}{X^2\varphi(q)(\log z)^2}\end{aligned}$$

where θ is defined in Lemma 2.3, and where we have used Lemma 4.1. Whence, when $z^2\leq X/100$ and $z\geq 100$, we find that

$$\mathscr{M}(q,\eta^*,X)\leq\frac{2X}{9\varphi(q)\log z}\biggl(1.00303+\frac{9\times 0.28\,z^6}{2X^3(\log z)^3}\biggr)\leq 1.004\frac{2X/9}{\varphi(q)\log z}. \tag{20}$$

Contribution of the non-principal characters. We now concentrate on $E(a)$ for which we seek an L^2-average upperbound. We first shift the line of integration to $\Re s=1/2$ and get

$$\varphi(q)^2\sum_{a\bmod^* q}E(a)^2=\sum_{a\bmod^* q}\biggl|\sum_{\substack{1<r|q,\\ \chi\bmod^* r}}\overline{\chi(a)}\frac{1}{2i\pi}\int_{\frac12-i\infty}^{\frac12+i\infty}L(s,\chi)K(s,\chi)X^s\check{\eta^*}(s)ds\biggr|^2.$$

We note that, for arbitrary complex numbers $(f(\chi))_\chi$, we have

$$\sum_{a\bmod^* q}\biggl|\sum_{\substack{1<r|q,\\ \chi\bmod^* r}}\overline{\chi(a)}f(\chi)\biggr|^2=\varphi(q)\sum_{\substack{1<r|q,\\ \chi\bmod^* r}}|f(\chi)|^2$$

simply because, when a is prime to q, there is no difference between the primitive character modulo r and its induced version modulo q. We continue with Cauchy's inequality and Lemma 9.2 to write successively

$$\begin{aligned}\varphi(q)\sum_{a\bmod^* q}E(a)^2 &\leq\frac{X}{4\pi^2}\sum_{\substack{1<r|q,\\ \chi\bmod^* r}}\int_{\frac12-i\infty}^{\frac12+i\infty}|(s+1)\check{\eta^*}(s)L(s,\chi)|^2|ds|\int_{\frac12-i\infty}^{\frac12+i\infty}\bigl|K(s,\chi)\bigr|^2\frac{dt}{|s+1|^2}\\ &\leq\frac{1.1\,X\sqrt{q}}{2\pi}\sum_{\substack{1<r|q,\\ \chi\bmod^* r}}\int_{\frac12-i\infty}^{\frac12+i\infty}\bigl|K(s,\chi)\bigr|^2\frac{dt}{|s+1|^2}\\ &\leq 1.1\,X\sqrt{q}\int_0^\infty\sum_{\substack{1<r|q,\\ \chi\bmod^* r}}\biggl|v\sum_{n\leq 1/v}nu_n\chi(n)\biggr|^2dv\end{aligned}$$

since

$$\frac{K(s)}{s+1}=\int_0^\infty v\sum_{n\leq t}nu_n\,v^{s-1}\frac{dv}{v}.$$

We continue by using the orthogonality of the characters and get

$$\begin{aligned}\varphi(q)\sum_{a \bmod^* q} E(a)^2 &\le 1.1\, X\sqrt{q}\int_0^\infty \sum_{n\le t} n^2 u_n^2 (t+q)\frac{dt}{t^4}\\ &\le 1.1\, X\sqrt{q}\sum_{n\le z^2} n^2u_n^2\biggl(\int_n^{z^2}\frac{dt}{t^3} + z^2\int_{z^2}^\infty \frac{dt}{t^4} + q\int_n^\infty\frac{dt}{t^4}\biggr)\\ &\le 1.1\, X\sqrt{q}\sum_{n\le t} u_n^2\biggl(\frac{1}{2}-\frac{n^2}{6z^4}+\frac{q}{3n}\biggr).\end{aligned}$$

On neglecting the term $n^2/6z^4$, we reach

$$\frac{\varphi(q)}{X\sqrt{q}}\sum_{a \bmod^* q} E(a)^2 \le 0.55\sum_{n\ge 1}|u_n|^2 + 0.367q\sum_{n\ge 1}\frac{|u_n|^2}{n}.$$

Let

$$\Delta = \frac{1}{\varphi(q)}\sum_{a \bmod^* q}\epsilon(a)^2 \quad\text{where}\quad \epsilon(a) = \varphi(q)E(a)\frac{\log z}{2X/9}.$$

By Theorems 6.1 and 8.2, we find that, when $z \ge 10^{27}$,

$$\Delta \le \left(0.55\frac{278000\, z^2\left(\frac{6}{\pi^2}\log z + 1.166\right)}{(\log z)^4} + 0.367q\left(7\frac{\log^9 z}{10^7}+\frac{6}{\pi^2}\right)\right)\frac{\sqrt{q}\log^2 z}{4X/81}.$$

We use

$$z = \sqrt{\frac{X}{10^9\sqrt{q}}},\quad X \ge (10^7 q)^{5/2}. \tag{21}$$

The hypothesis $q \ge 10^{30}$ ensures that $z \ge 10^{32}$, hence

$$\begin{aligned}\Delta \le\ & \frac{1.95\cdot 10^6}{10^9\log(10^{35/4}q/10^{9/2})}\\ &+ \frac{248\log^2(10^6 q/10^{9/2})}{10^{35/2}q}\left(2\cdot 10^{-7}\left(2\log\frac{10^{35/4}q}{10^{9/2}}\right)^9+\frac{6}{\pi^2}\right).\end{aligned}$$

A numerical application concludes that $\Delta \le 1/42804$. We finally have to take care of the additional factor $(zq^{-1}+1)/2$. We introduce

$$\delta = (z+q)(\log z)/(4X/9) \ge \varphi(q)\frac{zq^{-1}+1}{2}\frac{\log z}{2X/9}.$$

We have $\delta \le 10^{-50}$. We find that

$$\frac{1}{\varphi(q)}\sum_{a \bmod^* q}(\epsilon(a)+\delta)^2 \le \Delta + 2\delta\sqrt{\Delta}+\delta^2 \le 1/42500$$

as claimed, by renaming $\epsilon(a)+\delta$ in $\epsilon(a)$. $\square$

10. A Brun–Titchmarsh inequality for cosets

Theorem 1.6 of [15] has the condition $q \leq y^{1/3}/900$. This was introduced only to get better numerics, as the proof indeed allows for y^{ξ} for any $\xi \leq 2/5$. At this level we can even save a bit, as shown in the next result.

In this section, we use

$$\eta(t) = \begin{cases} 2t, & 0 \leq t \leq 1/2, \\ 2(1-t), & 1/2 \leq t \leq 1, \\ 0, & \text{otherwise.} \end{cases} \tag{22}$$

Lemma 10.1. *When $z \geq 6$, we have $\sum_{d \leq z} \mu^2(d)/\varphi(d) \geq \log z + 1.24$.*

The bound follows from [17, Theorem 3.1] when $z \geq 700$. We complete the proof by a direct inspection. The constant 1.24 is forced by the case $z = 29$.

Lemma 10.2. *Let $y^{1/5} \geq \sqrt{q} \geq 3000$ and η be as in (22). Let $G_q = (\mathbb{Z}/q\mathbb{Z})^{\times}$ and let $H \subseteq G_q$ be a subgroup of index 5. Then*

$$\sum_{\substack{p \leq y, \\ p \in uH}} \eta(p/y) \leq \frac{y}{5 \log \frac{y}{30\sqrt{q}}}$$

for any class u in G_q.

If we guarantee $\frac{y}{30\sqrt{q}} > y^{4/5}$, and since the total weight (given by Lemma 10.3) is indeed $y/(2\log y)$, this ensures an accumulation per coset which is strictly less than half the total number of primes. From this we shall deduce that at least *three* cosets contain primes.

Proof. We modify the end of the proof of [15, Theorem 1.6] starting from equation (17) therein as follows and using Lemma 10.1 above rather than $G(z) \geq \log z$ as in [15, Lemma 4.2]. We first deduce a more precise inequality from these assumptions:

$$\frac{Y \log y}{y/2} \pi_\eta(y; q, uH) \leq \frac{\log y}{\log z + 1.24} + \frac{2zY \log y}{y} + \sqrt{q} \frac{2Y z^2 \log y}{y(\log z + 1.24)^2} \left(\frac{15}{\pi^2} + \frac{30}{\sqrt{z}} \right)^2.$$

We take $Y = 5$ and

$$z^2 = \frac{y}{30\sqrt{q}}. \tag{23}$$

This gives us

$$\frac{Y \log y}{y/2} \pi_\eta(y; q, uH) \frac{1.24 + \log z}{2 \log y} \leq$$
$$1 + \left(\frac{10 \log y (1.24 + \log z)^2}{30\sqrt{qy}} + \frac{10}{30} \left(\frac{15}{\pi^2} + \frac{30}{\sqrt{z}} \right)^2 \right) \frac{1}{1.24 + \log z}.$$

Notice that $z^2 \geq y^{4/5} \geq (3000)^4$. The above inequality implies that

$$\frac{Y \log y}{y/2} \pi_\eta(y; q, uH) \frac{1 + \log z}{2 \log y} \leq 1 + \frac{1}{1 + \log z}.$$

With $\log z = Z$, we find that

$$\frac{1}{Z+1} \left(1 + \frac{1}{Z+1} \right) \leq \frac{1}{Z}$$

as this last inequality is equivalent to $Z(Z+2) \le (Z+1)^2$. This completes the proof of Lemma 10.2. □

We also recall part of [15, Lemma 2.5].

Lemma 10.3. *For any $1 \le q < x$ and any $x \ge 4 \cdot 10^7$, we have*

$$\sum_{(p,q)=1} \eta(p/x) \ge \frac{x/2}{\log x}.$$

11. Proof of Theorem 1.1

Let us recall [1, Lemma 4.3].

Lemma 11.1. *Let $\mathcal{A}$ be a subset of a finite abelian group G such that $|\mathcal{A}| \ge \eta|G|$, with $\eta > 1/3$. Define, for any integer Y,*

$$\lambda(Y) = \begin{cases} \lceil \eta Y \rceil + 1 & \text{when } Y \equiv 2[3] \text{ and } 2 \le Y \le 1/(3\eta - 1), \\ \lceil \eta Y \rceil & \text{otherwise.} \end{cases}$$

For any subgroup H of index Y, assume $\mathcal{A}$ meets at least $\lambda(Y)$ cosets. Then $\mathcal{A} + \mathcal{A} + \mathcal{A} = G$.

Proof of Theorem 1.1. Let $\mathcal{P}(y)$ be the set of primes below y that do not divide q and let $\mathcal{A}$ be the image of $\mathcal{P}(y)$ in $G_q = (\mathbb{Z}/q\mathbb{Z})^\times$. We seek to show that $\mathcal{A} \cdot \mathcal{A} \cdot \mathcal{A} = G_q$. Recall that we assume that

$$10^{30} \le q \le y^{2/5}/10^7. \tag{24}$$

Let us obtain a lower bound for $|\mathcal{A}|$. From Lemma 10.3 and Theorem 1.4, we get

$$\frac{2y/9}{\log y} \le \sum_{\substack{p \le y \\ (p,q)=1}} \eta^*(p/y) = \sideset{}{^*}\sum_{a (\mathrm{mod}\ q)} \sum_{p \equiv a[q]} \eta^*(p/y) \le \sum_{a \in \mathcal{A}} \frac{(U + \xi(a)) 2y/9}{\varphi(q) \log y} \tag{25}$$

where

$$J = \frac{\log y}{\log \frac{y}{10^9 \sqrt{q}}} \le 1.303, \quad U = 2.008 J, \quad \xi(a) = 2\epsilon(a) J \tag{26}$$

whence, with $c = |\mathcal{A}|/\varphi(q)$,

$$\begin{aligned} \varphi(q) &\le \sum_{a \in \mathcal{A}} (U + \xi(a)) \le 2.0008 \cdot J|\mathcal{A}| + |\mathcal{A}|^{1/2} \Big(\sum_{a \in \mathcal{A}} \xi(a)^2 \Big)^{1/2} \\ &\le c\varphi(q) 2.008 J + \sqrt{c\varphi(q) \frac{\varphi(q)}{42500} J^2} \le (2.008c + 0.00486\sqrt{c}) \varphi(q) J. \end{aligned}$$

This and (26) imply that $c \ge 0.380$. Lemma 11.1 tells us that we have to check that $\mathcal{A}$ contains at least $\lceil 0.380 Y \rceil + 1$ cosets of any subgroup of index $Y \in \{2, \cdots, 7\}$ and congruent to 2 modulo 3. This leaves only the cases $Y = 2$ and $Y = 5$. In the case $Y = 2$, corresponding to a subgroup H say, any integer that falls in $G_q \setminus H$ has a prime factor that is also in $G_q \setminus H$. So $\mathcal{A} \cap (G_q \setminus H) \ne \emptyset$. And Lemma 3.8 implies that $\mathcal{A} \cap H \ne \emptyset$. Hence $\mathcal{A}$ meets 2 cosets modulo H which is enough.

Let us consider the case when $Y = 5$. If we were able to replace 0.380 by $\frac{2}{5} - o(1)$, this would guarantee us only two classes in a subgroup of index 5, which is not enough (they could be $\{0, 1\}$ in $(\mathbb{Z}/5\mathbb{Z}, +)$, whose three-fold sum does not cover $\mathbb{Z}/5\mathbb{Z}$). We need to go simply beyond 2/5 but we do not need to increase the exponent, since a bound of the shape $0.4 + \epsilon(q)$ for some *positive* $\epsilon(q)$ is enough, even if $\epsilon(q)$ tends to 0, and this is exacty what Lemma 10.2 gives us. On reasoning as in [15], we get $\lambda \geq 3$ which is enough to conclude this proof. □

12. Some computations for small values of the modulus

For a modulus q, let $P(q)$ be the largest over the invertible residue class a modulo q, of the smallest prime congruent to a modulo q. The paper [9] contains in Appendix A the report of some numerical computations that implies that $P(q) \leq 2.2q(\log q)^2$ when $q \leq 10^6$.

Our problem is somewhat distinct, and we consider three situations.

12.1. **Products of two primes below q.** Let $P_2(q)$ to be the smallest X of the largest over the invertible residue class a modulo q such that product of two primes in the class a modulo q, each prime being at most X in size. A brute force Pari/GP script simply producing products of two primes and collecting the classes obtained yields the next lemma.

Lemma 12.1. *For every $q \in [402, 50\,000]$, we have $P_2(q) \leq q$. The class 385 modulo the prime 401 does not contain any product of two primes, each being below 400.*

When $q \geq [3404, 50\,000]$, we have $P_2(q) \leq q/2$.

When $q \geq [11\,742, 50\,000]$, we have $P_2(q) \leq q/3$.

When $q \geq [35\,072, 50\,000]$, we have $P_2(q) \leq q/4$.

The reader should notice that this is not only important theorerically but also algorithmically: every time we decrease the bound we aim for (hence aiming at a better theoretical result), we shorten the computation. As the bad cases are rare enough, we first aim at $q/5$ for instance when $q \geq 20\,000$ and, if it does not work, we try $q/4$ and so on. A conjecture of Erdős recalled in [6, Section 2] says that, when $q > q_0$ for some q_0, every invertible class modulo q contains a product of two primes, each of which is of size at most q. The above lemma hints that one can maybe take $q_0 = 401$.

12.2. **Products of three primes below q.** A Pari/GP script simply producing products of three primes and collecting the classes obtained yields the initial step of our ladder.

Lemma 12.2. *For every $q \leq 200$ and $q \notin \{2, 3, 4, 5, 6, 10\}$, every invertible class modulo q contains a product of three primes, each of which of size at most q.*

If we replace the bound q by $2q$, then no exception would occur. We then go farther step by step to take care of the irregularities occuring at small values.

Lemma 12.3. *For every $q \in [200, 30\,000]$, every invertible class modulo q contains a product of three primes, each of which being of size at most $B(q)$, where*

Step-1. $B(q) = q/2$ *when* $q \in [200, 600]$,
Step-2. $B(q) = q/4$ *when* $q \in [600, 1000]$,
Step-3. $B(q) = q/5$ *when* $q \in [1000, 2000]$,
Step-4. $B(q) = q/8$ *when* $q \in [2000, 3000]$,
Step-5. $B(q) = q/10$ *when* $q \in [3000, 4500]$,
Step-6. $B(q) = q/14$ *when* $q \in [4500, 5200]$,
Step-7. $B(q) = q^{5/7}$ *when* $q \in [5200, 10\,000]$,
Step-8. $B(q) = q^{2/3}$ *when* $q \in [10\,000, 24\,000]$,

Step-9. $B(q) = q^{13/20}$ *when* $q \in [24\,000, 30\,000]$.

It is expected that one should be able to reach $B(q) = q^{1/3+\delta}$ for any arbitrary small positive δ. If this is true, our numerical trials show that the asymptotic regime is slow to become dominant. This explains why this line of approach yields less slower results than the one with products of two primes. A better algorithm would aim at at a lesser bound for $B(q)$ and handle the exceptional q's and the classes not represented in a special way, but we did not venture in this kind of very dedicated programming.

12.3. **Products of three primes below** $q(\log q)^2$**. Proof of Theorem 1.2.** The previous strategies require large tables and are impratical when q starts to be somewhat large. We relax these conditions in Theorem 1.2. Our algorithmic strategy is to first find the smallest prime congruent to 1 modulo q and then to check that each class modulo q contains an integer that has at most three prime factors and is indeed of size at most $q(\log q)^2$. We then precompute the integers than have more than four many prime factors and use the `Set` data-structure to determine quickly whether an integer belongs or not to this list. With the Pari/GP script below compiled with `gp2c`, we reach the bound $2 \cdot 10^7$ in about 30 minutes of a usual laptop. Increasing the bound $2 \cdot 10^7$ would require an increase of the memory by the same factor, and we already use nearly 8Gb.

The bound for the primes required outputed by this script varies slowly but does not tend to decrease and for instance, we need primes up to more that $0.95\, q(\log q)^2$ for some $q \in [19900000, 19950000]$. If we need a statement valid for every $q \geq 3$ (resp. $q \geq 2$), we only have to replace $q(\log q)^2$ by $2q(\log q)^2$ (resp. $4q(\log q)^2$).

```
{check(borneinf, bornesup, whentotell = 50000) =
   my(pmax, maxgot = 0, multi = 3, btt,
      locmax = 0, curindex, nbbadn = 0, setbadn);

   for(n = 1, multi*bornesup, if(bigomega(n) > 3, nbbadn++));
   setbadn = Set(Vecsmall([1.. nbbadn]));
   curindex = 1;
   for(n = 1, multi*bornesup,
      if(bigomega(n) > 3, setbadn[curindex] = n; curindex++));

   for(q = borneinf, bornesup,
      if(q % whentotell == 0,
         print("When ", max(borneinf, q - whentotell) ,
               " <= q < ", q, " --> locmax = ", locmax);
         maxgot = max(maxgot, locmax); locmax = 0);
      btt = q*log(q)^2; pmax = 0;
      /* Ensure we can find a prime congruent to 1 */
      forprimestep(p = q + 1, q^2, q, if(p % q == 1, pmax = p; break));
      if(pmax == 0,  print("Error!!"); return(0),
         locmax = max( locmax, pmax / btt));
      /* Find a product of at most three primes in each class: */
      for(n = 16, q - 1,
         if(gcd(n, q) == 1,
            while(((n <= multi*bornesup)&&(setsearch(setbadn, n)))
```

```
              ||(bigomega(n) >= 4), n += q);
          locmax = max( locmax, n / btt))));
if(bornesup % whentotell != 0,
   print("When ",
          max(borneinf-1, floor(bornesup/whentotell)*whentotell),
          " <= q <= ", bornesup, " --> locmax = ", locmax));
maxgot = max(maxgot, locmax);
return(maxgot);}
```

13. Changing the constants in Theorems 1.4 and 1.1

On reading patiently the proof of Theorem 1.4, we obtain the following.

Theorem 13.1. *Let $q \geq q_0 \geq 2\cdot 10^7$ be an integer. Let $X \geq (Cq)^{5/2}$ be some real number. For every invertible class a modulo q, we have*

$$\sum_{p\equiv a[q]} \eta^*(p/X) \leq \frac{2(1.004+\epsilon(a))\cdot 2X/9}{\varphi(q)\log\frac{X}{R\sqrt{q}}} \quad \textit{where} \quad \sum_{a \bmod^* q} \epsilon(a)^2 \leq \varphi(q)/4250,$$

where the constants (q_0, C, R) may be chosen from table.

q_0	$2\cdot 10^7$	$2\cdot 10^7$	$2\cdot 10^7$	10^{10}	10^{15}	10^{20}	10^{30}	10^{30}
C	10^{13}	3000	1	10^{10}	10^6	90	10^8	1/1000
R	10^9	10^{10}	10^{13}	10^9	10^9	10^9	$5\cdot 10^8$	10^9

Table 2

Proof. We follow the proof of Theorem 1.4. Lemma 9.2 can be used as done there since $q_0 \geq 10^7$. Inequality (20) requires $10^4 \leq z^2 \leq X/100$. Theorems 6.1 and 8.2 ask for the stronger condition: $z \geq 1000$. Proceeding, we modify (21) in

$$z = \sqrt{\frac{X}{R\sqrt{q}}}, \quad X \geq (Cq)^{5/2}. \tag{27}$$

We thus get $z^2 \geq (C^{5/2}/R)q^2$. We have to take care of some monotonicity condition. In particular, we want $\sqrt{q}(\log z)^{11}/X$ to be non-increasing in $z^2 = X/(R\sqrt{q})$. We may rewrite this expression as $(11/2)^{11}((\log(z^{2/11})/z^{2/11})^{11/2}$ so it is enough to assume that $z \geq \exp(11/2)$ which is small enough. The upper bound for Δ reduces to, with $L = \log z$ and $X = (Cq)^{5/2}$,

$$\Delta \leq \frac{3096225\,(\frac{6}{\pi^2}L + 1.166)}{RL^2} + \frac{7.44L^2}{qC^{5/2}}\left(2\frac{L^9}{10^7} + \frac{6}{\pi^2}\right)$$

which we want to be at most 1/42800. The first term is dominant and asks for $\frac{\pi^2}{6}RL \geq 42800\times 3096225$. When $\log z \geq 10^{35}$, the value $R = 10^9$ is enough. This explains the choice made in Theorem 1.4. In general we need to satisfy the inequality

$$\frac{3096225\,(\frac{6}{\pi^2}L + 1.166)}{RL^2} + \frac{7.44L^2}{q_0C^{5/2}}\left(7\frac{L^9}{10^7} + \frac{6}{\pi^2}\right) \leq \frac{1}{42800} \quad \left(L = \frac{1}{2}\log\frac{C^{5/2}q_0^2}{R}\right)$$

provided that $C^{5/2}q_0^2 \geq 10^6 R$. A numerical application yields the values of Table 2. $\square$

Theorem 13.2. *For any $q \geq q_0$ and any invertible residue class a modulo q, there exists a natural number that is congruent to a modulo q and that is the product of exactly three primes, all of which are below $(Cq)^{5/2}$, where q_0 and C may be obtained from Table 3.*

q_0	$2 \cdot 10^7$	10^{10}	10^{15}	10^{20}	10^{30}	10^{88}
C	10^{15}	10^{15}	10^{14}	10^{13}	10^{11}	30
R	10^9	10^9	10^9	10^9	$5 \cdot 10^8$	$2 \cdot 10^8$

Table 3

The present method would not allow to take primes only $\geq q^{5/2}$ however large q, due the usage of Lemma 10.2.

Proof. We follow the proof of Theorem 1.1 and assume that the constants q_0, C and R are admissible in Theorem 13.1. The condition that emerges is

$$J = \frac{\log y}{\log \frac{y}{R\sqrt{q}}} \leq 1.3054 \tag{28}$$

where $y = (Cq)^{5/2}$. This is required so that we may reach $c \geq 0.380$. This condition reads

$$1.3054 \log R - \left(2 \times 1.3054 - \tfrac{5}{2}\right) \log q \leq 0.3054 \frac{5}{2} \log C.$$

We next swiftly build Table 3, though it is to be noticed that C has to be taken as large as 30 for using Lemma 10.2. The proof is complete. $\square$

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