The Kolkata Lectures

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On the Transference Principle for Primes

Enveloping Sieve, Majorant Property and some Examples

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Introduction

Some historical background

The paper [12] of B.J. Green introduces several tools and one of them is a *restriction estimate*, by which we mean here the following inequality, valid for any sequence $(u_p)_{p\leqslant N}$ of complex numbers and any $\ell > 2$,

$$\left(\int_0^1 \left|\sum_{p\leqslant N} u_p e(p\alpha)\right|^\ell d\alpha\right)^{1/\ell} \ll_\ell N^{-1/\ell}\left(\frac{N}{\log N}\sum_{p\leqslant N}|u_p|^2\right)^{1/2}. \tag{1}$$ {iniBJG}

Though the usage of this inequality in number theoretical context was new, such an inequality had been already proved by J. Bourgain in [4, Proof of Theorem 3] by using very specific information on the primes. B.J. Green & T. Tao found in [10] another proof that reduces to sieve properties, enabling a wide generalization of this inequality. We strengthened further the proof in [23].

The heart of the matter

Theorem $\mathscr{A}$

Let $\mathcal{X}$ be a δ-well spaced subset of $\mathbb{R}/\mathbb{Z}$ and $N \geqslant 10^{11}$. Let $(u_p)_{p\leqslant N}$ be a sequence of complex numbers. We have

$$\sum_{x\in\mathcal{X}}\left|\sum_{p\leqslant N} u_p e(xp)\right|^2 \leqslant 10^6\,\frac{N+\delta^{-1}}{\log N}\log(2|\mathcal{X}|)\sum_{p\leqslant N}|u_p|^2.$$

Notice that in most applications, δ^{-1} is smaller than N. B.J. Green & T. Tao's result relate to a similar inequality though with a larger dependence in $|\mathcal{X}|$ that the $\log(2|\mathcal{X}|)$ we have here*.

Origin

This theorem follows by using an *enveloping sieve*, and our novelty with respect to [25] is to incorporate a preliminary *unsieving* into this sieving process. We shall spend some time to describe properly this enveloping sieve.

In some sense, *sieving*, and this is all the more true in the context of the large sieve, relies on describing a sequence through congruence properties. As a

[12] B. Green, 2005, "Roth's theorem in the primes".
[4] J. Bourgain, 1989, "On $\Lambda(p)$-subsets of squares".
[10] B. Green and T. Tao, 2006, "Restriction theory of the Selberg sieve, with applications".
[23] O. Ramaré, 2021, "Notes on restriction theory in the primes".

*In case $\mathcal{X} = \{a/q, (a,q) = 1, 1 \leqslant a \leqslant q \leqslant Q_0\}$, this inequality can be found for instance in [27, Theorem 5] (its ancestor being around [25, Eq. (7.10)]), a better version is given in [21, Theorem 5.3].

consequence, properties of arithmetical sequences may well shared by sequences properly described by sieves. The terminology *transference* refers to this idea; In these lectures, we inspect how the enveloping sieve may provides with the technology to prove such an inheritance.

Analytical Usage

Theorem $\mathscr{B}$

Assume $N \geqslant 10^{11}$ and let $\ell \geqslant 2$. We have

$$\left(\int_0^1 \left|\sum_{p\leqslant N} u_p e(p\alpha)\right|^\ell d\alpha\right)^{1/\ell} \leqslant 10^8 \left(\int_0^1 \left|\sum_{p\leqslant N} e(p\alpha)\right|^\ell d\alpha\right)^{1/\ell}$$

as soon as $\sum_{p\leqslant N} |u_p|^2 \leqslant \sum_{p\leqslant N} 1$.

The value $\ell = 2$ is singular: Inequality (1) with $\ell = 2$ does *not* hold, for instance when $u_p = 1$; the inequality of the theorem however *does* hold, as noticed by B.J. Green in [12]. Before the work of B.J. Green, it was customary in prime number theory to restrict our attention to the case $\ell = 2$, while Green used $\ell = 5/2$. Theorem $\mathscr{B}$ shows that one may vary boundedly from the case $\ell = 2$ to the case $\ell > 2$. Furthermore, and as the proof will disclose, this result is a direct consequence of the L^2-theorem $\mathscr{A}$. There is however a maximal property hidden in this theorem: the possibility to choose freely the set of points $\mathcal{X}$. See for instance Corollary 19. This gives us access to maximal estimates and to distributional properties that shall enable us to control the L^ℓ-norm.

An auxiliary problem is to get a better constant than the 10^8 above, and to guess what should be the optimal one.

Arithmetical Usage

Let us now change of tack and present arithmetical problems that will help us understand the situation. We denote by $\mathcal{P}$ the set of primes. We shall prove three theorems. The first one is to illustrate the link between the envelopping sieve and the large sieve inequality.

Theorem $\mathscr{C}$

{thmC} For every $X_0 \geqslant 2$, there exists a $c > 0$ such that the following holds. For any $\delta \in (0, 1/2]$ and any subset $\mathcal{P}_1 \subset \mathcal{P}$ such that $|\mathcal{P}_1 \cap [1, X]| \geqslant \delta X/\log X$

[12] B. Green, 2005, "Roth's theorem in the primes".

when $X \geqslant X_0$, we have, for every $n \geqslant N_0(\delta)$,

$$\sum_{\substack{p_1+p_2+p_3=n,\\ p_1,p_2,p_3\in\mathcal{P}_1}} 1 \leqslant c\delta^2 \log(1/\delta) \frac{n^2}{(\log n)^3}.$$

In the next theorem, we put stricter conditions on one variable and reach an inequality in a very simple proof.

Theorem $\mathscr{D}$

Let N be an odd integer. Let $K \geqslant [2, \log\log N]$ be a parameter. We set $M(K) = \prod_{p\leqslant K} p$. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two sets of primes. We have

$$\sum_{\substack{p+p_2+p_3=N,\\ p_1\in\mathcal{P}_1,p_2\in\mathcal{P}_2}} 1 = \prod_{p\leqslant K}\left(1-\frac{1}{p}\right) \sum_{\substack{(N-(p_2+p_3),M(K))=1,\\ p_1\in\mathcal{P}_1,p_2\in\mathcal{P}_2}} 1 + \mathcal{O}\left(\frac{\log K}{K^{1/4}}\frac{N^2}{(\log N)^3}\right).$$

It is possible to keep also the prime p in an arbitrary sequence of density.

Theorem $\mathscr{E}$

We can prove Vinogradov three primes theorem without using L-functions. {thmE}

We recall that historically, X. Shao gave in [33] a proof that every large enough odd integer is a sum of three primes, without using L-functions, though the distribution of primes in a finite number of arithmetic progressions is used. The above proof is simpler though using the same basis. It has the advantage of yielding the proper asymptotics for the number of representations.

The reader may consult the books [1] and [18]. Some other excellent books on our topics : [37], [7], [3] and [9].

[1] T. Apostol, 1976, *Introduction to analytic number theory.*
[18] H. Montgomery and R. Vaughan, 2006, *Multiplicative Number Theory: I. Classical Theory.*
[37] I. M. Vinogradov, 1954, *Elements of number theory.*
[7] H. Davenport, 2000, *Multiplicative Number Theory.*
[3] O. Bordellès, 2012, *Arithmetic Tales.*
[9] W. Ellison, 1975, *Les nombres premiers.*

[33] X. Shao, 2014, "An L-function-free proof of Vinogradov's three primes theorem".

1 On the Selberg Sieve for Primes

1.1. A lightning introduction

Let us consider the primes between z and X, where z is some parameter $\leqslant \sqrt{X}$ at our disposal. Given *any* real sequence $(\lambda_d)_{d\leqslant z}$ with $\lambda_1 = 1$, the following upper bound holds true:

$$1_{z<\mathscr{P}\leqslant X}(n) \leqslant \beta(n) = \Bigl(\sum_{d|n}\lambda_d\Bigr)^2 \tag{1.1}$$ {baseSelberg}

where the readers will have understood that $1_{z<\mathscr{P}\leqslant X}$ stands for the characteristic function of the primes from the interval $(z, X]$. Indeed, when n is a prime in $(z, X]$, both left and right-hand side take the value 1, while otherwise, the left-hand side vanishes while the right-hand side is non-negative. The square is here for this only purpose! Once this fundamental remark is made (it is due to A. Selberg in [32]), we may turn to summatory functions and write

$$S = \sum_{z<p\leqslant X} 1 \leqslant \sum_{n\leqslant X}\beta(n) = S^*.$$

We develop the square, interchange the summations and approximate the inner sum to infer that

$$S^* = \sum_{d_1,d_2\leqslant z}\lambda_{d_1}\lambda_{d_2}\Bigl(\frac{X}{[d_1,d_2]} + \mathcal{O}^*(1)\Bigr) = XS_0^* + \mathcal{O}^*\Bigl(\sum_d |\lambda_d|\Bigr)^2$$

where $S_0^* = \sum_{d_1,d_2\leqslant z}\lambda_{d_1}\lambda_{d_2}/[d_1,d_2]$ and $[d_1,d_2]$ is the lcm of d_1 and d_2. Since the λ_d's are at our disposal (save for λ_1), we now seek to minimize this quadratic form and keep the paramater z to control the "error term", or more precisely, to turn this second term into an error term. The way to handle this main term is famous and an important tool even outside sieve theory. Here is how it goes. We first introduce the gcd to replace the lcm, since we kow that gcd's are small on average, getting

$$S_0^* = \sum_{d_1,d_2\leqslant z}\frac{\lambda_{d_1}}{d_1}\frac{\lambda_{d_2}}{d_2}(d_1,d_2).$$

The variables d_1 and d_2 are still linked by the factor (d_1,d_2). We write

$$(d_1,d_2) = (1\star\varphi)((d_1,d_2)) = \sum_{\delta|(d_1,d_2)}\varphi(\delta) = \sum_{\substack{\delta|d_1,\\ \delta|d_2}}\varphi(\delta)$$

[32] A. Selberg, 1947, "On an elementary method in the theory of primes".

where d_1 and d_2 are now separated! On using this decomposition, we reach

$$S_0^* = \sum_{\delta \leqslant z} \varphi(\delta) \biggl(\sum_{\delta | d} \frac{\lambda_d}{d}\biggr)^2 = \sum_{\delta \leqslant z} \varphi(\delta) y_\delta^2 \quad \text{where} \quad y_\delta = \sum_{\delta | d} \frac{\lambda_d}{d}.$$

We may revert from the variables y_δ to the variables λ_d by the formula

$$\frac{\lambda_\ell}{\ell} = \sum_{\ell | \delta} \mu(\delta/\ell) y_\delta. \tag{1.2}$$

Proof. Indeed, the right-hand side reads

$$\sum_{\ell | \delta} \mu(\delta/\ell) y_\delta = \sum_{\ell | d} \frac{\lambda_d}{d} \sum_{\ell | \delta | d} \mu(\delta/\ell)$$

and the last sum vanishes when $\ell \neq d$ and takes the value 1 otherwise, as wanted. □

This inversion formula implies in particular that

{condSelberg}
$$1 = \lambda_1 = \sum_{\delta \leqslant z} \mu(\delta) y_\delta. \tag{1.3}$$

We are left with the problem of minimizing the quadratic form $\sum_{\delta \leqslant z} \varphi(\delta) y_\delta^2$ under the linear condition (1.3). There are several way to proceed. Let us follow Y. Motohashi and write

$$1 = \biggl(\sum_{\delta \leqslant z} \frac{\mu(\delta)}{\sqrt{\varphi(\delta)}} \sqrt{\varphi(\delta)} y_\delta\biggr)^2 \leqslant \sum_{\delta \leqslant z} \frac{\mu^2(\delta)}{\varphi(\delta)} S_0^*$$

with equality if (and only if)

$$y_\delta = C \frac{\mu(\delta)}{\varphi(\delta)}$$

for some constant C that is chosen to satisfy (1.3): we take $C = 1/G(z)$ with $G(z) = \sum_{\delta \leqslant z} \mu^2(\delta)/\varphi(\delta)$.

Squarefree condition and summary

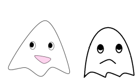
The above analysis shows that we may take $\lambda_d = 0$ when d is not squarefree, a condition we now assume. We define

{defGdz}
$$G_d(y) = \sum_{\substack{\ell \leqslant y, \\ (\ell, d) = 1}} \frac{\mu^2(\ell)}{\varphi(\ell)}, \quad G(y) = G_1(y). \tag{1.4}$$

The above leads to

{eq:18}
$$\lambda_d = \mu(d) \frac{\frac{d}{\varphi(d)} G_d(z/d)}{G(z)}. \tag{1.5}$$

Here are three lemmas that helps us clear the situation.



{vanLR} **Lemma 1**. We have $G(z/q) \leqslant \dfrac{q}{\varphi(q)} G_q(z/q) \leqslant G(z)$.

This comes from [17] by J. van Lint and H.E. Richert. As a consequence, we find that $|\lambda_d| \leqslant 1$.

Lemma 2. We have $\log z \leqslant G(z) \leqslant 1.4709 + \log z$. {vanLR-2}

The lower bound is classical and can for instance be found in the book [2] of E. Bombieri. The upper bound is a tad more difficult and can be found in [25, Lemma 3.5] (see [20] and [30] for more precise expansions).

Lemma 3

Let $z > 1$ be a real number. We have

$$\sum_{d \leqslant z} |\lambda_d| \leqslant \frac{z}{\log z}\left(\frac{15}{\pi^2} + \frac{30}{\sqrt{z}}\right).$$

See [28, Lemma 4.2]. This is more precise than the obvious $\sum_{d \leqslant z} |\lambda_d| \leqslant z$. Let us assume these results and resume our main line of enquiry. We get, when $z \geqslant 340$,

$$S \leqslant X S_0^* + \left(\sum_{d \leqslant z} |\lambda_d|\right)^2 \leqslant \frac{X}{\log z} + \frac{10 z^2}{(\log z)^2}. \tag{1.6}$$ {eq:19}

We select $z = \sqrt{X}$ and get

$$S \leqslant \frac{2(1 + o(1))X}{\log X}. \tag{1.7}$$ {bound2}

Such a bound is of course known and even better is available in this special case. Notice however the extraordinary flexiblity of the process!

1.2. Enveloping sieve and transference philosophy

[17] J. van Lint and H. Richert, 1965, "On primes in arithmetic progressions".
[2] E. Bombieri, 1987/1974, *Le grand crible dans la théorie analytique des nombres.*
[25] O. Ramaré, 1995, "On Snirel'man's constant".
[20] A. P. and O. Ramaré, 2017, "Explicit averages of non-negative multiplicative functions: going beyond the main term".
[30] O. Ramaré, 2019, "Explicit average orders: news and problems".
[28] O. Ramaré and P. Srivastav, 2020, "Products of primes in arithmetic progressions".

The first remark is that the system of weights $\beta(n)$ defined in (1.1) does not only give an upper bound for the summatory function S but provides us also with a point-wise upper bound. We may in fact consider $(\beta(n))$ as a full-fledged (though weighted) sequence. And looking more closely at (1.7), we see that this sequence is *twice* larger than the sequence of primes. In fact, it is $(\log X)/G(z)$-larger, and we may select z to be smaller. At the price of loosing only a constant, we may thus replace the sequence of primes by a sequence that is very flexible. This is the *enveloping sieve* part.

On looking more closely at what our sieve does and more specifically, at the shape of $\beta(n)$, we see that we aim at replacing primes by the solutions of a system of congruences, which is to say by an arithmetic progression. This would be true if the moduli of the implied congruences were fixed, but this is not quite true. However the *transference philosophy* says that the properties of the integers may be shared by or *transferred to* the sequence of primes.

The books [2] and [13] are two essential references.

[2] E. Bombieri, 1987/1974, *Le grand crible dans la théorie analytique des nombres*.
[13] H. Halberstam and H.-E. Richert, 1974, *Sieve methods*.

2 An Enveloping Sieve

We define $P(z_0) = \prod_{p<z_0} p$ and we generalize the definition (1.4) by

$$G_d(y;z_0) = \sum_{\substack{\ell\leqslant y,\\ (\ell,dP(z_0))=1}} \frac{\mu^2(\ell)}{\varphi(\ell)}, \quad G(y;z_0) = G_1(y;z_0). \tag{2.1}$$ {defGdyz0}

2.1. Some averages of multiplicative functions

{AMF}

Here is a version of the celebrated Levin-Faǐnleib Theorem from [16] that is [21, Theorem 21.1].

Theorem 4 {LFT}

Let g be a non-negative multiplicative function. Let κ, L and A be three non-negative real parameters such that

$$\begin{cases} \displaystyle\sum_{\substack{p\geqslant 2,\nu\geqslant 1\\ p^\nu\leqslant Q}} g(p^\nu)\log\big(p^\nu\big) = \kappa\log Q + \mathcal{O}^*(L) \qquad (Q\geqslant 1),\\ \displaystyle\sum_{p\geqslant 2}\sum_{\nu,k\geqslant 1} g\big(p^k\big)g\big(p^\nu\big)\log\big(p^\nu\big)\leqslant A\,. \end{cases}$$

Then, when $D \geqslant \exp(2(L+A))$, we have

$$\sum_{d\leqslant D} g(d) = C\left(\log D\right)^\kappa\left(1+\mathcal{O}^*(B/\log D)\right)\,,$$

with

$$\begin{cases} C = \dfrac{1}{\Gamma(\kappa+1)}\displaystyle\prod_{p\geqslant 2}\left\{\left(1+\sum_{\nu\geqslant 1} g\big(p^\nu\big)\right)\left(1-\frac{1}{p}\right)^\kappa\right\}\,,\\ B = 2(L+A)\big(1+2(\kappa+1)e^{\kappa+1}\big)\,. \end{cases}$$

Lemma 5. When $z_0 < z$, we have $G(z;z_0) = \dfrac{e^{-\gamma}\log z}{\log z_0}\big(1+\mathcal{O}(\log z_0/\log z)\big)$. {EstGdownLF}

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[16] B. Levin and A. Fainleib, 1967, "Application of some integral equations to problems of number theory".
[21] O. Ramaré, 2009, *Arithmetical aspects of the large sieve inequality*.

This conclusion can also be reached via [13, Lemma 5.4] by H. Halberstam and H.-E. Richert. Beware the change of notation between this monograph and these notes. Lemma 5 together with the lower bound $G(z;z_0) \geqslant 1$ is enough to ensure that $G(z;z_0) \gg (\log z)/\log z_0$.

Proof. Let us use Theorem 4 with $g(n) = \mu^2(n)1_{(n,P(z_0))=1}/\varphi(n)$ and $\kappa = 1$. As

$$\sum_{z_0 \leqslant p \leqslant Q} \frac{\log p}{p-1} = \log Q + \mathcal{O}(\log z_0)$$

we may select $L = c \log z_0$ for some large enough constant c such that $A \leqslant c$. □

2.2. An enveloping sieve

We fix two real parameters $z_0 \leqslant z$ and first consider the sole case of prime numbers. It is easy to reproduce the analysis of [27, Section 3] as far as exact formulae are concerned, but one gets easily sidetracked towards slightly different formulae. The reader may for instance compare [25, Lemma 4.2] and [27, (4.1.14)]. Similar material is also the topic of [21, Chapter 12]. So we present a path leading to [27, (4.1.14)] in our special case.

Following the notation of Section 2.1, we set

{defbetan}
$$\beta_{z_0,z}(n) = \Big(\sum_{d|n} \lambda_d\Big)^2, \qquad \lambda_d = 1_{(d,P(z_0))=1} \frac{\mu(d) d G_d(z/d;z_0)}{\varphi(d) G(z;z_0)}. \tag{2.2}$$

Section 3 of [27] corresponds to $z_0 = 1$.

{Fourierbetan}

Theorem 6

The coefficients $\beta_{z_0,z}(n)$ admits the expansion

$$\beta_{z_0,z}(n) = \sum_{\substack{q \leqslant z^2, \\ q | P(z)/P(z_0)}} w_q(z;z_0) c_q(n)$$

[13] H. Halberstam and H.-E. Richert, 1974, *Sieve methods*.
[27] O. Ramaré and I. Ruzsa, 2001, “Additive properties of dense subsets of sifted sequences”.
[25] O. Ramaré, 1995, “On Snirel’man’s constant”.
[21] O. Ramaré, 2009, *Arithmetical aspects of the large sieve inequality*.

where $c_q(n)$ is the Ramanujan sum and where

$$G(z;z_0)^2 w_q(z;z_0) = \frac{\mu(q)}{\varphi(q)} \sum_{\substack{\ell\leqslant z/\sqrt{q},\\ (\ell,qP(z_0))=1}} \frac{\mu^2(\ell)}{\varphi(\ell)} \xi_q(z/\ell)$$

with the definitions

$$\xi_q(y) = \sum_{\substack{q_1q_2q_3=q,\\ q_1q_3\leqslant y,\\ q_2q_3\leqslant y}} \frac{\mu(q_3)\varphi_2(q_3)}{\varphi(q_3)} \quad \text{and} \quad \varphi_2(q_3) = \prod_{p|q}(p-2).$$

It is worth mentioning that, when developing the theory of *local models*, we show the much neater expression:

$$\alpha(n) = \sum_{d|n} \lambda_d = \sum_{\substack{d\leqslant z,\\ (d,P(z_0))=1}} \frac{\mu(d)}{G(z;z_0)\varphi(d)} c_q(n).$$

In the theory of local models, we realize $\alpha(n)$ as being (close to) the best approximation of the characteristic function of the primes, while in the theory of the Selberg sieve, we also ask for a pointwise upper bound.

Proof. We develop the square above and get

$$\begin{aligned}\beta_{z_0,z}(n) &= \sum_{d_1,d_2} \lambda_{d_1}\lambda_{d_2} 1_{[d_1,d_2]|n} = \sum_{d_1,d_2} \frac{\lambda_{d_1}\lambda_{d_2}}{[d_1,d_2]} \sum_{q|[d_1,d_2} \sum_{a \bmod^* q} e(na/q) \\ &= \sum_{\substack{q\leqslant z^2,\\ (q,P(z_0))=1}} w_q(z;z_0) c_q(n)\end{aligned}$$

where

$$w_q(z;z_0) = \sum_{q|[d_1,d_2]} \frac{\lambda_{d_1}\lambda_{d_2}}{[d_1,d_2]}. \tag{2.3}$$

We introduce the definition the λ_d's, see (2.2), and obtain

$$G(z;z_0)^2 w_q(z;z_0) = \sum_{\substack{\ell_1,\ell_2\leqslant z,\\ (\ell_1\ell_2,P(z_0))=1}} \frac{\mu^2(\ell_1)}{\varphi(\ell_1)} \frac{\mu^2(\ell_2)}{\varphi(\ell_2)} \sum_{\substack{q|[d_1,d_2],\\ d_1|\ell_1,d_2|\ell_2}} \frac{d_1\mu(d_1)d_2\mu(d_2)}{[d_1,d_2]}.$$

The inner sum vanishes is ℓ_1 has a prime factor prime to $q\ell_2$, and similarly for ℓ_2. Furthermore, we need to have $q|[\ell_1,\ell]$ for the inner sum not be empty. Whence we may write $\ell_1 = q_1q_3\ell$ and $\ell_2 = q_2q_3\ell$ where $(\ell,q)=1$ and $q = q_1q_2q_3$. The

part of the inner sum corresponding to ℓ has value $\prod_{p|\ell}(p-2+1)=\varphi(\ell)$. We have reached

$$G(z;z_0)^2 w_q(z;z_0) = \sum_{\substack{\ell\leqslant z,\\ (\ell,qP(z_0))=1}} \frac{\mu^2(\ell)}{\varphi(\ell)} \sum_{\substack{q_1q_2q_3=q,\\ q_1q_3\ell\leqslant z,\\ q_2q_3\ell\leqslant z}} \frac{1}{\varphi(q)\varphi(q_3)} \sum_{\substack{q|[d_1,d_2],\\ d_1|q_1q_3,\\ d_2|q_2q_3}} \frac{d_1\mu(d_1)d_2\mu(d_2)}{[d_1,d_2]}.$$

In this last inner sum, we have necessarily $d_1=q_1d_1'$ and $d_2=q_2d_2'$, so $q_3=[d_1',d_2']$. Here is the expression we have obtained

$$G(z;z_0)^2 w_q(z;z_0) = \sum_{\substack{\ell\leqslant z,\\ (\ell,qP(z_0))=1}} \frac{\mu^2(\ell)}{\varphi(\ell)} \sum_{\substack{q_1q_2q_3=q,\\ q_1q_3\ell\leqslant z,\\ q_2q_3\ell\leqslant z}} \frac{\mu(q)\mu(q_3)}{\varphi(q)\varphi(q_3)} \sum_{q_3=[d_1',d_2']} \frac{d_1'\mu(d_1')d_2'\mu(d_2')}{[d_1',d_2']}.$$

This last inner sum has value $\varphi_2(q_3)$, whence

$$G(z;z_0)^2 w_q(z;z_0) = \frac{\mu(q)}{\varphi(q)} \sum_{\substack{\ell\leqslant z,\\ (\ell,qP(z_0))=1}} \frac{\mu^2(\ell)}{\varphi(\ell)} \sum_{\substack{q_1q_2q_3=q,\\ q_1q_3\ell\leqslant z,\\ q_2q_3\ell\leqslant z}} \frac{\mu(q_3)\varphi_2(q_3)}{\varphi(q_3)}$$

as announced. The size conditions are readily seen to imply that $\ell\leqslant z/\sqrt{q}$. □

As these coefficients w_d are of extreme importance, it is a good idea to identify them geometrically. As can be guessed from Theorem 6 and up to some renormalisation, they are simply the Fourier coefficients of the (weighted) sequence $\beta(n)$, as shown in the next lemma.

{geomwd} **Lemma 7**

We have, for any a coprime with d,

$$G(z)w_d(z,z_0) = \lim_{X\to\infty}\frac{G(z)}{X}\sum_{n\leqslant X}\beta(n)e(na/d).$$

{Courageous} **Lemma 8**

When $2\leqslant z_0\leqslant z$ and $z\geqslant 320$, we have

$$\left|\frac{w_q(z;z_0)}{\varphi(q)}\right| \leqslant 11500\,\frac{\log z_0}{\sqrt{q}\log z}.$$

Proof. We deduce from the definition the estimate $|\xi_q(y)|\leqslant 3^{\omega(q)}$, and thus

{eq:7} $$|G(z;z_0)w_q(z;z_0)|\leqslant 3^{\omega(q)}/\varphi(q). \tag{2.4}$$

We use Lemma 12 to get

$$|G(z; z_0)w_q(z; z_0)| \leqslant \prod_p \max\left(\frac{3\sqrt{p}}{p-1}, 1\right)\frac{1}{G(z; z_0)\sqrt{q}} \leqslant \frac{11500 \log z_0}{\sqrt{q}\log z}.$$

□

2.3. Explicit estimates for $G(z; z_0)$

In this section, we investigate explicit lower estimates for $G(z; z_0)$.

We first bound it from below via some steps based of Rankin's trick. Let us start with some estimates due to J.B. Rosser & L. Schoenfeld in [31, Theorem 1, Corollary 2, Theorem 6-8].

Lemma 9 {RS}

We have

$$\sum_{p\leqslant x}\frac{\log p}{p} < \log x + E + \frac{1}{2\log x} \quad \text{when } x \geqslant 319,$$

$$\sum_{p\leqslant x}\frac{\log p}{p} > \log x + E - \frac{1}{2\log x} \quad \text{when } x > 1,$$

where $E = -1.33258\,22757\,33220\,87\cdots$. Also

$$\prod_{p\leqslant x}\frac{p}{p-1} \geqslant e^{\gamma}(\log x)\left(1 - \frac{1}{2\log^2 x}\right) \quad \text{when } x > 1,$$

$$\prod_{p\leqslant x}\frac{p-1}{p} > \frac{e^{-\gamma}}{\log x}\left(1 - \frac{1}{2\log^2 x}\right) \quad \text{when } x \geqslant 285,$$

and

$$\prod_{p\leqslant x}\frac{p}{p-1} \leqslant e^{\gamma}(\log x)\left(1 + \frac{1}{2\log^2 x}\right) \quad \text{when } x \geqslant 286,$$

$$\prod_{p\leqslant x}\frac{p-1}{p} \leqslant \frac{e^{-\gamma}}{\log x}\left(1 - \frac{1}{2\log^2 x}\right) \quad \text{when } x > 1.$$

Futhermore $\pi(x) = \sum_{p\leqslant x} 1 \leqslant \frac{x}{\log x}(1 + \frac{3}{2\log x})$ and $\pi(x) \leqslant \frac{5x}{4\log x}$, both valid when $x \geqslant 114$. Finally, $\pi(x) \geqslant x/(\log x)$ when $x \geqslant 17$.

[31] J. Rosser and L. Schoenfeld, 1962, "Approximate formulas for some functions of prime numbers".

We next recall [6, Lemma 4] by H. Daboussi & J. Rivat, which is inspired by [8, around page 81] by P.D.T.A. Elliott.

Lemma 10

{DR} Let $z \geqslant 2$ and f be a non-negative multiplicative function. Set

$$S = \sum_{p<z} \frac{f(p)}{1+f(p)} \log p.$$

We assume that $\log y \geqslant S > 0$. Then

$$\sum_{\substack{n>y,\\ k(n)|P(z)}} \mu^2(n) f(n) \leqslant \prod_{p \leqslant z} (1+f(p)) \exp\left\{-\frac{\log y}{\log z} K\left(\frac{\log y}{S}\right)\right\},$$

$$\sum_{\substack{n\leqslant y,\\ k(n)|P(z)}} \mu^2(n) f(n) \geqslant \prod_{p \leqslant z} (1+f(p)) \left(1-\exp\left\{-\frac{\log y}{\log z} K\left(\frac{\log y}{S}\right)\right\}\right)$$

where $k(n) = \prod_{p|n} p$ and $K(t) = \log t - 1 + 1/t$. When $\log y \geqslant 7S$, we have $K(t) \geqslant 1$.

Proof. Let $\eta \geqslant 0$ be some real parameter to choose. We start with

$$\sum_{\substack{d\geqslant y,\\ d|P(z)}} \mu^2(d) f(d) \leqslant \sum_{\substack{d\geqslant y,\\ d|P(z)}} \mu^2(d) f(d) \left(\frac{d}{y}\right)^{\eta} \leqslant y^{-\eta} \prod_{p<z} \bigl(1+f(p)p^{\eta}\bigr)\ .$$

We next use the following identity

$$1+f(p)p^{\eta} = \bigl(1+f(p)\bigr)\left(1+\frac{f(p)}{1+f(p)}(p^{\eta}-1)\right)\ .$$

We further notice that the function $(e^x-1)/x$ is non-decreasing when $x \geqslant 0$ (its series expansion has only non-negative coefficients), so that, when p is not more than z,

$$p^{\eta} - 1 \leqslant \log p \frac{z^{\eta}-1}{\log z}\ .$$

[6] H. Daboussi and J. Rivat, 2001, “Explicit upper bounds for exponential sums over primes”.
[8] P. D. T. A. Elliott, 1979, *Probabilistic number theory. I.*.

Therefore

$$\sum_{\substack{d\geqslant y,\\ d|P(z)}} \mu^2(d)f(d) \leqslant y^{-\eta}\prod_{p<z}(1+f(p))\exp\sum_{p<z}\frac{f(p)\log p}{1+f(p)}\frac{z^\eta-1}{\log z}$$

$$\leqslant \prod_{p<z}(1+f(p))\exp\biggl(\frac{(z^\eta-1)S}{\log z}-\eta\log y\biggr).$$

We change parameter and set $\nu = \eta\log z$. The argument of the exponential reads

$$\frac{S}{\log z}\biggl(e^\nu-1-\nu\frac{\log y}{S}\biggr).$$

The parameter ν can be chosen as we want provided it remains non-negative. We select

$$\nu = \log\frac{\log y}{S}$$

and this gives our result. We next check that the function K is non-decreasing. The last Pari/GP script explains the value 7.

```
K(t) = log(t) - 1 + 1/t;
solve( t = 2, 7, K(t)-1)
```

The proof of the lemma is complete. □

Lemma 11 {PR}

When $z\geqslant 1$, we have $\displaystyle\sum_{d\leqslant z}\frac{\mu^2(d)}{\varphi(d)} = \log z + c_0 + \mathcal{O}^*(3.95/\sqrt{z})$, the constant c_0 being given by $c_0 = \gamma + \sum_{p\geqslant 2}\frac{\log p}{p(p-1)}$. Also $\sum_{d\leqslant z}\frac{\mu^2(d)}{\varphi(d)} \leqslant \log z + 1.4709$.

Proof. The first estimate is taken from [20, Theorem 1.2] while the second one is [25, Lemma 3.5, (1)] □

Lemma 12 {EstGdown}

When $320\leqslant z_0\leqslant z$, we have $G(z;z_0)\geqslant \dfrac{3}{40}\dfrac{\log z}{\log z_0}$.

When $2\leqslant z_0\leqslant z$ and $z\geqslant 320$, we have $G(z;z_0)\geqslant \dfrac{1}{111}\dfrac{\log z}{\log z_0}$.

[20] A. P. and O. Ramaré, 2017, "Explicit averages of non-negative multiplicative functions: going beyond the main term".
[25] O. Ramaré, 1995, "On Snirel'man's constant".

Proof. We define the auxiliary function

$$\tilde{G}(y, z; z_0) = \sum_{\substack{d \leqslant y, \\ d | P(z)/P(z_0)}} \frac{\mu^2(d)}{\varphi(d)}.$$

We readily find that

$$\tilde{G}(\infty, z; z_0) = \prod_{z_0 \leqslant p < z} \left(1 - \frac{1}{p}\right)^{-1}.$$

By Lemma 9, we obtain a lower estimate for this product:

$$\tilde{G}(\infty, z; z_0) \geqslant \frac{\log z}{\log z_0} \left(1 - \frac{1}{2(\log z_0)^2}\right)^2 \geqslant 0.97 \frac{\log z}{\log z_0}.$$

In Lemma 10 with $f(n) = \mu^2(n) 1_{k(n) \geqslant z_0} / \varphi(n)$, we find that

$$S = \sum_{z_0 \leqslant p < z} \frac{\log p}{p} = \log \frac{z}{z_0} + \mathcal{O}^*\left(\frac{2}{\log z_0}\right)$$

where we appealed to Lemma 9 for the approximate estimate. Notice that $S \leqslant \log z$ since $-\log z_0 + 2/\log z_0 \leqslant 0$. We thus get

$$\tilde{G}(y, z; z_0) \geqslant \tilde{G}(\infty, z; z_0)\left(1 - \exp\left(-\frac{\log z}{\log y} K(3)\right)\right)$$

as soon as $\log y \geqslant 3 \log z$ since then, $K((\log y)/S) \geqslant K(2) = 0.4319\cdots$, and thus, in particular,

$$\tilde{G}(y, z; z_0) \geqslant \tilde{G}(\infty, z; z_0)(1 - e^{-0.4319}) \geqslant 0.34 \frac{\log z}{\log z_0}.$$

We complete this estimate with

$$\tilde{G}(z^3, z; z_0) - G(z; z_0) = \sum_{\substack{z < d \leqslant z^3, \\ z_0 \leqslant k(n) < z}} \frac{\mu^2(d)}{\varphi(d)} \leqslant \sum_{z < d \leqslant z^3} \frac{\mu^2(d)}{\varphi(d)} \leqslant 3 + \frac{8}{\sqrt{z}} \leqslant 3.45$$

by Lemma 11 and since $\tilde{G}(z, z; z_0) = G(z; z_0)$. Whence

$$G(z; z_0) \geqslant 0.34 \frac{\log z}{\log z_0} - 3.45.$$

Since also $G(z; z_0) \geqslant 1$ (by considering the contribution of $d = 1$), we find that

$$G(z; z_0) \geqslant \min_{c \geqslant 1} \max\left(0.34 - \frac{3.45}{c}, \frac{1}{c}\right) \frac{\log z}{\log z_0} \geqslant \frac{3}{40} \frac{\log z}{\log z_0}.$$

This yields the first estimate of the lemma. The second one is simply infered from this through

$$G(z;z_0) \geqslant G(z;320) \geqslant \frac{3\log 2}{40\log 320}\frac{\log z}{\log z_0}.$$

□

The proof of Lemma 12 may look unnatural. It is however efficient even when z_0 is a power of z. We work out the details of an approach via the Convolution Method in this section, getting more precise estimates, but in a narrower range. This more narrow range is nonetheless the one that is most interesting.

Lemma 13 {AR}

When $C \geqslant 1$, we have

$$\sum_{\substack{c\leqslant C,\\(c,2)=1}} \frac{\mu^2(c)c}{\varphi(c)} \leqslant C$$

and

$$\sum_{\substack{c> C,\\(c,2)=1}} \frac{\mu^2(c)}{c\varphi(c)} \leqslant \frac{0.592}{C}.$$

Proof. The second estimate comes from [20, Lemma 7.4]. The first one is proved in [20, Lemma 5.2] when $C \geqslant 50$ (with constant 1.066/2 rather than 1, and readily extended to $C \geqslant 1$. □

Lemma 14 {yo}

For any positive real number K and any modulus q^*, we have

$$\Bigg|\sum_{\substack{k>K,\\(k,q^*)=1}} \frac{\mu(k)}{\varphi(k)k}\Bigg| \leqslant \frac{1.26}{K}.$$

This is [20, Lemma 5.7].

Lemma 15 {EstGdownE}

[20] A. P. and O. Ramaré, 2017, “Explicit averages of non-negative multiplicative functions: going beyond the main term”.

When $7 < z_0 \leqslant z$, we have

$$G(z;z_0) = e^{-\gamma}\frac{\log z}{\log z_0} + \mathcal{O}\big((\log z_0)^2\big).$$

Proof. We shall proceed by the Convolution Method. Define

$$\begin{aligned}
&\forall p < z_0, \quad f(p) = \frac{-1}{p},\ f(p^k) = 0 \text{ when } k \geqslant 2,\\
&\forall p \geqslant z_0, \quad f(p) = \frac{1}{p(p-1)},\ f(p^2) = \frac{-1}{p(p-1)},\ f(p^k) = 0 \text{ when } k \geqslant 3,
\end{aligned}$$

so that

{eq:20} $$\frac{\mu^2(n)}{\varphi(n)}1_{(n,P(z_0))=1} = \sum_{\ell m = n}\frac{f(\ell)}{m}. \tag{2.5}$$

This identity is readily checked either by computing the relevant Dirichlet series, or by noticing that both left-hand side and right-hand side being multiplicative functions, it is enough to check the claimed identity on prime powers; it is straightforward to do so. The Dirichlet series of f will be required:

{eq:23} $$D(f,s) = \prod_{p<z_0}\left(1-\frac{1}{p^{s+1}}\right)\prod_{p\geqslant z_0}\left(1+\frac{1}{(p-1)p^{s+1}}-\frac{1}{(p-1)p^{2s+1}}\right). \tag{2.6}$$

Since, for every real number $M \geqslant 1$, we have

{eq:22} $$\sum_{m\leqslant M}\frac{1}{m} = \log M + \gamma + \mathcal{O}^*\left(\frac{7}{12M}\right) \tag{2.7}$$

we find that

$$G(z;z_0) = \sum_{\ell\leqslant z} f(\ell)\left(\log\frac{z}{\ell} + \gamma + \mathcal{O}^*\left(\frac{7\ell}{12z}\right)\right)$$

whence

$$\begin{aligned}
G(z;z_0) &= \sum_{\ell\geqslant 1} f(\ell)\left(\log\frac{z}{\ell}+\gamma\right) + \mathcal{O}^*(E)\\
&= D(f,0)\left(\log z + \gamma + \frac{D'(f,0)}{D(f,0)}\right) + \mathcal{O}^*(E)
\end{aligned}$$

where

{defE} $$E = \left|\sum_{\ell>z} f(\ell)\left(\log\frac{z}{\ell}+\gamma\right)\right| + \frac{7}{12z}\sum_{\ell\leqslant z}\ell|f(\ell)|. \tag{2.8}$$

Main term

We readily find that

$$D(f,0) = \prod_{p<z_0}\left(1-\frac{1}{p}\right).$$

$$\begin{aligned}\frac{D'(f,0)}{D(f,0)} &= \sum_{p<z_0}\frac{\log p}{p-1} + \sum_{p\geqslant z_0}\frac{\log p}{p(p-1)}\\ &= \sum_{p<z_0}\frac{(p-2)\log p}{(p-1)^2} + \sum_{p\geqslant 2}\frac{\log p}{p(p-1)}.\end{aligned} \tag{2.9}$$

Error term

We find that

$$\sum_{\ell\leqslant z}\ell|f(\ell)| = \sum_{\substack{abc^2\leqslant z,\\ a|P(z_0)}}\frac{\mu(abc)^2 c}{\varphi(b)\varphi(c)} \leqslant \sqrt{z}\sum_{\substack{ab\leqslant z,\\ a|P(z_0)}}\frac{\mu(ab)^2}{\sqrt{ab}\varphi(b)}$$

by Lemma 13 and on using $z_0 > 2$. We next use $\sum_{a\leqslant A}\mu^2(a)/\sqrt{a} \leqslant 2\sqrt{A}$, getting the upper bound

$$\sum_{\ell\leqslant z}\ell|f(\ell)| \leqslant 2z\prod_{p>z_0}\left(1+\frac{1}{p(p-1)}\right) \leqslant 2.1\ z$$

on using $z_0 > 7$. We next check that

$$\sum_{\ell>z} f(\ell)\log\frac{\ell}{z} = \int_z^\infty \sum_{\ell>t} f(\ell)\frac{dt}{t}$$

so our task reduces to finding an upper bound for $F(t) = \sum_{\ell>t} f(\ell)$. We readily find that

$$F(t) = \sum_{\substack{abc^2>t,\\ a|P(z_0),\\ (bc,P(z_0))=1,\\ (b,c)=1}}\frac{\mu(a)\mu^2(b)\mu(c)}{ab\varphi(b)c\varphi(c)} = F(t; a>\sqrt{t}) + F(t; a\leqslant\sqrt{t})$$

with an obvious notation. When $a\leqslant\sqrt{t}$, the summation over 1 is bounded in absolute value by 1 and either $b\geqslant t^{1/6}$ or $c\geqslant t^{1/6}$. Therefore

$$|F(t; a\leqslant\sqrt{t})| \leqslant 2\sum_{\substack{b>t^{1/6},\\ (b,P(z_0))=1}}\frac{\mu^2(b)}{b\varphi(b)}\prod_{p>z_0}\left(1+\frac{1}{p(p-1)}\right) \leqslant \frac{1.184}{t^{1/6}}\prod_{p>z_0}\left(1+\frac{1}{p(p-1)}\right) \leqslant \frac{1.3}{t^{1/6}}$$

by Lemma 13. Concerning the second contribution to $F(t)$, we first use Lemma 14 on the summation over c to find that

$$|F(t; a > \sqrt{t})| \leqslant 1.26 \sum_{\substack{a>\sqrt{t},\\ a|P(z_0),\\ (b,P(z_0))=1}} \frac{\mu^2(a)\mu^2(b)}{ab\varphi(b)} \leqslant 1.31 \sum_{\substack{a>\sqrt{t},\\ a|P(z_0)}} \frac{\mu^2(a)}{a}$$

and Lemma 10 now applies with $z = z_0$ and $y = \sqrt{t}$. In the notation of this lemma, $S = \sum_{p<z_0} \frac{\log p}{p+1} \leqslant \log z_0$. Hence, when $\frac{1}{2}\log t \geqslant \log z_0$, we have

$$\sum_{\substack{a>\sqrt{t},\\ a|P(z_0)}} \frac{\mu^2(a)}{a} \leqslant \prod_{p<z_0}\left(1+\frac{1}{p}\right)\exp\left\{-\frac{\log t}{2\log z_0}\log\frac{\log t}{2\log z_0}+\frac{\log t}{2\log z_0}\right\}$$

$\square$

....

3 An upper estimate with primes

In this chapter, we illustrate the use and effect of the enveloping sieve, and its excellent adequation with the large sieve inequality. The proof that follows is taken from my PhD memoir [22].

Theorem $\mathscr{C}$

For every $X_0 \geqslant 2$, there exists a $c > 0$ such that the following holds. For any $\delta \in (0, 1/2]$ and any subset $\mathcal{P}_1 \subset \mathcal{P}$ such that $|\mathcal{P}_1 \cap [1, X]| \geqslant \delta X/\log X$ when $X \geqslant X_0$, we have, for every $n \geqslant N_0(\delta)$,

$$\sum_{\substack{p_1+p_2+p_3=n,\\ p_1,p_2,p_3\in\mathcal{P}_1}} 1 \leqslant c\delta^2 \log(1/\delta) \frac{n^2}{(\log n)^3}.$$

Setting the stage

We fix some large enough X and assume that $X < n \leqslant 2X$. We assume that $\delta \leqslant 1/4$, else the result is a trivial consequence of the sieve (it may also be obtained by the method we develop, but let us simplify the stage). We also assume that $\delta(X/2)/\log X \leqslant |\mathcal{P}_1 \cap [1, X]| \leqslant 2\delta X/\log X$, which can be done by suppressing some elements from $\mathcal{P}_1$. We furthermore assume that $\mathcal{P}_1$ has no elements less than $\sqrt{X}$. We also set

$$r(n) = \sum_{\substack{p_1+p_2+p_3=n,\\ p_1,p_2,p_3\in\mathcal{P}_1}} 1 \tag{3.1}$$ {defrn}

Large sieve external bounds

We will require two large sieve inequalities. The first one is the classical large sieve inequality.

Theorem 16 {LS}

Let $\mathcal{X}$ be a finite set of points of $\mathbb{R}/\mathbb{Z}$. Set $\delta = \min\{\|x - x'\|, x \neq x' \in \mathcal{X}\}$. For any sequence of complex numbers $(u_n)_{1\leqslant n\leqslant N}$, we have

$$\sum_{x\in\mathcal{X}} \Bigl|\sum_n u_n e(nx)\Bigr|^2 \leqslant \sum_n |u_n|^2 (N - 1 + \delta^{-1}).$$

[22] O. Ramaré, 1991, "Contribution au problème de Goldbach : tout entier > 1 est d'au plus 13 nombres premiers".

The L.H.S. can be thought as a Riemann sum over the points in $\mathcal{X}$; at least when the set $\mathcal{X}$ is dense enough. The spacing between two consecutive points being at least δ, this L.H.S. multiplied by δ can thought as approximating

$$\int_0^1 \Big|\sum_n u_n e(n\alpha)\Big|^2 d\alpha = \sum_n |u_n|^2.$$

This is essentially so if δ^{-1} is much greater than N, but it turns out that the case of interest in number theory is the opposite one. The theorem in this version is due to A. Selberg. The same year and by a different method, a marginally weaker version (without the -1 on the right) was proved by H. Montgomery and R.C. Vaughan in [19].

A second ingredient is an improved large sieve inequality for primes. This is a ready consequence of Theorem $\mathscr{A}$, but can be proved more easily.

{EIR} **Lemma 17**

If $(u_n)_{n\leqslant N}$ is such that u_n vanishes as soon as n has a prime factor less than $\sqrt{N}$, then

$$\sum_{q\leqslant Q_0}\sum_{a \bmod^* q}\Big|\sum_n u_n e(na/q)\Big|^2 \leqslant 7\,\frac{N\log Q_0}{\log N}\sum_n |u_n|^2$$

for any $Q_0 \leqslant \sqrt{N}$ and provided $N \geqslant 100$.

This is [21, Theorem 5.3], but is already contained in [27, Theorem 5], save for the numerical constants. Specializing $\mathcal{X}$ to $\{a/q, q \leqslant Q_0, (a,q) = 1\}$ in Theorem $\mathscr{A}$ essentially gives this result, though in a slightly different phrasing.

The basic proof

We set $z_0 = 1$, $z = X^{1/10}$ and use our enveloping sieve $\beta(m) = \beta_{z_0,z}(m)$ from (2.2). The first set is to notice that

$$r(n) \leqslant r^\sharp(n) = \sum_{\substack{p_1+p_2+m=n,\\ p_1,p_2\in\mathcal{P}_1}} \beta(m).$$

We then appeal to Theorem 6 to write

$$r^\sharp(n) = \sum_{p_1,p_2\in\mathcal{P}_1}\beta(n-p_1-p_2) = \sum_{q\leqslant z^2} w_q(z;1)\sum_{a \bmod^* q}\sum_{p_1,p_2\in\mathcal{P}_1} e\Big(\frac{a(p_1+p_2-n)}{q}\Big)$$
$$= \sum_{q\leqslant z^2} w_q(z;1)\sum_{a \bmod^* q} S\Big(\mathcal{P}_1;\frac{a}{q}\Big)^2 e\Big(\frac{-na}{q}\Big).$$

[19] H. Montgomery and R. Vaughan, 1973, "The large sieve".
[21] O. Ramaré, 2009, *Arithmetical aspects of the large sieve inequality.*
[27] O. Ramaré and I. Ruzsa, 2001, "Additive properties of dense subsets of sifted sequences".

Once we have reached this expression, we first dispense with the large q's by using Lemma 8 and the Large Sieve Inequality of Theorem 16 (together with some integration by parts). This leads to, with an obvious notation,

$$\begin{aligned} r^\sharp(n; q > Q_0) &\ll \frac{1}{\log z} \sum_{Q_0 < q \leqslant z^2} \frac{1}{\sqrt{q}} \sum_{a \bmod^* q} \left| S\left(\mathcal{P}_1; \frac{a}{q}\right)\right|^2 \\ &\ll \frac{1}{\log X}\left(\frac{X}{\sqrt{Q_0}} + z^2\right)|\mathcal{P}_1| \\ &\ll \left(\frac{\log X}{\delta\sqrt{Q_0}} + \frac{z^2 \log X}{\delta X}\right)\frac{(\delta X)^2}{(\log X)^3} \ll \frac{(\delta X)^2}{(\log X)^3} \end{aligned}$$

with $Q_0 = (\log X)^3$ and provided X is large enough in terms of δ. The part $r^\sharp(n; q \leqslant Q_0)$ is treated with our refined large sieve inequality from Lemma 17. Indeed we infer from this inequality that

$$\sum_{q \leqslant Q_1} \sum_{a \bmod^* q} \left| S\left(\mathcal{P}_1; \frac{a}{q}\right)\right|^2 \ll \frac{\delta X^2}{(\log X)^2} \log(2Q_1). \tag{3.2}$$ {eq:11}

By summation by parts, we deduce from that the inequality

$$\sum_{A < q \leqslant Q_0} \sum_{a \bmod^* q} \left| S\left(\mathcal{P}_1; \frac{a}{q}\right)\right|^2 \ll \frac{\delta X^2}{(\log X)^2} \frac{\log(2A)}{A}. \tag{3.3}$$ {eq:12}

and we select $A = 1/\delta^2$.

Better analysis for the initial Farey fractions

Now that q is reduced in size, we may investigate our sum more closely. Let us start by a better apprximation for $w_q(z; 1)$.

Lemma 18. We have $w_q(z;1) = \dfrac{\mu(q)}{\varphi(q)G(z)} + \mathcal{O}\left(\dfrac{3^{\omega(q)}\log q}{\varphi(q)G(z)^2}\right)$. {wdasymp}

The reader will find in [25, Lemma 4.3] a more precise asymptotic development for $w_q(z; 1)$.

Proof. We start from Lemma 6 and check there that $\xi_q(y) = 1$ when $y \geqslant q$.

$$w_q(z;1) = \frac{\mu(q)}{\varphi(q)} \frac{\frac{q}{\varphi(q)} G_q(z/q)}{G(z)^2} + \mathcal{O}\left(\frac{3^{\omega(q)}\log q}{\varphi(q)G(z)^2}\right).$$

Lemmas 1 and 2 imply that

$$\frac{q}{\varphi(q)} G_q(z/q) = \log z + \mathcal{O}(\log q).$$

[25] O. Ramaré, 1995, "On Snirel'man's constant".

We infer from this expression that

$$w_q(z;1) = \frac{\mu(q)}{\varphi(q)G(z)} + \mathcal{O}\biggl(\frac{3^{\omega(q)}\log q}{\varphi(q)G(z)^2}\biggr). \tag{3.4}$$ {eq:15}

□

On appealing to Lemma 18 leads to the main expression

{eq:16} $$r^\sharp(n) = \sum_{q|P(1/\delta^2)} \frac{\mu(q)}{\varphi(q)G(z)} \sum_{a \bmod^* q} S\Bigl(\mathcal{P}_1; \frac{a}{q}\Bigr)^2 e\Bigl(\frac{-na}{q}\Bigr) + \mathcal{O}\biggl(\frac{(\delta X)^2}{(\log X)^3}\biggr) \tag{3.5}$$

provided $X \geqslant X_0(\delta)$. We set $M = P(1/\delta^2)$ and we define

$$T(b/M) = \sum_{m \bmod^* M} e(mb/M).$$

Note that when $b/M = a/q$ with $(a,q) = 1$, we have $T(a/q) = \varphi(M/q)\mu(q)$ and thus

$$r^\sharp(n) = \frac{1}{\varphi(M)G(z)} \sum_{q|M} \sum_{a \bmod^* q} T(a/q) S\Bigl(\mathcal{P}_1; \frac{a}{q}\Bigr)^2 e\Bigl(\frac{-na}{q}\Bigr) + \mathcal{O}\biggl(\frac{(\delta X)^2}{(\log X)^3}\biggr)$$

which is readily to equal

$$r^\sharp(n) = \frac{M}{\varphi(M)G(z)} \sum_{\substack{p_1,p_2\in\mathcal{P}_1,\\ (p_1+p_2-n,M)=1}} 1 + \mathcal{O}\biggl(\frac{(\delta X)^2}{(\log X)^3}\biggr) \ll \frac{\log(2/\delta)}{\log z}|\mathcal{P}_1|^2 + \frac{(\delta X)^2}{(\log X)^3}.$$

Our theorem follows readily from this estimate. It is likely, that, on using the technique developed in [27], the factor $\log(2/\delta)$ may be brought down to $\log\log(3/\delta)$.

[27] O. Ramaré and I. Ruzsa, 2001, "Additive properties of dense subsets of sifted sequences".

4 Restriction Theory in the Primes

4.1. Introduction and some results

Let us recall that a set $\mathcal{X} \subset \mathbb{R}/\mathbb{Z}$ is said to be δ-well spaced when $\min_{x \neq x' \in \mathcal{X}} |x - x'|_{\mathbb{Z}} \geqslant \delta$, where $|y|_{\mathbb{Z}} = \min_{k \in \mathbb{Z}} |y - k|$ denotes in a rather unusual manner the distance to the nearest integer.

At the heart of B. Green & T. Tao's result lies an estimate close to our next theorem. The main difference (aside from the fact that we state it in dual format, Theorem 22 below being its true analogue) is that the dependence in $|\mathcal{X}|$ is not $|\mathcal{X}|^{\varepsilon}$, but $\log(2|\mathcal{X}|)$. This positive ε came from some average of restricted divisors. A more precise proof leads to $\exp \frac{c \log |\mathcal{X}|}{\log\log |\mathcal{X}|}$ for some constant $c > 0$, but to nothing better.

Theorem $\mathscr{A}$

Let $\mathcal{X}$ be a δ-well spaced subset of $\mathbb{R}/\mathbb{Z}$ and $N \geqslant 10^{11}$. Let $(u_p)_{p \leqslant N}$ be a sequence of complex numbers. We have

$$\sum_{x \in \mathcal{X}} \Big| \sum_{1 \leqslant p \leqslant N} u_p e(xp) \Big|^2 \leqslant 10^6 \frac{N + \delta^{-1}}{\log N} \sum_{p \leqslant N} |u_p|^2 \log(2|\mathcal{X}|).$$

This result should be compared with Lemma 17. A way to compare both results is to state the maximal estimate we can now get.

Corollary 19 {Extensionprecisebis}

Let $N \geqslant 10^{11}$ and $Q_0 \in [2, \sqrt{N}]$. Let $(u_p)_{p \leqslant N}$ be a sequence of complex numbers. We have

$$\sum_{q \leqslant Q_0} \sum_{a \bmod^* q} \max_{|\alpha - \frac{a}{q}| \leqslant \frac{1}{qQ_0}} \Big| \sum_{1 \leqslant p \leqslant N} u_p e(\alpha p) \Big|^2 \leqslant 10^7 \frac{N \log Q_0}{\log N} \sum_{p \leqslant N} |u_p|^2.$$

Some analytical consequences

Here is our first result.

Theorem 20 {Extension}

Let $\mathscr{X}$ be a δ-well spaced subset of $\mathbb{R}/\mathbb{Z}$. Assume $N \geqslant 10^{11}$ and let $h > 0$. We have

$$\sum_{x\in\mathscr{X}}\left|\sum_{p\leqslant N} u_p e(xp)\right|^{2+h} \leqslant 10^8\big((1+\tfrac{3}{2\log N})^h+1/h\big)\left(\frac{N+\delta^{-1}}{\log N}\sum_{p\leqslant N}|u_p|^2\right)^{1+h/2}.$$

On taking $\mathscr{X} = \{\beta + k/N, 0 \leqslant k \leqslant N-1\}$ and integrating over β in $[0, 1/N]$, we get the corollary we advertised above.

{mainCor} **Corollary 21**

Assume $N \geqslant 10^{11}$ and let $h > 0$. We have

$$\int_0^1\left|\sum_{p\leqslant N} u_p e(p\alpha)\right|^{2+h} d\alpha \leqslant 10^8\frac{(1+\frac{3}{2\log N})^h+1/h}{N}\left(\frac{2N}{\log N}\sum_{p\leqslant N}|u_p|^2\right)^{1+h/2}.$$

This result offers an optimal (save for the implied constants) transition to the case $h = 0$. Indeed, on selecting $h = 1/\log N$, this corollary implies that, when $|u_p| \leqslant 1$, we have te optimal

$$\int_0^1\left|\sum_{p\leqslant N} u_p e(p\alpha)\right|^{2+h} d\alpha \ll \sum_{p\leqslant N}|u_p|^2.$$

All of that leads to already stated Theorem $\mathscr{B}$.

4.2. The fundamental estimate. Proof of Theorem $\mathscr{A}$

Let us now state and prove our main lemma.

{Bel} **Lemma 22**

Let $N \geqslant 2\cdot 10^{10}$. Let B be a δ-well spaced subset of $\mathbb{R}/\mathbb{Z}$. For any function f on B, we have

$$\sum_{1\leqslant p\leqslant N}\left|\sum_{b\in B} f(b)e(bp)\right|^2 \leqslant 330\,000(N+\delta^{-1})\|f\|_2^2\frac{\log(2\|f\|_1^2/\|f\|_2^2)}{\log N}.$$

Proof. Let $z = N^{1/4} \geqslant 320$ and

$$z_0 = \left(2\frac{\|f\|_1^2}{\|f\|_2^2}\right)^2 \geqslant 2.$$

We have $z_0 \leqslant z$ when $\|f\|_1^2/\|f\|_2^2 \leqslant N^{1/8}/2$. When this condition is not met, we use the dual of the usual large sieve inequality (i.e. Theorem 16) to infer that

$$\begin{aligned}\sum_{1\leqslant p\leqslant N}\Big|\sum_{b\in B} f(b)e(bp)\Big|^2 &\leqslant (N+\delta^{-1})\|f\|_2^2\\ &\leqslant (N+\delta^{-1})\|f\|_2^2\frac{\log(2\|f\|_1^2/\|f\|_2^2)}{\log(N^{1/8}/2)}.\end{aligned}$$

Some numerical analysis shows that this establishes our inequality in this case. Henceforth, we assume that $z_0 \leqslant z$. We first notice that

$$\begin{aligned}\sum_{1\leqslant p\leqslant z}\Big|\sum_{b\in B} f(b)e(bp)\Big|^2 &\leqslant z\|f\|_1^2 \leqslant N^{3/8}\|f\|_2^2/2\\ &\leqslant N\frac{\|f\|_2^2\log(2\|f\|_1^2/\|f\|_2^2)}{\log N}\frac{\log N}{N^{5/8}\log 2}\\ &\leqslant \frac{1}{53000}N\frac{\|f\|_2^2\log(2\|f\|_1^2/\|f\|_2^2)}{\log N}.\end{aligned}$$

Let us now call W the quantity to be studied with $z < p \leqslant N$. We bound above the characteristic function of those primes by our enveloping sieve and further majorize the characteristic function of the interval $[1, N]$ by a function ψ of non-negative Fourier transform supported by $[-\delta_1, \delta_1]$ where $\delta_1 = \min(\delta, 1/(2z^4))$, i.e. we write

$$W \leqslant \sum_n \beta_{z_0,z}(n)\psi(n)\Big|\sum_{b\in B} f(b)e(bn)\Big|^2.$$

We then develop the square and use the Fourier expansion for $\beta_{z_0,z}(n)$ provided by Theorem 6 to get

$$W \leqslant \sum_{\substack{q\leqslant z^2,\\ (q,P(z_0))=1}} w_q(z;z_0)\sum_{a \bmod^* q}\sum_{b_1,b_2} f(b_1)\overline{f(b_2)}\sum_{n\in\mathbb{Z}} e((b_1-b_2)n)e(an/q)\psi(n).$$

We split this quantity according to whether $q < z_0$ or not:

$$W = W(q < z_0) + W(q \geqslant z_0).$$

When $q \geqslant z_0$, Poisson summation formula tells us that the inner sum is also $\sum_{m\in\mathbb{Z}}\hat{\psi}(b_1 - b_2 - (a/q) + m)$. The sum over b_1, b_2 and n is thus

$$\leqslant (N+\delta_1^{-1})\sum_{b_1,b_2}|f(b_1)||f(b_2)|\#\big\{(a/q)/\|b_1-b_2+a/q\| < \delta_1\big\}.$$

Given (b_1, b_2), at most one a/q may work, since $1/z^4 > 2\delta_1$. By bounding above $w_q(z;z_0)$ by Lemma 8, we see that

$$\begin{aligned}W(q\geqslant z_0) &\leqslant 11500(N+\delta_1^{-1})\frac{\|f\|_1^2\log z_0}{\sqrt{z_0}\log z}\\ &\leqslant \frac{11500}{\sqrt{2}}(N+\delta_1^{-1})\frac{\|f\|_2^2\log z_0}{\log z}.\end{aligned}$$

When $q < z_0$, only $q = 1$ remains. This yields

$$W(q < z_0) \leqslant (N + \delta_1^{-1}) \frac{111\|f\|_2^2 \log z_0}{\log z}.$$

We check that $(N + \delta_1^{-1}) \leqslant \frac{N+4N}{N}(N + \delta^{-1})$. We finally get

$$\sum_{1 \leqslant p \leqslant N} \left| \sum_{b \in B} f(b) e(bn) \right|^2 \leqslant \left(\frac{1}{53000} + 5 \times 2 \times 4 \times \left(\frac{11500}{\sqrt{2}} + 111 \right) \right) \\ \times (N + \delta^{-1}) \|f\|_2^2 \frac{\log(2\|f\|_1^2/\|f\|_2^2)}{\log N}.$$

The proof of the theorem follows readily. □

Proof of Theorem $\mathscr{A}$. We write

$$W = \sum_{x \in \mathcal{X}} \left| \sum_{1 \leqslant p \leqslant N} u_p e(xp) \right|^2 = \sum_{x \in \mathcal{X}} \sum_{p \leqslant N} u_p \overline{S(x)}$$

where $S(x) = \sum_{1 \leqslant p \leqslant N} u_p e(xp)$. On using the Cauchy-Schwarz inequality, we get

$$W^2 \leqslant \sum_{p \leqslant N} |u_p|^2 \sum_{p \leqslant N} \Big| \sum_{x \in \mathcal{X}} \overline{S(x)} e(xp) \Big|^2.$$

We invoque Lemma 22 and notice that

$$\Big(\sum_{x \in \mathcal{X}} |\overline{S(x)}| \Big)^2 \leqslant |\mathcal{X}| \sum_{x \in \mathcal{X}} |\overline{S(x)}|^2.$$

This leads to

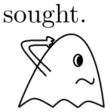
$$W^2 \leqslant 330\,000 \frac{N + \delta^{-1}}{\log N} \sum_{p \leqslant N} |u_p|^2 \sum_{x \in \mathcal{X}} |S(x)|^2 \log(2|\mathcal{X}|).$$

On simplifying by $\sum_{x \in \mathcal{X}} |S(x)|^2$ (after discussing whether it vanishes or not), we get our estimate. □

4.3. On moments. Proof of Theorem 20

{easy} **Lemma 23.** When $y/\log y \leqslant t$, $y \geqslant 2$ and $t \geqslant 10^7$, we have $y \leqslant 2t \log t$.

Proof. Our property is trivial when $y \leqslant 10^7$. Notice that the function $f : y \mapsto y/\log y$ is non-increasing when $y \geqslant e$. We find that $f(2t \log t) \geqslant t \geqslant f(y)$, whence $2t \log t \geqslant y$ as sought. □



Proof of Theorem 20. For typographical simplification, we define

{defB}
$$B = \left(\frac{N+\delta^{-1}}{\log N}\sum_{p\leqslant N}|u_p|^2\right)^{1/2}. \tag{4.1}$$

We also set $\ell = 2+h$. For any $\xi > 0$, we examine the set

$$\mathcal{X}_\xi = \left\{x\in\mathcal{X}/\left|\sum_{p\leqslant N}u_pe(xp)\right|\geqslant \xi B\right\}. \tag{4.2}$$
{defXclaxi}

By Lemma 9, we see that $\xi\leqslant c_1 = \min(5/4, 1+\frac{3}{2\log N})$ or else, the set $\mathcal{X}_\xi$ is empty. We consider (as in [14] by A.J. Harper, bottom of page 1141)

$$\Gamma(\xi) = \sum_{x\in\mathcal{X}_\xi}\left|\sum_{p\leqslant N}u_pe(xp)\right|. \tag{4.3}$$
{defGammaxi}

We apply Cauchy's inequality to this expression to get

$$\xi^2|\mathcal{X}_\xi|^2B^2\leqslant\Gamma(\xi)^2\leqslant 10^7\,B^2|\mathcal{X}_\xi|\log(2|\mathcal{X}_\xi|)$$

by Theorem $\mathscr{A}$. It follows that

$$2|\mathcal{X}_\xi|/\log(2|\mathcal{X}_\xi|)\leqslant 2\cdot 10^7\,/\xi^2.$$

which we convert with Lemma 23 in $2|\mathcal{X}_\xi|\leqslant 10^7\xi^{-2}\log(10^7/\xi^2)$.

We can now turn towards the proof of the stated inequality and select $\xi_j = c_1/c^j$ for some $c>1$ that we will select later. We get

$$\begin{aligned}\sum_{x\in\mathcal{X}}\left|\sum_{p\leqslant N}u_pe(xp)\right|^\ell/B^\ell &\leqslant \sum_{j\geqslant 0}\frac{c_1^\ell}{c^{\ell j}}(|\mathcal{X}_{\xi_j}|-|\mathcal{X}_{\xi_{j+1}}|)\\ &\leqslant\frac{10^7}{2}\sum_{j\geqslant 0}\frac{c_1^{\ell-2}(\log(10^7)-2\log c_1+2j\log c)}{c^{(\ell-2)j}}.\\ &\leqslant\frac{10^7}{2}\sum_{j\geqslant 0}\frac{c_1^{\ell-2}(14+2j\log c)}{c^{(\ell-2)j}}.\end{aligned}$$

We note that

$$\frac{10^7}{2}\sum_{j\geqslant 0}\frac{c_1^{\ell-2}\times 14}{c^{(\ell-2)j}} = \frac{10^7\times 14\times c_1^{\ell-2}}{1-c^{2-\ell}}$$

and

$$\begin{aligned}\frac{10^7}{2}\sum_{j\geqslant 1}\frac{c_1^{\ell-2}j\,2\log c}{c^{(\ell-2)j}} &\leqslant 10^7\frac{(\log c)}{c^{\ell-2}}c_1^{\ell-2}\sum_{j\geqslant 1}\frac{j}{c^{(\ell-2)(j-1)}}\\ &\leqslant\frac{10^7\times(c_1/c)^{\ell-2}\log c}{(1-c^{2-\ell})^2}.\end{aligned}$$

[14] A. J. Harper, 2016, "Minor arcs, mean values, and restriction theory for exponential sums over smooth numbers".

When $\ell \geqslant 3$, we select $c = 2$, getting

$$\sum_{x\in\mathcal{X}}\biggl|\sum_{p\leqslant N} u_p e(xp)\biggr|^{2+h} \leqslant \bigl(14\cdot 10^7(1+\tfrac{3}{2\log N})^h + 10^7\bigr)\biggl(\frac{N+\delta^{-1}}{\log N}\sum_{p\leqslant N}|u_p|^2\biggr)^{1+h/2}.$$

When $\ell \in (2,3)$, we select $c = \exp(1/h)$, getting

$$\sum_{x\in\mathcal{X}}\biggl|\sum_{p\leqslant N} u_p e(xp)\biggr|^{2+h} \leqslant \biggl(10^7(1+\tfrac{3}{2\log N})^h + \frac{10^7}{h}\biggr)\biggl(\frac{N+\delta^{-1}}{\log N}\sum_{p\leqslant N}|u_p|^2\biggr)^{1+h/2}.$$

Our theorem follows readily. □

4.4. Optimality and uniform boundedness. Proof of Theorem $\mathscr{B}$

{appGh}

Lemma 24
Let $h > 0$. We have

$$\sum_{d\leqslant D}\frac{\mu^2(d)}{\varphi(d)^{1+h}} \geqslant \frac{1-D^{-h}}{h}.$$

Proof. We first notice that

$$\begin{aligned}\sum_{d\leqslant D}\frac{\mu^2(d)}{\varphi(d)^{1+h}} &\geqslant \sum_{d\leqslant D}\frac{\mu^2(d)}{d^{1+h}}\prod_{p|d}\biggl(\sum_{k\geqslant 0}\frac{1}{p^k}\biggr)^{1+h}\\ &\geqslant \sum_{d\leqslant D}\frac{\mu^2(d)}{d^{1+h}}\prod_{p|d}\biggl(\sum_{k\geqslant 0}\frac{1}{p^{k(1+h)}}\biggr) = \sum_{\substack{q\geqslant 1,\\ k(q)\leqslant D}}\frac{1}{q^{1+h}} \geqslant \sum_{q\leqslant D}\frac{1}{q^{1+h}}.\end{aligned}$$

Concerning this last quantity, we write

$$\begin{aligned}\sum_{q\leqslant D}\frac{1}{q^{1+h}} &= \int_1^D\sum_{q\leqslant t}1\frac{(1+h)dt}{t^{2+h}} + \frac{[D]}{D^{1+h}}\\ &\geqslant \int_1^D (t-1)\frac{(1+h)dt}{t^{2+h}} + \frac{D-1}{D^{1+h}} = \frac{h+1}{h}\biggl(1-\frac{1}{D^h}\biggr) + \frac{D-1}{D^{1+h}}\\ &\geqslant \frac{1-D^{-h}}{h} + 1 - \frac{1}{D^{1+h}} \geqslant \frac{1-D^{-h}}{h}\end{aligned}$$

as required. □

We assume that $N \geqslant 10^{11}$ and set

{eq:1}
$$S(\alpha) = \sum_{p\leqslant N} e(p\alpha). \tag{4.4}$$

The argument employed at the bottom of page 1626 of [12] by B. Green is not enough for us. Instead, we got our inspiration from the argument developped by R.C. Vaughan in [35]. It runs as follows. We first notice that

$$\Bigl|\sum_{a \bmod^* q} S\Bigl(\frac{a}{q}+\beta\Bigr)\Bigr| \leqslant \Bigl(\sum_{a \bmod^* q}\Bigl|S\Bigl(\frac{a}{q}+\beta\Bigr)\Bigr|^{\ell}\Bigr)^{1/\ell}\Bigl(\sum_{a \bmod^* q} 1\Bigr)^{(\ell-1)/\ell}.$$

A direct inspection shows that

$$\sum_{a \bmod^* q} S\Bigl(\frac{a}{q}+\beta\Bigr) = \mu(q)S(\beta) + T(q,\beta)$$

where $T(q,\beta) = \sum_{p|q} e(p\beta)(c_q(p)-\mu(q))$. The bound $|c_q(n)| \leqslant \varphi((n,q))$ for the Ramanujan sum $c_q(n)$ (use for instance the von Sterneck expression for $c_q(n)$) gives us

$$|T(q,\beta)| \leqslant \sum_{p|q}(p-1+1) \leqslant q. \tag{4.5}$$ {eq:10}

The last inequality follows from the trivial property that a sum of positive integers is certainly not more than its product. We next get a lower bound for $S(\beta)$ by writing

$$1 - e(\beta p) = 2i\pi\int_0^{\beta p} e(t)dt$$

whence

$$|S(\beta)| \geqslant S(0) - 2\pi\beta N S(0) \geqslant (1-2\pi\beta N)S(0) \geqslant (1-2\pi\beta N)\frac{N}{\log N} \tag{4.6}$$ {you}

by Lemma 9. When $|\beta| \leqslant 1/(7N)$, this leads to $|S(\beta)| \geqslant c_2 N/\log N$ with $c_2 = 1-2\pi/7$, and, when q is squarefree and not more than $\sqrt{N}$, to

$$\bigl|\mu(q)S(\beta) + T(q,\beta)\bigr| \geqslant c_2\frac{N}{\log N} - \sqrt{N} \geqslant \frac{N}{10\log N}. \tag{4.7}$$ {eq:13}

We thus get, when $|\beta| \leqslant 1/(7N)$,

$$\sum_{a \bmod^* q}\Bigl|S\Bigl(\frac{a}{q}+\beta\Bigr)\Bigr|^{\ell} \geqslant \frac{\mu^2(q)}{\varphi(q)^{\ell-1}}|S(\beta)+T(q,\beta)|^{\ell} \geqslant \frac{\mu^2(q)}{\varphi(q)^{\ell-1}}\Bigl(\frac{N}{10\log N}\Bigr)^{\ell}.$$

Thus

$$\begin{aligned}\int_0^1 |S(\alpha)|^{\ell} d\alpha &\geqslant \sum_{q\leqslant\sqrt{N}}\sum_{a \bmod^* q}\mu^2(q)\int_{\frac{a}{q}-\frac{1}{7N}}^{\frac{a}{q}+\frac{1}{7N}}\Bigl|S\Bigl(\frac{a}{q}+\beta\Bigr)\Bigr|^{\ell} d\beta \\ &\geqslant \frac{2}{7N}\sum_{q\leqslant\sqrt{N}}\frac{\mu^2(q)}{\varphi(q)^{\ell-1}}\Bigl(\frac{N}{10\log N}\Bigr)^{\ell}.\end{aligned}$$

[12] B. Green, 2005, "Roth's theorem in the primes".
[35] R. C. Vaughan, 1988, "The L^1 mean of exponential sums over primes".

By Lemma 24, we conclude that

$$\int_0^1 |S(\alpha)|^\ell d\alpha \geqslant \frac{1-\sqrt{N}^{2-\ell}}{\ell-2}\frac{2}{7N}\biggl(\frac{N}{10\log N}\biggr)^\ell .$$

We thus find that

{refK}

$$\int_0^1 \biggl|\sum_{p\leqslant N} u_p e(p\alpha)\biggr|^\ell d\alpha \leqslant K(\ell)\biggl(\frac{\log N}{N}\sum_{p\leqslant N}|u_p|^2\biggr)^{\ell/2}\int_0^1 \biggl|\sum_{p\leqslant N} e(p\alpha)\biggr|^\ell d\alpha \tag{4.8}$$

where $\ell = 2+h$ and

$$\begin{aligned}K(2+h) &= 10^7\frac{(1+\frac{3}{2\log N})^h + 1/h}{N}\biggl(\frac{2N^2}{(\log N)^2}\biggr)^{\ell/2}\frac{h}{1-\sqrt{N}^{-h}}\frac{7N}{2}\biggl(\frac{N}{10\log N}\biggr)^{-\ell}\\ &= 10^7\Bigl((1+\tfrac{3}{2\log N})^h + 1/h\Bigr)2^{\ell/2}\frac{h}{1-\sqrt{N}^{-h}}\frac{7}{2}10^\ell .\end{aligned}$$

When $h\geqslant 1$, this is readily bounded above by $10^8\cdot 20^\ell$. Else, it is bounded above by

$$\frac{3\cdot 10^{12}}{h}\frac{h}{1-\sqrt{N}^{-h}} = \frac{3\cdot 10^{12}}{1-\sqrt{N}^{-h}}.$$

This is bounded above by 10^{13} when $h\geqslant 1/\log N$. When $0\leqslant h\leqslant 1/\log N$, we use

$$\begin{aligned}\int_0^1 \biggl|\sum_{p\leqslant N} u_p e(p\alpha)\biggr|^\ell d\alpha &\leqslant \biggl(\pi(N)\sum_{p\leqslant N}|u_p|^2\biggr)^{h/2}\int_0^1 \biggl|\sum_{p\leqslant N} u_p e(p\alpha)\biggr|^2 d\alpha\\ &\leqslant \sqrt{5/4}\biggl(\frac{\log N}{N}\sum_{p\leqslant N}|u_p|^2\biggr)^{\ell/2}\biggl(\frac{N}{\log N}\biggr)^\ell N^{-1}\log N\end{aligned}$$

which leads to (4.8) with

$$\begin{aligned}K(2+h) &= \frac{\sqrt{5/4}}{N}\biggl(\frac{N}{\log N}\biggr)^\ell\frac{h\log N}{1-\sqrt{N}^{-h}}\frac{7N}{2}\biggl(\frac{N}{10\log N}\biggr)^{-\ell}\\ &\leqslant \frac{7\sqrt{5/4}}{2(1-\exp(-1/2))}\leqslant 10^{12}.\end{aligned}$$

Theorem $\mathscr{B}$ follows readily.

4.5. A maximal large sieve estimate. Proof of Corollary 19

Proof of Corollary 19. The split the Farey sequence

{eq:2}
$$F(Q_0) = \left\{\frac{a}{q}, 1 \leqslant a \leqslant q \leqslant Q_0, (a,q)=1\right\} \tag{4.9}$$
$$= \left\{0 < x_1 < x_2 < \ldots < x_K = 1\right\}$$

in $F_1(Q_0) = \{x_{2i}, 1 \leqslant i \leqslant K/2\}$ union $F_2(Q_0) = \{x_{2i+1}, 1 \leqslant 0 \leqslant (K-1)/2\}$. We recall that the distance between two consecutive points a/q and a'/q' in $F(Q_0)$ is $1/(qq')$; this is at least as large as $\frac{1}{qQ_0} + \frac{1}{q'Q_0}$ by the known property $q+q' \geqslant Q_0$. Hence two intervals $[\frac{a_1}{q_1} - \frac{1}{q_1Q_0}, \frac{a_1}{q_1} + \frac{1}{q_1Q_0}]$ and $[\frac{a_2}{q_2} - \frac{1}{q_2Q_0}, \frac{a_2}{q_2} + \frac{1}{q_2Q_0}]$ with $\frac{a_1}{q_1}, \frac{a_2}{q_2} \in F_1(Q_0)$ are separated by at least $1/Q_0^2$. We check this is also true when seen on the unit circle: the largest point of $F(Q_0)$ is 1 and its smallest is $\frac{1}{[Q_0]}$. The same applies to $F_2(Q_0)$. We finally notice that $|F(Q_0)| \leqslant Q_0(Q_0+1)/2 \leqslant Q_0^2$.

To prove our corollary, for every $x_{2i} \in F_1(Q_0)$, we select a point $\tilde{x}_{2i}$ such that

$$\Big|\sum_{p\leqslant N} u_p e(p\tilde{x_{2i}})\Big| = \max_{|x-x_{2i}|\leqslant \frac{1}{qQ_0}} \Big|\sum_{p\leqslant N} u_p e(px)\Big| \tag{4.10}$$ {eq:9}

and apply Theorem $\mathscr{A}$ to the set $\tilde{X}_1 = \{\tilde{x}_{2i}\}$. We proceed similarly with $F_2(Q_0)$. The last details are left to the readers. □

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5 Vinogradov Theorem without L-functions

5.1. Introduction and some results

In [33], the author gives a proof that every large enough odd integer is a sum of three primes, without using L-functions, though the distribution of primes in a finite number of arithmetic progressions is used. We propose a much simpler proof, using the same basis, and which has the advantage of yielding the proper asymptotic for the number of representations. Here is what we prove.

Theorem $\mathscr{D}$

Let N be an odd integer. Let $K \geqslant [2, \frac{1}{2}\log\log N]$ be a parameter. We set $P(K) = \prod_{p<K} p$. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two sets of primes. We have

$$\sum_{\substack{p+p_2+p_3=N,\\ p_1\in\mathcal{P}_1, p_2\in\mathcal{P}_2}} 1 = \prod_{p\leqslant K}\left(1-\frac{1}{p}\right)^{-1} \sum_{\substack{(N-(p_2+p_3),P(K))=1,\\ p_1\in\mathcal{P}_1, p_2\in\mathcal{P}_2}} 1 + \mathcal{O}\left(\frac{\log K}{K^{1/4}}\frac{N^2}{(\log N)^3}\right).$$

Theorem 25 {th1}

Let N be an odd integer. Let $K \geqslant [2, \frac{1}{2}\log\log N]$ be a parameter. We have

$$\sum_{p_1+p_2+p_3=N} 1 = \frac{\mathfrak{S}_3(N)N^2}{(\log N)^3} + \mathcal{O}\left(\frac{\log K}{K^{1/4}}\frac{N^2}{(\log N)^3}\right).$$

As a slightly unusual notation, we mention that we use

$$S(\Lambda,\alpha) = \sum_{n\leqslant N}\Lambda(n)e(n\alpha). \tag{5.1}$$ {eq:4}

5.2. Lemmas for Theorem $\mathscr{D}$

Lemma 26 {L21}

[33] X. Shao, 2014, "An L-function-free proof of Vinogradov's three primes theorem".

For any $|\alpha - a/q| \leqslant 1/(qQ)$, we have

$$|S(\Lambda,\alpha)| \ll (\log N)^3\big(Nq^{-1/2} + N^{4/5} + N^{1/2}q^{1/2}\big).$$

This is [36, Theorem 3.1] by R.C. Vaughan in the form given in [15, Theorem 13.6] by H. Iwaniec and E. Kowalski.

{L22} **Lemma 27**

For any $|\alpha - a/q| \leqslant N^{-1/30}$ and $q \leqslant N^{1/30}$, we have

$$|S(\Lambda,\alpha)| \ll N\sqrt{q}/\varphi(q).$$

This is a consequence of [29, Theorem 4]. See also [24, Theorem 3]. In the last chapter, we prove Theorem 32 that is enough to prove this lemma but for a more restricted range and with a weaker error term. This would however be enough to prove the Lemma 28 below with $A = 1/2$, which in turn is enough for our final goal.

We use the above two lemmas to infer the next one.

{L2} **Lemma 28**

For any $A \geqslant 1/2$, any $Q = N(\log N)^{-B}$ for some $B \geqslant 2A+6$, for $|\alpha - a/q| \leqslant 1/(qQ)$ and $q \leqslant Q$, we have

$$|S(\Lambda,\alpha)| \ll_{A,B} \frac{N\sqrt{q}}{\varphi(q)} + \frac{N}{(\log N)^A}.$$

Proof. When $q \geqslant (\log N)^{2A+6}$, this estimate is a consequence of Lemma 26. When $q \leqslant (\log N)^{2A+6}$ Lemma 27. $\square$

{L3} **Lemma 29**

Let $Q = N/(\log N)^B$ and $v(\beta) = \sum_{n\leqslant N} e(\beta m)$. When $|\alpha - a/q| \leqslant 1/(qQ)$ and $q \leqslant (\log N)^B$, we have

$$S(\Lambda,\alpha) = \frac{\mu(q)}{\varphi(q)} v\Big(\alpha - \frac{a}{q}\Big) + \mathcal{O}_B(N/(\log N)^B).$$

[36] R. Vaughan, 1981, *The Hardy-Littlewood method.*
[15] H. Iwaniec and E. Kowalski, 2004, *Analytic number theory.*
[29] O. Ramaré and G. K. Viswanadham, 2021, "Modular Ternary Additive Problems with Irregular or Prime Numbers".
[24] O. Ramaré, 2010, "On Bombieri's asymptotic sieve".

The proof of [36, Lemma 3.1] by R.C. Vaughan applies.

5.3. Proof of Theorem 25

We consider the Dirichlet dissection of the torus $\mathbb{R}/\mathbb{Z}$:

$$\mathfrak{M}(a/q) = \{\alpha/|\alpha - a/q| \leqslant 1/(qQ)\}$$

for $Q = N/(\log N)^B$. We set $\mathfrak{m} = \cup_{K \leqslant q \leqslant Q} \mathfrak{M}(a/q)$ where the constant $K \geqslant 2$ is introduced in the statement of Theorem 25.

We employ the circle method in Vinogradov's format:

$$r(N) = \int_0^1 S(\Lambda, \alpha)^3 e(-N\alpha) d\alpha \tag{5.2}$$

{eq:2}

and write

$$r(N) = r_0(N) + r'(N), \quad r'(N) = \int_{\mathfrak{m}} S(\Lambda, \alpha)^3 e(-N\alpha) d\alpha. \tag{5.3}$$

{eq:1}

To evaluate $r'(N)$, we proceed as follows. Let $\ell \in (2, 3)$. We notice that

$$\frac{1}{\ell} + \frac{1}{\ell} + \frac{\ell - 2}{\ell} = 1, \quad \frac{\ell}{\ell - 2} = \ell + \frac{3 - \ell}{\ell - 2}\ell \tag{5.4}$$

{eq:3}

and use Hölder inequality to infer that

$$|r'(N)| \leqslant \left(\int_{\mathfrak{m}} |S(\Lambda, \alpha)|^\ell d\alpha\right)^{1/\ell} \left(\int_{\mathfrak{m}} |S(\Lambda, \alpha)|^\ell d\alpha\right)^{1/\ell} \left(\int_{\mathfrak{m}} |S(\Lambda, \alpha)|^\ell |S(\Lambda, \alpha)|^{\frac{\ell(3-\ell)}{\ell-2}} d\alpha\right)^{(\ell-2)/\ell}.$$

We use Lemma 28 on $|S(\Lambda, \alpha)|^{\frac{\ell(3-\ell)}{\ell-2}}$, extend the remaining integrals to the full torus and appeal to Theorem $\mathscr{B}$. This gives us

$$r'(N) \ll_\ell N^{1-\frac{1}{\ell}} \times N^{1-\frac{1}{\ell}} \times \left(\frac{\log K}{\sqrt{K}}\right)^{3-\ell} N^{3-\ell} N^{\frac{(\ell-1)(\ell-2)}{\ell}},$$

by using

$$\max_{q \geqslant K} \frac{\sqrt{q}}{\varphi(q)} \ll \frac{\log K}{\sqrt{K}}.$$

Of course, we have

$$1 - \frac{1}{\ell} + 1 - \frac{1}{\ell} + 3 - \ell + \frac{(\ell - 1)(\ell - 2)}{\ell} = 2.$$

We study the major arcs as usual. We finally select $\ell = 5/2$, $A = 1$ and $B = 7$.

5.4. Additional lemmas for Theorem 25

We introduce the function

$$g_M(n) = 1_{(P(K),n)=1}. \tag{5.5}$$ {defgM}

{L1b}

Lemma 30

For any $\ell > 2$ and any subset $\mathcal{P}$ of primes, we have

$$\int_0^1 \left| \sum_{n \leqslant N} g_M(n)\, e(n\alpha) \right|^\ell d\alpha \ll \left(\frac{\varphi(M)}{M} N \right)^{\ell - 1}.$$

Proof. We simply use Parseval:

$$\int_0^1 \left| \sum_{n \leqslant N} g_M(n)\, e(n\alpha) \right|^\ell d\alpha \leqslant \left(\sum_{n \leqslant N} g_M(n) \right)^{\ell - 2} \sum_{n \leqslant N} g_M(n)^2.$$

□

{L3b}

Lemma 31

Let $Q = P(K)^3$ and $v(\beta) = \sum_{n \leqslant N} e(\beta m)$. When $|\alpha - a/q| \leqslant 1/(qQ)$ and $q|P(K)$, we have

$$S(g_M, \alpha) = \frac{\mu(q)}{\varphi(q)} \frac{\varphi(M)}{M} v\left(\alpha - \frac{a}{q}\right) + \mathcal{O}(P(K)^4).$$

When $|\alpha - a/q| \leqslant 1/(qQ)$ and $q \nmid P(K)$, we have $S(g_M, \alpha) = \mathcal{O}(P(K)^4)$.

Proof. We readily find that

{gMexact}

$$S(g_M, \alpha) = \sum_{d|P(K)} \mu(d) \frac{e(d[N/d]\alpha) - 1}{e(d\alpha) - 1}. \tag{5.6}$$

When $d \leqslant P(K)$ we have $\|d\alpha\| \geqslant \|da/q\| - d/P(K)^2 \geqslant 1/(2q)$ except when $q|d$. Whence

$$S(g_M, \alpha) = \sum_{q|d|P(K)} \mu(d) \frac{e(d[N/d]\alpha) - 1}{e(d\alpha) - 1} + \mathcal{O}(qP(K)).$$

If we specialize this to $\alpha = a/q$, we get

$$S(g_M, a/q) = \sum_{q|d|P(K)} \mu(d) \left[\frac{N}{d} \right] + \mathcal{O}(qP(K)) = \frac{\mu(q) N}{\varphi(q)} \frac{\varphi(M)}{M} + \mathcal{O}(P(K)^4).$$

The general case in α follows by summation by parts. □

5.5. Proof of Theorem $\mathscr{E}$

We proceed as for Theorem 25. We consider the Dirichlet dissection of the torus $\mathbb{R}/\mathbb{Z}$:

$$\mathfrak{M}(a/q) = \{\alpha/|\alpha - a/q| \leqslant 1/(qQ)\}$$

for $Q = N/(\log N)^B$. We set

$$\mathfrak{m} = \bigcup_{\substack{q \leqslant Q,\\ q \nmid P(K)}} \mathfrak{M}(a/q)$$

where the constant $K \geqslant 2$ is introduced in the statement of Theorem $\mathscr{E}$.

We set

$$S_1(\alpha) = \sum_{\substack{p_1 \leqslant N,\\ p_1 \in \mathcal{P}_1}} (\log p_1) e(p_1 \alpha), \quad S_2(\alpha) = \sum_{\substack{p_2 \leqslant N,\\ p_1 \in \mathcal{P}_2}} (\log p_2) e(p_2 \alpha), \tag{5.7}$$ {eq:5}

and

$$S(\alpha) = \sum_{p \leqslant N} (\log p) e(p\alpha), \quad S^*(\alpha) = \sum_{\substack{n \leqslant N,\\ (n, P(K))=1}} e(n\alpha). \tag{5.8}$$ {eq:6}

We employ the circle method in Vinogradov's format:

$$\begin{cases} r(N) = \displaystyle\int_0^1 S(\alpha) S_1(\alpha) S_2(\alpha) e(-N\alpha) d\alpha, \\ r^*(N) = \displaystyle\int_0^1 S^*(\alpha) S_1(\alpha) S_2(\alpha) e(-N\alpha) d\alpha \end{cases} \tag{5.9}$$ {eq:2}

and write

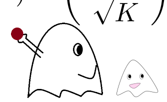
$$r(N) = r_0(N) + r'(N), \quad r'(N) = \int_{\mathfrak{m}} S(\alpha) S_1(\alpha) S_2(\alpha) e(-N\alpha) d\alpha \tag{5.10}$$ {eq:1}

and, correspondingly,

$$r^*(N) = r_0^*(N) + r^{*\prime}(N), \quad r^{*\prime}(N) = \int_{\mathfrak{m}} S^*(\alpha) S_1(\alpha) S_2(\alpha) e(-N\alpha) d\alpha. \tag{5.11}$$ {eq:1}

On imitating the proof of Theorem 25, we find that

$$r'(N) \ll \left(\frac{\log K}{\sqrt{K}} \right)^{1/2} N^2.$$



The same proof applies for $r^{*\prime}(N)$, tough the estimates for $S^*(\alpha)$ are different. We obtain:

$$r^{*\prime}(N) \ll N^{2-\frac{2}{\ell}} \left(\frac{\log K}{K} \frac{\varphi(M)}{M} \right)^{3-\ell} N^{3-\ell} \left(\frac{\varphi(M)}{M} N \right)^{\frac{(\ell-1)(\ell-2)}{\ell}}$$

which reduces to

$$r^{*\prime}(N) \ll N^2 \left(\frac{\log K}{K} \right)^{3-\ell} \left(\frac{\varphi(M)}{M} \right)^{3-\ell+\frac{(\ell-1)(\ell-2)}{\ell}} = N^2 \left(\frac{\log K}{K} \right)^{3-\ell} \left(\frac{\varphi(M)}{M} \right)^{2/\ell}.$$

Since $\varphi(M)/M \asymp 1/\log K$, and we select $\ell = 5/2$, this gives

$$r^{*\prime}(N) \ll \frac{\varphi(M)}{M} N^2 \frac{(\log K)^{\frac{1}{2}+\frac{1}{5}}}{\sqrt{K}}.$$

On the major arcs, $S(\alpha)$ and $(M/\varphi(M))S^*(\alpha)$ are similar, hence the result.

....

6 The Exponential Sum over Primes

In 1937, Vinogradov introduced in [38] a very innovative technique to handle sums over primes (his book [39] is an excellent source). We refer to the survey paper [26] for a historical introduction on the subject. Lemma 26 originates from this line of development. In problems using a lemma like Lemma 26, the contribution of small q's is often handled by examining the distribution in arithmetic progressions to a small modulus (meaning $q \leqslant (\log N)^C$ for some C), and this is achieved through L-functions. Diminishing the value of C has thus become an issue, most notably because any choice $C \geqslant 2$ renders the result ineffective, due the use of a theorem of C. Siegel.

In the same way we prove (1), we may also prove that

$$\sum_{a \leqslant N} \left| \sum_{p \leqslant N} u_p e(pa/N) \right|^{\ell} \ll_{\ell} \left(\frac{N}{\log N} \sum_{p \leqslant N} |u_p|^2 \right)^{\ell/2}. \tag{6.1}$$ {biniBG}

On taking for instance $\ell = 4$, the above inequality* implies that $|S(\Lambda; a/N) \gg \varepsilon N$ on a set of a's of cardinality $\ll 1/\varepsilon^2$. So, if we need to save only a small constant $\varepsilon > 0$, we need worry only on a finite number (though depending of ε) of points. The next theorem shows that the corresponding rationals a/N may also assumed to be well approximated by rationals of small denominator, or more precisely, whose denominator is bounded above in terms of ε.

Theorem 32 {SFA}

Let N be a large real parameter, let $\delta > 0$, let a and $q \in [1, \exp(\log\log N)^2]$ be coprime integers and let $|q\alpha - a| \ll N^{-\frac{1}{2}-\delta}$. We have

$$\sum_{p \leqslant N} e(p\alpha) \ll \left(\frac{\sqrt{q}}{\varphi(q)} + \frac{(\log\log N)^3}{\log N} \right) \frac{N}{\log N}.$$

Lemma 27 is stronger than the above, but we propose a different and simpler proof, which is the core of the approach used in [24]. It has the advantage, from the viewpoint of these lectures, on relying only on an enveloping sieve. This proof relies of the Selberg Formula from [**49**] (see also [34] by T. Tatuzawa and

[38] I. Vinogradov, 1937, "Representation of an odd number as a sum of three primes".
[39] I. Vinogradov, 2004, *The method of trigonometrical sums in the theory of numbers*.
[26] O. Ramaré, 2013, "Prime numbers: emergence and victories of bilinear forms decomposition".

*It is worth mentionaing that, in that case, the inequality we need is a consequence of the orthogonality of the additive characters $n \mapsto e(na/N)$ combined with the sieve bound $\sum_{p_1+p_2=n} 1 \ll (n\varphi(n))n/(\log n)^2$, see [13, Theorem 3.11] by H. Halberstam and H.-E. Richert.
[24] O. Ramaré, 2010, "On Bombieri's asymptotic sieve".
[34] T. Tatuzawa and K. Iseki, 1951, "On Selberg's elementary proof of the prime-number theorem".

K. Iseki), namely the identity

$$\Lambda \log + \Lambda \star \Lambda = \mu \star \log^2, \tag{6.2}$$ {SelbergFormula}

i.e.

$$\Lambda(n)\log n + \sum_{\ell m = n} \Lambda(\ell)\Lambda(m) = \sum_{\ell m = n} \mu(\ell)(\log m)^2.$$

We shall also borrow an idea of I.M. Vinogradov that is also used by H. Daboussi in [5]. This last paper contains also an estimate for $\frac{\log N}{N}\sum_{p\leqslant N} e(p\alpha)$ that goes to 0 when q goes to infinity, but which relies on different sieve ingredient. We stick to the envelopping sieve since this is the theme of these notes. We set

$$\begin{aligned} S(\Lambda\log;\alpha) &= \sum_{n\leqslant N} \Lambda(n)(\log n)e(n\alpha) \\ &= \sum_{\ell m\leqslant N} \mu(\ell)(\log m)^2 e(\ell m\alpha) - \sum_{\ell m\leqslant N} \Lambda(\ell)\Lambda(m)e(\ell m\alpha) = S_1(\alpha) - S_2(\alpha) \end{aligned}$$

say. We readily transfer results from $S(\Lambda\log;\alpha)$ to $S(\Lambda;\alpha)$.

6.1. A variation on the Selberg Formula

Identity (6.2) is the consequence of the differentiation formula:

$$\frac{\zeta''}{\zeta} = \left(\frac{-\zeta'}{\zeta}\right)' + \left(\frac{-\zeta'}{\zeta}\right)^2.$$

Let us define

{defzetaq} $$\zeta_q(s) = \sum_{\substack{n\geqslant 1,\\ (n,q)=1}} \frac{1}{n^s} = \prod_{p|q}\left(1 - \frac{1}{p^s}\right)\zeta(s) = D_q(s)^{-1}\zeta(s) \tag{6.3}$$

say, so that

{eq:17} $$\frac{\zeta''}{\zeta_q} = D_q(s)\left(\frac{-\zeta'}{\zeta}\right)' + D_q(s)\left(\frac{-\zeta'}{\zeta}\right)^2. \tag{6.4}$$

Note that

$$D_q(s) = \sum_{t|q^\infty} \frac{1}{t^s}$$

where the notation '$t|q^\infty$' means that the prime factors of t divide q. As a consequence, we find that

{mthelp} $$\begin{aligned} \sum_{\substack{\ell m\leqslant Y,\\ (\ell,q)=1}} \mu(\ell)(\log m)^2 &= \sum_{t|q^\infty}\sum_{n\leqslant Y/t} \Lambda(n)\log n + \sum_{t|q^\infty}\sum_{\ell m\leqslant Y/t} \Lambda(\ell)\Lambda(m) \\ &\ll \sum_{\substack{t|q^\infty,\\ t\leqslant Y}} \frac{Y}{t}\log Y + \sum_{\substack{t|q^\infty,\\ t\leqslant Y}} \frac{Y}{t}\log Y \ll \frac{q}{\phi(q)} Y\log Y. \end{aligned} \tag{6.5}$$

[5] H. Daboussi, 2001, "Brun's fundamental lemma and exponential sums over primes".

6.2. Some auxiliaries

Lemma 33 {Vino}

Let a, q be two coprime positive integer and $\alpha = (a+\beta)/q$ where $|\beta| \leqslant 1/(3y)$. We have

$$\sum_{\substack{\ell \leqslant y,\\ (\ell,q)=1}} \frac{1}{\|\alpha\ell\|} \ll (y+q)\tau(q)\log(2q)$$

where $y \geqslant 1$ is an arbitrary positive real parameters and $\tau(q)$ is the number of divisors of q.

Proof. We split the integer interval $[1, y]$ in at most $1 + yq^{-1}$ intervals of length at most q. When ℓ is in a typical interval $[1+kq, q+kq]$ and is prime to q, the point $\alpha\ell$ remains close enough to $a\ell/q$. The points $(a\ell/q)_\ell$ remains $1/(3q)$-well-spaced and avoid 0. The corresponding contribution to our sum is $\ll q\log 2q$. However, when $(\ell, q) = q/d$, we need to proceed differently: group those the a's in d groups of q/d elements: where $a\ell$ is constant modulo d. The contribution is

$$\ll (q/d) \sum_{1 \leqslant b \leqslant d-1} \frac{1}{\|b/d\|} \ll q\log(2d).$$

The lemma follows readily. □

6.3. Study of $S_1(\alpha)$

In Vinogradov's terminology, $S_1(\alpha)$ is a 'Type I' sum, also called later the 'linear part'.

Truncation

We simply write

$$\begin{aligned} S_1(\alpha) &= \sum_{\ell \leqslant L} \sum_{m \leqslant N/\ell} \mu(\ell)(\log m)^2 e(\alpha\ell m) + \mathcal{O}\Big(\sum_{m \leqslant N/L} (\log m)^2 N/m\Big) \\ &= \sum_{\ell \leqslant L} \mu(\ell) \sum_{m \leqslant N/\ell} (\log m)^2 e(\alpha\ell m) + \mathcal{O}\big(N\log^3(N/L)\big). \end{aligned}$$

Cancellation due to the phase

We readily check that

$$\sum_{m\leqslant M}(\log m)^2 e(\alpha\ell m) \ll \min\left(M, \frac{1}{\|\alpha\ell\|}\right)(\log M)^2.$$

We use the decomposition

$$S_1(\alpha) = \sum_{\substack{\ell m\leqslant N,\\ q|\ell}} \mu(\ell)(\log m)^2 e(\alpha\ell m) \\ + \sum_{\substack{\ell\leqslant L,\\ q\nmid\ell}} \mu(\ell) \sum_{m\leqslant N/\ell} (\log m)^2 e(\alpha\ell m) + \mathcal{O}\big(N\log^3(N/L)\big).$$

On using Lemma 33, we infer that

$$S_1(\alpha) = \mu(q) \sum_{\substack{\ell m\leqslant N/q,\\ (\ell,q)=1}} \mu(\ell)(\log m)^2 e(q\alpha\ell m) \\ + \mathcal{O}\Big((L+q)\log(2q)(\log L)^2 + N\log^3(N/L)\Big).$$

We then use Eq. (6.5) and infer the bound

$$S_1(\alpha) \ll \frac{N\log N}{\varphi(q)} + (L+q)\log(2q)(\log L)^2 + N\log^3(N/L).$$

We take $L = N/(\log N)^3$ and get

{majS1}
$$S_1(\alpha) \ll \frac{N\log N}{\varphi(q)} + N(\log\log N)^3. \tag{6.6}$$

6.4. Study of $S_2(\alpha)$

In Vinogradov's terminology, $S_2(\alpha)$ is a 'Type II' sum, also called later the 'bilinear part'.

Preparation I

Let us first discard powers of primes. Let us momentarily set

{defLambdasharp}
$$\Lambda^\sharp(n) = \begin{cases} 0 & \text{when } \Omega(n) = 1, \\ \Lambda(n) & \text{otherwise.} \end{cases} \tag{6.7}$$

We readily check that $\sum_{n\leqslant y}\Lambda^\sharp(n) \ll \sqrt{y}$, hence

$$\sum_{n\leqslant N} (\Lambda\star\Lambda^\sharp)(n)e(n\alpha) \ll N.$$

Preparation II

Let us now discard the small primes. Let $N_0 = \exp(\log\log N)^3$. We have

$$\sum_{p_1 \leqslant N_0} \sum_{p_2 \leqslant N/p_1} (\log p_1)(\log p_2) e(\alpha p_1 p_2) \ll N(\log\log N)^3.$$

Sharp localization

To handle the noise from a/q to α, we study interval version of the sum to study. We follow I.M. Vinogradov and H. Daboussi in [5] to do so. First we can reduce our problem to studying

$$\sum_{N < p_1 p_2 \leqslant 2N} (\log p_1)(\log p_2) e(p_1 p_2 \alpha).$$

To do so, we consider

$$\Sigma\Big(N^*, Y, \frac{a}{q}\Big) = \sum_{\substack{N^* < p_1 p_2 \leqslant N^* + Y,\\ p_1, p_2 > N_0}} (\log p_1)(\log p_2) e(p_1 p_2 a/q) \tag{6.8}$$ {defSigma}

for $N \leqslant N^* \leqslant N^* + Y \leqslant 2N$.

Contribution of a diadic slice

We further reduce our study to consider

$$\Sigma_0(N^*, Y) = \sum_{p_1 \sim P} \log p_1 \sum_{\substack{N^* < p_1 p_2 \leqslant N^* + Y,\\ p_2 > p_1}} (\log p_2) e(p_1 p_2 a/q) \tag{6.9}$$ {defSigma0}

for some $P \in [N_0, \sqrt{N}]$. Let us select some parameter $z = P^4$ and use an enveloping sieve with parameters z though we discard the divisors of q, i.e.

$$\beta_q(n) = \Bigg(\sum_{\substack{d|n,\\ (d,q)=1}} \lambda_d\Bigg)^2, \quad \lambda_d = \mu(d) \frac{\frac{d}{\varphi(d)} G_{dq}(z/d)}{G_q(z)}. \tag{6.10}$$ {eq:21}

We thus get

$$|\Sigma_0(N^*, Y)|^2 \ll P(\log P) \sum_{n \sim P} \beta_q(n) \Bigg| \sum_{\substack{N^* < p_1 p_2 \leqslant N^* + Y,\\ p_2 > n}} (\log p_2) e(p_1 p_2 a/q) \Bigg|^2$$

and therefore

$$\frac{|\Sigma_0(N^*, Y)|^2}{P \log P} \ll \sum_{\frac{N^*}{2P} < p_2, p_2' \leqslant \frac{N^* + Y}{P}} \sum_{\substack{d_1, d_2 \leqslant z,\\ (d_1 d_2, q) = 1}} \lambda_{d_1} \lambda_{d_2} \sum_{m \in I(p_2, p_2', [d_1, d_2])} e\big(a(p_2 - p_2')[d_1, d_2] m/q\big)$$

[5] H. Daboussi, 2001, "Brun's fundamental lemma and exponential sums over primes".

where $I(p_2, p_2', d)$ is the interval obtained by the intersection of the intervals

$$\begin{aligned}
&P < dm \leqslant 2P, \quad dm < p_2, p_2'.\\
&\frac{N^*}{p_2} < dm \leqslant \frac{N^*+Y}{p_2}, \quad \frac{N^*}{p_2'} < dm \leqslant \frac{N^*+Y}{p_2'}.
\end{aligned}$$

It is of length $L(p_2, p_2')/d + \mathcal{O}(1)$ for some $L(p_2, p_2') \ll \min(P, Y/P)$.

Diagonal contribution

When $p_2 \equiv p_2'[q]$, we use

$$\sum_{\substack{d_1, d_2 \leqslant z,\\ (d_1 d_2, q) = 1}} \frac{\lambda_{d_1}\lambda_{d_2}}{[d_1, d_2]} = \frac{1}{G_q(z)}$$

to get the contribution

$$\frac{Y}{P\log(Y/P)} \frac{Y}{\varphi(q)P\log(Y/(qP))} \frac{1}{G_q(z)} \min(P, Y/P) \ll \frac{q}{\varphi(q)^2} \frac{Y^2}{P(\log N)^2 \log P}.$$

Off-diagonal contribution

Let us, on counting p_2 trivially by the Brun-Titchmarsh Theorem and discarding the primality condition for p_2', we readily get the bound

$$\sum_{\substack{p_2 - p_2' = u,\\ \frac{N^*}{2P} < p_2, p_2' \leqslant \frac{N^*+Y}{P}}} 1 \ll \frac{Y}{P\log(Y/P)}.$$

By Lemma 33, we get the contribution

$$\frac{Y}{P\log(Y/P)} z^2 \left(\frac{Y}{P} + q\right) \tau(q) \log q \ll \tau(q) \log q \frac{Y^2}{P^{3/2} \log N}.$$

since $q \ll N^\epsilon \leqslant Y/P$.

Contribution of a full slice

We have reached

$$|\Sigma_0(N^*, Y)|^2 \ll \frac{q}{\varphi(q)^2} \frac{Y^2}{(\log N)^2} + \tau(q)(\log q)(\log P) \frac{Y^2}{P^{1/2} \log N}$$

and, on recalling (6.8), this leads to

$$\Sigma\Big(N^*, Y, \frac{a}{q}\Big) \ll \frac{\sqrt{q}}{\varphi(q)} Y$$

valid for $q \leqslant N_0^{1/10}$ and $Y \geqslant N^{\frac{1}{2}+\varepsilon}$.

Adding an analytical offset

We find that

$$\Sigma\Big(N^*, Y, \frac{a+\beta}{q}\Big) = e\Big(\frac{N^*\beta}{q}\Big) \sum_{\substack{N^* < p_1p_2 \leqslant N^* + Y, \\ p_1, p_2 > N_0}} (\log p_1)(\log p_2) e\Big(\frac{p_1p_2a}{q}\Big) e\Big(\frac{(p_1p_2 - N^*)\beta}{q}\Big)$$
$$\ll \frac{\sqrt{q}}{\varphi(q)} Y \log N + \frac{Y|\beta|}{q} Y \log N \ll (1 + Y|\beta|) \frac{\sqrt{q}}{\varphi(q)} Y \log N.$$

Summary for S_2

On gathering the error terms coming from the preparations, and the final estimate, we readily reach

$$S_2(\alpha) \ll_\varepsilon (1 + N^{\frac{1}{2}+\varepsilon}|\beta|) \frac{\sqrt{q}}{\varphi(q)} Y \log N + N(\log\log N)^3. \tag{6.11}$$ {majS2}

....

Notation

The notation used throughout these notes is standard ... in one way or the other! Here is a guideline:

— $e(y) = \exp(2i\pi y)$.

— The use of the letter p for a variable always implies this variable is a prime number.

— $[d, d']$ stands for the lcm and (d, d') for the gcd of d and d'.

— $|\mathcal{A}|$ stands for the cardinality of the set $\mathcal{A}$ while $1_{\mathcal{A}}$ stands for its characteristic function.

— 1 denotes a characteristic function in one way or another. For instance, $1_{\mathcal{K}_d}$ is 1 if $n \in \mathcal{K}_d$ and 0 otherwise, but we could also write it as $1_{n\in\mathcal{K}_d}$, closer to what is often called the Dirac δ-symbol. We also use $1_{(n,d)=1}$ and $1_{q=q'}$.

— $\mathcal{P}$ is the set of prime numbers.

— $q\|d$ means that q divides d in such a way that q and d/q are coprime. In words we shall say that q *divides* d *exactly*.

— The squarefree kernel of the integer $d = \prod_i p_i^{\alpha_i}$ is $\prod_i p_i$, the product of all prime factors of d.

— $\omega(d)$ is the number of prime factors of d, counted without multiplicity.

— $\varphi(d)$ is the Euler totient, i.e. the cardinality of the multiplicative group of $\mathbb{Z}/d\mathbb{Z}$.

— $\tau(d)$ is the number of positive divisors of d.

— $\tau_k(d)$ is the number of k-tuples of (positive) integers $(d_1, \cdots, d_k)$ such that $d_1 \cdots d_k = d$, so that $\tau_2 = \tau$.

— $\mu(d)$ is the Moebius function, that is 0 when d is divisible by a square > 1 and otherwise $(-1)^r$ otherwise, where r is the number of prime factors of d.

— $c_q(n)$ is the Ramanujan sum. It is the sum of $e(an/q)$ over all a modulo q that are prime to q.

— $\Lambda(n)$ is van Mangoldt function: which is $\log p$ is n is a power of the prime p and 0 otherwise.

— The notation $f = \mathcal{O}_A(g)$ means that there exists a constant B such that $|f| \leqslant Bg$ but that this constant may depend on A. When we put in several parameters as subscripts, it simply means the implied constant depends on all of them.

— The notation $f = \mathcal{O}^*(g)$ means that $|f| \leqslant g$, that is a $\mathcal{O}$-like notation, but with an implied constant equal to 1.

— The notation $f \star g$ denotes the arithmetic convolution of f and g, that is to say the function h on positive integers such that $h(d) = \sum_{q|d} f(q)g(d/q)$. exists for every real number x.

— $\mathcal{U}$ is the compact set $(\mathcal{U}_d)_d$ where, for each d, $\mathcal{U}_d$ is the set of invertible elements modulo d.

— The letter ψ is used in two different context: either to denote the summatory function of the van Mangoldt function, that is to say $\psi(x) = \sum_{n \leqslant x} \Lambda(n)$, with the variation $\psi(x, \chi) = \sum_{n \leqslant x} \chi(n)\Lambda(n)$.

— We used the Chebyshev functions ϑ and ψ as well as their variations $\vartheta(x; \chi)$, $\vartheta(x; q, a)$, $\psi(x, \chi)$ and $\psi(x; q, a)$.

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We present in this series of lectures the idea of the *enveloping sieve* and the way its leads to transfer some properties of the sequence of integers to the sequence of prime. Termed the Transference Principle for Primes, this penomenom is for instance a leading idea of the proof by B.J. Green and T. Tao in [11] of the equivalent of Szemerédi's Theorem in the sequence of primes. Still following these two authors, we shall also prove a Hardy-Littelwood majorant property for the primes. The transition from the L^2-setting to the L^ℓ-setting will particularly detailled. As consequences, we shall investigate some ternary problems with primes that now admit particularly simple solutions. If time permits, we will conclude this series by proving a sharp exponential sum estimate for the trigonometric polynomial on the primes, that complements the artillery at our disposal.

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