EXPLICIT COUNT OF INTEGRAL IDEALS OF AN IMAGINARY QUADRATIC FIELD

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Abstract. We provide explicit bounds for the number of integral ideals of norms at most X is $\mathbb{Q}[\sqrt{d}]$ when $d<0$ is a fundamendal discriminant with an error term of size $\mathcal{O}(X^{1/3})$. In particular, we prove that, when χ is the non-principal character modulo 3 and $X\ge 1$, we have $\sum_{n\le X}(1\star\chi)(n)=\frac{\pi X}{3\sqrt{3}}+\mathcal{O}^*(1.94\,X^{1/3})$, and that , when χ is the non-principal character modulo 4 and $X\ge 1$, we have $\sum_{n\le X}(1\star\chi)(n)=\frac{\pi X}{4}+\mathcal{O}^*(1.4\,X^{1/3})$.

1. Introduction and results

General context. Let $\mathbb{K}$ be a number field, of degree $n_{\mathbb{K}}$, discriminant $\Delta(\mathbb{K})$, associated Dedekind zeta-function $\zeta_{\mathbb{K}}$ of residue $\kappa_{\mathbb{K}}$ at one. Counting the number of integral ideals of norm below some bound is a fundamental question that has been addressed by numerous authors. The explicit angle has been less popular and three pieces of works emerge: the paper [2] by K. Debaene, the PhD memoir [8] by J. Sunley and its upgraded version [7] by E.S. Lee. The first goes by lattice point counting, gets a dependence on the size of order $x^{1-1/n_{\mathbb{K}}}$ and treats the dependence in the field very finely. This approach is reused in [3] to enumerate integral ideals in ray classes. The second and third approaches follow the analytic treatment proposed by E. Landau: they get a better dependence on the size of order $x^{1-2/(n_{\mathbb{K}}+1)}$ but only rely on the discriminant of the field, an invariant which is notoriously large. The constants obtained have the clear advantage of being explicit but they remain gigantic: when $n_{\mathbb{K}}=2$, the error term in [7] reads

$$\mathcal{O}^*\left(8.81\cdot 10^{11}|\Delta_{\mathbb{K}}|^{\frac{1}{n_{\mathbb{K}}+1}}(\log|\Delta_{\mathbb{K}}|)^{n_{\mathbb{K}}-1}x^{1-\frac{2}{n_{\mathbb{K}}+1}}\right).$$

We aim here at being less demanding in generality but to gain in numerical precision.

Our results for imaginary quadratic number fields. Let d be a squarefree integer. We associate to this integer its fundamental discriminant defined by

$$\Delta_d=\begin{cases} d & \text{when } d\equiv 1[4],\\ 4d & \text{when } d\equiv 2,3[4].\end{cases} \tag{1}$$

The associated character is given in terms of the Kronecker symbol by the formula $\chi(n)=(\frac{\Delta_d}{n})$.

Theorem 1.1. *When $X\ge\max(|\Delta_d|,2c_0(d))$ and d is a negative squarefree integer, we have*

$$\sum_{n\le X}(1\star\chi)(n)=XL(1,\chi)+\frac{1}{2|\Delta_d|}\sum_{1\le r\le|\Delta_d|}r\chi(r)+\mathcal{O}^*\left(0.76\,L(1,\chi)c_0(d)X^{1/3}\right)$$

2010 *Mathematics Subject Classification.* Primary: 11N37, 11N45, Secondary: 11R11.
Key words and phrases. Quadratic character, imaginary quadratic fields.

where

$$c_0(d) = \max(c(3/4), c(5/4))^{2/3}, \tag{2}$$

and

$$c(3/4) = \max_{M\geq 1} \sum_{m\leq M} \frac{(1\star\chi)(m)}{m^{3/4}M^{1/4}L(1,\chi)},\quad c(5/4) = \frac{M^{1/4}}{L(1,\chi)} \sum_{m\geq M} \frac{(1\star\chi)(m)}{m^{5/4}}. \tag{3}$$

When $X \geq \max(130^2|\Delta_d|, 2c_0(d))$*, the constant* 0.76 *may be replaced by* 0.67*.*

Lemmas 3.2 and 3.3 propose upper bounds for $c(3/4)$ and $c(5/4)$. It is in particular proved that $\min(c(3/4), c(5/4)) \geq 4$. As the reader sees, imposing larger bounds reduces only marginally the final constant. It may still be of interest for very small values of d, where the range in X can be completed by direct computations.

Special cases. We start by some numerical verification done with a GP-Pari script.

Theorem 1.2. *Let χ be the non principal character modulo 4. For $X \in [1, 10^8]$, we have*

$$\sum_{n\leq X} (1\star\chi)(n) = \frac{\pi X}{4} + \mathcal{O}^*\big(2.08\, X^{1/4}\big).$$

The constant in the big-O seems to increase slightly when X increases.

Theorem 1.3. *Let χ be the non principal character modulo 3. For $X \in [1, 10^8]$, we have*

$$\sum_{n\leq X} (1\star\chi)(n) = \frac{\pi X}{3\sqrt{3}} + \mathcal{O}^*\big(1.63\, X^{1/4}\big).$$

Corollary 1.4. *Let χ be the non principal character modulo 4. For $X \geq 1$, we have*

$$\sum_{n\leq X} (1\star\chi)(n) = \frac{\pi X}{4} + \mathcal{O}^*\big(1.4\, X^{1/3}\big).$$

Corollary 1.5. *Let χ be the non principal character modulo 3. For $X \geq 1$, we have*

$$\sum_{n\leq X} (1\star\chi)(n) = \frac{\pi X}{3\sqrt{3}} + \mathcal{O}^*\big(1.94\, X^{1/3}\big).$$

Corollary 1.6. *Let χ be the quadratic character of $\mathbb{Q}[\sqrt{d}]$, where $-19 \leq d \leq -1$. For $X \geq 68$, we have*

$$\sum_{n\leq X} (1\star\chi)(n) = XL(1,\chi) + \frac{1}{2|\Delta|} \sum_{1\leq r\leq |\Delta_d|} r\chi(r) + \mathcal{O}^*\big(3.4\, X^{1/3}\big).$$

Methodology. E. Landau's approach in [5] (See also [6, Satz 210]) relies on several ingredients, but the first and main one is the functional equation of the associated Dedekind zeta function. E. Landau sends the line of integration to $\Re s = -1/2$, then uses the functional equation to study the last integral. This Landau's approach is described in modern language in [7]. The process used has become known as the Voronoï Summation Formula(s), based on [9, 10], though this latter is more commonly used for the divisor function. In fact, though the papers of Voronoï largely predates the ones of Landau, and it cannot be assumed that Landau did not know of them, Landau does not mention the Voronoï approach, a surprising fact as this author has most of the time been very prompt in explaining the genesis of ideas. This absence may be due to the combination of two facts: Landau worked in great generality, with intricate Gamma-factors, and a general view of the Voronoï process was missing at the time. This process is now well understood and is for instance well-documented in Chapter 10 of the book [1] by H. Cohen. The addition of Voronoï is to recognize the involved Mellin transform as a Bessel function and to consider a functional transform of the initial weight function, see Lemma 4.1 below. We follow this approach here.

Two more ingredients are being used: a non-negative smoothing device and an apriori trivial upper bound for the number of integral ideals below some bound to avoid divisor functions of x in the remainder term, see Lemma 3.2 and 3.3 below.

We do not examine what happens for the small values of the size with respect to the discriminant.

Thanks. Thanks are due to Olivier Bordellès for some helpful discussions on this subject.

2. On the Bessel functions

Lemma 2.1. *When $a > 0$, we have*

$$\int_0^X J_0(a\sqrt{t})dt = \frac{2\sqrt{X}}{a} J_1(a\sqrt{X})$$

and

$$\int_0^X tJ_0(a\sqrt{t})dt = \frac{4X}{a^2} J_2(a\sqrt{X}) + \frac{2X^{3/2}}{a} J_1(a\sqrt{X}).$$

Proof. Indeed, when $\nu \geq 1$, we have $J_\nu(t)' = J_{\nu-1}(t) - J_\nu(t)\nu/t$. Hence

$$(\sqrt{t}J_1(a\sqrt{t}))' = \frac{1}{2\sqrt{t}} J_1(a\sqrt{t}) + \frac{a}{2}\left(J_0(a\sqrt{t}) - \frac{1}{a\sqrt{t}} J_1(a\sqrt{t})\right) = \frac{a}{2} J_0(a\sqrt{t})$$

which proves the first formula. For the second one, we notice similarly that

$$\begin{aligned}(t^{3/2}J_1(a\sqrt{t}))' &= \frac{3\sqrt{t}}{2} J_1(a\sqrt{t}) + \frac{at}{2}\left(J_0(a\sqrt{t}) - \frac{1}{a\sqrt{t}} J_1(a\sqrt{t})\right) \\ &= \frac{a}{2} tJ_0(a\sqrt{t}) - \sqrt{t}J_1(a\sqrt{t})\end{aligned}$$

and

$$(tJ_2(a\sqrt{t}))' = J_2(a\sqrt{t}) + \frac{a\sqrt{t}}{2}\left(J_1(a\sqrt{t}) - \frac{2}{a\sqrt{t}} J_2(a\sqrt{t})\right) = \frac{a\sqrt{t}}{2} J_1(a\sqrt{t})$$

and therefore

$$\frac{d}{dx}\left(\frac{at^{3/2}}{2} J_1(a\sqrt{t}) + tJ_2(a\sqrt{t})\right) = \frac{a^2}{4} tJ_0(a\sqrt{t})$$

□

Lemma 2.2. *When $\nu > 0$ and $x \geq 0$, we have*

$$\left|J_\nu(x) - \sqrt{\frac{2}{\pi x}} \cos(x - (2\nu+1)\tfrac{\pi}{4})\right| \leq \frac{4|\nu^2 - 1/4|}{5x^{3/2}}.$$

When $\nu > 1/2$ and $x \geq 0$, we have $|x^2 - \nu^2 + \frac{1}{4}|^{1/4}|J_\nu(x)| \leq \sqrt{2/\pi}$.

Proof. The first inequality is given in a consequence of [4, Theorem 4] by Krasikov while the second one comes from [4, Theorem 3]. □

Lemma 2.3. *With*

$$T(z;a) = \frac{(1+z)J_2(a\sqrt{1+z}) - J_2(a)}{z} \tag{4}$$

we have, when $a \geq 4\pi$ and $z \in (0, 1/3]$:

$$|T(z,a)| \leq \min\left(0.53\sqrt{a}, \frac{7}{3z\sqrt{a}}\right).$$

When $a \ge 130 \cdot 4\pi$ and $z \in (0, 1/10)$:

$$|T(z,a)| \le \min\biggl(0.4\sqrt{a}, \frac{2.1}{z\sqrt{a}}\biggr).$$

Proof. By the mean value theorem and when $z > 0$, we find that

$$\begin{aligned}|T(z;a)| &\le \frac{a(\sqrt{1+z}-1)}{z}\max_{a\le t\le a\sqrt{1+z}}|J_2'(t)| + |J_2(a\sqrt{1+z})| \\ &\le \frac{a}{2}\max_{a\le t\le a\sqrt{1+z}}\biggl(|J_1(t)| + \frac{2}{t}|J_2(t)|\biggr) + |J_2(a\sqrt{1+z})|.\end{aligned}$$

The map $x \mapsto |x^2 - 3/4|^{1/4}$ is non-decreasing when $x \ge \sqrt{3}/2$ and the map $x \mapsto |x^2 - 7/4|^{1/4}$ is also non-decreasing when $x \ge \sqrt{7}/2$. On assuming that $a \ge \sqrt{7}/2$, Lemma 2.2 thus gives us that

$$\sqrt{\pi/2}|T(z;a)| \le \frac{a}{2|a^2 - 3/4|^{1/4}} + \frac{2}{|a^2 - 7/4|^{1/4}}.$$

When $a \ge 4\pi$, a rapid plot shows that $\sqrt{\pi/2}|T(z;a)| \le 0.66\sqrt{a}$, while, when $a \ge 130 \cdot 4\pi$, we find that $\sqrt{\pi/2}|T(z;a)| \le 0.5013\sqrt{a}$. This establishes the first bound. When a is large, it is better to simply use

$$\begin{aligned}|T(z;a)| &\le \frac{|J_2(a\sqrt{1+z})| + |J_2(a)|}{z} + |J_2(a\sqrt{1+z})| \\ &\le \frac{2}{z|a^2 - 7/4|^{1/4}} + \frac{1}{|a^2(1+z) - 7/4|^{1/4}} \le \frac{7}{3z\sqrt{a}}\end{aligned}$$

which we prove first for $z \le 1/4$ by using $|a^2(1+z) - 7/4|^{1/4} \ge |a^2 - 7/4|^{1/4}$ and then on discretizing the interval (such precision is not required, it only leads to be better looking estimate). When $z \le 1/10$ and $a \ge 130 \cdot 4\pi$, we find that $|T(z;a)| \le 2.1/(z\sqrt{a})$. □

3. Some a priori estimates

Let us start with a well-known estimate.

Lemma 3.1. *When $s > 0$ and $s \ne 1$, we have*

$$\sum_{m\le M}\frac{1}{m^s} = \frac{M^{1-s}}{1-s} + \zeta(s) + \mathcal{O}^*(1/M^s).$$

Proof. Indeed, we find that

$$\begin{aligned}\sum_{m\le M}\frac{1}{m^s} &= s\int_1^M [t]\frac{dt}{t^{s+1}} + \frac{[M]}{M^s} = \frac{s}{s-1} - s\int_1^M \{t\}\frac{dt}{t^{s+1}} + \frac{M^{1-s}}{1-s} - \frac{\{M\}}{M^s} \\ &= \frac{s}{s-1} - s\int_1^\infty \{t\}\frac{dt}{t^{s+1}} + \frac{M^{1-s}}{1-s} + s\int_M^\infty \{t\}\frac{dt}{t^{s+1}} - \frac{\{M\}}{M^s} \\ &= \zeta(s) + \frac{M^{1-s}}{1-s} + s\int_M^\infty \{t\}\frac{dt}{t^{s+1}} - \frac{\{M\}}{M^s}.\end{aligned}$$

We finally check that

$$s\int_M^\infty \{t\}\frac{dt}{t^{s+1}} - \frac{\{M\}}{M^s} = s\int_M^\infty (\{t\} - \{M\})\frac{dt}{t^{s+1}}$$

and the lemma follows readily. □

The next lemma gives an upper bound as well as a mean to approximate $c(s)$, when $s \in (0, 1)$.

Lemma 3.2. *When $s \in (0,1)$, we have*

$$\sum_{n\le N}\frac{(1\star\chi)(n)}{n^sN^{1-s}} \le \frac{L(1,\chi)}{1-s}+\frac{|\zeta(s)L(s,\chi)|}{N^{1-s}}+\frac{(\frac14+|\zeta(s)|+\frac{5}{1-s})\Omega(\chi)}{\sqrt{N}}$$

where

$$\Omega(\chi)=\max_{L\ge1}\Big|\sum_{\ell\le L}\chi(\ell)\Big|. \tag{5}$$

Moreover, the quantity considered is asymptotic to $L(1,\chi)/(1-s)$.

Proof. By using Lemma 3.1 and the Dirichlet hyperbola formula, we find that the sum S to be computed equals (with parameters $L\ge1$ and $M\ge1$ such that $LM=N$)

$$\begin{aligned}
S&=\sum_{\ell\le L}\frac{\chi(\ell)}{\ell^s}\left(\zeta(s)+\frac{(N/\ell)^{1-s}}{1-s}+\mathcal{O}^*\left(\frac{\ell^s}{4N^s}\right)\right)+\sum_{m\le M}\frac{1}{m^s}\mathcal{O}^*\big(2\Omega(\chi)/L^s\big)\\
&=\zeta(s)\sum_{\ell\le L}\frac{\chi(\ell)}{\ell^s}+\frac{N^{1-s}}{1-s}\sum_{\ell\le L}\frac{\chi(\ell)}{\ell}+\mathcal{O}^*\left(\frac{L}{4N^s}+2\Omega(\chi)\frac{1-s+M^{1-s}}{(1-s)L^s}\right)\\
&=\zeta(s)L(s,\chi)+\frac{N^{1-s}L(1,\chi)}{1-s}\\
&\quad+\mathcal{O}^*\left(\frac{N^{1-s}\Omega(\chi)}{(1-s)L}+\frac{L}{4N^s}+2\Omega(\chi)\frac{1-s+M^{1-s}}{(1-s)L^s}+|\zeta(s)|\frac{\Omega(\chi)}{L^s}\right)
\end{aligned}$$

so that

$$\begin{aligned}
S/N^{1-s}\le&\frac{|\zeta(s)L(s,\chi)|}{N^{1-s}}+\frac{L(1,\chi)}{1-s}+\frac{\Omega(\chi)}{(1-s)L}+\frac{L}{4N}\\
&+2\Omega(\chi)\frac{1-s+M^{1-s}}{(1-s)L^sN^{1-s}}+|\zeta(s)|\frac{\Omega(\chi)}{L^sN^{1-s}}.
\end{aligned}$$

The simplistic choice $L=M=N^{1/2}$ leads to

$$S/N^{1-s}\le\frac{|\zeta(s)L(s,\chi)|}{N^{1-s}}+\frac{L(1,\chi)}{1-s}+\frac{5\Omega(\chi)+\frac14}{(1-s)\sqrt{N}}+|\zeta(s)|\frac{\Omega(\chi)}{N^{1-s/2}}.$$

□

Our final tool in this section is the next lemma which gives an upper bound as well as a mean to approximate $c(s)$, when $s>1$.

Lemma 3.3. *When $s>1$, we have*

$$N^{s-1}\sum_{n\ge N}\frac{(1\star\chi)(n)}{n^s}\le\frac{L(1,\chi)}{s-1}+\frac{(3\zeta(s)+\frac{1}{s-1})\Omega(\chi)+\frac14}{\sqrt{N}}.$$

Moreover, the quantity considered is asymptotic to $L(1,\chi)/(s-1)$.

Proof. We proceed as in Lemma 3.2. On setting $S=\sum_{n\le N}(1\star\chi)(n)/n^s$, we find that (with parameters $L\ge1$ and $M\ge1$ such that $LM=N$)

$$\begin{aligned}
S&=\sum_{\ell\le L}\frac{\chi(\ell)}{\ell^s}\left(\zeta(s)-\frac{(\ell/N)^{s-1}}{s-1}+\mathcal{O}^*\left(\frac{\ell^s}{4N^s}\right)\right)+\sum_{m\le M}\frac{1}{m^s}\mathcal{O}^*\big(2\Omega(\chi)/L^s\big)\\
&=\zeta(s)\sum_{\ell\le L}\frac{\chi(\ell)}{\ell^s}-\frac{1}{(s-1)N^{s-1}}\sum_{\ell\le L}\frac{\chi(\ell)}{\ell}+\mathcal{O}^*\left(\frac{L}{4N^s}+2\Omega(\chi)\frac{\zeta(s)}{L^s}\right)\\
&=\zeta(s)L(s,\chi)-\frac{L(1,\chi)}{(s-1)N^{s-1}}+\mathcal{O}^*\left(\frac{\Omega(\chi)}{(s-1)LN^{1-s}}+\frac{L}{4N^s}+3\Omega(\chi)\frac{\zeta(s)}{L^s}\right)
\end{aligned}$$

so that

$$(\zeta(s)L(s,\chi)-S)N^{s-1}\le \frac{L(1,\chi)}{s-1}+\frac{\Omega(\chi)}{(s-1)L}+\frac{L}{4N}+3\Omega(\chi)\frac{\zeta(s)N^{s-1}}{L^s}.$$

We take $L=M=N^{1/2}$ and get

$$(\zeta(s)L(s,\chi)-S)N^{s-1}\le \frac{L(1,\chi)}{s-1}+\frac{(3\zeta(s)+\frac{1}{s-1})\Omega(\chi)+\frac14}{\sqrt{N}}.$$

(explain the shift $N+1\mapsto N$). □

Proof. We readily find that

$$\sum_{\ell\le L}\frac{\chi(\ell)}{\ell^s}=s\int_1^L\sum_{\ell\le t}\chi(\ell)\frac{dt}{t^{s+1}}+\frac{\sum_{\ell\le L}\chi(\ell)}{L^s}$$

and thus

$$L^s\sum_{\ell\ge L}\frac{\chi(\ell)}{\ell^s}=sL^s\int_L^\infty\sum_{\ell\le t}\chi(\ell)\frac{dt}{t^{s+1}}-\sum_{\ell\le L}\chi(\ell)$$

□

4. Around the Voronoï Summation Formula

Here is the Voronoï Summation Formula we want to use.

Lemma 4.1. *Let $f:[0,\infty)\mapsto\mathbb{C}$ be a function that we assume to have a finite number of simple discontinuities (i.e. with finite left and right limits), to be is piecewise C^∞ and piecewise monotonic and such that f and all its derivatives tend to zero faster than any power of x at infinity. We have*

$$\begin{aligned}\sum_{n\ge1}(1\star\chi)(n)f(n)=&L(1,\chi)\check{f}(0)+\frac{f(0)}{2|\Delta_d|}\sum_{1\le r\le|\Delta_d|}r\chi(r)\\&+\frac{2\pi}{\sqrt{|\Delta_d|}}\sum_{m\ge1}(1\star\chi)(m)\int_0^\infty f(t)J_0\big(4\pi\sqrt{nt/|\Delta_d|}\big)dt.\end{aligned}$$

Proof. Let us denote by ζ_d the Dedekind zeta-function of the imaginary quadratic field $\mathbb{Q}[\sqrt{d}]$. It satisfies the functional equation

$$\gamma_d(1-s)\zeta_d(1-s)=\gamma_d(s)\zeta_d(s)\quad\text{where}\quad\gamma_d(s)=\left(\frac{\sqrt{|\Delta_d|}}{2\pi}\right)^s\Gamma(s).\tag{6}$$

We use [1, Theorem 10.2.17] by H. Cohen. The kernel to be considered is

$$\begin{aligned}K_d(x)&=\frac{1}{2i\pi}\int_{\frac12-i\infty}^{\frac12+i\infty}\left(\frac{\sqrt{|\Delta_d|}}{2\pi}\right)^s\left(\frac{\sqrt{|\Delta_d|}}{2\pi}\right)^{s-1}\frac{\Gamma(s)}{\Gamma(1-s)}x^{-s}ds\\&=\frac{1}{2i\pi}\frac{2\pi}{\sqrt{|\Delta_d|}}\int_{\frac12-i\infty}^{\frac12+i\infty}\frac{4^s\Gamma(s)}{\Gamma(1-s)}\left(\frac{16\pi^2x}{|\Delta_d|}\right)^{-s}ds.\end{aligned}$$

The formula

$$4^s\frac{\Gamma(s)}{\Gamma(1-s)}=\int_0^\infty t^{s-1}J_0(\sqrt{t})dt$$

gives us

$$K_d(x)=\frac{2\pi}{\sqrt{|\Delta_d|}}J_0\big(4\pi\sqrt{x/|\Delta_d|}\big).$$

Concerning the value at 0, we use $\zeta(0)=-1/2$ and

$$L(0,\chi)=\frac{-1}{|\Delta_d|}\sum_{1\le r\le|\Delta_d|}r\chi(r)\tag{7}$$

as per [1, Corollary 10.3.2]. □

Lemma 4.2. *We have*

$$\sum_{n\le X}\Bigl(1-\frac{n}{X}\Bigr)(1\star\chi)(n) = \frac{XL(1,\chi)}{2} + \frac{1}{2|\Delta_d|}\sum_{1\le r\le|\Delta_d|} r\chi(r) + \frac{\sqrt{|\Delta_d|}}{2\pi}\sum_{m\ge1}\frac{(1\star\chi)(m)}{m}J_2\bigl(4\pi\sqrt{mX/|\Delta_d|}\bigr).$$

Proof. By using Lemma 4.1, we readily find that

$$\sum_{n\le X}\Bigl(1-\frac{n}{X}\Bigr)(1\star\chi)(n) = \frac{XL(1,\chi)}{2} + \frac{1}{2|\Delta_d|}\sum_{1\le r\le|\Delta_d|} r\chi(r) + \frac{2\pi}{\sqrt{|\Delta_d|}}\sum_{m\ge1}(1\star\chi)(m)\int_0^X\Bigl(1-\frac{t}{X}\Bigr)J_0\bigl(4\pi\sqrt{mt/|\Delta_d|}\bigr)dt.$$

By using Lemma 2.1, we find that

$$\sum_{n\le X}\Bigl(1-\frac{n}{X}\Bigr)(1\star\chi)(n) = \frac{XL(1,\chi)}{2} + \frac{1}{2|\Delta_d|}\sum_{1\le r\le|\Delta_d|} r\chi(r) + \frac{2\pi}{\sqrt{|\Delta_d|}}\sum_{m\ge1}(1\star\chi)(m)\frac{|\Delta_d|}{4\pi^2 m}J_2\bigl(4\pi\sqrt{mX/|\Delta_d|}\bigr).$$

Our lemma follows swiftly from this last expression. □

5. Main engine

Lemma 5.1. *When $Y\in[0,X/3]$ and $X\ge|\Delta_d|$, we have*

$$\sum_{n\le X}(1\star\chi)(n) + \sum_{X<n\le X+Y}\frac{X+Y-n}{Y}(1\star\chi)(n) = \frac{(2X+Y)L(1,\chi)}{2} + \frac{1}{2|\Delta_d|}\sum_{1\le r\le|\Delta_d|} r\chi(r) + \mathcal{O}^*\Bigl(0.36\,C_0(d)\sqrt{X|\Delta_d|/Y}\Bigr)$$

where $C_0(d)=L(1,\chi)c_0(d)$. When $Y\in[0,X/10]$ and $X\ge 130^2|\Delta_d|$, the constant 0.36 may be reduced to 0.292.

Proof. By using Lemma 4.2 twice, we find that

$$(1/Y)\sum_{n\le X}\Bigl[(X+Y)\Bigl(1-\frac{n}{X+Y}\Bigr)-X\Bigl(1-\frac{n}{X}\Bigr)\Bigr](1\star\chi)(n) = \frac{(2X+Y)L(1,\chi)}{2} + \frac{1}{2|\Delta_d|}\sum_{1\le r\le|\Delta_d|} r\chi(r) + \frac{\sqrt{|\Delta_d|}}{2\pi}\sum_{m\ge1}\frac{(1\star\chi)(m)}{m}T(Y/X;4\pi\sqrt{mX/|\Delta_d|}).$$

We now majorize the last sum by appealing to Lemma 2.3. Recall that we assume that $X\ge|\Delta_d|$ (resp. $x\ge130^2|\Delta_d|$). We use the first estimate of Lemma 2.3 when

$$4\pi\sqrt{mX/|\Delta_d|}\le\frac{7X}{3Y\cdot0.53}\quad\Bigl(\text{resp. } 4\pi\sqrt{mX/|\Delta_d|}\le\frac{2.1X}{Y\cdot0.4}\Bigr)$$

i.e. when

$$m\ge M=\frac{49|\Delta_d|X}{144(0.53\pi)^2Y^2}\quad\Bigl(\text{resp. } m\ge M=\frac{2.1^2|\Delta_d|X}{16(0.4\pi)^2Y^2}\Bigr).$$

We thus get

$$\frac{\sqrt{|\Delta_d|}}{2\pi}\sum_{m\ge1}\frac{(1\star\chi)(m)}{m}|T(Y/X;4\pi\sqrt{mX/|\Delta_d|})|$$
$$\le 0.53\frac{|\Delta_d|^{1/4}X^{1/4}}{\sqrt{\pi}}\sum_{m\le M}\frac{(1\star\chi)(m)}{m^{3/4}}+\frac{7X^{3/4}}{3Y}\frac{|\Delta_d|^{3/4}}{4\pi^{3/2}}\sum_{m>M}\frac{(1\star\chi)(m)}{m^{5/4}}.$$

We recall taht $C_0(d)=L(1,\chi)c_0(d)$ where $c_0(d)$ is being defined in (2). On appealing to Lemmas 3.2 and 3.3, this leads to

$$\frac{\sqrt{|\Delta_d|}}{2\pi}\sum_{m\ge1}\frac{(1\star\chi)(m)}{m}|T(Y/X;4\pi\sqrt{mX/|\Delta_d|})|$$
$$\le 0.53C_0(d)\frac{|\Delta_d|^{1/4}X^{1/4}}{\sqrt{\pi}}M^{1/4}+C_0(d)\frac{7X^{3/4}}{3Y}\frac{|\Delta_d|^{3/4}}{4\pi^{3/2}M^{1/4}}$$
$$\le C_0(d)\frac{X^{1/2}\sqrt{0.53}\sqrt{7/3}}{Y^{1/2}\pi}|\Delta_d|^{1/2}\le 0.36\,C_0(d)\sqrt{X|\Delta_d|/Y}.$$

When $Y\in[0,X/10]$ and $X\ge 130^2|\Delta_d|$, the constant 0.36 can be replaced by an upper bound for $\sqrt{2.1\cdot0.4}/\pi$, e.g. 0.292. The proof is complete. □

Final proof. We use Lemma 5.1 with $X-Y+Y$ and $X+Y$. Thus

$$\frac{-Y}{2}L(1,\chi)-0.36\,C_0(d)\sqrt{(X-Y)|\Delta_d|/Y}\le S-XL(1,\chi)+\frac{1}{2|\Delta_d|}\sum_{1\le r\le|\Delta_d|}r\chi(r)$$
$$\le\frac{Y}{2}L(1,\chi)+0.36\,C_0(d)\sqrt{X|\Delta_d|/Y}.$$

We select

$$Y=\left(\frac{0.36\,C_0(d)\sqrt{X}}{L(1,\chi)}\right)^{2/3}\tag{8}$$

getting the error term

$$X^{1/3}C_0(d)^{2/3}L(1,\chi)^{1/3}\left(\frac{0.36^{2/3}}{2}+0.36^{2/3}\right).$$

We find that $Y\le X/3$ when

$$3^{3/2}\frac{0.36\,C_0(d)}{L(1,\chi)}\le X.$$

The theorem follows readily in that case. The case $X\ge130^2|\Delta_d|$ is treated similarly. □

6. Computing $C_0(d)$

We use the script `ConvolutionAndVoronoi-01.gp` and the function `run(1000000, d)` therein to build the next table.

d	Δ_d	$\Omega(\chi)$	$L(1,\chi)c_0(d) \le$
-1	-4	1	2.04
-2	-8	2	2.89
-3	-3	1	1.58
-5	-20	4	3.66
-6	-24	4	3.35
-7	-7	2	3.09
-10	-40	4	2.60
-11	-11	3	2.48
-13	-52	5	2.30
-14	-56	8	4.40
-15	-15	3	4.21
-17	-68	8	4.01
-19	-19	3	1.90

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