The TME-EMT Project

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The PDF Notes

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Théorie Multiplicative Explicite des nombres

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Explicit Multiplicative number Theory

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Introduction

Fully explicit results in multiplicative number theory are often scattered through the litterature. The aim of this site is to present an annoted bibliography in order to keep track of the current knowledge. By the way, the acronym TME-EMT stands for

Théorie Multiplicative Explicite des nombres
—
Explicit Multiplicative number Theory

Being up-to date is a difficulty, so please do not consider this site as presenting the best available, but as a first line on which to strengthen more specialized inquiries.

Contents

Notation

Notation is standard, except maybe for the following one: we write $f = O^*(g)$ to say that $|f| \leq g$. This is simply a Landau-bigO symbol with an implied constant equal to one. Furthermore, the letter p always denotes a prime variable.

Part I

Averages of arithmetical functions

Chapter 1

Explicit bounds on primes

Corresponding html file: `../Articles/Art01.html`

Collecting references: [P. Dusart, 58], [P. Dusart, 63],

1.1 Bounds on primes, in special ranges

The paper [J. Rosser and L. Schoenfeld, 180], contains several bounds valid only when the variable is small enough.

In [J. Büthe, 25], the author proves the next theorem.

Theorem (2016) Assume the Riemann Hypothesis has been checked up to height H_0. Then when x satisfies $\sqrt{x/\log x} \le H_0/4.92$, we have

- $|\psi(x) - x| \le \frac{\sqrt{x}}{8\pi} \log^2 x$ when $x > 59$,
- $|\theta(x) - x| \le \frac{\sqrt{x}}{8\pi} \log^2 x$ when $x > 599$,
- $|\pi(x) - \mathrm{li}(x)| \le \frac{\sqrt{x}}{8\pi} \log x$ when $x > 2657$.

If we use the value $H_0 = 30\,610\,046\,000$ obtained by D. Platt in [D. J. Platt, 152], these bounds are thus valid for $x \le 1.8 \cdot 10^{21}$.

In [J. Büthe, 24], the following bounds are also obtained.

Theorem (2018) We have

- $|\psi(x) - x| \le 0.94\sqrt{x}$ when $11 < x \le 10^{19}$,
- $0 < \mathrm{li}(x) - \pi(x) \le \frac{\sqrt{x}}{\log x}\left(1.95 + \frac{3.9}{\log x} + \frac{19.5}{\log^2 x}\right)$ when $2 \le x \le 10^{19}$.

[58] P. Dusart, 1998, "Autour de la fonction qui compte le nombre de nombres premiers".
[63] P. Dusart, 2016, "Estimates of ψ, θ for large values of x without the Riemann hypothesis".
[180] J. Rosser and L. Schoenfeld, 1962, "Approximate formulas for some functions of prime numbers".
[25] J. Büthe, 2016, "Estimating $\pi(x)$ and related functions under partial RH assumptions".
[152] D. J. Platt, 2017, "Isolating some non-trivial zeros of zeta".
[24] J. Büthe, 2018, "An analytic method for bounding $\psi(x)$".

1.2 Bounds on primes, without any congruence condition

The subject really started with the four papers [J. Rosser, 176], [J. Rosser and L. Schoenfeld, 180], [J. Rosser and L. Schoenfeld, 181] and [L. Schoenfeld, 186]. We recall the usual notation: $\pi(x)$ is the number of primes up to x (so that $\pi(3) = 2$), the function $\psi(x)$ is the summatory function of the van Mangold function Λ, i.e. $\psi(x) = \sum_{n\le x} \Lambda(n)$, while we also define $\vartheta(x) = \sum_{p\le x} \log p$. Here are some elegant bounds that one can find in these papers.

Theorem (1962)

- For $x > 0$, we have $\psi(x) \le 1.03883x$ and the maximum of $\psi(x)/x$ is attained at $x = 113$.
- When $x \ge 17$, we have $\pi(x) > x/\log x$.
- When $x > 1$, we have $\displaystyle\sum_{p\le x} 1/p > \log\log x$.
- When $x > 1$, we have $\displaystyle\sum_{p\le x} (\log p)/p < \log x$.

There are many other results in these papers. In [P. Dusart, 61], on can find among other things the inequality

$$\text{When } x \ge 17, \text{ we have } \pi(x) > \frac{x}{\log x - 1}.$$

And also

Theorem (1999)

- When $x \ge e^{22}$, we have $\psi(x) = x + O^*\Big(0.006409\dfrac{x}{\log x}\Big)$.
- When $x \ge 10\,544\,111$, we have $\vartheta(x) = x + O^*\Big(0.006788\dfrac{x}{\log x}\Big)$.
- When $x \ge 3\,594\,641$, we have $\vartheta(x) = x + O^*\Big(0.2\dfrac{x}{\log^2 x}\Big)$.
- When $x > 1$, we have $\vartheta(x) = x + O^*\Big(515\dfrac{x}{\log^3 x}\Big)$.

[176] J. Rosser, 1941, "Explicit bounds for some functions of prime numbers".
[180] J. Rosser and L. Schoenfeld, 1962, "Approximate formulas for some functions of prime numbers".
[181] J. Rosser and L. Schoenfeld, 1975, "Sharper bounds for the Chebyshev Functions $\vartheta(X)$ and $\psi(X)$".
[186] L. Schoenfeld, 1976, "Sharper bounds for the Chebyshev Functions $\vartheta(X)$ and $\psi(X)$ II"..
[61] P. Dusart, 1999, "Inégalités explicites pour $\psi(X)$, $\theta(X)$, $\pi(X)$ et les nombres premiers".

This is improved in [P. Dusart, 60], and in particular, it is shown that the 515 above can be replaced by 20.83 and also that

$$\text{When } x \geq 89\,967\,803, \text{ we have } \vartheta(x) = x + O^*\Big(\frac{x}{\log^3 x}\Big).$$

Bounds of the shape $|\psi(x) - x| \leq \epsilon x$ have started appearing in [J. Rosser and L. Schoenfeld, 180]. The latest paper is [L. Faber and H. Kadiri, 66] with its corrigendum [**Kadiri-Faber*18**], where the explicit density estimate from [H. Kadiri, 94] is put to contribution, even for moderate values of the variable. In particular

$$\text{When } x \geq 485\,165\,196, \text{ we have } \psi(x) = x + O^*(0.00053699\,x).$$

Refined bounds for $\pi(x)$ are to be found in [L. Panaitopol, 148] and in [C. Axler, 2].

By comparing in an efficient manner with $\psi(x) - x$, [O. Ramaré, 163], obtained the next two results.

Theorem (2013) For $x > 1$, we have $\sum_{n\leq x} \Lambda(n)/n = \log x - \gamma + O^*(1.833/\log^2 x)$. When $x \geq 23$, we can replace the error term by $O^*(0.0067/\log x)$. Furthermore, when $1 \leq x \leq 10^{10}$, this error term can be replaced by $O^*(1.31/\sqrt{x})$.

Theorem (2013) For $x \geq 8950$, we have

$$\sum_{n\leq x} \Lambda(n)/n = \log x - \gamma + \frac{\psi(x) - x}{x} + O^*\Big(\frac{1}{2\sqrt{x}} + 1.75 \cdot 10^{-12}\Big)$$

.

[R. Mawia, 128] developed the method to incorporate more functions (and corrected the initial work), extending results of [J. Rosser and L. Schoenfeld, 180].

Here are some of his results.

Theorem (2017) For $x \geq 2$, we have

$$\sum_{p\leq x} \frac{1}{p} = \log\log x + B + O^*\Big(\frac{4}{\log^3 x}\Big).$$

When $x \geq 1000$, one can replace the 4 in the error term by 2.3, and when $x \geq 24284$, by 1. The constant B is the expected one.

[60] P. Dusart, 2018, “Estimates of some functions over primes”.
[66] L. Faber and H. Kadiri, 2015, “New bounds for $\psi(x)$”.
[94] H. Kadiri, 2013, “A zero density result for the Riemann zeta function”.
[148] L. Panaitopol, 2000, “A formula for $\pi(x)$ applied to a result of Koninck-Ivić”.
[2] C. Axler, 2016, “New bounds for the prime counting function”.
[163] O. Ramaré, 2013, “Explicit estimates for the summatory function of $\Lambda(n)/n$ from the one of $\Lambda(n)$”.
[128] R. Mawia, 2017, “Explicit estimates for some summatory functions of primes”.

Theorem (2017) For $\log x \geq 4635$, we have

$$\sum_{p\leq x}\frac{1}{p} = \log\log x + B + O^*\Big(1.1\frac{\exp(-\sqrt{0.175\log x})}{(\log x)^{3/4}}\Big).$$

When truncating sums over primes, Lemma 3.2 of [O. Ramaré, 158] is handy.

Theorem (2016) Let f be a C^1 non-negative, non-increasing function over $[P,\infty[$, where $P \geq 3\,600\,000$ is a real number and such that $\lim_{t\to\infty} tf(t) = 0$. We have

$$\sum_{p\geq P} f(p)\log p \leq (1+\epsilon)\int_P^\infty f(t)dt + \epsilon Pf(P) + Pf(P)/(5\log^2 P)$$

with $\epsilon = 1/914$. When we can only ensure $P \geq 2$, then a similar inequality holds, simply replacing the last $1/5$ by a 4.

The above result relies on (5.1*) of [L. Schoenfeld, 186] because it is easily accessible. However on using Proposition 5.1 of [P. Dusart, 63], one has access to $\epsilon = 1/36260$.

Here is a result due to [E. Treviño, 196].

Theorem (2012) For x a positive real number. If $x \geq x_0$ then there exist c_1 and c_2 depending on x_0 such that

$$\frac{x^2}{2\log x} + \frac{c_1 x^2}{\log^2 x} \leq \sum_{p\leq x} p \leq \frac{x^2}{2\log x} + \frac{c_2 x^2}{\log^2 x}.$$

The constants c_1 and c_2 are given for various values of x_0 in the next table.

x_0	c_1	c_2
315437	0.205448	0.330479
468577	0.211359	0.32593
486377	0.212904	0.325537
644123	0.21429	0.322609
678407	0.214931	0.322326
758231	0.215541	0.321504
758711	0.215939	0.321489
10544111	0.239818	0.29251

[158] O. Ramaré, 2016, “An explicit density estimate for Dirichlet L-series”.
[186] L. Schoenfeld, 1976, “Sharper bounds for the Chebyshev Functions $\vartheta(X)$ and $\psi(X)$ II”..
[63] P. Dusart, 2016, “Estimates of ψ,θ for large values of x without the Riemann hypothesis”.
[196] E. Treviño, 2012, “The least inert prime in a real quadratic field”.

1.3 Bounds on the n-th prime

Denote by p_n the n-th prime. Thus $p_1 = 2,\ p_2 = 3,\ p_4 = 5, \cdots$.

The classical form of prime number theorem yields easily $p_n \sim n \log n$.

[J. Rosser, 179] shows that this equivalence does not oscillate by proving that p_n is greater than $n \log n$ for $n \geq 2$.

The asymptotic formula for p_n can be developped as shown in [M. Cipolla, 31]:

$$p_n \sim n \left(\log n + \log\log n - 1 + \frac{\log\log n - 2}{\log n} - \frac{(\ln\ln n)^2 - 6\log\log n + 11}{2\log^2 n} + \cdots \right).$$

This asymptotic expansion is the inverse of the logarithmic integral $\mathrm{Li}(x)$ obtained by series reversion.

But [J. Rosser, 179] also proved that for every $n > 1$:

$$n(\log n + \log\log n - 10) < p_n < n(\log n + \log\log n + 8).$$

He improves these results in [J. Rosser, 176] : for every $n \geq 55$,

$$n(\log n + \log\log n - 4) < p_n < n(\log n + \log\log n + 2).$$

This result was subsequently improved by Rosser and Schoenfeld [J. Rosser and L. Schoenfeld, 180] in 1962 to

$$n(\log n + \log\log n - 3/2) < p_n < n(\log n + \log\log n - 1/2),$$

for $n > 1$ and $n > 19$ respectively.

The constants were subsequently reduced by [G. Robin, 174]. In particular, the lower bound

$$n(\log n + \log\log n - 1.0072629) < p_n$$

is valid for $n > 1$ and the constant 1.0072629 can be replaced by 1 for $p_k \leq 10^{11}$. Then [J.-P. Massias and G. Robin, 126] showed that the lower bound constant equals to 1 was admissible for $p_k < e^{598}$ and $p_k > e^{1800}$. Finally, [P. Dusart, 62] showed that

$$n(\log n - \log\log n - 1) < p_n$$

for all $n > 1$, and also that

$$p_n < n(\log n + \log\log n - 0.9484)$$

[179] J. Rosser, 1938, "The n-th prime is greater than $n \log n$".
[31] M. Cipolla, 1902, "La determinatzione assintotica dell'n^{imo} numero primo".
[176] J. Rosser, 1941, "Explicit bounds for some functions of prime numbers".
[180] J. Rosser and L. Schoenfeld, 1962, "Approximate formulas for some functions of prime numbers".
[174] G. Robin, 1983, "Estimation de la fonction de Tchebychef θ sur le k-ième nombres premiers et grandes valeurs de la fonction $\omega(n)$ nombre de diviseurs premiers de n".
[126] J.-P. Massias and G. Robin, 1996, "Bornes effectives pour certaines fonctions concernant les nombres premiers".
[62] P. Dusart, 1999, "The kth prime is greater than $k(\ln k + \ln\ln k - 1)$ for $k \geq 2$".

for $n > 39017$ i.e. $p_n > 467\,473$.

In [E. Carneiro, M. Milinovich, and K. Soundararajan, 26], the authors prove the next result.

Theorem (2019) Under the Riemann Hypothesis we have $p_{n+1} - p - n \leq \frac{22}{25}\sqrt{p_n}\log p_n$.

1.4 Bounds on primes in arithmetic progressions

Collecting references: [K. McCurley, 130], [K. McCurley, 129], [O. Ramaré and R. Rumely, 172], [P. Dusart, 59], Lemma 10 of [P. Moree, 137], section 4 of [P. Moree and H. te Riele, 138].

A consequence of Theorem 1.1 and 1.2 of [M. A. Bennett et al., 11] states that

Theorem (2018) We have $\max_{3\leq q\leq 10^4} \max_{x\geq 8\cdot 10^9} \max_{\substack{1\leq a\leq q,\\(a,q)=1}} \frac{\log x}{x}\Big|\sum_{\substack{n\leq x,\\ n\equiv a[q]}} \Lambda(n) - \frac{x}{\varphi(q)}\Big| \leq 1/840$.

When $q \in (10^4, 10^5]$, we may replace $1/840$ by $1/160$ and when $q \geq 10^5$, we may replace $1/840$ by $\exp(0.03\sqrt{q}\log^3 q)$.

Furthermore, we may replace $\sum_{\substack{n\leq x,\\ n\equiv a[q]}} \Lambda(n)$ by $\sum_{\substack{p\leq x,\\ p\equiv a[q]}} \log p$ with no changes in the constants.

Similarly, as a consequence of Theorem 1.3 of [M. A. Bennett et al., 11] states that

Theorem (2018) We have $\max_{3\leq q\leq 10^4} \max_{x\geq 8\cdot 10^9} \max_{\substack{1\leq a\leq q,\\(a,q)=1}} \frac{\log^2 x}{x}\Big|\sum_{\substack{p\leq x,\\ p\equiv a[q]}} 1 - \frac{\mathrm{Li}(x)}{\varphi(q)}\Big| \leq 1/840$.

When $q \in (10^4, 10^5]$, we may replace $1/840$ by $1/160$ and when $q \geq 10^5$, we may replace $1/840$ by $\exp(0.03\sqrt{q}\log^3 q)$.

Concerning estimates with a logarithmic density, in [O. Ramaré, 171] and

[26] E. Carneiro, M. Milinovich, and K. Soundararajan, 2019, *Fourier optimization and prime gaps.*
[130] K. McCurley, 1984, "Explicit estimates for the error term in the prime number theorem for arithmetic progressions".
[129] K. McCurley, 1984, "Explicit estimates for $\theta(x; 3, \ell)$ and $\psi(x; 3, \ell)$".
[172] O. Ramaré and R. Rumely, 1996, "Primes in arithmetic progressions".
[59] P. Dusart, 2002, "Estimates for $\theta(x; k, \ell)$ for large values of x".
[137] P. Moree, 2004, "Chebyshev's bias for composite numbers with restricted prime divisors".
[138] P. Moree and H. te Riele, 2004, "The hexagonal versus the square lattice".
[11] M. A. Bennett et al., 2018, "Explicit bounds for primes in arithmetic progressions".
[171] O. Ramaré, 2002, "Sur un théorème de Mertens".

in [D. Platt and O. Ramaré, 151], estimates for the functions $\sum_{\substack{n\le x,\\ n\equiv a[q]}} \Lambda(n)/n$ are considered. Extending computations from the former, the latter paper Theorem 8.1 reads as follows.

Theorem (2016) We have $\max_{q\le 1000}\max_{q\le x\le 10^5}\max_{\substack{1\le a\le q,\\ (a,q)=1}}\sqrt{x}\left|\sum_{\substack{n\le x,\\ n\equiv a[q]}}\frac{\Lambda(n)}{n}-\frac{\log x}{\varphi(q)}-C(q,a)\right|\in$ $(0.8533, 0.8534)$ and the maximum is attained with $q=17$, $x=606$ and $a=2$.

The constant $C(q,a)$ is the one expected, i.e. such that $\sum_{\substack{n\le x,\\ n\equiv a[q]}}\frac{\Lambda(n)}{n}-\frac{\log x}{\varphi(q)}-C(q,a)$ goes to zero when x goes to infinity.

When q belongs to "Rumely's list", i.e. in one of the following set:

- $\{k\le 72\}$
- $\{k\le 112, k \text{ non premier}\}$
- $\{116, 117, 120, 121, 124, 125, 128, 132, 140, 143, 144, 156, 163, 169, 180, 216, 243, 256, 360, 420, 432\}$

Theorem 2 of [O. Ramaré, 171] gives the following.

Theorem (2002) When q belongs to Rumely's list and a is prime to q, we have $\sum_{\substack{n\le x,\\ n\equiv a[q]}}\frac{\Lambda(n)}{n}=\frac{\log x}{\varphi(q)}+C(q,a)+O^*(1)$ as soon as $x\ge 1$.

More precise bounds are given.

1.5 Least prime verifying a condition

[E. Bach and J. Sorenson, 3], [H. Kadiri, 97],

Last updated on January 1rst, 2019, by Olivier Ramaré

[151] D. Platt and O. Ramaré, 2017, "Explicit estimates: from $\Lambda(n)$ in arithmetic progressions to $\Lambda(n)/n$".
[3] E. Bach and J. Sorenson, 1996, "Explicit bounds for primes in residue classes".
[97] H. Kadiri, 2008, "Short effective intervals containing primes in arithmetic progressions and the seven cube problem".

Chapter 2

Explicit bounds on the Moebius function

Corresponding html file: `../Articles/Art02.html`

Collecting references: [H. Diamond and P. Erdös, 49], [M. Deléglise and J. Rivat, 46], [P. Borwein, R. Ferguson, and M. Mossinghoff, 21].

2.1 Bounds on $M(D) = \sum_{d \le D} \mu(d)$

The first explicit estimate for $M(D)$ is due to [R. von Sterneck, 205] where the author proved that $|M(D)| \le \frac{1}{9}D + 8$ for any $D \ge 0$. A popular estimate is the one of [R. Mac Leod, 124].

Theorem (1967) When $D \ge 0$, we have $|M(D)| \le \frac{1}{80}D + 5$. When $D \ge 1119$, we have $|M(D)| \le D/80$.

We mention at this level the annoted bibliography contained at the end of [F. Dress, 52]. [N. Costa Pereira, 37] shows that

Theorem (1993) When $D \ge 120\,727$, we have $|M(D)| \le D/1036$.

On elaborating on this method, [F. Dress and M. El Marraki, 53] showed that

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[49] H. Diamond and P. Erdös, 1980, "On sharp elementary prime number estimates".

[46] M. Deléglise and J. Rivat, 1996, "Computing the summation of the Möbius function".

[21] P. Borwein, R. Ferguson, and M. Mossinghoff, 2008, "Sign changes in sums of the Liouville function".

[205] R. von Sterneck, 1898, "Bemerkung über die Summierung einiger zahlentheorischen Funktionen".

[124] R. Mac Leod, 1967, "A new estimate for the sum $M(x) = \sum_{n \le x} \mu(n)$".

[52] F. Dress, 1983/84, "Théorèmes d'oscillations et fonction de Möbius".

[37] N. Costa Pereira, 1989, "Elementary estimates for the Chebyshev function $\psi(X)$ and for the Möbius function $M(X)$".

[53] F. Dress and M. El Marraki, 1993, "Fonction sommatoire de la fonction de Möbius 2. Majorations asymptotiques élémentaires".

Theorem (1993) When $D \geq 617\,973$, we have $|M(D)| \leq D/2360$.

One of the argument is the estimate from [F. Dress, 51]

Theorem (1993) When $33 \leq D \leq 10^{12}$, we have $|M(D)| \leq 0.571\sqrt{D}$.

This has been extended by [T. Kotnik and J. van de Lune, 102] to 10^{14} and recently in [G. Hurst, 85] to 10^{16}, i.e.

Theorem (2018) When $33 \leq D \leq 10^{16}$, we have $|M(D)| \leq 0.571\sqrt{D}$.

Another tool is [H. Cohen and F. Dress, 34] where refined explicit estimates for the remainder term of the counting functions of the squarefree numbers in intervals are obtained.

The latest best estimate of this shape comes from [H. Cohen, F. Dress, and M. El Marraki, 35]. This preprint being difficult to get, it has been republished in [H. Cohen, F. Dress, and M. El Marraki, 36].

Theorem (1996) When $D \geq 2\,160\,535$, we have $|M(D)| \leq D/4345$.

These results are used in [F. Dress, 50] to study the discrepancy of the Farey series.

Concerning upper bounds that tend to 0, [L. Schoenfeld, 185] is the pioneer and shows among other estimates that

Theorem (1969) When $D > 0$, we have $|M(D)|/D \leq 2.9/\log D$.

[M. El Marraki, 64] improves that into

Theorem (1995) When $D \geq 685$, we have $|M(D)|/D \leq 0.10917/\log D$.

The latest bound coming from [O. Ramaré, 166] improves that:

Theorem (2012) When $D \geq 1\,100\,000$, we have $|M(D)|/D \leq 0.013/\log D$.

[51] F. Dress, 1993, "Fonction sommatoire de la fonction de Möbius 1. Majorations expérimentales".
[102] T. Kotnik and J. van de Lune, 2004, "On the order of the Mertens function".
[85] G. Hurst, 2018, "Computations of the Mertens function and improved bounds on the Mertens conjecture".
[34] H. Cohen and F. Dress, 1988, "Estimations numériques du reste de la fonction sommatoire relative aux entiers sans facteur carré".
[35] H. Cohen, F. Dress, and M. El Marraki, 1996, "Explicit estimates for summatory functions linked to the Möbius μ-function".
[36] H. Cohen, F. Dress, and M. El Marraki, 2007, "Explicit estimates for summatory functions linked to the Möbius μ-function".
[50] F. Dress, 1999, "Discrépance des suites de Farey".
[185] L. Schoenfeld, 1969, "An improved estimate for the summatory function of the Möbius function".
[64] M. El Marraki, 1995, "Fonction sommatoire de la fonction μ de Möbius, majorations asymptotiques effectives fortes".
[166] O. Ramaré, 2013, "From explicit estimates for the primes to explicit estimates for the Moebius function".

In [O. Ramaré, 164], bounds including coprimality conditions are proved and here is a typical example.

Theorem (2013) When $1 \le q < D$, we have $\left|\sum_{\substack{d \le D,\\ (d,q)=1}} \mu(d)\right|/D \le \frac{q}{\varphi(q)}/(1 + \log(D/q))$.

2.2 Bounds on $m(D) = \sum_{d \le D} \mu(d)/d$

[R. Mac Leod, 124] shows that the sum $m(D)$ takes its minimal value at $D = 13$. A folklore result is generalized in [A. Granville and O. Ramaré, 77] and reads

Theorem (1996) When $D \ge 0$ and for any integer $r \ge 1$, we have $\left|\sum_{\substack{d \le D,\\ (d,r)=1}} \mu(d)/d\right| \le 1$.

In fact, Lemma 1 of [H. Davenport, 42] already contains the requisite material.

The next result is proved in [O. Ramaré, 164].

Theorem (2013) When $D \ge 7$, we have $|\sum_{d \le D} \mu(d)/d| \le 1/10$. We can replace the couple (7, 1/10) by (41, 1/20) or (694, 1/100).

This is further extended in [O. Ramaré, 170] where it is shown that

Theorem (2012) When $D \ge 0$ and for any integer $r \ge 1$ and any real number $\varepsilon \ge 0$, we have $\left|\sum_{\substack{d \le D,\\ (d,r)=1}} \mu(d)/d^{1+\varepsilon}\right| \le 1 + \varepsilon$.

Concerning upper bounds that tend to 0, [M. El Marraki, 65] is the first to get such an estimate.

Theorem (1996) When $D \ge 33$ we have $|m(D)| \le 0.2185/\log D$.
When $D > 1$ we have $|m(D)| \le 726/(\log D)^2$.

This second bound is improved in [O. Bordellès, 20].

Theorem (2015) When $D > 1$ we have $|m(D)| \le 546/(\log D)^2$.

[164] O. Ramaré, 2015, "Explicit estimates on several summatory functions involving the Moebius function".
[124] R. Mac Leod, 1967, "A new estimate for the sum $M(x) = \sum_{n \le x} \mu(n)$".
[77] A. Granville and O. Ramaré, 1996, "Explicit bounds on exponential sums and the scarcity of squarefree binomial coefficients".
[42] H. Davenport, 1937, "On some infinite series involving arithmetical functions".
[170] O. Ramaré, 2013, "Some elementary explicit bounds for two mollifications of the Moebius function".
[65] M. El Marraki, 1996, "Majorations de la fonction sommatoire de la fonction $\frac{\mu(n)}{n}$".
[20] O. Bordellès, 2015, "Some explicit estimates for the Möbius function".

[O. Ramaré, 166] proves several bounds of the shape $m(D) \ll 1/\log D$. This is improved in [O. Ramaré, 164] by using [M. Balazard, 8]. which provides us with a better manner to convert bounds on $M(D)$ into bounds for $m(D)$. Here is one result obtained.

Theorem (2015) When $D \geq 463\,421$ we have $|m(D)| \leq 0.0144/\log D$.

We can for instance replace the couple (463 421, 0.0144)by any of (96 955, 1/69), (60 298, 1/65), (1426, 1/20) or (687, 1/12).

In [O. Ramaré, 165] and [O. Ramaré, 164], the problem of adding coprimality conditions is further addressed. Here is one of the results obtained.

Theorem (2015) When $1 \leq q < D$ we have $\left|\sum_{\substack{d\leq D,\\ (d,q)=1}} \mu(d)/d\right| \leq \frac{q}{\varphi(q)} 0.78/\log(D/q)$. When $D/q \geq 24233$, we can replace 0.78 by 17/125.

2.3 Bounds on $\check{m}(D) = \sum_{d\leq D} \mu(d)\log(D/d)/d$

The initial investigations on this function go back to [R. Daublebsky von Sterneck, 41]. In [O. Ramaré, 164] it is proved that

Theorem (2015) When $3846 \leq D$ we have $|\check{m}(D)-1| \leq 0.00257/\log D$. When $D > 1$, we have $|\check{m}(D) - 1| \leq 0.213/\log D$.

This implies in particular that

Theorem (2015) When $222 \leq D$ we have $|\check{m}(D) - 1| \leq 1/1250$. When $D > 1$, the optimal bound 1 holds.

These bounds are a consequence of the identity:

$$|\check{m}(D) - 1| \leq \frac{\frac{7}{4} - \gamma}{D^2} \int_1^D |M(t)|dt + \frac{2}{D}.$$

It is also proved that, for any $D \geq 1$, we have

$$0 \leq \sum_{\substack{d\leq D,\\ (d,q)=1}} \mu(d)\log(D/d)/d \leq 1.00303q/\varphi(q).$$

[166] O. Ramaré, 2013, “From explicit estimates for the primes to explicit estimates for the Moebius function”.
[164] O. Ramaré, 2015, “Explicit estimates on several summatory functions involving the Moebius function”.
[8] M. Balazard, 2012, “Elementary Remarks on Möbius’ Function”.
[165] O. Ramaré, 2014, “Explicit estimates on the summatory functions of the Moebius function with coprimality restrictions”.
[41] R. Daublebsky von Sterneck, 1902, “Ein Analogon zur additiven Zahlentheorie.”

2.4 Miscellanae

Here is an elegant wide ranging estimate, taken from Claim 3.1 of [E. Treviño, 195].

Theorem (2015) When $D \geq 1$ we have $|\sum_{d>D} \mu(d)/d^2| \leq 1/D$.

2.5 The Moebius function and arithmetic progressions

The results in this section are scarce. We mention a Theorem of [O. Bordellès, 20].

Theorem (2015) Let χ be a non-principal Dirichlet character mmodulo $q \geq 37$ and let $k \geq 1$ be some integer. Then

$$\Bigg| \sum_{\substack{n \leq x, \\ (n,k)=1}} \frac{\mu(n)\chi(n)}{n} \Bigg| \leq \frac{k}{\varphi(k)} \frac{2\sqrt{q}\log q}{L(1,\chi)}.$$

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[195] E. Treviño, 2015, "The least k-th power non-residue".
[20] O. Bordellès, 2015, "Some explicit estimates for the Möbius function".

Chapter 3

Averages of non-negative multiplicative functions

Corresponding html file: `../Articles/Art10.html`

3.1 Asymptotic estimates

When looking for averages of functions that look like 1 or like the divisor function, Lemma 3.2 of [O. Ramaré, 169] offers an efficient easy path. The technique of comparison of two arithmetical function via their Dirichlet series is known as the Convolution method and is for instance decribed at length in [P. Berment and O. Ramaré, 13], and in the course that can be found here[1].

Theorem (1995) Let $(g_n)_{n\geq 1}$, $(h_n)_{n\geq 1}$ and $(k_n)_{n\geq 1}$ be three complex sequences. Let $H(s)=\sum_n h_n n^{-s}$, and $\overline{H}(s)=\sum_n |h_n|n^{-s}$. We assume that $g=h\star k$, that $\overline{H}(s)$ is convergent for $\Re(s)\geq -1/3$ and further that there exist four constants A, B, C and D such that

$$\sum_{n\leq t} k_n = A\log^2 t + B\log t + C + \mathcal{O}^*(Dt^{-1/3}) \text{ for } t>0.$$

Then we have for all $t>0$:

$$\sum_{n\leq t} g_n = u\log^2 t + v\log t + w + \mathcal{O}^*(Dt^{-1/3}\overline{H}(-1/3))$$

with $u=AH(0)$, $v=2AH'(0)+BH(0)$ and $w=AH''(0)+BH'(0)+CH(0)$. We have also

$$\sum_{n\leq t} ng_n = Ut\log t + Vt + W + \mathcal{O}^*(2.5Dt^{2/3}\overline{H}(-1/3))$$

[169] O. Ramaré, 1995, "On Snirel'man's constant".
[13] P. Berment and O. Ramaré, 2012, "Ordre moyen d'une fonction arithmétique par la méthode de convolution".

[1] https://ramare-olivier.github.io/CoursNouakchott/index.html

with

$$\begin{aligned}U &= 2AH(0), V = -2AH(0) + 2AH'(0) + BH(0),\\ W &= A(H''(0) - 2H'(0) + 2H(0)) + B(H'(0) - H(0)) + CH(0).\end{aligned}$$

This Lemma says that one derives information from g_n from informations on the model k_n. When this model is $k_n = 1$, the values concerning A, B and C are given by the first half of Lemma 3.3 of [O. Ramaré, 169]:

Lemma (1995) $\sum_{n\le t} 1/n = \log t + \gamma + \mathcal{O}^*(0.9105 t^{-1/3})$.

When this model is $k_n = \tau(n)$, the number of divisors of n, the values concerning A, B and C are given by Corollary 2.2 of [D. Berkane, O. Bordellès, and O. Ramaré, 12]. Please note the "$\gamma^2 - 2\gamma_1$" which is wrongly typed as "$\gamma^2 - \gamma_1$" in the aforementioned paper (and thanks to Tim Trudgian and David Platt for spotting this typo):

Lemma (2011) $\sum_{n\le t} \tau(n)/n = \frac{1}{2}\log^2 t + 2\gamma \log t + \gamma^2 - 2\gamma_1 + \mathcal{O}^*(1.16/t^{1/3})$ where γ_1 is the second Laurent-Stieljes constant – for instance [R. Kreminski, 103] and [M. Coffey, 33]. In particular, we have $\gamma_1 = -0.0728158454836767248605863758749013191377 + \mathcal{O}^*(10^{-40})$.

The constants $H(0)$, $H'(0)$ and $H''(0)$ are to be computed. In most cases, the Dirichlet series has an Euler product, in which case, (see section 3 of [O. Ramaré, 169]) we have

$H(0) = \prod_p (1 + \sum_m h_{p^m})$, then $\dfrac{H'(0)}{H(0)} = \displaystyle\sum_p \frac{\sum_m m h_{p^m}}{1 + \sum_m h_{p^m}}(-\log p)$, and also

$$\frac{H''(0)}{H(0)} = \left(\frac{H'(0)}{H(0)}\right)^2 + \sum_p \left\{\frac{\sum_m m^2 h_{p^m}}{1 + \sum_m h_{p^m}} - \left[\frac{\sum_m m h_{p^m}}{1 + \sum_m h_{p^m}}\right]^2\right\} \log^2 p.$$

It is sometimes more expedient to use the same convolution method but by comparing the function to the function $q \mapsto q$. In such a case, the next lemma, Lemma 4.3 from [O. Ramaré, 158], is handy.

Theorem (2015) We have, for any real number $x \ge 0$ and any real number $c \in [1, 2]$, $\displaystyle\sum_{q\le x} q = \frac{1}{2}x^2 + O^*(x^c/2)$.

This leads to the next theorem.

[169] O. Ramaré, 1995, "On Snirel'man's constant".
[12] D. Berkane, O. Bordellès, and O. Ramaré, 2012, "Explicit upper bounds for the remainder term in the divisor problem".
[103] R. Kreminski, 2003, "Newton-Cotes integration for approximating Stieltjes (generalized Euler) constants".
[33] M. Coffey, 2006, "New results on the Stieltjes constants: asymptotic and exact evaluation".
[158] O. Ramaré, 2016, "An explicit density estimate for Dirichlet L-series".

Theorem Let $(h_n)_{n\geq 1}$ be a complex sequences. Let $H(s) = \sum_n h_n n^{-s}$, and $\overline{H}(s) = \sum_n |h_n| n^{-s}$. We assume that $\overline{H}(s)$ is convergent for $\Re(s) \geq c$, for some $c \in [1, 2]$. Then we have for all $t > 0$:

$$\sum_{n\leq t}\sum_{d|n}\frac{n}{d}h(d) = \frac{t^2}{2}H(2) + O^*(t^c\overline{H}(c)/2).$$

A typical usage is to evaluate $\sum_{n\leq t}\phi(n)$, with $h(d) = \mu(d)$.

The convolution method has been brought one step further in [A. P. and O. Ramaré, 147] where the following theorem is proved.

Theorem (2017) Let $(g(m))_{m\geq 1}$ be a sequence of complex numbers such that both series $\sum_{m\geq 1} g(m)/m$ and $\sum_{m\geq 1} g(m)(\log m)/m$ converge. We define $G^\sharp(x) = \sum_{m>x} g(m)/m$ and assume that $\int_1^\infty |G^\sharp(t)|dt/t$ converges. Let $A_0 \geq 1$ be a real parameter. We have

$$\sum_{n\leq D}\frac{(g\star 1\!\!1)(n)}{n} = \sum_{m\geq 1}\frac{g(m)}{m}\Big(\log\frac{D}{m}+\gamma\Big) + \int_{D/A_0}^\infty G^\sharp(t)\frac{dt}{t} + O^*(\mathfrak{R})$$

where $\mathfrak{R}$ is defined by

$$\mathfrak{R} = \left|\sum_{1\leq a\leq A_0}\frac{1}{a}G^\sharp\left(\frac{D}{a}\right) + G^\sharp\left(\frac{D}{A_0}\right)\left(\log\frac{A_0}{[A_0]} - R([A_0])\right)\right| + \frac{6/11}{D}\sum_{m\leq D/A_0}|g(m)|$$

where $[A_0]$ is the integer part of A_0, while the remainder R is defined by $R(X) = \sum_{n\leq X} 1/n - \log X - \gamma$.

The remainder $R(X)$ is shown in Lemma to verify $|R(X)| \leq \gamma/X$ for every $X > 0$, and $|R(X)| \leq (6/11)/X$ when $X \geq 1$.

Theorem 21.1 of [O. Ramaré, 161] offers a fully explicit estimate for the average of a general non-negative multiplicative function, but it is often numerically rather poor. It relies on the technique developped by [B. Levin and A. Fainleib, 111].

Theorem (2009) Let g be a non-negative multiplicative function. Let A and κ be three positive real parameters such that, for any $Q \geq 1$, one has

$$\sum_{\substack{p\geq 2,\nu\geq 1\\ p^\nu\leq Q}} g(p^\nu)\log(p^\nu) = \kappa\log Q + \mathcal{O}^*(L)$$

[147] A. P. and O. Ramaré, 2017, "Explicit averages of non-negative multiplicative functions: going beyond the main term".
[161] O. Ramaré, 2009, *Arithmetical aspects of the large sieve inequality*.
[111] B. Levin and A. Fainleib, 1967, "Application of some integral equations to problems of number theory".

and $\sum_{p\geq 2}\sum_{\nu,k\geq 1} g(p^k)g(p^\nu)\log(p^\nu) \leq A$. Then, when $D \geq \exp(2(L+A))$, we have

$$\sum_{d\leq D} g(d) = C\,(\log D)^\kappa\,\bigl(1+\mathcal{O}^*\bigl(B/\log D\bigr)\bigr)$$

where $B = 2(L+A)\bigl(1+2(\kappa+1)e^{\kappa+1}\bigr)$ and

$$C = \frac{1}{\Gamma(\kappa+1)}\prod_{p\geq 2}\left\{\left(\sum_{\nu\geq 0} g(p^\nu)\right)\left(1-\frac{1}{p}\right)^\kappa\right\}.$$

3.2 Upper bounds

When looking for an upper bound, it is common to compare sums to an Euler product, via,

$$\sum_{n\leq y} f(n)/n \leq \prod_{p\leq y}\left(1+\sum_{1\leq m\leq \log y/\log p} f(p^m)\right)$$

valid when f is non-negative and multiplicative. Lemma 4 of [H. Daboussi and J. Rivat, 40] extends this. Let z be a parameter and $v_z(n)$ be the characteristic function of those integers that have all their prime factors $p \leq z$.

Theorem (2000) Let $z \geq 2$, f a multiplicative function with $f \geq 0$ and $S = \sum_{p\leq z}\frac{f(p)}{1+f(p)}\log p$. We assume that $S > 0$ and write $K(t) = \log t - 1 - (1/t)$. For any y such that $\log y \geq S$, we have

$$\sum_{n>y} v_z(n)\mu^2(n)f(n) \leq \prod_{p\leq z}(1+f(p))\exp\left(-\frac{\log y}{\log z}K\left(\frac{\log y}{S}\right)\right)$$

$$\sum_{n\leq y} v_z(n)\mu^2(n)f(n) \geq \prod_{p\leq z}(1+f(p))\left\{1-\exp\left(-\frac{\log y}{\log z}K\left(\frac{\log y}{S}\right)\right)\right\}$$

and in particular, when $\log y \geq 7S$, we have

$$\sum_{n>y} v_z(n)\mu^2(n)f(n) \leq \prod_{p\leq z}(1+f(p))\exp\left(-\frac{\log y}{\log z}\right)$$

$$\sum_{n\leq y} v_z(n)\mu^2(n)f(n) \geq \prod_{p\leq z}(1+f(p))\left\{1-\exp\left(-\frac{\log y}{\log z}\right)\right\}.$$

It is sometimes required to compare a function close to 1 (or more generally to the divisor function τ_k) to a function close to $1/n$ or $\tau_k(n)/n$. Theorem 01 of [R. Hall and G. Tenenbaum, 81] offers a fast way of doing so.

[40] H. Daboussi and J. Rivat, 2001, "Explicit upper bounds for exponential sums over primes".
[81] R. Hall and G. Tenenbaum, 1988, *Divisors*.

Theorem (1988) Let f be a non-negative multiplicative function such that, for some A and B,

$$\sum_{p\le y} f(p)\log p \le Ay \quad (y\ge 0), \qquad \sum_p \sum_{\nu\ge 2} \frac{f(p^\nu)}{p^\nu}\log p^\nu \le B.$$

Then, for $x>1$,

$$\sum_{n\le x} f(n) \le (A+B+1)\frac{x}{\log x}\sum_{n\le x}\frac{f(n)}{n}$$

See also Section 4.6, and for instance Theorem 4.22, of [O. Bordellès, 17]. In particular, in case a further condition is assumed, we have Theorem 4.28 of [O. Bordellès, 17] at our disposal.

Theorem (2012) Let f be a non-negative multiplicative function such that, for every prime p and every non-negative power a the condition $f(p^{a+1}) \ge f(p^a)$ holds, we have for $x\ge 1$

$$\sum_{n\le x} f(n) \le x\prod_{p\le x}\Big(1-\frac{1}{p}\Big)\Big(1+\sum_{a\ge 1}\frac{f(p^a)}{p^a}\Big).$$

The next lemma is handy to remove coprimality conditions. It originates from [J. van Lint and H. Richert, 112].

Theorem (1965) Let f be a non-negative multiplicative function and let d be a positive integer. We have for $x\ge 0$

$$\sum_{n\le x}\mu^2(n)f(n) \le \prod_{p|d}(1+f(p))\sum_{\substack{n\le x,\\ (n,d)=1}}\mu^2(n)f(n) \le \sum_{n\le xd}\mu^2(n)f(n).$$

Though it is somewhat difficult to get, this lemma has been further generalized in Lemma 4.1 of [O. Ramaré, 168].

3.3 Estimates of some special functions

[H. Cohen and F. Dress, 34] contains the following Theorem.

Theorem (1988) Let $R(x)=\sum_{n\le x}\mu^2(n)-6x/\pi^2$. We have $|R(x+y)-R(x)| \le 1.6749\sqrt{y}+0.6212x/y$ and $|R(x+y)-R(x)| \le 0.7343y/x^{1/3}+1.4327x^{1/3}$ for $x,y\ge 1$.

[17] O. Bordellès, 2012, *Arithmetic Tales*.
[112] J. van Lint and H. Richert, 1965, "On primes in arithmetic progressions".
[168] O. Ramaré, 2012, "On long κ-tuples with few prime factors".
[34] H. Cohen and F. Dress, 1988, "Estimations numériques du reste de la fonction sommatoire relative aux entiers sans facteur carré".

See also [N. Costa Pereira, 37]. [L. Moser and R. A. MacLeod, 139] and [H. Cohen, F. Dress, and M. El Marraki, 36] contains:

Theorem (2008) We have $|\sum_{n\le x}\mu^2(n)-6x/\pi^2|\le 0.02767\sqrt{x}$ for $x\ge 438653$. One can replace $(0.02767, 438653)$ by $(0.036438, 82005)$, by $(0.1333, 1004)$, by $(1/2, 8)$ or by $(1, 1)$.

Lemma 3.4 of [O. Ramaré, 158] gives us:

Theorem (2013) We have $\frac{6}{\pi^2}\log x+0.578\le\sum_{n\le x}\mu^2(n)/n\le\frac{6}{\pi^2}\log x+1.166$ for $x\ge 1$

When $x\ge 1000$, one can also replace the couple $(0.578, 1.166)$ by $(1.040, 1.048)$.

In fact, in the same paper, the asymptotic

$$\sum_{n\le x}\frac{\mu^2(n)}{n}=\frac{6}{\pi^2}\Big(\log x+2\sum_{p\ge 2}\frac{\log p}{p^2-1}+\gamma\Big)+O^*(3/x^{1/3})$$

valid for $x\ge 1$ is proved. A script using SAGE and another one using GP/PARI are then displayed to explain how to cover the initial range in x. See also Lemma 1 of [L. Schoenfeld, 185] for an earlier version.

The main result [D. Berkane, O. Bordellès, and O. Ramaré, 12] reads as follows.

Theorem (2012) We define $\Delta(x)=\sum_{n\le x}\tau(n)-x(\log x+2\gamma-1)$. We have

- When $x\ge 1$, we have $|\Delta(x)|\le 0.961\,x^{1/2}$.
- When $x\ge 1\,981$, we have $|\Delta(x)|\le 0.482\,x^{1/2}$.
- When $x\ge 5\,560$, we have $|\Delta(x)|\le 0.397\,x^{1/2}$.
- When $x\ge 5$, we have $|\Delta(x)|\le 0.764\,x^{1/3}\log x$.

For evaluation of the average of the divisor function on integers belonging to a fixed residue class modulo 6, see Corollary to Proposition 3.2 of [J.-M. Deshouillers and F. Dress, 47].

For more complicated sums and when x is large with respect to k, one can use [C. Mardjanichvili, 125].

[37] N. Costa Pereira, 1989, "Elementary estimates for the Chebyshev function $\psi(X)$ and for the Möbius function $M(X)$".
[139] L. Moser and R. A. MacLeod, 1966, "The error term for the squarefree integers".
[36] H. Cohen, F. Dress, and M. El Marraki, 2007, "Explicit estimates for summatory functions linked to the Möbius μ-function".
[158] O. Ramaré, 2016, "An explicit density estimate for Dirichlet L-series".
[185] L. Schoenfeld, 1969, "An improved estimate for the summatory function of the Möbius function".
[12] D. Berkane, O. Bordellès, and O. Ramaré, 2012, "Explicit upper bounds for the remainder term in the divisor problem".
[47] J.-M. Deshouillers and F. Dress, 1988, "Sommes de diviseurs et structure multiplicative des entiers".
[125] C. Mardjanichvili, 1939, "Estimation d'une somme arithmétique".

Theorem (1939) Let k and ℓ be two positive integers. We have for any real number $x \geq 1$

$$\sum_{m \leq x} \tau_k^\ell(m) \leq x \frac{k^\ell}{(k!)^{\frac{k^\ell - 1}{k-1}}} (\log x + k^\ell - 1)^{k^\ell - 1}.$$

See [J.-M. Deshouillers and F. Dress, 47] for some upper bounds linked with τ_3.

[O. Bordellès, 18] contains the following bounds, better than the above when x is small with respect to k.

Theorem (2002) Let $k \geq 1$ be a positive integer.

- When $x \geq 1$ is a real number, we have $\sum_{m \leq x} \tau_k(m) \leq x(\log x + \gamma + (1/x))^{k-1}$.
- When $x \geq 6$ is a real number, we have $\sum_{m \leq x} \tau_k(m) \leq 2x(\log x)^{k-1}$.

In [K. Lapkova, 106], we find the next result.

Theorem (2015) Let b and c be two integers such that $\delta = b^2 - c$ is non-zero, square-free and not congruent to 1 modulo 4. Assume further that the function $n^2 + 2bn + c$ is positive and non-decreasing when $n \geq 1$.Then, for $N \geq 1$, we have

$$\sum_{n \leq N} \tau(n^2 + 2bn + c) \leq C_1 N \log N + C_2 + C_3$$

where the constants C_1, C_2 and C_3 are defined as follows. We first define $\xi = \sqrt{1 + 2|b| + |c|}$ and $\kappa = \frac{4}{\pi^2}\sqrt{4|\delta|}(\log(4|\delta|) + 0.648)$. Then

$$C_1 = \frac{12}{\pi^2}(\log\kappa + 1), C_2 = 2\left[\kappa + (\log\kappa + 1)\left(\frac{6}{\pi^2}\log\xi + 1.166\right)\right], C_3 = 2\kappa(\max(|b|, |c|^{1/2}) + 1).$$

See [K. Lapkova, 107] for the number of divisors of a reducible quadratic polynomial.

Evaluations of Lemma 4.3 of [M. Cipu, 32] are improved in Lemma 12 of [T. Trudgian, 200]. Only upper bounds are given, but the proof given there gives the lower bounds as well. This gives the first two estimates, while the third one comes from Lemma 4.3 of [M. Cipu, 32].

Theorem (2015) Let $x \geq 1$ be a real number. We have

[18] O. Bordellès, 2002, "Explicit upper bounds for the average order of $d_n(m)$ and application to class number".
[106] K. Lapkova, 2016, "Explicit upper bound for an average number of divisors of quadratic polynomials".
[107] K. Lapkova, 2017, "On the average number of divisors of reducible quadratic polynomials".
[32] M. Cipu, 2015, "Further remarks on Diophantine quintuples".
[200] T. Trudgian, 2015, "Bounds on the number of Diophantine quintuples".

- $0.786x - 0.3761 - 8.14x^{2/3} \leq \sum_{n\leq x} 2^{\omega(n)} - \frac{6}{\pi^2}x\log x \leq 0.787x - 0.3762 + 8.14x^{2/3}$
- $1.3947\log x + 0.4106 - 3.253x^{-1/3} \leq \sum_{n\leq x}\frac{2^{\omega(n)}}{n} - \frac{3}{\pi^2}(\log x)^2 \leq 1.3948\log x + 0.4107 + 3.253x^{-1/3}$,
- $\sum_{n\leq x}\frac{2^{\omega(2n-1)}}{2n-1} \leq \frac{3}{2\pi^2}(\log x)^2 + 3.123\log x + 3.569 + \frac{0.525}{x}$.

We take the next lemma from [E. Treviño, 194], Lemma 2.

Theorem (2015) Let $x \geq 1$ be a real number. We have $\sum_{n\leq x}\phi(n)/n \leq \frac{6}{\pi^2}x + \log x + 1$.

Lemma 3 of the same paper is as follows.

Theorem (2015) Let $x \geq 1$ be a real number. We have $\sum_{n\leq x} n\phi(n) \leq \frac{2}{\pi^2}x^3 + \frac{1}{2}x^2\log x + x^2$.

Several estimates are proved in [A. P. and O. Ramaré, 147]. For instance Theorem 1.2 gives the following.

Theorem (2017) Let $x \geq 1$ be a real number. We have $\sum_{n\leq x}\mu^2(n)/\phi(n) = \log x + c_0 + O^*(3.95/\sqrt{x})$ where $c_0 = \gamma + \sum_{p\geq 2}\frac{\log p}{p(p-1)}$. When $x > 1$, this O^* can be replaced by $O^*(21/\sqrt{x\log x})$.

The function $\sum_{n\leq x}\mu^2(n)/\phi(n)$ has been the subject of several estimates, see for instance Lemma 7 of [H. Montgomery and R. Vaughan, 135], Lemma 3.4-3.5 of [O. Ramaré, 169], the earlier paper [D. R. Ward, 206] and Lemma 4.5 of [J. Büthe, 23] where the error term $O^*(58/\sqrt{x})$ is achieved. The constant c_0 is evaluated precisely in (2.11) of [J. Rosser and L. Schoenfeld, 180].

[194] E. Treviño, 2015, "The Burgess inequality and the least kth power non-residue".
[147] A. P. and O. Ramaré, 2017, "Explicit averages of non-negative multiplicative functions: going beyond the main term".
[135] H. Montgomery and R. Vaughan, 1973, "The large sieve".
[169] O. Ramaré, 1995, "On Snirel'man's constant".
[206] D. R. Ward, 1927, "Some Series Involving Euler's Function".
[23] J. Büthe, 2014, "A Brun-Titchmarsh inequality for weighted sums over prime numbers".
[180] J. Rosser and L. Schoenfeld, 1962, "Approximate formulas for some functions of prime numbers".

3.4 Euler products

[J. Rosser and L. Schoenfeld, 180] contains estimates regarding $\prod_p(1-1/p)$ and its inverse. In particular we find the next results therein.

Theorem (1962)
- When $x > 1$, we have $1 - \dfrac{1}{\log^2 x} \leq e^\gamma (\log x) \displaystyle\prod_{p\leq x}\Big(1 - \dfrac{1}{p}\Big) \leq 1 + \dfrac{1}{2\log^2 x}$.
- When $x > 1$, we have $1 - \dfrac{1}{2\log^2 x} \leq e^{-\gamma} \displaystyle\prod_{p\leq x}\Big(1 - \dfrac{1}{p}\Big)^{-1} / \log x \leq 1 + \dfrac{1}{\log^2 x}$.

Several other estimates are proven. In [P. Dusart, 60], it is proved that

Theorem (2016)
- When $x > 2\,278\,382$, we have $1 - \dfrac{1}{5\log^3 x} \leq e^\gamma (\log x) \displaystyle\prod_{p\leq x}\Big(1 - \dfrac{1}{p}\Big) \leq 1 + \dfrac{1}{5\log^3 x}$.
- When $x > 2\,278\,382$, we have $1 - \dfrac{1}{5\log^3 x} \leq e^{-\gamma} \displaystyle\prod_{p\leq x}\Big(1 - \dfrac{1}{p}\Big)^{-1} / \log x \leq 1 + \dfrac{1}{5\log^3 x}$.

In [O. Bordellès, 19], the reader will find explicit upper bounds for $\displaystyle\prod_{\substack{p\leq x,\\ p\equiv a[q]}}\Big(1 - \dfrac{1}{p}\Big)^{-1}$

Theorem 5 of [R. Mawia, 128] contains the next result.

Theorem (2017) Let ϵ be a complex number such that $|\epsilon| \leq 2$. When $x \geq \exp(22)$, we have $\displaystyle\prod_{p\leq x}\Big(1 + \frac{\epsilon}{p}\Big) = e^{\gamma(\epsilon)+\epsilon B}(\log x)^\epsilon \Big\{1 + O^*\Big(\frac{0.841}{\log^3 x}\Big)\Big\}$ where $\gamma(\epsilon) = \displaystyle\sum_{p\geq 2}\sum_{n\geq 2}(-1)^{n+1}\frac{\epsilon^n}{np^n}$ and $B = \gamma + \displaystyle\sum_{p\geq 2}\big(\log(1-1/p) + (1/p)\big)$.

Equation (2.2) of [J. Rosser and L. Schoenfeld, 180] gives an approximate value for B.

Last updated on June 10th, 2019, by Olivier Ramaré

[60] P. Dusart, 2018, “Estimates of some functions over primes”.
[19] O. Bordellès, 2005, “An explicit Mertens’ type inequality for arithmetic progressions”.
[128] R. Mawia, 2017, “Explicit estimates for some summatory functions of primes”.

Chapter 4

Explicit upper bounds for some special arithmetic functions

Corresponding html file: `../Articles/Art12.html`

The following bounds may be useful is applications.

From [G. Robin, 174]:

Theorem (1983) For any integer $n \geq 3$, the number of prime divisors $\omega(n)$ of n satisfies:

$$\omega(n) \leq 1.3841 \frac{\log n}{\log \log n}.$$

From [J.-L. Nicolas and G. Robin, 144]:

Theorem (1983) For any integer $n \geq 3$, the number $\tau(n)$ of divisors of n satisfies:

$$\tau(n) \leq n^{1.538 \, \log 2 / \log \log n}.$$

From page 51 of [G. Robin, 175]:

Theorem (1983) For any integer $n \geq 3$, we have

$$\tau_3(n) \leq n^{1.59141 \, \log 3 / \log \log n}$$

where $\tau_3(n)$ is the number of triples (d_1, d_2, d_3) such that $d_1 d_2 d_3 = n$.

[174] G. Robin, 1983, “Estimation de la fonction de Tchebychef θ sur le k-ième nombres premiers et grandes valeurs de la fonction $\omega(n)$ nombre de diviseurs premiers de n”.
[144] J.-L. Nicolas and G. Robin, 1983, “Majorations explicites pour le nombre de diviseurs de n”.
[175] G. Robin, 1983, “Grandes valeurs de fonctions arithmétiques et problèmes d'optimisation en nombres entiers”.

The PhD memoir [J.-L. Duras, 56] contains result concerning the maximum of $\tau_k(n)$, i.e. the number of k-tuples $(d_1, d_2, \ldots, d_k)$ such that $d_1 d_2 \cdots d_k = n$, when $3 \le k \le 25$.

From [J.-L. Duras, J.-L. Nicolas, and G. Robin, 57]:

Theorem (1999) For any integer $n \ge 1$, any real number $s > 1$ and any integer $k \ge 1$, we have

$$\tau_k(n) \le n^s \zeta(s)^{k-1}$$

where $\tau_k(n)$ is the number of k-tuples $(d_1, d_2, \cdots, d_k)$ such that $d_1 d_2 \cdots d_k = n$.

The same paper also announces the bound for $n \ge 3$ and $k \ge 2$

$$\tau_k(n) \le n^{a_k \log k / \log\log k}$$

where $a_k = 1.53797 \log k / \log 2$ but the proof never appeared.

From [J.-L. Nicolas, 143]:

Theorem (2008) For any integer $n \ge 3$, we have

$$\sigma(n) \le 2.59791\, n \log\log(3\tau(n)),$$

$$\sigma(n) \le n\{e^\gamma \log\log(e\tau(n)) + \log\log\log(e^e\tau(n)) + 0.9415\}.$$

The first estimate above is a slight improvement of the bound

$$\sigma(n) \le 2.59 n \log\log n \quad (n \ge 7)$$

obtained in [A. Ivić, 90]. In this same paper, the author proves that

$$\sigma^*(n) \le \frac{28}{15} n \log\log n \quad (n \ge 31)$$

where $\sigma^*(n)$ is the sum of the unitary divisors of n, i.e. divisors d of n that are such that d and n/d are coprime.

On this subject, the readers may consult the web site

Computation about the paper The sum of divisors function and the Riemann hypothesis .

Last updated on September 2nd, 2021, by Olivier Ramaré

[56] J.-L. Duras, 1993, "Etude de la fonction nombre de façons de représenter un entier comme produit de k facteurs".
[57] J.-L. Duras, J.-L. Nicolas, and G. Robin, 1999, "Grandes valeurs de la fonction d_k".
[143] J.-L. Nicolas, 2008, "Quelques inégalités effectives entre des fonctions arithmétiques".
[90] A. Ivić, 1977, "Two inequalities for the sum of the divisors functions".

Part II

Exact computations

Chapter 5

Exact computations of the number of primes

Corresponding html file: `../Articles/Art03.html`

Collecting references: [M. Deléglise and J. Rivat, 44], [M. Deléglise and J. Rivat, 45], [D. Platt, 149].

Last updated on July 14th, 2012, by Olivier Ramaré

[44] M. Deléglise and J. Rivat, 1996, "Computing $\pi(x)$: The Meissel, Lehmer, Lagarias, Miller, Odlyzko method".
[45] M. Deléglise and J. Rivat, 1998, "Computing $\psi(x)$".
[149] D. Platt, 2011, "Computing degree 1 L-function rigorously".

Chapter 6

Computations of arithmetical constants

Corresponding html file: `../Articles/Art04.html`

Collecting references: [J. Cazaran and P. Moree, 27].

6.1 Euler products of rational functions

The computation of Euler product of rational function is dealt with in [P. Moree, 136]. The reader may also consult the following web page[1].

6.2 Some special sums over prime values that are derivatives

Last updated on July 14th, 2012, by Olivier Ramaré

[27] J. Cazaran and P. Moree, 1999, "On a claim of Ramanujan in his first letter to Hardy".
[136] P. Moree, 2000, "Approximation of singular series constant and automata. With an appendix by Gerhard Niklasch."

[1] http://guests.mpim-bonn.mpg.de/moree/Moree.en.html

Part III

General analytical tools

Chapter 7

Tools on Fourier transforms

Corresponding html file: `../Articles/Art16.html`

7.1 The large sieve inequality

The best version of the large sieve inequality from [H. Montgomery and R. Vaughan, 134] and [H. Montgomery and R. Vaughan, 135] (obtained at the same time by A. Selberg) is as follows.

Theorem (1974) Let M and $N \geq 1$ be two real numbers. Let X be a set of points of $[0, 1)$ such that

$$\min_{x,y\in X}\min_{k\in\mathbb{Z}}|x-y+k| \geq \delta > 0.$$

Then, for any sequence of complex numbers $(a_n)_{M<n\leq M+N}$, we have

$$\sum_{x\in X}\left|\sum_{M<n\leq M+N} a_n \exp(2i\pi nx)\right|^2 \leq \sum_{M<n\leq M+N}|a_n|^2(N-1+\delta^{-1}).$$

It is very often used with part of the Farey dissection.

Theorem (1974) Let M and $N \geq 1$ be two real numbers. Let $Q \geq 1$ be a real parameter. For any sequence of complex numbers $(a_n)_{M<n\leq M+N}$, we have

$$\sum_{q\in Q}\sum_{\substack{a \mod q,\\(a,q)=1}}\left|\sum_{M<n\leq M+N} a_n \exp(2i\pi na/q)\right|^2 \leq \sum_{M<n\leq M+N}|a_n|^2(N-1+Q^2).$$

The summation over a runs over all invertible classes a modulo q.

Last updated on July 14th, 2013, by Olivier Ramaré

[134] H. Montgomery and R. Vaughan, 1974, “Hilbert’s inequality”.
[135] H. Montgomery and R. Vaughan, 1973, “The large sieve”.

Chapter 8

Tools on Mellin transforms

Corresponding html file: `../Articles/Art17.html`

8.1 Explicit truncated Perron formula

Here is Theorem 7.1 of [O. Ramaré, 162].

Theorem (2007) Let $F(z) = \sum_n a_n/n^z$ be a Dirichlet series that converges absolutely for $\Re z > \kappa_a$, and let $\kappa > 0$ be strictly larger than κ_a. For $x \geq 1$ and $T \geq 1$, we have

$$\sum_{n\leq x} a_n = \frac{1}{2i\pi}\int_{\kappa-iT}^{\kappa+iT} F(z)\frac{x^z dz}{z} + \mathcal{O}^*\left(\int_{1/T}^{\infty}\sum_{|\log(x/n)|\leq u}\frac{|a_n|}{n^\kappa}\frac{2x^\kappa du}{Tu^2}\right).$$

See [O. Ramaré, 167] for different versions.

8.2 L^2-means

We start with a majorant principle taken for instance from [H. Montgomery, 133], chapter 7, Theorem 3.

Theorem Let $\lambda_1, \cdots, \lambda_N$ be N real numbers, and suppose that $|a_n| \leq A_n$ for all n. Then

$$\int_{-T}^{T}\Bigl|\sum_{1\leq n\leq N} a_n e(\lambda_n t)\Bigr|^2 dt \leq 3\int_{-T}^{T}\Bigl|\sum_{1\leq n\leq N} A_n e(\lambda_n t)\Bigr|^2 dt$$

.

[162] O. Ramaré, 2007, “Eigenvalues in the large sieve inequality”.
[167] O. Ramaré, 2016, “Modified truncated Perron formulae”.
[133] H. Montgomery, 1994, *Ten lectures on the interface between analytic number theory and harmonic analysis.*

The constant 3 has furthermore been shown to be optimal in [B. F. Logan, 114] where the reader will find an intensive discussion on this question. The next lower estimate is also proved there:

Theorem Let $\lambda_1, \cdots, \lambda_N$ be N be real numbers, and suppose that $a_n \geq 0$ for all n. Then
$$\int_{-T}^{T}\Big|\sum_{1\leq n\leq N} a_n e(\lambda_n t)\Big|^2 dt \geq T \sum_{n\leq N} a_n^2.$$
.

We follow the idea of Corollary 3 of [H. Montgomery and R. Vaughan, 134] but rely on [E. Preissmann, 155] to get the following.

Theorem (2013) Let $(a_n)_{n\geq 1}$ be a series of complex numbers that are such that $\sum_n n|a_n|^2 < \infty$ and $\sum_n |a_n| < \infty$. We have, for $T \geq 0$,

$$\int_{0}^{T}\Big|\sum_{n\geq 1} a_n n^{it}\Big|^2 dt = \sum_{n\leq N} |a_n|^2\big(T + \mathcal{O}^*(2\pi c_0(n+1))\big),$$

where $c_0 = \sqrt{1+\frac{2}{3}\sqrt{\frac{6}{5}}}$. Moreover, when a_n is real-valued, the constant $2\pi c_0$ may be reduced to πc_0.

This is Lemma 6.2 from [O. Ramaré, 158].

Corollary 6.3 and 6.4 of [O. Ramaré, 158] contain explicit versions of a Theorem of [P. Gallagher, 74]

Theorem (2013) Let $(a_n)_{n\geq 1}$ be a series of complex numbers that are such that $\sum_n n|a_n|^2 < \infty$ and $\sum_n |a_n| < \infty$. We have, for $T \geq 0$,

$$\sum_{q\leq Q}\frac{q}{\varphi(q)}\sum_{\substack{\chi \bmod q,\\ \chi \text{ primitive}}}\int_{-T}^{T}\Big|\sum_n a_n\chi(n)n^{it}\Big|^2 dt \leq 7\sum_n |a_n|^2(n + Q^2\max(T,3)).$$

Theorem (2013) Let $(a_n)_{n\geq 1}$ be a series of complex numbers that are such that $\sum_n n|a_n|^2 < \infty$ and $\sum_n |a_n| < \infty$. We have, for $T \geq 0$,

$$\sum_{q\leq Q}\frac{q}{\varphi(q)}\sum_{\substack{\chi \bmod q,\\ \chi \text{ primitive}}}\int_{-T}^{T}\Big|\sum_n a_n\chi(n)n^{it}\Big|^2 dt \leq \sum_n |a_n|^2(43n + \tfrac{33}{8}Q^2\max(T,70)).$$

Last updated on July 14th, 2013, by Olivier Ramaré

[114] B. F. Logan, 1988, "An interference problem for exponentials".
[134] H. Montgomery and R. Vaughan, 1974, "Hilbert's inequality".
[155] E. Preissmann, 1984, "Sur une inégalité de Montgomery et Vaughan".
[158] O. Ramaré, 2016, "An explicit density estimate for Dirichlet L-series".
[74] P. Gallagher, 1970, "A large sieve density estimate near $\sigma = 1$."

Part IV

Exponential sums / points close to curves

Chapter 9

Explicit results on exponential sums

Corresponding html file: `../Articles/Art05.html`

Collecting references: [A. Granville and O. Ramaré, 77], [H. Daboussi and J. Rivat, 40].

Last updated on July 14th, 2012, by Olivier Ramaré

[77] A. Granville and O. Ramaré, 1996, "Explicit bounds on exponential sums and the scarcity of squarefree binomial coefficients".
[40] H. Daboussi and J. Rivat, 2001, "Explicit upper bounds for exponential sums over primes".

Chapter 10

Integer Points near Smooth Plane Curves

Corresponding html file: `../Articles/Art11.html`

In what follows, $N \geqslant 1$ is an arbitrary large integer, $\delta \in \left(0, \frac{1}{2}\right)$ and if $f : [N, 2N] \longrightarrow \mathbb{R}$ is any positive function, then let $\mathcal{R}(f, N, \delta)$ be the number of integers $n \in [N, 2N]$ such that $\|f(n)\| < \delta$, where as usual $\|x\|$ denotes the distance from $x \in \mathbb{R}$ to its nearest integer. Note that, since δ is very small, $\mathcal{R}(f, N, \delta)$ roughly counts the number of integer points very close to the arc $y = f(x)$ with $N \leqslant x \leqslant 2N$. Hence the trivial estimate is given by $\mathcal{R}(f, N, \delta) \leqslant N + 1$.

The number $\mathcal{R}(f, N, \delta)$ arises fairly naturally in a large collection of problems in number theory, e.g. [M. Filaseta, 67], [M. Filaseta and O. Trifonov, 68], [M. Huxley, 86], [M. Huxley and P. Sargos, 87], [M. Huxley and P. Sargos, 88], [M. Huxley and O. Trifonov, 89] and bibref("Huxley*07""). We deal with either getting an asymptotic formula of the shape

$$\mathcal{R}(f, N, \delta) = N\delta + \text{Error terms}$$

where the remainder terms depend on the derivatives of f but not on δ, or finding an upper bound for $\mathcal{R}(f, N, \delta)$ as accurate as possible.

[67] M. Filaseta, 1990, "Short interval results for squarefree numbers".
[68] M. Filaseta and O. Trifonov, 1996, "The distribution of fractional parts with applications to gap results in number theory."
[86] M. Huxley, 1996, *Area, Lattice Points and Exponential Sums.*
[87] M. Huxley and P. Sargos, 1995, "Integer points close to a plane curve of class C^n. (Points entiers au voisinage d'une courbe plane de classe C^n.)"
[88] M. Huxley and P. Sargos, 2006, "Integer points in the neighborhood of a plane curve of class C^n. II. (Points entiers au voisinage d'une courbe plane de classe C^n. II.)".
[89] M. Huxley and O. Trifonov, 1996, "The square-full numbers in an interval".

10.1 Bounds using elementary methods

The basic result of the theory is well-known and may be found in [I. Vinogradov, 204]. The proof follows from a clever use of the mean-value theorem (see Theorem 5.6 of [O. Bordellès, 17] for instance).

Theorem (First derivative test) Let $f \in C^1[N, 2N]$ such that there exist $\lambda_1 > 0$ and $c_1 \geqslant 1$ such that, for all $x \in [N, 2N]$, we have

$$\lambda_1 \leqslant \left|f'(x)\right| \leqslant c_1 \lambda_1.$$

Then

$$\mathcal{R}(f, N, \delta) \leqslant 2c_1 N\lambda_1 + 4c_1 N\delta + \frac{2\delta}{\lambda_1} + 1.$$

This result is useful when λ_1 is very small, so that the condition is in general too restrictive in the applications. Using a rather neat combinatorial trick, [M. Huxley, 86] succeeded in passing from the first derivative to the second derivative. This reduction step enables him to apply this theorem to a function being approximatively of the same order of magnitude as f'. This provides the following useful result.

Theorem (Second derivative test) Let $f \in C^2[N, 2N]$ such that there exist $\lambda_2 > 0$ and $c_2 \geqslant 1$ such that, for all $x \in [N, 2N]$, we have

$$\lambda_2 \leqslant \left|f''(x)\right| \leqslant c_2 \lambda_2 \quad \text{and} \quad N\lambda_2 \geqslant c_2^{-1}.$$

Then

$$\mathcal{R}(f, N, \delta) \leqslant 6 \left\{ (3c_2)^{1/3} N\lambda_2^{1/3} + (12c_2)^{1/2} N\delta^{1/2} + 1 \right\}.$$

Both hypotheses above are often satisfied in practice, so that this result may be considered as the first useful tool of the theory. A proof of this Theorem may be found in Theorem 5.8 of [O. Bordellès, 17].

Using a kth version of Huxley's reduction principle may allow us to generalize the above results. A better way is to split the integer points into two classes, namely the major arcs in which the points belong to a same algebraic curve of degree $\leq k-1$, and the minor arcs. The points coming from the minor arcs are treated by divided differences techniques, generalizing the proof of both theorems above and, by a careful analysis of the points belonging to major arcs, [M. Huxley and P. Sargos, 87] and [M. Huxley and P. Sargos, 88] succeeded in proving the following fundamental result. A proof of an explicit version may be found in Theorem 5.12 of [O. Bordellès, 17].

[204] I. Vinogradov, 2004, *The method of trigonometrical sums in the theory of numbers.*
[17] O. Bordellès, 2012, *Arithmetic Tales.*
[86] M. Huxley, 1996, *Area, Lattice Points and Exponential Sums.*
[87] M. Huxley and P. Sargos, 1995, "Integer points close to a plane curve of class C^n. (Points entiers au voisinage d'une courbe plane de classe C^n.)"
[88] M. Huxley and P. Sargos, 2006, "Integer points in the neighborhood of a plane curve of class C^n. II. (Points entiers au voisinage d'une courbe plane de classe C^n. II.)".

Theorem (kth derivative test) Let $k \geqslant 3$ be an integer and $f \in C^k[N, 2N]$ such that there exist $\lambda_k > 0$ and $c_k \geqslant 1$ such that, for all $x \in [N, 2N]$, we have

$$\lambda_k \leqslant \left|f^{(k)}(x)\right| \leqslant c_k \lambda_k.$$

Let $\delta \in \left(0, \frac{1}{4}\right)$. Then

$$\mathcal{R}(f, N, \delta) \leqslant \alpha_k N \lambda_k^{\frac{2}{k(k+1)}} + \beta_k N \delta^{\frac{2}{k(k-1)}} + 8k^3 \left(\frac{\delta}{\lambda_k}\right)^{1/k} + 2k^2 \left(5e^3 + 1\right)$$

where

$$\alpha_k = 2k^2 c_k^{\frac{2}{k(k+1)}} \quad \text{and} \quad \beta_k = 4k^2 \left(5e^3 c_k^{\frac{2}{k(k-1)}} + 1\right).$$

10.2 Bounds using exponential sums techniques

The next result leads us to estimate $\mathcal{R}(f, N, \delta)$ with the help of exponential sums (see [S. W. Graham and G. Kolesnik, 76] for instance), which have been extensively studied in the 20th century by many specialists, such as van der Corput, Weyl or Vinogradov. Nevertheless, even using the finest exponent pairs to date, the result generally does not significantly improve on the previous estimates seen above. A simple proof of the following inequality may be found in [M. Filaseta, 67].

Theorem (kth derivative test) Let $f : [N, 2N] \longrightarrow \mathbb{R}$ be any function and $\delta \in \left(0, \frac{1}{4}\right)$. Set $K = \left\lfloor (8\delta)^{-1} \right\rfloor + 1$. Then, for any positive integer $H \leqslant K$, we have

$$\mathcal{R}(f, N, \delta) \leqslant \frac{4N}{H} + \frac{4}{H} \sum_{h=1}^{H} \left| \sum_{N \leqslant n \leqslant 2N} e(hf(n)) \right|.$$

10.3 Integer points on curves

This last part is somewhat out of the scope of the TME-EMT project, but may help the reader in orienting him/herself in the litterature.

When $\delta \longrightarrow 0$, we are led to counting the number of integer points lying on curves, and we denote this number by $\mathcal{R}(f, N, 0)$. This problem goes back to Jarník [V. Jarník, 93] who proved that a strictly convex arc $y = f(x)$ with length L has at most

$$\leqslant \frac{3}{(2\pi)^{1/3}} L^{2/3} + O\left(L^{1/3}\right)$$

integer points and this is a nearly best possible result under the sole hypothesis of convexity. However, [H. Swinnerton-Dyer, 192] proved that if $f \in C^3[0, N]$

[76] S. W. Graham and G. Kolesnik, 1991, *Van der Corput's Method of Exponential Sums*.
[67] M. Filaseta, 1990, "Short interval results for squarefree numbers".
[93] V. Jarník, 1925, "Über die Gitterpunkte auf konvexen Kurven".
[192] H. Swinnerton-Dyer, 1974, "The number of lattice points on a convex curve".

is such that $|f(x)| \leqslant N$ and $f'''(x) \neq 0$ for all $x \in [0, N]$, then the number of integer points on the arc $y = f(x)$ with $0 \leqslant x \leqslant N$ is $\ll N^{3/5+\varepsilon}$. This result was later generalized by [E. Bombieri and J. Pila, 14] who showed among other things the following estimate.

Theorem (1989) Let $N \geqslant 1$, $k \geqslant 4$ be integers and define $K = \binom{k+2}{2}$. Let $\mathcal{I}$ be an interval with length N and $f \in C^K(\mathcal{I})$ satisfying $|f'(x)| \leqslant 1$, $f''(x) > 0$ and such that the number of solutions of the equation $f^{(K)}(x) = 0$ is $\leqslant m$. Then there exists a constant $c_0 = c_0(k) > 0$ such that

$$\mathcal{R}(f, N, 0) \leqslant c_0(m+1)N^{1/2+3/(k+3)}.$$

Last updated on July 23rd, 2012, by Olivier Bordellès

[14] E. Bombieri and J. Pila, 1989, "The number of integral points on arcs and ovals".

Part V

Size of $L(1, \chi)$ and character sums

Chapter 11

Size of $L(1, \chi)$

Corresponding html file: `../Articles/Art07.html`

Collecting references: [S. Louboutin, 119],

11.1 Upper bounds for $|L(1, \chi)|$

[S. Louboutin, 120], [A. Granville and K. Soundararajan, 80], [A. Granville and K. Soundararajan, 78]. [O. Ramaré, 159], [O. Ramaré, 160], [S. Louboutin, 121],

11.2 Lower bounds for $|L(1, \chi)|$

[S. Louboutin, 115] announces the following lower bound proved in [S. R. Louboutin, 123] .

Theorem (2013) For any non-quadratic primitive Dirichlet character χ of conductor f, we have $|L(1, \chi)| \geq 1/(10 \log(f/\pi))$.

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[119] S. Louboutin, 1993, “Majorations explicites de $|L(1, \chi)|$”.
[120] S. Louboutin, 1996, “Majorations explicites de $|L(1, \chi)|$ (suite)”.
[80] A. Granville and K. Soundararajan, 2003, “The distribution of values of $L(1, \chi)$”.
[78] A. Granville and K. Soundararajan, 2004, “Errata to: The distribution of values of $L(1, \chi)$, in GAFA 13:5 (2003).”
[159] O. Ramaré, 2001, “Approximate Formulae for $L(1, \chi)$”.
[160] O. Ramaré, 2004, “Approximate Formulae for $L(1, \chi)$, II”..
[121] S. Louboutin, 1998, “Majorations explicites du résidu au point 1 des fonctions zêta”.
[115] S. Louboutin, 2013, “An explicit lower bound on moduli of Dichlet L-functions at $s = 1$”.
[123] S. R. Louboutin, 2015, “An explicit lower bound on moduli of Dirichlet L-functions at $s = 1$”.

Chapter 12

Character sums

Corresponding html file: `../Articles/Art15.html`

12.1 Explicit Polya-Vinogradov inequalities

The main Theorem of [Z. M. Qiu, 156] implies the following result.

Theorem (1991) For χ a primitive character to the modulus $q > 1$, we have $\left|\sum_{a=M+1}^{M+N} \chi(a)\right| \le \frac{4}{\pi^2}\sqrt{q}\log q + 0.38\sqrt{q} + \frac{0.637}{\sqrt{q}}$.

When χ is not especially primitive, but is still non-principal, we have $\left|\sum_{a=M+1}^{M+N} \chi(a)\right| \le \frac{8\sqrt{6}}{3\pi^2}\sqrt{q}\log q + 0.63\sqrt{q} + \frac{1.05}{\sqrt{q}}$.

This was improved later by [G. Bachman and L. Rachakonda, 4] into the following.

Theorem (2001) For χ a non-principal character to the modulus $q > 1$, we have $\left|\sum_{a=M+1}^{M+N} \chi(a)\right| \le \frac{1}{3\log 3}\sqrt{q}\log q + 6.5\sqrt{q}$.

These results are superseded by [D. Frolenkov, 71] and more recently by [D. A. Frolenkov and K. Soundararajan, 72] into the following.

Theorem (2013) For χ a non-principal character to the modulus $q \ge 1000$, we have $\left|\sum_{a=M+1}^{M+N} \chi(a)\right| \le \frac{1}{\pi\sqrt{2}}\sqrt{q}(\log q + 6) + \sqrt{q}$.

[156] Z. M. Qiu, 1991, “An inequality of Vinogradov for character sums”.
[4] G. Bachman and L. Rachakonda, 2001, “On a problem of Dobrowolski and Williams and the Pólya-Vinogradov inequality”.
[71] D. Frolenkov, 2011, “A numerically explicit version of the Pólya-Vinogradov inequality”.
[72] D. A. Frolenkov and K. Soundararajan, 2013, “A generalization of the Pólya–Vinogradov inequality”.

In the same paper they improve upon estimates of [C. Pomerance, 154] and get the following.

Theorem (2013) For χ a primitive character to the modulus $q \geq 1200$, we have

$$\max_{M,N} \left| \sum_{a=M+1}^{M+N} \chi(a) \right| \leq \begin{cases} \frac{2}{\pi^2}\sqrt{q}\log q + \sqrt{q}, & \chi \text{ even}, \\ \frac{1}{2\pi}\sqrt{q}\log q + \sqrt{q}, & \chi \text{ odd}. \end{cases}$$

This latter estimates holds as soon as $q \geq 40$.

In case χ odd, the constant $1/(2\pi)$ has already been asymptotically obtained in [E. Landau, 105] and is still unsurpassed. When χ is odd and $M = 1$, the best asymptotical constant up to now is $1/(3\pi)$ from Theorem 7 of [A. Granville and K. Soundararajan, 79],

In case χ even, we have

$$\max_{M,N} \left| \sum_{a=M}^{N} \chi(a) \right| = 2 \max_{N} \left| \sum_{a=1}^{N} \chi(a) \right|.$$

(The LHS is always less than the RHS. Equality is then easily proved). The asymptotical best constant is $23/(35\pi\sqrt{3})$ from Theorem 7 of [A. Granville and K. Soundararajan, 79].

12.2 Burgess type estimates

The following from [E. Treviño, 194] is an explicit version of Burgess with the only restriction being $p \geq 10^7$.

Theorem (2015) Let p be a prime such that $p \geq 10^7$. Let χ be a non-principal character mod p. Let r be a positive integer, and let M and N be non-negative integers with $N \geq 1$. Then

$$\left| \sum_{a=M+1}^{M+N} \chi(a) \right| \leq 2.74 N^{1-\frac{1}{r}} p^{\frac{r+1}{4r^2}} (\log p)^{\frac{1}{r}}.$$

From the same paper, we get the following more specific result.

Theorem (2015) Let p be a prime. Let χ be a non-principal character mod p. Let M and N be non-negative integers with $N \geq 1$, let $2 \leq r \leq 10$ be a positive

[154] C. Pomerance, 2011, "Remarks on the Pólya-Vinogradov inequality".
[105] E. Landau, 1918, "Abschätzungen von Charaktersummen, Einheiten und Klassenzahlen".
[79] A. Granville and K. Soundararajan, 2007, "Large character sums: pretentious characters and the Pólya-Vinogradov theorem".
[194] E. Treviño, 2015, "The Burgess inequality and the least kth power non-residue".

integer, and let p_0 be a positive real number. Then for $p \geq p_0$, there exists $c_1(r)$, a constant depending on r and p_0 such that

$$\left|\sum_{a=M+1}^{M+N} \chi(a)\right| \leq c_1(r) N^{1-\frac{1}{r}} p^{\frac{r+1}{4r^2}} (\log p)^{\frac{1}{r}}$$

where $c_1(r)$ is given by

r	$p_0 = 10^7$	$p_0 = 10^{10}$	$p_0 = 10^{20}$
2	2.7381	2.5173	2.3549
3	2.0197	1.7385	1.3695
4	1.7308	1.5151	1.3104
5	1.6107	1.4572	1.2987
6	1.5482	1.4274	1.2901
7	1.5052	1.4042	1.2813
8	1.4703	1.3846	1.2729
9	1.4411	1.3662	1.2641
10	1.4160	1.3495	1.2562

We can get a smaller exponent on log if we restrict the range of N or if we have $r \geq 3$.

Theorem (2015) Let p be a prime. Let χ be a non-principal character mod p. Let M and N be non-negative integers with $1 \leq N \leq 2p^{\frac{1}{2}+\frac{1}{4r}}$ or $r \geq 3$. Let $r \leq 10$ be a positive integer, and let p_0 be a positive real number. Then for $p \geq p_0$, there exists $c_2(r)$, a constant depending on r and p_0 such that

$$\left|\sum_{a=M+1}^{M+N} \chi(a)\right| \leq c_2(r) N^{1-\frac{1}{r}} p^{\frac{r+1}{4r^2}} (\log p)^{\frac{1}{2r}},$$

where $c_2(r)$ is given by

r	$p_0 = 10^7$	$p_0 = 10^{10}$	$p_0 = 10^{20}$
2	3.7451	3.5700	3.5341
3	2.7436	2.5814	2.4936
4	2.3200	2.1901	2.1071
5	2.0881	1.9831	1.9037
6	1.9373	1.8504	1.7748
7	1.8293	1.7559	1.6843
8	1.7461	1.6836	1.6167
9	1.6802	1.6262	1.5638
10	1.6260	1.5786	1.5210

Kevin McGown in [K. J. McGown, 132] has slightly worse constants in a slightly larger range of N for smaller values of p.

[132] K. J. McGown, 2012, “Norm-Euclidean cyclic fields of prime degree”.

Theorem (2012) Let $p \geq 2 \cdot 10^4$ be a prime number. Let M and N be non-negative integers with $1 \leq N \leq 4p^{\frac{1}{2}+\frac{1}{4r}}$. Suppose χ is a non-principal character mod p. Then there exists a computable constant $C(r)$ such that

$$\left|\sum_{a=M+1}^{M+N} \chi(a)\right| \leq C(r) N^{1-\frac{1}{r}} p^{\frac{r+1}{4r^2}} (\log p)^{\frac{1}{2r}},$$

where $C(r)$ is given by

r	$C(r)$	r	$C(r)$
2	10.0366	9	2.1467
3	4.9539	10	2.0492
4	3.6493	11	1.9712
5	3.0356	12	1.9073
6	2.6765	13	1.8540
7	2.4400	14	1.8088
8	2.2721	15	1.7700

Finally, if the character is quadratic (and with a more restrictive range), we have slightly stronger results due to Booker in [A. Booker, 15].

Theorem (2006) Let $p > 10^{20}$ be a prime number with $p \equiv 1 \pmod 4$. Let $r \in \{2, 3, 4, \ldots, 15\}$. Let M and N be real numbers such that $0 < M, N \leq 2\sqrt{p}$. Let χ be a non-principal quadratic character mod p. Then

$$\left|\sum_{a=M+1}^{M+N} \chi(a)\right| \leq \alpha(r) N^{1-\frac{1}{r}} p^{\frac{r+1}{4r^2}} \left(\log p + \beta(r)\right)^{\frac{1}{2r}},$$

where $\alpha(r)$ and $\beta(r)$ are given by

r	$\alpha(r)$	$\beta(r)$	r	$\alpha(r)$	$\beta(r)$
2	1.8221	8.9077	9	1.4548	0.0085
3	1.8000	5.3948	10	1.4231	-0.4106
4	1.7263	3.6658	11	1.3958	-0.7848
5	1.6526	2.5405	12	1.3721	-1.1232
6	1.5892	1.7059	13	1.3512	-1.4323
7	1.5363	1.0405	14	1.3328	-1.7169
8	1.4921	0.4856	15	1.3164	-1.9808

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[15] A. Booker, 2006, “Quadratic class numbers and character sums”.

Part VI

Zeros and zero-free regions

Chapter 13

Bounds for $|\zeta(s)|$, $|L(s,\chi)|$ and related questions

Corresponding html file: `../Articles/Art06.html`

Collecting references: [T. Trudgian, 199], [H. Kadiri and N. Ng, 100],

13.1 Size of $|\zeta(s)|$ and of L-series

Theorem 4 of [H. Rademacher, 157] gives the convexity bound. See also section 4.1 of [T. S. Trudgian, 202].

Theorem (1959) In the strip $-\eta \le \sigma \le 1+\eta$, $0<\eta\le 1/2$, the Dedekind zeta function $\zeta_K(s)$ belonging to the algebraic number field K of degree n and discriminant d satisfies the inequality

$$|\zeta_K(s)| \le 3\left|\frac{1+s}{1-s}\right|\left(\frac{|d||1+s|}{2\pi}\right)^{\frac{1+\eta-\sigma}{2}}\zeta(1+\eta)^n.$$

On the line $\Re s = 1/2$, Lemma 2 of [R. Lehman, 109] gives a better result, namely

Theorem (1970) If $t \ge 1/5$, we have $|\zeta(\frac{1}{2}+it)| \le 4(t/(2\pi))^{1/4}$.

In fact, Lehman states this Theorem for $t \ge 64/(2\pi)$, but modern means of computations makes it easy to check that it holds as soon as $t \ge 0.2$. See also equation (56) of [R. J. Backlund, 5] reproduced below.

[199] T. Trudgian, 2011, "Improvements to Turing's method".
[100] H. Kadiri and N. Ng, 2012, "Explicit zero density theorems for Dedekind zeta functions".
[157] H. Rademacher, 1959, "On the Phragmén-Lindelöf theorem and some applications".
[202] T. S. Trudgian, 2014, "An improved upper bound for the argument of the Riemann zeta-function on the critical line II"..
[109] R. Lehman, 1970, "On the distribution of zeros of the Riemann zeta-function".
[5] R. J. Backlund, 1918, "Über die Nullstellen der *Riemann*schen Zetafunktion."

For Dirichlet L-series, Theorem 3 of [H. Rademacher, 157] gives the corresponding convexity bound.

Theorem (1959) In the strip $-\eta \le \sigma \le 1+\eta$, $0 < \eta \le 1/2$, for all moduli $q > 1$ and all primitive characters χ modulo q, the inequality

$$|L(s,\chi)| \le \left(q\frac{|1+s|}{2\pi}\right)^{\frac{1+\eta-\sigma}{2}} \zeta(1+\eta)$$

holds.

This paper contains other similar convexity bounds.

Corollary to Theorem 3 of [Y. Cheng and S. Graham, 29] goes beyond convexity.

Theorem (2001) For $0 \le t \le e$, we have $|\zeta(\frac12 + it)| \le 2.657$. For $t \ge e$, we have $|\zeta(\frac12 + it)| \le 3t^{1/6}\log t$. Section 5 of [T. S. Trudgian, 202] shows that one can replace the constant 3 by 2.38.

This is improved in [G. A. Hiary, 84].

Theorem (2016) When $t \ge 3$, we have $|\zeta(\frac12 + it)| \le 0.63t^{1/6}\log t$.

Concerning L-series, the situation is more difficult but [G. A. Hiary, 83] manages, among other and more precise results, to prove the following.

Theorem (2016) Assume χ is a primitive Dirichlet character modulo $q > 1$. Assume further that q is a sixth power. Then, when $|t| \ge 200$, we have

$$|L(\tfrac12 + it, \chi)| \le 9.05 d(q)(q|t|)^{1/6}(\log q|t|)^{3/2}$$

where $d(q)$ is the number of divisors of q.

It is often useful to have a representation of the Riemann zeta function or of L-series inside the critical strip. In the case of L-series, [R. Spira, 189] and [R. Rumely, 182] proceed via decomposition in Hurwitz zeta function which they compute through an Euler-MacLaurin development. We have a more efficient approximation of the Riemann zeta function provided by the Riemann Siegel

[157] H. Rademacher, 1959, "On the Phragmén-Lindelöf theorem and some applications".
[29] Y. Cheng and S. Graham, 2004, "Explicit estimates for the Riemann zeta function".
[202] T. S. Trudgian, 2014, "An improved upper bound for the argument of the Riemann zeta-function on the critical line II"..
[84] G. A. Hiary, 2016, "An explicit van der Corput estimate for $\zeta(1/2+it)$".
[83] G. A. Hiary, 2016, "An explicit hybrid estimate for $L(1/2+it,\chi)$".
[189] R. Spira, 1969, "Calculation of Dirichlet L-functions".
[182] R. Rumely, 1993, "Numerical Computations Concerning the ERH"..

formula, see for instance equations (3-2)–(3.3) of [A. Odlyzko, 145]. This expression is due to [W. Gabcke, 73]. See also equations (2.4)-(2.5) of [R. Lehman, 110], a corrected version of Theorem 2 of [E. Titchmarsh, 193].

In general, we have the following estimate taken from equations (53)-(54), (56) and (76) of [R. J. Backlund, 5] (see also [R. Backlund, 6]).

Theorem (1918)
- When $t \geq 50$ and $\sigma \geq 1$, we have $|\zeta(\sigma+it)| \leq \log t - 0.048$.
- When $t \geq 50$ and $0 \leq \sigma \leq 1$, we have $|\zeta(\sigma+it)| \leq \frac{t^2}{t^2-4}\left(\frac{t}{2\pi}\right)^{\frac{1-\sigma}{2}} \log t$.
- When $t \geq 50$ and $-1/2 \leq \sigma \leq 0$, we have $|\zeta(\sigma+it)| \leq \left(\frac{t}{2\pi}\right)^{\frac{1}{2}-\sigma} \log t$.

On the line $\Re s = 1$, one can rely on [T. Trudgian, 197].

Theorem (2012) When $t \geq 3$, we have $|\zeta(1+it)| \leq \frac{3}{4}\log t$.

Asymptotically better bounds are available since the huge work of [K. Ford, 69].

Theorem (2002) When $t \geq 3$ and $1/2 \leq \sigma \leq 1$, we have $|\zeta(\sigma+it)| \leq 76.2 t^{4.45(1-\sigma)^{3/2}} (\log t)^{2/3}$.

The constants are still too large for this result to be of use in any decent region. See [M. Kulas, 104] for an earlier estimate.

13.2 On the total number of zeroes

The first explicit estimate for the number of zeros of the Riemann ζ-function goes back to [R. Backlund, 6]. An elegant consequence of the result of Backlund is the following easy estimate taken from Lemma 1 of [R. S. Lehman, 108].

Theorem (1966) If φ is a continuous function which is positive and monotone decreasing for $2\pi e \leq T_1 \leq t \leq T_2$, then

$$\sum_{T_1 < \gamma \leq T_2} \varphi(\gamma) = \frac{1}{2\pi}\int_{T_1}^{T_2} \varphi(t) \log\frac{t}{2\pi} dt + O^*\left(4\varphi(T_1)\log T_1 + 2\int_{T_1}^{T_2} \frac{\varphi(t)}{t} dt\right)$$

where the summation is over all zeros of the Riemann ζ-function of imaginary part between T_1 and T_2, with multiplicity.

[145] A. Odlyzko, 1987, "On the distribution of spacings between zeros of the zeta function".
[73] W. Gabcke, 1979, "Neue Herleitung und explizite Restabschaetzung der Riemann-Siegel-Formel".
[110] R. Lehman, 1966, "Separation of zeros of the Riemann zeta-function".
[193] E. Titchmarsh, 1947, "On the zeros of the Riemann zeta function".
[5] R. J. Backlund, 1918, "Über die Nullstellen der *Riemann*schen Zetafunktion."
[6] R. Backlund, 1914, "Sur les zéros de la fonction $\zeta(s)$ de Riemann".
[197] T. Trudgian, 2012, "A new upper bound for $|\zeta(1+it)|$".
[69] K. Ford, 2002, "Vinogradov's integral and bounds for the Riemann zeta function".
[104] M. Kulas, 1994, "Some effective estimation in the theory of the Hurwitz-zeta function".
[108] R. S. Lehman, 1966, "On the difference $\pi(x) - \mathrm{li}(x)$".

Theorem 19 of [J. Rosser, 176] gives a bound for the total number of zeroes.

Theorem (1941) For $T \geq 2$, we have

$$N(T) = \sum_{\substack{\rho,\\ 0<\gamma\leq T}} 1 = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + O^*\Big(0.137\log T + 0.443\log\log T + 1.588\Big)$$

where the summation is over all zeros of the Riemann ζ-function of imaginary part between 0 and T, with multiplicity.

It is noted in Lemma 1 of [O. Ramaré and Y. Saouter, 173] that the O-term can be replaced by the simpler $O^*(0.67\log\frac{T}{2\pi})$ when $T \geq 10^3$.

This is improved in Corollary 1 of [T. S. Trudgian, 202] into

Theorem (1941) For $T \geq e$, we have

$$N(T) = \sum_{\substack{\rho,\\ 0<\gamma\leq T}} 1 = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + O^*\big(0.112\log T + 0.278\log\log T + 2.510 + \frac{1}{5T}\big)$$

where the summation is over all zeros of the Riemann ζ-function of imaginary part between 0 and T, with multiplicity.

13.3 L^2-averages

We can find in [H. Helfgott, 82] the proof of the following estimate. Though it is unpublished yet, the full proof is available.

Theorem (1941) Let $0 < \sigma \leq 1$ and $T \geq 3$. Then

$$\frac{1}{2\pi}\left(\int_{\sigma-i\infty}^{\sigma-iT} + \int_{\sigma+iT}^{\sigma+i\infty}\right)\frac{|\zeta(s)|^2}{|s|^2}ds \leq \kappa_{\sigma,T}\begin{cases}\frac{c_{1,\sigma}}{T} + \frac{c^\flat_{1,\sigma}}{T^{2\sigma}} & \text{when } \sigma > 1/2,\\ \frac{\log T}{2T} + \frac{c^\flat_{2,\sigma}}{T} & \text{when } \sigma = 1/2,\\ c_{3,\sigma}/T^{2\sigma} & \text{when } \sigma < 1/2.\end{cases}$$

where

$$c_{1,\sigma} = \zeta(2\sigma)/2, c^\flat_{1,\sigma} = c^2\frac{3^{2\sigma}}{2\sigma}, c^\flat_{2,\sigma} = 3c^2 + \frac{1-\log 3}{2}, c = 9/16,$$

$$c_{3,\sigma} = \left(\frac{c^2}{2\sigma} + \frac{1/6}{1-2\sigma}\right)\left(1+\frac{1}{\sigma}\right)^{2\sigma}, \kappa_{\sigma,T} = \begin{cases}\frac{9/4}{\left(1-\frac{9/2}{T^2}\right)^2} & \text{when } 1/2 \leq \sigma \leq 1,\\ \frac{(1+\sigma)^2}{\left(1-\frac{(1+\sigma)^2}{\sigma T^2}\right)^2} & \text{when } 0 < \sigma < 1/2.\end{cases}$$

[176] J. Rosser, 1941, "Explicit bounds for some functions of prime numbers".
[173] O. Ramaré and Y. Saouter, 2003, "Short effective intervals containing primes".
[202] T. S. Trudgian, 2014, "An improved upper bound for the argument of the Riemann zeta-function on the critical line II"..
[82] H. Helfgott, 2017, "L^2 bounds for tails of $\zeta(s)$ on a vertical line".

13.4 Bounds on the real line

After some estimates from [G. Bastien and M. Rogalski, 9], Lemma 5.1 of [O. Ramaré, 158] shows the following.

Theorem (2013) When $\sigma > 1$ and t is any real number, we have $|\zeta(\sigma + it)| \le e^{\gamma(\sigma-1)}/(\sigma - 1)$.

Here is the Theorem of [H. Delange, 43]. See also Lemma 2.3 of [K. Ford, 70] for a slightly weaker version.

Theorem (1987) When $\sigma > 1$ and t is any real number, we have

$$-\Re\frac{\zeta'}{\zeta}(\sigma + it) \le \frac{1}{\sigma - 1} - \frac{1}{2\sigma^2}.$$

Last updated on June 18th, 2019, by Olivier Ramaré

[9] G. Bastien and M. Rogalski, 2002, “Convexité, complète monotonie et inégalités sur les fonctions zêta et gamma, sur les fonctions des opérateurs de Baskakov et sur des fonctions arithmétiques”.
[158] O. Ramaré, 2016, “An explicit density estimate for Dirichlet L-series”.
[43] H. Delange, 1987, “Une remarque sur la dérivée logarithmique de la fonction zêta de Riemann”.
[70] K. Ford, 2000, “Zero-free regions for the Riemann zeta function”.

Chapter 14

Explicit zero-free regions for the ζ and L functions

Corresponding html file: `../Articles/Art08.html`

14.1 Numerical verifications of the Generalized Riemann Hypothesis

Numerical verifications of the Riemann hypothesis for the Riemann ζ-function have been pushed extremely far. B. Riemann himself computed the first zeros. Concerning more recent published papers, we mention [J. Van de Lune, H. te Riele, and D.T.Winter, 203] who proved that

Theorem (1986) Every zero ρ of ζ that have a real part between 0 and 1 and an imaginary part not more, in absolute value, than $\le T_0 = 545\,439\,823$ are in fact on the critical line, i.e. satisfy $\Re\rho = 1/2$.

The bound $545\,439\,823$ is increased to $1\,000\,000\,000$ in [D. Platt, 149]. In [D. J. Platt, 152], this bound is further increased to $30\,610\,046\,000$. Between these results, [S. Wedeniwski, 183] announced that, he and many collaborators proved, using a network method:

Theorem (2002) $T_0 = 29\,538\,618\,432$ is admissible in the theorem above.

And [X. Gourdon and P. Demichel, 75] went one step further

[203] J. Van de Lune, H. te Riele, and D.T.Winter, 1986, "On the zeros of the Riemann zeta-function in the critical strip. IV"..
[149] D. Platt, 2011, "Computing degree 1 L-function rigorously".
[152] D. J. Platt, 2017, "Isolating some non-trivial zeros of zeta".
[183] S. Wedeniwski, 2002, "On the Riemann hypothesis".
[75] X. Gourdon and P. Demichel, 2004, "The 10^{13} first zeros of the Riemann Zeta Function and zeros computations at very large height".

Theorem (2004) $T_0 = 2.445 \cdot 10^{12}$ is admissible in the theorem above.

These two last announcements have not been subject to any academic papers.

One of the key ingredient is an explicit Riemann-Siegel formula due to [W. Gabcke, 73] (the preprint of Gourdon mentionned above gives a version of Gabcke's result) and such a formula is missing in the case of Dirichlet L-function.

Let us introduce some terminology. We say that a modulus $q \geq 1$ (i.e. an integer!) satisfies $GRH(H)$ for some numerical value H when every zero ρ of the L-function associated to a primitive Dirichlet character of conductor q and whose real part lies within the critical line (i.e. has a real part lying inside the open interval $(0, 1)$) and whose imaginary part is below, in absolute value, H, in fact satisfies $\Re\rho = 1/2$.

By employing an Euler-McLaurin formula, [R. Rumely, 182] has proved that

Theorem (1993)

- Every $q \leq 13$ satisfies $GRH(10\,000)$.
- Every q belonging to one of the sets
  - $\{k \leq 72\}$
  - $\{k \leq 112, k \text{ non premier}\}$
  - $\{116, 117, 120, 121, 124, 125, 128, 132, 140, 143, 144, 156, 163, 169, 180, 216, 243, 256, 360, 420, 432\}$

  satisfies $GRH(2\,500)$.

These computations have been extended by [M. Bennett, 10] by using Rumely's programm. All these computations have been superseded by the work of D. Platt. [D. Platt, 149] and [D. Platt, 150] use two fast Fourier transforms, one in the t-aspect and one in the q-aspect, as well as an approximate functionnal equation to prove via extremely rigorous computations that

Theorem (2011-2013) Every modulus $q \leq 400\,000$ satisfies $GRH(100\,000\,000/q)$.

We mention here the algorithm of [S. Omar, 146] that enables one to prove efficiently that some L-functions have no zero within the rectangle $1/2 \leq \sigma \leq 1$ et $2\sigma - |t| = 1$ though this algorithm has not been put in practice.

There are much better results concerning real zeros of Dirichlet L-functions associated to real characters.

[73] W. Gabcke, 1979, "Neue Herleitung und explizite Restabschaetzung der Riemann-Siegel-Formel".
[182] R. Rumely, 1993, "Numerical Computations Concerning the ERH"..
[10] M. Bennett, 2001, "Rational approximation to algebraic numbers of small height: the Diophantine equation $|ax^n - by^n| = 1$".
[149] D. Platt, 2011, "Computing degree 1 L-function rigorously".
[150] D. Platt, 2013, "Numerical computations concerning the GRH"..
[146] S. Omar, 2001, "Localization of the first zero of the Dedekind zeta function".

14.2 Asymptotical zero-free regions

The first fully explicit zero free region for the Riemann zeta-function is due to [J. Rosser, 179] in Lemma 19 (essentially with $R_0 = 19$ in the notations below). This is next imporved upon in Theorem 1 of [J. Rosser and L. Schoenfeld, 181] by using a device of [S. Stechkin, 191] (getting essentially $R_0 = 9.646$). The next step is in [O. Ramaré and R. Rumely, 172] where the second order term is improved upon, relying on [S. Stechkin, 190].

Next, in [H. Kadiri, 99] and later in [H. Kadiri, 98], the following result is proven.

Theorem (2002) The Riemann ζ-function has no zeros in the region

$$\Re s \geq 1 - \frac{1}{R_0 \log(|\Im s| + 2)} \quad \text{with } R_0 = 5.70175.$$

[W.-J. Jang and S.-H. Kwon, 92] improved the value of R_0 by showing that $R_0 = 5.68371$ is admissible. By plugging a better trigonometric polynomial in the same method, it is proved in [M. J. Mossinghoff and T. S. Trudgian, 140] that

Theorem (2015) The Riemann ζ-function has no zeros in the region

$$\Re s \geq 1 - \frac{1}{R_0 \log(|\Im s| + 2)} \quad \text{with } R_0 = 5.573412.$$

Concerning Dirichlet L-function, the first explicit zero-free region has been obtained in [K. McCurley, 131] by adaptating [J. Rosser and L. Schoenfeld, 181]. [H. Kadiri, 99] (cf also [H. Kadiri, 95]) improves that into:

Theorem (2002) The Dirichlet L-functions associated to a character of conductor q has no zero in the region:

$$\Re s \geq 1 - \frac{1}{R_1 \log(q \max(1, |\Im s|))} \quad \text{with } R_1 = 6.4355,$$

to the exception of at most one of them which would hence be attached to a real-valued character. This exceptional one would have at most one zero inside the forbidden region (and which is loosely called a "Siegel zero").

[179] J. Rosser, 1938, "The n-th prime is greater than $n \log n$".
[181] J. Rosser and L. Schoenfeld, 1975, "Sharper bounds for the Chebyshev Functions $\vartheta(X)$ and $\psi(X)$".
[191] S. Stechkin, 1970, "Zeros of Riemann zeta-function".
[172] O. Ramaré and R. Rumely, 1996, "Primes in arithmetic progressions".
[190] S. Stechkin, 1989, "Rational inequalities and zeros of the Riemann zeta-function".
[99] H. Kadiri, 2002, "Une région explicite sans zéros pour les fonctions L de Dirichlet".
[98] H. Kadiri, 2005, "Une région explicite sans zéros pour la fonction ζ de Riemann".
[92] W.-J. Jang and S.-H. Kwon, 2014, "A note on Kadiri's explicit zero free region for Riemann zeta function".
[140] M. J. Mossinghoff and T. S. Trudgian, 2015, "Nonnegative trigonometric polynomials and a zero-free region for the Riemann zeta-function".
[131] K. McCurley, 1984, "Explicit zero-free regions for Dirichlet L-functions".
[95] H. Kadiri, 2009, "An explicit zero-free region for the Dirichlet L-functions".

In [**Kadiri*18**], the next theorem is proved.

Theorem (2016) The Dirichlet L-functions associated to a character of conductor $q \in [3, 400\,000]$ has no zero in the region:

$$\Re s \geq 1 - \frac{1}{R_2 \log(q \max(1, |\Im s|))} \quad \text{with } R_1 = 5.60.$$

Concerning the Vinogradov-Korobov zero-free region, [K. Ford, 70] shows that

Theorem (2001) The Riemann ζ-function has no zeros in the region

$$\Re s \geq 1 - \frac{1}{58(\log|\Im s|)^{2/3}(\log\log|\Im s|)^{1/3}} \quad (|\Im s| \geq 3).$$

Concerning the Dedekind ζ-function, see [H. Kadiri, 96].

14.3 Real zeros

[J. Rosser, 177], [J. Rosser, 178], [K. S. Chua, 30], [M. Watkins, 207],

14.4 Density estimates

After initial work of [J. Chen and T. Wang, 28] and [M. Liu and T. Wang, 113], here are the latest two best results. We first define

$$N(\sigma, T, \chi) = \sum_{\substack{\rho=\beta+i\gamma, \\ L(\rho,\chi)=0, \\ \sigma\leq\beta, |\gamma|\leq T}} 1$$

which thus counts the number of zeroes ρ of $L(s, \chi)$, zeroes whose real part is denoted by β (and assumed to be larger than σ), and whose imaginary part in absolute value γ is assumed to be not more than T. For the Riemann ζ-function (i.e. when $\chi = \chi_0$ the principal character modulo 1), it is customary to count only the zeroes with positive imaginary part. The relevant number is usually denoted by $N(\sigma, T)$. We have $2N(\sigma, T) = N(\sigma, T, \chi_0)$.

For low values, we start with the main Theorem of [H. Kadiri and N. Ng, 100]. We reproduce only a special case.

[70] K. Ford, 2000, "Zero-free regions for the Riemann zeta function".
[96] H. Kadiri, 2012, "Explicit zero-free regions for Dedekind zeta functions".
[177] J. Rosser, 1949, "Real roots of Dirichlet L-series".
[178] J. Rosser, 1950, "Real roots of Dirichlet L-series".
[30] K. S. Chua, 2005, "Real zeros of Dedekind zeta functions of real quadratic field".
[207] M. Watkins, 2004, "Real zeros of real odd Dirichlet L-functions".
[28] J. Chen and T. Wang, 1989, "On the distribution of zeros of Dirichlet L-functions".
[113] M. Liu and T. Wang, 2002, "Distribution of zeros of Dirichlet L-functions and an explicit formula for $\psi(t, \chi)$".
[100] H. Kadiri and N. Ng, 2012, "Explicit zero density theorems for Dedekind zeta functions".

Theorem (2013) Let $T \geq 3.061 \cdot 10^{10}$. We have $2N(17/20, T, \chi_0) \leq 0.5561T + 0.7586 \log T - 268658$ where χ_0 is the principal character modulo 1.

Otherwise, here is the result of [O. Ramaré, 158].

Theorem (2016) For $T \geq 2\,000$ and $T \geq Q \geq 10$, as well as $\sigma \geq 0.52$, we have

$$\sum_{q \leq Q} \frac{q}{\varphi(q)} \sum_{\chi \mod {}^*q} N(\sigma, T, \chi) \leq 20\big(56\,Q^5 T^3\big)^{1-\sigma} \log^{5-2\sigma}(Q^2 T) + 32\,Q^2 \log^2(Q^2 T)$$

where $\chi \mod {}^*q$ denotes a sum over all primitive Dirichlet character χ to the modulus q. Furthermore, we have

$$N(\sigma, T, \chi_0) \leq 6T \log T \log\left(1 + \frac{6.87}{2T}(3T)^{8(1-\sigma)/3} \log^{4-2\sigma}(T)\right) + 103(\log T)^2$$

where χ_0 is the principal character modulo 1.

In [H. Kadiri, A. Lumley, and N. Ng, 101]. this result is improved upon, we refer to their paper for their result by quote a corollary.

For $T \geq 1$, we have $N(0.9, T) \leq 11.5\,T^{4/14} \log^{16/5}(T) + 3.2 \log^2(T)$ where $N(\sigma, T) = N(\sigma, T, \chi_0)$ and χ_0 is the principal character modulo 1.

14.5 Miscellanae

The LMFDB[1] database contains the first zeros of many L-functions. A part of Andrew Odlyzko's website[2] contains extensive tables concerning zeroes of the Riemann zeta function.

Last updated on June 18th, 2019, by Olivier Ramaré

[158] O. Ramaré, 2016, "An explicit density estimate for Dirichlet L-series".
[101] H. Kadiri, A. Lumley, and N. Ng, 2018, "Explicit zero density for the Riemann zeta function".

[1] http://www.lmfdb.org
[2] http://www.dtc.umn.edu/~odlyzko/zeta_tables/index.html

Part VII

Sieve and short interval results

Chapter 15

Short intervals containing primes

Corresponding html file: `../Articles/Art09.html`

15.1 Interval with primes, without any congruence condition

The story seems to start in 1845 when Bertrand conjectured after numerical trials that the interval $]n, 2n-3]$ contains a prime as soon as $n \geq 4$. This was proved by Čebyšev in 1852 in a famous work where he got the first good quantitative estimates for the number of primes less than a given bound, say x. By now, analytical means combined with sieve methods (see [R. Baker, G. Harman, and J. Pintz, 7]) ensures us that each of the intervals $[x, x + x^{0.525}]$ for $x \geq x_0$ contains at least one prime. This statement concerns only for the (very) large integers.

It falls very close to what we can get under the assumption of the Riemann Hypothesis: the interval $[x - K\sqrt{x}\log x, x]$ contains a prime, where K is an effective large constant and x is sufficiently large (cf [D. Wolke, 208] for an account on this subject). A theorem of Schoenfeld [L. Schoenfeld, 186] also tells us that the interval

$$[x - \sqrt{x}\log^2 x/(4\pi), x]$$

contains a prime for $x \geq 599$ under the Riemann Hypothesis. These results are still far from the conjecture in [H. Cramer, 38] on probabilistic grounds: the

[7] R. Baker, G. Harman, and J. Pintz, 2001, "The difference between consecutive primes, III"..

[208] D. Wolke, 1983, "On the explicit formula of Riemann-von Mangoldt, II"..

[186] L. Schoenfeld, 1976, "Sharper bounds for the Chebyshev Functions $\vartheta(X)$ and $\psi(X)$ II"..

[38] H. Cramer, 1936, "On the order of magnitude of the difference between consecutive prime numbers".

interval $[x - K\log^2 x, x]$ contains a prime for any $K > 1$ and $x \geq x_0(K)$. Note that this statement has been proved for almost all intervals in a quadratic average sense in [A. Selberg, 187] assuming the Riemann Hypothesis and replacing K by a function $K(x)$ tending arbitrarily slowly to infinity.

[L. Schoenfeld, 186] proved the following.

Theorem (1976) Let x be a real number larger than 2 010 760. Then the interval
$$\Big]x\Big(1 - \frac{1}{16\,597}\Big), x\Big]$$
contains at least one prime.

The main ingredient is the explicit formula and a numerical verification of the Riemann hypothesis.

From a numerical point of view, the Riemann Hypothesis is known to hold up to a very large height (and larger than in 1976). [S. Wedeniwski, 183] and the Zeta grid project verified this hypothesis till height $T_0 = 2.41 \cdot 10^{11}$ and [X. Gourdon and P. Demichel, 75] till height $T_0 = 2.44 \cdot 10^{12}$ thus extending the work [J. Van de Lune, H. te Riele, and D.T.Winter, 203] who had conducted such a verification in 1986 till height 5.45×10^8. This latter computations has appeared in a refereed journal, but this is not the case so far concerning the other computations; section 4 of the paper [Y. Saouter and P. Demichel, 184] casts some doubts on whether all the zeros where checked. Discussions in 2012 with Dave Platt from the university of Bristol led me to believe that the results of [S. Wedeniwski, 183] can be replicated in a very rigorous setting, but that it may be difficult to do so with the results of [X. Gourdon and P. Demichel, 75] with the hardware at our disposal.

In [O. Ramaré and Y. Saouter, 173], we used the value $T_0 = 3.3 \cdot 10^9$ and obtained the following.

Theorem (2002) Let x be a real number larger than 10 726 905 041. Then the interval
$$\Big]x\Big(1 - \frac{1}{28\,314\,000}\Big), x\Big]$$
contains at least one prime.

If one is interested in somewhat larger value, the paper [O. Ramaré and Y. Saouter, 173] also contains the following.

[187] A. Selberg, 1943, "On the normal density of primes in small intervals, and the difference between consecutive primes".
[186] L. Schoenfeld, 1976, "Sharper bounds for the Chebyshev Functions $\vartheta(X)$ and $\psi(X)$ II"..
[183] S. Wedeniwski, 2002, "On the Riemann hypothesis".
[75] X. Gourdon and P. Demichel, 2004, "The 10^{13} first zeros of the Riemann Zeta Function and zeros computations at very large height".
[203] J. Van de Lune, H. te Riele, and D.T.Winter, 1986, "On the zeros of the Riemann zeta-function in the critical strip. IV"..
[184] Y. Saouter and P. Demichel, 2010, "A sharp region where $\pi(x) - \mathrm{li}(x)$ is positive".
[173] O. Ramaré and Y. Saouter, 2003, "Short effective intervals containing primes".

Theorem (2002) Let x be a real number larger than $\exp(53)$. Then the interval

$$\Big]x\Big(1-\frac{1}{204\,879\,661}\Big),x\Big]$$

contains at least one prime.

Increasing the lower bound in x only improves the constant by less than 5 percent. If we rely on [X. Gourdon and P. Demichel, 75], we can prove that

Theorem (2004, conditional) Let x be a real number larger than $\exp(60)$. Then the interval

$$\Big]x\Big(1-\frac{1}{14\,500\,755\,538}\Big),x\Big]$$

contains at least one prime.

Note that all prime gaps have been computed up to 10^{15} in [T. Nicely, 142], extending a result of [A. Young and J. Potler, 209].

In [T. Trudgian, 201], we find

Theorem (2016) Let x be a real number larger than $2\,898\,242$. The interval

$$\Big[x,x\Big(1+\frac{1}{111(\log x)^2}\Big)\Big]$$

contains at least one prime.

In [P. Dusart, 60], we find

Theorem (2016) Let x be a real number larger than $468\,991\,632$. The interval

$$\Big[x,x\Big(1+\frac{1}{5000(\log x)^2}\Big),x\Big]$$

contains at least one prime.

Let x be a real number larger than $89\,693$. The interval

$$\Big[x,x\Big(1+\frac{1}{\log^3 x}\Big)\Big]$$

contains at least one prime.

The proof of these latter results has an asymptotical part, for $x\ge 10^{20}$ where we used the numerical verification of the Riemann hypothesis together with two other arguments: a (very strong) smoothing argument and a use of the Brun-Titchmarsh inequality.

[142] T. Nicely, 1999, "New maximal primes gaps and first occurences".
[209] A. Young and J. Potler, 1989, "First occurence prime gaps".
[201] T. Trudgian, 2016, "Updating the error term in the prime number theorem".
[60] P. Dusart, 2018, "Estimates of some functions over primes".

The second part is of algorithmic nature and covers the range $10^{10} \leq x \leq 10^{20}$ and uses prime generation techniques [U. Maurer, 127]: we only look at families of numbers whose primality can be established with one or two Fermat-like or Pocklington's congruences. This kind of technique has been already used in a quite similar problem in [J.-M. Deshouillers, H. te Riele, and Y. Saouter, 48]. The generation technique we use relies on a theorem proven in [J. Brillhart, D. Lehmer, and J. Selfridge, 22] and enables us to generate dense enough families for the upper part of the range to be investigated. For the remaining (smaller) range, we use theorems of [G. Jaeschke, 91] that yield a fast primality test (for this limited range).

Let us recall here that a second line of approach following the original work of Čebyšev is still under examination though it does not give results as good as analytical means (see [N. Costa Pereira, 37] for the latest result).

For very large numbers [A. W. Dudek, 54] proved the following.

Theorem (2014) The interval $(x, x+3x^{2/3}]$ contains a prime for $x \geq \exp(\exp(34.32))$.

This is improved in [M. Cully-Hugill, 39] as follows.

Theorem (2021) The interval $(x, x+3x^{2/3}]$ contains a prime for $x \geq \exp(\exp(33.99))$.

15.2 Interval with primes under RH, without any congruence condition

Theorem (2002) Under the Riemann Hypothesis, the interval $\left]x-\frac{8}{5}\sqrt{x}\log x, x\right]$ contains a prime for $x \geq 2$.

This is improved upon in [A. W. Dudek, 55] into:

Theorem (2015) Under the Riemann Hypothesis, the interval $\left]x-\frac{4}{\pi}\sqrt{x}\log x, x\right]$ contains a prime for $x \geq 2$.

In [E. Carneiro, M. Milinovich, and K. Soundararajan, 26], the authors go one step further and prove the next result.

Theorem (2019) Under the Riemann Hypothesis, the interval $\left]x-\frac{22}{25}\sqrt{x}\log x, x\right]$ contains a prime for $x \geq 4$.

[127] U. Maurer, 1995, "Fast Generation of Prime Numbers and Secure Public-Key Cryptographic Parameters".
[48] J.-M. Deshouillers, H. te Riele, and Y. Saouter, 1998, "New experimental results concerning the Goldbach conjecture".
[22] J. Brillhart, D. Lehmer, and J. Selfridge, 1975, "New primality crietria and factorizations for $2^m \pm 1$".
[91] G. Jaeschke, 1993, "On strong pseudoprimes to several bases".
[37] N. Costa Pereira, 1989, "Elementary estimates for the Chebyshev function $\psi(X)$ and for the Möbius function $M(X)$".
[54] A. W. Dudek, 2016, "An explicit result for primes between cubes".
[39] M. Cully-Hugill, 2021, "Primes between consecutive powers".
[55] A. W. Dudek, 2015, "On the Riemann hypothesis and the difference between primes".
[26] E. Carneiro, M. Milinovich, and K. Soundararajan, 2019, *Fourier optimization and prime gaps*.

15.3 Interval with primes, with congruence condition

Collecting references: [K. McCurley, 130], [K. McCurley, 129], [H. Kadiri, 97].

Last updated on September 1rst, 2021, by Charles Greathouse.

[130] K. McCurley, 1984, "Explicit estimates for the error term in the prime number theorem for arithmetic progressions".
[129] K. McCurley, 1984, "Explicit estimates for $\theta(x;3,\ell)$ and $\psi(x;3,\ell)$".
[97] H. Kadiri, 2008, "Short effective intervals containing primes in arithmetic progressions and the seven cube problem".

Chapter 16

Sieve bounds

Corresponding html file: `../Articles/Art14.html`

16.1 Some upper bounds

Theorem 2 of [H. Montgomery and R. Vaughan, 135] contains the following explicit version of the Brun-Tichmarsh Theorem.

Theorem (1973) Let x and y be positive real numbers, and let k and ℓ be relatively prime positive integers. Then $\pi(x+y;k,\ell)-\pi(x;k,\ell)<\frac{2y}{\phi(k)\log(y/k)}$ provided only that $y>k$.

Here as usual, we have used the notation

$$\pi(z;k,\ell)=\sum_{\substack{p\le z,\\ p\equiv\ell[k]}}1,$$

i.e. the number of primes up to z that are coprime to ℓ modulo k. See [J. Büthe, 23] for a generic weighted version of this inequality.

Here is a bound concerning a sieve of dimension 2 proved by [H. Siebert, 188].

Theorem (1976) Let a and b be coprime integers with $2|ab$. Then we have, for $x>1$,

$$\sum_{\substack{p\le x,\\ ap+b\text{ prime}}}1\le 16\omega\frac{x}{(\log x)^2}\prod_{\substack{p|ab,\\ p>2}}\frac{p-1}{p-2}\qquad \omega=\prod_{p>2}(1-(p-1)^{-2}).$$

[135] H. Montgomery and R. Vaughan, 1973, "The large sieve".
[23] J. Büthe, 2014, "A Brun-Titchmarsh inequality for weighted sums over prime numbers".
[188] H. Siebert, 1976, "Montgomery's weighted sieve for dimension two".

16.2 Combinatorial sieve estimates

The combinatorial sieve is known to be difficult from an explicit viewpoint. For the linear sieve, the reader may look at Chapter 9, Theorem 9.7 and 9.8 from [M. B. Nathanson, 141].

Last updated on July 18th, 2013, by Olivier Ramaré

[141] M. B. Nathanson, 1996, *Additive number theory – The classical bases.*

Part VIII

Analytic Number Theory in Number Fields

Chapter 17

Bounds on the Dedekind zeta-function

Corresponding html file: `../Articles/Art18.html`

17.1 Size

The knowledge on the general Dedekind zeta is less accomplished than the one of the Riemann zeta-function, but we still have interesting results. Theorem 4 of [H. Rademacher, 157] gives the convexity bound. See also section 4.1 of [T. S. Trudgian, 202].

Theorem (1959) In the strip $-\eta \le \sigma \le 1+\eta$, $0 < \eta \le 1/2$, the Dedekind zeta function $\zeta_K(s)$ belonging to the algebraic number field K of degree n and discriminant d satisfies the inequality

$$|\zeta_K(s)| \le 3 \left|\frac{1+s}{1-s}\right| \left(\frac{|d||1+s|}{2\pi}\right)^{\frac{1+\eta-\sigma}{2}} \zeta(1+\eta)^n.$$

17.2 Zeroes and zero-free regions

We denote by $N_K(T)$ the number of zeros ρ, of the Dedekind zeta-function of the number field K of degree n and discriminant d_K, zeros that lie in the critical strip $0 < \Re\rho = \sigma < 1$ and which verify $|\Im\rho| \le T$. After a first result in [H. Kadiri and N. Ng, 100], we find in [T. Trudgian, 198] The following result.

[157] H. Rademacher, 1959, "On the Phragmén-Lindelöf theorem and some applications".
[202] T. S. Trudgian, 2014, "An improved upper bound for the argument of the Riemann zeta-function on the critical line II"..
[100] H. Kadiri and N. Ng, 2012, "Explicit zero density theorems for Dedekind zeta functions".
[198] T. Trudgian, 2015, "An improved upper bound for the error in the zero-counting formulae for Dirichlet L-function and Dedekind zeta-function on the critical line".

Theorem (1959) When $T \geq 1$, we have $N_K(T) = \frac{T}{\pi}\log\Bigl(|d_K|\bigl(\frac{T}{2\pi e}\bigr)^n\Bigr) + O^*\bigl(0.316(\log|d_K| + n\log T) + 5.872n + 3.655\bigr)$.

In [H. Kadiri, 96], a zero-free region is proved.

Theorem (1959) Let K be a number field of degree n over $\mathbb{Q}$ and of discriminant $d \geq 2$. The associated Dedekind zeta-function ζ_K has no zeros in the region

$$\sigma \geq 1 - \frac{1}{12.55\log|d_K| + n(9.69\log|t| + 3.03) + 58.63}, |t| \geq 1$$

and at most one zero in the region

$$\sigma \geq 1 - \frac{1}{12.74\log|d_K|}, |t| \leq 1.$$

The exceptional zero, if it exists, is simple and real.

See [J.-H. Ahn and S.-H. Kwon, 1] for a result for Hecke L-series.

Last updated on February 14th, 2017, by Olivier Ramaré

[96] H. Kadiri, 2012, "Explicit zero-free regions for Dedekind zeta functions".
[1] J.-H. Ahn and S.-H. Kwon, 2014, "Some explicit zero-free regions for Hecke L-functions".

Part IX

Applications

Chapter 18

Explicit bounds for class numbers

Corresponding html file: `../Articles/Art13.html`

Let K be a number field of degree $n \geq 2$, signature (r_1, r_2), absolute value of discriminant d_K, class number h_K, regulator $\mathcal{R}_K$ and w_K the number of roots of unity in K. We further denote by κ_K the residue at $s = 1$ of the Dedekind zeta-function $\zeta_K(s)$ attached to K.

Estimating h_K is a long-standing problem in algebraic number theory.

18.1 Majorising $h_K \mathcal{R}_K$

One of the classic way is the use of the so-called analytic class number formula stating that

$$h_K \mathcal{R}_K = \frac{w_K \sqrt{d_K}}{2^{r_1}(2\pi)^{r_2}} \kappa_K$$

and to use Hecke's integral representation of the Dedekind zeta function to bound κ_K. This is done in [S. Louboutin, 116] and in [S. Louboutin, 118] with additional properties of log-convexity of some functions related to ζ_K and enabled Louboutin to reach the following bound:

$$h_K \mathcal{R}_K \leq \frac{w_K}{2} \left(\frac{2}{\pi}\right)^{r_2} \left(\frac{e \log d_K}{4n - 4}\right)^{n-1} \sqrt{d_K}.$$

Furthermore, if $\zeta_K(\beta) = 0$ for some $\frac{1}{2} \leq \beta < 1$, then we have

$$h_K \mathcal{R}_K \leq (1 - \beta) w_K \left(\frac{2}{\pi}\right)^{r_2} \left(\frac{e \log d_K}{4n}\right)^{n} \sqrt{d_K}.$$

[116] S. Louboutin, 2000, "Explicit bounds for residues of Dedekind zeta functions, values of L-functions at $s = 1$, and relative class numbers".

[118] S. Louboutin, 2001, "Explicit upper bounds for residues of Dedekind zeta functions and values of L-functions at $s = 1$, and explicit lower bounds for relative class numbers of CM-fields".

When K is abelian, then the residue κ_K may be expressed as a product of values at $s = 1$ of L-functions associated to primitive Dirichlet characters attached to K. On using estimates for such L-functions from [O. Ramaré, 159], we get for instance

$$h_K \mathcal{R}_K \le \frac{w_K}{2}\left(\frac{2}{\pi}\right)^{r_2}\left(\frac{\log d_K}{4n-4} + \frac{5-\log 36}{4}\right)^{n-1}\sqrt{d_K}.$$

Note that the constant $\frac{1}{4}(5-\log 36) = 0.354\cdots$ can be improved upon in many cases. For instance, when K is abelian and totally real (i.e. $r_2 = 0$), a result from [O. Ramaré, 159] implies that the constant may be replaced by 0, so that

$$h_K \mathcal{R}_K \le \left(\frac{\log d_K}{4n-4}\right)^{n-1}\sqrt{d_K}.$$

18.2 Majorising h_K

One may also estimate h_K alone, without any contamination by the regulator since this contamination is often difficult to control, see [M. Pohst and H. Zassenhaus, 153].

In this case, one rather uses explicit bounds for the Piltz-Dirichlet divisor functions τ_n (see [O. Bordellès, 18] and [O. Bordellès, 16]) and get

$$h_K \le \frac{M_K}{(n-1)!}\left(\frac{\log\big(M_K^2 d_K\big)}{2} + n - 2\right)^{n-1}\sqrt{d_K}$$

as soon as

$$n \ge 3, \quad d_K \ge 139 M_K^{-2} \quad \text{where} \quad M_K = (4/\pi)^{r_2} n!/n^n.$$

The constant M_K is known as the Minkowski constant of K.

18.3 Using the influence of small primes

It is explained in [S. Louboutin, 122] how the behavior of certain small primes may subtantially improve on the previous bounds. To make things more significant, define, for a rational prime p,

$$\Pi_K(p) = \prod_{\mathfrak{p}|p}\left(1 - \frac{1}{\mathcal{N}_K(\mathfrak{p})}\right)^{-1}.$$

[159] O. Ramaré, 2001, "Approximate Formulae for $L(1,\chi)$".
[153] M. Pohst and H. Zassenhaus, 1989, *Algorithmic algebraic number theory*.
[18] O. Bordellès, 2002, "Explicit upper bounds for the average order of $d_n(m)$ and application to class number".
[16] O. Bordellès, 2006, "An inequality for the class number".
[122] S. Louboutin, 2005, "On the use of explicit bounds on residues of Dedekind zeta functions taking into account the behavior of small primes".

From [S. Louboutin, 122], we have among other things

$$h_K \mathcal{R}_K \leq \frac{w_K}{2} \left(\frac{2}{\pi}\right)^{r_2} \frac{\Pi_K(2)}{\Pi_{\mathbb{Q}}(2)^n} \left(\frac{e \log d_K}{4n-4} \times e^{n \log 4/\log d_K}\right)^{n-1} \sqrt{d_K}$$

where K is any number field of degree $n \geq 3$. In particular, when 2 is inert in K, then

$$h_K \mathcal{R}_K \leq \frac{w_K}{2(2^n-1)} \left(\frac{2}{\pi}\right)^{r_2} \left(\frac{e \log d_K}{4n-4} \times e^{n \log 4/\log d_K}\right)^{n-1} \sqrt{d_K}.$$

18.4 The h_K^- of CM-fields

Let K be here a CM-field of degree $2n > 2$, i.e. a totally complex quadratic extension K of its maximal totally real subfield K^+. it is well known that h_{K^+} divides h_K. The quotient is denoted by h_K^- and is called the relative class number of K. The analytic class number formula yields

$$h_K^- = \frac{Q_K w_K}{(2\pi)^n} \left(\frac{d_K}{d_{K^+}}\right)^{1/2} \frac{\kappa_K}{\kappa_{K^+}} = \frac{Q_K w_K}{(2\pi)^n} \left(\frac{d_K}{d_{K^+}}\right)^{1/2} L(1, \chi)$$

where χ is the quadratic character of degree 1 attached to the extension K/K^+ and $Q_K \in \{1, 2\}$ is the Hasse unit index of K. Here are three results originating in this formula.

From [S. Louboutin, 116]:

Theorem (2000) We have

$$h_K^- \leq 2 Q_K w_K \left(\frac{d_K}{d_{K^+}}\right)^{1/2} \left(\frac{e \log(d_K/d_{K^+})}{4\pi n}\right)^n.$$

From [S. Louboutin, 117]:

Theorem (2003) Assume that $(\zeta_K/\zeta_{K^+})(\sigma) \geq 0$ whenever $0 < \sigma < 1$. Then we have

$$h_K^- \geq \frac{Q_K w_K}{\pi e \log d_K} \left(\frac{d_K}{d_{K^+}}\right)^{1/2} \left(\frac{n-1}{\pi e \log d_K}\right)^{n-1}.$$

Again from [S. Louboutin, 117]:

Theorem (2003) Let $c = 6 - 4\sqrt{2} = 0.3431\cdots$. Assume that $d_K \geq 2800^n$ and that either K does not contain any imaginary quadratic subfield, or that the real zeros in the range $1 - \frac{c}{\log d_N} \leq \sigma < 1$ of the Dedekind zeta-functions of

[116] S. Louboutin, 2000, "Explicit bounds for residues of Dedekind zeta functions, values of L-functions at $s = 1$, and relative class numbers".
[117] S. Louboutin, 2003, "Explicit lower bounds for residues at $s = 1$ of Dedekind zeta functions and relative class numbers of CM-fields".

the imaginary quadratic subfields of K are nor zeros of $\zeta_K(s)$, where N is the normal closure of K. Then we have

$$h_K^- \geq \frac{cQ_K w_K}{4ne^{c/2}[N:\mathbb{Q}]} \left(\frac{d_K}{d_{K^+}}\right)^{1/2} \left(\frac{n}{\pi e \log d_K}\right)^n .$$

And a third result from [S. Louboutin, 117]:

Theorem (2003) Assume $n > 2$, $d_K > 2800^n$ and that K contains an imaginary quadratic subfield F such that $\zeta_F(\beta) = \zeta_K(\beta) = 0$ for some β satisfying $1 - \frac{2}{\log d_K} \leq \beta < 1$. Then we have

$$h_K^- \geq \frac{6}{(\pi e)^2} \left(\frac{d_K}{d_{K^+}}\right)^{1/2-1/n} \left(\frac{n}{\pi e \log d_K}\right)^{n-1} .$$

Last updated on August 23rd, 2012, by Olivier Bordellès

[117] S. Louboutin, 2003, "Explicit lower bounds for residues at $s = 1$ of Dedekind zeta functions and relative class numbers of CM-fields".

Chapter 19

Primitive Roots

Corresponding html file: `../Articles/Art19.html`

Last updated on July 14th, 2012, by Olivier Ramaré

Part X

Development

Chapter 20

README

20.1 How to write

This part is technical and destined at the development team only.

1. Sections are coded via `<div class="section">1.  Title</div>`. The numbering is done by hand.

2. Theorems, Lemmas, Propositions are coded via

```
<span class="THM">Theorem (1986)</span>
<blockquote class="outer-thm">
<div class="thm">
    Every zero $\rho$ of $\zeta$ that have a
    real part between 0 and 1 and an imaginary part not more,
    in absolute value, than $T_0=545439823$ are in fact on
    the critical line, i.e. satisfy $\Re \rho=1/2$.
</div></blockquote>
```

   Note that the part `</div></blockquote>` should be on a single line.

3. Mathematics are entered latex style and processed via MathJax. Macros are to be avoided, of course.

4. A reference is introduced on one line in the form `<script language="javascript">bibref("Ramare*12")</script>` where Ramare*12 is the key of the bibtex entry, which has to be introduced in the Local-TME-EMT.bib file.

The file `Latex/booklet.tex` is the master file of the PDF booklet and has to be edited by hand. The Perl script `Biblio/UpDateBiblio.pl` will create tex-files that are to be used as chapter for each html-file found in `Articles/`, save the file `Template_Article.html` of course.

20.2 How to contribute

Everyone is most welcome to help us keep track of the results. You can do so by simply sending to the development team a mail with the proper information (in bibtex format for the relevant part). You may also propose a new annoted bibliography, for instance for "Explicit results in the combinatorial sieve", or any other missing entry. There are other ways to contribute, like modifying the CSS so that this site would be readable under windows, a fact we do not guarantee. Or by proposing a rewrite of some already present bibliography. Or anything else we did not think about.

20.3 Adding a part or a chapter

Adding a chapter requires several steps.

- Modify the file `accueil.html`.
- Modify in the file `../MiseEnPage.js` the variable `Architecture_TME_EMT`. This modification changes the numbers and each file `Articles/Art**.html` has to be subsequently modified at the level of the command `BandeauGeneral(2, "../../", [0, 2])`.
- Modify accordingly the file `Latex/booklet.tex`.

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