AN L^2-BOUND FOR THE BARBAN–VEHOV WEIGHTS

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Abstract. Let λ be the Barban–Vehov weights, defined in (1). Let $X \geq z_1 \geq 100$ and $z_2 = z_1^2$. We prove that

$$\sum_{n\leq X}\frac{1}{n}\Big(\sum_{d|n}\lambda_d\Big)^2 \leq 30\frac{\log X}{\log(z_2/z_1)},$$

saving more than a factor of 5 on what was the best known constant in such inequality. Two related estimates are also provided for $X \geq z_1 \geq 100$ and $z_2 = z_1^\tau$ for some $\tau > 1$.

1. Introduction and results

In [1], M. Barban and P. Vehov introduced the weights

$$\lambda_d = \lambda_d(z_2, z_1) = \mu(d)\frac{\log^+(z_2/d) - \log^+(z_1/d)}{\log(z_2/z_1)}, \tag{1}$$

defined for fixed $z_2 > z_1 > 0$, where $\log^+ = \max\{\log, 0\}$. They discovered that the L^2-mean of $\sum_{d|n}\lambda_d$ is remarkably small, even when n is smaller than z_2^2.

The Barban–Vehov weights were further studied for instance by S. Graham [2], Y. Motohashi [6, § 1.3] and M. Haye Betah [4].

In particular, Y. Motohashi found that, in order to use such weights to obtain log-free density estimates, it is enough to upper bound their harmonic L^2-average. In fact, he found that it is sufficient to bound

$$\sum_{n\geq 1}\frac{1}{n^{1+\varepsilon}}\Big(\sum_{d|n}\lambda_d\Big)^2$$

for any $\varepsilon > 0$. It is this reason that motivates us to study the above expression via Theorem 1.1.

On the other hand, the use of the Barban–Vehov weights and some modified weights has been made by H. Helfgott [5] for the proof of the ternary Goldbach conjecture, whence their major importance.

Here is our first result.

Theorem 1.1. *Let $\varepsilon > 0$ and $z_1 \geq 100$. Consider $z_2 = z_1^\tau$ for some $\tau > 1$. Then,*

$$\sum_{n\geq 1}\frac{1}{n^{1+\varepsilon}}\Big(\sum_{d|n}\lambda_d\Big)^2 \leq \frac{e^{\gamma\varepsilon}}{\varepsilon\log(z_2/z_1)}\frac{A(\tau+1) + [B - \frac{C}{z_1^\varepsilon} + (B - \frac{C}{z_2^\varepsilon})\tau^2]\varepsilon\log z_1}{\tau - 1},$$

where $A = 1.084$, $B = 1.301$, $C = 0.318$.

The next is a result that enables one to compare Theorem 1.1 to previous work.

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Corollary 1.2. *When $X \geq z_1 \geq 100$ and $z_2 = z_1^\tau$, for some $\tau > 1$, we have*

$$\sum_{n \leq X} \frac{1}{n} \biggl(\sum_{d|n} \lambda_d \biggr)^2 \leq f(\tau) \frac{\log X}{\log(z_2/z_1)},$$

where

$$f(\tau) = \frac{3.09\ [1.084(\tau+1) + 1.301(1+\tau^2) - 0.116]}{\tau - 1}.$$

In particular, we may take $f(2) = 30$.

The above estimate not only generalises the result of [4], but also saves more than a factor of 5 on the special case considered therein, namely when $\tau = 2$. Note that a more precise explicit study of these weights when $X \geq z_2^2$ has been the subject of [10] by the second author.

On the other hand, while studying zeros of the Riemann zeta function, S. Graham also used the Barban–Vehov weights. Here is a corollary that may be compared to [3, Lemma 9].

Corollary 1.3. *Let $X \geq z_1 \geq 100$ and $z_2 = z_1^\tau$, for some $\tau > 1$. Then, for any $\alpha \in [1/2, 1]$, we have*

$$\sum_{n \leq X} \frac{1}{n^{2\alpha-1}} \biggl(\sum_{d|n} \lambda_d \biggr)^2 \leq 3.09 X^{2-2\alpha} \frac{\log X}{\log(z_2/z_1)} \frac{1.084(\tau+1) + 1.301(1+\tau^2) - 0.116}{\tau - 1}.$$

As the reader will see, the proofs of the above results are remarkably simple, a fact that is reflected thanks to the very moderate constants that appear throughout our bounds. It is significant that, giving our historical remarks, it has taken quite a while to reach the reasonable bounds we present in Theorem 1.1, Corollary 1.2 and Corollary 1.3.

Notation. We shall use throughout the notation $\sigma = 1 + \varepsilon$ — we will use both notations interchangeably. Further, and in accordance with earlier work on the subject, we use the family of functions

$$\check{m}_q(X, \sigma) = \sum_{\substack{n \leq X \\ (n,q)=1}} \frac{\mu(n)}{n^\sigma} \log\biggl(\frac{X}{n}\biggr). \tag{2}$$

We shall also require the *Euler φ_s-function* defined, for any complex number s, by $\varphi_s(q) = q^s \prod_{p|q} (1 - 1/p^s)$. Finally, the variable p always corresponds to a prime number.

2. Auxiliary results

The following result may be found in [7, Lemma 5.1].

Lemma 2.1. *Let $\varepsilon > 0$. Then $\zeta(1+\varepsilon) \leq e^{\gamma\varepsilon}/\varepsilon$.*

On the other hand, we have

Lemma 2.2. *Suppose that $Y \geq 100$. Then*

$$\sum_{p \leq Y} \frac{\log p}{p} \log\biggl(\frac{Y}{p}\biggr) \geq 0.318 \log^2 Y.$$

The above sum is in fact asymptotic to $\frac{1}{2}\log^2 Y$, converging very slowly.

Proof. We verify the claimed inequality for $Y \in [100, 10^8]$ by using the Pari-GP script `GP/MertensAux.gp`.

Suppose now that $Y \geq 10^8$. By summation by parts, $\sum_{p\leq Y} \frac{\log p}{p}\log(\frac{Y}{p})$ equals

$$\log Y \sum_{p\leq 100} \frac{\log p}{p} - \sum_{p\leq 100} \frac{\log^2(p)}{p} + \int_{100}^{Y} (\vartheta(t)-\vartheta(100))\frac{1+\log(Y/t)}{t^2}dt$$

$$\geq 3.369\log Y - 8.739 + \int_{100}^{Y} (\vartheta(t)-\vartheta(100))\frac{1+\log(Y/t)}{t^2}dt, \tag{3}$$

where $\vartheta(t) = \sum_{p\leq t}\log p$ is the Chebyshev function. Now, by [9, Thm. 6], we have that $\vartheta(t) > 0.998697\,t$ for any $t \geq 1155901$. On the other hand, one can verify that $\vartheta(t) > 0.835\,t$ provided that $100 \leq t \leq 1155901$. Hence,

$$\int_{100}^{Y} (\vartheta(t)-\vartheta(100))\frac{1+\log(Y/t)}{t^2}dt \geq \int_{100}^{Y} (0.835\,t-\vartheta(100))\frac{1+\log(Y/t)}{t^2}dt$$

$$\geq \log^2(Y)\left[\frac{1}{2} + \frac{0.845}{\log Y} - \frac{5.058}{\log^2 Y}\right]. \tag{4}$$

By combining (3) and (4), and then using that $Y \geq 10^8$, we arrive at

$$\sum_{p\leq Y}\frac{\log p}{p}\log\left(\frac{Y}{p}\right) \geq \log^2(Y)\left[\frac{1}{2} - \frac{13.8}{\log^2(10^8)}\right] \geq 0.376\log^2 Y.$$

As $0.376 > 0.318$, we derive the inequality of the statement.

Finally, thanks to the prime number theorem, $\vartheta(t)\sim t$, so we indeed have that $\sum_{p\leq Y}\frac{\log p}{p}\log(\frac{Y}{p}) \sim \frac{1}{2}\log^2 Y$. □

3. Some lemmas involving $\check{m}_q(X,\sigma)$

The following result is crucial for our improvement on [4]. It can be found in [8, Theorem 1.1].

Lemma 3.1. *Let $X > 0$. When $k \geq 1$ and $\sigma \geq 1$, we have*

$$0 \leq \sum_{\substack{n\leq X\\(n,q)=1}} \frac{\mu(n)}{n^\sigma}\log^k\left(\frac{X}{n}\right) \leq 1.00303\frac{q}{\varphi(q)}\big(k+(\sigma-1)\log X\big)\log^{k-1} X.$$

When $k \geq 2$, we may replace 1.00303 *by 1.*

We also have the following result.

Lemma 3.2. *Let $Y \geq 100$. Then*

$$U(Y) = \sum_{\delta\leq Y}\frac{\mu^2(\delta)\varphi_\sigma(\delta)}{\delta^{2\sigma}}\check{m}_\delta\Big(\frac{Y}{\delta};\sigma\Big)^2 \leq \big(A+(\sigma-1)\left(B-\tfrac{C}{Y^\varepsilon}\log Y\right)\big)\log Y.$$

where $A = 1.084$, $B = 1.301$, $C = 0.318$.

Proof. Each $\check{m}_\delta(y/\delta;\sigma)$ is non-negative, thanks to Lemma 3.1 with $k = 1$. Thus, we have the following estimation

$$U(Y) \leq 1.00303\sum_{\delta\leq Y}\frac{\mu^2(\delta)\varphi_\sigma(\delta)}{\delta^{2\sigma}}\frac{\delta}{\varphi(\delta)}\check{m}_\delta\Big(\frac{Y}{\delta};\sigma\Big)\left(1+\varepsilon\log\left(\frac{Y}{\delta}\right)\right). \tag{5}$$

Let us define the auxiliary non-negative multiplicative function f on primes p by $f(p) = \frac{1-p^{-\sigma}}{1-1/p} > 0$ and on prime powers by $f(p^k) = 0$ for $k\in\mathbb{Z}_{>1}$. We observe that

$$\frac{\mu^2(\delta)\varphi_\sigma(\delta)}{\delta^\sigma}\frac{\delta}{\varphi(\delta)} = \mu^2(\delta)\sum_{a|\delta} f(a). \tag{6}$$

On using identity (6) on estimation (5), and recalling definition (2) , we derive

$$\begin{aligned}
U(Y) \le 1.00303 \sum_{a\le Y} \mu^2(a)\frac{f(a)}{a^\sigma} &\sum_{\substack{b\le Y/a\\(b,a)=1}} \frac{\mu^2(b)}{b^\sigma}\check{m}_{ab}\Big(\frac{Y}{ab};\sigma\Big)\Big(1+\varepsilon\log\Big(\frac{Y}{ab}\Big)\Big)\\
\le 1.00303 \sum_{a\le Y} \mu^2(a)\frac{f(a)}{a^\sigma}&\Big(1+\varepsilon\log\Big(\frac{Y}{a}\Big)\Big)\sum_{\substack{d\le Y/a\\(d,a)=1}}\frac{\mu^2(d)}{d^\sigma}\log\Big(\frac{Y/a}{d}\Big)\sum_{bc=d}\mu(c)\\
&- 1.00303\varepsilon \sum_{a\le Y}\mu^2(a)\frac{f(a)}{a^\sigma}\sum_{\substack{d\le Y/a\\(d,a)=1}}\frac{\mu^2(d)}{d^\sigma}\log\Big(\frac{Y/a}{d}\Big)\sum_{bc=d}\mu(c)\log b.
\end{aligned}$$

By Möbius inversion, the innermost sum of the above first term reduces to $d=1$. Moreover, as $\sum_{bc=d}\mu(c)\log(b)=\Lambda(d)$, where Λ is the von Mangoldt function, the innermost sum of the above second term is supported on prime numbers. Hence,

$$\begin{aligned}
\frac{U(Y)}{1.00303} \le \sum_{a\le Y}\mu^2(a)\frac{f(a)}{a^\sigma}&\left(1+\varepsilon\log\left(\frac{Y}{a}\right)\right)\log\left(\frac{Y}{a}\right)\\
&-\varepsilon\sum_{a\le Y}\mu^2(a)\frac{f(a)}{a^\sigma}\sum_{\substack{p\le Y/a\\(p,a)=1}}\frac{\log p}{p^\sigma}\log\left(\frac{Y/a}{p}\right).
\end{aligned}$$

In the above inequality, we consider the two terms coming from $a=1$. Then, by using the inequality $p^{-\sigma}\ge y^{-\varepsilon}p^{-1}$, valid whenever $p\le y$, we derive

$$\begin{aligned}
\frac{U(Y)}{1.00303} \le &(1+\varepsilon\log Y)\log Y + \sum_{2\le a\le Y}\mu^2(a)\frac{f(a)}{a^\sigma}\left(1+\varepsilon\log\left(\frac{Y}{a}\right)\right)\log\left(\frac{Y}{a}\right)\\
&-\varepsilon\sum_{p\le Y}\frac{\log p}{p^\sigma}\log\left(\frac{Y}{p}\right)-\varepsilon\sum_{2\le a\le Y}\mu^2(a)\frac{f(a)}{a^\sigma}\sum_{\substack{p\le Y/a\\(p,a)=1}}\frac{\log p}{p^\sigma}\log\left(\frac{Y/a}{p}\right).
\end{aligned}$$

Next, by trivially bounding the above last term and using Lemma 2.2, we have

$$\begin{aligned}
\frac{U(Y)}{1.00303} &\le (1+\varepsilon\log Y)\log Y - \frac{0.318\varepsilon}{Y^\varepsilon}\log^2 Y\\
&\quad+\left[-1+\prod_{p\ge 2}\left(1+\frac{p^{-\varepsilon}(1-p^{-\varepsilon})}{p(p-1)}\right)\right]\left(1+\varepsilon\log\left(\frac{Y}{2}\right)\right)\log\left(\frac{Y}{2}\right).
\end{aligned}$$

Furthermore, by using that $t\in\mathbb{R}_{>0}\mapsto t^{-1}(1-t^{-1})$ has a maximum $1/4$, we deduce that

$$\begin{aligned}
-1+\prod_{p\ge 2}\left(1+\frac{p^{-\varepsilon}(1-p^{-\varepsilon})}{p(p-1)}\right) &\le -1+\left(1+\frac{2^{-\varepsilon}(1-2^{-\varepsilon})}{2}\right)\prod_{p\ge 3}\left(1+\frac{1}{4p(p-1)}\right)\\
&\le -1+\left(1+\frac{2^{-\varepsilon}(1-2^{-\varepsilon})}{2}\right)1.08 \le 0.08+0.54\cdot 2^{-\varepsilon}(1-2^{-\varepsilon}),
\end{aligned}$$

so that

$$\begin{aligned}
&\big(0.08+0.54\cdot 2^{-\varepsilon}(1-2^{-\varepsilon})\big)(1+\varepsilon\log Y)\\
&\quad\le 0.08+\varepsilon\left(0.08+0.54\cdot 2^{-\varepsilon}(1-2^{-\varepsilon})+0.54\frac{2^{-\varepsilon}(1-2^{-\varepsilon})}{\varepsilon\log Y}\right)\log Y\\
&\quad\le 0.08+0.297\,\varepsilon\log Y,
\end{aligned}$$

where we have used that $t \in \mathbb{R}_{\geq 0} \mapsto (1-2^{-t})/t$ is decreasing and has as maximum $\log 2$ at $t=0$. We conclude the result by observing that

$$\begin{aligned}\frac{U(Y)}{1.00303} &\leq (1+\varepsilon\log Y)\log Y - \frac{0.318\varepsilon}{Y^\varepsilon}\log^2 Y + \left(0.08 + 0.297\varepsilon\log Y\right)\log\left(\frac{Y}{2}\right)\\ &\leq 1.08\log Y + \varepsilon\log^2 Y\left(1.297 - \tfrac{0.318}{Y^\varepsilon}\right).\end{aligned}$$

□

4. Main result

Proof of Theorem 1.1. Recall definition (1). Let us define

$$\Sigma(\sigma) = \Sigma(\sigma, z_2, z_1) = \sum_n \frac{1}{n^\sigma}\Big(\sum_{d|n}\lambda_d\Big)^2. \tag{7}$$

By expanding the square, we observe that

$$\Sigma(\sigma) = \sum_{d,e}\lambda_d\lambda_e\sum_{[d,e]|n}\frac{1}{n^\sigma} = \zeta(\sigma)\sum_{d,e}\frac{\lambda_d\lambda_e}{[d,e]^\sigma}, \tag{8}$$

where the variables d and e may be restricted to $d, e \leq z_2$.

Let us use now the Selberg diagonalisation process. For any pair of square-free numbers d, e, we have

$$\frac{d^\sigma e^\sigma}{[d,e]^\sigma} = (d,e)^\sigma = \sum_{\delta|(d,e)}\varphi_\sigma(\delta). \tag{9}$$

As the support of λ is contained in the set of square-free numbers, we may derive from (9) the following expression

$$\sum_{d,e}\frac{\lambda_d\lambda_e}{[d,e]^\sigma} = \sum_\delta\frac{\varphi_\sigma(\delta)}{\delta^{2\sigma}}\Bigg(\sum_{\substack{\ell\\(\ell,\delta)=1}}\frac{\lambda_{\delta\ell}}{\ell^\sigma}\Bigg)^2. \tag{10}$$

Thus, by definition (1) and equation (8), we obtain

$$\Sigma(\sigma) \leq \frac{\zeta(\sigma)}{\log^2(z_2/z_1)}\sum_{\delta\leq z_2}\frac{\mu^2(\delta)\varphi_\sigma(\delta)}{\delta^{2\sigma}}\left[\check{m}_\delta\Big(\frac{z_2}{\delta};\sigma\Big) - \check{m}_\delta\Big(\frac{z_1}{\delta};\sigma\Big)\right]^2, \tag{11}$$

where we have used the definition (2). Furthermore, as $\check{m}_q(X;\sigma) \geq 0$, thanks to Lemma 3.1, we find that

$$\left[\check{m}_\delta\left(\frac{z_2}{\delta};\sigma\right) - \check{m}_\delta\left(\frac{z_1}{\delta};\sigma\right)\right]^2 \leq \check{m}_\delta^2\left(\frac{z_2}{\delta};\sigma\right) + \check{m}_\delta^2\left(\frac{z_1}{\delta};\sigma\right)$$

with equality if and only if $\check{m}_\delta(z_2/\delta;\sigma)\check{m}_\delta(z_1/\delta;\sigma) = 0$. Therefore, we see from (11) that

$$\frac{\log^2(z_2/z_1)}{\zeta(\sigma)}\Sigma(\sigma) \leq \sum_{\delta\leq z_2}\frac{\mu^2(\delta)\varphi_\sigma(\delta)}{\delta^{2\sigma}}\check{m}_\delta^2\Big(\frac{z_2}{\delta};\sigma\Big) + \sum_{\delta\leq z_1}\frac{\mu^2(\delta)\varphi_\sigma(\delta)}{\delta^{2\sigma}}\check{m}_\delta^2\Big(\frac{z_1}{\delta};\sigma\Big),$$

so that

$$\Sigma(\sigma) \leq \frac{\zeta(\sigma)}{\log^2(z_2/z_1)}(U(z_2)+U(z_1)) = \frac{\zeta(\sigma)}{\log(z_2/z_1)}\frac{U(z_2)+U(z_1)}{(\tau-1)\log z_1}.$$

Theorem 1.1 follows by applying Lemma 3.2 twice, recalling that $z_2 = z_1^\tau$ and using then Lemma 2.1.

□

With the help of (10), it is worth noticing that, for any $\varepsilon > 0$ and any weight λ, the sum $\sum_{d,e} \frac{\lambda_d \lambda_e}{[d,e]^\sigma}$ is always non-negative.

5. Corollaries

Proof of Corollary 1.2. By using Rankin's trick, we have

$$\Sigma(0) = \sum_{n \le X} \frac{1}{n} \Big(\sum_{d|n} \lambda_d \Big)^2 \le \sum_n \frac{X^\varepsilon}{n^{1+\varepsilon}} \Big(\sum_{d|n} \lambda_d \Big)^2$$

for any parameter ε that we shall soon choose. Now, by using Theorem 1.1, we obtain the bound

$$\Sigma(0) \le \frac{X^\varepsilon e^{\gamma\varepsilon}}{\varepsilon \log(z_2/z_1)} \frac{A(\tau+1) + [B - \frac{C}{z_1^\varepsilon} + (B - \frac{C}{z_2^\varepsilon})\tau^2]\varepsilon \log z_1}{\tau - 1}, \tag{12}$$

where $A = 1.084$, $B = 1.301$, $C = 0.318$.

Set $\varepsilon = 1/\log X$. Since there is no further assumptions on z_2, we use the bound $-Cz_2^{-\varepsilon} \le 0$. Thus, by writing $u = \varepsilon \log z_1$, we obtain

$$\begin{aligned} \Sigma(0) &\le \frac{e^{1+\gamma/\log X} \log X}{\log(z_2/z_1)} \frac{A(\tau+1) + [B(1+\tau^2) - \frac{C}{e^u}]u}{\tau - 1} \\ &\le \frac{e^{1+\gamma/\log 100} \log X}{\log(z_2/z_1)} \frac{A(\tau+1) + B(1+\tau^2) - \frac{C}{e}}{\tau - 1} \end{aligned}$$

where we have used that $X \ge 100$ and $0 < u \le 1$.

In the specific and common situation when $\tau = 2$, unlike the general case, we may take advantage of the factor $Cz_2^{-\varepsilon} = Ce^{-2u}$ appearing in (12). Hence, by using again that $X \ge 100$ and $0 < u \le 1$, we finally obtain

$$\Sigma(0, z_1^2, z_1) \le e^{1+\gamma/\log 100}[A(\tau+1) + B(1+\tau^2) - \tfrac{C}{e} - \tfrac{C\tau^2}{e}] \frac{\log X}{\log(z_2/z_1)} \le \frac{29.18 \log X}{\log(z_2/z_1)}.$$

□

Proof of Corollary 1.3. We simply write

$$\sum_{n \le X} \frac{1}{n^{2\alpha-1}} \Big(\sum_{d|n} \lambda_d \Big)^2 = \sum_{n \le X} \frac{(\sum_{d|n} \lambda_d)^2}{n} n^{2-2\alpha} \le X^{2-2\alpha} \sum_{n \le X} \frac{1}{n} \Big(\sum_{d|n} \lambda_d \Big)^2,$$

since $2 - 2\alpha \ge 0$. Then the result follows by using Corollary 1.2. □

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