

Parabyzantine Protocols

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This memo is for friends and colleagues to explain the beginnings of my study of a category of distributed computing protocols which I call Parabyzantine. Below, I will begin with a motivation for the study. I'll then provide a trivial combinatorial example of a protocol assumed to be in the category. Finally, I'll suggest more abstract constructions with which the study is properly concerned.

Most terms and notation follow [HKR]. The final section introduces provisional and more suggestive terms.

1 Motivation

My motivation for studying Parabyzantine protocols, as I have dubbed them, is derived primarily from two phenomena in the world of distributed computing: the need for fault-tolerant order and compute inefficiencies of Byzantine fault-tolerant replica-based protocols.

1.1 The General Need for Order

In computing, we typically reason about partial functions. These are functions which deterministically map one value to another, but may not be defined for all inputs.

$\hat{f} : S \rightarrow S \cup \{\perp\}$ is a lifted partial function.
 $f : S \rightharpoonup S$ is a partial function in its native category.

When composing these partial functions, we axiomatize that undefinedness propagates through the composition.

$f, g : S \rightharpoonup S$
 $f \circ g$ is defined if and only if
 $g(x)$ is defined and
 $f(g(x))$ is defined

We find this to be a suitable model for programming modern machines, because we know how to design hardware which can logically represent functions with these properties and which is rarely corrupted by the surrounding environment.

However, for general computing tasks, we often have partial functions which are non-commutative.

$$f \circ g \neq g \circ f$$

A familiar example is subtraction on \mathbb{N} , which is naturally partial:

$$\begin{aligned} \text{sub} : \mathbb{N} \times \mathbb{N} &\rightarrow \mathbb{N} \\ \text{sub}(a, b) &= \begin{cases} a - b & \text{if } a \geq b, \\ \text{undefined} & \text{otherwise.} \end{cases} \end{aligned}$$

And is also non-commutative:

$$\text{sub}(5, 3) = 2 \neq \text{sub}(3, 5) \text{ (undefined).}$$

In general, for a family f_1, \dots, f_n of partial functions, the composite is determined by the ordered sequence (f_1, \dots, f_n) not the unordered set $\{f_1, \dots, f_n\}$. In local hardware cases—such as a single hart—we can often form reasonable assumptions or conventions, such that something like a state machine reading through an assembly is equivalent to a sequence of partial functions. In the context of general distributed computing tasks, however, we tend to make these assumptions more minimal.

In distributed computing, we may instead refer to a set of processes P which are able to share information with one another via messages $m \in M$ to varying degrees of success. Then, within this context, we hope to achieve results up to some notion of correctness and consistency, i.e., up to some notion of equivalence with the local hardware case.

Put more provocatively, a distributed computing protocol often has to decide from a chaotic world of messages what a reasonable order of partial functions should be. Signatures can be placed on messages to ensure they are authentic. Prover systems can be built to attest to the correctness of the value of a partial function. But, these may mean little without a reasonable order.

1.2 2f + 1

Often the success of a distributed computing protocol is framed in terms of its ability to induce agreement or consensus. And, in the abundant cases where order matters, this necessarily means consensus on said order.

If we take set of processes $|P| = 3f + 1$ and ask for the unique messages $M_i \subseteq M$ which indicate computing the partial function at index i , we may wonder what different requirements on the agreement of these messages might mean.

A simple majority $|M_i| > \frac{|P|}{2}$ would ensure by the Generalized Pigeonhole Principle that any two subsets of messages M_i, M_j would intersect in at least one process. In a sense, at least one process can attest to each ordering step. By recurrence, this reads out a total order.

Let $Q : M \rightarrow 2^P$ be the function which maps a message to the set of processes which have sent it, representing a quorum. Then, we may define order $Order(M_i)$ as follows:

$$\begin{aligned} Q(M_i) &= \bigsqcup_{m \in M_i} Q(m) \\ O(M_i, M_j) &:= (Q(M_i) \cap Q(M_j) \neq \emptyset) \\ Order(M_i) &= Order(M_{i-1}) \wedge O(M_{i-1}, M_i) \\ &= Order(M_0) \wedge O(M_0, M_1) \wedge \dots \wedge O(M_{i-1}, M_i) \end{aligned}$$

Importantly, this simple majority model does not account for the possibility that a process may fault. As it turns out, the most aggressive assumption we can make and still retain the

possibility of these intersecting sets amongst non-faulty processes is that at most f processes may fault and quorums must be formed by at least $2f + 1$ processes. This is the Byzantine fault model.

Let H be the set of honest processes and F be the set of faulty processes. Any two quora, under the Byzantine fault model, must then intersect in at least $f + 1$ processes:

$$\begin{aligned}
 |Q(M_1) \cap Q(M_2)| &= \\
 &|Q(M_1)| + |Q(M_2)| - |Q(M_1) \cup Q(M_2)| \\
 &\geq (2f + 1) + (2f + 1) - (3f + 1) \\
 &= f + 1
 \end{aligned}$$

We can further deduce that any two quora must intersect in at least one honest process $h \in H$ in the presence of f faulty processes:

$$\begin{aligned}
 &|Q(M_1) \cap Q(M_2)| \\
 &= |Q(M_1) \cap Q(M_2) \cap H| + |Q(M_1) \cap Q(M_2) \cap F| \\
 &|Q(M_1) \cap Q(M_2) \cap H| \\
 &= |Q(M_1) \cap Q(M_2)| - |Q(M_1) \cap Q(M_2) \cap F| \\
 &f \geq |Q(M_1) \cap Q(M_2) \cap F| \\
 &|Q(M_1) \cap Q(M_2) \cap H| \geq f + 1 - f = 1
 \end{aligned}$$

A straightforward interpretation of this fact renders the ubiquitous family of Byzantine fault-tolerant state machine replication protocols (BFT SMR). Without a prover system—which is common in practice owing to the computational expense of systems like ZKPs—BFT SMRs require at least $2f + 1$ of the honest processes to be engaged in the computing and broadcasting a partial function f_i, f_{i+1}, \dots, f_n . With prover systems, this requirement holds up to the task of ordering.

Roughly speaking, if a Byzantine fault-tolerant SMR protocol tolerates f faulty processes, then it uses at best $\frac{1}{2f+1}$ of the available non-faulty compute.

2 A Parabyzantine Protocol

If we want to access more compute in a Byzantine fault-tolerant setting, an initial thought is to simply relax the requirement on order. Perhaps we can include fewer processes and allow our total order to fail with an acceptably low likelihood. Indeed, as I will show, even a fairly straightforward application of this concept yields a protocol which is computationally efficient and low loss.

However, before we endeavor this protocol, let's first consider what computationally efficient and a low loss would mean.

As referenced above, SMR requires at least $2f + 1$ processes to certify each partial function. Thus, each logical step costs $\Theta(2f + 1)$ replicated work. So, in abstract, if we use fewer than $2f + 1$ processes, we will have improved compute efficiency.

What is meant by low loss may be less obvious. In the Byzantine fault-tolerant setting, we have described a discrete mathematical limit that tells us we cannot have a property we

called $Order(M_i)$ without involving at least $2f + 1$ processes. So, the notion that we can have protocol which is computationally more efficient seems to contradict this limit.

However, we may consider the following objective function which is minimized for M_i when $Order(M_i) = 1$:

$$\begin{aligned} Obj(M_i) = 0 &\iff Order(M_i) = 1 \\ Obj(M_i) = 1 &\iff Order(M_i) = 0 \end{aligned}$$

In its discrete case, we cannot improve. But, if we constrain on the average case $E[Obj(M_i)] = Pr[Obj(M_i) = 1] \leq \epsilon$, we have much more flexibility.

In a sense, it is precisely this sort of assumption which gives rise to the name Parabyzantine. These are protocols which act like Byzantine fault-tolerant protocols up to some well-understood point. In the case of the RESAMPLE protocol, this point will be an average-case guarantee. While we are under the veil of average case, RESAMPLE will be Byzantine fault-tolerant. Once we remove the veil, RESAMPLE will not be.

2.1 The Resample Protocol

We define the following Byzantine fault-allowing protocol RESAMPLE:

- For a given index, i on M , pick a random subcommittee $K_i \subseteq P$ s.t. $|K_i| = 3k + 1$.
- Accept M_i if and only if $|Q_{K_i}(M_i)| \geq 2k + 1$. Otherwise, pick another random subcommittee and repeat.

Let...

$$f' = |K_i \cap F|, h' = |K_i \cap H|, f' + h' = |K_i| = 3k + 1$$

Observe the following possible outcomes for selection of a random subcommittee:

- $|K_i \cap H| \geq 2k + 1 \iff \text{Divine} = D$, representing a subcommittee which has an honest supermajority which computes M_i correctly.
- $|K_i \cap H| \leq k \iff \text{Corrupt} = C$, representing a subcommittee which has a dishonest supermajority and may compute M_i incorrectly.
- $k < |K_i \cap H| < 2k + 1 \iff \text{Undecided} = U$, representing all other cases wherein neither an honest nor dishonest supermajority exists and may either compute M_i correctly or disagree internally and not render a supermajority.

2.2 The Loss of the Resample Protocol

Let $\mathcal{S}(f, k)$ represent the total number of ways to select the subcommittee without replacement:

$$\mathcal{S}(f, k) = \binom{3f + 1}{3k + 1}$$

The total number of ways to select a Corrupt subcommittee is:

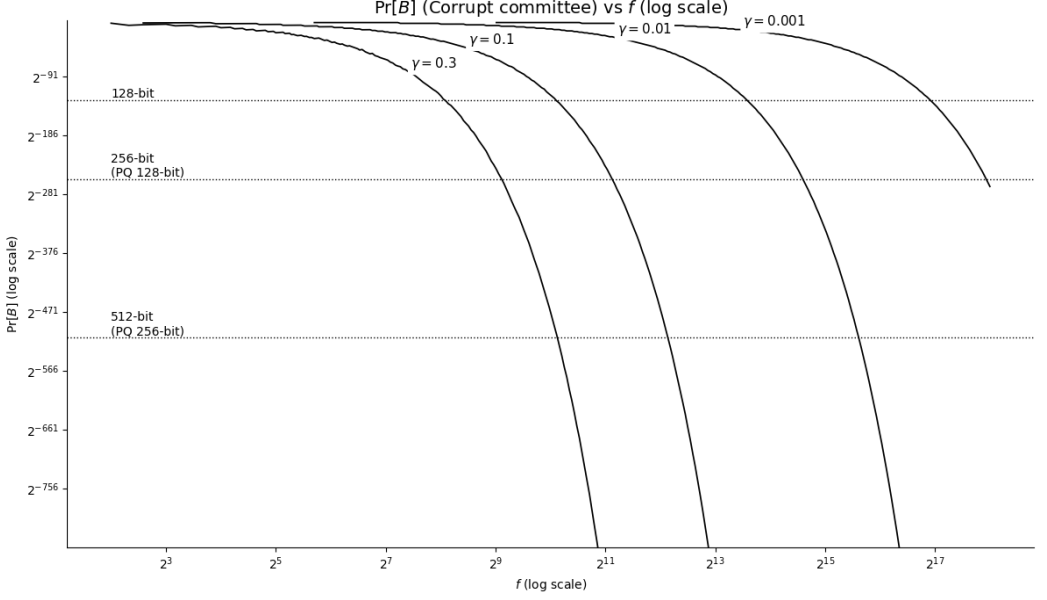


Fig. 1. $\Pr[B]$ representing the probability of selecting a corrupt subcommittee as a function of f for different sampling ratios $\gamma = \frac{k}{f}$.

$$\mathcal{S}_C(f, k) = \sum_{f'=2k+1}^{\min(3k+1, f)} \binom{f}{f'} \cdot \binom{2f+1}{3k+1-f'}$$

The total number of ways to select a Divine subcommittee is:

$$\mathcal{S}_D(f, k) = \sum_{h'=2k+1}^{\min(3k+1, 2f+1)} \binom{2f+1}{h'} \cdot \binom{f}{3k+1-h'}$$

All other outcomes are by definition Undecided, so the total number of ways to select an Undecided subcommittee is:

$$\mathcal{S}_U(f, k) = \mathcal{S}(f, k) - \mathcal{S}_C(f, k) - \mathcal{S}_D(f, k)$$

Let S represent the transient state in which the RESAMPLE is sampling, i.e., has just started or produced a Undecided subcommittee. Let G represent the absorbing state in which RESAMPLE makes a "good" decision to select a Divine subcommittee. Let B represent the absorbing state in which RESAMPLE makes a "bad" decision to select a Corrupt subcommittee. The Markov Chain is then given by the state space Ω and transition matrix T .

$$\Omega = \{S, G, B\}$$

$$T = \begin{pmatrix} \Pr[U] & \Pr[D] & \Pr[C] \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The probability of the algorithm making a "bad" decision to select a Corrupt subcommittee $\Pr[B]$ is then:

Let $x := \Pr[\text{eventually hitting } B \mid \text{start in } S]$

$$x = \Pr[C] \cdot 1 + \Pr[D] \cdot 0 + \Pr[U] \cdot x$$

$$x - \Pr[U] \cdot x = \Pr[C] \implies x \cdot (1 - \Pr[U]) = \Pr[C] \implies x = \frac{\Pr[C]}{1 - \Pr[U]}$$

Observe that...

$$1 - \Pr[U] = \Pr[D] + \Pr[C] \implies \Pr[B] = \frac{\Pr[C]}{\Pr[D] + \Pr[C]}$$

Conversely, the probability of the algorithm making a "good" decision to select a Divine subcommittee $\Pr[G]$ is then:

$$\Pr[G] = 1 - \Pr[B] = 1 - \frac{\Pr[C]}{\Pr[D] + \Pr[C]} = \frac{\Pr[D]}{\Pr[D] + \Pr[C]}$$

As I will show, the probability $\Pr[C]$ drops off quickly for a fixed sampling ratio $\gamma = \frac{k}{f}$ (with, say, $k = \lfloor \gamma f \rfloor$). First observe that the number of faulty replicas in a sampled subcommittee follows a hypergeometric distribution:

$$X := |K_i \cap F| \sim \text{Hypergeometric}(N, K, n),$$

$$C \equiv \{X \geq 2k + 1\},$$

$$\Pr[C](\gamma, f) = \frac{\mathcal{S}_C(f, \lfloor \gamma f \rfloor)}{\mathcal{S}(f, \lfloor \gamma f \rfloor)}.$$

Standard large-deviation bounds for sampling without replacement give a KL-based tail bound:

$$\Pr[X \geq qn] \leq \exp(-n \cdot D(q \parallel p)),$$

$$D(q \parallel p) = q \log\left(\frac{q}{p}\right) + (1 - q) \log\left(\frac{1 - q}{1 - p}\right).$$

In our case $p = \frac{f}{3f+1} \approx \frac{1}{3}$, $q = \frac{2k+1}{3k+1} \approx \frac{2}{3}$, and $n = 3k + 1$. Hence

$$D\left(\frac{2}{3} \parallel \frac{1}{3}\right) = \frac{2}{3} \log 2 + \frac{1}{3} \log \frac{1}{2} = \frac{1}{3} \log 2.$$

Therefore,

$$\begin{aligned}\Pr[C] &\lesssim \exp\left(-(3k+1) \cdot \frac{1}{3} \log 2\right) = 2^{-(3k+1)/3}, \\ \Pr[C](\gamma, f) &\lesssim 2^{-(3\gamma f+1)/3} = 2^{-\gamma f-1/3} = 2^{-\Omega(f)}.\end{aligned}$$

In particular, for any fixed $\gamma \in (0, 1)$,

$$\lim_{f \rightarrow \infty} \Pr[C](\gamma, f) = 0 \forall \gamma \in (0, 1)$$

2.3 The Cost of the Resample Protocol

The expected cost of RESAMPLE, measured in the total number of processes engaged across all sampling attempts, remains on the order $\Theta(k)$, but admits a constant-factor improvement over the $2f + 1$ quorum required by standard SMR.

Fix an index i and write $\Pr[U] := \Pr[\mathcal{U}]$ for the probability that a uniformly sampled subcommittee of size $3k + 1$ is *Undecided*. Each sampling attempt terminates (i.e., yields a decision) precisely when the outcome is not \mathcal{U} . Thus, if T denotes the number of sampling attempts until termination, then T is geometric with success probability $1 - \Pr[U]$:

$$\begin{aligned}\Pr[T = t] &= \Pr[U]^{t-1} \cdot (1 - \Pr[U]) \quad \text{for } t \in \mathbb{N}, \\ \mathbb{E}[T] &= \sum_{t=1}^{\infty} t \cdot \Pr[U]^{t-1} (1 - \Pr[U]) = \frac{1}{1 - \Pr[U]}.\end{aligned}$$

Each attempt engages at most $3k + 1$ processes. Therefore the expected number of process-participations consumed per decision is

$$\mathbb{E}[\text{Processes Used}] = (3k + 1) \cdot \mathbb{E}[T] = \frac{3k + 1}{1 - \Pr[U]}.$$

In regimes of interest (e.g., for a fixed sampling ratio $\gamma = \frac{k}{f}$ with large f), $\Pr[U]$ is typically bounded away from 1; in particular, if $\Pr[U] \approx \frac{1}{2}$, then

$$\begin{aligned}\mathbb{E}[T] &\approx \frac{1}{1 - \frac{1}{2}} = 2, \\ \mathbb{E}[\text{Processes Used}] &\approx 2(3k + 1) = \Theta(k).\end{aligned}$$

Thus, RESAMPLE is Parabyzantine in the sense described above: it replaces a worst-case quorum requirement with an average-case sampling cost, remaining computationally efficient while permitting a tunably low probability of failure.

3 An Expanded View of Parabyzantine Protocols

Now that we have found one Parabyzantine protocol, we might wonder if we can find others and under what conditions these others act like Byzantine fault-tolerant protocols. This is, however, a big universe, and it helps to explore it in a structured manner.

To this end, I have chosen to align my views along two vantages. The first is the rich perspective of topological distributed computing. The second is the category theoretic consideration of relationships between Parabyzantine protocols.

My exploration is further guided by the search for a non-trivial bridge between topological descriptions of protocols and non-topological descriptions of protocols—one that can be expressed in the morphisms of a category of Parabyzantine protocols.

3.1 Topological Protocols

The fields of topology and distributed computing have a well-established and fruitful intersection. We know, for example, that we can formulate Byzantine agreement as a problem of barycentric subdivision on a simplicial complex via protocols inheriting from Herlihy and Shavit, 1999 such as [HKR's] BARYAGREE. Further, we know that BARYAGREE achieves $Obj(M_i) = 0$ in the Byzantine fault-tolerant setting.

Much of this intersection is, in a word, unavoidable: it is hard to model agreement problems without something that looks like the sort of complex geometric object with which topology is concerned. And, indeed I have spent most of my own study digesting the various topological tools and constructions which are already invoked in the catalog of Byzantine fault-tolerant protocols.

Rather the spending the time reciting various twists and turns which I have not yet synthesized into a coherent contribution, I'd instead like to point at two reasons why the topological formulation of Byzantine agreement is acutely interesting to my study of Parabyzantine protocols.

The first is that Byzantine agreement is often concerned with some kind of connected structure. Consider our recursive definition of $Order(M_i)$, which required consecutive quorums to intersect. Intuitively, these intersections form a connected structure in the process set P such that we can move from one quorum to the next.

To make this intuition precise, consider the following nerve construction. Let P be the set of processes, and for each index i let $Q_i \subseteq P$ denote the quorum of processes whose messages certify the i th step.¹

Consider the family of subsets $\mathcal{U} = \{Q_i\}_{i \in I}$. Viewing \mathcal{U} as a cover of

$$X := \bigcup_{i \in I} Q_i \subseteq P,$$

we may form the *nerve* simplicial complex $N(\mathcal{U})$ defined by:

$$\text{Vert}(N(\mathcal{U})) = \mathcal{U},$$

$$\{Q_{i_0}, \dots, Q_{i_d}\} \text{ spans a } d\text{-simplex} \iff \bigcap_{j=0}^d Q_{i_j} \neq \emptyset.$$

In this language, our intersection predicate $O(M_{i-1}, M_i)$ asserts precisely that consecutive vertices in the nerve share an edge:

$$O(M_{i-1}, M_i) \iff Q_{i-1} \cap Q_i \neq \emptyset \iff \{Q_{i-1}, Q_i\} \in N(\mathcal{U})^{(1)}.$$

Thus, a run satisfying $Order(M_i)$ determines a path in the 1-skeleton of the nerve. Or, in other words, it is the connectedness of a space with a topology describing the intersection of quorums with which the fundamental Byzantine assumption is concerned.

The second reason for my interest in the topological formulation of Byzantine agreement is that topology is a rich discipline for studying non-standard equivalence relations.

¹More generally, one may take \mathcal{Q} to be a family of quorums and regard $\{Q\}_{Q \in \mathcal{Q}}$ as a cover of the participating process set.

Recall that non-commutativity was a key consideration in arriving at our Byzantine fault-tolerant state machine replica protocol.

$$f \circ g \neq g \circ f$$

When studying topologically, however, we may be able to show that fg and gf are equivalent up to homotopy:

$$f \circ g \simeq g \circ f$$

In topology generally, we can access many of these non-standard equivalence relations. Homotopy, indistinguishability under a quotient space, and various more specific deformation relations, give us a lot of ways to relate two protocols which are not equivalent in the standard sense. As we used the average case to find the Parabyzantine RESAMPLE protocol, we may be able to infer other protocols which act like Byzantine fault-tolerant protocols up to some well-understood topological relationship and hence are Parabyzantine.

Thus, in addition to its extensive corpus, the topological formulation of Byzantine agreement is both a fitting tool for describing the sorts of properties we would like to preserve and source of instruments for relaxing requirements into Parabyzantine protocols.

3.2 A Category of Parabyzantine Protocols

Acknowledgments