CSE 321 HW#1 Solutions

1- a) Let $f(n) = 5n^2 - 7n + 9$ and
$g(n) = n^2 - 4n + 1$

To show $f(n) = O(g(n))$ we should find constants $c, n_0 > 0$ such that

$$f(n) \leq c\, g(n) \text{ for all } n \geq n_0.$$

for $c = 6$ we have

$$5n^2 - 7n + 9 \leq 6(n^2 - 4n + 1)$$
$$0 \leq n^2 - 17n - 3$$

roots of $n^2 - 17n - 3$ is -0.15 and 17.15

that is

so $c = 6$ and $n_0 = 18$ satisfy the definition.

1-b) Let $2n^3-6n^2+n-3 = f(n)$

To show $f(n) = \Theta(n^3)$

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we should find constants $c_1, c_2, n_0 > 0$

such that $0 \leq c_1 n^3 \leq f(n) \leq c_2 n^3$

for $c_1 = 1$ and $c_2 = 2$ we have

$$n^3 \leq 2n^3 - 6n^2 + n - 3 \leq 2n^3$$

$0 \leq 6n^2 - n + 3$

$\Delta = 1 - 4.6.3 < 0$

so there is no root and because $a = 6$ in $ax^2 + bx + c$ is positive the above inequality is positive for all n.

$0 \leq n^3 - 6n^2 + n - 3$

In this part we will use Induction to prove correctness.

for $n = 6$ the above inequality is satisfied.

So we will now show that ~~for any k≥~~ for $k+1$ whenever the above inequality is true for $k > 6$. the equality is true

assume $0 \leq k^3 - 6k^2 + k - 3$ is true for $k > 6$.

we will investigate if $0 \leq (k+1)^3 - 6(k+1)^2 + (k+1) - 3$

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1-6) continue . . .

when we subtract the first inequality from the first one we get

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$3k^2 - 3k + 1$ ~~which roots are~~

$\Delta = 9 - 4.3.1 = -3$ which is always positive.

so we get if

$0 \leq k^3 - 6k^2 + k - 3$ is true for $k > 6$

it is true for $(k+1)$ also.

so $\cancel{f(n) = \Theta(n^3)}$ $c_1 = 1$ $c_2 = 2$

$n_0 = 6$ satisfies the definition.

1-c) Let's first prove that $(\log n)^3$ is $O(n)$. We will prove this by using limits. We will use L'Hôpital 3 times.

$$\lim_{n\to\infty} \frac{n}{(\log n)^3} = \lim_{n\to\infty} \frac{n}{3(\log n)^2} = \lim_{n\to\infty} \frac{n}{6\log n} = \lim_{n\to\infty} = \frac{n}{6} = \infty$$

So $(\log n)^3$ is $O(n)$.

$7n^2 + 5n(\log n)^3 \leq 12n^2$ because

$(\log n)^3 < n$ for all ~~[illegible]~~ $n > 4$

So for $c = 12$ and $n_0 = 2$

$7n^2 + 5n(\log n)^3$ is $O(n^2)$.

2- a) $\sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} 1 = \sum_{i=0}^{n-2} n-i+1 = 1+2+3+\cdots+n-1$

$= \frac{n^2-n}{2}$

$$\boxed{f(n) = \frac{n^2-n}{2}}$$

2- b) $\sum_{i=0}^{i=N} 1 + \sum_{i=0}^{i=N/2} 1 + \sum_{i=0}^{i=N/4} 1 + \cdots \sum_{i=0}^{i=1} 1$

$= N + N/2 + N/4 + \cdots + 1 = 2N-1$

$$\boxed{f(N) = 2N-1}$$

3-a) Solution can be found on the internet.

3-b) To show that $(n+a)^b = \Theta(n^b)$

we should find constants $c_1, c_2, n_0 > 0$ such that

$$0 \leq c_1 n^b \leq (n+a)^b \leq c_2 n^b \text{ for all } n \geq n_0.$$

Note that:

$$n+a \leq n + |a| \leq 2n \quad \text{when } |a| \leq n.$$

and

$$n+a \geq n - |a| \geq \frac{1}{2} n \quad \text{when } |a| \leq \frac{1}{2} n$$

Thus, when $n \geq 2|a|$,

$$0 \leq \frac{1}{2} n \leq n+a \leq 2n.$$

Since $b > 0$, the inequality still holds when all parts are raised to the power b.

$$0 \leq \left(\frac{1}{2} n\right)^b \leq (n+a)^b \leq (2n)^b$$

$$0 \leq \left(\frac{1}{2}\right)^b n^b \leq (n+a)^b \leq 2^b n^b$$

Thus $c_1 = (1/2)^b$, $c_2 = 2^b$ and $n_0 = 2|a|$

satisfy the definition.

4-)

a) $$\lim_{n\to\infty} \frac{(n^2+1)^{10}}{n^{20}} = \lim_{n\to\infty}\left(1+\frac{1}{n^2}\right) = 1$$

so $(n^2+1)^{10} = \Theta(n^{20})$

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b) $$\lim_{n\to\infty} \frac{\sqrt{10n^2+7n+3}}{n} = \sqrt{10}$$

so $\sqrt{10n^2+7n+3} = \Theta(n)$

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c) $2n\lg(n+2)^2 + (n+2)^2 \lg\frac{n}{2} = 2n\,2\lg(n+2) + (n+2)^2(\lg n - 1)$

$\in \Theta(n\lg n) + \Theta(n^2 \lg n) = \Theta(n^2 \lg n).$

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d) $2^{n+1} + 3^{n-1} = 2\,2^n + 3^n \cdot \frac{1}{3} \in \Theta(2^n) + \Theta(3^n) = \Theta(3^n)$

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e) $\lg_2 n - 1 \leq \lfloor \log_2 n \rfloor \leq \log_2 n$

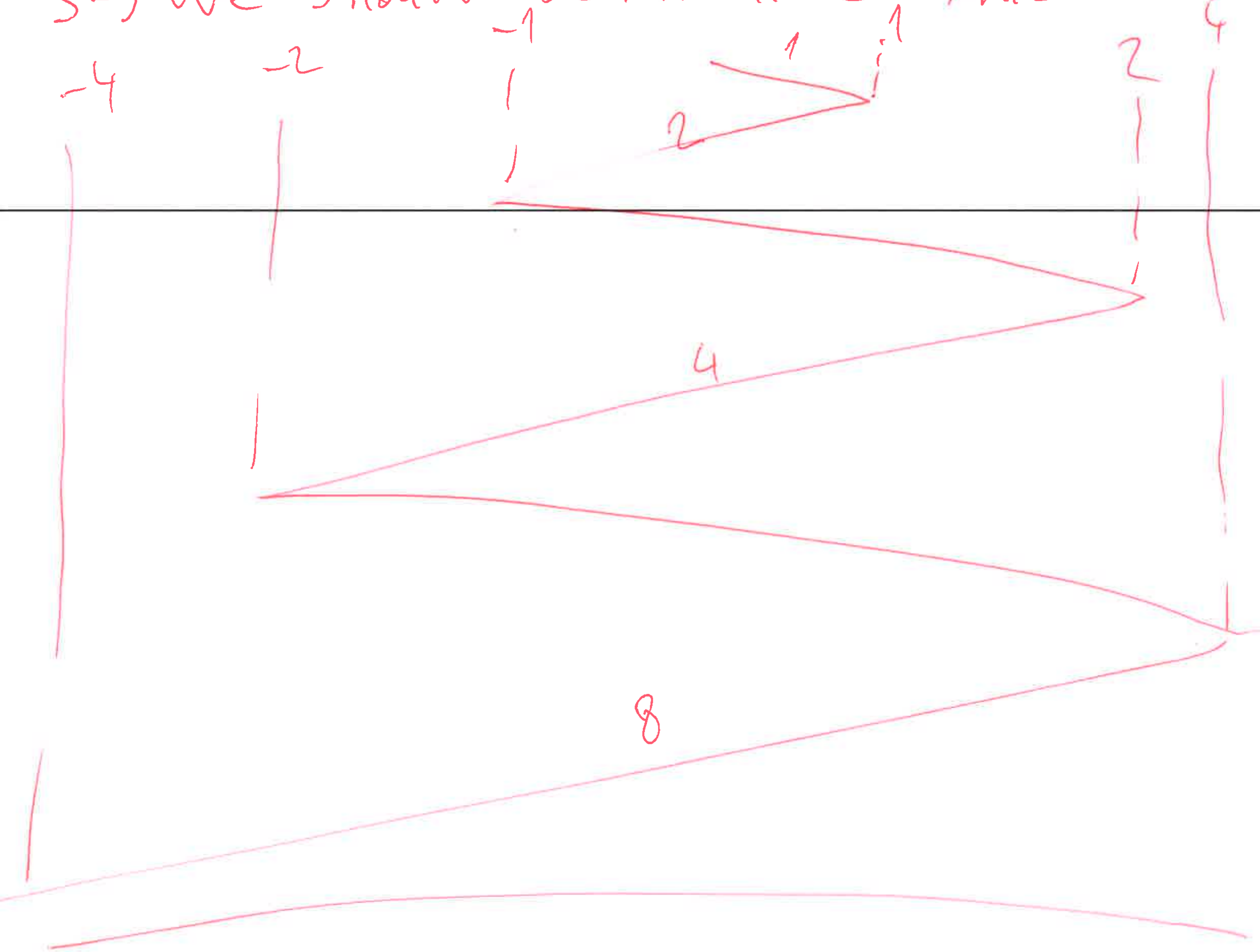
$\frac{1}{2}\log_2 n \leq \log_2 n - 1 \leq \log_2 n$

$\log_2 n - 1 \in \Theta(\log_2 n)$

$\log_2 n \in \Theta(\log_2 n)$

so $\lfloor \log_2 n \rfloor \in \Theta(\log_2 n)$

5-) We should search like this:



So total stepts is less then

$$2(2+4+8+\dots+n/2+n)$$

which is equal to $4\cdot(1+2+\dots+n/2)$

$= 4\cdot(n-1) = 4n-4$

total stepts is less then

$4n-4$ which is $O(n)$.

6-)

a) $\sum_{i=0}^{n-1} (i^2+1)^2 = \sum_{i=0}^{n-1} (i^4+2i^2+1) = \sum_{i=0}^{n-1} i^4 + 2\sum_{i=0}^{n-1} i^2 + \sum_{i=0}^{1} 1$

$\in \Theta(n^5) + \Theta(n^3) + \Theta(n) = \Theta(n^5)$

b) $\sum_{i=2}^{n-1} \log_2 i^2 = \sum_{i=2}^{n-1} 2\log_2 i = 2\sum_{i=2}^{n-1} \log_2 i = 2\sum_{i=1}^{n} \log_2 i - 2.\log_2 n$

$2\sum_{i=1}^{n} \log_2 i = 2(\log 1 + \log 2 + \log 3 + \dots + \log n) < 2 \times n . \log n$,

$\frac{1}{2} n \log n < 2\sum_{i=1}^{n} \log_2 i$

so $2\sum_{i=1}^{n} \log_2 i \in \Theta(n\log n)$ and $2\log_2 n \in \Theta(n\log n)$

hence $\sum_{i=2}^{n-1} \log_2 i^2 \in \Theta(n\log n)$

c) $\sum_{i=1}^{n} (i+1) . 2^{i-1} = \sum_{i=1}^{n} i . 2^{i-1} + \sum_{i=1}^{n} 2^{i-1} = \frac{1}{2}\sum_{i=1}^{n} i.2^i + \sum_{j=0}^{n-1} 2^j$

$\in \Theta(n2^n) + \Theta(2^n) = \Theta(n2^n)$

d) $\sum_{i=0}^{n-1} \sum_{j=0}^{i-1} (i+j) = \sum_{i=0}^{n-1} \left(\sum_{j=0}^{i-1} i + \sum_{j=0}^{i-1} j \right) = \sum_{i=0}^{n-1} \left(i^2 + \frac{(i-1)i}{2} \right)$

$= \sum_{i=0}^{n-1} \left(\frac{3}{2} i^2 - \frac{1}{2} i \right) = \frac{3}{2} \sum_{i=0}^{n-1} i^2 - \frac{1}{2} \sum_{i=0}^{n-1} i \in \Theta(n^3) - \Theta(n^2) = \Theta(n^3)$