



## TTK4130 Modeling and Simulation Assignment 5

### Introduction

In this assignment we will study the modeling of complex mechanical systems using the constrained Lagrange equations. Since the calculations of the partial derivatives of the Lagrangian can be involved, we will once more outsource this task by using the Matlab Symbolic Math Toolbox™.

### Problem 1 (Hovering mass)

We consider a helicopter hovering a mass. We model this system as two point masses coupled by a rigid link with length  $L$ . The masses are  $m_1$  and  $m_2$ , and their positions in space are given by  $p_1 \in \mathbb{R}^3$  and  $p_2 \in \mathbb{R}^3$ . We assume that an external force  $u \in \mathbb{R}^3$  is applied to  $m_1$  (helicopter).

- (a) We will first model this system by using the classical Lagrange approach, where the number of coordinates is equal to the number of degrees of freedom. In this setting, the position of the helicopter is described by  $p_1 \in \mathbb{R}^3$ , and the position of the hovering mass is described by the two angles  $\theta, \phi$ , which give the orientation of the rigid link/cable (spherical coordinates). Hence, the generalized coordinates are

$$q = \begin{bmatrix} p_1 \\ \theta \\ \phi \end{bmatrix} \in \mathbb{R}^5. \quad (1)$$

As seen in the course and in the previous assignment, the fully assembled model has the form:

$$\dot{q} = v \quad (2a)$$

$$M(q)\dot{v} = b(q, \dot{q}, u). \quad (2b)$$

Complete the delivered template `HoveringMassUnconstraintTemplate.m` by doing the following tasks:

1. Write the expression for the position of mass 2.
2. Write the expression for the generalized forces.
3. Write the expression for the kinetic energy.
4. Write the expression for the potential energy.
5. Write the expression for the Lagrangian.
6. Run the routine in order to obtain the expressions for  $M$  and  $b$ .

Add the implemented code to your answer.

- (b) We will now use the constrained Lagrange approach to model the dynamics of the system. The generalized coordinates in this case are

$$q = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \in \mathbb{R}^6, \quad (3)$$

and the scalar constraint is

$$C = \frac{1}{2} (e^\top e - L^2), \quad \text{where } e = p_1 - p_2. \quad (4)$$

The dynamics are then given by

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}}(q, \dot{q}) - \frac{\partial \mathcal{L}}{\partial q}(q, \dot{q}) - \nabla C(q) = Q \quad (5a)$$

$$C(q) = 0. \quad (5b)$$

In this case, the fully assembled model should have the form:

$$\dot{q} = v \quad (6a)$$

$$M(q)\dot{v} = b(q, z, u) \quad (6b)$$

$$0 = C(q). \quad (6c)$$

Perform the following tasks:

1. Find the model matrices  $M(q)$  and  $b(q, \dot{q}, u)$ .

Note that these matrices are not the same as in the previous part.

In this case, it is fairly easy to derive the model equations by hand, by following the steps of the previous task.

However, if you prefer, you can use Matlab's symbolic toolbox to calculate them by completing the delivered template `HoveringMassConstraintTemplate.m`.

2. Compare the complexity of the models from part a) and b), i.e. the complexity of the symbolic expressions for  $M$  and  $b$ . What do you conclude?

### Problem 2 (Explicit vs. Implicit model)

- (a) Write the hovering mass model of task 1(b) in the form:

$$\begin{bmatrix} M & a(q) \\ a(q)^\top & 0 \end{bmatrix} \begin{bmatrix} \ddot{q} \\ z \end{bmatrix} = c(q, \dot{q}, u). \quad (7)$$

Specify what the functions  $a$  and  $c$  are.

- (b) Compare the model in the form (7) to its explicit counterpart:

$$\begin{bmatrix} \ddot{q} \\ z \end{bmatrix} = \begin{bmatrix} M & a(q) \\ a(q)^\top & 0 \end{bmatrix}^{-1} c(q, \dot{q}, u). \quad (8)$$

Make sure to use Matlab's symbolic toolbox for finding the right-hand side of (8).

The last part of `HoveringMassConstraintTemplate.m` provides a template for this.

What do you observe?

Which form of the ODE would you rather evaluate when solving it numerically?

### Problem 3 ( $\Delta$ -robot)

$\Delta$ -robots are common in ultra-fast packaging applications. Figure 1 illustrates a  $\Delta$ -robot. The three yellow arms of length  $l$  are actuated and can pivot in their vertical planes. These arms drive the three double thin rods of length  $L$  (typically made of ultra-light carbon fiber), connected to the nacelle (triangular shape at the bottom). The geometry imposes that the nacelle always remains horizontal. Moreover, the pivots on the nacelle are forced to remain at a distance  $L$  from the pivot at the extremities of the yellow arms.

In order to express the Lagrange function and the constraints, the following parameters and variables are defined: The cartesian frame has its origin at the center of the upper platform, with the  $x$ -axis aligned with the frontal robot arm, and the  $z$  axis pointing up. The position of the pivots of the yellow arms in this cartesian frame,  $p_{1,2,3}$ , are given by

$$p_k = R_k \begin{bmatrix} d + l \cos \alpha_k \\ 0 \\ -l \sin \alpha_k \end{bmatrix}, \quad R_k = \begin{bmatrix} \cos \gamma_k & -\sin \gamma_k & 0 \\ \sin \gamma_k & \cos \gamma_k & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (9)$$

where  $\gamma_{1,2,3} = \{0, \frac{2\pi}{3}, \frac{4\pi}{3}\}$ ,  $d$  is the constant distance from the center of the upper platform to the axis of the motors (black in the figure), and  $\alpha_k$  are the angles of the yellow arms with respect to the

horizontal plane. The yellow arms together with the motors have an inertia  $J$ , i.e. their kinetic energy is  $T_k = \frac{1}{2}J\dot{\alpha}_k^2$ . The nacelle has a mass  $m$ . For simplicity, we will assume that the nacelle is just a point where the long arms are all connected. This is a simplification.

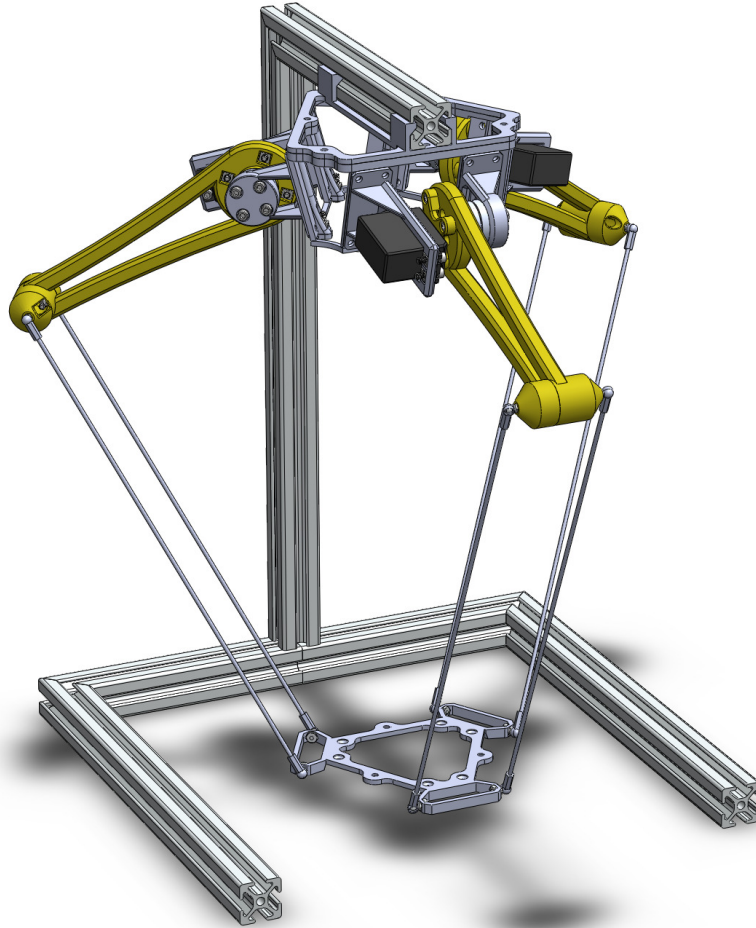


Figure 1: Illustration of the  $\Delta$ -robot. The yellow arms (length  $l$ ) are actuated by the motors on the upper platform (black boxes).

- (a) Assume that we use the classical unconstrained Lagrange approach with the angles  $\alpha_{1,2,3}$  as the generalized coordinates.

Explain what is the challenge with this approach in this particular case.

*Hint: How would you find the position of the nacelle as a function of the generalized coordinates?*

We choose the following generalized coordinates for the  $\Delta$ -robot:

$$\mathbf{q} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \mathbf{p} \end{bmatrix},$$

where  $\mathbf{p} \in \mathbb{R}^3$  is the position of the nacelle.

- (b) Write down the Lagrange function  $\mathcal{L}$  of the  $\Delta$ -robot, as well as the associated constraints  $c$ .
- (c) What is the differential index of the DAE that results from  $\mathcal{L}$  and  $c$ ?
- (d) What are the consistency conditions of the  $\Delta$ -robot? The explicit expression is not needed, only the abstract form.