Chapter 2

RIGID MOTIONS AND HOMOGENEOUS TRANSFORMATIONS

A large part of robot kinematics is concerned with establishing various coordinate frames to represent the positions and orientations of rigid objects, and with transformations among these coordinate frames. Indeed, the geometry of three-dimensional space and of rigid motions plays a central role in all aspects of robotic manipulation. In this chapter we study the operations of rotation and translation, and introduce the notion of homogeneous transformations.[1] Homogeneous transformations combine the operations of rotation and translation into a single matrix multiplication, and are used in Chapter 3 to derive the so-called forward kinematic equations of rigid manipulators.

We begin by examining representations of points and vectors in a Euclidean space equipped with multiple coordinate frames. Following this, we introduce the concept of a rotation matrix to represent relative orientations among coordinate frames. Then we combine these two concepts to build homogeneous transformation matrices, which can be used to simultaneously represent the position and orientation of one coordinate frame relative to another. Furthermore, homogeneous transformation matrices can be used to perform coordinate transformations. Such transformations allow us to represent various quantities in different coordinate frames, a facility that we will often exploit in subsequent chapters.

[1]Since we make extensive use of elementary matrix theory, the reader may wish to review Appendix B before beginning this chapter.

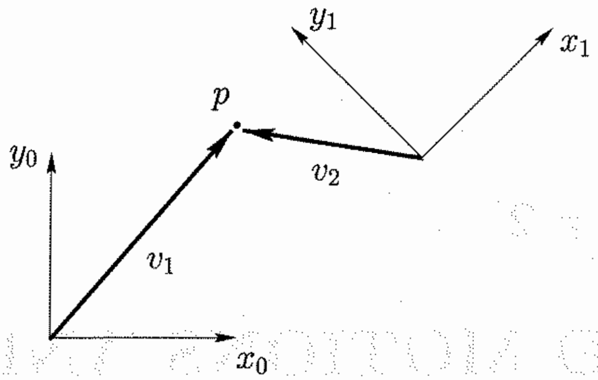

Figure 2.1: Two coordinate frames, a point p, and two vectors v_1 and v_2.

2.1 REPRESENTING POSITIONS

Before developing representation schemes for points and vectors, it is instructive to distinguish between the two fundamental approaches to geometric reasoning: the **synthetic** approach and the **analytic** approach. In the former, one reasons directly about geometric entities (e.g., points or lines), while in the latter, one represents these entities using coordinates or equations, and reasoning is performed via algebraic manipulations. The latter approach requires the choice of a reference coordinate frame. A coordinate frame consists of an origin (a single point in space), and two or three orthogonal coordinate axes, for two- and three-dimensional spaces, respectively.

Consider Figure 2.1, which shows two coordinate frames that differ in orientation by an angle of 45°. Using the synthetic approach, without ever assigning coordinates to points or vectors, one can say that x_0 is perpendicular to y_0, or that $v_1 \times v_2$ defines a vector that is perpendicular to the plane containing v_1 and v_2, in this case pointing out of the page.

In robotics, one typically uses analytic reasoning, since robot tasks are often defined using Cartesian coordinates. Of course, in order to assign coordinates it is necessary to specify a reference coordinate frame. Consider again Figure 2.1. We could specify the coordinates of the point p with respect to either frame $o_0x_0y_0$ or frame $o_1x_1y_1$. In the former case, we might assign to p the coordinate vector $[5, 6]^T$, and in the latter case $[-2.8, 4.2]^T$. So that the reference frame will always be clear, we will adopt a notation in which a superscript is used to denote the reference frame. Thus, we would write

$$p^0 = \begin{bmatrix} 5 \\ 6 \end{bmatrix}, \qquad p^1 = \begin{bmatrix} -2.8 \\ 4.2 \end{bmatrix}$$

Geometrically, a point corresponds to a specific location in space. We stress here that p is a geometric entity, a point in space, while both p^0 and p^1 are coordinate vectors that represent the location of this point in space with respect to coordinate frames $o_0x_0y_0$ and $o_1x_1y_1$, respectively.

Since the origin of a coordinate frame is just a point in space, we can assign coordinates that represent the position of the origin of one coordinate frame with respect to another. In Figure 2.1, for example, we have

$$o_1^0 = \begin{bmatrix} 10 \\ 5 \end{bmatrix}, \qquad o_0^1 = \begin{bmatrix} -10.6 \\ 3.5 \end{bmatrix}$$

In cases where there is only a single coordinate frame, or in which the reference frame is obvious, we will often omit the superscript. This is a slight abuse of notation, and the reader is advised to bear in mind the difference between the geometric entity called p and any particular coordinate vector that is assigned to represent p. The former is independent of the choice of coordinate frames, while the latter obviously depends on the choice of coordinate frames.

While a point corresponds to a specific location in space, a *vector* specifies a direction and a magnitude. Vectors can be used, for example, to represent displacements or forces. Therefore, while the point p is not equivalent to the vector v_1, the displacement from the origin o_0 to the point p is given by the vector v_1. In this text, we will use the term *vector* to refer to what are sometimes called *free vectors*, that is, vectors that are not constrained to be located at a particular point in space. Under this convention, it is clear that points and vectors are not equivalent, since points refer to specific locations in space, but a vector can be moved to any location in space. Under this convention, two vectors are equal if they have the same direction and the same magnitude.

When assigning coordinates to vectors, we use the same notational convention that we used when assigning coordinates to points. Thus, v_1 and v_2 are geometric entities that are invariant with respect to the choice of coordinate frames, but the representation by coordinates of these vectors depends directly on the choice of reference coordinate frame. In the example of Figure 2.1, we would obtain

$$v_1^0 = \begin{bmatrix} 5 \\ 6 \end{bmatrix}, \qquad v_1^1 = \begin{bmatrix} 7.77 \\ 0.8 \end{bmatrix}, \qquad v_2^0 = \begin{bmatrix} -5.1 \\ 1 \end{bmatrix}, \qquad v_2^1 = \begin{bmatrix} -2.89 \\ 4.2 \end{bmatrix}$$

In order to perform algebraic manipulations using coordinates, it is essential that all coordinate vectors be defined with respect to the same coordinate frame. In the case of free vectors, it is enough that they be defined with respect to "parallel" coordinate frames, that is, frames whose respective coordinate axes are parallel, since only their magnitude and direction are specified and not their absolute locations in space.

Using this convention, an expression of the form $v_1^1 + v_2^2$, where v_1^1 and v_2^2 are as in Figure 2.1, is not defined since the frames $o_0x_0y_0$ and $o_1x_1y_1$ are not parallel. Thus, we see a clear need not only for a representation system that allows points to be expressed with respect to various coordinate frames, but also for a mechanism that allows us to transform the coordinates of points from one coordinate frame to another. Such coordinate transformations are the topic for much of the remainder of this chapter.

2.2 REPRESENTING ROTATIONS

In order to represent the relative position and orientation of one rigid body with respect to another, we will attach coordinate frames to each body, and then specify the geometric relationships between these coordinate frames. In Section 2.1 we saw how one can represent the position of the origin of one frame with respect to another frame. In this section, we address the problem of describing the orientation of one coordinate frame relative to another frame. We begin with the case of rotations in the plane, and then generalize our results to the case of orientations in a three-dimensional space.

2.2.1 Rotation in the Plane

Figure 2.2 shows two coordinate frames, with frame $o_1x_1y_1$ being obtained by rotating frame $o_0x_0y_0$ by an angle θ. Perhaps the most obvious way to represent the relative orientation of these two frames is to merely specify the angle of rotation θ. There are two immediate disadvantages to such a representation. First, there is a discontinuity in the mapping from relative orientation to the value of θ in a neighborhood of $\theta = 0$. In particular, for $\theta = 2\pi - \epsilon$, small changes in orientation can produce large changes in the value of θ, for example, a rotation by ϵ causes θ to "wrap around" to zero. Second, this choice of representation does not scale well to the three-dimensional case.

A slightly less obvious way to specify the orientation is to specify the coordinate vectors for the axes of frame $o_1x_1y_1$ with respect to coordinate frame $o_0x_0y_0$:

$$R_1^0 = [x_1^0 \mid y_1^0]$$

in which x_1^0 and y_1^0 are the coordinates in frame $o_0x_0y_0$ of unit vectors x_1

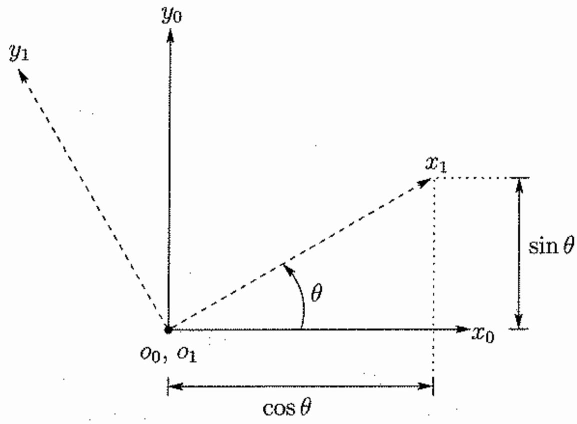

Figure 2.2: Coordinate frame $o_1x_1y_1$ is oriented at an angle θ with respect to $o_0x_0y_0$.

and y_1, respectively.[2] A matrix in this form is called a **rotation matrix**. Rotation matrices have a number of special properties that we will discuss below.

In the two-dimensional case, it is straightforward to compute the entries of this matrix. As illustrated in Figure 2.2,

$$x_1^0 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \qquad y_1^0 = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

which gives

$$R_1^0 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{2.1}$$

Note that we have continued to use the notational convention of allowing the superscript to denote the reference frame. Thus, R_1^0 is a matrix whose column vectors are the coordinates of the unit vectors along the axes of frame $o_1x_1y_1$ expressed relative to frame $o_0x_0y_0$.

Although we have derived the entries for R_1^0 in terms of the angle θ, it is not necessary that we do so. An alternative approach, and one that scales nicely to the three-dimensional case, is to build the rotation matrix by projecting the axes of frame $o_1x_1y_1$ onto the coordinate axes of frame $o_0x_0y_0$. Recalling that the dot product of two unit vectors gives the projection of

[2] We will use x_i, y_i to denote both coordinate axes and unit vectors along the coordinate axes depending on the context.

one onto the other, we obtain

$$x_1^0 = \begin{bmatrix} x_1 \cdot x_0 \\ x_1 \cdot y_0 \end{bmatrix}, \qquad y_1^0 = \begin{bmatrix} y_1 \cdot x_0 \\ y_1 \cdot y_0 \end{bmatrix}$$

which can be combined to obtain the rotation matrix

$$R_1^0 = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 \end{bmatrix}$$

Thus, the columns of R_1^0 specify the direction cosines of the coordinate axes of $o_1x_1y_1$ relative to the coordinate axes of $o_0x_0y_0$. For example, the first column $[x_1 \cdot x_0, x_1 \cdot y_0]^T$ of R_1^0 specifies the direction of x_1 relative to the frame $o_0x_0y_0$. Note that the right-hand sides of these equations are defined in terms of geometric entities, and not in terms of their coordinates. Examining Figure 2.2 it can be seen that this method of defining the rotation matrix by projection gives the same result as was obtained in Equation (2.1).

If we desired instead to describe the orientation of frame $o_0x_0y_0$ with respect to the frame $o_1x_1y_1$ (that is, if we desired to use the frame $o_1x_1y_1$ as the reference frame), we would construct a rotation matrix of the form

$$R_0^1 = \begin{bmatrix} x_0 \cdot x_1 & y_0 \cdot x_1 \\ x_0 \cdot y_1 & y_0 \cdot y_1 \end{bmatrix}$$

Since the dot product is commutative, (that is, $x_i \cdot y_j = y_j \cdot x_i$), we see that

$$R_0^1 = (R_1^0)^T$$

In a geometric sense, the orientation of $o_0x_0y_0$ with respect to the frame $o_1x_1y_1$ is the inverse of the orientation of $o_1x_1y_1$ with respect to the frame $o_0x_0y_0$. Algebraically, using the fact that coordinate axes are mutually orthogonal, it can readily be seen that

$$(R_1^0)^T = (R_1^0)^{-1}$$

The column vectors of R_1^0 are of unit length and mutually orthogonal (Problem 2-4). Such a matrix is said to be **orthogonal**. It can also be shown (Problem 2-5) that $\det R_1^0 = \pm 1$. If we restrict ourselves to right-handed coordinate frames, as defined in Appendix B, then $\det R_1^0 = +1$ (Problem 2-5). It is customary to refer to the set of all such $n \times n$ matrices by the symbol $SO(n)$, which denotes the **Special Orthogonal** group of order n.

For any $R \in SO(n)$ the following properties hold.

- $R^T = R^{-1} \in SO(n)$
- The columns (and therefore the rows) of R are mutually orthogonal
- Each column (and therefore each row) of R is a unit vector
- $\det R = 1$

To provide further geometric intuition for the notion of the inverse of a rotation matrix, note that in the two-dimensional case, the inverse of the rotation matrix corresponding to a rotation by angle θ can also be easily computed simply by constructing the rotation matrix for a rotation by the angle $-\theta$:

$$\begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^T$$

2.2.2 Rotations in Three Dimensions

The projection technique described above scales nicely to the three-dimensional case. In three dimensions, each axis of the frame $o_1x_1y_1z_1$ is projected onto coordinate frame $o_0x_0y_0z_0$. The resulting rotation matrix is given by

$$R_1^0 = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 & z_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 & z_1 \cdot y_0 \\ x_1 \cdot z_0 & y_1 \cdot z_0 & z_1 \cdot z_0 \end{bmatrix}$$

As was the case for rotation matrices in two dimensions, matrices in this form are orthogonal, with determinant equal to 1. In this case, 3×3 rotation matrices belong to the group $SO(3)$.

Example 2.1

Suppose the frame $o_1x_1y_1z_1$ is rotated through an angle θ about the z_0-axis, and we wish to find the resulting transformation matrix R_1^0. By convention, the right hand rule (see Appendix B) defines the positive sense for the angle θ to be such that rotation by θ about the z-axis would advance a right-hand threaded screw along the positive z-axis. From Figure 2.3 we see that

$$x_1 \cdot x_0 = \cos\theta, \qquad y_1 \cdot x_0 = -\sin\theta,$$
$$x_1 \cdot y_0 = \sin\theta, \qquad y_1 \cdot y_0 = \cos\theta$$

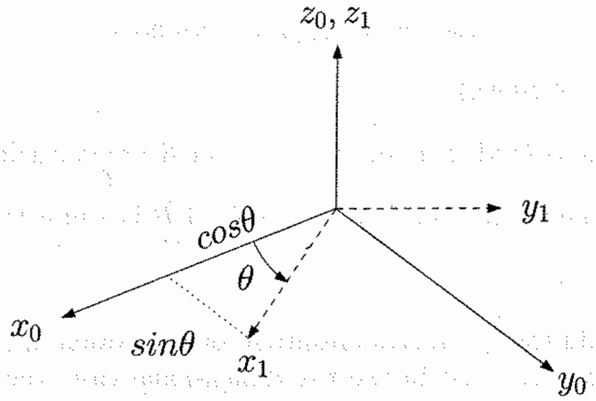

Figure 2.3: Rotation about z_0 by an angle θ.

and

$$z_0 \cdot z_1 = 1$$

while all other dot products are zero. Thus, the rotation matrix R_1^0 *has a particularly simple form in this case, namely*

$$R_1^0 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.2}$$

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The rotation matrix given in Equation (2.2) is called a **basic rotation matrix** (about the z-axis). In this case we find it useful to use the more descriptive notation $R_{z,\theta}$ instead of R_1^0 to denote the matrix. It is easy to verify that the basic rotation matrix $R_{z,\theta}$ has the properties

$$R_{z,0} = I \tag{2.3}$$
$$R_{z,\theta}R_{z,\phi} = R_{z,\theta+\phi} \tag{2.4}$$

which together imply

$$\left(R_{z,\theta}\right)^{-1} = R_{z,-\theta} \tag{2.5}$$

Similarly, the basic rotation matrices representing rotations about the x

and y-axes are given as (Problem 2-8)

$$R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \tag{2.6}$$

$$R_{y,\theta} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \tag{2.7}$$

which also satisfy properties analogous to Equations (2.3)–(2.5).

Example 2.2

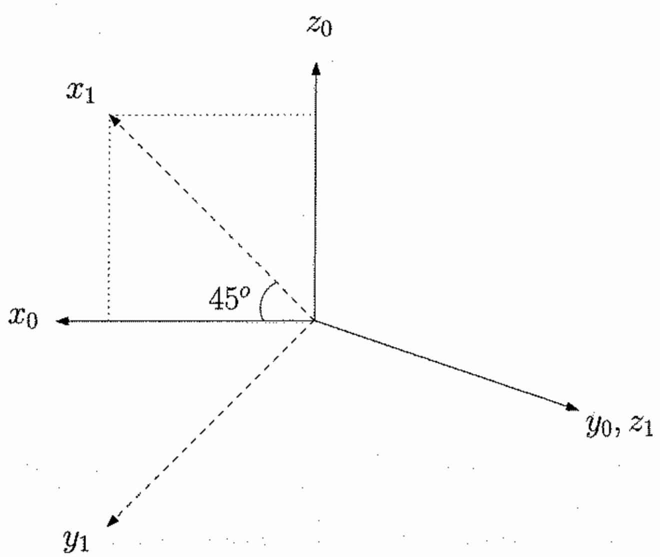

Figure 2.4: Defining the relative orientation of two frames.

Consider the frames $o_0x_0y_0z_0$ and $o_1x_1y_1z_1$ shown in Figure 2.4. Projecting the unit vectors x_1, y_1, z_1 onto x_0, y_0, z_0 gives the coordinates of x_1, y_1, z_1 in the $o_0x_0y_0z_0$ frame. We see that the coordinates of x_1, y_1, and z_1 are given by

$$x_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \quad y_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{-1}{\sqrt{2}} \end{bmatrix}, \quad z_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

The rotation matrix R_1^0 specifying the orientation of $o_1x_1y_1z_1$ relative to

$o_0x_0y_0z_0$ has these as its column vectors, that is,

$$R_1^0 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \end{bmatrix}$$

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2.3 ROTATIONAL TRANSFORMATIONS

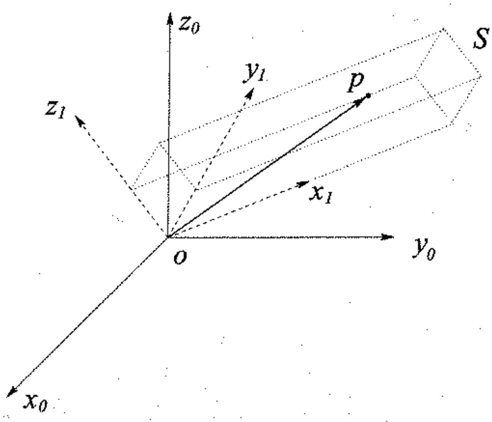

Figure 2.5: Coordinate frame attached to a rigid body.

Figure 2.5 shows a rigid object S to which a coordinate frame $o_1x_1y_1z_1$ is attached. Given the coordinates p^1 of the point p (in other words, given the coordinates of p with respect to the frame $o_1x_1y_1z_1$), we wish to determine the coordinates of p relative to a fixed reference frame $o_0x_0y_0z_0$. The coordinates $p^1 = [u, v, w]^T$ satisfy the equation

$$p = ux_1 + vy_1 + wz_1$$

In a similar way, we can obtain an expression for the coordinates p^0 by projecting the point p onto the coordinate axes of the frame $o_0x_0y_0z_0$, giving

$$p^0 = \begin{bmatrix} p \cdot x_0 \\ p \cdot y_0 \\ p \cdot z_0 \end{bmatrix}$$

Combining these two equations we obtain

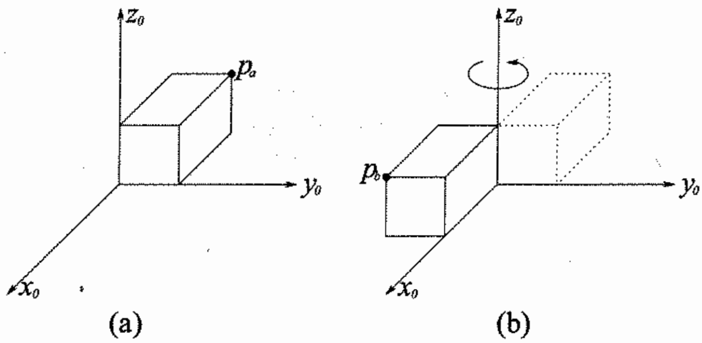

Figure 2.6: The block in (b) is obtained by rotating the block in (a) by π about z_0.

$$
\begin{aligned}
p^0 &= \begin{bmatrix} (ux_1 + vy_1 + wz_1) \cdot x_0 \\ (ux_1 + vy_1 + wz_1) \cdot y_0 \\ (ux_1 + vy_1 + wz_1) \cdot z_0 \end{bmatrix} \\
&= \begin{bmatrix} ux_1 \cdot x_0 + vy_1 \cdot x_0 + wz_1 \cdot x_0 \\ ux_1 \cdot y_0 + vy_1 \cdot y_0 + wz_1 \cdot y_0 \\ ux_1 \cdot z_0 + vy_1 \cdot z_0 + wz_1 \cdot z_0 \end{bmatrix} \\
&= \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 & z_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 & z_1 \cdot y_0 \\ x_1 \cdot z_0 & y_1 \cdot z_0 & z_1 \cdot z_0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}
\end{aligned}
$$

But the matrix in this final equation is merely the rotation matrix R_1^0, which leads to

$$p^0 = R_1^0 p^1 \tag{2.8}$$

Thus, the rotation matrix R_1^0 can be used not only to represent the orientation of coordinate frame $o_1x_1y_1z_1$ with respect to frame $o_0x_0y_0z_0$, but also to transform the coordinates of a point from one frame to another. If a given point is expressed relative to $o_1x_1y_1z_1$ by coordinates p^1, then $R_1^0 p^1$ represents the **same point** expressed relative to the frame $o_0x_0y_0z_0$.

We can also use rotation matrices to represent rigid motions that correspond to pure rotation. Consider Figure 2.6. One corner of the block in Figure 2.6(a) is located at the point p_a in space. Figure 2.6(b) shows the same block after it has been rotated about z_0 by the angle π. In Figure 2.6(b), the same corner of the block is now located at point p_b in space. It is possible to derive the coordinates for p_b given only the coordinates for p_a

and the rotation matrix that corresponds to the rotation about z_0. To see how this can be accomplished, imagine that a coordinate frame is rigidly attached to the block in Figure 2.6(a), such that it is coincident with the frame $o_0x_0y_0z_0$. After the rotation by π, the block's coordinate frame, which is rigidly attached to the block, is also rotated by π. If we denote this rotated frame by $o_1x_1y_1z_1$, we obtain

$$R_1^0 = R_{z,\pi} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In the local coordinate frame $o_1x_1y_1z_1$, the point p_b has the coordinate representation p_b^1. To obtain its coordinates with respect to frame $o_0x_0y_0z_0$, we merely apply the coordinate transformation Equation (2.8), giving

$$p_b^0 = R_{z,\pi}p_b^1$$

It is important to notice that the local coordinates p_b^1 of the corner of the block do not change as the block rotates, since they are defined in terms of the block's own coordinate frame. Therefore, when the block's frame is aligned with the reference frame $o_0x_0y_0z_0$ (that is, before the rotation is performed), the coordinates p_b^1 equals p_a^0, since before the rotation is performed, the point p_a is coincident with the corner of the block. Therefore, we can substitute p_a^0 into the previous equation to obtain

$$p_b^0 = R_{z,\pi}p_a^0$$

This equation shows how to use a rotation matrix to represent a rotational motion. In particular, if the point p_b is obtained by rotating the point p_a as defined by the rotation matrix R, then the coordinates of p_b with respect to the reference frame are given by

$$p_b^0 = Rp_a^0$$

This same approach can be used to rotate vectors with respect to a coordinate frame, as the following example illustrates.

Example 2.3
The vector v with coordinates $v^0 = [0, 1, 1]^T$ is rotated about y_0 by $\frac{\pi}{2}$ as shown in Figure 2.7. The resulting vector v_1 has coordinates given by

$$v_1^0 = R_{y,\frac{\pi}{2}}v^0 \tag{2.9}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \tag{2.10}$$

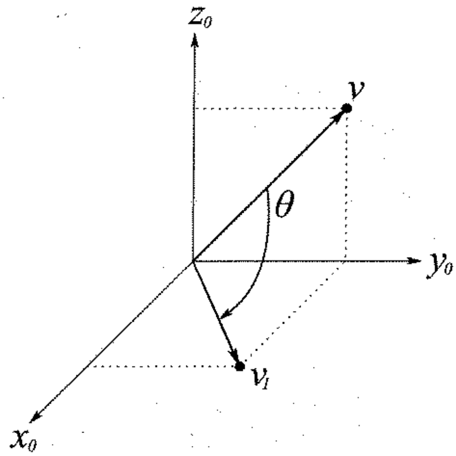

Figure 2.7: Rotating a vector about axis y_0.

Thus, a third interpretation of a rotation matrix R is as an operator acting on vectors in a fixed frame. In other words, instead of relating the coordinates of a fixed vector with respect to two different coordinate frames, Equation (2.9) can represent the coordinates in $o_0x_0y_0z_0$ of a vector v_1 that is obtained from a vector v by a given rotation.

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As we have seen, rotation matrices can serve several roles. A rotation matrix, either $R \in SO(3)$ or $R \in SO(2)$, can be interpreted in three distinct ways:

1. It represents a coordinate transformation relating the coordinates of a point p in two different frames.

2. It gives the orientation of a transformed coordinate frame with respect to a fixed coordinate frame.

3. It is an operator taking a vector and rotating it to give a new vector in the same coordinate frame.

The particular interpretation of a given rotation matrix R will be made clear by the context.

2.3.1 Similarity Transformations

A coordinate frame is defined by a set of **basis vectors**, for example, unit vectors along the three coordinate axes. This means that a rotation matrix, as a coordinate transformation, can also be viewed as defining a change of basis from one frame to another. The matrix representation of a general linear transformation is transformed from one frame to another using a so-called **similarity transformation.**[3] For example, if A is the matrix representation of a given linear transformation in $o_0x_0y_0z_0$ and B is the representation of the same linear transformation in $o_1x_1y_1z_1$ then A and B are related as

$$B = (R_1^0)^{-1} A R_1^0 \tag{2.11}$$

where R_1^0 is the coordinate transformation between frames $o_1x_1y_1z_1$ and $o_0x_0y_0z_0$. In particular, if A itself is a rotation, then so is B, and thus the use of similarity transformations allows us to express the same rotation easily with respect to different frames.

Example 2.4

Henceforth, whenever convenient we use the shorthand notation $c_\theta = \cos\theta$, $s_\theta = \sin\theta$ *for trigonometric functions. Suppose frames* $o_0x_0y_0z_0$ *and* $o_1x_1y_1z_1$ *are related by the rotation*

$$R_1^0 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

If $A = R_{z,\theta}$ *relative to the frame* $o_0x_0y_0z_0$, *then, relative to frame* $o_1x_1y_1z_1$ *we have*

$$B = (R_1^0)^{-1} A R_1^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\theta & s_\theta \\ 0 & -s_\theta & c_\theta \end{bmatrix}$$

In other words, B *is a rotation about the* z_0*-axis but expressed relative to the frame* $o_1x_1y_1z_1$. *This notion will be useful below and in later sections.*

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2.4 COMPOSITION OF ROTATIONS

In this section we discuss the composition of rotations. It is important for subsequent chapters that the reader understand the material in this section thoroughly before moving on.

[3] See Appendix B.

2.4.1 Rotation with Respect to the Current Frame

Recall that the matrix R_1^0 in Equation (2.8) represents a rotational transformation between the frames $o_0x_0y_0z_0$ and $o_1x_1y_1z_1$. Suppose we now add a third coordinate frame $o_2x_2y_2z_2$ related to the frames $o_0x_0y_0z_0$ and $o_1x_1y_1z_1$ by rotational transformations. A given point p can then be represented by coordinates specified with respect to any of these three frames: p^0, p^1, and p^2. The relationship among these representations of p is

$$p^0 = R_1^0 p^1 \tag{2.12}$$
$$p^1 = R_2^1 p^2 \tag{2.13}$$
$$p^0 = R_2^0 p^2 \tag{2.14}$$

where each R_j^i is a rotation matrix. Substituting Equation (2.13) into Equation (2.12) gives

$$p^0 = R_1^0 R_2^1 p^2 \tag{2.15}$$

Note that R_1^0 and R_2^0 represent rotations relative to the frame $o_0x_0y_0z_0$ while R_2^1 represents a rotation relative to the frame $o_1x_1y_1z_1$. Comparing Equations (2.14) and (2.15) we can immediately infer

$$R_2^0 = R_1^0 R_2^1 \tag{2.16}$$

Equation (2.16) is the composition law for rotational transformations. It states that, in order to transform the coordinates of a point p from its representation p^2 in the frame $o_2x_2y_2z_2$ to its representation p^0 in the frame $o_0x_0y_0z_0$, we may first transform to its coordinates p^1 in the frame $o_1x_1y_1z_1$ using R_2^1 and then transform p^1 to p^0 using R_1^0.

We may also interpret Equation (2.16) as follows. Suppose that initially all three of the coordinate frames coincide. We first rotate the frame $o_2x_2y_2z_2$ relative to $o_0x_0y_0z_0$ according to the transformation R_1^0. Then, with the frames $o_1x_1y_1z_1$ and $o_2x_2y_2z_2$ coincident, we rotate $o_2x_2y_2z_2$ relative to $o_1x_1y_1z_1$ according to the transformation R_2^1. The resulting frame, $o_2x_2y_2z_2$ has orientation with respect to $o_0x_0y_0z_0$ given by $R_1^0R_2^1$. We call the frame relative to which the rotation occurs the **current frame**.

Example 2.5

Suppose a rotation matrix R represents a rotation of angle ϕ about the current y-axis followed by a rotation of angle θ about the current z-axis as

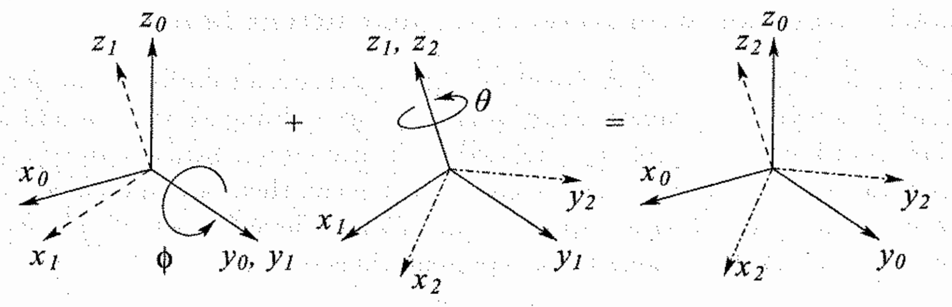

Figure 2.8: Composition of rotations about current axes.

shown in Figure 2.8. Then the matrix R is given by

$$
\begin{aligned}
R &= R_{y,\phi}R_{z,\theta} \\
&= \begin{bmatrix} c_\phi & 0 & s_\phi \\ 0 & 1 & 0 \\ -s_\phi & 0 & c_\phi \end{bmatrix} \begin{bmatrix} c_\theta & -s_\theta & 0 \\ s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} c_\phi c_\theta & -c_\phi s_\theta & s_\phi \\ s_\theta & c_\theta & 0 \\ -s_\phi c_\theta & s_\phi s_\theta & c_\phi \end{bmatrix}
\end{aligned} \tag{2.17}
$$

◇

It is important to remember that the order in which a sequence of rotations is performed, and consequently the order in which the rotation matrices are multiplied together, is crucial. The reason is that rotation, unlike position, is not a vector quantity and so rotational transformations do not commute in general.

Example 2.6

Suppose that the above rotations are performed in the reverse order, that is, first a rotation about the current z-axis followed by a rotation about the current y-axis. Then the resulting rotation matrix is given by

$$
\begin{aligned}
R' &= R_{z,\theta}R_{y,\phi} \\
&= \begin{bmatrix} c_\theta & -s_\phi & 0 \\ s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\phi & 0 & s_\phi \\ 0 & 1 & 0 \\ -s_\phi & 0 & c_\phi \end{bmatrix} \\
&= \begin{bmatrix} c_\theta c_\phi & -s_\theta & c_\theta s_\phi \\ s_\theta c_\phi & c_\theta & s_\theta s_\phi \\ -s_\phi & 0 & c_\phi \end{bmatrix}
\end{aligned} \tag{2.18}
$$

Comparing Equations (2.17) and (2.18) we see that $R \neq R'$.
◇

2.4.2 Rotation with Respect to the Fixed Frame

Many times it is desired to perform a sequence of rotations, each about a given fixed coordinate frame, rather than about successive current frames. For example we may wish to perform a rotation about x_0 followed by a rotation about y_0 (and not y_1!). We will refer to $o_0x_0y_0z_0$ as the **fixed frame.** In this case the composition law given by Equation (2.16) is not valid. It turns out that the correct composition law in this case is simply to multiply the successive rotation matrices *in the reverse order* from that given by Equation (2.16). Note that the rotations themselves are not performed in reverse order. Rather they are performed about the fixed frame instead of about the current frame.

To see this, suppose we have two frames $o_0x_0y_0z_0$ and $o_1x_1y_1z_1$ related by the rotational transformation R_1^0. If $R \in SO(3)$ represents a rotation relative to $o_0x_0y_0z_0$, we know from Section 2.3.1 that the representation for R in the **current** frame $o_1x_1y_1z_1$ is given by $(R_1^0)^{-1}RR_1^0$. Therefore, applying the composition law for rotations about the current axis yields

$$R_2^0 \;=\; R_1^0\left[(R_1^0)^{-1}RR_1^0\right] = RR_1^0 \tag{2.19}$$

Thus, when a rotation R is performed with respect to the world coordinate frame, the current rotation matrix is *premultiplied* by R to obtain the desired rotation matrix.

Example 2.7 Rotations about Fixed Axes

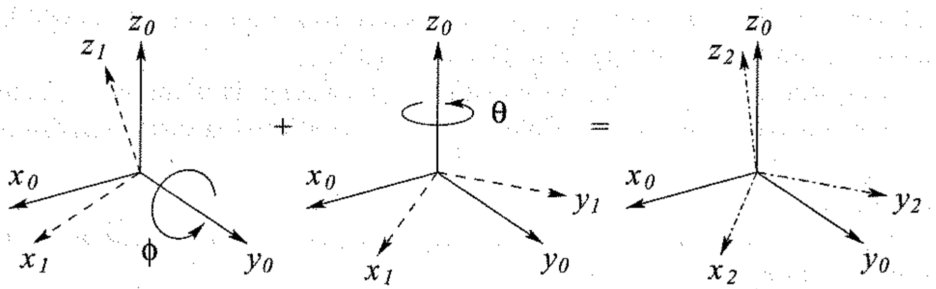

Figure 2.9: Composition of rotations about fixed axes.

Referring to Figure 2.9, suppose that a rotation matrix R represents a rotation of angle ϕ about y_0 followed by a rotation of angle θ about the *fixed z_0. The second rotation about the fixed axis is given by $R_{y,-\phi}R_{z,\theta}R_{y,\phi}$, which is the basic rotation about the z-axis expressed relative to the frame $o_1x_1y_1z_1$ using a similarity transformation. Therefore, the composition rule for rotational transformations gives us*

$$R = R_{y,\phi}\left[R_{y,-\phi}R_{z,\theta}R_{y,\phi}\right] = R_{z,\theta}R_{y,\phi} \tag{2.20}$$

It is not necessary to remember the above derivation, only to note by comparing Equation (2.20) with Equation (2.17) that we obtain the same basic rotation matrices, but in the reverse order.

◇

2.4.3 Rules for Composition of Rotational Transformations

We can summarize the rule of composition of rotational transformations by the following recipe. Given a fixed frame $o_0x_0y_0z_0$ and a current frame $o_1x_1y_1z_1$, together with rotation matrix R_1^0 relating them, if a third frame $o_2x_2y_2z_2$ is obtained by a rotation R performed relative to the **current frame** then **postmultiply** R_1^0 by $R = R_2^1$ to obtain

$$R_2^0 \quad = \quad R_1^0R_2^1 \tag{2.21}$$

If the second rotation is to be performed relative to the **fixed frame** then it is both confusing and inappropriate to use the notation R_2^1 to represent this rotation. Therefore, if we represent the rotation by R, we **premultiply** R_1^0 by R to obtain

$$R_2^0 \quad = \quad RR_1^0 \tag{2.22}$$

In each case R_2^0 represents the transformation between the frames $o_0x_0y_0z_0$ and $o_2x_2y_2z_2$. The frame $o_2x_2y_2z_2$ that results from Equation (2.21) will be different from that resulting from Equation (2.22).

Using the above rule for composition of rotations, it is an easy matter to determine the result of multiple sequential rotational transformations.

Example 2.8

Suppose R is defined by the following sequence of basic rotations in the order specified:

1. *A rotation of θ about the current x-axis*

2. *A rotation of ϕ about the current z-axis*

3. *A rotation of α about the fixed z-axis*

4. A rotation of β about the current y-axis

5. A rotation of δ about the fixed x-axis

In order to determine the cumulative effect of these rotations we simply begin with the first rotation $R_{x,\theta}$ and pre- or postmultiply as the case may be to obtain

$$R = R_{x,\delta} R_{z,\alpha} R_{x,\theta} R_{z,\phi} R_{y,\beta} \tag{2.23}$$

◇

2.5 PARAMETERIZATIONS OF ROTATIONS

The nine elements r_{ij} in a general rotational transformation $R \in SO(3)$ are not independent quantities. Indeed a rigid body possesses at most three rotational degrees of freedom, and thus at most three quantities are required to specify its orientation. This can be easily seen by examining the constraints that govern the matrices in $SO(3)$:

$$\sum_i r_{ij}^2 = 1, \quad j \in \{1,2,3\} \tag{2.24}$$

$$r_{1i}r_{1j} + r_{2i}r_{2j} + r_{3i}r_{3j} = 0, \quad i \neq j \tag{2.25}$$

Equation (2.24) follows from the fact that the columns of a rotation matrix are unit vectors, and Equation (2.25) follows from the fact that columns of a rotation matrix are mutually orthogonal. Together, these constraints define six independent equations with nine unknowns, which implies that there are three free variables.

In this section we derive three ways in which an arbitrary rotation can be represented using only three independent quantities: the **Euler-angle** representation, the **roll-pitch-yaw** representation, and the **axis/angle** representation.

2.5.1 Euler Angles

A common method of specifying a rotation matrix in terms of three independent quantities is to use the so-called **Euler angles**. Consider the fixed coordinate frame $o_0x_0y_0z_0$ and the rotated frame $o_1x_1y_1z_1$ shown in Figure 2.10. We can specify the orientation of the frame $o_1x_1y_1z_1$ relative to the frame $o_0x_0y_0z_0$ by three angles (ϕ, θ, ψ), known as Euler angles, and obtained by three successive rotations as follows. First rotate about the

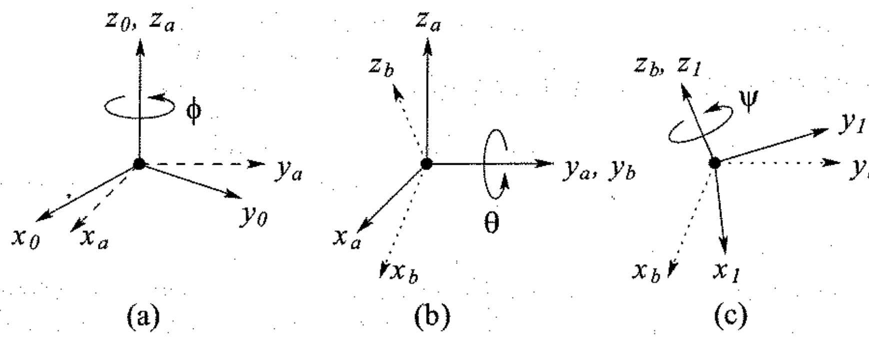

Figure 2.10: Euler angle representation.

z-axis by the angle ϕ. Next rotate about the current y-axis by the angle θ. Finally rotate about the current z-axis by the angle ψ. In Figure 2.10, frame $o_a x_a y_a z_a$ represents the new coordinate frame after the rotation by ϕ, frame $o_b x_b y_b z_b$ represents the new coordinate frame after the rotation by θ, and frame $o_1 x_1 y_1 z_1$ represents the final frame, after the rotation by ψ. Frames $o_a x_a y_a z_a$ and $o_b x_b y_b z_b$ are shown in the figure only to help visualize the rotations.

In terms of the basic rotation matrices the resulting rotational transformation can be generated as the product

$$\begin{aligned}
R_{ZYZ} &= R_{z,\phi} R_{y,\theta} R_{z,\psi} \\
&= \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix} \qquad (2.26)
\end{aligned}$$

The matrix R_{ZYZ} in Equation (2.26) is called the ZYZ-**Euler angle transformation**.

The more important and more difficult problem is to determine for a particular $R = (r_{ij})$ the set of Euler angles ϕ, θ, and ψ, that satisfy

$$R = \begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix} \qquad (2.27)$$

for a matrix $R \in SO(3)$. This problem will be important later when we address the inverse kinematics problem for manipulators in Section 3.3.

To find a solution for this problem we break it down into two cases. First, suppose that not both of r_{13}, r_{23} are zero. Then from Equation (2.26) we deduce that $s_\theta \neq 0$, and hence that not both of r_{31}, r_{32} are zero. If not both r_{13} and r_{23} are zero, then $r_{33} \neq \pm 1$, and we have $c_\theta = r_{33}$, $s_\theta = \pm\sqrt{1 - r_{33}^2}$ so

$$\theta = \text{Atan2}\left(r_{33}, \sqrt{1 - r_{33}^2}\right) \tag{2.28}$$

or

$$\theta = \text{Atan2}\left(r_{33}, -\sqrt{1 - r_{33}^2}\right) \tag{2.29}$$

where the function Atan2 is the **two-argument arctangent function** defined in Appendix A.

If we choose the value for θ given by Equation (2.28), then $s_\theta > 0$, and

$$\phi = \text{Atan2}(r_{13}, r_{23}) \tag{2.30}$$

$$\psi = \text{Atan2}(-r_{31}, r_{32}) \tag{2.31}$$

If we choose the value for θ given by Equation (2.29), then $s_\theta < 0$, and

$$\phi = \text{Atan2}(-r_{13}, -r_{23}) \tag{2.32}$$

$$\psi = \text{Atan2}(r_{31}, -r_{32}) \tag{2.33}$$

Thus, there are two solutions depending on the sign chosen for θ.

If $r_{13} = r_{23} = 0$, then the fact that R is orthogonal implies that $r_{33} = \pm 1$, and that $r_{31} = r_{32} = 0$. Thus, R has the form

$$R = \begin{bmatrix} r_{11} & r_{12} & 0 \\ r_{21} & r_{22} & 0 \\ 0 & 0 & \pm 1 \end{bmatrix} \tag{2.34}$$

If $r_{33} = 1$, then $c_\theta = 1$ and $s_\theta = 0$, so that $\theta = 0$. In this case, Equation (2.26) becomes

$$\begin{bmatrix} c_\phi c_\psi - s_\phi s_\psi & -c_\phi s_\psi - s_\phi c_\psi & 0 \\ s_\phi c_\psi + c_\phi s_\psi & -s_\phi s_\psi + c_\phi c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_{\phi+\psi} & -s_{\phi+\psi} & 0 \\ s_{\phi+\psi} & c_{\phi+\psi} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, the sum $\phi + \psi$ can be determined as

$$\phi + \psi = \text{Atan2}(r_{11}, r_{21}) = \text{Atan2}(r_{11}, -r_{12}) \tag{2.35}$$

Since only the sum $\phi+\psi$ can be determined in this case, there are infinitely many solutions. In this case, we may take $\phi = 0$ by convention. If $r_{33} = -1$, then $c_\theta = -1$ and $s_\theta = 0$, so that $\theta = \pi$. In this case Equation (2.26) becomes

$$\begin{bmatrix} -c_{\phi-\psi} & -s_{\phi-\psi} & 0 \\ s_{\phi-\psi} & c_{\phi-\psi} & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & 0 \\ r_{21} & r_{22} & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{2.36}$$

The solution is thus

$$\phi - \psi = \text{Atan2}(-r_{11}, -r_{12}) \tag{2.37}$$

As before there are infinitely many solutions.

2.5.2 Roll, Pitch, Yaw Angles

A rotation matrix R can also be described as a product of successive rotations about the principal coordinate axes x_0, y_0, and z_0 taken in a specific order. These rotations define the **roll**, **pitch**, and **yaw** angles, which we shall also denote ϕ, θ, ψ, and which are shown in Figure 2.11.

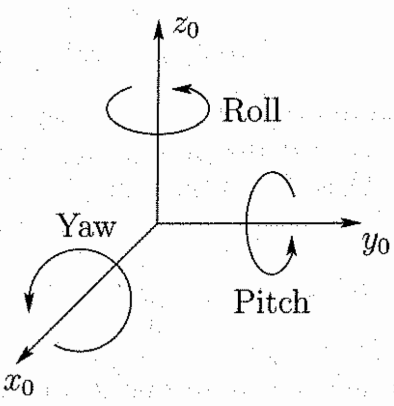

Figure 2.11: Roll, pitch, and yaw angles.

We specify the order of rotation as $x-y-z$, in other words, first a yaw about x_0 through an angle ψ, then pitch about the y_0 by an angle θ, and finally roll about the z_0 by an angle ϕ.[4] Since the successive rotations are

[4]It should be noted that other conventions exist for naming the roll, pitch, and yaw angles.

relative to the fixed frame, the resulting transformation matrix is given by

$$
\begin{aligned}
R &= R_{z,\phi}R_{y,\theta}R_{x,\psi} \\
&= \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\psi & -s_\psi \\ 0 & s_\psi & c_\psi \end{bmatrix} \\
&= \begin{bmatrix} c_\phi c_\theta & -s_\phi c_\psi + c_\phi s_\theta s_\psi & s_\phi s_\psi + c_\phi s_\theta c_\psi \\ s_\phi c_\theta & c_\phi c_\psi + s_\phi s_\theta s_\psi & -c_\phi s_\psi + s_\phi s_\theta c_\psi \\ -s_\theta & c_\theta s_\psi & c_\theta c_\psi \end{bmatrix} \qquad (2.38)
\end{aligned}
$$

Of course, instead of yaw-pitch-roll relative to the fixed frames we could also interpret the above transformation as roll-pitch-yaw, in that order, each taken with respect to the current frame. The end result is the same matrix as in Equation (2.38).

The three angles ϕ, θ, and ψ can be obtained for a given rotation matrix using a method that is similar to that used to derive the Euler angles above.

2.5.3 Axis/Angle Representation

Rotations are not always performed about the principal coordinate axes. We are often interested in a rotation about an arbitrary axis in space. This provides both a convenient way to describe rotations, and an alternative parameterization for rotation matrices. Let $k = [k_x, k_y, k_z]^T$, expressed in the frame $o_0x_0y_0z_0$, be a unit vector defining an axis. We wish to derive the rotation matrix $R_{k,\theta}$ representing a rotation of θ about this axis.

There are several ways in which the matrix $R_{k,\theta}$ can be derived. One approach is to note that the rotational transformation $R = R_{z,\alpha}R_{y,\beta}$ will bring the world z-axis into alignment with the vector k. Therefore, a rotation about the axis k can be computed using a similarity transformation as

$$
\begin{aligned}
R_{k,\theta} &= RR_{z,\theta}R^{-1} &\quad (2.39)\\
&= R_{z,\alpha}R_{y,\beta}R_{z,\theta}R_{y,-\beta}R_{z,-\alpha} &\quad (2.40)
\end{aligned}
$$

From Figure 2.12 we see that

$$
\sin\alpha = \frac{k_y}{\sqrt{k_x^2 + k_y^2}} \qquad \cos\alpha = \frac{k_x}{\sqrt{k_x^2 + k_y^2}} \qquad (2.41)
$$

$$
\sin\beta = \sqrt{k_x^2 + k_y^2} \qquad \cos\beta = k_z \qquad (2.42)
$$

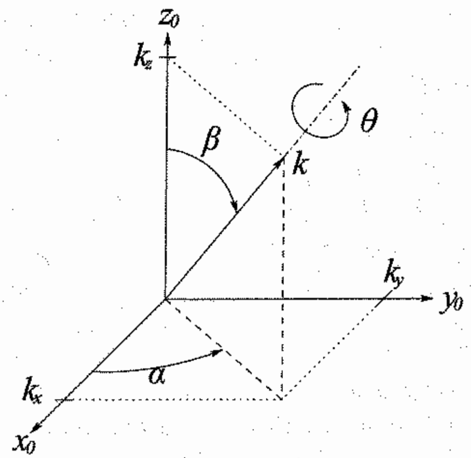

Figure 2.12: Rotation about an arbitrary axis.

Note that the final two equations follow from the fact that k is a unit vector. Substituting Equations (2.41) and (2.42) into Equation (2.40), we obtain after some lengthy calculation (Problem 2-17)

$$R_{k,\theta} = \left[\begin{array}{c|c|c} k_x^2 v_\theta + c_\theta & k_x k_y v_\theta - k_z s_\theta & k_x k_z v_\theta + k_y s_\theta \\ k_x k_y v_\theta + k_z s_\theta & k_y^2 v_\theta + c_\theta & k_y k_z v_\theta - k_x s_\theta \\ k_x k_z v_\theta - k_y s_\theta & k_y k_z v_\theta + k_x s_\theta & k_z^2 v_\theta + c_\theta \end{array}\right] \tag{2.43}$$

where $v_\theta = \text{vers}\,\theta = 1 - c_\theta$.

In fact, any rotation matrix $R \in S0(3)$ can be represented by a single rotation about a suitable axis in space by a suitable angle,

$$R = R_{k,\theta} \tag{2.44}$$

where k is a unit vector defining the axis of rotation, and θ is the angle of rotation about k. The pair (k, θ) is called the **axis/angle representation** of R. Given an arbitrary rotation matrix R with components r_{ij}, the equivalent angle θ and equivalent axis k are given by the expressions

$$\theta = \cos^{-1}\left(\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right)$$

and

$$k = \frac{1}{2\sin\theta}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix} \tag{2.45}$$

These equations can be obtained by direct manipulation of the entries of the matrix given in Equation (2.43). The axis/angle representation is not unique since a rotation of $-\theta$ about $-k$ is the same as a rotation of θ about k, that is,

$$R_{k,\theta} = R_{-k,-\theta} \tag{2.46}$$

If $\theta = 0$ then R is the identity matrix and the axis of rotation is undefined.

Example 2.9

Suppose R is generated by a rotation of 90° about z_0 followed by a rotation of 30° about y_0 followed by a rotation of 60° about x_0. Then

$$\begin{aligned} R &= R_{x,60}R_{y,30}R_{z,90} \\ &= \begin{bmatrix} 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{4} & -\frac{3}{4} \\ \frac{\sqrt{3}}{2} & \frac{1}{4} & \frac{\sqrt{3}}{4} \end{bmatrix} \end{aligned} \tag{2.47}$$

We see that $Tr(R) = 0$ and hence the equivalent angle is given by Equation (2.45) as

$$\theta = \cos^{-1}\left(-\frac{1}{2}\right) = 120^\circ \tag{2.48}$$

The equivalent axis is given from Equation (2.45) as

$$k = \left[\frac{1}{\sqrt{3}}, \frac{1}{2\sqrt{3}} - \frac{1}{2}, \frac{1}{2\sqrt{3}} + \frac{1}{2}\right]^T \tag{2.49}$$

◇

The above axis/angle representation characterizes a given rotation by four quantities, namely the three components of the equivalent axis k and the equivalent angle θ. However, since the equivalent axis k is given as a unit vector only two of its components are independent. The third is constrained by the condition that k is of unit length. Therefore, only three independent quantities are required in this representation of a rotation R. We can represent the equivalent axis/angle by a single vector r as

$$r = [r_x, r_y, r_z]^T = [\theta k_x, \theta k_y, \theta k_z]^T \tag{2.50}$$

Note, since k is a unit vector, that the length of the vector r is the equivalent angle θ and the direction of r is the equivalent axis k.

One should be careful to note that the representation in Equation (2.50) does not mean that two axis/angle representations may be combined using standard rules of vector algebra, as doing so would imply that rotations commute which, as we have seen, is not true in general.

2.6 RIGID MOTIONS

We have now seen how to represent both positions and orientations. We combine these two concepts in this section to define a **rigid motion** and, in the next section, we derive an efficient matrix representation for rigid motions using the notion of homogeneous transformation.

Definition 2.1 *A rigid motion is an ordered pair* (d, R) *where* $d \in \mathbb{R}^3$ *and* $R \in SO(3)$*. The group of all rigid motions is known as the* **Special Euclidean Group** *and is denoted by* $SE(3)$*. We see then that* $SE(3) = \mathbb{R}^3 \times SO(3)$*.*

A rigid motion is a pure translation together with a pure rotation.[5] Let R_1^0 be the rotation matrix that specifies the orientation of frame $o_1x_1y_1z_1$ with respect to $o_0x_0y_0z_0$, and d be the vector from the origin of frame $o_0x_0y_0z_0$ to the origin of frame $o_1x_1y_1z_1$. Suppose the point p is rigidly attached to coordinate frame $o_1x_1y_1z_1$, with local coordinates p^1. We can express the coordinates of p with respect to frame $o_0x_0y_0z_0$ using

$$p^0 = R_1^0 p^1 + d^0 \tag{2.51}$$

Now consider three coordinate frames $o_0x_0y_0z_0$, $o_1x_1y_1z_1$, and $o_2x_2y_2z_2$. Let d_1 be the vector from the origin of $o_0x_0y_0z_0$ to the origin of $o_1x_1y_1z_1$ and d_2 be the vector from the origin of $o_1x_1y_1z_1$ to the origin of $o_2x_2y_2z_2$. If the point p is attached to frame $o_2x_2y_2z_2$ with local coordinates p^2, we can compute its coordinates relative to frame $o_0x_0y_0z_0$ using

$$p^1 = R_2^1 p^2 + d_2^1 \tag{2.52}$$

and

$$p^0 = R_1^0 p^1 + d_1^0 \tag{2.53}$$

The composition of these two equations defines a third rigid motion, which we can describe by substituting the expression for p^1 from Equation (2.52) into Equation (2.53)

$$p^0 = R_1^0 R_2^1 p^2 + R_1^0 d_2^1 + d_1^0 \tag{2.54}$$

[5]The definition of rigid motion is sometimes broadened to include **reflections**, which correspond to $\det R = -1$. We will always assume in this text that $\det R = +1$ so that $R \in SO(3)$.

Since the relationship between p^0 and p^2 is also a rigid motion, we can equally describe it as

$$p^0 = R_2^0 p^2 + d_2^0 \tag{2.55}$$

Comparing Equations (2.54) and (2.55) we have the relationships

$$R_2^0 = R_1^0 R_2^1 \tag{2.56}$$
$$d_2^0 = d_1^0 + R_1^0 d_2^1 \tag{2.57}$$

Equation (2.56) shows that the orientation transformations can simply be multiplied together and Equation (2.57) shows that the vector from the origin o_0 to the origin o_2 has coordinates given by the sum of d_1^0 (the vector from o_0 to o_1 expressed with respect to $o_0x_0y_0z_0$) and $R_1^0 d_2^1$ (the vector from o_1 to o_2, expressed in the orientation of the coordinate frame $o_0x_0y_0z_0$).

2.7 HOMOGENEOUS TRANSFORMATIONS

One can easily see that the calculation leading to Equation (2.54) would quickly become intractable if a long sequence of rigid motions were considered. In this section we show how rigid motions can be represented in matrix form so that composition of rigid motions can be reduced to matrix multiplication as was the case for composition of rotations.

In fact, a comparison of Equations (2.56) and (2.57) with the matrix identity

$$\begin{bmatrix} R_1^0 & d_1^0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_2^1 & d_1^2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_1^0 R_2^1 & R_1^0 d_1^2 + d_1^0 \\ 0 & 1 \end{bmatrix} \tag{2.58}$$

where 0 denotes the row vector $(0, 0, 0)$, shows that the rigid motions can be represented by the set of matrices of the form

$$H = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix}, \quad R \in SO(3), \quad d \in \mathbb{R}^3 \tag{2.59}$$

Transformation matrices of the form given in Equation (2.59) are called **homogeneous transformations**. A homogeneous transformation is therefore nothing more than a matrix representation of a rigid motion and we will use $SE(3)$ interchangeably to represent both the set of rigid motions and the set of all 4×4 matrices H of the form given in Equation (2.59).

Using the fact that R is orthogonal it is an easy exercise to show that the inverse transformation H^{-1} is given by

$$H^{-1} = \begin{bmatrix} R^T & -R^T d \\ 0 & 1 \end{bmatrix} \tag{2.60}$$

In order to represent the transformation given in Equation (2.51) by a matrix multiplication, we must augment the vectors p^0 and p^1 by the addition of a fourth component of 1 as follows,

$$P^0 = \begin{bmatrix} p^0 \\ 1 \end{bmatrix} \tag{2.61}$$

$$P^1 = \begin{bmatrix} p^1 \\ 1 \end{bmatrix} \tag{2.62}$$

The vectors P^0 and P^1 are known as **homogeneous representations** of the vectors p^0 and p^1, respectively. It can now be seen directly that the transformation given in Equation (2.51) is equivalent to the (homogeneous) matrix equation

$$P^0 = H_1^0 P^1 \tag{2.63}$$

A set of **basic homogeneous transformations** generating $SE(3)$ is given by

$$\text{Trans}_{x,a} = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \text{Rot}_{x,\alpha} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_\alpha & -s_\alpha & 0 \\ 0 & s_\alpha & c_\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.64}$$

$$\text{Trans}_{y,b} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \text{Rot}_{y,\beta} = \begin{bmatrix} c_\beta & 0 & s_\beta & 0 \\ 0 & 1 & 0 & 0 \\ -s_\beta & 0 & c_\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.65}$$

$$\text{Trans}_{z,c} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \text{Rot}_{z,\gamma} = \begin{bmatrix} c_\gamma & -s_\gamma & 0 & 0 \\ s_\gamma & c_\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.66}$$

for translation and rotation about the x, y, z-axes, respectively.

The most general homogeneous transformation that we will consider may be written now as

$$H_1^0 = \begin{bmatrix} n_x & s_x & a_x & d_x \\ n_y & s_y & a_y & d_y \\ n_z & s_x & a_z & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} n & s & a & d \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.67}$$

In the above equation $n = [n_x, n_y, n_z]^T$ is a vector representing the direction of x_1 in the $o_0x_0y_0z_0$ frame, $s = [s_x, s_y, s_z]^T$ represents the direction of y_1, and $a = [a_x, a_y, a_z]^T$ represents the direction of z_1. The vector $d = [d_x, d_y, d_z]^T$ represents the vector from the origin o_0 to the origin o_1 expressed in the frame $o_0x_0y_0z_0$. The rationale behind the choice of letters n, s, and a is explained in Chapter 3.

The same interpretation regarding composition and ordering of transformations holds for 4×4 homogeneous transformations as for 3×3 rotations. Given a homogeneous transformation H_1^0 relating two frames, if a second rigid motion, represented by $H \in SE(3)$ is performed relative to the current frame, then

$$H_2^0 = H_1^0 H$$

whereas if the second rigid motion is performed relative to the fixed frame, then

$$H_2^0 = H H_1^0$$

Example 2.10

The homogeneous transformation matrix H that represents a rotation by angle α about the current x-axis followed by a translation of b units along the current x-axis, followed by a translation of d units along the current z-axis, followed by a rotation by angle θ about the current z-axis, is given by

$$\begin{aligned} H &= Rot_{x,\alpha} Trans_{x,b} Trans_{z,d} Rot_{z,\theta} \\ &= \begin{bmatrix} c_\theta & -s_\theta & 0 & b \\ c_\alpha s_\theta & c_\alpha c_\theta & -s_\alpha & -ds_\alpha \\ s_\alpha s_\theta & s_\alpha c_\theta & c_\alpha & dc_\alpha \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

◇

2.8 SUMMARY

In this chapter, we have seen how matrices in $SE(n)$ can be used to represent the relative position and orientation of two coordinate frames for $n = 2, 3$. We have adopted a notional convention in which a superscript is used to indicate a reference frame. Thus, the notation p^0 represents the coordinates of the point p relative to frame 0.

The relative orientation of two coordinate frames can be specified by a rotation matrix, $R \in SO(n)$, with $n = 2, 3$. In two dimensions, the orientation of frame 1 with respect to frame 0 is given by

$$R_1^0 = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

in which θ is the angle between the two coordinate frames. In the three-dimensional case, the rotation matrix is given by

$$R_1^0 = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 & z_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 & z_1 \cdot y_0 \\ x_1 \cdot z_0 & y_1 \cdot z_0 & z_1 \cdot z_0 \end{bmatrix}$$

In each case, the columns of the rotation matrix are obtained by projecting an axis of the target frame (in this case, frame 1) onto the coordinate axes of the reference frame (in this case, frame 0).

The set of $n \times n$ rotation matrices is known as the special orthogonal group of order n, and is denoted by $SO(n)$. An important property of these matrices is that $R^{-1} = R^T$ for any $R \in SO(n)$.

Rotation matrices can be used to perform coordinate transformations between frames that differ only in orientation. We derived rules for the composition of rotational transformations as

$$R_2^0 = R_1^0 R$$

for the case where the second transformation, R, is performed relative to the current frame and

$$R_2^0 = R R_1^0$$

for the case where the second transformation, R, is performed relative to the fixed frame.

In the three-dimensional case, a rotation matrix can be parameterized using three angles. A common convention is to use the Euler angles (ϕ, θ, ψ), which correspond to successive rotations about the z, y, and z axes. The corresponding rotation matrix is given by

$$R(\phi, \theta, \psi) = R_{z,\phi} R_{y,\theta} R_{z,\psi}$$

Roll, pitch, and yaw angles are similar, except that the successive rotations are performed with respect to the fixed, world frame instead of being performed with respect to the current frame.

Homogeneous transformations combine rotation and translation. In the three-dimensional case, a homogeneous transformation has the form

$$H = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix}, R \in SO(3),\ d \in \mathbb{R}^3$$

The set of all such matrices comprises the set $SE(3)$, and these matrices can be used to perform coordinate transformations, analogous to rotational transformations using rotation matrices.

Homogeneous transformation matrices can be used to perform coordinate transformations between frames that differ in orientation and translation. We derived rules for the composition of rotational transformations as

$$H_2^0 = H_1^0 H$$

for the case where the second transformation, H, is performed relative to the current frame and

$$H_2^0 = H H_1^0$$

for the case where the second transformation, H, is performed relative to the fixed frame.

PROBLEMS

2-1 Using the fact that $v_1 \cdot v_2 = v_1^T v_2$, show that the dot product of two free vectors does not depend on the choice of frames in which their coordinates are defined.

2-2 Show that the length of a free vector is not changed by rotation, that is, that $\|v\| = \|Rv\|$.

2-3 Show that the distance between points is not changed by rotation, that is, $\|p_1 - p_2\| = \|Rp_1 - Rp_2\|$.

2-4 If a matrix R satisfies $R^T R = I$, show that the column vectors of R are of unit length and mutually perpendicular.

2-5 If a matrix R satisfies $R^T R = I$, then
a) Show that $\det R = \pm 1$
b) Show that $\det R = +1$ if we restrict ourselves to right-handed coordinate frames.

2-6 Verify Equations (2.3)–(2.5).

2-7 A **group** is a set X together with an operation $*$ defined on that set such that

- $x_1 * x_2 \in X$ for all $x_1, x_2 \in X$
- $(x_1 * x_2) * x_3 = x_1 * (x_2 * x_3)$
- There exists an element $I \in X$ such that $I * x = x * I = x$ for all $x \in X$
- For every $x \in X$, there exists some element $y \in X$ such that $x * y = y * x = I$

Show that SO(n) with the operation of matrix multiplication is a group.

2-8 Derive Equations (2.6) and (2.7).

2-9 Suppose A is a 2×2 rotation matrix. In other words $A^T A = I$ and $\det A = 1$. Show that there exists a unique θ such that A is of the form

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

2-10 Consider the following sequence of rotations:

1. Rotate by ϕ about the world x-axis.
2. Rotate by θ about the current z-axis.
3. Rotate by ψ about the world y-axis.

Write the matrix product that will give the resulting rotation matrix (do not perform the matrix multiplication).

2-11 Consider the following sequence of rotations:

1. Rotate by ϕ about the world x-axis.
2. Rotate by θ about the world z-axis.
3. Rotate by ψ about the current x-axis.

Write the matrix product that will give the resulting rotation matrix (do not perform the matrix multiplication).

2-12 Consider the following sequence of rotations:

1. Rotate by ϕ about the world x-axis.

2. Rotate by θ about the current z-axis.
3. Rotate by ψ about the current x-axis.
4. Rotate by α about the world z-axis.

Write the matrix product that will give the resulting rotation matrix (do not perform the matrix multiplication).

2-13 Consider the following sequence of rotations:

1. Rotate by ϕ about the world x-axis.
2. Rotate by θ about the world z-axis.
3. Rotate by ψ about the current x-axis.
4. Rotate by α about the world z-axis.

Write the matrix product that will give the resulting rotation matrix (do not perform the matrix multiplication).

2-14 If the coordinate frame $o_1x_1y_1z_1$ is obtained from the coordinate frame $o_0x_0y_0z_0$ by a rotation of $\frac{\pi}{2}$ about the x-axis followed by a rotation of $\frac{\pi}{2}$ about the fixed y-axis, find the rotation matrix R representing the composite transformation. Sketch the initial and final frames.

2-15 Suppose that three coordinate frames $o_1x_1y_1z_1$, $o_2x_2y_2z_2$, and $o_3x_3y_3z_3$ are given, and suppose

$$R_2^1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}, \quad R_3^1 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Find the matrix R_3^2.

2-16 Derive equations for the roll, pitch, and yaw angles corresponding to the rotation matrix $R = (r_{ij})$.

2-17 Verify Equation (2.43).

2-18 Verify Equation (2.45).

2-19 If R is a rotation matrix show that $+1$ is an eigenvalue of R. Let k be a unit eigenvector corresponding to the eigenvalue $+1$. Give a physical interpretation of k.

2-20 Let $k = \frac{1}{\sqrt{3}}[1, 1, 1]^T$, $\theta = 90°$. Find $R_{k,\theta}$.

2-21 Show by direct calculation that $R_{k,\theta}$ given by Equation (2.43) is equal to R given by Equation (2.47) if θ and k are given by Equations (2.48) and (2.49), respectively.

2-22 Compute the rotation matrix given by the product

$$R_{x,\theta} R_{y,\phi} R_{z,\pi} R_{y,-\phi} R_{x,-\theta}$$

2-23 Suppose R represents a rotation of 90° about y_0 followed by a rotation of 45° about z_1. Find the equivalent axis/angle to represent R. Sketch the initial and final frames and the equivalent axis vector k.

2-24 Find the rotation matrix corresponding to the Euler angles $\phi = \frac{\pi}{2}$, $\theta = 0$, and $\psi = \frac{\pi}{4}$. What is the direction of the x_1 axis relative to the base frame?

2-25 Section 2.5.1 described only the Z-Y-Z Euler angles. List all possible sets of Euler angles. Is it possible to have Z-Z-Y Euler angles? Why or why not?

2-26 Unit magnitude complex numbers $a + ib$ with $a^2 + b^2 = 1$ can be used to represent orientation in the plane. In particular, for the complex number $a + ib$, we can define the angle $\theta = \text{Atan2}(a, b)$. Show that multiplication of two complex numbers corresponds to addition of the corresponding angles.

2-27 Show that complex numbers together with the operation of complex multiplication define a group. What is the identity for the group? What is the inverse for $a + ib$?

2-28 Complex numbers can be generalized by defining three independent square roots for -1 that obey the multiplication rules

$$\begin{aligned}
-1 &= i^2 = j^2 = k^2, \\
i &= jk = -kj, \\
j &= ki = -ik, \\
k &= ij = -ji
\end{aligned}$$

Using these, we define a **quaternion** by $Q = q_0 + iq_1 + jq_2 + kq_3$, which is typically represented by the 4-tuple (q_0, q_1, q_2, q_3). A rotation by θ about the unit vector $n = [n_x, n_y, n_z]^T$ can be represented by the unit quaternion $Q = \left(\cos\frac{\theta}{2}, n_x \sin\frac{\theta}{2}, n_y \sin\frac{\theta}{2}, n_z \sin\frac{\theta}{2}\right)$. Show that such a quaternion has unit norm, that is, $q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$.

2-29 Using $Q = \left(\cos\frac{\theta}{2}, n_x \sin\frac{\theta}{2}, n_y \sin\frac{\theta}{2}, n_z \sin\frac{\theta}{2}\right)$, and the results from Section 2.5.3, determine the rotation matrix R that corresponds to the rotation represented by the quaternion (q_0, q_1, q_2, q_3).

2-30 Determine the quaternion Q that represents the same rotation as given by the rotation matrix R.

2-31 The quaternion $Q = (q_0, q_1, q_2, q_3)$ can be thought of as having a scalar component q_0 and a vector component $q = [q_1, q_2, q_3]^T$. Show that the product of two quaternions, $Z = XY$ is given by

$$\begin{aligned} z_0 &= x_0 y_0 - x^T y \\ z &= x_0 y + y_0 x + x \times y, \end{aligned}$$

Hint: Perform the multiplication $(x_0 + ix_1 + jx_2 + kx_3)(y_0 + iy_1 + jy_2 + ky_3)$ and simplify the result.

2-32 Show that $Q_I = (1, 0, 0, 0)$ is the identity element for unit quaternion multiplication, that is, $QQ_I = Q_I Q = Q$ for any unit quaternion Q.

2-33 The conjugate Q^* of the quaternion Q is defined as

$$Q^* = (q_0, -q_1, -q_2, -q_3)$$

Show that Q^* is the inverse of Q, that is, $Q^*Q = QQ^* = (1, 0, 0, 0)$.

2-34 Let v be a vector whose coordinates are given by $[v_x, v_y, v_z]^T$. If the quaternion Q represents a rotation, show that the new, rotated coordinates of v are given by $Q(0, v_x, v_y, v_z)Q^*$, in which $(0, v_x, v_y, v_z)$ is a quaternion with zero as its real component.

2-35 Let the point p be rigidly attached to the end effector coordinate frame with local coordinates (x, y, z). If Q specifies the orientation of the end effector frame with respect to the base frame, and T is the vector from the base frame to the origin of the end effector frame, show that the coordinates of p with respect to the base frame are given by

$$Q(0, x, y, z)Q^* + T \tag{2.68}$$

in which $(0, x, y, z)$ is a quaternion with zero as its real component.

2-36 Verify Equation (2.60).

2-37 Compute the homogeneous transformation representing a translation of 3 units along the x-axis followed by a rotation of $\frac{\pi}{2}$ about the current z-axis followed by a translation of 1 unit along the fixed y-axis. Sketch the frame. What are the coordinates of the origin o_1 with respect to the original frame in each case?

2-38 Consider the diagram of Figure 2.13. Find the homogeneous transformations H_1^0, H_2^0, H_2^1 representing the transformations among the three frames shown. Show that $H_2^0 = H_1^0, H_2^1$.

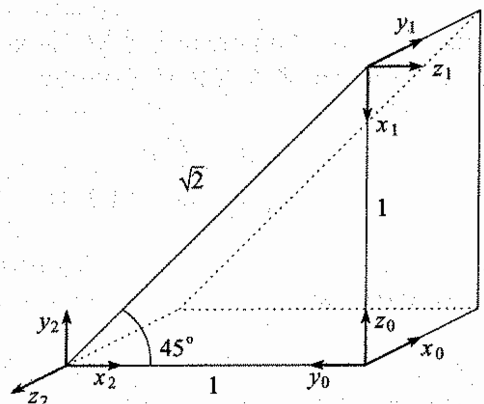

Figure 2.13: Diagram for Problem 2-38.

2-39 Consider the diagram of Figure 2.14. A robot is set up 1 meter from a table. The table top is 1 meter high and 1 meter square. A frame $o_1x_1y_1z_1$ is fixed to the edge of the table as shown. A cube measuring 20 cm on a side is placed in the center of the table with frame $o_2x_2y_2z_2$ established at the center of the cube as shown. A camera is situated directly above the center of the block 2 meters above the table top with frame $o_3x_3y_3z_3$ attached as shown. Find the homogeneous transformations relating each of these frames to the base frame $o_0x_0y_0z_0$. Find the homogeneous transformation relating the frame $o_2x_2y_2z_2$ to the camera frame $o_3x_3y_3z_3$.

2-40 In Problem 2-39, suppose that, after the camera is calibrated, it is rotated 90° about z_3. Recompute the above coordinate transformations.

2-41 If the block on the table is rotated 90° about z_2 and moved so that its center has coordinates $[0, .8, .1]^T$ relative to the frame $o_1x_1y_1z_1$, compute the homogeneous transformation relating the block frame to the camera frame; the block frame to the base frame.

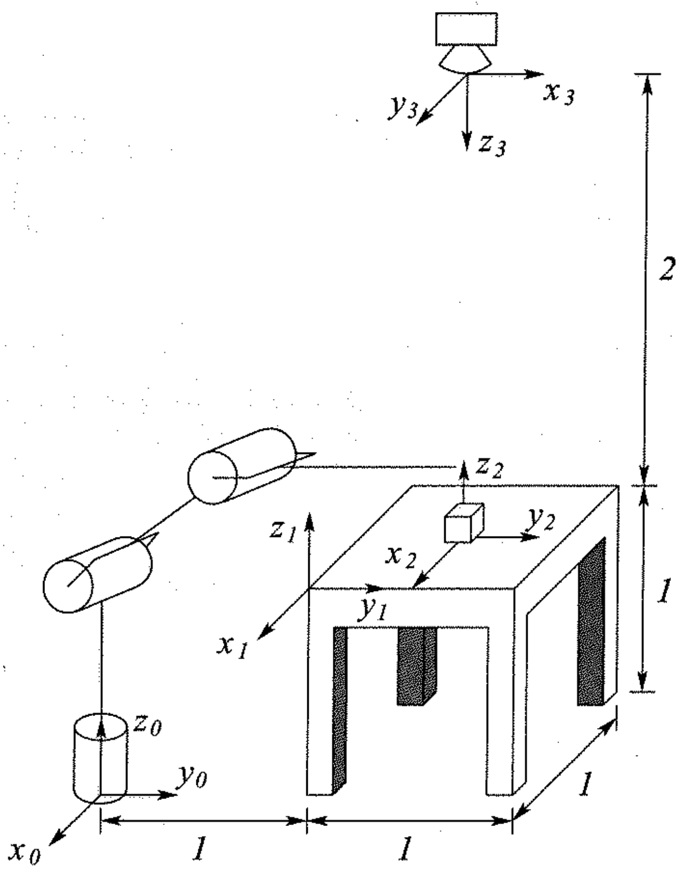

Figure 2.14: Diagram for Problem 2-39.

2-42 Consult an astronomy book to learn the basic details of the Earth's rotation about the sun and about its own axis. Define for the Earth a local coordinate frame whose z-axis is the Earth's axis of rotation. Define $t = 0$ to be the exact moment of the summer solstice, and the global reference frame to be coincident with the Earth's frame at time $t = 0$. Give an expression $R(t)$ for the rotation matrix that represents the instantaneous orientation of the earth at time t. Determine as a function of time the homogeneous transformation that specifies the Earth's frame with respect to the global reference frame.

2-43 In general, multiplication of homogeneous transformation matrices is not commutative. Consider the matrix product

$$H = \text{Rot}_{x,\alpha}\text{Trans}_{x,b}\text{Trans}_{z,d}\text{Rot}_{z,\theta}$$

Determine which pairs of the four matrices on the right hand side commute. Explain why these pairs commute. Find all permutations of these four matrices that yield the same homogeneous transformation matrix, H.

NOTES AND REFERENCES

Rigid body motions and the groups $SO(n)$ and $SE(n)$ are often addressed in mathematics books on the topic of linear algebra. Standard texts for this material include [8], [23], and [40]. These topics are also often covered in applied mathematics texts for physics and engineering, such as [108], [119], and [139]. In addition to these, a detailed treatment of rigid body motion developed with the aid of exponential coordinates and Lie groups is given in [93].