Assignment 6 in OptReg

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Task 1

a)

From Newtons second law $F = ma$, where $F = u$ and $m = 1$, we know that $a = u$. Writing this on state space form we have that:

- $x_1 = x$
- $x_2 = \dot{x} = v$
- $\dot{x}_1 = x_2$
- $\dot{x}_2 = \ddot{x} = a$

where x, v and a are position, velocity and acceleration, respectively. On continuous state space form this is:

- $\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$

b)

By exact descretization we have that the descrete system matrix and vector A and b are given by:

- $A = e^{A_c T}$ and $b = \left(\int_0^T e^{A_c \tau} d\tau \right) b_c$

where $e^{A_c T}$ is given by $e^{A_c T} = \sum_{k=0}^{\infty} A_c^k T^k$, where the first element in the series is defined as the identity matrix. We don't care about the higher terms and calculate the matrix exponential as:

- $e^{A_c T} = I + A_c T = I + A_c \cdot 0.5 = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix} = A$

and the b vector as:

- $b = \left(\int_0^T e^{A_c \tau} d\tau \right) b_c = \left(\int_0^T \begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix} \right) b_c = \begin{bmatrix} T & \frac{1}{2} T^2 \\ c & T \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} T^2 \\ T \end{bmatrix} = \begin{bmatrix} 0.125 \\ 0.5 \end{bmatrix}$

c)

For a finite-horizon LQR the Riccati equation is given by:

- $P_t = Q_t + A_t^T P_{t+1} (I + B_t R_t^{-1} B_t^T P_{t+1})^{-1} A_t, \quad \text{for } t = 0, \dots, N-1$

where $P_N = Q_N$. By using this, the LQR feedback gain can be found:

- $K_t = R_t^{-1} B_t^T P_{t+1} (I + B_t R_t^{-1} B_t^T P_{t+1})^{-1} A_t, \quad \text{for } t = 0, \dots, N-1$

using this we get the state feedback:

- $u_t = -K_t x_t$

A sketch of the controller is given below

d)

We know that if $N \to \infty$ that the system becomes an infinite horizon LQR with a corresponding Riccati equation and way of finding K:

- $P = Q + A^T P (I + BR^{-1} B^T P)^{-1} A$
- $K = R^{-1} B^T P (I + BR^{-1} B^T P)^{-1} A$

we therefore choose to omit the horizon in our implementation of **dlqr**.

```
% Initialising system
A = [1 0.5; 0 1];
b = [0.125; 0.5];

Q  = [2 0; 0 2];
R = 2;

% We divide by two because this hasn't been done
Q = Q/2;
R = R/2;

% Finding K and P
[K, P, CLP] = dlqr(A, b, Q, R);
P
```

```
P = 2×2
    4.0350    2.0616
    2.0616    4.1438
```

```
CLP
```

```
CLP = 2×1 complex
   0.6307 + 0.1628i
   0.6307 - 0.1628i
```

We see that we get the expected result. The closed-loop eigenvalues are $\lambda = 0.63 \pm 0.16i$ which is less than one and the closed-loop system is therefore stable.

e)

In general LQR is stable if the matrices (A, B) are stabilizable and (A, D) are detectable. Here $Q = D^T D$. A system is stabilizable if it is controllable and detectable if its observable.

Task 2

a)

We have seen the Riccati equation for infinite-horizon LQR earlier in this assignment, the scalar version of this is given by:

- $p = q + ap\dfrac{1}{1 + b\frac{1}{r}bp}a = q + \dfrac{a^2pr}{r + b^2p}$

for $q = 2$, $a = 3$, $b = 2$ and $r = 1$ we have:

- $p = 2 + \dfrac{9 \cdot p}{1 + 4 \cdot p} \Rightarrow p^2 - 4p - \dfrac{1}{2}$

```
roots([1 -4 -0.5])
```

```
ans = 2×1
    4.1213
   -0.1213
```

Since p must be positive definite we have that $p = 4.12$.

b)

The optimal feedback coefficient k is given by:

- $k = \dfrac{bpa}{r + pb^2} = \dfrac{2 \cdot 4.12 \cdot 3}{1 + 4.12 \cdot 4} \approx \sqrt{2}$

This gives and optimal feed back term $u = -\sqrt{2}\,x$.

c)

This is the same as for the finite time horizon case if the system is detectable and stabilizable, then the closed-loop solution will be asymptotically stable for a linear system.

Task 3

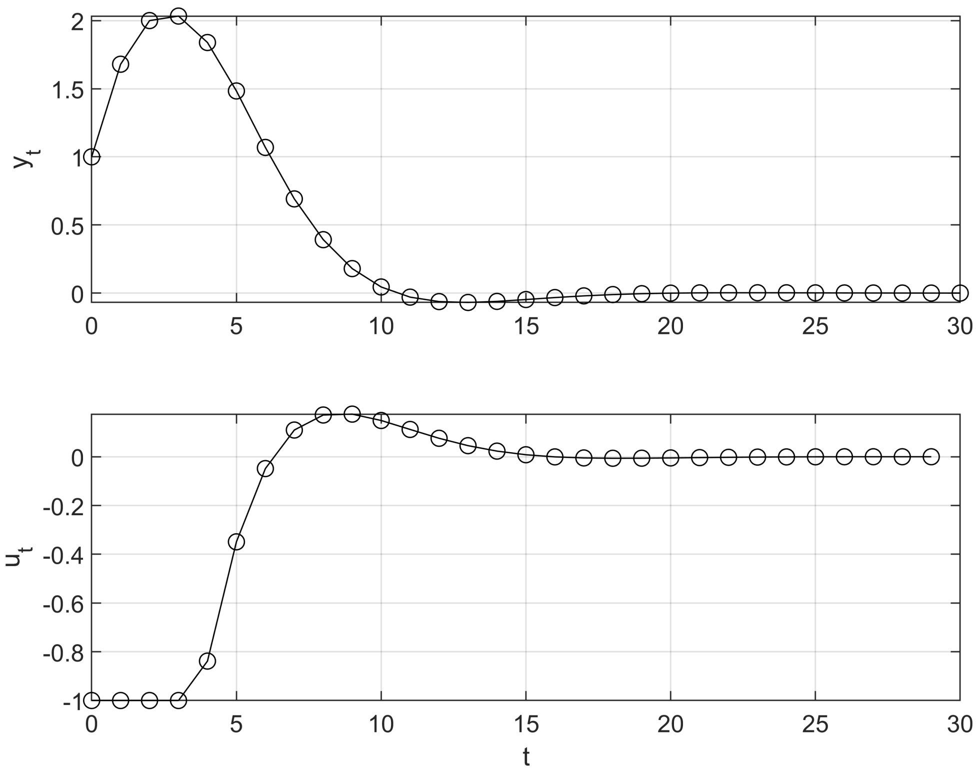
a)

Here I choose to use the code provided on Blackboard:

```
%% Assignment 5, Problem 1 f)
%  Solves the QP with quadprog, with lower and upper bounds on u.
%  Tor Aksel N. Heirung, April 2013.

% System matrices
A = [0     0      0    ;
     0     0      1    ;
     0.1  -0.79   1.78];
B = [1 0 0.1]';
C = [0 0 1];

x0 = [0 0 1]'; % Initial state

N = 30; % Length of time horizon
nx = size(A,2); % nx: number of states (equals the number of rows in A)
nu = size(B,2); % nu: number of controls (equals the number of rows in B)

% Cost function
I_N = eye(N);
Qt = 2*diag([0, 0, 1]);
Q = kron(I_N, Qt);
Rt = 2*1;
R = kron(I_N, Rt);
G = blkdiag(Q, R);

% Equality constraint
Aeq_c1 = eye(N*nx);                            % Component 1 of A_eq
Aeq_c2 = kron(diag(ones(N-1,1),-1), -A);    % Component 2 of A_eq
Aeq_c3 = kron(I_N, -B);                        % Component 3 of A_eq
Aeq = [Aeq_c1 + Aeq_c2, Aeq_c3];

beq = [A*x0; zeros((N-1)*nx,1)];

% Inequality constraints
x_lb = -Inf(N*nx,1);    % Lower bound on x
x_ub =  Inf(N*nx,1);    % Upper bound on x
u_lb = -ones(N*nu,1);   % Lower bound on u
u_ub =  ones(N*nu,1);   % Upper bound on u
lb = [x_lb; u_lb];      % Lower bound on z
ub = [x_ub; u_ub];      % Upper bound on z

% Solving the equality- and inequality-constrained QP with quadprog
opt = optimset('Display','notify', 'Diagnostics','off', 'LargeScale','off', 'Algorithm', 'inter
[z,fval,exitflag,output,lambda] = quadprog(G,[],[],[],Aeq,beq,lb,ub,[],opt);
```

```
Minimum found that satisfies the constraints.

Optimization completed because the objective function is non-decreasing in
feasible directions, to within the value of the optimality tolerance,
and constraints are satisfied to within the value of the constraint tolerance.

<stopping criteria details>
```

```
% Extracting variables
y = [x0(3); z(nx:nx:N*nx)]; % y = x3
u = z(N*nx+1:N*nx+N*nu);    % Control
% Time vector
t = 1:N;

% Plot optimal trajectory
figure(4);
subplot(2,1,1);
plot([0,t],y,'-ko'); % Plot on 0 to N
grid('on');
ylabel('y_t')
subplot(2,1,2);
plot(t-1,u,'-ko'); % Plot on 0 to N-1
grid('on');
xlabel('t');
ylabel('u_t');
```



b)

```
% Previous code modified
clear all;

% System matrices
A = [0     0      0    ;
     0     0      1    ;
```

```
     0.1  -0.79   1.78];
B = [1 0 0.1]';
C = [0 0 1];

x0 = [0 0 1]'; % Initial state

N = 30; % Length of time horizon
n_pieces = 6;
samples_in_block = N/n_pieces;
nx = size(A,2); % nx: number of states (equals the number of rows in A)
nu = size(B,2); % nu: number of controls (equals the number of rows in B)

% Cost function
I_N = eye(N);
Qt = 2*diag([0, 0, 1]);
Q = kron(I_N, Qt);
Rt = 2*1;
R = kron(samples_in_block*eye(n_pieces), Rt);
G = blkdiag(Q, R);

% Equality constraint
Aeq_c1 = eye(N*nx);                              % Component 1 of A_eq
Aeq_c2 = kron(diag(ones(N-1,1),-1), -A);    % Component 2 of A_eq
Aeq_c3 = kron(kron(eye(n_pieces),ones(samples_in_block,1)), -B);                    % Componen
Aeq = [Aeq_c1 + Aeq_c2, Aeq_c3];
beq = [A*x0; zeros((N-1)*nx,1)];

size(Aeq)
```

```
ans = 1×2
    90    96
```

```
size(G)
```

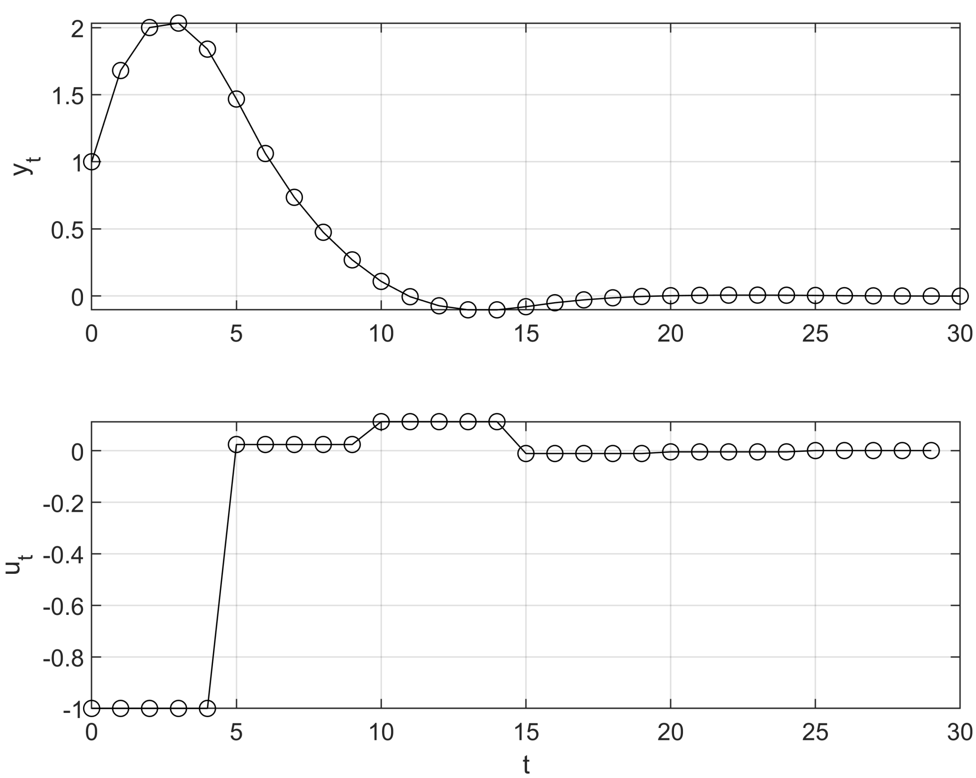
```
ans = 1×2
    96    96
```

```

% Inequality constraints
x_lb = -Inf(N*nx,1);    % Lower bound on x
x_ub =  Inf(N*nx,1);    % Upper bound on x
u_lb = -ones(N*nu,1);   % Lower bound on u
u_ub =  ones(N*nu,1);   % Upper bound on u
lb = [x_lb; u_lb];      % Lower bound on z
ub = [x_ub; u_ub];      % Upper bound on z

% Solving the equality- and inequality-constrained QP with quadprog
opt = optimset('Display','notify', 'Diagnostics','off', 'LargeScale','off', 'Algorithm', 'inter
[z,fval,exitflag,output,lambda] = quadprog(G,[],[],[],Aeq,beq,lb,ub,[],opt);
```

```
Warning: Length of lower bounds is > length(x); ignoring extra bounds.
Warning: Length of upper bounds is > length(x); ignoring extra bounds.
Minimum found that satisfies the constraints.

Optimization completed because the objective function is non-decreasing in
feasible directions, to within the value of the optimality tolerance,
```

```
and constraints are satisfied to within the value of the constraint tolerance.

<stopping criteria details>
```

```
% Extracting variables
y = [x0(3); z(nx:nx:N*nx)]; % y = x3
u_temp = z(N*nx+1:N*nx+n_pieces*nu);    % Control
u = kron(eye(n_pieces),ones(samples_in_block,1))*u_temp;
% Time vector
t = 1:N;

% Plot optimal trajectory
figure(5);
subplot(2,1,1);
plot([0,t],y,'-ko'); % Plot on 0 to N
grid('on');
ylabel('y_t')
subplot(2,1,2);
plot(t-1,u,'-ko'); % Plot on 0 to N-1
grid('on');
xlabel('t');
ylabel('u_t');
```




```
output
```

```
output = struct with fields:
```

```
          message: '↵Minimum found that satisfies the constraints.↵↵Optimization completed because the objectiv
        algorithm: 'interior-point-convex'
    firstorderopt: 4.0019e-09
  constrviolation: 1.3089e-13
       iterations: 5
     linearsolver: 'dense'
     cgiterations: []
```

This works as inteded and iti is clear that we have fewer values for u. This seems to have little effects on the system that is being controlled as the plots for **a)** and **b)** are very similar. quadprog does this in 5 iterations.

c)

```
% Previous code modified
clear all;

% System matrices
A = [0     0      0    ;
     0     0      1    ;
     0.1  -0.79   1.78];
B = [1 0 0.1]';
C = [0 0 1];

x0 = [0 0 1]'; % Initial state

N = 30; % Length of time horizon
n_pieces = 6;
samples_in_block = [1; 1; 2; 4; 8; 14];
nx = size(A,2); % nx: number of states (equals the number of rows in A)
nu = size(B,2); % nu: number of controls (equals the number of rows in B)

% Cost function
I_N = eye(N);
Qt = 2*diag([0, 0, 1]);
Q = kron(I_N, Qt);
Rt = 2*1;
R = kron(samples_in_block.*eye(n_pieces), Rt);
G = blkdiag(Q, R);

% Equality constraint
ones_and_ones = blkdiag(ones(samples_in_block(1),1), ones(samples_in_block(2),1),ones(samples_i
Aeq_c1 = eye(N*nx);                            % Component 1 of A_eq
Aeq_c2 = kron(diag(ones(N-1,1),-1), -A);       % Component 2 of
Aeq_c3 = kron(ones_and_ones, -B);                        % Component 3 of A_eq
Aeq = [Aeq_c1 + Aeq_c2, Aeq_c3];
beq = [A*x0; zeros((N-1)*nx,1)];

size(Aeq)
```

```
ans = 1×2
    90    96
```

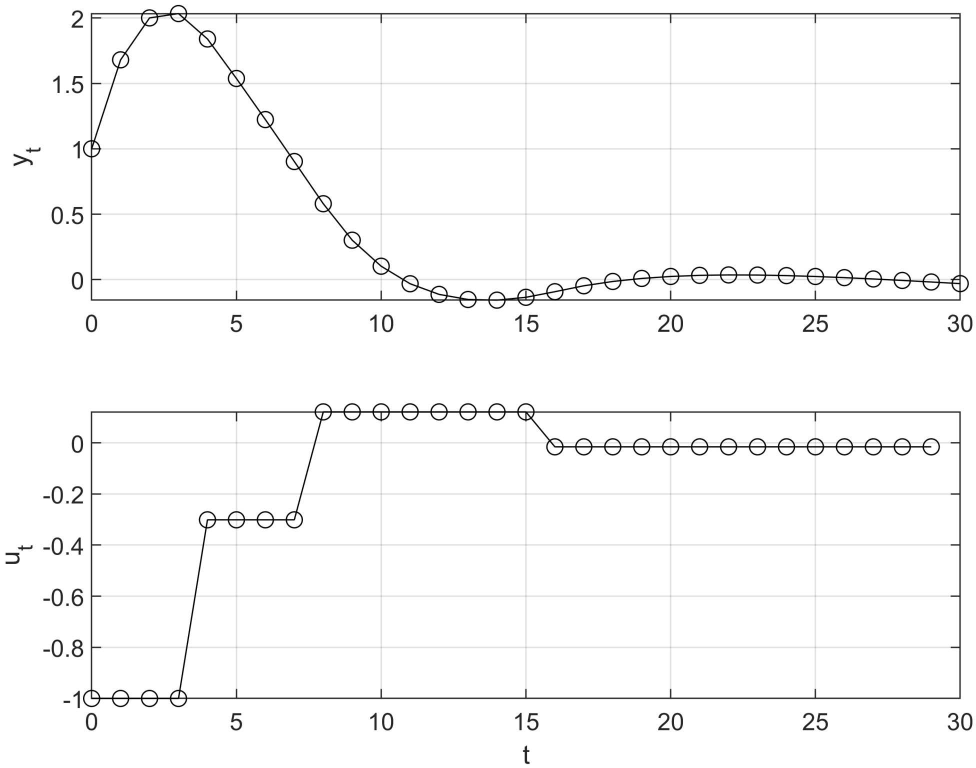
```
size(G)
```

```
ans = 1×2
    96    96
```

```
% Inequality constraints
x_lb = -Inf(N*nx,1);    % Lower bound on x
x_ub =  Inf(N*nx,1);    % Upper bound on x
u_lb = -ones(N*nu,1);   % Lower bound on u
u_ub =  ones(N*nu,1);   % Upper bound on u
lb = [x_lb; u_lb];      % Lower bound on z
ub = [x_ub; u_ub];      % Upper bound on z

% Solving the equality- and inequality-constrained QP with quadprog
opt = optimset('Display','notify', 'Diagnostics','off', 'LargeScale','off', 'Algorithm', 'inter
[z,fval,exitflag,output,lambda] = quadprog(G,[],[],[],Aeq,beq,lb,ub,[],opt);
```

```
Warning: Length of lower bounds is > length(x); ignoring extra bounds.
Warning: Length of upper bounds is > length(x); ignoring extra bounds.
Minimum found that satisfies the constraints.

Optimization completed because the objective function is non-decreasing in
feasible directions, to within the value of the optimality tolerance,
and constraints are satisfied to within the value of the constraint tolerance.

<stopping criteria details>
```

```
% Extracting variables
y = [x0(3); z(nx:nx:N*nx)]; % y = x3
u_temp = z(N*nx+1:N*nx+n_pieces*nu);    % Control
u = ones_and_ones*u_temp;
% Time vector
t = 1:N;

% Plot optimal trajectory
figure(6);
subplot(2,1,1);
plot([0,t],y,'-ko'); % Plot on 0 to N
grid('on');
ylabel('y_t')
subplot(2,1,2);
plot(t-1,u,'-ko'); % Plot on 0 to N-1
grid('on');
xlabel('t');
ylabel('u_t');
```




```
output
```

```
output = struct with fields:
          message: '↵Minimum found that satisfies the constraints.↵↵Optimization completed because the objective
        algorithm: 'interior-point-convex'
    firstorderopt: 1.9902e-08
  constrviolation: 2.8288e-13
       iterations: 5
     linearsolver: 'dense'
     cgiterations: []
```

We can still see little effects in the strep response of the system, but we see that this gives a higher resolution for the input and will handle systems experiencing larger or multiple pertubations better.