Assignment 6 in OptReg

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Task 1

a)

From Newtons second law $F = ma$, where $F = u$ and $m = 1$, we know that $a = u$. Writing this on state space form we have that:

- $x_1 = x$
- $x_2 = \dot{x} = v$
- $\dot{x}_1 = x_2$
- $\dot{x}_2 = \ddot{x} = a$

where x, v and a are position, velocity and acceleration, respectively. On continuous state space form this is:

- $\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$

b)

By exact descretization we have that the descrete system matrix and vector A and b are given by:

- $A = e^{A_c T}$ and $b = \left(\int_0^T e^{A_c \tau} d\tau \right) b_c$

where $e^{A_c T}$ is given by $e^{A_c T} = \sum_{k=0}^{\infty} A_c^k T^k$, where the first element in the series is defined as the identity matrix. We don't care about the higher terms and calculate the matrix exponential as:

- $e^{A_c T} = I + A_c T = I + A_c \cdot 0.5 = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix} = A$

and the b vector as:

- $b = \left(\int_0^T e^{A_c \tau} d\tau \right) b_c = \left(\int_0^T \begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix} \right) b_c = \begin{bmatrix} T & \frac{1}{2} T^2 \\ c & T \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} T^2 \\ T \end{bmatrix} = \begin{bmatrix} 0.125 \\ 0.5 \end{bmatrix}$

c)

For a finite-horizon LQR the Riccati equation is given by:

- $P_t = Q_t + A_t^T P_{t+1} (I + B_t R_t^{-1} B_t^T P_{t+1})^{-1} A_t, \quad \text{for } t = 0, \ldots, N-1$

where $P_N = Q_N$. By using this, the LQR feedback gain can be found:

- $K_t = R_t^{-1} B_t^T P_{t+1} (I + B_t R_t^{-1} B_t^T P_{t+1})^{-1} A_t, \quad \text{for} \;\; t = 0, \ldots, N-1$

using this we get the state feedback:

- $u_t = -K_t x_t$

A sketch of the controller is given below

d)

We know that if $N \to \infty$ that the system becomes an infinite horizon LQR with a corresponding Riccati equation and way of finding K:

- $P = Q + A^T P (I + BR^{-1} B^T P)^{-1} A$
- $K = R^{-1} B^T P (I + BR^{-1} B^T P)^{-1} A$

we therefore choose to omit the horizon in our implementation of **dlqr**.

```
% Initialising system
A = [1 0.5; 0 1];
b = [0.125; 0.5];

Q  = [2 0; 0 2];
R = 2;

% We divide by two because this hasn't been done
Q = Q/2;
R = R/2;

% Finding K and P
[K, P, CLP] = dlqr(A, b, Q, R);
P
```

```
P = 2×2
    4.0350    2.0616
    2.0616    4.1438
```

```
CLP
```

```
CLP = 2×1 complex
   0.6307 + 0.1628i
   0.6307 - 0.1628i
```

We see that we get the expected result. The closed-loop eigenvalues are $\lambda = 0.63 \pm 0.16i$ which is less than one and the closed-loop system is therefore stable.

e)

In general LQR is stable if the matrices (A, B) are stabilizable and (A, D) are detectable. Here $Q = D^T D$. A system is stabilizable if it is controllable and detectable if its observable.

Task 2

a)

We have seen the Riccati equation for infinite-horizon LQR earlier in this assignment, the scalar version of this is given by:

- $p = q + ap\dfrac{1}{1 + b\frac{1}{r}bp}a = q + \dfrac{a^2pr}{r + b^2p}$

for $q = 2$, $a = 3$, $b = 2$ and $r = 1$ we have:

- $p = 2 + \dfrac{9 \cdot p}{1 + 4 \cdot p} \Rightarrow p^2 - 4p - \dfrac{1}{2}$

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roots([1 -4 -0.5])
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ans = 2×1
    4.1213
   -0.1213
```

Since p must be positive definite we have that $p = 4.12$.

b)

The optimal feedback coefficient k is given by:

- $k = \dfrac{bpa}{r + pb^2} = \dfrac{2 \cdot 4.12 \cdot 3}{1 + 4.12 \cdot 4} \approx \sqrt{2}$

This gives and optimal feed back term $u = -\sqrt{2}\,x$.

c)

This is the same as for the finite time horizon case if the system is detectable and stabilizable, then the closed-loop solution will be asymptotically stable for a linear system.

Task 3

a)