Assignment 1, the KKT conditions

Problem 1

$$\min_x \; x_1 + 2x_2 \quad \text{s.t.} \quad 2 - x_1^2 - x_2^2 \geq 0$$
$$x_2 \geq 0$$

a)

This point is where $x_1 = 0$ in the left half-plane

$2 - 0 - x_2^2 = 0$

$x_2^* = \pm\sqrt{2}$

the point is at $x^* = (-\sqrt{2}, 0)$ witch is the minimum vale of $f(x)$ given the constraints

b) We have the following Lagragian eq:

$$\mathcal{L}(x, \lambda) = f(x) - \lambda_1 c_1(x) - \lambda_1 c_1(x)$$

where $c_1 = 2 - x_1^2 - x_2^2$,

$c_2 = x_2$, $\mathcal{I} = \{1, 2\}$

then we check if $\nabla_x \mathcal{L}(x^*, \lambda^*) = 0$:

$$\mathcal{L}(x, \lambda) = x_1 + 2x_2 - 2\lambda_1 + \lambda_1 x_1^2 + \lambda_1 x_2^2 - \lambda_2 x_2$$

$$\nabla_x \mathcal{L}(x, \lambda) = \begin{bmatrix} \frac{\partial \mathcal{L}(x, \lambda)}{\partial x_1} \\ \frac{\partial \mathcal{L}(x, \lambda)}{\partial x_2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 + 2\lambda_1 x_1 \\ 2 + 2\lambda_1 x_2 - \lambda_2 \end{bmatrix}$$

$$\nabla_x \mathcal{L}(x^*, \lambda^*) = \begin{bmatrix} 1 + 2\lambda_1^*(-\sqrt{2}) \\ 2 + 2\lambda_1^* 0 - \lambda_2^* \end{bmatrix} = 0$$

This holds

with Lagrangian multipliers:

$$\lambda^* = \begin{bmatrix} \lambda_1^* \\ \lambda_2^* \end{bmatrix} = \begin{bmatrix} 1/2\sqrt{2} \\ 2 \end{bmatrix}$$

this also means that $\lambda^* \geq 0$

and the last condition to check

is $c_1(x^*) \geq 0$ and $c_2(x^*) \geq 0$

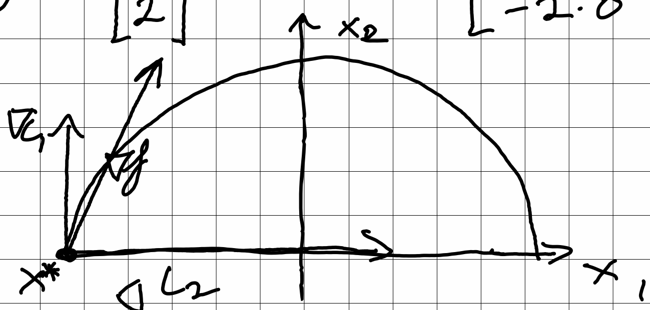
$$c_1(x^*) = 2 - (-\sqrt{2})^2 - 0^2 \geq 0$$
$$= 2 - 2 = 0$$

$c_2(x^*) = 0$ $\boxed{\lambda_i^* c_i(x^*) = 0}$ Also holds

Both conditions hold, proving that x^* is the optimal solution.

c)
$$\nabla c_1 = \begin{bmatrix} -2x_1 \\ -2x_2 \end{bmatrix}, \quad \nabla c_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\nabla f = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \nabla c_1(x^*) = \begin{bmatrix} -2 \cdot (-\sqrt{2}) \\ -2 \cdot 0 \end{bmatrix} = \begin{bmatrix} 2\sqrt{2} \\ 0 \end{bmatrix}$$



d) The Lagrange multipliers are positive because this is the only way to satisfy the first condition $\mathcal{L}(x^*, \lambda^*) = 0$

Problem 2

$\min 2x_1 + x_2 \quad s.t \quad x_1^2 + x_2^2 - 2 = 0$

a) $x_1 = \frac{1}{2} x_2 \qquad x_1^2 + \frac{1}{4} x_1^2 - 2 = 0$

$\frac{5}{4} x_1^2 = 2$

$x_2^2 + \frac{8}{5} = 2$

$x_1 = \pm \sqrt{\frac{8}{5}}$

$x_2 = \pm \sqrt{\frac{2}{5}}$

$= \pm \frac{2\sqrt{2}}{\sqrt{5}}$

We have 4 possible extremepoints

$\left(-\frac{2\sqrt{2}}{5}, -\sqrt{\frac{2}{5}}\right), \left(-\frac{2\sqrt{2}}{5}, \sqrt{\frac{2}{5}}\right), \left(\frac{2\sqrt{2}}{5}, -\sqrt{\frac{2}{5}}\right),$
$\left(\frac{2\sqrt{2}}{5}, \sqrt{\frac{2}{5}}\right)$

b) $\mathcal{L}(x, \lambda) = 2x_1 + x_2 - \lambda x_1^2 - \lambda x_2^2 - \lambda 2$

$\nabla_x \mathcal{L}(x^*, \lambda^*) = 2 + 1 - \lambda \frac{8}{5} - \lambda \frac{2}{5} - \lambda 2$

$= 3 - 4\lambda = 0$

This holds with $\lambda = \frac{3}{4}$

$c_1(x^*) = \frac{8}{5} + \frac{2}{5} - 2 = 0$

$\lambda^* c_1^* = \frac{3}{4} \cdot 0 = 0 \qquad$ Both hold

KKT conditions hold for all four points because all four points yield the same gradient $\nabla_x \mathcal{L}(x^*, \lambda^*)$, $c_1(x^*)$ and therefore same λ^*

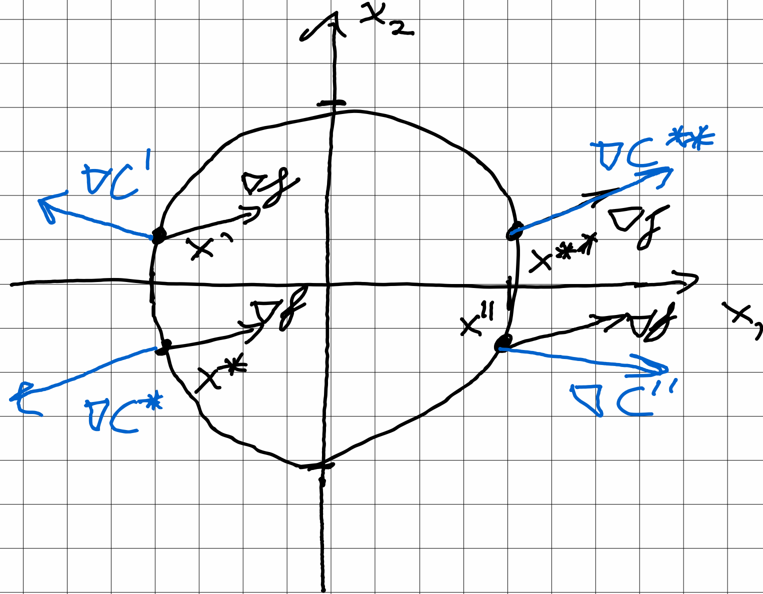
c) $x^* = \begin{bmatrix} -\frac{2\sqrt{2}}{\sqrt{5}} \\ -\sqrt{\frac{2}{5}} \end{bmatrix}$, $x' = \begin{bmatrix} -\frac{2\sqrt{2}}{5} \\ \sqrt{\frac{2}{5}} \end{bmatrix}$

$x'' = \begin{bmatrix} \frac{2\sqrt{2}}{5} \\ -\sqrt{\frac{2}{5}} \end{bmatrix}$, $x^{**} = \begin{bmatrix} \frac{2\sqrt{2}}{5} \\ \sqrt{\frac{2}{5}} \end{bmatrix}$

$$\nabla c_1(x^*) = \begin{bmatrix} 2x_1^* \\ 2x_2^* \end{bmatrix} = \begin{bmatrix} -4\frac{\sqrt{2}}{\sqrt{5}} \\ -2\sqrt{\frac{2}{5}} \end{bmatrix}$$

$\nabla c_1(x') = \begin{bmatrix} -4\sqrt{\frac{2}{5}} \\ 2\sqrt{\frac{2}{5}} \end{bmatrix}$, $\nabla c_1(x'') = \begin{bmatrix} 4\sqrt{\frac{2}{5}} \\ -2\sqrt{\frac{2}{5}} \end{bmatrix}$

$\nabla c_1(x^{**}) = \begin{bmatrix} 4\sqrt{\frac{2}{5}} \\ 2\sqrt{\frac{2}{5}} \end{bmatrix}$, $\nabla f(x) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$



We can quite clearly see from this that x^* is the optimal solution for this problem.

d) The Lagrangian is given by:

$$\mathcal{L}(x,\lambda) = f(x) - \lambda_1 c_1$$
$$= 2x_1 + x_2 - \lambda_1 (x_1^2 + x_2^2 - 2)$$

We check the stationarity condition

$$\nabla_x \mathcal{L}(x^*,\lambda^*) = \begin{bmatrix} 2 - \lambda_1^* 2x_1^* \\ 1 - \lambda_1^* 2x_2^* \end{bmatrix} = \begin{bmatrix} 2 + \lambda_1^* 4\sqrt{\frac{2}{5}} \\ 1 + \lambda_1^* 2\sqrt{\frac{2}{5}} \end{bmatrix}$$

$$\lambda_1^* = -\frac{2}{4\sqrt{\frac{2}{5}}} = -\frac{1}{2\sqrt{\frac{2}{5}}}$$

$$\nabla_x \mathcal{L}(x^*, \lambda^*) = \begin{bmatrix} 2 + \lambda_1^* 4\sqrt{\frac{2}{5}} \\ 1 + \lambda_1^* 2\sqrt{\frac{2}{5}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The stationarity condition holds. now we check primal feasability condition for the equality condition

$$c_1(x^*) = x_1^{*2} + x_2^{*2} - 2 = \frac{8}{5} + \frac{2}{5} - 2 = 0$$

this holds, it's then trivial that $\lambda_1^* c_1(x^*) = 0$.

Since we only have one equality constraint we are allowed to have a negative λ^*. $I = \emptyset$.

e)

We are going to check the 2nd order conditions for x^* and x^{**}

First we look for a critical cone:

$$\nabla c_1(x^*) = \begin{bmatrix} -4\sqrt{\frac{2}{5}} \\ -2\sqrt{\frac{2}{5}} \end{bmatrix}, \nabla c_1(x^{**}) = \begin{bmatrix} 4\sqrt{\frac{2}{5}} \\ 2\sqrt{\frac{2}{5}} \end{bmatrix}$$

$$\mathcal{C}(x^*, \lambda^*) = \nabla c_1(x^*)^T w = 0$$

$$= -4\sqrt{\tfrac{2}{5}}\, w_1 - 2\sqrt{\tfrac{2}{5}}\, w_2 = 0$$

$$= -4\sqrt{\tfrac{2}{5}}\, w_1 + 4\sqrt{\tfrac{2}{5}}\, w_1 = 0$$

where $w_2 = -2w_1$

$$\mathcal{C}(x^*, \lambda^*) = \{ w = (w_1, -2w_1) \mid w_1 \geq 0 \}$$

$$\nabla^2_{xx} \mathcal{L}(x, \lambda) = \begin{bmatrix} -2\lambda_1 & 0 \\ 0 & -2\lambda_1 \end{bmatrix}$$

$$\nabla^2_{xx} \mathcal{L}(x^*, \lambda^*) = \begin{bmatrix} 1/\sqrt{\frac{2}{5}} & 0 \\ 0 & 1/\sqrt{\frac{2}{5}} \end{bmatrix}$$

$w \in \mathcal{C}(x^*, \lambda^*)$:

$$w^T \nabla^2_{xx} \mathcal{L}(x^*, \lambda^*) w = \begin{bmatrix} w_1 \\ -2w_1 \end{bmatrix}^T \begin{bmatrix} 1/\sqrt{\frac{2}{5}} & 0 \\ 0 & 1/\sqrt{\frac{2}{5}} \end{bmatrix} \begin{bmatrix} w_1 \\ -2w_1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{\frac{2}{5}}} w_1 & -\frac{2}{\sqrt{\frac{2}{5}}} w_1 \end{bmatrix} \begin{bmatrix} w_1 \\ -2w_1 \end{bmatrix} = \frac{1}{\sqrt{\frac{2}{5}}} w_1^2 + \frac{4}{\sqrt{\frac{2}{5}}} w_2^2 \geq 0$$

The sufficient condition holds

$$\mathcal{C}(x^{**}, \lambda^{**}) = \nabla c_1(x^{**})^T w = 0$$

$$= 4\sqrt{\tfrac{2}{5}}\, w_1 + 2\sqrt{\tfrac{2}{5}}\, w_2 = 0$$

$$= 4\sqrt{\tfrac{2}{5}}\, w_1 - 4\sqrt{\tfrac{2}{5}}\, w_1 = 0$$

where $w_2 = -2w_1$

$$\mathcal{C}(x^{**}, \lambda^{**}) = \{ w = (w_1, -2w_1) \mid w_2 \geq 0 \}$$

$$w^T \nabla^2_{xx} \mathcal{L}(x^{**}, \lambda^{**}) w = \begin{bmatrix} w_1 \\ -2w_1 \end{bmatrix}^T \begin{bmatrix} -1/\sqrt{\frac{2}{5}} & 0 \\ 0 & -1/\sqrt{\frac{2}{5}} \end{bmatrix} \begin{bmatrix} w_1 \\ -2w_1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{\sqrt{\frac{2}{5}}} w_1 & \frac{2}{\sqrt{\frac{2}{5}}} w_1 \end{bmatrix} \begin{bmatrix} w_1 \\ -2w_1 \end{bmatrix} = -\frac{1}{\sqrt{\frac{2}{5}}} w_1^2 - \frac{4}{\sqrt{\frac{2}{5}}} w_1^2 \not\geq 0$$

the sufficient condition does not hold.

the critical point in both points point the same way. And we see by testing these two points using the 2nd order conditions that x^* indeed is the optimal point.

f) The problem is not convex eventhough the objective function is linear, because the equality constraint is not.

Problem 3

$$\min f(x) = -2x_1 + x_2 \quad \text{s.t.} \begin{cases} (1-x_1)^3 - x_2 \geq 0 \\ x_2 + 0.25x_1^2 - 1 \geq 0 \end{cases}$$

Optimal solution is $x^* = (0,1)^T$

a) Since both constraints are active:

$A(x) = \{1, 2\}$ we only need to check if the gradients of the constraints are linearly independent.

$$\nabla c_1(x) = \begin{bmatrix} -3x_1^2 + 2x_1 - 3 \\ -1 \end{bmatrix}, \nabla c_2(x) = \begin{bmatrix} 0.5x_1 \\ 1 \end{bmatrix}$$

$$\nabla c_1(x^*) = \begin{bmatrix} -3 \\ -1 \end{bmatrix}, \nabla c_2(x^*) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

the two gradients are linearly independent at x^* and LICQ holds.

b) $\mathcal{L}(x,\lambda) = -2x_1 + x_2 - \lambda_1((1-x_1)^3 - x_2)$
$\qquad - \lambda_2(x_2 + \frac{1}{4}x_1^2 - 1)$

$c_1 = (1-x_1)^3 - x_2$

$c_2 = x_2 + \frac{1}{4}x_1^2 - 1$,

$\mathcal{I} = \{1, 2\}$

$$\nabla_x \mathcal{L}(x^*, \lambda^*) = \begin{bmatrix} -2 + \lambda_1^* 3 \\ 1 + \lambda_1^* - \lambda_2^* \end{bmatrix} = 0$$

$\lambda_1^* = \frac{2}{3}$, $\lambda_2^* = \frac{5}{3}$

the stationary condition holds

and $\lambda_i^* \geq 0$, $\forall i \in \mathcal{I}$ does too.

$c_1(x^*) = (1-0)^3 - 1 = 0$

$c_2(x^*) = 1 + \frac{1}{4}0^2 - 1 = 0$

Primal feasability conditions hold, since $c_i = 0$ $\forall i \in \mathcal{I}$ it is trivial that $\lambda_i^* c_i(x^*) = 0$

All KKT conditions hold.

c)

$$\mathcal{F}(x^*) = \{ d \mid d^T \nabla c_i(x^*) \geq 0 \quad \forall i \in \mathcal{A}(x) \cup \mathcal{I} \}$$

$$\Rightarrow \begin{cases} d^T \nabla c_1(x^*) \geq 0 \\ d^T \nabla c_2(x^*) \geq 0 \end{cases}$$

$$\Rightarrow \begin{array}{l} -3d_1 - d_2 \geq 0 \Rightarrow -3d_1 \geq d_2 \\ \underline{d_2 \geq 0} \qquad\qquad d_1 \leq -\frac{1}{3} d_2 \end{array}$$

$$\mathcal{F}(x^*) = \{ d \mid d_2 \geq 0,\ d_1 \leq -\frac{1}{3} d_2 \}$$

$w \in \mathcal{C}(x^*, \lambda^*) \Leftrightarrow \nabla c_i(x^*)^T w = 0,$

$\forall i \in \mathcal{A}(x^*) \cap \underline{\mathcal{I}}$ because both λ_1^* and λ_2^* are positive

$$\Rightarrow \begin{cases} \nabla c_1(x^*)^T w = 0 = -3w_1 - \cancel{w_2}^{\,0} \\ \nabla c_2(x^*)^T w = 0 = w_2 \end{cases}$$

$$w = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

d) Since $\omega = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ the necessary condition can be satisfied, but not the sufficient condition.

Problem 4

$\max x_1 x_2$ s.t. $1 - x_1^2 - x_2^2 \geq 0$

$x = (0, 0)^T$ is a saddle point and the maximum solution must be at the edges of the constraing circle

We find these points using the KKT conditions, for this we change the problem to finding the minimum of $-x_1 x_2$:

$\min x_1 x_2$ s.t. $1 - x_1^2 - x_2^2 \geq 0$

KKT:

$$\mathcal{L}(x,\lambda) = -x_1 x_2 - \lambda_1 (1 - x_1^2 - x_2^2)$$

I $\quad \dfrac{\partial \mathcal{L}(x^*,\lambda^*)}{\partial x_1^*} = -x_2^* + 2\lambda_1^* x_1^* = 0$

II $\quad \dfrac{\partial \mathcal{L}(x^*,\lambda^*)}{\partial x_2^*} = -x_1^* + 2\lambda_1^* x_2^* = 0$

Since the constraint is active at the probable opimal points we also have:

III $\quad \lambda_1^* c_1(x^*) = \lambda_1^* - \lambda_1 x_1^{*2} - \lambda_1 x_2^{*2} = 0$

I $\quad 2\lambda_1^* x_1^* = x_2^*$

$$x_1^* = \frac{x_2^*}{2\lambda_1^*}$$

II $\quad 2\lambda_1^* x_2^* = x_1^*$

$$2\lambda_1^* x_2^* = \frac{x_2^*}{2\lambda_1}$$

$$\lambda_1^{*2} = \frac{\cancel{x_2^*}}{\cancel{x_2^*}4} = \frac{1}{4} \Rightarrow \lambda_1^* = \pm\frac{1}{2}$$

since the constraint is an inequality λ^* must be nonnegative, meaning $\lambda_1^* = \frac{1}{2}$

I $\quad x_1^* = \frac{x_2^*}{2\lambda_1^*} = x_2^*$

II $\quad 2\lambda_1^* x_2^* = x_1^* = x_2^*$

III $\quad \frac{1}{2} - \frac{1}{2}x_1^{*2} - \frac{1}{2}x_1^{*2} = 0$

$$x_1^{*2} = \frac{1}{2}$$

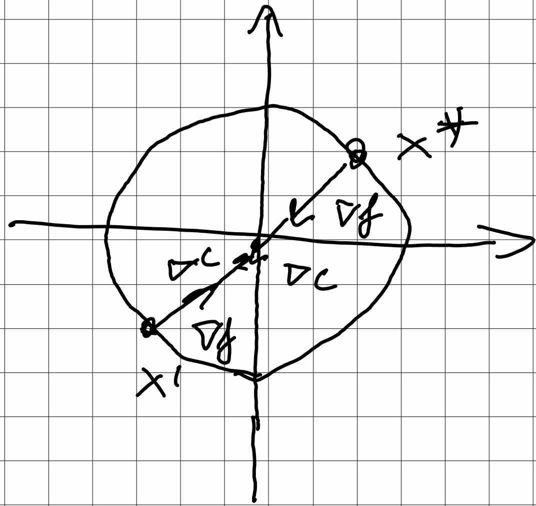
$$\underline{x_1^* = x_2^* = \pm\frac{1}{\sqrt{2}}}$$

we have two optimal points one at $x^* = \left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]^T$ and one at $x' = \left[-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right]$.

$$\nabla c_1(x^*) = \begin{bmatrix} -2x_1^* \\ -2x_2^* \end{bmatrix} = \begin{bmatrix} -\frac{2}{\sqrt{2}} \\ -\frac{2}{\sqrt{2}} \end{bmatrix}$$

$$\nabla c_1(x') = \begin{bmatrix} \frac{2}{\sqrt{2}} \\ \frac{2}{\sqrt{2}} \end{bmatrix}, \quad \nabla f(x^*) = \begin{bmatrix} -x_2^* \\ -x_1^* \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\nabla f(x') = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$



x^*
∇f
∇c
∇c
∇f
x'