Assignment 1

Problem 7 a)

The mean value Theorem for multivariate functions state that

$$f(x+p) = f(x) + \nabla f(x+\alpha p)^T p$$

with $f : \mathbb{R}^n \rightarrow \mathbb{R}$

From the example we have the function $f(x) = x_1^3 + 3x_1x_2^2$ and we use $x = [0,0]^T$ and $p = [2,1]^T$.

This gives:

$f(x) = 0$ and

$$f(x+p) = (x_1+p_1)^3 + 3(x_1+p_1)(x_2+p_2)^2$$
$$= 2^3 + 3 \cdot 2 \cdot 1 = 14$$

The gradient is given by:

$$\nabla f(x+\alpha p) = \begin{bmatrix} 3(x_1+\alpha p_1)^2 + 3(x_2+\alpha p_2)^2 \\ 6(x_1+\alpha p_1)(x_2+\alpha p_2) \end{bmatrix}$$

$$= \begin{bmatrix} 3(\alpha p_1)^2 + 3(\alpha p_2)^2 \\ 6(\alpha p_1)(\alpha p_2) \end{bmatrix}$$

$$= \begin{bmatrix} 12\alpha^2 + 3\alpha^2 \\ 12\alpha^2 \end{bmatrix} = \begin{bmatrix} 15\alpha^2 \\ 12\alpha^2 \end{bmatrix}$$

$$\nabla f(x+\alpha p)^T p = \begin{bmatrix} 15\alpha^2 & 12\alpha^2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= 30\alpha^2 + 12\alpha^2 = 42\alpha^2$$

$$f(x+p) = f(x) + \nabla f(x+\alpha p)^T p$$

$$= 0 + 42\alpha^2 = 14$$

$$\Rightarrow \alpha = \pm\sqrt{\frac{14}{42}} = \pm\frac{1}{\sqrt{3}}$$

We have ~~ore~~ $\alpha \in (0,1) \Rightarrow \alpha \frac{1}{\sqrt{3}}$

b) $f(x) = x^{\frac{1}{2}}$, $f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2x^{\frac{1}{2}}}$

This is not Lipschitz in $x=0$ because there can not exist a constant L that is the upper bound of the function. Meaning the derivative $f'(x)$ goes to infinity if $x=0$

Problem 2

$$\min_x c^T x \text{ s.t } Ax = b, \; x \geq 0,$$

with $c \in \mathbb{R}^n$, $x \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$

$$\mathcal{L}(x, \lambda, s) = c^T x - \lambda^T (Ax - b) - s^T x$$

We find the KKT conditions by first finding the gradient of $\mathcal{L}$ with regards to x, then setting this ~~to~~ zero.

$$\nabla_x \mathcal{L}(x, \lambda, s) = \frac{\partial}{\partial x} c^T x - \frac{\partial}{\partial x} \lambda^T A x + \frac{\partial}{\partial x} \lambda^T b - \frac{\partial}{\partial x} s^T x$$

$$= c - (\lambda^T A)^T - s$$

We set this to zero and set

$$\lambda^{*T} A^T + s^* = c$$

The rest of the KKT conditions follow:

$$Ax^* = b$$

$$x^* \geq 0$$

$$s^* \geq 0$$

$$x_i s_i^* = 0 \quad \forall\, i \in \mathbb{N}$$

as per theorem 12.1 in N&W

Problem 3

a) To solve this maximation problem as an LP we need to solve it as a minimisation of the negative cost. We also choose x_1, x_2 and x_3 to represent R, S and T respectivly. We want the problem on the form

$$\min_x c^T x, \quad Ax = b, \quad x \geq 0$$

to achieve this we use $c^T = [c_1, c_2, c_3]$ $= [-100, -75, -55]$

Furthermore, we have that the A constraint is realized using:

$$3x_1 + 2x_2 + 1x_3 = 7200$$

where the coefficients of the polynomial is given by the hours needed at the A production stage.

Similarly the B constraint is realized with:

$$2x_1 + 2x_2 + 3x_3 = 6000$$

Entering this into the LP standard form we have that:

$$Ax = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = b = \begin{bmatrix} 7200 \\ 6000 \end{bmatrix}$$

it naturally follows that we can not produce negative product therefore $x \geq 0$ holds as well.

b)

To find the basic feasible points we first need to determine the basis set $\mathcal{B}$, but since we have 3 variables and only two constraints $\mathcal{B}$ is a subset of the index set $\{1, 2, 3\}$. There are therefore 3 Bases to choose from and three basic feasible points.

The first basis is given by:

$$\mathcal{B} = \{1, 2\} \Rightarrow B = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix} \Rightarrow x_3 = 0$$

If we then substitute A for B in the constraints we get:

$$Ax = b \Rightarrow Bx = b \Rightarrow \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7200 \\ 6000 \end{bmatrix}$$

This results in the BFP:

$$x^* = [1200 \quad 1800 \quad 0]^T$$

$\mathcal{B} = \{1, 3\}$:

$$B = \begin{bmatrix} 3 & 1 \\ 2 & 3 \end{bmatrix} \Rightarrow x_2 = 0$$

$$\Rightarrow \begin{bmatrix} 3 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7200 \\ 6000 \end{bmatrix}$$

$$\Downarrow$$

$$x^* = \frac{1}{7}[15600, 0, 3600]^T$$

$\mathcal{B} = \{2, 3\}$:

$$B = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7200 \\ 6000 \end{bmatrix}$$

$$x^* = [0, 3900, -600]^T$$

c) Since LPs are convex we can simply find the optimal point through the KKT conditions. Since the third point we found was not feasible we will try using the other two points. For the first point the KKT conditions are:

$$\begin{bmatrix} 3 & 2 \\ 2 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ s_3 \end{bmatrix} = \begin{bmatrix} -100 \\ -75 \\ -55 \end{bmatrix}$$

Here $s_1 = s_2 = 0$ because $x_i s_i = 0$ $\forall i \in \{1, 2, 3\}$. This gives:

$$\lambda = -\begin{bmatrix} 25 \\ \frac{25}{2} \end{bmatrix} \text{ and } s_3 = 7.5 > 0$$

fullfilling all the KKT conditions for the point $x^* = [1200, 1800, 0]^T$.

the maximal profit is therefore

$$c^T x^* = -100 \cdot 1200 - 75 \cdot 1800$$

$$= -250\,000 \text{ NOK}$$

b) The dual problem is defined as:

$$\max b^T \lambda \text{ s.t. } A^T \lambda \leq c$$

this is supossed to give the same result as $\min c^T x$:

$$b^T \lambda^* = 7200 \cdot (-25) + 6000 \cdot \left(-\frac{25}{2}\right)$$

$$= -255\,000$$

It holds.

e) $c^T x^* = -255\,000 = b^T \lambda^*$ $\blacksquare$

d) Looking at our lagrangian multipliers λ_1^* and λ_2^* we know that an increase in A would result in a bigger impact to profit because $\lambda_1^* = 2\lambda_2^*$.