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Andrej Ščedrov

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Abstract. We give a general method of forcing over categories as a category-theoretic universal construction which subsumes, on one hand, all known instances of forcing in set theory, Boolean and Heyting valued models and sheaf interpretations for both classical and intuitionistic formal systems; and, on the other hand, constructions of classifying topoi in topos theory (Grothendieck's generalization of classifying spaces considered in algebraic topology, algebraic geometry). The generic object obtained by forcing is shown to have a clear cohomological meaning. Furthermore, we show that iterated forcing in set theory, and Grothendieck's construction of a lax limit of a fibred topos are the same up to Godel's negative interpretation of classical into intuitionistic logic. This suggests possibilities of interapplications between logic, and algebraic geometry and algebraic topology.

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IN MEMORIAM

ING. MIRA ŠČEDROV

1924 - 1981

INTRODUCTION

This monograph is a revised version of the author's doctoral dissertation, defended at the State University of New York at Buffalo in August, 1981. The main bulk of results was obtained in the spring of 1980. The results given in chapters 1-3 were presented by the author at a meeting of the New York Topos Seminar at Columbia University in New York City in December, 1980, and at the Cambridge Summer Meeting in Category Theory at the University of Cambridge, England, in July, 1981. Some results given in chapter 4 were presented by the author at the New Mexico Research Conference on Constructive Mathematics, held at Las Cruces in August, 1980.

The core of this work (section 1.1.) is a general method of forcing over categories which subsumes, on one hand, all known instances of forcing in set theory, Boolean and Heyting valued models and sheaf interpretations for both classical and intuitionistic systems, and on the other hand, constructions of classifying topoi in topos theory. Moreover, the generic object obtained is shown to have a clear cohomological meaning. We give a unifying, general theory with many examples, both new and known (cf. [FSe], [BSc4] for other new applications). We consider forcing over categories as a way of constructing objects by geometric approximation, including a construction of a generic model of a geometric theory ([L4], [Ti2], [Jol], [MR]) as its special case. When determining decisive properties of a required object, one first singles out simple geometric properties (often in $\wedge, \vee, \exists$-fragment), thus defining a geometric theory T_1. The category C of forcing conditions is given as a small (not necessarily full) subcategory $C \to Mod(S,T_1)$ of the category of set-models of T_1, where morphisms in C preserve additional structure given in the problem, e.g. C may be reduced to a poset. Evaluations at objects of C have both adjoints, so $C \hookrightarrow Mod(S,S^C)$. All objects of C are finitely presented models ([MR] appendix, [GU]). Further geometric properties of the required object ($\bigvee, \exists$-properties) give a Grothendieck topology on C^{op}

([SGA4],[Jo 1]), i.e. a geometric theory T_2 classified by the sheaf category $\mathcal{E} \hookrightarrow S^C$. The object one is looking for is then (a subobject of the associated sheaf of) $G = \sum_{c\in|C|} \mathrm{Hom}(c,-)$. Due to genericity, G may satisfy further, non-geometric properties (e.g. preservation of cardinals in forcing in set theory). As a simplicial set, G is the positive part of the augmented complex $\mathrm{Dec}^1(\mathrm{Nerv}(C))$ (for simplicial methods, cf. [Dus]). A generic object G thus constructed may not fall into cumulative hierarchy $0 \hookrightarrow P(0) \hookrightarrow P(P(0)) \hookrightarrow \ldots \hookrightarrow \varinjlim_n P^n(0) = P^{\omega}(0) \hookrightarrow \ldots \hookrightarrow P^{\alpha}(0) \hookrightarrow \ldots$ in the classifying topos $\mathcal{E}$. However, working in $\mathcal{E}$, one can force G into cumulative hierarchy in a topos over $\mathcal{E}$ by constructing a generic embedding $G \hookrightarrow P^{\alpha}(0)$ for α large enough. E.g. to construct $G \hookrightarrow 2^N$ one looks at the poset $[N,G;2]$ of finite partial functions $f\colon N\times G \dashrightarrow 2$ as forcing conditions in $\mathcal{E}$ ordered by inclusion, with an appropriate Grothendieck topology which enables properties of G in $\mathcal{E}$ to be inherited to G as a subobject of 2^N in a topos over $\mathcal{E}$. E.g., one can look at sheaves over $2^{N\times G}$ in S^C, or at $\neg\neg$-topology on $[N,G;2]$ in $\mathcal{E}$. In this way, consistency and independence results w.r.t. higher-order arithmetic (i.e. type theory) are strengthened to ones w.r.t. ZF set theory.

As first examples, we construct two classifying topoi by using our general forcing method. In both examples, instances of a very general duality are used to compute categories of forcing conditions. This duality is important in applications to model theory as well.

Permutation models are recently being studied with renewed interest as (atomic) topoi ([BD],[Fre 1]). In chapter 2 we use our general method of forcing to study them as classifying topoi. In particular, all examples in [BD] are shown to be of one kind, i.e. topoi of continuous H-sets for various topological groups H. (General features of these topoi are used in chapter 4 in applications to models of intuitionistic set theory). In particular, in one of those examples, we show how to construct the etale topos by forcing.

We also reexamine forcing in set theory (chapter 3). Our main new result there (which goes beyond [Ti 1], and [F]) is that the standard set-theoretic notion of a generic subset of a poset is actually classified by the topos of double negation sheaves. More precisely, if C is a poset, $G \hookrightarrow \Delta C$ described above reduces to the fuctor $G(c) = \{c' \varepsilon |C| \mid c' \leq c\}$. $G \hookrightarrow \Delta C$ is $\neg\neg$-closed iff $\neg\neg$-topology on C^{op} is subcanonical. If C has a least element, and a sup of each two compatible elements, $Sh_{\neg\neg}(C) \hookrightarrow S^C$ classifies the theory of a "generic subset of C" in the standard set-theoretic sense. $Sh_{\neg\neg}(C)$ is equivalent as a category to the topos of sheaves on a cBa $RO(C^{op})$.

Iterations of this construction through the ordinals are considered in the second part of chapter 3. There we prove another of our main new results, that iterated forcing in set theory (in particular, the one given by Solovay and Tennenbaum, cf. e.g. [Ku]) and Grothendieck's construction of a lax limit of a fibred topos in [SGA 4], Exp. VI, §§6-8 (which was motivated by notions and problems in algebraic geometry and algebraic topology), are the same up to Gödel's negative interpretation of classical into intuitionistic logic. This suggests possibilities of interapplications between logic, and algebraic geometry and algebraic topology.

For concreteness, we consider the Solovay-Tennenbaum iterated forcing although a general iterated forcing construction (cf. e.g. [Ku, chapter VIII] is an example of the lax limit of a fibred topos.

More precisely, let $\mathcal{B}$ be a cBa in a topos of sheaves on a cBa B . It then follows from a general sheaf-representation theorem [FSo] that the set $\Gamma(\mathcal{B})$ of global sections of $\mathcal{B}$ is a complete Heyting algebra, (and here it is in fact a cBa), that $sh_{Sh(B)}(\mathcal{B}) \simeq Sh(\Gamma(\mathcal{B}))$, and that $B \hookrightarrow \Gamma(\mathcal{B})$ is a complete Boolean inclusion. If λ is a limit ordinal, and $B_0 \hookrightarrow B_1 \hookrightarrow \ldots \hookrightarrow B_\alpha \hookrightarrow \ldots$ $(\alpha<\lambda)$ are complete Boolean inclusions, let E_λ be the lax limit of the fibred topos $Sh(B_0) \leftrightarrows Sh(B_1) \leftrightarrows \ldots \leftrightarrows Sh(B_\alpha) \leftrightarrows \ldots$ $(\alpha<\lambda)$. Then $sh_{\neg\neg}(E_\lambda) \simeq Sh(B_\lambda)$, where B_λ is the Boolean completion of $\bigcup_{\alpha<\lambda} B_\alpha$.

Applications to intuitionistic systems are given in chapter 4. In section 4.1 we extend a model of second-order intuitionistic arithmetic given in [Kr] to a symmetric extension of a Heyting-valued model of intuitionistic set theory ZFI with Collection (roughly speaking, a topos localic over a topos of continuous H-sets), showing that ZFI is consistent with $\forall\alpha\exists n$-Continuity Principle with parameters, Kripke's Scheme KS , Uniformity Principle, Relativized Dependent Choice RDC , and Bar-Induction. This result goes beyond [Gr2] (appendix A3, where a consistency proof is suggested w.r.t. higher-order intuitionistic arithmetic): namely, all principles considered contain parameters, and we allow them to vary over the whole universe, not only over given types. Moreover, RDC was not considered in [Kr],[Gr2].

In section 4.2 we present Joyal's forcing of spoiling the Heine-Borel theorem, as an example of our general approach given in section 1.1. In section 4.3. we force $\mathbb{N}^{\mathbb{N}}$ to be subcountable, i.e. we generically introduce a surjection from a subset of $\mathbb{N}$ onto $\mathbb{N}^{\mathbb{N}}$. In section 4.4., we give a new topological model of ZFI with RDC, KS, and parameterless $\forall\alpha\exists n$-Continuity, and the Fan Theorem, in which Bar-Induction fails. (Recently, the author has obtained an independence proof of the Fan Theorem from ZFI with RDC and parameterless $\forall\alpha\exists n$-Continuity [Sc2]).

Applications of forcing in non-set-theoretic situations (e.g. model-theoretic Robinson forcing and the closely connected notion of a model completion, cf. [R], [Kei])are studied in [BSc1,2]. Applications as [Fe], [Bl] and the recent application to recursion theory [Ma]) are also examples of the approach given in 1.1; we will address this in the future. It seems there is no need to restrict outselves to forcing over models of set theory (alternatively, looking at classifying topoi); the method given in 1.1. should work on other levels as well.

The point of view taken here is to make the metatheory as weak as possible. It turns out again and again that certain constructions naturally "carry on" assumed additional structure (Axioms of Foundation and Replacement are a case in point - cf. our remarks above on ZF-systems). In particular, the Law of Excluded Middle makes its appearance rather more rarely then it would be generally assumed. We always proceed by specifying data (i.e. hypotheses of the theorem) and then apply a metamathematically simple construction to such a given structure. We assume Excluded Middle and Axiom of Choice for (the category of) Sets, but we also specify crucial mathematical assumptions on which a construction really depends.

I would like to express my deep gratitude to my advisor, Professor John Myhill, for his advice, encouragement, guidance and illuminating conversations on intuitionism and other subjects. I owe a special debt to Professor F. William Lawvere for his help and discussions which brought substantial insights to my understanding of the subject of chapters 1-3. It was through these discussions that the principles espoused in section 1.1. became clear to me.

Many other people helped improve my understanding of topics discussed in this monograph. Conversations with A. Joyal on his construction of spoiling local compactness of the reals (presented with his permission in 4.2) first brought a connection with Cohen forcing to my attention. I have furthermore benefited from valuable remarks and suggestions of S. MacLane, P. Freyd, A. Blass, S. Schanuel, M. Barr, G. Reyes, J. Duskin, J. Isbell, M. Fourman, and R. Grayson.

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Chapter 0

PRELIMINARIES

To begin with, we state all basic definitions and theorems which we refer to in the sequel, but with which a non-specialist is not likely to be familiar. This does not make this work self-contained, for we will not give their proofs. Rather, it is intended as a guide to detailed references, which will be given for each theorem mentioned in this chapter, and which will be kept visible in the sequel. Thus, the best way to use this chapter is to consider it as a quick reference: not to read it at the beginning, but refer to it as the need arises, and then possibly consult other references.

0.1. Adjoint Functors. Equivalence of Categories.

Let $\mathbb{C},\mathbb{D}$ be two categories, $\mathbb{C} \underset{G}{\overset{F}{\leftrightarrows}} \mathbb{D}$ two functors. We say that F is a left adjoint of G (i.e. G is a right adjoint of F) if there is a 1-1 correspondence:

$$\frac{F(X) \to Y \qquad \text{in } \mathbb{C}}{X \to G(Y) \qquad \text{in } \mathbb{D}}$$

natural in X and Y. We write $F \dashv G$, and say that (F,G) is an adjoint pair of functors. It is called an equivalence of categories if FG is naturally isomorphic to $1_{\mathbb{D}}$ and GF is naturally isomorphic to $1_{\mathbb{C}}$. Categories $\mathbb{C}$ and $\mathbb{D}$ are said to be equivalent if there is an equivalence of categories (F,G) between them.

Left adjoints preserve colimits, right adjoints preserve limits. (cf. [Fre 0], [ML]).

0.2. Presheaf Categories. Yoneda Lemma.

Let S be the category of sets and mappings. For a small category $\mathbb{C}$, we write $S^{\mathbb{C}^{op}}$ for the category of presheaves on $\mathbb{C}$, i.e. contravariant functors from $\mathbb{C}$ to S. $S^{\mathbb{C}^{op}}$ has all (small) limits and co-

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limits (pointwise). Each object X of $\mathbb{C}$ gives rise to the <u>representable functor</u> h_X, defined by $h_X(Y) = \mathrm{Hom}_{\mathbb{C}}(Y,X)$ = the set of all morphisms $Y \to X$ in $\mathbb{C}$. On morphisms, h_X is given by composition. (We also mention the covariant representable functor h^X given by $h^X(Y) = \mathrm{Hom}_{\mathbb{C}}(X,Y)$).

In this way, to each object X of $\mathbb{C}$, we associate a uniquely given object h_X of $S^{\mathbb{C}^{op}}$. Moreover, each morphism $X \xrightarrow{\alpha} X'$ in $\mathbb{C}$ defines a natural transformation $h_X \xrightarrow{h_\alpha} h_{X'}$ by composition. The reader can check that this gives a functor $\mathbb{C} \to S^{\mathbb{C}^{op}}$, which is furthermore both <u>full</u> (meaning that each morphism $h_X \to h_{X'}$ in $S^{\mathbb{C}^{op}}$ comes from a morphism $X \to X'$ in $\mathbb{C}$) and faithful (meaning that any two morphisms $h_X \overset{h_{\alpha'}}{\underset{h_\alpha}{\rightrightarrows}} h_{X'}$ in $S^{\mathbb{C}^{op}}$ are equal iff $X \overset{\alpha'}{\underset{\alpha}{\rightrightarrows}} X'$ are equal as morphisms in $\mathbb{C}$). This functor is called the <u>Yoneda embedding</u>. One has the following:

<u>Yoneda Lemma</u>. For objects X in $\mathbb{C}$, F in $S^{\mathbb{C}^{op}}$, there is a 1-1 correspondence (natural in both X and F) between morphisms $h_X \to F$ in $S^{\mathbb{C}^{op}}$, and elements of the set $F(X)$.

The above facts are sometimes rephrased as:

<u>Lemma</u>. Any object of $S^{\mathbb{C}^{op}}$ can be expressed as a colimit of a diagram whose vertices are representable functors.

More detailed discussion is e.g. in [Fre 0] or in [ML].

0.3. <u>Elementary Topoi</u>.

A category $\mathcal{E}$ is called an <u>elementary topos</u> if:

(i) $\mathcal{E}$ has finite limits (it suffices to require pullbacks and a terminal object).

(ii) $\mathcal{E}$ is <u>cartesian closed</u>, i.e. for any objects X,Y, there is an object Y^X so that there is a 1-1 correspondence

$$\frac{Z \to Y^X}{Z \times X \to Y}$$

of morphisms in E , natural in Z,Y. Equivalently, for each object X , the functor $(-)\times X: E \to E$ has a right adjoint $(-)^X: E \to E$.

(iii) E has a subobject classifier, i.e. an object Ω and a morphism $1 \xrightarrow{t} \Omega$ (called "true") such that, for each monomorphism $Y \overset{\sigma}{\hookrightarrow} X$ in E , there is a unique morphism $\phi_\sigma: X \to \Omega$ in E (the classifying map of σ) making

$$\begin{tikzcd} Y \arrow[r, hook] \arrow[d] & X \arrow[d, "\phi_\sigma"] \\ 1 \arrow[r, "t"] & \Omega \end{tikzcd}$$

a pullback diagram.

[Jo 1 , §1.1] gives an outline of the proof that each presheaf category $S^{\mathbb{C}^{op}}$ is an elementary topos.

A natural number object in an elementary topos E is an object N together with morphisms $1 \xrightarrow{0} N \xrightarrow{s} N$ such that for any diagram $1 \xrightarrow{x} X \xrightarrow{u} X$ in E , there exists a unique $N \xrightarrow{f} X$ such that the following diagram commutes:

$$\begin{tikzcd} 1 \arrow[r, "0"] \arrow[dr, "x"'] & N \arrow[r, "s"] \arrow[d, dashed, "f"] & N \arrow[d, "f"] \\ & X \arrow[r, "u"] & X \end{tikzcd}$$

In a presheaf topos $S^{\mathbb{C}^{op}}$, the natural object is given by the constant presheaf $\mathbb{N}$.

In an elementary topos E , the lattice of subobjects of each object is a Heyting algebra. If, in addition, for every $A \hookrightarrow B$ in E there is $A' \hookrightarrow B$ with $A \wedge A' = 0$, $A \vee A' = B$ (so the lattice of subobjects of each object of E is a Boolean algebra), we say that E is a Boolean topos.

0.4. Presheaf Semantics.

Any elementary topos interprets intuitionistic higher-order logic, and any elementary topos with a natural number object interprets intuitionistic higher-order arithmetic (Boolean topoi interpret classical logic)

[Jo 1, §5.4] [BJ], [Os]. A particular case of this interpretation, for a topos of presheaves, merits attention, and which when $\mathbb{C}$ is a poset, reduces to the Kripke semantics for higher-order intuitionistic theories.

Let $\mathbb{C}$ be a small category. Each object of $S^{\mathbb{C}^{op}}$ is considered as a sort of the language. For each object C of $\mathbb{C}$, formula $\phi(x_1,\dots,x_n)$ of the language (whose free variables are among $x_1,\dots,x_n$ of sorts $F_1,\dots,F_n$, respectively), and a choice of elements $a_i \in F_i(C)$, $1 \leq i \leq n$, we define "C forces $\phi(x_1,\dots,x_n)$ for $(a_1,\dots,a_n)$", written as:

$$C \Vdash \phi(x_1,\dots,x_n)[a_1,\dots,a_n] \quad ,$$

by induction on the complexity of ϕ. For simplicity, let us assume $F_1 = F_2 = \dots = F_n = F$.

0) $C \Vdash \top$, $C \nVdash \bot$.

1) For each n-any predicate symbol $P(x_1,\dots,x_n)$ let $\overline{P} \hookrightarrow F^n$ be a given subfunctor of the n-fold cartesian product F^n. Then $C \Vdash P(x_1,\dots,x_n)[a_1,\dots,a_n]$ iff $\langle a_1,\dots,a_n\rangle \in \overline{P}(C)$.

2) $C \Vdash x_1 = x_2\ [a_1,a_2]$ iff $a_1 = a_2$ as element of $F(C)$.

3) $C \Vdash (\phi\wedge\psi)(x_1,\dots,x_n)[a_1,\dots,a_n]$ iff $C \Vdash \phi(x_1,\dots,x_n)[a_1,\dots,a_n]$ and $C \Vdash \psi(x_1,\dots,x_n)[a_1,\dots,a_n]$.

4) $C \Vdash (\phi\vee\psi)(x_1,\dots,x_n)[a_1,\dots,a_n]$ iff $C \Vdash \phi(x_1,\dots,x_n)[a_1,\dots,a_n]$ or $C \Vdash \psi(x_1,\dots,x_n)[a_1,\dots,a_n]$.

5) $C \Vdash (\phi\to\psi)(x_1,\dots,x_n)[a_1,\dots,a_n]$ iff for each morphism $C' \xrightarrow{f} C$ in $\mathbb{C}$, if $C' \Vdash \phi(x_1,\dots,x_n)[f(a_1),\dots,(a_n)]$, then $C' \Vdash \psi(x_1,\dots,x_n)[f(a_1)\dots f(a_n)]$ (the meaning of $f(a_i)$ will be explained below).

6) $C \Vdash \exists y\phi(x_1,\dots,x_n,y)[a_1,\dots,a_n]$ iff for some $b \in F(C)$, $C \Vdash \phi(x_1,\dots,x_n,y)[a_1,\dots,a_n,b]$.

7) $C \Vdash \forall y\phi(x_1,\dots,x_n,y)[a_1,\dots,a_n]$ iff for each morphism $C' \xrightarrow{f} C$ in $\mathbb{C}$, and each $b \in F(C')$, $C' \Vdash \phi(x_1,\dots,x_n,y)[f(a_1),\dots,f(a_n),b]$,

where $f(a_i)$ denotes $F(f)(a_i)$, the value of the mapping $F(C) \xrightarrow{F(f)} F(C')$ at a_i. The higher-order aspect of the language is interpreted as follows: the sort Ω^F is the power-sort of the sort F.

We say that a sentence ϕ is true in $S^{\mathbb{C}^{op}}$, and write $S^{\mathbb{C}^{op}} \models \phi$, if $C \Vdash \phi$ at each object C of $\mathbb{C}$. The truth-value $\|\phi\|$ of a formula $\phi(x_1,\ldots,x_n)$ is the subfunctor $\|\phi\| \hookrightarrow F^n$ given by $\|\phi\|(C) = \{\langle a_1,\ldots,a_n\rangle \in F^n(C) | C \Vdash \phi(x_1,\ldots,x_n)[a_1,\ldots,a_n]\}$.

When $\mathbb{C}$ is the trivial one-object category, the presheaf semantics in $S^{\mathbb{C}^{op}}$ specializes to the ordinary Tarski semantics.

0.5. Grothendieck Topologies. Sheaves. Grothendieck Topoi.

Let $\mathbb{C}$ be a small category. A Grothendieck topology on $\mathbb{C}$ is defined by specifying a set $J(X)$ of sieves on X (i.e. subfunctors of h_X) called covering sieves of the topology, for each object X of $\mathbb{C}$, such that:

(i) For any X, the maximal sieve $\{\alpha | \text{codomain}(\alpha) = X\}$ is an element of $J(X)$.

(ii) If $R \in J(X)$, and $Y \xrightarrow{f} X$ is a morphism of $\mathbb{C}$, then the sieve

$$f^*(R) = \{Z \xrightarrow{\alpha} Y | f\alpha \in R\}$$

is an element of $J(Y)$ (closure under pullbacks).

(iii) If $R \in J(X)$, and S is a sieve on X such that for each $Y \xrightarrow{f} X$ in R we have $f^*(S) \in J(Y)$, then $S \in J(X)$.

A small category equipped with a Grothendieck topology is called a site. The following lemma is useful in the applications.

Lemma (folklore). Let K be a family of sieves in a small category $\mathbb{C}$, and assume K is closed under pullbacks. Then the Grothendieck topology J generated by K is the smallest family f sieves in $\mathbb{C}$ that includes K and satisfies (i) and (iii) above.

Let $(\mathbb{C}, J)$ be a site, F a presheaf on $\mathbb{C}$. We say that F is a sheaf (for the topology J), if for every object X of $\mathbb{C}$, and every $R \in J(X)$:

$$\begin{tikzcd} R \arrow[r, hook] \arrow[dr, "\forall"'] & h_X \arrow[d, dashed, "\exists!"] \arrow[dl, phantom, "//"] \\ & F \end{tikzcd} .$$

$Sh_J(\mathbb{C})$ is then the full subcategory of $S^{\mathbb{C}^{op}}$ whose objects are sheaves for the Grothendieck topology J . By a <u>Grothendieck topos</u> we mean the category of sheaves on a site (this includes presheaf categories, the topology being trivial). Every Grothendieck topos is an elementary topos with the natural number object (outline is in [Jo 1 , §1.1]), and it moreover has all (small) limits and colimits. The best, but very detailed reference to Grothendieck topoi is [SGA4], cf. also [MR], [Jo 1].

The <u>canonical topology</u> on $\mathbb{C}$ is the largest topology for which all the representable functors are sheaves. We say that on topology J is <u>subcanonical</u> if it is smaller than the canonical topology, i.e. if all the representable functors are J-sheaves.

The inclusion $Sh_J(\mathbb{C}) \hookrightarrow S^{\mathbb{C}^{op}}$ has a left adjoint, for any site $(\mathbb{C},J)$. It is called the <u>associated sheaf functor</u>, which also preserves finite limits. The natural number object in a Grothendieck topos is the associated sheaf of the constant presheaf $\mathbb{N}$.

The following lemma is useful in the applications (cf. [SGA4],Exp.II, Cor.2.3):

<u>Lemma</u>. Let K be a family of sieves on a small category $\mathbb{C}$, and assume that K is closed under pullbacks. Let J be the Grothendieck topology on $\mathbb{C}$ generated by K . Then a presheaf F on $\mathbb{C}$ is a J-sheaf iff for each object X of $\mathbb{C}$, and each $R \in K(X)$:

$$\begin{tikzcd} R \arrow[r, hook] \arrow[dr, dashed, "\forall"'] & X \arrow[d, dashed, "\exists!"] \arrow[dl, phantom, "//"] \\ & F \end{tikzcd} .$$

In a Grothendieck topos, the lattice of subobjects of any object is a complete Heyting algebra (i.e. locale). Given $F \hookrightarrow G$ in $S^{\mathbb{C}^{op}}$, and a Grothendieck topology J , we say that $\overline{F} \to G$ in $S^{\mathbb{C}^{op}}$ is the J-closure of $F \hookrightarrow G$ iff for each object X of $\mathbb{C}$:

$$\bar{F}(X) = \{a \varepsilon G(X) \mid \exists R \ \varepsilon \ J(X).\forall (Y \overset{\alpha}{\to} X) \ \varepsilon \ R.\alpha(a) \ \varepsilon \ F(Y)\} ,$$

where the meaning of $\alpha(a)$ is as in 0.4. above.

We say that F is a J-closed subobject of G in $S^{\mathbb{C}^{op}}$ iff $\bar{F} = F$. It is easy to see that a closed subobject of a sheaf is a sheaf. Moreover, an important example of a Grothendieck topos is the category of canonical sheaves on a complete Heyting algebra (i.e. locale) (cf. [FSo]). More precisely, $R \hookrightarrow h_p$ is a covering iff $p = \bigvee R$ in the complete Heyting algebra structure. Every topos of sheaves on a poset is equivalent as a category to a topos of (canonical) sheaves on a locale (cf. e.g. [Jo 1, §5.37].

0.6. Sheaf Semantics.

To describe the elementary topos semantics for intuitionistic higher-order arithmetic in the case of a topos of sheaves, one needs to reconsider the presheaf semantics. We are now interested in sentences true "locally", "on a cover". To this purpose, the clauses 1), 2), 4), and 6) for the presheaf semantics are modified as below, and the clauses 0), 3), 5), and 7) are left intact. Furthermore, only sheaves are allowed as sorts of the language.

1) $C \Vdash P(x_1,\dots,x_n)[a_1,\dots,a_n]$ iff

$\exists R \ \varepsilon \ J(C).\forall (C' \overset{\alpha}{\to} C) \ \varepsilon \ R.\langle\alpha(a_1),\dots,\alpha(a_n)\rangle \ \varepsilon \ \bar{P}(C')$,

where the meaning of $\alpha(a_i)$ is as in 0.4. above.

2) $C \Vdash x_1 = x_2[a_1,a_2]$ iff $\exists R \ \varepsilon \ J(C).\forall (C' \overset{\alpha}{\to} C) \ \varepsilon \ R.\alpha(a_1) = \alpha(a_2)$.

4) $C \Vdash (\phi \vee \psi)(x_1,\dots,x_n)[a_1,\dots,a_n]$ iff

$\exists R \ \varepsilon \ J(C).\forall (C' \overset{\alpha}{\to} C) \ \varepsilon \ R.\ C' \Vdash \phi(x_1,\dots,x_n)[\alpha(a_1),\dots,\alpha(a_n)]$
or $C' \Vdash \psi(x_1,\dots,x_n)[\alpha(a_1),\dots,\alpha(a_n)]$.

6) $C \Vdash \exists y \phi(x_1,\dots,x_n)[a_1,\dots,a_n]$ iff

$\exists R \ \varepsilon \ J(C).\forall (C' \overset{\alpha}{\to} C) \ \varepsilon \ R.\exists b \ \varepsilon \ F(C').C' \Vdash \phi(x_1,\dots,x_n,y)$
$[\alpha(a_1),\dots,\alpha(a_n),b]$.

Fourman [F] and Hayashi have given extensions of this interpretation to ZF set theory (in a Grothendieck topos).

0.7. Geometric Morphisms. Geometric Theories. Classifying Topoi.

Let $\mathcal{E},\mathcal{F}$ be two Grothendieck topoi. By a geometric morphism $\mathcal{E} \xrightarrow{f} \mathcal{F}$ we mean an adjoint pair $f = (f^*,f_*)$, where $\mathcal{E} \xrightarrow{f_*} \mathcal{F}$, and f^* preserves finite limits (f^* is called the inverse image of f).

A geometric formula of a first-order, multi-sorted infinitary language with equalities is obtainable from atomic formulas by finite conjunction (including the empty conjunction, $\top$), possibly infinite disjunction (including the empty disjunction, $\perp$), and existential quantification. If M is an interpretation in a Grothendieck topos $\mathcal{F}$, $\mathcal{E} \xrightarrow{f} \mathcal{F}$ a geometric morphism, and ϕ a geometric formula, then the truth values of ϕ in M and f^*M satisfy $\|\phi\|_{f^*M} = f^*\|\phi\|_M$.

A geometric sequent is a sentence of the kind $\forall x_1 \ldots \forall x_n(\phi(x_1,\ldots,x_n) \rightarrow \psi(x_1,\ldots,x_n))$ (we often write $\phi(x_1,\ldots,x_n) \vdash \psi(x_1,\ldots,x_n)$), where ϕ and ψ are geometric formulas and every variable free in ϕ or ψ occurs in the list $x_1,\ldots,x_n$. A geometric theory is a theory in the multi-sorted first-order language with equalities, with classical logic, axiomatized by geometric sequents. If $\mathcal{E} \xrightarrow{f} \mathcal{F}$ is a geometric morphism, and M a $\mathcal{F}$-model of a geometric theory $\mathcal{T}$ (i.e. a sheaf interpretation in $\mathcal{F}$ satisfying the axioms of $\mathcal{T}$), then f^*M is an $\mathcal{E}$-model of $\mathcal{T}$. Barr's theorem (cf. e.g. [Jo 1, §7.5]) implies that if a geometric sequent is deducible from certain others in classical logic, then it is deducible from them in intuitionistic logic as well. Hence, the geometric sequents provable in a geometric theory $\mathcal{T}$ hold in all models of $\mathcal{T}$ in all Grothendieck topoi.

If $\mathcal{T}$ is a geometric theory, then there exists a Grothendieck topos $\mathcal{E}$, called the classifying topos of $\mathcal{T}$, and there exists a model M of $\mathcal{T}$ in $\mathcal{E}$, called the universal model of $\mathcal{T}$, such that, for any Grothendieck topos $\mathcal{F}$, the category of geometric morphisms $\mathcal{F} \xrightarrow{f} \mathcal{E}$ and natural transformations $\eta\colon f^* \to g^*$ is equivalent to the category of models of $\mathcal{T}$ in $\mathcal{F}$ and homomorphisms, the identity morphism of $\mathcal{E}$ corresponding to the model M. Thus, M has the universal property that every model of $\mathcal{T}$ in any Grothendieck topos $\mathcal{F}$ can be obtained (up to isomorphism) as f^*M for a unique (up to natural isomorphism) geometric morphism $\mathcal{F} \xrightarrow{f} \mathcal{E}$.

For a geometric theory T and a Grothendieck topos F, we let $\mathrm{Mod}(F,T)$ be the category of models of T in F and homomorphisms. For a Grothendieck topos E, we often write $\mathrm{Mod}(F,E)$ for the category of geometric morphisms $F \xrightarrow{f} E$ and natural transformation $\eta\colon f^* \to g^*$. E is a classifying topos of T iff $\mathrm{Mod}(F,T) \simeq \mathrm{Mod}(F,E)$ for all Grothendieck topoi F. A geometric morphism $F \xrightarrow{f} E$ (with its inverse image f^* in mind) is often called a F-model of (a geometric theory classified by) E. Indeed, for every Grothendieck topos E there exists a geometric theory whose classifying topos is E.

0.8. Flat functors. Diaconescu's Theorem.

We conclude the preliminaries by stating a most important and useful theorem of Diaconescu ([D],[Jo 1 ,§4.3.]) on characterization of geometric morphisms into $S^{\mathbb{C}^{op}}$ in terms of functors on $\mathbb{C}$, and some related lemmas.

Let F be a functor on a category $\mathbb{C}$ (to *Sets*). We say that F is flat if the following conditions are met:

(1) given finitely many (possibly zero) objects A_i of $\mathbb{C}$ and elements $a_i \in F(A_i)$, there is an object B of $\mathbb{C}$, morphisms $B \xrightarrow{\alpha_i} A_i$ of $\mathbb{C}$, and an element $b \in F(B)$ such that $F(\alpha_i)(b) = a_i$ for all i,

(2) given $a \in F(A)$ and finitely many morphisms $A \xrightarrow{\alpha_i} A'$ of $\mathbb{C}$ such that all of the $F(\alpha_i)(a)$ are equal, we have a morphism $B \xrightarrow{\beta} A$ of $\mathbb{C}$ such that all of the composites $B \xrightarrow{\alpha_i\beta} A'$ are equal, and a is in the image of $F(\beta)$.

Notice that (1) and (2) imply that given finitely many morphisms $A_i \xrightarrow{\alpha_i} A'$ of $\mathbb{C}$, and elements $a_i \in F(A_i)$ such that all of the $F(\alpha_i)(a_i)$ are equal, there is an object B of $\mathbb{C}$, morphisms $B \xrightarrow{\beta_i} A_i$ of $\mathbb{C}$, and an element $b \in F(B)$ such that all of the composites $\alpha_i\beta_i$ are equal and $F(\beta_i)(b) = a_i$ for all i.

It is also easy to see that a flat functor preserves finite limits.

Theorem (Diaconescu). Let $\mathbb{C}$ be a small category with a set of objects. Let F be a Grothendieck topos. Then there is an equivalence between the category $\mathrm{Mod}(F, S^{\mathbb{C}^{op}})$ and the category of F-internal flat functors on $\mathbb{C}$ (as a constant object in F) and F-internal natural transformations. Moreover, this equivalence is natural in F.

Remark. A few comments on the statement of the theorem are in order. If F is the category of S, it just says that "the category $\mathrm{Mod}(F, S^{\mathbb{C}^{op}})$ is equivalent to the full subcategory of $S^{\mathbb{C}}$ whose objects are flat functors on $\mathbb{C}$". The statement of the theorem means that the above sentence in quotation marks is true when interpreted in F. In practice, it often suffices to look at the case $F = S$, and check that the facts about the situation are proved intuitionistically.

Corollary (Diaconescu). Under the hypotheses of the theorem, let J be a Grothendieck topology on $\mathbb{C}$. Then there is an equivalence between the category $\mathrm{Mod}(F, \mathrm{Sh}_J(\mathbb{C}))$ and the category of F-internal flat continuous functors on $\mathbb{C}$ (i.e. F-internal flat functors on $\mathbb{C}$ taking J-covers to epimorphic families in F).

In practice, J is the Grothendieck topology generated by a family of sieves on $\mathbb{C}$ closed w.r.t. pullbacks:

Lemma (folklore). Let K be a family of sieves in a category $\mathbb{C}$, and assume that K is closed w.r.t. pullbacks. Then for the Grothendieck topology J generated by K, a flat functor from $\mathbb{C}$ to a Grothendieck topos F is continuous for J iff it sends every sieve in K to an epimorphic family in F.

Chapter 1

FORCING AND CLASSIFYING TOPOI

In this chapter we study a very general method of forcing over categories, which subsumes as particular cases set-theoretic forcing as developed by P. J. Cohen [Coh] and, on the other hand, construction of a classifying topos of a geometric theory. In 1.1. we give general principles which will be used in this and later chapters to understand unifying threads in seemingly different constructions in (seemingly) different areas of mathematics (cf. [FSe], [BSc4] for more applications).

1.1. Category of Forcing Conditions as a Category of Models.

Let us begin with a simple problem. Given a commutative ring R with a unit, see whether $P_1(s) = 0$ implies $P_2(s) = 0$ for any element s of a ring extension S of R, and for given integral polynomials P_1, P_2. Well, one takes for s an element X about which "nothing is known" (i.e. one passes from the ring R to the polynomial ring $R[X]$), then one "forces" it to satisfy $P_1(X) = 0$ (i.e. one passes from the ring $R[X]$ to the quotient ring $R[X]/P_1(X)$) and then one simply checks whether X in the quotient ring satisfies $P_2(X) = 0$. This is a good example of "universal thinking": one always takes only the bare essentials needed to satisfy the hypothesis (i.e. a universal example).

For a more involved example, recall the proof of the Cayley-Hamilton theorem e.g. in Lang's book [La, XV, §4]. If k is a commutative ring, E a free module of dimension n over k, and M a $n\times n$ matrix of a linear map $A: E \to E$, $P_M(t) = \det(tI_n - M)$, then showing $P_M(M) = 0$ at once is too hard. Rather, think of $tI_n - M$ as a matrix over $k[t]$, E as a module over $k[t]$ (by substituting M for t in polynomials in k) and show $P_M(t)E = 0$ in this setting. Then, in particular, $P_M(M) = 0$.

Constructions like these abound in mathematics (e.g. a group on X objects, a ring of polynomials with coefficients in a given ring, a category on a given graph,...). Notice that in general, <u>two steps</u> are involved: passing from a given setting to a more variable one, increasing a degree of variation, i.e. introducing new variable objects which interpolate properties of a (given) collection of (relatively) constant ones. The second step involves "cutting down" the degree of variation to conform with additional conditions.

With this in mind, let us take a fresh look at the consistency and independence problems in logic. The usual view is that a model for a collection of sentences is called for. Instead, we shall think of the consistency and independence problems as problems of constructing a certain object from a given collection of objects. Thus e.g. to establish the independence of the Continuum Hypothesis, one constructs a family of $\aleph_2$ subsets of $\mathbb{N}$; to establish the independence of the Suslin's Hypothesis, one constructs a Suslin tree; to establish the consistency of the Suslin's Hypothesis, one constructs an uncountable chain for every candidate for a Suslin tree.

As in the preceding examples, we shall construct such a required object universally. More precisely, we shall construct its universal, geometric (cf. section 0.7.) approximation, which will (due to universality) satisfy further, non-geometric properties.

Recognizing the decisive properties of an object we want to construct will require inventiveness in each case, as well as formulating those properties in a suitable way. Nevertheless, we are guided by the requirement to construct <u>geometric approximations</u> to the needed properties of an object to be constructed. Thus, when determining decisive properties of a required object, one should distinguish the following steps:

(i) One first tries to single out simple geometric conditions, notably finitary ones (expressed by $\wedge$, $\vee$ and $\exists$ from atomic formulae) - thereby defining a geometric theory T_1 .

(ii) Let $\mathrm{Mod}(S,T_1)$ be the category of set-models of T_1 (cf.0.4.,and [MR , Chapter 7]). The category C of forcing conditions is always given as a small subcategory:

$$C \hookrightarrow \mathrm{Mod}(S,T_1) \quad ,$$

which is not necessarily a full subcategory, one distinguishes morphisms which preserve some additional structure. E.g. C

may be a poset due to inclusion of each object of C into a fixed set (given as part of the structure) or due to a fixed enumeration - we speak of <u>rigidifying conditions</u>.

(iii) Look at the topos S^C of (covariant) functors $C \to Sets$. (Eventual additional geometric requirements will be considered as density conditions determining a Grothendieck topology in C^{op}; cf. 0.5 [SGA4, Exp. II], [Jo 1, §0.32]). The topos S^C classifies the theory (given by) C^{op} (cf. e.g. [MR], 3.5.) whose universal model is the Yoneda embedding

$$C^{op} \hookrightarrow S^C$$

defined as the assignment $c \rightsquigarrow \mathrm{Hom}_C(c,-)$. Then the evaluation functor

$$C \times S^C \to S$$

given by $\langle c,F\rangle \rightsquigarrow F(c)$, allows us to consider C as a subcategory of a category of set-models of C^{op} (where $\mathrm{Mod}(S,C^{op})$ is the full subcategory of $S^{C^{op}}$ of <u>flat</u> functors (cf. 0.8., [Jo 1, 4.31])) as classified by S^C. Indeed, every object c in C gives a functor

$$S^C \xrightarrow{ev_c} S$$

of evaluating at c, which actually has both adjoints. Since it is a right adjoint, it preserves limits (in particular, finite limits). It also has a right adjoint, so it is an inverse image of a geometric morphism from S to $S^{\mathbb{C}}$. Thus for any non-degenerate Grothendieck topos F (cf. 0.5., [SGA4] Exp. 4) we have:

$$C \hookrightarrow \mathrm{Mod}(S,S^C) \hookrightarrow \mathrm{Mod}(F,S^C) \quad ,$$

where $\mathrm{Mod}(F,S^C)$ is the full subcategory of all geometric morphisms $F \to S^C$).

<u>Remark</u>. For a sheaf category $E \hookrightarrow S^C$ we have:

$$\mathrm{Mod}(F,E) \hookrightarrow \mathrm{Mod}(F,S^C) \quad ,$$

but we could have $\mathrm{Mod}(S,E) = \emptyset$ (cf. e.g. 4.2. below).

Additional geometric requirements (expressed through possibly infinitary disjunctions of formulae) define a Grothendieck topology on C^{op} (alternatively, a geometric theory T_2, cf. [MR], Chapter 9). This corresponds to "cutting down" in the examples at the beginning of this section, cf. [Jo 1, §3.4.]. Let $E \hookrightarrow S^C$ be the topos of sheaves (it classifies T_2). The consistency of geometric requirements is then expressed as $E \neq 1$, so one has to make sure there are plenty of sheaves (e.g. when the topology is subcanonical, i.e. every representable presheaf is a sheaf). The object we need is then (as a universal model of T_2) a subobject of (the associated sheaf of) the "diagonal" functor G in S^C, G arising from the evaluation functor by the Yoneda embedding:

$$G = \sum_{c' \in C} \mathrm{Hom}_C(c',-) ,$$

the generic object in S^C.

(iv) Since G is a <u>universal</u> object with the required geometric properties, it might satisfy further non-geometric requirements (e.g. preservation of cardinals in set-theoretic forcing).

One may think of G as a variable object interpolating the category C of approximations. More precisely, $G \hookrightarrow \Delta C$ is a variable subobject of the constant functor $\Delta C(c) = \mathrm{Ar}(C)$, whose subobjects generate S^C. (G is the Yoneda profunctor of Chapter 2 of [Jo 1].) This functor G can be given as a category $\underline{G}$ whose objects are pairs $\langle g,p \rangle$, where g is a morphism of the category C with the codomain p; and whose morphisms:

$$\langle g,p \rangle \xrightarrow{\pi} \langle g',p' \rangle$$

are 5-tuples $\langle g,p,\pi,g',p' \rangle$, where π is a morphism $p \xrightarrow{\pi} p'$ such that $\pi g = g'$ in C. Composition of morphisms in $\underline{G}$ is defined naturally. It is easy to see that the codomain functor makes $\underline{G}$ into a discrete opfibration over C (cf. [Jo 1], 2.1.): the object G_0 of objects in $\underline{G}$

is the object C_1 of morphisms in C, the object G_1 of morphisms in $\underset{\sim}{G}$ is the object C_2 of composable pairs of morphisms in C, etc.:

$$\begin{tikzcd}
C_1 \arrow[d, "d_1"'] \arrow[dr, phantom, "p.b."] & C_2 \arrow[l, shift right, "\pi_1"'] \arrow[l, shift left, "m"'] \arrow[d, "\pi_2"] \arrow[dr, phantom, "p.b."] & C_3 \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] \arrow[d] \\
C_0 & C_1 \arrow[l, shift right, "d_0"'] \arrow[l, shift left, "d_1"] & C_2 \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2]
\end{tikzcd}$$

Our convention here is to write:

$$\begin{tikzcd}
C_2 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "p.b."] & C_1 \arrow[d, "d_0"] \\
C_1 \arrow[r, "d_1"'] & C_0 = C
\end{tikzcd}$$

so that for $<\sigma,\tau> \varepsilon\ C_2$, codomain (σ) = domain (τ). m is the composition in C, d_0 is domain, d_1 codomain in C.

In fact, as a simplicial set, G is obtained from (the simplicial set determined by) C by a shift for one place and dropping the lowest arrow down:

$$\begin{tikzcd}
C_1 \arrow[d, "d_1"'] & C_2 \arrow[l, shift right, "\pi_1"'] \arrow[l, shift left, "m"'] \arrow[d, "\pi_2"] & C_3 \arrow[l, shift right=2, "\alpha"'] \arrow[l, "\beta"] \arrow[l, shift left=2, "\gamma"] \arrow[d, "\delta"] & C_4 \cdots \arrow[l, shift right=3] \arrow[l, shift right=1] \arrow[l, shift left=1] \arrow[l, shift left=3] \arrow[d] \\
C_0 & C_1 \arrow[l, shift right, "d_0"'] \arrow[l, shift left, "d_1"] & C_2 \arrow[l, shift right=2, "\pi_1"'] \arrow[l, "m"] \arrow[l, shift left=2, "\pi_2"] & C_3 \cdots \arrow[l, shift right=3, "\alpha"'] \arrow[l, shift right=1, "\beta"] \arrow[l, shift left=1, "\gamma"] \arrow[l, shift left=3, "\delta"]
\end{tikzcd}$$

and all diagrams are pullbacks. I.e., G is the positive part of $\mathrm{Dec}^1(\mathrm{Nerv}(C))$, cf. [Dus, 0.14].

<u>Example</u>. <u>Forcing an object into the cumulative hierarchy</u>.

Historically, the cumulative hierarchy was first introduced by J. von Neumann <u>as a relative consistency proof</u> of the axiom of Foundation. One may mimick his construction in any Grothendieck topos E : start with the initial object 0, take $P(0) = \Omega^0$, $P(P(0)),\ldots, P^n(0),\ldots$ $P^\omega(0) = \varinjlim_{n\varepsilon\omega} P^n(0)$ w.r.t. the system $0 \hookrightarrow P(0) \hookrightarrow \ldots \hookrightarrow P^n(0) \hookrightarrow \ldots$, and so on: at each limit ordinal α, let $P^\alpha(0) = \varinjlim_{\beta\varepsilon\alpha} P^\beta(0)$, w.r.t. the system $0 \hookrightarrow P(0) \hookrightarrow \ldots \hookrightarrow P^\beta(0) \hookrightarrow \ldots$ $(\beta<\alpha)$. The full subcategory of E of subobjects of these is then a subtopos of E , it is what Freyd [Fre 1,2] calls the minimal exponential variety in E . Getting a consistency or an independence result w.r.t. ZF set theory requires constructing the desired object G in the cumulative hierarchy. The construction of G given above does not guarantee that. However, this problem itself can be solved by forcing. Simply consider E as the ground model, and construct a universal embedding

$$G \hookrightarrow P^{\alpha}(0)$$

for α large enough. The situation is quite general: e.g. given an object, introduce a universal monomorphism $A \hookrightarrow \Omega^{\mathbb{N}}$. According to the construction above, one takes the poset $\mathbb{P}$ of finite functions $X \to \Omega$ (with X a finite subset of $\mathbb{N} \times A$), and introduces an appropriate Grothendieck topology on $S^{\mathbb{P}}$ (working internally in E). Observe that in the case of $A = \aleph_2$, and the $\neg\neg$-topology, this is just the forcing of the independence of the Continiuum Hypothesis. Similarly, Cohen's first model for the independence of the Axiom of Choice can be obtained in this way, if E is the first Fraenkel-Mostowski model (cf. chapter 2, [BSc3]).

The universality of the embedding

$$G \hookrightarrow P^{\alpha}(0)$$

then may imply the preservation of non-geometric properties of G as well.

In short, by applying the construction twice (if necessary), one does obtain the results w.r.t. ZF set theory.

1.2. Partially Ordered Sets vs. Distributive Lattices

Now we give a few examples of purely geometric forcing, i.e. constructions of classifying toposes, over categories of forcing conditions (which may not be posets). In turn, the categories of models we will be considering here have natural "duals" (after Stone), which leads to particular representations of classifying toposes. We shall consider "almost algebraic" theories which are (because of their simplicity) classified by toposes of presheaves.

Dualities given in this section are examples of a general situation ([Is],[LR]) and one ought be able to prove general characterization theorems of the kind given in 1.2.1 and 1.2.2 (say when there is one "dualizing object" living in both categories). J. Isbell's remarks on this and on Proposition 1.2.1.2 are gratefully acknowledged.

1.2.1. Partially Ordered Sets vs. Distributive Lattices

The notion of the partially ordered set is geometric (really, Horn):

$$\begin{aligned} &\vdash x \leq x \\ x \leq y \wedge y \leq x &\vdash x = y \\ x \leq y \wedge y \leq z &\vdash x \leq z \end{aligned}$$

(together with axioms for equality), so to construct a classifying topos for this theory (after 1.1), one just ought to look at the presheaf topos $S^{\mathcal{P}_{fin}}$, where $\mathcal{P}$ $(\mathcal{P}_{fin})$ is the category of all (finite) partially ordered sets end order-preserving maps. There is another presentation of $\mathcal{P}_{fin}$, however: let $\mathcal{D}$ $(\mathcal{D}_{fin})$ be the category of all (finite) distributive lattices (from now on we assume that a distributive lattice has specified $\top$, $\bot$ as part of the structure) and lattice homomorphisms (again, we require that $\top$, $\bot$ are preserved).

Proposition 1.2.1.1. Let $\mathcal{D}_{fin} \underset{\overline{G}}{\overset{\overline{F}}{\rightleftarrows}} \mathcal{P}_{fin}$ be contravariant functors defined by:

$\overline{F}(D) = \mathrm{Hom}_{\mathcal{D}}(D,2)$ with pointwise order

$\overline{G}(P) = \mathrm{Hom}_{\mathcal{P}}(P,2)$ with pointwise lattice structure.

The contravariant adjointness of $\overline{F}$ and $\overline{G}$ is an equivalence.

<u>Proof</u>. One checks easily that $\overline{F}(D)$ is the set of all prime filters of D ordered by inclusion (every non-empty finite partial order has a maximal element so for every $d \in D$, $d \neq \bot$ there is a maximal (hence prime) filter F with $d \in F$). For each $P \in \mathcal{P}_{fin}$, look at the map $\mathcal{E}_P: P \to \overline{F}\,\overline{G}(P)$ given by $\mathcal{E}_P(p) = \tilde{p} \in \overline{F}\,\overline{G}(P)$, the functional defined by $\tilde{p}(f) = f(p)$ for every $f \in \overline{G}(P)$. We first show that this defines a natural transformation $\mathcal{E}: I_{\mathcal{P}_{fin}} \to \overline{F}\,\overline{G}$. Indeed, each $\tilde{p}$ is a lattice homomorphism, since e.g. $\tilde{p}(f_1 \vee f_2) = (f_1 \vee f_2)(p) = f_1(p) \vee f_2(p) = \tilde{p}(f_1) \vee \tilde{p}(f_2)$. Also, each $\mathcal{E}_P$ is orderpreserving, since $p \leq q$ in P implies $\tilde{p}(f) = f(p) \leq f(q) = \tilde{q}(f)$ for each $f \in \overline{G}(P)$, hence $\tilde{p} \leq \tilde{q}$. Furthermore, for a diagram in $\mathcal{P}_{fin}$:

$$\begin{tikzcd} P \arrow[r] \arrow[d, "h"'] & \overline{FG}(P) \arrow[d, "\overline{FG}(h)"] \\ Q \arrow[r] & \overline{FG}(Q) \end{tikzcd}$$

We have for each $p \in P$: $\mathcal{E}_Q h(p) = \widetilde{h(p)}$ and on the other side $\overline{F}\,\overline{G}(h)\mathcal{E}_P(p) = \overline{F}\,\overline{G}(h)(\tilde{p}) = \tilde{p} \circ \overline{G}(h)$. But for each $f \in \overline{G}(Q)$: $(\tilde{p} \circ \overline{G}(h))(f) = \tilde{p}(\overline{G}(h) \circ f) = \tilde{p}(f \circ h) = (f \circ h)(p) = f(h(p)) = (\widetilde{h(p)})(f)$, so the above diagram commutes. Hence we have a natural transformation. We now claim that each $\mathcal{E}_P$ is an isomorphism. It is clearly a monomorphism: given $p_1 \neq p_2$, look at $f_{p_i} \in \overline{G}(P)$ given by

$$f_{p_i}(p) = \begin{cases} 0, & \text{if } p \leq p_i \\ 1, & \text{otherwise} \end{cases},$$

so $\tilde{p}_1 \neq \tilde{p}_2$ since they differ on some f_{p_i} $(i=1,2)$. Finally, we have to show that for each $\pi \in \overline{F}\,\overline{G}(P)$, $\pi = \tilde{p}$ for some $p \in P$. To see that, let $f^{\pi} = \bigvee\{f \in \overline{G}(P) \mid \pi(f) = 0\}$ and let $\mathring{\pi} = \{p \in P \mid f^{\pi}(p) = 0\}$. $\mathring{\pi}$ is nonempty since f^{π} is not constant. We claim that $\mathring{\pi}$ has a greatest element p_0. (Then it readily follows that $\pi = \tilde{p}_0$). Let

$X = \{p_1,\ldots,p_n\} \subseteq \overset{\circ}{\pi}$ be the set of all maximals elements of $\overset{\circ}{\pi}$. Thus for any $p \in \overset{\circ}{\pi}$, $p \leq p_i$ for some $i = 1,\ldots,n$. Hence $\bigwedge_{i=1}^{n} f_{p_i} \leq f^{\pi}$, and since the ideal (f^{π}) is prime, for some i_o we have $f_{p_{i_o}} \leq f^{\pi}$. Thus $n=1$ and π has a greatest element.

One can similarly show that the Stone duality $\eta_D\colon D \to \overline{G}\,\overline{F}(D)$ is an isomorphism for each $D \in \mathcal{D}_{fin}$ and that it gives a natural isomorphism $\eta\colon I_{\mathcal{D}_{fin}} \quad \overline{G}\,\overline{F}$.

Actually, this also characterizes finite distributive lattices (partial orders):

Proposition 1.2.1.2. Assume the Axiom of Choice. Then:

$$\mathcal{D} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{P}$$

contravariant functors $\mathrm{Hom}(-,2)$ are an adjoint pair. Furthermore, $\eta_D\colon D \to G\,F(D)$ for $D \in \mathcal{D}$ is an isomorphism iff D is finite and $\varepsilon_P\colon P \to F\,G(P)$ for $P \in \mathcal{P}$ is an isomorphism iff P is finite.

Proof. Rather than checking the adjunction (cf. similar situation in Theorem 1.2.2.2) we prove that a nonempty partial order P is finite if $\varepsilon_P\colon P \hookrightarrow F\,G(P)$ is onto (it is always a monomorphism). That will suffice, since an object is finite iff its dual is finite. Thus, suppose ε_P is onto, i.e. for every functional $\pi \in F\,G(P)$ there is (exactly one) $p \in P$ such that $\pi = \tilde{p}$. The proof proceeds through series of claims:

Claim 1. Every chain in P has a lower bound.

Proof. Let $C \subseteq P$ be a chain. Look at $\tilde{q} \in F\,G(P)$ for each $q \in C$, each of them determines a prime ideal $\mathcal{T}_q = \{f \in G(P) \mid f(q) = 0\}$. Let $\mathcal{T} = \bigcup_{q \in C} \mathcal{T}_q$. $\mathcal{T}$ is closed downward and $r \wedge s \in \mathcal{T}$ implies that $r \in \mathcal{T}$ or $s \in \mathcal{T}$. Also, if $r,s \in \mathcal{T}$, then $r \in \mathcal{T}_{q_1}$, $s \in \mathcal{T}_{q_2}$ for some $q_1,q_2 \in C$. Since C is a chain, say $q_1 \leq q_2$. Then $s \in \mathcal{T}_{q_1}$, so $r \vee s \in \mathcal{T}_{q_1} \subseteq \mathcal{T}$. Thus $\mathcal{T}$ is a prime ideal in $G(P)$. Let $\pi \in F\,G(P)$ be given by
$$\pi(f) = \begin{cases} 0, & \text{if } f \in \mathcal{T} \\ 1, & \text{if } f \notin \mathcal{T} \end{cases}$$
. By assumption, we have $\pi = \tilde{p}$ for $p \in P$.

Since ε_p is a monomorphism, $p \leq q$ for each $q \in C$.

<u>Claim 2</u>. There are at most finitely many minimal elements of P.

<u>Proof</u>. Suppose there are K many minimal elements of P (K an infinite cardinal). (So we can assume that K is the set of all minimal elements of P). Let $X = \{f \in G(P) \mid (f{\restriction}K)^{-1}\{0\}$ is finite$\}$. X is a (non-trivial) filter on $G(P)$, so it is contained in a prime filter F (given by $\pi \in F\,G(P)$ so that $f \in F$ iff $\pi(f) = 1$ for each $f \in G(P)$). By assumption, $\pi = \tilde{p}$ for some $p \in P$. But $p \neq \alpha$ for all $\alpha < K$.

<u>Claim 3</u>. Let C_p be a chain in $\uparrow_p = \{q \in P \mid p < q\}$ for $p \in P$. Then C_p has a minimal element in $\uparrow_p$.

<u>Proof</u>. As in the proof of Claim 1, there is $s \in P$ so that $\tilde{s} = \bigcup_{q \in C_p} T_q$ and $s \leq q$ for all $q \in C_p$. Also, $p < s$, since for $f_p \in G(P)$ given by $f_p(r) = \begin{cases} 0, & \text{if } r \leq p \\ 1, & \text{otherwise} \end{cases}$ we have $\tilde{p}(f_p) = 0$, $\tilde{s}(f_p) = 1$, since $f_p(q) = 1$ for all $q > p$.

<u>Claim 4</u>. Each element $p \in P$ has at most finitely many immediate successors in P.

<u>Claim 5</u>. Every chain in P has an upper bound in P.

<u>Claim 6</u>. Let D_p be a chain in $\downarrow_p = \{q \in P \mid q < p\}$, $p \in P$. Then D_p has a maximal element in $\downarrow_p$.

These claims imply (under Zorn's Lemma) that P has a maximal element (and finitely many of them), that for each $p \in P$, the partial order $\uparrow_p = \{q \in P \mid q > p\}$ has a minimal element (and finitely many of them). Then claims 5,6 imply that P is finite, for if not, there is a minimal element p_0 of P with infinitely many successors. Then there exists its immediate successor p_1, with infinitely many successors, ..., getting an infinite chain $\{p_i\}_{i \in N} \subseteq P$, which contradicts claims 5 and 6.

Proposition 1.2.1.1. allows us to prove:

Theorem 1.2.1.3. Topos $S^{\mathcal{D}^{op}_{fin}}$ classifies partial orders and topos $S^{P^{op}_{fin}}$ classifies distributive lattices (with $\top, \bot$).

Proof. We have $S^{\mathcal{D}^{op}_{fin}} \simeq S^{P_{fin}}$ and $S^{\mathcal{D}_{fin}} \simeq S^{P^{op}_{fin}}$. We will show that $S^{P_{fin}}$ classifies partial orders (cf. Horn theory at the beginning of 1.2.1), the other statement is then proved similarly. Since both $\mathcal{D}_{fin}$ and P_{fin} are obviously left exact, they also have all finite colimits (after Proposition 1.2.1.1). Now:

(a) The forgetful functor U in $S^{P_{fin}}$ is a given partially ordered object in $S^{P_{fin}}$ (by "remembering" partial order on $U(P) = P$). Then for the functor $P^{op}_{fin} \xrightarrow{\phi} S^{P_{fin}}$ given by: $\phi_U(P)$ = object of order-preserving morphisms $\Delta P \to U$ (as a subobject of $U^{(\Delta P)}$ in $S^{P_{fin}}$); one has $\phi_U(P)(Q) = \mathrm{Hom}_{P_{fin}}(P,Q) = \mathcal{Y}(P)(Q)$, so ϕ_U is the Yoneda embedding.

(b) (We work for $E = S$, but the argument holds for any Grothendieck topos E). Given a partially ordered set $\mathbb{P}$, the functor $P^{op}_{fin} \longrightarrow \mathrm{Hom}(-,\mathbb{P})$ is left exact.

Therefore, the category of partially ordered objects in E is equivalent to the category of left exact functors $P^{op}_{fin} \to E$, so $S^{P_{fin}}$ classifies.

Remark. The theory of nontrivial distributive lattices (i.e. $\top \neq \bot$) is classified by $S^{C^{op}}$, where $C \hookrightarrow P_{fin}$ is the full subcategory of nonempty finite partial orders. One can really repeat the analogue of the above proof even if, strictly speaking, C is not left exact (anyhow, one shows flatness of certain functors by recalling pullbacks in P_{fin}).

1.2.2. Linearly Ordered Sets vs. Intervals.

The geometric theory describing "an interval" (i.e. a linearly ordered set with specified first and last element, which are different) is given below. A. Joyal proved that it is classified by the topos $S^{\Delta^{op}}$ where Δ is the simplicial category (of non-empty finite linearly ordered sets and order-preserving maps), cf. §7 of [Jo 2]. We concentrate on the aspects of this fact suggested by the general theory in 1.1 (cf. 1.2.2.1-1.2.2.2 below), which in turn allows one to present a simpler proof that $S^{\Delta^{op}}$ is the classifying topos.

We say that an <u>interval</u> (in a topos) is an object I equipped with a binary relation $\leq$ and two constants 0,1 which satisfy the geometric axioms:

$$\begin{aligned}
T &\vdash i \leq i \\
i \leq j \wedge j \leq k &\vdash i \leq k \\
i \leq j \wedge j \leq i &\vdash i = j \\
T &\vdash i \leq j \vee j \leq i \\
T &\vdash 0 \leq i \\
T &\vdash i \leq 1 \\
0 = 1 &\vdash \perp
\end{aligned}$$

Thus the object we wish to classify (construct in a generic way) is completely given by its geometric properties. Moreover, we could regard it as simple, i.e. $T_1 = T_2$, and thus try to construct a generic interval in a <u>presheaf</u> category (cf. 1.1).

Let M be the category of all <u>finite</u> set-models of the geometric theory T_1 described above: objects of are finite linearly ordered sets $(X ; x_0, x_1)$ with first element x_0 and last element x_1 such that $x_0 \neq x_1$; morphisms in M are maps which preserve linear order, first and last elements. We show (Proposition 1.2.2.1 below) that $M \simeq \Delta^{op}$. The Yoneda embedding $\Delta \hookrightarrow S^{\Delta^{op}}$ gives, as usual, the generic presheaf:

$$G = \sum_{X \in \Delta} \mathrm{Hom}_\Delta(-,X) = \sum_{n \in N} \mathrm{Hom}_\Delta(-,n+1) \quad ,$$

in this case the object of all finite ordered tuples of "elements" of the generic interval $\mathrm{Hom}_\Delta(-,2) \in S^{\Delta^{op}}$, in which the total order is given by:

$$f \leq g \quad \text{in} \quad \mathrm{Hom}_\Delta(n+1,2) \quad \text{iff} \quad f(p) \leq g(p) \quad \text{for all} \quad p \in n+1 \;,$$

the first element is given by the constant 0, the last element is the constant 1.

Proposition 1.2.2.1. Consider the following two contravariant functions:

$$\begin{tikzcd} M \arrow[r, bend left, "\overline{F}"] & \Delta \arrow[l, bend left, "\overline{G}"] \end{tikzcd}$$

$\overline{F}(X;x_0,x_1) = M((X;x_0,x_1) , (2;0,1))$,

$\overline{G}(L) = (\Delta(L,2);$ constant 0, constant $1)$,

and naturally on morphisms, with order in hom-sets given pointwise. Then $\overline{F}$ and $\overline{G}$ give an equivalence:

$$M \simeq \Delta^{op} \quad .$$

One will be able to give the proof of this proposition after the proof of the following more interesting fact:

Proposition 1.2.2.2 (characterization of Δ). Let $R_{0,1}$ be the category whose objects are sets $(X;x_0,x_1)$ with a binary relation ρ and two specified elements $x_0,x_1 \in X$ such that $x_0 \;\rho\; x_1$; and morphisms are maps preserving binary relation and specified elements. Let R be the category whose objects are nonempty sets with a binary relation, and whose morphisms are relation-preserving maps. Consider the contravariant functors:

$$\begin{tikzcd} R_{0,1} \arrow[r, bend left, "F"] & R \arrow[l, bend left, "G"] \end{tikzcd}$$

given by:

$F(X;x_0,x_1) = R_{0,1}((X;x_0,x_1),(2;0,1))$

$G(L) = (R(L,2);\ \xi_0,\xi_1)$

and naturally on morphisms (where $(2;0,1)$ and 2 are as in Proposition 1.2.1.1 ξ_0 is the constant 0 , ξ_1 is the constant 1 ; relation is the pointwise partial order). Then:

1 There is an adjunction $F \dashv G$.

2 Let $F_{0,1} \hookrightarrow R_{0,1}$ and $F \hookrightarrow R$ be maximal full subcategories such that for $F' = F|F_{0,1}$, $G' = G|F$ the adjunction $F' \dashv G'$ is an equivalence. Then $F_{0,1} \simeq M$, $F \simeq \Delta$.

Proof. We may assume that a binary relation on an object in $\mathcal{R}_{0,1}$ or $\mathcal{R}$ is a partial order, and that x_0 is the first, x_1 the last element of an object $(X;x_0,x_1)$ in $\mathcal{R}_{0,1}$ w.r.t. its partial order. Then given $x \in X$ (for $(X;x_0,x_1) \in \mathcal{R}_{0,1}$), let $\tilde{x}\colon F(X;x_0,x_1) \to 2$ be the functional defined by $\tilde{x}(g) = g(x)$ for each $g \in F(X;x_0,x_1)$. Since $g_1 \leq g_2$ in $F(X;x_0,x_1)$ means $g_1(x) \leq g_2(x)$ for each $x \in X$, $\tilde{x}$ is order-preserving. (Also, for $x \leq x'$ in X we have:

$$\tilde{x}(g) = g(x) \leq g(x') = \tilde{x}'(g)$$

for each $g \in F(X;x_0,x_1)$ since g is order-preserving, and:

$$\tilde{x}_0(g) = g(x_0) = 0\ , \quad \tilde{x}_1(g) = g(x_1) = 1\ ,$$

since each such g preserves first and last element. All this shows that $x \rightsquigarrow \tilde{x}$ is a $\mathcal{R}_{0,1}$-morphism $(X;x_0,x_1) \xrightarrow{\eta_X} GF(X;x_0,x_1)$. To see that this gives a natural transformation $I \xrightarrow{\eta} GF$, look at the diagram:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "h"'] & GF(X) \arrow[d, "GF(h)"] \\ Y \arrow[r] & GF(Y) \end{tikzcd}$$

For $x \in X$, we have $\eta_Y h(x) = \widetilde{h(x)}$, and on the other hand $(GF(h) \circ \eta_X)(x) = GF(h)(\tilde{x}) = \tilde{x}\circ F(h)$. Then for each $k \in F(Y)$:

$$(\tilde{x} \circ F(h))(k) = \tilde{x}(F(h)(k)) = \tilde{x}(k\circ h) = (k\circ h)(x) =$$
$$= k(h(x)) = \widetilde{h(x)}(k)\ ,$$

so $\tilde{x} \circ F(h) = \widetilde{h(x)}$, i.e. the above diagram commutes.

In the similar way one defines morphisms $L \xrightarrow{E_L} FG(L)$ for each object L in $\mathcal{R}$, and shows that they give a natural transformation $I \xrightarrow{E} FG$.

To verify the adjunction $F \dashv G$ we show that for each object $L \in \mathcal{R}$, $G(E_L) \circ \eta_{G(L)} = id_{G(L)}$; verification of the other adjunction identity is similiar, bearing in mind the contravariance. Since $x \in G(L)$, we have $(G(E_L)\circ(\eta_{G(L)})(x) = G(E_L)(\eta_{G(L)}(x)) = \tilde{x}\circ E_L$. For every $\ell \in L$, $(\tilde{x}\circ E_L)(\ell) = \tilde{x}(\tilde{\ell}) = \tilde{\ell}(x) = x(\ell)$ thus $\tilde{x}' \circ E_L = x$. So:

$$\begin{tikzcd} G(L) \arrow[r, "\eta_{G(L)}"] \arrow[dr, "id"'] & GFG(L) \arrow[d, "G(\varepsilon_L)"] \\ & G(L) \end{tikzcd}$$

To show the second part of the proposition, assume $X \xrightarrow{\eta} GF(X)$ is an isomorphism in $\mathcal{R}_{0,1}$. We first claim that the order on $(X;x_0,x_1)$ is linear. For, suppose there are incomparable elements $x, x' \varepsilon X$. Look at $F(X) \xrightarrow{\chi} 2$ defined by $\chi(h) = \max\{h(x), h(x')\}$. χ is obviously order-preserving. Since η_X was assumed to be an isomorphism, there is $y \varepsilon X$ such that $\chi = \tilde{y}$. Now let $h_1, h_2, h_3 \colon (X;x_0,x_1) \to (2;0,1)$ be $\mathcal{R}_{0,1}$-morphisms defined as follows:

(i) $h_1(z) = 0$ iff $z \leq x$

(ii) $h_2(z) = 0$ iff $z \leq x'$

(iii) $h_3(z) = \min\{h_1(z), h_2(z)\}$.

Then $h_1(x') = h_2(x) = 1$, so $\tilde{y}(h_1) = h_1(y) = 1 = h_2(y) = \tilde{y}(h_2)$. Hence by (i), $y \nleq x$; and by (ii) $y \nleq x'$. On the other hand $h_1(x) = h_2(x') = 0$, so $h_3(x) = h_3(x') = 0$ and hence $\tilde{y}(h_3) = h_3(y) = 0$, therefore $y \leq x$ or $y \leq x'$, a contradiction. Thus X is linearly ordered.

Next we show that X is well-ordered. A similar reasoning would then show that every nonempty subset $A \subseteq X$ has a last element, and hence that X is finite, i.e. an object in $\mathcal{M}$. So let $A \subseteq X$, $A \neq \emptyset$, $x_0 \notin A$. Look at $\mathcal{R}$-morphism $F(X) \xrightarrow{\phi} 2$ given by:

$$\phi(h) = \inf_{a \varepsilon A}\{h(a)\}$$

for each $h \varepsilon F(X)$. η_X was assumed to be an isomorphism, so for some $a_o \varepsilon X$, $\phi = \tilde{a}_o$. Look at $g \varepsilon F(X)$ such that $g(x) = 1$ iff $\exists a \varepsilon A$ such that $a \leq x$. Then $a_o(g) = g(a_0) = 1$, so $\exists a \varepsilon A$, $a \leq a_o$. If $a_o \notin A$, $a < a_o$, so look at $g_a \varepsilon F(X)$ given by $g_a(x) = 0$ iff $x \leq a$. We have $g_a(a) = 0$, $g_a(a_o) = 1$, $\inf_{b \varepsilon A}\{g_a(b)\} = 0$, a contradiction. So $a_o \in A$ and the same reasoning shows that a_o is the first element of A.

Remark. As in 1.2.1, we can allow trivial objects and consider categories Δ^+ and M^+, retaining analogues of Propositions 1.2.2.1 and 1.2.2.2. Although categories of this section badly fail to have limits and colimits, we still have:

Theorem 1.2.2.3.

(a) $S^{\Delta^{op}}$ $(\simeq S^M)$ classifies intervals.

(b) $S^{(\Delta^+)^{op}}$ $(\simeq S^{M^+})$ classifies linear orders with specified (not necessarily different) first and last element.

(c) $S^{(M^+)^{op}}$ $(\simeq S^{\Delta^+})$ classifies linear orders.

Proof. We show (c), the rest is similar. As in 1.2.1, the forgetful functor in S^{Δ^+} is the generic interval. Also, given a linear order L in a Grothendieck topos F, we have to show (after a theorem of Diaconescu, cf. [Jo 1], 4.34) that the functor $F_L(n)$ = order-preserving morphisms $\Delta n \to L$ (as a subobject of $L^{(\Delta n)}$ in F) is a flat functor from $(\Delta^+)^{op}$ to F, i.e. that the corresponding fibred category $\mathbb{F}_L \to \Delta^+$ is filtered. We do this for $F = S$ but the argument is general. Objects of $\mathbb{F}_L$ are order-preserving maps $n \xrightarrow{\alpha} L$, fibers are discrete, and an arrow $\alpha \xrightarrow{f} \beta$ in $\mathbb{F}$ above an arrow $n \xrightarrow{f} m$ in $\mathbb{F}_L$ is a triple (α, f, β) such that $n \xrightarrow{\alpha} L$, $m \xrightarrow{\beta} L$ are order-preserving and $\beta f = \alpha$, $\mathbb{F}_L$ is filtered since one can (locally) take unions of words (by interlacing them suitably) and (locally) identify a word with an object in Δ^+.

Remark. One can modify theorem 1.2.2.3 by requiring specification of first (last) element only, etc.

Chapter 2

MORE GEOMETRIC FORCING: CONTINUOUS ACTIONS

Permutation models were introduced by Fraenkel and Mostowski (cf. e.g. [Je]) to show that the Axiom of Choice can fail, and were later combined with (set-theoretic) forcing by P. J. Cohen. Fourman [F] made explicit the connection with familiar toposes of continuous G-sets. Here we want to study some examples of those toposes to present permutation models as forcing (in the sense of 1.1.).

Given a topological group G, a continuous G-set X is a set X (thought of as a discrete space) with a continuous G-action $G \times X \xrightarrow{\alpha} X$. Category Cont (G) of continuous G-sets has as objects all continuous G-sets and as morphisms $(X,\alpha) \to (Y,\beta)$ G-equivariant maps $f: X \to Y$ (i.e. such that $f(\alpha(g,x)) = \beta(g,f(x))$ for all $x \in X$, $g \in G$). It is a Grothendieck topos: the subobject classifier is a two-element set, the natural number object is N, (both with the trivial action). Y^X consists of all maps $f: X \to Y$ whose stabilizers are open (G-action on maps being $(\pi f)(\pi x) = \pi(fx))$. The inclusion Cont $(G) \hookrightarrow S^G$ has a right adjoint. The global-sections functor Cont $(G) \xrightarrow{\Gamma} S$ has a left adjoint which gives a trivial action to every set.

So Cont (G) is a Boolean (actually, 2-valued) Grothendieck topos. Since every (continuous) G-set splits into orbits, it is an atomic topos (in the sense of [BD]), atoms (objects with no nontrivial subobject) being the orbits. Thus, according to [BD], it is (equivalent to) a topos of sheaves on an atomic site C (a category C with every nonempty sieve considered as a covering sieve). In particular, it is shown in [BD] that (a small category equivalent to) the category of atoms is an atomic site with the topology being induced by the canonical topology on a given atomic topos (any morphism from an atom to an atom is an epimorphism). In an atomic topos, every constant presheaf is a sheaf. Every atomic topos is 2-valued (cf. [BD]).

In the case of Cont (G), every transitive continuous G-set is isomorphic to G/H for an open subgroup $H \leq G$. One can thus present a site as a category A_G whose objects are open subgroups of G and whose morphisms $H \to K$ are equivalence classes of elements $g \in G$ such that $H \subseteq g K g^{-1}$, where one considers g and g' equivalent as morphisms $H \to K$ iff $gK = g'K$ (cf. also [Fre]). According to a general remark on generating families [SGA 4] Exp. II, Prop. 3.0.4., one can alternatively look at the full subcategory A'_G whose objects are subgroups ranging over a neighborhood basis of 1 in G (cf. also proof of Prop. 9 in [BD]).

The examples we study below have been proposed in [BD] as interesting examples of atomic topoi and were afterwards recognized by many people as being (equivalent to) topoi Cont (G) for certain groups G. We want to show here how all that follows if we consider them as classifying topoi arising through a process given in 1.1.

The atomic topology on a category exists (obviously there is at most one) iff the pullback condition is satisfied, i.e. the pullback of any dense sieve along any morphism is a dense sieve, i.e. every pair of morphisms with the same codomain can be completed to a commutative square. Such a topology is subcanonical (i.e. every representable functor is a sheaf) iff every morphism is a joint coequalizer (of all pairs of morphisms that it coequalizes, cf. [BD]).

2.1. On Infinite Sets

The topos we present in this section was recently studied by S. Schanuel in connection with combinatorics, the most recent of its guises (as we shall see shortly).

One way of saying that an object X is infinite is to require that for every n different elements $x_1,\dots,x_n$ there is yet another x_{n+1} different from them all. This gives a geometric theory:

(1) $\top \vdash \mathrm{Mon}_n(x_1,\dots,x_n) \vee \bigvee_{1\le i\ne j\le n} x_i = x_j$

(2) $\mathrm{Mon}_n(x_1,\dots,x_n) \wedge x_i = x_y \vdash \perp$ for each $1\le i\ne j\le n$

(3) $\top \vdash \exists x \cdot x = x$

(4) $\mathrm{Mon}_n(x_1,\dots,x_n) \vdash \exists x_{n+1}\dots x_k \mathrm{Mon}_k(x_1,\dots,x_n,x_{n+1},\dots,x_k)$, for $1\le n\le k$

where for each $n > 1$, Mon_n is n-ary predicate. The geometric theory (1)-(4) describes an infinite decidable set. In categorical terms, these conditions can be expressed by requiring:

$$\begin{tikzcd} X_k \arrow[r, hook] \arrow[d, dashed] & X^k \arrow[d] \\ X_n \arrow[r, hook] & X^n \end{tikzcd}$$

for every $k>n\geq 0$ and every projection $X^k \to X^n$, where for each $k>1$, $Mon_k = X_k \hookrightarrow X^k$ is the subobject $\{<x_1,\ldots,x_k> \mid \bigwedge_{1\leq i\neq j\leq k} x_i \neq x_j\}$, $X_1 = X$, $X_0 = 1$.

We consider requirements (1)-(3) as defining a category C , of forcing conditions, with (4) being density conditions. So let T_1 be the theory given by (1)-(3), let

$$C \hookrightarrow Mod(S,T_1)$$

be the subcategory of all (nonempty) finite set-models of T_1 , in other words C is the category of all nonempty finite sets and injections. There is the atomic topology on C : first of all, for any two maps $\ell \leftarrow n \rightarrow k$ in C , there is a commutative square:

$$\begin{tikzcd} n \arrow[r] \arrow[d] & \ell \arrow[d] \\ k \arrow[r] & k \underset{n}{+} \ell \end{tikzcd}$$

where $k \underset{n}{+} \ell$ is the pushout in the category of finite sets and mappings (it is not the pushout in C). Also, in C , given $n \xrightarrow{f} k$, let $m \xrightarrow{g} k$ be such that $h_1 \circ g = h_2 \circ g$ for any $k \overset{h_1}{\underset{h_2}{\rightrightarrows}} \ell$ such that $h_1 \circ f = h_2 \circ f$. If $n < k$, we have $m \leq n$, for otherwise it is possible to define h_1 and h_2 on $k \setminus f(n)$ so that $h_1 \circ g \neq h_2 \circ g$. This argument also shows that $g(m) \subseteq f(n)$. But then there exists $m \xrightarrow{h} n$ such that $f \circ h = g$. Thus the atomic topology on C^{op} is subcanonical.

Let $T \rightarrow S^C$ be the category of sheaves for the atomic topology on C^{op} . As expected from the above, we have:

<u>Theorem 2.1.1.</u> $\mathcal{T}$ classifies infinite decidable sets.

<u>Proof.</u> The atomic topology is subcanonical, so the Yoneda embedding factors:

$$\begin{tikzcd} C^{op} \arrow[rr, hook] \arrow[dr] \arrow[rr, phantom, shift right=3ex, "///"] & & S^{C} \\ & \mathcal{T} \arrow[ur, hook] & \end{tikzcd}$$

The underlying-set functor $X = \mathcal{Y}(1)$ is the generic infinite decidable set, and $X_n = \mathrm{Mono}((\Delta n),X)$ (as a subobject of monomorphisms in $X^{(\Delta n)}$) is given by $\mathcal{Y}(n)$, Δn being the constant presheaf n . It is also easy to see that any "infinite decidable" object S in a Grothendieck topos $\mathcal{F}$ (i.e. an $\mathcal{F}$-model of (1)-(4)) defines a continuous flat functor $F(n) = \mathrm{Mono}(\Delta n,S) \to S^{(\Delta n)}$ from C^{op} to $\mathcal{F}$ so that one has $S = f^*(X)$ for the corresponding geometric morphism $\mathcal{F} \xrightarrow{f} \mathcal{T}$.

Why would one be interested in finding a generic infinite set? Well, for example, to see whether it is also Dedekind infinite:

<u>Theorem 2.1.2.</u> Let $X = \mathcal{Y}(1)$ be the underlying-set functor in $\mathcal{T}$. Then X is also infinite in the sense that $\mathrm{Epi}(\Delta n,X) \hookrightarrow X^{(\Delta n)}$, the object of epimorphisms from Δn to X , is empty for each $n \in N$. On the other hand, X is Dedekind finite in the sense that $\mathrm{Mono}(\mathbb{N},X)$ $X^{\mathbb{N}}$, subobject of monomorphisms from $\mathbb{N}$ to X ($\mathbb{N}$ being the constant presheaf), is empty.

<u>Proof.</u> We use sheaf semantics (cf. e.g. [JR],[BJ],[Os]) in:

$$\mathrm{Epi}(\Delta n,X)(k) = \{f \in X^{(\Delta n)}(k) \mid k \Vdash \forall x \in X.\ \exists i \in n.\ f(i) = x\} ,$$

and $k \Vdash \forall x \in X.\ \exists i \in n. f(i) = x$ comes down to:

- for every $k \xrightarrow{\psi} \ell$ in C and for each $j \leq \ell$, there is $i \leq n$ (here we think of n as $\{1,2,\ldots,n\}$) such that: $\psi(f(i)) = j$; which is clearly false for $\ell > k$.

On the other hand, one sees similarly that:

$$X^{\mathbb{N}}(k) = \{\text{maps } f\colon \mathbb{N} \to k\} ,$$

hence:

$$\mathrm{Mono}(\mathbb{N},X)(k) = \{f\colon \mathbb{N} \to k \mid k \Vdash f \text{ is 1-1}\} ,$$

where $k \Vdash f$ is 1-1 comes down to:

- for each $k \xrightarrow{\psi} \ell$ in C and each $m,n \in \mathbb{N}$,

  $\psi(fm) = \psi(fn)$ implies $m=n$,

i.e. to:

- f is 1-1 ,

which is impossible.

The topos T is a Fraenkel-Mostowski model in disguise. Let G be the group of permutations of natural numbers, topologized by letting the subgroup $H \leq G$ be open iff it contains a subgroup $H_n = \{\pi \in G | \pi(i) = i$ for all $i = 1,\ldots,n\}$ for some $n \in N$.

Theorem 2.1.3. $T \simeq \text{Cont}(G)$.

Proof 1. We refer to the remarks on atomic toposes at the beginning of this chapter. $\{H_n\}_{n\geq 1}$ is a neighborhood basis of 1 in G , so we look at the full subcategory $A'_G \hookrightarrow A_G$ whose objects are subgroups H_n ; $n \geq 1$. But $H_n \xrightarrow{\pi} H_m$ is an arrow in A'_G if $m \leq n$ and $\pi(m) \subseteq n$. Moreover, the restriction to m is a bijection between $\text{Hom}_{A'_G}(H_n,H_m)$ and $\text{Hom}_C(m,n)$.

Proof 2. N with the obvious nontrivial action is a $\text{Cont}(G)$ - model of (1)-(4), i.e. an infinite decidable object in $\text{Cont}(G)$, N_n , being the subobject $\text{Mono}(\Delta n,N) \hookrightarrow N^{(\Delta n)}$ of monomorphisms in $N^{(\Delta n)}$, is easily seen to be the set of 1-1 maps from n to N , which is isomorphic to G/H_n . The corresponding geometric morphism $\text{Cont}(G) \xrightarrow{f} T$ is an equivalence, since $f^*(X_n) = N_n$ for each $n \geq 1$.

Remark. The subgroup $G' \leq G$ of permutations of finite support is dense in G , so by a general remark (cf. e.g. [Fre]), $\text{Cont}(G') \simeq \text{Cont}(G)$. $\text{Cont}(G')$ was studied by Fourman [F] in connection with Fraenkel-Mostowski models. He credits the theorem above to A. Joyal.

2.2 A Generic Dense Linear Ordering

Here we again deal with fully geometric conditions:

(1) $i < i \vdash \bot$

(2) $i < j \wedge j < k \vdash i < k$

(3) $T \vdash i < j \vee i = j \vee j < i$

(4) $T \vdash \exists i . i = i$

(5) $i < j \vdash \exists k . i < k \wedge k < j$

(6) $T \vdash \exists \ell . \ i < \ell$

(7) $T \vdash \exists m . m < i$

As before, let $C \hookrightarrow \mathrm{Mod}(S,\{(1)-(4)\})$ be the category of all (nonempty) finite models of the geometric theory given by (1)-(4), i.e. the category of all nonempty finite strictly linearly ordered sets and orderpreserving injections (because of (1)). To see that there is the atomic topology on C^{op}, look at any two morphisms $Y \xleftarrow{f} X \xrightarrow{g} Z$ in C with the same domain. Then one can define a strict linear order on $Y \underset{X}{+} Z$ (between any two closest "common" elements let all y's come before all z's). The proof that the atomic topology is subcanonical is similar to 2.1.

Theorem 2.2.1. The topos DLO of atomic sheaves classifies a dense linear ordering without endpoints.

Proof. The Yoneda embedding again gives a model of (1)-(7), $\mathcal{Y}(2)$ is a dense linear order on $\mathcal{Y}(1)$: for a k-element strictly linearly ordered set K, $\mathcal{Y}(1)(K) \cong K$, so it has a natural strict linear order, and it is not hard to show that for any m-element strictly linearly ordered set M, $\mathcal{Y}(M) \hookrightarrow \mathcal{Y}(1)^m$ is the subobject $\{<x_1,\ldots,x_n> \mid x_1<x_2<\ldots<x_n$ in $\mathcal{Y}(1)\}$. As in 6.1., $\mathcal{Y}(M) \xrightarrow{\mathcal{Y}(f)} \mathcal{Y}(K)$ is epi for every morphism $K \xrightarrow{f} M$ in C, since $\mathcal{Y}$ is continuous (maps covers to covers).

Again, the topos DLO of atomic sheaves is equivalent to a topos of continuous G-sets: let G be the group of order-preserving permutations of rationals in $(0,1)$, topologized by defining the neighborhood basis of 1 of G as the set of all subgroups $H_n = \{\pi \in G \mid \pi \restriction Q_n = \mathrm{id}\}$, $n \geq 1$, where $Q_i \subseteq Q_j \subseteq \mathbb{Q} \cap (0,1)$ for $i < j$ and Q_n has exactly n elements, and $\bigcup Q_n = \mathbb{Q} \cap (0,1)$, i.e. the pointwise convergence topology [Fre].

Theorem 2.2.2. $\mathrm{DLO} \simeq \mathrm{Cont}(G)$.

Proof is similar to the proof of Theorem 2.1.2.

Similar discussion is possible for dense linear orders with and without different endpoints.

Remark. In DLO, $\mathcal{V}(1)$ is a Dedekind finite dense linear order.

2.3. Separable Closure of a Field

Here we discuss how to construct a separable closure of a given field k by forcing. We restrict ourselves to the case when one is given a field k in Sets. It is understood now that a generic separable closure can be constructed over a topos $\mathcal{E}$ by a different construction([Ken 1],[Ken 2], [W]) and that the construction really works on the level of local rings - giving the connection with strict henselization of a local ring (cf. [W]). The topos we study here - the etale topos of a field k - and the connection with the Galois theory in a sense go back to Krull. A duality (cf. proof of theorem 2.3.1. below) which implies that it is a topos of continuous G-sets, was certainly recognized in [SGA 1] Exp. V.4. More recently, it has been studied by Wraith [W] and, as an atomic topos, in [BD]. Indeed, it was through study of this and related examples in M. Hakim's book [Ha] that notion of a classifying topos really gained its importance - thus we consider it worthwhile to see that it is an example of the considerations in 1.1.

A separable closure of a field k is a field extension of k, separable over k and separably closed. All these conditions are geometric (carefully written out in [W]) - one can e.g. give a field condition by requiring:

$$\vdash x = 0 \vee \exists y(xy = 1) ;$$

a polynomial is separable iff its g.c.d. with its derivative is 1 ; every element of a separable exgension of k is a root of a separable polynomial in k (this does involve infinitary disjunctions). It is more complex to say that a field K is separably closed; given any separable polynomial in K (by its coefficients in K), there is an element of K which is its root. Now, this is rather like the infinity requirement in 2.1. or the density requirement in 2.2, so we organize the presentation as follows: we look at the category $C(k)$ of finitely presented S-models of the geometric theory of separable field extensions of k (i.e. the category of finite separable field extensions and k-linear field homomorphisms). Then the condition of X being a separably closed field extension of k is expressed (as in 2.1. and 2.2.) by requiring (for all natural numbers $m > n \geq 1$) that every subfield of X of dimension n over k has an extension in X of dimension m over k. In other words, for morphism $K \to L$ in $C(k)$ (with K of dimension n and L of dimension m) we have an epimorphism $L \to K$ as subfields of X.

Thus we are forced to consider the atomic topology on $C(k)$: it exists, since (over S) every polynomial over k splits into the product of irreducibles, so for:

$$\begin{tikzcd} & L_1 \\ K \arrow[ur, hook] \arrow[dr, hook] & \\ & L_2 \end{tikzcd}$$

in $C(k)$, $L_1 \otimes_K L_2$ splits (as a finite separable k-algebra) into finite product of fields F_i , each of which suffices to complete the above to a commutative square in $C(k)$. It is also easily seen that the atomic topology is subcanonical (by taking Galois extensions). Let

$$\mathrm{Sep}(k) \hookrightarrow S^{C(k)}$$

be the atomic topos of sheaves for the atomic topology on $C(k)^{op}$. It can also be presented as a topos of continuous G-sets: given a separable closure $\tilde{k}$ of k , let $G = G(k)$ be the group of k-automorphisms of $\tilde{k}$, topologized as a profinite group (cf. e.g. [Jo 1] 8.4.), i.e. a subgroup is open iff it contains a subgroup $G_K = \{\pi \in G \mid \pi \upharpoonright K = id\}$ for a subfield K of $\tilde{k}$, Galois over k . We have $G(k) = \varprojlim_{K/k \text{ Galois}} G(K/k)$, which on the level of topological spaces means that $G(k)$ (as a space) is a limit of (finite) discrete spaces; hence $G(k)$ is compact. So every transitive continuous $G(k)$-set is finite, since it is isomorphic to $G(k)/H$ for an open subgroup $H \leq G(k)$.

<u>Theorem 2.3.1.</u> $\mathrm{Sep}(k) \simeq \mathrm{Cont}\,(G(k))$.

<u>Proof</u>: For each finite separable field extension K of k, the set $\mathrm{Hom}_k(K,\tilde{k})$ of all k-linear field-homomorphisms is a transitive continuous $G(k)$-set under composition. Also, for a transitive continuous $G(k)$-set X, the set $\mathrm{Hom}_{G(k)}(X,k)$ of all $G(k)$-equivariant maps $X \to \tilde{k}$ is a finite separable field extension of k under pointwise operations. One readily checks that the functors $\mathrm{Hom}_k(-,\tilde{k})$ and $\mathrm{Hom}_{G(k)}(-,\tilde{k})$ give an equivalence of the category $C(k)^{op}$ and the category of all transitive $G(k)$-sets. These being atoms in $\mathrm{Cont}\,(G(k))$, it follows that toposes $\mathrm{Sep}(k)$ and $\mathrm{Cont}\,(G(k))$ are equivalent (cf. our reference to [BD] in remarks at the beginning of this chapter).

<u>Theorem 2.3.2.</u> $\mathrm{Sep}(k)$ classifies separable closures of k.

<u>Proof</u>. Either using Theorem 2.3.1. and Wraith's observation in [W] that $\tilde{k}$ is a generic separable closure as an object in $\mathrm{Cont}\,(G(k))$; or, alternatively, observing that the forgetful functor $U = y(k) = \mathrm{Hom}_C(k,-)$ is a field object in $\mathrm{Sep}(k)$ (by remembering a field structure on $U(K) = K$ over k). Moreover U is separable over the constant field Δk in $\mathrm{Sep}(k)$, and is separably closed since the Yoneda embedding is continuous. $U_L \to U_K$ is an epimorphism in $\mathrm{Sep}(k)$ for each morphism $K \to L$ in C. Here for each K in C: U_K = object of all Δk-linear field homomorphisms $\Delta K \to U$ (as a subobject of $U^{(\Delta K)}$ in $\mathrm{Sep}(k)$).

<u>Remark</u>. Alternatively, one can prove theorem 2.3.1. from the fact that both $\mathrm{Sep}(k)$ and $\mathrm{Cont}\,(G(k))$ classify separable closure of k.

The inclusion of $\mathrm{Sep}(k)$ (as $\mathrm{Cont}(G(k))$) into $S^{G(k)}$ has a right adjoint, the inclusion of $\mathrm{Sep}(k)$ into $S^{C(k)}$ has a left adjoint:

$$\begin{tikzcd} & S^{G(k)} \arrow[dl, shift left] \arrow[dd, shift right, "f_*"'] \\ \mathrm{Sep}(k) \arrow[ur, hook, shift left] \arrow[dr, hook, shift right] & \\ & S^{C(k)} \arrow[ul, dotted, shift right] \arrow[uu, shift right, "f^*"'] \end{tikzcd}$$

The resulting geometric morphism $S^{G(k)} \xrightarrow{f} S^{C(k)}$ has $\mathrm{Sep}(k)$ as its image.

Remark. Let $\overline{k}$ be an algebraic closure of k in Sets. $\Delta\overline{k}$ is not a subobject of U in $\mathrm{Sep}(k)$.

Chapter 3

FORCING IN SET THEORY

3.1. $\neg\neg$-topology

We now discuss how original P. J. Cohen's method of forcing and its variant in terms of Boolean-valued models fit with our general outlook given in 1.1. The connection with toposes, for cases 3.1.2. and 3.1.3. below, has been discussed in the literature ([Til], [Bu], [F]). However, there are still some important things worth noticing (Theorems 3.1.1.2.-3) and even for those particular cases, it is worth stating them precisely as examples of the method discussed in 1.1. Moreover, our discussion in section 3.1.1. will be quite general and we will refer to it in 3.2 as well.

3.1.1. $\neg\neg$-sheaves on a poset P

Standard set-theoretic textbooks (e.g. [Je],[Bel]) discuss the method of forcing over a partial order P. In this case, the object G in S^P given in 1.1. reduces to the functor:

$$G(p) = \{p' \in P \mid p' \leq P\} \quad ,$$

The $\neg\neg$-topology on P consists of the following sieves: for each $p \in P$, let $R \hookrightarrow p$ be a covering sieve iff $\forall q \geq p . \exists r \geq q . r \in R$ (here we use identification of sieves $R \hookrightarrow p$ with upward closed subsets of $\{q \in P \mid q \geq p\}$ and identify each $p \in P$ with the representable functor given by $p(q) = 1$ iff $p \leq q$. The category of sheaves $Sh_{\neg\neg}(P) \hookrightarrow S^P$ for this topology is generated by subobjects of 1, so it is equivalent to the topos of (canonical) sheaves on the complete Heyting algebra B of these subobjects ([Jo 1], 5.37) (in this case B is actually Boolean). Namely, B consists of $\neg\neg$-closed subjects of 1 in S^P. Assuming that 1 is representable (i.e. that P has the least element) and checking with [Ti 1] that for a subfunctor $R \hookrightarrow p$ (with the above identification) we have

$$\neg R = \{q \geq p \mid \forall q' \geq q . q' \notin R\}$$

$$\neg\neg R = \{q \geq p \mid \forall q' \geq q . \exists r \geq q' . r \in R\}$$

(hence the name $\neg\neg$-topology), and so $\neg\neg R = R$ (i.e. R is $\neg\neg$-closed) iff

$$\forall q \geq p [\forall q' \geq q . \exists r \geq q' . r \in R \rightarrow q \in R] ,$$

and $R \hookrightarrow p$ is $\neg\neg$-dense (i.e. a covering sieve in $\neg\neg$- topology) iff $\neg\neg R = p$. Comparing this with the language of [Je] §16, $R \hookrightarrow 1$ is a covering sieve for the Grothendieck topology given by $\neg\neg$ in S^P iff R is dense in the (topological) topology on P given by the basis consisting of sets $\{q \in P \mid q \geq p\}$ for $p \in P$, and $R \hookrightarrow 1$ is $\neg\neg$-closed iff it is regular open set in that topology. Thus we have shown:

Proposition 3.1.1.1. The topos $Sh_{\neg\neg}(P)$ is equivalent to the topos $Sh(RO(P^{op}))$ of (canonical) sheaves over a complete Boolean algebra $RO(P^{op})$, the Boolean completion of a partial order P .

It is sometimes said that it is easier to think in terms of forcing than in terms of Boolean-valued models. Forcing semantics being sheaf semantics (cf. e.g. [JR], [BJ], [Os]) in $Sh_{\neg\neg}(P)$, and Boolean-valued semantics being an interpretation in $Sh(RO(P^{op}))$ (cf. [FSo],[F]), the above Proposition assures us that we can work in one setting and "think in another", and vice versa. Fourman [F] gives the precise relationship between the topos $Sh(RO(P^{op}))$ and the Boolean-valued universe $V^{(RO(P^{op}))}$ as defined e.g. in [Bel]. An ultrafilter on $RO(P^{op})$ gives us a 2-valued model.

Another requirement in §16 of [Je] (or cf. Chapter 2 of [Bel]) becomes clear from the point of view taken in 1.1: in [Je], P is called separative iff:

for every $p,q \in P$ $p \not\leq q$ implies $\exists r \geq q . \neg\exists s \geq r . s \geq p$

(refined, in terms of [Bel]; also, both [Je] and [Bel] talk about P^{op} , cf. our 1.1.).

Theorem 3.1.1.2. The following are equivalent:

(i) The functor G in S^P given by $G(p) = \{p' \in P \mid p' \leq p\}$ is $\neg\neg$-closed as a subfunctor of the constant functor ΔP , i.e. $\neg\neg G = G$ as $G \hookrightarrow \Delta P$.

(ii) P is separative

(iii) $\neg\neg$-topology is subcanonical

(iv) P^{op} is order-isomorphic to a dense subset of the complete Boolean algebra $RO(P^{op})$.

Proof.

(i)<=>(ii): $(\neg\neg G)(q) = \{p \in P \mid \forall r \geq q . \exists s \geq r . p \in G(s)\} = \{p \in P \mid \forall r \geq q . \exists s \geq r . s \geq p\}$. So $\neg\neg G = G$ iff for every $p, q \in P$, $\forall r \geq q . \exists s \geq r . s \geq p$ implies $p \leq q$ (*) which is equivalent to the separativity of P (classically).

(i) => (iii): Assuming (*), we want to show that every (representable functor) p is a sheaf. So, given any q and a $\neg\neg$-dense $R \hookrightarrow q$, we need to show that every natural transformation $R \hookrightarrow p$ extends (necessarily uniquely) as:

$$\begin{tikzcd} R \arrow[r, hook] \arrow[dr] \arrow[dr, phantom, "///"] & q \arrow[d] \\ & p \end{tikzcd} .$$

But the components of any such $R \xrightarrow{\lambda} p$ are identities, hence for each $s \in R$, $s \geq p$. Since $R \hookrightarrow q$ is $\neg\neg$-dense, $\forall r \geq q . \exists s \in R . s \geq r$, so for that s we have $s \geq p$. By (i), $p \leq q$, i.e. there exists a morphism $q \to p$ in S^P .

(iii) => (i). Given p and q , assume $\forall r \geq q . \exists s \geq r . s \geq p$. Hence the subfunctor $* \to q$ determined by the following pullback in S^P :

$$\begin{tikzcd} * \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\text{p.b.}"] & p \arrow[d, hook] \\ q \arrow[r, hook] & 1 \end{tikzcd}$$

is $\neg\neg$-dense. Since p is a $\neg\neg$-sheaf, $* \hookrightarrow p$ extends to:

$$\begin{tikzcd} * \arrow[r, hook] \arrow[dr, hook] \arrow[dr, phantom, "///"] & q \arrow[d, "!"] \\ & p \end{tikzcd}$$

hence $p \leq q$.

(iii) <=> (iv). (iii) means that the Yoneda embedding $P^{op} \hookrightarrow S^P$ factors through $Sh_{\neg\neg}(P)$, which is rephrased by (iv).

Remark. (ii) <=> (iv) is standard in set theory ([Je], [Bel]). Here we give a different proof connecting them with (i), (iii). Notice also that our proof is constructive, except for taking the contrapositive of (*) to get separativity. But separativity is usually used in the form (*) (cf. e.g. [Je]), and it should be stated this way.

Conditions expressed in Theorem 3.1.1.2 are the consistency conditions on $Sh_{\neg\neg}(P)$ (cf. 1.1.). An even stronger condition is to require, in addition, that $Sh_{\neg\neg}(P)$ has no points (i.e. that the cBa $RO(P^{op})$ is atomless). This runs parallel to the usual forcing construction in set theory which starts with a suitable model M of ZF set theory, and for a partially ordered set $P \in M$ wants a "generic set" $G \subseteq P$, i.e. $G \subseteq P$ satisfying the following conditions:

(a) $q \leq p \in G \Rightarrow q \in G$

(b) $p,q \in G \Rightarrow \exists r \in G \cdot r \geq p$ and $r \geq q$

(c) G intersects every dense open (in the topology defined immediately before Proposition 3.1.1.1.) $X \subseteq P$, $X \in M$;

and then gets a new model $M[G]$ of ZF with $G \in M[G]$, which properly contains M (if $G \notin M$) . Instead of M we have (the Boolean topos of) Sets, and $M \subsetneq M[G]$ becomes the above strong nontriviality requirement. Let us assume, furthermore, that in P there are sups of every p,q which are bounded above. Now, notice that (a)-(c) are geometric requirements giving a geometric theory for which there is a canonical (not to overuse the word "generic") model in the classifying topos. We can think of this canonical generic (variable) set as being given by the truth values $a_p = [\![p \in G]\!]$. So our geometric language will have propositional constants a_p for $p \in P$ and the theory T will have the following groups of axioms:

$$
\begin{array}{lll}
(0) & a_p \vdash a_q & \text{for } q \le p \\
 & a_p \wedge a_q \vdash a_{p \vee q}\ , & \text{if } \exists r \in P \cdot r \ge p \text{ and } r \ge q \\
(1) & & \\
 & a_p \wedge a_q \vdash \perp\ , & \text{otherwise} \\
(2) & T \vdash \bigvee_{p \in X} a_p\ , & \text{for every } \neg\neg\text{-dense } X \hookrightarrow 1
\end{array}
$$

(in the light of the above, (2) is just a restatement of (c)).

The following theorem is the main result of this section.

<u>Theorem 3.1.1.3</u>. Let P be a partially ordered set with the least element such that every $p,q \in P$ bounded above in P have a sup. Let $\neg\neg$-topology on P be subcanonical. Then the topos $Sh_{\neg\neg}(P)$ classifies the geometric theory T whose axioms are (0)-(2) above.

<u>Proof</u>. We can assume that P^{op} is left exact (i.e. all finite limits exist), since in S^P (and in $Sh_{\neg\neg}(P)$) every non-representable pullback of representables is 0. But this is described precisely by axioms of group (1). It remains to show that pullbacks of sieves $X \hookrightarrow 1$ mentioned in axioms (2) generate $\neg\neg$-topology. Indeed, given $\neg\neg$-dense $R \hookrightarrow p$, we define a $\neg\neg$-dense $S \hookrightarrow 1$ by $S = R \cup \{q \in P | q \text{ incomparable with } p\}$. Obviously:

$$
\begin{tikzcd}
R \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\text{p.b.}"] & S \arrow[d, hook] \\
p \arrow[r, hook] & 1
\end{tikzcd}
$$

is a pullback. So given any Grothendieck topos F, F-model M of T is just a left exact continuous functor on P (cf. [MR], Chapters 1 and 9; [SGA 4], Exp. III.1; or [Jo 1], 7.13). Moreover, the functor G given in (1) of Theorem 3.1.1.2. is a $Sh_{\neg\neg}(P)$-model of T, whose inverse image is M. Alternatively, one describes the canonical model in $Sh(RO(P^{op}))$ as the $RO(P^{op})$-valued set $G(p) = p$.

<u>Remark</u>. Both constructions presented in the remainder of this chapter will be, in effect, cases of Theorem 3.1.1.3.

3.1.2. Independence of Continuum Hypothesis

The method of forcing was born out of pioneering work of P.J. Cohen [Coh] on this problem. This particular construction has been widely discussed in set-theoretic textbooks (e.g. [Je],[Bel]) and from topos-theoretic point of view (cf. [Ti 1]). Here we want to emphasize it as a case of the general approach given in 1.1 and consideration of the previous section.

The problem is to find $\aleph_2$ different subsets of N, i.e. an enumeration

$$\aleph_2 \hookrightarrow 2^N \quad .$$

One has to keep in mind that we want a Boolean model, (i.e. with classical logic) so these subests $P_\alpha (\alpha \varepsilon \aleph_2)$ should be given together with their "complements" R_α as primitive notions. Booleanness then includes the decidability condition. We thus first look at the geometric theory T_1 of $\aleph_2$ "complemented" predicates (the requirement that they be distinct will be thought of as a density condition later on). T_1 is given by:

$$P_\alpha(x^\alpha) \wedge R_\alpha(x^\alpha) \vdash \perp$$

The rigidification in the category of S-models of T_1 is given by the requirement that P_α's and R_α's be predicates on subsets of natural numbers (and that the morphisms in Mod (S,T_1) should preserve that additional structure). This reduces us to a poset given by inclusion. The simplest models of T_1 of this kind are the finite ones: the poset P of the forcing conditions will be the set of all finite functions $N \times \aleph_2 \to \{0,1\}$ ordered by inclusion. Every such function f is a model of T_1 in S, reading $f(n,\alpha) = 0$ as $R_\alpha(n)$ and $f(n,\alpha) = 1$ as $P_\alpha(n)$.

As before, we define the generic object G by values $p_{\alpha,n}$ and $r_{\alpha,n}$ of statements "$n \varepsilon P_\alpha$" and "$n \varepsilon R_\alpha$" respectively: we look at a variable subobject G' of the constant presheaf $\Delta(N \times \aleph_2)$ in S^P given by:

$$G'(p) = \{\langle n,\alpha\rangle \varepsilon N \times \aleph_2 \mid \langle\langle n,\alpha\rangle,1\rangle \subseteq p \text{ in } P\} \quad .$$

(Alternatively, we may look at the object G given by finite conjunctions of statements "$n \in P_\alpha$" and "$n \notin P_\alpha$" :

$$G\ (p) = \{p' \in P \mid p' \subseteq p \text{ in } P\}\ .)$$

To get a Boolean topos, we take $\neg\neg$-sheaves in S^P . Notice that for this poset P , $\neg\neg$-topology is subcanonical. It is easy to check that both G and G' are $\neg\neg$-closed, and thus correspond uniquely to sheaves in $Sh_{\neg\neg}(P)$.

As shown in 3.1.1., $Sh_{\neg\neg}(P)$ classifies the propositional geometric theory T_2 given by,

(0) $a_p \vdash a_q$ for $q \leq p$ in P

(1) (a) $a_p \wedge a_q \vdash a_{p \cup q}$ if $p \cup q \in P$

(1) (b) $a_p \wedge a_q \vdash \bot$ if $p \cup q \notin P$

(2) $T \vdash \bigvee_{p \in X} a_p$ for every $\neg\neg$-dense crible $X \hookrightarrow 1$ in S^P (i.e. "open dense" subset of P , in set theoretic terms).

Notice that (2) takes care of decidability, since:

$$X_{\alpha,n} = \{p \in P \mid p \supseteq p_{\alpha,n} \text{ or } p \supseteq r_{\alpha,n}\}$$

is a $\neg\neg$-dense sieve for any $\alpha \in \aleph_2$, $n \in N$. It also takes care of the requirement that $P_\alpha \neq P_\beta$ for $\alpha \neq \beta$, since for such α,β :

$$Y_{\alpha,\beta} = \{p_{\alpha,n} \cup r_{\beta,n} \mid n \in N\}$$

generates a $\neg\neg$-dense sieve.

Since P is c.c.c., for constant sheaves $\check{\aleph}_i$ in $Sh_{\neg\neg}(P)$ we have $[\![\exists \text{ epimorphism } \check{\aleph}_i \to \check{\aleph}_j]\!]_{Sh_{\neg\neg}(P)} = 0$ for $i < j \leq 2$, cf. [Til] pp.32-37; [Je], p.65, and we see that G gives $\aleph_2$ different subsets of N in $Sh_{\neg\neg}(P)$ (a non-geometric requirement that the "cardinals are preserved").

3.1.3. Independence of Suslin's Hypothesis

Here we are interested in constructing a Suslin tree, i.e. a partial order $\leq$ on $\aleph_1$ such that for every $\alpha \in \aleph_1$, $\{\beta \in \aleph_1 \mid \beta \leq \alpha\}$ is a linearly ordered set ($\leq$ is a tree), and every chain and antichain is countable. Solutions to this problem were given independently by Tennenbaum, [Ten] and Jech [Je]. For a topos-theoretic treatment, cf. Bunge [Bu]. There are many equivalent formulations of this problem. In section 3.2. (but not here) we will work with an alternative definition of a Suslin tree which requires, in addition to the conditions listed above, that $\{\beta \in \aleph_1 | \alpha \leq \beta\}$ is uncountable for every $\alpha \in \aleph_1$. Although this is not equivalent to the previous definition, it can be easily seen that in every Suslin tree $\langle \aleph_1, \leq \rangle$ (on the first definition) there is $\alpha \in \aleph_1$ such that $\{\beta \in \aleph_1 | \alpha \leq \beta\}$ satisfies the additional condition w.r.t. the induced ordering (it obviously satisfies the other ones), cf. Appendix in [S-T].

We need a Boolean model, so we work with "complemented notions" as before. Let us consider the tree property as a simple geometric property, i.e. T_1 is given by:

$$\begin{aligned}
\pi(x,y) \wedge \rho(x,y) &\vdash \bot \\
\top &\vdash \pi(x,x) \\
\pi(x,y) \wedge \pi(y,z) &\vdash \pi(x,z) \\
\pi(x,y) \wedge \pi(y,x) &\vdash x=y \\
\pi(y,x) \wedge \pi(z,x) &\vdash \pi(y,z) \vee \pi(z,y)
\end{aligned}$$

(we think of $\pi(x,y)$ as $x \leq y$ and $\rho(x,y)$ as $x \nleq y$). First of all we are interested only in a category of S-models of T_1 in which equality in interpreted as actual identity. The rigidifying conditions are given by the requirements that morphisms of models preserve a given enumeration by $\aleph_1$ which "agrees with π ", i.e. we are looking at trees in $\aleph_1$ such that $\alpha \leq \beta$ implies $\alpha \leq \beta$ in a "natural" order; morphisms of models are therefore inclusions. The category of all finite such models is the pre-order P of forcing conditions.

As in 3.1.2, we look at the topos $Sh_{\neg\neg}(P)$ (additional geometric conditions). The generic presheaf $G' \hookrightarrow \Delta(\aleph_1 \times \aleph_1)$ given by:

$$G'(p) = \{\langle\alpha,\beta\rangle \in \aleph_1 \times \aleph_1 | \pi_p(\alpha,\beta)\}$$

is easily seen to be $\neg\neg$-closed, and the corresponding sheaf is a Suslin tree (cf. [Ten]; [Bu] §4), and "cardinals are preserved" since P is c.c.c. For a different analysis of forcing conditions, cf. [Je] §21.

3.2. Fibred Categories and Iterated Forcing.

We have seen in Chapter 1 how the notions of a sheaf and of a classifying topos, developed by A. Grothendieck and his school through study of algebraic geometry and algebraic topology, underlie the method of forcing. Another central (in a sense, more elaborate) notion in their work is one of descent or of fibred category (cf. [SGA 1] Exp. VI, VIII; [SGA 4], Exp. VI §§6-8). In this section we show a strong connection between categorical constructions with fibred categories and iterated forcing, on the example of Solovay-Tennenbaum forcing of (consistency of) Suslin's hypothesis [ST]. In particular, we show that this forcing construction follows Grothendieck's construction of the (lax) limit of a fibred topos ([SGA 4], Exp. VI, §8). Thus the connection between algebraic geometry and logic has turned up even on this more refined level.

We are convinced that the constructions of completions (limits and colimits) of fibred categories is fundamental itself, so that Solovay-Tennenbaum iterated forcing, and Grothendieck's more general construction mentioned above should be thought of as its cases in the sense that they show that a particular additional structure (in those cases: model of set theory; Grothendieck topos) is preserved by such a construction.

3.2.0. The problem we are concerned with here is to ruin all possible Suslin trees (cf. 3.1.3). Given a Suslin tree $T = \langle \aleph_1, \leq \rangle$, it is easy to construct a generic uncountable branch on T in a suitable Boolean topos E (we will do this shortly). But E may also have more Suslin trees than S, and we will have to take care of those "after" eliminating all Suslin trees in S: this calls for iteration. There are at most $2^{\aleph_1}$ Suslin trees in S and if we arrange matters in such a way that this still holds in any topos E obtained in the iterated construction of generic uncountable branches (this is the case if a preorder of forcing conditions is c.c.c., cf. [Je], p.65, [Ti 1], p.32), we may hope that it suffices to iterate this construction in $2^{\aleph_1}$ steps. For the bookkeeping in the iteration process, the assignment $x \rightsquigarrow \langle px, qx \rangle$, such that for all $x \in 2^{\aleph_1}$ except the first one, $qx < x$ in a well-ordering of $2^{\aleph_1}$ (cf. 3.1.3.). Notice that usual set-theoretic construction (cf. [Je] p.11) of such an isomorphism by the lexicographic well-ordering on $2^{\aleph_1} \times 2^{\aleph_1}$ does not use Replacement (which is only used to show that such an ordering is canonically given for all ordinals; but here we are interested only in a particular "ordinal" at a time, namely $2^{\aleph_1} \times 2^{\aleph_1}$).

3.2.1. We first briefly describe how to construct a generic branch. Let us analyse the problem following the principles discussed in 1.1: we are given a Suslin tree $T = \langle \aleph_1, \leq \rangle$; to construct an uncountable chain it suffices to look for an uncountable ideal C in T, i.e. an uncountable subset C of T such that:

(i) $\alpha \in C \wedge \beta \leq \alpha \vdash \beta \in C$

(ii) $\alpha \in C \wedge \beta \in C \vdash \exists \delta \in C . \delta \geq \alpha, \delta \geq \beta$

(with axioms for $\leq$),
and since we are interested in a classical setting for this, we have to introduce $(\alpha \notin C)$ as primitive to describe it geometrically:

(iii) $\alpha \in C \wedge \alpha \notin C \vdash \perp$

(iv) $\top \vdash \alpha \in C \vee \alpha \notin C$

(these two requirements are trivially satisfied in set models, since S is Boolean). Naturally, (i) and (ii) generate a theory T_1. The category P of forcing conditions is determined, as usual, as a subcategory of the category of set-models of T_1. The rigidifying condition is already mentioned above, namely that C be an ideal in T, thus introducing a poset (since elements of ideals are fixed) as the inclusion of ideals. As before, we consider principal ideals: these will be forcing conditions. Then, the order on P described above coincides with the one in T. Conditions (iii) and (iv) secure Booleanness. We are led to the geometric theory given in terms of propositions $a_\alpha = "\alpha \in C"$ and $b_\alpha = "\alpha \notin C"$, for $\alpha \in T$:

(1) $a_\alpha \vdash a_\beta$, for $\beta \leq \alpha$

(2) $a_\alpha \wedge a_\beta \vdash \bot$, if α and β are not on the same branch

(3) $a_\alpha \wedge b_\alpha \vdash \bot$

(4) $\top \vdash a_\alpha \vee b_\alpha$

The uncountability is a non-geometric requirement that will have to be verified at the end. To make this easier, and to show consistency at the same time, we notice that (1)-(4) are satisfied for sheaves associated to representable presheaves in $Sh_{\neg\neg}(P) \hookrightarrow S^P$. Furthermore, in working with $Sh_{\neg\neg}(P)$ we can rely on the fact that $\aleph_1$ is uncountable, since $P = T$ is a Suslin tree, therefore c.c.c. tree.

It is easy to see that the associated sheaf of the generic object $G \in S^P$ (also to be called G ; its characteristic function on T is given by $G(\alpha) = [\alpha]$, cf. 3.1) is a chain in T . To see that it is uncountable, recall that T was a Suslin tree, so we can assume that $\{\beta \in T \mid \beta \geq \alpha\}$ is uncountable for every $\alpha \in T$ (cf. 3.1.3.). Also, since we are in a Boolean topos, it suffices to refute that G is countable.

Thus, suppose that G is countable, then for some $\lambda \in \aleph_1$ (by density): $[\![\forall \alpha \in G.\, \alpha < \lambda]\!] \supseteq [\delta] \neq 0$. Since: $[\![\forall \alpha \in G.\alpha<\lambda]\!] = \bigwedge_{\alpha \in \aleph_1} \{\alpha \to [\![\alpha<\lambda]\!]\}$, we would have:

$$[\delta] \cap [\alpha] \subseteq [\![\alpha<\lambda]\!] = \begin{cases} 1 , & \text{if } \alpha<\lambda \text{ ("ordinary" well-ordering on } \aleph_1 \text{)} \\ \emptyset , & \text{otherwise.} \end{cases}$$

But take α with $\delta \leq \alpha$. Then $[\delta] \cap [\alpha] \neq \emptyset$, so $\alpha < \lambda$, and hence $\{\alpha \mid \alpha \geq \delta\}$ is not uncountable, a contradiction. We urge the reader to compare this computation in $Sh_{\neg\neg}(P)$ with the corollary given in [ST] and to notice similarities and differences between the two. (Cf. remarks at the end of 1.1., and recall that G is also "$\mathcal{P}(P)$-generic" in set-theoretic terms: it is true in $Sh_{\neg\neg}(P)$ that it intersects any (constant) "open dense" subset $X \subseteq P$, i.e. any $\neg\neg$-dense crible $X \hookrightarrow 1$).

3.2.2. We now describe how to iterate the above construction. Let us fix a well-ordering on $2^{\aleph_1}$ and the isomorphism $2^{\aleph_1} \overset{\approx}{\to} 2^{\aleph_1} \times 2^{\aleph_1}$ described in 3.2.0. The following procedure is defined by recursion on $2^{\aleph_1}$; we define a $2^{\aleph_1}$-sequence of c.c.c. complete Boolean algebras (cBa's):

$$B_0 \hookrightarrow B_1 \hookrightarrow B_2 \hookrightarrow \dots \hookrightarrow B_\omega \hookrightarrow \dots \hookrightarrow B_{\aleph_1} \hookrightarrow \dots$$

such that for every $\alpha \in 2^{\aleph_1}$, B_α is a complete Boolean subalgebra of $B_{\alpha+1}$ and there is a surjection $2^{\aleph_1} \to B_\alpha$.

<u>Stage 0</u>. Let $B_0 = 2$.

<u>Successor stage $\alpha+1$</u>. Let B_α be c.c.c. cBa with a surjection $2^{\aleph_1} \to B_\alpha$ (so $B_\alpha \to 2^{\aleph_1}$ as sets). The powerset $P_\alpha(\aleph_1)$ of $\check{\aleph}_1$ in $E_\alpha = Sh(B_\alpha)$ consists of some maps $\aleph_1 \to B_\alpha$. Since B_α is c.c.c., $\check{\aleph}_1$ is $\aleph_1$. Thus $P_\alpha(\aleph_1) \hookrightarrow (2^{\aleph_1})^{\aleph_1} \simeq 2^{\aleph_1 \times \aleph_1} \simeq 2^{\aleph_1}$ as sets. This also gives estimate on $P_\alpha((\check{\aleph}_1 \times \check{\aleph}_1))$. Let R be a c.c.c. partial order on $\aleph_1$ in $Sh(B_\alpha)$. By the above estimates applied to $B_{q_\alpha} \hookrightarrow B_\alpha$ let P_α be $p_\alpha{}^{th}$ such $R \in Sh(B_{q_\alpha})$. Notice that we require $R \in Sh(B_{q_\alpha})$, but $[\![R$ is a c.c.c. partial order of $\aleph_1]\!] B_\alpha = 1$. It follows that $[\![\dots\dots]\!] B_{q_\alpha} = 1$ (as shown below by lemmata). Look at $F_\alpha = P_\alpha(\aleph_1)$, and repeat the construction from 3.2.1. in $Sh(B_\alpha)$. Let E_α be the topos of $\neg\neg$-sheaves in P_α in $Sh(B_\alpha)$. Thus E_α is a topos over $E_\alpha = Sh(B_\alpha)$ and (working in E_α):

$$E_\alpha \models \text{ there is } F_\alpha \text{ - generic ideal on } P_\alpha .$$

We may think of E_α as the topos of internal sheaves on an internal cBa $\mathbb{B}_\alpha$ in E_α. Taking $B_{\alpha+1} = \Gamma(\mathbb{B}_\alpha)$ (the global sections of the sheaf $\mathbb{B}_\alpha \in E_\alpha$) we get cBa $B_{\alpha+1}$ such that:

$$\begin{array}{ccc}
Sh_{E_\alpha}(\mathbb{B}_\alpha) & \simeq & Sh(B_{\alpha+1}) \\
\Big\downarrow \Gamma_E & /\!/\!/ & \Gamma_\alpha \Big\downarrow \Big\uparrow i_\alpha \\
E_\alpha & \simeq & Sh(B_\alpha)
\end{array}$$

commutes; B_α is a complete Boolean subalgebra of $B_{\alpha+1}$, and Γ_α is the right adjoint to inclusion i_α. Indeed, general theory tells us (cf. e.g. [FSo] §9.9) that B_α is a complete Heyting algebra, that $i_\alpha \dashv \Gamma_\alpha$ and that the diagram commutes. Moreover, $i_\alpha(X)$ generates $\Gamma(X)$ in $Sh(\Gamma(\mathbb{B}_\alpha))$ for any $X \in Sh(B_\alpha)$. Thus we just have to see that $B_{\alpha+1}$ is Boolean and that i_α is a complete Boolean inclusion, and we will show this in lemmata below, as well as that $B_{\alpha+1}$ is c.c.c., and of the appropriate "size".

In other words, we have for $E_{\alpha+1} = Sh(B_{\alpha+1})$:

$$E_{\alpha+1} \models \text{there is } F_\alpha\text{-generic ideal of } P_\alpha \ .$$

<u>Limit stage λ</u>. A limit stage involves limits $\varprojlim_{\alpha<\lambda} Sh(B_\alpha)$. To make this precise let $\Sigma_\lambda = \sum_{\alpha<\lambda} B_\alpha$, and consider each B_α as a site with the canonical Grothendieck topology. We want to show that Σ_λ can be organized into a category such that:

$$\Sigma_\lambda \xrightarrow{\text{index}} \lambda^{op}$$

is a fibration. Morphisms in B_α are determined by the preorder in B_α, and morphisms over $\beta \to \alpha$ (i.e. $\alpha < \beta$) are all maps $b_\beta \xrightarrow{f} a_\alpha$ (for $b_\beta \in B_\beta$, $a_\alpha \in B_\alpha$) which factor through $i(a_\alpha)$, where i is the inclusion $B_\alpha \xhookrightarrow{i} B_{\alpha+1}$:

$$\begin{tikzcd}
b_\beta \arrow[dr, dashed, "//"'] \arrow[drr] & & \\
& i(a_\alpha) \arrow[r] & a_\alpha
\end{tikzcd}$$

$$\beta \longrightarrow \alpha$$

One can easily convince himself that this defines a preorder on Σ_λ. Cartesian morphisms in Σ_λ (cf. [SGA 4], Exp. VI. 6) are precisely morphisms $i(a_\alpha) \to a_\alpha$: indeed, all those morphisms are cartesian; on the other hand, if $b_\beta \to a_\alpha$ is cartesian, then:

$$\begin{tikzcd}
i(a_\alpha) \arrow[dr, dashed] \arrow[drr] & & \\
& b_\beta \arrow[r] & a_\alpha
\end{tikzcd}$$

$$\beta \longrightarrow \alpha$$

so $b_\beta = i(a_\alpha)$ by definition of morphisms in Σ_λ. If $\beta = \alpha$, $i(a_\alpha) \to a_\alpha$ is the identity. Objects over α in Σ_α are precisely the elements of B_α, and inverse image functors are inclusions $B_\alpha \to B_\beta$ for $\alpha<\beta<\lambda$. Thus Σ_λ is a fibred category over λ^{op}. Moreover, since the inclusions are complete, Σ_λ is a <u>fibred site</u> in the sense of [SGA 4] Exp. VI. 7.2.

λ is a filtered category, so λ^{op} satisfies the properties L1)-L2) from [SGA 4] Exp. VI. 6.4. Thus by 6.8.2) there, we have:

$$\varinjlim_{\alpha<\lambda} B_\alpha \simeq \Sigma_\lambda\,[\mathrm{cart}]^{-1}$$

(i.e. the category of fractions obtained by formally inverting all cartesian morphisms in Σ_λ), where $\varinjlim$ is taken in the sense of a <u>functor</u> $\lambda \to \mathrm{Cat}$ given by the assignment $\alpha \rightsquigarrow B_\alpha$. (This is indeed a functor, and all isomorphisms involved in descent data are identities.) Obviously,

$$\varinjlim_{\alpha<\lambda} B_\alpha \simeq \bigcup_{\alpha<\lambda} B_\alpha$$

($B_{<\lambda} = \bigcup_{\alpha<\lambda} B_\alpha$ is a Boolean algebra). Let the <u>total topology</u> on Σ_λ be the smallest Grothendieck topology making all inclusions $B_\alpha \hookrightarrow \Sigma_\lambda$ continuous. Its covering sieves $R \hookrightarrow a_\alpha \in B_\alpha$ are exactly those which contain a B_α-covering sieve $R^\alpha \hookrightarrow a_\alpha$ in B_α (cf. [SGA 4] Exp. VI. 7.4.2.)

Let:

$$\Sigma_\lambda \xrightarrow{\pi_\lambda} \Sigma_\lambda\,[\mathrm{cart}]^{-1}$$

be the canonical functor. Let J_λ be the smallest topology on $B_{<\lambda}$ making π_λ continuous. Since

$$\begin{tikzcd}
B_\alpha \arrow[r, hook] \arrow[dr, "i_\alpha"'] & \Sigma_\lambda \arrow[d, "\pi_\lambda"] \\
& B_{<\lambda}
\end{tikzcd}$$

commutes for all $\alpha<\lambda$ (with i_α canonical inclusions), J_λ is the smallest topology on $B_{<\lambda}$ making all canonical inclusions i_α continuous. Then the main result of [SGA 4] Exp. VI.8 says that the topos $E_{<\lambda}$ of J_λ-sheaves on $B_{<\lambda}$ is the $\varprojlim$ of the system

$$S \underset{\Gamma}{\overset{i}{\leftrightarrows}} Sh(B_1) \underset{\Gamma}{\overset{i}{\leftrightarrows}} \cdots\cdots \underset{\Gamma}{\overset{i}{\leftrightarrows}} Sh(B_\alpha) \underset{\Gamma}{\overset{i}{\leftrightarrows}} \cdots \ , \ \alpha<\lambda$$

which can be thought of as a fibred topos $F_\lambda \xrightarrow{\text{index}} \lambda^{op}$ associated to the fibred site Σ_λ by the canonical topologies on each B_α (i.e. each fiber of F_λ is the topos $Sh(B_\alpha)$) . More precisely,

$$\Sigma_\lambda \xrightarrow{<\pi_\lambda\,,\ \text{index}>} B_{<\lambda} \times \lambda^{op}$$

is a morphism from a fibred site $B_{<\lambda} \times \lambda^{op} \xrightarrow{\text{projection}} \lambda^{op}$ to a fibred site Σ_λ , which induces a morphism:

$$E_{<\lambda} \times \lambda^{op} \xrightarrow{\mu_\lambda} F_\lambda$$

of fibred topoi which preserves cartesian morphisms, universal among all such morphisms of fibred topoi:

$$G \times \lambda^{op} \xrightarrow{g_\lambda} F_\lambda$$

for a Grothendieck topos G ; i.e. for any such g_λ there is a unique (up to a unique isomorphism) geometric morphism

$$G \xrightarrow{g} E_{<\lambda}$$

such that:

$$\begin{tikzcd} E_{<\lambda} \times \lambda^{op} \arrow[r, "\mu_\lambda"'] & F_\lambda \\ G \times \lambda^{op} \arrow[u, "<g \times id>"] \arrow[ur, "g_\lambda"'] & \end{tikzcd}$$

commutes (up to isomorphism). To determine the topology J_λ on $B_{<\lambda}$, observe that J_λ contains the collection J^λ of sieves in B given by:

$R \hookrightarrow p \in J^\lambda(p)$ iff $\exists \alpha<\lambda$ with $p \in B_\alpha$, such that B_α-covering sieve $R^\alpha \hookrightarrow p$ which factors through R , since J_λ makes π_λ continuous. We claim that the collection J^λ generates J_λ as a Grothendieck topology. Indeed, let T^λ be the Grothendieck topology on $B_{<\lambda}$ generated by J^λ. Since π_λ (and hence all i_α's) is continuous w.r.t. T^λ, we have $J_\lambda \leq T^\lambda$. But since J_λ contains J^λ, we have $T^\lambda \leq J_\lambda$.

We next observe that the collection J^λ is <u>closed under pullbacks</u>, so that it suffices to check sheaf-condition w.r.t. J^λ (cf. [SGA 4] Exp. II, Corollary 2.3): let $R = (p_i \to p)_{i \in I}$ be a J^λ-sieve in $B_{<\lambda}$, so let $R^\alpha = \{r_k \to p\}_{k \in K}$ be a B_α-covering sieve where α is such that r_k, $p \in B_\alpha$ (i.e. we have $p = \bigvee r_k$ in B_α). Let $q \overset{f}{\to} p$ in B_α. We want to show that:

$$f^*(R) = \{s \to q \mid (s \to q \to p) \in R\} \hookrightarrow q$$

is a J^λ-sieve in $B_{<\lambda}$. To see this, it suffices to find $\delta < \lambda$ such that $q \in B_\delta$ and a B_δ-covering sieve $S^\delta \hookrightarrow q$ such that $S^\delta \hookrightarrow f^*(R)$. Say $q \in B_\beta$. Look at pullbacks (in $B_{<\lambda}$):

$$\begin{tikzcd} t_k \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{p.b.}"] & r_k \arrow[d] \\ q \arrow[r, "f"] & p \end{tikzcd} \qquad (k \in K) \quad .$$

If $\alpha < \beta$, then B_α is a complete subalgebra of B_β. Since $p = \bigvee r_k$ in B_β, we have $p = \bigvee r_k$ in B_β. We have $t_k = q \wedge r_k$ in B_β, and thus $\bigvee t_k = \bigvee\{q \wedge r_k\} = q \wedge \bigvee r_k = q \wedge p = q$. Hence let $S^\beta \hookrightarrow q$ $\beta < a$, then also $q \in B_\alpha$, so repeat the above computations in B_α.

Since $E_{<\lambda}$ is a topos of sheaves on a preorder, it is equivalent to a topos $Sh(\Omega^\lambda)$ of canonical sheaves on Ω^λ, where Ω^λ is the cHa of global sections of the subobject classifier in $E_{<\lambda}$ (e.g. cf. [Jo. 1],5.37), i.e. Ω^λ consists of sieves of $B_{<\lambda}$ which are J_λ-closed. The topos of $\neg\neg$-sheaves in $E_{<\lambda}$ is a Boolean topos containing (cf. [F]) the Boolean-valued model which appears in [ST]:

<u>Proposition 3.2.2.1</u>. Let $P^\lambda = (B_{<\lambda} \setminus \{0\})^{op}$. Then:

$$sh_{\neg\neg}(E_{<\lambda}) \simeq Sh_{\neg\neg}(P^\lambda) \quad .$$

Proof. To distinguish between two topoi, we write $\sim$ for the negation in $E_{<\lambda}$, for the negation in S^{P^λ} . First notice that:

$$sh_{\sim\sim}(E_{<\lambda}) \simeq Sh(\Omega^\lambda{}_{\sim\sim}) ,$$

where $\Omega^\lambda{}_{\sim\sim}$ is the cBa of $\sim\sim$-closed elements of Ω^λ , and that:

$$Sh_{\neg\neg}(P^\lambda) \simeq Sh(\pi^\lambda{}_{\neg\neg}) ,$$

where $\pi^\lambda_{\neg\neg}$ is the cBa of $\neg\neg$-closed sieves in P^λ . In this proof, $\leq$ refers to the preorder of $B_{<\lambda}$.

Let U be a $\neg\neg$-closed sieve in P^λ . We claim that $\overline{U} = U \cup \{0\}$ is a J_λ-sheaf. It suffices to check this against J^λ : let $p > 0$, $R \hookrightarrow p$ a J^λ-sieve in Ω^λ , and $R \xrightarrow{\rho} \overline{U}$ a natural transformation. All of its nonempty components are identities. Since $R \in J^\lambda(p)$, there is $\alpha < \lambda$ such that $p \in B_\alpha$, and a B_α-covering sieve $R^\alpha \hookrightarrow p$ such that $R^\alpha \hookrightarrow R$:

$$\begin{tikzcd} R^\alpha \arrow[r, hook] & R \arrow[r, hook] \arrow[d, "\rho"'] & p \\ & \overline{U} & \end{tikzcd} .$$

We claim that $R \setminus \{0\}$ is $\neg\neg$-dense in P^λ . Indeed, if $R^\alpha = (p_i \to p)_{i\varepsilon I}$ we can show that $\forall q \in P^\lambda$, $q \leq p \Rightarrow \exists r \in P .\ r \leq q \wedge r \in R$: take any $q \leq p$, $q \neq 0$. Then $q = q \wedge p = q \wedge \bigvee p_i = \bigvee\{q \wedge p_i\}$. Thus for some i_o , $q \wedge p_{i_o} \neq 0$, so let $r = q \wedge p_{i_o}$. Since we obviously have a natural transformation $R \setminus \{0\} \overset{\rho'}{\to} U$ restricting ρ , and since U is a $\neg\neg$-sheaf, ρ' extends uniquely to:

$$\begin{tikzcd} R\setminus\{0\} \arrow[r, hook] \arrow[dr, "\rho'"'] & p \arrow[d, "\exists!"] \arrow[dl, phantom, "///"] \\ & U \end{tikzcd}$$

in S^{P^λ} , and thus obviously:

$$\begin{tikzcd} R \arrow[r, hook] \arrow[dr, "\rho"'] & p \arrow[d, "\exists!"] \arrow[dl, phantom, "///"] \\ & \overline{U} \end{tikzcd}$$

in $S^{B^{op}_{<\lambda}}$, so $\overline{U}$ is a J^λ-sheaf. To show that it is $\sim\sim$-closed, let us look at negation in Ω^λ : for $V \in \Omega^\lambda$ we have:

$$\sim V = \bigvee^{\Omega^\lambda}\{V \in \Omega | V \cap V' = 0\} \quad .$$

Let $\hat{V} = V \setminus \{0\}$. Then:

$$\neg \hat{V} = \{q \in P^\lambda | \forall q' \in P^\lambda . q' \leq q = q' \notin \hat{V}\} \; .$$

Since $\neg\neg\neg \hat{V} = \neg \hat{V}$, $\neg \hat{V}$ is a $\neg\neg$-sheaf in P^λ , so $(\neg \hat{V}) \cup \{0\}$ is J_λ-sheaf by the above. At the same time:

$$V \cap [(\neg \hat{V}) \cup \{0\}] = \{q \in B_{<\lambda} | q \in V \text{ and}$$
$$\forall q' \leq q \Rightarrow q' \in V\} = \emptyset \quad ,$$

since V is a sieve. Thus $(\neg \hat{V}) \cup \{0\} \subseteq \sim V$. But on the other hand, take any sieve U such that $U \cap V = \emptyset$. Then $\hat{U} \subseteq \neg \hat{V}$, because in P^λ we have for any $q \in U$: $\forall q' \leq q$, $q' \notin \hat{V}$ (since $\hat{U}$ is a sieve in P^λ). Thus also $U = \hat{U} \cup \{0\} \subseteq (\neg \hat{V}) \cup \{0\}$, hence:

$$\sim V = (\neg \hat{V}) \cup \{0\} \quad .$$

This first of all shows that if a sieve U in P^λ is a $\neg\neg$-sheaf in P^λ , then $\overline{U} = U \cup \{0\}$ is an element of Ω^λ . But it also shows that if a sieve V in $B_{<\lambda}$ (i.e. an element of Ω^λ) is $\sim\sim$-closed, then $\hat{V} = V \setminus \{0\}$ is $\neg\neg$-closed in P^λ , hence a $\neg\neg$-sheaf.

One can now see how sups look in $\Omega^\lambda_{\sim\sim}$ and $\pi^\lambda_{\neg\neg}$ and thus conclude that $\Omega^\lambda_{\sim\sim} \cong \pi^\lambda_{\neg\neg}$, finishing the proof.

We let $B^\lambda = \Omega^\lambda_{\sim\sim}$, and $E_\lambda = \mathrm{Sh}(B_\lambda)$; so E_λ is the Boolean topos described by the Proposition 3.2.2.1. In particular, B_λ is the completion of a Boolean algebra $B_{<\lambda}$.

To complete the construction of a Boolean topos in which there are no Suslin trees, let $E = \mathrm{sh}_{\neg\neg}(\underset{\alpha \in (2^{\aleph_1})}{\underleftarrow{Lim}} E_\alpha)$.

The following lemma is proved in [ST], 6.6. (or cf. [Je],pp. 105-106), by approximating $q \in B_{\alpha+1}$ by B_α in $\Sigma_{<2^{\aleph_1}}$, taking $\{p \in B_\alpha | q \leq p \text{ in } \Sigma_{<2^{\aleph_1}}\}$. It is crucial.

<u>Lemma 3.2.2.2.</u> Let λ be a limit ordinal in $2^{\aleph_1}$ and let:

$$B_0 \hookrightarrow B_1 \hookrightarrow \ldots \hookrightarrow B_\alpha \hookrightarrow \ldots \qquad (\alpha < \lambda)$$

be complete c.c.c. Boolean algebras such that B_α is a complete Boolean subalgebra of $B_{\alpha+1}$ and such that for each limit ordinal $\delta < \lambda$, B_δ is the completion of the Boolean algebra $\bigcup_{\beta<\delta} B_\beta$. Then the completion B_λ of the Boolean algebra $\bigcup_{\alpha<\lambda} B_\alpha$ is c.c.c.

Note that the statement of the above lemma is trivial for $\lambda = 2^{\aleph_1}$, and in fact also for any λ, for which there is no cofinal map $\{x \in 2^{2^{\aleph_1}} | x < \aleph_1\} \to \{x \in 2^{2^{\aleph_1}} | x < \lambda\}$. Lemma 3.2.2.2, is applied at the limit stages of the construction described above, together with the following lemmata:

Lemma 3.2.2.3. Let B be a complete c.c.c. Boolean algebra, P a sheaf over B such that:

$$Sh(B) \models P \text{ is a c.c.c. partial order.}$$

Let $\mathcal{T}$ be the topos of $\neg\neg$-sheaves on P in $Sh(B)$. Then there is a complete c.c.c. Boolean algebra C such that $\mathcal{T} \simeq Sh(C)$, and B is a complete Boolean subalgebra of C .

Proof. Let D be an internal cBa determining $\mathcal{T}$: i.e. D is an internal cBa of $\neg\neg$-closed sieves of P , in $Sh(B)$. In particular, D is a sheaf over B . Let C be the set of all global sections of D as a sheaf. Negation on D is a sheaf morphism, so negation of a global section is a global section. General theory tells us, as we mentioned before, that C is a complete Heyting algebra, and it is straightforward to check that C is Boolean, since D is. Moreover, there is a complete Heyting-algebra morphism $B \xrightarrow{i} C$ given by

$$p \rightsquigarrow \bigvee\{1 | p\}$$

(cf. [FSo], §§8.12, 9.9.), so i preserves complements (which are determined equationally)

$$[\![i(p) = 1]\!]_B = p \quad ,$$

$$[\![i(p) = 0]\!]_B = -p \quad .$$

Moreover, i is a monomorphism (in S) .

It is an easy exercise to see that:

$$Sh(B) \models D \text{ is c.c.c. cBa}$$

(cf. e.g. [ST], 7.8.), working in $Sh(B)$. Now, assume that C is not c.c.c., i.e. there is a pairwise disjoint family $\{p_\alpha | \alpha < \aleph_1\}$ of non-zero elements of C . It suffices to show:

$$[\![\forall\alpha \in \check{\aleph}_1 . \exists\beta \in \check{\aleph}_1 . \beta > \alpha \text{ and } p_\beta \neq 0]\!]_B > 0$$

since then for $X \in Sh(B)$ defined by:

$$[\![\alpha \in X]\!] = \bigvee_{\beta<\aleph_1} [\![\alpha = p_\beta]\!]_B$$

we would have:

$$[\![X \text{ is an uncountable antichain in } D]\!]_B > 0 \quad .$$

Thus assume that:

$$[\![\exists\alpha \in \check{\aleph}_1 . \forall\beta \in \check{\aleph}_1 (\beta > \alpha \Rightarrow p_\beta = 0)]\!]_B = 1 \quad .$$

Since B is Boolean and $\check{\aleph}_1$ is a sheaf, there is a global section ξ_0 of $\check{\aleph}_1$ such that $[\![\forall\beta \in \check{\aleph}_1 . \beta > \xi_0 \Rightarrow p_\beta = 0]\!]_B = 1$. But $\check{\aleph}_1$ is the simple sheaf (generated by the constant presheaf), so $1 = [\![\xi_0 = \xi_0]\!] = \bigvee_{\gamma<\aleph_1} [\![\xi_0 = \gamma]\!]$ (cf. [FSo] §4.15). By extensionality,

$$[\![\xi_0 = \gamma]\!] \wedge [\![\xi_0 = \delta]\!] \leq [\![\gamma = \delta]\!] \quad ,$$

so $A = \{[\![\xi_0 = \gamma]\!] > 0 | \gamma < \aleph_1\}$ is a pairwise disjoint family in B . Since B is c.c.c., this family is at most countable. Thus there is $\eta < \aleph_1$, such that $[\![\eta > \xi_0]\!]_B = 1$. Thus we have $[\![p_\eta = 0]\!] = 1$, so $[\![p_\eta = 0]\!] = 1$ in C , a contradiction.

The following lemma can be proved as in [ST], 5.2.5.:

<u>Lemma 3.2.2.4</u>. Let B be a cBa , D a cBa in $Sh(B)$, C the cBa of global sections of D . Let $X \subseteq C$, and let X' be the subsheaf of D determined by X . Let:

$$[\![X' \subseteq D \text{ generates } D \text{ as a cBa}]\!]_B = 1 \ .$$

Then $X \cup B$ generates C as a cBa .

This lemma gives us an estimate on the size of $B_{\alpha+1}$ from such an estimate on B_α. For, if $B_\alpha \hookrightarrow 2^{\aleph_1}$ and $P_\alpha \in Sh(B_\alpha)$ such that:

$$Sh(B_\alpha) \models P_\alpha \subseteq (\check{\aleph}_1 \times \check{\aleph}_1) \quad \text{is a c.c.c. partial order of } \aleph_1 ,$$

then, since $(\check{\aleph}_1 \times \check{\aleph}_1)$ can be represented as a set of certain maps $\aleph_1 \times \aleph_1 \to B_\alpha$ (cf. [FSo], §4.) and P_α as a subset $|P_\alpha|$ of such a set, we have $|P_\alpha| \hookrightarrow B_\alpha^{\aleph_1} \simeq (2^{\aleph_1})^{\aleph_1} \simeq 2^{\aleph_1}$. Moreover, since B_α is Boolean, P_α is generated by the set of X of its global sections (reflexivity assures us of $[\![\exists x \cdot x \in P_\alpha]\!]_B = 1$). Since P_α generates $\mathbb{B}_\alpha$ as a cBa in $Sh(B_\alpha)$ and in turn is generated by the set X_α as a sheaf over B_α, lemma 3.2.2.4. tells us that $X_\alpha \cup B_\alpha$ generates $\Gamma(\mathbb{B}_\alpha) = B_{\alpha+1}$ as a cBa. Since B_α is assumed to be c.c.c. by the hypothesis, $B_{\alpha+1}$ is c.c.c. by the lemma 3.2.2.3. But $X_\alpha \cup B_\alpha$ is dense in $B_{\alpha+1}$, so every element of $B_{\alpha+1}$ can be represented as a countable disjoint sup of elements of $X_\alpha \cup B_\alpha$. Thus $B_{\alpha+1} \hookrightarrow (2^{\aleph_1})^{\aleph_0} \simeq 2^{\aleph_1}$.

Since the size of B_α's is preserved at limit stages by the construction, we have:

<u>Corollary 3.2.2.5.</u> $B_{2^{\aleph_1}} \hookrightarrow 2^{\aleph_1}$ in $\mathcal{S}$.

We now show that there are no Suslin trees in $\mathcal{E}$. To begin with:

<u>Proposition 3.2.2.6.</u> In $\mathcal{E} = Sh(B_{2^{\aleph_1}})$, for every c.c.c. partial order $\leq$ of $\check{\aleph}_1$, and any $F \subseteq P(\check{\aleph}_1)$ with $\check{\aleph}_1 \twoheadrightarrow F$, there is a F-generic ideal in $\langle \check{\aleph}_1, \leq \rangle$.

(we recall that for a partial order $\langle x, \leq \rangle$ and a family y of subsets of x, $z \subseteq x$ is y-generic ideal in $\langle x, \leq \rangle$ if:

(i) $q \leq p \in z \Rightarrow q \in z$

(ii) $p, q \in z \Rightarrow \exists r \in z . r \geq p, r \geq q$

(iii) $\forall t \in y$. t open dense in $\langle x, \leq \rangle \Rightarrow \exists p \cdot p \in t$, $p \in z$,

where $t \subseteq x$ is open dense in $\leq$ if:

(a) $q \geq p \in t \Rightarrow q \in t$

(b) $p \in x \Rightarrow \exists q \in t\,.\ q \geq p\,.)$

Proof. Since the ordering on $\check{\aleph}_1$ inherited from a well-ordering of $\aleph_1$ in S satisfies in E all assumptions of the claim to be proved in E (with F the family of singletons), it suffices to assume

(*) $E \models \langle \check{\aleph}_1, \leq \rangle$ is a c.c.c. partial order and $F \subseteq P(\check{\aleph}_1)$ is such that there is $\check{\aleph}_1 \twoheadrightarrow F$,

(since there are such objects) and to show:

$E \models$ there is F-generic ideal in $\langle \check{\aleph}_1, \leq \rangle$.

Notice that for $X \in E$, $X \subseteq \check{\aleph}_1$; we actually have $X \in Sh(B_\gamma)$ for some $\gamma < 2^{\aleph_1}$. Namely, X is given by the values $b_\alpha = [\![\alpha \in X]\!] \in B_{2^{\aleph_1}}$ for $\alpha < \aleph_1$, and at the same time $B_{2^{\aleph_1}}$ is c.c.c. and generated by $\bigcup_{\beta<2^{\aleph_1}} B_\beta$, so each $b = \bigvee_{n<\omega} b_{\alpha,n}$ with $b_{\alpha,n} \in \bigcup_{\beta<2^{\aleph_1}} B_\beta$. But then there is $\gamma < 2^{\aleph_1}$ such that $b_{\alpha,n} \in B_\gamma$ for all $\alpha < \aleph_1$, $n < \omega$. However, given $F \in E$, $F \subseteq P(\check{\aleph}_1)$ with $\check{\aleph}_1 \twoheadrightarrow F$, we may think of F as a subset of $(\check{\aleph}_1 \times \check{\aleph}_1)$. In the similar way, we may thus conclude that $F \in Sh(B_\gamma)$ for some $\gamma < 2^{\aleph_1}$ such that $\leq$ is a partial order of $\check{\aleph}_1$ in $Sh(B_\gamma)$ (we will show in lemmata below that $\leq$ is c.c.c. for any β such that $\leq \in Sh(B_\beta)$) . By the construction, there is $\alpha \geq \gamma$ such that $\leq = P_\alpha$; so that for $F' = P_\alpha(\check{\aleph}_1)$ we have:

$Sh(B_{\alpha+1}) \models$ there is an F'-generic ideal in $P_\alpha = \langle \check{\aleph}_1, \leq \rangle$.

We will show in lemmata below that also:

$E \models$ there is an F'-generic ideal in P_α ,

so that:

$E \models$ there is an F-generic ideal in P_α .

This immediately gives:

Theorem 3.2.2.7. $E \models$ there are no Suslin trees.

Proof. Say $T = \langle \check{\aleph}_1, \leq \rangle$ is a Suslin tree. Then T is c.c.c. partial order on $\check{\aleph}_1$, and for each $\alpha \in \check{\aleph}_1$, $\{\beta \varepsilon T | \beta \geq \alpha\}$ is uncountable. Thus for each $\alpha \in \check{\aleph}_1$, look at:

$$D_\alpha = \{y \in T | \exists x \in T.\ x > \alpha \text{ as an ordinal and } y \geq x \text{ in } T\},$$

and let $F = \{D_\alpha | \alpha < \aleph_1\}$. By the Proposition 3.2.2.6., there is a F-generic ideal C in T. C is a chain and since D_α's are open dense in T, C is uncountable, a contradiction.

We still have to show the following lemmata:

Lemma 3.2.2.8. Let B and C be complete c.c.c. Boolean algebras such that B is a complete Boolean subalgebra of C. Let $R \in Sh(B)$ be a subsheaf of $(\check{\aleph}_1 \times \check{\aleph}_1)$. Consider:

$$\phi(R) := \text{"R is a c.c.c. partial order on } \check{\aleph}_1\text{"}.$$

Then $[\![\phi(R)]\!]_C \leq [\![\phi(R)]\!]_B$.

Proof. Let $\phi_1(R) :=$ "R is a partial order", $\phi_2(R) :=$ "R is c.c.c.". Obviously $[\![\phi_1(R)]\!]_B = [\![\phi_1(R)]\!]_C$. However, ϕ_2 involves arbitrary subsheaves of $\check{\aleph}_1$, so we have to be careful. $\phi_2(R)$ is:

$$\forall S \in P(\check{\aleph}_1) [\underbrace{\forall \alpha, \beta, \gamma \varepsilon \check{\aleph}_1 (\alpha, \beta \varepsilon S, \alpha \neq \beta \to \neg(\alpha R \gamma) \vee \neg(\beta R \gamma))}_{S \text{ is an antichain}} \to$$

$$\to \exists f^{N \to S}.\ \forall \alpha \in S.\ \exists n \in N. f(n) = \alpha] .$$

Since there are antichains in R, the subsheaf of antichains over B (over C) is generated by its global sections, so we have (both in B and C):

$$[\![\phi_2(R)]\!] = \bigwedge_{\substack{S \varepsilon P(\check{\aleph}_1) \\ \wedge \\ [\![S \text{ is an antichain}]\!] = 1}} [\![\exists f^{N \to S}.\ \forall \alpha \varepsilon S. \exists n \varepsilon N. f(n) = \alpha]\!] .$$

We will show below that (both in B and C):

$$[\![\neg\phi_2(R)]\!] = \bigvee_{S\in P(\check{\aleph}_1)} [\![\exists f^{\check{\aleph}_1\to S}.\forall\alpha,\beta\in\check{\aleph}_1.\forall\gamma\in S.f(\alpha) = f(\beta) = \gamma \to \alpha = \beta]\!] \ .$$

$$\wedge\, [\![S \text{ is an antichain}]\!] = 1$$

We show that $[\![\neg\phi_2(R)]\!]_B \leq [\![\neg\phi_2(R)]\!]_C$. Remember that it suffices to take sups and infs over generators only. Also, for $S \in Sh(B)$ with $S \in P(\check{\aleph}_1)$, $[\![S$ is an antichain$]\!]_B = [\![S$ is an antichain$]\!]_C$, since B is a complete Boolean subalgebra of C . For each such S we have (cf. [FSo] §§5.6.,5.7.,7.5):

$$[\![\neg\phi_2(R)_S]\!]_B = \bigvee_{f:\check{\aleph}_1\to S \text{ in } Sh(B),} \{Ef \wedge \bigwedge_{\substack{\alpha,\beta\in\aleph_1\\ \gamma\in S}} \{f(\alpha,\gamma)\wedge f(\beta,\gamma) \to [\![\alpha=\beta]\!]\}\}.$$

For every such f , inf is the same in C as in B . Also, every such f is also C-morphism. Thus $[\![\neg\phi_2(R)_S]\!]_B \leq [\![\neg\phi_2(R)_S]\!]_C$ for every antichain in $Sh(B)$. Hence $[\![\neg\phi_2(R)]\!]_B \leq [\![\neg\phi_2(R)]\!]_C$.

Finally, to show that $\neg\phi_2(R)$ is of the kind needed, reason as usual, but in $Sh(B)$ (or $Sh(C)$, resp.). Namely, since B and C are c.c.c., $[\![\check{\aleph}_1$ is uncountable$]\!] = 1$ (in both categories). Also notice that one does not need Replacement to prove comparability of two given well-ordered sets. But for every sheaf $X \in Sh(B)$ (or in $Sh(C)$, resp.) $[\![$there is a well ordering on $X]\!] = 1$ in both categories, since they satisfy choice.

<u>Lemma 3.2.2.9</u>. Let B be a complete Boolean subalgebra of a cBa C . Let $\phi(\langle x,\leq\rangle,y,z)$ be the formula:

"$\langle x,\leq\rangle$ is a partial order, y a family of subsets of x , $z \subseteq x$ is y-generic ideal in $\langle x,\leq\rangle$ ".

Then for $\langle x,\leq\rangle,y,z$ in $Sh(B)$, $\phi(\langle x,\leq\rangle,y,z)$ is true in $Sh(B)$ iff it is true in $Sh(C)$.

Proof. ϕ is geometric (cf. the statement of Proposition 3.2.2.6.) But B is a complete Boolean subalgebra of C .

Remark. One may read this paragraph as the proof of the consistency of Zermelo set theory with Choice (ZC) with the Suslin's Hypothesis, from the assumption that ZC is consistent. The proof itself does not use Replacement in the metatheory: we were taking unions of sets of bounded cardinality, so we may think of them as subsets of a suitable set.

Remark. Assuming GCH in S , this construction shows consistency of Martin's Axiom (cf. [ST] or [Je]).

Chapter 4

APPLICATIONS TO INTUITIONISTIC THEORIES

4.1. Classifying Infinite Sequences.

Consider Brouwer's theorem "Every function $f: [0,1] \to \mathbb{R}$ is uniformly continuous", which he proves (cf. [Du]) from the $\forall\alpha\exists n$ Continuity Principle:

$$\forall\alpha \in \mathbb{N}^{\mathbb{N}} . \exists n \in \mathbb{N}. A(\alpha,n) \to \exists \text{continuous } f: \mathbb{N}^{\mathbb{N}} \to \mathbb{N} . \forall\alpha \in \mathbb{N}^{\mathbb{N}} A(\alpha,f(\alpha)),$$

and Bar Induction:

$$\left[\begin{array}{l} \forall u \in \mathbb{N}^* (u \in X \vee \neg(u\varepsilon X)) \\ \wedge \\ \forall u \in \mathbb{N}^* (u \in X \to u \in Y) \\ \wedge \\ \forall\alpha \in \mathbb{N}^{\mathbb{N}} . \exists n \in \mathbb{N} . \overline{\alpha}(n) \in X \\ \wedge \\ \forall u \in \mathbb{N}^* ((\forall n \in \mathbb{N}. u*\langle n\rangle \in Y) \to u \in Y) \end{array}\right] \longrightarrow \langle\ \rangle \in Y ,$$

where X,Y are any given subsets of the set $\mathbb{N}^*$ of all finite sequences of natural numbers, $\langle\ \rangle$ is the empty sequence, $\overline{\alpha}(n)$ is the sequence $\langle\alpha(0),\alpha(1),\ldots,\alpha(n-1)\rangle$, and the finite sequence $u*\langle n\rangle$ is obtained by concatenating the number n to the right of the finite sequence u.

The discussion of intuitionistic motivation for (classically false) $\forall\alpha\exists n$ Continuity Principle is given in [KV], p. 72, and its main thrust is as follows: an infinite sequence (of natural numbers) is a growing, but possibly never complete, entity (e.g. it is impossible to have a complete knowledge of an infinite sequence obtained by throwing a dice), so a construction which gives a number depending on an infinite sequence, has to depend actually only on its finite initial segment (i.e. it has to be continuous w.r.t. Baire topology on $\mathbb{N}^{\mathbb{N}}$).

In this section we see how this argument can actually be made into a classically valid consistency proof for $\forall\alpha\exists n$ Continuity. The approach suggested by 1.1 is clear: construct a universal infinite sequence and then see what can be said about it (more precisely, about the Baire space object in the classifying topos). In particular, we shall see that both $\forall\alpha\exists n$ Continuity Principle and Bar Induction hold in the classifying topos.

Considering finite initial segments as approximations to an infinite sequence F, we will take the category of forcing conditions to be the poset P of finite sequences of natural numbers under inclusion as finite functions. The geometric theory "of an infinite sequence of natural numbers" is then given in propositional symbols a_p, $p \in P$ (to be interpreted as a_p = "p is an initial segment of an infinite sequence F".

(0) $a_p \vdash a_q$, if $q \subseteq p$

(1a) $a_p \wedge a_q \vdash a_{p \cup q}$, if $p \cup q \in P$

(1b) $a_p \wedge a_q \vdash \perp$, if $p \cup q \notin P$.

(2) $a_p \vdash \bigvee_{q \in X} a_q$, where $X \subseteq \{q \in P \mid q \supseteq p\}$ is any upward closed set such that every path $\alpha \in \mathbb{N}^{\mathbb{N}}$ through p passes through a node in X .

Axioms (0) express that we are considering <u>initial</u> segments of F, axioms (1a,b) that F is a function, i.e. that its initial segments are compatible, and axioms (2) require F to be an <u>infinite</u> sequence by requiring that each of its initial segments have a proper extension. In (2), we say that X is a <u>bar</u> of p .

<u>Proposition 4.1.1</u>. The family $\mathcal{B}$ of sieves in $\mathcal{S}^P$ given by

$$\mathcal{B}(p) = \{X \hookrightarrow p \mid X \text{ is a bar of } p\}$$

is a Grothendieck topology on P^{op} .

<u>Proof</u>. Using the identification of upward closed subsets of $\{q \in P \mid q \supseteq p\}$ and sieves of (the representable functor) p: obviously p is a bar of p, so $p \in \mathcal{B}(p)$. Re closure w.r.t. pullback, let $X \hookrightarrow p$ be bar of p ,

and let $q \supseteq p$. Let X_q^* be a pullback of X along this extension: $X_q^* = \{r \supseteq q \mid (r \supseteq p) \in X\}$. We have to show that X_q^* is a bar of q . Indeed, let α be a path through q . It is a path through p , since $q \supseteq p$. So, there is $s \supseteq p$, $s \in X$ so that α passes through s . Note that s and q are compatible in P , since α passes through q by assumption. If $s \supseteq q$, $s \in X_q^*$. If $q \supseteq s$, then $q \in X$ (so $X_q^* = q$) since X is a sieve (i.e. upward closed). In any case, α passes through a node in X_q^* , hence X_q^* is a bar of q . Finally, re third condition for a Grothendieck topology. Let X_q^* be a bar of $q \supseteq p$ for every q in a bar $Y \hookrightarrow p$. We have to show that X is a bar of p . So let α be a path through p . Since Y is a bar of p , there is $s \in Y$ so that α passes through s . But X_s^* is a bar of s , so there is $r \supseteq s$, $r \in X_s^*$ so that α passes through r . By definition of X_s^* , $r \in X$.

<u>Proposition 4.1.2</u>. Every representable functor in S^P is a sheaf w.r.t. topology $\mathcal{B}$.

<u>Proof</u>. Given a diagram:

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr, "\lambda"'] & p \\ & q \end{tikzcd}$$

in S^P , where X is a bar of p , notice that every nonempty component of the natural transformation λ is an identity (since P is a poset). Thus $q \subseteq r$ for each $r \in X$. Since P is a tree, q and p are comparable. Suppose $q \supsetneq p$, say $q = p*u*\langle n \rangle$ for a finite sequence u . Since X is a bar of p , there is $r \in X$ such that $r \supseteq p*u*\langle n+1 \rangle$. But then r and q are incompatible, since P is a tree, and in particular $r \not\supseteq q$. Thus $q \subseteq p$, and the given diagram is completed to

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr, "\lambda"'] & p \arrow[d] \arrow[dl, phantom, "///"] \\ & q \end{tikzcd} .$$

<u>Theorem 4.1.3</u>. The topos of sheaves $Sh_{\mathcal{B}}(P) \hookrightarrow S^P$ (w.r.t. topology $\mathcal{B}$) classifies the geometric theory (0)-(2).

<u>Proof</u>. In S^P, 1 is representable by the empty sequence $< >$. We first show that interpreting a_p as the representable functor $p \hookrightarrow 1$, one gets a model of the geometric theory (0)-(2) in $Sh_{\mathcal{B}}(P)$. Indeed, by axioms (0) we can think of this interpretation as the Yoneda embedding $P^{op} \hookrightarrow Sh_{\mathcal{B}}(P)$ (cf. Proposition 4.1.2.). This model will be the universal one. Axioms (1a,b) describe pullbacks of representable functors in $Sh_{\mathcal{B}}(P)$ precisely. For axioms (2), consider the sup: $\bigvee_{q\varepsilon X} a_q$ (where X is a bar of p) in the complete Heyting algebra Ω of subobjects of 1 in $Sh_{\mathcal{B}}(P)$. It is the smallest sieve $S \hookrightarrow 1$ containing the sieve $\bigcup_{q\varepsilon X} q$, and which is a sheaf (using the identification of the representable functor $q \hookrightarrow 1$ with the set $\{r\varepsilon P \mid r \geq q\}$). To show $S = p$ it suffices to show $p \hookrightarrow S$, since $(\bigcup_{q\varepsilon X} q) \hookrightarrow p$, and p is a sheaf by Proposition 4.1.2. Since $(\bigcup_{q\varepsilon X} q) \hookrightarrow S$, we have a natural transformation $X \xrightarrow{\lambda} S$. But then the diagram in S^P:

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr, "\lambda"'] & p \\ & S \end{tikzcd}$$

can be completed to:

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr, "\lambda"'] & p \arrow[d] \\ & S \end{tikzcd}$$

since S is a sheaf.

Now given any flat, continuous functor $P^{op} \xrightarrow{F} F$ for a Grothendieck topos F, consider the corresponding geometric morphism $F \to Sh_{\mathcal{B}}(P)$ given by Disconescu's Theorem [Jol, §4.3]. Its inverse image takes the model of (0)-(2) in $Sh_{\mathcal{B}}(P)$ described above to a model of (0)-(2) in F. In other words, the interpretation (given by the functor) F is a model of (0)-(2) in the topos F.

For the other direction, let ϕ be an interpretation of the (propositional symbols of the) geometric theory (0)-(2) in a Grothendieck topos F, which is a model of (0)-(2). Since axioms (0) are true in F under ϕ, ϕ is a functor $P^{op} \to F$. Since axioms (1a,b) are true in F under ϕ, ϕ is a flat functor. Since axioms (2) are true in F under ϕ, the functor ϕ takes covering sieves in P^{op} to epimorphic families in F, i.e. it is continuous. The topos $Sh_{\mathcal{B}}(P)$ is a topos of sheaves on a poset so by a general fact (cf. [Jo 1, §5.37]) it is equivalent as a category to the topos $Sh(\Omega)$ of (canonical) sheaves on the complete Heyting algebra Ω of subobjects of 1 in $Sh_{\mathcal{B}}(P)$. We claim that Ω is here actually spatial:

<u>Theorem 4.1.4</u>. The topos $Sh_{\mathcal{B}}(P)$ is equivalent to the spatial topos $Sh(\mathbb{N}^{\mathbb{N}})$ of sheaves over the Baire space.

<u>Proof 1</u>. Given $p \in P$, let $\mathcal{U}_p = \{\alpha \in \mathbb{N}^{\mathbb{N}} | \exists n \in \mathbb{N}.\ p = \bar{\alpha}(n)\}$, i.e. $\mathcal{U}_p \subseteq \mathbb{N}^{\mathbb{N}}$ is the set of all paths through P which pass through p. $\{\mathcal{U}_p | p \in P\}$ is the basis for the Baire topology on $\mathbb{N}^{\mathbb{N}}$, and $\mathcal{U}_p \subseteq \mathcal{U}_q$ iff $q \subseteq p$ in P. Hence P^{op} is (isomorphic to) a basis for the complete Heyting algebra $\mathcal{O}(\mathbb{N}^{\mathbb{N}})$ of all open sets in the Baire topology. One also has the correspondence of $\bigvee^{\Omega}_{i \in I} q_i$ and $\bigcup_{i \in I} \mathcal{U}_{q_i}$, i.e. Ω and $\mathcal{O}(\mathbb{N}^{\mathbb{N}})$ are isomorphic as <u>complete</u> Heyting algebras.

<u>Proof 2</u>. The proof of Theorem 4.1.3 can be repeated for $Sh(\mathbb{N}^{\mathbb{N}})$ instead of $Sh_{\mathcal{B}}(P)$, the universal model being the above assignment $p \rightsquigarrow \mathcal{U}_p$ given in the proof 1 of Theorem 4.1.4. Since $Sh(\mathbb{N}^{\mathbb{N}})$ and $Sh_{\mathcal{B}}(P)$ then both classify the geometric theory (0)-(2), they are equivalent as categories.

Thus we have the following:

Corollary 4.1.5. The higher-order Beth semantics is equivalent to the Scott-Moschovakis topological interpretation:

$$p \Vdash A \quad \text{iff} \quad [\![A]\!] \geq p \quad .$$

Proof. The higher-order Beth semantics given in [Da] is the sheaf semantics in $Sh_B(P)$. The Scott-Moschovakis topological interpretation ([Sco 1,2],[Mo]) is the sheaf semantics in $Sh(\mathbb{N}^{\mathbb{N}})$.

Historically, D. Scott first showed [Sco 1,2] that Brouwer's theorem can be interpreted classically by the topological interpretation in the Baire space. J.R. Moschovakis later showed that one actually has the topological interpretation of intuitionistic second-order arithmetic, which satisfies the parameterless $\forall\alpha\exists n$-Continuity Principle, Bar Induction, Kripke's Schema $\exists\alpha \in \mathbb{N}^{\mathbb{N}}((\exists n.\alpha(n) \neq 0) \leftrightarrow A)$ (for any formula A), parameterless Uniformity Principle $\forall X \in P(\mathbb{N}).\exists n\in\mathbb{N}.A(n,X) \to \exists n\in\mathbb{N}.\forall X\in P(\mathbb{N}).A(n,X)$ and Relationed Dependent Choice:

$$\forall x^{\sigma}(A(x) \to \exists y^{\sigma}(A(y) \wedge B(x,y))) \to$$
$$\to \forall x^{\sigma}(A(x) \to \exists f \in \sigma^{\mathbb{N}}(f(0) = x \wedge \forall n\in\mathbb{N}(A(f(n)) \wedge B(f(n),f(n+1))))) ,$$

where σ is the appropriate type. The work on topos theory recognized this a fragment of the sheaf semantics in the topos $Sh(\mathbb{N}^{\mathbb{N}})$ (cf. e.g. historical discussion in [FSo]).

Summing up, we have:

Corollary 4.1.6. In $Sh_B(P)$, the following hold:

(i) Parameterless $\forall\alpha\exists n$-Continuity Principle

(ii) Bar Induction

(iii) Kripke's Schema

(iv) Relativized Dependent Choice (w.r.t. any type)

(v) Brouwer's Theorem

(vi) Uniformity Principle (without parameters) .

Remarks on metatheory are in order. We are working over the Boolean topos of Sets, and the Booleanness is necessary in order to have (i) in Corollary 4.1.6. It also means that this model of intuitionistic analysis is not helpful in iterations of relative consistency proofs.

Another drawback of this model is the lack of parameters in (i),(vi) of Corollary 4.1.6. In fact, $\forall\alpha\exists n$-Continuity Principle with parameters allowed in the formula A (cf. beginning of this section) fails in $Sh_{\mathcal{B}}(P)$ (cf. e.g. [Da]). For second-order intuitionistic arithmetic, a more refined model satisfying (i)-(vi) with parameters allowed in (i),(vi) was given in [Kr] . In [Sc2] we extended his model to a model of intuitionistic ZF satisfying (i)-(vi) with parameters allowed in (i),(vi). In the remainder of this section, we describe the topos in question.

We consider the Cantor discontinuum C, i.e. the space of paths in the tree of finite 0-1-sequences (ordered by inclusion), the basis of the topology being given by sets $U_p = \{\alpha\varepsilon 2^{\mathbb{N}} | \exists n\varepsilon\mathbb{N}.\bar{\alpha}(n) = p\}$, where p is a finite 0-1-sequence. We say that an infinite sequence $f: \mathbb{N} \to \mathbb{N}$ is rare if $f(n) \neq f(m)$ for $m \neq n$, and $\lim_n \frac{n}{f(n)} = 0$. Given a rare sequence $f: \mathbb{N} \to \mathbb{N}$, we say that an automorphism $\phi: C \to C$ is a f-automorphism if $(\phi t)(fn) = t(fn)$ for all $t \varepsilon C$, $n \varepsilon \mathbb{N}$. Let G be the set of all homeomorphisms $\phi: C \to C$ for which there exist functions a_ϕ, b_ϕ mapping rare sequences to rare sequences so that for each rare sequence $f: \mathbb{N} \to \mathbb{N}$ the following holds:

(a) for every $a_\phi(f)$-automorphism $\sigma: C \to C$, $\phi^{-1}\sigma\phi$ is a f-automorphism,

(b) for every $b_\phi(f)$-automorphism $\tau: C \to C$, $\phi\tau\phi^{-1}$ is a f-automorphism.

Topologize the group G (w.r.t. composition) by letting a subgroup $H \leq G$ be open iff there exists a rare sequence $f: \mathbb{N} \to \mathbb{N}$ so that H contains all f-automorphisms in G. Let Cont(G) be the topos of continuous G-sets (cf. chapter 2). It is a Boolean topos. Think of it as the metatheory for the construction of the topos $Sh_{\mathcal{B}}(P)$: i.e. let $\mathcal{E}$ be the topos of internal sheaves over the internal Baire space in Cont(G) . In other words, we are considering a 2-step iterated forcing. The relevant objects in $\mathcal{E}$ are given as follows: let Σ be the complete Heyting algebra of open sets in C . The internal $\mathbb{N}^{\mathbb{N}}$ in $\mathcal{E}$ consists of continuous functions $\xi\colon C \to \mathbb{N}^{\mathbb{N}}$ (in S) such that for some rare sequence $f\colon \mathbb{N} \to \mathbb{N}$ (depending on ξ), $\xi \circ \phi = \xi$ for all f-automorphisms $\phi \in G$. The internal $P(\mathbb{N})$ in E consists of all "characteristic functions" (in S) $P\colon \mathbb{N} \to \Sigma$ such that for some rare sequence $f\colon \mathbb{N} \to \mathbb{N}$ (depending on P) $\phi P(n) = P(n)$ for each $n \in \mathbb{N}$, and each f-automorphism $\phi \in G$ (cf. also [Fre 1]). Since is a Grothendieck topos, the internal object of natural numbers is standard (the one from S).

The following theorem is proved in [Sc2]:

<u>Theorem 4.1.7</u>. In E, the following hold:

(i) $\forall\alpha\exists n$-Continuity Principle

(ii) Bar Induction

(iii) Kripke's Schema

(iv) Relativized Dependent Choice (w.r.t. any type)

(v) Brouwer's Theorem

(vi) Uniformity Principle •

4.2. Spoiling Local Compactness of the Reals

The notion of compactness (every open cover has a finite subcover) loses its importance without the presence of logical law of excluded middle. It actually fails in recursive constructive analysis (which leads to various independence results, cf. [Be], [Sc 1]), and is replaced by a different notion in Bishop's approach. However, in Brouwer's development of intuitionistic analysis, compactness of 2^N (which implies local compactness of the reals) plays an important role and is stated as the Fan Theorem, which is itself a consequence of Bar Induction. The intuitionistic provability of Bar Induction is doubtful, to say the least, (since Bar Induction may fail in sheaf models [FH], [Sc 2]), and we mention that the Fan Theorem itself may fail in sheaf models even in the presence of principles denying recursiveness of all functions $f: N \to N$ (cf. [Sc 3]).

We now wish to present, from a point of view developed in 1.1., a forcing construction due to André Joyal, which spoils compactness of $[0,1]$ by constructing a generic open cover with certain properties which forbid finite subcovers. Although it does not give as strong as independence result as a (rather artificial) sheaf model given in [Sc 3], we think it sheds light on the problem of compactness in constructive mathematics, because of its simplicity and elegance. I am grateful to André Joyal for permission to include this construction (which precedes [Sc 3]) here.

The object we are looking for is an open covering G of the reals in $[0,1]$ with no finite subcovers. Since rationals are in any case preserved under geometric morphisms (there will be no "new rationals"), it is convenient to restrict ourselves to regular open sets (in the usual topology on $[0,1]$) which are finite unions of disjoint subintervals with rational endpoints. If we require that every such open set in G have a Lebesgue measure $< \frac{1}{2}$ and that G is closed under finite unions, G cannot have any finite subcovers. We might as well assume that G is an ideal. Since $[0,1]$ is compact in (Boolean) *Sets*, such an object G does not exist in *Sets*. However, the requirements listed above are geometric, so we look at the generic G in the classifying topos. The covering property is geometric, but Dedekind reals are not preserved by geometric morphisms, so G is a covering only of constant (i.e. "old") reals in $[0,1]$. We will have to check that the Dedekind reals of the classifying topos E are the constant ones (a non-geometric requirement, like the preservation of cardinals). Thus, in effect, although there are no new reals in E, there are plenty of new open subsets of reals in E.

Let T_1 be the geometric theory of an ideal in an upper semilattice:

$$\begin{array}{rl} & \vdash \; x \vee x = x \\ & \vdash \; x \vee y = y \vee x \\ & \vdash \; x \vee (y \vee z) = (x \vee y) \vee z \\ x \in J \,\&\, y \in J & \vdash \; x \vee y \in J \\ y \in J \,\&\, x \vee y = y & \vdash \; x \in J \end{array}$$

The category of forcing conditions will be obtained by "rigidification" as a certain subcategory of $\mathrm{Mod}(S,T_1)$: namely, we are interested in ideals of the upper semi-lattice (under union) of open subsets of $[0,1]$, so we require morphisms of models to preserve inclusion in $[0,1]$. Hence these morphisms are inclusions of ideals, so we have a subcategory $C \hookrightarrow \mathrm{Mod}(S,T_1)$ which is a preorder. To take care of the other conditions given above, we look at the full subcategory $P \hookrightarrow C$ which consists of principal ideals generated by regular finite unions of disjoint subintervals of $[0,1]$ with rational endpoints (open in the topology of $[0,1]$, so we may have, say, $[0,\frac{1}{3})$ in P), with Lebesgue measure (i.e. length) less than $\frac{1}{2}$. Alternatively, we may think of P as a preorder of those sets, ordered by inclusion in $[0,1]$.

We still have to take care of the requirement that an ideal G cover $[0,1]$. This requirement gives the Grothendieck topology on (the theory) P^{op}: we describe G in terms of statements "$J \in G$" for every $J \in P$. Thus atomic sentences of the geometric theory T_2 will be propositional symbols a_J for all $J \in P$:

$$
\begin{array}{lll}
(0) & a_{J_2} \vdash a_{J_1} & , \text{ for all } J_1 \subseteq J_2 \\
(1) & a_{J_1} \wedge a_{J_2} \vdash a_{J_1 \cup J_2} & , \text{ if } J_1 \vee J_2 \in P \\
 & a_{J_1} \wedge a_{J_2} \vdash \bot & , \text{ if } J_1 \vee J_2 \notin P \\
(2) & T \vdash \bigvee_{r \in J} a_J & , \text{ for each real } r \in [0,1]
\end{array}
$$

(we may think of T as $a_\emptyset$) . Here $J_1 \vee J_2 = \mathrm{int}\, c\ell(J_1 \cup J_2)$.

Theorem 4.2.1. T_2 is classified by the topos $\mathcal{E} \to S^P$ of sheaves given by the Grothendieck topology generated by the following family $\mathcal{J}$ of sieves:

$$\mathcal{J}(J) = \{\text{all sieves } R \hookrightarrow J \mid \exists \text{real } r \in [0,1] \text{ such that } R = \{J' \in P \mid J' \supseteq J \text{ and } r \in J'\}\} ,$$

for each $J \in P$.

Proof. To be brief, we write $\mathrm{Cov}_r(J) = \{J' \in P \mid J' \supseteq J \text{ and } r \in J'\}$ for $J \in P$ and a real $r \in [0,1]$. Next, by a very useful trick (cf. [SGA4], Exp. II, Corollary 2.3.), we only have to check that $\mathcal{J}$ is closed under pullbacks, for then the sheaf condition w.r.t. $\mathcal{J}$ is the same as the sheaf condition w.r.t. Grothendieck topology generated by $\mathcal{J}$.

We shall actually show that $\mathrm{Cov}_r(K)$ is the pullback in S^P of $\mathrm{Cov}_r(J)$ along $K \overset{f}{\supseteq} J$.

$$
\begin{tikzcd}
\mathrm{Cov}_r(K) \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\text{p.b.}"] & \mathrm{Cov}_r(J) \arrow[d, hook] \\
K \arrow[r, phantom, "\underset{f}{\supseteq}"] & J
\end{tikzcd}
$$

To see this, we note that $f^*(\mathrm{Cov}_r(J)) = \{L \supseteq K \mid (L \supseteq K \supseteq J) \in \mathrm{Cov}_r(J)\}$. Then for any $L \in f^*(\mathrm{Cov}_r(J))$ we have $L \supseteq K \supseteq J$ and $r \in L$, so $L \in \mathrm{Cov}_r(K)$. Vice versa, for any $L \in \mathrm{Cov}_r(K)$ we have $L \supseteq K$ and $r \in L$, and thus $L \supseteq K \supseteq J$ and $r \in L$, i.e. $L \in f^*(\mathrm{Cov}_r(J))$.

Let $\mathcal{E}$ be the topos of $\mathcal{J}$-sheaves. $\mathcal{E}$ is nontrivial, since the topology generated by $\mathcal{J}$ is subcanonical (i.e. all representable presheaves are sheaves). Indeed, let $J, K \varepsilon P$ and let $Cov_r(J) \overset{\rho}{\to} K$ be a natural transformation. Since P is a preorder, all nonempty components $L \varepsilon Cov_r(J) \overset{\rho_L}{\to} Hom_P(L,K)$ are identities. Since $\inf\{J' | J' \varepsilon Cov_r(J)\} = J$, we see that $J \supseteq K$. But then obviously:

$$\begin{tikzcd} Cov_r(J) \arrow[r, hook] \arrow[dr, "\rho"'] & J \arrow[d] \arrow[dl, phantom, "///"] \\ & K \end{tikzcd}$$

Finally, to show that $\mathcal{E}$ classifies T_2, notice first of all that T_2 can be thought of as P^{op}, with axioms of group (2) as being covering families of $\emptyset$. Note that they are exactly families $Cov_r(\emptyset) = \{J \varepsilon P | r \varepsilon J\}$. Since the topology on P is subcanonical, the Yoneda embedding factors as:

$$\begin{tikzcd} P^{op} \arrow[rr, hook] \arrow[dr] & \arrow[d, phantom, "///"] & S^P \\ & \mathcal{E} \arrow[ur, hook] & \end{tikzcd}$$

and it is easy to show that it preserves axioms of T_2. Indeed, representable presheaves in S^P can be thought of as sieves $[J] = \{J' \varepsilon P | J' \supseteq J\}$ for $J \varepsilon P$. Moreover, pullbacks in $\mathcal{E}$ are computed in S^P, so $[J_1] \wedge [J_2] = [J_1] \cap [J_2] = \{K_1 \epsilon P | K_1 \supseteq J_1\} \cap \{K_2 \epsilon P | K_2 \supseteq J_2\} = \{K \varepsilon P | K \supseteq J_1 \cup J_2\}$, hence axioms (1) are satisfied, as are obviously also the axioms (0). To check the axioms (2), let r be any real in $[0,1]$. We want to show that $C_r = \bigvee\{[J] | r \varepsilon J, J \varepsilon P\} = 1$ (where the sup is computed in $\mathcal{E}$), i.e. $C_r = P$. We certainly have $C_r \hookrightarrow 1$ in $\mathcal{E}$, and for every $J \varepsilon P$ with $r \varepsilon J$, $[J] \subseteq C_r$, hence $J \varepsilon C_r$. For any $K \varepsilon P$ with $r \notin K$, look at $Cov_r(K)$. There is a natural transformation $Cov_r(K) \overset{\mu}{\to} C_r$ (because of the previous remark about $J \varepsilon P$ with $r \varepsilon J$). Now, C_r is a sheaf, so there exists a natural transformation $K \to C_r$ such that:

$$\begin{tikzcd} Cov_r(K) \arrow[r, hook] \arrow[dr, "\mu"'] & K \arrow[d] \arrow[dl, phantom, "///"] \\ & C_r \end{tikzcd},$$

hence $K \varepsilon C_r$. Thus $C_1 = P$.

Observe that the topology on P^{op} generated by the families given by the axioms (2) is the same as the topology generated by the family J, since $\mathrm{Cov}_r(J) = \mathrm{Cov}_r(\emptyset) \times J$ in S^P, as shown above. So, E classifies T_2.

Remark: We have given some additional information here not needed for the proof of the Theorem 4.2.1., since we wanted to give all computations at the same place and then to refer to them later as needed.

Let G be the functor in S^P given by $G(p) = \{p' \varepsilon P | p' \subseteq p\}$.

Proposition 4.2.2.: G is a sheaf.

Proof. Let $J \varepsilon P$, $r \varepsilon [0,1]$ a real. Let $\mathrm{Cov}_r(J) \overset{\mu}{\to} G$ be a natural transformation, i.e. for $K,L \varepsilon \mathrm{Cov}_r(J)$, $K \subseteq L$ we have:

$$\begin{tikzcd} \mathrm{Cov}_r(J) \ni K \arrow[r, "\mu_K"] \arrow[d] \arrow[dr, phantom, "///"] & G(K) = \{M \varepsilon P | M \subseteq K\} \arrow[d, hook] \\ \mathrm{Cov}_r(J) \ni L \arrow[r, "\mu_L"] & G(L) = \{M \varepsilon P | M \subseteq L\} \end{tikzcd}.$$

Such a natural transformation μ corresponds to $M \varepsilon P$ such that for all $J' \varepsilon \mathrm{Cov}_r(J)$, $M \subseteq J'$. Hence $M \subseteq J$, so μ extends to

$$\begin{tikzcd} \mathrm{Cov}_r(J) \arrow[r, hook] \arrow[dr, "\mu"'] \arrow[dr, phantom, shift left=3, "///"] & J \arrow[d] \\ & G \end{tikzcd},$$

uniqueness is trivial by the structure of G. General theory tells us that $E \simeq \mathrm{Sh}(\Omega)$ (cf. [Jo 1], 5.37.), where Ω is the complete Heyting algebra of global sections of the truth value object of E, i.e. all J-closed sieves. Meets and joins in Ω were computed above. Relying on this and on the interpretation of the intuitionistic type theory in sheaves over a cHa as developed in [FSo] §§ 5 and 7, and on the above computations, one can easily check (for $[0,1]^*$ the constant reals in $[0,1]$ in E) that the following geometric statements about G are true in $E \simeq \mathrm{Sh}(\Omega)$:

- $\forall J \in G$, J is open.
- $\forall r \in [0,1]^* .\ J \in G .\ r \in J$.
- $\forall J \in G .\ \ell(J) < \frac{1}{2}$.
- $\forall J_1, J_2 \in G .\ J_1 \vee J_2 \in G$.

Rather, we concentrate on showing that the Dedekind reals in E are constant (i.e. the reals from S) . In [FH] it is shown that every Dedekind real in $Sh(\Omega)$ is a $\wedge\bigvee$-map $\mathcal{O}(\mathbb{R}) \xrightarrow{a} \Omega$. We want to show that for each such a there is a unique $p \in \mathbb{R}$ such that $U \in \mathcal{O}(\mathbb{R})$, $p \in U$ iff $a(U) = \mathrm{true}_\Omega$. We start with a

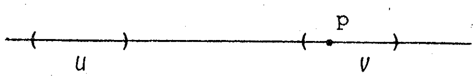
Claim: Given an $\wedge\bigvee$-map $\mathcal{O}(\mathbb{R}) \xrightarrow{a} \Omega$,

$\exists! p \in \mathbb{R} .\ \forall U \in \mathcal{O}(\mathbb{R}) .\ p \in U \to a(U) \neq 0_\Omega$.

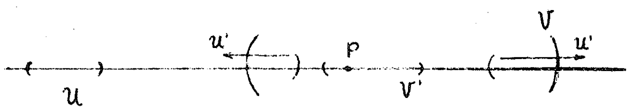
Proof:

Existence: Suppose there is no such $p \in \mathbb{R}$, i.e. suppose that $\forall p \in \mathbb{R} . \exists U_p \in \mathcal{O}(\mathbb{R})$ with $p \in U_p$ and $a(U_p) = 0$. Obviously $\mathbb{R} = \bigcup_{p\in\mathbb{R}} U_p$, so we would have $T = a(T_{\mathcal{O}(\mathbb{R})}) = a(\mathbb{R}) = a(\bigcup_{p\in\mathbb{R}} U_p) = \bigvee_{p\in\mathbb{R}} a(U_p) = \bigvee 0 = 0$, since a is $\wedge\bigvee$-map; a contradiction.

Uniqueness. Say $p \in \mathbb{R}$ is such that for every open $U \subseteq \mathbb{R}$, $p \in U$ implies $a(U) \neq 0$. It suffices to show that for all open $U \subseteq \mathbb{R}$, $a(U) \neq 0$ implies $p \in U$. Suppose there exists an open set $U \subseteq \mathbb{R}$ with $a(U) \neq 0$ and $p \notin U$. We can assume that the situation is as on the picture:



since a is $\wedge\bigvee$-map. Let $V \subseteq \mathbb{R}$ be an open set such that $p \in V$ and $U \cap V = \emptyset$. We have $a(U) \wedge a(V) = 0$, hence for any $K \in a(U)$, $L \in a(V)$, $K \cup L \notin P$, since $a(U)$ and $a(V)$ are sieves. Therefore, lengths of intervals in both $a(U)$ and $a(V)$ are bounded below by a positive rational. Now choose open sets $U', V' \subseteq \mathbb{R}$:



so that $p \in V'$, $U' \cap V' = \emptyset$, $V \cup U' = \mathbb{R}$, $U \subseteq U'$. First of all, $a(U') \neq 0$ so lengths of intervals in both $a(U')$ and $a(V')$ are bounded below by a positive rational, as above. On the other hand, $a(V) \vee a(U')$ is the smallest sieve containing $a(V) \cup a(U')$ which is a sheaf. Since the Grothendieck topology is generated by sieves $\mathrm{Cov}_r(J)$, lengths of intervals in $a(V) \vee a(U')$ are also bounded below by a positive rational. Thus $T_\Omega \neq a(V) \vee a(U')$, contrary to the fact that a is $\wedge\bigvee$-map, since $V \cup U' = \mathbb{R}$.

Thus we have a unique real p such that $a(U) \neq 0$ iff $p \in U$, for any open set $U \subseteq \mathbb{R}$. But then $a(U) = T_\Omega$ iff $p \in U$ as well.

Thus $a(U)$ depends just on whether $p \in U$, so it is completely determined by p . We recall that <u>a point</u> of a Grothendieck topos X is a geometric morphism $S \xrightarrow{f} X$. Its left adjoint f^* preserves conjunctions, (infinitary) disjunctions and existential quantifiers, and thus is a set-model of (a geometric theory classified by) X . In short, $X \xrightarrow{f^*} S$ preserves geometric properties.

Letting $X = E$, the topos constructed above, we see that for any point $S \xrightarrow{f} E$, its left adjoint $E \xrightarrow{f^*} S$ would preserve all geometric properties of a generic object G given above. Moreover, f maps the constant sheaf $[0,1]^*$ to $[0,1]$ in S , and thus $f^*(G)$ would be an open covering of $[0,1]$ in S with no finite subcover, impossible. Hence E has no points. Since E is (equivalent to) a topos of sheaves over a cHa Ω (cf. above), we conclude that Ω has no points either (i.e. there are no $\wedge\bigvee$-preserving maps $\Omega \to P(1)$). Let R be the object of Dedekind reals in E , R^* the object of constant reals in E . The results of this paragraph may seem less surprising if viewed as an assertion that for the internal cHa of open sets we might have $O(R^*) \neq O(R)$ even if they have the same points (internal $P(1)$ is Ω).

4.3. Making Baire space subcountable.

In recursive constructive mathematics, every function $f\colon \mathbb{N} \to \mathbb{N}$ is recursive, so recursive indices give a surjection $A \to \mathbb{N}^{\mathbb{N}}$ for a subset $A \subseteq \mathbb{N}$ (i.e. $\mathbb{N}^{\mathbb{N}}$ is subcountable). The 1945 recursive realizability topos [HJP], built as an extension of Kleene's 1945 realizability to higher order intuitionistic arithmetic (cf. also extensions to intuitionistic ZF [Fri], [Bel], [Scl]), verifies this internally. However, it is not a Grothendieck topos. In particular, its natural number object is not standard.

We see in this section how the methods given in 1.1. allow one to construct a universal example of a surjection

$$\begin{tikzcd} & \mathbb{N} \\ A \arrow[ur, hook] \arrow[rr, two heads, "F"'] & & \mathbb{N}^{\mathbb{N}} \end{tikzcd}$$

in a Grothendieck topos.

We need a partial function from $\mathbb{N}$ to $\mathbb{N}^{\mathbb{N}}$, which is surjective. Read this as follows: "partial function" is a geometric theory T_1, "from $\mathbb{N}$ to $\mathbb{N}^{\mathbb{N}}$" is the rigidifying condition, and "surjective" defines a Grothendieck topology on the category of forcing conditions:

$$\mathbb{C} \hookrightarrow \mathrm{Mod}(S, T_1) \quad .$$

The geometric theory T_1 is given by:

$$u \in f \wedge v \in f \wedge \pi_1(u) = \pi_1(v) \vdash \pi_2(u) = \pi_2(v) \ ,$$

where u, v are of sort $X \times Y$, and π_1, π_2 are projections. We rigidify the full subcategory of finitely presented set-models of T_1 as follows: we are interested in partial functions from $\mathbb{N}$ to $\mathbb{N}^{\mathbb{N}}$, so look only at finitely presented set models of T_1 in which X is $\mathbb{N}$ and Y is $\mathbb{N}^{\mathbb{N}}$. Morphisms are supposed to preserve f as a subset of $\mathbb{N} \times \mathbb{N}^{\mathbb{N}}$, those are inclusions only. Thus we have $\mathbb{C} = P$, the poset of finite partial functions from $\mathbb{N}$ to $\mathbb{N}^{\mathbb{N}}$, ordered by inclusion.

We think of finite functions in P as pieces of the desired surjection F, and thus have the following requirements expressed in propositional symbols a_f (intuitively, for "$f \subseteq F$"):

0) $a_{f_2} \vdash a_{f_1}$, for all $f_1 \subseteq f_2$

1a) $a_{f_1} \wedge a_{f_2} \vdash a_{f_1 \cup f_2}$, if $f_1 \cup f_2$ is a function

1b) $a_{f_1} \wedge a_{f_2} \vdash \bot$, if $f_1 \cup f_2$ is not a function

2) $T \vdash \bigvee_{\alpha \varepsilon \mathrm{Rge}(f)} a_f$, for each $\alpha : \mathbb{N} \to \mathbb{N}$.

(Condition 2) states surjectivity: it is true that for each α there exists $f \subseteq F$ such that $\alpha \varepsilon \mathrm{Rge}(f)$) . Axioms 0) - 2) give the geometric theory T_2 .

The corresponding Grothendieck topology on P^{op} (we will prove shortly that it is in fact the corresponding one - it classifies!) is given as follows.

For each $f \varepsilon P$, and each $\alpha \varepsilon \mathbb{N}^{\mathbb{N}}$, let:

$$\mathrm{Cov}_\alpha(f) = \{g \varepsilon P \mid f \subseteq g \text{ and } \alpha \varepsilon \mathrm{Rge}(g)\} ,$$

and, for each $f \varepsilon P$, let:

$$J(f) = \{\mathrm{Cov}_\alpha(f) \mid \alpha \varepsilon \mathbb{N}^{\mathbb{N}}\} .$$

J is not a Grothendieck topology on P^{op} , and one could consider the Grothendieck topology generated by J . However, as in 4.2., we recall Corollary 2.3. in Exp. II of [SGA4] together with the following:

<u>Proposition 4.3.1</u>. The family J is closed under pullbacks in S^P .

<u>Proof</u>. We show that:

$$\begin{tikzcd} \mathrm{Cov}_\alpha(g) \arrow[r] \arrow[d, hook] & \mathrm{Cov}_\alpha(f) \arrow[d, hook] \\ g \arrow[r, "\lambda"'] & f \end{tikzcd}$$

is a pullback in S^P for any $f \subseteq g$ in P , and any $\alpha \varepsilon \mathbb{N}^{\mathbb{N}}$. Indeed, $\lambda^*(\mathrm{Cov}_\alpha(f)) = \{h \supseteq g \mid (h \supseteq g \supseteq f) \varepsilon \mathrm{Cov}_\alpha(f)\}$. So for any $h \varepsilon \lambda^*(\mathrm{Cov}_\alpha(f))$ we have $h \supseteq g$ and $\alpha \varepsilon \mathrm{Rge}(h)$, i.e. $h \varepsilon \mathrm{Cov}_\alpha(g)$. Conversely, if $h \varepsilon \mathrm{Cov}_\alpha(g)$, we have $h \supseteq g \supseteq f$ and $\alpha \varepsilon \mathrm{Rge}(h)$, so $h \varepsilon \mathrm{Cov}_\alpha(f)$ as an extension of f , i.e. $h \varepsilon \lambda^*(\mathrm{Cov}_\alpha(f))$ since h is an extension of g .

As usual, we look at the interpolation object $G \hookrightarrow \Delta P$ in S^P given by:

$$G = \sum_{f \varepsilon P} \mathrm{Hom}_P(f,-) \quad ,$$

in this case:

$$G(f) = \{f' \varepsilon P | f' \subseteq f\} \quad ,$$

since P is a partially ordered set as a category.

<u>Proposition 4.3.2.</u> G is a J-sheaf.

<u>Proof</u>. For any $f \varepsilon P$, $\alpha \varepsilon \mathbb{N}^{\mathbb{N}}$, and natural transformation λ :

$$\begin{tikzcd}
\mathrm{Cov}_\alpha(f) \arrow[r, hook] \arrow[dr, "\lambda"'] & f \\
& G
\end{tikzcd} \quad ,$$

we have for $g \subseteq h$ in P :

$$\begin{tikzcd}
\mathrm{Cov}_\alpha(f) \ni \{g\} \arrow[r, "\lambda_g"] \arrow[d, "g \rightsquigarrow h"'] \arrow[dr, phantom, "///"] & G(g) = \{k \varepsilon P | k \quad g\} \arrow[d, hook] \\
\mathrm{Cov}_\alpha(f) \ni \{h\} \arrow[r, "\lambda_h"] & G(h) = \{k \varepsilon P | k \quad h\}
\end{tikzcd} \quad .$$

Such a natural transformation λ is given by $k \varepsilon P$ such that for all g in $\mathrm{Cov}_\alpha(f)$, $k \subseteq g$. If $\alpha \varepsilon \mathrm{Rge}(f)$, this clearly says that $k \subseteq f$. If $\alpha \notin \mathrm{Rge}(f)$, there are incompatible extensions of f with α in their ranges, so again $k \subseteq f$. Thus we have a natural transformation $\overline{\lambda}: f \to G$ such that:

$$\begin{tikzcd}
\mathrm{Cov}_\alpha(f) \arrow[r, hook] \arrow[dr, "\lambda"'] \arrow[dr, phantom, shift left=3, "///"] & f \arrow[d, "\overline{\lambda}"] \\
& G
\end{tikzcd}$$

as required ($\overline{\lambda}$ is necessarily unique).

The topos of J-sheaves $\mathrm{Sh}_J(P) \hookrightarrow S^P$ is not trivial, since:

<u>Proposition 4.3.3.</u> Every representable functor is a J-sheaf.

Proof. For any $f, g \in P$, $\alpha \in \mathbb{N}^{\mathbb{N}}$, and natural transformation λ :

$$\begin{tikzcd} Cov_\alpha(f) \arrow[r, hook] \arrow[dr, "\lambda"'] & f \\ & g \end{tikzcd},$$

all nonempty components of λ :

$$Cov_\alpha(f) \ni \{h\} \xrightarrow{\lambda_h} Hom_P(g,h)$$

are identities. Thus $g \subseteq h$ for each $h \in Cov_\alpha(f)$. Since $\bigcap Cov_\alpha(f) = f$, we have $g \subseteq f$, and hence a (necessarily unique) natural transformation $\overline{\lambda}: f \to g$ such that:

$$\begin{tikzcd} Cov_\alpha(f) \arrow[r, hook] \arrow[dr, "\lambda"'] & f \arrow[d, "\overline{\lambda}"] \arrow[dl, phantom, "///"] \\ & g \end{tikzcd}$$

as required.

The desired surjection F is obtained as a pullback in S^P :

$$\begin{tikzcd} F \arrow[r, hook] \arrow[d, hook] & G \arrow[d, hook] \\ \Delta(\mathbb{N}\times\mathbb{N}^{\mathbb{N}}) \arrow[r, hook] & \Delta P \end{tikzcd}$$

Indeed, we have:

Theorem 4.3.4. G is the generic model of T_2 in its classifying topos $Sh_J(P)$.

Proof. Let Ω be the complete Heyting algebra of subobjects of 1 in $Sh_J(P)$. It is a folklore (cf. [Jol] 5.37) that $Sh_J(P)$ and $Sh(\Omega)$ are equivalent categories. As an Ω-subset of the constant Ω-set P, G is given by $[\![f \in G]\!] = f$ as a representable functor, for each $f \in P$ (after Proposition 4.3.3.). Axioms 0) are obviously satisfied for $a_f = [\![f \in G]\!]$. Re axioms 1), observe that $f_1 \wedge f_2 = \{g \in P | f_1 \subseteq g\} \cap \{h \in P | f_2 \subseteq h\} = \{k \in P | k \supseteq f_1 \cup f_2\}$. Re axioms 2), we want to show that for each $\alpha \in \mathbb{N}^{\mathbb{N}}$, $C_\alpha = \bigvee\{f | f \in P, \alpha \in Rge(f)\} = 1$ in Ω. Obviously, $C_\alpha \hookrightarrow 1$ and $g \hookrightarrow C_\alpha$ for each $g \in P$ with $\alpha \in Rge(g)$. This gives us a natural transformation λ which extends to:

As usual, we look at the interpolation object $G \hookrightarrow \Delta P$ in S^P given by:

$$G = \sum_{f \varepsilon P} \mathrm{Hom}_P(f,-) \quad ,$$

in this case:

$$G(f) = \{f' \ \varepsilon \ P \mid f' \subseteq f\} \quad ,$$

since P is a partially ordered set as a category.

Proposition 4.3.2. G is a J-sheaf.

Proof. For any $f \ \varepsilon \ P$, $\alpha \ \varepsilon \ \mathbb{N}^{\mathbb{N}}$, and natural transformation λ :

$$\begin{tikzcd} \mathrm{Cov}_\alpha(f) \arrow[r, hook] \arrow[dr, "\lambda"'] & f \\ & G \end{tikzcd} \quad ,$$

we have for $g \subseteq h$ in P :

$$\begin{tikzcd} & \mathrm{Cov}_\alpha(f) \ni \{g\} \arrow[r, "\lambda_g"] \arrow[d] \arrow[dr, phantom, "///"] & G(g) = \{k \varepsilon P \mid k \quad g\} \arrow[d, hook] \\ g \rightsquigarrow h & \mathrm{Cov}_\alpha(f) \ni \{h\} \arrow[r, "\lambda_h"] & G(h) = \{k \varepsilon P \mid k \quad h\} \end{tikzcd} \quad .$$

Such a natural transformation λ is given by $k \ \varepsilon \ P$ such that for all g in $\mathrm{Cov}_\alpha(f)$, $k \subseteq g$. If $\alpha \ \varepsilon \ \mathrm{Rge}(f)$, this clearly says that $k \subseteq f$. If $\alpha \notin \mathrm{Rge}(f)$, there are incompatible extensions of f with α in their ranges, so again $k \subseteq f$. Thus we have a natural transformation $\bar{\lambda}\colon f \to G$ such that:

$$\begin{tikzcd} \mathrm{Cov}_\alpha(f) \arrow[r, hook] \arrow[dr, "\lambda"'] \arrow[dr, phantom, shift left=3, "///"] & f \arrow[d, "\bar{\lambda}"] \\ & G \end{tikzcd}$$

as required ($\bar{\lambda}$ is necessarily unique).

The topos of J-sheaves $\mathrm{Sh}_J(P) \hookrightarrow S^P$ is not trivial, since:

Proposition 4.3.3. Every representable functor is a J-sheaf.

Proof. For any $f,g \in P$, $\alpha \in \mathbb{N}^{\mathbb{N}}$, and natural transformation λ :

$$\begin{tikzcd} \mathrm{Cov}_\alpha(f) \arrow[r, hook] \arrow[dr, "\lambda"'] & f \\ & g \end{tikzcd},$$

all nonempty components of λ :

$$\mathrm{Cov}_\alpha(f) \ni \{h\} \xrightarrow{\lambda_h} \mathrm{Hom}_P(g,h)$$

are identities. Thus $g \subseteq h$ for each $h \in \mathrm{Cov}_\alpha(f)$. Since $\bigcap \mathrm{Cov}_\alpha(f) = f$, we have $g \subseteq f$, and hence a (necessarily unique) natural transformation $\overline{\lambda}: f \to g$ such that:

$$\begin{tikzcd} \mathrm{Cov}_\alpha(f) \arrow[r, hook] \arrow[dr, "\lambda"'] & f \arrow[d, "\overline{\lambda}"] \arrow[dl, phantom, "///"] \\ & g \end{tikzcd}$$

as required.

The desired surjection F is obtained as a pullback in S^P :

$$\begin{tikzcd} F \arrow[r, hook] \arrow[d, hook] & G \arrow[d, hook] \\ \Delta(\mathbb{N}\times\mathbb{N}^{\mathbb{N}}) \arrow[r, hook] & \Delta P \end{tikzcd}$$

Indeed, we have:

Theorem 4.3.4. G is the generic model of T_2 in its classifying topos $Sh_J(P)$.

Proof. Let Ω be the complete Heyting algebra of subobjects of 1 in $Sh_J(P)$. It is a folklore (cf. [Jol] 5.37) that $Sh_J(P)$ and $Sh(\Omega)$ are equivalent categories. As an Ω-subset of the constant Ω-set P , G is given by $[\![f \in G]\!] = f$ as a representable functor, for each $f \in P$ (after Proposition 4.3.3.). Axioms 0) are obviously satisfied for $a_f = [\![f \in G]\!]$. Re axioms 1), observe that $f_1 \wedge f_2 = \{g\in P \mid f_1 \subseteq g\} \cap \{h \in P \mid f_2 \subseteq h\} = \{k \in P \mid k \supseteq f_1 \cup f_2\}$. Re axioms 2), we want to show that for each $\alpha \in \mathbb{N}^{\mathbb{N}}$, $C_\alpha = \bigvee \{f \mid f \in P, \alpha \in \mathrm{Rge}(f)\} = 1$ in Ω . Obviously, $C_\alpha \hookrightarrow 1$ and $g \hookrightarrow C_\alpha$ for each $g \in P$ with $\alpha \in \mathrm{Rge}(g)$. This gives us a natural transformation λ which extends to:

$$\begin{tikzcd}
\mathrm{Cov}_\alpha(\emptyset) \arrow[r, hook] \arrow[dr, "\lambda"'] \arrow[r, phantom, shift right=4ex, "///"] & 1 \arrow[d, "\bar{\lambda}"] \\
& C_\alpha
\end{tikzcd}$$

since C_α is a sheaf, as a sup in Ω . Thus $C_\alpha = 1$, and so 2) hold. Hence, $G \hookrightarrow \Delta P$ is a model of T_2 .

To show that T_2 classifies, by Diaconescu's theorem [Jol], 4.3 it suffices to show that for every Grothendieck topos F , every model $\{\overline{a_f} \hookrightarrow 1 \mid f \in P\}$ of T_2 in F defines a flat, continuous functor $P^{op} \to F$. The assignment $f \rightsquigarrow \overline{a_f}$ from P^{op} to subobjects of 1 in F does define a flat functor, since it is a model of 0) and 1). Since it is a model of 2), it takes all $\mathrm{Cov}_\alpha(\phi) \hookrightarrow 1$ in S^P to epimorphic families in F , and hence by distributivity all $\mathrm{Cov}_\alpha(f) \hookrightarrow f$ in S^P to epimorphic families, by Proposition 4.3.1. By analogy with [SGA4], Exp. II, Corollary 2.3. we conclude that it is therefore continuous, cf. Lemma in section 0.8.

Since G is a model of T_2 F is a partial function from $\mathbb{N}$ onto the constant sheaf $(\mathbb{N}^{\mathbb{N}})$. Since $\mathrm{Sh}_J(P)$ is the classifying topos, we may hope for non-geometric "preservation properties". Indeed:

<u>Lemma 4.3.6</u>. The internal $\mathbb{N}^{\mathbb{N}}$ in $\mathrm{Sh}_J(P)$ is constant.

<u>Proof</u>. The internal $\mathbb{N}^{\mathbb{N}}$ in $\mathrm{Sh}_J(P)$ is given by $\wedge\bigvee$- maps a: Baire space $\to \Omega$ in S [FH]. We show that for each such a there exist a unique $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that for every open set $U \subseteq \mathbb{N}^{\mathbb{N}}$ (in the Baire topology), $\alpha \in U$ if and only if $a(U) = T_\Omega$. It suffices to show $\alpha \in U$ if and only if $a(U) \neq 0_\Omega$ for all such U , since a is $\wedge\bigvee$-map. For this, it suffices to show that for each $\wedge\bigvee$-map a there exists $\alpha \in \mathbb{N}^{\mathbb{N}}$ so that for each open set $U \subseteq \mathbb{N}^{\mathbb{N}}$, $\alpha \in U$ if and only if $a(U) \neq 0_\Omega$ (uniqueness then follows since Baire space is Hausdorff). Given a , we show that there exists $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that for each open set $U \subseteq \mathbb{N}^{\mathbb{N}}$, $\alpha \in U$ implies $a(U) \neq 0_\Omega$. Then we will show that for that α , we also have that $a(U) \neq 0_\Omega$ implies $\alpha \in U$ for each open set $U \subseteq \mathbb{N}^{\mathbb{N}}$. Suppose there is no α with the properties above, i.e.

that for each $\alpha \in \mathbb{N}^{\mathbb{N}}$ there exists an open set $\mathcal{U}_\alpha \subseteq \mathbb{N}^{\mathbb{N}}$ such that $\alpha \in \mathcal{U}_\alpha$ and $a(\mathcal{U}_\alpha) = 0_\Omega$. Since $\mathbb{N}^{\mathbb{N}} = \bigcup_{\alpha \in \mathbb{N}^{\mathbb{N}}} \mathcal{U}_\alpha$, and since a is $\wedge\bigvee$-map, we would have:

$$T_\Omega = a(\mathbb{N}^{\mathbb{N}}) = a(\bigcup_{\alpha \in \mathbb{N}^{\mathbb{N}}} \mathcal{U}_\alpha) = \bigvee_{\alpha \in \mathbb{N}^{\mathbb{N}}} a(\mathcal{U}_\alpha) = \bigvee 0_\Omega = 0_\Omega \quad ,$$

a contradiction. Hence there exists $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that $\alpha \in \mathcal{U}$ implies $a(\mathcal{U}) \neq 0_\Omega$ for each open set $\mathcal{U}$ $\mathbb{N}^{\mathbb{N}}$. To show that any such α has the required further property, suppose that there exists an open set $\mathcal{U} \subseteq \mathbb{N}^{\mathbb{N}}$ with $a(\mathcal{U}) \neq 0_\Omega$ and $\alpha \notin \mathcal{U}$. $\mathcal{U}$ is a union of basic open sets, so without loss of generality we may assume that $\mathcal{U}$ itself is a basic open set (since a preserves maps), hence closed (Baire space is 0-dimensional). Let $\mathcal{V} = \mathbb{N}^{\mathbb{N}} \setminus \mathcal{U}$. Since a is $\wedge\bigvee$-map, we would have $a(\mathcal{U}) \cap a(\mathcal{V}) = 0_\Omega$, $T_\Omega = a(\mathcal{U}) \vee a(\mathcal{V})$. So, sieves $a(\mathcal{U})$ and $a(\mathcal{V})$ are disjoint, hence any two $f \in a(\mathcal{U})$, $g \in a(\mathcal{V})$ are incompatible, for otherwise $(f \cup g) \in a(\mathcal{U}) \cap a(\mathcal{V})$ ($a(\mathcal{V}) \neq 0_\Omega$ since $\alpha \in \mathcal{V}$) . In particular, $\mathrm{dom}(f) \cap \mathrm{dom}(g) \neq \emptyset$. Let $k, \ell > 0$ be the minimal cardinalities of $f \in a(\mathcal{U})$, $g \in a(\mathcal{V})$, respectively. It now suffices to show that $a(\mathcal{U}) \vee a(\mathcal{V})$, the smallest sheaf subobject of 1 in $\mathrm{Sh}_J(P)$ which contains the sieve $a(\mathcal{U}) \cup a(\mathcal{V})$, is a proper subobject of 1 . We in fact claim that the minimal cardinality of $h \in a(\mathcal{U}) \vee a(\mathcal{V})$ is not less than $\min\{k, \ell\}$. Let $f \in a(\mathcal{U})$, $\mathrm{card}(f) = k$. Its domain intersects $\mathrm{dom}(g)$ for each $g \in a(\mathcal{V})$ with $\mathrm{card}(g) = \ell$, hence the union of all such $\mathrm{dom}(g)$ is a finite subset of $\mathbb{N}$. Similarly, $\bigcup\{\mathrm{dom}(f) \mid f \in a(\mathcal{U}) , \mathrm{card}(f) = k\}$ is a finite subset of $\mathbb{N}$. Without loss of generality, suppose $k \leq \ell$. Suppose that for some $h \in P$ with $\mathrm{card}(h) < k$, and some $\beta \in \mathbb{N}^{\mathbb{N}}$, $\mathrm{Cov}_\beta(h) \subseteq a(\mathcal{U}) \cup a(\mathcal{V})$. But $\mathrm{Cov}_\beta(h) = \{h' \in P \mid h \subseteq h' , \beta \in \mathrm{Rge}(h')\}$, so the set $\bigcup\{\mathrm{dom}(h') \mid h' \in \mathrm{Cov}_\beta(h), \mathrm{card}(h') = k\}$ has to be infinite, a contradiction.

Thus we have proved:

<u>Theorem 4.3.7</u>. $\mathrm{Sh}_J(P) \models \mathbb{N}^{\mathbb{N}}$ is subcountable.

<u>Remark</u>. The construction presented here is analogous to Joyal's construction presented in section 4.2. We believe Joyal was aware of this construction as well.

4.4. Independence of Bar Induction in the Presence of Continuity Principles.

In section 4.1. we showed that in the classifying topos for an infinite sequence, several other principles of intuitionistic analysis hold (due to the universality, although we were not looking directly for a model in which they hold). In this section we will see a similar situation: trying to construct a universal counterexample to Bar Induction, we will actually get a stronger result. It turns out that Continuity Principles, Kripke's Schema, Uniformity Principles, and Relativized Dependent Choice all hold in the topos which classifies a counterexample to Bar Induction. In [Sc2], this topos was presented as a topological model of ZFI.

Let $\mathbb{N}^*$ be the tree of all finite sequences of natural numbers, partially ordered by inclusion, and let $A,B \subseteq \mathbb{N}^*$. Recall that Bar-Induction states that:

$$\left[\begin{array}{l} \forall\alpha \in \mathbb{N}^{\mathbb{N}}.\exists n \in \mathbb{N}.\overline{\alpha}(n) \in B \\ \wedge \\ \forall u \in \mathbb{N}^*[u\varepsilon B \vee \neg(u\varepsilon B)] \\ \wedge \\ \forall u \in \mathbb{N}^*[u\varepsilon B \to u\varepsilon A] \\ \wedge \\ \forall u \in \mathbb{N}^*[\forall n\varepsilon\mathbb{N}(u*\langle n\rangle \in A) \to u\varepsilon A] \end{array}\right] \longrightarrow \langle\ \rangle \in A \quad ,$$

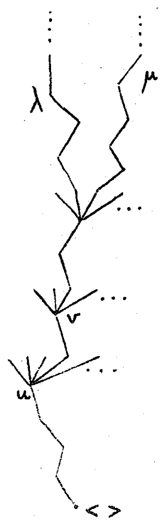
where $\overline{\alpha}(n)$ is the finite sequence $\langle\alpha(0),\ldots,\alpha(n-1)\rangle$, $\langle\ \rangle$ is the empty sequence, and $u*\langle n\rangle$ is the result of concatenating the one-element sequence $\langle n\rangle$ on the right to a finite sequence u.

Classically, Bar Induction is most easily justified by proving the contraposite. Intuitionistically, that of course would not work, but that is not the point. We shall see that a universal counterexample to the contraposite of Bar Induction is actually a counterexample to Bar Induction (this is also true for Kleene's recursive example [KV], pp. 112-113: more precisely, a recursive counterexample to König's Lemma he constructs is actually a counterexample to the Fan Theorem, its classical contraposite).

Let us study the case when $A=B$, so A is in particular a detachable subset of $\mathbb{N}^*$. Thus we might as well study its complement X. Suppose $\langle\ \rangle \in X$. Furthermore, say $u \in X$. If it is possible to induct on A downward (the last assumption of Bar Induction), for each extension $v \supseteq u$ there is a minimal $n \in \mathbb{N}$ so that $v*\langle n\rangle \in X$ (we are aware that this reasoning is classical and we ask the reader acquainted with the subtleties of intuitionistic logic to bear with us for a while, cf. our remarks above). In other words, there is a path λ through u such that only extensions $w \supseteq u$ "to the right of or on" λ are possibly in X : ("$w \in X$ is allowed by (u,λ)").

Schema 4.4.1.

As we move up the tree $\mathbb{N}^*$, we are specifying X more strongly: given a specification (u,λ) for X , a stronger specification is (v,μ) with $v \supseteq u$, μ a path through v "to the right of or equal to" λ :



Schema 4.4.2.

These rigidifying conditions for X as a subset of $\mathbb{N}^*$ give a partial order P : for $u,v \in \mathbb{N}^*$ and paths λ through u , μ through v we write $(u,\lambda) \leq (v,\mu)$ iff $v \supseteq u$ as finite sequences and μ is "to the right of or equal to" λ .

We now describe a theory of X as a theory of specifications for X as given above. First, if (v,μ) describes possible elements of X , and $(u,\mu) \leq (v,\mu)$, then so does (u,λ) (only more loosely). Furthermore, since X is being specified just as u moves up the tree $\mathbb{N}^*$, if (u,λ) are (v,μ) are two specifications of X , we have either $u \supseteq v$ or $v \supseteq u$, and we have a specification (w,κ) where w is the longer one of u,v (or $w=u=v$) and κ is the one of paths λ , μ which eventually veers to the right of the other one (or $\kappa=\mu=\lambda$) . Finally, given a specification (u,λ) of X which allows $w \in X$, there is a strictly better specification (v,μ) of X which allows $w \in X$. Let us then think of "(u,μ) is a specification of (possible elements of) X " as a propositional symbol $a_{u,\lambda}$, and the requirements just described as a geometric theory in these propositional symbols:

(0) $\quad a_{v,\mu} \vdash a_{u,\lambda}$, for $(u,\lambda) \leq (v,\mu)$

(1) $\quad a_{u,\lambda} \wedge a_{v,\mu} \vdash a_{w,\kappa}$, where w and κ were described above if (u,λ) and (v,μ) are compatible in P

$\quad a_{u,\lambda} \wedge a_{v,\mu} \vdash \bot$, if (u,λ) and (v,μ) are incompatible in P

(2) $\quad a_{u,\lambda} \vdash \bigvee_{(v,\mu)\in R} a_{v,\mu}$, where $R \subseteq P$ is any upward closed subset of extensions of (u,λ) such that each $s \in \mathbb{N}^*$ allowed into X by (u,λ) is allowed into X by some specification $(w,\chi) \in R$.

<u>Theorem 4.4.1.</u> The classifying topos of the geometric theory (0)-(2) is spatial, on the topological space on $\mathbb{N}^*$ given by the basis consisting of sets $\mathcal{U}_{u,\lambda} = \{u\} \cup \{v \supseteq u^*\langle m_1,\ldots,m_i,k\rangle \ , \ i \geq 1, k > m_{i+1}\} \cup \{\overline{\lambda}(n) \mid \overline{\lambda}(n) \supseteq u\}$, where $(u,\lambda) \in P$, and for $\overline{\lambda}(k) = u$ we have $\lambda(k+i) = m_i$ for each i (cf. Schema 4.4.1.).

Proof. P^{op} is the localic basis for the locale of open sets. Consider the canonical Grothendieck topology on P^{op}, and let $\mathcal{E} \hookrightarrow S^P$ be the topos of canonical sheaves. Since P^{op} is the localic basis, we have $\mathcal{E} \simeq \mathrm{Shv}(\mathbb{N}^*)$. We show that $\mathcal{E}$ classifies. Indeed, let the interpretation of $a_{u,\lambda}$ in $\mathcal{E}$ be the representable functor $(u,\lambda) \hookrightarrow 1$. 1 is representable by $(<\,>,\alpha)$, where α is the constant zero sequence. Meets and joins are given as intersections and unions of sieves, respectively. Thus we clearly have a model of (0)-(2), which is of course preserved by inverse images of geometric morphisms $\mathcal{F} \to \mathcal{E}$, $\mathcal{F}$ being any Grothendieck topos. By Diaconescu's Theorem [Jo 1,§4.3], it remains to show that every $\mathcal{F}$-model of (0)-(2) is a continuous flat functor $P^{op} \to \mathcal{F}$. Well, it is a functor because it is a model of (0). Next, notice that the axioms (1) describe pullbacks of representable functors in S^P. In particular, nonrepresentable pullbacks of representable functors are empty (as sieves). So the corresponding functor $P^{op} \to \mathcal{F}$ is flat, since it is left exact on the larger category $P^{op} \hookrightarrow \mathbb{C} \hookrightarrow S^P$ just described. Finally, it is obviously continuous as a model of (2).

Consider $G \hookrightarrow \Delta P$ in S^P given by $G = \sum_{(u,\lambda)\in P} P((u,\lambda),\_)$, in other words $G((u,\lambda)) = \{(v,\mu)\in P \mid (v,\mu) \le (u,\lambda)\}$. As a topological-valued set, G is given by the characteristic function $G((u,\lambda))$ = the representable sheaf $(u,\lambda) = \mathcal{U}_{u,\lambda}$. We have $\mathbb{N}^* \hookrightarrow P$ as posets, by $u \rightsquigarrow (u,\lambda_u^0)$, where λ_u^0 is the constant zero path after it goes through u. Let $X \hookrightarrow \Delta\mathbb{N}^*$ be the restriction of G:

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\text{p.b.}"] & G \arrow[d, hook] \\ \Delta\mathbb{N}^* \arrow[r, hook] & \Delta P \end{tikzcd}$$

As a topological-valued set, X is given by the characteristic function $X(u) = \mathcal{U}_{u,\lambda_u^0} = \{v \in \mathbb{N}^* \mid v \supseteq u\}$. Since all sets $\mathcal{U}_{u,\lambda}$ are clopen in the generated topology on the set $\mathbb{N}^*$, its complement A is given as $A(u) = \mathbb{N}^* \setminus \{v\in\mathbb{N}^* \mid v \supseteq u\}$.

Theorem 4.4.2. Bar induction fails in E w.r.t. A .

Proof. In $E \simeq \mathrm{Shv}(\mathbb{N}^*)$, internal object of finite sequences of natural numbers is given by (the associated sheaf of) $\Delta\mathbb{N}^*$. Internally, therefore, it suffices to work with finite sequences of natural numbers from S . Since $[\![<\ > \in A]\!] = \mathbb{N}^* \setminus \{v\in\mathbb{N} \mid v \supseteq <\ >\} = \mathbb{N}^* \setminus \mathbb{N}^* = \emptyset$, so the conclusion of Bar Induction fails. We check that the last hypothesis of Bar Induction holds in E , the others are obvious. For any $u \in \mathbb{N}^*$:

$$[\![\forall n\in\mathbb{N}. u*<n>\in A]\!] = \mathrm{int}\bigcap_{n\in\mathbb{N}} (\mathbb{N}^* \setminus \{v\in\mathbb{N}^* \mid v \supseteq u*<n>\}) =$$

$$= \mathrm{int}(\mathbb{N}^* \setminus \bigcup_{n\in\mathbb{N}} \{v\in\mathbb{N}^* \mid v \supseteq u*<n>\}) = \mathrm{int}(\{u\} \cup (\mathbb{N}^* \setminus \{v\in\mathbb{N}^* \mid v \supseteq u\})) =$$

$= \mathbb{N}^* \setminus \{v\in\mathbb{N}^* \mid v \supseteq u\} = [\![u \in A]\!]$, since every open set $\mathcal{O} \ni u$ intersects $\bigcup_{n\in\mathbb{N}} \{v\in\mathbb{N}^* \mid v \supseteq u*<n>\}$.

The following theorem is proved in §2 of [Sc2]:

Theorem 4.4.3. The following hold in $\mathrm{Shv}(\mathbb{N}^*)$:

(i) Local $\forall\alpha\exists n$-Continuity without parameters:

$$\forall\alpha\in\mathbb{N}^{\mathbb{N}}.\exists x\in\mathbb{N}.A(\alpha,x) \to \forall\alpha\in\mathbb{N}^{\mathbb{N}}.\exists y,b\in\mathbb{N}.\forall\gamma\in\mathbb{N}^{\mathbb{N}}[\overline{\alpha}(y) = \overline{\gamma}(y) \to A(\gamma,b)]$$

where α, x are the only parameters in $A(\alpha,x)$.

(ii) Kripke's Schema:

$$\exists\alpha \in \mathbb{N}^{\mathbb{N}}[\exists n\in\mathbb{N}.\alpha(n) \neq 0 \leftrightarrow A] ,$$

where α is not free in A .

(iii) Uniformity Principle:

$$\forall X\in P(\mathbb{N}).\exists n\in\mathbb{N}.A(n,X) \to \exists n\in\mathbb{N}.\forall X\in P(\mathbb{N}).A(n,X) .$$

(iv) Relativized Dependent Choice (for any set S):

$$\forall x\in S(A(x) \to \exists y\in S(A(y) \wedge B(x,y))) \to$$

$$\to \forall x\in S(A(x) \to \exists f\in S^{\mathbb{N}}(f(0) = x \wedge \forall n\in\mathbb{N}(A(f(n)) \wedge B(f(n),f(n+1))))) .$$

Remark: Fourman and Hyland [FH] showed that Bar Induction fails in a spatial topos, where the topology on $\mathbb{N}^*$ consists of all $\mathcal{O} \subseteq \mathbb{N}^*$ such that for each $u \in \mathbb{N}^*$:

$$u \in \mathcal{O} \rightarrow \exists m \in \mathbb{N}. \forall k > m.\ u*\langle k\rangle \in \mathcal{O} \quad ,$$

by the same calculation as in the proof of our Theorem 4.4.2. We took the coarsest topology (it is strictly coarser than the one just described) in which that calculation works, and this universal property then results in Theorem 4.4.3. It is not mentioned in [FH] whether any proofs of the principles mentioned in Theorem 4.4.3. hold in the spatial topos given in [FH].

Since our topos is spatial, "$2^{\mathbb{N}}$ is compact" (i.e. the Fan Theorem) holds in it, cf. [FSo, §26]. In [Sc3], we describe a localic topos in which Fan Theorem fails, but (mild restrictions of) principles mentioned in Theorem 4.4.3. all hold.

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Department of Mathematics
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