Elastic Constants of Layered Media*

P. C. CHOU AND J. CARLEONE

Mechanics and Structures, Drexel University, Philadelphia, PA 19104

AND

C. M. HSU

Chung Shan Research Institute, Taiwan, Republic of China

(Received November 1, 1971)

ABSTRACT

A three-dimensional laminated medium is studied by an approach which replaces the heterogeneous medium by an equivalent homogeneous material. A set of macroscopic elastic constants is developed in terms of the properties of the constituent layers by considering a representative small element of the laminated medium and imposing the condition of continuity of stress and displacement at the layer interfaces. An example boundary value problem is also considered. Based on the solution of the boundary value problem for the homogeneous material, stress and strain fields corresponding to each layer are then calculated. This individual layer solution contains certain discontinuous stresses and strains as does the exact solution for the layered material.

I. INTRODUCTION

THE DETERMINATION of gross elastic constants for composite materials has received much attention. For composites of arbitrary geometry, bounds of the overall elastic moduli have been presented by Hashin and Shtrikman [1] and Hill [2,3]. For a specific geometry, such as layered composites, various explicit formulas for the moduli have been offered in the literature.

In treating a layered medium as homogeneous, three obvious processes have been used, namely, the "law of mixture," Voigt's hypothesis and Reuss's hypothesis. In the "law of mixture," the Young's moduli of constituents are averaged by volume. For small specimens under uniaxial tension, the Young's modulus calculated by the

*Appreciation is given to the U.S. Army Ballistics Research Laboratories at Aberdeen, Maryland for their partial support of this work.

law of mixture is acceptable. But, as is well-known, the law of mixture cannot be applied to other elastic constants. Voigt's hypothesis, originally stated for a polycrystal, assumes that all of the strain components throughout the mixture are uniform. The counterpart to this approximation is Reuss's hypothesis which assumes that all the stress components throughout the mixture are uniform. Both Voigt's and Reuss's hypotheses are not consistent with the theory of elasticity. The stresses implied by Voigt's hypothesis are not in equilibrium at the interface between the layers, and the strains resulting from Reuss's hypothesis are such that the material could not remain bonded and thus the compatibility condition of the material is violated.

More recently, many investigators have proposed more sophisticated methods in deriving overall moduli for layered materials. White and Angona [4], Postma [5], Rytov [6] and Behrens [7] studied the alternately layered composite, in which each layer is composed of an isotropic medium. Salamon [8] studied the elastic moduli of a stratified rock mass. White and Angona used a static approach to determine the elastic constants by assuming certain stresses and strains in the medium. The stresses and strains assumed are those which would exist during the passage of an elastic wave through the medium. Rytov as well as Behrens used dispersion techniques to study the propagation of elastic waves in an alternately layered medium, and found the averaged elastic constants by first calculating the phase velocity in the long-wavelength limit for different directions of propagation and polarization. Salamon used a static approach and considered stratified rock to be a repeated n-layered medium. References [4] through [7] obtain the stiffnesses for an equivalent transversely isotropic material. Salamon, on the other hand, calculates the compliances.

It is interesting to note that the elastic constants derived in References [4-8], although different in appearance, can be shown to be identical, i.e. by proper algebraic manipulation the elastic constants in all these papers may be transformed into one identical set of formulas. This fact has not been mentioned by any of these authors; in fact, none of them have compared their results with any of the others. This is perhaps due to the differences in their basic assumptions and approaches.

For most loading conditions an equivalent homogeneous anisotropic material will yield satisfactory results as a model for a composite material. However, Sun, Achenbach and Herrmann [9] pointed out that under certain loading conditions, in particular dynamic loadings for which characteristic lengths of the deformation are small, an equivalent homogeneous anisotropic material does not give a satisfactory description of the mechanical behavior of a layered medium. They then proceed to develop a continuum theory with microstructure for an alternately layered medium. Subsequently, Achenbach [10] showed that by neglecting terms involving the microstructure, the elastic constants of References [4-8] can be obtained from the more general continuum theory with microstructure. Chou and Wang [11] recently used a control volume approach to study a one-dimensional wave front in composite materials. In their analysis of the case of an alternately layered medium, they obtained an overall elastic stiffness relating the average normal stress to the normal strain in the plane of the layering. This stiffness is also identical to the corresponding stiffness for the equivalent homogeneous material in References [4-8].

In this paper, a systematic approach is given for the derivation of the formulas relating the stiffness or compliance matrices of an equivalent homogeneous material to the properties of its layered constituents. The relevant constants of References [4-8] can be obtained by reduction of the present results. In addition, the present analysis includes the case of each layer being anisotropic, whereas those in References [4-8] are limited to isotropic layers. The more general case of repeated n-layered medium is also included. A procedure for the solution of boundary value problems utilizing the homogeneous moduli will also be explained. This procedure is similar to the one used in classical laminated plated theories (see, e.g., References [12-14]). In these theories a continuous solution is first obtained by applying classical plate assumptions and considering the total forces integrated across the layers. Then, through the use of the constitutive laws of each layer, stresses in the individual layers are obtained. The laminated plate theories are limited to two-dimensional state of stress; the third normal stress and the transverse shear stresses are not included. The procedure adopted in this paper may be considered as an extension of the procedure used in laminated plate theories to the general three-dimensional case.

The basic assumptions of the present approach are a combination of Voigt's and Reuss's hypotheses. We first isolate an element containing n layers from the composite material. This element is assumed to be small in comparison to the overall size of the composite and will be considered as the "small element" in studying the Hooke's law equations in classical elasticity. Thus we will consider this element to be under a uniform state of stress when the entire three-dimensional composite is loaded in any arbitrary manner. We further assume (1) that the normal strains in the element parallel to the layers and the shear strain in the plane of the layers, are uniform and the same for each ingredient, and we average the corresponding stresses; (2) the normal stress perpendicular to the layers and the shear stresses in the planes perpendicular to the layers are uniform and the same in each material, and we average the corresponding strains. With these assumptions both equilibrium at the interface and the compatibility of the material is satisfied. It is thus possible to effectively "smooth out" the layered composite and determine overall elastic constants of an "equivalent" homogeneous anisotropic material. As the thickness of each layer approaches zero, the solution of the present theory approaches the exact solution, since the basic element then becomes truly infinitesimal.

This paper treats the case of a laminated body composed of n-types of material where each material may be anisotropic. The degree of anisotropy of each layer is monoclinic with the plane of symmetry parallel to the layers. The equivalent material is found to be monoclinic and the stiffnesses and compliances are given.

An example boundary value problem, consisting of layered concentric rings under internal pressure, is solved by the present approach. The results are in good agreement with the exact elasticity solution.

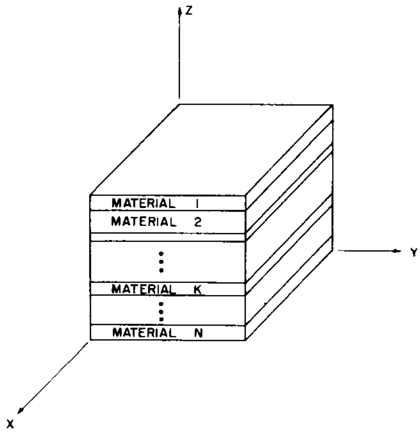

Figure 1. *Representative element of a repeated multilaminar composite.*

II. MULTILAMINAR COMPOSITE WITH ANISOTROPIC LAYERS

Consider a layered medium obtained by stacking together identical n-layered plates, where the material of each layer is anisotropic. The anisotropy is restricted to monoclinic, with planes of symmetry parallel to the layering. Of course, the cases of orthotropic, transversely isotropic and isotropic layers will also be included as special cases of the monoclinic layering. A representative element of the laminated medium is shown in Figure 1. This representative element is assumed to be much smaller than the overall composite. At the same time, we will consider a small element of the equivalent homogeneous medium. In order to obtain the constitutive law of the equivalent medium in terms of the properties of the individual layers, certain relations between the stresses and strains of the individual layers and those of the equivalent homogeneous medium will be assumed.

Contracted notation will be used; when using subscripts the summation convention will be employed. A superscripted quantity will refer to the particular layer but the superscripts will not adhere to the summation convention. Quantities without superscripts are quantities associated with the equivalent homogeneous material. Let us take the x, y-plane (1, 2-plane) to be parallel to the layering. We will assume stresses and strains in the layered element such that equilibrium and compatibility are satisfied. On the basis of continuity of displacement at the layer interfaces, we assume that the normal strains in the 1- and 2-directions and the shear strain in the 1, 2-plane are uniform and the same in each layer of the element and equal to the corresponding strains in the equivalent homogeneous element, i.e.

$$\epsilon_i = \epsilon_i^k \qquad (i = 1, 2, 6 \;;\; k = 1, 2, \ldots, n) \tag{1}$$

On the basis of continuity of stress at the layer interfaces, we assume that the normal stress in the 3-direction and the shear stress associated with the 3-direction are uniform and the same in each layer of the element and equal to the corresponding stresses in the equivalent homogeneous element, i.e.

$$\sigma_i = \sigma_i^k \qquad (i = 3, 4, 5 \;;\; k = 1, 2, \ldots, n) \tag{2}$$

The remaining stresses and strains of the equivalent homogeneous element are assumed to be the average of those in the layered element, or

$$\epsilon_i = \sum_{k=1}^{n} V^k \epsilon_i^k \qquad (i = 3, 4, 5) \tag{3}$$

$$\sigma_i = \sum_{k=1}^{n} V^k \sigma_i^k \qquad (i = 1, 2, 6) \tag{4}$$

where $V^k = \dfrac{\text{original volume of material } k \text{ in the element}}{\text{original volume of the composite element}}$.

The generalized Hooke's Law in each monoclinic material is given by

$$\sigma_i^k = C_{ij}^k \epsilon_j^k \qquad (i = 1, \ldots, 6 \;;\; k = 1, \ldots, n) \tag{5}$$

or

$$\epsilon_i^k = S_{ij}^k \sigma_j^k \qquad (i = 1, \ldots, 6 \;;\; k = 1, \ldots, n) \tag{6}$$

where

$$[C_{ij}^k] = \begin{bmatrix} C_{11}^k & C_{12}^k & C_{13}^k & 0 & 0 & C_{16}^k \\ C_{21}^k & C_{22}^k & C_{23}^k & 0 & 0 & C_{26}^k \\ C_{31}^k & C_{32}^k & C_{33}^k & 0 & 0 & C_{36}^k \\ 0 & 0 & 0 & C_{44}^k & C_{45}^k & 0 \\ 0 & 0 & 0 & C_{54}^k & C_{55}^k & 0 \\ C_{61}^k & C_{62}^k & C_{63}^k & 0 & 0 & C_{66}^k \end{bmatrix}$$

$$C_{ij}^k = C_{ji}^k \ , \ i \neq j$$

with the identical terms appearing in $[S_{ij}^k]$. Equations (1) through (5) represent $12n + 6$ linear equations in $12n + 12$ variables. Thus we can solve for the stresses of the equivalent material in terms of the strains of the equivalent material. The equivalent homogeneous elastic constants are thus expressed in terms of the constituent constants.

Performing the algebra, we find

$$\sigma_i = C_{ij} \, \epsilon_j \quad (i, j = 1, \ldots, 6)$$

where

$$C_{ij} = \sum_{k=1}^{n} V^k \left[C_{ij}^k - \frac{C_{i3}^k C_{3j}^k}{C_{33}^k} + \frac{C_{i3}^k \sum\limits_{\ell=1}^{n} \dfrac{V^\ell C_{3j}^\ell}{C_{33}^\ell}}{C_{33}^k \sum\limits_{\ell=1}^{n} \dfrac{V^\ell}{C_{33}^\ell}} \right] \quad (i, j = 1, 2, 3, 6) \quad (7)$$

$$C_{ij} = C_{ji} = 0 \quad (i = 1, 2, 3, 6; j = 4, 5) \qquad (8)$$

$$C_{ij} = \frac{\sum\limits_{k=1}^{n} \dfrac{V^k}{\Delta'_k} C_{ij}^k}{\sum\limits_{k=1}^{n} \sum\limits_{\ell=1}^{n} \dfrac{V^k V^\ell}{\Delta'_k \Delta'_\ell} (C_{44}^k C_{55}^\ell - C_{45}^k C_{54}^\ell)} \quad (i, j = 4, 5) \qquad (9)$$

$$\Delta'_k = \begin{vmatrix} C_{44}^k & C_{45}^k \\ C_{54}^k & C_{55}^k \end{vmatrix}$$

From these stiffnesses we see that the equivalent homogeneous material is monoclinic with the 1,2-plane as the plane of symmetry.

The compliances of the equivalent material can also be solved for, although the algebra involved is longer than in solving for the stiffnesses. To do this we take equations (1) through (4) and (6). This again yields $12n + 6$ linear equations in $12n + 12$ variables. Thus the strains can be solved for in terms of the stresses, or

$$\epsilon_i = S_{ij}\,\sigma_j \qquad (i, j = 1, \ldots, 6)$$

The compliances S_{ij} may be expressed as functions of the constituent compliances in two different forms. In the first, a convenient reference layer is used; in the second no reference layer is used. The expressions of the first form are,

$$S_{ij} = \frac{1}{\Delta}\left(S_{i1}^{m}\,\Delta_{j'1} + S_{i2}^{m}\,\Delta_{j'2} + S_{i6}^{m}\,\Delta_{j'3}\right) \tag{10}$$

$$i, j = 1, 2, 6; \quad j' = \begin{cases} j \text{ if } j = 1, 2 \\ 3 \text{ if } j = 6 \end{cases}$$

$$S_{i3} = S_{i3}^{m} - \frac{1}{\Delta}\sum_{k=1}^{n}\sum_{\ell=1}^{3}\frac{V^k\,\Delta_k^{\ell}}{\Delta_k}\left(S_{i1}^{m}\,\Delta_{\ell 1} + S_{i2}^{m}\,\Delta_{\ell 2} + S_{i6}^{m}\,\Delta_{\ell 3}\right) \tag{11}$$

$$i = 1, 2, 6$$

$$S_{3i} = \frac{1}{\Delta}\sum_{k=1}^{n}\sum_{\ell=1}^{3}\frac{V^k}{\Delta_k}\left(S_{31}^{k}\,\Delta_k^{1\ell} + S_{32}^{k}\,\Delta_k^{2\ell} + S_{36}^{k}\,\Delta_k^{3\ell}\right)\sigma_{i'\ell} \tag{12}$$

$$i = 1, 2, 6 \qquad i' = \begin{cases} i \text{ if } i = 1, 2 \\ 3 \text{ if } i = 6 \end{cases}$$

$$\begin{aligned} S_{33} = \sum_{k=1}^{n} V^k \Bigg\{ & S_{33}^{k} + \frac{V^k}{\Delta_k}\left(S_{31}^{k}\,\Delta_k^{1} + S_{32}^{k}\,\Delta_k^{2} + S_{36}^{k}\,\Delta_k^{3}\right) \\ & - \frac{1}{\Delta\Delta_k}\sum_{p=1}^{n}\sum_{\ell=1}^{3}\sum_{i=1}^{3}\frac{V^p\,\Delta_p^{\ell}}{\Delta_p}\left(S_{31}^{k}\,\Delta_k^{1i} + S_{32}^{k}\,\Delta_k^{2i} + S_{36}^{k}\,\Delta_k^{3i}\right)\Delta_{\ell i}\Bigg\} \end{aligned} \tag{13}$$

$$S_{ij} = \sum_{k=1}^{n} V^k\, S_{ij}^{k} \quad (j = 4, 5) \tag{14}$$

$$S_{ij} = S_{ji} = 0 \;\; (i = 1, 2, 3, 6;\; j = 4, 5) \tag{15}$$

where the superscript m in the above refers to any convenient reference layer. The other symbols in equations (10) to (14) are defined as

$$\Delta_k = \det D_k \tag{16}$$

where

$$D_k = \begin{bmatrix} S_{11}^k & S_{12}^k & S_{16}^k \\ S_{21}^k & S_{22}^k & S_{26}^k \\ S_{61}^k & S_{62}^k & S_{66}^k \end{bmatrix}$$

The symbol Δ_k^{ij} is the determinant of the matrix formed from D_k by replacing the i^{th} column of D_k by the j^{th} column of D_m.
The symbol Δ_k^i is the determinant of the matrix formed by replacing the i^{th} column of D_k by the column vector $[S_{13}^m - S_{13}^k, S_{23}^m - S_{23}^k, S_{63}^m - S_{63}^k]$.

And,
$$\Delta = \det E \tag{17}$$

where

$$E = \begin{bmatrix} \sum_{k=1}^{n} \frac{V^k \Delta_k^{11}}{\Delta_k} & \sum_{k=1}^{n} \frac{V^k \Delta_k^{12}}{\Delta_k} & \sum_{k=1}^{n} \frac{V^k \Delta_k^{13}}{\Delta_k} \\ \sum_{k=1}^{n} \frac{V^k \Delta_k^{21}}{\Delta_k} & \sum_{k=1}^{n} \frac{V^k \Delta_k^{22}}{\Delta_k} & \sum_{k=1}^{n} \frac{V^k \Delta_k^{23}}{\Delta_k} \\ \sum_{k=1}^{n} \frac{V^k \Delta_k^{31}}{\Delta_k} & \sum_{k=1}^{n} \frac{V^k \Delta_k^{32}}{\Delta_k} & \sum_{k=1}^{n} \frac{V^k \cdot \Delta_k^{33}}{\Delta_k} \end{bmatrix}$$

The symbol Δ_{ij} is the cofactor of the $i\,j^{th}$ element of E.

Notice that equations (10) to (14) give the compliances with respect to any convenient reference layer. We see that if one of the layers is isotropic by using that layer as a reference the calculation of the compliances can be greatly simplified. However, the form of the compliances given by equations (10) to (14) does not explicitly express the symmetry of the compliance matrix.

The second form of S_{ij} is a general expression for the compliances which is not based on any single reference layer. The two forms of expression can be shown to be equivalent. The following are the second form,

$$S_{ij} = \frac{1}{\Delta^*} \sum_{k=1}^{n} \sum_{\ell=1}^{n} \frac{V^k V^\ell}{\Delta_k \Delta_\ell} \left(\Delta_{pr}^{k} \Delta_{qs}^{\ell} - \Delta_{ps}^{k} \Delta_{qr}^{\ell} \right)$$

$$(i, p, q), (j, r, s) = (1, 2, 3), (2, 3, 1), (6, 1, 2) \tag{18}$$

$$S_{3i} = S_{i3} = \frac{1}{\Delta^*} \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} \sum_{k_3=1}^{n} \frac{V^{k_1} V^{k_2} V^{k_3}}{\Delta_{k_1} \Delta_{k_2} \Delta_{k_3}} \widetilde{\Delta}_{i'}$$

$$i = 1, 2, 6\,;\; j' = \begin{cases} i \text{ if } i = 1, 2 \\ 3 \text{ if } i = 6 \end{cases} \tag{19}$$

$$S_{33} = \sum_{k=1}^{n} V^k \left\{ S_{33}^{\,k} + \frac{1}{\Delta_k} \left[\left(S_{13} - S_{13}^{\,k} \right) \Delta_1^k + \right. \right.$$

$$\left. \left. \left(S_{23} - S_{23}^{\,k} \right) \Delta_2^k + \left(S_{63} - S_{63}^{\,k} \right) \Delta_3^k \right] \right\} \tag{20}$$

where

$$\Delta_{ij}^{r} \text{ is the cofactor of the } ij^{th} \text{ element of } D_r \tag{21}$$

$$\Delta^* = \frac{\Delta}{\Delta_m} = \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} \sum_{k_3=1}^{n} \frac{V^{k_1} V^{k_2} V^{k_3}}{\Delta_{k_1} \Delta_{k_2} \Delta_{k_3}} \widetilde{\Delta} \tag{22}$$

and

$$\widetilde{\Delta} = \det F\,; \tag{23}$$

$$F = \begin{bmatrix} \Delta_{11}^{k_1} & \Delta_{12}^{k_2} & \Delta_{13}^{k_3} \\ \Delta_{21}^{k_1} & \Delta_{22}^{k_2} & \Delta_{23}^{k_3} \\ \Delta_{31}^{k_1} & \Delta_{32}^{k_2} & \Delta_{33}^{k_3} \end{bmatrix}$$

$$\widetilde{\Delta}_{i'} \text{ is the determinant of the matrix formed from } F \text{ by replacing the } i'^{th} \text{ column of } F \text{ by the column vector } [\Delta_1^{k_{i'}},\ \Delta_2^{k_{i'}},\ \Delta_3^{k_{i'}}] \tag{24}$$

where

$$\Delta_j^{k_{i'}} = S_{31}^{k_{i'}} \Delta_{j1}^{k_{i'}} + S_{32}^{k_{i'}} \Delta_{j2}^{k_{i'}} + S_{36}^{k_{i'}} \Delta_{j3}^{k_{i'}}$$

Although the elastic constants of the present theory were derived for Cartesian coordinates, they can be directly applied to layered systems which are oriented in any orthogonal curvilinear coordinate system, as long as the coordinate system is an intrinsic coordinate system of the layering, i.e. one of the coordinate directions is perpendicular to the layering, and the other two lie in the plane of the layering. Of course, once the overall elastic constants in the intrinsic coordinates are found, these constants may be found in any other system by using the transformation equations for a fourth order tensor.

The stiffnesses given previously by other authors for an alternately layered medium with isotropic layers can be obtained from (7), (8) and (9) by setting n=2, and restricting the layer to be isotropic. For example, White and Angona [4] give wave speeds for different directions of propagation. These wave speeds were found by a static technique similar to the one just presented, hov/ever, the basic assumptions are not clearly stated. By simply dividing our stiffnesses for the case n=2 with isotropic layers by the average density

$$\rho_{AV} = V^1 \rho^1 + V^2 \rho^2$$

the square of the wave speeds given by [4] can be found after some algebraic manipulation.

Postma [5] uses a static technique similar to the approach given here to determine the stiffnesses. The stiffnesses given by equations (7) of [5] can be shown to be a special case of the stiffnesses given by equations (7), (8) and (9) here. Rytov [6] and Behrens [7] used dispersion techniques to study elastic waves in alternately layered composites. In the long wave-length limit they determine the overall static material constants of the layered medium. The stiffnesses listed in Table 1 of [7] and in equations (21) and (22) of [6] can also be shown to be a special case of equations (7), (8) and (9) given here. Salamon's result [8] is also a special case of the present theory. If we restrict each layer to be transversely isotropic with the isotropic plane parallel to the layers, it can be shown that equations (10) to (15) or (18) to (20), reduce to equations (15) of [8].

III. EXAMPLE BOUNDARY VALUE PROBLEM

In this section we shall solve a boundary value problem using a "smearing" and "unsmearing" technique and compare the results with those of the exact elasticity

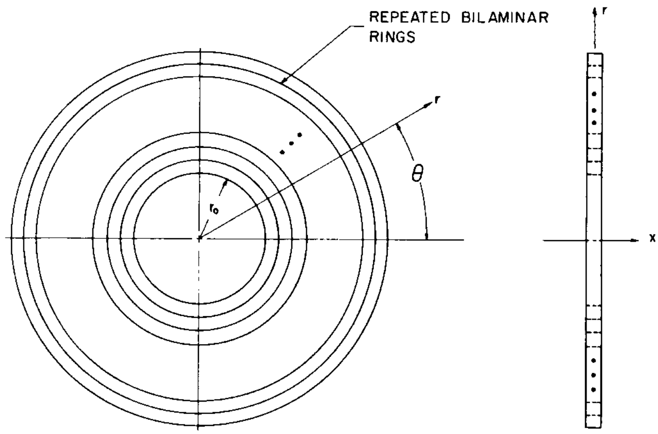

Figure 2. *Repeated bilaminar concentric rings.*

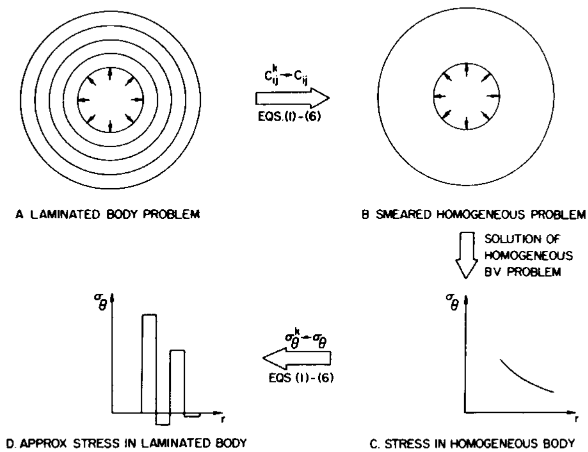

Figure 3. *The "smearing" and "unsmearing" technique for a laminated medium.*

solution. Consider a body composed of bilaminar concentric rings rigidly bonded together and loaded under internal pressure as shown in Figure 2. Figure 3 shows a schematic representation of this smearing and unsmearing process. Using the present theory the elastic constants of the equivalent homogeneous anisotropic material are found ("smearing" technique). The boundary value problem formed by replacing the layered medium by the smeared homogeneous material, as shown in Figure 3B, is then solved exactly. The stress and strain fields given by this solution, shown in Figure 3C, are equivalent to the average stress and strain in a bilaminar element of the present theory. Using these average stress and strain fields together with the present theory, we can "unsmear" the homogeneous body and calculate the stress and strain in the individual layers, as illustrated schematically in Figure 3D. These approximate results are then compared to the exact solution.

The exact elasticity solution is found by solving the elasticity equations in each layer and matching normal stress and radial displacement at the interfaces. This solution is given by Lekhnitskii [15]. The exact solution for a homogeneous anisotropic thick ring under internal pressure is also given in [15].

In this example, we consider a body composed of 19 bilaminar concentric rings (38 layers). Each bilaminar ring consists of two isotropic materials (material 1 and material 2). Material 1 has a Young's modulus $E^1 = 11 \times 10^6$ psi and a Poisson's ratio $\nu^1 = 0.326$. Material 2 has $E^2 = 0.5 \times 10^6$ psi and $\nu^2 = 0.350$. In each bilaminar ring the layers are 0.05 in. thick. The inside radius of the innermost ring is $r_o = 1$ in. The internal pressure for this problem is 1000 psi. The equivalent material for this body is a cylindrical transversely isotropic material with the x, θ-plane as the plane of isotropy and has a constitutive equation as follows,

$$\begin{bmatrix} \epsilon_x \\ \epsilon_\theta \\ \epsilon_r \\ \gamma_{r\theta} \\ \gamma_{rx} \\ \gamma_{x\theta} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{11} & S_{13} & 0 & 0 & 0 \\ S_{13} & S_{13} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(S_{11}-S_{12}) \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_\theta \\ \sigma_r \\ \tau_{r\theta} \\ \tau_{rx} \\ \tau_{x\theta} \end{bmatrix}$$

where, from equations (10) to (14), the compliances are found to be $S_{11} = 0.174 \times 10^{-6}$ in^2/lb, $S_{12} = -0.569 \times 10^{-7}$ in^2/lb, $S_{13} = -0.598 \times 10^{-7}$ in^2/lb, $S_{33} = 0.715 \times 10^{-6}$ in^2/lb, $S_{44} = 0.282 \times 10^{-5}$ in^2/lb.

The exact solution and the present approximate solution to this problem are compared in Figures 4,5,6, and 7. Figures 4 and 5 show plots of σ_θ versus $r - r_o$. Figure 4 is for the case where material 1 is the inner layer of each bilaminar ring and Figure 5 is for the case where material 2 is the inner layer of each bilaminar ring. Note that the solution for the equivalent material problem is the same for both cases since the elastic constants of the homogeneous material are not affected by the ordering of the layers. Once the equivalent homogeneous problem is solved, we choose values of this solution at points in the equivalent medium which correspond to the interface within each bilaminar ring. These values are taken as the values of average stress and strain throughout the corresponding bilaminar ring. We then use these average stresses and strains together with equations (1) to (6) to solve for the

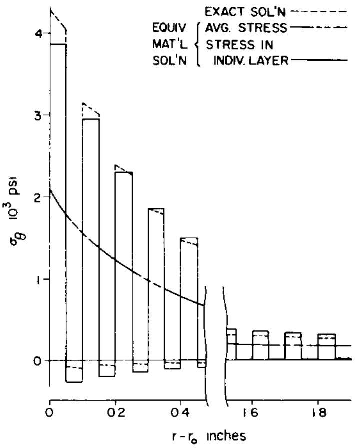

Figure 4. *Circumferential stress versus radial distance, inner layer material 1.*

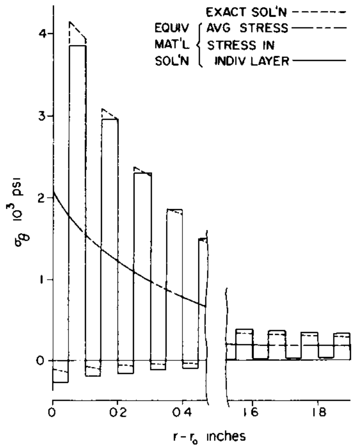

Figure 5. *Circumferential stress versus radial distance, inner layer material 2.*

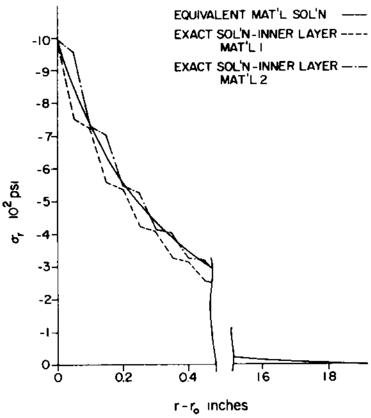

Figure 6. *Radial stress versus radial distance.*

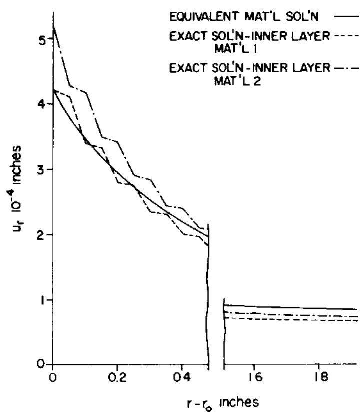

Figure 7. *Radial displacement versus radial distance.*

values of σ_θ corresponding to individual layers of each bilaminar ring. Good agreement is observed between this individual layer solution and the exact solution.

Figure 6 shows a plot of radial stress versus radial distance. The exact solution and the approximate solution using the present theory are compared, and good agreement is observed. Figure 7 shows a comparison of radial displacement given by the exact and approximate solutions, and again good agreement is observed. Note that since σ_r and u_r are continuous functions, there is no need to further calculate an individual layer solution for these variables.

The above problem was also solved for various layer thicknesses. The results show that as the layers become thinner (the overall size of the composite remaining the same) better agreement is obtained between the approximate solution and the exact solution.

IV. CONCLUDING REMARKS

The analysis in this paper is restricted to layered medium with repeating layer pattern, such as a medium built up by stacking up many identical n-layered plates. The same technique can be extended to a medium of arbitrary non-repeating layers. It would then involve the smoothing of a layered medium with discontinuous properties into a nonhomogeneous, but continuous, medium.

REFERENCES

1. Z. Hashin, and S. Shtrikman, "A Variational Approach to the Theory of the Elastic Behavior of Multiphase Materials," *J. Mech. Phys. Solids,* Vol. 11 (1963), p. 127.
2. R. Hill, "Elastic Properties of Reinforced Solids: Some Theoretical Principles," *J. Mech. Phys. Solids,* Vol. 11 (1963), p. 357.
3. R. Hill, "Theory of Mechanical Properties of Fibre-Strengthened Materials: I. Elastic Behavior," *J. Mech. Phys. Solids,* Vol. 12 (1964), p. 199.
4. J. E. White, and F. A. Angona, "Elastic Wave Velocities in Laminated Media " *J. Acous. Soc. Am.* Vol. 27 (1955), p. 311.
5. G. W. Postma, "Wave Propagation in a Stratified Medium," *Geophysics,* Vol. 20 (1955), p. 780.
6. S. M. Rytov, "Acoustical Properties of a Thinly Laminated Medium," *Soviet Phys. Acoustics,* Vol. 2 (1956), p. 68.
7. E. Behrens, "Sound Propagation in Lamellar Composite Materials and Averaged Elastic Constants," *J. Acous. Soc. Am.,* Vol. 42 (1967), p. 378.
8. M. D. G. Salamon, "Elastic Moduli of a Stratified Rock Mass," *Int. J. Rock Mech. Min. Sci.,* (1968), p. 519.
9. C.-T. Sun, J. D. Achenbach and G. Herrmann, "Continuum Theory for a Laminated Medium," *J. Appl. Mech.,* Vol. 35 (1968), p. 467.
10. J. D. Achenbach, "The Layered Medium as a Homogeneous Continuum with Microstructure," AFML-TR-70-275, (1970).
11. P. C. Chou and A. S. D. Wang, "Control Volume Analysis of Elastic Wave Front in Composite Materials," *J. Comp. Matl.,* Vol. 4 (1970), p. 444.
12. E. Reissner and Y. Stavsky, "Bending and Stretching of Certain Types of Heterogeneous Aeolotropic Elastic Plates," *J. Appl. Mech.,* Vol. 28 (1961), p. 402.
13. S. B. Dong, K. S. Pister and R. L. Taylor, "On the Theory of Laminated Anisotropic Shells and Plates," *J. Aero. Sci.,* Vol. 28 (1962), p. 969.
14. N. J. Pagano, "Exact Solutions for Composite Laminates in Cylindrical Bending," *J. Composite Materials,* Vol. 3 (1968), p. 398.
15. S. G. Lekhnitskii, *Anisotropic Plates,* Gordon and Breach (1968).