My Inequality Project

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Dedicated to Professor XY

Contents

Preface

This is the preface and it is created using a TeX field in a paragraph by itself containing `\chapter*{Preface}`. When the document is loaded, this appears if it were a normal chapter, but it is actually an unnumbered chapter.

CHAPTER 1

AM-GM Inequality

1. Arithmetic Mean - Geometric Mean

Lets go from scratch,

DEFINITION 1 (**Arithmetic Mean**). *Arithmetic mean of two non-negative real numbers a and b is defined as the average of the two numbers and is mathematically expressed as -*

$$A.M. = \frac{a+b}{2}$$

where ofcourse,
A.M. stands for the arithmetic mean of the two concerned non-negative real numbers - a and b.

Extending this idea to n-variables - $a_1, a_2, a_3, .., a_n$ we get that -

$$A.M. = \frac{a_1 + a_2 + a_3 + ... + a_n}{n} = \frac{\sum_{i=1}^{n} a_i}{n}$$

DEFINITION 2 (**Geometric Mean**). *Geometric Mean of two real numbers is the collection of positive real numbers is the nth root of the product of the numbers. Note that if it is even, we take the positive nth root and it is mathematically expressed as -*

$$G.M. = \sqrt{ab}$$

where,
G.M. stands for Geometric Mean of the two concerned non-negative real numbers - a and b

Extending this idea to n-variables - $a_1, a_2, a_3, .., a_n$ we get that -

$$G.M. = (a_1 a_2 a_3 ... a_n)^{\frac{1}{n}} = (\prod_{i=1}^{n} a_i)^{\frac{1}{n}}$$

Recall the fact that for any real number x , we know that $x^2 \geq 0$.
therfore we know that for all non-negative real numbers - $\sqrt{a}$ and $\sqrt{b}$ -

$$(\sqrt{a} - \sqrt{b})^2 \geq 0$$
$$a + b - 2\sqrt{ab} \geq 0$$
$$\frac{a+b}{2} \geq \sqrt{ab}$$

ring any bells?

yes you have probably spotted that LHS of the inequality is the arithmetic mean we discussed and ofcourse the RHS is the geometric mean. Now this encourages the following proposition -

THEOREM 1. *Arithmetic Mean of some 'n' non-negative real numbers is always greatern than or equal to the Geometric Mean of the same* is that true? if so where is the validity? by seeing that it is true for certaina two positive reals doesnt imply a wider truth for any number of variables.

2. Proof

PROOF. Lets proceede by proving the above statement for smaller numbers and eventually the general case-

Since we know that $A.M. \geq G.M.$ for two variables we have -

$$\frac{p+q}{2} \geq \sqrt{pq}$$

$$\frac{r+s}{2} \geq \sqrt{rs}$$

also we have -

$$\frac{\sqrt{pq}+\sqrt{rs}}{2} \geq \sqrt{\sqrt{pqrs}} = (pqrs)^{\frac{1}{4}}$$

combining we get that-

$$\frac{p+q+r+s}{4} = \frac{\frac{p+q}{2}+\frac{r+s}{2}}{2} \geq \frac{\sqrt{pq}+\sqrt{rs}}{2} \geq (pqrs)^{\frac{1}{4}}$$

or,..

$$\frac{p+q+r+s}{4} \geq (pqrs)^{\frac{1}{4}}$$

which is AM-GM inequality for 4 variables!!!

With a similar idea we can do the same as above for 8 - variables by splitting into 4-4 and using AM-GM for 4 variables.

Now something should be irking in your mind.... Can't we prove in the same way for any n?? - the answer is **NO**. Why not? - well this covers only 2-powers not even even integers.. so this proof is incomplete considering the general case. But a proof with the same idea can be given by induction for any 2-powers.It goes as follows -

Consider 2^{k+1} variables - $a_1, a_2, a_3, ..., a_{2^{k+1}}$

Assume the truth of the statement for 2^k, we shall prove it for 2^{k+1}

Since it is true for 2^k, we have -

$$\frac{a_1+a_2+a_3+...+a_{2^k}}{2^k} \geq (a_1a_2a_3..a_{2^k})^{\frac{1}{2^k}} \longrightarrow (1)$$

and,

$$\frac{a_{2^k+1}+a_{2^k+2}+a_{2^k+3}+...+a_{2^{k+1}}}{2^k} \geq (a_{2^k+1}a_{2^k+2}a_{2^k+3}...a_{2^{k+1}})^{\frac{1}{2^k}} \longrightarrow (2)$$

again,

$$\frac{(1)+(2)}{2} = \frac{a_1+a_2+...+a_{2^{k+1}}}{2^{k+1}} \geq \frac{(a_1a_2a_3..a_{2^k})^{\frac{1}{2^k}}+(a_{2^k+1}a_{2^k+2}a_{2^k+3}...a_{2^{k+1}})^{\frac{1}{2^k}}}{2} \geq (a_1a_2...a_{2^{k+1}})^{\frac{1}{2^{k+1}}}$$

or,

$$\frac{a_1+a_2+a_3+...+a_{2^{k+1}}}{2^{k+1}} \geq (a_1a_2a_3..a_{2^{k+1}})^{\frac{1}{2^{k+1}}}$$

thus by induction it is proved for all powers of 2!!! this is a bit closer to the full proof..

But then what about $6, 10..$ and $3, 5, 7...$ how do we prove them..?

with a little bit of trickery we get away with 3 - since the inequality is true for 4 we have -

$$\frac{a+b+c+(abc)^{\frac{1}{3}}}{4} \geq (abc)^{\frac{4}{3}\cdot\frac{1}{4}} = (abc)^{\frac{1}{3}}$$

and this re-arranges to -

$$\frac{a+b+c}{3} \geq (abc)^{\frac{1}{3}}$$

which is the AM-GM inequality for 3 reals!!!! this can also be done by the following way -

$$\frac{a+b+c+(\frac{a+b+c}{3})}{4} \geq [abc \cdot (\frac{a+b+c}{3}]^{\frac{1}{4}}$$

that is..

$$\frac{a+b+c}{3} \geq [abc \cdot (\frac{a+b+c}{3}]^{\frac{1}{4}}$$

which re-arranges to -

$$\frac{a+b+c}{3} \geq (abc)^{\frac{1}{3}}$$

The above two proofs crucially modifies our way of looking at AM- GM inequality. Let us re-define ourselves the meaning of AM,GM into -

A.M. = sum equaliser

G.M. = product equaliser

what does that mean ? It means that say A is the A.M. of 3 variables x, y, z and G the G.M. then -

$$x + y + z = 3A$$

and,

$$x.y.z = G^3$$

If we know that the inequality is true for a certain m , then for any n such that $m > n$ we can prove the validity of AM-GM inequality as follows-

$$\frac{a_1 + a_2 + ... + a_n + (m-n)A}{m} \geq (a_1 a_2 a_3 .. a_n)^{\frac{1}{m}} \cdot A^{\frac{m-n}{m}}$$

$$\frac{mA}{m} \geq (G)^{\frac{n}{m}} \cdot A^{\frac{m-n}{m}}$$

$$A^m \geq G^n \cdot A^{m-n}$$

or,

$$A^n \geq G^n$$

$$\implies A \geq G$$

thus we have prove the AM-GM inequality for any 'n' as we know it to be true for any 2^k.. or -

$$A.M. \geq G.M.$$

for any n This is trivial looking inequality is probably the most celebrated of all that we all shall discuss in this book.It has far and wide applications.. □

Proof. The inequality is true for $2n$ if it is true for n or it is true for all powers of 2 (already proved) Suppose that the inequality is true for n numbers.We then choose

$$a_n = \frac{s}{n-1}$$

where,

$$s = a_1 + a_2 + a_3 + ... + a_n$$

According to the inductive hypothesis, we get

$$s + \frac{s}{n-1} \geq n(\frac{a_1 a_2 a_2 a_3 ... a_n \cdot s}{n-1})^{\frac{1}{n}}$$

$$\iff s \geq (n-1)(a_1 a_2 a_3 .. a_n)^{\frac{1}{n-1}}$$

Therfore if the inequality is true for n numbers than it will be true for $n-1$ numbers and by induction (Cauchy Induction),the inequality is true for every natural number n .Equality occurs if and only if $a_1 = a_2 = a_3 = ... = a_n$ □

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3. Beginners' Practice Problems

1. For $a, b, c \in \mathbb{R}_0^+$.Prove that

$$(a+b)(b+c)(c+a) \geq 8abc$$

Solution

Note that by AM-GM,

$$a+b \geq 2\sqrt{ab}$$
$$b+c \geq 2\sqrt{bc}$$
$$c+a \geq 2\sqrt{ca}$$

multiplying the above inequalities we get the desired with equality for $a=b=c$

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2.For $a, b, c \in \mathbb{R}_0^+$. Prove that

$$a^2+b^2+c^2 \geq ab+bc+ca$$

Solution

$$a^2+b^2 \geq 2ab$$
$$b^2+c^2 \geq 2bc$$
$$c^2+a^2 \geq 2ac$$

adding this we get the desired with equality for $a=b=c$

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3. For $a, b, c \in \mathbb{R}^+$ such that - $a+b+c=2$.Prove that

$$abc \geq 8(1-a)(1-b)(1-c)$$

Solution
Set $1-a=x, 1-b=y, 1-c=z$
therfore by condition,

$$c=x+y, b=x+z, a=y+z$$

substituting this in the inequality we get that it is equivalent to -

$$(x+y)(y+z)(z+x) \geq 8xyz$$

which we just proved! with equality for $a=b=c=\frac{2}{3}$

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4. For $a, b, c \in \mathbb{R}^+$.Prove that -

$$(a^2b+b^2c+c^2a)(a^2c+b^2a+c^2b) \geq 9a^2b^2c^2$$

Solution Observe that by AM-GM-

$$a^2b+b^2c+c^2a \geq 3abc$$

and,

$$a^2c + b^2a + c^2b \geq 3abc$$

multiply the above two inequalities to get the desired with equality for $a = b = c$

$$\nabla$$

5.For $a, b, c \in \mathbb{R}_0^+$.Prove that -

$$a^4 + b^4 + c^4 \geq abc(\sqrt{ab} + \sqrt{bc} + \sqrt{ca})$$

(Source:Nagasaki University 1970)

Solution

This problem seems a step tougher to novice. Expanding the RHS will not lead in the correct direction. Let us try to transform this inequality into an equivalent one that for convenient sake looks simpler. One way of this being done is by taking abc to the LHS -

$$a^4 + b^4 + c^4 \geq abc(\sqrt{ab} + \sqrt{bc} + \sqrt{ca})$$

$$\Longleftrightarrow \frac{a^3}{bc} + \frac{b^3}{ac} + \frac{c^3}{ab} \geq \sqrt{ab} + \sqrt{bc} + \sqrt{ca}$$

$$\frac{a^3}{bc} + \frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab} \geq 4a$$

$$\frac{b^3}{ac} + \frac{b^3}{ac} + \frac{c^3}{ba} + \frac{a^3}{cb} \geq 4b$$

$$\frac{c^3}{ab} + \frac{c^3}{ba} + \frac{a^3}{bc} + \frac{b^3}{ca} \geq 4c$$

the three inequalities are true by AM-GM, add these inequality to get -

$$\frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab} \geq a + b + c \geq \sqrt{ab} + \sqrt{bc} + \sqrt{ca}$$

notice that the last inequality is true and is equivalent to Problem 2 thus we have proved the inequality with equality for $a = b = c$

(by M.Ramchandran)

aliter: Alternatively another ingenious AM-GM solution will be notice by AM-GM that -

$$\frac{3a^4 + 3b^4 + 2c^4}{8} \geq abc\sqrt{ab}$$
$$\frac{3b^4 + 3c^4 + 2a^4}{8} \geq abc\sqrt{bc}$$
$$\frac{3c^4 + 3a^4 + 2b^4}{8} \geq abc\sqrt{ca}$$

adding these inequalities we get the desired.

(by Mathias Tejs)

$$\nabla$$

Note It is not expected of the reader to get the above two proofs (if he/she is a newbie). Such proofs come due to some strong observation and several wrong tries(like mine.. :P). Both involved splitting the terms into terms with suitable **co-efficients** which shall come as time goes. So dont get disheartened or awed.

6. For $a, b, c \in \mathbb{R}_0^+$ such that $a+b+c+d=1$.Prove that -

$$ab+bc+cd \leq \frac{1}{4}$$

Solution Write the Inequality using the given condition as -

$$ab+bc+cd \leq \frac{(a+b+c+d)^2}{4}$$

By AM-GM,

$$\frac{(a+b+c+d)^2}{4} = (\frac{(a+c)+(b+d)}{2})^2 \geq (a+c)(b+d) = ab+bc+cd+da \geq ab+bc+cd$$

with the equality for $a=b=c, d=0$ or $d=c=b, a=0$

(by M.Ramchandran)

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7. For $a, b, c \in \mathbb{R}_0^+$.Prove that -

$$\frac{a^2+b^2}{a+b} + \frac{b^2+c^2}{b+c} + \frac{c^2+a^2}{c+a} \geq a+b+c$$

Solution Note that by AM-GM,

$$2(x^2+y^2) \geq (x+y)^2$$

for all non-negative reals x, y thus,

$$\frac{a^2+b^2}{a+b} + \frac{b^2+c^2}{b+c} + \frac{c^2+a^2}{c+a} \geq \frac{a+b}{2} + \frac{b+c}{2} + \frac{c+a}{2} = a+b+c$$

with equality for $a=b=c$

∇

8. For $a, b, c \in \mathbb{R}_0^+$ such that $a+b+c+d=1$.Prove that -

$$\frac{a^2}{a+b} + \frac{b^2}{b+c} + \frac{c^2}{c+d} + \frac{d^2}{d+a} \geq \frac{1}{2}$$

Solution We have -

$$(a-b)+(b-c)+(c-d)+(d-a)=0$$

$$\iff \frac{a^2-b^2}{a+b} + \frac{b^2-c^2}{b+c} + \frac{c^2-d^2}{c+d} + \frac{d^2-a^2}{d+a} = 0$$

$$\iff \frac{a^2}{a+b} + \frac{b^2}{b+c} + \frac{c^2}{c+d} + \frac{d^2}{d+a} = \frac{b^2}{a+b} + \frac{c^2}{b+c} + \frac{d^2}{c+d} + \frac{a^2}{d+a}$$

thus multiplying the original inequality to be proven by 2 we get that -

$$\frac{a^2+b^2}{a+b} + \frac{b^2+c^2}{b+c} + \frac{c^2+d^2}{c+d} + \frac{d^2+a^2}{d+a} \geq 1 = a+b+c+d$$

which can be proved by proceeding similar to the previous question.

(by R.Keerthan)

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Thus we close this section in the notion that the reader has atleast become familiar with the concepts that have been explained.

4. Geometric Interpretations

5. AM-GM Tautogrid Technique

While dealing with inequalities sometimes we might be having the denominators as the trouble terms and applying AM-GM in a tricky way we suprisingly are able to solve some rather difficult-looking inequalities. I shall demonstrate the method through the following inequalities. It certainly isnt rocket-science just some common sense which probably the reader might have arrived at with some 5 minutes of genuine thinking.

1. For $a, b, c \in \mathbb{R}^+$.Prove that -

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq a + b + c$$

Solution Note that by AM-GM,

$$\frac{a^2}{b} + b \geq 2a$$
$$\frac{b^2}{c} + c \geq 2b$$
$$\frac{c^2}{a} + a \geq 2c$$

adding the above inequalities we get the desired with equality for $a = b = c$

$$\nabla$$

2.For $a, b, c \in \mathbb{R}_0^+$.Prove that -

$$a^4 + b^4 + c^4 \geq abc(\sqrt{ab} + \sqrt{bc} + \sqrt{ca})$$

(Source:Nagasaki University 1970)

Solution Proceed as in my solution until you get -

$$\frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab} \geq a + b + c \geq \sqrt{ab} + \sqrt{bc} + \sqrt{ca}$$

The last but one inequality can also be proved in a simple way as follows -

$$\frac{a^3}{bc} + b + c \geq 3a$$
$$\frac{b^3}{ca} + c + a \geq 3b$$
$$\frac{c^3}{ab} + a + b \geq 3c$$

Note: What is the trick? - the idea is the remove the trouble terms and here they are present in the denominator so we do it by adding suuitably.

3. For $a, b, c \in \mathbb{R}_0^+$ such that $a + b + c + d = 1$.Prove that -

$$\frac{a^2}{a+b} + \frac{b^2}{b+c} + \frac{c^2}{c+d} + \frac{d^2}{d+a} \geq \frac{1}{2}$$

Solution BY AM-GM,

$$\frac{a^2}{a+b} + \frac{a+b}{4} \geq a$$
$$\frac{b^2}{b+c} + \frac{b+c}{4} \geq b$$
$$\frac{c^2}{c+d} + \frac{c+d}{4} \geq c$$
$$\frac{d^2}{d+a} + \frac{d+a}{4} \geq d$$

adding these we get the desired. with equality for $a = b = c = d = \frac{1}{4}$

(by M.Ramchandran)

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4. For $a, b, c \in \mathbb{R}_0^+$ such that $ab + bc + cd + da = 1$.Prove that-

$$\sum_{cyclic} \frac{a^3}{b+c+d} \geq \frac{1}{3}$$

Solution By AM-GM,

$$\begin{aligned}
\frac{a^3}{b+c+d} + \frac{b+c+d}{18} + \frac{1}{12} &\geq \frac{a}{2} \\
\frac{b^3}{a+c+d} + \frac{a+c+d}{18} + \frac{1}{12} &\geq \frac{b}{2} \\
\frac{c^3}{b+a+d} + \frac{b+a+d}{18} + \frac{1}{12} &\geq \frac{c}{2} \\
\frac{d^3}{b+c+a} + \frac{b+c+a}{18} + \frac{1}{12} &\geq \frac{d}{2}
\end{aligned}$$

Add these inequalities to get -

$$LHS \geq \frac{a+b+c+d-1}{3}$$

and also frolm AM-GM,

$$(a+b+c+d)^2 \geq 4(a+c)(b+d) = 4$$

or

$$a+b+c+d \geq 2$$

and the conclusion follows. With equallity for $a = b = c = d = \frac{1}{4}$.

(by mathlinks user : quykhtn-qa1)

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5. For $a, b, c \in \mathbb{R}^+$ such that $a + b + c = 2$.Prove that-

$$\frac{a}{b(a+b)} + \frac{b}{c(b+c)} + \frac{c}{a(a+c)} > 2$$

(Source:Own Inequality)

Solution By AM-GM,

$$\begin{aligned}
\frac{a}{b(a+b)} + (a+b)a + ab &\geq 3a \\
\frac{b}{c(c+b)} + (c+b)b + cb &\geq 3b \\
\frac{c}{a(a+c)} + (a+c)c + ac &\geq 3c
\end{aligned}$$

or ,

$$\sum_{cyclic} \frac{a}{b(a+b)} \geq 3(a+b+c) - (a+b+c)^2 = 6 - 4 = 2$$

but the equality case cant occur so the inequality sign becomes **strict**.

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6. For $x, y, z \in \mathbb{R}^+$ such that $xyz = 1$ .Prove that -

$$\frac{x^3}{(1+y)(1+z)} + \frac{y^3}{(1+x)(1+z)} + \frac{z^3}{(1+x)(1+y)} \geq \frac{3}{4}$$

(Source: IMO Shortlist 1998)

Solution By AM-GM

$$\frac{x^3}{(1+y)(1+z)} + \frac{1+y}{8} + \frac{1+z}{8} \geq \frac{3x}{4}$$

$$\implies \sum_{cyclic} \frac{x^3}{(1+y)(1+z)} \geq \frac{1}{4} \sum_{cyclic} (2x-1) \geq \frac{3}{4}$$

Equality for $x = y = z = 1$

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7. For $a, b \in \mathbb{R}_0^+$ such that $a + b = 1$ Ṗrove that -

$$\frac{a^2}{1+b} + \frac{b^2}{1+a} \geq 13$$

Solution By AM-GM,

$$\frac{a^2}{b+1} + \frac{b+1}{9} \geq \frac{2a}{3}$$
$$\frac{b^2}{a+1} + \frac{a+1}{9} \geq \frac{2b}{3}$$

add these to get the result with equality for $a = b = \frac{1}{2}$

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6. Nessbit's Inequality

This a very famous,well-known and well discussed inequality. Most problems are probably stronger than this (that is the job of the proposers - if they keep the questions down to elementary inequality then what is the fun?) but nevertheless it is a must to know this beautiful inequality-

Theorem 2. *For $a, b, c \in \mathbb{R}_0^+$ the following inequality holds -*

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$$

Proof

Note that for any positive real x, y, z we have by AM-GM -

$$\frac{x}{y} + \frac{y}{z} + \frac{z}{x} \geq 3$$

Consider the following expressions -

$$S = \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}$$
$$M = \frac{b}{b+c} + \frac{c}{c+a} + \frac{a}{a+b}$$
$$N = \frac{c}{b+c} + \frac{a}{c+a} + \frac{b}{a+b}$$

we have ofcourse: $M + N = 3$. According to the Lemma,

$$M + S = \frac{a+b}{b+c} + \frac{b+c}{c+a} + \frac{c+a}{a+b} \geq 3$$

$$N + S = \frac{a+c}{b+c} + \frac{b+a}{c+a} + \frac{c+b}{a+b} \geq 3$$

Therefore,

$$M + N + 2S \geq 3$$

or

$$2S \geq 3$$

or,

$$S \geq \frac{3}{2}$$

Practice

1. Extend the same idea to prove Nessbit's inequality for 4 non-negative reals -

$$\frac{a}{b+c}+\frac{b}{c+d}+\frac{c}{d+a}+\frac{d}{a+b}\geq 4$$

2. For $a,b,c\in\mathbb{R}_0^+$, Prove that -

$$\frac{a^2+1}{b+c}+\frac{b^2+1}{c+a}+\frac{c^2+1}{a+b}\geq 3$$

(Source : Indian RMO 2006)

3. Prove Nessbits inequality using the Tautogrid Technique
(Left to readers)

7. The Reverse Technique

In most inequalities AM-GM can be applied after if for a certain inequality we are to prove - $x\geq y$, it ultimately comes down to the fact that - $a^2\geq 0$ where $a^2=x-y$ so factorizations can help. But for factorizations, one certainly has to be something short of God to be able to prove all inequalities by factorizations into squares but.... algebraic manipulation is certainly a powerful tool. In certain inequalities by a tricky manipulation we oberve that solutions are obtained are more easily. In general it should be understood that *,for a positive real number which is rational , the value increases as we decrease the denominator and the value decreases as we increase the denominator* and the converse for a negative rational.

As always i shall explain this with an example -

For $a,b,c\in\mathbb{R}_0^+$, Prove that -

$$\frac{a^3}{a^2+b^2}+\frac{b^3}{b^2+c^2}+\frac{c^3}{a^2+c^2}\geq\frac{a+b+c}{2}$$

Solution Seeing the terms $a^2+b^2,\ldots$, instictively a student does the following -

$$\frac{a^3}{a^2+b^2}+\frac{b^3}{b^2+c^2}+\frac{c^3}{a^2+c^2}\geq\frac{a^3}{2ab}+\frac{b^3}{2bc}+\frac{c^3}{2ca}=\frac{1}{2}\left(\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\right)\geq\frac{1}{2}(a+b+c)$$

Ok, now is that correct? **NO** .. ok i guess this was anticipated by you.. but then what was the flaw? read the last lines of the paragraph and the fact that you conceived as elementary has been flawed on by the student.

Now an inquisitive student wouldnt move on to a certain different try , He would try to correct his flaw to make his idea right. The begining of that important last line said -"**Positive**" so that is the cause. This means for applying AM-GM for those terms we need to have the fraction negative, so why not express it as some $X-\frac{Y}{a^2+b^2}$? Yes that is the central idea behind this useful technique.Now the solution will be -

$$\sum_{cyclic}\frac{a^3}{a^2+b^2}=\sum_{cyclic}\left(a-\frac{ab^2}{a^2+b^2}\right)\geq\sum_{cyclic}\left(a-\frac{ab^2}{2ab}\right)=\sum_{cyclic}\left(a-\frac{b}{2}\right)=\frac{a+b+c}{2}$$

and *Voila!* we have a solution and it is correct and it uses AM-GM inequality.

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Here the main idea is -

$$\frac{a^t}{ka^{t-1}+lb^{t-1}}=\frac{a}{k}-\frac{\frac{l}{k}\cdot ab^{t-1}}{ka^{t-1}+lb^{t-1}}$$

This is only a random form. To justify my statement that we can do several types of problems and to get you used to this technique,
The strength and importance of this technique cant be more revealed than the following problems.

Problems

1. Let $a,b,c\in\mathbb{R}^+$, then prove that we have

$$\frac{a^3}{a^2+ab+b^2}+\frac{b^3}{b^2+bc+c^2}+\frac{c^3}{c^2+ca+a^2}\geq\frac{a+b+c}{3}$$

Solution

$$LHS = \frac{a^3}{a^2+ab+b^2} + \frac{b^3}{b^2+bc+c^2} + \frac{c^3}{c^2+ca+a^2}$$

$$= \sum_{cyclic} (a - \frac{ab(a+b)}{a^2+ab+b^2}) \geq \sum_{cyclic} (a - \frac{ab(a+b)}{3ab}) = \frac{a+b+c}{3}$$

with equaliy for $a = b = c$
(by M.Ramchandran)
∇

The application of this technique doesnt come by just seeing the solutions given by the Author.. for really *learning* it, the reader is advised to try the following problems before succumbing to seeing the solutions.

2. Let $a, b, c \in \mathbb{R}^+$ such that , $a + b + c = 3$. Prove that

$$\frac{a}{1+b^2} + \frac{b}{1+c^2} + \frac{c}{1+a^2} \geq \frac{3}{2}$$

Solution
We have;

$$\frac{a}{1+b^2} = a - \frac{ab^2}{1+b^2}$$

. Summing up cyclically we get that

$$\sum_{cyc} \frac{a}{1+b^2} = a+b+c - \sum_{cyc} \frac{ab^2}{1+b^2} \geq a+b+c - \sum_{cyc} \frac{ab^2}{2b}$$

$$= 3 - \frac{1}{2}(ab+bc+ca) \geq 3 - \frac{3}{2} = \frac{3}{2}$$

since from trivial inequality we have that

$$(ab+bc+ca) \leq \frac{1}{3}(a+b+c)^2 = 3$$

. Therefore we are done.

∇

Can we extend this problem to four variables? The answer is yes

3. For $a, b, c, d \in \mathbb{R}^+$ such that $a + b + c + d = 4$ .Show that we have

$$\frac{a}{1+b^2} + \frac{b}{1+c^2} + \frac{c}{1+d^2} + \frac{d}{1+a^2} \geq 2$$

Solution In the same manner as the previous problem, we have that

$$\frac{a}{1+b^2} \geq a - \frac{ab}{2}$$

Summing up we have-

$$\sum_{cyc} \frac{a}{1+b^2} \geq a+b+c+d - \frac{1}{2}(ab+bc+cd+ad) = 4 - \frac{1}{2}(a+c)(b+d)$$

$$\geq 4 - \frac{1}{2} \cdot \left(\frac{a+b+c+d}{2}\right)^2 = 4 - \frac{1}{2} \cdot 4 = 2$$

Equality holds for $a = b = c = d = 1$

∇

4. Let $a, b, c \in \mathbb{R}^+$ satisfy $a+b+c=3$. Prove that

$$\frac{a^2}{a+2b^2}+\frac{b^2}{b+2c^2}+\frac{c^2}{c+2a^2} \geq$$

(by Pham Kim Hung)

Solution Obviously

$$\frac{a^2}{a+2b^2}=a-\frac{2ab^2}{a+2b^2} \geq a-\frac{2ab^2}{3\sqrt[3]{a \cdot b^4}}=a-\frac{2}{3} \cdot a^{\frac{2}{3}} b^{\frac{2}{3}}$$

So we have

$$\sum_{cyc} \frac{a^2}{a+2b^2} \geq 3-\frac{2}{3}\left(a^{\frac{2}{3}} b^{\frac{2}{3}}+b^{\frac{2}{3}} c^{\frac{2}{3}}+c^{\frac{2}{3}} a^{\frac{2}{3}}\right)$$

Hence it suffices to show that

$$a^{\frac{2}{3}} b^{\frac{2}{3}}+b^{\frac{2}{3}} c^{\frac{2}{3}}+c^{\frac{2}{3}} a^{\frac{2}{3}} \leq 3$$

But,

$$\begin{aligned} a^{\frac{2}{3}} b^{\frac{2}{3}}+b^{\frac{2}{3}} c^{\frac{2}{3}}+c^{\frac{2}{3}} a^{\frac{2}{3}} &\leq \frac{1}{3}[ab+ab+1+bc+bc+1+ca+ca+1] \\ &=1+\frac{2}{3}(ab+bc+ca) \leq 1+\frac{2}{3} \cdot 3=1+2=3 \end{aligned}$$

because

$$(a+b+c)^2 \geq 3(ab+bc+ca) \Rightarrow ab+bc+ca \leq 3$$

. Hence we finish our proof here.

Note that this is also true for $ab+bc+ca=3$. However, here I pose a challenge for the readers.

∇

5. For $a, b, c \in \mathbb{R}^+$ such that $a+b+c=3$,Prove that -

$$\frac{a^2}{a+2b^3}+\frac{b^2}{b+2c^3}+\frac{c^2}{c+2a^3} \geq 1$$

Solution Obviously

$$\begin{aligned} \frac{a^2}{a+2b^3} &= a-\frac{2ab^3}{a+2b^3} \geq a-2 \cdot \frac{ab^3}{3\sqrt[3]{a} \cdot b^2} \\ &= a-\frac{2}{3}\left(b\sqrt[3]{a^2}\right) \end{aligned}$$

Hence we have

$$\begin{aligned} \sum_{cyc} \frac{a^2}{a+2b^3} &\geq a+b+c-\frac{2}{3}\left(b\sqrt[3]{a^2}+c\sqrt[3]{b^2}+a\sqrt[3]{c^2}\right) \\ &\geq a+b+c-\frac{2}{9}[b(a+a+1)+c(b+b+1)+a(c+c+1)] \\ &= 3-\frac{2}{9}\{2(ab+bc+ca)+3\} \geq 3-\frac{2}{9}\{6+3\}=1 \end{aligned}$$

The last inequality is true by AM-GM, and since we have

$$ab+bc+ca \leq \frac{1}{3}(a+b+c)^2=3.$$

Equality occurs if and only if $a=b=c=1$.

∇

6. For $a, b, c \in \mathbb{R}^+$ such that $a+b+c=3$, Prove that -

$$\frac{a+1}{b^2+1}+\frac{b+1}{c^2+1}+\frac{c+a}{a^2+1} \geq 3$$

Solution Since

$$\frac{a+1}{b^2+1} = a+1-\frac{b^2(a+1)}{b^2+1} \geq a+1-\frac{b^2(a+1)}{2b} = a+1-\frac{ab}{2}-\frac{b}{2}$$

. Therefore we have

$$\sum_{cyc}\frac{a+1}{b^2+1} \geq 3+\frac{a+b+c}{2}-\frac{1}{2}(ab+bc+ca)$$

$$\geq 3+\frac{3}{2}-\frac{3}{2}=3$$

Equality for $a=b=c=1$
Hence proved.

∇

7. For $a,b,c \in \mathbb{R}^+$ such that $a^2+b^2+c^2=3$, prove that

$$\frac{1}{a^3+2}+\frac{1}{b^3+2}+\frac{1}{c^3+2} \geq 1$$

(Pham Kim Hung)

Solution Note that

$$\sum_{cyc}\frac{1}{a^3+2} = \sum_{cyc}\frac{1}{2}\left[1-\frac{a^3}{a^3+2}\right]$$

$$\geq \frac{3}{2}-\frac{1}{2}\cdot\sum_{cyc}\frac{a^3}{3a} = \frac{3}{2}-\frac{1}{2}=1$$

∇

8. For $a,b,c>0$. Prove that -

$$\frac{a^3}{a+b}+\frac{b^3}{b+c}+\frac{c^3}{c+a} \geq \frac{1}{2}\left(a^2+b^2+c^2\right)$$

Solution

$$\sum\frac{a^3}{a+b} = \sum\left[a^2-\frac{a^2b}{a+b}\right]$$

$$= a^2+b^2+c^2-\sum\frac{a^2b}{a+b} \geq a^2+b^2+c^2-\sum\frac{1}{2}\cdot\frac{a^2b}{\sqrt{ab}}$$

$$= a^2+b^2+c^2-\sum\frac{1}{2}(a\sqrt{ab}) \geq a^2+b^2+c^2-\sum\frac{a}{4}(a+b)$$

$$= a^2+b^2+c^2-\frac{1}{4}(a^2+b^2+c^2+ab+bc+ca) \geq a^2+b^2+c^2-\frac{1}{2}(a^2+b^2+c^2)$$

$$= \frac{1}{2}(a^2+b^2+c^2)$$

Therefore we are done. Equality occurs if and only if $a=b=c$.

9. Let $a,b,c \in \mathbb{R}^+$ that sum up to 3. Prove that we always have -

$$\frac{1}{1+2b^2c}+\frac{1}{1+2c^2a}+\frac{1}{1+2a^2b} \geq 1$$

Solution Note that

$$\frac{1}{1+2b^2c} = 1-\frac{2b^2c}{1+2b^2c} \geq 1-\frac{2}{3}\frac{b^2c}{\sqrt[3]{b^4c^2}}$$

$$= 1-\frac{2}{3}\sqrt[3]{b^2c} \geq 1-\frac{2}{9}(2b+c)$$

Therefore we have that

$$\sum_{cyc} \frac{1}{1+2b^2c} \geq 3 - \frac{2}{9}\left[\sum_{cyc}(2b+c)\right]$$

$$= 3 - \frac{2}{9} \cdot 3(a+b+c) = 3 - 2 = 1$$

Therefore we are done. Equality occurs if and only if $a = b = c = 1$.

∇

Practice Problems

10. For $a, b, c, d \in \mathbb{R}^+$ such that $a + b + c + d = 1$.Prove that -

$$\frac{a^2}{a+b} + \frac{b^2}{b+c} + \frac{c^2}{c+d} + \frac{d^2}{d+a} \geq \frac{1}{2}$$

11. Given $a, b, c, d \in \mathbb{R}^+$, show that we have

$$\frac{a^4}{a^3+2b^3} + \frac{b^4}{b^3+2c^3} + \frac{c^4}{c^3+2d^3} + \frac{d^4}{d^3+2a^3} \geq \frac{a+b+c+d}{3}$$

12. For $a, b, c \in \mathbb{R}^+$ such that $a + b + c = 3$; show that we have

$$\frac{ab}{b^3+1} + \frac{bc}{c^3+1} + \frac{ca}{a^3+1} \leq \frac{3}{2}$$

(by Gibbenergy)

13. For given four positives a, b, c, d with sum 4; show that

$$\frac{a}{1+b^2c} + \frac{b}{1+c^2a} + \frac{c}{1+d^2a} + \frac{d}{1+a^2b} \geq 1$$

(by Pham Kim Hung)

15.Let $a, b, c, d > 0$ satisfy $a + b + c + d = 4$; show that

$$\frac{1+ab}{1+b^2c^2} + \frac{1+bc}{1+c^2d^2} + \frac{1+cd}{1+d^2a^2} + \frac{1+ad}{1+a^2b^2} \geq 4$$

(by Pham Kim Hung)

16. For all $a, b, c, d \in \mathbb{R}^+$ satisfying $a + b + c + d = 4$, Prove that we have

$$\frac{a+1}{b^2+1} + \frac{b+1}{c^2+1} + \frac{c+1}{d^2+1} + \frac{d+1}{a^2+1} \geq 4$$

17. For $a, b, c, d \in \mathbb{R}^+$ with sum 4. Prove that -

$$\frac{1}{a^2+1} + \frac{1}{b^2+1} + \frac{1}{c^2+1} + \frac{1}{d^2+1} \geq 2$$

∇

8. The Weighted AM-GM Inequality

THEOREM 3. *Suppose that $a_1, a_2, a_3, ..., a_n$ are positive real numbers.If n nonegative real numbers $x_1, x_2, x_3.., x_n$ have sum 1 then,*

$$x_1a_1 + x_2a_2 + ... + x_na_n \geq a_1^{x_1}a_2^{x_2}...a_n^{x_n}$$

In a way similar to the second proof provided for AM-GM. We have to prove that if $x, y \geq 0, x + y = 1$ and $a, b > 0$

$$ax + by \geq a^x b^y$$

Consider rational numbers x, y then take a limit. Certainly if $x, y,$ are rational numbers then - $x = \frac{m}{m+n}, y = \frac{n}{m+n}$, $m, n \in \mathbb{N}$, the problem is true according to AM-GM inequality -

$$ma + nb \geq (m + n)a^{\frac{m}{m+n}} b^{\frac{n}{m+n}} \implies ax + by \geq a^x b^y$$

If x, y are real numbers,there exist two sequences of rational numbers $(r_n)_{n\geq 0}$ and $(s_n)_{n\geq 0}$ for which $r_n \to x, s_n \to y, r_n + s_n = 1$.Certainly

$$ar_n + bs_n \geq a^{r_n} b^{s_n}$$

or

$$ar_n + b(1 - r_n) \geq a^{r_n} b^{1-r_n}$$

Taking the limit when $n \to \infty$, we have $ax + by \geq a^x b^y$

∇

Problems

1. Let a, b, c be the sidelengts of a triangle.Prove that -

$$(a + b - c)^a(b + c - a)^b(c + a - b)^c \leq a^a b^b c^c$$

Solution Applying the weighted AM-GM inequality, we conclude that

$$\left[\left(\frac{a+b-c}{a}\right)^a \cdot \left(\frac{b+c-a}{b}\right)^b \cdot \left(\frac{c+b-a}{c}\right)^c\right]^{\frac{1}{a+b+c}}$$

$$\leq \frac{1}{a+b+c}\left(a \cdot \frac{a+b-c}{a} + b \cdot \frac{b+c-a}{b} + c \cdot \frac{c+a-b}{c}\right) = 1$$

Or equivalenty,

$$(a + b - c)^a(b + c - a)^b(c + a - b)^c \leq a^a b^b c^c$$

Equality occurs for $a = b = c$

∇

2. Let $a, b, c \in \mathbb{R}^+$ such that $abc = 1$.Prove that -

$$a^{b+c} \cdot b^{c+a} \cdot c^{a+b} \leq 1$$

(Source : INMO 2001)

Solution 1 Write the LHS as - $(ab)^c(bc)^a(ca)^b$.So we have to prove that -

$$(ab)^c(bc)^a(ca)^b \leq 1$$

By Weighted AM-GM inequality we have -

$$\left[(ab)^c(bc)^a(ca)^b\right]^{\frac{1}{a+b+c}} \leq \frac{1}{a+b+c}(a.bc + b.ca + c.ab) = \frac{3}{a+b+c} \leq 1$$

(By Ramchandran)

Solution 2 WLOG:$c \geq b \geq a$ $abc = 1$, so $c \geq 1$,so $ab \leq 1$ and $a \leq 1$ Now:

$$a^{b+c} \cdot b^{c+a} \cdot c^{a+b}$$

$$= (abc)^a + ba^{c-a}b^{c-b}$$

$$= a^{c-a}b^{c-b}$$

$$= (ab)^{c-b}a^{b-a}$$

$$\leq 1^{c-b}1^{b-a}$$

$$= 1$$

Therefore $a^{b+c} \cdot b^{c+a} \cdot c^{a+b} \leq 1$

(by mathlinks user: rem)

Solution 3 Note that -

$$a^{b+c} \cdot b^{c+a} \cdot c^{a+b} = \frac{1}{a^a b^b c^c} \leq 1$$

the last inequality is true because for a, b, c individually greater or lesser than 1 we have -

$$a^a \geq a, b^b \geq b, c^c \geq c$$

(by mathlinks user:Maharjun)

$$\nabla$$

3. Let $a, b, c \in \mathbb{R}_0^+$, Prove that -

$$4(a+b+c) \geq 3(a + (ab)^{\frac{1}{2}} + (abc)^{\frac{1}{3}})$$

Solution

$$3a + (\frac{a}{4} + b) + (\frac{a}{2} + 2b) + (\frac{a}{4} + b + 4c) \geq 3a + \sqrt{ab} + 2\sqrt{ab} + 3\sqrt[3]{abc}$$

(by Aravind Srinivas)

9. Method Of Balancing Co-efficients by AM-GM

In most inequalities we have to group terms suitably so that classical inequalities like AM-GM , C-S etc.. can be aplied to get the result.This is not as easy as it looks, it requires proper terms and proper grouping. We usually need some additional variables to solve the equations for finding out the original variables.This is the Method of Balancing co-efficients. In this chapter we shall see the method using AM-GM Inequality.

Let me demonstrate it with a simple example -

1. If $x, y, z \in \mathbb{R}^+$ such that $xy + yz + zx = 1$,then find the minimum of the following expression-

$$k(x^2 + y^2) + z^2$$

Solution Lets experiment with some values of k shall we? Let $k = 10$, so we are now required to find the minimum of this non-symmetric expression-

$$10(x^2 + y^2) + z^2$$

How do we apply a classical basic inequality like AM-GM for this? Well it does seem horrendously difficult, so lets take a sneak-peek at the magical solution? By AM-GM we have the following inequalities,

$$2x^2 + 2y^2 \geq 4xy$$

$$8x^2 + \frac{1}{2}z^2 \geq 4yz$$

$$8y^2 + \frac{1}{2}z^2 \geq 4zx$$

and summing up these we have,

$$10(x^2 + y^2) + z^2 \geq 4(xy + yz + zx) = 4$$

Equality holds for

$$x = y$$
$$4x = z$$
$$4y = z$$
$$\implies x = y = \frac{1}{3}$$
$$z = \frac{4}{3}$$

Hurray ! We did it!!(don't get envious i like you was flabbergasted at this cause it aint mine) how can a guy arrive at this?? why not pair up 1 and 9 , 5 and 5 or something? All answers shall be revealed in the following lines. back to our general problem, Lets choose some $l \leq k$, then apply AM-GM this way,

$$lx^2 + ly^2 \geq 2lxy$$
$$(k-l)y^2 + \frac{1}{2}z^2 \geq yx\sqrt{2(k-l)}$$
$$(k-l)x^2 + \frac{1}{2}z^2 \geq xz\sqrt{2(k-l)}$$

summing up we get this -

$$k(x^2 + y^2) + z^2 \geq 2lxy + (yz + zx)\sqrt{2(k-)}$$

now we have a condition given - $xy + yz + zx = 1$ so for obtaining a numerical value we have to have the co-efficients in the RHS the same and without a variable hopefully. so intuitively lets just equate them- $2l = \sqrt{2(k-l)}$ and solving this we obtain that

$$l = \frac{-1 + \sqrt{1 + 8k}}{4}$$

and ofcourse substituting this in the equation we get the minimal value we are looking for to be -

$$\frac{-1 + \sqrt{1 + 8k}}{2}$$

so we observe that there is a unique pair of integers - $l, k - l$ that show us the way by AM-GM. Now that is the reason we were seeing the use of 8,2 and not 3,7 etc.. sure enough you can check the credibility of the pairings now!

Note

$$\nabla$$

2. Let x, y, z, t be real numbers satisfying $xy + yz + zt + tx = 1$. Find the minimum of the expression -

$$5x^2 + 4y^2 + 5z^2 + t^2$$

Solution Here $k = 5$, so we chooose $l \leq 5$,

$$lx^2 + 2y^2 \geq 2\sqrt{2l}xy$$
$$2y^2 + lz^2 \geq 2\sqrt{2l}yz$$
$$(5-l)z^2 + \frac{1}{2}t^2 \geq \sqrt{2(5-l)}tx$$

Summing up these results, We conclude that

$$5x^2 + 4y^2 + 5z^2 + t^2 \geq 2\sqrt{2l}(xy + tz) + \sqrt{2(5-l)}(zt + tx)$$

The condition $xy + yz + zt + tx = 1$ suggests us to choose a number $l(0 \leq l \leq 5)$ such that -

$$2\sqrt{2l} = \sqrt{2(5-l)}$$

. A simple calculation yields $l = 1$, thus the minimum of $5x^2 + 4y^2 + 5z^2$ is $2\sqrt{2}$

Note Using the same method, solve the following problem :
Let x, y, z, t be arbirary real numbers. Prove that -

$$x^2 + ky^2 + z^2 + lt^2 \geq \left(\frac{2kl}{k+l}\right)^{\frac{1}{2}} \cdot (xy + yz + zt + tx)$$

3. Let x, y, z be positive real numbers with sum 3. Find the minimum of the expression

$$x^2 + y^2 + z^3$$

(Pham Kim Hung)

Solution Let a and b be teo positive real numbers.. Then,by AM-GM inequality we have -

$$x^2 + a^2 \geq 2ax$$

$$y^2 + a^2 \geq 2ay$$

$$z^3 + b^2 + b^2 \geq 3zb^2$$

Combining these we have $x^2 + y^2 + z^3 + 2(a^2 + b^3) \geq 2a(x + y) + 3b^2 z$ with equality for $x = y = a$ and $z = b$.In this case, we could have $2a + b = x + y + z = 3(\star)$. Moreover, in order for $2a(x + y) + 3b^2 z$ to be represented as $x + y + z$, we must have $2a = 3b^2(\star\star)$.Accordin to $(\star)$ and $(\star\star)$ we can find out that,

$$b = \frac{-1 + \sqrt{37}}{6}, a = \frac{3 - b}{2} = \frac{19 - \sqrt{37}}{12}$$

Therfore the minimum of $x^2 + y^2 + z^3$ is $6a - 2(a^2 + b^3)$ where a, b are as determined.The proof is completed.

∇

4. For $x, y, z \in \mathbb{R}^+$ such that $xy + yz + zx = 1$, Prove that - $15x^2 + 7y^2 + 3z^2 \geq 6$

(Own Inequality)

Solution Rewrite the Inequality as -

$$5x^2 + \frac{7}{3}y^2 + z^2 \geq 2$$

By AM-GM we have ,

$$x^2 + y^2 \geq 2xy$$

$$4x^2 + \frac{1}{4}z^2 \geq 2xz$$

$$\frac{4}{3}y^2 + \frac{3}{4}z^2 \geq 2xz$$

now adding these we get the desired.

10. Quasiliearisation

This is a very intrigueing idea due to Russian problem proposer - Fedor Petrov

We know by AM-GM that ,

$$2ab \leq a^2 + b^2$$

Introduce a parameter and get the following:

$$2ab \leq ta^2 + \frac{b^2}{t}$$

Then read the last inequality from the other point: for any positive a, b there exist positive t such that

$$2ab = ta^2 + \frac{b^2}{t} \ (t = \frac{b}{a})$$

Or, we may write:

$$2ab = \min(ta^2 + \frac{b^2}{t})$$

How may this observation help? Put

$$a = \sum(x_i^2), b = \sum(y_i^2)$$

Then for appropriate t we have:

$$2\sqrt{a}\sqrt{b} = ta + \frac{b}{t} = \sum(tx_i^2 + \frac{y_i^2}{t}) \geq \sum(2x_i y_i)$$

So, we get the Cauchy-Schwarz Inequality. A lot of other inequalities also may be proved by this idea

Problem Prove that for any four nonnegative reals a, b, c, d the following inequality holds-

$$(ab)^{\frac{1}{3}} + (cd)^{\frac{1}{3}} \leq ((a+c+b)(a+c+d))^{\frac{1}{3}}$$

(Source:Proposed at 239 Lyceum Traditional Olympiad)
(Author : Fedor Petrov)

We have

$$3(AB)^{\frac{1}{3}} \leq Ax + By + \frac{1}{xy}$$

And for any positive A and B there exist appropriate x and y ,for which equality holds

$$x = \frac{(AB)^{\frac{1}{3}}}{A}, y = \frac{(AB)^{\frac{1}{3}}}{B}$$

Let

$$A = (a+c+b), B = (a+c+d)$$

in terms of the problem. For some positive x, y we have -

$$3(AB)^{\frac{1}{3}} = Ax + By + \frac{1}{xy} = (a+c+b)x + (a+c+d)y + \frac{1}{xy} =$$

$$(a+c+b)x+(a+c+d)y+\frac{1}{x(x+y)}+\frac{1}{y(x+y)} = \big(a(x+y)+bx+\frac{1}{x(x+y)}\big)+\big(c(x+y)+dy+\frac{1}{y(x+y)}\big) \geq 3(ab)^{\frac{1}{3}}+3(cd)^{\frac{1}{3}}$$

By AM-GM and we are done!

11. Equivalent Summation Technique

This is an interesting technique that helps us to solve elegantly many problems. It involves finding a suitable equivalent summation and using it prove the inequality given. Let me demonstrate it through the following already discussed example -

1. Prove that for $a, b, c \in \mathbb{R}^+$ -

$$\frac{a^3}{a^2+ab+b^2} + \frac{b^3}{b^2+bc+c^2} + \frac{c^3}{c^2+ca+a^2} \geq \frac{a+b+c}{3}$$

<u>Solution 1</u>

$$\frac{a^3-b^3}{a^2+ab+b^2} = a - b \Longrightarrow \sum_{cyc} \frac{a^3}{a^2+ab+b^2} = \sum_{cyc} \frac{b^3}{a^2+ab+b^2} = \frac{1}{2}\sum_{cyc} \frac{a^3+b^3}{a^2+ab+b^2} \geq \frac{a+b+c}{3}$$

But there exists another nice solution using the reverse technique. The solution runs as follows:

$$\sum_{cyc} \frac{a^3}{a^2+ab+b^2} = \sum_{cyc} \left[a - \frac{ab(a+b)}{a^2+ab+b^2}\right] \geq a+b+c - \sum_{cyc} \frac{ab(a+b)}{3ab}$$

Since we have $a^2+ab+b^2 \geq 3ab$ from AM-GM. Therefore we have that

$$\sum_{cyc} \frac{a^3}{a^2+ab+b^2} \geq a+b+c - \frac{2}{3}(a+b+c) = \frac{a+b+c}{3}$$

Hence proved. Equality occurs if and only if $a = b = c$.

Solution 2 By the same idea,

$$LHS = \frac{1}{2}\sum \frac{a^3+b^3}{a^2+b^2+ab}$$

. But

$$ab \leq \frac{a^2+b^2}{2}$$

and

$$2(a^3+b^3) \geq (a+b)(a^2+b^2)$$

Therefore

$$LHS \geq \frac{a+b+c}{3}$$

∇

2. For $a, b, c \in \mathbb{R}^+$ with sum 1,prove that -

$$\frac{ab}{\sqrt{ab+bc}} + \frac{bc}{\sqrt{bc+ca}} + \frac{ca}{\sqrt{ca+ab}} \leq \frac{1}{\sqrt{2}}$$

(MOSP 2007 3.2)

Solution

$$\frac{ab}{\sqrt{ab+bc}} + \frac{bc}{\sqrt{bc+ca}} + \frac{ca}{\sqrt{ca+ab}} \leq \frac{1}{\sqrt{2}}$$

$$\sum \frac{ab}{\sqrt{ab+bc}} \leq \sum \frac{\sqrt{2}ab}{\sqrt{ab}+\sqrt{bc}}$$

just need to prove

$$\sum \frac{ab}{\sqrt{ab}+\sqrt{bc}} \leq \frac{a+b+c}{2}$$

note that

$$\sum \frac{ab}{\sqrt{ab}+\sqrt{bc}} = \sum \frac{bc}{\sqrt{ab}+\sqrt{bc}}$$

this is true because :

$$\sum \frac{ab-bc}{\sqrt{ab}+\sqrt{bc}} = \sum \frac{(ab-bc)(\sqrt{ab}-\sqrt{bc})}{ab-bc} = \sum(\sqrt{ab}-\sqrt{bc}) = 0$$

so equivalent to

$$\sum \frac{ab+bc}{\sqrt{ab}+\sqrt{bc}} \leq a+b+c$$

and this is equivalent to -

$$\sum \frac{\sqrt{ab}}{(\sqrt{c}+\sqrt{a})(\sqrt{c}+\sqrt{b})}(\sqrt{a}-\sqrt{b})^2 \geq 0$$

and hence proved.

(by mathlinks user : kuing)

∇

12. The G function

This is a beautiful idea for which credit goes to inequality solver - Pham Kim Hung (hungkhtn).

DEFINITION 3. *For $a, b, c \in \mathbb{R}^+$, we define,*

$$G(a,b,c) = \frac{a}{b} + \frac{b}{c} + \frac{c}{a} - 3$$

It is trivial to oberve by AM-GM that always -

$$G(a,b,c) \geq 0$$

Some nice properties have been found and some tough inequalities have been solved by this idea.

Pham Kim Hung's Nice Factorisation

$$G(a,b,c) = \frac{a}{b} + \frac{b}{c} + \frac{c}{a} - 3 = \left(\frac{a}{b} + \frac{b}{a} - 2\right) + \left(\frac{b}{c} + \frac{c}{a} - \frac{b}{a} - 1\right) = \frac{(a-b)^2}{ab} + \frac{(b-c)(a-c)}{ca}$$

this factorization plays an important role in many proofs

Note

The inequality - $G(a,b,c) \geq 0$ is a cyclic inequality and thus no pair-wise order can be assumed.

I shall present some properties of this important function.

Properties

1. For $a, b, c, k \in \mathbb{R}^+$ we have -

$$G(a,b,c) \geq G(a+k, b+k, c+k)$$

Proof WLOG : $c = \min\{a,b,c\}$ We have ,

$$G(a,b,c) = \frac{(a-b)^2}{ab} + \frac{(b-c)(a-c)}{ca}$$

So it is enough to prove that -

$$\frac{(a-b)^2}{ab} + \frac{(b-c)(a-c)}{ca} \geq \frac{(a-b)^2}{(a+k)(b+k)} + \frac{(b-c)(a-c)}{(c+k)(a+k)}$$

This is true as $k > 0$ and by assumption - $(a-c)(b-c) \geq 0$. Hence proved with equality for $a = b = c$

∇

2. For $a, b, c, k \in \mathbb{R}^+$ we have -

$$G(a,b,c) \geq G(a+b, b+c, c+a)$$

Proof We only have to show that -

$$\frac{(a-b)^2}{ab} + \frac{(b-c)(a-c)}{ca} \geq \frac{(a-b)^2}{(a+c)(b+c)} + \frac{(b-c)(a-c)}{(a+b)(a+c)}$$

It is trivial to see that the inequality is true.

Note The following problem (equivalent to the property discusses above) was asked in the Mathlinks contest 2003 -

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{a+b}{b+c} + \frac{b+c}{c+a} + \frac{c+a}{a+b}$$

∇

The following properties(same conditions as above)can be solved using the same method -

3. For $a, b, c, k \in \mathbb{R}^+$ we have -

$$G\left((a-b)^2, (b-c)^2, (c-a)^2\right) \geq 2$$

(by Darij Grinberg)

4. For $a, b, c, k \in \mathbb{R}^+$ we have -

$$G(a^2, b^2, c^2) \geq 4G(a,b,c)$$

(by M.Ramchandran)

5. Let $a, b, c, k \in \mathbb{R}^+$, then If and only if the sum of any two of $\{a, b, c\}$ is less than 2

$$G(ab, bc, ac) \geq G(a, b, c)$$

(by M.Ramchandran)

6. For $a, b, c, k \in \mathbb{R}^+$ and let $a \geq b \geq c$ then,

$$G(\frac{a^2}{bc}, \frac{b^2}{ca}, \frac{c^2}{ab}) \geq G(a, b, c)$$

(by M.Ramchandran)

7. For $a, b, c, k \in \mathbb{R}^+$ and let $k \geq \max\{a^2, b^2, c^2\}$ then,

$$G(a, b, c) \geq G(a^2 + k, b^2 + k, c^2 + k)$$

(Pham Kim Hung)

8. For $a, b, c, k \in \mathbb{R}^+$ we have -

$$G(a^3, b^3, c^3) \geq 3G(a^2, b^2, c^2)$$

(by M.Ramchandran)

9. For $a, b, c, k \in \mathbb{R}^+$ we have -

$$G(a^2, b^2, c^2) \geq G(a, c, b)$$

(by M.Ramchandran)

If more properties are invited to be shared with the author by e-mail.

13. Problem Set

This section consists of problems a step more difficult then the problems already discussed.

1. Let $a, b, c \in \mathbb{R}^+$ such that, $a + b + c = 3$.Prove that -

$$\sqrt{a} + \sqrt{b} + \sqrt{c} \geq ab + bc + ca$$

(Source : Russian MO 2004)

<u>Solution</u> By AM-GM,

$$a^2 + \sqrt{a} + \sqrt{a} \geq 3a$$
$$b^2 + \sqrt{b} + \sqrt{b} \geq 3b$$
$$c^2 + \sqrt{c} + \sqrt{c} \geq 3c$$

Thus,by adding the above and using $a + b + c = 3$,

$$a^2 + b^2 + c^2 + 2(\sqrt{a} + \sqrt{b} + \sqrt{c}) \geq 3(a + b + c) = (a + b + c)^2$$
$$\implies 2(\sqrt{a} + \sqrt{b} + \sqrt{c}) \geq 2(ab + bc + ca)$$
$$\implies \sqrt{a} + \sqrt{b} + \sqrt{c} \geq ab + bc + ca$$

With equality for $a = b = c = 1$

$$\nabla$$

2.Let $x, y, z \in \mathbb{R}^+$.Prove that -

$$\left(1 + \frac{x}{y}\right)\left(1 + \frac{y}{z}\right)\left(1 + \frac{z}{x}\right) \geq 2 + \frac{2(x + y + z)}{(xyz)^{\frac{1}{3}}}$$

(Source : APMO 1998)

Solution After expanding, the inequality reduces to -

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\geq\frac{x+y+z}{(xyz)^{\frac{1}{3}}}$$

For the proof of this inequality see Lemma . Equality for $x=y=z$

3.Let a,b,c be positive real numbers.Prove that

$$\frac{1}{a^3+abc+b^3}+\frac{1}{b^3+abc+c^3}+\frac{1}{c^3+abc+a^3}\leq\frac{1}{abc}$$

(USA MO 1998)

Solution Using the lemma : $a^3+b^3\geq ab(a+b)$

$$\frac{abc}{a^3+b^3+abc}\leq\frac{abc}{ab(a+b)+abc}=\frac{c}{a+b+c}$$

Construction two more similar inequalities and adding we get the desired result with equality for $a=b=c$

∇

4. If $x_1,x_2,...x_n\in\mathbb{R}^+$ such that -

$$\frac{1}{1+x_1}+\frac{1}{1+x_2}+...+\frac{1}{1+x_n}=1$$

, Prove that -

$$x_1x_2x_3....x_n\geq(n-1)^n$$

Solution The condition is equivalent to -

$$\frac{1}{1+x_1}+\frac{1}{1+x_2}+...+\frac{1}{1+x_{n-1}}=\frac{x_n}{1+x_n}$$

Using AM-GM inequality for all the terms in the LHS we get -

$$\frac{x_n}{1+x_n}\geq\frac{n-1}{((1+x_1)(1+x_2)(1+x_3)...(1+x_n))^{\frac{1}{n-1}}}$$

Similarly constructing n more inequalities and multiplying all we get the desired. Equality for $x_i=n-1$ for $i\in\{1,2,3,...n\}$

∇

5. Suppose that $x,y,z\in\mathbb{R}^+$ and $x^5+y^5+z^5=3$.Prove that -

$$\frac{x^4}{y^3}+\frac{y^4}{z^3}+\frac{z^4}{x^3}\geq 3$$

Solution Notice that -

$$(x^5+y^5+z^5)^2=x^{10}+2x^5y^5+y^{10}+2y^5z^5+z^{10}+2z^5x^5=9$$

This suggests the use of AM-GM in this way -

$$10\cdot\frac{x^4}{y^3}+6x^5y^5+3x^{10}\geq 19x^{\frac{100}{19}}$$

Adding all the cyclic results we get,

$$10(\frac{x^4}{y^3}+\frac{y^4}{z^3}+\frac{z^4}{x^3})+3(x^5+y^5+z^5)^2\geq 19\left(x^{\frac{100}{19}}+y^{\frac{100}{19}}+z^{\frac{100}{19}}\right)$$

Therfore it is enough if we prove -

$$(x^{\frac{100}{19}}+y^{\frac{100}{19}}+z^{\frac{100}{19}}\geq x^5+y^5+z^5$$

which is true by AM-GM because -

$$3+19\sum_{cyclic}=\sum_{cyclic}(1+19x^{\frac{100}{19}})\geq 20\sum_{cyclic}x^5$$

Equality for $x = y = z = 1$

∇

6. Suppose that $x, y, z \in \mathbb{R}^+$ and $x^4 + y^4 + z^4 = 3$.Prove that -

$$x^{\frac{25}{6}} + y^{\frac{25}{6}} + z^{\frac{25}{6}} \geq 3$$

(Own Inequality)

Solution The inequality is equivalent to -

$$3 + 24 \sum_{cyclic} x^{\frac{25}{6}} \geq 25x^4$$

which is true by AM-GM :

$$\sum_{cyclic} (1 + 24 \sum_{cyclic} x^{\frac{25}{6}}) \geq 25 \sum_{cyclic} x^4$$

Equality for $x = y = z = 1$

∇

7. Let $a, b, c \in \mathbb{R}^+$ such that $abc = 1$.Prove that

$$\sqrt{\frac{a+b}{a+1}} + \sqrt{\frac{b+c}{b+1}} + \sqrt{\frac{c+a}{c+1}} \geq 3$$

(Source : Mathlinks Contest)

Solution After applying AM-GM to the left hand side we get -

$$(a+b)(b+c)(c+a) \geq (a+1)(b+1)(c+1)$$

and since $abc = 1$ it is equivalent to -

$$ab(a+b) + bc(b+c) + ca(c+a) \geq a + b + c + ab + bc + ca$$

By AM-GM,

$$2LHS + \sum_{cyclic} ab \geq \sum_{cyclic} \left(a^2b + a^2b + a^2c + a^2c + bc\right) \geq 5 \sum_{cyclic} [a^5 \cdot (abc)^{\frac{1}{3}}]^{\frac{1}{5}} = 5 \sum_{cyclic} a$$

$$2LHS + \sum_{cyclic} a = \sum_{cyclic} \left(a^2b + a^2b + b^2a + b^2a + c\right) \geq 4 \sum_{cyclic} [(ab)^5 \cdot .abc]^{\frac{1}{5}}$$

Therefore,

$$4LHS + \sum_{cyclic} ab + \sum_{cyclic} a \geq 5 \sum_{cyclic} a + 5 \sum_{cyclic} ab$$

$$\implies 4LHS \geq 4 \left(\sum_{cyclic} a + \sum_{cyclic} ab \right)$$

Hence Proved. Equality for $a = b = c = 1$

∇

8. Let $a, b, c, d \in \mathbb{R}_0^+$ such that $a + b + c + d = 4$. Prove that -

$$a^2 + b^2 + c^2 + d^2 - 4 \geq 4(a-1)(b-1)(c-1)(d-1)$$

(Pham Kim Hung)

Solution By AM-GM ,

$$a^2 + b^2 + c^2 + d^2 - 4 = (a-1)^2 + (b-1)^2 + (c-1)^2 + (d-1)^2 \geq 4\sqrt{|(a-1)(b-1)(c-1)(d-1)|}$$

If the RHS of the question is negative, then the question is meaningless . So we only have to consider the case when - $a \geq b \geq 1 \geq c \geq d$ Since $a + b \leq 4$ and $c, d \leq 1$ and by AM-GM,

$$(1-c)(1-d) \leq 1$$

$$(a-1)(b-1) \leq \frac{1}{4} \cdot (a+b-2)^2 \leq 1$$

Therefore -

$$(a-1)(b-1)(1-c)(1-d) \leq 1$$

and the conclusion follows. Equality for $a=b=c=d=1, a=b=2, c=d=0$ and cylic permutations.

∇

9. Let $a,b,c \in \mathbb{R}_0^+$ and $a+b+c=3$. Prove that -

$$a\sqrt{1+b^3}+b\sqrt{1+c^3}+c\sqrt{1+a^3} \leq 5$$

(Pham Kim Hung)

Solution We know by AM-GM that,

$$\sum_{cyclic} a\sqrt{1+b^3} = \sum_{cyclic} a\sqrt{(1+b)(1-b+b^2)} \leq \sum_{cyclic} \frac{1}{2} \cdot a(2+b^2)$$

Thus we are left to prove that -

$$ab^2+bc^2+ca^2 \leq 4$$

WLOG : Let b be the middle number in $\{a,b,c\}$ So we have ,

$$a(b-a)(b-c) \leq 0$$

$$\implies ab^2+a^2c \leq abc+a^2b$$

Thus is it is enough if we prove that -

$$abc+a^2b+bc^2 \leq 4 \Leftrightarrow b(a^2+ac+c^2) \leq 4$$

By AM-GM inequality,

$$b(a^2+ac+c^2) \leq b \cdot (a+c)^2 = 4 \cdot b \cdot \frac{a+c}{2} \cdot \frac{a+c}{2} \leq 4 \cdot \left(\frac{a+b+c}{3}\right)^3 = 4$$

This finished our proof with equality for $a=1, b=2, c=0$ and cyclic permutations.

∇

10. Let $a,b,c \in \mathbb{R}^+$.Prove that -

$$\frac{a^3}{b^2}+\frac{b^3}{c^2}+\frac{c^3}{a^2} \geq \frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}$$

(Source : JBMO Shortlist 2002)

Solution 1 Note that,

$$\frac{a^3}{b^2} \geq \frac{a^2}{b}+a-b$$

or,

$$a^3 \geq a^2b+ab^2-b^3$$

which is true by Lemma . Add all cyclic results to get the desired

(by mathlinks user : limes123)

Solution 2 By AM-GM, we know that

$$\sum_{cyc}\left(\frac{a^3}{b^2}+a\right) \geq \sum_{cyc}\frac{2a^2}{b}$$

We shall prove ,

$$\sum_{cyc}\frac{2a^2}{b}-a-b-c \geq \sum_{cyc}\frac{a^2}{b}$$

or,

$$\sum_{cyc}\frac{a^2}{b} \geq a+b+c$$

It is true by AM-GM,

$$\sum\left(\frac{a^2}{b}+b\right)\geq\sum_{cyc}2a$$

Hence Proved.

(by Johan Gunardi)

Solution 3
LEMMA For $a,b\in\mathbb{R}^+$,

$$a^3+b^3\geq ab(a+b)$$

We have,

$$\left(\frac{a^3}{b^2}+\frac{b^3}{c^2}+\frac{c^3}{a^2}\right)+(a+b+c)=\sum\left(\frac{a^3}{b^2}+b\right)=\sum\frac{a^3+b^3}{b^2}\geq\sum\frac{ab\,(a+b)}{b^2}=\sum\left(\frac{a^2}{b}+a\right)$$

$$\Longrightarrow\Rightarrow\frac{a^3}{b^2}+\frac{b^3}{c^2}+\frac{c^3}{a^2}\geq\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}$$

(by mathlinks user : trungk42sp)

Solution 4 Note by AM-GM that,

$$14\frac{a^3}{b^2}+3\frac{b^3}{c^2}+2\frac{c^3}{a^2}\geq\sqrt[19]{\frac{a^{38}}{b^{19}}}=\frac{a^2}{b}$$

(by mathlinks administrator : nsato)

∇

11. For $a,b,c\in\mathbb{R}^+$.Prove that -

$$\frac{a^6}{b^3}+\frac{b^6}{c^3}+\frac{c^6}{a^3}\geq\frac{b^4}{a}+\frac{c^4}{b}+\frac{a^4}{c}$$

(Vascile Cirtoaje)

Solution Note that by AM-GM we have -

$$3\sum\frac{a^6}{b^3}=\sum(\frac{b^6}{c^3}+\frac{b^6}{c^3}+\frac{c^6}{a^3})\geq 3\sum\frac{b^4}{a}$$

(by mathlinks user : karis)

∇

12. For $a,b,c,d\in\mathbb{R}^+$.Prove that -

$$\frac{a^{14}}{b^7}+\frac{b^{14}}{c^7}+\frac{c^{14}}{d^7}+\frac{d^{14}}{a^7}\geq\frac{b^8}{a}+\frac{c^8}{b}+\frac{d^8}{c}+\frac{a^8}{d}$$

(Vascile Cirtoaje)

Solution Note that by AM-GM we have -

$$7\sum\frac{a^{14}}{b^7}=\sum(4\frac{b^{14}}{c^7}+2\frac{c^{14}}{d^7}+\frac{d^{14}}{a^7})\geq 7\sum\frac{b^8}{a}$$

(by mathlinks user : karis)

∇

13. Let $a,b,c\in\mathbb{R}^+$ such that they are all pairwise distinct . Prove that -

$$\left|\frac{a+b}{a-b}+\frac{b+c}{b-c}+\frac{c+a}{c-a}\right|>1$$

(Source : Iranian National Olympiad (3rd Round) 2007)

Solution Set ,

$$\frac{a+b}{a-b}=x$$
$$\frac{b+c}{b-c}=y$$

$$\frac{a+c}{c-a} = z$$

$$\implies xy + yz + zx = 1$$

By AM-GM we have,

$$(x+y+z)^2 \geq 3(xy+yz+zx) = 3 \implies |x+y+z| \geq \sqrt{3} >$$

(by mathlinks user : Phm Thnh Quang)

∇

14.Prove the following inequality for all $a, b, c \in \mathbb{R}_0^+$ -

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} + \frac{3(abc)^{\frac{1}{3}}}{a+b+c} \geq 4$$

Solution LEMMA For $a, b, c \in \mathbb{R}^+$.

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{a+b+c}{(abc)^{\frac{1}{3}}}$$

Proof By AM-GM we have -

$$\sum_{cyclic} 2 \cdot \frac{a}{b} + \frac{b}{c} \geq 3 \sum_{cyclic} \left(\frac{a^2}{bc}\right)^{\frac{1}{3}} = 3 \sum_{cyclic} \frac{a}{(abc)^{\frac{1}{3}}}$$

we have to prove that -

$$\frac{a+b+c}{(abc)^{\frac{1}{3}}} + \frac{3(abc)^{\frac{1}{3}}}{a+b+c} \geq 4$$

By AM-GM,

$$\frac{a+b+c}{3(abc)^{\frac{1}{3}}} + \frac{a+b+c}{3(abc)^{\frac{1}{3}}} + \frac{a+b+c}{3(abc)^{\frac{1}{3}}} + \frac{3(abc)^{\frac{1}{3}}}{a+b+c} \geq 4 \cdot \sqrt{\frac{a+b+c}{3(abc)^{\frac{1}{3}}}} \geq 4$$

Hence proved with equality for $a = b = c$

(by mathlinks user : enndb0x)

∇

15. Let $a, b, c \in \mathbb{R}^+$ such that $abc = 1$.Prove that -

$$a^2 + b^2 + c^2 + 9(ab + bc + ca) \geq 10(a + b + c)$$

Solution This beautiful problem has many solutions - mostly involving some high-level methods like uvw method, mixing variables etc..But, the following extra-ordinary generalisation was given by an Indian - Aakansh Gupta -

$$\sum_{cyc} a^2 + k \sum_{cyc} ab \geq (k+1) \sum_{cyc} a \;\forall\; k \in R$$

Proof
We have

$$\sum_{cyc} a \geq 3 \;\; and \;\; \sum_{cyc} ab \geq 3$$

$$\Rightarrow (\sum_{cyc} a)^2 + (\sum_{cyc} ab)^2 \geq 3 \sum_{cyc} a + 3 \sum_{cyc} ab$$

$$\Rightarrow \sum_{cyc} a^2 + \sum_{cyc} a^2b^2 \geq \sum_{cyc} a + \sum_{cyc} ab$$

$$\Rightarrow (\sum_{cyc} a^2 + k \sum_{cyc} ab) + (\sum_{cyc} a^2b^2 + k \sum_{cyc} a) \geq (k+1) \sum_{cyc} a + (k+1) \sum_{cyc} ab$$

$$\Rightarrow \sum_{cyc} a^2 + k \sum_{cyc} ab \geq (k+1) \sum_{cyc} a$$

Or,

$$\sum_{cyc} a^2b^2 + k\sum_{cyc} a \geq (k+1)\sum_{cyc} ab$$

If first one occurs we are done and if the second one occurs then replace a by $\frac{1}{a}$; b by $\frac{1}{b}$; c by $\frac{1}{c}$ and we get the same expression as the first one and thus we have proved !!

$$\nabla$$

16.Prove that for $a, b, c \in \mathbb{R}^+$ -

$$\sqrt{\frac{2a}{a+b}} + \sqrt{\frac{2b}{b+c}} + \sqrt{\frac{2c}{c+a}} \geq \frac{(a+b+c)(ab+bc+ca)}{\sqrt{(a^2+b^2+c^2)(a^2b^2+b^2c^2+c^2a^2)}}$$

Solution

$$\sqrt{\frac{2b}{b+c}} \geq \frac{4b}{3b+c}$$
$$\sqrt{\frac{2c}{c+a}} \geq \frac{4c}{3c+a}$$
$$\sqrt{\frac{2a}{a+b}} \geq \frac{4a}{3a+b}$$

So that,

$$\sqrt{\frac{2a}{a+b}} + \sqrt{\frac{2b}{b+c}} + \sqrt{\frac{2c}{c+a}} \geq 4\left(\frac{a}{3a+b} + \frac{b}{3b+c} + \frac{c}{3c+a}\right)$$

From Titu's Lemma and the following well-known inequality $a^2+b^2+c^2 \geq ab+bc+ca$

$$\frac{a}{3a+b} + \frac{b}{3b+c} + \frac{c}{3c+a} = \frac{a^2}{3a^2+ab} + \frac{b^2}{3b^2+bc} + \frac{c^2}{3c^2+ca}$$
$$\geq \frac{(a+b+c)^2}{3(a^2+b^2+c^2)+ab+bc+ca} \geq \frac{(a+b+c)^2}{4(a^2+b^2+c^2)}$$

Therefore,

$$\sqrt{\frac{2a}{a+b}} + \sqrt{\frac{2b}{b+c}} + \sqrt{\frac{2c}{c+a}} \geq \frac{(a+b+c)^2}{a^2+b^2+c^2}$$

We also have, (just prove analogously)

$$\sqrt{\frac{2a}{a+b}} + \sqrt{\frac{2b}{b+c}} + \sqrt{\frac{2c}{c+a}} = \sqrt{\frac{2ac}{ac+bc}} + \sqrt{\frac{2ba}{ba+ca}} + \sqrt{\frac{2cb}{cb+ab}}$$
$$\geq \frac{(ab+bc+ca)^2}{a^2b^2+b^2c^2+c^2a^2}$$

So that,

$$\sqrt{\frac{2a}{a+b}} + \sqrt{\frac{2b}{b+c}} + \sqrt{\frac{2c}{c+a}} \geq \frac{1}{2}\left(\frac{(a+b+c)^2}{a^2+b^2+c^2} + \frac{(ab+bc+ca)^2}{a^2b^2+b^2c^2+c^2a^2}\right)$$
$$\geq \frac{(a+b+c)(ab+bc+ca)}{\sqrt{(a^2+b^2+c^2)(a^2b^2+b^2c^2+c^2a^2)}}$$

(from AM-GM inequality)

The proof is completed. Equality holds if and only if $a=b=c$

(by mathlinks user : leviethai)

$$\nabla$$

17. For $a, b, c \in \mathbb{R}_0^+$. Prove that -

$$(ab+bc+ca)\left(\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}\right)\geq\frac{9}{4}$$

(Source : Iran 1996)

Solution This is one the most famous, discussed and celebrated inequality of all times. In the year asked it was percieved as a very difficult inequality not solved by any elementary methods but this notion was wronged by this solution given by the Vietnamese Inequality Solver well known for his beautiful solutions - Va Quoc Ba Can (mathlinks uers id : canhang2007)

Without loss of generality, we may assume that $a \geq b \geq c$. Then, we will show that

$$\frac{1}{(a+b)^2}+\frac{1}{(a+c)^2}+\frac{1}{(b+c)^2}\geq\frac{1}{4ab}+\frac{2}{(a+c)(b+c)}$$

Indeed, this inequality is equivalent to

$$\frac{1}{(a+c)^2}+\frac{1}{(b+c)^2}-\frac{2}{(a+c)(b+c)}\geq\frac{1}{4ab}-\frac{1}{(a+b)^2}$$

or

$$\frac{(a-b)^2}{(a+c)^2(b+c)^2}\geq\frac{(a-b)^2}{4ab(a+b)^2}$$

This is true because

$$4ab\geq 4b^2\geq(b+c)^2$$

and

$$(a+b)^2\geq(a+c)^2$$

Now, using the above estimation, it is sufficient to prove that

$$(ab+bc+ca)\left[\frac{1}{4ab}+\frac{2}{(a+c)(b+c)}\right]\geq\frac{9}{4}$$

Since

$$\frac{ab+bc+ca}{4ab}=\frac{1}{4}+\frac{c(a+b)}{4ab}$$

and

$$\frac{2(ab+bc+ca)}{(a+c)(b+c)}=2-\frac{2c^2}{(a+c)(b+c)}$$

it is equivalent to

$$\frac{c(a+b)}{4ab}\geq\frac{2c^2}{(a+c)(b+c)}$$

or

$$(a+b)(b+c)(c+a)\geq 8abc$$

The last one is true according to the AM-GM Inequality, so our proof is completed

It stands out as one of the best solutions for the inequality.

∇

18. Let $a, b, c \in \mathbb{R}^+$ such that $abc = 1$. Prove that -

$$\frac{a^k}{a+b}+\frac{b^k}{b+c}+\frac{c^k}{c+a}\geq\frac{3}{2}$$

for any positive integer k

(Source : China Northern Mathematical Olympiad 2007)

Solution Re-write the LHS as -

$$a^{k-1}+b^{k-1}+c^{k-1}\geq\frac{3}{2}+\sum_{cyclic}\frac{a^{k-1}b}{a+b}$$

by applying AM-GM to the denominators in the RHS we get -

$$RHS \leq \sum_{cyclic} a^{k-\frac{3}{2}} b^{\frac{1}{2}}$$

Thus, it is enough if we prove that -

$$2(a^{k-1} + b^{k-1} + c^{k-1}) \geq 3 + a^{k-\frac{3}{2}} b^{\frac{1}{2}} + b^{k-\frac{3}{2}} c^{\frac{1}{2}} + c^{k-\frac{3}{2}} a^{\frac{1}{2}}$$

This follows directly from AM-GM as -

$$a^{k-1} + b^{k-1} + c^{k-1} \geq 3 \cdot (abc)^{\frac{k-1}{3}} = 3$$

And also ,

$$(2k-3)a^{k-1} + b^{k-1} \geq (2k-2)a^{k-\frac{3}{2}} b^{\frac{1}{2}}$$
$$(2k-3)b^{k-1} + c^{k-1} \geq (2k-2)b^{k-\frac{3}{2}} c^{\frac{1}{2}}$$
$$(2k-3)c^{k-1} + a^{k-1} \geq (2k-2)c^{k-\frac{3}{2}} a^{\frac{1}{2}}$$

Adding the above we get the desired.Equality for $a = b = c = 1$

(By Pham Kim Hung)

∇

19. Prove that for $a, b, c \in \mathbb{R}^+$, we have -

$$1 + \frac{3}{ab + bc + ca} \geq \frac{6}{a + b + c}$$

(Source : Macedonia Team Selection Test 2007)

Solution The inequality is equivalent to -

$$a + b + c + \frac{3(a + b + c)}{ab + bc + ca} \geq 6$$

By AM-GM inequality we have,

$$a + b + c + \frac{3(a + b + c)}{ab + bc + ca} \geq 2\sqrt{\frac{3(a + b + c)^2}{ab + bc + ca}} \geq 6$$

Equality for $a = b = c = 1$

(By Vo Danh)

∇

20. Let $a, b, c \geq 0$ such that $abc \geq 1$. Prove that -

$$\left(a + \frac{1}{a+1}\right)\left(b + \frac{1}{1+b}\right)\left(c + \frac{1}{1+c}\right) \geq \frac{27}{8}$$

(Source : Ukraine Mathematical Festival 2007)

Solution By AM-GM we have - ,

$$\frac{a+1}{4} + \frac{1}{1+a} \geq 1$$

and ,

$$\frac{3a}{4} + \frac{3}{4} \geq \frac{3}{2} \cdot \sqrt{a}$$

Adding the two inequalities we get -

$$a + \frac{1}{1+a} \geq \frac{3}{2} \cdot \sqrt{a}$$

Obtaining similarly all the cyclic inequalities and multiplying them we get -

$$\left(a + \frac{1}{a+1}\right)\left(b + \frac{1}{1+b}\right)\left(c + \frac{1}{1+c}\right) \geq \frac{27}{8} \cdot \sqrt{abc} \geq \frac{27}{8}$$

Equality for $a = b = c = 1$

(by Nguyen Dung TN)

∇