# MATHEMATICS

# CLASS XII

# INDEX

# CHAPTER-I

# RELATIONS AND FUNCTIONS

## 2 MARK QUESTIONS

**1.If R = {(a, $a^3$): a is a prime number less than 5} be a relation. Find the range of R .**

**Answer:**

Given, R = {{a, cd): a is a prime number less than 5}
We know that, 2 and 3 are the prime numbers less than 5.
So, a can take values 2 and 3.
Thus, R = {(2, $2^3$), (3, $3^3$)} = {(2, 8), (3, 27)}
Hence, the range of R is (8, 27}.

**2.If f: {1,3, 4} → {1, 2, 5} and g: {1,2, 5} → {1, 3} given by f = {(1,2), (3, 5), (4,1)} and g = {(1,3), (2, 3), (5,1)}. Write down gof.**

**Answer:**

Given, functions f:{1, 3, 4} → {1, 2, 5} and g : {1, 2, 5} → {1, 3} are defined as f = {(1, 2),(3, 5),(4, 1)} and g = {(1, 3),(2, 3),(5, 1)}
Therefore, f(1) = 2, f(3) = 5, f(4) = 1
and g(1) = 3, g (2) = 3, g(5) = 1
Now, gof: {1,3,4} → {1,3} and it is defined as
gof (1) = g[(f(1)] = g(2) = 3

gof(3) = g[f(3)] = g(5) = 1
gof (4) = g[f(4)] = g(1) = 3
∴ gof = {(1, 3), (3, 1), (4, 3)}

**3.Let R is the equivalence relation in the set A = {0,1, 2, 3, 4, 5} given by R = {(a, b) : 2 divides (a – b)}. Write the equivalence class [0].**

**Answer:**

Given, R = {(a, b):2 divides(a – b)}
and A = { 0,1, 2, 3, 4, 5}
Clearly, [0] = {b ∈ A : (0, b) ∈ R}
= {b ∈ A: 2 divides (0 – b)}
= {b ∈ A : 2divides (-b)} = {0, 2, 4}
Hence, equivalence class of [0] = {0,2,4}.

**4.If R = {(x, y): x + 2y = 8} is a relation on N, then write the range of R.**

**Answer:**

Given, the relation R is defined on the set of natural numbers, i.e. N as
R= {(x, y) : x + 2y = 8}
To find the range of R, x + 2y = 8 can be rewritten as y = 8−x2
On putting x = 2, we get y = 8−22 = 3
On puttmg x = 4, we get y = 8−42 = 2
On putting x = 6, we get y = 8−62 = 1
As, x, y ∈ N, therefore R = {(2, 3), (4, 2), (6, 1)}. Hence, the range of relation R is {3,2,1}.
Note: For x = 1, 3, 5, 7, 9, ……… we do not get y as natural number.

**5.If A = {1, 2, 3}, S = {4, 5,6, 7} and f = {(1, 4), (2, 5), (3, 6)} is a function from A to B. State whether f is one-one or not. (All India 2011)**

**Answer:**

Given, A = {1, 2, 3} , B = {4, 5, 6, 7}
and f:A → Bis defined as f = {(1, 4), (2, 5), (3, 6)}
i.e. f(1) = 4, f(2) = 5and f(3) = 6.
It can be seen that the images of distinct elements of A under f are distinct. So, f is one-one.

# 4 MARK QUESTIONS

**1. Show that the Signum Function f: R → R, given by**

**$f(x)$=**
**{1$for\ x$>00**
**$for\ x$=0 $is\ neither\ one{-}one\ nor\ onto$−1$for\ x$<0**

**Solution:**

**Check for one to one function:**

For example:

f(0) = 0

f(-1) = -1

f(1) = 1

f(2) = 1

f(3) = 1

Since, the different elements say f(1), f(2) and f(3), shows the same image, then the function is not one to one function.

**Check for Onto Function:**

For the function,f: R →R

$f(x)$={1$for\ x$>00$for\ x$=0−1$for\ x$<0
In this case, the value of f(x) is defined only if x is 1, 0, -1

For any other real numbers(for example $y = 2$, $y = 100$) there is no corresponding element x.

Thus, the function "f" is not onto function.

Hence, the given function "f" is neither one-one nor onto.

**2: If f: R → R is defined by $f(x) = x^2 - 3x + 2$, find $f(f(x))$.**

**Solution:**

Given function:

$f(x) = x^2 - 3x + 2.$

To find $f(f(x))$

$f(f(x)) = f(x)^2 - 3f(x) + 2.$

$= (x^2 - 3x + 2)^2 - 3(x^2 - 3x + 2) + 2$

By using the formula $(a-b+c)^2 = a^2 + b^2 + c^2 - 2ab + 2ac - 2ab$, we get

$= (x^2)^2 + (3x)^2 + 2^2 - 2x^2 (3x) + 2x^2(2) - 2x^2(3x) - 3(x^2 - 3x + 2) + 2$

Now, substitute the values

$= x^4 + 9x^2 + 4 - 6x^3 - 12x + 4x^2 - 3x^2 + 9x - 6 + 2$

$= x^4 - 6x^3 + 9x^2 + 4x^2 - 3x^2 - 12x + 9x - 6 + 2 + 4$

Simplify the expression, we get,

$f(f(x)) = x^4 - 6x^3 + 10x^2 - 3x$

# 7 MARK QUESTIONS

**1: Let A = N × N and * be the binary operation on A defined by (a, b) * (c, d) = (a + c, b + d). Show that * is commutative and associative. Find the identity element for * on A, if any.**

**Solution:**

**Check the binary operation * is commutative :**

We know that, * is commutative if (a, b) * (c, d) = (c, d) * (a, b) ∀ a, b, c, d ∈ R

L.H.S =(a, b) * (c, d)

=(a + c, b + d)

R. H. S = (c, d) * (a, b)

=(a + c, b + d)

Hence, L.H.S = R. H. S

Since (a, b) * (c, d) = (c, d) * (a, b) ∀ a, b, c, d ∈ R

* is commutative (a, b) * (c, d) = (a + c, b + d)

**Check the binary operation * is associative :**

We know that * is associative if (a, b) * ( (c, d) * (x, y) ) = ((a, b) * (c, d)) * (x, y) ∀ a, b, c, d, x, y ∈ R

L.H.S = (a, b) * ( (c, d) * (x, y) ) = (a+c+x, b+d+y)

R.H.S = ((a, b) * (c, d)) * (x, y) = (a+c+x, b+d+y)

Thus, L.H.S = R.H.S

Since (a, b) * ( (c, d) * (x, y) ) = ((a, b) * (c, d)) * (x, y) ∀ a, b, c, d, x, y ∈ R

Thus, the binary operation * is associative

**Checking for Identity Element:**

e is identity of * if (a, b) * e = e * (a, b) = (a, b)

where e = (x, y)

Thus, (a, b) * (x, y) = (x, y) * (a, b) = (a, b) (a + x, b + y)

= (x + a , b + y) = (a, b)

Now, (a + x, b + y) = (a, b)

Now comparing these, we get:

a+x = a

x = a -a = 0

Next compare: b +y = b

y = b-b = 0

Since A = N x N, where x and y are the natural numbers. But in this case, x and y is not natural number. Thus, the identity element does not exist.

Therefore, operation * does not have any identity element.

**2: Let f : N → Y be a function defined as f (x) = 4x + 3, where, Y = {y ∈ N: y = 4x + 3 for some x ∈ N}. Show that f is invertible. Find the inverse.**

**Solution:**

**Checking for Inverse:**

$f(x) = 4x + 3$

Let $f(x) = y$

$y = 4x + 3$

$y – 3 = 4x$

$4x = y – 3$

$x = (y - 3)/4$

Let $g(y) = (y - 3)/4$

where $g: Y \rightarrow N$

**Now find gof:**

gof**=** g(f(x))

$= g(4x + 3) = [(4x + 3) - 3]/4$

$= [4x + 3 - 3]/4$

$=4x/4$

$= x = I_N$

**Now find fog:**

fog= f(g(y))

$= f\ [(y - 3)/4]$

$=4[(y - 3)/4] +3$

$= y – 3 + 3$

$= y + 0$

$= y = I_y$

Thus, gof = $I_N$ and fog = $I_y$,

Hence, f is invertible

Also, the Inverse of f = g(y) = [$y$ – 3]/ 4

**3: Let A = R {3} and B = R – {1}. Consider the function f: A →B defined by f (x) = (x- 2)/(x -3). Is f one-one and onto? Justify your answer.**

**Solution:**

Given function:

f (x) = (x- 2)/(x -3)

**Checking for one-one function:**

f ($x_1$) = ($x_1$– 2)/ ($x_1$– 3)

f ($x_2$) = ($x_2$-2)/ ($x_2$-3)

Putting f ($x_1$) = f ($x_2$)

($x_1$-2)/($x_1$-3)= ($x_2$-2 )/($x_2$ -3)

($x_1$-2) ($x_2$– 3) = ($x_1$– 3) ($x_2$-2)

$x_1$ ($x_2$– 3)- 2 ($x_2$ -3) = $x_1$ ($x_2$– 2) – 3 ($x_2$– 2)

$x_1$ $x_2$ -3$x_1$ -2$x_2$ + 6 = $x_1$ $x_2$ – 2$x_1$ -3$x_2$ + 6

-3$x_1$– 2$x_2$ =- 2$x_1$ -3$x_2$

3$x_2$ -2$x_2$ = – 2$x_1$ + 3$x_1$

$x_1$= $x_2$

Hence, if f ($x_1$) = f ($x_2$), then $x_1$ = $x_2$

Thus, the function f is one-one function.

Checking for onto function:

$f(x) = (x-2)/(x-3)$

Let $f(x) = y$ such that y B i.e. $y \in R - \{1\}$

So, $y = (x-2)/(x-3)$
$y(x-3) = x-2$

$xy - 3y = x - 2$

$xy - x = 3y - 2$

$x(y-1) = 3y-2$

$x = (3y-2)/(y-1)$

For $y = 1$, x is not defined But it is given that. $y \in R - \{1\}$

Hence, $x = (3y-2)/(y-1) \in R - \{3\}$ Hence, f is onto.

# CHAPTER-II

# INVERSE TRIGONOMETRIC FUNCTIONS

## 2 MARK QUESTIONS

**1.Write the value of**
**$\tan^{-1}(\sqrt{3}) - \cot^{-1}(-\sqrt{3})$. (All India 2019,13)**
**Answer:**

We have, $\tan^{-1}(\sqrt{3}) - \cot^{-1}(-\sqrt{3})$
$= \tan^{-1}(\sqrt{3}) - \{\pi - \cot^{-1}(\sqrt{3})\}$ [$\because \cot^{-1}(-x) = \pi - \cot^{-1}x; x \in R$]
$= \tan^{-1}\sqrt{3} - \pi + \cot^{-1}\sqrt{3}$
$= (\tan^{-1}\sqrt{3} + \cot^{-1}\sqrt{3}) - \pi$
$= \pi2 - \pi = -\pi2$ [$\because \tan^{-1}x + \cot^{-1}x = \pi2; x \in R$]
Which is the required principal value.

**2.Find the principal value of**
**$\tan^{-1}\sqrt{3} - \sec^{-1}(-2)$. (CBSE 2018 C; All India 2012)**
**Answer:**
We have, $\tan^{-1}\sqrt{3} - \sec^{-1}(-2)$

$$= \tan^{-1}\left(\tan\frac{\pi}{3}\right) - \sec^{-1}\left(\sec\frac{2\pi}{3}\right)$$

$$\left[\because \tan\frac{\pi}{3} = \sqrt{3} \text{ and } \sec\frac{2\pi}{3} = -2\right]$$

$$= \frac{\pi}{3} - \frac{2\pi}{3} = -\frac{\pi}{3}$$

$$\left[\begin{array}{l} \because \tan^{-1}(\tan\theta) = \theta; \forall\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \text{and} \\ \sec^{-1}(\sec\theta) = \theta; \forall\theta \in [0, \pi] - \left\{\frac{\pi}{2}\right\} \end{array}\right]$$

Which is the required principal value.

**3: Determine the principal value of $\cos^{-1}(-1/2)$.**

**Solution:**

Let us assume that, $y = \cos^{-1}(-1/2)$

We can write this as:

$\cos y = -1/2$

$\cos y = \cos(2\pi/3)$.

Thus, the Range of the principal value of $\cos^{-1}$ is $[0, \pi]$.

Therefore, the principal value of $\cos^{-1}(-1/2)$ is $2\pi/3$.

**4: Find the value of $\cot(\tan^{-1}\alpha + \cot^{-1}\alpha)$.**

**Solution:**

Given that: $\cot(\tan^{-1}\boldsymbol{\alpha} + \cot^{-1}\boldsymbol{\alpha})$

$= \cot(\boldsymbol{\pi/2})$ (since, $\tan^{-1} x + \cot^{-1} x = \pi/2$)

$= \cot(180°/2)$ ( we know that $\cot 90° = 0$ )

$= \cot(90°)$

$= 0$

Therefore, the value of $\cot(\tan^{-1}\alpha + \cot^{-1}\alpha)$ is 0.

# 4 MARK QUESTIONS

**1.If sin ($\sin^{-1}15 + \cos^{-1}x$) = 1, then find the value of x. (Delhi 2014)**
**Answer:**

Given, $\sin\left(\sin^{-1}\frac{1}{5} + \cos^{-1}x\right) = 1$

$$\Rightarrow \quad \sin^{-1}\frac{1}{5} + \cos^{-1}x = \sin^{-1}(1)$$

$$[\because \sin\theta = x \Rightarrow \theta = \sin^{-1}x]$$

$$\Rightarrow \quad \sin^{-1}\frac{1}{5} + \cos^{-1}x = \sin^{-1}\left(\sin\frac{\pi}{2}\right) \quad \left[\because \sin\frac{\pi}{2} = 1\right]$$

$$\Rightarrow \quad \sin^{-1}\frac{1}{5} + \cos^{-1}x = \frac{\pi}{2}$$

$$\Rightarrow \quad \sin^{-1}\frac{1}{5} = \frac{\pi}{2} - \cos^{-1}x$$

$$\Rightarrow \quad \sin^{-1}\frac{1}{5} = \sin^{-1}x$$

$$\left[\because \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2};\ x \in [-1, 1]\right]$$

$$\therefore \quad x = \frac{1}{5}$$

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**2.Write the value of**
**$\tan^{-1}(\sqrt{3}) - \cot^{-1}(-\sqrt{3})$. (All India 2019,13)**

**Answer:**

We have, $\tan^{-1}(\sqrt{3}) - \cot^{-1}(-\sqrt{3})$
$= \tan^{-1}(\sqrt{3}) - \{\pi - \cot^{-1}(\sqrt{3})\}$ [$\because \cot^{-1}(-x) = \pi - \cot^{-1}x;\ x \in R$]
$= \tan^{-1}\sqrt{3} - \pi + \cot^{-1}\sqrt{3}$
$= (\tan^{-1}\sqrt{3} + \cot^{-1}\sqrt{3}) - \pi$
$= \pi 2 - \pi = -\pi 2$ [$\because \tan^{-1}x + \cot^{-1}x = \pi 2;\ x \in R$]
Which is the required principal value.

**3.Find the principal value of**
**$\tan^{-1}\sqrt{3} - \sec^{-1}(-2)$. (CBSE 2018 C; All India 2012)**

**Answer:**

We have, $\tan^{-1}\sqrt{3} - \sec^{-1}(-2)$

$$= \tan^{-1}\left(\tan\frac{\pi}{3}\right) - \sec^{-1}\left(\sec\frac{2\pi}{3}\right)$$

$$\left[\because \tan\frac{\pi}{3} = \sqrt{3} \text{ and } \sec\frac{2\pi}{3} = -2\right]$$

$$= \frac{\pi}{3} - \frac{2\pi}{3} = -\frac{\pi}{3}$$

$$\left[\begin{array}{l}\because \tan^{-1}(\tan\theta) = \theta; \forall\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \text{and} \\ \sec^{-1}(\sec\theta) = \theta; \forall\theta \in [0, \pi] - \left\{\frac{\pi}{2}\right\}\end{array}\right]$$

Which is the required principal value.

**Question 4.**
**If $\sin(\sin^{-1}\frac{1}{5} + \cos^{-1}x) = 1$, then find the value of x. (Delhi 2014)**
**Answer:**

Given, $\sin\left(\sin^{-1}\frac{1}{5} + \cos^{-1}x\right) = 1$

$$\Rightarrow \quad \sin^{-1}\frac{1}{5} + \cos^{-1}x = \sin^{-1}(1)$$

$$[\because \sin\theta = x \Rightarrow \theta = \sin^{-1}x]$$

$$\Rightarrow \quad \sin^{-1}\frac{1}{5} + \cos^{-1}x = \sin^{-1}\left(\sin\frac{\pi}{2}\right) \quad \left[\because \sin\frac{\pi}{2} = 1\right]$$

$$\Rightarrow \quad \sin^{-1}\frac{1}{5} + \cos^{-1}x = \frac{\pi}{2}$$

$$\Rightarrow \quad \sin^{-1}\frac{1}{5} = \frac{\pi}{2} - \cos^{-1}x$$

$$\Rightarrow \quad \sin^{-1}\frac{1}{5} = \sin^{-1}x$$

$$\left[\because \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}; x \in [-1, 1]\right]$$

$$\therefore \quad x = \frac{1}{5}$$

**5.If tan-1x + tan-1 y = π4; xy < 1, then write the value of x + y + xy. (All India 2014)**
**Answer:**

Given, $\tan^{-1} x + \tan^{-1} y = \frac{\pi}{4}, xy < 1$

We know that,

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right), xy < 1$$

$$\therefore \quad \tan^{-1}\left(\frac{x+y}{1-xy}\right) = \frac{\pi}{4} \Rightarrow \frac{x+y}{1-xy} = \tan\frac{\pi}{4}$$

$$\Rightarrow \quad \frac{x+y}{1-xy} = 1 \qquad \left[\because \tan\frac{\pi}{4} = 1\right]$$

$$\Rightarrow \quad x + y = 1 - xy$$

$$\therefore \quad x + y + xy = 1$$

**6.Write the principal value of the following.**
**[cos−13√2+cos−1(−12)] (Delhi 2013C)**
**Answer:**

We have, $\cos^{-1}\frac{\sqrt{3}}{2} + \cos^{-1}\left(-\frac{1}{2}\right)$

$$= \cos^{-1}\frac{\sqrt{3}}{2} + \left[\pi - \cos^{-1}\left(\frac{1}{2}\right)\right]$$

$$[\because \cos^{-1}(-x) = \pi - \cos^{-1} x, \forall x \in [-1, 1]]$$

$$= \cos^{-1}\left(\cos\frac{\pi}{6}\right) + \left[\pi - \cos^{-1}\left(\cos\frac{\pi}{3}\right)\right]$$

$$= \frac{\pi}{6} + \pi - \frac{\pi}{3} = \frac{\pi + 6\pi - 2\pi}{6} = \frac{5\pi}{6}$$

$$[\because \cos^{-1}(\cos\theta) = \theta; \forall \theta \in [0, \pi]]$$

which is the required principal value.

**7.Write the value of tan (2 tan$^{-1}$15). (Delhi 2013)**

**Answer:**

We have,

$$\tan\left(2\tan^{-1}\frac{1}{5}\right)=\tan\left[\tan^{-1}\left(\frac{2\times\frac{1}{5}}{1-\left(\frac{1}{5}\right)^2}\right)\right]$$

$$\left[\because 2\tan^{-1}x=\tan^{-1}\left(\frac{2x}{1-x^2}\right);-1<x<1\right]$$

$$=\tan\left[\tan^{-1}\left(\frac{2\times 5}{24}\right)\right]=\tan\left[\tan^{-1}\left(\frac{5}{12}\right)\right]=\frac{5}{12}$$

$$[\because \tan(\tan^{-1}x)=x;\forall x\in R]$$

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# 7 MARK QUESTIONS

**1: Prove that $\sin^{-1}(3/5) - \sin^{-1}(8/17) = \cos^{-1}(84/85)$.**

**Solution:**

Let $\sin^{-1}(3/5) = a$ and $\sin^{-1}(8/17) = b$

Thus, we can write $\sin a = 3/5$ and $\sin b = 8/17$

Now, find the value of cos a and cos b

**To find cos a:**

$\cos a = \sqrt{[1 - \sin^2 a]}$

$= \sqrt{[1 - (3/5)^2]}$

$= \sqrt{[1 - (9/25)]}$

$= \sqrt{[(25-9)/25]}$

$= 4/5$

Thus, the value of $\cos a = 4/5$

**To find cos b:**

$\cos b = \sqrt{[1 - \sin^2 b]}$

$= \sqrt{[1 - (8/17)^2]}$

$= \sqrt{[1 - (64/289)]}$

$= \sqrt{[(289-64)/289]}$

$= 15/17$

Thus, the value of $\cos b = 15/17$

We know that cos (a- b) = cos a cos b + sin a sin b

Now, substitute the values for cos a, cos b, sin a and sin b in the formula, we get:

cos (a – b) = (4/5)x (15/17) + (3/5)x(8/17)

cos (a – b) = (60 + 24)/(17x 5)

cos (a – b) = 84/85

(a – b) = $\cos^{-1}$ (84/85)

Substituting the values of a and b $\sin^{-1}$ (3/5)- $\sin^{-1}$ (8/7) = $\cos^{-1}$ (84/85)

Hence proved.

**2: Find the value of $\cos^{-1}$ (1/2) + 2 $\sin^{-1}$ (1/2).**

**Solution:**

**First, solve for $\cos^{-1}$ (1/2):**

Let us take, y = $\cos^{-1}$ (1/2)

This can be written as:

cos y = (1/2)

cos y = cos (π /3).

Thus, the range of principal value of $\cos^{-1}$ is [0, π ]

Therefore, the principal value of $\cos^{-1}$ (1/2) is π/3.

**Now, solve for $\sin^{-1}$ (1/2):**

Let y = $\sin^{-1}$ (1/2)

sin y = 1/2

sin y = sin ( π/6)

Thus, the range of principal value of $\sin^{-1}$ is $[(-\pi)/2, \pi/2]$

Hence, the principal value of $\sin^{-1}(1/2)$ is $\pi/6$.

Now we have $\cos^{-1}(1/2) = \pi/3$ & $\sin^{-1}(1/2) = \pi/6$

Now, substitute the obtained values in the given formula, we get:

$= \cos^{-1}(1/2) + 2\sin^{-1}(1/2)$

$= \pi/3 + 2(\pi/6)$

$= \pi/3 + \pi/3$

$= (\pi+\pi)/3$

$= 2\pi/3$

Thus, the value of $\cos^{-1}(1/2) + 2\sin^{-1}(1/2)$ is $2\pi/3$.

# CHAPTER-III

# MATRICES

## 2 MARK QUESTIONS

**1. Define Square Matrix.**

**Ans:** A square matrix is a matrix in which the number of rows is equal to the number of columns, ie., m=n.

**2. What is the Value of Every Diagonal Element of a Skew Matrix?**

**Ans:** Zero.

**3. What Are the Possible Orders If a Matrix Has 28 Elements?**

**Ans:** The possible orders are denoted by

1 x 28,

2 x 14,

4 x 7,

7 x 4,

14 x 2,

28 x 1

**4.If [2x+y03y4]=[6004], then find the value of x. (All India 2010C)**
**Answer:**
x = 3

**5.If [3y−x3−2x7]=[53−27], then find the value of y. (All India 2010C)**

**Answer:**
y = 2

# 4 MARK QUESTIONS

**1: Matrices *A* and *B* will be inverse of each other only if**

1. ***AB*=*BA***
2. ***AB*=0,*BA*=*I***
3. ***AB*=*BA*=0**
4. ***AB*=*BA*=*I***

**Answer:** We know that if $A$ is a square of order $m$, and if there exists another square matrix $B$ of the same order $m$, such that $AB=I$, then $B$ is said to be the inverse of $A$.

In this case, it is clear that $A$ is the inverse of $B$.
Thus , matrices $A$ and $B$ will be inverses of each other only if $AB=BA=I$.

**2: If a matrix has 24 elements, what are the possible order it can have? What, if it has 13 elements?**

**Answer:** We know that if a matrix is of the order $m\times n$, it has $mn$ elements.
Thus to find all the possible orders of a matrix having 24 elements, we have to find all the ordered pairs of natural numbers whose product is 24.
The ordered pairs are(1,24),(24,1),(2,12),(12,2),(3,8),(8,3),(4,6) and (6,4)
Hence, the possible orders of a matrix having 24 elements are:
$1\times24, 24\times1, 2\times12, 12\times2, 3\times8, 8\times3, 4\times6, 6\times4$
(1,13) and (13,1) are the ordered pairs of natural numbers whose product is 13.
Hence, the possible orders of a matrix having 13 elements are $1\times13$ and $13\times1$

**3: If $n=p$, then the order of the matrix $7X-5Z$ is:**

1. $p\times2$
2. $2\times n$
3. $n\times3$
4. $p\times n$

**Answer:** In this, order of $X=2\times n$

and order of $Z=2\times p$

Therefore, $n=p$

Hence order of $7X-5Z=2\times n$.

Thus option (B) is correct.

**4: If $A,B$ are symmetric matrices of same order, then $AB-BA$ is a ,**
**A. Skew symmetric matrix**

1. **Symmetric matrix**
2. **Zero matrix**
3. **Identity matrix**

**Answer:** Given, $A$ and $B$ are symmetric matrices, therefore, we have:
$A'=A$ and $B'=B$..........($i$)
Consider
$(AB-BA)'=(AB)'-(BA)', [\because (A-B)'=A'-B']$
$=B'A'-A'B', [\because (AB)'=B'A']$

$=BA-AB$ [by (i) ]

$=-(AB-BA)$
$\therefore (AB-BA)'=-(AB-BA)$
Thus, $(AB-BA)$ is a skew-symmetric matrix.

**5.Write the number of all possible matrices of order 2 × 2 with each entry 1, 2 or 3.**

**Answer:**

We know that, a matrix of order 2 × 2 has 4 entries. Since, each entry has 3 choices, namely 1, 2 or 3, therefore number of required matrices

$3^4 = 3 \times 3 \times 3 \times 3 = 81$.

# 7 MARK QUESTIONS

**1..If 3A – B = [5101] and B = [4235] then find the value of matrix A. (Delhi 2019)**

**Answer:**

$$\text{Given,} \quad 3A - B = \begin{bmatrix} 5 & 0 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$$

$$\Rightarrow \quad 3A - \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\Rightarrow \quad 3A = \begin{bmatrix} 5 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$$

$$\Rightarrow \quad 3A = \begin{bmatrix} 5+4 & 0+3 \\ 1+2 & 1+5 \end{bmatrix}$$

$$\Rightarrow \quad 3A = \begin{bmatrix} 9 & 3 \\ 3 & 6 \end{bmatrix}$$

$$\Rightarrow \quad A = \frac{1}{3}\begin{bmatrix} 9 & 3 \\ 3 & 6 \end{bmatrix}$$

$$\Rightarrow \quad A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$

**2.Find the value of x – y, if (Delhi 2019)**

**2[103x]+[y102]=[5168]**

**Answer:**

Given that,

$$2\begin{bmatrix} 1 & 3 \\ 0 & x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 2 & 6 \\ 0 & 2x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 2+y & 6+0 \\ 0+1 & 2x+2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$$

Here, both matrices are equal, so we equate the corresponding elements,

2 + y = 5 and 2x + 2 = 8

⇒ y = 3 and 2x = 6 ⇒ x = 3
Therefore, x – y = 3 – 3 = 0

**3.If A is a square matrix such that $A^2$ = I, then find the simplified value of $(A – I)^3 + (A + I)^3 – 7A$. (Delhi 2016)**
**Answer:**
Given, $A^2 = 7$ ……. (i)
Now, $(A – I)^3 + (A + I)^3 – 7A$
$= (A^3 – 3A^2I + 3AI^2 – I) + (A^3 + 3A^2I + 3AI^2 + I^3) – 7A$
$= A^3 – 3A^2 + 3AI – I + A^3 + 3A^2 + 3AI + I – 7A$
$[\because A^2I = A^2 \text{ and } I^3 = I^3 = I]$
$= 2A^3 + 6AI – 7A = 2A^2 A + 6A – 7A$ $[\because AI = A]$
$= 2IA – A$ [from Eq. (1)]

$= 2A – A = A$ $[\because IA = A]$

**4.If [2 1 3] ⎡⎣⎢ −1−10011−101 ⎤⎦⎥⎡⎣⎢ 10−1 ⎤⎦⎥ = A, then write the order of matrix A.**
**Answer:**

$$\text{We have, } A = [2 \;\; 1 \;\; 3]\begin{bmatrix} -1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$= [-2-1 \;\; 1+3 \;\; -2+3]\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = [-3 \;\; 4 \;\; 1]\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$= [- 3 – I] = [- 4]_{1 \times 1}$
∴ Order of matrix A is 1 × 1.

**5.Let A = [23−14], B = [5724], C = [2358], find a matrix D such that CD – AB = 0.**

**Answer:**

Given, $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 2 \\ 7 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix}$

Let matrix $D = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$

According to the questions, $CD - AB = 0$

$$\begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix}\begin{bmatrix} x & y \\ z & w \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 2 \\ 7 & 4 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 2x + 5z & 2y + 5w \\ 3x + 8z & 3y + 8w \end{bmatrix} - \begin{bmatrix} 10 - 7 & 4 - 4 \\ 15 + 28 & 6 + 16 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 2x + 5z & 2y + 5w \\ 3x + 8z & 3y + 8w \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 43 & 22 \end{bmatrix}$$

On equating corresponding elements both sides, we get

$$2x + 5z = 3, \; 3x + 8z = 43$$

and $2y + 5w = 0, \; 3y + 8w = 22$

After solving, we get

$$x = -191, \; y = -110, \; z = 77 \text{ and } w = 44$$

$$\therefore \quad D = \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} -191 & -110 \\ 77 & 44 \end{bmatrix}$$

# CHAPTER-IV

# DETERMINANTS

## 2 MARK QUESTIONS

**1.Find |AB|, if A = [00−12] and B = [3050]. (All India 2019)**

**Answer:**

Given, $A = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 \\ 0 & 0 \end{bmatrix}$

Now, $AB = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 0 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 0+0 & 0+0 \\ 0+0 & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore \quad |AB| = 0$

**2.In the interval it π/2 < x < π, find the value of x for which the matrix [2sinx132sinx] is singular. (All India 2015C)**

Answer:

Let $A = \begin{bmatrix} 2\sin x & 3 \\ 1 & 2\sin x \end{bmatrix}$

$\because A$ is a singular matrix.

$\therefore \quad |A| = 0 \Rightarrow \begin{vmatrix} 2\sin x & 3 \\ 1 & 2\sin x \end{vmatrix} = 0$

$\Rightarrow \quad 4\sin^2 x - 3 = 0 \Rightarrow \sin^2 x = \frac{3}{4}$

$\Rightarrow \quad \sin x = \frac{\sqrt{3}}{2} \qquad \left[\begin{matrix}\text{taking positive square} \\ \text{root because } \frac{\pi}{2} < x < \pi\end{matrix}\right]$

$\therefore \quad x = \frac{2\pi}{3}$

**3.If A is a square matrix satisfying A'A = I, write the value of |A|. (All India 2019)**

**Answer:**

We have, A'A = I

$\Rightarrow |A'A| = |I| \Rightarrow |A'||A| = 1$ [$\because$ $|AB| = |A|\,|B|$]

$\Rightarrow |A|^2 = 1$ [$\because$ $|A'| = |A|$]

$\Rightarrow |A| = \pm 1$

**4.If A and B are square matrices of the same order 3, such that |A| = 2 and AB = 27. Write the value of |B|. (Delhi 2019)**

**Answer:**

We know that,

(i) $|kA| = k^n|A|$, if A is square matrix of nth order

(ii) $|AB| = |A| \times |B|$

Here, we have AB = 21 and n – 3

$\therefore |AB| = |2I| = 2^3|I| = 8.1 = 8$ [$\because |I| = 1$]

$\Rightarrow |4||B| = 8$

$\Rightarrow 2.|B| = 8 \Rightarrow |B|=4$

# 4 MARK QUESTIONS

**1.Write the value of Δ = ||||x+yz−3y+zx−3z+xy−3||||. (All India 2015)**
**Answer:**

$$\text{Given,}\quad \Delta = \begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ -3 & -3 & -3 \end{vmatrix}$$

On applying $R_1 \to R_1 + R_2$, we get

$$\Delta = \begin{vmatrix} x+y+z & x+y+z & x+y+z \\ z & x & y \\ -3 & -3 & -3 \end{vmatrix}$$

On taking (x + y + z) common from $R_1$ and -3
Δ = (x + y + z)(-3)||||1z11x11y1||||
= (x + y + z)(-3) × 0 [∵ $R_1$ and $R_3$ are identical]
= 0

**2.Let A be a square matrix of order 3 × 3. Write the value of |2A|, where |A| = 4.**

**Answer:**

We know that, for a square matrix A of order n,
$|kA| = k^n - |A|$
Here, $|2A| = 2^3.|A|$ [∵ order of A is 3 × 3]
$= 2^3 \times 4 = 8 \times 4 = 32$ [put |A| = 4]

**3.If the determinant of matrix A of order 3 × 3 is of value 4, then write the value of |3A|.**

**Answer:**

We know that, for a square matrix A of order n,
$|kA| = k^n - |A|$
Here, $|3A| = 3^3.|A|$ [∵ order of A is 3 × 3]
= 108

**4.If A is a square matrix of order 3 such that |adjA| = 64,then find |A|.**

**Answer:**

We know that, for a square matrix of order n,
$|adj\ (A)| = |A|^{n-1}$
Here, the order of A is 3 × 3 therefore n- 3
Now, $|adj\ (A)| = |A|^{3-1} = |A|^2$
Given, $|adj\ (A)| = 64 \Rightarrow 64 = |A|^2$
$\Rightarrow (8)^2 = |A|^2$
$\Rightarrow |A| = \pm 8$ [taking square root]

# 7MARK QUESTIONS

**1.|A|=2≠0|$A$|=2≠0**
**Therefore $A^{-1}$ exists**

AB = I
$A^{-1}$ AB = $A^{-1}$I
B = $A^{-1}$
adjA=[2143]$adjA$=[2413]
A–1=1|A|(adjA)$A$–1=1|$A$|($adjA$)
=12[2143]=12[2413]
=[112232]=[121232]
Hence B=[112232]

**2.Consider the following system of linear equations; x + y + z = 6, x – y + z = 2, 2x + y + z = 1**
**(i) Express this system of equations in the Standard form AXB**
**(ii) Prove that A is non-singular.**
**(iii) Find the value of x, y and z satisfying the above equation.**

**Answer:**

(i) Let AX = B,

Where $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$

(ii) $|A| = 1(-1-1) - 1(1-2) + 1(1+2) = 2$

(iii) $C_{11} = -2, C_{12} = 1, C_{13} = 3, C_{21} = 0, C_{22} = -1$

$C_{23} = 1, C_{31} = 2, C_{32} = 0, C_{33} = -2$

$$adj(A) = \begin{bmatrix} -2 & 1 & 3 \\ 0 & -1 & 1 \\ 2 & 0 & -2 \end{bmatrix}^T = \begin{bmatrix} -2 & 0 & 2 \\ 1 & -1 & 0 \\ 3 & 1 & -2 \end{bmatrix}$$

$$A^{-1} = \frac{adj(A)}{|A|} = \frac{1}{2}\begin{bmatrix} -2 & 0 & 2 \\ 1 & -1 & 0 \\ 3 & 1 & -2 \end{bmatrix}$$

$$X = A^{-1}B = \frac{1}{2}\begin{bmatrix} -2 & 0 & 2 \\ 1 & -1 & 0 \\ 3 & 1 & -2 \end{bmatrix} \times \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} -10 \\ 4 \\ 18 \end{bmatrix} = \begin{bmatrix} -5 \\ 2 \\ 9 \end{bmatrix}$$

$$\Rightarrow x = -5, y = 2, z = 9$$

**3.(i) If |||x532|||=5, then x = ………..**
**(ii) Prove that**
**||||y+kyyyy+kyyyy+k||||=k2(3y+k)**
**(iii) Solve the following system of linear Equations, using matrix method; 5x + 2y = 3, 3x + 2y = 5 (March – 2012)**

**Answer:**

(i) $\begin{vmatrix} x & 3 \\ 5 & 2 \end{vmatrix} = 5 \Rightarrow 2x - 15 = 5 \Rightarrow x = 10$

(ii) $\begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & y & y+k \end{vmatrix} \overset{C_1 \to C_1 + C_2 + C_3}{=} \begin{vmatrix} 3y+k & y & y \\ 3y+k & y+k & y \\ 3y+k & y & y+k \end{vmatrix}$

$= (3y+k)\begin{vmatrix} 1 & y & y \\ 1 & y+k & y \\ 1 & y & y+k \end{vmatrix}$

$\overset{R_2 \to R_2 - R_1,\, R_3 \to R_3 - R_1}{=} (3y+k) \begin{vmatrix} 1 & y & y \\ 0 & k & 0 \\ 0 & 0 & k \end{vmatrix} = (3y+k)k^2$

(iii) Let AX = B,

Where $A = \begin{bmatrix} 5 & 2 \\ 3 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

$$|A| = 10 - 6 = 4$$

$$C_{11} = 2, C_{12} = -3, C_{21} = -2, C_{22} = 5$$

$$adj(A) = \begin{bmatrix} 2 & -3 \\ -2 & 5 \end{bmatrix}^T = \begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix}$$

$$A^{-1} = \frac{adj(A)}{|A|} = \frac{1}{4}\begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix}$$

$$X = A^{-1}B = \frac{1}{4}\begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix} \times \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} -4 \\ 16 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix} \Rightarrow x = -1, y = 4$$

**4.Consider the matrices A=[2435]**
**(i) Find $A^2$ – 7A – 21 = 0**
**(ii) Hence find A-1**
**(iii) Solve the following system of equations using matrix method 2x + 3y = 4; 4x + 5y = 6**

**Answer:**

(i) $A^2 = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 16 & 21 \\ 28 & 37 \end{bmatrix}$

$$A^2 - 7A - 2I = \begin{bmatrix} 16 & 21 \\ 28 & 37 \end{bmatrix} - 7\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & 21 \\ 28 & 37 \end{bmatrix} - \begin{bmatrix} 14 & 21 \\ 28 & 37 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(ii) We have; $A^2 - 7A = 2I$

$$\Rightarrow (A^2 - 7A)A^{-1} = 2I \times A^{-1}$$

$$\Rightarrow (A - 7I) = 2A^{-1} \Rightarrow A^{-1} = \frac{1}{2}(A - 7I)$$

$$\Rightarrow A^{-1} = \frac{1}{2}\left(\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} - 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} -5 & 3 \\ 4 & -2 \end{bmatrix}$$

(iii) The given system of equations can be converted into matrix form AX = B

$$X = A^{-1}B = \frac{1}{2}\begin{bmatrix} -5 & 3 \\ 4 & -2 \end{bmatrix} \times \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$x = -1, y = 2$

**5.(i) Let A be a square matrix of order 2 x 2 then |KA| is equal to**
**(a) K|A|**
**(b) $K^2$|A|**
**(c) $K^3$|A|**
**(d) 2K|A|**

**(ii) Prove that**
**||||a−b−c2 b2c2a b−c−a2c2a2 bc−a−b||||=(a+b+c)3**
**(iii) Examine the consistency of the system of Equations. 5x + 3y = 5; 2x + 6y = 8**

**Answer:**

(i) $K^2$ |A|

(ii) $$\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} \quad R_1 \to R_1 + R_2 + R_3$$

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ 2b & -b-c-a & 0 \\ 2c & 0 & -c-a-b \end{vmatrix} \quad \begin{matrix} C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1 \end{matrix}$$

$$= (a+b+c)(a+b+c)^2 - 0 = (a+b+c)^3$$

(iii) The given system of equation can be written in matrix form as

$$A=\begin{bmatrix}5 & 3\\ 2 & 6\end{bmatrix}; X=\begin{bmatrix}x\\ y\end{bmatrix}; B=\begin{bmatrix}5\\ 8\end{bmatrix}$$

$$|A|=\begin{vmatrix}5 & 3\\ 2 & 6\end{vmatrix}=30-6=24\neq 0$$

solution exist and hence it is consistent.

**6.(a) Choose the correct statement related to the matnces A=[1001],B=[0110]**

**(i) A3=A,B3≠B (ii) A3≠A,B3=B (iii) A3=A,B3=B (iv) A3≠A,B3≠B**

**(b) If M=[7253] then verity the equation $M^2$ – 10M + 11 $I_2$ = O**

**(c) Inverse of the matrix ⎡⎣⎢001110212⎤⎦⎥**

**Answer:**

(a) (iii) $A^3 = A, B^3 = B$

(b) $M^2 = \begin{bmatrix} 7 & 5 \\ 2 & 3 \end{bmatrix} \times \begin{bmatrix} 7 & 5 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 59 & 50 \\ 20 & 19 \end{bmatrix}$

$10M = \begin{bmatrix} 70 & 50 \\ 20 & 30 \end{bmatrix}; 11I = \begin{bmatrix} 11 & 0 \\ 0 & 11 \end{bmatrix}$

$M^2 - 10M + 11I_2$

$= \begin{bmatrix} 59 & 50 \\ 20 & 19 \end{bmatrix} - \begin{bmatrix} 70 & 50 \\ 20 & 30 \end{bmatrix} + \begin{bmatrix} 11 & 0 \\ 0 & 11 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(c) Cofactor matrix $= \begin{bmatrix} 2 & 1 & -1 \\ -2 & -2 & 1 \\ -1 & 0 & 0 \end{bmatrix}$

$AdjA = \begin{bmatrix} 2 & -2 & -1 \\ 1 & -2 & 0 \\ -1 & 1 & 0 \end{bmatrix}; |A| = \begin{vmatrix} 0 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{vmatrix} = -1$

$A^{-1} = \frac{AdjA}{|A|} = \frac{1}{-1}\begin{bmatrix} 2 & -2 & -1 \\ 1 & -2 & 0 \\ -1 & 1 & 0 \end{bmatrix}$

$= \begin{bmatrix} -2 & 2 & 1 \\ -1 & 2 & 0 \\ 1 & -1 & 0 \end{bmatrix}$

# CHAPTER-V

# CONTINUITY AND DIFFERENTIABILITY

## 2 MARK QUESTIONS

**1:Explain the continuity of the function f(x) = sin x . cos x**

**Solution:**

We know that sin x and cos x are continuous functions. It is known that the product of two continuous functions is also a continuous function.

Hence, the function f(x) = sin x . cos x is a continuous function.

**2:Determine the points of discontinuity of the composite function y = f[f(x)], given that, f(x) = 1/x-1.**

**Solution:**

Given that, f(x) = 1/x-1

We know that the function f(x) = 1/x-1 is discontinuous at x = 1

Now, for x ≠1,

f[f(x)]= f(1/x-1)

= 1/[(1/x-1)-1]

= x-1/ 2-x, which is discontinuous at the point x = 2.

Therefore, the points of discontinuity are x = 1 and x=2.

**3.Find the values of a and b such that the function defined as follows is continuous.**

$$f(x) = \begin{cases} x + 2\,, & x \leq 2 \\ ax + b\,, & 2 < x < 5 \\ 3x - 2, & x \geq 5 \end{cases}$$

Answer:
a = 3 and b = – 2

# 4 MARK QUESTIONS

**1:If f (x) = |cos x|, find f'(3π/4)**

**Solution:**

Given that, f(x) = |cos x|

When π/2 <x< π, cos x < 0,

Thus, |cos x| = -cos x

It means that, f(x) = -cos x

Hence, f'(x) = sin x

Therefore, f'(3π/4) = sin (3π/4) = 1/√2

f'(3π/4) = 1/√2

**2.Determine the value of 'k' for which the following function is continuous at x = 3:**

$$f(x) = \begin{cases} \dfrac{(x+3)^2 - 36}{x-3}, & x \neq 3 \\ k, & x = 3 \end{cases}$$

Answer:

Given, $f(x) = \begin{cases} \dfrac{(x+3)^2 - 36}{x-3}, & x \neq 3 \\ k, & x = 3 \end{cases}$

Let $f(x)$ is continuous at $x = 3$

Then, we have $\lim\limits_{x \to 3} f(x) = f(3)$

$\Rightarrow \quad \lim\limits_{x \to 3} \dfrac{(x+3)^2 - 36}{x-3} = k$

$\Rightarrow \quad \lim\limits_{x \to 3} \dfrac{(x+3)^2 - 6^2}{x-3} = k$

$\Rightarrow \quad \lim\limits_{x \to 3} \dfrac{(x+3-6)(x+3+6)}{x-3} = k$

$[\because a^2 - b^2 = (a-b)(a+b)]$

$\Rightarrow \quad \lim\limits_{x \to 3} \dfrac{(x-3)(x+9)}{(x-3)} = k$

$\Rightarrow \quad \lim\limits_{x \to 3} (x+9) = k$

$\Rightarrow \quad 3 + 9 = k \quad \Rightarrow \quad k = 12$

---

**3.**

**Find the value of k, so that the following function is continuous at x = 2. (Delhi**

$f(x) = \begin{cases} \dfrac{x^3 + x^2 - 16x + 20}{(x-2)^2}, & x \neq 2 \\ k, & x = 2 \end{cases}$

---

Answer:

$$\text{Let } f(x) = \begin{cases} \dfrac{x^3 + x^2 - 16x + 20}{(x-2)^2}; & x \neq 2 \\ k; & x = 2 \end{cases}$$

is continuous at $x = 2$.

Now, we have $f(2) = k$

and $\lim\limits_{x \to 2} f(x) = \lim\limits_{x \to 2} \dfrac{x^3 + x^2 - 16x + 20}{(x-2)^2}$

$$= \lim_{x \to 2} \frac{(x-2)(x^2 + 3x - 10)}{(x-2)^2}$$

$$= \lim_{x \to 2} \frac{(x-2)(x+5)(x-2)}{(x-2)^2}$$

$$= \lim_{x \to 2} (x+5) = 2 + 5 = 7$$

Since, $f(x)$ is continuous at $x = 2$

$\therefore \lim\limits_{x \to 2} f(x) = f(2) \Rightarrow 7 = k \quad \Rightarrow \quad k = 7$

---

**4.**

**Find the value of k, so that the function f defined by**

$$f(x) = \begin{cases} \dfrac{k \cos x}{\pi - 2x}, & \text{if } x \neq \dfrac{\pi}{2} \\ 3, & \text{if } x = \dfrac{\pi}{2} \end{cases}$$

is continuous at x = π2. (Delhi 2012C; Foregin 2011)

Answer:

Let $f(x) = \begin{cases} \dfrac{k \cos x}{\pi - 2x}, & \text{if } x \neq \dfrac{\pi}{2} \\ 3, & \text{if } x = \dfrac{\pi}{2} \end{cases}$

is continuous at $x = \pi/2$

Then, at $x = \dfrac{\pi}{2}$, LHL = RHL = $f\left(\dfrac{\pi}{2}\right)$ ...(i)

Now, $\text{LHL} = \lim\limits_{x \to \frac{\pi}{2}^-} f(x) = \lim\limits_{x \to \frac{\pi}{2}^-} \dfrac{k \cos x}{\pi - 2x}$

$$\Rightarrow \quad \text{LHL} = \lim_{h \to 0} \frac{k \cos\left(\frac{\pi}{2} - h\right)}{\pi - 2\left(\frac{\pi}{2} - h\right)}.$$

$$\left[\text{put } x = \frac{\pi}{2} - h;\ \text{when } x \to \frac{\pi^-}{2}, \text{ then } h \to 0\right]$$

$$= \lim_{h \to 0} \frac{k \sin h}{\pi - \pi + 2h} \quad \left[\because \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta\right]$$

$$= \lim_{h \to 0} \frac{k \sin h}{2h} = \frac{k}{2} \lim_{h \to 0} \frac{\sin h}{h}$$

$$\Rightarrow \text{LHL} = \frac{k}{2} \qquad \left[\because \lim_{h \to 0} \frac{\sin h}{h} = 1\right]$$

Also, from the given function, we get

$$f\left(\frac{\pi}{2}\right) = 3$$

Now, from Eq. (i), we have

$$\text{LHL} = f\left(\frac{\pi}{2}\right) \Rightarrow \frac{k}{2} = 3$$

$\therefore \qquad k = 6$

# 7 MARK QUESTIONS

**1:Verify the mean value theorem for the following function f (x) = (x – 3) (x – 6) (x – 9) in [3, 5]**

**Solution:**

$f(x)=(x-3)(x-6)(x-9)$

$=(x-3)(x^2-15x+54)$

$=x^3-18x^2+99x-162$

$fc\in(3,5)$

$f'(c)=f(5)-f(3)/5-3$

$f(5)=(5-3)(5-6)(5-9)$

$=2(-1)(-4)=-8$

$f(3)=(3-3)(3-6)(3-9)=0$

$f'(c)=8-0/2=4$

$\therefore f'(c)=3c^2-36c+99$

$3c^2-36c+99=4$

$3c^2-36c+95=0$

$ax^2+bx+c=0$

$a=3$

$b=-36$

$c=95$

$c=36\pm\sqrt{(36)^2-4(3)(95)}/2(3)$

$=36\pm\sqrt{1296-1140}/6$

$=36\pm12.496$

$c=8.8$ & $c=4.8$

$c\in(3,5)$

$f(x)=(x-3)(x-6)(x-9)$ on $[3,5]$

**2.Explain the continuity of the function f = |x| at x = 0.**

**Solution:**

From the given function, we define that,

$f(x) = \{-x$, if $x<0$ and $x$, if $x\geq0$

It is clearly mentioned that the function is defined at 0 and f(0) = 0. Then the left-hand limit of f at 0 is

$\text{Lim}_{x\to0^-} f(x)= \lim_{x\to0^-} (-x) = 0$

Similarly for the right hand side,

$\text{Lim}_{x\to0^+} f(x)= \lim_{x\to0^+} (x) = 0$

Therefore, for the both left hand and the right hand limit, the value of the function coincide at the point x = 0.

Therefore, the function f is continuous at the point x =0.

**3:If y= tan x + sec x , then show that $d^2y / dx^2 = \cos x / (1-\sin x)^2$**

**Solution:**

Given that, y= tan x + sec x

Now, the differentiate wih respect to x, we get

$dy/dx = \sec^2 x + \sec x \tan x$

$= (1/ \cos^2 x) + (\sin x/ \cos^2 x)$

$= (1+\sin x)/ (1+\sin x)(1-\sin x)$

Thus, we get.

$dy/dx = 1/(1-\sin x)$

Now, again differentiate with respect to x, we will get

$d^2y / dx^2 = -(-\cos x )/(1- \sin x)^2$

$d^2y / dx^2 = \cos x / (1-\sin x)^2$.

**4.Find the values of p and q for which**

$$f(x) = \begin{cases} \dfrac{1-\sin^3 x}{3\cos^2 x}, & \text{if } x < \dfrac{\pi}{2} \\ p, & \text{if } x = \dfrac{\pi}{2} \\ \dfrac{q(1-\sin x)}{(\pi - 2x)^2}, & \text{if } x > \dfrac{\pi}{2} \end{cases}$$

is continuous at x = π2. (Delhi 2016)
Answer:

Let $f(x) = \begin{cases} \dfrac{1-\sin^3 x}{3\cos^2 x}, & \text{if } x < \dfrac{\pi}{2} \\ p, & \text{if } x = \dfrac{\pi}{2} \\ \dfrac{q(1-\sin x)}{(\pi-2x)^2}, & \text{if } x > \dfrac{\pi}{2} \end{cases}$

is continuous at $x = \dfrac{\pi}{2}$.

Then, $(\text{LHL})_{x=\frac{\pi}{2}} = (\text{RHL})_{x=\frac{\pi}{2}} = f\left(\dfrac{\pi}{2}\right)$ ...(i)

Now, LHL $= \lim\limits_{x \to \frac{\pi}{2}^-} f(x) = \lim\limits_{h \to 0} f\left(\dfrac{\pi}{2} - h\right)$

$$\left[\text{put } x = \frac{\pi}{2} - h; \text{ when } x \to \frac{\pi^-}{2}, \text{ then } h \to 0\right]$$

$$= \lim_{h \to 0} \frac{1 - \sin^3\left(\frac{\pi}{2} - h\right)}{3\cos^2\left(\frac{\pi}{2} - h\right)} = \lim_{h \to 0} \frac{1-\cos^3 h}{3\sin^2 h}$$

$$\left[\because \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta, \sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta\right]$$

$$= \lim_{h \to 0} \frac{(1-\cos h)(1^2 + \cos^2 h + 1 \times \cos h)}{3(1-\cos^2 h)}$$

$$= \lim_{h \to 0} \frac{(1-\cos h)(1+\cos^2 h + \cos h)}{3(1-\cos h)(1+\cos h)}$$

$$= \lim_{h \to 0} \frac{(1+\cos^2 h + \cos h)}{3(1+\cos h)}$$

$$= \frac{1 + \cos^2 0 + \cos 0}{3(1+\cos 0)} = \frac{1+1+1}{3(1+1)} = \frac{3}{3 \times 2} = \frac{1}{2}$$

...(ii)

and RHL $= \lim\limits_{x \to \frac{\pi}{2}^+} f(x) = \lim\limits_{h \to 0} f\left(\dfrac{\pi}{2} + h\right)$

$$\left[\text{put } x = \frac{\pi}{2} + h; \text{ when } x \to \frac{\pi^+}{2}, \text{ then } h \to 0\right]$$

$$= \lim_{h \to 0} \frac{q\left[1 - \sin\left(\frac{\pi}{2} + h\right)\right]}{\left[\pi - 2\left(\frac{\pi}{2} + h\right)\right]^2}$$

$$= \lim_{h \to 0} \frac{q(1-\cos h)}{(\pi-\pi-2h)^2} = \lim_{h \to 0} \frac{q(1-\cos h)}{4h^2}$$

$$= \frac{q}{8} \lim_{h \to 0} \left[ \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \right]^2$$

$$= \frac{q}{8} \times 1 = \frac{q}{8} \qquad \left[\because \lim_{x \to 0} \frac{\sin x}{x} = 1\right] \ldots \text{(iii)}$$

On substituting the values from Eqs. (ii) and (iii) to Eq. (i), we get

$$\frac{1}{2} = \frac{q}{8} = f\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \qquad \frac{1}{2} = \frac{q}{8} = p \qquad \left[\because f\left(\frac{\pi}{2}\right) = p \text{ (given)}\right]$$

$$\Rightarrow \qquad \frac{1}{2} = \frac{q}{8} \text{ and } \frac{1}{2} = p$$

$$\therefore \qquad q = 4 \text{ and } p = \frac{1}{2}$$

**5.Find the value of k, so that the function**

$$f(x) = \begin{cases} \left(\dfrac{1 - \cos 4x}{8x^2}\right), & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases}$$

is continuous at x = 0. (All India 2014C).

Let $f(x) = \begin{cases} \left(\dfrac{1-\cos 4x}{8x^2}\right), & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases}$

is continuous at $x = 0$.

Then, $(\text{LHL})_{x=0} = (\text{RHL})_{x=0} = f(0)$ ...(i)

Now, $\text{LHL} = \lim_{x \to 0^-} f(x)$

$$= \lim_{x \to 0^-} \frac{1-\cos 4x}{8x^2}$$

$$= \lim_{h \to 0} \frac{1-\cos 4(0-h)}{8(0-h)^2}$$

[put $x = 0-h$; when $x \to 0^-$, then $h \to 0$]

$$= \lim_{h \to 0} \frac{1-\cos 4h}{8h^2} \quad [\because \cos(-\theta) = \cos\theta]$$

$$= \lim_{h \to 0} \frac{2\sin^2 2h}{8h^2} \quad [\because 1-\cos 2\theta = 2\sin^2\theta]$$

$$= \lim_{h \to 0} \frac{\sin^2 2h}{4h^2} = \lim_{h \to 0} \left(\frac{\sin 2h}{2h}\right)^2 = 1$$

$$\left[\because \lim_{x \to 0} \frac{\sin x}{x} = 1\right]$$

On substituting this value in Eq. (i), we get

$$1 = f(0) \Rightarrow 1 = k \quad [\because f(0) = k, \text{ (given)}]$$

Hence, for $k = 1$, the given function $f(x)$ is continuous at $x = 0$.

Alternate Method:

Let $f(x) = \begin{cases} \left(\dfrac{1-\cos 4x}{8x^2}\right), & \text{if } x = 0 \\ k, & \text{if } x = 0 \end{cases}$ is

continuous at $x = 0$.

Here, $f(0) = k$

and $\lim_{x\to 0} f(x) = \lim_{x\to 0} \dfrac{1-\cos 4x}{8x^2} = \lim_{x\to 0} \dfrac{2\sin^2 2x}{8x^2}$

$$= \lim_{x\to 0}\left(\frac{\sin 2x}{2x}\right)^2 = 1$$

$\because$ $f(x)$ is continuous at $x = 0$.

$\therefore$ $\lim_{x\to 0} f(x) = f(0) \Rightarrow 1 = k \Rightarrow k = 1$

**6.Find the value of k, for which**

$$f(x) = \begin{cases} \dfrac{\sqrt{1+kx}-\sqrt{1-kx}}{x}, & \text{if} -1 \le x < 0 \\ \dfrac{2x+1}{x-1}, & \text{if } 0 \le x < 1 \end{cases}$$

is continuous at x = 0. (All India 2013)

Answer:

$$\text{Let } f(x) = \begin{cases} \dfrac{\sqrt{1+kx} - \sqrt{1-kx}}{x}, & \text{if } -1 \le x < 0 \\[2ex] \dfrac{2x+1}{x-1}, & \text{if } 0 \le x < 1 \end{cases}$$

is continuous at $x = 0$.

Now, $f(0) = \dfrac{2 \cdot 0 + 1}{0 - 1} = \dfrac{1}{-1} = -1$

and $\text{LHL} = \lim_{h \to 0} f(0 - h)$

$$= \lim_{h \to 0} \frac{\sqrt{1-kh} - \sqrt{1+kh}}{-h}$$

$$= \lim_{h \to 0} \frac{\sqrt{1-kh} - \sqrt{1+kh}}{-h} \times \frac{(\sqrt{1-kh} + \sqrt{1+kh})}{(\sqrt{1-kh} + \sqrt{1+kh})}$$

$$= \lim_{h \to 0} \frac{(1-kh) - (1+kh)}{-h(\sqrt{1-kh} + \sqrt{1+kh})}$$

$$[\because (a+b)(a-b) = a^2 - b^2]$$

$$= \lim_{h \to 0} \frac{-2kh}{-h(\sqrt{1-kh} + \sqrt{1+kh})}$$

$$= \lim_{h \to 0} \frac{2k}{\sqrt{1-kh} + \sqrt{1+kh}} = \frac{2k}{1+1} = \frac{2k}{2} = k$$

Since, $f(x)$ is continuous at $x = 0$.

$\therefore \quad f(0) = \text{LHL} \Rightarrow -1 = k$

$\Rightarrow \quad k = -1$

# CHAPTER-VI

# APPLICATIONS AND DERIVATIVES

## 2 MARK QUESTIONS

**1:For the given curve: $y = 5x – 2x^3$, when x increases at the rate of 2 units/sec, then how fast is the slope of curve changes when $x = 3$?**

**Solution:**

Given that, $y = 5x – 2x^3$

Then, the slope of the curve, $dy/dx = 5-6x^2$

$\Rightarrow d/dt\ [dy/dx]= -12x.\ dx/dt$

$= -12(3)(2)$

$= -72$ units per second

Hence, the slope of the curve is decreasing at the rate of 72 units per second when x is increasing at the rate of 2 units per second.

**2.The total revenue received from the sale of x units of a product is given by $R(x) = 3x^2 + 36x + 5$ in rupees. Find the marginal revenue when $x = 5$, where by marginal revenue we mean the rate of change of total revenue with respect to the number of items sold at an instant.**

**Answer:**
Marginal Rcvcnuc (MR) = dRdx=ddx$(3x^2 + 36x + 5)$
$= 6x + 36$
∴ When $x = 5$
Marginai Revenue (MR) $= 6 \times 5 + 36 = 66$

**3.The volume of a cube is increasing at the rate of 8 $cm^3/s$. How fast is the surface area increasing when the length of its edge is 12 cm?**

**Answer:**

Let x be the length of an edge of the cube, V be the volume and S be the surface area at any time t.
Then, $V = x^3$ and $S = 6x^2$.

It is given that,

$$\frac{dV}{dt} = 8 \text{ cm}^3/\text{sec} \Rightarrow \frac{d}{dt}(x^3) = 8$$

$$\Rightarrow \quad 3x^2 \frac{dx}{dt} = 8 \Rightarrow \frac{dx}{dt} = \frac{8}{3x^2}$$

Now, $S = 6x^2$

$$\Rightarrow \quad \frac{dS}{dt} = 12x \frac{dx}{dt} \Rightarrow \frac{dS}{dt} = 12x \times \frac{8}{3x^2}$$

$$\Rightarrow \quad \frac{dS}{dt} = \frac{32}{x}$$

$$\Rightarrow \left(\frac{dS}{dt}\right)_{x=12} = \frac{32}{12} \text{ cm}^2/\text{sec} = \frac{8}{3} \text{ cm}^2/\text{sec}$$

**4.Find the intervals in which the function given by ;**
**$f(x) = 310\,x^4 - 45x^3 - 3x^2 + 36x5 + 11$ is**
**(i) strictly increasing.**
**(ii) strictly decreasing.**

**Answer:**

(i) Strictly increasing in (-2, 1) and (3, ∞).
(ii) Strictly decreasing in (-∞,- 2) and (1, 3).

**5.The sides of an equilateral triangle are increasing at the rate of 2 cm/s. Find the rate at which the area increases, when the side is 10 cm? (All India 2014C)**

**Answer:**

$10\sqrt{3}\ cm^2/s$

# 4 MARK QUESTIONS

**1:Show that the function $f(x) = \tan x - 4x$ is strictly decreasing on $[-\pi/3, \pi/3]$**

**Solution:**

Given that, $f(x) = \tan x - 4x$

Then, the differentiation of the function is given by:

$f'(x) = \sec^2 x - 4$

When $-\pi/3 < x\ \pi/3$, $1 < \sec x < 2$

Then, $1 < \sec^2 x < 4$

Hence, it becomes $-3 < (\sec^2 x - 4) < 0$

Hence, for $-\pi/3 < x\ \pi/3$, $f'(x) < 0$

Therefore, the function "f" is strictly decreasing on $[-\pi/3, \pi/3]$

**2:A stone is dropped into a quiet lake and waves move in the form of circles at a speed of 4 cm/sec. At the instant, when the radius of the circular wave is 10 cm, how fast is the enclosed area increasing?**

**Solution:**

We know that the area of a circle with radius "r" is given by $A = \pi r^2$.

Hence, the rate of change of area "A' with respect to the time "t" is given by:

$dA/dt = (d/dt)\ \pi r^2$

By using the chain rule, we get:

$(d/dr)(\pi r^2) . (dr/dt) = 2\pi r.(dr/dt)$

It is given that, $dr/dt = 4$ cm/sec

Therefore, when r = 10 cm,

dA/dt = 2π. (10). (4)

dA.dt = 80 π

Hence, when r = 10 cm, the enclosing area is increasing at a rate of 80π $cm^2$/sec.

**3:What is the equation of the normal to the curve y = sin x at (0, 0)?**

**(a)x =0 (b) y=0 (c)x+y =0 (d)x-y=0**

**Solution:**

A correct answer is an option (c)

**Explanation:**

Given that, y = sin x

Hence, dy/dx = cos x

Thus, the slope of the normal = $(-1/\cos x)_{x=0}$ = -1

Therefore, the equation of the normal is y-0 = -1(x-0) or x+y=0

Hence, the correct solution is option c.

.

**4.Show that the function f(x) = $4x^3 - 18x^2 + 27x - 7$ is always increasing on R.**

Answer:

We have, f(x) = $4x^3 - 18x^2 + 27x - 7$
On differentiating both sides w.r.t. x, we get
f(x) = $12x^2 - 36x + 27$
⇒ f'(x) = $3(4x^2 - 12x + 9)$
⇒ f'(x) = $3(2x - 3)^2$

$\Rightarrow f(x) > 0$
$\Rightarrow$ For any $x \in R$, $(2x - 3)^2 > 0$
Since, a perfect square number cannot be negative.
$\therefore$ Given function f(x) is an increasing function on R.

**5.Using differentials, find the approximate value of $(3.968)^{3/2}$. (Delhi 2014C)**

**Answer:**

Let $y = f(x) = (x)^{3/2}$
On differentiating both sides w.r.t. x, we get
Let $x = 4$ and $x + \Delta x = 3.968$
Then, $\Delta x = -0.032$
Now, $f(x + \Delta x)^{3/2} \approx f(x) + f'(x)\Delta x$
$(x + \Delta x)^{3/2} \approx (x)^{3/2} + 32.(x)^{1/2}.(-0.032)$
$\Rightarrow (4 - 0.032)^{3/2} \approx (4)^{3/2} + 32(4)^{1/2}(-0.032)$ [put x = 4]
$\Rightarrow (3.968)^{3/2} \approx 8 + 32\ .2.(-0.032)$
$\Rightarrow (39368)^{3/2} \approx 8 - 0.096$
$\Rightarrow (3.968)^{3/2} \approx 7.904$

**6.Find the approximate value of f(3.02), upto 2 places of decimal, where**
**$f(x) = 3x^2 + 15x + 3$. (Foreign 2014)**

**Answer:**

First, split 3.02 into two parts x and Ax, so that $x + \Delta x = 3.02$ and $f(x + \Delta x) = f(3.02)$
Now, write $f(x + \Delta x) = f(x) + \Delta x\ .\ f'(x)$ and use this result to find the required value.
Given function is $f(x) = 3x^2 + 15x + 3$
On differentiating both sides w.r.t. x, we get $f'(x) = 6x + 15$
Let $x = 3$ and $\Delta x = 0.02$
So that $f(x + \Delta x) - f(302)$
By using $f(x + \Delta x) \sim f(x) + \Delta x\ f'(x)$, we get
$f(x + \Delta x) = 3x^2 + 15x + 3 + (6x + 15)\ \Delta x$

$f(3 + 0.02) = 3(3)^2 + 15(3) + 3 + [6(3) + 15]\ (0.02)$

$= 27 + 45 + 3 + 33(0.02)$

$= 75 + 0.66$

$= 75.66$

Hence, $f(3.02) \approx 75.66$

**7.If the radius of sphere is measured as t 9 cm with an error of 0.03 cm, then find the approximate error in calculating its surface area.**

**Answer:**

Let S be the surface area, r be the radius of the sphere.
Given, r = 9 cm
Then, dr = Approximate error in radius r = 0.03 cm
and dS = Approximate error in surface area
Now, we know that surface area of sphere is given by
$S = 4\pi r^2$
On differentiating both sides w.r.t. r, we get
$\frac{dS}{dr} = 4\pi \times 2r = 8\pi r$
$dS = 8\pi r \times dr$
$\Rightarrow dS = 8\pi \times 9 \times 0.03$ [$\because$ r = 9 cm and dr = 0.03 cm ]
$\Rightarrow dS = 72 \times 0.03\pi$
$\therefore dS = 2.16\pi\ cm^2/cm$
Hence, approximate error in surface area is $2.16\pi\ cm^2/cm$.

# 7 MARK QUESTIONS

**1:Determine all the points of local maxima and local minima of the following function: $f(x) = (-\frac{3}{4})x^4 - 8x^3 - (45/2)x^2 + 105$**

**Solution:**

Given function: $f(x) = (-\frac{3}{4})x^4 - 8x^3 - (45/2)x^2 + 105$

Thus, differentiate the function with respect to x, we get

$f'(x) = -3x^3 - 24x^2 - 45x$

Now take, -3x as common:

$= -3x(x^2 + 8x + 15)$

Factorise the expression inside the bracket, then we have:

$= -3x(x+5)(x+3)$

$f'(x) = 0$

$\Rightarrow x = -5, x = -3, x = 0$

Now, again differentiate the function:

$f''(x) = -9x^2 - 48x - 45$

Take -3 outside,

$= -3(3x^2 + 16x + 15)$

Now, substitue the value of x in the second derivative function.

$f''(0) = -45 < 0$. Hence, x = 0 is point of local maxima

$f''(-3) = 18 > 0$. Hence, x = –3 is point of local minima

$f''(-5) = -30 < 0$. Hence, x = –5 is point of local maxima.

**2:A circular disc of radius 3 cm is being heated. Due to expansion, its radius increases at a rate of 0.05 cm per second. Find the rate at which its area is increasing if the radius is 3.2 cm.**

**Solution:**

Let us assume that "r" be the radius of the given disc and "A" be the area, then the area is given as:

$A = \pi r^2$

By using the chain rule,

Then $dA/dt = 2\pi r(dr/dt)$

Thus, the approximate rate of increase of radius = $dr = (dr/dt)\ \Delta t = 0.05$ cm per second

Hence, the approximate rate of increase in area is:

$dA = (dA/dt)(\Delta t) = 2\pi r[(dr/dt)\ \Delta t\ ]$

$= 2\pi\ (3.2)\ (0.05)$

$= 0.320\pi\ cm^2$ per second.

Therefore, when r= 3.2 cm, then the area is increasing at a rate of $0.320\pi\ cm^2$/second.

**3.Find the points on the curve**
**y = $[x(x – 2)]^2$, where the tangent is parallel to X-axis.**

**Answer:**

We have to find the points on the given curve where the tangent is parallel to X-axis. We know that, when a tangent is parallel to X-axis, then
dydx = 0
⇒ ddx$(x^2 – 2x)^2$ = 0
⇒ $2(x^2 – 2x)(2x – 2) = 0$
⇒ x = 0, 1, 2
When x = 0, then y = $[0(-2)]^2$ = 0
When x = 1, then y = $[1 – 2(1)]^2$ = 1
When x = 2, then y = $[2^2 – 2 \times 2]^2$ = 0
Hence, the tangent is parallel to X-axis at the points (0, 0), (1, 1) and (2, 0).

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{(x^2 - 5x + 6) \cdot 1 - (x - 7)(2x - 5)}{(x^2 - 5x + 6)^2}$$

$$\left[\because \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{\left[(x^2 - 5x + 6) - y(x^2 - 5x + 6)(2x - 5)\right]}{(x^2 - 5x + 6)^2}$$

$$\left[\begin{array}{l}\because \quad y = \dfrac{x - 7}{x^2 - 5x + 6} \\ \therefore (x - 7) = y(x^2 - 5x + 6)\end{array}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1 - (2x - 5)y}{x^2 - 5x + 6} \quad \ldots(ii)$$

[dividing numerator and denominator by $x^2 – 5x + 6$]

Also, given that curve cuts X-axis, so its y-coordinate is zero.
Put y = 0 in Eq. (i), we get
x−7x2−5x+6 = 0
⇒ x = 7
So, curve passes through the point (7, 0).
Now, slope of tangent at (7, 0) is
m = (dydx)(70)=1−049−35+6=120
Hence, the required equation of tangent passing through the point (7, 0) having slope 1/20 is
y – o = 120(x – 7)
⇒ 20y = x – 7
∴ x – 20y = 7

# CHAPTER-VII

# INTEGRALS

## 2 MARK QUESTIONS

**1:Write the anti-derivative of the following function: $3x^2+4x^3$**

**Solution:**

Given: $3x^2+4x^3$

The antiderivative of the given function is written as:

$\int 3x^2+4x^3\, dx = 3(x^3/3) + 4(x^4/4)$

$= x^3 + x^4$

Thus, the antiderivative of $3x^2+4x^3 = x^3 + x^4$

**2.Write the value of ∫dxx2+16**

**Answer:**

$$\text{Let } I = \int \frac{dx}{x^2+16} = \int \frac{dx}{x^2+(4)^2}$$

$$= \frac{1}{4}\tan^{-1}\frac{x}{4} + C$$

$$\left[\because \int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C\right]$$

# 4 MARK QUESTIONS

**1:Evaluate: $\int 3ax/(b^2+c^2x^2)\, dx$**

**Solution:**

To evaluate the integral, $I = \int 3ax/(b^2+c^2x^2)\, dx$

Let us take $v = b^2+c^2x^2$, then

$dv = 2c^2x\, dx$

Thus, $\int 3ax/(b^2+c^2x^2)\, dx$

$= (3ax/2c^2x)\int dv/v$

Now, cancel x on both numerator and denominator, we get

$= (3a/2c^2)\int dv/v$

$= (3a/2c^2) \log |b^2+c^2x^2| + C$

Where C is an arbitrary constant

**2:Determine $\int \tan^8x \sec^4 x\, dx$**

**Solution:**

Given: $\int \tan^8x \sec^4 x\, dx$

Let $I = \int \tan^8x \sec^4 x\, dx$ — (1)

Now, split $\sec^4x = (\sec^2x) (\sec^2x)$

Now, substitute in (1)

$I = \int \tan^8x (\sec^2x) (\sec^2x)\, dx$

$= \int \tan^8x (\tan^2 x +1) (\sec^2x)\, dx$

It can be written as:

$= \int \tan^{10}x \sec^2 x\, dx + \int \tan^8 x \sec^2 x\, dx$

Now, integrate the terms with respect to x, we get:

$I = (\tan^{11} x /11) + (\tan^9 x /9) + C$

Hence, $\int \tan^8 x \sec^4 x\, dx = (\tan^{11} x /11) + (\tan^9 x /9) + C$

**3.Write the value of ∫2−3sinxcos2x dx.**

**Answer:**

Let I = ∫2−3sinxcos2x dx
= ∫(2cos2x−3sinxcos2x) dx
$= \int (2 \sec^2 x - 3 \sec x \tan x)\, dx$
$= 2\int \sec^2 x\, dx - 3\int \sec x \tan x\, dx$
$= 2 \tan x - 3 \sec x + C$

# 7 MARK QUESTIONS

**1.Determine the antiderivative F of “f” , which is defined by f (x) = $4x^3 – 6$, where F (0) = 3.**

**Solution:**

Given function: $f(x) = 4x^3 – 6$

Now, integrate the function:

$\int 4x^3 – 6\, dx = 4(x^4/4)-6x + C$

$\int 4x^3 – 6\, dx = x^4 – 6x + C$

Thus, the antiderivative of the function, F is $x^4 – 6x + C$, where C is a constant

Also, given that, F(0) = 3,

Now, substitute x = 0 in the obtained antiderivative function, we get:

$(0)^4 – 6(0) + C = 3$

Therefore, C = 3.

Now, substitute C = 3 in antiderivative function

Hence, the required antiderivative function is $x^4 – 6x + 3$.

**2.Integrate the given function using integration by substitution: $2x \sin(x^2+1)$ with respect to x:**

**Solution:**

Given function: $2x \sin(x^2+1)$

We know that, the derivative of $x^2 + 1$ is $2x$.

Now, use the substitution method, we get

$x^2 + 1 = t$, so that $2x\, dx = dt$.

Hence, we get $\int 2x \sin(x^2+1)\, dx = \int \sin t\, dt$

$= -\cos t + C$

$= -\cos(x^2 + 1) + C$

Where C is an arbitrary constant

Therefore, the antiderivative of $2x \sin(x^2+1)$ using integration by substitution method is $= -\cos(x^2 + 1) + C$

**3.Integrate: $\int \sin^3 x \cos^2 x\, dx$**

**Solution:**

Given that, $\int \sin^3 x \cos^2 x\, dx$

This can be written as:

$\int \sin^3 x \cos^2 x\, dx = \int \sin^2 x \cos^2 x (\sin x)\, dx$

$=\int (1 - \cos^2 x) \cos^2 x (\sin x)\, dx$ —(1)

Now, substitute $t = \cos x$,

Then $dt = -\sin x\, dx$

Now, equation can be written as:

Thus, $\int \sin^3 x \cos^2 x\, dx = -\int (1-t^2)t^2\, dt$

Now, multiply $t^2$ inside the bracket, we get

$= -\int (t^2-t^4)\, dt$

Now, integrate the above function:

$= -[(t^3/3) - (t^5/5)] + C$ —(2)

Where C is a constant

Now, substitute t = cos x in (2)

$= -(⅓)\cos^3 x + (1/5)\cos^5 x + C$

Hence, $\int \sin^3 x \cos^2 x\, dx = -(⅓)\cos^3 x + (1/5)\cos^5 x + C$

**4.Find ∫sin2x−cos2xsinxcosx dx.**

**Answer:**

$$\text{Let } I = \int \frac{\sin^2 x - \cos^2 x}{\sin x \cdot \cos x}\, dx$$

$$= \int \left[ \frac{\sin^2 x}{\sin x \cdot \cos x} - \frac{\cos^2 x}{\sin x \cdot \cos x} \right] dx$$

$$= \int \left[ \frac{\sin x}{\cos x} - \frac{\cos x}{\sin x} \right] dx$$

$= \int(\tan x - \cot x)\, dx$

$= \int \tan x\, dx - \int \cot x\, dx$

$= \log|\sec x| - [-\log|\text{cosec } x|] + C$

$= \log|\sec x| + \log|\text{cosec } x| + C$

$= \log|\sec x \cdot \text{cosec } x| + C$

**5.Given, $\int e^x$ (tan x + 1) sec x dx = $e^x$ f(x) + C.**
**Write f(x) satisfying above. (All India 2012;**

**Answer:**

Use the relation $\int e^x$ [f(x) + f'(x)dx = $e^x$ f(x) + C and simplify it.

Given that $\int e^x$ (tan x + 1) sec x dx = $e^x \cdot$ f(x) + C
$\Rightarrow \int$ $e^x$ (sec x + sec x tan x)dx = $e^x$ f(x) + C
$\Rightarrow$ $e^x$ . sec x + C = $e^x$ f(x) + C
[$\because$ $e^x$ {f(x) + f'(x)}dx = $e^x$ f(x) + C and here ddx (sec x) = sec x tan x]
On comparing both sides, we get
f(x) = sec x

# CHAPTER-VIII

# APPLICATIONS OF THE INTEGRALS

## 2 MARK QUESTIONS

**1.Using integration, find the area of triangle whose vertices are (2, 3), (3, 5) and (4, 4).**

**Answer:**

32 sq units.

**2.Using integration, find the area of region bounded by the triangle whose vertices are (- 2, 1), (0, 4) and (2, 3).**

**Answer:**

4 sq units

**3.Using the method of integration, find the area of the ΔABC, coordinates of whose vertices are A (4, 1), B(6, 6) and C (8, 4).**

**Answer:**

7 sq units

**4.Using integration, find the area of the triangular region whose vertices are (2, – 2), (4, 3) and (1, 2).**

**Answer:**

132 sq units

**5.Using integration, find the area of the region in the first quadrant enclosed by the X-axis, the line y = x and the circle $x^2 + y^2 = 18$.**

**Answer:**

9π4 sq units

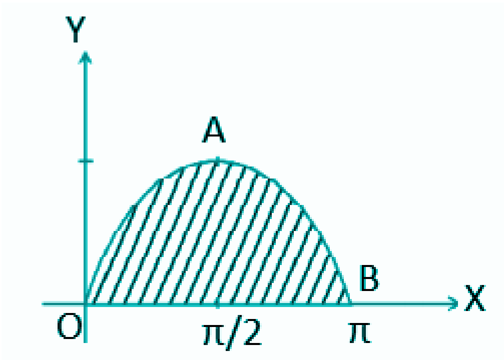
# 4 MARK QUESTIONS

**1: Find the area of the curve y = sin x between 0 and π.**

**Solution:**

Given,

y = sin x




Area of OAB

$$= \int_0^{\pi} y\, dx$$

$$= \int_0^{\pi} \sin x\, dx$$

$$= [-\cos x]_0^{\pi}$$

= – [cos π – cos 0]

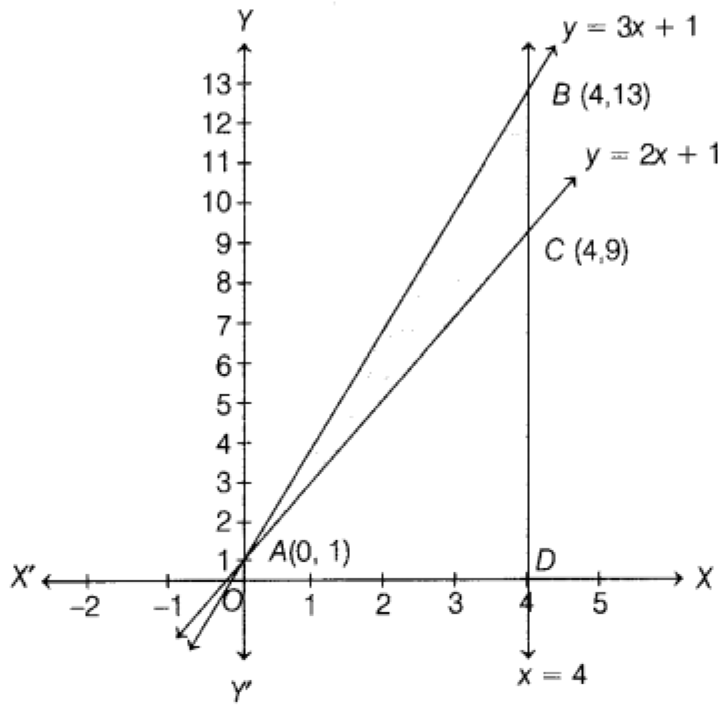
= -(-1 -1)

= 2 sq. units

**2.Using integration, find the area of the triangular region whose sides have the equations y = 2x + 1, y = 3x + 1 and x = 4.**

**Answer:**

Given, equation of sides are
y = 2x + 1, y = 3x + 1 and x = 4
On drawing the graph of these equations, we get the following triangular region




By solving these equations we get the vertices of triangle as A(0, 1), B(4, 13) and C(4, 9).
∴ Required area = Area (OABDO) – area (OACDO)

$$= \int_0^4 (3x+1)\,dx - \int_0^4 (2x+1)\,dx$$

$$= \left[\frac{3x^2}{2} + x\right]_0^4 - [x^2 + x]_0^4$$

$$= \frac{3 \times 4^2}{2} + 4 - 0 - (4^2 + 4 - 0)$$

$$= 24 + 4 - 20$$

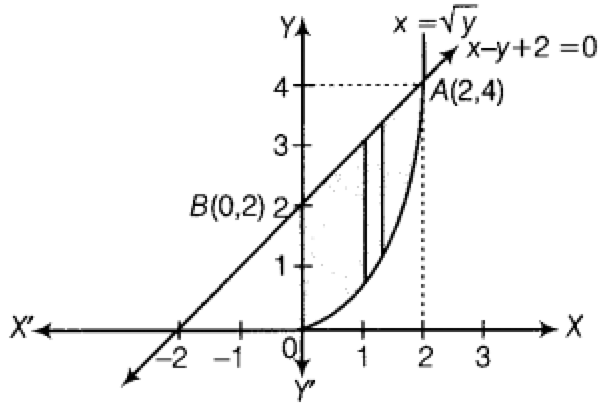
$$= 8 \text{ sq units}$$

**3.Using integration, find the area of the region bounded by the line $x - y + 2 = 0$, the curve $x = \sqrt{y}$ and Y-axis.**

**Answer:**

Given curves are
$x - y + 2 = 0$ ….. (i)
and $x = \sqrt{y}$ ….. (ii)
Consider $x = \sqrt{y} \Rightarrow x^2 = y$, which represents the parabola whose vertex is (0, 0) and axis is Y-axis.
Now, the point of intersection of Eqs.(i) and (ii)
is given by x = x+2−−−−√
$\Rightarrow x^2 = x + 2$
$\Rightarrow x^2 - x - 2 = 0$
$\Rightarrow (x - 2)(x + 1) = 0$
$\Rightarrow x = -1, 2$
But $x = -1$ does not satisfy the Eq. (ii).
$\therefore x = 2$
Now, putting $x = 2$ in Eq. (ii), we get
$2 = \sqrt{y} \Rightarrow y = 4$
Hence, the point of intersection is (2, 4).
But actual equation of given parabola is $x = \sqrt{y}$, it means a semi-parabola which is on right side of Y – axis.

The graph of given curves are shown below:




Clearly, area of bounded region

$$= \text{Area of region } OABO$$

$$= \int_0^2 [y_{\text{(line)}} - y_{\text{(parabola)}}]dx$$

$$= \int_0^2 (x+2)dx - \int_0^2 x^2 dx$$

$$= \left[\frac{x^2}{2} + 2x\right]_0^2 - \left[\frac{x^3}{3}\right]_0^2 = \left[\frac{4}{2} + 4 - 0\right] - \left[\frac{8}{3} - 0\right]$$

$$= 6 - \frac{8}{3} = \frac{18-8}{3} = \frac{10}{3} \text{ sq units}$$

**4.Find the area of the region {(x, y): $y^2 \le 4x$, $4x^2 + 4y^2 \le 9$}, using method of integration.**
**Answer:**

First, find the intersection points of given curves and then draw a rough diagram to represent the required area. If it is symmetrical about X-axis or Y-axis, then we first find area of only one portion , from them and then required area is twice of that area.

Given curves are
$y^2 = 4x$ …… (i)
and $4x^2 + 4y^2 = 9$
⇒ $x^2 + y^2 = 94$ …… (ii)
Eq. (i) represents a parabola having vertex (0, 0) and axis is X-axis and Eq. (ii)

represents a circle having centre (0, 0) and radius 32.

On substituting $y^2 = 4x$ in Eq. (ii), we get

$x^2 + 4x = 94$

$\Rightarrow 4x^2 + 18x – 2x – 9 = 0$

$\Rightarrow 2x(2x + 9) – 1 (2x + 9) = 0$

$\Rightarrow (2x + 9) (2x – 1) = 0$

$\Rightarrow x = 12,−92$

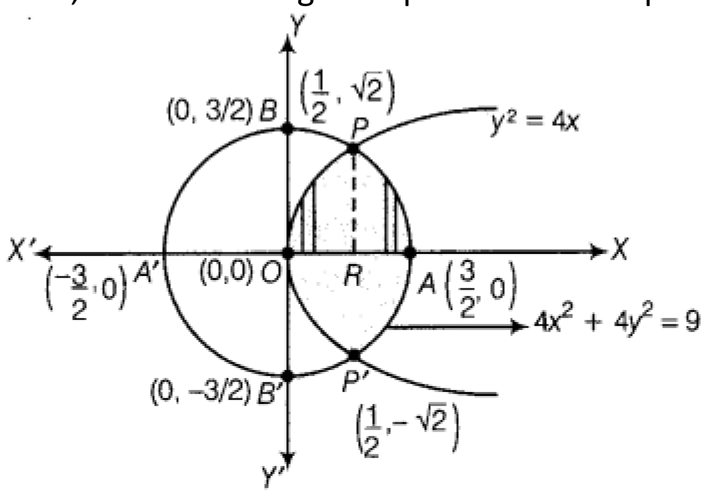
On putting x = 12 in Eq. (i), we get

$y = \pm\sqrt{2}$

At x = –92, y have imaginary values.

So, intersection points are p(12,2–√) and p(12,−2–√).

Now, the shaded region represents the required region as shown below:



∴ Required area = 2[Area of the region ORPO + Area of the region RAPR]

$$= 2\left[\int_0^{1/2} y_{\text{(parabola)}}\,dx + \int_{1/2}^{3/2} y_{\text{(circle)}}\,dx\right]$$

$$= 2\left[\int_0^{1/2} \sqrt{4x}\,dx + \int_{1/2}^{3/2} \sqrt{\frac{9}{4} - x^2}\,dx\right]$$

$$= 2\left\{\left[2\left(\frac{2}{3}x^{3/2}\right)\right]_0^{1/2} + \left[\frac{x}{2}\sqrt{\frac{9}{4} - x^2} + \frac{9}{8}\sin^{-1}\left(\frac{2x}{3}\right)\right]_{1/2}^{3/2}\right\}$$

$$\left[\because \int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$$

$$= 2\left[\frac{4}{3}\left\{\left(\frac{1}{2}\right)^{3/2} - 0\right\} + \left\{0 + \frac{9}{8}\sin^{-1}(1) - \frac{1}{4}\cdot\sqrt{2} - \frac{9}{8}\sin^{-1}\frac{1}{3}\right\}\right]$$

$$= 2\left[\frac{4}{3}\times\frac{1}{2\sqrt{2}} + \frac{9}{8}\times\frac{\pi}{2} - \frac{\sqrt{2}}{4} - \frac{9}{8}\sin^{-1}\frac{1}{3}\right]$$

$$= 2\left[\frac{2}{3\sqrt{2}} + \frac{9\pi}{16} - \frac{1}{2\sqrt{2}} - \frac{9}{8}\sin^{-1}\frac{1}{3}\right]$$

$$= 2\left[\frac{4-3}{6\sqrt{2}} + \frac{9\pi}{16} - \frac{9}{8}\sin^{-1}\frac{1}{3}\right]$$

$$= 2\left[\frac{1}{6\sqrt{2}} + \frac{9\pi}{16} - \frac{9}{8}\sin^{-1}\frac{1}{3}\right]$$

$$= \left[\frac{1}{3\sqrt{2}} + \frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\frac{1}{3}\right] \text{ sq units}$$

# 7 MARK QUESTIONS

**1: Find the area enclosed by the ellipse $x^2/a^2 + y^2/b^2 = 1$.**

**Solution:**

Given,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$\frac{y^2}{b^2} = \frac{a^2 - x^2}{a^2}$$

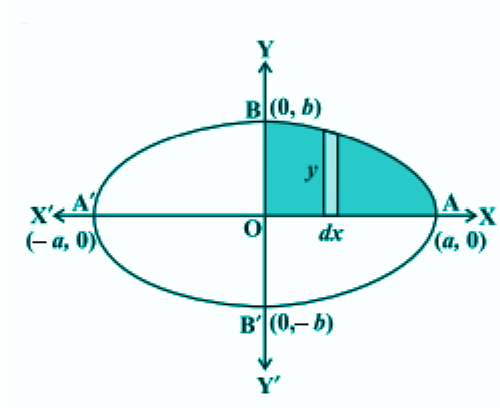
$$y^2 = \frac{b^2}{a^2}(a^2 - x^2)$$

$$y = \pm\sqrt{\frac{b^2}{a^2}(a^2 - x^2)}$$

$$y = \pm\frac{b}{a}\sqrt{(a^2 - x^2)}$$

We know that,

Ellipse is symmetrical about both x-axis and y-axis.




Area of ellipse = 4 × Area of AOB

$= 4 \times \int_0^a y\, dx$

Substituting the positive value of y in the above expression since OAB lies in the first quadrant.

$$= 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}dx$$

$$= \frac{4b}{a}\int_0^a \sqrt{a^2 - x^2}dx$$

$$= \frac{4b}{a}\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= \frac{4b}{a}\left[\left(\frac{a}{2}\sqrt{a^2 - a^2} + \frac{a^2}{2}\sin^{-1}\frac{a}{a}\right) - \left(\frac{0}{2}\sqrt{a^2 - 0} - \frac{a^2}{2}\sin^{-1}(0)\right)\right]$$

$$= \frac{4b}{a}\left[0 + \frac{a^2}{2}\sin^{-1}(1) - 0 - 0\right]$$

$$= \frac{4b}{a} \times \frac{a^2}{2}\sin^{-1}(1)$$

$= 2ab \times \sin^{-1}(1)$

$= 2ab \times \pi/2$

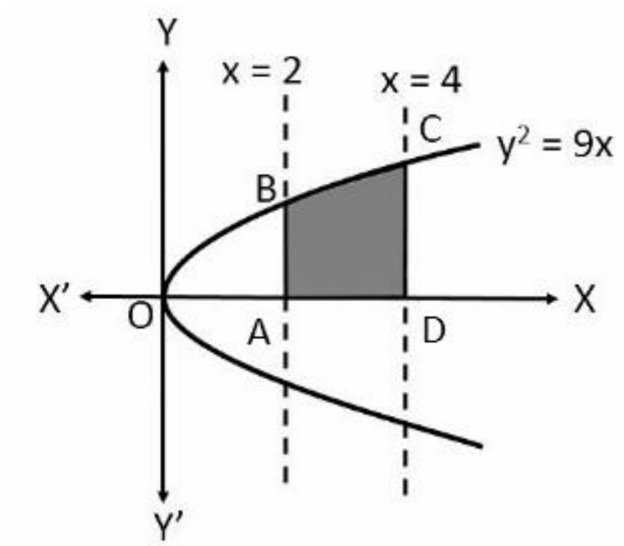
$= \pi ab$

Hence, the required area is πab sq.units.

**2: Find the area of the region bounded by $y^2 = 9x$, $x = 2$, $x = 4$ and the x-axis in the first quadrant.**

**Solution:**

We can draw the figure of $y^2 = 9x$; $x = 2$, $x = 4$ and the x-axis in the first curve as below.




$y^2 = 9x$

$y = \pm\sqrt{(9x)}$

$y = \pm 3\sqrt{x}$

We can consider the positive value of y since the required area is in the first quadrant.

The required area is the shaded region enclosed by ABCD.

$= \int_2^4 y \, . dx$

$= 3\int_2^4 \sqrt{x}\, dx$

$= 3\int_2^4 (x)^{\frac{1}{2}} dx$

$= 3\left[\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_2^4$

$= 3\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_2^4$

$= 3 \times \frac{2}{3}\left[x^{\frac{3}{2}}\right]_2^4$

$= 2\left[(4)^{\frac{3}{2}} - (2)^{\frac{3}{2}}\right]$

$= 2\left[\left((4)^{\frac{1}{2}}\right)^3 - \left((2)^{\frac{1}{2}}\right)^3\right]$

$= 2\,[(2)^3 - (\sqrt{2})^3]$

$= 2[8 - 2\sqrt{2}]$

$= 16 - 4\sqrt{2}$

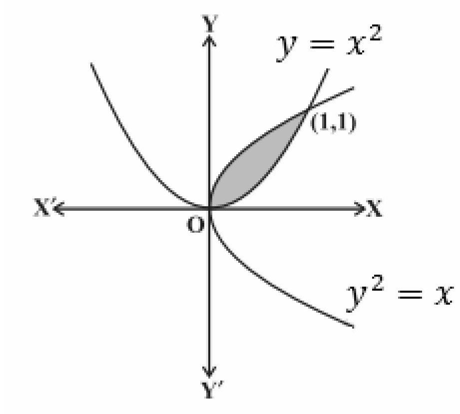
Hence, the required area is $16 - 4\sqrt{2}$ sq.units.

**3: Find the area of the region bounded by the two parabolas $y = x^2$ and $y^2 = x$.**

**Solution:**

Given two parabolas are $y = x^2$ and $y^2 = x$.

The point of intersection of these two parabolas is O (0, 0) and A (1, 1) as shown in the below figure.




Now,

$y^2 = x$

$y = \sqrt{x} = f(x)$

$y = x^2 = g(x)$, where, $f(x) \geq g(x)$ in [0, 1].

Area of the shaded region

$$= \int_0^1 [f(x) - g(x)]dx$$

$$= \int_0^1 \left[\sqrt{x} - x^2\right]dx$$

$$= \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{x^3}{3}\right]_0^1$$

= (⅔) – (⅓)

= ⅓

Hence, the required area is ⅓ sq.units.

**4: Smaller area enclosed by the circle $x^2 + y^2 = 4$ and the line $x + y = 2$ is**

**(A) $2(\pi - 2)$**

**(B) $\pi - 2$**

**(C) $2\pi - 1$**

**(D) $2(\pi + 2)$**

**Solution:**

Option (B) is the correct answer.

Explanation:

Given,

Equation of circle is $x^2 + y^2 = 4$……….(i)

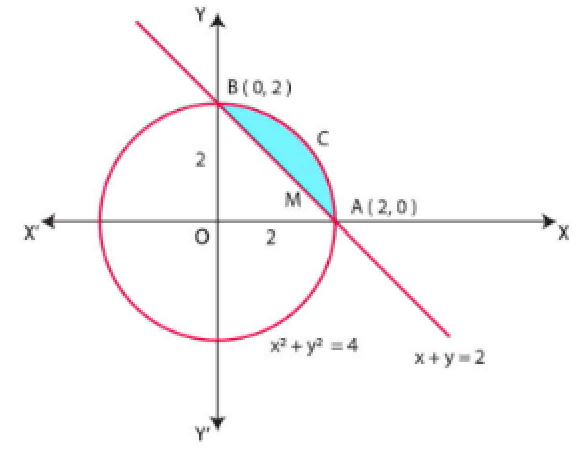
$x^2 + y^2 = 2^2$

$y = \sqrt{(2^2 - x^2)}$ …………(ii)

Equation of a lines is $x + y = 2$ ………(iii)

$y = 2 - x$

| | | |
|---|---|---|
| X | 0 | 2 |
| Y | 2 | 0 |




Therefore, the graph of equation (iii) is the straight line joining the points (0, 2) and (2, 0).

From the graph of a circle (i) and straight-line (iii), it is clear that points of intersections of circle

(i) and the straight line (iii) is A (2, 0) and B (0.2).

Area of OACB, bounded by the circle and the coordinate axes is

$$= \int_0^2 \sqrt{2^2 - x^2}\, dx$$

$$= \left[\frac{x}{2}\sqrt{2^2 - x^2} + \frac{2^2}{2}\sin^{-1}\frac{x}{2}\right]_0^2$$

$$= \left[\frac{x}{2}\sqrt{4 - x^2} + 2\sin^{-1}\frac{x}{2}\right]_0^2$$

$$= \left[\frac{2}{2}\sqrt{4 - 2^2} + 2\sin^{-1}\frac{2}{2}\right] - \left[\frac{0}{2}\sqrt{4 - 0^2} + 2\sin^{-1}\frac{0}{2}\right]$$

$= [\ 1 \times \sqrt{0} + 2 \sin^{-1}(1) – 0\sqrt{4} – 2 \times 0]$

$= 2 \sin^{-1}(1)$

$= 2 \times \pi/2$

$= \pi$ sq. units

Area of triangle OAB, bounded by the straight line and the coordinate axes is

$= \int_0^2 y\, dx$

$= \int_0^2 (2 - x) dx$

$= \left[2x - \frac{x^2}{2}\right]_0^2$

$= 2 \times 2 - \frac{2^2}{2} - \left[2 \times 0 - \frac{0}{2}\right]$

$= 4 - 2 - 0 + 0$

= 2 sq.units

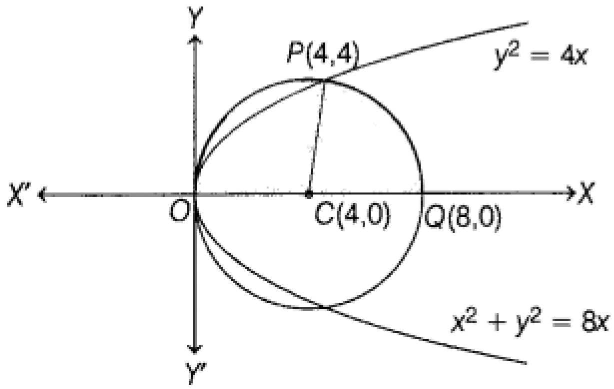
Hence, the required area = Area of OACB – Area of triangle OAB

= $(\pi - 2)$ sq.units

**5.Find the area of the region lying above X-axis and included between the circle $x^2 + y^2 = 8x$ and inside the parabola $y^2 = 4x$**
**Answer:**

The equation of circle is
$x^2 + y^2 = 8x$ …… (i)
and the equation of parabola is
$y^2 = 4x$ …… (ii)
Eq. (i) can be written as
$(x^2 - 8x) + y^2 = 0$
$\Rightarrow (x^2 - 8x + 16) + y2 = 16$
$\Rightarrow (x - 4)^2 + y^2 = (4)^2$ …… (iii)
which is a circle with centre C(4, 0) and radius = 4.
From Eqs. (i) and (ii), we get
$x^2 + 4x = 8x$
$\Rightarrow x^2 - 4x = 0$
$\Rightarrow x(x - 4) = 0$

⇒ x = 0, 4
Now, from Eq. (ii), we get
y = 0, 4
∴ Points of intersection of circle (i) and parabola (ii), above the A-axis, are 0(0, 0) and P(4, 4).




Now, required area = area of region OPQCO = (area of region OCPQ + (area of region PCQP)

$$= \int_0^4 y(\text{Parabola})\,dx + \int_4^8 y(\text{Circle})\,dx$$

$$= 2\int_0^4 \sqrt{x}\,dx + \int_4^8 \sqrt{(4)^2 - (x-4)^2}\,dx$$

[from Eqs. (ii), (iii)]

$$= 2\left[\frac{x^{3/2}}{3/2}\right]_0^4$$

$$+ \left[\frac{(x-4)}{2}\sqrt{(4)^2-(x-4)^2} + \frac{(4)^2}{2}\cdot\sin^{-1}\frac{x-4}{4}\right]_4^8$$

$$\left[\because \int\sqrt{a^2-x^2}dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C\right]$$

$$= \frac{4}{3}[x^{3/2}]_0^4$$

$$+ \left\{\left[\left(\frac{8-4}{2}\right)\sqrt{16-16} + 8\sin^{-1}(1)\right] - [0 + 8\sin^{-1}0]\right\}$$

$$= \frac{4}{3}[(4)^{3/2} - 0] + \left[0 + 8\times\frac{\pi}{2}\right] - [0+0]$$

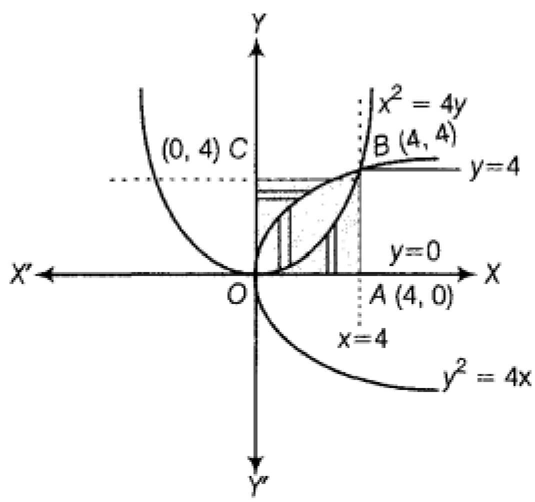
$$= \frac{4}{3}\times 8 + 4\pi = \frac{32}{3} + 4\pi = \frac{4}{3}(8+3\pi) \text{ sq units}$$

**6.Using integration, prove that the curves $y^2 = 4x$ and $x^2 = 4y$ divide the area of the square bounded by x = 0, x = 4, y = 4 and y = 0 into three equal parts.**

**Answer:**

First, we draw a square formed by the lines x = 0, x = 4, y = 4, and y = 0 and after that, we draw given parabolas which intersect each other on the square such that the whole region divided into three parts. Now, we find separately area of each part and show that area of each part is equal.

Let OABC be the square whose sides are represented by following equations
Equation of OA is y = 0
Equation of AB is x = 4
Equation of BC is y = 4
Equation of CO is x = 0




On solving equations $y^2 = 4x$ and $x^2 = 4y$, we get A(0, 0) and B(4, 4) as their points of intersection. Now, area bounded by these curves is given by.

$$\int_0^4 [y_{(\text{parabola } y^2=4x)} - y_{(\text{parabola} x^2=4y)}]\, dx$$

$$= \int_0^4 \left(2\sqrt{x} - \frac{x^2}{4}\right) dx = \left[2 \cdot \frac{2}{3} x^{3/2} - \frac{x^3}{12}\right]_0^4$$

$$= \left[\frac{4}{3} x^{3/2} - \frac{x^3}{12}\right]_0^4 = \frac{4}{3} \cdot (4)^{3/2} - \frac{64}{12}$$

$$= \frac{4}{3} \cdot (2^2)^{3/2} - \frac{64}{12} = \frac{4}{3} \cdot (2)^3 - \frac{64}{12}$$

$$= \frac{32}{3} - \frac{16}{3} = \frac{16}{3} \text{ sq units}$$

Hence, area bounded by curves $y^2 = 4x$ and $x^2 = 4y$ is \frac{16}{3} sq units ……. (i)
Now, area bounded by curve $x^2 = 4y$ and the lines x = 0, x = 4 and X-axis

$$= \int_0^4 y_{(\text{parabola } x^2=4y)}\, dx = \int_0^4 \frac{x^2}{4}\, dx$$

$$= \left[\frac{x^3}{12}\right]_0^4 = \frac{64}{12} = \frac{16}{3} \text{ sq units} \quad \ldots\text{(ii)}$$

Similarly, the area bounded by curve $y^2 = 4x$, the lines y = 0, y = 4 and Y-axis

$$= \int_0^4 x_{(\text{parabola } y^2=4x)}\, dy = \int_0^4 \frac{y^2}{4}\, dy$$

$$= \left[\frac{y^3}{12}\right]_0^4 = \frac{64}{12} = \frac{16}{3} \text{ sq units} \quad \ldots\text{(iii)}$$

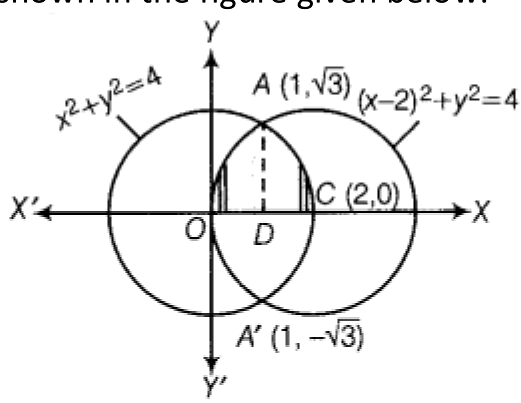
From Eqs. (i), (ii) and (iii), it is clear that area bounded by the parabolas $y^2 = 4x$ and $x^2 = 4y$ divides the area of square into three equal parts.
Hence proved.

**7.Using integration, find the area of the region enclosed between the two circles $x^2 + y^2 = 4$ and $(x - 2)^2 + y^2 = 4$.**

**Answer:**

First, find the intersecting points of two circles and then draw a rough sketch of these two circles. The common shaded region is symmetrical about X-axis. So, we find area of one part only, i.e. upper part of X-axis.
After that required area is twice of that area.
Given circles are
$x^2 + y^2 = 4$ ……… (i)
and $(x - 2)^2 + y^2 = 4$ …… (ii)

Eq. (i) is a circle with centre origin and radius 2, Eq. (ii) is a circle with centre C (2, 0) and radius 2.
On solving Eqs. (i) and (ii), we get
$(x - 2)^2 + y^2 = x^2 + y^2$
$\Rightarrow x^2 - 4x + 4 + y^2 = x^2 + y^2$
$\Rightarrow x = 1$
On putting x = 1 in Eq. (i), we get
$y = \pm \sqrt{3}$
Thus, the points of intersection of the given circles are A $(1, \sqrt{3})$ and A'$(1, -\sqrt{3})$ as shown in the figure given below:




Clearly, required area = Area of the enclosed region OACA'O between circles
= 2 [Area of the region ODCAO]

= 2 [Area of the region ODAO + Area of the region DCAD]

$$= 2\left[\int_0^1 y_2\,dx + \int_1^2 y_1\,dx\right]$$

[where, $y_1 = \sqrt{4-x^2}$ and $y_2 = \sqrt{4-(x-2)^2}$]

$$= 2\left[\int_0^1 \sqrt{4-(x-2)^2}\,dx + \int_1^2 \sqrt{4-x^2}\,dx\right]$$

$$= 2\left[\frac{1}{2}(x-2)\sqrt{4-(x-2)^2} + \frac{1}{2}\times 4\sin^{-1}\left(\frac{x-2}{2}\right)\right]_0^1$$

$$+ 2\left[\frac{1}{2}x\sqrt{4-x^2} + \frac{1}{2}\times 4\sin^{-1}\frac{x}{2}\right]_1^2$$

$$\left[\because \int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$$

$$= \left[(x-2)\sqrt{4-(x-2)^2} + 4\sin^{-1}\left(\frac{x-2}{2}\right)\right]_0^1$$

$$+ \left[x\sqrt{4-x^2} + 4\sin^{-1}\frac{x}{2}\right]_1^2$$

$$= \left[\left\{-\sqrt{3} + 4\sin^{-1}\left(\frac{-1}{2}\right)\right\} - 0 - 4\sin^{-1}(-1)\right]$$

$$+ \left[0 + 4\sin^{-1}1 - \sqrt{3} - 4\sin^{-1}\frac{1}{2}\right]$$

$$= \left[\left(-\sqrt{3} - 4\times\frac{\pi}{6}\right) + 4\times\frac{\pi}{2}\right] + \left[4\times\frac{\pi}{2} - \sqrt{3} - 4\times\frac{\pi}{6}\right]$$

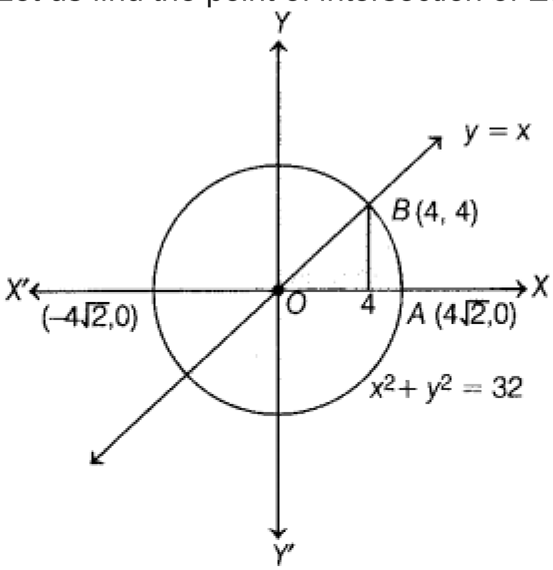
$$= \left(-\sqrt{3} - \frac{2\pi}{3} + 2\pi\right) + \left(2\pi - \sqrt{3} - \frac{2\pi}{3}\right)$$

$$= \frac{8\pi}{3} - 2\sqrt{3} \text{ sq units}$$

**8.Find the area of the region in the first quadrant enclosed by the X-axis, the line $y = x$ and the circle $x^2 + y^2 = 32$.**

**Answer:**

Given the circle $x^2 + y^2 = 3^2$ ….. (i)
having centre (0, 0) and radius $4\sqrt{2}$ and the line $y = x$ ….. (ii)
Let us find the point of intersection of Eqs. (i) and (ii).




On substituting $y = x$ in Eq. (i), we get
$x^2 + x^2 = 32$
$\Rightarrow 2x^2 = 32$
$\Rightarrow x^2 = 16$
$\Rightarrow x = \pm 4$
Thus, the points of intersection are (4, 4) and (-4, -4). [$\because y = x$]
Clearly, the required area

= Area of shaded region OABO

$$= \int_0^4 y(\text{line})\,dx + \int_4^{4\sqrt{2}} y(\text{circle})\,dx$$

$$= \int_0^4 x\,dx + \int_4^{4\sqrt{2}} \sqrt{32-x^2}\,dx$$

$$[\because x^2 + y^2 = 32 \Rightarrow y = \pm\sqrt{32-x^2} \text{ and } y > 0]$$

$$= \left[\frac{x^2}{2}\right]_0^4 + \int_4^{4\sqrt{2}} \sqrt{(4\sqrt{2})^2 - x^2}\,dx$$

$$= \frac{1}{2}[16-0] + \frac{1}{2}\left[x\sqrt{(4\sqrt{2})^2 - x^2} + (4\sqrt{2})^2 \sin^{-1}\left(\frac{x}{4\sqrt{2}}\right)\right]_4^{4\sqrt{2}}$$

$$= 8 + \frac{1}{2}\left[(0 + 32\sin^{-1}(1)) - \left(4\sqrt{32-16} + 32\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)\right)\right]$$

$$= 8 + \frac{1}{2}\left[32\sin^{-1}(1) - 16 - 32\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)\right]$$

$$= 8 + \frac{1}{2}\left[32\cdot\frac{\pi}{2} - 16 - 32\cdot\frac{\pi}{4}\right]$$

$$= 8 + \frac{1}{2}[16\pi - 16 - 8\pi]$$

$$= 8 + \frac{1}{2}[8\pi - 16] = 8 + 4\pi - 8$$

$$= 4\pi \text{ sq units}$$

# CHAPTER-IX

# DIFFERENTIAL EQUATIONS

## 2 MARK QUESTIONS

**1: Find the differential equation of the family of lines through the origin.**

**Solution:**

Let y = mx be the family of lines through the origin.

Therefore, dy/dx = m

Eliminating m, (substituting m = y/x)

y = (dy/dx) . x

or

x. dy/dx – y = 0

**2.Find the differential equation of the family of circles in the first quadrant which touch the coordinate axes.**

**Answer:**

Equation of family of circles in the first quadrant which touch the coordinate axes is $(x - a)^2 + (y - a)^2 = a^2$
$(x - y)^2 [(y')^2 + 1] = (x + yy')^2$

**3.Solve the differential equation**
**cos(dydx) = a,(a ∈ R)**

**Answer:**

Given equation is cos(dydx) = a
which can be rewritten as dydx = $\cos^{-1}a$
⇒ dy = $\cos^{-1}a$
⇒ ∫dy = ∫$\cos^{-1}a$ dx
⇒ y = $\cos^{-1}a$. x + C
which is the required solution.

# 4 MARK QUESTIONS

**1: Determine order and degree (if defined) of differential equation $(y''')^2 + (y'')^3 + (y')^4 + y^5 = 0$**

**Solution:**

Given differential equation is $(y''')^2 + (y'')^3 + (y')^4 + y^5 = 0$

The highest order derivative present in the differential equation is $y'''$.

Therefore, its order is 3.

The given differential equation is a polynomial equation in $y'''$, $y''$, and $y'$.

The highest power raised to $y'''$ is 2.

Hence, its degree is 2.

**2: Verify that the function $y = a \cos x + b \sin x$, where, $a, b \in R$ is a solution of the differential equation $d^2y/dx^2 + y=0$.**

**Solution:**

The given function is $y = a \cos x + b \sin x$ … (1)

Differentiating both sides of equation (1) with respect to x,

$dy/dx = - a \sin x + b \cos x$

$d^2y/dx^2 = - a \cos x - b \sin x$

LHS = $d^2y/dx^2 + y$

$= - a \cos x - b \sin x + a \cos x + b \sin x$

$= 0$

= RHS

Hence, the given function is a solution to the given differential equation.

**3: The number of arbitrary constants in the general solution of a differential equation of fourth order is:**

**(A) 0 (B) 2 (C) 3 (D) 4**

**Solution:**

We know that the number of constants in the general solution of a differential equation of order n is equal to its order.

Therefore, the number of constants in the general equation of the fourth-order differential equation is four.

Hence, the correct answer is D.

Note: The number of constants in the general solution of a differential equation of order n is equal to zero.

**4: Form the differential equation representing the family of curves y = a sin (x + b), where a, b are arbitrary constants.**

**Solution:**

Given,

$y = a \sin (x + b)$ … (1)

Differentiating both sides of equation (1) with respect to x,

$dy/dx = a \cos (x + b)$ … (2)

Differentiating again on both sides with respect to x,

$d^2y/dx^2 = -a \sin (x + b)$ … (3)

Eliminating a and b from equations (1), (2) and (3),

$d^2y/dx^2 + y = 0$ … (4)

The above equation is free from the arbitrary constants a and b.

This the required differential equation.

**5: Form the differential equation of the family of circles having a centre on y-axis and radius 3 units.**

**Solution:**

The general equation of the family of circles having a centre on the y-axis is $x^2 + (y - b)^2 = r^2$

Given the radius of the circle is 3 units.

The differential; equation of the family of circles having a centre on the y-axis and radius 3 units is as below:

$x^2 + (y - b)^2 = 3^2$

$x^2 + (y - b)^2 = 9$ ……(i)

Differentiating (i) with respect to x,

$2x + 2(y - b).y' = 0$

$\Rightarrow (y - b).\ y' = -x$

$\Rightarrow (y - b) = -x/y'$ …….(ii)

Substituting (ii) in (i),

$x^2 + (-x/y')^2 = 9$

$\Rightarrow x^2[1 + 1/(y')^2] = 9$

$\Rightarrow x^2\ [(y')^2 + 1) = 9\ (y')^2$

$\Rightarrow (x^2 - 9)\ (y')^2 + x^2 = 0$

Hence, this is the required differential equation.

**6: Find the general solution of the differential equation dy/dx =1+$y^2$/1+$x^2$.**

**Solution:**

Given differential equation is dy/dx =1+$y^2$/1+$x^2$

Since 1 + $y^2 \neq 0$, therefore by separating the variables, the given differential equation can be written as:

dy/1+$y^2$ = dx/1+$x^2$ …….(i)

Integrating equation (i) on both sides,

$$\int \frac{dy}{1+y^2} = \int \frac{dx}{1+x^2}$$

$\tan^{-1}y = \tan^{-1}x + C$

This is the general solution of the given differential equation.

# 7 MARK QUESTIONS

**1: For each of the given differential equation, find a particular solution satisfying the given condition:**

**dy/dx = y tan x ; y = 1 when x = 0**

**Solution:**

dy/dx = y tan x

dy/y = tan x dx

Integrating on both sides,

$$\int \frac{dy}{y} = \int tan\ x\ dx$$

log y = log (sec x) +C

log y = log (C sec x)

⇒ y = C sec x ……..(i)

Now consider y = 1 when x= 0.

1 = C sec 0

1 = C (1)

C = 1

Substituting C = 1 in (i)

y = sec x

Hence, this is the required particular solution of the given differential equation.

**2: Find the equation of a curve passing through $(1, \pi/4)$ if the slope of the tangent to the curve at any point P $(x, y)$ is $y/x - \cos^2(y/x)$.**

**Solution:**

According to the given condition,

$dy/dx = y/x - \cos^2(y/x)$ ………….(i)

This is a homogeneous differential equation.

Substituting $y = vx$ in (i),

$v + (x)\, dv/dx = v - \cos^2 v$

$\Rightarrow (x) dv/dx = - \cos^2 v$

$\Rightarrow \sec^2 v\, dv = - dx/x$

By integrating on both the sides,

$\Rightarrow \int \sec^2 v\, dv = - \int dx/x$

$\Rightarrow \tan v = - \log x + c$

$\Rightarrow \tan (y/x) + \log x = c$ ……….(ii)

Substituting $x = 1$ and $y = \pi/4$,

$\Rightarrow \tan (\pi/4) + \log 1 = c$

$\Rightarrow 1 + 0 = c$

$\Rightarrow c = 1$

Substituting $c = 1$ in (ii),

$\tan (y/x) + \log x = 1$

**3: Integrating factor of the differential equation $(1 – x^2)dy/dx – xy = 1$ is**

**(A) -x**

**(B) $x/ (1 + x^2)$**

**(C) $\sqrt{(1- x^2)}$**

**(D) ½ log $(1 – x^2)$**

**Solution:**

Given differential equation is $(1 – x^2)dy/dx – xy = 1$

$(1 – x^2)dy/dx = 1 + xy$

$dy/dx = (1/1 – x^2) + (x/1 – x^2)y$

$dy/dx – (x/1 – x^2)y = 1/1-x^2$

This is of the form $dy/dx + Py = Q$

We can get the integrating factor as below:

$$I.F = e^{\int P\, dx}$$

$$= e^{\int \frac{-x}{1-x^2} dx}$$

Let $1 – x^2 = t$

Differentiating with respect to x

$-2x\ dx = dt$

$-x\ dx = dt/2$

Now,

$$I.F = e^{\int \frac{dt}{2t}}$$

$$= e^{\frac{1}{2}\int \frac{dt}{t}}$$

$$= e^{\frac{1}{2}\log t}$$

$$= e^{\log \sqrt{t}}$$

I.F = $\sqrt{t} = \sqrt{(1 - x^2)}$

Hence, option C is the correct answer.

**4.Solve the differential equation**
**$(1 + x)^2 + 2xy – 4x^2 = 0$, subject to the initial condition $y(0) = 0$.**

**Answer:**

Given differential equation is
$(1 + x)^2 + 2xy – 4x^2 = 0$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{2x}{1+x^2}\,y = \frac{4x^2}{1+x^2}$$

which is the equation of the form

$$\frac{dy}{dx} + Py = Q,$$

where $P = \dfrac{2x}{1+x^2}$ and $Q = \dfrac{4x^2}{1+x^2}$

Now, $\text{IF} = e^{\int \frac{2x}{1+x^2}dx} = e^{\log(1+x^2)} = 1 + x^2$

The general solution is

$$y \cdot (1+x^2) = \int (1+x^2)\frac{4x^2}{(1+x^2)}\,dx + C$$

$$\Rightarrow \quad (1+x^2)\,y = \int 4x^2\,dx + C$$

$$\Rightarrow \quad (1+x^2)\,y = \frac{4x^3}{3} + C$$

$$\Rightarrow \quad y = \frac{4x^3}{3(1+x^2)} + C(1+x^2)^{-1} \quad \text{...(i)}$$

Now, $y(0) = 0$

$$\Rightarrow \quad 0 = \frac{4 \cdot 0^3}{3(1+0^2)} + C(1+0^2)^{-1} \Rightarrow C = 0$$

Put the value of $C$ in Eq. (i), we get

$$y = \frac{4x^3}{3(1+x^2)},$$

which is the required solution.

**5.**

**Solve the following differential equation.**

**x dydx = y – x tan(yx). (All IndIa 2019)**

**Answer:**

Given differential equation is

$$x\frac{dy}{dx} = y - x\tan\left(\frac{y}{x}\right) \Rightarrow \frac{dy}{dx} = \frac{y - x\tan\left(\frac{y}{x}\right)}{x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x} - \tan\left(\frac{y}{x}\right) \quad \ldots(i)$$

which is a homogeneous differential equation as
$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$.

On putting $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ in Eq. (i),

we get

$$v + x\frac{dv}{dx} = v - \tan v \Rightarrow x\frac{dv}{dx} = -\tan v$$

$$\Rightarrow \quad \frac{dv}{\tan v} = -\frac{dx}{x}$$

$$\Rightarrow \quad \cot v\, dv = -\frac{dx}{x} \quad \left[\because \frac{1}{\tan v} = \cot v\right]$$

On integrating both sides, we get

$$\int \cot v\, dv = -\int \frac{dx}{x}$$

$$\Rightarrow \quad \log|\sin v| = -\log|x| + C$$

$$\left[\because \int \cot v\, dv = \log|\sin v|\right]$$

$$\Rightarrow \quad \log|\sin v| + \log|x| = C$$

$$\Rightarrow \quad \log|x\sin v| = C$$

$$[\because \log m + \log n = \log mn]$$

$$\therefore \quad \log\left|x\sin\frac{y}{x}\right| = C \quad \left[\text{put } v = \frac{y}{x}\right]$$

$$\Rightarrow x\sin\frac{y}{x} = e^C$$

$$\Rightarrow x\sin\frac{y}{x} = A \quad [\because e^C = A]$$

$$\Rightarrow \sin\frac{y}{x} = \frac{A}{x} \Rightarrow y = x\sin^{-1}\left(\frac{A}{x}\right).$$

which is the required solution.

**6.Solve the differential equation. (All India 2019)**
**dydx=−[x+ycosx1+sinx]**

**Answer:**

Given, $\frac{dy}{dx} = -\frac{x}{1+\sin x} - \frac{y\cos x}{1+\sin x}$

or $\frac{dy}{dx} + \frac{y\cos x}{1+\sin x} = -\frac{x}{1+\sin x}$ ...(i)

which is in the linear form, $\frac{dy}{dx} + Py = Q$, where

$$P = \frac{\cos x}{1+\sin x}, Q = -\frac{x}{1+\sin x}$$

Now, $\text{IF} = e^{\int \frac{\cos x}{1+\sin x}dx} = e^{\log(1+\sin x)} = 1+\sin x$

and the general solution is

$$y(1+\sin x) = \int -x\,dx + C$$

$$[\because y\cdot(\text{IF}) = \int Q\cdot(\text{IF})\,dx + C]$$

$$\Rightarrow \quad y(1+\sin x) = -\frac{x^2}{2} + C$$

**7.Solve the following differential equation**
**cosec x log |y| dydx + $x^2y^2$ = 0.**

**Answer:**

First, separate the variables, then integrate by using integration by parts.
Given differential equation is
cosec x log |y| dydx + $x^2y^2$ = 0

$$\text{cosec } x \log|y| \frac{dy}{dx} = -x^2y^2$$

On separating the variables, we get

$$\frac{\log|y|}{y^2}\, dy = \frac{-x^2}{\text{cosec } x}\, dx$$

On integrating both sides, we get

$$\int \frac{\log|y|}{y^2}\, dy = -\int \frac{x^2}{\text{cosec}\, x}\, dx$$

$\Rightarrow$ $$I_1 = -I_2 \qquad \text{...(ii)}$$

where, $I_1 = \int \frac{\log|y|}{y^2}\, dy$

and $\quad I_2 = \int \frac{x^2}{\text{cosec}\, x}\, dx = \int x^2 \sin x\, dx$

Consider, $I_1 = \int \frac{\log|y|}{y^2} dy$

Put $\quad \log y = t \Rightarrow y = e^t$, then $\frac{dy}{y} = dt$

$\therefore\ I_1 = \int \underset{\text{I}}{t}\, \underset{\text{II}}{e^{-t}}\, dt = t \int e^{-t}\, dt - \int \left[\frac{d}{dt}(t) \int e^{-t}\, dt\right] dt$

[using integration by parts]

$$= -t\, e^{-t} - \int (-e^{-t})\, dt$$

$$= -t\, e^{-t} + \int e^{-t} dt = -t\, e^{-t} - e^{-t} + C_1$$

$$= -\frac{\log|y|}{y} - \frac{1}{y} + C_1 \qquad \text{...(iii)}$$

$$\left[\because t = \log|y| \text{ and } e^{-t} = \frac{1}{y}\right]$$

and $I_2 = \int \underset{\text{I}}{x^2}\, \underset{\text{II}}{\sin} x\, dx$

$$= x^2 \int \sin x\, dx - \int \left[\frac{d}{dx}(x^2) \int \sin x\, dx\right] dx$$

[ using integration by parts]

$$= x^2(-\cos x) - \int [2x(-\cos x)]\, dx$$

$$= -x^2 \cos x + 2\int \underset{\text{I}}{x}\, \underset{\text{II}}{\cos}\, x\, dx$$

$$= -x^2 \cos x + 2\left[x \int \cos x\, dx \right.$$

$$\left. - \int \left\{\frac{d}{dx}(x) \int \cos x\, dx\right\} dx\right]$$

$= -x^2 \cos x + 2 [x \sin x - \int \sin x \, dx]$
$= -x^2 \cos x + 2x \sin x + 2 \cos x + C_2$ …(iv)

On putting the values of $I_1$ and $I_2$ from Eqs.(iii) and (iv) in Eq. (ii), we get

$$-\frac{\log|y|}{y} - \frac{1}{y} + C_1 = x^2 \cos x - 2x \sin x - 2\cos x - C_2$$

$$\Rightarrow \quad -\frac{(1 + \log|y|)}{y} = x^2 \cos x - 2x \sin x - 2\cos x - C_2 - C_1$$

$$\Rightarrow \quad -\frac{(1 + \log|y|)}{y} = x^2 \cos x - 2x \sin x - 2\cos x + C$$

where, $C = -C_2 - C_1$
which is the required solution of given differential equation.

**8.Solve the following differential equation**
**$2x^2$ dydx – 2xy + $y^2$ = 0.**

**Answer:**
**Given differential equation is**

$$2x^2\frac{dy}{dx} - 2xy + y^2 = 0 \Rightarrow \frac{dy}{dx} = \frac{y}{x} - \frac{y^2}{2x^2} \quad \dots(\text{i})$$

which is a homogeneous differential equation as $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$.

On putting $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ in Eq. (i),

we get

$$\Rightarrow v + x\frac{dv}{dx} = v - \frac{v^2}{2} \Rightarrow x\frac{dv}{dx} = \frac{-v^2}{2} \Rightarrow \frac{2dv}{v^2} = -\frac{1}{x}dx \quad \textbf{(1)}$$

On integrating both sides, we get

$$2\int v^{-2}dv = -\log|x| + C$$

$$\Rightarrow \quad \frac{2v^{-1}}{-1} = -\log|x| + C$$

$$\Rightarrow \quad \frac{-2}{v} = -\log|x| + C$$

$$\Rightarrow \quad \frac{-2x}{y} = -\log|x| + C \qquad \left[\text{put } v = \frac{y}{x}\right]$$

$$\Rightarrow \quad -2x = y(-\log|x| + C)$$

$$\therefore \quad y = \frac{-2x}{-\log|x| + C}$$

# CHAPTER-X

# VECTORS

## 2 MARK QUESTIONS

**1.Write the direction ratios of the vector $3\vec{a} + 2\vec{b}$, where $\vec{a} = \hat{i} + \hat{j} - 2\hat{k}$ and $\vec{b} = 2\hat{i} - 4\hat{j} + 5\hat{k}$**

**Answer:**

Clearly, $3\vec{a} + 2\vec{b} = 3(\hat{i} + \hat{j} - 2\hat{k}) + 2(2\hat{i} - 4\hat{j} + 5\hat{k})$
$= (3\hat{i} + 3\hat{j} - 6\hat{k}) + (4\hat{i} - 8\hat{j} + 10\hat{k})$
$= 7\hat{i} - 5\hat{j} + 4\hat{k}$
Hence, direction ratios of vectors $3\vec{a} + 2\vec{b}$ are 7, – 5 and 4.

**2.Find the unit vector in the direction of the sum of the vectors $2\hat{i} + 3\hat{j} - \hat{k}$ and $4\hat{i} - 3\hat{j} + 2\hat{k}$.**

**Answer:**

Let $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{b} = 4\hat{i} - 3\hat{j} + 2\hat{k}$
Now, sum of two vectors,
$\vec{a} + \vec{b} = (2\hat{i} + 3\hat{j} - \hat{k}) + (4\hat{i} - 3\hat{j} + 2\hat{k}) = 6\hat{i} + \hat{k}$

$\therefore$ Required unit vector $= \dfrac{\vec{a} + \vec{b}}{|\vec{a} + \vec{b}|}$

$$= \frac{6\hat{i} + \hat{k}}{\sqrt{6^2 + 1^2}} = \frac{6\hat{i} + \hat{k}}{\sqrt{36 + 1}} = \frac{6\hat{i} + \hat{k}}{\sqrt{37}} = \frac{6}{\sqrt{37}}\hat{i} + \frac{\hat{k}}{\sqrt{37}}$$

**3.Write a vector in the direction of the vector $\hat{i} - 2\hat{j} + 2\hat{k}$ that has magnitude 9 units.**

**Answer:**

$3\hat{i} - 6\hat{j} + 6\hat{k}$

**4.If $\vec{a} = x\hat{i} + 2\hat{j} - z\hat{k}$ and $\vec{b} = 3\hat{i} - y\hat{j} + \hat{k}$ are two equal vectors, then write the value of x + y + z. (Delhi 2013)**

**Answer:**

Two vectors are equal, if coefficients of their components are equal.
Given, $\vec{a} = \vec{b} \Rightarrow x\hat{i} + 2\hat{j} - z\hat{k} = \hat{i} - y\hat{j} + \hat{k}$

On comparing the coefficient of components, we get
$x = 3, y = -2, z = -1$
Now, $x + y + z = 3 - 2 - 1 = 0$

**5.P and Q are two points with position vectors $3\vec{a} - 2\vec{b}$ and $\vec{a} + \vec{b}$, respectively. Write the position vector of a point R which divides the line segment PQ in the ratio 2 : 1 externally.**

**Answer:**

$-\vec{a} + 4\vec{b}$

**6.L and M are two points with position vectors $2\vec{a} - \vec{b}$ and $\vec{a} + 2\vec{b}$, respectively. Write the position vector of a point N which divides the line segment LM in the ratio 2 : 1 externally.**

**Answer:**

$5\vec{b}$

**7.Find the sum of the following vectors. $\vec{a} = \hat{i} - 3\hat{k}$, $\vec{b} = 2\hat{j} - \hat{k}$, $\vec{c} = 2\hat{i} - 3\hat{j} + 2\hat{k}$.**

**Answer:**

$3\hat{i} - \hat{j} - 2\hat{k}$

# 4 MARK QUESTIONS

**1: Find the unit vector in the direction of the sum of the vectors**

***a*→=2*i*^−*j*^+2*k*^**

***b*→=−*i*^+*j*^+3*k*^**

**Solution:**

Let $c$→ be the sum of $a$→ and $b$→.

$$\vec{c} = (2\hat{\imath} - \hat{\jmath} + 2\hat{k}) + (-\hat{\imath} + \hat{\jmath} + 3\hat{k}) = \hat{\imath} + 5\hat{k}$$

$$|\vec{c}| = \sqrt{1^2 + 5^2} = \sqrt{26}$$

The unit vector is:

$$\hat{c} = \frac{\vec{c}}{|\vec{c}|} = \frac{\hat{\imath} + 5\hat{k}}{\sqrt{26}} = \frac{1}{\sqrt{26}}\hat{\imath} + \frac{5}{\sqrt{26}}\hat{k}$$

**2: Find the vector joining the points P(2, 3, 0) and Q(– 1, – 2, – 4) directed from P to Q.**

**Solution:**

Since the vector is to be directed from P to Q, clearly P is the initial point and Q is the terminal point.

P(2, 3, 0) = ($x_1$, $y_1$, $z_1$)

Q(-1, -2, -4) = ($x_2$, $y_2$, $z_2$)

Vector joining the points P and Q is:

$$\overrightarrow{PQ} = (-1-2)\hat{\imath} + (-2-3)\hat{\jmath} + (-4-0)\hat{k}$$

$$\overrightarrow{PQ} = -3\hat{\imath} - 5\hat{\jmath} - 4\hat{k}$$

**3. No. 6: Show that the points A, B and C with position vectors**

***a*→=3*i*^−4*j*^−4*k*^,*b*→=2*i*^−*j*^+*k*^**
***c*→=*i*^−3*j*^−5*k*^**
**form the vertices of a right-angled triangle.**
**Solution:**

**Solution:**

Position vectors of points A, B and C are respectively given as below.

*a*→=3*i*^−4*j*^−4*k*^,
*b*→=2*i*^−*j*^+*k*^
*c*→=*i*^−3*j*^−5*k*^

$$\overrightarrow{AB} = \vec{b} - \vec{a} = (2-3)\hat{\imath} + (-1+4)\hat{\jmath} + (1+4)\hat{k} = -\hat{\imath} + 3\hat{\jmath} + 5\hat{k}$$

$$\overrightarrow{BC} = \vec{c} - \vec{b} = (1-2)\hat{\imath} + (-3+1)\hat{\jmath} + (-5-1)\hat{k} = -\hat{\imath} - 2\hat{\jmath} - 6\hat{k}$$

$$\overrightarrow{CA} = \vec{a} - \vec{c} = (3-1)\hat{\imath} + (-4+3)\hat{\jmath} + (-4+5)\hat{k} = 2\hat{\imath} - \hat{\jmath} + \hat{k}$$

$$\therefore \left|\overrightarrow{AB}\right|^2 = (-1)^2 + 3^2 + 5^2 = 1 + 9 + 25 = 35$$

$$\left|\overrightarrow{BC}\right|^2 = (-1)^2 + (-2)^2 + (-6)^2 = 1 + 4 + 36 = 41$$

$$\left|\overrightarrow{CA}\right|^2 = 2^2 + (-1)^2 + 1^2 = 4 + 1 + 1 = 6$$

$$\left|\overrightarrow{AB}\right|^2 + \left|\overrightarrow{CA}\right|^2 = 35 + 6 = 41 = \left|\overrightarrow{BC}\right|^2$$

Therefore, ABC is a right-angled triangle.

**4: Find the area of the triangle with vertices A(1, 1, 2), B(2, 3, 5) and C(1, 5, 5).**

**Solution:**

Vertices of a triangle ABC are A(1, 1, 2), B(2, 3, 5) and C(1, 5, 5).

Let AB and BC be the adjacent sides of triangle ABC.

$$\overrightarrow{AB} = (2-1)\hat{\imath} + (3-1)\hat{\jmath} + (5-2)\hat{k} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$$

$$\overrightarrow{BC} = (1-2)\hat{\imath} + (5-3)\hat{\jmath} + (5-5)\hat{k} = -\hat{\imath} + 2\hat{\jmath}$$

$$ar(\Delta\, ABC) = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{BC}\right|$$

$$\left|\overrightarrow{AB} \times \overrightarrow{BC}\right| = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & 3 \\ -1 & 2 & 0 \end{vmatrix} = \hat{\imath}(-6) - \hat{\jmath}(3) + \hat{k}(2+2) = -6\hat{\imath} - 3\hat{\jmath} + 4\hat{k}$$

$$\overrightarrow{AB} \times \overrightarrow{BC} = \sqrt{(-6)^2 + (-3)^2 + 4^2} = \sqrt{36+9+16} = \sqrt{61}$$

Hence, the area of triangle ABC is √61/2 sq.units

**5.Find the sum of the vectors $\vec{a} = \hat{\imath} - 2\hat{\jmath} + \hat{k}$ $\vec{b} = -2\hat{\imath} + 4\hat{\jmath} + 5\hat{k}$ and $\vec{c} = \hat{\imath} - 6\hat{\jmath} - 7\hat{k}$.**

**Answer:**

Given vectors are $\vec{a} = \hat{\imath} - 2\hat{\jmath} + \hat{k}$ $\vec{b} = -2\hat{\imath} + 4\hat{\jmath} + 5\hat{k}$ and $\vec{c} = \hat{\imath} - 6\hat{\jmath} - 7\hat{k}$.
Sum of the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is
$\vec{a} + \vec{b} + \vec{c} = (\hat{\imath} - 2\hat{\jmath} + \hat{k}) + (-2\hat{\imath} + 4\hat{\jmath} + 5\hat{k}) + (\hat{\imath} - 6\hat{\jmath} - 7\hat{k})$
$= -4\hat{\jmath} - \hat{k}$

**6.Write the direction cosines of vector -2î + ĵ – 5k̂.**

**Answer:**

Direction cosines of the vector aî + bĵ + ck̂ are

$$\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}}.$$

Let $\vec{a} = -2\hat{i} + \hat{j} - 5\hat{k}$

$\therefore$ Direction cosines of $\vec{a}$ are

$$\frac{-2}{\sqrt{(-2)^2+(1)^2+(-5)^2}}, \frac{1}{\sqrt{(-2)^2+(1)^2+(-5)^2}} \text{ and } \frac{-5}{\sqrt{(-2)^2+(1)^2+(-5)^2}}$$

i.e. $\frac{-2}{\sqrt{30}}, \frac{1}{\sqrt{30}}, \frac{-5}{\sqrt{30}}$

# 7 MARK QUESTIONS

**1.Find the magnitude of each of the two vectors a⃗ and b⃗ , having the same magnitude such that the angle between them is 60° and their scalar product is 92.**

**Answer:**

Given, two vectors $\vec{a}$ and $\vec{b}$ such that $|\vec{a}|=|\vec{b}|$,

$\vec{a}\cdot\vec{b}=\frac{9}{2}$ and angle between them is $60^o$.

We know that

$$\vec{a}\cdot\vec{b}=|\vec{a}|\,|\vec{b}|\cos\theta,$$

where $\theta$ is angle between $\vec{a}$ and $\vec{b}$.

$\therefore \quad \frac{9}{2}=|\vec{a}|\cdot|\vec{a}|\cos 60^o$ **(1/2)**

$\Rightarrow \quad \frac{1}{2}\cdot|\vec{a}|^2=\frac{9}{2} \qquad \left[\because \cos 60^o=\frac{1}{2}\right]$

$\Rightarrow \quad |\vec{a}|^2=9$

$\Rightarrow \quad |\vec{a}|=3$

[$\because$ magnitude cannot be negative]

Thus, $\quad |\vec{a}|=|\vec{b}|=3$ **(1/2)**

**2.If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three mutually perpendicular vectors of the same magnitude, then prove that $\vec{a}+\vec{b}+\vec{c}$ is equally inclined with the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$.**

**Answer:**

If three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are mutually perpendicular to each other, then $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{c} = \vec{c}\cdot\vec{a} = 0$ and if all three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are equally inclined with the vector $(\vec{a}+\vec{b}+\vec{c})$ that means each vector $\vec{a}$, $\vec{b}$ and $\vec{c}$ makes equal angle with $(\vec{a}+\vec{b}+\vec{c})$ by using formula cos θ = a⃗ ·b⃗ |a⃗ ||b⃗ |.

Given, $|\vec{a}| = |\vec{b}| = |\vec{c}| = \lambda$ (say) ...(i)

and $\vec{a}\cdot\vec{b} = 0, \vec{b}\cdot\vec{c} = 0$ and $\vec{c}\cdot\vec{a} = 0$ ...(ii)

Now, $|\vec{a}+\vec{b}+\vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2$

$$+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})$$

$$= \lambda^2 + \lambda^2 + \lambda^2 + 2(0+0+0) = 3\lambda^2$$

$\Rightarrow \quad |\vec{a}+\vec{b}+\vec{c}| = \sqrt{3}\lambda$

[length cannot be negative]

Suppose $(\vec{a}+\vec{b}+\vec{c})$ is inclined at angles $\theta_1, \theta_2$ and $\theta_3$ respectively with vectors $\vec{a}, \vec{b}$ and $\vec{c}$, then

$$(\vec{a}+\vec{b}+\vec{c})\cdot\vec{a} = |\vec{a}+\vec{b}+\vec{c}|\ |\vec{a}|\cos\theta_1$$

$[\because \vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta]$

$\Rightarrow \quad |\vec{a}|^2 + \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c} = \sqrt{3}\lambda \times \lambda\cos\theta_1$

$\Rightarrow \quad \lambda^2 + 0 + 0 = \sqrt{3}\,\lambda^2 \cos\theta_1$

[from Eqs. (i) and (ii)]

$\therefore \quad \cos\theta_1 = \dfrac{1}{\sqrt{3}}$

$$(\vec{a}+\vec{b}+\vec{c})\cdot\vec{b} = |\vec{a}+\vec{b}+\vec{c}||\vec{b}|\cos\theta_2$$

$\Rightarrow \quad \vec{a}\cdot\vec{b} + |\vec{b}|^2 + \vec{c}\cdot\vec{b} = \sqrt{3}\lambda\cdot\lambda\cos\theta_2$

$\Rightarrow \quad 0 + \lambda^2 + 0 = \sqrt{3}\,\lambda^2\cos\theta_2$

[from Eqs. (i) and (ii)]

$\Rightarrow \quad \cos\theta_2 = \dfrac{1}{\sqrt{3}}$

Similarly, $(\vec{a}+\vec{b}+\vec{c})\cdot\vec{c} = |\vec{a}+\vec{b}+\vec{c}||\vec{c}|\cos\theta_3$

$\Rightarrow \quad \cos\theta_1 = \dfrac{1}{\sqrt{3}}$

Thus, $\cos\theta_1 = \cos\theta_2 = \cos\theta_3 = \dfrac{1}{\sqrt{3}}$

Hence, it is proved that $(\vec{a}+\vec{b}+\vec{c})$ is equally inclined with the vectors $\vec{a}, \vec{b}$ and $\vec{c}$.

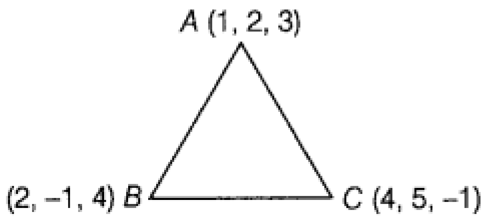
**3.Using vectors, find the area of the ΔABC, whose vertices are A(1, 2, 5), 5(2, -1, 4) and C(4, 5, -1).**

**Answer:**

Let the position vectors of the verices A, B and C of ΔABC be

$$\overrightarrow{OA} = \hat{i} + 2\hat{j} + 3\hat{k},\ \overrightarrow{OB} = 2\hat{i} - \hat{j} + 4\hat{k}$$

and $\overrightarrow{OC} = 4\hat{i} + 5\hat{j} - \hat{k}$, respectively.




Then, $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$$= (2\hat{i} - \hat{j} + 4\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k})$$

$$= \hat{i} - 3\hat{j} + \hat{k}$$

and $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$

$$= (4\hat{i} + 5\hat{j} - \hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k})$$

$$= (3\hat{i} + 3\hat{j} - 4\hat{k})$$

$$\text{Now, } \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 1 \\ 3 & 3 & -4 \end{vmatrix}$$

$$= \hat{i}(12-3) - \hat{j}(-4-3) + \hat{k}(3+9)$$

$$= 9\hat{i} + 7\hat{j} + 12\hat{k}$$

and $|\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{(9)^2 + (7)^2 + (12)^2}$

$$= \sqrt{81 + 49 + 144} = \sqrt{274}$$

$$\therefore \text{ Area of } \Delta ABC = \frac{1}{2} |\overrightarrow{AB} \times \overrightarrow{AC}|$$

$$= \frac{1}{2}\sqrt{274} \text{ sq units}$$

**4.Show that the points A, B, C with position vectors $2\hat{i} - \hat{j} + \hat{k}$, $\hat{i} - 5\hat{j} - 5\hat{k}$ and $5\hat{i} - 4\hat{j} - 4\hat{k}$ respectively, are the vertices of a right-angled triangle. Hence find the**

We have,

$\overrightarrow{AB}$ = (Position vector of $B$) − (Position vector of $A$)

$$= (\hat{i} - 3\hat{j} - 5\hat{k}) - (2\hat{i} - \hat{j} + \hat{k})$$

$$= -\hat{i} - 2\hat{j} - 6\hat{k}$$

$$\overrightarrow{BC} = (3\hat{i} - 4\hat{j} - 4\hat{k}) - (\hat{i} - 3\hat{j} - 5\hat{k})$$

$$= 2\hat{i} - \hat{j} + \hat{k}$$

and $\overrightarrow{CA} = (2\hat{i} - \hat{j} + \hat{k}) - (3\hat{i} - 4\hat{j} - 4\hat{k})$

$$= -\hat{i} + 3\hat{j} + 5\hat{k}$$

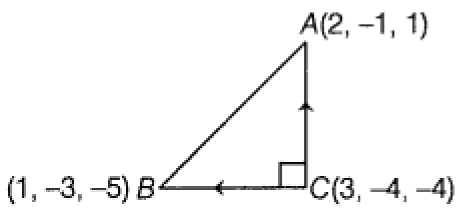
Here, $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = 0$

$\Rightarrow$ $A$, $B$ and $C$ are the vertices of a triangle.

Now, $\overrightarrow{BC} \cdot \overrightarrow{CA} = (2\hat{i} - \hat{j} + \hat{k}) \cdot (-\hat{i} + 3\hat{j} + 5\hat{k})$

$$= -2 - 3 + 5 = 0$$

$\Rightarrow$ $\overrightarrow{BC} \perp \overrightarrow{CA}$ $\Rightarrow$ $\angle C = 90°$




Now, area of $\Delta ABC = \frac{1}{2} |\overrightarrow{CB} \times \overrightarrow{CA}|$

$= \frac{1}{2} |(-8\hat{i} - 11\hat{j} + 5\hat{k})| = \frac{1}{2}\sqrt{210}$ sq units

# CHAPTER-XI

# THREE DIMENSIONAL GEOMETRY

## 2 MARK QUESTIONS

**1: If a line makes angles 90°, 135°, 45° with the x, y and z-axes respectively, find its direction cosines.**

**Solution:**

Let the direction cosines of the line be l, m, and n.

$l = \cos 90° = 0$

$m = \cos 135° = -1/\sqrt{2}$

$n = \cos 45° = 1/\sqrt{2}$

Hence, the direction cosines of the line are 0, $-1/\sqrt{2}$, and $1/\sqrt{2}$.

**2.Write the vector equation of the line given by x−53=y+47=z−62 (Delhi 2011)**

**Answer:**

$\vec{r} = (5\hat{i} - 4\hat{j} + 6\hat{k}) + \lambda(3\hat{i} + 7\hat{j} + 2\hat{k})$

**3.Find the equation of the plane passing through the line of intersection of the planes $\vec{r} \cdot (\hat{i} + 3\hat{j}) - 6 = 0$ and $\vec{r} \cdot (3\hat{i} - \hat{j} - 4\hat{k}) = 0$, whose perpendicular distance from origin is unity. (All India 2013C)**

**Answer:**

$\vec{r} \cdot (2\hat{i} + \hat{j} - 2\hat{k}) - 3 = 0$

and $\vec{r} \cdot (-\hat{i} + 2\hat{j} + 2\hat{k}) - 3 = 0$

**4 .Find the coordinates of the point, where the line through (3, – 4, – 5) and (2, – 3, 1) crosses the plane, passing through the points (2, 2, 1), (3, 0, 1) and (4, – 1, 0).**

**Answer:**

(1, – 2, 7)

# 4 MARK QUESTIONS

**1: Show that the points A (2, 3, – 4), B (1, – 2, 3) and C (3, 8, – 11) are collinear.**

**Solution:**

We know that the direction ratios of the line passing through two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are given by:

$x_2 - x_1, y_2 - y_1, z_2 - z_1$ or $x_1 - x_2, y_1 - y_2, z_1 - z_2$

Given points are A (2, 3, – 4), B (1, – 2, 3) and C (3, 8, – 11).

Direction ratios of the line joining A and B are:

1 – 2, – 2 – 3, 3 + 4

i.e. – 1, – 5, 7.

The direction ratios of the line joining B and C are:

3 –1, 8 + 2, – 11 – 3

i.e., 2, 10, – 14.

From the above, it is clear that direction ratios of AB and BC are proportional.

That means AB is parallel to BC. But point B is common to both AB and BC.

Hence, A, B, C are collinear points.

**2: Find the angle between the pair of lines given by**

$$\vec{r} = 3\hat{\imath} + 2\hat{\jmath} - 4\hat{k} + \lambda\left(\hat{\imath} + 2\hat{\jmath} + 2\hat{k}\right)$$

$$\vec{r} = 5\hat{\imath} - 2\hat{\jmath} + \mu(3\hat{\imath} + 2\hat{\jmath} + 6\hat{k})$$

**Solution:**

From the given,

$$\overrightarrow{b_1} = (\hat{\imath} + 2\hat{\jmath} + 2\hat{k})$$

$$\overrightarrow{b_2} = (3\hat{\imath} + 2\hat{\jmath} + 6\hat{k})$$

Let θ be the angle between the given pair of lines.

$$\cos\theta = \left|\frac{\vec{b}_1 \cdot \vec{b}_2}{|\vec{b}_1||\vec{b}_2|}\right| = \left|\frac{(\hat{i}+2\hat{j}+2\hat{k})\cdot(3\hat{i}+2\hat{j}+6\hat{k})}{\sqrt{1+4+4}\,\sqrt{9+4+36}}\right|$$

$$= \left|\frac{3+4+12}{3\times 7}\right| = \frac{19}{21}$$

$$\theta = \cos^{-1}\left(\frac{19}{21}\right)$$

**3: Show that the lines (x – 5)/7 = (y + 2)/-5 = z/1 and x/1 = y/2 = z/3 are perpendicular to each other.**

**Solution:**

Given lines are:

(x – 5)/7 = (y + 2)/-5 = z/1 and x/1 = y/2 = z/3

The direction ratios of the given lines are 7, -5, 1 and 1, 2, 3, respectively.

We know that,

Two lines with direction ratios $a_1$, $b_1$, $c_1$ and $a_2$, $b_2$, $c_2$ are perpendicular to each other if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

Therefore, 7(1) + (-5) (2) + 1 (3)

= 7 – 10 + 3

=0

Hence, the given lines are perpendicular to each other.

**4: Find the intercepts cut off by the plane 2x + y – z = 5.**

**Solution:**

Given plane is 2x + y – z = 5 ……(i)

Dividing both sides of the equation (i) by 5,

(2/5)x + (y/5) – (z/5) = 1

$$\frac{x}{\frac{5}{2}} + \frac{y}{5} + \frac{z}{-5} = 1 \ldots\ldots (ii)$$

We know that,

The equation of a plane in intercept form is (x/a) + (y/b) + (z/c) = 1, where a, b, c are intercepts cut off by the plane at x, y, z-axes respectively.

For the given equation,

a = 5/2, b = 5, c = -5

Hence, the intercepts cut off by the plane are 5/2, 5 and -5.

**5.If a line makes angles 90°, 60° and θ with X, Y and Z-axis respectively, where θ is acute angle, then find θ. (Delhi 2015)**

**Answer:**

Let l, m and n be the direction cosines of the given line. Then, we have
l = cos 90° = 0,
m = cos 60° = 12
and n = cos θ
∵ $l^2 + m^2 + n^2 = 1$

$$\therefore \quad 0 + \left(\frac{1}{2}\right)^2 + \cos^2\theta = 1$$

$$\Rightarrow \qquad \cos^2\theta = 1 - \frac{1}{4} = \frac{3}{4}$$

$$\Rightarrow \qquad \cos\theta = \pm\frac{\sqrt{3}}{2}$$

[∵ cos θ cannot be negative as θ is an acute angle]
⇒ cos θ = cos 30°
∴ θ = 30°

# 7 MARK QUESTIONS

**1: Find the equations of the planes that passes through three points (1, 1, 0), (1, 2, 1), and (– 2, 2, – 1).**

**Solution:**

Given points are (1, 1, 0), (1, 2, 1), and (– 2, 2, – 1).

$$\begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ -2 & 2 & -1 \end{vmatrix} = 1(-2-2) - 1(-1+2) + 0 = -5 \neq 0$$

Therefore, the plane will pass through the given three points.

We know that,

The equation of the plane through the points $(x_1, y_1, z_1)$, $(x_2, y_2, z_2)$ and $(x_3, y_3, z_3)$ is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

$$\begin{vmatrix} x-1 & y-1 & z \\ 0 & 1 & 1 \\ -3 & 1 & -1 \end{vmatrix} = 0$$

(x – 1)(-2) -(y – 1) (3) + z (3) = 0

-2x + 2 – 3y + 3 + 3z = 0

-2x – 3y + 3z + 5 = 0

-2x – 3y + 3z = -5

Therefore, 2x + 3y – 3z = 5 is the required Cartesian equation of the plane.

**2.The equations of a line is 5x – 3 = 15y + 7 = 3 – 10z. Write the direction cosines of the line.**

**Answer:**

Given equation of a line is
5x – 3 = 15y + 17 = 3 – 10z ……. (i)
Lei us first convert the equation in standard form
x−x1a= y−y1b= z−z1c ……. (iii)
Let us divide Eq. (i) by LCM (coefficients of x, y and z), i.e. LCM (5, 15, 10) = 30
Now, the Eq. (i) becomes

$$\frac{5x-3}{30}=\frac{15y+7}{30}=\frac{3-10z}{30}$$

$$\Rightarrow \frac{5\left(x-\frac{3}{5}\right)}{30}=\frac{15\left(y+\frac{7}{15}\right)}{30}=\frac{-10\left(z-\frac{3}{10}\right)}{30}$$

$$\Rightarrow \frac{x-\frac{3}{5}}{6}=\frac{y+\frac{7}{15}}{2}=\frac{z-\frac{3}{10}}{-3}$$

On comparing the above equation with Eq.(ii), we get 6, 2, – 3 are the direction ratios of the given line.
Now, the direction cosines of given line are

$$\frac{6}{\sqrt{6^2+2^2+(-3)^2}},\frac{2}{\sqrt{6^2+2^2+(-3)^2}} \text{ and } \frac{-3}{\sqrt{6^2+2^2+(-3)^2}}$$

i.e. $\frac{6}{7},\frac{2}{7},\frac{-3}{7}$.

**3.If a line makes angles a,p,y with the position direction of coordinate axes, then write the value of $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$.**

**Answer:**

Given, if a line makes angles a,y with the coordinate axes.
Then, direction cosine of a line are
$\cos\alpha, \cos\beta, \cos\gamma$
$\therefore \sin^2\alpha + \sin^2\beta + \sin^2\gamma$
$= 1 - \cos^2\alpha + 1 - \cos^2\beta + 1 - \cos^2\gamma$
$= 3 - (\cos^2\alpha + \cos^2\beta + \cos^2\gamma)$
$= 3 - 1 = 2$ [$\because \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$]

**4.Write the equation of the straight line through the point (α β γ) and parallel to Z-axis.**

**Answer:**

The vector equation of a line parallel to Z-axis is $\vec{m} = 0\hat{i} + 0\hat{j} + 0\hat{k}$ Then, the required line passes through the point A(α β γ) whose position vector is $\vec{r}1 = \alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}$ and is parallel to the vector $\vec{m} = (0\hat{i} + 0\hat{j} + \hat{k})$.
$\therefore$ The equation is $\vec{r} = \vec{r1} + \lambda\vec{m}$
$= (\alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}) + \lambda(0\hat{i} + 0\hat{j} + 0\hat{k})$
$= (\alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}) + \lambda(\hat{k})$

# CHAPTER-XII

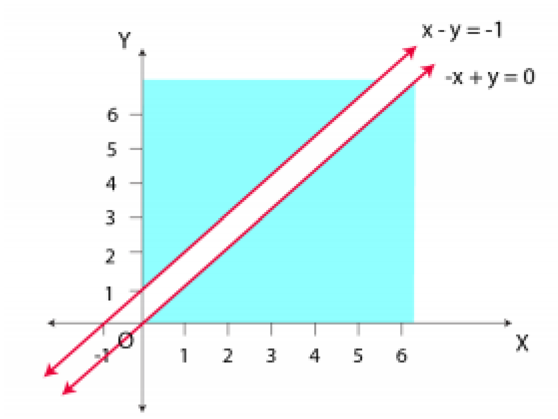
# LINEAR PROGRAMMING

## 2 MARK QUESTIONS

**1. Maximise Z = *x* + *y*, subject to** $x - y \leq -1,\ -x + y \leq 0,\ x, y \geq 0$**.**

**Solution:**

The region determined by the constraints $x - y \leq -1,\ -x + y \leq 0,\ x, y \geq 0$ is given below.




There is no feasible region, and therefore, z has no maximum value.

**2. How many packets of each food should be used to maximise the amount of vitamin A in the diet? What is the maximum amount of vitamin A in the diet?**

**Solution:**

Let the diet contain x and y packets of foods P and Q, respectively. Hence,

$x \geq 0$ and $y \geq 0$

The mathematical formulation of the given problem is given below.

Maximise $z = 6x + 3y$ …………. (i)

Subject to the constraints,

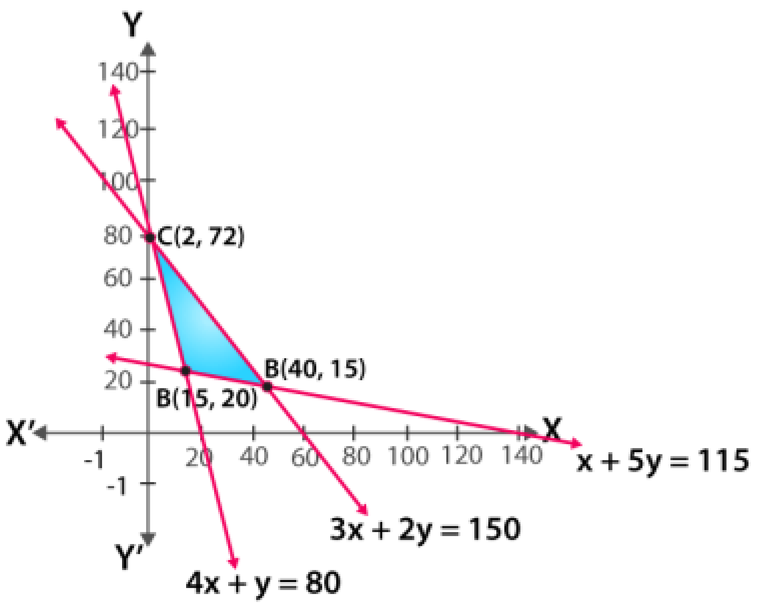
$4x + y \geq 80$ ………… (ii)

$x + 5y \geq 115$ ………. (iii)

$3x + 2y \leq 150$ ……….. (iv)

$x, y \geq 0$ …………… (v)

The feasible region determined by the system of constraints is given below.




A (15, 20), B (40, 15) and C (2, 72) are the corner points of the feasible region.

The values of z at these corner points are given below.

| Corner point | z = 6x + 3y | |
|---|---|---|
| A (15, 20) | 150 | |
| B (40, 15) | 285 | Maximum |
| C (2, 72) | 228 | |

So, the maximum value of z is 285 at (40, 15).

Hence, to maximise the amount of vitamin A in the diet, 40 packets of food P and 15 packets of food Q should be used.

The maximum amount of vitamin A in the diet is 285 units.

**3. A farmer mixes two brands, P and Q, of cattle feed. Brand P, costing Rs 250 per bag, contains 3 units of nutritional element A, 2.5 units of element B and 2 units of element C. Brand Q costing Rs 200 per bag, contains 1.5 units of nutritional element A, 11.25 units of element B, and 3 units of element C. The minimum requirements of nutrients A, B and C are 18 units, 45 units and 24 units respectively. Determine the number of bags of each brand which should be mixed in order to produce a mixture having a minimum cost per bag. What is the minimum cost of the mixture per bag?**

**Solution:**

Let the farmer mix x bags of brand P and y bags of brand Q, respectively.

The given information can be compiled in a table, as given below.

| | Vitamin A (units/kg) | Vitamin B (units/kg) | Vitamin C (units/kg) | Cost (Rs/kg) |
|---|---|---|---|---|
| Food P | 3 | 2.5 | 2 | 250 |

| | | | | |
|---|---|---|---|---|
| Food Q | 1.5 | 11.25 | 3 | 200 |
| Requirement (units/kg) | 18 | 45 | 24 | |

The given problem can be formulated as given below.

Minimise z = 250x + 200y ………… (i)

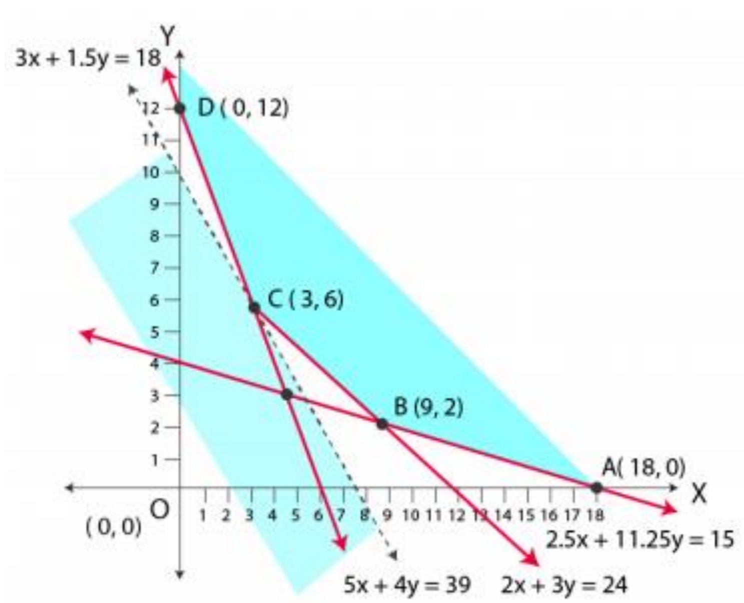
3x + 1.5y ≥ 18 ………….. (ii)

2.5x + 11.25y ≥ 45 ……….. (iii)

2x + 3y ≥ 24 ………….. (iv)

x, y ≥ 0 ………. (v)

The feasible region determined by the system of constraints is given below.




A (18, 0), B (9, 2), C (3, 6) and D (0, 12) are the corner points of the feasible region.

The values of z at these corner points are given below.

| | | |
|---|---|---|
| Corner point | z = 250x + 200y | |

| | | |
|---|---|---|
| A (18, 0) | 4500 | |
| B (9, 2) | 2650 | |
| C (3, 6) | 1950 | Minimum |
| D (0, 12) | 2400 | |

Here, the feasible region is unbounded; hence, 1950 may or may not be the minimum value of z.

For this purpose, we draw a graph of the inequality, $250x + 200y < 1950$ or $5x + 4y < 39$, and check whether the resulting half-plane has points in common with the feasible region or not.

Here, it can be seen that the feasible region has no common point with $5x + 4y < 39$

Hence, at points (3, 6), the minimum value of z is 1950.

Therefore, 3 bags of brand P and 6 bags of brand Q should be used in the mixture to minimise the cost to Rs 1950.

**4.A small firm manufactures necklaces and bracelets. The total number of necklaces and bracelets that it can handle per day is at most 24. It takes one hour to make a bracelet and half an hour to make a necklace. The maximum number of hours available per day is 16. If the profit on a necklace is ₹ 100 and that on a bracelet is ₹ 300. Formulate on L.P.P. for finding how many of each should be produced daily to maximise the profit? It is being given that at least one of each must be produced.**

**Answer:**

Let number of necklaces and bracelets produced by firm per day be x and y, respectively.
Clearly, $x \geq 0$, $y \geq 0$
∵ Total number of necklaces and bracelets that the firm can handle per day is atmost 24.
∴ $x + y \leq 24$
Since it takes one hour to make a bracelet and half an hour to make a necklace and maximum number of hours available per day is 16.
∴ $12x + y \leq 16$
⇒ $x + 2y \leq 32$
Let Z be the profit function.
Then, $Z = 100x + 300y$
∴ The given LPP reduces to
Maximise $Z = 100x + 300y$ subject to,
$x + y \leq 24$
$x + 2y \leq 32$
and $x, y \geq 0$

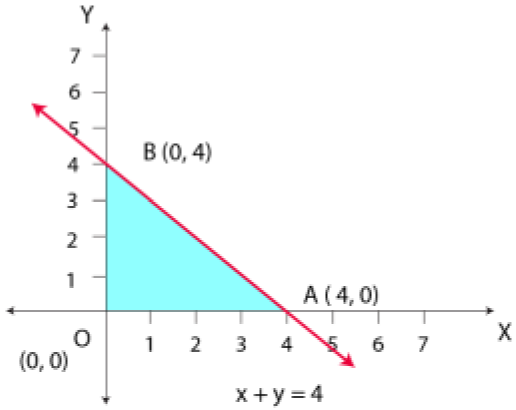
# 4 MARK QUESTIONS

**1. Maximise Z = 3*x* + 4*y***

**Subject to the constraints:** $x + y \leq 4, x \geq 0, y \geq 0.$

**Solution:**

The feasible region determined by the constraints, $x + y \leq 4$, $x \geq 0$, $y \geq 0$, is given below.




O (0, 0), A (4, 0), and B (0, 4) are the corner points of the feasible region. The values of Z at these points are given below.

| Corner point | Z = 3x + 4y | |
|---|---|---|
| O (0, 0) | 0 | |
| A (4, 0) | 12 | |
| B (0, 4) | 16 | Maximum |

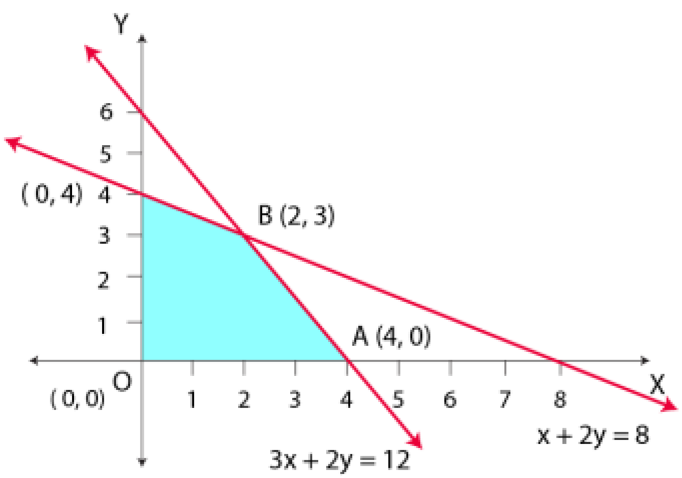
Hence, the maximum value of Z is 16 at the point B (0, 4).

**2.Minimise Z = −3*x* + 4*y***

**subject to** $x+2y \le 8,\ 3x+2y \le 12,\ x \ge 0,\ y \ge 0$.

**Solution:**

The feasible region determined by the system of constraints, $x+2y \le 8,\ 3x+2y \le 12,\ x \ge 0,\ y \ge 0$ is given below.




O (0, 0), A (4, 0), B (2, 3) and C (0, 4) are the corner points of the feasible region.

The values of Z at these corner points are given below.

| Corner point | Z = – 3x + 4y | |
|---|---|---|
| O (0, 0) | 0 | |
| A (4, 0) | -12 | Minimum |
| B (2, 3) | 6 | |
| C (0, 4) | 16 | |

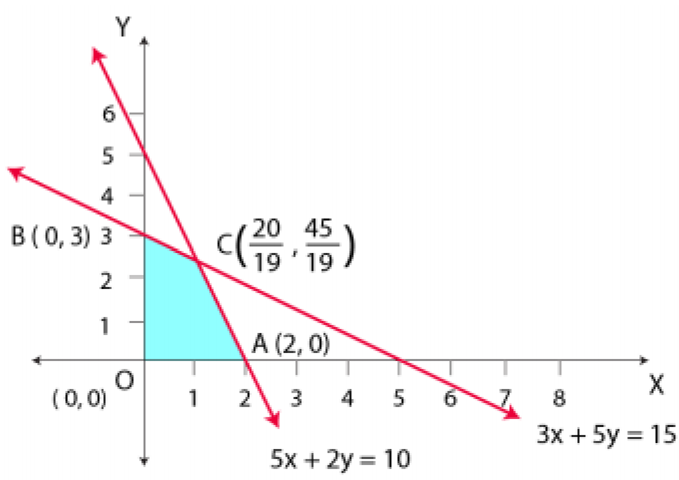
Hence, the minimum value of Z is – 12 at the point (4, 0).

**3. Maximise Z = 5*x* + 3*y***

**subject to** $3x+5y\leq 15,\ 5x+2y\leq 10,\ x\geq 0,\ y\geq 0$.

**Solution:**

The feasible region determined by the system of constraints, $3x + 5y \leq 15$, $5x + 2y \leq 10$, $x \geq 0$, and $y \geq 0$, is given below.




O (0, 0), A (2, 0), B (0, 3) and C (20 / 19, 45 / 19) are the corner points of the feasible region. The values of Z at these corner points are given below.

| Corner point | Z = 5x + 3y | |
|---|---|---|
| O (0, 0) | 0 | |
| A (2, 0) | 10 | |
| B (0, 3) | 9 | |
| C (20 / 19, 45 / 19) | 235 / 19 | Maximum |

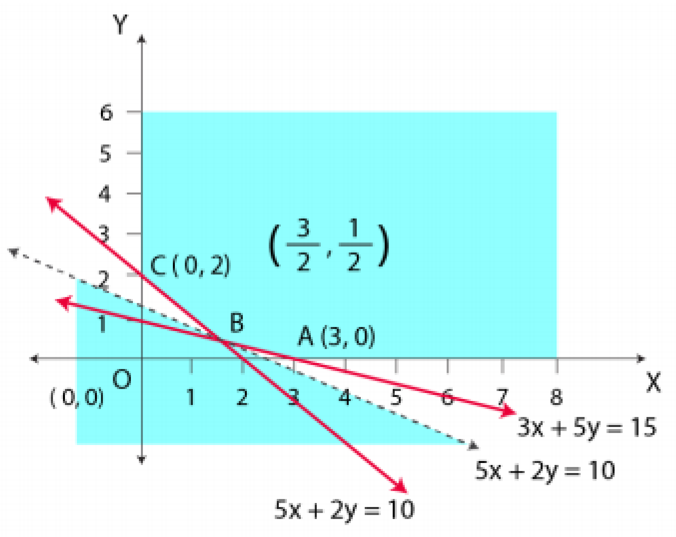
Hence, the maximum value of Z is 235 / 19 at the point (20 / 19, 45 / 19).

**4. Minimise Z = 3*x* + 5*y***

**such that** $x+3y \geq 3,\ x+y \geq 2,\ x, y \geq 0$ **.**

**Solution:**

The feasible region determined by the system of constraints, x + 3y ≥ 3, x + y ≥ 2, and x, y ≥ 0, is given below.




It can be seen that the feasible region is unbounded.

The corner points of the feasible region are A (3, 0), B (3 / 2, 1 / 2) and C (0, 2).

The values of Z at these corner points are given below.

| Corner point | Z = 3x + 5y | |
|---|---|---|
| A (3, 0) | 9 | |
| B (3 / 2, 1 / 2) | 7 | Smallest |
| C (0, 2) | 10 | |

7 may or may not be the minimum value of Z because the feasible region is unbounded.

For this purpose, we draw the graph of the inequality, $3x + 5y < 7$ and check whether the resulting half-plane has common points with the feasible region or not.

Hence, it can be seen that the feasible region has no common point with $3x + 5y < 7$.

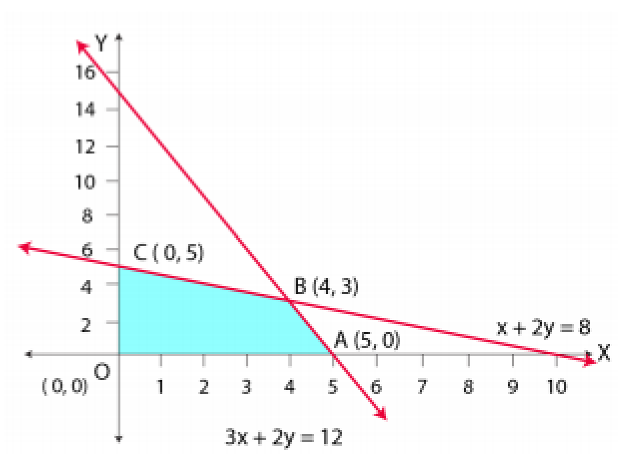
Thus, the minimum value of Z is 7 at point B (3 / 2, 1 / 2).

**5. Maximise $Z = 3x + 2y$**

**subject to** $x+2y \le 10, 3x+y \le 15, x, y \ge 0$.

**Solution:**

The feasible region determined by the constraints, $x + 2y \le 10$, $3x + y \le 15$, $x \ge 0$, and $y \ge 0$, is given below.




A (5, 0), B (4, 3), C (0, 5) and D (0, 0) are the corner points of the feasible region.

The values of Z at these corner points are given below.

| Corner point | Z = 3x + 2y | |
|---|---|---|
| A (5, 0) | 15 | |

| | | |
|---|---|---|
| B (4, 3) | 18 | Maximum |
| C (0, 5) | 10 | |

Hence, the maximum value of Z is 18 at points (4, 3).

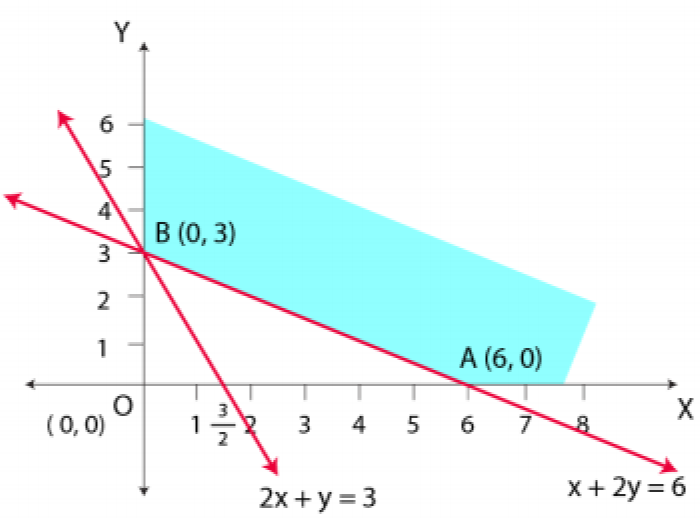
**6. Minimise Z = *x* + 2*y***

**subject to**

$2x + y \geq 3,\ x + 2y \geq 6,\ x, y \geq 0$.

**Solution:**

The feasible region determined by the constraints, $2x + y \geq 3$, $x + 2y \geq 6$, $x \geq 0$, and $y \geq 0$, is given below.




A (6, 0) and B (0, 3) are the corner points of the feasible region.

The values of Z at the corner points are given below.

| Corner point | Z = x + 2y |
|---|---|

| | |
|---|---|
| A (6, 0) | 6 |
| B (0, 3) | 6 |

Here, the values of Z at points A and B are same. If we take any other point, such as (2, 2) on line $x + 2y = 6$, then $Z = 6$.

Hence, the minimum value of Z occurs for more than 2 points.

Therefore, the value of Z is minimum at every point on the line $x + 2y = 6$.

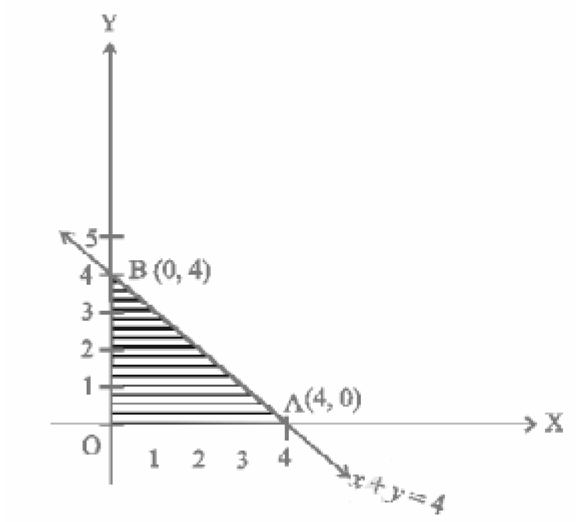
# 7 MARK QUESTIONS

**1: Solve the following LPP graphically:**

**Maximise Z = 2x + 3y, subject to x + y ≤ 4, x ≥ 0, y ≥ 0**

**Solution:**

Let us draw the graph pf x + y =4 as below.




The shaded region (OAB) in the above figure is the feasible region determined by the system of constraints x ≥ 0, y ≥ 0 and x + y ≤ 4.

The feasible region OAB is bounded and the maximum value will occur at a corner point of the feasible region.

Corner Points are O(0, 0), A (4, 0) and B (0, 4).

Evaluate Z at each of these corner points.

| Corner Point | Value of Z |
|---|---|
| O(0, 0) | 2 (0) + 3(0) = 0 |

| | |
|---|---|
| A (4, 0) | 2 (4) + 3(0) = 8 |
| B (0, 4) | 2 (0) + 3 (4)= 12 ← maximum |

Hence, the maximum value of Z is 12 at the point (0, 4).

**2: Solve the following linear programming problem graphically:**

**Minimise Z = 200 x + 500 y subject to the constraints:**

**$x + 2y \geq 10$**

**$3x + 4y \leq 24$**

**$x \geq 0, y \geq 0$**

**Solution:**

Given,

Minimise Z = 200 x + 500 y … (1)

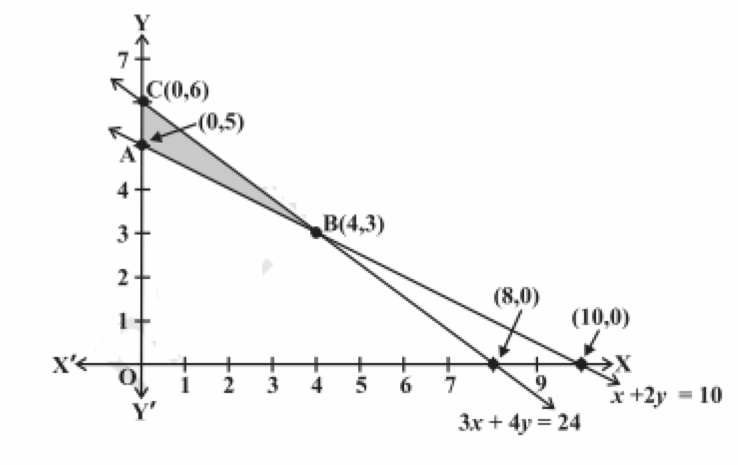
subject to the constraints:

$x + 2y \geq 10$ … (2)

$3x + 4y \leq 24$ … (3)

$x \geq 0, y \geq 0$ … (4)

Let us draw the graph of $x + 2y = 10$ and $3x + 4y = 24$ as below.




The shaded region in the above figure is the feasible region ABC determined by the

system of constraints (2) to (4), which is bounded. The coordinates of corner point A, B and C are (0,5), (4,3) and (0,6) respectively.

Calculation of Z = 200x + 500y at these points.

| Corner point | Value of Z |
|---|---|
| (0, 5) | 2500 |
| (4, 3) | 2300← Minimum |
| (0, 6) | 3000 |

Hence, the minimum value of Z is 2300 is at the point (4, 3).

**3: A manufacturing company makes two types of television sets; one is black and white and the other is colour. The company has resources to make at most 300 sets a week. It takes Rs 1800 to make a black and white set and Rs 2700 to make a coloured set. The company can spend not more than Rs 648000 a week to make television sets. If it makes a profit of Rs 510 per black and white set and Rs 675 per coloured set, how many sets of each type should be produced so that the company has a maximum profit? Formulate this problem as a LPP given that the objective is to maximise the profit.**

**Solution:**

Let x and y denote, respectively, the number of black and white sets and coloured sets made each week.

Thus $x \geq 0, y \geq 0$

The company can make at most 300 sets a week, therefore, $x + y \leq 300$.

Weekly cost (in Rs) of manufacturing the set is $1800x + 2700y$ and the company can spend up to Rs. 648000.

Therefore, $1800x + 2700y \leq 648000$

or

$2x + 3y \leq 720$

The total profit on x black and white sets and y coloured sets is Rs $(510x + 675y)$.

Let the objective function be $Z = 510x + 675y$.

Therefore, the mathematical formulation of the problem is as follows.

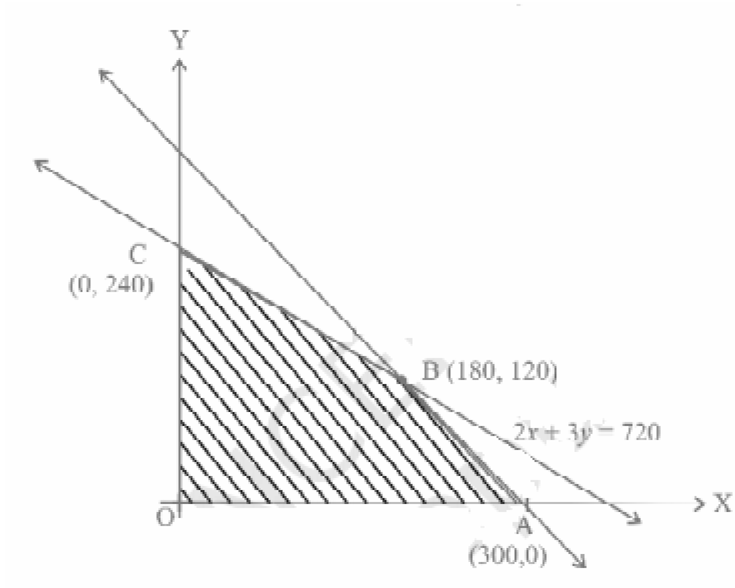
Maximise $Z = 510x + 675y$ subject to the constraints :

$x + y \leq 300$

$2x + 3y \leq 720$

$x \geq 0, y \geq 0$

The graph of x + y = 30 and 2x + 3y = 720 is given below.




| Corner point | Value of Z |
|---|---|
| A(300, 0) | 153000 |
| B(180, 120) | 172800 = Maximum |
| C(0, 240) | 162000 |

Hence, the maximum profit will occur when 180 black & white sets and 120 coloured sets are produced.

**4: A dietician wishes to mix two types of foods in such a way that vitamin contents of the mixture contain atleast 8 units of vitamin A and 10 units of vitamin C. Food 'I' contains 2 units/kg of vitamin A and 1 unit/kg of vitamin C. Food 'II' contains 1 unit/kg of vitamin A and 2 units/kg of vitamin C. It costs Rs 50 per kg to purchase Food 'I' and Rs 70 per kg to purchase Food 'II'. Formulate this problem as a linear programming problem to minimise the cost of such a mixture.**

**Solution:**

Let the mixture contain x kg of Food 'I' and y kg of Food 'II'.

Clearly, $x \geq 0$, $y \geq 0$.

Tabulate the given data as below.

| Resources | Food | | Requirement |
|---|---|---|---|
| | I (x) | II (y) | |
| Vitamin A (units/kg) | 2 | 1 | 8 |
| Vitamin C (units/kg) | 1 | 2 | 10 |
| Cost (Rs/kg) | 50 | 70 | |

Given that, the mixture must contain at least 8 units of vitamin A and 10 units of vitamin C.

Thus, the constraints are:

$2x + y \geq 8$

$x + 2y \geq 10$

Total cost Z of purchasing x kg of food 'I' and y kg of Food 'II' is $Z = 50x + 70y$

Hence, the mathematical formulation of the problem is:

Minimise Z = 50x + 70y … (1)

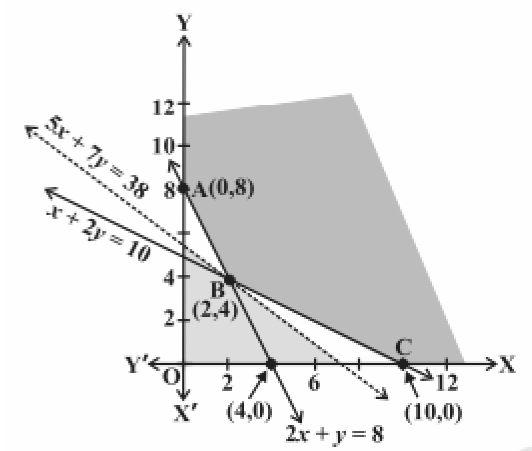
subject to the constraints:

2x + y ≥ 8 … (2)

x + 2y ≥ 10 … (3)

x, y ≥ 0 … (4)

Let us draw the graph of 2x + y = 8 and x + 2y = 10 as given below.




Here, observe that the feasible region is unbounded.

Let us evaluate the value of Z at the corner points A(0,8), B(2,4) and C(10,0).

| Corner point | Value of Z |
|---|---|
| A(0, 8) | 560 |
| B(2, 4) | 380 = Minimum |
| C(10, 0) | 500 |

Therefore, the minimum value of Z is 380 obtained at the point (2, 4).

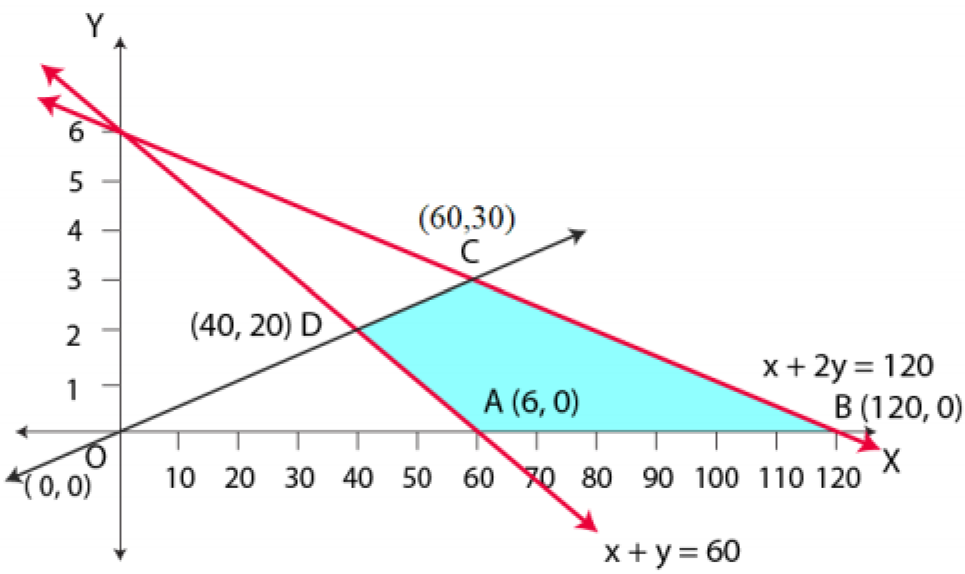
Hence, the optimal mixing strategy for the dietician would be to mix 2 kg of Food 'I' and 4 kg of Food 'II', and with this strategy, the minimum cost of the mixture will be Rs 380.

**5. Minimise and Maximise Z = 5*x* + 10*y***

**subject to** $x+2y\le 120, x+y\ge 60, x-2y\ge 0, x, y\ge 0$.

**Solution:**

The feasible region determined by the constraints, $x + 2y \le 120$, $x + y \ge 60$, $x - 2y \ge 0$, $x \ge 0$, and $y \ge 0$, is given below.




A (60, 0), B (120, 0), C (60, 30), and D (40, 20) are the corner points of the feasible region. The values of Z at these corner points are given below.

| Corner point | Z = 5x + 10y | |
|---|---|---|
| A (60, 0) | 300 | Minimum |
| B (120, 0) | 600 | Maximum |

| | | |
|---|---|---|
| C (60, 30) | 600 | Maximum |
| D (40, 20) | 400 | |

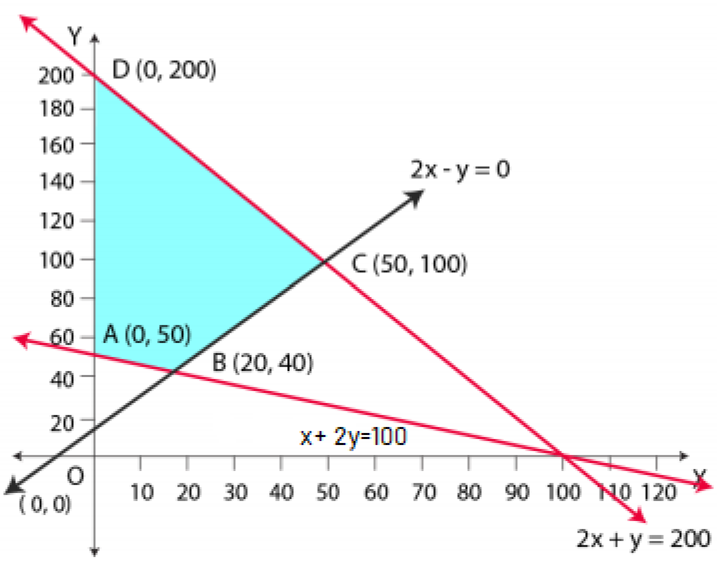
The minimum value of Z is 300 at (60, 0) and the maximum value of Z is 600 at all the points on the line segment joining (120, 0) and (60, 30).

**6. Minimise and Maximise Z = *x* + 2*y***

**subject to** $x+2y\geq 100,\ 2x-y\leq 0,\ 2x+y\leq 200;\ x,y\geq 0$ **.**

**Solution:**

The feasible region determined by the constraints, $x + 2y \geq 100$, $2x - y \leq 0$, $2x + y \leq 200$, $x \geq 0$, and $y \geq 0$, is given below.




A (0, 50), B (20, 40), C (50, 100) and D (0, 200) are the corner points of the feasible region. The values of Z at these corner points are given below.

| Corner point | Z = x + 2y | |
|---|---|---|
| A (0, 50) | 100 | Minimum |

| | | |
|---|---|---|
| B (20, 40) | 100 | Minimum |
| C (50, 100) | 250 | |
| D (0, 200) | 400 | Maximum |

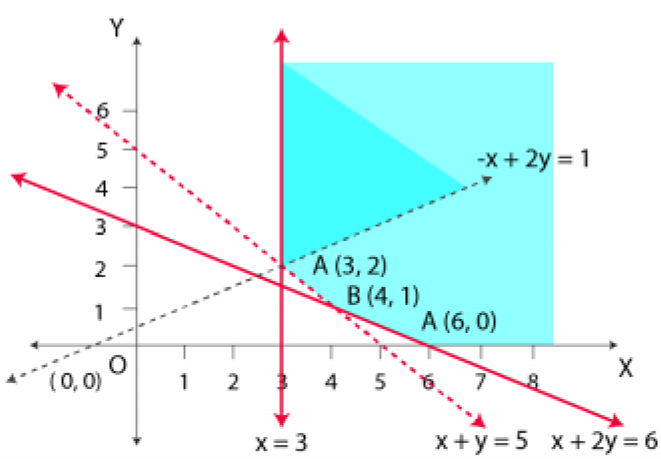
The maximum value of Z is 400 at points (0, 200), and the minimum value of Z is 100 at all the points on the line segment joining the points (0, 50) and (20, 40).

**7. Maximise Z = – *x* + 2*y*, subject to the constraints.**

$x \geq 3,\ x+y \geq 5,\ x+2y \geq 6,\ y \geq 0$.

**Solution:**

The feasible region determined by the constraints, $x \geq 3,\ x+y \geq 5,\ x+2y \geq 6,\ y \geq 0$ is given below.




Here, it can be seen that the feasible region is unbounded.

The values of Z at corner points A (6, 0), B (4, 1) and C (3, 2) are given below.

| Corner point | Z = – x + 2y |
|---|---|

| | |
|---|---|
| A (6, 0) | Z = – 6 |
| B (4, 1) | Z = – 2 |
| C (3, 2) | Z = 1 |

Since the feasible region is unbounded, z = 1 may or may not be the maximum value.

For this purpose, we graph the inequality, – x + 2y > 1, and check whether the resulting half-plane has points in common with the feasible region or not.

Here, the resulting feasible region has points in common with the feasible region.

Hence, z = 1 is not the maximum value.

Z has no maximum value.

**8.Reshma wishes to mix two types of food, P and Q, in such a way that the vitamin contents of the mixture contain at least 8 units of vitamin A, and 11 units of vitamin B. Food P costs Rs 60/kg, and Food Q costs Rs 80/kg. Food P contains 3 units /kg of vitamin A and 5 units /kg of vitamin B while food Q contains 4 units /kg of vitamin A and 2 units /kg of vitamin B. Determine the minimum cost of the mixture?**

**Solution:**

Let the mixture contain x kg of food P and y kg of food Q.

Hence, $x \geq 0$ and $y \geq 0$

The given information can be compiled in a table, as given below.

| | Vitamin A (units/kg) | Vitamin B (units/kg | Cost (Rs/kg) |
|---|---|---|---|
| Food P | 3 | 5 | 60 |

| | | | |
|---|---|---|---|
| Food Q | 4 | 2 | 80 |
| Requirement (units/kg) | 8 | 11 | |

The mixture must contain at least 8 units of vitamin A and 11 units of vitamin B. Hence, the constraints are

$3x + 4y \geq 8$

$5x + 2y \geq 11$

The total cost of purchasing food is Z = 60x + 80y.

So, the mathematical formulation of the given problem can be written as

Minimise Z = 60x + 80y (i)

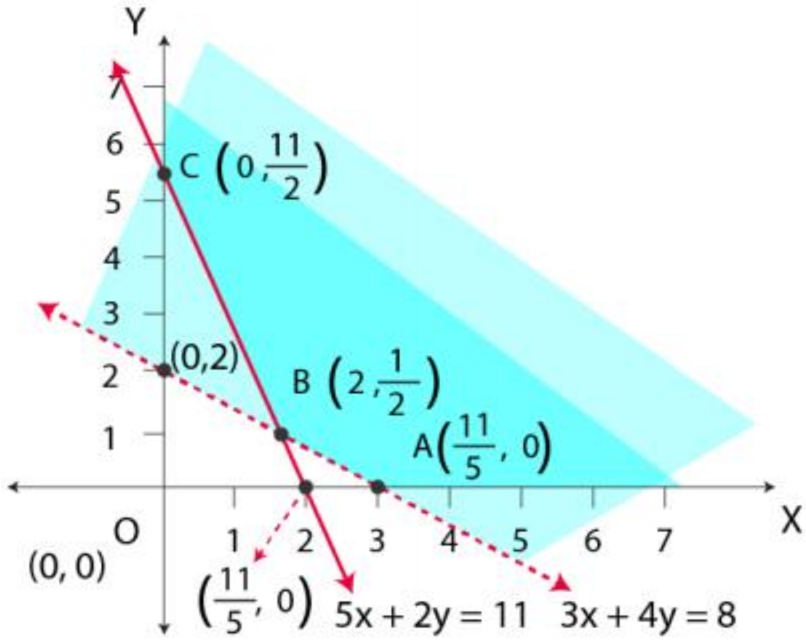
Now, subject to the constraints,

$3x + 4y \geq 8$ … (2)

$5x + 2y \geq 11$ … (3)

$x, y \geq 0$ … (4)

The feasible region determined by the system of constraints is given below.




Clearly, we can see that the feasible region is unbounded.

A (8 / 3, 0), B (2, 1 / 2) and C (0, 11 / 2)

The values of Z at these corner points are given below.

| Corner point | Z = 60x + 80 y | |
|---|---|---|
| A (8 / 3, 0) | 160 | Minimum |
| B (2, 1 / 2) | 160 | Minimum |
| C (0, 11 / 2) | 440 | |

Here, the feasible region is unbounded; therefore, 160 may or may not be the minimum value of Z.

For this purpose, we graph the inequality, $60x + 80y < 160$ or $3x + 4y < 8$, and check whether the resulting half-plane has points in common with the feasible region or not.

Here, it can be seen that the feasible region has no common point with $3x + 4y < 8$.

Hence, at the line segment joining the points (8 / 3, 0) and (2, 1 / 2), the minimum cost of the mixture will be Rs 160.

**9.A factory manufacturers two types of screws, A and B. Each type of screw requires the use of two machines, an automatic and a hand operated. It takes 4 minutes on the automatic and 6 minutes on hand operated machines to manufacture a package of screws A, while it takes 6 minutes on automatic and 3 minutes on the hand operated machines to manufacture a package of screws B. Each machine is available for at the most 4 hours on any day. The manufacturer can sell a package of screws A at a profit of Rs 7 and screws B at a profit of Rs 10. Assuming that he can sell all the screws he manufactures, how many packages of each type should the factory owner produce in a day in order to maximise his profit? Determine the maximum profit.**

**Solution:**

On each day, let the factory manufacture x screws of type A and y screws of type B.

Hence,

$x \geq 0$ and $y \geq 0$

The given information can be compiled in a table, as given below.

| | Screw A | Screw B | Availability |
|---|---|---|---|
| Automatic Machine (min) | 4 | 6 | $4 \times 60 = 240$ |
| Hand Operated Machine (min) | 6 | 3 | $4 \times 60 = 240$ |

The profit on a package of screws A is Rs 7 and on the package screws, B is Rs 10.

Hence, the constraints are

$4x + 6y \leq 240$

$6x + 3y \leq 240$

Total profit, $Z = 7x + 10y$

The mathematical formulation of the given problem can be written as

Maximise $Z = 7x + 10y$ ………… (i)

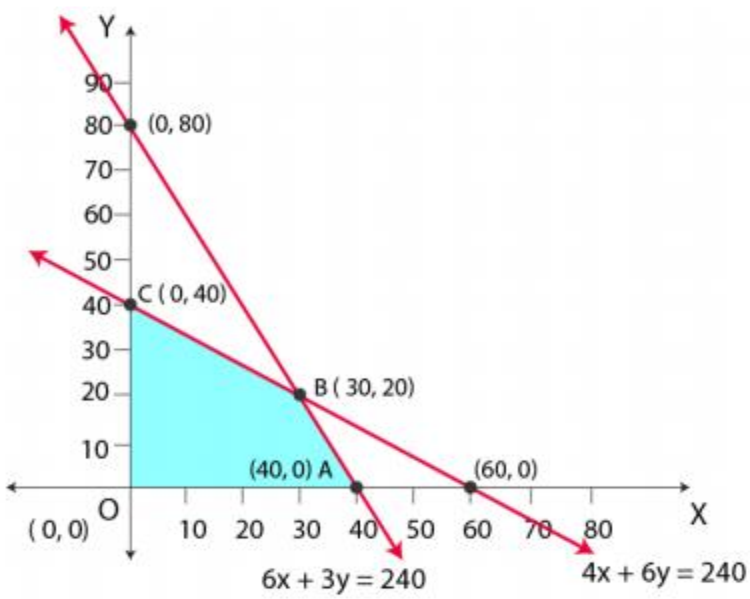
Subject to the constraints,

$4x + 6y \leq 240$ ………… (ii)

$6x + 3y \leq 240$ ………… (iii)

$x, y \geq 0$ ……………… (iv)

The feasible region determined by the system of constraints is given below.




A (40, 0), B (30, 20) and C (0, 40) are the corner points.

The value of Z at these corner points is given below.

| Corner point | $Z = 7x + 10y$ | |
|---|---|---|
| A (40, 0) | 280 | |

| | | |
|---|---|---|
| B (30, 20) | 410 | Maximum |
| C (0, 40) | 400 | |

The maximum value of Z is 410 at (30, 20).

Hence, the factory should produce 30 packages of screws A and 20 packages of screws B to get the maximum profit of Rs 410.

**10.A cottage industry manufactures pedestal lamps and wooden shades, each requiring the use of a grinding/cutting machine and a sprayer. It takes 2 hours on the grinding/cutting machine and 3 hours on the sprayer to manufacture a pedestal lamp. It takes 1 hour on the grinding/cutting machine and 2 hours on the sprayer to manufacture a shade. On any day, the sprayer is available for at the most 20 hours and the grinding/cutting machine for at the most 12 hours. The profit from the sale of a lamp is Rs 5 and that from a shade is Rs 3. Assuming that the manufacturer can sell all the lamps and shade that he produces, how should he schedule his daily production in order to maximise his profit?**

**Solution:**

Let the cottage industry manufacture x pedestal lamps and y wooden shades, respectively.

Hence,

$x \geq 0$ and $y \geq 0$

The given information can be compiled in a table, as given below.

| | Lamps | Shades | Availability |
|---|---|---|---|
| Grinding/Cutting Machine (h) | 2 | 1 | 12 |
| Sprayer (h) | 3 | 2 | 20 |

The profit on a lamp is Rs 5 and on the shades is Rs 3. Hence, the constraints are

$2x + y \leq 12$

$3x + 2y \leq 20$

Total profit, $Z = 5x + 3y$ ................ (i)

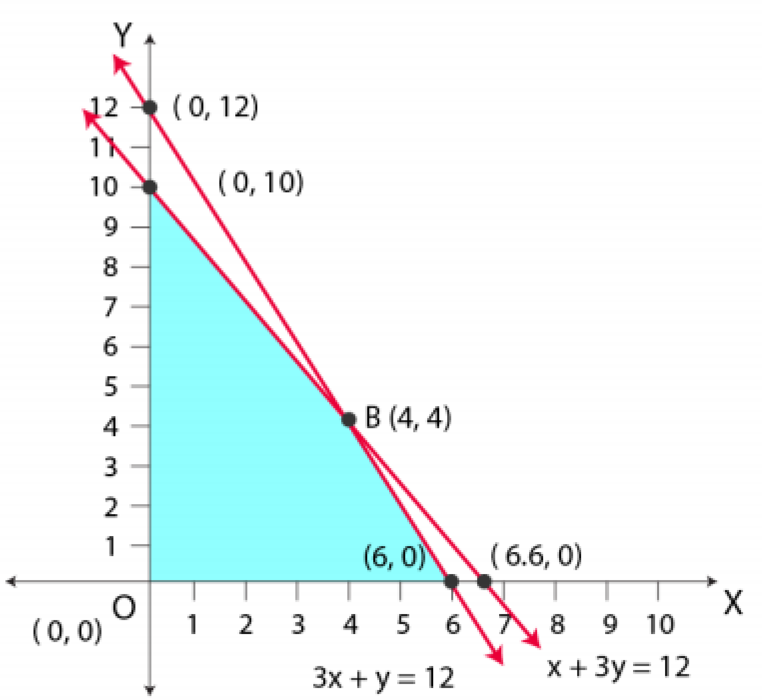
Subject to the constraints,

$2x + y \leq 12$ ............ (ii)

$3x + 2y \leq 20$ ........... (iii)

$x, y \geq 0$ ............ (iv)

The feasible region determined by the system of constraints is given below.




A (6, 0), B (4, 4) and C (0, 10) are the corner points.

The value of Z at these corner points is given below.

| Corner point | $Z = 5x + 3y$ | |
|---|---|---|

| | | |
|---|---|---|
| A (6, 0) | 30 | |
| B (4, 4) | 32 | Maximum |
| C (0, 10) | 30 | |

The maximum value of Z is 32 at points (4, 4).

Therefore, the manufacturer should produce 4 pedestal lamps and 4 wooden shades to maximise his profits.

**11. A company manufactures two types of novelty souvenirs made of plywood. Souvenirs of type A require 5 minutes each for cutting and 10 minutes each for assembling. Souvenirs of type B require 8 minutes each for cutting and 8 minutes each for assembling. There are 3 hours 20 minutes available for cutting and 4 hours for assembling. The profit is Rs 5 each for type A and Rs 6 each for type B souvenirs. How many souvenirs of each type should the company manufacture in order to maximise the profit?**

**Solution:**

Let the company manufacture x souvenirs of type A and y souvenirs of type B, respectively.

Hence,

$x \geq 0$ and $y \geq 0$

The given information can be compiled in a table, as given below.

| | Type A | Type B | Availability |
|---|---|---|---|
| Cutting (min) | 5 | 8 | $3 \times 60 + 20 = 200$ |
| Assembling (min) | 10 | 8 | $4 \times 60 = 240$ |

The profit on type A souvenirs is Rs 5 and on type B souvenirs is Rs 6. Hence, the constraints are

$5x + 8y \leq 200$

$10x + 8y \leq 240$ i.e.,

$5x + 4y \leq 120$

Total profit, $Z = 5x + 6y$

The mathematical formulation of the given problem can be written as

Maximise $Z = 5x + 6y$ …………… (i)

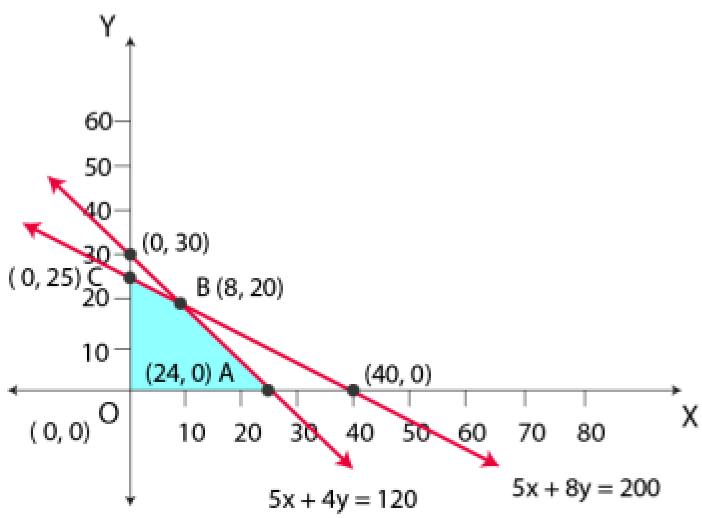
Subject to the constraints,

$5x + 8y \leq 200$ ……………. (ii)

$5x + 4y \leq 120$ …………. (iii)

$x, y \geq 0$ …………. (iv)

The feasible region determined by the system of constraints is given below.




A (24, 0), B (8, 20) and C (0, 25) are the corner points.

The values of Z at these corner points are given below.

| Corner point | Z = 5x + 6y | |
|---|---|---|
| A (24, 0) | 120 | |
| B (8, 20) | 160 | Maximum |
| C (0, 25) | 150 | |

The maximum value of Z is 200 at (8, 20).

Hence, 8 souvenirs of type A and 20 souvenirs of type B should be produced each day to get the maximum profit of Rs 160.

**12.A merchant plans to sell two types of personal computers – a desktop model and a portable model that will cost Rs 25,000 and Rs 40,000, respectively. He estimates that the total monthly demand of computers will not exceed 250 units. Determine the number of units of each type of computers which the merchant should stock to get maximum profit if he does not want to invest more than Rs 70 lakhs and if his profit on the desktop model is Rs 4,500 and on the portable model is Rs 5,000.**

**Solution:**

Let the merchant stock x desktop models and y portable models, respectively.

Hence,

$x \geq 0$ and $y \geq 0$

Given that the cost of desktop model is Rs 25,000 and of a portable model is Rs 40,000.

However, the merchant can invest a maximum of Rs 70 lakhs.

Hence, $25000x + 40000y \leq 7000000$

$5x + 8y \leq 1400$

The monthly demand of computers will not exceed 250 units.

Hence, $x + y \le 250$

The profit on a desktop model is 4500, and the profit on a portable model is Rs 5000.

Total profit, $Z = 4500x + 5000y$

Therefore, the mathematical formulation of the given problem is

Maximum $Z = 4500x + 5000y$ ………… (i)

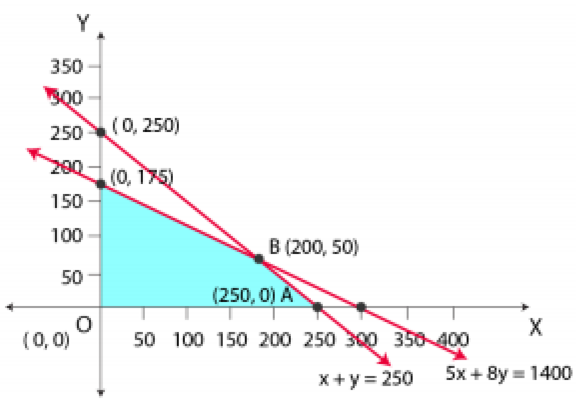
Subject to the constraints,

$5x + 8y \le 1400$ ………… (ii)

$x + y \le 250$ ………….. (iii)

$x, y \ge 0$ …………. (iv)

The feasible region determined by the system of constraints is given below.




A (250, 0), B (200, 50) and C (0, 175) are the corner points.

The values of Z at these corner points are given below.

| Corner point | $Z = 4500x + 5000y$ | |
|---|---|---|

| | | |
|---|---|---|
| A (250, 0) | 1125000 | |
| B (200, 50) | 1150000 | Maximum |
| C (0, 175) | 875000 | |

The maximum value of Z is 1150000 at (200, 50).

Therefore, the merchant should stock 200 desktop models and 50 portable models to get the maximum profit of Rs 11,50,000.

**13. A diet is to contain at least 80 units of vitamin A and 100 units of minerals. Two foods $F_1$ and $F_2$ are available. Food $F_1$ costs Rs 4 per unit food and $F_2$ costs Rs 6 per unit. One unit of food $F_1$ contains 3 units of vitamin A and 4 units of minerals. One unit of food $F_2$ contains 6 units of vitamin A and 3 units of minerals. Formulate this as a linear programming problem. Find the minimum cost for the diet that consists of mixture of these two foods and also meets the minimal nutritional requirements.**

**Solution:**

Let the diet contain x units of food $F_1$ and y units of food $F_2$. Hence,

$x \geq 0$ and $y \geq 0$

The given information can be compiled in a table, as given below.

| | Vitamin A (units) | Mineral (units) | Cost per unit (Rs) |
|---|---|---|---|

| | | | |
|---|---|---|---|
| Food $F_1$ (x) | 3 | 4 | 4 |
| Food $F_2$ (y) | 6 | 3 | 6 |
| Requirement | 80 | 100 | |

The cost of food $F_1$ is Rs 4 per unit, and of food, $F_2$ is Rs 6 per unit.

Hence, the constraints are

$3x + 6y \geq 80$

$4x + 3y \geq 100$

$x, y \geq 0$

The total cost of the diet, $Z = 4x + 6y$

The mathematical formulation of the given problem can be written as

Minimise $Z = 4x + 6y$ …………… (i)

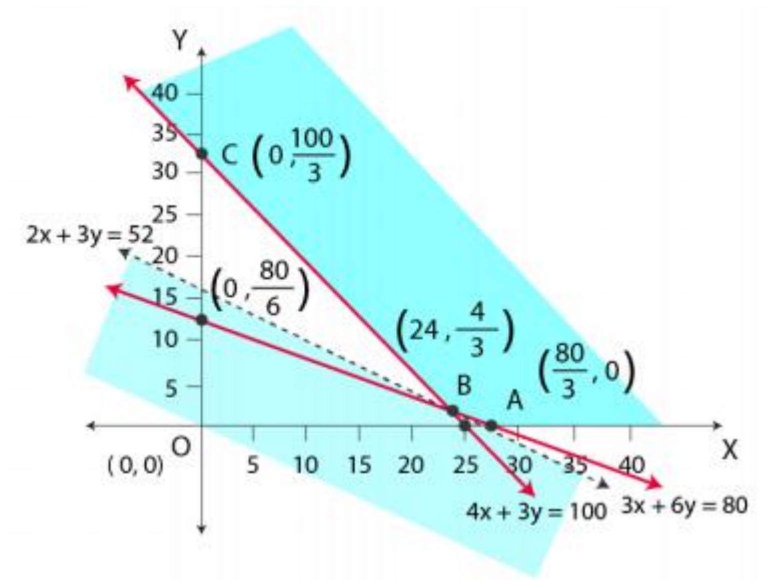
Subject to the constraints,

$3x + 6y \geq 80$ ………… (ii)

$4x + 3y \geq 100$ ………. (iii)

$x, y \geq 0$ …………. (iv)

The feasible region determined by the constraints is given below.




We can see that the feasible region is unbounded.

A (80 / 3, 0), B (24, 4 / 3), and C (0, 100 / 3) are the corner points.

The values of Z at these corner points are given below.

| Corner point | Z = 4x + 6y | |
|---|---|---|
| A (80 / 3, 0) | 320 / 3 = 106.67 | |
| B (24, 4 / 3) | 104 | Minimum |
| C (0, 100 / 3) | 200 | |

Here, the feasible region is unbounded, so 104 may or not be the minimum value of Z.

For this purpose, we draw a graph of the inequality, 4x + 6y < 104 or 2x + 3y < 52, and check whether the resulting half-plane has points in common with the feasible region or not.

Here, it can be seen that the feasible region has no common point with 2x + 3y < 52.

Hence, the minimum cost of the mixture will be Rs 104.

**14.There are two types of fertilisers, $F_1$ and $F_2$. $F_1$ consists of 10% nitrogen and 6% phosphoric acid, and $F_2$ consists of 5% nitrogen and 10% phosphoric acid. After testing the soil conditions, a farmer finds that she needs atleast 14 kg of nitrogen and 14 kg of phosphoric acid for her crop. If $F_1$ costs Rs 6/kg and $F_2$ costs Rs 5/kg, determine how much of each type of fertiliser should be used so that nutrient requirements are met at a minimum cost. What is the minimum cost?**

**Solution:**

Let the farmer buy x kg of fertiliser $F_1$ and y kg of fertiliser $F_2$. Hence,

$x \geq 0$ and $y \geq 0$

The given information can be compiled in a table, as given below.

| | Nitrogen (%) | Phosphoric Acid (%) | Cost (Rs / kg) |
|---|---|---|---|
| $F_1$ (x) | 10 | 6 | 6 |
| $F_2$ (y) | 5 | 10 | 5 |
| Requirement (kg) | 14 | 14 | |

$F_1$ consists of 10% nitrogen, and $F_2$ consists of 5% nitrogen.

However, the farmer requires at least 14 kg of nitrogen.

So, 10% of x + 5% of y ≥ 14

$x / 10 + y / 20 \geq 14$

By L.C.M we get

$2x + y \geq 280$

$F_1$ consists of 6% phosphoric acid, and $F_2$ consists of 10% phosphoric acid.

However, the farmer requires at least 14 kg of phosphoric acid.

So, 6% of x + 10 % of y ≥ 14

$6x / 100 + 10y / 100 \geq 14$

$3x + 5y \geq 700$

The total cost of fertilisers, $Z = 6x + 5y$

The mathematical formulation of the given problem can be written as

Minimise $Z = 6x + 5y$ ………….. (i)

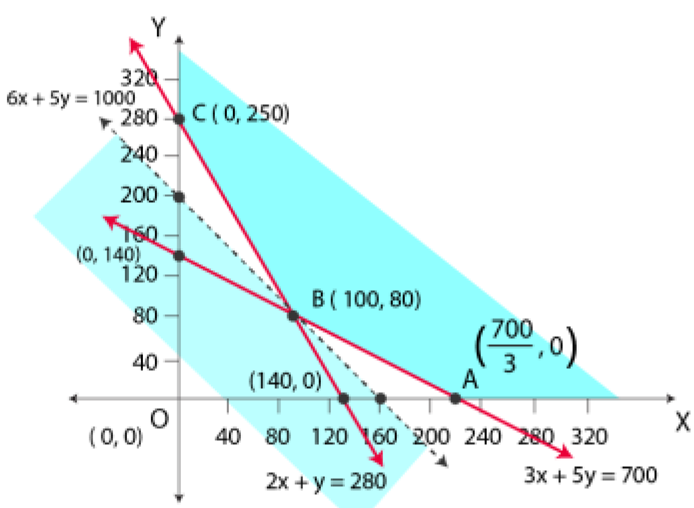
Subject to the constraints,

$2x + y \geq 280$ ……… (ii)

$3x + 5y \geq 700$ ………. (iii)

$x, y \geq 0$ …………. (iv)

The feasible region determined by the system of constraints is given below.




Here, we can see that the feasible region is unbounded.

A (700 / 3, 0), B (100, 80) and C (0, 280) are the corner points.

The values of Z at these points are given below.

| Corner point | $Z = 6x + 5y$ | |
|---|---|---|
| A (700 / 3, 0) | 1400 | |
| B (100, 80) | 1000 | Minimum |
| C (0, 280) | 1400 | |

Here, the feasible region is unbounded; hence, 1000 may or may not be the minimum value of Z.

For this purpose, we draw a graph of the inequality, $6x + 5y < 1000$, and check whether the resulting half-plane has points in common with the feasible region or not.

Here, it can be seen that the feasible region has no common point with $6x + 5y < 1000$

Hence, 100 kg of fertiliser $F_1$ and 80 kg of fertiliser $F_2$ should be used to minimise the cost. The minimum cost is Rs 1000.

**15. The corner points of the feasible region are determined by the following system of linear inequalities:**

**$2x + y \le 10$, $x + 3y \le 15$, $x, y \ge 0$ are (0, 0), (5, 0), (3, 4) and (0, 5). Let $Z = px + qy$, where $p, q > 0$. Condition on p and q so that the maximum of Z occurs at both (3, 4) and (0, 5) is**

**(A) $p = q$**

**(B) $p = 2q$**

**(C) $p = 3q$**

**(D) $q = 3p$**

**Solution:**

The maximum value of Z is unique.

Here, it is given that the maximum value of Z occurs at two points, (3, 4) and (0, 5).

Value of Z at (3, 4) = Value of Z at (0, 5)

$p(3) + q(4) = p(0) + q(5)$

$3p + 4q = 5q$

$3p = 5q - 4q$

$3p = q$ or $q = 3p$

Therefore, the correct answer is option (D).

# CHAPTER-XIII

# PROBABILITY

## 2 MARK QUESTIONS

**1. Given that E and F are events, such that P (E) = 0.6, P (F) = 0.3 and P (E ∩ F) = 0.2, find P (E|F) and P (F|E).**

**Solution:**

Given P (E) = 0.6, P (F) = 0.3 and P (E ∩ F) = 0.2

We know that by the definition of conditional probability,

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

By substituting the values we get

$$\Rightarrow P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{0.2}{0.3} = \frac{2}{3}$$

And $\Rightarrow P(F|E) = \frac{P(E \cap F)}{P(E)} = \frac{0.2}{0.6} = \frac{2}{6} = \frac{1}{3}$

**2. Compute P (A|B), if P (B) = 0.5 and P (A ∩ B) = 0.32**

**Solution:**

Given: P (B) = 0.5 and P (A ∩ B) = 0.32

We know that by definition of conditional probability,

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Now by substituting the values we get

$$\Rightarrow P(A|B) = \frac{0.32}{0.5} = \frac{32}{50} = \frac{16}{25}$$

**3. If P (A) = 0.8, P (B) = 0.5 and P (B|A) = 0.4, find**
**(i) P (A ∩ B)**

**(ii) P (A|B)**

**(iii) P (A ∪ B)**

**Solution:**

Given P (A) = 0.8, P (B) = 0.5 and P (B|A) = 0.4

(i) We know that by definition of conditional probability,

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

⇒ P (A ∩ B) = P (B|A) P (A)

⇒ P (A ∩ B) = 0.4 × 0.8

⇒ P (A ∩ B) = 0.32

(ii) We know that by definition of conditional probability,

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Now substituting the values we get

$$\Rightarrow P(A|B) = \frac{0.32}{0.5} = 0.64$$

$\Rightarrow$ P (A|B) = 0.64

(iii) Now, $\because$ P (A ∪ B) = P (A) + P (B) – P (A ∩ B)

Substituting the values we get

$\Rightarrow$ P (A ∪ B) = 0.8 + 0.5 – 0.32 = 1.3 – 0.32

$\Rightarrow$ P (A ∪ B) = 0.98

**4. Evaluate P (A ∪ B), if 2P (A) = P (B) = 5/13 and P (A|B) = 2/5.**

**Solution:**

Given $2P(A) = P(B) = \frac{5}{13}$ and $P(A|B) = \frac{2}{5}$

$\Rightarrow P(B) = \frac{5}{13}, P(A) = \frac{5}{13\times2} = \frac{5}{26}, P(A|B) = \frac{2}{5}$ .......... (i)

We know that by definition of conditional probability,

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

⇒ P (A ∩ B) = P (A|B) P (B)

$$\Rightarrow \; P(A \cap B) = \frac{2}{5} \times \frac{5}{13} = \frac{2}{13} \text{ .......... (ii)}$$

Now, ∵ P (A * B) = P (A) + P (B) – P (A ∩ B)

$$\Rightarrow P(A \cup B) = \frac{5}{26} + \frac{5}{13} - \frac{2}{13} = \frac{5 + 10 - 4}{26} = \frac{15 - 4}{26}$$

$$\Rightarrow P(A \cup B) = \frac{11}{26}$$

# 4 MARK QUESTIONS

**1: A die is thrown twice and the sum of the numbers rising is noted to be 6. Calculate the is the conditional probability that the number 4 has arrived at least once?**

**Solution:**

If a dice is thrown twice, then the sample space obtained is:

S = {(1,1)(1,2)(1,3)(1,4)(1,5)(1,6)

(2,1)(2,2)(2,3)(2,4)(2,5)(2,6)

(3,1)(3,2)(3,3)(3,4)(3,5)(3,6)

(4,1)(4,2)(4,3)(4,4)(4,5)(4,6)

(5,1)(5,2)(5,3)(5,4)(5,5)(5,6)

(6,1)(6,2)(6,3)(6,4)(6,5)(6,6)}

From the given data, it is needed to find the Probability that 4 has appeared at least once, given the sum of nos. is observed to be 6

Assume that, F: Addition of numbers is 6

and take E: 4 has appeared at least once

So, that, we need to find P(E|F)

**Finding P (E):**

The probability of getting 4 atleast once is:

E = {(1, 4), (2, 4), (3, 4), (4, 4), (5, 4), (6, 4), (4, 1), (4, 2), (4, 3), (4, 5), (4, 6)}

Thus , P(E) = 11/ 36

**Finding P (F):**

The probability to get the addition of numbers is 6 is:

F = {(1, 5), (5, 1), (2, 4), (4, 2), (3, 3)}

Thus, P(F) = 5/ 36

Also, $E \cap F = \{(2,4), (4,2)\}$

$P(E \cap F) = 2/36$

Thus, $P(E|F) = (P(E \cap F) ) / (P (F) )$

Now, subsbtitute the probability values obtained= (2/36)/ (5/36)

Hence, Required probability is 2/5.

**2: The probability of solving the specific problem independently by the persons' A and B are 1/2 and 1/3 respectively. In case, if both the persons try to solve the problem independently, then calculate the probability that the problem is solved.**

**Solution:**

Given that, the two events say A and B are independent if $P(A \cap B) = P(A). P(B)$

From the given data, we can observe that P(A) = 1/2 & P(B) = 1/3

The probability that the problem is solved = Probability that person A solves the problem or the person B solves the Problem

This can be written as:

$= P(A \cup B) = P(A) + P(B) – P(A \cap B)$

If A and B are independent, then $P(A \cap B) = P(A). P(B)$

Now, substitute the values,

= (1/2) × (1/3)

$P(A \cap B) = 1/6$

Now, the probability of problem solved is written as

P(Problem is solved) = P(A) + P(B) – P(A ∩ B)

= (1/2) + (1/3) – (1/6)

= (3/6) + (2/6) – (1/6)

= 4/6

= 2/3

Hence, the probability of the problem solved is 2/3.

**3: 5 cards are drawn successively from a well-shuffled pack of 52 cards with replacement. Determine the probability that (i) all the five cards should be spades? (ii) only 3 cards should be spades? (iii) none of the cards is a spade?**

**Solution:**

Let us assume that X be the number of spade cards

Using the Bernoulli trial, X has a binomial distribution

$P(X = x) = {}^{n}C_{x}\, q^{n-x}\, p^{x}$

Thus, the number of cards drawn, n = 5

Probability of getting spade card, p = 13/52 = 1/4

Thus the value of the q can be found using

q = 1 – p = 1 – (1/4)= 3/4

Now substitute the p and q values in the formula,

Hence, $P(X = x) = {}^{5}C_{x}\, (3/4)^{5-x}(1/4)^{x}$

**(1) Probability of Getting all the spade cards:**

P(all the five cards should be spade) = ${}^5C_5$ $(1/4)^5(3/4)^0$

= $(1/4)^5$

= 1/1024

**(2) Probability of Getting only three spade cards:**

P(only three cards should be spade) = ${}^5C_3$ $(1/4)^3(3/4)^2$

= (5!/3! 2!) × (9/1024)

= 45/ 512

**(3) Probability of Getting no spades:**

P(none of the cards is a spade) = ${}^5C_0(1/4)^0(3/4)^5$

= $(3/4)^5$

= 243/ 1024

**4: An fair die is thrown double times. Assume that the event A is "odd number on the first throw" and B the event "odd number on the second throw". Compare the independence of the events A and B.**

**Solution:**

Let us consider two independent events A and B, then P(A ∩ B) = P(A). P(B)

when an unbiased die is thrown twice

S = {(1,1)(1,2)(1,3)(1,4)(1,5)(1,6)

(2,1)(2,2)(2,3)(2,4)(2,5)(2,6)

(3,1)(3,2)(3,3)(3,4)(3,5)(3,6)

(4,1)(4,2)(4,3)(4,4)(4,5)(4,6)

(5,1)(5,2)(5,3)(5,4)(5,5)(5,6)

(6,1)(6,2)(6,3)(6,4)(6,5)(6,6)}

Let us describe two events as

A: odd number on the first throw

B: odd number on the second throw

**To find P(A)**

A = {(1, 1), (1, 2), (1, 3), …, (1, 6)

(3, 1), (3, 2), (3, 3), …, (3, 6)

(5, 1), (5, 2), (5, 3), …, (5, 6)}

Thus, P (A) = 18/36 = 1/2

**To find P(B)**

B = {(1, 1), (2, 1), (3, 1), …, (6, 1)

(1, 3), (2, 3), (3, 3), …, (6, 3)

(1, 5), (2, 5), (3, 5), …, (6, 5)}

Thus, P (B) = 18/36 = 1/2

A ∩ B = odd number on the first & second throw = { (1, 1), (1, 3), (1, 5), (3, 1), (3, 3), (3, 5), (5, 1), (5, 3), (5, 5)}

So, P(A ∩ B) = 9/36 = 1/ 4

Now, P(A). P(B) = (1/2) × (1/2) = 1/4

As P(A ∩ B) = P(A). P(B),

Hence, the two events A and B are independent events.

**5. If P (A) = 6/11, P (B) = 5/11 and P (A ∪ B) = 7/11, find**
**(i) P (A∩B)**

**(ii) P (A|B)**

**(iii) P (B|A)**

**Solution:**

Given: $P(A) = \frac{6}{11}, P(B) = \frac{5}{11}, P(A \cup B) = \frac{7}{11}$

(i) We know that P (A * B) = P (A) + P (B) – P (A ∩ B)

⇒ P (A ∩ B) = P (A) + P (B) – P (A ∪ B)

$$\Rightarrow P(A \cap B) = \frac{6}{11} + \frac{5}{11} - \frac{7}{11} = \frac{11-7}{11}$$

$$\Rightarrow P(A \cap B) = \frac{4}{11}$$

(ii) Now, by definition of conditional probability,

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$\Rightarrow P(A|B) = \frac{4/11}{5/11}$$

$$\Rightarrow P(A|B) = \frac{4}{5}$$

(iii) Again, by definition of conditional probability,

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

$$\Rightarrow P(B|A) = \frac{4/11}{6/11} = \frac{4}{6} = \frac{2}{3}$$

$$\Rightarrow P(B|A) = \frac{2}{3}$$

**Determine P (E|F) in Exercises 6 to 9.**

**6. A coin is tossed three times, where**
**(i) E : head on the third toss, F : heads on first two tosses**
**(ii) E : at least two heads, F : at most two heads**
**(iii) E : at most two tails, F : at least one tail**

**Solution:**

The sample space of the given experiment will be:

S = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}

(i) Here, E: head on third toss

And F: heads on first two tosses

⇒ E = {HHH, HTH, THH, TTH} and F = {HHH, HHT}

⇒ E ∩ F = {HHH}

So, $P(E) = \frac{4}{8} = \frac{1}{2}, P(F) = \frac{2}{8} = \frac{1}{4}, P(E \cap F) = \frac{1}{8}$

Now, we know that by definition of conditional probability,

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

$$\Rightarrow P(E|F) = \frac{1/8}{1/4} = \frac{4}{8} = \frac{1}{2}$$

$$\Rightarrow P(E|F) = \frac{1}{2}$$

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

$$\Rightarrow P(E|F) = \frac{1/8}{1/4} = \frac{4}{8} = \frac{1}{2}$$

$$\Rightarrow P(E|F) = \frac{1}{2}$$

(ii) Here, E: at least two heads

And F: at most two heads

⇒ E = {HHH, HHT, HTH, THH} and F = {HHT, HTH, THH, HTT, THT, TTH, TTT}

⇒ E ∩ F = {HHT, HTH, THH}

So, $P(E) = \frac{4}{8} = \frac{1}{2}, P(F) = \frac{7}{8}, P(E \cap F) = \frac{3}{8}$

Now, we know that

By definition of conditional probability,

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

$$\Rightarrow P(E|F) = \frac{3/8}{7/8} = \frac{3}{7}$$

(iii) Here, E: at most two tails

And F: at least one tail

⇒ E = {HHH, HHT, HTH, THH, HTT, THT, TTH}

And F = {HHT, HTH, THH, HTT, THT, TTH, TTT}

So, $P(E) = \frac{7}{8}, P(F) = \frac{7}{8}, P(E \cap F) = \frac{6}{8} = \frac{3}{4}$

Now, we know that

By definition of conditional probability,

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

$$\Rightarrow P(E|F) = \frac{3/4}{7/8} = \frac{6}{7}$$

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

$$\Rightarrow P(E|F) = \frac{3/4}{7/8} = \frac{6}{7}$$

# 7 MARK QUESTIONS

**1: Given that the events A and B are such that P(A) = 1/2, P (A ∪ B) = 3/5, and P(B) = p. Find p if they are**

**(i) mutually exclusive**

**(ii) independent**

**Solution:**

Given, P(A) = 1/2 ,

P (A ∪ B) = 3/5

and P(B) = p.

**(1) For Mutually Exclusive**

Given that, the sets A and B are mutually exclusive.

Thus, they do not have any common elements

Therefore, P(A ∩ B) = 0

We know that P(A ∪ B) = P(A) + P(B) – P(A ∩ B)

Substitute the formulas in the above-given formula, we get

3/5 = (1/2) + p – 0

Simplify the expression, we get

(3/5) – (1/2) = p

(6 − 5)/10 = p

1/10 = p

Therefore, p = 1/10

Hence, the value of p is 1/10, if they are mutually exclusive.

**(ii) For Independent events:**

If the two events A & B are independent,

we can write it as $P(A \cap B) = P(A) P(B)$

Substitute the values,

$= (1/2) \times p$

$= p/2$

Now, $P(A \cup B) = P(A) + P(B) – P(A \cap B)$

Now, substitute the values in the formula,

$(3/5) = (1/2)+ p – (p/2)$

$(3/2)– (1/2)= p – (p/2)$

$(6 – 5)/10 = p/2$

$1/10 = p/2$

$p= 2/10$

$P = 1/5$

Thus, the value of p is 1/5, if they are independent.

**2. Two coins are tossed once, where**
**(i) E: tail appears on one coin, F: one coin shows the head**
**(ii) E: no tail appears, F: no head appears**

**Solution:**

The sample space of the given experiment is S = {HH, HT, TH, TT}

(i) Here, E: tail appears on one coin

And F: one coin shows head

⇒ E = {HT, TH} and F = {HT, TH}

⇒ E ∩ F = {HT, TH}

So, $P(E) = \frac{2}{4} = \frac{1}{2}, P(F) = \frac{2}{4} = \frac{1}{2}, P(E \cap F) = \frac{2}{4} = \frac{1}{2}$

Now, we know that by definition of conditional probability,

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

Substituting the values we get

$$\Rightarrow P(E|F) = \frac{1/2}{1/2}$$

⇒ P (E|F) = 1

(ii) Here, E: no tail appears

And F: no head appears

⇒ E = {HH} and F = {TT}

⇒ E ∩ F = φ

So, $P(E) = \frac{1}{4}, P(F) = \frac{1}{4}, P(E \cap F) = \frac{0}{4} = 0$

Now, we know that by definition of conditional probability,

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

Substituting the values we get

$$\Rightarrow P(E|F) = \frac{0}{1/4}$$

$\Rightarrow$ P (E|F) = 0

**3. A die is thrown three times, E: 4 appears on the third toss, F: 6 and 5 appear, respectively, on the first two tosses.**

**Solution:**

The sample space has 216 outcomes, where each element of the sample space has 3 entries and is of the form (x, y, z) where $1 \le x, y, z \le 6$.

Here, E: 4 appears on the third toss

$$\Rightarrow E = \left\{\begin{matrix}(1,1,4),(1,2,4),(1,3,4),(1,4,4),(1,5,4),(1,6,4),\\(2,1,4),(2,2,4),(2,3,4),(2,4,4),(2,5,4),(2,6,4),\\(3,1,4),(3,2,4),(3,3,4),(3,4,4),(3,5,4),(3,6,4),\\(4,1,4),(4,2,4),(4,3,4),(4,4,4),(4,5,4),(4,6,4),\\(5,1,4),(5,2,4),(5,3,4),(5,4,4),(5,5,4),(5,6,4),\\(6,1,4),(6,2,4),(6,3,4),(6,4,4),(6,5,4),(6,6,4)\end{matrix}\right\}$$

Now, F: 6 and 5 appears respectively on first two tosses

$\Rightarrow$ F = {(6, 5, 1), (6, 5, 2), (6, 5, 3), (6, 5, 4), (6, 5, 5), (6, 5, 6)}

$\Rightarrow$ E ∩ F = {(6, 5, 4)}

So, $P(E) = \frac{36}{216}, P(F) = \frac{6}{216}, P(E \cap F) = \frac{1}{216}$

Now, we know that by definition of conditional probability,

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

$$\Rightarrow P(E|F) = \frac{1/216}{6/216} = \frac{1}{6}$$

$$\Rightarrow P(E|F) = \frac{1}{6}$$

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

Now by substituting the values we get

$$\Rightarrow P(E|F) = \frac{1/216}{6/216} = \frac{1}{6}$$

$$\Rightarrow P(E|F) = \frac{1}{6}$$

**4. Mother, father and son line up at random for a family picture**
**E: son on one end, F: father in the middle.**

**Solution:**

Let M denotes mother, F denotes father, and S denotes son.

Then, the sample space for the given experiment will be

S = {MFS, SFM, FSM, MSF, SMF, FMS}

Here, E: Son on one end

And F: Father in the middle

⇒ E = {MFS, SFM, SMF, FMS} and F = {MFS, SFM}

⇒ E ∩ F = {MFS, SFM}

So, $P(E) = \frac{4}{6} = \frac{2}{3}, P(F) = \frac{2}{6} = \frac{1}{3}, P(E \cap F) = \frac{2}{6} = \frac{1}{3}$

Now, we know that by definition of conditional probability,

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

Now by substituting the values we get

$$\Rightarrow P(E|F) = \frac{1/3}{1/3} = 1$$

⇒ P (E|F) = 1

**5. A fair die is rolled. Consider events E = {1,3,5}, F = {2,3} and G = {2,3,4,5}**
**Find**
**(i) P (E|F) and P (F|E)**
**(ii) P (E|G) and P (G|E)**
**(iii) P ((E ∪ F)|G) and P ((E ∩ F)|G)**

**Solution:**

The sample space for the given experiment is S = {1, 2, 3, 4, 5, 6}

Here, E = {1, 3, 5}, F = {2, 3} and G = {2, 3, 4, 5} .......... (i)

$\Rightarrow P(E) = \frac{3}{6} = \frac{1}{2}, P(F) = \frac{2}{6} = \frac{1}{3}, P(G) = \frac{4}{6} = \frac{2}{3}$ .......... (ii)

Now, E ∩ F = {3}, F ∩ G = {2, 3}, E ∩ G = {3, 5} .......... (iii)

$\Rightarrow P(E \cap F) = \frac{1}{6}, P(F \cap G) = \frac{2}{6} = \frac{1}{3}, P(E \cap G) = \frac{2}{6} = \frac{1}{3}$ .......... (iv)

(i) We know that by definition of conditional probability,

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

$\Rightarrow P(E|F) = \frac{1/6}{1/3} = \frac{3}{6} = \frac{1}{2}$ [Using (II) and (IV)]

$$\Rightarrow P(E|F) = \frac{1}{2}$$

Similarly, we have

$P(F|E) = \frac{P(F \cap E)}{P(E)} = \frac{1/6}{1/2} = \frac{2}{6} = \frac{1}{3}$ [Using (ii) and (iv)]

$$\Rightarrow P(F|E) = \frac{1}{3}$$

(ii) We know that by definition of conditional probability,

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

$$\Rightarrow \ P(E|G) = \frac{P(E \cap G)}{P(G)} = \frac{1/3}{2/3} = \frac{1}{2}$$

$$\Rightarrow P(E|G) = \frac{1}{2}$$

Similarly, we have

$$P(G|E) = \frac{P(G \cap E)}{P(E)} = \frac{1/3}{1/2} = \frac{2}{3}$$

$$\Rightarrow P(G|E) = \frac{2}{3}$$

(iii) Clearly, from (i), we have

E = {1, 3, 5}, F = {2, 3} and G = {2, 3, 4, 5}

⇒ E ∪ F = {1, 2, 3, 5}

⇒ (E ∪ F) ∩ G = {2, 3, 5}

(iii) Clearly, from (i), we have

E = {1, 3, 5}, F = {2, 3} and G = {2, 3, 4, 5}

⇒ E ∪ F = {1, 2, 3, 5}

⇒ (E ∪ F) ∩ G = {2, 3, 5}

$$\Rightarrow P((E \cup F) \cap G) = \frac{3}{6} = \frac{1}{2}$$

$\Rightarrow \ P((E \cup F) \cap G) = \frac{1}{2}$ .......... (v)

Now, we know that

By definition of conditional probability,

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

$\Rightarrow P((E \cup F)|G) = \frac{P((E \cup F) \cap G)}{P(G)} = \frac{1/2}{2/3} = \frac{3}{4}$ [Using (ii) and (v)]

$$\Rightarrow \ P((E \cup F)|G) = \frac{3}{4}$$

Similarly, we have E ∩ F = {3} [Using (iii)]

And G = {2, 3, 4, 5} [Using (i)]

⇒ (E ∩ F) ∩ G = {3}

$\Rightarrow P((E \cap F) \cap G) = \frac{1}{6}$ .......... (vi)

So,

$P((E \cap F)|G) = \frac{P((E \cap F) \cap G)}{P(G)} = \frac{1/6}{2/3} = \frac{1}{4}$ [Using (ii) and (vi)]

$$\Rightarrow \ P((E \cap F)|G) = \frac{1}{4}$$