ALGEBRA MATRICIAL Y OPTIMIZACIÓN

posgrado.maec@cimat.mx

Rosy Dávalos: rosa@cimat.mx

$$(X^T X)^{-1} X^T y = \hat{\beta}$$

vectores

matrices X

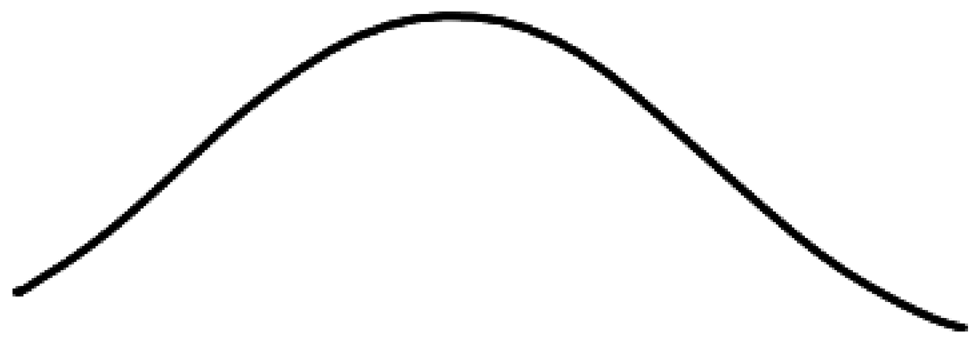
operaciones X^T

$X^T X$

$(\ \ )^{-1}$

A. Mat: → Matrices como estructura básica

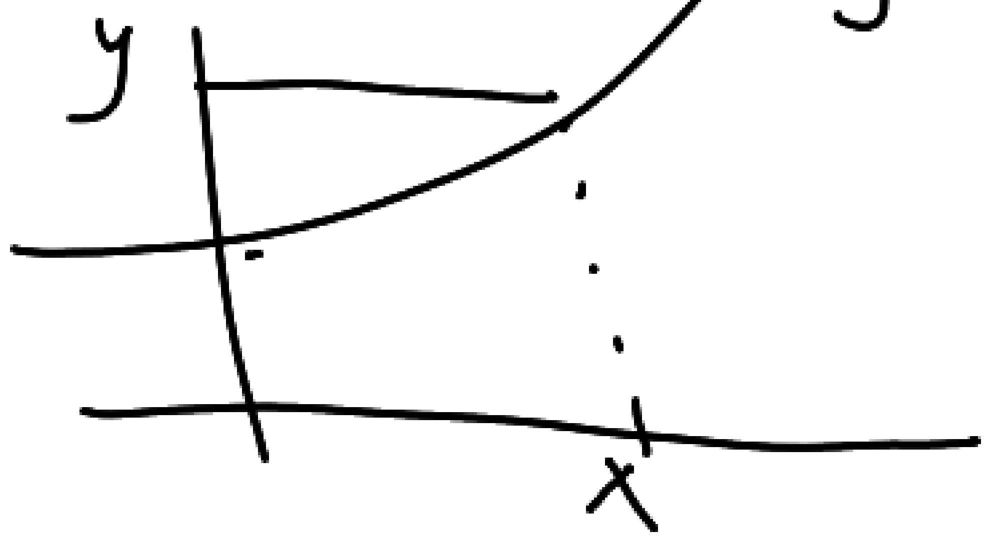
Opt:



Espacios Vectoriales

función:

$y = e^x$



$y = f(x)$

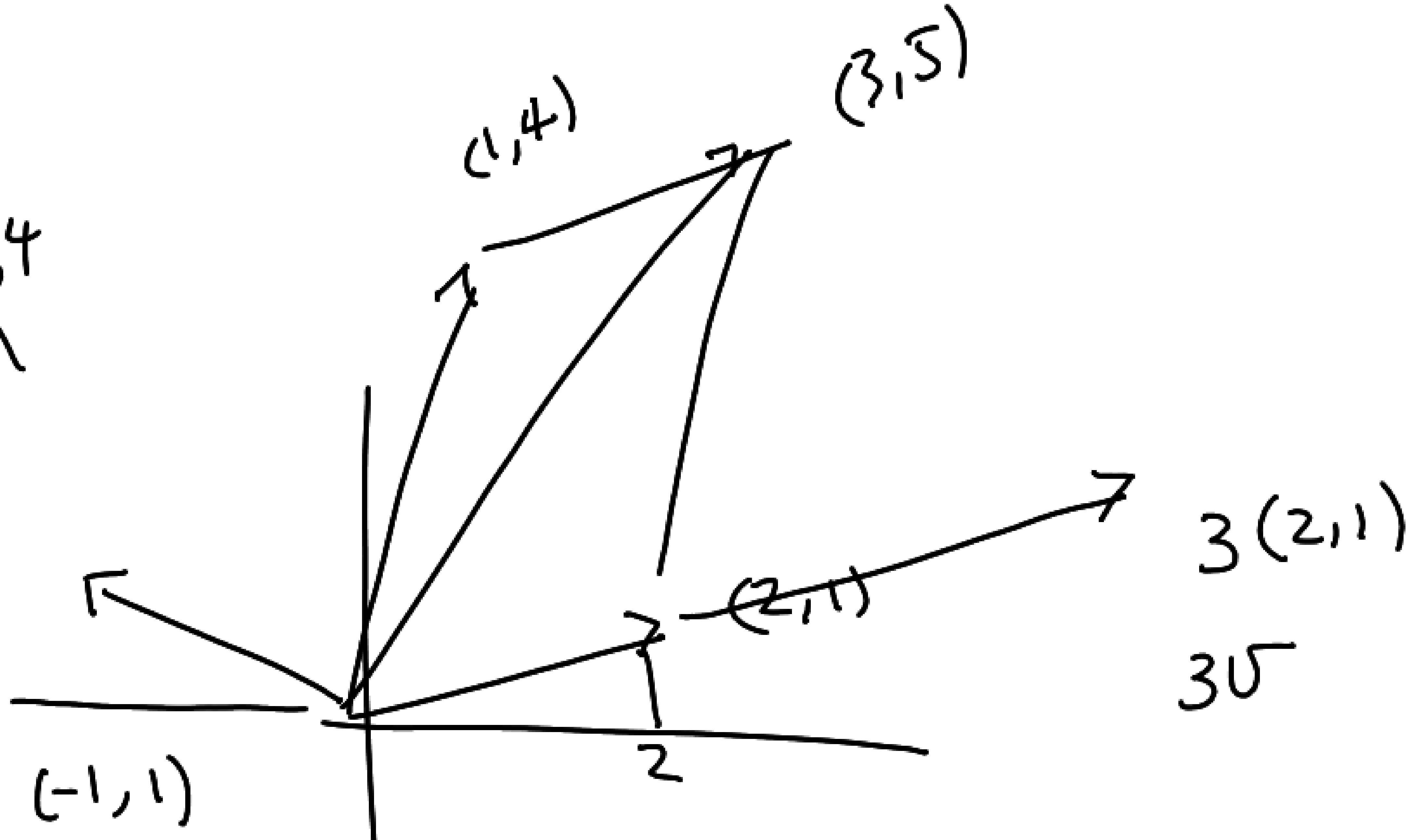
$f: \mathbb{R} \to \mathbb{R}$

Ejemplo de Espacio Vectorial

$$V = \mathbb{R}^n$$

$$v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{bmatrix} \qquad v = \begin{bmatrix} 3 \\ 0 \\ -1 \\ 0 \end{bmatrix} \in \mathbb{R}^4$$

$$u + v = \begin{bmatrix} \cdot \\ \\ \\ \end{bmatrix} + \begin{bmatrix} \cdot \\ \\ \\ \end{bmatrix} = \begin{bmatrix} \cdot \\ \\ \\ \end{bmatrix}$$



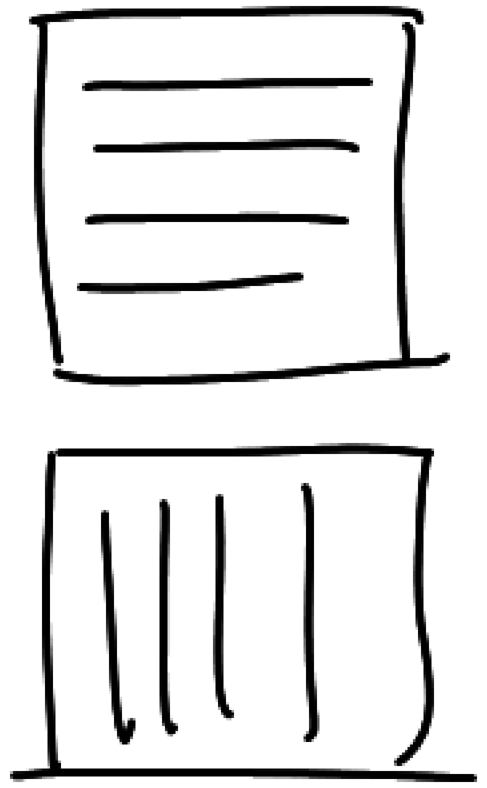
$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$\big((2,1) + (1,4)\big) + (-1,1) = (2,1) + \big((1,4) + (-1,1)\big)$$

Espacio de Matrices $n \times m$

n rengl.

m col.



M

$M + N$

$M + O = M$

$\alpha M = M$

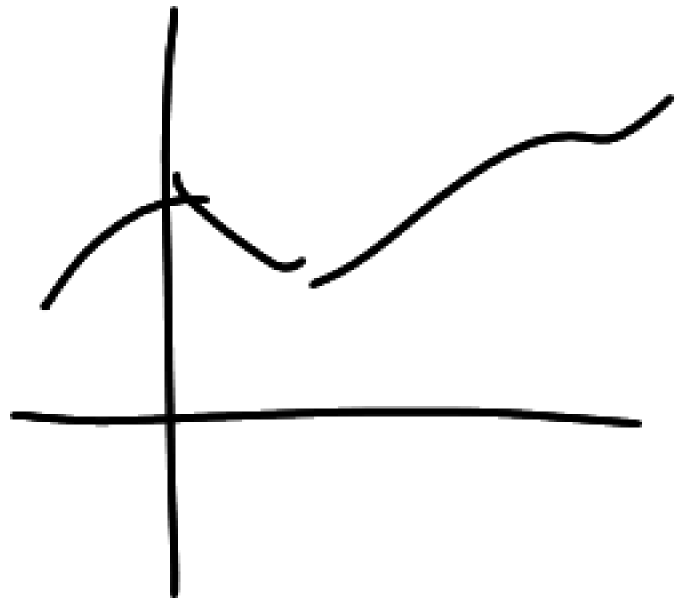
$V = \{ f : \mathbb{R} \to \mathbb{R} \}$

$(f+g)(x) = f(x) + g(x) = 3x^2 - \frac{1}{2}x^3$

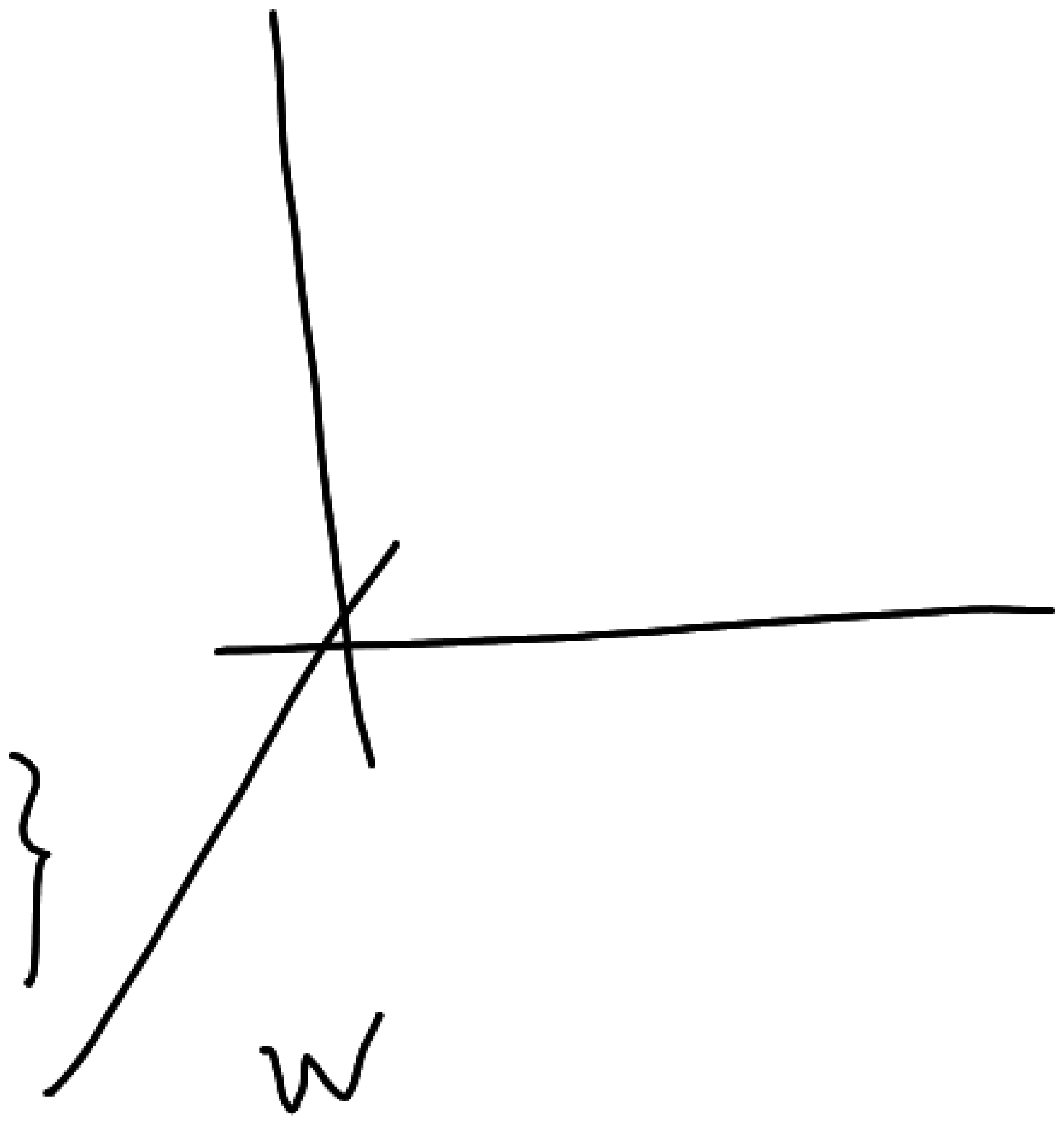
$f(x) = 3x^2$

$g(x) = -\frac{1}{2}x^3$

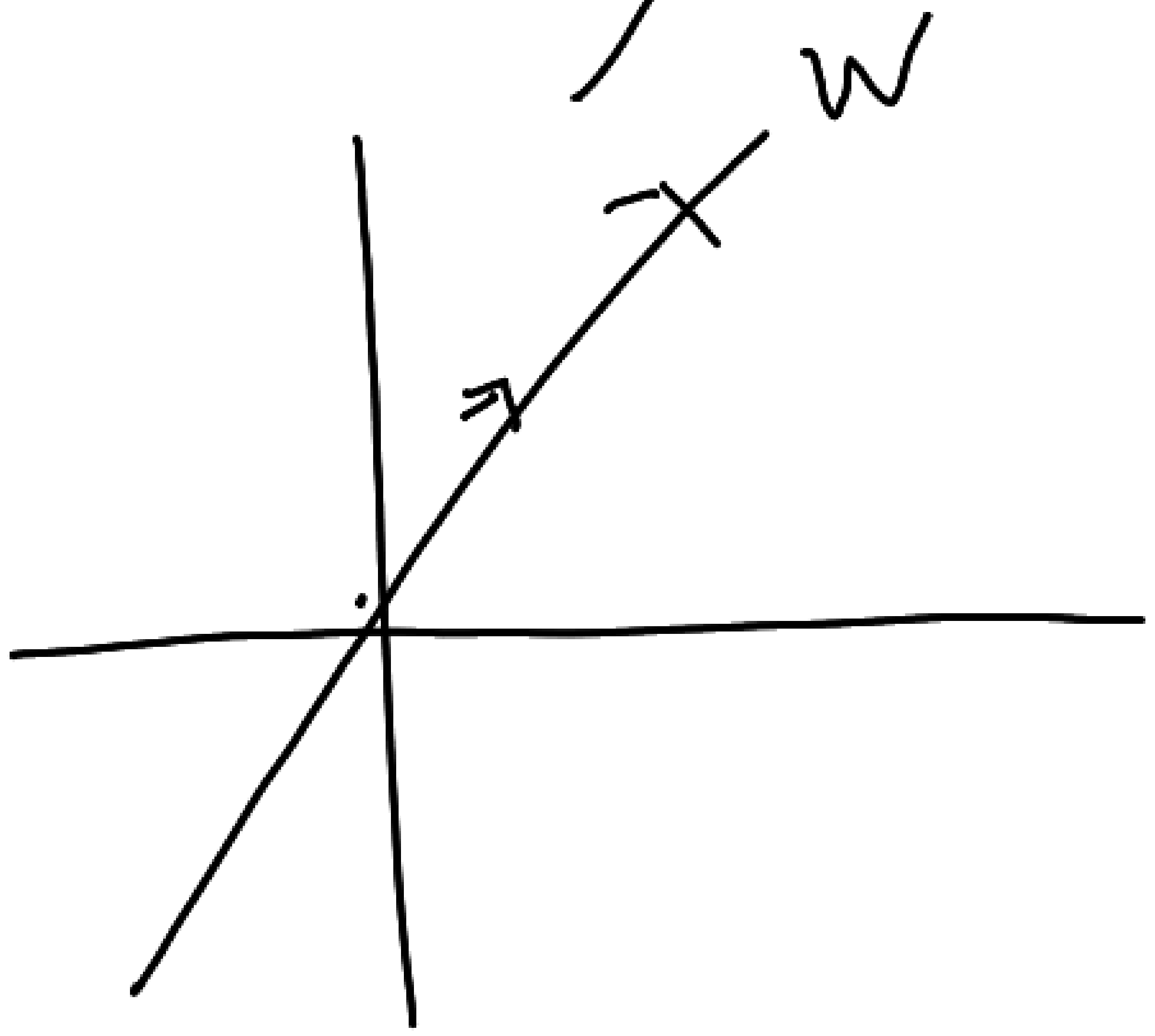
```
x = seq(-10,10,len=50)
y = 3*x^2 - x^3/2
plot(x,y, type="l")
```



$\mathbb{R}^3$



$$W = \left\{ v \in \mathbb{R}^2 \;\middle|\; v = \begin{bmatrix} v_1 \\ 3v_1 \end{bmatrix} \right\}$$



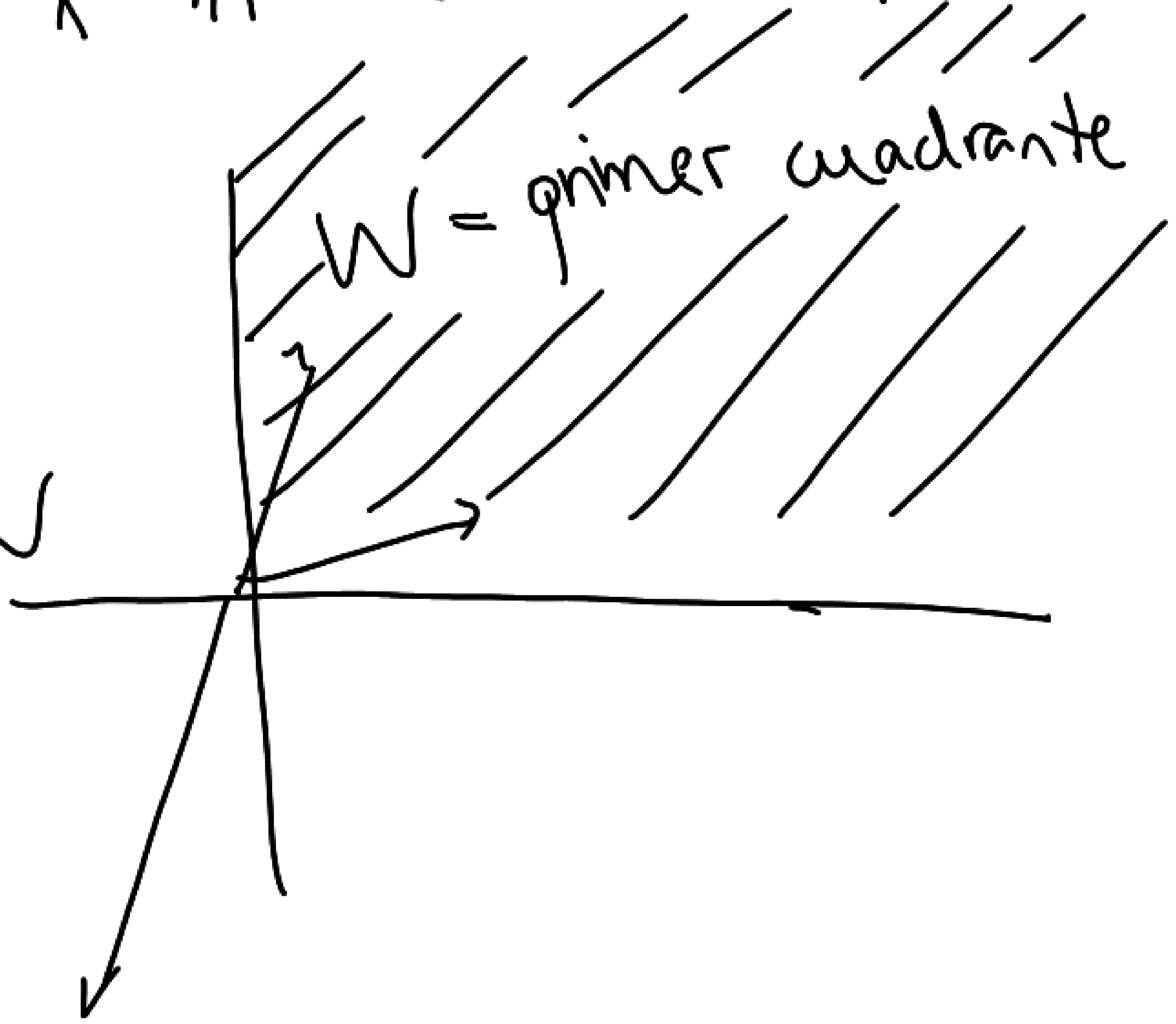
$$V = \mathbb{R}^3 = \left\{ v \;\middle|\; v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \right\}$$

$$X = \mathbb{R}^2 = \left\{ v \;\middle|\; v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\}$$

$W \subset \mathbb{R}^2$

$v \in W$

$-2v \notin W$



$$W = \{(v_1, 3v_1)\}$$

$$(v_1, 3v_1) + (v_2, 3v_2) = (v_1 + v_2, 3(v_1 + v_2))$$
$$= (x, 3x)$$

$$\alpha(v_1, 3v_1) = (\alpha v_1, 3\alpha v_1) \equiv (x, 3x)$$

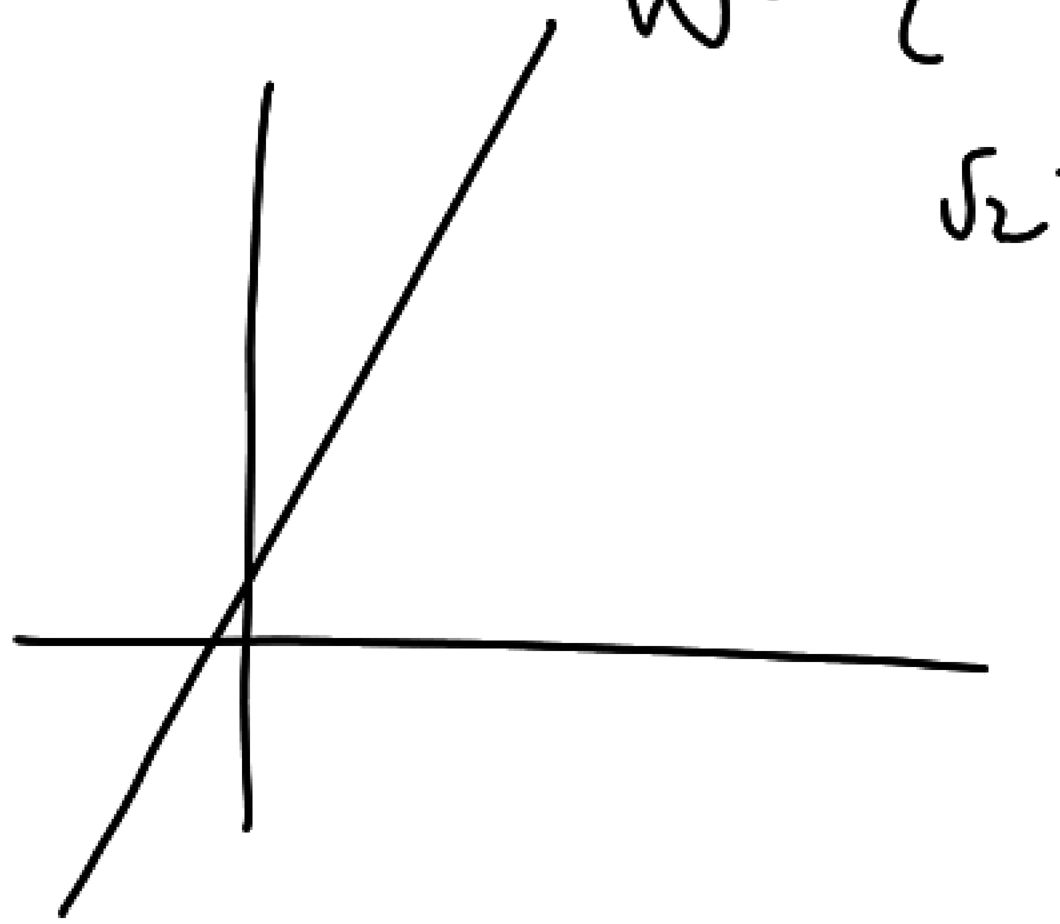
$$(8, 24)$$

M = matrices 3X3

$$\begin{bmatrix} a & b & 0 \\ c & d & 0 \\ e & f & 0 \end{bmatrix} \subset M_{3\times3}$$

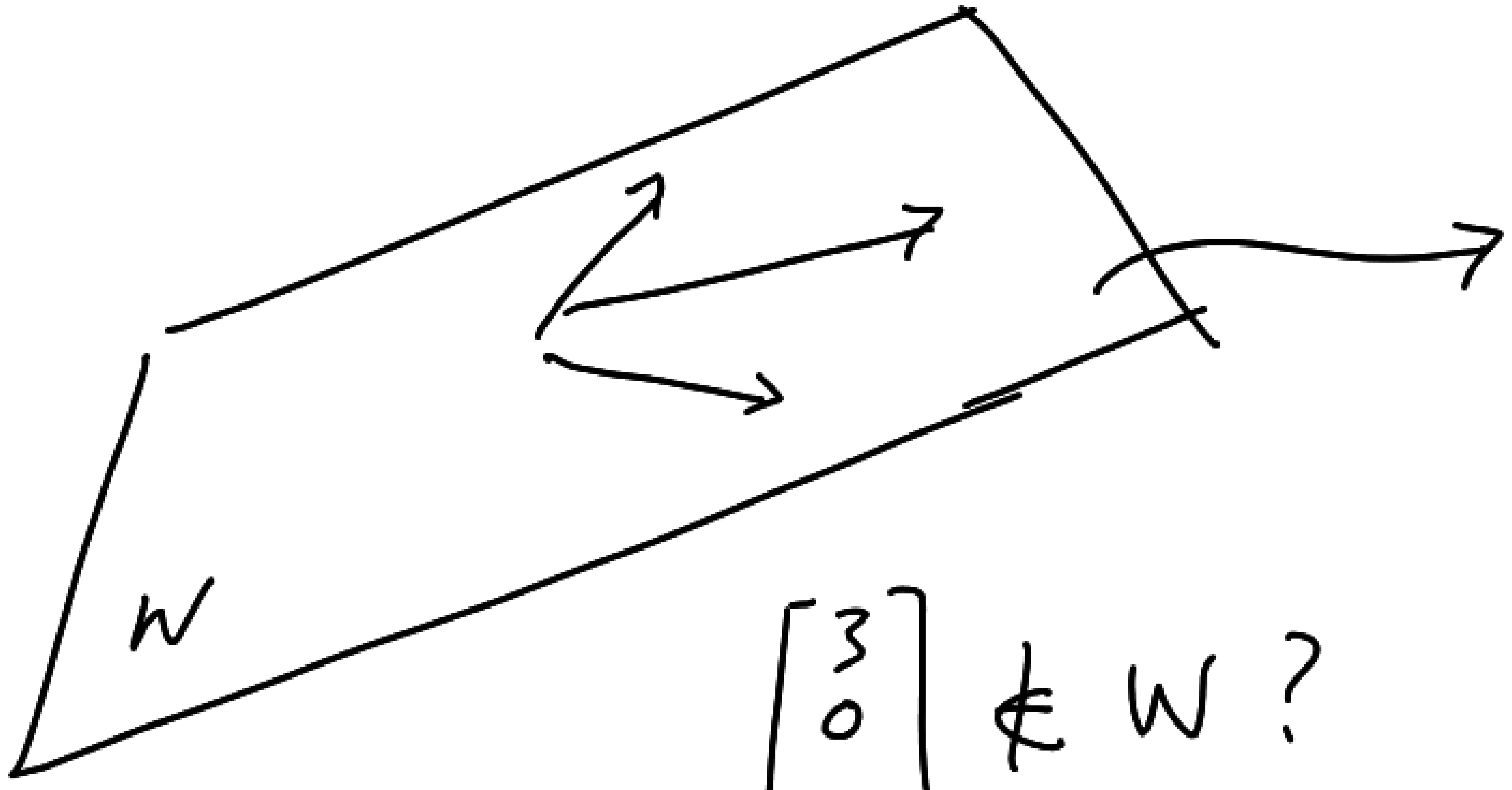
$W =$

$$W = \{(v_1, v_2) \mid v_2 = 3v_1\}$$



W es un subespacio

$\mathbb{R}^3$



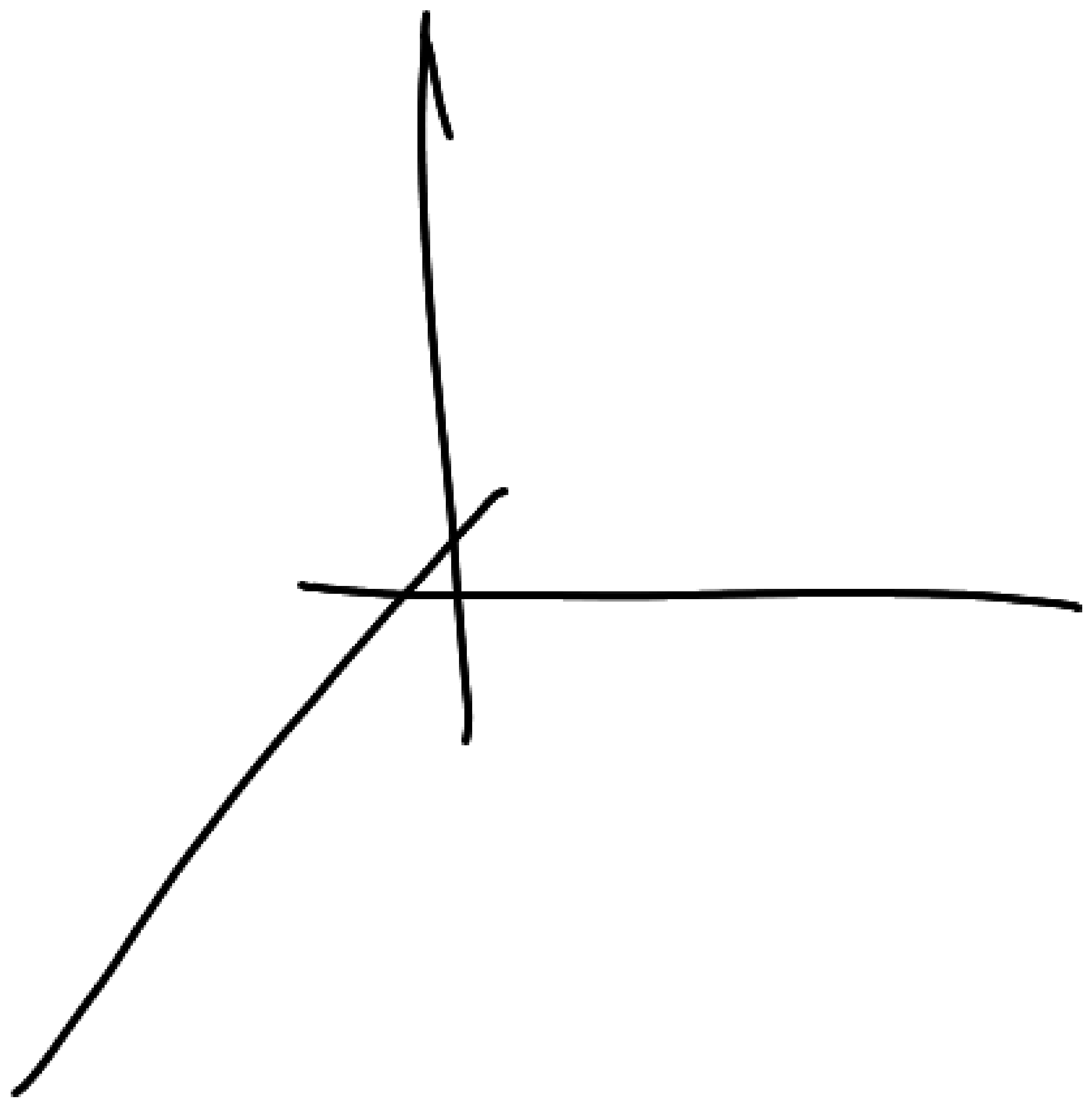
$aX_1 + bX_2 + cX_3 = 0$

$\begin{bmatrix}3\\0\\0\end{bmatrix} \notin W$?

(x_1, x_2, x_3) + (u_1, u_2, u_3) $\Rightarrow$ $(x_1+u_1, x_2+u_2, x_3+u_3)$

$W \subset \mathbb{R}^3$

$\begin{bmatrix}0\\2\\8\end{bmatrix}$



$\mathbb{R}^n \to W = \{v \mid a_1v_1 + a_2v_2 + \dots + a_nv_n = 0\}$

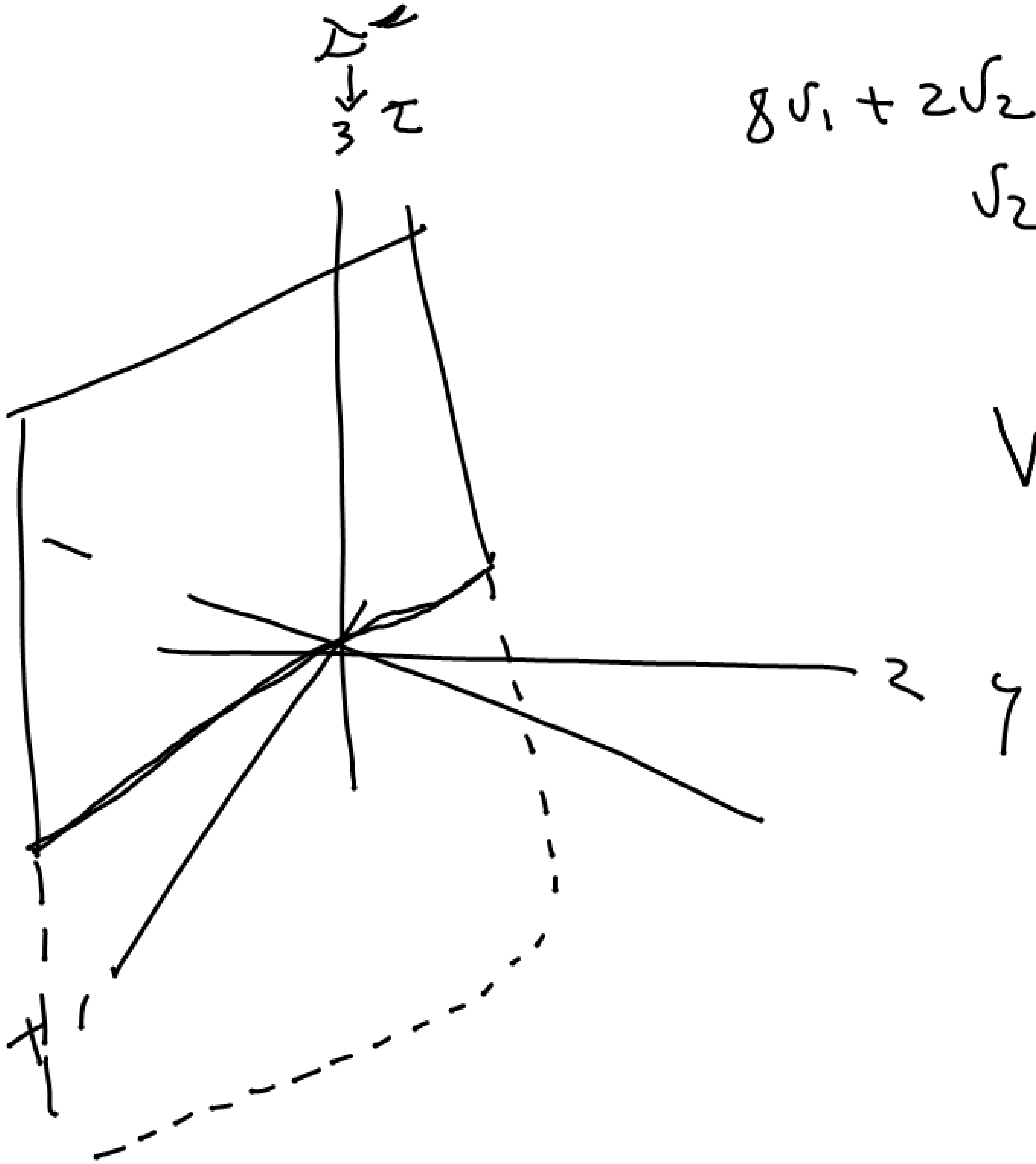
$\mathbb{R}^3 \to W = \{v \mid a_1v_1 + \overset{a_2}{0}v_2 + \overset{a_3}{0}v_3 = 0\}$

$a_1v_1 = 0$

$v = \begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} = \begin{bmatrix}0\\v_2\\v_3\end{bmatrix}$

$W \subset \mathbb{R}^3$

$$W = \left\{ v \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} 0v_1 + 2v_2 + 5v_3 = 0 \\ 8v_1 + 2v_2 + 0v_3 = 0 \end{array} \right\}$$



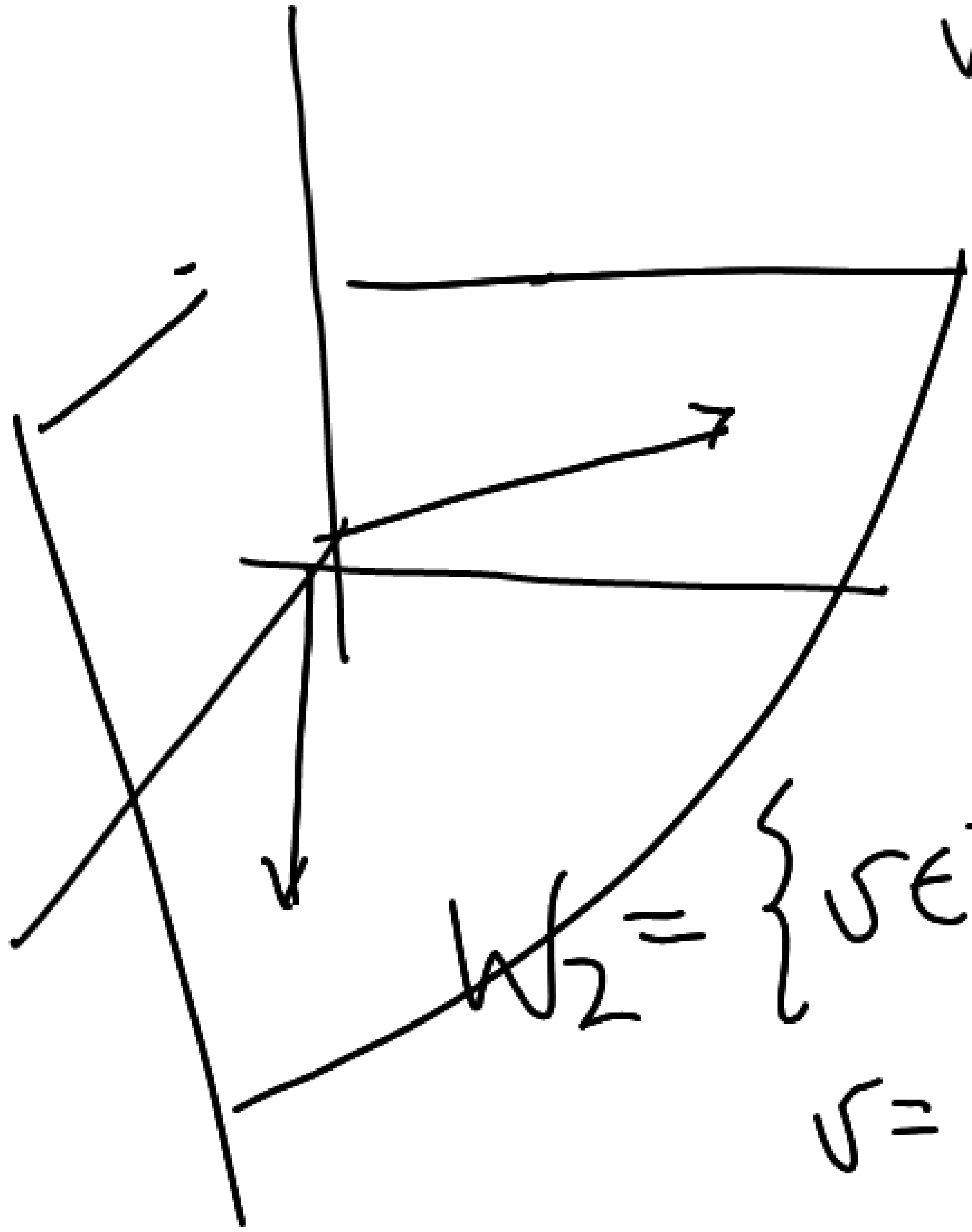
$8v_1 + 2v_2 = 0$

$v_2 = -4v_1$

Si $v \in W$ entonces

$$v = \begin{bmatrix} v_1 \\ -4v_1 \\ v_3 \end{bmatrix}$$

$W_1 + W_2$



$$W_1 = \left\{ v \in \mathbb{R}^3 \;\middle|\; v = \alpha \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right\}$$

$$v = \alpha \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$W_2 = \left\{ v \in \mathbb{R}^3 \;\middle|\; v = \beta \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Si W_1 y W_2 son subespacios de V

y $W_1 \cap W_2 = \{0\}$

Consideremos $S = W_1 \oplus W_2$

$v \in S$ $\quad v \underset{?}{=} w_1 + w_2$

$v = v_1 + v_2$

$$w_1 + w_2 = v_1 + v_2$$

$$0 = w_1 - v_1 = v_2 - w_2 = 0$$

$\in W_1$ $\quad \in W_2$

$w_1 = v_1$ $\quad v_2 = w_2$

$W \leftarrow$ debe ser cerrado para mult. por escalar

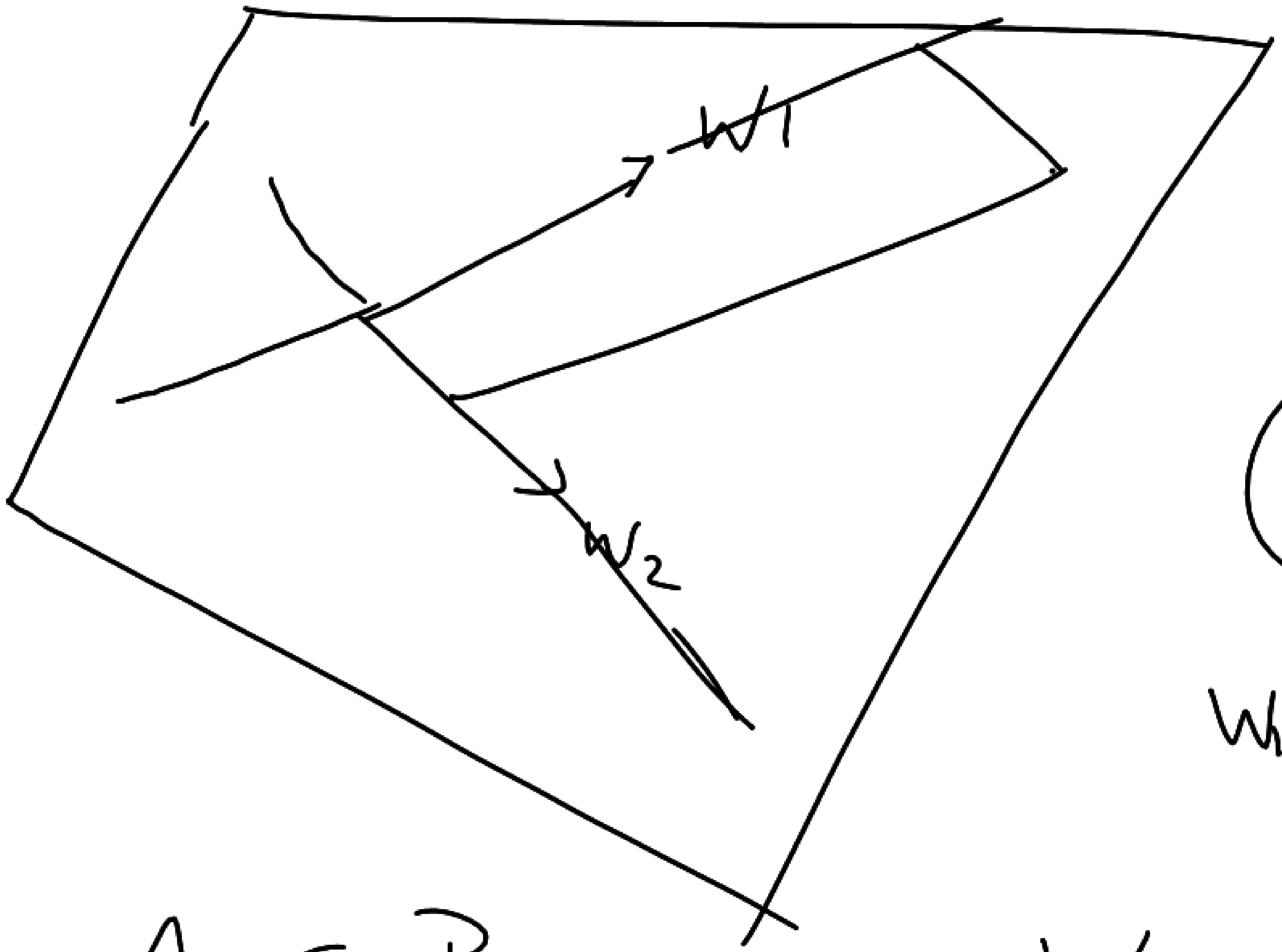
$v \in$

$\alpha v \in W$

$0 = 0v \in W$

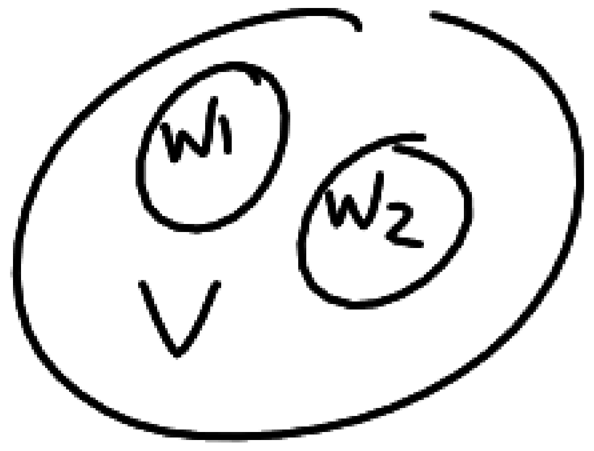
NOTA:

Todos los subespacios **tienen** que contener al vector ~~$\vec{0}$~~ 0



$W_1 + W_2$ es el subespacio más pequeño que contiene a W_1 y W_2

ó $W_1 \cup W_2$



$$W_1+W_2 = \left\{ v \;\middle|\; \begin{array}{l} v = w_1 + w_2 \\ w_1 \in W_1 \\ w_2 \in W_2 \end{array} \right\}$$

$A \subset B$

V que contiene a W_1 y W_2

es tal que $W_1 + W_2 \subset V$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 5 \\ 0 & -16 & -1 & -15 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 0 & \frac{1}{2} & -\frac{5}{2} \\ 0 & -16 & -1 & -15 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & \frac{1}{2} & -\frac{5}{2} \\ 0 & 1 & \frac{1}{16} & \frac{15}{16} \end{bmatrix}$$

$v_1 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ $\quad v_2 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

$S = \{v_1, v_2\}$

$\alpha_1 + \frac{1}{2}\alpha_3 = -\frac{5}{2}$

$\alpha_2 + \frac{1}{16}\alpha_3 = \frac{15}{16}$

$\alpha_1 = -\frac{5}{2} - \frac{1}{2}\alpha_3$

$\alpha_2 = \frac{15}{16} - \frac{1}{16}\alpha_3$

$\alpha_1 v_1 + \alpha_2 v_2$

$\beta_1 v_1 + \beta_2 v_2$

$(\alpha_1 + \beta_1) v_1 + (\alpha_2 + \beta_2) v_2$

$v \in gen(S)$

$v = 3v_1 + 2v_2 - v_3$

$v = 2v_1 + 8v_2 + 2v_3$

$$gen(S) = \{ w \in \mathbb{R}^3 \mid w = \alpha_1 v_1 + \alpha_2 v_2 \}$$

$\alpha_1 = cte$

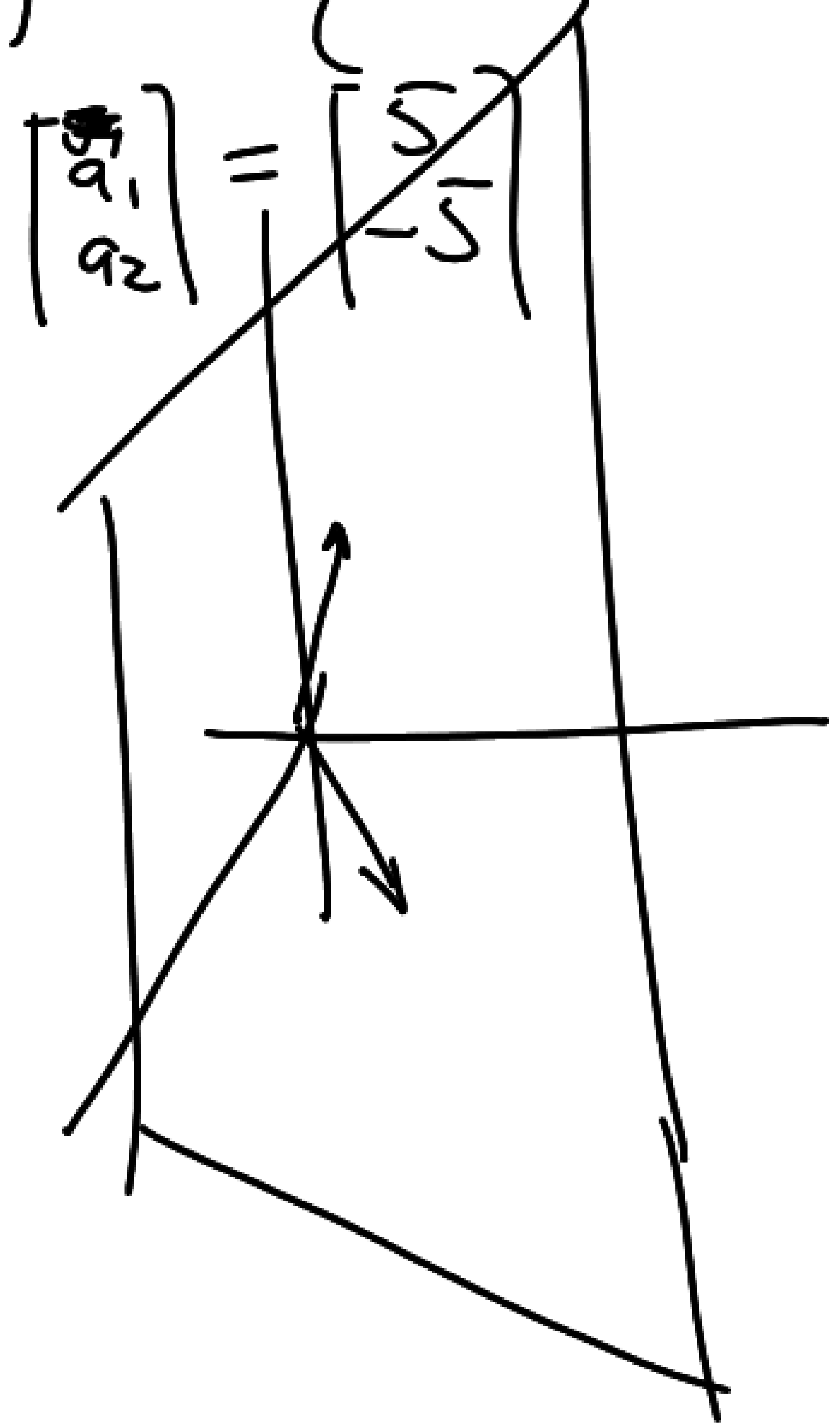
$\alpha_2 = cte$ escalar

$\alpha_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \alpha_2 \begin{bmatrix} 8 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \end{bmatrix}$

$\alpha_1 + 8\alpha_2 + \alpha_3 = 5$

$2\alpha_1 + 0\alpha_2 + \alpha_3 = -5$

$$-2 \begin{bmatrix} 1 & 8 & 1 & 5 \\ 2 & 0 & 1 & -5 \end{bmatrix} \sim *$$



$v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\quad v_2 = \begin{bmatrix} 8 \\ 0 \end{bmatrix}$ $\quad v_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$S = \{v_1, v_2, v_3\}$

$gen(S) = \mathbb{R}^2$

$S \subset V$

$\downarrow$

subconjunto (no necesariamente subespacio)

gen(S) es un subespacio y contiene a S

$S = \{ v_1, v_2, v_3, v_4 \}$

$w_1 + w_2 \overset{?}{\in} \text{gen}(S)$

Resultado. gen(S) es el subespacio más pequeño que contiene a S

W <u>otro</u> subespacio que contiene a S

otro cualquiera

otro arbitrario

$\text{gen}(S) \overset{?}{\subset} W$

$$\text{gen}\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}3\\8\end{bmatrix},\begin{bmatrix}8\\0\end{bmatrix}\right\} = \text{gen}\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}8\\0\end{bmatrix}\right\}$$

$$\begin{bmatrix}3\\8\end{bmatrix} = 4\begin{bmatrix}1\\2\end{bmatrix} + \frac{-1}{8}\begin{bmatrix}8\\0\end{bmatrix}$$

$S = \{v_1, v_2, \dots, v_n\}$

sabemos que $\{v_{i_1}, v_{i_2}, \dots, v_{i_k}\}$ son dependientes

$(i_1, i_2, \dots, i_k)$ son k índices de $1, 2, \dots n$

$k \leq n$

Porqué S es dependiente?

$\alpha_{i_1} v_{i_1} + \dots + \alpha_{i_k} v_{i_k} = 0$ para algunos α_{i_r}'s donde no todos son cero

entonces $0 v_1 + 0 v_2 + \dots + \alpha_{i_1} v_{i_1} + \dots + \alpha_{i_k} v_{i_k} + \dots + 0 v_n = 0$

$v_1, v_2, \dots, v_n$ n vectores en $\mathbb{R}^m$ con $n > m$

$\rho(A) + \nu(A) = n$

$\nu(A) = n - \rho(A) \geq n - m > 0$

¿Porqué son dependientes?

$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = 0$

Porqué esto es cierto para α's no todas iguales a 0?

$\rho(A) \leq m$

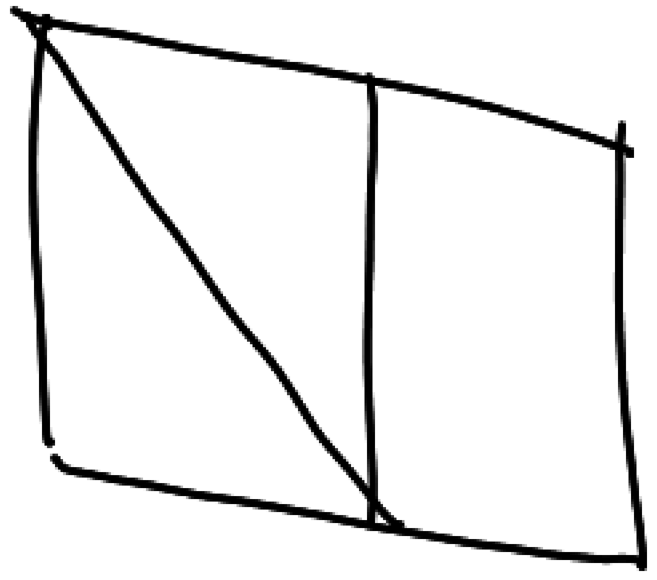
$-\rho(A) \geq -m$

$n - \rho(A) \geq n - m$

$$\alpha_1 \begin{bmatrix} v_{11} \\ v_{12} \\ \vdots \\ v_{1m} \end{bmatrix} + \alpha_2 \begin{bmatrix} v_{21} \\ v_{22} \\ \vdots \\ v_{2m} \end{bmatrix} + \dots + \alpha_n \begin{bmatrix} v_{n1} \\ v_{n2} \\ \vdots \\ v_{nm} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\begin{aligned} v_{11}\alpha_1 + v_{21}\alpha_2 + \dots + v_{n1}\alpha_n &= 0 \\ v_{12}\alpha_1 + v_{22}\alpha_2 + \dots + v_{n2}\alpha_n &= 0 \\ &\vdots \\ v_{1m}\alpha_1 + v_{2m}\alpha_2 + \dots + v_{nm}\alpha_n &= 0 \end{aligned}$$

Sistema de m ecuaciones homogéneas en n incógnitas



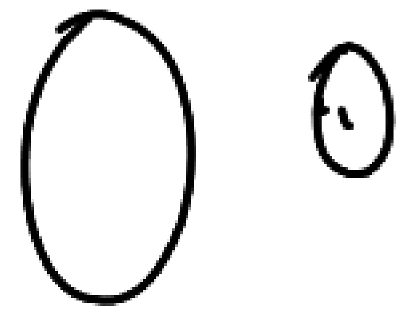
$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$A\alpha = 0$

$m \times n$

$A : \mathbb{R}^n \to \mathbb{R}^m$

$m < n$



Sup $W = \text{gen}\{v_1, v_2, v_3\}$

$w_1, w_2, w_3, w_4, w_5 \in W$ y son independents

$\underline{v_1 \ldots v_n} \in \mathbb{R}^m \quad m < n$

$\Rightarrow$ son dep.

Base de $\mathbb{R}^3$ $\quad v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

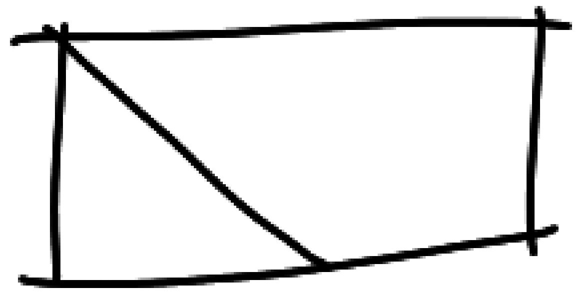
$$w = \begin{bmatrix} 3 \\ 8 \\ 1 \end{bmatrix} = 3v_1 + 8v_2 + 1v_3$$

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0 \iff \begin{array}{l} \alpha_1 = 0 \\ \alpha_2 = 0 \\ \alpha_3 = 0 \end{array}$$

$W = \{ v \mid 2w_1 + 3w_2 + \dots + 5w_n = 0 \}$

$\cancel{S = \{ v \mid 3v_1 +}$

$W = \{ v \mid 2w_1 + w_2 - 2w_3 = 0 \}$



1) dim = ?

2) base ? $= \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\}$

$$v = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

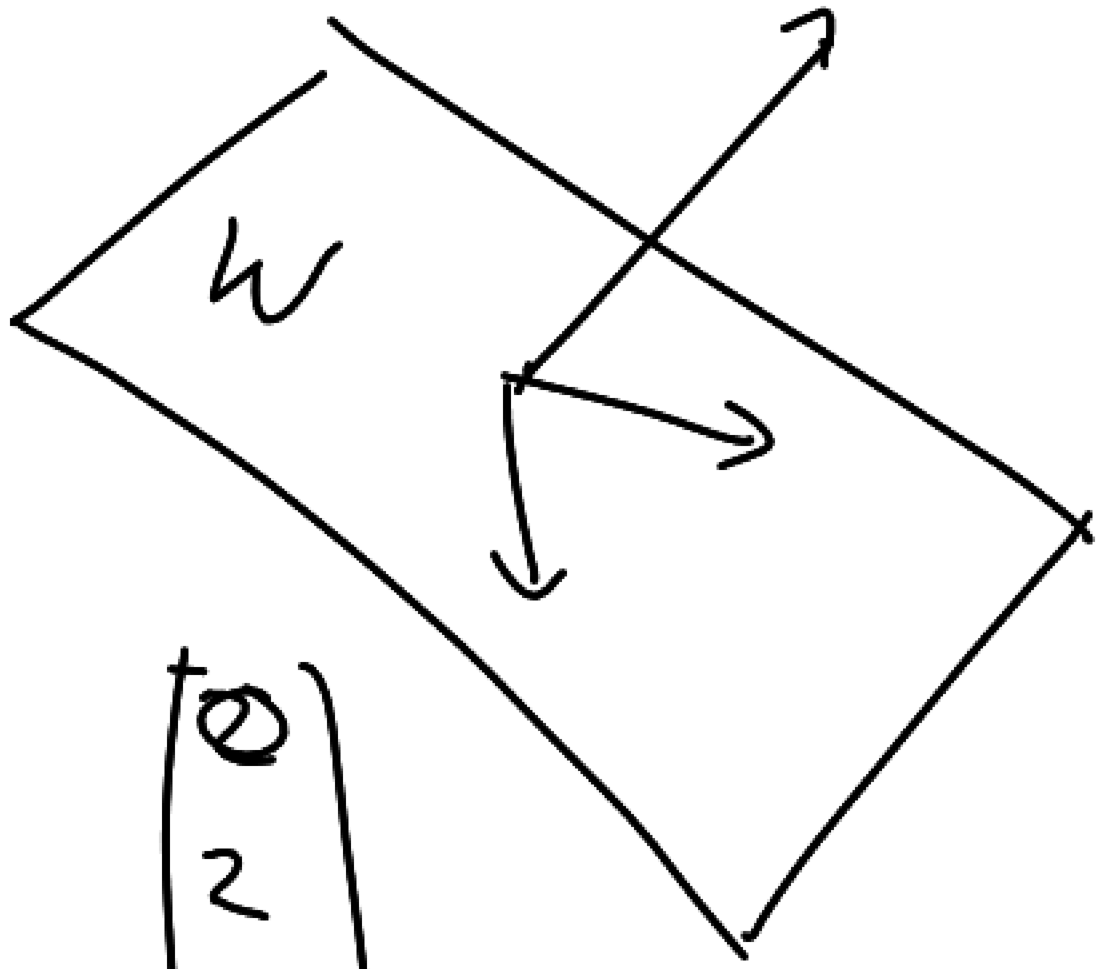
$(2,1,-2) \cdot (1,0,1)$

$w_1 \; w_2 \; w_3$

$\cdot \, (0,2,1)$

$+$

$2w_1 + w_2 - 2w_3 = 0$



$(2,1,-2) \cdot (w_1, w_2, w_3) = 0$

$2w_1 + w_2 - 2w_3 = 0$ $\quad \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$

$\cancel{w_1}$ $\quad w_2 = 2w_3 - 2w_1$ $\quad \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$