# T2 Solución

*IE*

*14/2/2020*

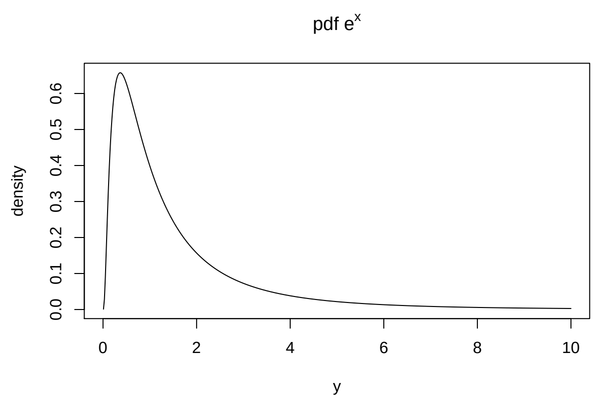
# EJ1

## Densidad de $e^X$

Tenemos que $Y = e^X$, entonces la función inversa $w(y) = \ln(y)$. Aplicando teorema de cambio de variable:

$$\begin{aligned} f_Y(y) &= f_X(w(y)) \cdot |w'(y)| \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{(\ln x)^2}{2}} \cdot \frac{1}{y} \\ &= \frac{1}{y\sqrt{2\pi}} e^{-\frac{(\ln x)^2}{2}} \end{aligned}$$

```r
d_y<-function(x){dnorm(log(x))/x}
y_i<-seq(-10,10,length.out = 1000)
plot(y_i,d_y(y_i),type='l',main=expression('pdf e'^x),xlim = c(0,10),xlab = 'y',ylab='density')
```




## E(Y) V(Y)

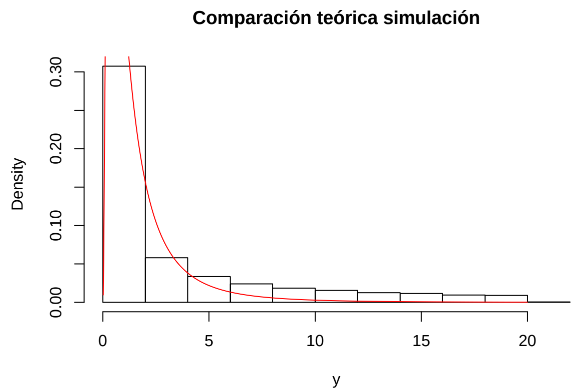
$$\begin{aligned} E(Y) &= \int_{-\infty}^{\infty} e^x \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2-2x}{2}} \\ &= e^{\frac{1}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-1)^2}{2}} \\ &= e^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned} E(Y^2) &= \int_{-\infty}^{\infty} e^{2x} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2-4x}{2}} \\ &= e^2 \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-2)^2}{2}} \\ &= e^2 \end{aligned}$$

Entonces $E(Y) = e^{1/2}$;$V(Y) = E(Y^2) - E(Y)^2 = e^2 - e$

## Simulación

```
x<-seq(-3,3,length.out = 1000)
y<-exp(x)
y_i<-seq(0,20,length.out = 1000)
hist(y,freq = F,main='Comparación teórica simulación')
lines(y_i,d_y(y_i),col='red')
```




# EJ3

Dado que $X$ y $Y$ son v.a Poisson independientes

$$
\begin{aligned}
P(X=x|X+Y=n) &= \frac{P(X=x)P(Y=n-x)}{P(X+Y=n)} \\
&= \frac{\frac{e^{-\lambda_x}\lambda_x^x}{x!}\frac{e^{-\lambda_y}\lambda_y^{n-x}}{(n-x)!}}{\sum_{k=0}^{n}\frac{e^{-\lambda_x}\lambda_x^k}{k!}\frac{e^{-\lambda_y}\lambda_y^{n-k}}{(n-k)!}} \\
&= \frac{\frac{e^{-\lambda_x}\lambda_x^x}{x!}\frac{e^{-\lambda_y}\lambda_y^{n-x}}{(n-x)!}}{\frac{e^{-\lambda_x-\lambda_y}(\lambda_x+\lambda_y)^n}{n!}} \\
&= \frac{n!}{x!(n-x)!}\cdot\frac{\lambda_x^x\lambda_y^{n-x}}{(\lambda_x+\lambda_y)^n} \\
&= \frac{n!}{x!(n-x)!}\cdot\frac{\lambda_x^x}{(\lambda_x+\lambda_y)^x}\cdot\frac{\lambda_y^{n-x}}{(\lambda_x+\lambda_y)^{n-x}}
\end{aligned}
$$

Se nota que $(X=x|X+Y=n)\sim \text{Bin(n, }\frac{\lambda_x}{\lambda_x+\lambda_y})$

# EJ4

Hallamos $f_Y(y)$ :

$$f_Y(y)=\int_0^1 c(x+y^2)dx = c(\frac{1}{2}+y^2)$$

La densidad condicional es:

$$f_{X|Y}(X|Y) = \frac{f(x,y)}{f_y(y)} = \frac{c(x+y^2)}{c(\frac{1}{2}+y^2)} = \frac{(x+y^2)}{(\frac{1}{2}+y^2)}$$

$$\begin{array}{rcl} P(X<\frac{1}{2}|y=\frac{1}{2}) & = & \int_0^{\frac{1}{2}} \frac{x+\frac{1}{4}}{\frac{1}{2}+\frac{1}{4}} dx \\ & = & \frac{4}{3}\int_0^{\frac{1}{2}}(x+\frac{1}{4})dx \\ & = & \frac{4}{3}\cdot\frac{1}{4} \\ & = & \frac{1}{3} \end{array}$$

# EJ7

Obtenemos las marginales:

$$f_X(x) = \frac{1}{3}\int_0^2 (x+y)dy = \frac{2(x+1)}{3}$$
$$f_Y(y) = \frac{1}{3}\int_0^1 (x+y)dx = \frac{2y+1}{6}$$

Se obtienen las esperanzas de $X$,$Y$,$X^2$,$Y^2$,$XY$:

$$E(X) = \int_0^1 x\frac{2(x+1)}{3} = \frac{5}{9}$$
$$E(X^2) = \int_0^1 x^2\frac{2(x+1)}{3} = \frac{7}{18}$$

$$E(Y) = \int_0^2 y\frac{2y+1}{6} = \frac{11}{9}$$

$$E(Y^2) = \int_0^2 y^2\frac{2y+1}{6} = \frac{16}{9}$$

$$E(XY) = \frac{1}{3}\int_0^1\int_0^2 xy(x+y)dxdy = \frac{2}{3}$$

Considerando que $V(X) = E(X^2) - (E(X))^2$, $\mathrm{cov}(X,Y) = E(XY) - E(X)E(Y)$ y $V(aX+bY) = a^2V(X) + b^2V(Y) + 2ab\mathrm{cov(X,Y)}$, con las sustituciones correspondientes se obtiene:

$$V(2X-3Y+8) = \frac{245}{81}$$

## EJ11

### Tabla

| | $x$ | | | |
|---|---|---|---|---|
| $y$ | 0 | 1 | 2 | $p_Y(y)$ |
| $-1$ | 1/6 | 1/6 | 1/6 | 1/2 |
| 1 | 0 | 1/2 | 0 | 1/2 |
| $p_X(x)$ | 1/6 | 2/3 | 1/6 | 1 |

### Independencia

Basta con comprobar que $p_X(0)p_Y(-1) \neq p_{XY}(0,1)$ para notar que son dependientes

## EJ 14

### Distribución marginales

$$F_X(x) = \lim_{y\to\infty}(1 - e^{-2x} - e^{-y} + e^{-(2x+y)}) = 1 - e^{-2x}$$

$$F_Y(y) = \lim_{x\to\infty}(1 - e^{-2x} - e^{-y} + e^{-(2x+y)}) = 1 - e^{-y}$$

### Densidad conjunta

$$f_{X,Y}(x,y) = \frac{\partial^2}{\partial x \partial y}(1 - e^{-2x} - e^{-y} + e^{-(2x+y)}) = 2e^{-(2x+y)}$$

### Densidades marginales

Derivamos la distribuciones marginales

$$f_X(x) = \frac{\partial}{\partial x}(1 - e^{-2x}) = 2e^{-2x}$$
$$f_Y(y) = \frac{\partial}{\partial y}(1 - e^{-y}) = e^{-y}$$

### Independencia

Basta con comprobar que $f_X(x) \cdot f_Y(y) = f_{X,Y}(x,y)$ para determinar que son independientes

## EJ 15

### $P(1/4 \leq X \leq 1/2, 1/3 \leq Y \leq 2/3)$

$$P(1/4 \leq X \leq 1/2, 1/3 \leq Y \leq 2/3) = \frac{12}{5}\int_{1/4}^{1/2}\int_{1/3}^{2/3} xy(1+y)dxdy = 41/720$$

$F(a,b)$

$$F(a,b) = P(X \leq a, Y \leq b) = \frac{12}{5}\int_0^a\int_0^b xy(1+y)dxdy = \frac{a^2b^2}{5}(3+2b)$$

$F_X(a)$

$$F_X(a) = P(X \leq a, Y \leq 1) = F(a,1) = a^2$$

$f_X(x)$

$$f_X(x) = \frac{12}{5}\int_0^1 xy(1+y)dy = 2x$$

### Independencia

Hallamos la marginal de y $f_Y(y)$

$$f_Y(y) = \frac{12}{5}\int_0^1 xy(1+y)dx = \frac{6y}{5}(y+1)$$

Basta con comprobar que $f_X(x) \cdot f_Y(y) = f_{X,Y}(x,y)$ para determinar que son independientes

## EJ 20

Se obtienen las esperanzas de $X$,$Y$,$X^2$,$Y^2$,$XY$:

$$E(X) = \int_0^1 x\frac{2(9x^2+7x)}{225} = \frac{109}{50}$$
$$E(X^2) = \int_0^1 x^2\frac{2(9x^2+7x)}{225} = \frac{1287}{250}$$

$$E(Y) = \int_0^2 y\frac{3y^2+12y}{25} = \frac{157}{100}$$

$$E(Y^2) = \int_0^2 y^2\frac{3y^2+12y}{25} = \frac{318}{125}$$

$$E(XY) = \frac{2}{75}\int_0^3 x\int_1^2 y(2x^2y+xy^2)dxdy = \frac{171}{150}$$

$$E(X+Y) = E(X) + E(Y) = \frac{15}{4}$$

$$E((X+Y)^2) = E(X^2) + 2E(XY) + E(Y^2) = \frac{3633}{250}$$

Realizando las sustituciones correspondientes:

$$V(X) = E(X^2) + (E(X))^2 = \frac{989}{2500}$$

$$V(Y) = E(Y^2) + (E(Y))^2 = \frac{791}{10000}$$

$$\text{Cov}(X, Y) = E(XY) - E(X)E(Y) = -\frac{13}{5000}$$

$$V(X + Y) = V(X) + V(Y) + 2\text{cov}(X, Y) = \frac{939}{2000}$$

o tambien :

$$V(X + Y) = E((X + Y)^2) + (E(X + Y))^2 = \frac{939}{2000}$$

Se nota que

$$V(X) + V(Y) \neq V(X + Y)$$