Inferencia Estadística

Maestría en Análisis Estadístico y Computación

Tarea 2

Fecha de entrega: Martes 11 de febrero, 2020

Problemas a entregar: 1, 3, 4, 7, 11, 14, 15, 20.

1. **Wasserman, Cap 2 y Cap 3.** Let $X \sim N(0,1)$ and let $Y = e^X$

   a. Find the PDF for Y. Plot it.

   b. Find $\mathbb{E}(Y)$ and $\mathbb{V}(Y)$.

   c. (Computer Experiment.) Generate a vector $x = (x_1, \ldots, x_{10,000})$ consisting of $10,000$ random standard Normals. Let $y = (y_1, \ldots, y_{10,000})$ where $y_i = e^{x_i}$. Draw a histogram of y and compare it to the PDF you found in part a.

2. **Wasserman, Cap 2.** Let (X, Y) be uniformly distributed on the unit disk $\left\{(x, y) : x^2 + y^2 \leq \right\}$. Let $R = \sqrt{X^2 + Y^2}$. Find the CDF and PDF of R

3. **Wasserman, Cap 2.** Let $X \sim$ Poisson (λ) and $Y \sim$ Poisson (μ) and assume that X and Y are independent. Show that the distribution of X given that $X + Y = n$ is Binomial (n, π) where $\pi = \lambda/(\lambda + \mu)$

   Hint 1: You may use the following fact: If $X \sim$ Poisson (λ) and $Y \sim$ Poisson(μ), and X and Y are independent, then $X + Y \sim$ Poisson $(\mu + \lambda)$

   Hint 2: Note that $\{X = x, X + Y = n\} = \{X = x, Y = n - x\}$

4. **Wasserman, Cap 2.** Let

$$f_{X,Y}(x, y) = \begin{cases} c\left(x + y^2\right) & 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

   Find $P\left(X < \frac{1}{2} | Y = \frac{1}{2}\right)$

5. **Wasserman, Cap 3.** Suppose we play a game where we start with c dollars. On each play of the game you either double or halve your money, with equal probability. What is your expected fortune after n trials?

6. **Wasserman, Cap 3.** (Computer Experiment: Simulating the Stock Market.) Let $Y_1, Y_2, \ldots$ be independent random variables such that $P(Y_i = 1) = P(Y_i = -1) = 1/2$. Let $X_n = \sum_{i=1}^{n} Y_i$. Think of $Y_i = 1$ as "the stock price increased by one dollar" , $Y_i = -1$ as "the stock price decreased by one dollar", and X_n as the value of the stock on day n.

   a. Find $\mathbb{E}(X_n)$ and $\mathbb{V}(X_n)$

   b. Simulate X_n and plot X_n versus n for $n = 1, 2, \ldots, 10,000$. Repeat the whole simulation several times. Notice two things. First, it's easy to "see" patterns in the sequence even though it is random. Second, you will find that the four runs look very different even though they were generated the same way. How do the calculations in (a) explain the second observation?

7. **Wasserman, Cap 3.** Let

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{3}(x+y) & 0 \leq x \leq 1, 0 \leq y \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

Find $\mathbb{V}(2X - 3Y + 8)$.

8. **Wasserman, Cap 3.** Let $X_1, \ldots, X_n \sim \text{Exp}(\beta)$. Find the moment generating function of X_i Prove that $\sum_{i=1}^{n} X_i \sim \text{Gamma}(n, \beta)$

9. **Wasserman, Cap 3.** *Let $X_1, \ldots, X_n \sim$ Uniform $(0,1)$ and let $Y_n = \max\{X_1, \ldots, X_n\}$. Find $\mathbb{E}(Y_n)$

10. **Wasserman, Cap 3.** *Suppose we generate a random variable X in the following way. First we flip a fair coin. If the coin is heads, take X to have a Unif$(0,1)$ distribution. If the coin is tails, take X to have a Unif$(3,4)$ distribution. (a) Find the mean of X. (b) Find the standard deviation of X.

11. **Dekking, Cap 9** The joint probability distribution of two discrete random variables X and Y is partly given in the following table.

	a			
b	0	1	2	P(Y=b)
-1	?	?	?	1/2
1	?	1/2	?	1/2
P(X=a)	1/6	2/3	1/6	1

a. Complete the table.

b. Are X and Y dependent or independent?

12. **Dekking, Cap 9** X and Y be two independent Bernoulli (1/2) random variables. Define random variables U and V by:

$$U = X + Y \quad \text{and} \quad V = |X - Y|$$

a. Determine the joint and marginal probability distributions of U and V

b. Find out whether U and V are dependent or independent.

13. **Dekking, Cap 9** Het X and Y be independent random variables with probability distributions given by

$$\mathrm{P}(X=0) = \mathrm{P}(X=1) = \frac{1}{2} \quad \text{and} \quad \mathrm{P}(Y=0) = \mathrm{P}(Y=2) = \frac{1}{2}$$

Compute the distribution of $Z = X + Y$.

14. **Dekking, Cap 9** Suppose that the joint distribution function of X and Y is given by

$$F(x,y) = 1 - \mathrm{e}^{-2x} - \mathrm{e}^{-y} + \mathrm{e}^{-(2x+y)} \quad \text{if } x > 0, y > 0$$

and $F(x,y) = 0$ otherwise.

a. Determine the marginal distribution functions of X and Y

b. Determine the joint probability density function of X and Y.

c. Determine the marginal probability density functions of X and Y.

d. Find out whether X and Y are independent.

15. **Dekking, Cap 9** Let X and Y be two continuous random variables with joint probability density function

$$f(x, y) = \frac{12}{5} xy(1 + y) \quad \text{for } 0 \le x \le 1 \text{ and } 0 \le y \le 1$$

and $f(x, y) = 0$ otherwise.

a. Find the probability $\mathrm{P}\left(\frac{1}{4} \le X \le \frac{1}{2}, \frac{1}{3} \le Y \le \frac{2}{3}\right)$
b. Determine the joint distribution function of X and Y for a and b between 0 and 1
c. Use your answer from b to find $F_X(a)$ for a between 0 and 1
d. Apply the rule on page 122 to find the probability density function of X from the joint probability density function $f(x, y)$. Use the result to verify your answer from c.
e. Find out whether X and Y are independent.

16. **Dekking, Cap 9** The joint probability density function f of the pair (X, Y) is given by

$$f(x, y) = K\left(3x^2 + 8xy\right) \quad \text{for } 0 \le x \le 1 \text{ and } 0 \le y \le 2$$

and $f(x, y) = 0$ for all other values of x and y. Here K is some positive constant.

a. Find K
b. Determine the probability $\mathrm{P}(2X \le Y)$

17. **Dekking, Cap 9 y 10** To investigate the relation between hair color and eye color, the hair color and eye color of 5383 persons was recorded. The data are given in the following table:

	Hair Color		
Eye Color	Fair/red	Medium	Dark/Black
Light	1168	825	305
Dark	573	1312	1200

Eye color is encoded by the values 1 (Light) and 2 (Dark), and hair color by 1 (Fair/red), 2 (Medium), and 3 (Dark/black). By dividing the numbers in the table by 5383, the table is turned into a joint probability distribution for random variables X (hair color) taking values 1 to 3 and Y (eye color) taking values 1 and 2.

a. Determine the joint and marginal probability distributions of X and Y .
b. Find out whether X and Y are dependent or independent.
c. Compute $\mathrm{Cov}(X, Y)$. Are X and Y positively correlated, negative correlated, or uncorrelated?
d. Compute the correlation coefficient between X and Y

18. **Dekking, Cap 10** Let U and V be the two random variables from Exercise 9.6. We have seen that U and V are dependent with joint probability distribution

	a			
b	0	1	2	$P(V = b)$
0	1/4	0	1/4	1/2
1	0	1/2	0	1/2
$P(U = a)$	1/4	1/2	1/4	1

Determine the covariance $\mathrm{Cov}(U, V)$ and the correlation coefficient $\rho(U, V)$

19. **Dekking, Cap 10** Let X and Y be random variables such that

$$\mathrm{E}[X]=2, \quad \mathrm{E}[Y]=3, \quad \text{and} \quad \operatorname{Var}(X)=4$$

a. Show that $\mathrm{E}\left[X^2\right]=8$.

b. Determine the expectation of $-2X^2+Y$.

20. **Dekking, Cap 10** consider the variables X and Y from the example in Section 9.2 with joint probability density

$$f(x,y)=\frac{2}{75}\left(2x^2y+xy^2\right) \quad \text{for } 0 \leq x \leq 3 \text{ and } 1 \leq y \leq 2$$

and marginal probability densities

$$\begin{aligned} f_X(x) &= \tfrac{2}{225}\left(9x^2+7x\right) && \text{for } 0 \leq x \leq 3 \\ f_Y(y) &= \tfrac{1}{25}\left(3y^2+12y\right) && \text{for } 1 \leq y \leq 2 \end{aligned}$$

a. Compute $\mathrm{E}[X], \mathrm{E}[Y]$, and $\mathrm{E}[X+Y]$

b. Compute $\mathrm{E}\left[X^2\right], \mathrm{E}\left[Y^2\right], \mathrm{E}[XY]$, and $\mathrm{E}\left[(X+Y)^2\right]$

c. Compute $\operatorname{Var}(X+Y), \operatorname{Var}(X)$, and $\operatorname{Var}(Y)$ and check that $\operatorname{Var}(X+Y) \neq \operatorname{Var}(X)+\operatorname{Var}(Y)$

21. **Dekking, Cap 10** Let X and Y be two random variables and let r, s, t, and u be arbitrary real numbers.

a. Derive from the definition that $\operatorname{Cov}(X+s, Y+u)=\operatorname{Cov}(X,Y)$

b. Derive from the definition that $\operatorname{Cov}(rX, tY)=rt\operatorname{Cov}(X,Y)$

c. Combine parts a and b to show $\operatorname{Cov}(rX+s, tY+u)=rt\operatorname{Cov}(X,Y)$