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THROUGH THE Λ GLASS

The topological path to Nonstandard Analysis

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> "Fairy tales are more than true: not because they tell us that dragons exist, but because they tell us that dragons can be beaten."
>
> — Neil Gaiman

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Introduction

> "She was already learning that if you ignore the rules people will, half the time, quietly rewrite them so that they don't apply to you."
>
> Terry Pratchett

There are a few moments, in a maths' student's career, more baffling and terrifying than the first meeting with a physics' course; those who are spared from this kind of encounter lose an interesting point of view, but are also exempted from the inevitable and painful process of adjusting to the way mathematics is done in these courses. Basic physics involves solving ODEs, and solving ODEs by separating variables amounts to simplifying differentials, something any student will initially abhor and then, slowly, come to terms with — under the evergreen tacit agreement that 'such things have a formal counterpart'. This sentence is, to some extent, the tl;dr[1] of this thesis. Non-Standard Analysis was born with the precise intent of making reasoning with infinitesimals precise, and sound, and at least a little less terrifyingly hand-wavey. Such a noble pursuit was, at least at the beginning of the story we are going to tell, explicitly internal to the mathematical logic community. Ever since the first steps in this direction, Non-Standard Analysis has spread in other areas of mathematics, proving itself to be an useful tool. A least expected application of Non-Standard Analysis will be the one approached in chapter three: Non-Archimedean Probability. Even though the use of Non-Standard Analysis in probability dates back to the first 'giants' of the field, the ideas behind the work of Wenmackers, Benci and Horsten in [**infprob**] are quite new, and lead to the possibility of modelling infinite, fair lotteries. The first chapter will be devoted to showcasing some of the most basic and known approaches to Non-Standard Analysis, while the second chapter will mostly consist of material from [**NAM**], where a topological approach to Non-Standard Analysis is developed in an effort to provide a non-logical (and thus apparently more mathematician-friendly) approach to these methods.

The titles of the thesis, of the chapters and of the sections all come from one or another opera of Carroll (mostly Alice's adventures). His works show to what extent creativity can go, how many worlds it can create and how many unexpected links will

[1] **Urban Dictionary:** "Too long; didn't read.", meaning a post, article, or anything with words was too long, and whoever used the phrase didn't read it for that reason.

show up from entirely different such worlds. This kind of free creativity, of unbounded exploration of what can happen in universes that aren't our own is, I believe, precisely the core of mathematics and mathematical logic. I do not believe Cantor wanted us to stay in the paradise he has created; I believe he wanted us to build our own.

A few words on notation

$\mathcal{P}(X)$ is the set of all subsets of X (the **power set of** X). $\mathcal{P}_{\mathsf{fin}}(X)$ is the set of all *finite* subsets of X. $\subseteq$ allows for the possibility of equality, $\subset$ does not. X^c is the complement of X with respect to an ambient set; whenever the latter is not obvious we use $I \setminus X$. We adhere to the religion whose gospel says that $0 \in \mathbb{N}$. In an ordered field of characteristic zero $(\mathbb{F}, \leqslant)$ an **infinitesimal** is an element smaller than all fractions $\frac{1}{n}$, where n is understood to be $\underbrace{1_{\mathbb{F}} + \ldots + 1_{\mathbb{F}}}_{n \text{ times}}$. Similarly, an **infinite** element is an element bigger than any n.

An ordered field with no infinitesimals is said to be **Archimedean**.

1 Ghosts of departed quantities

"The trouble with having an open mind, of course, is that people will insist on coming along and trying to put things in it."

Terry Pratchett

The need for a formal theory of infinitesimals dates back to George Berkeley's famous comment on the foundations of calculus: «*And what are these same evanescent increments? They are neither finite quantities, nor quantities infinitely small, nor yet nothing. May we not call them the* ghosts of departed quantities*?*». This harsh critique did not stop Euler, amongst others, from developing a huge body of literature on calculus and infinitesimals, and it wasn't until Weierstrass' theory of $\varepsilon - \delta$s that Berkeley's ghost — pun not intended — was put to rest. Still, the formal development of calculus wasn't enough; something had to be done for the infinitesimals, and more than two hundred years after Leibniz and Newton's time the first true hit to the *evanescent* nature of infinitesimals was blown. In a series of spectacular papers, Abraham Robinson built a sound and indisputable base for a theory of infinitesimal and infinite numbers: the sixties marked the birth of **Nonstandard Analysis**.

Robinson's work relied heavily on a logical formalism, and was thus quite indigestible for the *working mathematicians* (who, rephrasing a famous quote of Feynman's, are interested in mathematical logic just as much as birds are interested in ornithology). Considerable work has been done since the sixties in order to make Nonstandard Analysis more mathematician-friendly: what follows is a sightseeing tour through the vast land of Nonstandard Analysis or, more precisely, of *nonstandard methods*: while the original purpose of Robinson's — giving a sound status to Leibniz and Newton's legacy — is indeed important for historical and philosophical reasons, the methods and tools developed studying Nonstandard Analysis have ever since spread throughout mathematics (for an example of the use of nonstandard methods in other branches of mathematics, see [**dinasso**]). We will start with trying to understand more precisely what nonstandard methods *are*; we will then give a few examples of possible frameworks for nonstandard methods: a very *elementary* one, albeit powerful enough to develop basic calculus; a more concrete one, through the use of ultrapowers; and a sneaky peek into the theory of superstructures.

1.1 | A farewell to standardness

Following [**eightfold**], we will say that nonstandard methods are made up by three main tools: a **star map**, a **transfer principle** and **saturation**. We begin with a universe $\mathbb{U}$, which will usually (for an example of exceptions, consider the section right after this one) be a set large enough to contain all the *important* objects we need to perform mathematics: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, functions between these sets, subsets of these sets, families of subsets of these sets (for example, topologies) and so on. We will refer to $\mathbb{U}$ as the **standard universe**. We wish to enlarge $\mathbb{U}$ into a bigger universe $\mathbb{V}$, which will contain **nonstandard** counterparts of the elements of $\mathbb{U}$ (for example, ${}^*\mathbb{R}$). Here enters the first tool of nonstandard methods: we call **star map** the function $* : \mathbb{U} \to \mathbb{V}$ that sends every element x of the standard universe into its nonstandard counterpart ${}^*x$. We ask that natural numbers are sent into natural numbers (i.e., that ${}^*n = n$) and that $\mathbb{N} \subset {}^*\mathbb{N}$ (**properness**). We call **internal** the objects that are images of standard objects; we call **external** everything else. Our star map needs to obey an important law, the **transfer principle**: we ask that for any property $p(x_1, ...x_n)$, $p(x_1, ...x_n)$ is true in $\mathbb{U}$ if and only if $p({}^*x_1, ...{}^*x_n)$ is true in $\mathbb{V}$. This is, arguably, the central point of Nonstandard Analysis: the tool that lets us *transfer* properties of standard objects onto their nonstandard counterparts. For example, we could talk about *hyperfinite* sets, which are the nonstandard counterparts of finite sets: they aren't finite, but enjoy many properties of finite sets. A final word about **saturation**: this property is essential in proving more advanced results, for example in functional analysis or measure theory, but its nature is beyond the scope of this thesis.

1.2 | Elementary, my dear Robinson

Model theory lends us a notion that, albeit remarkably more contained than the more general setting introduced in the previous paragraph, will give us an abstract example of nonstandard methods.

DEFINITION **1.2.1** Let L be a first-order signature, M, N be $L-$structures and $L(N)$ be the set of first-order sentences with parameters in N. We say that M is an **elementary extension** of N, and write $N \preceq M$, if $N \subseteq M$ and for every $\varphi \in L(N)$,

$$N \vDash \varphi \iff M \vDash \varphi.$$

One may imagine an elementary extension of a ”universe” N as a bigger universe containing N where the old inhabitants maintain the same properties, while possibly gaining new ones (and meeting new inhabitants as well).

DEFINITION **1.2.2** Let $\mathbb{R}$ be the structure of the real numbers in the language of ordered rings. Consider an elementary extension ${}^*\mathbb{R}$: we call it the **hyperreal field**.

LEMMA **1.2.3** *${}^*\mathbb{R}$ is a field.*

PROOF. All the properties of addition and multiplication can be written down in first order sentences, which are then true in the new structure. □

We now consider $\mathbb{R}$ to be our standard universe, the identity map on $\mathbb{R}$ to be the star map and ${}^*\mathbb{R}$ to be the nonstandard universe where nonstandard methods are performed. While being extremely far from concrete, this setting allows us to prove basic theorems such as:

THEOREM **1.2.4** *${}^*\mathbb{R}$ contains infinitesimals and infinite elements.*

PROOF. We are going to show the existence of an infinitesimal number ε: after doing so, ε^{-1} is going to be an infinite number. Suppose $d \in {}^*\mathbb{R} \setminus \mathbb{R}$ is neither infinite nor infinitesimal and consider $\{y \in \mathbb{R} : y < d\} \subseteq \mathbb{R}$. Being a upper-bounded non-empty subset of $\mathbb{R}$, completeness grants the existence of a supremum $k = \sup\{y \in \mathbb{R} : y < d\}$. If $d - k > 0$, then call $\varepsilon = d - k$ and suppose $\varepsilon = d - k > c$ for some standard positive real c; then in particular $c + k < d$, meaning $k < k + c < d$, against the definition of (least) upper-bound; if, on the other hand, $k - d > 0$, then call $\varepsilon = k - d$ and suppose $\varepsilon > c$ for some standard positive real c: this means $d < k - c$, so $k - c < k$ is still an upper-bound, against the definition of supremum. □

We will now introduce a few notions that will be the same throughout the thesis, modulo appropriate corrections.

DEFINITION **1.2.5** Let $a, b \in {}^*\mathbb{R}$. We say that a and b are **infinitesimally close** if $|a - b|$ is infinitesimal. We denote this by $a \sim b$.

One could verify that $\sim$ is an equivalence relationship. Note that it is not definable in the structure and thus neither are definable the following sets:

DEFINITION **1.2.6** Let $x \in {}^*\mathbb{R}$. We call **monad of** x the set $\mu(x) = \{y \in {}^*\mathbb{R} : y \sim x\}$.

As an example of the nonstandard methods that can be done in this setting, we prove one of the basic results about continuity, which formally embodies our intuition about "close" points staying "close" after the action of a continuous function: let us extend the language by adding a symbol for a function $f : \mathbb{R} \to \mathbb{R}$, and denote by ${}^*f$ its interpretation in ${}^*\mathbb{R}$.

THEOREM **1.2.7** *Let $f : \mathbb{R} \to \mathbb{R}$. Let ${}^*f$ denote its interpretation in ${}^*\mathbb{R}$. Then f is continuous at $b \in \mathbb{R}$ if and only if for every $x \in \mu(b)$, ${}^*f(x) \in \mu({}^*f(b))$.*

REMARK The fact that this characterization of continuity sounds very much like the topological definition through the use of open sets is not a coincidence. Indeed, one could show that a set $A \subseteq \mathbb{R}$ is open iff for every standard $x \in {}^*A$ we have that $\mu(x) \subseteq {}^*A$. Monads provide thus an alternative way of looking at *nearness*.

PROOF. Let f be continuous at b, and let $x \in \mu(b)$: then $|x - b|$ is infinitesimal. We translate the definition of continuity into a first order formula that holds in $\mathbb{R}$:

$$\forall \varepsilon > 0 \exists \delta > 0 (\forall y \in \mathbb{R}(|x - b| < \delta \to |f(x) - f(b)| < \varepsilon)).$$

A few remarks are due: some shorthands are in use, so $(\forall x > 0)\varphi$ truly means $\forall x(x > 0 \to \varphi)$, and the same goes for $(\forall x \in \mathbb{R})\varphi$ that, remembering that our language contains a predicate $\mathbb{R}(x)$, translates as $\forall x(\mathbb{R}(x) \to \varphi)$. A similar shorthand is in use for $\exists$ (where $\wedge$ substitutes $\to$). Finally, the absolute value function $|\cdot|$ can be defined in our language, so we use its symbol freely. Let $\bar{\varepsilon} > 0$ be a real positive number, and let $\bar{\delta} > 0$ be the positive number obtained from the truth of the sentence. By transfer, the following sentence is true in ${}^*\mathbb{R}$:

$$\forall y \in {}^*\mathbb{R}(|x - b| < \bar{\delta} \to |{}^*f(x) - {}^*f(b)| < \bar{\varepsilon}).$$

Now let $x \in \mu(b)$: clearly, $|x-b| < \bar{\delta}$, since $\bar{\delta}$ is a positive real number, so $|{}^*f(x) - {}^*f(b)| < \bar{\varepsilon}$. Since $|{}^*f(x) - {}^*f(b)|$ is smaller than any positive real number, it is infinitesimal and thus ${}^*f(x) \in \mu({}^*f(b))$.

Viceversa, suppose f is not continuous at b: then the following sentence is true in $\mathbb{R}$:

$$\exists \varepsilon > 0(\forall \delta > 0(\exists y \in \mathbb{R}(|y - b| < \delta \wedge |f(y) - f(b)| \geqslant \varepsilon)).$$

Let's fix a standard real number $\bar{\varepsilon}$ as a witness for the existential quantifier; then by transfer in ${}^*\mathbb{R}$ it is true that

$$\forall \delta > 0(\exists y \in {}^*\mathbb{R}(|y - b| < \delta \wedge |f(y) - f(b)| \geqslant \bar{\varepsilon}).$$

This is in particular true for an infinitesimal δ, leading to a contradiction. □

While this is powerful, it is also unsatisfactory. We don't have a real grasp of what these infinitesimals are: we only know they are there. A more concrete example of this type of setting for nonstandard methods will be provided in the next paragraph.

1.3 | The merry ultrapowers of $\mathbb{R}$

The central notion is that of an ultrafilter: one might think of it as a way of deciding which sets are *large* and which are *small* (indeed, one could think of ultrafilters as $\{0, 1\}$-valued finitely additive measures on $\mathcal{P}(X)$). This choice nonetheless needs to be rationally sound, so the intersection of large sets must be large (as if there wasn't enough space for two large sets to exists "indipendently"), any superset of a large set must be large and if a set is large, then its complement is small (and viceversa). This motivates the following definition:

DEFINITION 1.3.1 Let I be a set. An **ultrafilter over** I is a family $\mathcal{U}$ of subsets of I closed under intersection (for all $A, B \in \mathcal{U}$, $A \cap B \in \mathcal{U}$), upwards (for all $A \in \mathcal{U}$, $B \subseteq I$, if $A \subseteq B$ then $B \in \mathcal{U}$) and, most importantly, that has the **ultra** property, i.e. for all $A \subseteq I$, either $A \in \mathcal{U}$ or $A^c \in \mathcal{U}$.

Ultrafilters come in two different flavours: there are **principal** ultrafilters, that is ultrafilters of the form $\{A \subseteq I : x \in A\}$ for some $x \in I$, and **nonprincipal** ultrafilters. For reasons that will become clearer later on, we will focus ourselves on nonprincipal ultrafilters (oftentimes also called **free**). Beforehand, though, it would be necessary to prove the existence of such ultrafilters. This is a classical result, whose proof can be found in any introductory set theory book (for example, see [**schimmerling**]).

THEOREM 1.3.2 *Let* I *be an infinite non-empty set, then there is an ultrafilter* $\mathcal{U}$ *over* I *that contains every cofinite subset of* I*; in particular, it is nonprincipal.*

We now begin with the set of real-valued sequences; we denote it by $\mathbb{R}^\omega$. It can be naturally embodied with a ring structure, which will later on be useful. As of (1.3.2), there exists a nonprincipal ultrafilter over $\mathbb{N}$: let us call it $\mathcal{U}$. We define an equivalence relationship on $\mathbb{R}^\omega$, saying that two sequences are the same (and denoting it by $r \equiv s$) if the set of natural numbers over which they coincide is *large*, i.e. it belongs to the ultrafilter. Using a suggestive logical notation, we define

$$[\![r = s]\!] := \{n \in \mathbb{N} : r(n) = s(n)\}.$$

We now say that $r \equiv s$ if and only if $[\![r = s]\!] \in \mathcal{U}$. We are now ready to build our concrete example of nonstandard universe:

DEFINITION **1.3.3** The **ultrapower of** $\mathbb{R}$ **by** $\mathcal{U}$, denoted $\mathbb{R}_\mathcal{U}$, is the quotient set $\mathbb{R}^\omega/\equiv$.

We extend addition and multiplication from $\mathbb{R}^\omega$ to $\mathbb{R}_\mathcal{U}$ naturally, i.e. $[r]+[s] = [r+s]$ and $[r]\cdot[s] = [r\cdot s]$.

REMARK The choice of a nonprincipal ultrafilter is instrumental in getting a **proper** ultrapower of $\mathbb{R}$. In fact, suppose we had built our ultrapower from a principal ultrafilter, say (without loss of generality) the set of subsets of $\mathbb{N}$ that contain 1. Let $r \in \mathbb{R}^\omega$ such that $r(1) = c \in \mathbb{R}$, and denote by $\mathbf{c}$ the constant sequence in c; then $r \equiv \mathbf{c}$. Every equivalence class would be, then, the equivalence class of a constant sequence, and the ultrapower would be isomorphic to the real numbers. Nothing new would be gained.

We identify every real number $a \in \mathbb{R}$ with the equivalence class of the constant sequence $[\mathbf{a}]$. This means that $\mathbb{R} \subset \mathbb{R}_\mathcal{U}$; not only that, but it can be shown that $\mathbb{R}_\mathcal{U}$ is a field of which $\mathbb{R}$ is a subfield. The natural ordering on real numbers can be extended as well:

DEFINITION **1.3.4** Let $[r], [s] \in \mathbb{R}_\mathcal{U}$. We say that $[r] \leqslant [s]$ if and only if $[\![r \leqslant s]\!] := \{n \in \mathbb{N} : r(n) \leqslant s(n)\}$ is large, i.e. is a member of $\mathcal{U}$. This extends the ordering because for any $a \leqslant b$ real numbers, $[\![\mathbf{a} \leqslant \mathbf{b}]\!] = \mathbb{N} \in \mathcal{U}$.

We now give an example of an infinitesimal number; this leads to the intuition that infinitesimals are the equivalence classes of sequences that converge to zero, while infinite numbers are the equivalence classes of divergent sequences.

EXAMPLE Let $r(n) = \frac{1}{n}$. Then $\varepsilon := [r]$ is an infinitesimal: for any natural number N, we have that

$$[\![r \leqslant \frac{1}{N}]\!] = \{n \in \mathbb{N} : n \geqslant N\} \in \mathcal{U},$$

since $\mathcal{U}$ contains every cofinite subset of $\mathbb{N}$.

We take the quotient projection as a star map, and the following fundamental theorem provides as a special case the transfer principle:

THEOREM **1.3.5** (ŁOŚ) *Let $\mathcal{U}$ be a nonprincipal ultrafilter over $\mathbb{N}$, and $\varphi(x_1, ...x_n)$ a first-order formula with free variables ranging amongst $x_1, ...x_n$. Then*

$$\mathbb{R}_\mathcal{U} \vDash \varphi([a_1], ...[a_k]) \iff [\![\varphi(a_1, ...a_k)]\!] := \{n \in \mathbb{N} : \mathbb{R} \vDash \varphi(a_1(n), ...a_k(n))\} \in \mathcal{U}.$$

In the special case where φ is true of certain real numbers $r_1, ...r_k$, then $[\![\varphi(\mathbf{r_1}, ...\mathbf{r_k})]\!] = \mathbb{N} \in \mathcal{U}$.

REMARK One could wonder whether the choice of $\mathcal{U}$ has any influence on the structure of the ultrapower $\mathbb{R}_{\mathcal{U}}$. The answer is, as it often is in mathematics, *it depends*. Assuming the Continuum Hypothesis, all ultrapowers are isomorphic regardless of the choice of the ultrafilter; for further details, see [**mse**].

1.4 | Refrain: superstructures (I)

While the ultrapower construction is indeed more concrete and intelligible, it is still somewhat unsatisfactory. What about topology? What about measure theory? Are we really stuck with real analysis forever? *Deo gratias*, no. More can be said and done using nonstandard methods; in order to do so, however, a bigger theory must be developed. We will barely scratch the surface of it: for a deeper analysis of the following objects, see the second chapter of [**introductionloeb**].

DEFINITION 1.4.1 Let X be an infinite, non-empty set. We define recursively:

$$\begin{aligned} V_0(X) &= X, \\ V_n(X) &= \mathcal{P}(V_{n-1}(X)) \cup V_{n-1}(X). \end{aligned}$$

The **superstructure over** X is the set

$$V_\omega(X) = \bigcup_{n\in\mathbb{N}} V_n(X).$$

We call X a set of **individuals**.

Superstructures allow us to formalize the otherwise vague notion of "universe" used when describing nonstandard methods. The superstructure over a set contains every mathematical object that can be built from the set using set-theoretic operations; for example, if we set $X = \mathbb{R}$ then the resulting superstructure $V_\omega(\mathbb{R})$ contains real numbers, subsets of real numbers, real-valued functions of real variable, but also complex numbers, subsets of the complex field, topologies on the complex plane, projective spaces, Banach spaces, measure spaces and so on: almost everything that might come to the mind of a mathematician can be found at a certain step of this hierarchy of sets, and thus belongs to the superstructure. This makes the superstructure the ideal home for performing mathematics, and it provides an excellent framework for nonstandard methods. Following the approach in the second chapter of [**introductionloeb**], we consider a first-order signature $\mathcal{L}_X = \{\in, =\} \cup \{a : a \in V_\omega(X)\}$; from now onwards, we will adopt the usual model-theoretic abuse of identifying the language with the set of its sentences, which will be then called $\mathcal{L}_X$. We now consider two sets of individuals X, Y both containing $\mathbb{N}$. Their respective superstructures, $V_\omega(X)$ and $V_\omega(Y)$, can be seen as structures respectively in $\mathcal{L}_X$ and $\mathcal{L}_Y$. Thus, a truth relation is well-defined. We are now ready to define what a **monomorphism** between superstructures is:

DEFINITION 1.4.2 An injective function $* : V_\omega(X) \to V_\omega(Y)$ is called a **monomorphism** if

I. ${}^*\emptyset = \emptyset$,

II. if $a \in X$, then ${}^*A \in Y$, and for every $n \in \mathbb{N}$, ${}^*n = n$,

III. if $a \in V_{n+1}(X) \setminus V_n(X)$, then ${}^*A \in V_{n+1}(Y) \setminus V_n(Y)$,

IV. if $a \in {}^*V_n(X)$, and $b \in a$, then $b \in {}^*V_{n-1}(X)$,

V. for all $\varphi \in \mathcal{L}_X$, let ${}^*\varphi \in \mathcal{L}_Y$ be the sentence obtained by replacing each constant in φ with its image under $*$: then $V_\omega(X) \models \varphi$ iff $V_\omega(Y) \models {}^*\varphi$.

Property [V] is precisely the transfer principle, so a triple $\langle V_\omega(X), V_\omega(Y), *\rangle$ is the perfect framework for nonstandard methods. This triple will usually be called a **nonstandard universe**. Depending on the source, the monomorphism might also be called a superstructure embedding (for example in [**keisler**]). The existence of a nonstandard universe is guaranteed by the possibility of extending the ultrapower construction to superstructures, but the details are once again beyond the scope of this thesis.

2 Alice in Λ-land

> "Why do you go away? So that you can come back. So that you can see the place you came from with new eyes and extra colors. And the people there see you differently, too. Coming back to where you started is not the same as never leaving."
>
> Terry Pratchett

Nonstandard Analysis and, more in general, nonstandard methods have always been considered a part of mathematical logic, mainly due to historical reasons. While nonstandard methods have, ever since the 60s, somehow detached themselves from the logical formalism Robinson had used, the *unpleasant* logical flavour still permeates the theory, thus leading other mathematicians away. The consequences of this kind of stereotype (towards nonstandard methods and, more generally, towards mathematical logic) are uncountable, and while defeating the general suspicions mathematicians have towards logic might take more than a couple papers, something can (and ought to) be done for nonstandard methods. What follows is a characterization of nonstandard methods and, more in general, non-Archimedean mathematics that rests on topological methods and techniques instead of logical ones. $\Lambda-$limits will be introduced, and the natural setting for non-Archimedean mathematics — a construction that parallels the construction of the reals as a completion of $\mathbb{Q}$ — will be built. All of the content of this chapter comes from [**NAM**].

2.1 | All in the $\Lambda-$afternoon: Λ-limits

DEFINITION 2.1.1 Let $(\mathfrak{X}, \tau)$ be an Hausdorff topological space, and let $\mathfrak{I}$ be a set, sometimes referred to as the **parameter space**. Let $\mathcal{U}$ be a non-principal ultrafilter over $\mathfrak{I}$, and $f : \mathfrak{I} \to \mathfrak{X}$ be a function. We say that $L \in \mathfrak{X}$ is the **Λ-limit of** f, and write

$$\lim_{\lambda \uparrow \Lambda} f(\lambda) = L,$$

if for every V open neighbourhood of L there exists a $Q \in \mathcal{U}$ such that $f[Q] \subseteq V$.

This notion of $\Lambda-$limit is built as a natural generalization of a notion very common in general topology, that of convergence of a net.

REMARK A poset $(\mathcal{I}, \preceq)$ is called a **directed set** if for any $A, B \in \mathcal{I}$ there exists a $C \in \mathcal{I}$ such that $A, B \preceq C$. If $(\mathfrak{X}, \tau)$ is a topological space, then a **net over** $\mathfrak{X}$ is a function $f : \mathcal{I} \to \mathfrak{X}$. Let $f : \mathcal{I} \to \mathfrak{X}$ be a net over $(\mathfrak{X}, \tau)$. Then f **converges to** $L \in \mathfrak{X}$, in symbols $f \uparrow L$, if for every V open neighbourhood of L, f **is eventually in** V, i.e. there exists a $\mu_0 \in \mathcal{I}$ such that for every $\mu \succeq \mu_0$, $f(\mu) \in V$. Λ–limits, when taken with a fine[1] ultrafilter $\mathcal{U}$, of nets generalize this notion (sometimes called **Cauchy limit**). In fact, suppose $f : \mathcal{I} \to \mathfrak{X}$ is a net over $\mathfrak{X}$. If $f \uparrow L$, then for any V open neighbourhood of L there exists a $\mu_0 \in \mathcal{I}$ such that $f[\{\mu \in \mathcal{I} : \mu_0 \subseteq \mu\}] \subseteq V$. Since $\mathcal{U}$ is fine, $\{\mu \in \mathcal{I} : \mu_0 \subseteq \mu\} \in \mathcal{U}$, so $\lim_{\lambda \uparrow \Lambda} f(\lambda) = L$.

REMARK Λ–limits could be defined topologically by giving $\mathcal{I} \cup \{\Lambda\}$, where $\Lambda \notin \mathcal{I}$, a topology that is discrete on $\mathcal{I}$ and that has as neighbourhoods of Λ sets of the form $Q \cup \{\Lambda\}$ with $Q \in \mathcal{U}$. In this case,

$$\lim_{\lambda \uparrow \Lambda} f(\lambda) = \lim_{\lambda \to \Lambda} f(\lambda).$$

2.2 | Down the Λ-hole: $\mathcal{I}$-completions

We will now define a class of topological spaces that, due to the similarity with the '$\mathbb{R}$ as a completion of $\mathbb{Q}$' case, will be called $\mathcal{I}$–completions of the reals. These spaces will be, together with the intertwined notion of Λ–limit, the core of this theory and will provide an example of Non-Archimedean 'universe' where analysis (and other branches of mathematics) can be performed.

DEFINITION **2.2.1** Fix a non-principal ultrafilter $\mathcal{U}$ on $\mathcal{I}$. We will call a $\mathcal{I}$**-completion of the reals** (and omit any reference to $\mathcal{U}$ whenever its specific properties are irrelevant) an Hausdorff topological space $(\mathbb{R}_\Lambda, \tau)$ that satisfies the following conditions:

Λ1. $\mathcal{I} \times \mathbb{R}$ is a dense subspace of $\mathbb{R}_\Lambda$,

Λ2. $\mathbb{R} \subseteq \mathbb{R}_\Lambda$ and for any $r \in \mathbb{R}$,
$$\lim_{\lambda \uparrow \Lambda} (\lambda, r) = r,$$

Λ3. for any function $f : \mathcal{I} \to \mathbb{R}$ there exists the Λ-limit of $F(\lambda) = (\lambda, f(\lambda))$,

Λ4. any two functions $f, g : \mathcal{I} \to \mathbb{R}$ coincide on an element of $\mathcal{U}$ if and only if
$$\lim_{\lambda \uparrow \Lambda} F(\lambda) = \lim_{\lambda \uparrow \Lambda} G(\lambda).$$

THEOREM **2.2.2** *There exists a $\mathcal{I}$-completion of the reals.*

PROOF. We build a concrete example of $\mathcal{I}$-completion. Let $\mathcal{F}(\mathcal{I}, \mathbb{R})$ be the algebra of functions over $\mathbb{R}$. Let

$$I = \{f \in \mathcal{F}(\mathcal{I}, \mathbb{R}) : \exists Q \in \mathcal{U}\ (\forall x \in Q\ (f(x) = 0))\}$$

be the ideal of functions that vanish on an element of the ultrafilter.

CLAIM I is a maximal ideal in $\mathcal{F}(\mathcal{I}, \mathbb{R})$.

[1] a ultrafilter on a directed set is fine if every set of the form $\{\mu : \mu_0 \preceq \mu\}$ belongs to the ultrafilter

Let $J \supset I$ be another ideal, and let $g \in J \setminus I$. Call $Z(g) = \{x \in \mathcal{I} : g(x) = 0\}$: since $g \notin I$, $Z(g) \notin \mathcal{U}$. Define

$$h(x) = \begin{cases} \frac{1}{g(x)} & \text{if} x \in Z(g)^c \\ 0 & \text{if} x \in Z(g) \end{cases},$$

then since J is an ideal $gh \in J$, and we have

$$(gh)(x) = \begin{cases} 1 & \text{if } x \in Z(g)^c \\ 0 & \text{if } x \in Z(g) \end{cases}.$$

$\mathcal{U}$ is an ultrafilter, so if $Z(g) \notin \mathcal{U}$ then $Z(g)^c \in \mathcal{U}$, thus the following function

$$m(x) = \begin{cases} 0 & \text{if } x \in Z(g)^c \\ 1 & \text{if } x \in Z(g) \end{cases}$$

belongs to I, since it vanishes on $Z(g)^c \in \mathcal{U}$. $J \supset I$, being an ideal, is closed under sum, so $(gh + m)(x) \in J$; the latter function is the identical function, so $J = \mathcal{F}(\mathcal{I}, \mathbb{R})$, showing I is maximal.

Call $\mathbb{K} = \mathcal{F}(\mathcal{I}, \mathbb{R})/I$. As a consequence of the latter claim, it is a field (with the inherited operations). We are now ready to define $\mathbb{R}_\Lambda$:

$$\mathbb{R}_\Lambda = (\mathcal{I} \times \mathbb{R}) \cup \mathbb{K}.$$

We give a basis for a topology on $\mathbb{R}_\Lambda$:

$$\beta_\Lambda = \mathcal{P}(\mathcal{I} \times \mathbb{R}) \cup \{\mathfrak{B}_f^Q : Q \in \mathcal{U},\ f \in \mathcal{F}(\mathcal{I}, \mathbb{R})\},$$

where for any $Q \in \mathcal{U}$ and $\varphi \in \mathcal{F}(\mathcal{I}, \mathbb{R})$,

$$\mathfrak{B}_f^Q = \{(x, f(x)) : x \in Q\} \cup \{[f]_I\}.$$

CLAIM β_Λ is the basis for an Hausdorff topology on $\mathbb{R}_\Lambda$.

We first show it is the basis for a topology: let $A, B \in \beta_\Lambda$ and $x \in A \cap B$. If $x \in \mathcal{I} \times \mathbb{R}$, then $(\mathcal{I} \times \mathbb{R}) \cap A \cap B$ is a basis open set and $(\mathcal{I} \times \mathbb{R}) \cap A \cap B \subseteq A \cap B$. On the other hand, if $x \in \mathbb{K}$ then $x = [f]_I$ for some function f and $A = \mathfrak{B}_g^Q$, $B = \mathfrak{B}_h^P$ with $[h]_I = [g]_I = [f]_I$. Call $V \in \mathcal{U}$ the set on which these three functions coincide. Then $x \in \mathfrak{B}_f^{V \cap Q \cap P} \subseteq A \cap B$.

To show that it is an Hausdorff topology, it is enough to show that $x \neq y \in \mathbb{K}$ can be separated by basis open sets. Let $x = [f]_I$ and $y = [g]_I$: since $x \neq y$, there exists a $Q \in \mathcal{U}$ such that $f(z) \neq g(z)$ for all $z \in \mathcal{U}$. It follows that $\mathfrak{B}_f^Q \cap \mathfrak{B}_g^Q = \emptyset$.

We identify real numbers with the quotient class of constant functions.

CLAIM For any function f with real values,

$$\lim_{\lambda \uparrow \Lambda} F(\lambda) = [f]_I.$$

Let $\mathfrak{B}_g^Q$ be an open basis neighbourhood of $[f]_I$, so $[f]_I = [g]_I$. Call $P \in \mathcal{U}$ the set of parameters on which the two functions coincide. Then $F[Q \cap P] \subseteq \mathfrak{B}_g^Q$.

With this claim in mind, we have that:

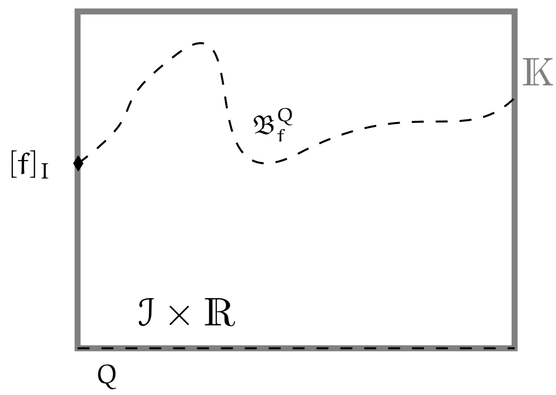

Figure 2.1: $\mathbb{K}$ can be thought of as "points at infinity" of $\mathcal{I} \times \mathbb{R}$, and open sets of the form $\mathfrak{B}_f^Q$ in $\mathbb{R}_\Lambda$ as the union of the image of $Q \in \mathcal{U}$ under $(\lambda, f(\lambda))$ with the equivalence class $[f]_I$. Here Q is a cofinite set that belongs to any non-principal ultrafilter.

Λ1. $\mathcal{I} \times \mathbb{R}$ is dense in $\mathbb{R}_\Lambda$, since any basis open set overlaps with it,

Λ2. follows from the identification of the real numbers with the equivalence classes of constant functions and the latter claim,

Λ3. follows from the latter claim,

Λ4. from the latter claim follows that if the limits coincide, $[f]_I = [g]_I$ and thus they coincide on an element of $\mathcal{U}$.

□

2.2.1 | IDEALS FROM A CATERPILLAR

The nature of a $\mathcal{I}$-completion of the reals strongly depends on the choice of the ultrafilter over $\mathcal{I}$. As we have seen in the proof of (2.2.2), an ultrafilter induces a maximal ideal in the algebra of real-valued functions $\mathcal{F}(\mathcal{I}, \mathbb{R})$ (the ideal of functions that vanish on elements of the ultrafilter). The converse is true — fixing a maximal ideal $M \subseteq \mathcal{F}(\mathcal{I}, \mathbb{R})$ one can recover an ultrafilter on $\mathcal{I}$,

$$\mathcal{U}_M = \{f^{-1}(0) : f \in M\}.$$

THEOREM **2.2.3** *Let $M \subseteq \mathcal{F}(\mathcal{I}, \mathbb{R})$ be a maximal ideal; then $\mathcal{U}_M$ is an ultrafilter.*

This sort of duality implies that in order to define a notion of Λ-limit and $\mathcal{I}$-completion one can start from an ultrafilter or from a maximal ideal: this remark will be useful in Chapter 3, when we will build NAP spaces using ideals instead of filters.

2.2.2 | THE QUEEN'S HYPERREAL-FIELD

We now consider a fixed $\mathcal{I}$—completion of the reals, $\mathbb{R}_\Lambda$. Inside any such space there is a special subspace that we will denote by $\mathbb{K} = \{\lim_{\lambda \uparrow \Lambda} (\lambda, f(\lambda)) : f \in \mathcal{F}(\mathcal{I}, \mathbb{R})\}$. We call this subspace the **hyperreal field**, since it can be equipped with two complete operations:

$$\lim_{\lambda\uparrow\Lambda}(\lambda, f(\lambda)) * \lim_{\lambda\uparrow\Lambda}(\lambda, g(\lambda)) = \lim_{\lambda\uparrow\Lambda}(\lambda, f(\lambda) * g(\lambda)), \tag{2.1}$$

with $* \in \{+, \cdot\}$. $\mathbb{K}$ naturally contains $\mathbb{R}$ as a subfield.

THEOREM **2.2.4** *The set $\mathbb{K}$ with the operations defined in (2.1) is a field.*

PROOF. The only interesting proof is that of the existence of multiplicative inverses. Namely, if $x \neq 0$ is an element of $\mathbb{K}$, then $x = \lim_{\lambda\uparrow\Lambda}(\lambda, f(\lambda))$ and since $\mathbb{R}_\Lambda$ is Hausdorff there exists a (basis) open set separating x and 0 — in particular, there exists a $Q \in \mathcal{U}$ such that for every $\lambda \in Q$, $f(\lambda) \neq 0$. This means we can define a function

$$g(\lambda) = \begin{cases} 1, & \text{if } x \notin Q, \\ \frac{1}{f(\lambda)}, & \text{if } x \in Q, \end{cases}$$

whose limit will be the inverse of x (in fact, $\lim_{\lambda\uparrow\Lambda}(\lambda, f(\lambda)\cdot g(\lambda)) = 1$). □

$\mathbb{K}$ can be ordered in the natural way, i.e. saying that

$$\lim_{\lambda\uparrow\Lambda}(\lambda, f(\lambda)) \leqslant \lim_{\lambda\uparrow\Lambda}(\lambda, g(\lambda))$$

if and only if $\{\lambda \in \mathcal{I} : f(\lambda) \leqslant g(\lambda)\} \in \mathcal{U}$.

We now show that the idea of $\mathbb{K}$ as disjoint "points at infinity" is preserved in the general setting.

LEMMA **2.2.5** *Let $\mathbb{R}_\Lambda$ be a $\mathcal{I}$—completion of the reals, then $\mathcal{P}_{\mathsf{fin}}(\mathbb{R}) \times \mathbb{R} \cap \mathbb{K} = \emptyset$.*

PROOF. Suppose not, so there exists a function $f : \mathcal{I} \to \mathbb{R}$ such that $\lim_{\lambda\uparrow\Lambda}(\lambda, f(\lambda)) = (\bar{\lambda}, \bar{r}) \in \mathcal{I}\times\mathbb{R}$. By definition of Λ—limit, there exists a $Q \in \mathcal{U}$ such that for every $\lambda \in Q$ $(\lambda, f(\lambda)) = (\bar{\lambda}, \bar{r})$. In particular, $Q = \{\bar{\lambda}\}$, which is absurd since $\mathcal{U}$ is non-principal. □

Call **minimal** every $\mathcal{I}$—completion of the reals of the form $(\mathcal{I} \times \mathbb{R}) \sqcup \mathbb{K}$.

As it is use in topology, we kept writing $\mathbb{R}_\Lambda$ to denote a $\mathcal{I}$—completion of the reals but we shouldn't forget that the same set could be a $\mathcal{I}$—completion of the reals for different choices of topologies. There is, however, a choice that we wish to isolate:

DEFINITION **2.2.6** Let $\mathbb{R}_\Lambda$ be a $\mathcal{I}$—completion of the reals for a certain topology τ. We will call **slim topology** the topology τ_Λ generated by

$$\mathcal{P}(\mathcal{I} \times \mathbb{R}) \cup \{\mathfrak{B}^Q_f : Q \in \mathcal{U},\ f \in \mathcal{F}(\mathcal{I}, \mathbb{R})\},$$

where the open sets $\mathfrak{B}^Q_f$ are defined precisely as in (2.2.2). Call $(\mathbb{R}_\Lambda, \tau_\Lambda)$ a **canonical** $\mathcal{I}$—completion of the reals.

Canonical $\mathcal{I}$—completions are well-behaved in the sense that just like $\overline{\mathcal{I} \times \mathbb{R}} = \mathbb{R}_\Lambda$, the closure of subsets of $\mathcal{I} \times \mathbb{R}$ is made up by the subset and the limits of functions that are eventually in the subset:

THEOREM **2.2.7** *Let $\mathbb{R}_\Lambda$ be a canonical $\mathcal{I}$—completion of the reals, then for every $B \subseteq \mathcal{I}\times\mathbb{R}$, we have*

$$\overline{B} = B \cup \{\lim_{\lambda\uparrow\Lambda}(\lambda, f(\lambda)) : \exists Q \in \mathcal{U}\ (\forall z \in Q\ (z, f(z)) \in B)\}.$$

PROOF. Let $f : \mathcal{I} \to \mathbb{R}$ and $B \subseteq \mathcal{I} \times \mathbb{R}$: call $B(f) = \{\lambda \in \mathcal{I} : (\lambda, f(\lambda)) \in B\}$. Call $\varphi = \lim_{\lambda \uparrow \Lambda} (\lambda, f(\lambda))$.

If $B(f) \in \mathfrak{U}$, then every open neighbourhood of φ meets, by definition, B, so $\varphi \in \overline{B}$; on the other hand, if $B(f) \notin \mathfrak{U}$ then $\mathfrak{B}_f^{B(f)}$ is disjoint from B, thus $\varphi \notin \overline{B}$. □

If we consider a minimal canonical $\mathcal{I}$−completion of the reals, it turns out this topological path is equivalent to a variant of a well-known construction we considered in the previous chapter.

DEFINITION 2.2.8 Let $\mathbb{R}_{\mathfrak{U}}$ be the quotient set of $\mathbb{R}^{\mathcal{I}}$ by the equivalence relationship $f \equiv g$ if and only if $\{\lambda \in \mathcal{I} : f(\lambda) = g(\lambda)\} \in \mathfrak{U}$.

THEOREM 2.2.9 *$\mathbb{K}$ and $\mathbb{R}_{\mathfrak{U}}$ are isomorphic as fields.*

PROOF. Let $i : \mathbb{K} \to \mathbb{R}_{\mathfrak{U}}$ be defined by $i(\lim_{\lambda \uparrow \Lambda} (\lambda, f(\lambda)) = [f]$. Then i respects operations by their very definitions, it is injective due to $\Lambda 4$ and surjective due to $\mathbb{K}$ being exactly the set of all Λ−limits. □

2.3 | Refrain: superstructures (II)

Classical nonstandard methods can be extended to superstructures, in order to allow for a more general use of mathematical objects. The same happens for Λ−limits:

DEFINITION 2.3.1 Let $\varphi : \mathcal{I} \to V_\omega(\mathbb{R})$ be a **bounded** function if there exists a maximal rank $n \in \mathbb{N}$, i.e. a natural number such that for every $\lambda \in \mathcal{I}$, $\varphi(\lambda) \in V_n(\mathbb{R})$. We define Λ−limits for bounded functions by induction. If $n = 0$, these were already defined. Let φ be bounded of maximal rank n, then

$$\lim_{\lambda \uparrow \Lambda} (\lambda, \varphi(\lambda)) = \left\{ \lim_{\lambda \uparrow \Lambda} (\lambda, f(\lambda)) : f : \mathcal{I} \to V_{n-1}(\mathbb{R}) \text{ and } \forall \lambda \in \mathcal{I}, f(\lambda) \in \varphi(\lambda) \right\} \in V_\omega(\mathbb{K}).$$

Λ−limits of bounded functions are called **internal**, while every other element is called **external**. $\mathbb{K}$ is internal, while $\mathbb{R}$ and $\mathbb{K} \setminus \mathbb{R}$ are external. This approach to superstructures ends up being strictly related to the general one exposed in the previous chapter:

THEOREM 2.3.2 *Let $\star : V_\omega(\mathbb{R}) \to V_\omega(\mathbb{K})$ be defined by*

$$^*x = \lim_{\lambda \uparrow \Lambda} (\lambda, x)$$

where the Λ−limit is taken at the rank of x. Then $\langle V_\omega(\mathbb{R}), V_\omega(\mathbb{K}), \star \rangle$ is a nonstandard universe.

3 How probable is forever?

> "Suppose that a dart is thrown, using the unit interval as a target; then what is the probability of hitting a point? Clearly this probability cannot be a positive real number, yet to say that it is zero violates the intuitive feeling that, after all, there is some chance of hitting the point."
>
> *Nonstandard Measure Theory* — Bernstein and Wattenberg

Classical probability, i.e. probability *à la Kolmogorov*, has proved to be a very fruitful mathematical field and nevertheless extremely useful in applications; its foundations are mathematically sound, and provide a framework for very interesting results. Unfortunately, the same can't be said for its *philosophical* foundations: in the following, we will give a brief resume of classical probability and then show how a very simple model can't be built under Kolmogorov's axiomatization. We then argue that a possible way out of this is through the use of nonstandard methods, giving a possible axiomatization relying on Λ-theory.

3.1 | Begin at the beginning

We say that a **probability space** is a triple $\langle \Omega, \Sigma, \mathbf{P} \rangle$ where Ω is a set, Σ is a σ-algebra on Ω and $\mathbf{P} : \Sigma \to \mathbb{R}$ is a real-valued set function that satisfies axioms K1, K2, K3.

K1. for all $A \in \Sigma$, $\mathbf{P}(A) \geqslant 0$,

K2. $\mathbf{P}(\Omega) = 1$,

K3. for any $\langle A_i : i \in \mathbb{N} \rangle \subseteq \Sigma$ such that for $i \neq j$, $A_i \cap A_j = \emptyset$, then

$$\mathbf{P}\left(\bigcup_{i \in \mathbb{N}} A_i\right) = \sum_{i=0}^{+\infty} \mathbf{P}(A_i).$$

Axiom K3 is called **σ-additivity** and is equivalent to K3$^-$ (**finite additivity**) + K$_\infty$ (**continuity**):

K3$^-$. for any $A, B \in \Sigma$ such that $A \cap B = \emptyset$, then $\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B)$,

K_∞. let $\langle A_n : n \in \mathbb{N}\rangle \subseteq \Sigma$ be an increasing sequence of sets, i.e. for all n we have $A_n \subseteq A_{n+1}$, then limits commute with **P**, that is

$$\mathbf{P}\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \lim_{n\to+\infty} \mathbf{P}(A_n).$$

The proof of this equivalence can be found on any introductory probability book.

3.1.1 | A FAIR LOTTERY ON THE NATURALS: A NEGATIVE RESULT

We now consider the natural numbers $\mathbb{N}$ and imagine we want to model — through the use of Kolmogorov's axioms — a fair lottery, i.e. a game where very ticket $n \in \mathbb{N}$ has the same probability $\mathbf{P}(\{n\}) = \varepsilon \in [0,1]$. By K3, since every singleton is disjoint from another one, we have that

$$\mathbf{P}\left(\bigcup_{n\in\mathbb{N}} \{n\}\right) = \sum_{n=0}^{+\infty} \mathbf{P}(\{n\}) = \sum_{n=0}^{+\infty} \varepsilon,$$

while on the other hand by K2

$$\mathbf{P}\big(\bigcup_{n\in\mathbb{N}} \{n\}\big) = \mathbf{P}(\mathbb{N}) = 1.$$

We have two possibilities: if $\varepsilon = 0$, then the above series converges to 0, which contradicts K2; on the other hand, if $\varepsilon > 0$ then the series diverges, again against K2. The obstruction to this model comes from K2 and K3, so in order to allow the possibility of such probabilistic scenarios — in order to build a **weakly Laplacian** theory of probability — one of them has to go. Our choice falls on K3, with a caveat: not *all* of K3 has to go. We can keep $\mathsf{K3}^-$, while simultaneously dropping the more controversial K_∞. This will be the aim of NAP, **Non-Archimedean Probability**.

3.2 | One can't believe infinitesimal things

A **NAP space** is a triple $\langle \Omega, \mathcal{R}, \mathbf{P}\rangle$ where Ω is the **event space**, $\mathcal{R}$ is a superreal field, $\mathbf{P} : \mathcal{P}(\Omega) \to \mathcal{R}$ is the **probability function**. We recall a basic definition from classical probability: for every $A, B \subseteq \Omega$ such that $B \neq \emptyset$, the **conditional probability** $\mathbf{P}(A \mid B)$ is the quantity

$$\mathbf{P}(A \mid B) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(B)}.$$

NAP1. for every $A \subseteq \Omega$, $\mathbf{P}(A) \geqslant 0$,

NAP2. for every $A \subseteq \Omega$, $\mathbf{P}(A) = 1 \iff A = \Omega$,

NAP3. for every disjoint $A, B \subseteq \Omega$, $\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B)$,

NAP4a. for every $X \in \mathcal{P}_{\mathsf{fin}}(\Omega) \setminus \{\emptyset\}$, $\mathbf{P}(A \mid X) \in \mathbb{R}$,

NAP4b. there exists an algebra homomorphism

$$j : \mathcal{F}(\mathcal{P}_{\mathsf{fin}}(\Omega), \mathbb{R}) \to \mathcal{R}$$

such that $\mathbf{P}(A) = j(\mathbf{P}(A \mid \cdot\,))$ for any $A \subseteq \Omega$.

In the context of classical probability, the axiom K_∞ is equivalent to the **Conditional Probability Principle**, that is, the statement

CPP If $\mathcal{A}_n$ is an increasing family of events such that $\bigcup_{n\in\mathbb{N}} \mathcal{A}_n = \Omega$, then there exists an $N > 0$ such that for every $n \geqslant N$, $\mathbf{P}(\mathcal{A}_n) > 0$ and for every event A, $\mathbf{P}(A) = \lim_{n\to+\infty} \mathbf{P}(A \mid \mathcal{A}_n)$.

The axioms NAP4a and NAP4b are a direct reformulation of the Conditional Probability Principle in a non-Archimedean setting.

3.2.1 | FROM Λ-LIMITS TO NAP SPACES

fix a $\mathcal{P}_{\mathsf{fin}}(\Omega)$–completion $\mathbb{R}_\Lambda$ of the reals built over a fine non-principal ultrafilter $\mathcal{U}$. As shown in the previous chapter, we could use $^*\mathbb{R}$ to denote the hyperreal field inside $\mathbb{R}_\Lambda$. $^*\mathbb{R}$ will be the superreal field required in our axioms. As of NAP4b, we need an algebra homomorphism into $^*\mathbb{R}$ that assigns an hyperreal number to every function $f : \mathcal{P}_{\mathsf{fin}}(\Omega) \to \mathbb{R}$ — a natural possibility is the following:

$$j(f) = \lim_{\lambda\uparrow\Lambda} (\lambda, f(\lambda)).$$

To provide a model of NAP space, though, it isn't enough to provide this homomorphism; we will need to define, at least on finite subsets of Ω, a probability function. To do so, we start by deciding the desired probability of singletons:

$$\forall\omega \in \Omega,\ \mathbf{P}(\{\omega\}) := p(\omega) \in \mathbb{K}.$$

We now fix an arbitrary point $\omega_0 \in \Omega$ and define a **weight function**

$$\forall\omega \in \Omega,\ w(\omega) = \frac{p(\omega)}{p(\omega_0)}.$$

We are now ready to define the conditional probability over finite events,

$$\mathbf{P}(A \mid \lambda) = \frac{\sum\limits_{\omega\in A\cap\lambda} w(\omega)}{\sum\limits_{\omega\in\lambda} w(\omega)},$$

and then saying that

$$\mathbf{P}(A) = \lim_{\lambda\uparrow\Lambda} (\lambda, \mathbf{P}(A \mid \lambda)). \tag{3.1}$$

The latter equation could be rewritten by first agreeing on a definition of **infinite sum** through Λ–limits. If we take a function $u : A \to \mathbb{R}$, and set that

$$\sigma_u(\lambda) = \sum_{x\in A\cap\lambda} u(x),$$

then we can define the infinite sum as follows:

$$\sum_{x\in A} u(x) = \lim_{\lambda\uparrow\Lambda} (\lambda, \sigma_u(\lambda)).$$

First, we observe that Λ–limits commute with algebraic operations, division included. So the definition of probability, as defined in equation (3.1), can be rewritten as

$$\mathbf{P}(A) = \frac{\sum\limits_{x\in A} w(x)}{\sum\limits_{x\in\Omega} w(x)}. \tag{3.2}$$

LEMMA 3.2.1 *The following equality holds:*

$$p(\omega_0) = \sum_{\omega \in \Omega} w(\omega).$$

PROOF. First, observe that $w(\omega_0) = 1$. Then, by definition

$$p(\omega_0) = \mathbf{P}(\{\omega_0\}) = \lim_{\lambda \uparrow \Lambda} (\lambda, \mathbf{P}(\{\omega_0\} \mid \lambda)).$$

We have that

$$\mathbf{P}(\{\omega_0\}|\lambda) = \frac{w(\omega_0)\chi_\lambda(\omega_0))}{\sum\limits_{\omega \in \lambda} w(\omega)},$$

where χ_λ is the characteristic function of λ. By applying the Λ–limits,

$$p(\omega_0) = \frac{\lim\limits_{\lambda \uparrow \Lambda} (\lambda, \chi_\lambda(\omega_0)}{\sum\limits_{\omega \in \Omega} w(\omega)}. \tag{3.3}$$

We now show that $\lim\limits_{\lambda \uparrow \Lambda} (\lambda, \chi_\lambda(\omega_0)$ has only two possible values: 0 or 1. If we observe that, for any $\omega \in \Omega$, the two following equations hold

$$\begin{aligned} \chi_\lambda(\omega)[1 - \chi_\lambda(\omega)] &= 0, \\ \chi_\lambda(\omega) + [1 - \chi_\lambda(\omega)] &= 1, \end{aligned}$$

so by applying Λ–limits to both equations,

$$\begin{aligned} \lim_{\lambda \uparrow \Lambda} (\lambda, \chi_\lambda(\omega))[1 - \lim_{\lambda \uparrow \Lambda} (\lambda, \chi_\lambda(\omega))] &= 0, \\ \lim_{\lambda \uparrow \Lambda} (\lambda, \chi_\lambda(\omega)) + [1 - \lim_{\lambda \uparrow \Lambda} (\lambda, \chi_\lambda(\omega))] &= 1, \end{aligned}$$

so the only possible values for $\lim\limits_{\lambda \uparrow \Lambda} (\lambda, \chi_\lambda(\omega)$ are 0 or 1.

Furthermore, we have that $\mathbf{P}(B) = 0 \iff B = \emptyset$ (essentially as a consequence of NAP2 and NAP3), so $p(\omega_0) > 0$ and thus $\lim\limits_{\lambda \uparrow \Lambda} (\lambda, \chi_\lambda(\omega_0)) > 0$ — this means that $\lim\limits_{\lambda \uparrow \Lambda} (\lambda, \chi_\lambda(\omega_0)) = 1$. Back to (3.3),

$$p(\omega_0) = \frac{1}{\sum\limits_{\omega \in \Omega} w(\omega)},$$

which is the thesis. □

This allows to further simplify (3.2),

$$\mathbf{P}(A) = \frac{1}{p(\omega_0)} \sum_{\omega \in A} w(\omega). \tag{3.4}$$

Since, as we have shown, what is really necessary in building a NAP space through Λ–limits is assigning a weight function, $w : \Omega \to \mathbb{R}^+$. We will say that this weight function **generates** the NAP space.

3.2.2 | A FAIR LOTTERY ON THE NATURALS: A POSITIVE RESULT

As shown in (2.2.1), building an $\mathcal{J}$-completion from an ultrafilter is perfectly equivalent to building one from a maximal ideal of the algebra of functions. In this example, we will do precisely so.

First, we fix $\Omega = \mathbb{N}$ and let

$$\lambda_{[n]} = \{1, 2, ...n\} \subseteq \mathbb{N},$$

and denote by $\Lambda = \{\lambda_{[n]} : n \in \mathbb{N}\} \subseteq \mathcal{P}_{\mathsf{fin}}(\mathbb{N})$. If we let

$$I_\Lambda = \{f : \mathcal{P}_{\mathsf{fin}}(\mathbb{N}) \to \mathbb{R} \mid \forall x \in \Lambda,\ f(x) = 0\} \subseteq \mathcal{F}(\mathcal{P}_{\mathsf{fin}}(\mathbb{N}), \mathbb{R}),$$

then I_Λ is a proper ideal and thus, by the Prime Ideal Theorem, can be extended to a maximal ideal $\widetilde{I_\Lambda}$. Let $\mathcal{U}$ denote the ultrafilter over $\mathcal{P}_{\mathsf{fin}}(\mathbb{N})$ obtained from $\widetilde{I_\Lambda}$, and let $\mathbb{R}_\Lambda$ be the $\mathcal{P}_{\mathsf{fin}}(\mathbb{N})$-completion of the reals obtained in the manner described in the proof of (2.2.2). In this case, for every $f : \mathcal{P}_{\mathsf{fin}}(\mathbb{N}) \to \mathbb{R}$ we set

$$j(f) = \lim_{\lambda \uparrow \Lambda} (\lambda,\ f(\lambda)) = [f].$$

Let $\mathfrak{c}(\lambda) = |\lambda|$, and let $\alpha = \lim_{\lambda \uparrow \Lambda} (\lambda,\ \mathfrak{c}(\lambda))$.

LEMMA **3.2.2** *α is an infinite number.*

PROOF. For any natural number $N > 0$,

$$\{\lambda \in \mathcal{P}_{\mathsf{fin}}(\mathbb{N}) : |\lambda| > N\} \supseteq \Lambda \cap (\mathcal{P}_{\mathsf{fin}}(\mathbb{N}) \setminus \{\lambda \in \mathcal{P}_{\mathsf{fin}}(\mathbb{N}) : \max \lambda \leqslant N\}),$$

where the former set belongs to $\mathcal{U}$ since $\{\lambda \in \mathcal{P}_{\mathsf{fin}}(\mathbb{N}) : \max \lambda \leqslant N\}$ is finite and $\mathcal{U}$ is non-principal. This implies that $\{\lambda \in \mathcal{P}_{\mathsf{fin}}(\mathbb{N}) : |\lambda| > N\} \in \mathcal{U}$, and thus that $\alpha > N$ for every $N \in \mathbb{N}$. □

Denote by $\varepsilon = \alpha^{-1}$, which is then an infinitesimal number. If we let $\mathbf{P}(\{n\}) = \varepsilon$ for every $n \in \mathbb{N}$, then we both satisfy the intuition that the probability of the winning ticket must be positive, yet infinitesimal, and also

$$\mathbf{P}(\mathbb{N}) = \frac{1}{\alpha} \sum_{m \in \mathbb{N}} w(m) = \frac{1}{\alpha} \lim_{\lambda \uparrow \Lambda} (\lambda,\ \sum_{m \in \lambda} w(m)) = \frac{1}{\alpha} \lim_{\lambda \uparrow \Lambda} (\lambda,\ \mathfrak{c}(\lambda)) = 1,$$

since $w \equiv 1$.