

DEFINABILITY OF HENSELIAN VALUATIONS IN POSITIVE (RESIDUE) CHARACTERISTIC

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THE ROAD AHEAD

$\underbrace{\text{model theory}}_{\text{MATHEMATICAL LOGIC}}$ of $\underbrace{\text{valued fields}}_{\text{ALGEBRA}}.$

Today, we will try to:

- ▶ Tell you what valued fields are.
- ▶ Give you an idea of what our results look like.
- ▶ Tell you about an obstacle in this area and how we turned it into a tool.

VALUATIONS AND WHERE TO FIND THEM

Definition

A valuation on a field K is a surjective map $v: K^\times \rightarrow \Gamma$, where $(\Gamma, +, \leq, 0)$ is an ordered abelian group, such that:

- ▶ $v(xy) = v(x) + v(y)$, *multiplying two elements sums their valuations*
- ▶ $v(x + y) \geq \min\{v(x), v(y)\}$. *all triangles are isosceles*

(Counter)intuition: an element $r \in K^\times$ is *large* if its valuation $v(r) \in \Gamma$ is *small*, i.e. close to 0. Along this intuition, we usually set $v(0) := \infty$.

The ordered abelian group Γ is called the *value group*. We also denote it by vK .

OUR FAVOURITE EXAMPLE

Fix a prime number p .

- If $a \in \mathbb{Z} \setminus \{0\}$, then

$$v_p(a) := \max\{n \in \mathbb{N} : p^n \text{ divides } a\}.$$

For example, $v_3(6560) = 0$. According to v_3 , then, 6560 is “big”. But $v_3(6561) = 8$, which is then “smaller” than 6560. If $a, b \in \mathbb{Z} \setminus \{0\}$ are coprime, then

$$v_p\left(\frac{a}{b}\right) := v_p(a) - v_p(b).$$

- This defines a valuation $v_p: \mathbb{Q} \setminus \{0\} \rightarrow \mathbb{Z}$, called *the p -adic valuation*. With it, we can define a distance on \mathbb{Q} by setting $d_p(a, b) := p^{-v_p(a-b)}$.
- If we *complete* the corresponding metric space, we obtain a (new) valued field called \mathbb{Q}_p , with its own valuation v_p . These are the *p -adic numbers*.

WHY YOU SHOULD LIKE THE p -ADICS

- ▶ (\mathbb{Q}_p, v_p) is crucial for algebraic purposes. But we are logicians (allegedly)!
- ▶ A valuation is “the same” as its valuation ring, i.e. the subring

$$\mathcal{O}_v = \{x \in K \mid v(x) \geq 0\}.$$

This is the part where we should tell you that ∞ is larger than all elements of Γ , and thus $0 \in \mathcal{O}_v$.

- ▶ In the case of \mathbb{Q}_p , this subring is called \mathbb{Z}_p (guess why!). Julia Robinson pointed out something remarkable about \mathbb{Z}_p (for $p \neq 2$):

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p \mid \exists Y (Y^2 = 1 + px^2)\}.$$

There is a similar formula for $p = 2$.

- ▶ \mathbb{Z}_p is given, as a subset of \mathbb{Q}_p , by a polynomial equation together with some quantifiers. We say that it is a *definable* set in the language of rings.

LOGICIANS, ASSEMBLE! CONT'D

Big question: Is this common? When is some valuation ring definable in the language of rings?

THE PROBLEM OF HENSELIANITY

Not all valuations are created equal.

- ▶ Take a field K with a valuation v . We give you an algebraic extension L of K , e.g. $L = K(\alpha)$ where α is the root of some polynomial over K . Can you extend v to L ?
Yes, but often in several different ways.
- ▶ v is *henselian* if there is a **unique** way to extend v to any algebraic extension of K . A henselian valuation is a bit like a *fill the gaps* exercise in a textbook.
- ▶ v_p is henselian. *We will only care about henselian valuations.*

THE BIG QUESTION, TAKE 2

Big question: when is
a **henselian** valuation ring definable in the language of rings?

TWO FIELDS IN DISGUISE

- To any valued field (K, v) we can associate another “smaller” field, called the *residue field*,

$$Kv := \{x \in K : v(x) \geq 0\} / \{x \in K : v(x) > 0\}.$$

Indeed, $\mathfrak{m}_v := \{x \in K : v(x) > 0\}$ is the unique maximal ideal of $\mathcal{O}_v = \{x \in K : v(x) \geq 0\}$.

- **Example:** (\mathbb{Q}, v_p) and (\mathbb{Q}_p, v_p) both have residue field \mathbb{F}_p , the finite field with p elements
In fact, $\mathbb{Q} \subseteq \mathbb{Q}_p$ is an *immediate extension*: They have the same value groups and residue fields.
- So a valued field consists of *two fields*: the “big” valued field and the “smaller” residue field. If we talk about the characteristic of a valued field, we talk about the characteristics of the two fields
 - **equicharacteristic zero:** $\text{char}(K) = \text{char}(Kv) = 0$
 - **mixed characteristic:** $\text{char}(K) = 0 < p = \text{char}(Kv)$, where p is prime
 - **positive characteristic:** $\text{char}(K) = \text{char}(Kv) = p$, where p is prime

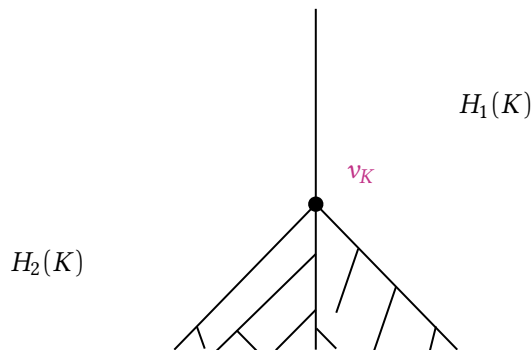
A CANONICAL FRIEND

- Henselian valuations on a given field K arrange themselves nicely according to whether their residue field is separably closed or not,

$$H_1(K) := \{v: Kv \text{ is not separably closed}\} \text{ vs. } H_2(K) := \{v: Kv \text{ is separably closed}\}.$$

- $H_1(K)$ is linearly ordered by inclusion.

The “middle point” between $H_1(K)$ and $H_2(K)$ is the *canonical henselian valuation* v_K .



THE GIST OF IT

$$\underbrace{\exists \text{ definable (non-trivial) henselian valuation}}_{\text{Logic question}} \iff \underbrace{\text{Conditions on the canonical henselian valuation}}_{\text{(Almost) algebra answer}}$$

WHAT WE PROVED

Theorem (Jahnke, Koenigsmann, 2017; Ketelsen, R., Szewczyk, 2023)

Let K be a non-separably closed henselian field.

If $\text{char}(K) = p > 0$, then assume that K is perfect.

If $\text{char}(K) = 0 < p = \text{char}(Kv_K)$, then assume that \mathcal{O}_{v_K}/p is semi-perfect.

Then,

$$K \text{ admits a definable non-trivial henselian valuation} \iff \begin{cases} Kv_K = Kv_K^{\text{sep}}, & \text{or} \\ Kv_K \text{ is not } t\text{-henselian}, & \text{or} \\ \exists L \succeq Kv_K \text{ with } v_L L \text{ divisible}, & \text{or} \\ v_K K \text{ is not divisible}, & \text{or} \\ (K, v_K) \text{ is not defectless}, & \text{or} \\ \exists L \succeq Kv_K \text{ with } (L, v_L) \text{ not defectless.} \end{cases}$$

WHAT WE HAD BEFORE

Theorem (Jahnke, Koenigsmann, 2017; Ketelsen, R., Szewczyk, 2023)

Let K be a non-separably closed henselian field, $\text{char}(Kv) = 0$.

If $\text{char}(K) = p > 0$, then assume that K is perfect.

If, further, $\text{char}(K) = 0 < p = \text{char}(Kv_K)$, then further assume that \mathcal{O}_{v_K}/p is semi-perfect.

Then,

$$K \text{ admits a definable non-trivial henselian valuation} \iff \begin{cases} Kv_K = Kv_K^{\text{sep}}, & \text{or} \\ Kv_K \text{ is not } t\text{-henselian}, & \text{or} \\ \exists L \succeq Kv_K \text{ with } v_L L \text{ divisible}, & \text{or} \\ v_K K \text{ is not divisible}, & \text{or} \\ (K, v_K) \text{ is not defectless}, & \text{or} \\ \exists L \succeq Kv_K \text{ with } (L, v_L) \text{ not defectless.} \end{cases}$$

WE DON'T TALK ABOUT DEFECT

Actually, we do now.

- ▶ Given a henselian valuation v and a finite field extension $K \subseteq L$, then there is a unique extension of v to L , which we denote by v again. Then, we have

$$[L : K] \geq [Lv : Kv](vL : vK).$$

More precisely,

$$[L : K] = p^d [Lv : Kv](vL : vK),$$

where $p = \text{char}(Kv)$, if the latter is positive, and $p = 1$ if $\text{char}(Kv) = 0$.

- ▶ We say that $(K, v) \subseteq (L, v)$ is *defectless* if

$$[L : K] = [Lv : Kv](vL : vK).$$

In particular, then, if $\text{char}(Kv) = 0$, then $p = 1$ and so equality holds. Otherwise, not being defectless (= *having defect*) is a problem.

- ▶ For us, however, defect is a **source of information!** (At least when it is “of independent type”).