étale methods in the model theory of fields

- Simone Ramello -

Source: "étale-open topology and the stable fields conjecture"
Johnson, Tran, Walsberg, Ye - 2009.02319

Goal: stable large fields are separably closed.
- L étale maps
- L large fields
- L stability theory

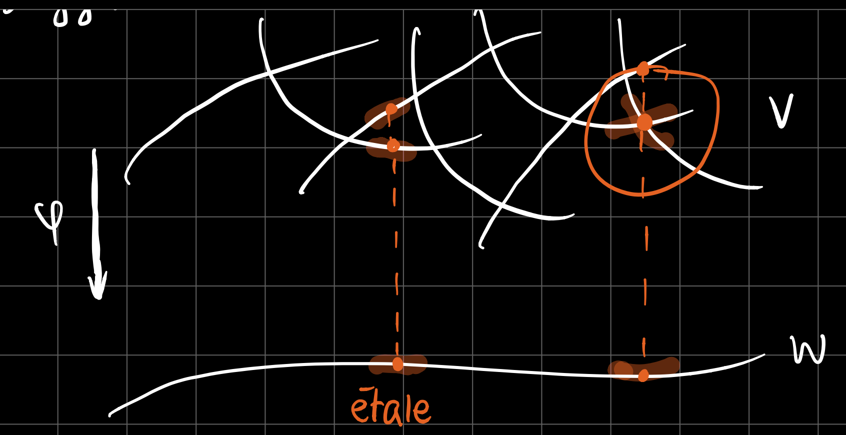
I. ÉTALE MAPS

MOTIVATION: $\varphi : M \to N$ C^∞ map of smooth mflds
$\varphi_{*,p} : T_pM \to T_{\varphi(p)}N$

Implicit function theorem:
$\varphi_{*,p}$ linear iso $\Rightarrow$ locally around p φ is a diffeo

Def. a map $\varphi : V \to W$ k-varieties is étale at $p \in V$ if $\varphi_{*,p}$ is a linear iso.

Suggestion:



An example:

$k \models ACF$, $\mathrm{char}(k) \neq p$

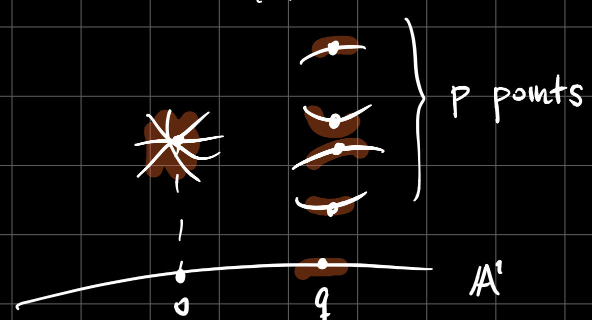
$$\varphi : \begin{cases} \underset{k}{\mathbb{A}^1} \to \underset{k}{\mathbb{A}^1} \\ x \mapsto x^p \end{cases}$$

$q \in \mathbb{A}^1$, $\varphi_{*,q} = p q^{p-1} dx$ linear iso

$\iff$

$q \neq 0$

$\rightsquigarrow \varphi|_{\mathbb{A}^1 \setminus \{0\}}$ étale



Def. k-variety = separated k-scheme of finite type

$\varphi : V \to W$ étale if it is smooth of rel. dim. zero

Def: V k-variety

an étale image in V : $U \subseteq V(k)$ of the form
$$U = \varphi(X(k))$$
for $\varphi : X \to V$ étale

Fact: finite unions & finite inters. of étale images are étale image.

$\rightsquigarrow$ étale images are basis for a topology on $V(k)$ $\mathcal{E}_V$: étale-open topology on V

$\rightsquigarrow$ functor: $\mathcal{E} : \underline{Var}_k \to \underline{Top}$ (system of topologies)

(*) Theorem: k is not separably closed iff $\mathcal{E}_V$ is Hausdorff on every q.p. V

($k \models SCF$, $\mathcal{E}$ is Zariski topology)

Facts: étale maps are finite-to-one, local coordinates are étale.

II. LARGE FIELDS

1996, Pop

Def: a field k is large if for every smooth k-curve C s.t. $C(k) \neq \emptyset$, then $|C(k)| \geq \omega$.

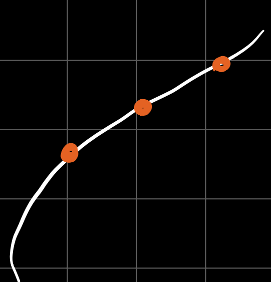
Theorem: TFAE:

1. k is large,
2. $k \preceq_1 k((t))$,
3. $(\mathbb{A}^1(k), \mathcal{E}_{\mathbb{A}^1})$ is not discrete.

PROOF SKETCH $(1 \Leftrightarrow 3)$:

k is not large $\rightsquigarrow$ C smooth k-curve

$1 \leq |C(k)| < \omega$



$p \in C(k)$, $\pi \in \mathcal{O}_{C,p}$ local coord.

$\rightsquigarrow$ $\pi : U \to \mathbb{A}^1$ étale

$\ni$ p

$\rightsquigarrow$ $\pi(U(k)) \subseteq \mathbb{A}^1(k)$ finite étale image

* $\mathcal{E}_{\mathbb{A}^1}$ is T_1
* affine grp acts transitively

$\rightsquigarrow$ $\mathcal{E}_{\mathbb{A}^1}$ discrete.

$\mathcal{E}_{\mathbb{A}^1}$ discrete $\Rightarrow$ $\{0\}$ étale image

$\rightsquigarrow$ $\varphi : X \longrightarrow \mathbb{A}^1$ étale map,

$\varphi(X(k)) = \{0\}$

$\rightsquigarrow$ X smooth k-curve
$X(k) \neq \emptyset$ but finite.

$\rightsquigarrow$ k not large. $\blacksquare$

Examples: ACF, PAC, RCF, (k,v) henselian valued fields

Non-examples: $\mathbb{Q}$, number fields, function fields

Open: $\mathbb{Q}^{ab}$?

III. STABILITY THEORY (Chernikov)

- $(G,\cdot)$ grp, X definable
  X generic if $\exists a_1, \dots a_n \in G$,
  $G = a_1 X \cup \dots \cup a_n X$

- k field, X definable
  additively generic: generic for $(k,+)$
  multiplicatively generic: —— —— $(k^\times, \cdot)$

a type is a additive generic / multiplicative generic if it only contains add./mult. generic formulae.

Theorem: $\mathbb{k}$ stable
There is a unique add. generic p_+,
mult. generic $p_\times$,
$p_+ = p_\times$
and a definable X is generic iff $p_+ = p_\times$ concentrates on X.

IV. STABLE FIELDS CONJECTURE

Stable infinite fields are separably closed.

Scanlon: $\mathbb{C}(t)$ stable?

Theorem: stable large fields are separably closed.
PROOF:

$\mathbb{k}$ stable, large, not sep. closed

$\mathcal{E}_{/A}^n$ Hausdorff

U, V disjoint étale images
definable

$\Downarrow$

U or V is not generic

$k^\times \setminus U$ generic

$a_1, \dots a_n \in k^\times$,

$$k^\times = a_1(k^\times \setminus U) \cup \dots \cup a_n(k^\times \setminus U)$$

$$\{0\} = a_1 U \cap \dots \cap a_n U$$

affine transformations are étale

$\rightsquigarrow$ $\{0\}$ étale image

$\rightsquigarrow$ $\mathcal{E}_{\mathbb{A}^1}$ discrete $\Rightarrow$ k not large. $\boxed{}$

V. GENERALIZATION

finite ext. large

Theorem: virtually large fields with stable existential formulae are separably closed.

PROOF STRATEGY:

$k \subseteq L$ large of min. deg.
not separably closed

$L \subseteq L'$ finite

1) $L' \to L'$ Artin-Schreier
2) $(L')^\times \to (L')^\times$ p-th power
not surjective

P image
étale image
ex. definable

$\rightsquigarrow$ φ defines cosets of P
Q coset of P, $Q \cap P = \emptyset$ defined by φ

if φ stable:
LOCAL STABILITY THEORY: P or Q not φ-generic
(Conant, Pillay, Terry)

$P' = P - c$, $c \in P$
$\quad \psi$
$\quad 0$ not φ-generic

$\{0\}$ étale image

L not large, $\perp$.

Beyond: pseudofinite fields (char $\neq 2$)

éz fields (Walsburg, Ye)

definable sets are finite unions of definable étale-open subsets of Zariski closed subsets

k psf, char $k \neq 2$,

$\mathcal{E}_V$ not a field topology