understanding valued fields via model theory

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I. WHY VALUATIONS

Work in $\mathcal{L}_{ring} := \{+, \cdot, -, 0, 1\}$.

$\mathbb{Q}$ (in $\mathcal{L}_{ring}$) is hard.

* Defines $\mathbb{Z} \subseteq \mathbb{Q}$ (J. Robinson)
* Has an undecidable theory
* Figuring out if polynomials have roots is complicated!

On the other hand,

$\mathbb{R}$ (in $\mathcal{L}_{ring}$) is ~~easy~~ not so bad!

completion of $\mathbb{Q}$ along an absolute value

* The order is definable : $x \geq y \iff \exists z \left(x - y = z^2 \right)$

  $\Rightarrow$ not a stable theory, but still NIP

* Definable sets are well-understood

$\underline{\underline{Q}}$:

how else can we "complete" $\mathbb{Q}$?

$\underbrace{\text{complete" } \mathbb{Q}}_{\text{local information on } \mathbb{Q}}$

Def. fix a prime p. If $a \in \mathbb{Z} \setminus \{0\}$,

$$v_p(a) := \max\{ n \in \mathbb{N} \mid p^n \mid a \},$$

$$v_p(0) := \infty$$

We extend v_p to $\mathbb{Q}$: if $a, b \in \mathbb{Z} \setminus \{0\}$ coprime,

$$v_p\left(\frac{a}{b}\right) := a - b.$$

$\Rightarrow$ if $a \in \mathbb{Q}$, $|a|_p := p^{-v_p(a)} \in \mathbb{R}$.

Theorem. (Ostrowski)

Up to equivalence, the only absolute values on $\mathbb{Q}$ are $|\cdot|$ and $|\cdot|_p$, for all p.

p-adic absolute value

Def. The completion of $\mathbb{Q}$ along $|\cdot|_p$ is denoted by $\mathbb{Q}_p$.

Inside $\mathbb{Q}_p$ we define $\mathbb{Z}_p := \{ a \in \mathbb{Q}_p : |a|_p \leq 1 \}$.

$\mathcal{L}_{ring} = \{+, \cdot, -, 0, 1\}$

$\mathbb{Q}_p$ is not so bad!

* Defines $\mathbb{Z}_p$ (J. Robinson)
* It is not stable, but NIP
* Definable sets can be understood

⚠ An absolute value is a map $|\cdot| : K \longrightarrow \mathbb{R}$

some field

and so, when I take elementary extensions (in some reasonable language) I get $|\cdot|^* : K^* \to \mathbb{R}^*$,

i.e. $|\cdot|^*$ will take infinite & infinitesimal values.

Def. Let K be a field and $(\Gamma, +, \leq)$ be an ordered abelian group. A valuation (on K with value group Γ) is a surjective map

$$v : \underset{= K \setminus \{0\}}{K^\times} \longrightarrow \underset{\mathbb{R},\, \mathbb{Z},\, \mathbb{Z} \otimes_{lex} \mathbb{Z},\, \dots}{\Gamma}$$

s.t.

- $v(ab) = v(a) + v(b)$,
- $v(a+b) \geq \min\{v(a), v(b)\}$.

Define $v(0) := \infty > \Gamma$.

Ex. v_p defined before on $\mathbb{Q}$ (or $\mathbb{Q}_p$) is a valuation with values in $\mathbb{Z}$.

$v_p(\underset{\mathbb{Z}}{\underset{\cap}{a}}) = \max\{n \in \mathbb{N} : p^n \mid a\}$

Given v, we let $\underset{v,}{\mathcal{O}_K} = \mathcal{O}_v = \{x \in K \mid v(x) \geq 0\}$ subring of K

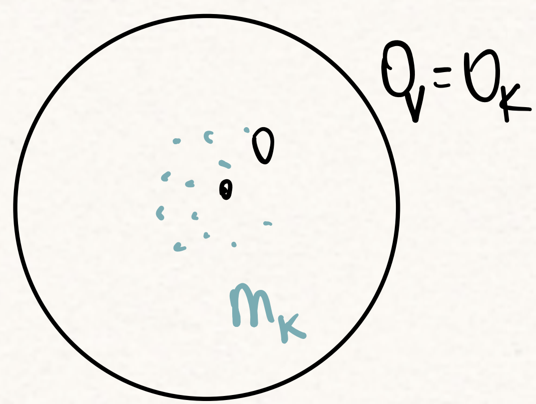
$\mathfrak{m}_K = \{x \in K \mid v(x) > 0\}$ max. ideal of $\mathcal{O}_K$

and call the quotient $k_K := \mathcal{O}_K / \mathfrak{m}_K$ - the residue field of v.

<u>Ex.</u> On $\mathbb{Q}_p$, $v_p(a) := -\log_p |a|_p$ is a valuation,

$\mathcal{O}_{v_p} = \mathbb{Z}_p$, $\mathfrak{m}_{v_p} = p\mathbb{Z}_p$,

the residue field is $\mathbb{F}_p$.



II. ENTER MODEL THEORY

Def. Consider the 3-sorted language $\mathcal{L}_{val}$ given by

$$\underset{\text{field sort}}{(K,+,\cdot,0,1,-)},\ \underset{\text{value gp sort}}{(\Gamma_K,0,+,\leq,\infty)},\ \underset{\text{residue field sort}}{(k,+,\cdot,0,1,-)}$$

v

ac

$$\mathrm{res}: \mathcal{O}_K \twoheadrightarrow \mathcal{O}_K/\mathfrak{m}_K = k_K$$

where ac is interpreted as a group hom.

$$ac: K^\times \longrightarrow k_K^\times$$

s.t. if u has valuation zero, then

$$ac(u) = \mathrm{res}(u).$$

Now, we can study a valued field (K,v) as an $\mathcal{L}_{val}$-structure and ask:

<u>How do we understand $Th(K,v)$?</u>

We saw before that $\mathbb{Q}$ is hard, but $(\mathbb{Q}_p, v_p)$ is not: the reason is that $\mathbb{Q}_p$ is complete, but this is not a first-order property.

<u>Def.</u> (K,v) is henselian if $\forall f \in \mathcal{O}_K[x]$, $a \in \mathcal{O}_K$,

$$\text{if } v(f(a)) > 0 \ \& \ v(f'(a)) = 0$$

$$\text{then } \exists b \in \mathcal{O}_K \text{ s.t. } f(b) = 0 \ \& \ v(b-a) > 0.$$

Theorem. (Pas) Let $\mathrm{Hen}_{0,0}$ be the $\mathcal{L}_{val}$-theory of valued fields (K,v) which are henselian and such that $\mathrm{char}(K) = \mathrm{char}(k_K) = 0$.

valued field   residue field

Then, every $\mathcal{L}_{val}$-formula is equivalent (modulo $\mathrm{Hen}_{0,0}$) to a formula where quantifiers only range over k_K, Γ_K.

$\parallel$

$\mathbb{Q}$

⇒ Upshot: all the (first-order) info about (K, v) is encoded in Γ_K and k.

Ax-Kochen/Ershov philosophy

This philosophy goes a long way, in weaker or stronger forms...

Corollary. Let $(K, v), (L, v) \models \mathrm{Hen}_{0,0}$. Then,

$$(K, v) \equiv (L, v) \iff k_K \equiv k_L \ \& \ \Gamma_K \equiv \Gamma_L .$$

complete! ⟶ $\mathrm{Hen}_{0,0} \cup \mathrm{Th}(k_K) \cup \mathrm{Th}(\Gamma_K)$

III. FOR SOMETHING COMPLETELY DIFFERENCE

Def. A valued difference field is the data of a valued field (K,v) together with a distinguished $\sigma \in \mathrm{End}(K,v)$.

Given $\sigma \in \mathrm{End}(K,v)$, one gets $\bar\sigma \in \mathrm{End}(k)$ and $\sigma_\Gamma \in \mathrm{End}(\Gamma_K)$.

Ex. $\mathrm{Aut}(\mathbb{Q}_p) \cong \mathrm{Aut}(\mathbb{F}_p) = 1$, so boring.

Better example:

$$\mathbb{C}((t)) := \left\{ \sum_{n \geq N} c_n t^n : (c_n)_{n \geq N} \subseteq \mathbb{C}, \; N \in \mathbb{Z} \right\}$$

$$\frac{1}{t} + 1 + t^2 + \dots \in \mathbb{C}((t))$$

$$\sigma\left(\sum_{n \geq N} c_n t^n \right) = \sum_{n \geq N} \bar{c}_n t^n \qquad v_t\left(\sum_{n \geq N} c_n t^n \right) = \min\{ n \in \mathbb{Z} : c_n \neq 0 \}$$

Let $\mathcal{L}^{\sigma}_{val}$ be the expansion of $\mathcal{L}_{val}$ given by

$$\begin{tikzcd}
(K,+,\cdot,-,0,1,\sigma) \arrow[r, bend right, "ac"'] \arrow[rr, bend right=40, "v"'] & (k,+,\cdot,-,0,1,\bar{\sigma}) & (\Gamma_K,+,\leq,0,\infty,\sigma_{\Gamma})
\end{tikzcd}$$

where now ac is meant to respect σ.

(Durhan-Onay)

difference

Theorem. Let $T\mathrm{Hen}_{0,0}$ be the $\mathcal{L}^{\sigma}_{val}$-theory of valued fields (K, v, σ) which are σ-henselian and such that char(K) = char(k) = 0.

σ is surjective

Then, every $\mathcal{L}^{\sigma}_{val}$-formula is equivalent to one where the quantifiers only range over Γ_K and k.

their respective automorphisms

$\sigma(K) \subseteq K$: the (λ_n^i)s parametrize linear indep. $/\sigma(K)$

Let $\mathcal{L}_{val}^{\sigma,\lambda}$ be the expansion of $\mathcal{L}_{val}^{\sigma}$ given by

$$\begin{tikzcd}
(K, +, \cdot, -, 0, 1, (\lambda_n^i)_{i,n\in\mathbb{N}}, \sigma) \arrow[r, bend right, "ac"'] \arrow[rr, bend right, "v"'] & (k, +, \cdot, -, 0, 1, \bar{\sigma}) & (\Gamma_K, +, \leq, 0, \infty, \sigma_\Gamma)
\end{tikzcd}$$

where now ac is meant to respect σ.

Theorem. (R.) Let $\mathrm{WTHen}_{0,0}$ be the $\mathcal{L}_{val}^{\sigma,\lambda}$-theory of valued difference fields (K,v,σ) which are weakly σ-hens. and such that $\sigma(K)\subseteq K$ is rel. alg. closed, $char(K)=char(k)=0$.

Then, every $\mathcal{L}_{val}^{\sigma,\lambda}$-formula is equivalent to one where the quantifiers only range over Γ_K and k.

with respective σ

$$(K,v,\sigma) \rightsquigarrow (k_K,\bar{\sigma}) \ \& \ (\Gamma_K,\sigma_\Gamma)$$

Thank you!