Tame valued fields reading group

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Introduction

Setting

(K, v) valued field of residue characteristic exponent $p \geq 1$, value group is vK and residue field if Kv.

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Inequalities

1. Fundamental inequality $[L : K] \leq \sum_{i \leq r} e_i f_i p^{d_i}$
2. Abhyankar inequality $\mathrm{trdeg}(L/K) \leq \mathrm{trdeg}(Lw/Kv) + \mathrm{rrk}(wL/vK)$.

Inequality in 2. is called 'transcendence defect'

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Background

AKE for

1. separably closed valued fields (Robinson, ...)
2. Henselian of equal characteristic 0 (Ax–Kochen/Ershov)
3. p-adically closed fields (Ax–Kochen/Ershov, Prestel–Roquette)
4. finitely ramified henselian valued fields (Ershov, Ziegler, van den Dries, A.–Jahnke, A.–Dittmann–Jahnke)
5. separable algerbaically maximal Kaplansky (Delon, Bélair)

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AKE principles

1. $\mathrm{AKE}^{\equiv}$: $(K, v) \equiv (L, w) \Leftrightarrow (vK \equiv wL) \& (Kv \equiv Lw)$
2. $\mathrm{AKE}^{\exists}$: $(K, v) \preceq_{\exists} (L, w) \Leftrightarrow (vK \preceq_{\exists} wL) \ \& \ (Kv \preceq_{\exists} Lw)$ for $(K, v) \subseteq (L, w)$
3. $\mathrm{AKE}^{\preceq}$: $(K, v) \preceq (L, w) \Leftrightarrow (vK \preceq wL) \ \& \ (Kv \preceq Lw)$ for $(K, v) \subseteq (L, w)$

Tame valued fields

Definition (Tame valued fields)

(K, v) is *tame* if

1. (K, v) is algebraically maximal,
2. vK is p-divisible, and
3. Kv is perfect

In positive characteristic, (K, v) is tame if and only if it is henselian, defectless, and perfect.

Examples and non-examples

1. $(F((\Delta)), v_t)$ iff F perfect and Δ p-divisible
2. $(F((t)), v_t)$ iff $p = 1$
3. $(F(t)^h, v_t)$ iff $p = 1$
4. $(F(t), v_t)$ not tame
5. $(\mathbb{Q}, v_p)$ not tame.

Tame valued fields

Definition (Tame extensions)

Algebraic extension $(L, w)/(K, v)$ is *tame* if

1. $(wL : vK)$ coprime to p,
2. Lw/Kv is separable, and
3. defectless.

Definition (Purely wild)

$(L, w)/(K, v)$ is *purely wild* if it is linearly disjoint from every tame extension of (K, v).

Proposition (Thm 3.2)

TFAE

1. (K, v) is tame,
2. K^r is algebraically closed,
3. no proper purely wild extensions.

Main theorems

Theorem (Theorem 1.4)

Class of tame fields satisfies $\mathrm{AKE}^{\exists}$ *and* $\mathrm{AKE}^{\preceq}$*. Class of tame fields of equal characteristic satisfies* $\mathrm{AKE}^{\equiv}$*. Class of tame fields satisfies a certain relative version of* $\mathrm{AKE}^{\equiv}$*.*

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Theorem (Theorem 1.4)

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Relative subcompleteness

Let $(L, w), (K, v)$ be two tame extensions of defectless (F, u) with vK/uF torsion free and Kv/Fu separable. Then $(K, v) \equiv_{(F,u)} (L, w) \Leftrightarrow (vK \equiv_{uF} wL) \& (Kv \equiv_{uF} Lw)$

Tamification, KPR

There exists an algebraic extension $(K^t, v^t)/(K, v)$ such that

1. (K^t, v^t) is tame,
2. $v^t K^t$ is the p-divisible hull of vK, and
3. $K^t v^t$ is the perfect hull of Kv.

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Pop's Lemma

Let (K, v) be tame, and suppose that (F, u) is a relatively algebraically closed valued subfield with Kv/Fu algebraic. Then (F, u) is tame, uF is pure in vK , and $Fu = Kv$.