Failure of Kaplansky theory for non-surjective isometries

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Contents

Abstract

The goal of this note is to construct an equicharacteristic zero valued field (K, v) endowed with a non-surjective isometry σ that admits two immediate σ-algebraic extensions $K\langle\alpha\rangle_{\mathbb{N}}$ and $K\langle\beta\rangle_{\mathbb{N}}$ such that $K\langle\alpha, \beta\rangle_{\mathbb{N}}$ is not immediate over K. As a consequence, (K, v, σ) admits non-isomorphic σ-algebraically maximal immediate extensions.

1 Preliminaries on valued difference fields

Given a valued field (K, v), we denote by k its residue field and by Γ_K its value group.

Definition 1.1. *A field endomorphism $\sigma\colon K \to K$ will be called an* isometry *if, for all $x \in K$, $v(\sigma(x)) = v(x)$. Any isometry induces a field endomorphism $\overline{\sigma}\colon k \to k$. We will call $(k, \overline{\sigma})$ the* residue difference field*.*

Let $(K, \sigma) \subseteq (L, \tau)$ be an extension of fields with an endomorphism (i.e., $K \subseteq L$ is a field extensions and $\tau|_K = \sigma$). For any $\alpha \in L$, we denote by

$$K\langle\alpha\rangle_{\mathbb{N}} = K(\tau^n(\alpha) \mid n \in \mathbb{N}) \subseteq L$$

and by

$$K\langle\alpha\rangle_{\mathbb{Z}} = K(\tau^n(\alpha) \mid n \in \mathbb{Z}).$$

Note that the latter might not be contained in L (depending on whether τ is surjective or not), but rather in its *inversive closure* $L^{\text{inv}} = \bigcup_{n\in\mathbb{N}} \tau^{-n}(L)$.

Definition 1.2. *Let $(K, \sigma) \subseteq (L, \tau)$ be an extension of fields with an endomorphism. An element $\alpha \in L \setminus K$ is said to be* σ-algebraic *if* $\operatorname{trdeg}_K(K\langle\alpha\rangle_{\mathbb{N}}/K) < \infty$*; it is said to be* σ-transcendental *otherwise.*

Definition 1.3. *A sequence* $(a_\rho)_{\rho<\lambda} \subseteq K$ *is said to be* pseudo-Cauchy *if there is* $\rho_0 < \lambda$ *such that, for all* $\rho_0 \leq \rho_1 < \rho_2 < \rho_3 < \lambda$,

$$v(a_{\rho_3} - a_{\rho_2}) > v(a_{\rho_2} - a_{\rho_1}).$$

In this case, we denote by $\gamma_\rho := v(a_{\rho'} - a_\rho)$ *for any* $\rho < \rho' < \lambda$, *which is independent of* ρ'. *An element* $a \in K$ *such that* $v(a - a_\rho) = \gamma_\rho$ *for all* ρ *is said to be a* pseudolimit *of* $(a_\rho)_{\rho<\lambda}$, *in symbols* $a_\rho \implies a$.

Given a polynomial $P(X_0, X_1, \dots X_n) \in K[X_0, X_1, \dots X_n]$, we can build a *σ-polynomial*

$$p(X) = P(X, \sigma(X), \dots \sigma^n(X)).$$

Definition 1.4. *Two pseudo-Cauchy sequences* $(a_\rho)_{\rho<\lambda}$ *and* $(b_\rho)_{\rho<\lambda}$ *are said to be* equivalent *if, for all extensions* L *of* K *and* $a \in L$, $a_\rho \implies a$ *if and only if* $b_\rho \implies a$.

Definition 1.5. *A pseudo-Cauchy sequence* $(a_\rho)_{\rho<\lambda}$ *is said to be* of algebraic type *(respectively, of* σ-algebraic type*) if there is a polynomial* $Q(X)$ *(respectively, a σ-polynomial* $p(X)$*) such that* $(v(Q(b_\rho)))_{\rho<\lambda}$ *(respectively,* $(v(p(b_\rho)))_{\rho<\lambda}$*) is strictly increasing in* Γ_K *for some equivalent* $(b_\rho)_{\rho<\lambda}$. *It will be* of transcendental type *(respectively,* of σ-transcendental type*) otherwise.*

Note that algebraic type implies σ-algebraic type.

Definition 1.6. *An extension* $(K, v, \sigma) \subseteq (L, w, \tau)$ *will be* σ-algebraic *if all elements in* L *are σ-algebraic over* K *(equivalently, if they are roots of σ-polynomials over* K*), and σ-*transcendental *otherwise. It will be called* immediate *if* $\Gamma_K = \Gamma_L$ *and* $k = \ell$ *along the natural embeddings.*

By *valued difference field* we will always mean valued field endowed with a (not necessarily surjective) isometry. We will use *inversive* to mean that the field endomorphism in question is surjective. All valued fields will be in equicharacteristic zero.

Definition 1.7. *A valued difference field will be called* (σ-algebraically) maximal *if it admits no proper (σ-algebraic) immediate extension.*

Definition 1.8. *A valued difference field has* enough constants *if for every* $\gamma \in \Gamma_K$, *there is* $c \in \mathrm{Fix}(K) = \{x \in K \mid \sigma(x) = x\}$ *such that* $v(c) = \gamma$.

Definition 1.9 (cf. section 3 in [BMS07])**.** *A field with an endomorphism* $(k, \overline{\sigma})$ *satisfies (R4) if for all* $\Lambda \in k[X_0, X_1, \dots X_n] \setminus \{0\}$ *there is some* $a \in k$ *such that* $\Lambda(a, \overline{\sigma}(a), \dots \overline{\sigma}^n(a)) \neq 0$.

Useful facts

Lemma 1.10 (cf. 2.2.2. in [EP05])**.** *Suppose* (K, v, σ) *is a valued difference field. Let* α *in some extension be σ-transcendental over* K. *Then there is a unique valuation on* $L = K\langle\alpha\rangle_{\mathbb{N}}$, *extending* v, *where* $v(\alpha) = 0$ *and* $\operatorname{res}\alpha$ *is σ-transcendental over* $(k, \bar{\sigma})$.

Proof. First, existence. Start with elements of L of the form $p(\alpha) = \sum_I a_I \alpha^I$, where $p(X) \in K[X]_\sigma$. Let

$$v(p(\alpha)) := \min_I v(a_I).$$

Further, we let

$$v\left(\frac{p(\alpha)}{q(\alpha)}\right) = v(p(\alpha)) - v(q(\alpha)).$$

This gives an extension of v to L with value group $\Gamma_L = \Gamma_K$ and residue difference field $k\langle \operatorname{res}\alpha\rangle_{\mathbb{N}}$. Note that $\operatorname{res}\alpha$ is σ-transcendental over $(k, \bar{\sigma})$: if $\sum_I \operatorname{res} a_I \operatorname{res}\alpha^I = 0$, then

$$v\left(\sum_I a_i \alpha^I\right) = \min_I v(a_I) > 0,$$

hence $\operatorname{res} a_I = 0$ for all I.

For the uniqueness part, let w be another such. For any $p(\alpha) = \sum_I a_I \alpha^I$, let J be such that $v(a_J) = \min_I v(a_I)$. Thus $p(\alpha) = a_J \sum_I b_I \alpha^I$, with $v(b_I) = v(a_I/a_J) \geq 0$. In the new valuation w, then, $w(p(\alpha)/a_J) \geq 0$, as $w(\alpha) = 0$ by definition. On the other hand, $w(p(\alpha)/a_J) = 0$, since $\operatorname{res}_w \alpha$ is $\bar{\sigma}$-transcendental over k. Then it follows that

$$w(p(\alpha)) = w(a_J) + w(p(\alpha)/a_J) = w(a_J) = \min_I v(a_I) = v(p(\alpha)).$$

□

Lemma 1.11. *Suppose (K, v) is a valued field which admits a pseudo-Cauchy sequence $(a_\rho)_{\rho<\lambda}$ of transcendental type. Let δ be transcendental over K and, for some $\Gamma \geq \Gamma_K$, let $\gamma \in \Gamma$. Extend v to $L = K(\delta)$ with the Gauss valuation, so that $v(\delta) = \gamma$. Then $(a_\rho)_{\rho<\lambda} \subseteq L$ is still of transcendental type. In particular it has no pseudolimit in L.*

Proof. Suppose it is, i.e. there is $p(X) \in L[X]$ such that $p(a_\rho) \implies 0$. Without loss of generality, we can write

$$p(X) = \sum_{i=0}^{m} p_i(X)\delta^i,$$

for p_i over K. Note that, for $i = 0, \dots m$, $v(p_i(a_\rho))$ is eventually constant, say equal to μ_i. Then, since v is extended to L as a Gauss valuation,

$$\begin{aligned} v(p(a_\rho)) &= v\left(\sum_{i=0}^{m} p_i(a_\rho)\delta^i)\right) \\ &= \min_{i=0,\dots m} \mu_i + iv(\delta), \end{aligned}$$

and the latter is clearly not strictly increasing. □

The proof of the following lemma goes through in the non-surjective case.

Lemma 1.12 (cf. 5.6 in [BMS07]). *Suppose (K, v, σ) is a valued difference field with enough constants and whose residue difference field satisfies (R4). Let $(a_\rho)_{\rho<\lambda}$ be a pseudo-Cauchy sequence in K, with pseudolimit a in some extension of K. If $p(X)$ is a σ-polynomial over K, then there is an equivalent pseudo-Cauchy sequence $(b_\rho)_{\rho<\lambda}$ such that $p(b_\rho) \implies p(a)$.*

Lemma 1.13 (cf. 7.1 and 7.2 in [BMS07]). *Suppose (K, v, σ) is a valued difference field with enough constants and whose residue difference field satisfies (R4). Suppose $(a_\rho)_{\rho<\lambda} \subseteq K$ is a pseudo-Cauchy sequence with no pseudo-limit in K. Let $(K, v, \sigma) \subseteq (L, w, \tau)$ be an extension and let $\alpha \in L \setminus K$ be such that $a_\rho \implies \alpha$. If $(a_\rho)_{\rho<\lambda}$ is of σ-algebraic type, as witnessed by $p(X)$ over K of minimal complexity, assume further that $p(\alpha) = 0$. Then $K \subseteq K\langle\alpha\rangle_{\mathbb{N}}$ is an immediate extension.*

Proof. We distinguish two cases.

$(a_\rho)_{\rho<\lambda}$ is of σ-transcendental type. Then α must be σ-transcendental over K: otherwise, upon switching to an equivalent pseudo-Cauchy sequence $(b_\rho)_{\rho<\lambda}$, we would have $p(b_\rho) \implies p(\alpha) = 0$ for some $p(X)$ over K. Now, take $\beta \in K\langle\alpha\rangle_{\mathbb{N}}$. Assume first that $\beta = p(\alpha)$, for some σ-polynomial $p(X)$ over K. Since $(a_\rho)_{\rho<\lambda}$ is of σ-transcendental type, $(v(p(a_\rho)))_{\rho<\lambda}$ is eventually constant in Γ_K. Upon moving to an equivalent sequence $(b_\rho)_{\rho<\lambda}$, we may assume that $v(p(b_\rho)) = v(p(\alpha))$ eventually. Thus $v(p(\alpha)) \in \Gamma_K$. If now $\beta = p(\alpha)/q(\alpha)$, then we have $v(\beta) = v(p(\alpha)) - v(q(\alpha)) \in \Gamma_K$. Similarly, the residue field does not change.

$(a_\rho)_{\rho<\lambda}$ is of σ-algebraic type. Let $\beta \in K\langle\alpha\rangle_{\mathbb{N}}$. As before, we may assume that β is equal to $h(\alpha)$ for some $h(X)$ over K and, upon operating division between the corresponding multivariate polynomials, we can write

$$q(X)h(X) = f(X) + r(X)p(X)$$

for $f(X), q(X)$ of lower complexity than $p(X)$. Evaluating at α gives

$$q(\alpha)h(\alpha) = f(\alpha),$$

thus, it suffices to work with $f(\alpha)$, for $f(X)$ of lower complexity than $p(X)$. Since $v(f(a_\rho))$ is eventually constant, by minimality of $p(X)$, the same argument as in the σ-transcendental case applies. □

2 The construction

Let $(k, \overline{\sigma})$ be a difference field that satisfies (R4), $\overline{\sigma}$ non-surjective, k of characteristic zero. Suppose k is of the form $k = k_0\langle t\rangle_{\mathbb{N}}$, where t is σ-transcendental over k_0.

Let (K, v, σ) be a valued difference field over $(k, \overline{\sigma})$ with enough constants, (K, v) henselian, $\Gamma_K = \mathbb{Z} \oplus_{\text{lex}} \mathbb{Z}$ and admitting a subfield $F \subseteq \mathcal{O}_K$ such that $(F, \sigma|_F) \simeq (k, \overline{\sigma})$ via the residue map. For example, we may assume that we are working with $K = k((\mathbb{Z} \oplus_{\text{lex}} \mathbb{Z}))$ with the t-adic valuation and the lift of $\overline{\sigma}$ by acting on coefficients. Call F' the difference subfield of F isomorphic to k_0 and write $F = F'\langle\tau\rangle_{\mathbb{Z}}$. Assume further that $(F, \sigma|_F) \subseteq (K, \sigma)$ is not purely σ-algebraic.

Let $\varepsilon \in \text{Fix}(K)$ be such that $v(\varepsilon) = (0, 1)$, and consider $F_0 = F'(\varepsilon)$, with value group $\mathbb{Z}$ and residue difference field $(k_0, \bar{\sigma}|_{k_0})$. Let $\xi = \frac{1}{\varepsilon} \in F_0$, so that $v(\xi) < 0$, and let $\alpha = \xi\tau$. Then, since τ is σ-transcendental over F_0, so is α. Let $c = \sigma(\alpha) - \varepsilon\alpha$. Note that c is still σ-transcendental over F_0. Consider $F_1 = F_0\langle c\rangle^h_{\mathbb{N}}$, with value group $\mathbb{Z}$ and residue difference field $(k, \bar{\sigma})$ (note that $F_0\langle c\rangle_{\mathbb{N}}$ is the same extension as the one obtained by adjoining $\frac{c}{\xi}$, which has valuation zero and σ-transcendental residue, hence we can apply 1.10).

Note α is σ-algebraic over F_1, but not algebraic. Now $F_1 \subseteq F_1\langle\alpha\rangle_{\mathbb{N}}$ is immediate: indeed, $F_1\langle\alpha\rangle_{\mathbb{N}} = F_1(\alpha)$ and the extension $F_1 \subseteq F_1(\alpha)$ is the same as the extension $F_1 \subseteq F_1(\frac{\alpha}{\xi})$, which is a Gauss valuation.

Thus, there is a pseudo-Cauchy sequence $(a_\rho)_{\rho<\lambda} \subseteq F_1$ witnessing this. Note that the sequence is of σ-algebraic type over F_1, as witnessed by $h_\alpha(X) = \sigma(X) - \varepsilon X - c$: we have

$$v(h_\alpha(a_\rho)) = v(h_\alpha(\alpha) - h_\alpha(a_\rho)) = v(\sigma(\alpha - a_\rho) - \varepsilon(\alpha - a_\rho)) = v(\alpha - a_\rho) = \gamma_\rho,$$

for all $\rho < \lambda$. It is, however, not of algebraic type over F_1 (as the latter is algebraically maximal, since we are in equicharacteristic zero).

Now, pick $\delta \in \mathrm{Fix}(K)$ with $v(\delta) = (1,1)$, and consider $F_2 = F_1(\delta)$, with value group $\mathbb{Z} \oplus_{\mathrm{lex}} \mathbb{Z}$. Note that α is still σ-algebraic over F_2 and not algebraic, by 1.11.

Lemma 2.1. *(F_2, v, σ) has enough constants.*

Proof. Note that $\delta, \varepsilon \in \mathrm{Fix}(F_2)$, hence $\mu = \frac{\delta}{\varepsilon} \in \mathrm{Fix}(F_2)$ with value equal to the other generator $v(\mu) = (1,1) - (0,1) = (1,0)$. Then, any element $(n,m) \in \mathbb{Z} \oplus_{\mathrm{lex}} \mathbb{Z}$ can be realized as the value of $\mu^n \varepsilon^m$. □

Now, $h_\alpha(X) = \sigma(X) - \varepsilon X - c \in F_2[X]_\sigma$ is a minimal σ-polynomial for $(a_\rho)_{\rho<\lambda}$ over F_2, the latter has no pseudolimit in F_2, $a_\rho \implies \alpha$, and $h(\alpha) = 0$. Thus, $F_2 \subseteq F_2\langle\alpha\rangle_{\mathbb{N}}$ is an immediate extension by 1.13 (which we can apply since F_2 has enough constants).

Consider now $z \in F$ such that $\operatorname{res} z \notin \overline{\sigma}(k)$, and let $h_\beta(X) = \sigma(X) - \varepsilon X - c - \delta z$. We now build an extension of F_2 where $h_\beta(X)$ admits a root β.

Let $L \supseteq F_2$ be an inversive, σ-henselian valued difference field (for example, a σ-algebraically maximal extension of F_2^{inv}). Then $h_\beta(X)$ is in σ-Hensel configuration at 0: indeed,

$$\min\left\{v\left(\frac{\partial h_\beta}{\partial X_0}(0)\right), v\left(\frac{\partial h_\beta}{\partial X_1}(0)\right)\right\} = \min\{v(\varepsilon), v(1)\} = 0,$$

and higher order derivatives vanish, so

$$v(h_\beta(0)) = v(c) + 0 < |J|v(c) + \infty = \infty$$

for all multi-indices J. Thus, by σ-henselianity, we can find an element $\beta \in L$ with $h_\beta(\beta) = 0$. Now,

1. by additivity, $v(\beta - \alpha) = v(\delta) > \gamma_\rho$ for all $\rho < \lambda$, thus $a_\rho \implies \beta$,
2. $v(h_\beta(a_\rho)) = v(h_\alpha(a_\rho) - \delta z) = v(h_\alpha(a_\rho))$ eventually, so $h_\beta(a_\rho) \implies 0$,
3. $h_\beta(\beta) = 0$,

thus $F_2 \subseteq F_2\langle\beta\rangle_{\mathbb{N}}$ is an immediate extension by 1.13.

Corollary 2.2. *(F_2, v, σ) admits two (σ-algebraically) maximal immediate extensions which are not isomorphic over F_2.*

Proof. The two extensions $F_2\langle\alpha\rangle_+$ and $F_2\langle\beta\rangle_{\mathbb{N}}$ cannot amalgamate in the immediate category. If L contains $F_2\langle\alpha\rangle_{\mathbb{N}}$ and $F_2\langle\beta\rangle_{\mathbb{N}}$, then $F_2 \subseteq L$ is not immediate: indeed, if it were, then we would have

$$\operatorname{res} z = \overline{\sigma}\left(\operatorname{res}\left(\frac{\beta-\alpha}{\delta}\right)\right) \in \overline{\sigma}(k),$$

a contradiction. Thus any two (σ-algebraically) maximal immediate extensions E_1 and E_2 of F_2 such that $F_2\langle\alpha\rangle_{\mathbb{N}} \subseteq E_1$ and $F_2\langle\beta\rangle_{\mathbb{N}} \subseteq E_2$ will not be isomorphic over F_2. □

References

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[EP05] Antonio J. Engler and Alexander Prestel. *Valued Fields*. Springer Berlin, 2005.