Definability of henselian valuations in positive (residue) characteristic

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MM Mathematics Münster Cluster of Excellence

The road ahead

$$\underbrace{\text{model theory}}_{\text{mathematical logic}} \text{ of } \underbrace{\text{valued fields}}_{\text{algebra}}.$$

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Today, we will try to:

- Tell you what valued fields are.
- Give you an idea of what our results look like.
- Tell you about an obstacle in this area and how we turned it into a tool.

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The ordered abelian group Γ is called the *value group*. We also denote it by vK.

Our favourite example

Fix a prime number p.

- If $a \in \mathbb{Z} \setminus \{0\}$, then

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- This defines a valuation $v_p \colon \mathbb{Q} \setminus \{0\} \to \mathbb{Z}$, called *the p-adic valuation*. With it, we can define a distance on $\mathbb{Q}$ by setting $d_p(a, b) \coloneqq p^{-v_p(a-b)}$.

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- If we *complete* the corresponding metric space, we obtain a (new) valued field called $\mathbb{Q}_p$, with its own valuation v_p. These are the *p-adic numbers*.

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Logicians, assemble! cont'd

Big question: Is this common? When is some valuation ring definable in the language of rings?

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- v_p is henselian. *We will only care about henselian valuations.*

The big question, take 2

Big question: when is
a henselian valuation ring definable in the language of rings?

Two fields in disguise

- To any valued field (K, v) we can associate another "smaller" field, called the *residue field*,

$$Kv := {}^{\{x \in K \colon v(x) \geqslant 0\}} \big/ {}_{\{x \in K \colon v(x) > 0\}}.$$

Indeed, $\mathfrak{m}_v := \{x \in K \colon v(x) > 0\}$ is the unique maximal ideal of $\mathcal{O}_v = \{x \in K \mid v(x) \geqslant 0\}$.

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In fact, $\mathbb{Q} \subseteq \mathbb{Q}_p$ is an *immediate extension*: They have the same value groups and residue fields.
- So a valued field consists of *two fields*: the "big" valued field and the "smaller" residue field. If we talk about the characteristic of a valued field, we talk about the characteristics of the two fields
  - equicharacteristic zero: $\mathrm{char}(K) = \mathrm{char}(Kv) = 0$
  - mixed characteristic: $\mathrm{char}(K) = 0 < p = \mathrm{char}(Kv)$, where p is prime
  - positive characteristic: $\mathrm{char}(K) = \mathrm{char}(Kv) = p$, where p is prime

A canonical friend

- Henselian valuations on a given field K arrange themselves nicely according to whether their residue field is separably closed or not,

$$H_1(K) \coloneqq \{v\colon Kv \text{ is not separably closed}\} \text{ vs. } H_2(K) \coloneqq \{v\colon Kv \text{ is separably closed}\}.$$

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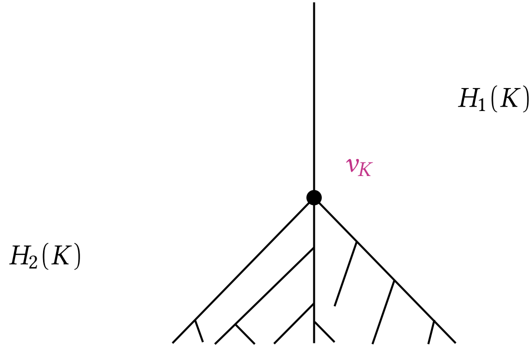
- $H_1(K)$ is linearly ordered by inclusion.
The "middle point" between $H_1(K)$ and $H_2(K)$ is the *canonical henselian valuation* v_K.



The gist of it

$$\underbrace{\exists\text{ definable (non-trivial) henselian valuation}}_{\text{Logic question}} \iff \underbrace{\text{Conditions on the canonical henselian valuation}}_{\text{(Almost) algebra answer}}$$

What we proved

Theorem (Jahnke, Koenigsmann, 2017; Ketelsen, R., Szewczyk, 2023)

Let K be a non-separably closed henselian field.
If $\mathrm{char}(K) = p > 0$*, then assume that* K *is perfect.*
If $\mathrm{char}(K) = 0 < p = \mathrm{char}(Kv_K)$*, then assume that* $\mathcal{O}_{v_K}/p$ *is semi-perfect.*
Then,

$$K \text{ admits a definable non-trivial henselian valuation} \iff \begin{cases} Kv_K = Kv_K^{\mathrm{sep}}, & \text{or} \\ Kv_K \text{ is not t-henselian}, & \text{or} \\ \exists L \succeq Kv_K \text{ with } v_L L \text{ divisible}, & \text{or} \\ v_K K \text{ is not divisible}, & \text{or} \\ (K, v_K) \text{ is not defectless}, & \text{or} \\ \exists L \succeq Kv_K \text{ with } (L, v_L) \text{ not defectless}. & \end{cases}$$

What we had before

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Let K be a non-separably closed henselian field, $\mathrm{char}(Kv) = 0$.
~~*If $\mathrm{char}(K) = p > 0$, then assume that K is perfect.*~~
~~*If, further, $\mathrm{char}(K) = 0 < p = \mathrm{char}(Kv_K)$, then further assume that $\mathcal{O}_{v_K}/p$ is semi-perfect.*~~
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  More precisely,

  $$[L : K] = p^d [Lv : Kv](vL : vK),$$

  where $p = \mathrm{char}(Kv)$, if the latter is positive, and $p = 1$ if $\mathrm{char}(Kv) = 0$.

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- For us, however, defect is a **source of information**! (At least when it is "of independent type").