WORKING GROUP ON *CONTRACTING ENDOMORPHISMS OF VALUED FIELDS*

This is all taken from [2].

1. **Notation.** We consider a field **of characteristic zero** K with a distinguished endomorphism $\sigma \in \mathrm{End}(K)$. We will write $\mathbb{N}[\sigma]$ for the semi-ring of formal finite sums $\lambda_n\sigma^n + \cdots + \lambda_1\sigma + \lambda_0$, where $\lambda_i \in \mathbb{N}$ for all i. If $I \in \mathbb{N}[\sigma]$ can be written as $I = \lambda_n\sigma^n + \cdots + \lambda_1\sigma + \lambda_0$, then by a^I we mean the element $\sigma^n(a)^{\lambda_n}\sigma^{n-1}(a)^{\lambda_{n-1}} \cdots \sigma(a)^{\lambda_1} a^{\lambda_0}$. By $K(a^{\mathbb{N}[\sigma]})$ we mean the difference field extension obtained as $K(a, \sigma(a), \sigma^2(a), \cdots)$. We will say that an element a in an extension (L, σ) of (K, σ) is *transformally algebraic* over K if there are $\{c_I \mid I \in \mathbb{N}[\sigma]\} \subseteq K$, all but finitely many zero, such that $\sum_I c_I a^I = 0$.

If (K, v) is a valued field, $a \in K$ and $\gamma \in \Gamma_{>0}$, then

$$B_\gamma(a) = \{b \in K \mid v(b - a) > \gamma\} \subseteq B_\gamma[a] = \{b \in K \mid v(b - a) \geq \gamma\}.$$

2. **The theory** FE**.** Given a difference field (K, σ), we will consider it in the language $\mathcal{L}_\sigma = \{+, \times, -, 0, 1\} \cup \{\sigma\}$ and consider the $\mathcal{L}_\sigma$-theory FE saying that (K, σ) is a difference field with $\sigma(K)^{\mathrm{alg}} \cap K = \sigma(K)$.

3. **Transformally separable extensions.** We will say that an extension of difference fields $(K, \sigma) \subseteq (L, \sigma)$ is *transformally separable* if K is linearly disjoint from $\sigma(L)$ over $\sigma(K)$. Equivalently, if the inversive closure $K^{\mathrm{inv}} \coloneqq \bigcup_{n\geq 0} \sigma^{-n}(K)$ is linearly disjoint from L over K.

This notion is not transitive in towers: as an example, consider a tower of difference fields $K \subseteq M \subseteq L$ where K is inversive, M is not, and $L = M^{\mathrm{inv}}$. Trivially, $K \subseteq L$ and $K \subseteq M$ are transformally separable, but $M \subseteq L$ is not.

Lemma 3.1 (Proposition 4.18)**.** If A is a model of FE, which is algebraically closed, and B, C are transformally separable extensions of A which are linearly disjoint over A, then $B \otimes_A C$ is a model of FE.

4. **Transitivity in towers, when the base is transformally algebraic.** The idea is that since we are in characteristic zero, if $(K, \sigma) \vDash$ FE, then $\sigma(K) \subseteq K$ is not just primary (as imposed by FE) but actually regular, and thus K is linearly disjoint from $\sigma(L)$ over $\sigma(K)$ if and only if K is *algebraically free* from $\sigma(L)$ over $\sigma(K)$ ([2] call this *almost transformally separable*).

Proposition 4.1 (Propositions 4.30 and 4.31)**.** Let $K \subseteq M \subseteq L$ be a tower of models of FE, where $K \subseteq M$ is transformally algebraic. Then $K \subseteq L$ is transformally separable if and only if both $K \subseteq M$ and $M \subseteq L$ are.

Proof. It is enough to prove that

$$K \underset{\sigma(K)}{\overset{\mathrm{alg}}{\perp}} \sigma(L) \iff K \underset{\sigma(K)}{\overset{\mathrm{alg}}{\perp}} \sigma(M) \wedge M \underset{\sigma(M)}{\overset{\mathrm{alg}}{\perp}} \sigma(L).$$

The $\Longleftarrow$ direction is clear. As for the $\Longrightarrow$ direction, one immediately has that K is algebraically free from $\sigma(M)$ over $\sigma(K)$. It is thus enough to show that M is algebraically free from $\sigma(L)$ over $\sigma(M)$.

Since $(K, \sigma) \subseteq (M, \sigma)$ and $(K, \sigma) \subseteq (L, \sigma)$ are both transformally separable, we can take the tensor products $M_0 \coloneqq M \otimes_K K^{\mathrm{inv}}$ and $L_0 \coloneqq L \otimes_K K^{\mathrm{inv}}$, and upon replacing K with K^{inv}, M with M_0 and L with L_0, we can assume that we are working with an inversive K.

Now, we argue that if K is inversive and $K \subseteq M$ is transformally algebraic, then actually $\sigma(M) \subseteq M$ is an algebraic extension. Indeed, given any $a \in M$, there are $\{c_I \mid I \in \mathbb{N}[\sigma]\} \subseteq K$, all but finitely many zero, such that $\sum_I c_I a^I = 0$. Upon applying σ^{-1} enough times (since K is inversive), we reduce this equation to an algebraic equation for a over $\sigma(a), \sigma^2(a), \ldots$ and K.

As $\sigma(M) \subseteq M$ is algebraic, one gets trivially that M is algebraically free from $\sigma(L)$ over $\sigma(M)$. □

5. **Simple roots.** We call expression of the form $\sum_I c_I X^I$, where X is a variable and $\{c_I \mid I \in \mathbb{N}[\sigma]\} \subseteq K$, a *difference polynomial* (or *difference polynomial*, or *transformal polynomial*) over K. Alternatively, we can see a difference polynomial $p(X)$ as obtained from a multivariate polynomial $P(X_0, \ldots X_n) \in K[X_0, \ldots X_n]$ by

$$p(X) \coloneqq P(X, \sigma(X), \ldots \sigma^n(X)).$$

Under this identification, we can define

$$p'(X) \coloneqq \frac{\partial P}{\partial X_0}(X, \sigma(X), \ldots \sigma^n(X)).$$

We will say that $a \in L$, where L is some extension of K, is a *simple root* of $p(X)$ if $p(a) = 0$ and $p'(a) \neq 0$.

Proposition 5.1 (Proposition 4.39)**.** We let $(K, \sigma) \vDash \mathrm{FE}$ and $L = K(a^{\mathbb{N}[\sigma]})$ be a transformally algebraic extension. Then $K \subseteq L$ is transformally separable if and only if there is a difference polynomial $p(X)$ over K such that $p(a) = 0$ and $p'(a) \neq 0$.

Proof. Assume that there is $p(X)$ over K such that $p(a) = 0$ and $p'(a) \neq 0$. This means that a is algebraic over $K(\sigma(a)^{\mathbb{N}[\sigma]})$, i.e. $K\sigma(L) \subseteq L$ is algebraic. As the base is FE, it is enough to check that K and L are algebraically free over $\sigma(K)$. Thus, we compute

$$\operatorname{trdeg}_K(L) = \operatorname{trdeg}_K(K\sigma(L)) \leq \operatorname{trdeg}_{\sigma(K)} \sigma(L) = \operatorname{trdeg}_K L,$$

so we obtain $\operatorname{trdeg}_K(K\sigma(L)) = \operatorname{trdeg}_{\sigma(K)} \sigma(L)$, as needed.

For the reverse implication, we choose a difference polynomial $p(X) = \sum_I c_I X^I$ over K such that $p(a) = 0$. First, we may assume that $p' \neq 0$, i.e. that there is some $I = \lambda_n \sigma^n + \cdots + \lambda_1 \sigma + \lambda_0 \in \mathbb{N}[\sigma]$ such that $c_I \neq 0$ and $\lambda_0 \neq 0$. Otherwise, we could apply σ^{-1} and obtain

$$\sum_I c_I^{\frac{1}{\sigma}} a^{\frac{I}{\sigma}} = 0,$$

giving a linear dependence relation for $(a^{\frac{I}{\sigma}})_I$ over $\sigma^{-1}(K)$. But because $K \subseteq L$ is transformally separable, this means that the $(a^{\frac{I}{\sigma}})_I$ had to be K-linearly dependent already, i.e. there are $(b_I)_I \subseteq K$ such that $\sum_I b_I a^{\frac{I}{\sigma}} = 0$. Upon iterating this process, we may assume that we land on a difference polynomial $q(X)$ with $q(a) = 0$ and $q' \neq 0$. Now, if $q'(a) = 0$, then we replace q with q' and iterate this process. Eventually, we must find a difference polynomial $r(X)$ with $r(a) = 0$ and $r'(a) \neq 0$. □

Proposition 5.2 (Proposition 4.39)**.** Let $K \subseteq L$ be a transformally algebraic extension of difference fields, with K a model of FE. Then the extension is transformally separable if and only if every element of L is the simple root of a difference polynomial over K.

6. **The** FE **closure.** If K is a difference field, then there is a difference field extension $K \subseteq E$ which is a model of FE and satisfies the universal property: whenever $K \subseteq F$ is a difference field extension and F is a model of FE, then E embeds uniquely in F over K. We can define $E \coloneqq K^{\mathrm{alg}} \cap K^{\mathrm{inv}}$, and quickly verify all conditions.

7. **The relative transformally separable transformally algebraic closure.** This is mostly Theorem 4.46.

(1) If $E \subseteq F$ is an extension of models of FE, then there is a transformally separable, transformally algebraic extension $\tilde{E} \vDash$ FE with $E \subseteq \tilde{E} \subseteq F$ that satisfies the following property: if $E \subseteq L$ is a transformally separable, transformally algebraic extension of E that is a model of FE, then any embedding of L in F over E has image contained in $\tilde{E}$. One has $a \in \tilde{E}$ if and only if it is the simple root of a difference polynomial over E.

(2) If $E \subseteq \tilde{E} \subseteq F$ is a tower of models of FE as above, then F is transformally separable over $\tilde{E}$ if and only if it is transformally separable over E.

Indeed, to prove (1) consider the set $X \subseteq F$ of simple roots of difference polynomials over E. The difference field $E' \coloneqq E(X^{\mathbb{N}[\sigma]})$ is a transformally separable, transformally algebraic extension of E. As F is a model of FE, the FE closure E'' of E' sits inside of F over E', so we have a tower of inclusions $E \subseteq X \subseteq E' \subseteq E''$. Now, we argue that E'' is a transformally separable, transformally algebraic extension of E: indeed, it is transformally algebraic and moreover, since $E \vDash \mathrm{FE}$, it is enough to check that E^{inv} is algebraically free from E'' over E. However, since this is true for E' in the place of E'', and $E' \subseteq E''$ is algebraic, this follows. This means in particular that $E'' \subseteq X$, meaning that the tower of inclusions was a tower of equalities, X is a difference field, and we can define $\widetilde{E} \coloneqq X$.

8. **The theory** VFE**.** We will consider structures in the language

$$\mathcal{L}_0 = \mathcal{L}_\sigma \cup \{\mathcal{O}, \mathfrak{m}\},$$

where $\mathcal{O}$ and $\mathfrak{m}$ are unary predicates. We will denote by VFE the $\mathcal{L}_0$-theory that says of a model (K, v, σ) that:

(1) (K, v) is a valued field with valuation ring $\mathcal{O}_v$ and maximal ideal $\mathfrak{m}_v$,
(2) $(K, \sigma) \vDash \mathrm{FE}$,
(3) $\sigma^{-1}(\mathcal{O}_v) = \mathcal{O}_v$ and $\sigma^{-1}(\mathfrak{m}_v) = \mathfrak{m}_v$,
(4) σ is ω-increasing (relative to v), i.e. for any $\alpha \in \mathfrak{m}_v$ and any $n \geq 1$, $v(\sigma(\alpha)) > nv(\alpha)$.

In this language, the model companion is inversive (and it is $\widetilde{\mathrm{VFA}}$ as in [3]). To avoid this, we expand the language to

$$\mathcal{L}_1 = \mathcal{L}_0 \cup \{R_n \mid n \in \mathbb{N}\},$$

where we interpret $R_n(x_1, \ldots x_n)$ to mean that $x_1, \ldots x_n$ are $\sigma(K)$-linearly independent. Note that this expansion of the language is the one already used in [1] to obtain the model companion of non-inversive difference fields.

9. **Transformal henselianity.** We say that $(K, v, \sigma) \vDash \mathrm{VFE}$ is *transformally henselian* if for every difference polynomial $f(X)$ over $\mathcal{O}_v$ and $a \in \mathcal{O}_v$ such that $v(f(a)) > 0$ and $v(f'(a)) = 0$, there is $b \in \mathcal{O}_v$ such that $f(b) = 0$ and $v(b - a) > 0$.

Take an extension $K \subseteq L$ of models of VFE. If $K \subseteq L$ is transformally separably transformally algebraically closed, and L is transformally henselian, then K is transformally henselian. Moreover, k is transformally separably transformally algebraically closed in l: indeed, if $\alpha \in l$ is transformally separably transformally algebraic over k, say as witnessed by some $g(X)$ over k with $g(\alpha) = 0$ and $g'(\alpha) \neq 0$. Then we can lift it to $f(X)$ over K satisfying $v(f(\alpha)) > 0$ and $v(f'(\alpha)) = 0$. By transformal henselianity, we find $a \in L$ with $f(a) = 0$ and residue α. Note that then $f'(a) \neq 0$, since otherwise we would have $g'(\alpha) = 0$. The element a is then transformally separable and transformally algebraic over K, thus $a \in K$. It then follows that $\alpha \in k$.

Remark 9.1. If $K \models$ VFE is henselian, then k is a model of FE.

We say that K is *strictly transformally henselian* if it is transformally henselian, and k is a model of SCFE.

10. **The theory $\widetilde{\mathrm{VFE}}$.** We will extend VFE to $\widetilde{\mathrm{VFE}}$ by further imposing on models K that:

(1) K is strictly transformally henselian,
(2) $k \models$ SCFE,
(3) $\Gamma \neq 0$ and it is *tamely transformally divisible*, i.e. for every $\eta \in \mathbb{Z}[\sigma]$ with non-zero constant term, $\eta\Gamma = \Gamma$,
(4) if $\tau \in K[x^{\mathbb{N}[\sigma]}]$ is an additive operator with $\tau' \neq 0$, then τ is onto on K.

We let $\widetilde{\mathrm{VFE}}_e$ be the expansion of $\widetilde{\mathrm{VFE}}$ that further requires that K is not inversive and it has imperfection degree equal to e. (For us, $e = 0$).

Remark 10.1. Later, we will see that in fact, for models of $\widetilde{\mathrm{VFE}}$, the residue field is a model of ACFA, because the model is dense in its inversive hull.

11. **Descent, generation and amalgamation.** This is the second part of Theorem 4.46.

(1) If E is a model of FE, then there is a canonical equivalence of categories

$$\{E \subseteq F \text{ transformally separable transformally algebraic}\}$$
$$\simeq$$
$$\{E^{\mathrm{inv}} \subseteq L \text{ inversive transformally algebraic}\}.$$

(2) If $E \subseteq F$ is a transformally separable extension of models of FE, then there is a transformal transcendence basis $b \subseteq F$ of F over E such that $E(b^{\mathbb{N}[\sigma]}) \subseteq F$ is transformally separable, transformally algebraic (i.e., $E \subseteq F$ is *transformally separably generated*.)

(3) If $E = E^{\mathrm{alg}}$ is a model of FE, and $E \subseteq E_1, E_2$ are two transformally separable extensions which are models of FE, then $E_1 \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{l.d.}}_{E} E_2$ and $E_1 \otimes_E E_1 \models \mathrm{FE}$.

12. **Transformal henselianity, again.** Take an extension $K \subseteq L$ of models of VFE. If $K \subseteq L$ is transformally separably transformally algebraically closed, and L is transformally henselian, then K is transformally henselian. Moreover, k is transformally separably transformally algebraically closed in l: indeed, if $\alpha \in l$ is transformally separably transformally algebraic over k, say as witnessed by some $g(X)$ over k with $g(\alpha) = 0$ and $g'(\alpha) \neq 0$. Then we can lift it to $f(X)$ over K satisfying $v(f(\alpha)) > 0$ and $v(f'(\alpha)) = 0$. By transformal henselianity, we find $a \in L$ with $f(a) = 0$ and residue α. Note that then $f'(a) \neq 0$, since otherwise we would have $g'(\alpha) = 0$. The element a is then transformally separable and transformally algebraic over K, thus $a \in K$. It then follows that $\alpha \in k$.

Remark 12.1. If $K \models \mathrm{VFE}$ is henselian, then k is a model of FE.

13. **The transformally henselian hull.** We consider an extension of models of VFE $K \subseteq L$. We will say that L is a *algebraically closed transformally henselian hull* of K if L is algebraically closed and transformally henselian, and further if $K \subseteq K' \subseteq L$ is an algebraically closed, transformally henselian subfield, then $K' = L$. We will say that K is a *strict amalgamation basis* if its inversive henselian hull has no non-trivial finite σ-invariant Galois extension (i.e. no non-trivial difference extension which is finite and Galois as a field extension).

Lemma 13.1. If K is a model of VFE and a strict amalgamation basis, then there is up to isomorphism a unique algebraically closed and transformally henselian hull of K, transformally separable over K.

Proof. If K is inversive, then Corollary 5.4 from [3] gives a unique algebraically closed and transformally henselian hull. Then we take the relative transformally separable closure. □

14. **Amalgamation.** Let K be a model of VFE, algebraically closed and transformally henselian. Let L_1, L_2 be transformally separable extensions of K and models of VFE. Then there is a model L of VFE, transformally separable over K, in which L_1 and L_2 jointly embed over K. We can take L_1 and L_2 to be linearly disjoint over K in L, and L to be transformally separable over L_1 and L_2.

We know amalgamation for models of FE; to equip the resulting $L = L_1 \otimes_K L_2$ with an VFE structure, we work with ${L_1}^{\mathrm{inv}} \otimes_{K^{\mathrm{inv}}} {L_2}^{\mathrm{inv}}$, for which the result is true by amalgamation in VFA, and then use descent.

15. **The strict transformal henselization.** Using descent from the same result in VFA, we obtain the following.

Proposition 15.1 (Proposition 5.20)**.** If K is a model of VFE whose residue field k is a model of FE, and k' is a model of FE which is transformally separably transformally algebraic over k, then there is a transformally henselian model K' of VFE which is transformally separably transformally algebraic over K and induces the embedding $k \subseteq k'$, that satisfies the following property: if $K \subseteq L$ is a transformally henselian model of VFE, then every embedding of k into l lifts uniquely to an embedding of K' in L over K. The extension $K \subseteq K'$ is purely inertial, and if K is a strict amalgamation basis and k is algebraically closed, then K' is also a strict amalgamation basis.

If K is an henselian model of VFE, then an extension K' of K is a *strict transformal henselization of K* if it is strictly transformally henselian and it is the K' mentioned in the Proposition above.

16. **Some further considerations on $\widetilde{\mathrm{VFE}}$.** Recall that $\widetilde{\mathrm{VFE}}$ was the theory of strictly transformally henselian models of VFE with further

(1) $k \models \mathrm{SCFE}$,
(2) $\Gamma \neq 0$ and it is *tamely transformally divisible*, i.e. for every $\eta \in \mathbb{Z}[\sigma]$ with non-zero constant term, $\eta\Gamma = \Gamma$,
(3) if $\tau \in K[x^{\mathbb{N}[\sigma]}]$ is an additive operator with $\tau' \neq 0$, then τ is onto on K.

We write VFE and $\widetilde{\mathrm{VFE}}_0$ if we further require σ to be non-surjective.

Proposition 16.1 (Proposition 5.25)**.** Let $K \models \mathrm{VFE}$. Then $K \models \widetilde{\mathrm{VFE}}$ if and only if $K^{\mathrm{inv}} \models \widetilde{\mathrm{VFA}}$. More generally, if L is a purely transformally inseparably algebraic extension of K, then $K \models \widetilde{\mathrm{VFE}}$ if and only if $L \models \widetilde{\mathrm{VFE}}$. In particular, if $K \models \widetilde{\mathrm{VFE}}$ and $E \subseteq K$ is non-trivially valued and relatively transformally separably algebraically closed, then $E \models \widetilde{\mathrm{VFE}}$.

Proof. The residue field of K^{inv} is k^{inv}, which is then a model of ACFA. Moreover, K^{inv} is a directed union of transformally henselian models of VFE, and thus it is transformally henselian. In other words, K^{inv} is strictly transformally henselian if K is. Viceversa, K is transformally separably, transformally algebraically closed in K^{inv}. □

We will say that a model of VFE is *deeply transformally ramified* if it is non-trivially valued and dense in its inversive hull.

Proposition 16.2 (Lemma 5.27)**.** Models of $\widetilde{\mathrm{VFE}}$ are deeply transformally ramified.

Proof. Let $F \models \widetilde{\mathrm{VFE}}$. By induction, it is enough to prove that $\sigma(F) \subseteq F$ is dense. In other words, given $a \in F$ and $\gamma \in \Gamma_{>0}$, we want to show that there is $b \in F$ such that $\sigma(b) \in B_\gamma(a)$.

Case 1: $v(a) \geq 0$. Take $c \in K^\times$ with $v(c) > \max(v(a), \gamma)$. By axiom (3) above, there is $b \in K$ with $\sigma(b) - cb - a = 0$. Now, $v(b) \geq 0$, otherwise $v(\sigma(b)) < \min(v(cb), v(a))$, a contradiction. Thus, $v(\sigma(b) - a) = v(cb) \geq v(c) > \gamma$, i.e. $\sigma(b) \in B_\gamma(a)$.

Case 2: $v(a) < 0$. Then we take $c \in K^\times$ such that $v(\sigma(c)) \geq -v(a)$. Then we take $a' = a\sigma(c)$, $\gamma' = \gamma + v(\sigma(c))$, and reduce to case 1. □

Remark 16.3. What we are really using here is axiom (3), i.e. the fact that all linear difference operators τ with $\tau' \neq 0$ are surjective; indeed, this is exactly the proof that if K is a model of SCVF, then K is dense in K^{alg}.

17. **The road ahead.** Our goal is proving that $\widetilde{\mathrm{VFE}}_0$ is the model companion of VFE in the language

$$\mathcal{L}_2 = \mathcal{L}_0 \cup \{\lambda_n \mid n \in \mathbb{N}\},$$

where λ_n takes $(a_1, \ldots a_n, b)$ as input and outputs 0 if $a_1, \ldots a_n$ are $\sigma(K)$-linearly dependent, or if $b \notin \langle a_1, \ldots a_n \rangle_{\sigma(K)}$. Otherwise, it outputs the unique $c_1, \ldots c_n \in \sigma(K)$ such that $\sum_{i=1}^n a_i c_i = b$.

We will first prove that models of $\widetilde{\mathrm{VFE}}_0$ are existentially closed, namely the following theorem.

Theorem 17.1 (Theorem 7.3)**.** Let K be a saturated model of $\widetilde{\mathrm{VFE}}_0$ and let $F \subseteq K$ be a small, strictly amalgamative model of VFE, with K transformally separable over F. If we have a small model E of VFE that is transformally separable over F, then there is an F-embedding of E in K, with K transformally separable over the image.

$$\begin{tikzcd}
 & & K \models \widetilde{\mathrm{VFE}}_0 \\
E \arrow[urr, hook, dashed] \arrow[dr, dash, "\text{t.sep.}"'] & & \\
 & F \arrow[uur, dash, "\text{t.sep.}"'] &
\end{tikzcd}$$

Strategy. (1) First, we may replace E with a transformally separable model of $\widetilde{\mathrm{VFE}}_0$ (Proposition 18.1).

(2) Next, we may assume that F is relatively transformally separably algebraically closed in E (Proposition 18.2). In particular, F is a model of $\widetilde{\mathrm{VFE}}_0$ (Proposition 16.1).

(3) Thus, any element $a \in E \setminus F$ is generic over F in a F-definable ball or a properly infinite intersection of F-definable balls (Proposition 19.6).
(4) By compactness, we may assume that E is transformally separably generated over F, and moreover (by repeating step 1) that $E = F(a^{\mathbb{N}[\sigma]})$. Then E is strictly amalgamative and we can apply Proposition 19.7.

□

18. **The steps.**

Proposition 18.1 (Proposition 7.2)**.** Let E be a model of VFE. Then there is a model of $\widetilde{\mathrm{VFE}}_0$ transformally separable over E.

Proof. Work inside a big $\mathcal{U} \models \widetilde{\mathrm{VFA}}$ containing F.

(1) We may assume that F is not inversive. Indeed, let $x \in \mathcal{U}$ be transformally transcendental over F and let $E = F(x^{\mathbb{N}[\sigma]})^{\mathrm{alg}}$. Then E is a model of VFE which fails to be inversive.
(2) We may assume that F is non-trivially valued. Indeed, let $x \in \mathcal{U}$ be transformally transcendental over F with $v(x) > 0$. Then $E = F(x^{\mathbb{N}[\sigma^{\pm 1}]})$ is a model of VFE transformally separable over F.
(3) Now we take E to be the relative transformal separable algebraic closure of F in $\mathcal{U}$. Then E fails to be inversive and is non-trivially valued and transformally separably algebraically closed in $\mathcal{U}$, and thus it is a model of $\widetilde{\mathrm{VFE}}_0$.

□

Proposition 18.2 (Proposition 5.28)**.** Suppose $K \models \widetilde{\mathrm{VFE}}$ is saturated. Let $F \subseteq K$ be a strict amalgamation basis, with the extension being transformally separable. If $F \subseteq E \models \mathrm{VFE}$ is transformally separably transformally algebraic, then E embeds in K over F, with $E \subseteq K$ transformally separable.

Proof. If there is an embedding, then by (the converse of) transitivity in towers K is automatically transformally separable over E.

So we build the embedding, i.e. for any finite tuple $a \in E$ and quantifier-free formula $\varphi(X)$ in the language $\mathcal{L}_0$, if $E \models \varphi(a)$, then there is $a' \in K$ with $K \models \varphi(a')$. We may switch K with the relative transformally separable closure of F in K, thus $F \subseteq K$ is transformally separable and transformally algebraic.

For simplicity, $|a| = 1$. Now, since a is transformally separable transformally algebraic over F, there is a difference polynomial $f(X) \in F[X]_\sigma$ such that $f(a) = 0$ and $f'(a) \neq 0$. Switching $\varphi(X)$ with

$$\varphi(X) \wedge (f(X) = 0) \wedge (f'(X) \neq 0),$$

we may assume all solutions of $\varphi(X)$ are simple roots of $f(X)$. As F is a strict amalgamation basis, the theory of models of $\widetilde{\mathrm{VFA}}$ over F is complete ([3, Proposition 4.29]). Since K^{inv} is one such model, by model completeness $\varphi(X)$ has a solution in K^{inv}. But all simple roots in K^{inv} of difference polynomials over K are elements of K already, hence $\varphi(X)$ has a solution in K. □

19. **Genericity in a ball.** We work in a large enough saturated model $\mathcal{U} \models \widetilde{\mathrm{VFA}}$.

Definition 19.1. We work in the three-sorted language of valued fields enlarged with a symbol for the action of σ. We denote by VF the valued field sort, and say that a definable set $B \subseteq \mathrm{VF}^1$ is a *closed ball* if $B = a + \gamma\mathcal{O}$ for some $a \in \mathrm{VF}$, $\gamma \in \Gamma$. We say it is *an open ball* if $B = \mathrm{VF}$ or $B = a + \gamma\mathfrak{m}$ as before. By a (possibly degenerate) *ball* we mean a ball as above, or a singleton. A ∞-*definable ball* is the intersection of a (small) chain of balls, regarded as a partial quantifier free type.

We work over an algebraically closed, transformally henselian model F of VFE, so that the theory of $\mathcal{U}$ is independent of the choice of $\mathcal{U}$ (and thus definability of balls over F is intrinsic, independent of $\mathcal{U}$). A ball $B \subseteq \mathrm{VF}$ over F is *split* if there are $a \in F$ and $\gamma \in \Gamma_F$ such that $B = a + \gamma\mathcal{O}$.

Proposition 19.2 (Proposition 6.2)**.** Let $F \models \widetilde{\mathrm{VFE}}$. Then all F-definable balls are split over F.

Proof. Let $E = F^{\mathrm{inv}}$. Then every element of E is F-definable and thus E-definable balls and F-definable balls are the same. As $F \subseteq E$ is dense, being split is also independent of working over F or E. We can thus work over E, and use model completeness to argue that all E-definable balls split over E, since $E \models \widetilde{\mathrm{VFA}}$. □

Lemma 19.3. Let F be an algebraically closed, transformally henselian model of VFE. Let $f(X)$ be a difference polynomial over F and $B \subseteq \mathrm{VF}$ be a closed ball, definable over F; then the function $\theta_f\colon x \mapsto v(f(x))$ has a minimum on B.

Proof. By descent, we may assume that F is a model of $\widetilde{\mathrm{VFA}}$. Then B is split and thus it is affinely isomorphic to $\mathcal{O}$. We may then assume $B = \mathcal{O}$. If $f = 0$, we are done; then we assume $f \neq 0$. Upon rescaling (which doesn't change the thesis), we may assume that f has coefficients in $\mathcal{O}_F$, at least one of them with valuation zero: then $v(f(a)) \geq 0$ for all $a \in \mathcal{O}$. □

Let B be a F-definable ball or a properly infinite intersection. If B is closed, we call $a \in B$ *generic in* B *over* F if for every difference polynomial $f(X)$ over F, $v(f(a))$ is the minimum of θ_f on B. If B is an open ball or a properly infinite intersection, then a is *generic in* B *over* F if whenever $C \subsetneq B$ is a F-definable closed ball, $a \notin C$.

Proposition 19.4 (Proposition 6.5)**.** Let F be an algebraically closed, transformally henselian model of VFE. Let B be an F-definable ball or a properly infinite intersection. Then being generic in B over F gives a complete, consistent, quantifier free type over F. A realization of this type is transformally transcendental over F, and if a is a generic of B over F, then $E = F(a^{\mathbb{N}[\sigma]})$ is a strictly amalgamative model of VFE.

Remark 19.5. Given any $a \in E \setminus F$, a is not F-definable, i.e. it is not in any degenerate F-definable subball: the only elements definable over F but not in F are elements of the inversive hull of F, but $F \subseteq E$ is transformally separable. Moreover, k_F is inversive because F is deeply transformally ramified, so no element of k_E is transforamlly algebraic over k_F.

Proposition 19.6 (Proposition 6.6)**.** Let F be a model of $\widetilde{\mathrm{VFE}}$. Let E be a model of VFE which is transformally separable over F. If k_F is transformally separably algebraically closed in k_E, then F is transformally separably algebraically closed in E. Every element $a \in E \setminus F$ is generic over F in an F-definable ball or a properly infinite intersection.

Proof. Let $\mathcal{B}$ be the family of all F-definable balls containing a, and $B = \bigcap_{b \in \mathcal{B}} b$. If $\mathcal{B}$ has an open maximal element under reverse inclusion, or no minimal element under reverse inclusion, then a is generic over F in B. We may therefore assume that B is closed, so it is split and we can assume $B = \mathcal{O}$. Now, $\operatorname{res} a$ must be transformally transcendental over k_F, i.e. a is generic in F over $\mathcal{O}$. □

Proposition 19.7 (Proposition 7.1)**.** Let K be a saturated model of $\widetilde{\mathrm{VFE}_0}$. Let $F \subseteq K$ be a small model of VFE which is algebraically closed and transformally henselian, with K transformally separable over F. Let B be a F-definable ball or a properly infinite intersection of F-definable balls. Pick a generic over F in B and let $E = F(a^{\mathbb{N}[\sigma]})$. Then, there is an embedding of E in K over F, with K transformally separable over the image.

20. A few consequences of Theorem 17.1.

Theorem 20.1 (Theorem 7.8)**.** Let $K \models \widetilde{\mathrm{VFE}_0}$ and $F \subseteq K$ a model of VFE. Then $F = \operatorname{acl}(F)$ if and only if it is algebraically closed, transformally henselian, and closed under transformal λ-functions of K.

Theorem 20.2 (Theorem 7.11)**.** The residue field and value group are stably embedded and fully orthogonal in models of $\widetilde{\mathrm{VFE}}_0$, with induced structure of pure difference field and pure ordered transformal module.

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