Portfolio Choice 1: Risk and Diversification

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Diversification

- The first key principle of finance is the time value of money

- The second key principle of finance is diversification
  - Do not put all your eggs in one basket
  - Implications for portfolio choice
- First we need to model risk

Expected Return and Variance

- Risk modeling involves <u>scenarios</u> and <u>probabilities</u>
- Scenarios: ***s = 1,2.. S*** to consider
  - *R(s)* is the realized return in scenario *s*
- Probabilities ***p(s)*** for scenario *s*
- The most important summary statistics are
  - the **expected return (= mean return)**
  - the **standard deviation (= volatility)**

Expected return and volatility

- Expected return

$$E[R] = \sum_{s=1}^{S} p(s)R(s)$$

- Variance = expected squared deviation

$$\sigma^2[R] = \sum_{s=1}^{S} p(s)(R(s) - E[R])^2$$

- Standard deviation (sigma, **volatility**)

$$\sigma = \sqrt{Variance}$$

Example: Scenarios and Risk Measures

Scenario	Return	Probability	Deviation from Exp. R.	Squared Deviation
Good	25%	0.35	16.75%	0.02806
Neutral	10%	0.45	1.75%	0.00031
Bad	-25%	0.2	-33.25%	0.11056
Expected return	**8.25%**			
Variance				0.03207
Standard deviation	**17.91%**			

Multiple assets: Covariance

- The *covariance* between two returns is the average of the products of their deviations from the mean:

$$Cov(R_i, R_j) = E\left([R_i - E(R_i)][R_j - E(R_j)]\right)$$

$$= \sum_{s=1}^{S} [R_i(s) - E(R_i)][R_j(s) - E(R_j)]p(s)$$

- The covariance is
  - **Positive** if the random variables tend to be unusually high *at the same time*
  - **Negative** if the one variable tends to be high when the other is low

Correlation

The correlation is a rescaled version of the covariance

$$Corr[R_i, R_j] = \rho_{ij} = \frac{Cov[R_i, R_j]}{\sigma_i \sigma_j}$$

- Advantage: ρ always between -1 and 1
- ρ=1 means perfect correlation, e.g. identical risks
- ρ=0 means uncorrelated, e.g. independent risks
- ρ=-1 means perfect negative correlation, e.g. opposite risks

What is the <u>correlation</u> between the returns of Amazon and Tesla?

Scenario	R_A	R_T	**Probability**
Recession	0%	-10%	10%
Normal	10%	10%	50%
Economic boom	20%	30%	40%

Steps:

1. Compute each expected return: $E[R_A]$=13%, $E[R_T]$=16%
2. Compute each volatility: σ_A = 6.4%, σ_T = 12.8%
3. Compute covariance and then correlation

Portfolio

- A combination of N assets
- Portfolio *p*, with portfolio weights w_1 ,…, w_N:
  - w_i is percentage of wealth invested in asset *i:*

$$w_{\mathrm{i}} = \frac{\$ \text{ value of stock i's position}}{\text{total } \$ \text{ value of portfolio}}$$

  - A negative weight indicates a short position.
  - **Important:** Portfolio weights sum to one: $w_1 + \ldots + w_N = 1$.

Realized Returns, Expected Returns

- The realized return on the portfolio is the weighted average return

$$R = w_A R_A + w_B R_B$$

  - When either R_A or R_B is risky, so is R. You do not know *today* what R will be *tomorrow.* If there is good news, R might be high. If not, R might be negative.

- But you can compute the **expected return**

$$E[R] = w_A E[R_A] + w_B E[R_B]$$

  - This is what you expect today regarding tomorrow's return

Volatility of Portfolio

- Expected return of portfolio is weighted average of expected returns
- Volatility is more complicated. Consider portfolio weights w_A and w_B.
  - It depends on volatilities of each component but also on correlation between the components

$$\sigma^2 = (w_A\sigma_A)^2 + (w_B\sigma_B)^2 + 2\rho w_A\sigma_A\, w_B\sigma_B$$

Example: 2 Risky Assets

- Suppose we have two assets, US and JP, with:

	Mean	Volatility
US	$E[R_1]$ = 13.6%	σ_1 = 15.4%
JP	$E[R_2]$ = 15.0%	σ_2 = 23.0%

and with correlation ρ_{12} = 27%.

- If an investor holds w_1 = 60% in the US and w_2 = 40% in JP, what is the expected return and the volatility of the portfolio?

Example continued

- Portfolio mean:

$$E\left[R_p\right] = 0.6 \times 0.136 + 0.4 \times 0.150 = 14.2\%$$

- Portfolio variance:

$$Var\left(R_p\right) = (0.6)^2 \times (0.154)^2 + (0.4)^2 \times (0.230)^2 + 2 \times 0.6 \times 0.4 \times 0.27 \times 0.154 \times 0.230 = 0.022$$

$$\sigma_p = \sqrt{0.022} = 14.7\% < 15.4\%$$

This portfolio has higher expected return and lower risk than the US market alone!

- Illustrates the **gains from diversification.**

Volatility: Important Special Cases

$$\sigma^2 = (w_A\sigma_A)^2 + (w_B\sigma_B)^2 + 2\rho w_A\sigma_A\, w_B\sigma_B$$

- If A and B are perfectly positively correlated, then
$$\sigma = |w_A\sigma_A + w_B\sigma_B|$$
- If A and B are perfectly negatively correlated, then **maximal gains from diversification**
$$\sigma = |w_A\sigma_A - w_B\sigma_B|$$

Zero risk portfolio with perfect negative correlation

As before $\sigma_{US} = 15.4\%$, $\sigma_{JP} = 23.0\%$.

But now assume the correlation between US and JP is $\rho = -1$. What portfolio weight *w* (on US) gives a zero risk portfolio?

Risk and Return with Varying Weights

- Let $\boldsymbol{w}$ be the weight on US, and $1 - \boldsymbol{w}$ the weight on JP.
- The expected return of the portfolio is:

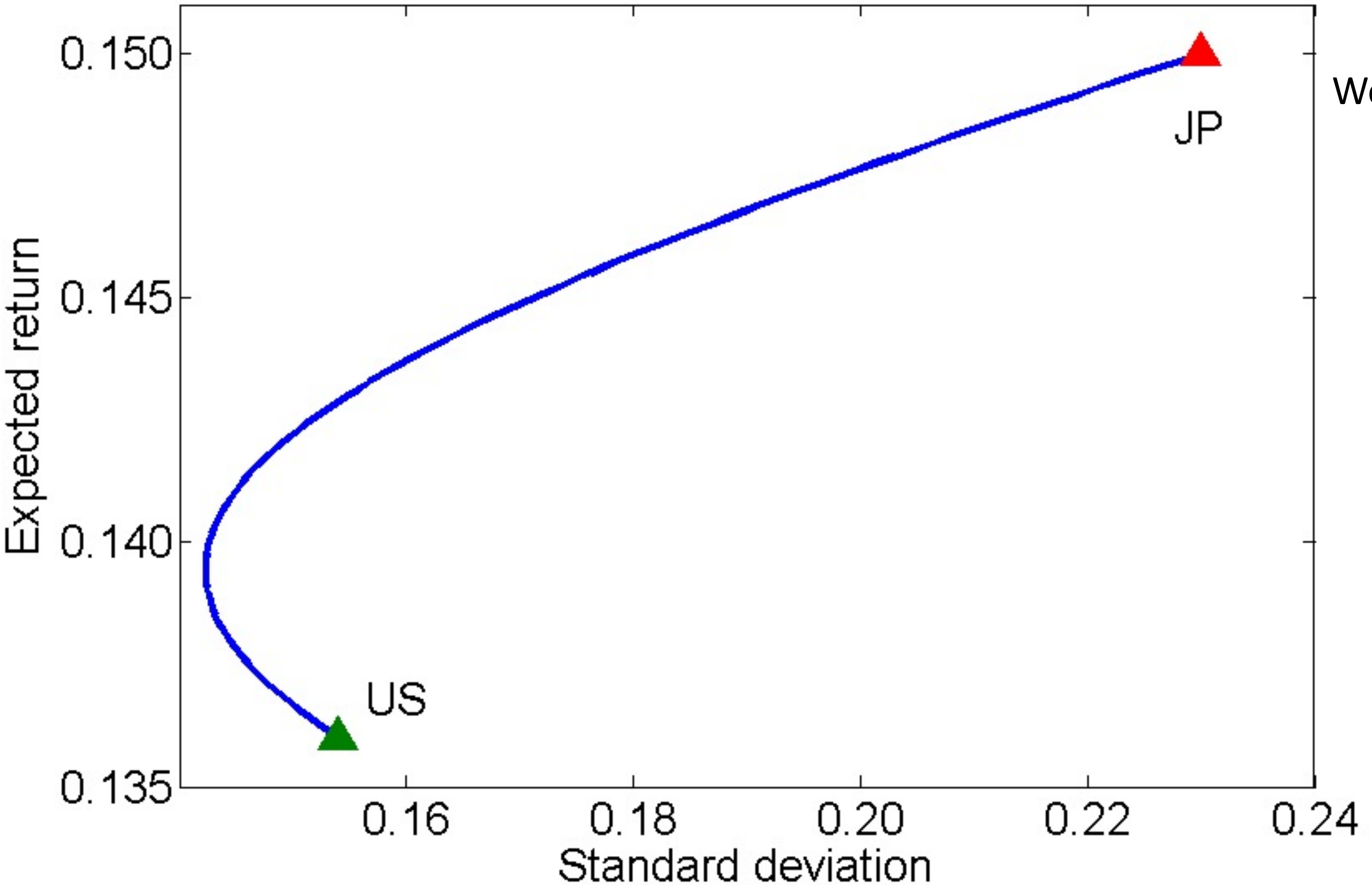
$$E\left[R_p\right] = \boldsymbol{w} \times 0.136 + (\mathbf{1} - \boldsymbol{w}) \times 0.150$$

- Back to ρ = 27%. The variance of the portfolio return is:

$$Var\left(R_p\right) = \boldsymbol{w}^{\mathbf{2}} \times (0.154)^2 + (\mathbf{1} - \boldsymbol{w})^{\mathbf{2}} \times (0.230)^2 \\ + 2 \times \boldsymbol{w} \times (\mathbf{1} - \boldsymbol{w}) \times 0.27 \times 0.154 \times 0.230$$

What happens when we vary weight $\boldsymbol{w}$?

Varying the portfolio weights gives: <u>The Investment Opportunity Set</u>

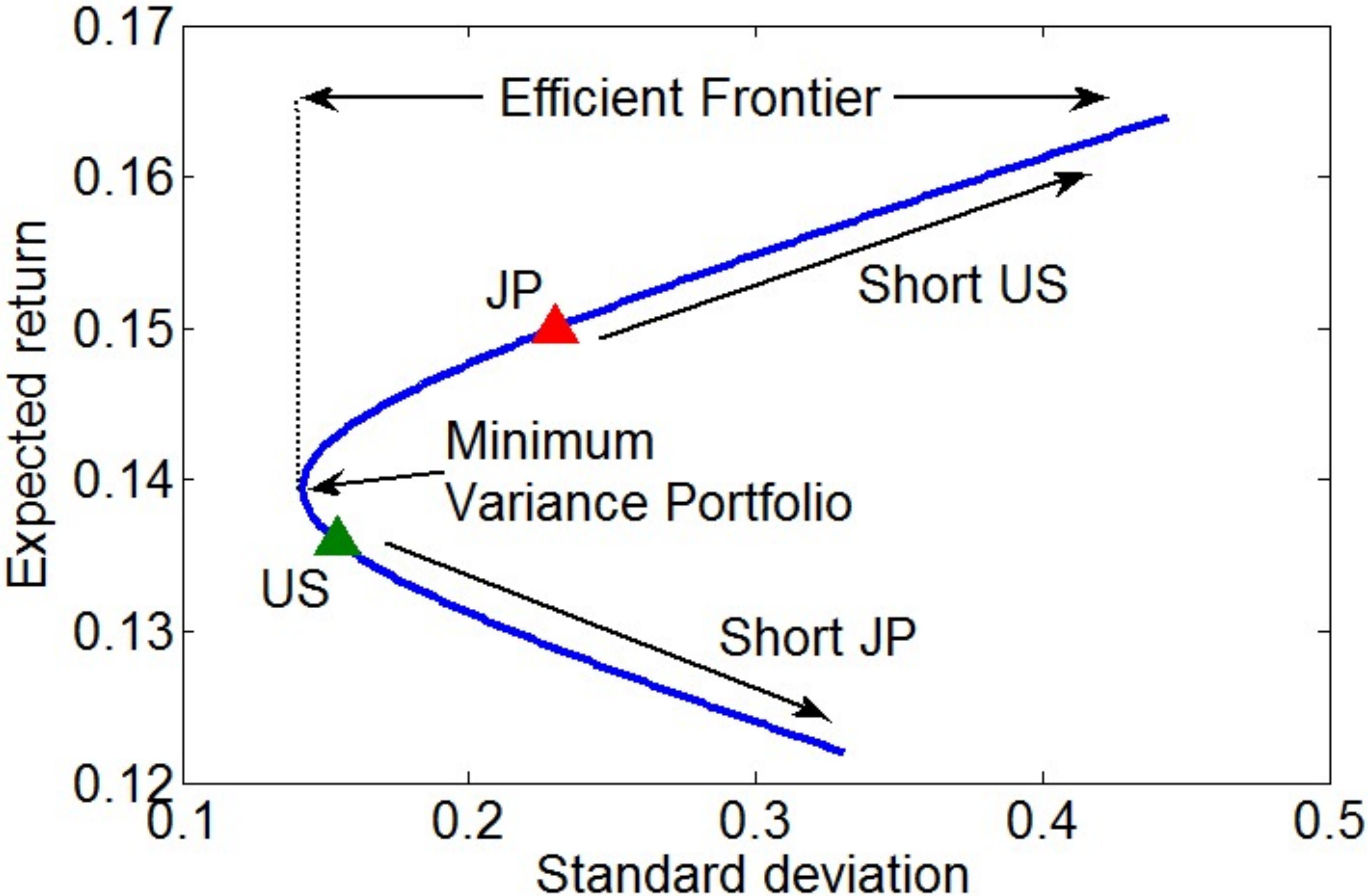


US Weight w	Mean $E[R_p]$	Volatility σ_p
0.0	0.150	0.230
0.1	0.149	0.212
0.2	0.147	0.195
0.3	0.146	0.179
0.4	0.144	0.166
0.5	0.143	0.155
0.6	0.142	0.147
0.7	0.140	0.143
0.8	0.139	0.143
0.9	0.137	0.146
1.0	0.136	0.154

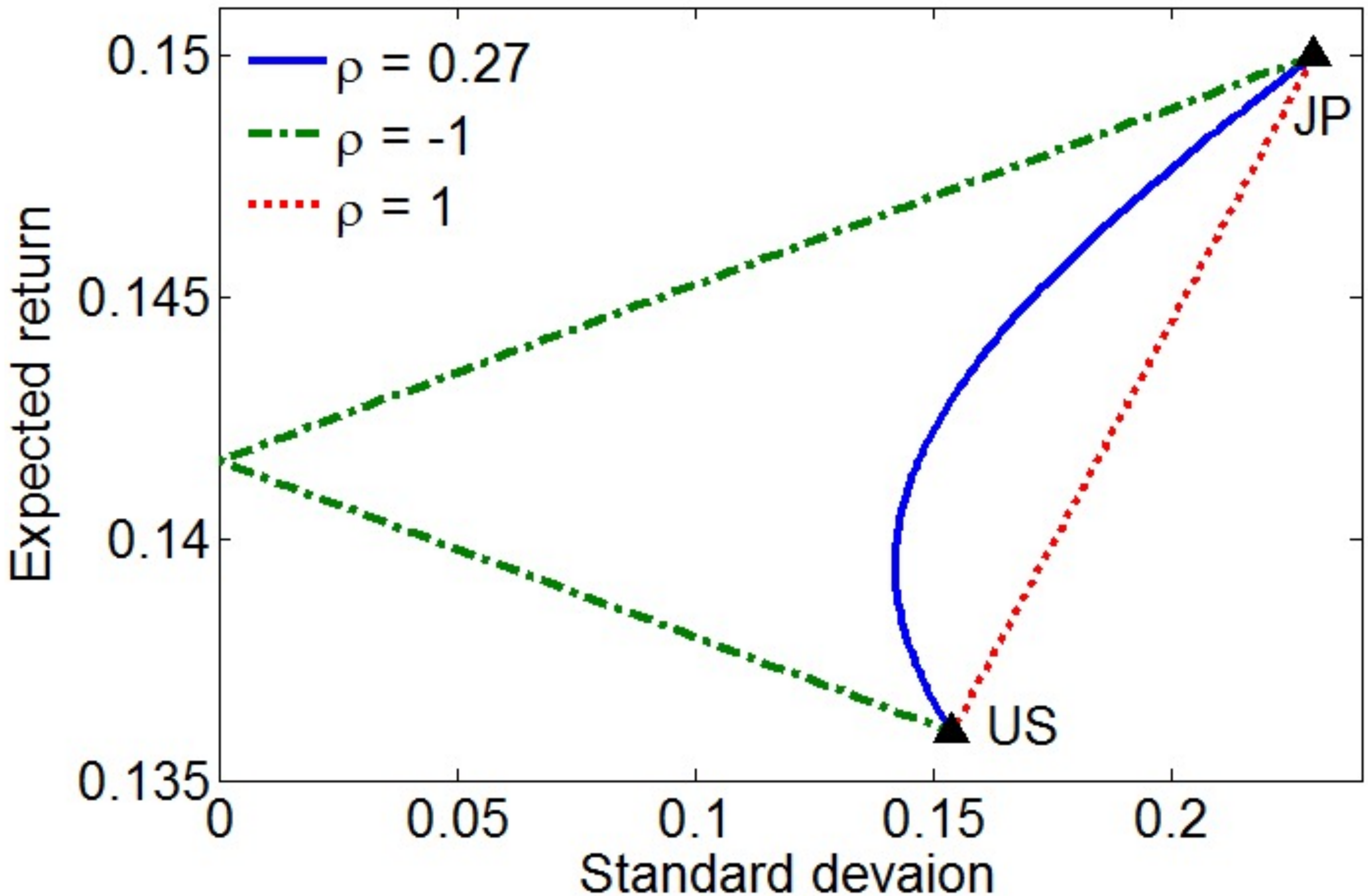
Portfolio Terminology

- The investment opportunity set consists of all available risk-return combinations.
- An efficient portfolio is a portfolio that has the highest possible expected return for a given standard deviation
- The efficient frontier is the set of efficient portfolios. It is the upper portion of the minimum variance frontier starting at the minimum variance portfolio.
- The minimum variance portfolio (mvp) is the portfolios that provides the lowest variance (standard deviation) among all possible portfolios of risky assets.

Efficient Frontier



Investment Opportunity Set with Different Correlations



A website to experiment with parameters

https://www.econgraphs.org/graphs/finance/diversification/correlation_and_diversification?textbook=varian