H0
Statistics Review
Foundations of Finance

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1 Introduction

Please study this handout carefully and as early as possible. It is instrumental for your understanding of portfolio theory. This is supposed to be a review, but from past experience I know that it will be new to some of you, and a much needed reminder for everybody else.

In finance we think of a stock return as a random variable. We think of a portfolio as a combination of stocks, and hence as a combination of random variables. Therefore, the return on a portfolio is a random variable itself. We will need to find the *expected portfolio return* and the *variance of the portfolio return* when the portfolio consists of two risky assets (two random variables) and when the portfolio consists of many risky assets (many random variables). This handout will guide you through the calculations. It starts with the simplest case of one random variable (section 2). Then it moves two the case of two random variables (section 3) and finally it discusses the case of many random variables (section 4). The case of many random variables builds on the rules for two random variables. In section 5, we will review the normal distribution, which is very frequently used in finance.

2 One Random Variable

Probability Distribution Let R_1 be a random variable. Let there be S different values (scenarios or events) that R_1 can take on: R_1(1), R_1(2), ..., R_1(s), ..., R_1(S-1), R_1(S). Associated with these values (scenarios, events) are probabilities p(1), p(2), ..., p(s), ..., p(S-1) and p(S). By the definition of a probability distribution, $\sum_{s=1}^{S} p(s) = 1$ and $p(s) \geq 0$ for each $s = 1, 2, ..., S$. You can think of R_1 as a stock return. As an example with $S = 3$, the events s could be a boom, normal times and recession. The notation $\sum_{s=1}^{S}$ means that the sum goes from $s = 1$ to $s = S$.

Expected Value The expected value of R_1, $E[R_1]$ or μ_1, tells you what the most likely outcome of the random variable R_1 is. The definition of an expected value is in equation (1):

$$\mu_1 = E[R_1] = \sum_{s=1}^{S} R_1(s)p(s) \tag{1}$$

It is the value of the stock in each event multiplied by the likelihood of that event, and then summed over all possible events.

Variance and Standard Deviation The variance of R_1, $Var[R_1]$ or σ_1^2, is a measure of the dispersion or variability of the outcomes of R_1 around its mean. The definition of the variance is in equation (2):

$$\sigma_1^2 = Var[R_1] = E[(R_1 - E[R_1])^2] = \sum_{s=1}^{S} (R_1(s) - \mu_1)^2 p(s) \tag{2}$$

$$Var[R_1] = E[R_1^2] - (E[R_1])^2 = \sum_{s=1}^{S} (R_1(s))^2 p(s) - \mu_1^2. \tag{3}$$

The last row (equation 3) gives you another way for calculating the variance. It equals the expectation of the random variable squared, minus the square of the expectation. The second equality then uses the definition of an expectation, given in equation (1). Check for yourself that the second line gives the same answer as the first line.

The standard deviation σ_1 is the square root of the variance: $\sigma_1 = \sqrt{\sigma_1^2}$. In a financial context, it is also referred to as the volatility. In the class we will use the words volatility and standard deviation interchangeably.

3 Two Random Variables

Let R_1 and R_2 be two different random variables with means μ_1 and μ_2 and standard deviations σ_1 and σ_2. You can think of R_1 and R_2 as two different stock returns, for example IBM and Dell.

Covariance and Correlation The covariance between two random variables R_1 and R_2 measures their co-movement, i.e., whether they move up and down together or in opposite directions. The definition of the covariance is in equation (4):

$$\begin{aligned} Cov[R_1, R_2] &= E[(R_1 - \mu_1)(R_2 - \mu_2)] = \sum_{s=1}^{S} (R_1(s) - \mu_1)(R_2(s) - \mu_2)p(s) && (4) \\ Cov[R_1, R_2] &= E[R_1 R_2] - E[R_1]E[R_2] = E[R_1 R_2] - \mu_1 \mu_2 && (5) \end{aligned}$$

The second line (equation 5) gives another way of computing the covariance. It is the expectation of the product minus the product of the expectations. Check for yourself that it gives you the same answer as the formula on the first line.

The variance of a random variable R_1 is the covariance of R_1 with itself:

$$Var[R_1] = Cov[R_1, R_1]. \tag{6}$$

So, you now understand that equation (3) is a special case of equation (5). For this reason, a number you get from a typical variance computation will be comparable to a number you get from a typical covariance computation.

Covariances are symmetric:

$$Cov[R_1, R_2] = Cov[R_2, R_1] \tag{7}$$

Convince yourself that this is true using the definition for covariance in equation (4).

The correlation ρ_{12} measures the same co-movement as the covariance, but has the property that it is unit-free. It is **always between -1 and 1**. The correlation is defined as the ratio of the covariance to the product of the standard deviations:

$$\rho_{12} = Corr[R_1, R_2] = \frac{Cov[R_1, R_2]}{\sigma_1 \sigma_2} \tag{8}$$

Rules Let ω_1 and ω_2 be two constants (just numbers, not random variables). Here are some very useful rules for means, variances, and covariances, which we will repeatedly use our classes on portfolio theory:

$$
\begin{aligned}
E[R_1 + R_2] &= E[R_1] + E[R_2] && (9)\\
E[\omega_1 R_1] &= \omega_1 E[R_1] && (10)\\
Var[R_1 + R_2] &= Var[R_1] + Var[R_2] + 2Cov[R_1, R_2] && (11)\\
Var[\omega_1 R_1] &= \omega_1^2 Var[R_1] && (12)\\
Cov[\omega_1 R_1, R_2] &= \omega_1 Cov[R_1, R_2] && (13)\\
Cov[\omega_1 R_1, \omega_2 R_2] &= \omega_1 \omega_2 Cov[R_1, R_2] && (14)
\end{aligned}
$$

Convince yourself of these formulas by using the definitions of expectation (equation 1), variance (equation 2) and covariance (equation 4). For example, the first rule is true because $\sum_{s=1}^{S}(R_1(s) + R_2(s))p(s) = \sum_{s=1}^{S} R_1(s)p(s) + \sum_{s=1}^{S} R_2(s)p(s)$. The second rule is true because $\sum_{s=1}^{S}(\omega_1 R_1(s))p(s) = \omega_1 \sum_{s=1}^{S} R_1(s)p(s)$. Go through similar arguments yourself for the other rules. The third rule is crucial. **The variance of the sum of two random variables is NOT equal to the sum of their variances!** Rather there is a third term which is twice their covariance. Please, don't forget that one.

You can think of the fourth rule as a special case of the sixth rule:

$$Var[\omega_1 R_1] = Cov[\omega_1 R_1, \omega_1 R_1] = \omega_1^2 Cov[R_1, R_1] = \omega_1^2 Var[R_1]. \qquad (15)$$

Combining these rules, we can now find the mean and variance of a linear combination of two random variables:

$$
\begin{aligned}
E[\omega_1 R_1 + \omega_2 R_2] &= E[\omega_1 R_1] + E[\omega_2 R_2] = \omega_1 E[R_1] + \omega_2 E[R_2] && (16)\\
Var[\omega_1 R_1 + \omega_2 R_2] &= Var[\omega_1 R_1] + Var[\omega_2 R_2] + 2Cov[\omega_1 R_1, \omega_2 R_2] &&\\
&= \omega_1^2 Var[R_1] + \omega_2^2 Var[R_2] + 2\omega_1 \omega_2 Cov[R_1, R_2] && (17)
\end{aligned}
$$

These two rules (16) and (17) are crucial for portfolio theory. Think of ω_1 and ω_2 as two portfolio weights, for example $\omega_1 = 0.2$ and $\omega_2 = 0.8$ means that you invest twenty percent of your wealth in stock R_1 and eighty percent of your wealth in stock R_2. The mean and variance we computed are then the mean and variance of the portfolio that consists of combining stocks R_1 and R_2 with weights 20% and 80%.

The last rule is for the covariance. Let R_1, R_2 and R_3 be 3 random variables. Then:

$$Cov[R_1, R_2 + R_3] = Cov[R_1, R_2] + Cov[R_1, R_3]. \qquad (18)$$

4 Multiple Variables

Let R_1, R_2,..., R_N be N different random variables with means μ_1 through μ_N and standard deviations σ_1 through σ_N. For concreteness, think of a portfolio with $N = 20$ different stocks; R_n is the (random) return on the n^{th} stock. Your goal is to compute the mean and variance of the portfolio return.

Linear Combination We can form a new random variable R_p which is a linear combination of the N random variables. Let ω_1, ω_2, ...,ω_N be N constants. R_p is defined as:

$$R_p = \omega_1 R_1 + \omega_2 R_2 + ... + \omega_N R_N = \sum_{n=1}^{N} \omega_n R_n$$

The summation sign $\sum_{n=1}^{N} R_n$ means: start with $n = 1$, the first term in the sum is R_1. Then go to $n = 2$, the corresponding term is R_2. Add R_2 to R_1. Now go to $n = 3$, add R_3 to $R_2 + R_1$. And so forth until you reach $n = N$. In the example of portfolio shares, we interpret R_n as the return on security n and we interpret ω_n as the fraction of your total wealth invested in asset n. Portfolio shares always sum to one:

$$\sum_{n=1}^{N} \omega_n = 1.$$

Expected Value of the Return on a Portfolio of N Stocks The expected value of the linear combination can be computed using the two rules for expected values that were given in the previous section:

$$\begin{aligned} E[R_p] &= E[\omega_1 R_1 + \omega_2 R_2 + ... + \omega_N R_N] \\ &= E[\omega_1 R_1] + E[\omega_2 R_2] + ... + E[\omega_N R_N] \\ &= \omega_1 E[R_1] + \omega_2 E[R_2] + ... + \omega_N E[R_N] \\ &= \sum_{n=1}^{N} \omega_n E[R_n] \end{aligned}$$

The expected value of a portfolio of many risky assets is the sum of the expected values of the individual risky asset returns, each weighted by their portfolio weight.

Variance of the Return on a Portfolio of N Stocks The variance of the linear combination can be computed using the three rules for variances and covariances that were given in the previous section:

$$\begin{aligned}
V[R_p] &= V[\omega_1 R_1 + \omega_2 R_2 + ... + \omega_N R_N] \\
&= V[\omega_1 R_1] + V[\omega_2 R_2] + ... + V[\omega_N R_N] + 2Cov[\omega_1 R_1, \omega_2 R_2] + ... + 2Cov[\omega_1 R_1, \omega_N R_N] + ... \\
&= \sum_{n=1}^{N} V[\omega_n R_n] + 2 \sum_{n=1}^{N} \sum_{m>n}^{N} Cov[\omega_n R_n, \omega_m R_m] \\
&= \sum_{n=1}^{N} \sum_{m=1}^{N} Cov[\omega_n R_n, \omega_m R_m] \\
&= \sum_{n=1}^{N} \sum_{m=1}^{N} \omega_n \omega_m Cov[R_n, R_m] \\
&= \sum_{n=1}^{N} \sum_{m=1}^{N} \omega_n \omega_m \rho_{nm} \sigma_n \sigma_m
\end{aligned}$$

The first line uses the definition of the portfolio return R_p. The second line uses the rule for the variance of the sum of N random variables. This is a straightforward extension of rule (11) above. This line includes all covariance terms of each R_n with all other R_m variables. The third line rewrites the second line using sum operators. The fourth line uses the fact that $Var[\omega_n R_n] = Cov[\omega_n R_n, \omega_n R_n]$ (see equation 15). Note that the 2 in front of the sum disappears because of the symmetry of the covariance $Cov[\omega_n R_n, \omega_m R_m] = Cov[\omega_m R_m, \omega_n R_n]$ (see equation 7). The fifth line uses the rule for $Cov[\omega_n R_n, \omega_m R_m] = \omega_n \omega_m Cov[R_n, R_m]$ (rule 14). That's it. **The variance of a portfolio of many risky assets is the sum of the covariances between the returns on each pair of assets in the portfolio, multiplied by the portfolio weights of that pair.** In other words, it is a weighted average of covariances, weighted by the importance of these assets in the portfolio.

If you find it more natural to think in terms of correlations instead of covariances, you can rewrite that expression once more. The last line does this using the definition of correlation $\rho_{nm} = \frac{Cov[R_n, R_m]}{\sigma_n \sigma_m}$ (equation 8).

Covariance between an Individual Asset Return and a Portfolio Return The last rule for the covariance is a generalization of the rule we discussed in equation (**??**):

$$\begin{aligned} Cov[R_i, \sum_{n=1}^{N} \omega_n R_n] &= Cov[R_i, \omega_1 R_1 + \omega_2 R_2 + ... + \omega_N R_N] \\ &= Cov[R_i, \omega_1 R_1] + ... + Cov[R_i, \omega_N R_N] \\ &= \omega_1 Cov[R_i, R_1] + ... + \omega_N Cov[R_i, R_N] \\ &= \sum_{n=1}^{N} \omega_n Cov[R_i, R_n] \end{aligned}$$

The covariance of an individual risky asset R_i (for example IBM) with a portfolio of risky assets $R_p = \sum_{n=1}^{N} \omega_n R_n$ (for example the S&P 500, a basket of the 500 largest stocks in the US) equals the sum of the covariances of the individual risky asset with each and every other risky asset, appropriately weighted by the portfolio weights.

5 The Normal Distribution

The material in this section is optional. We will touch upon it in class, but you will not be held responsible for any of the derivations in this section. However, you are responsible for knowing how to work with the normal distribution and use the probability tables (see handout H7).

Probability Distribution Let R_1 be a *normally distributed* random variable. There are an infinite number of values that R_1 can take on between $-\infty$ and $+\infty$. Let's call one of those R_1(s). Associated with these values are probabilities f(R_1(s)). By the definition of a probability distribution, $\int_{s=-\infty}^{+\infty} f(R_1(s)) = 1$ and $f(R_1(s)) \geq 0$, $\forall s$. The normal probability distribution function is given by

$$f(R_1(s)) = \frac{1}{\sigma_1 \sqrt{2\pi}} \exp\left(\frac{(R_1(s) - \mu_1)^2}{2\sigma_1^2} \right)$$

$f(R_1(s))$ simply tells you the probability that the random variable R_1 takes on the value $R_1(s)$ under the normal distribution.

Expected Value The expected value of R_1, $E[R_1]$ or μ_1, tells you what the most likely outcome of the random variable R_1 is.

$$\mu_1 = E[R_1] = \int_{s=-\infty}^{+\infty} R_1(s) f\left(R_1(s)\right) ds$$

Note the similarity with the definition of expected value for a discretely distributed random variable (first section). Here $f(\cdot)$ plays the role of the probabilities $p(\cdot)$ and the integral plays the role of the summation sign.

Variance The variance of R_1, $Var[R_1]$ or σ_1^2, is a measure of the dispersion of the outcomes of R_1 around its mean.

$$\sigma_1^2 = Var[R_1] = E[(R_1 - E[R_1])^2] = \int_{s=-\infty}^{+\infty} (R_1(s) - \mu_1)^2 f(R_1(s)) ds$$

Again, note the similarity with the definition of expected value for a discretely distributed random variable (first section).

A nice feature of the normal distribution is that it is completely described by two numbers: the expected value and the variance. That's all we need to know. A short way of writing that the variable R_1 is normally distributed with mean μ_1 and variance σ_1^2 is

$$R_1 \sim \mathcal{N}\left(\mu_1, \sigma_1^2\right)$$

The standard normal distribution is a special normal distribution. It has mean zero and variance 1. If a random variable Z is standard normally distributed, we know:

$$Z \sim \mathcal{N}\left(0, 1\right)$$

Two Random Variables When two random variables R_1 and R_2 are *jointly normally distributed*, we write

$$\begin{bmatrix} R_1 \\ R_2 \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 \\ \rho_{12}\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} \right)$$

The first entry describes the two expected values, the second entry is the *variance-covariance matrix*. It contains the two variances on the diagonal and the covariance on the off-diagonal. I used the definition of the correlation to rewrite the off-diagonal elements in terms of the correlation instead of the covariance. The bivariate normal distribution is described completely by 5 moments: the two means (μ_1, μ_2), the two variances (σ_1^2, σ_2^2) and the correlation ρ_{12}.

Multiple Variables Let R_1, R_2,..., R_N be N different normally distributed random variables with means μ_1 through μ_N, standard deviations σ_1 through σ_N and correlations ρ_{12}, ρ_{13}, ..., ρ_{1N}, ρ_{23}, ρ_{24}, ..., ρ_{2N}, ρ_{34},..., ρ_{3N},.... Together, the N means, the N variances and the $\frac{N(N-1)}{2}$ correlations completely describe the joint normal distribution of the variables R_1, R_2,..., R_N.