

H5

Geometric Average Versus Arithmetic Average

Foundations of Finance

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The Question Suppose you invest \$435 in a zero coupon bond for one year and earn a return of 8%. You then reinvest the proceeds at 12% for a second year. How do you describe your average annual return over the two-year period?

A Simple Answer One possibility is to calculate the simple arithmetic average of the annual returns,

$$\frac{0.08 + 0.12}{2} = 10\%.$$

We suspect that it might be more complicated than that because we know that compounding is involved whenever the proceeds of year 1 are reinvested in year 2. So we check to make sure the arithmetic average tells the truth.

Brute Force Answer We know how to calculate the average annual return, assuming annual compounding, of an initial sum V_0 that grows to V_t over t years:

$$ann.R = \left(\frac{V_t}{V_0} \right)^{1/t} - 1. \quad (1)$$

We can also calculate exactly what V_2 will be after 2 years in our case since we know that $V_0 = 435$. In particular,

$$V_2 = 435(1.08)(1.12) = 526.176.$$

*Based on the notes of Profs. Alexi Savov and William Silber.

Therefore, we know that in this case,

$$ann.R = \left(\frac{526.176}{435} \right)^{12} - 1 = 0.0998.$$

Thus, we see that the arithmetic mean is bigger than the true annual average return because we know that 0.0998 is correct since we calculated it from “first principles”. That is, we calculated it using the proper definition of annual returns.

A General Answer

Let’s try to formulate an answer for calculating the average annual return over the two year period using the annual rates themselves, 8% and 12%, or more generally, r_1 for year 1 and r_2 for year 2. We know the following is true

$$\frac{V_2}{V_0} = \left(\frac{V_2}{V_1} \right) \left(\frac{V_1}{V_0} \right). \quad (2)$$

We also know that by the definition of annual return $V_1/V_0 = 1 + r_1$ and $V_2/V_1 = 1 + r_2$. Hence, by substitution, expression (2) becomes:

$$\frac{V_2}{V_0} = (1 + r_2) (1 + r_1).$$

Given equation (1) above, this means

$$ann.R = \left[\left(\frac{V_2}{V_1} \right) \left(\frac{V_1}{V_0} \right) \right]^{1/2} - 1 \quad (3)$$

$$= [(1 + r_1) (1 + r_2)]^{1/2} - 1. \quad (4)$$

The last expression is called the geometric average of r_1 and r_2 . Note that if you substitute 0.08 for r_1 and 0.12 for r_2 , the calculation in (4) produces 0.0998, which is the correct answer. More generally, if r_1 is the return in year 1, r_2 is the return in year 2 and r_t is the return in year t , then the correct formula for calculating the average annual return, assuming the proceeds of year 1 and reinvested in year 2 and so on, is the geometric average of r_1, r_2, \dots, r_t , given by the following formula:

$$ann.R = [(1 + r_1) (1 + r_2) \cdots (1 + r_t)]^{1/t} - 1. \quad (5)$$

Highlighting the Difference

The difference between the geometric average and the arithmetic average seems almost too small to worry about in our current example (although 2 basis points is not always so trivial). In fact, the two numbers would be identical in the case where $r_1 = r_2 = \dots = r_t$. However, when r_1 and r_t differ substantially, then the geometric and arithmetic averages can produce very different answers. Suppose you invested in an emerging market mutual fund and earned 100% in year 1 and then lost 50% in year 2. What is your average annual return over the two-year period? The arithmetic average is clearly +25%, while if you plug in +1 for r_1 and -0.5 for r_2 and then calculate the geometric average according to formula (5) your geometric return is zero.

Although we know that the geometric average is correct it is useful to provide a simple numerical example showing why the return over the two years is zero. If you start with \$100 in year 1 and earn 100%, you begin year 2 with \$200. You then lose 50%, which brings you back to \$100. Thus, you earn a return of zero over the two-year period. (Notice that reversing the sequence makes no difference: Start with \$100 and lose 50%, brings you to \$50; which then doubles to \$100 in the second year when you earn 100%. The return over the two years is still zero.)

Is the Arithmetic Average Ever Correct? The arithmetic average is the appropriate way to calculate the average return over more than one period if instead of reinvesting whatever you have at the end of each year, you start each year with the same amount of money, e.g. \$100. Note that starting each year with \$100 means that if you lose money at the end of a year you must add and if you make money you must withdraw. Here is the numerical example. Start with \$100. If you make 100% you have \$200. Withdraw \$100. Reinvest only \$100. If you lose 50% you have \$50 at the end and you must put in \$50 so you start the next year with \$100. Add up the withdrawals (\$100 - \$50) and you accumulated \$50 over 2 years. Thus you made \$25 per annum on \$100, or 25% per year, which is the arithmetic average of 100% and -50%. We prefer the geometric average because it tells us how an initial sum grows “untouched by human hands”. But if you like to add and subtract at the end of each year to maintain the same dollar investment (you probably won’t like the adding part), then the arithmetic mean tells the truth.