Portfolio Choice 2: Risk Aversion and Optimal Portfolio

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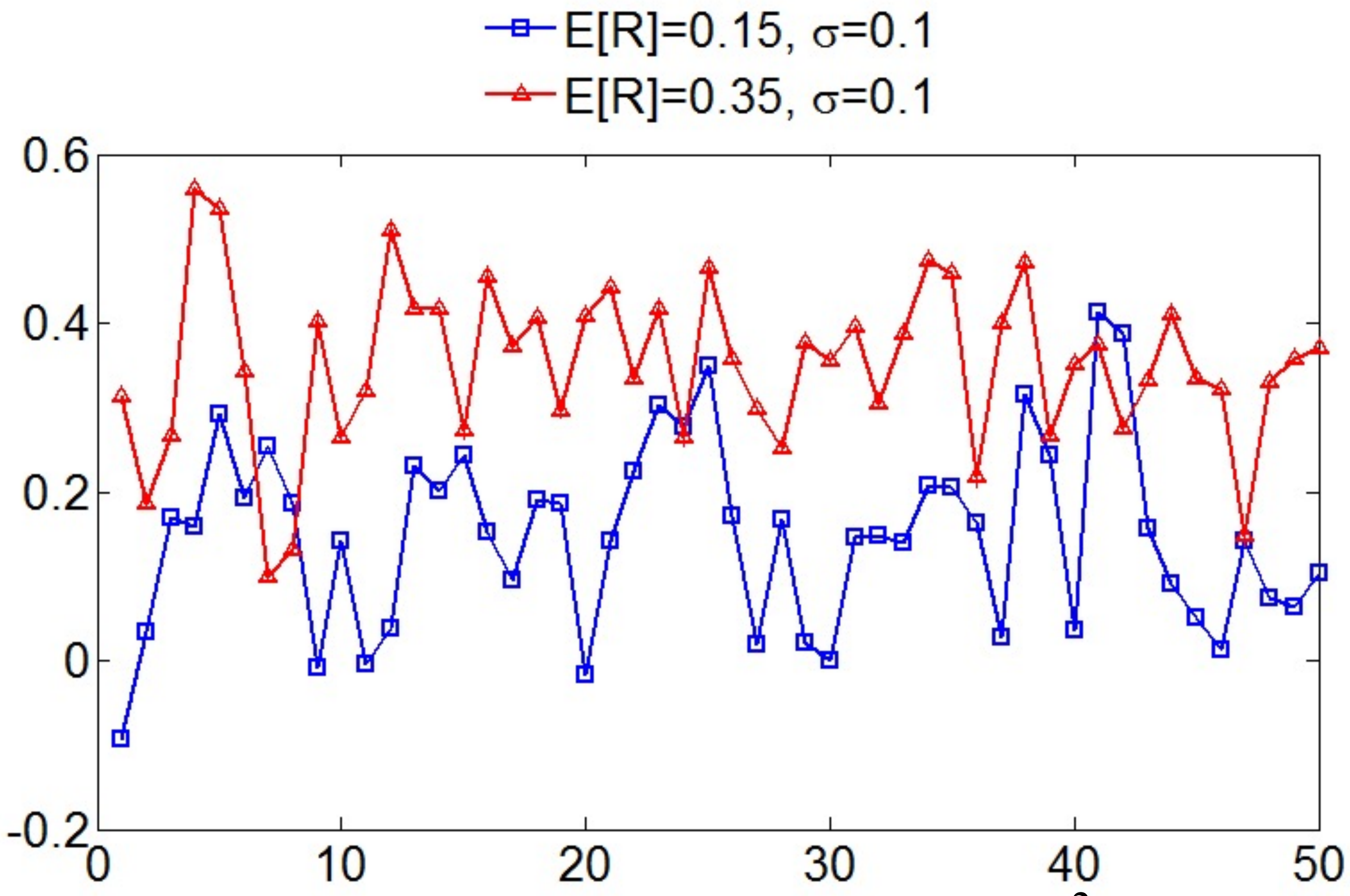
Optimal Portfolio Choice

- Any investor should choose an *efficient* portfolio to benefit from diversification.
- Which efficient portfolio is *optimal* depends on the investor's preferences, in particular her risk aversion.
- We turn to modeling investor preferences next.

Risk-Return Tradeoff

- Suppose you could only pick one of the following return patterns:



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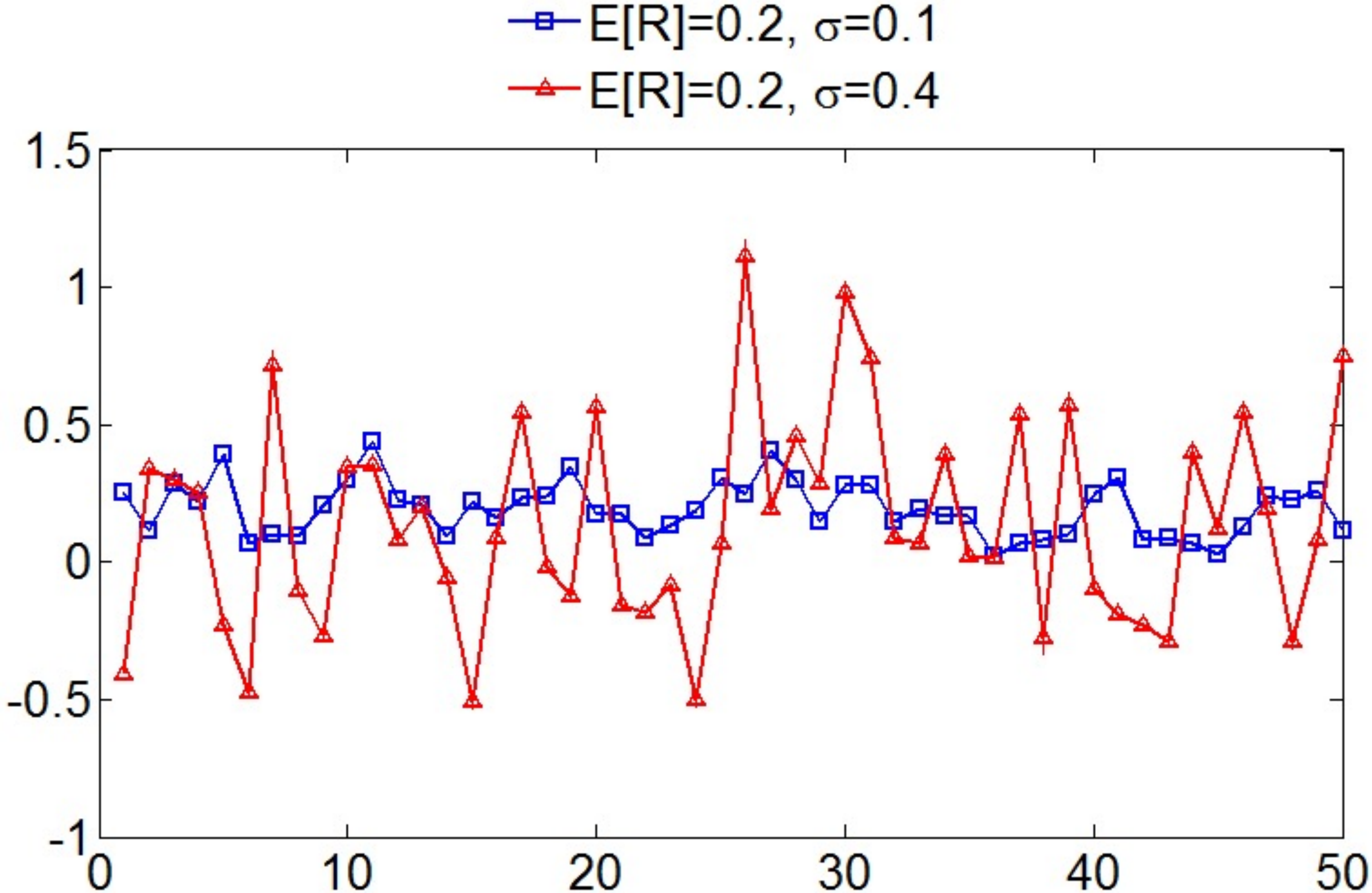

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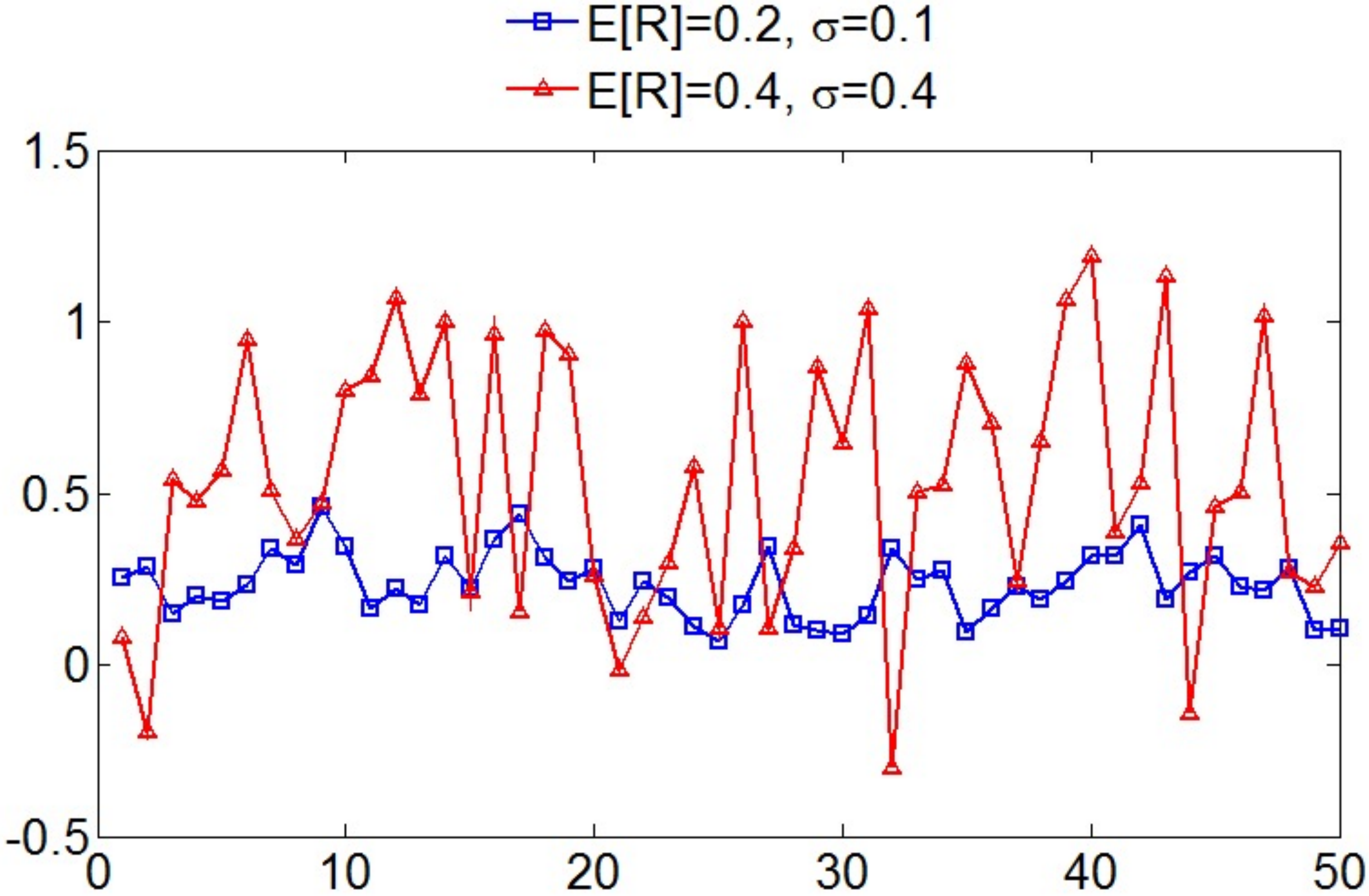
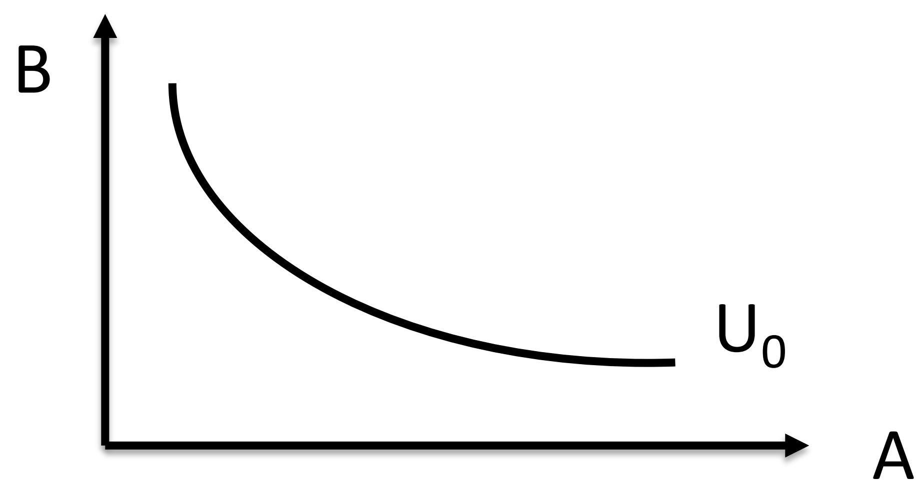


Risk-Return Tradeoff

- Recall two of the Finance Axioms:
  - Investors prefer more to less
  - Investors are risk-averse
- This means that investors prefer an investment i:
  - with a higher expected return $E(R_i)$
  - with a lower variance and standard deviation, σ_i
- Investors optimally **trade off** risk and return in order to <u>maximize their expected utility.</u>

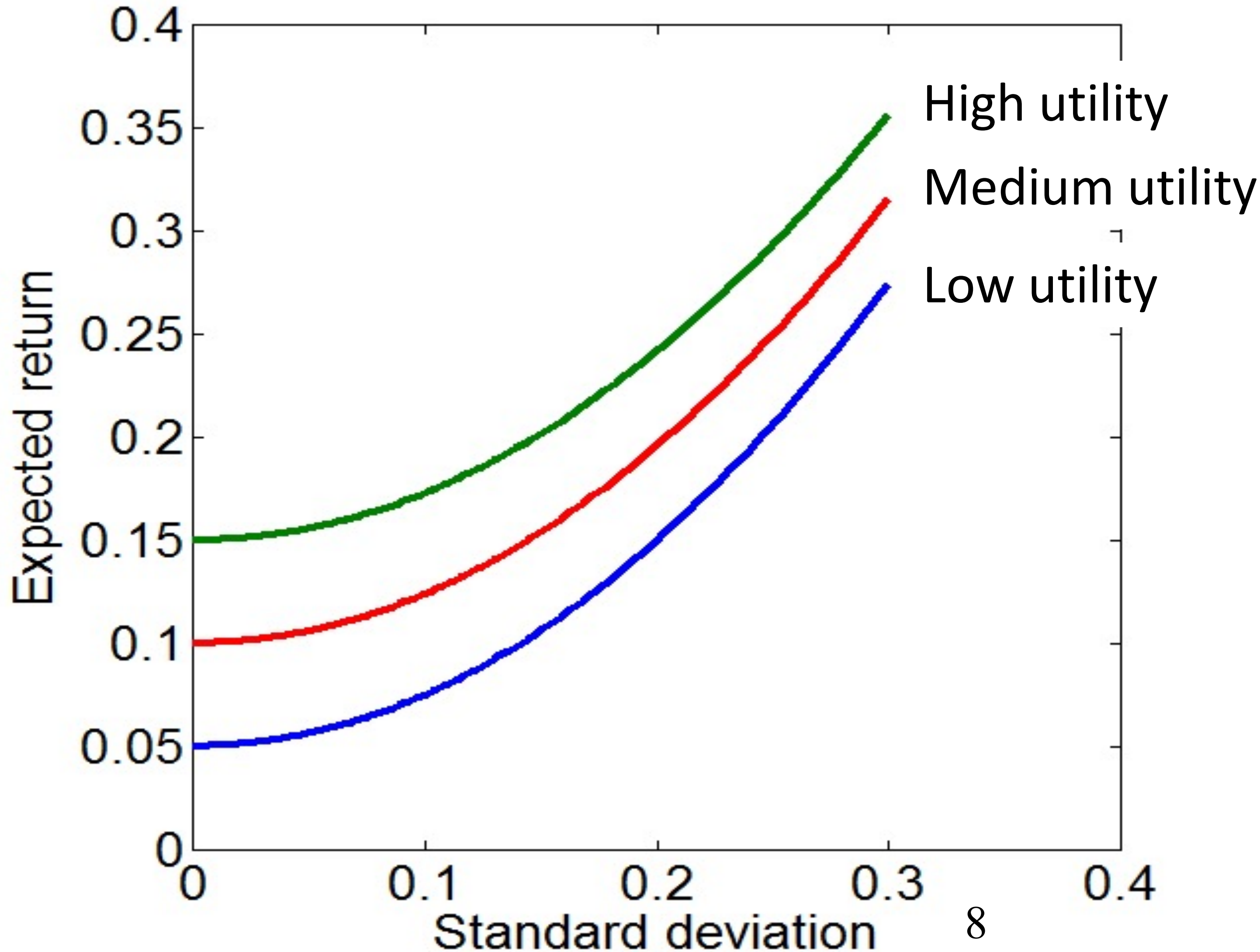
Indifference Curves: Review

- A person likes 2 goods: apples (A) and bananas (B).
- An indifference curve gives all the combinations of A and B that give the same utility level $U_0 = U(A,B)$.
- People like to be on the highest possible indifference curve (people prefer more to less).



Indifference Curves in Finance

- Indifference curve: A set of $(E(R_p),\sigma_p)$ combinations that give an investor the same expected utility



Example of utility function

- One utility function that is easy to work with is the **mean-variance** utility:

$$U(R_p) = E(R_p) - 0.5AVar(R_p)$$

- Portfolio with higher utility score *U* has a more attractive risk-return profile

- Parameter A>0 measures **risk aversion**
  – If A=0, investor is risk-neutral: does not care about risk, only about expected return

Example of utility function

$$U(R_p) = E(R_p) - 0.5AVar(R_p)$$

- Utility score of risky portfolios is a **certainty equivalent rate of return**
  - Investor would be indifferent between portfolio *p* and an asset with a risk-free return $U(R_p)$
- Are indifference curves steeper or flatter for more risk-averse investors?

Which is the most attractive portfolio?

Consider the following 2 investments:

Asset	Expected return	Standard Deviation
Low risk stock L	7%	5%
High risk stock H	13%	20%

- Calculate the utility scores for two investors:

Asset	Investor 1 (A=2)	Investor 2 (A=5)
Low risk stock L	U =	U =
High risk stock H	U =	U =

Putting it all together

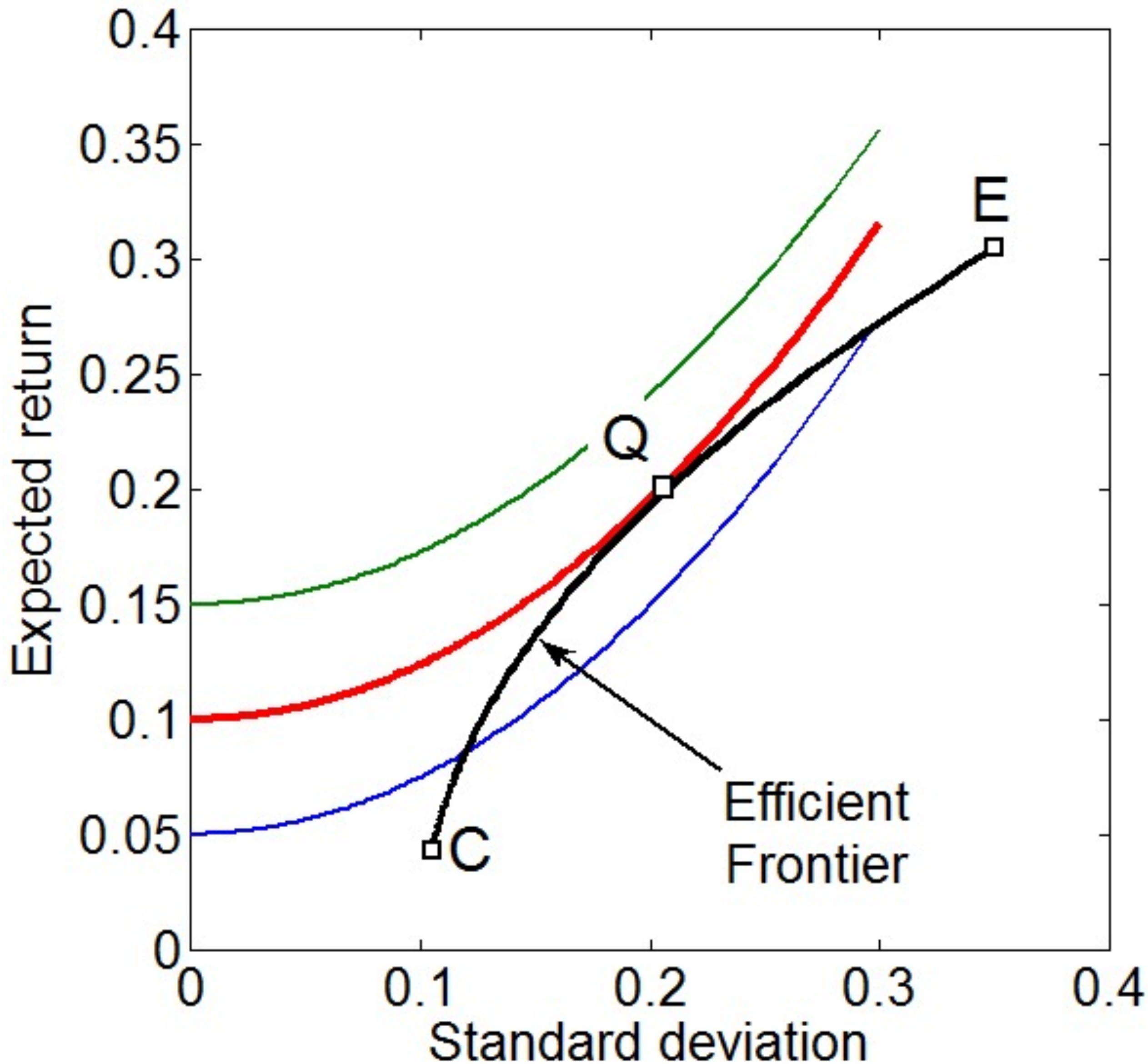
Optimal portfolio:

Within the feasible portfolios – described by the upper portion of the investment opportunity set, the *efficient frontier*

Pick the

Most desirable - highest *indifference curve*

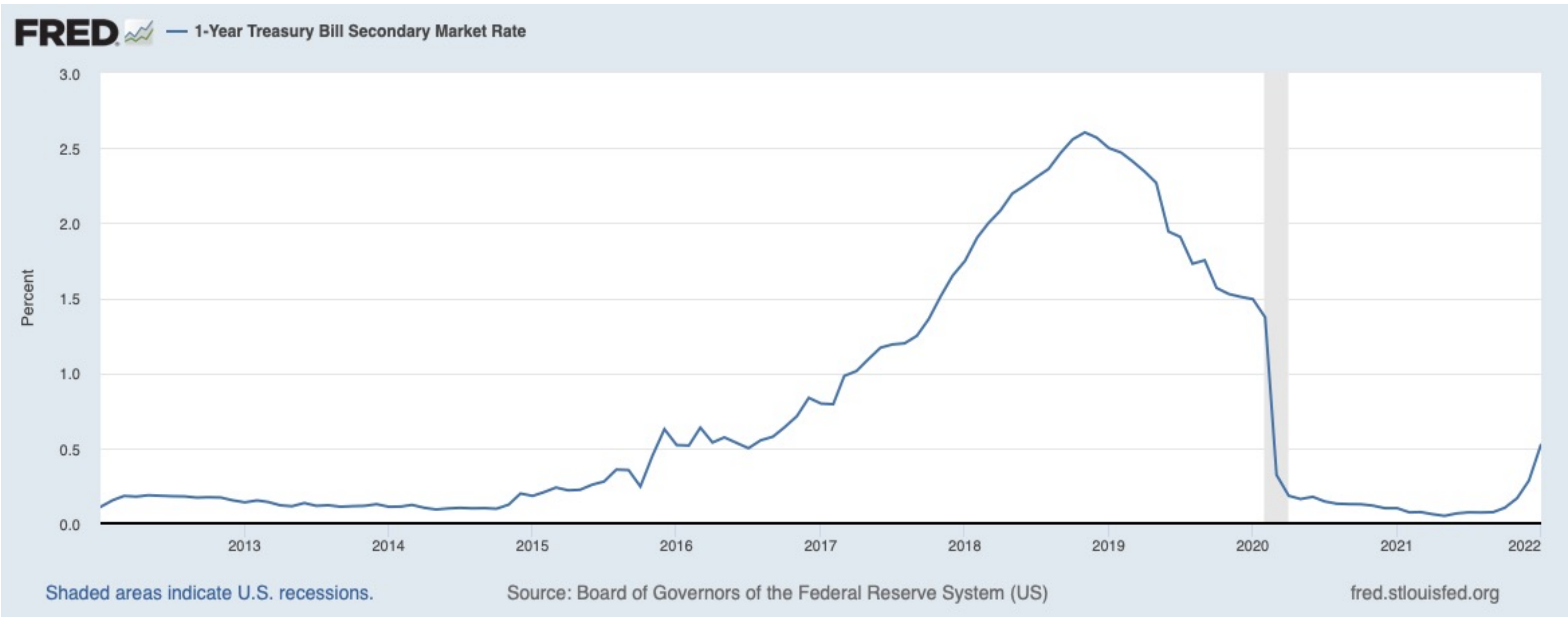
Optimal Portfolio:



Next: Adding a Riskless Security

The Risk-Free Asset

The risk-free return is denoted R_f. Think of T-bills (short-term Treasuries)



Properties of the Risk-Free Asset

The risk-free return has 3 key properties

- $E[R_f] = R_f$
- $Var[R_f] = \sigma_f^2 = 0$
- $Cov(R_f, R_i) = 0$ for any asset i.

This makes it very simple to compute volatility of a portfolio containing the risk-free asset.

Portfolio with Risk-Free + 1 Risky Asset

- Let w be the fraction of wealth invested in the risky asset (the rest is invested in the risk-free asset)

- Expected portfolio return:

$$\mathrm{E}[\mathrm{R}_\mathrm{p}] = w \cdot \mathrm{E}[\mathrm{R}_i] + (1-w)\mathrm{R}_\mathrm{f} = \mathrm{R}_\mathrm{f} + w \cdot \underbrace{\mathrm{E}[\overbrace{\mathrm{R}_\mathrm{i} - \mathrm{R}_\mathrm{f}}^{\text{excess return}}]}_{\text{risk premium}}$$

- Variance of portfolio return:

$$\sigma_\mathrm{p}^2 = w^2\sigma_i^2 + (1-w)^2\sigma_f^2 + 2w(1-w)\sigma_i\sigma_f\rho_{if} = w^2\sigma_i^2$$

- The standard deviation is:

$$\sigma_\mathrm{p} = |w| \cdot \sigma_i$$

Investment Opportunity Set with a Risk-free and a Risky Asset

- Consider various portfolios *p* (which are long in the risky asset and long or short in the risk-free asset).

- What is the risk-return relationship ?
  (combine the expected return and volatility formulas)

$$
\begin{aligned}
\mathrm{E}[\mathrm{R_p}] &= \mathrm{R_f} + \frac{\mathrm{E}[\mathrm{R_i} - \mathrm{R_f}]}{\sigma_\mathrm{i}} \sigma_\mathrm{p} \\
&= \mathrm{R_f} + (\text{Sharpe ratio of i})\ \sigma_\mathrm{p} \\
&= \mathrm{R_f} + \mathrm{SR_i}\ \sigma_\mathrm{p}
\end{aligned}
$$

- The <u>Capital Allocation Line</u>.

Investment Opportunity Set: the Capital Allocation Line

- Example:
  - Risky asset, US stock market:

    $$E[R_{US}] \;=\; 12\%, \sigma_{US} = 20\%$$

  - Risk-free, US T-bill:

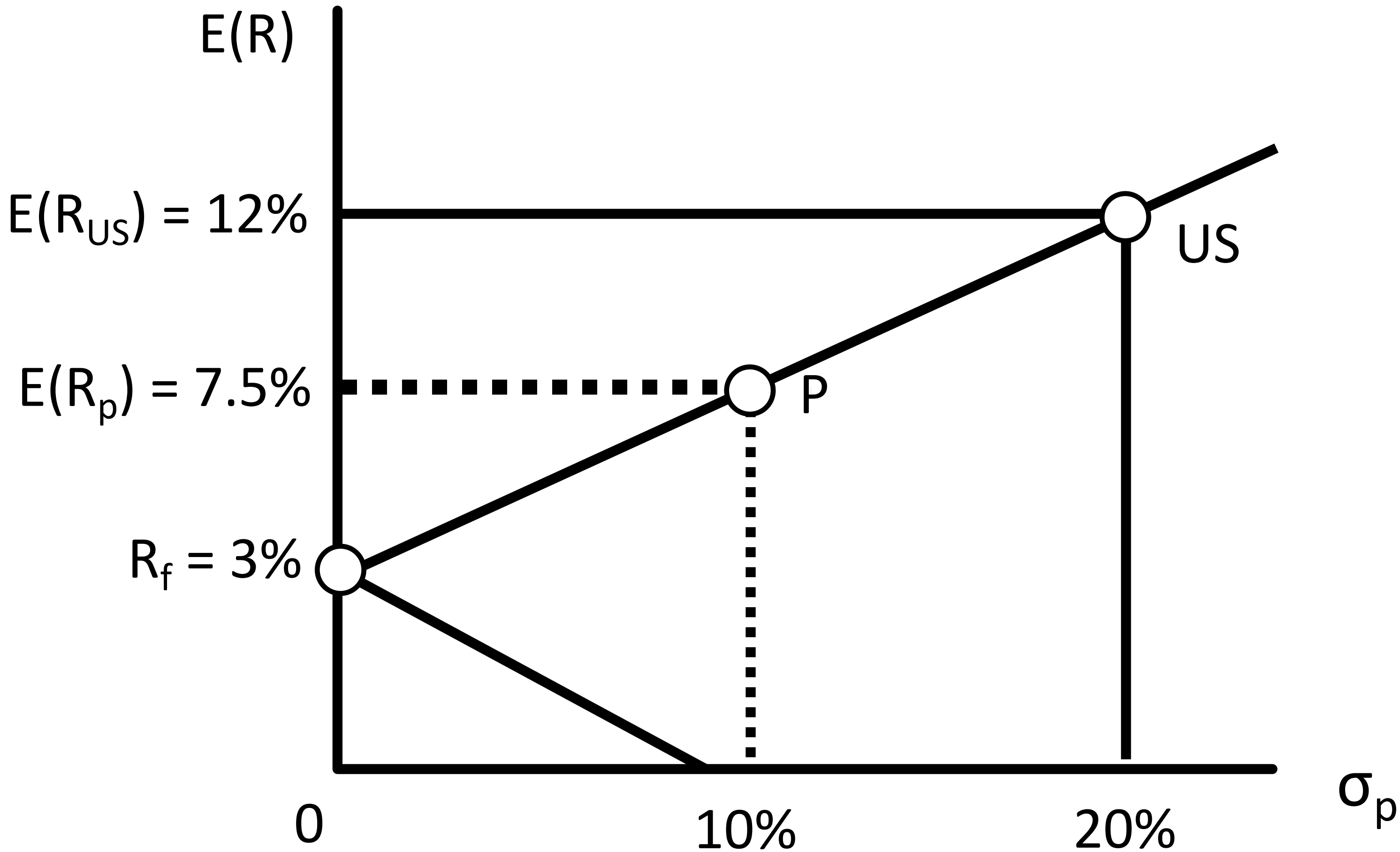
    $$R_f = 3\%$$

- Hence: $E\left[R_p\right] = R_f + SR \times \sigma_p$

$$SR = \frac{E[R_{US}] - R_f}{\sigma_{US}} = \frac{0.12 - 0.03}{0.2} = 0.45$$

- Sharpe Ratio (SR) = return premium per unit of risk

Investment Opportunity Set: the Capital Allocation Line



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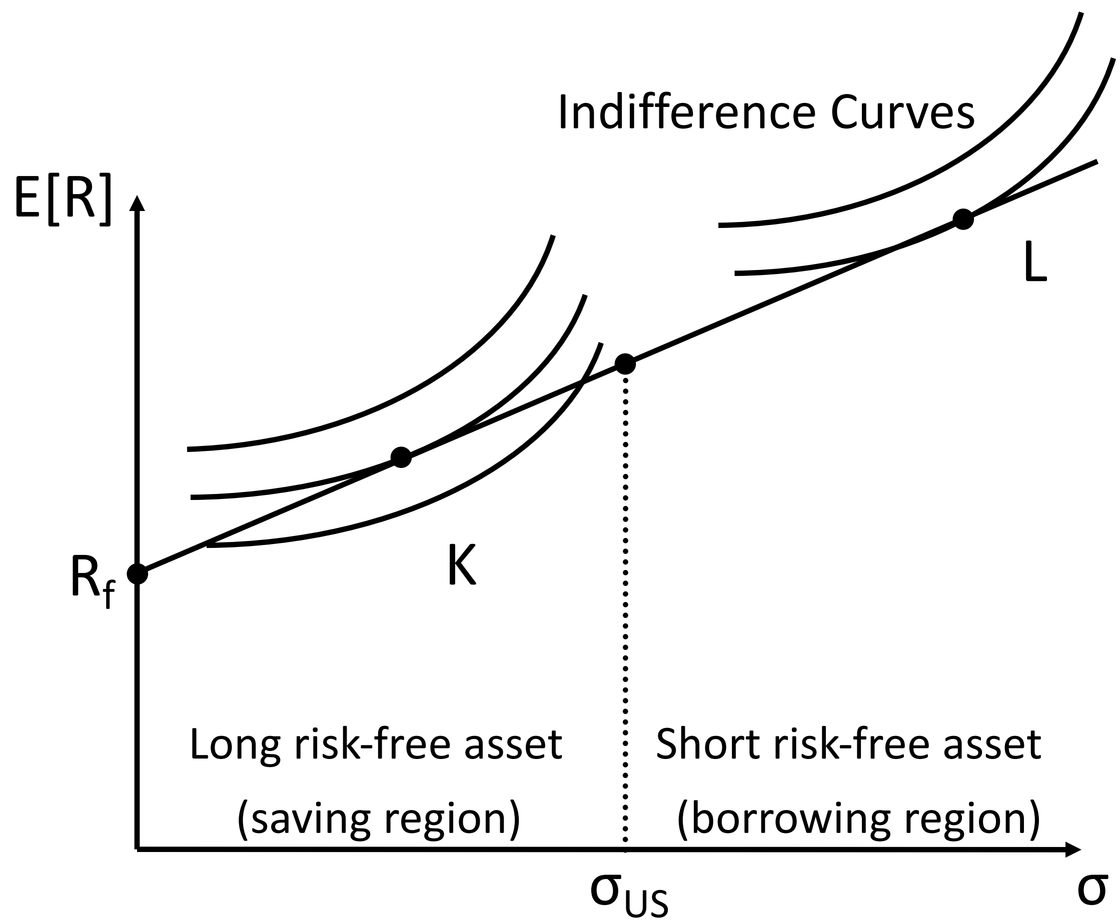
$$R_f = 3\%,\ E[R_{US}] = 12\%, \sigma_{US} = 20\%$$

1. How can I get a 15.6% expected return by investing in US stocks and the risk free asset?

2. What is the standard deviation of this portfolio?

Optimal Portfolio Choice with a risk-free and a risky asset



Portfolio K:

Optimal choice for risk-averse investor

Long in the risky asset
Long in the risk-free asset

Portfolio L:

Optimal choice for risk-tolerant investor

Short risk-free asset (=borrow) to buy risky asset “on margin”