H3
Continuous Compounding: Some Basics

Foundations of Finance

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Because you may encounter continuously compounded growth rates elsewhere, and because you will encounter continuously compounded discount rates when we examine the Black-Scholes option pricing formula, here is a brief introduction to what happens when something grows at r percent per annum, compounded continuously. We know that as $n \to \infty$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \quad (1)$$

$$= 2.71828183\ldots. \quad (2)$$

In our context, this means that if \$1 is invested at 100% interest, continuously compounded, for one year, it produces \$2.71828 at the end of the year.

It is also true that if the interest rate is r percent, then \$1 produces e^r dollars after 1 year. For example, if $r = 0.06$ we have

$$\$1 \times e^{0.06} = \$1.0618365.$$

After two years, we would have:

$$e^{0.06} \times e^{0.06} = e^{0.06\times 2} = \$1.127497.$$

More generally, investing $\$P$ at $r\%$, continuously compounded, over t years, produces (grows

*Based on the notes of Profs. Alexi Savov and William Silber.

to) the amount F according to the following formula:

$$Pe^{rt} = F. \tag{3}$$

For example, \$100 invested at 6%, continuously compounded, for 5 years produces

$$\$100 \times e^{0.06\times 5} = \$134.98588$$

We can use equation (3) to solve for the present value of F dollars paid after t years, assuming the interest rate is r percent, continuously compounded. In particular,

$$P = \frac{F}{e^{rt}}$$

or

$$P = Fe^{-rt}. \tag{4}$$

The term e^{-rt} in expression (4) is nothing more than a discount factor like $1/\left(1+r\right)^t$, except that r is continuously compounded (rather than compounded annually). For example, suppose $r = 0.06$ and $t = 1$.

$$\frac{1}{\left(1+r\right)^t} = \frac{1}{\left(1+0.06\right)^1} = 0.9434 \tag{5}$$

$$e^{-rt} = e^{-0.06\times 1} = 0.9417. \tag{6}$$

This last result is slightly surprising. Why is the present value of \$1 less (0.9417) under continuous compounding compared with annual compounding (0.9434)? The answer is: With a fixed dollar amount (\$1) at the end of one year, continuous compounding allows you to put away fewer dollars (0.9417 rather than 0.9434) because it grows at a faster (continuously compounded) rate.

A note on EAR: It is quite straightforward to calculate the EAR if you are given a continuously compounded rate. We saw above that \$1 compounded continuously at 6% produces \$1.061836 at the end of one year:

$$e^{0.06} = 1.061836.$$

Subtracting one from the right hand side of the above shows that a simple annual rate (without compounding) of 6.1836% would be equivalent to 6% continuously compounded.

And that is what we mean by the EAR.

What if you were told that the annual rate without compounding was 6%, could you derive the continuously compounded rate that produces a 6% EAR? The answer is given by solving the following expression for x:

$$e^x = 1.06.$$

Taking the natural log (ln) of both sides produces:

$$x = \ln(1.06) = 0.0582689.$$

Thus, 6% simple interest is equivalent to 5.82689% continuously compounded. In general, taking the natural log of one plus a simple rate produces the corresponding continuously compounded rate. File away this last point until we discuss options towards the end of the semester.