H2
Annuities and Perpetuities: Present Value

Foundations of Finance

Prof. Olivier Wang*

I. The present value of an annuity, PV, can be written as the sum of the present values of each component annual payment, C, as follows:

$$PV = \frac{C}{1+r} + \frac{C}{(1+r)^2} + \cdots + \frac{C}{(1+r)^t}, \tag{1}$$

where r is the single average interest rate per annum and t is the number of years the annuity is paid. This can be simplified as follows:

$$PV = C\left[\frac{1}{1+r} + \frac{1}{(1+r)^2} + \cdots + \frac{1}{(1+r)^t}\right]. \tag{2}$$

Using a formula for the sum of a geometric progression (as long as $r > 0$), we have:

$$PV = C\left[\frac{1 - \frac{1}{(1+r)^t}}{r}\right], \tag{3}$$

which is the same as:

$$PV = C\left[\frac{1}{r} - \frac{1}{r(1+r)^t}\right]. \tag{4}$$

II. Thus if you have a three-year annuity ($t = 3$) that pays \$100 per annum ($C = \$100$)

*Based on the notes of Profs. Alexi Savov and William Silber.

and the average annual interest rate, is $r = 6\%$, then from equation (4), we have:

$$PV = \$100\left[\frac{1}{0.06} - \frac{1}{0.06\,(1+0.06)^3}\right] = \$267.30.$$

You can check that this is correct by calculating as in equation (1):

$$PV = \frac{\$100}{1+0.06} + \frac{\$100}{(1+0.06)^2} + \frac{\$100}{(1+0.06)^3} = \$267.30.$$

III. More interesting is what happens to the present value formula when the annual payments, C, continue forever. The annuity becomes a perpetuity as $t \to \infty$ and the formula in (4) becomes:

$$\begin{aligned} PV &= \lim_{t\to\infty} C\left[\frac{1}{r} - \frac{1}{r\,(1+r)^t}\right] \qquad (5) \\ &= \frac{C}{r}. \qquad (6) \end{aligned}$$

(To get the last line, note that $(1+r)^t$ gets bigger and bigger with t since $r > 0$. As $t \to \infty$, $(1+r)^t \to \infty$ and so $1/(1+r)^t \to 0$.)

Equation (6) is very simple. It says that the present value of an annuity of C dollars per annum is C divided by r, where r is the average interest rate per annum. This makes considerable sense once you provide a numerical example. Suppose $C = \$10$ per annum and the interest rate is $5\% = 0.05$. How many dollars, designated by the letter P, would you have to put away today so that it produces \$10 every year forever? The answer is given by solving the following equation for P:

$$\begin{aligned} P \times 0.05 &= \$10 \\ P &= \frac{\$10}{0.05} = \$200. \end{aligned}$$

Investing \$200 at 5% generates \$10 in interest per year and continues to do so forever. Thus, if an annuity promises to pay \$10 forever and the annual interest rate is 5%, the value of that infinite stream of payments is \$200. If the annuity were priced in a competitive market its price should be \$200.