The oldest open problem in mathematics

NEU Math Circle, December 2, 2007, Oliver Knill

Perfect numbers

The integer $n = 6$ has the proper divisors $1, 2, 3$. The sum of these divisors is 6, the number itself. A natural number n for which the sum of proper divisors is n is called a **perfect number**. So, 6 is a perfect number.

All presently known perfect numbers are even. Here are the smallest 12 perfect numbers:

$$6, 28, 496, 8128, 33550336, 8589869056, 137438691328$$
$$2305843008139952128, 2658455991569831744654692615953842176$$
$$191561942608236107294793378084303638130997321548169216$$
$$13164036458569648337239753460458722910223472318386943117783728128,$$
$$14474011154664524427946373126085988481573677491474835889066354349131199152128$$

Whether there are odd perfect numbers is the oldest known open problem in mathematics: [10]:

Is there an odd perfect number?

Also unknow is the answer to the question:

Are there infinitely many perfect numbers?

One often abbreviates the term **odd perfect number** with OPN. Define the function $\sigma(n)$ as the sum over all divisors of n including n. A number n is **perfect**, if $\sigma(n) = 2n$.

If $n = p$ is a prime number, then the only divisors are 1 and p. Therefore $\sigma(p) = 1 + p$. More generally, if $n = p^k$ is a power of a prime number p, then

$$\sigma(p^k) = 1 + p + p^2 + \cdots + p^k = \frac{p^{k+1} - 1}{p - 1} .$$

Problem 1.1 Verify that 28 is a perfect number.

Problem 1.2 Verify in the case $18 = 2 \cdot 3^2 = p^k q^l$ that the sum $\sigma(n)$ of all divisors satisfies the formula

$$\sigma(n) = (1 + p + p^2 + \cdots + p^k)(1 + q + q^2 + \cdots + q^l) .$$

Moral: Open problems can last for millenia.

The σ-function

If $n = ab$, where a and b have no common divisor, every divisor of n is the product of a divisor of a and a divisor of b. In other words

$$\sigma(ab) = \sigma(a)\sigma(b) \ .$$

A function satisfying this relation for all pairs (a, b) with no common divisor is called a **multiplicative function**.

Here are more examples of multiplicative functions:

- $\iota_s(n) = n^s$
- $\phi(n)$, the Euler phi function counting $\{1 \leq m \leq n \mid \gcd(m, n) = 1 \}$.
- $\sigma_s(n) = \sum_{d|n} d^s$, where $s \in \mathbf{R}$ genearalizes $\sigma(n) = \sigma_1(n)$.
- $\tau(n) = \sigma_0(n)$ the number of prime factors of n.

Using the multiplicativity property, we get now a formula for $\sigma(n)$, where $n = p_1^{k_1} p_2^{k_2} \cdots p_l^{k_l}$.

$$\sigma(n) = \frac{p_1^{k_1+1} - 1}{p_1 - 1} \cdots \frac{p^{k+l} - 1}{p_l - 1}$$

Here is a simple program in Mathematica which finds all perfect number below one million.

```
T[l_]:=DivisorSigma[1,l]-l; k=0; Do[If[T[l]==l,k++; Print[{k,l}]],{l,10^6}]
```

Note that the 12'th largest perfect number has already 76 digits. The above program would never find it. Our universe is believed to be 10^{17} seconds old. Even if we checked a billion numbers per second since then, we would still reach only have reached numbers with 26 digits.

Problem 2.1. Verify the multiplicative property of the σ function by hand for $n = 6 \cdot 35$ that $\sigma(210) = \sigma(6)\sigma(35)$.

Problem 2.2. Verify the geometric summation formula

$$1 + p + p^2 + \cdots + p^k = \frac{p^{k+1} - 1}{p - 1} \ .$$

Moral: Algebraic insight allows to come up with concrete formulas. Brute force search for perfect numbers is pointless for large numbers.

Modular considerations

Putnam [26] attributes the following formula to Bourlet in 1896:

Show that if n is a perfect number, then $\sum_d \frac{1}{d} = 2$, where the sum is over all factors d of n.

Proof. Divide $\sum_d d = 2n$ by n. This gives $\sum_d \frac{d}{n} = 2$, but if d is a factor, then also n/d is a factor.

One defines more $\sigma_s(n) = \sum_{d|n} d^s$, where d runs over all divisors of n. We have $\sigma(n) = \sigma_1(n)$. The previous computation also shows $\sigma_{-1}(n) = \frac{\sigma(n)}{n}$. This function is called the **index function**.

An other consequence is is the statement [15] that if n is a non-square perfect number, then $\sum_{d<\sqrt{n}}(d + \frac{n}{d}) = 2n$.

Proof. Divisors of n come in two classes. Either $d < \sqrt{n}$ or $d > \sqrt{n}$. In the second case, we can write it as $d = n/d'$, where d' is a divisor smaller than $\sqrt{n}$.

No perfect number n can be of the form $6k - 1$. [15]

Proof. If $n = 6k - 1$, then $n = -1$ modulo 3 and $2n = 1$ modulo 3. If d is a divisor of n, then $d(n/d) = n$ shows that $d = 1$ modulo 3 and $n/d = -1$ modulo 3 or $d = -1$ modulo 3 and $n/d = 1$ modulo 3. In any case $d + \frac{n}{d} = 0$ modulo 3 and

$$\sigma(n) = \sum_{d<\sqrt{n}} (d + \frac{n}{d})$$

is divisible by 3. But $2n = 1$ modulo 3.

An OPN n is of the form $n = 12m + 1$ or $n = 36m + 9$. (Touchard, 1953)

Proof. n is congruent to 1 modulo 4 and according to the previous result congruent to 1 or 3 modulo 6. Every solution to $x = 1(4), x = 1(6)$ is of the form $12m + 1$. Every solution to $x = 1(4), x = 3(6)$ is $12m + 9$. If 3 divides m, then 3 divides n and $\sigma(n) = \sigma(3)\sigma(4m + 3)$ so that $\sigma(n) = 2(mod4)$ while $2n = 2(12m + 9) = 2(mod4)$ which is not possible. Therefore 3 does not divide m and n has the form $36m + 9$. [15] ([27] mentions [28]).

Problem 3.1. Is it possible that $n = p_1p_2 \cdots p_k$ is a perfect number, where the numbers p_i are all different primes?

Problem 3.2. Show that a square number can not be a perfect number.

Moral: It is possible to exclude classes of numbers from being perfect.

History

J. O'Connor and E. Robertson tell in their online article [21] that it is not known who studied perfect numbers first but that "the first studies may go back to the earliest times when numbers first aroused curiosity." Perfect numbers were definitely known by Pythagoras and his followers. It seems that they were interested in their mystical properties. The name "perfect" was introduced by the Pythagoreans [18] and there are speculations that there could be religious or astrological origins: because the earth was created in 6 and the moon needs 28 days to circle the earth, mystical associations are natural. Perfect numbers were also studied by the early Hebrews: Rabbi Josef ben Jehuda Ankin in the 12'th century recommended their study in the book "Healing of Souls" [7, 18].

The Greek mathematician **Euclid of Alexandria** (300-275 BC) saw that if $2^p - 1$ is prime then $2^{p-1}(2^p - 1)$ is a perfect number. It needed almost 2000 years to prove that these are all the even perfect numbers.

Around 100 AD, **Nicomachus of Gerasa** (60-120) gave in the book "Introduction to Arithmetic" a classification of numbers based on the concept of perfect numbers. Nicomachus lists the first four perfect numbers. He also defines **superabundant numbers** $\sigma(n) > n$ and **deficient numbers** $\sigma(\mathbf{n}) < \mathbf{n}$. Also the Greek philosopher **Theon of Smyrna** (70-135) around 130 AD distinguished between perfect, abundant and deficient numbers.

In the second millennium, many mathematicians studied perfect numbers: **René Descartes**, **Leonard Euler**, **James Joseph Sylvester** (1814-1897) were among the first to do so seriously.

Problem 4.1. If p is a prime and k is an integer then p^k can not be perfect.

Problem 4.2. If p, q are two primes and k, l are integers, then $p^k q^l$ can not be perfect.

Moral: A lot of history of mathematics still needs to be explored.

The index function

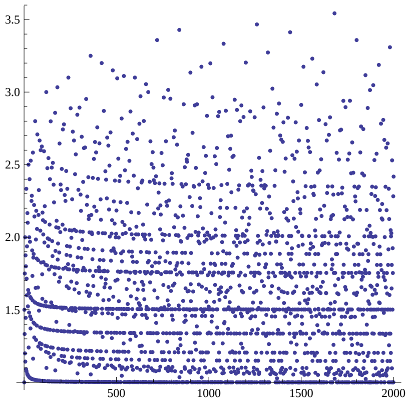
The multiplicative function

$$h(n) = \frac{\sigma(n)}{n}$$

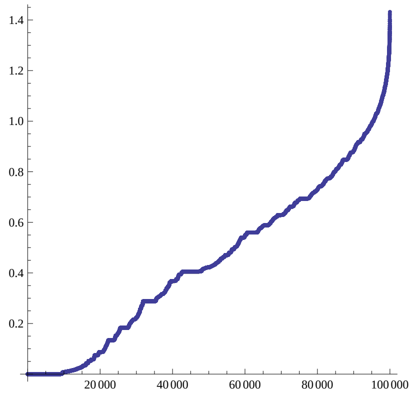
is called the **index** of n. A number n is perfect if and only if $h(n) = 2$. The function h takes values larger than 1 for all $n > 1$. If $h(n) < 2$, the number is called **deficient**, if $h(n) > 2$, the integer n is called **abundant**. The function h is an unbounded function. For every c there is a n for which $h(n) > c$: Take $n = 235711...p_k$, the product of the first k primes. Now

$$\frac{\sigma(n)}{n} = (1 + \frac{1}{2})(1 + \frac{1}{3})(1 + \frac{1}{5}) \dots (1 + \frac{1}{p_k}) \; .$$

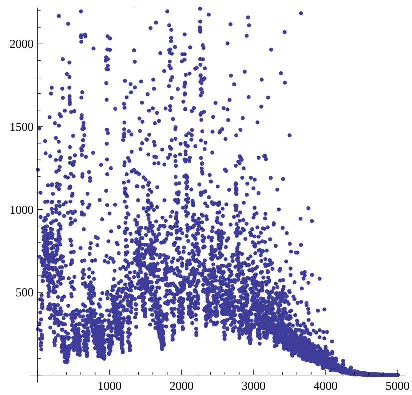
This can be bounded below by $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{p_k}$, a sum which grows like $\log(\log(k))$. One can also see it as follows: if one evaluates $1/h(n)$ where n is the product $n(k) = \prod_{j=1}^{k} p_j^k$. then $h(n(k)) \to \prod_p (1 - \frac{1}{p})^{-1} = \zeta(1) = \infty$. The identity $\prod_p (1 - \frac{1}{p^s})^{-1} = \zeta(s) = \sum_n \frac{1}{n^s}$ a formula which is called the **golden key** which can be found on the bookcover of [6]. The golden key has been used by Euler to show the divergence of the sum $\sum_p \frac{1}{p}$. How can we find the limiting distribution function? Is more and more weight going to numbers which have a large number of prime factors? An integer n is called **index champion** if $h(n) > h(k)$ for all $h < n$. The first index champions are 2, 4, 6, 12, 24, 36 ...



The index function $h(n) = \sigma(n)/n$.



The distribution function of h.



The density function of h.

$$\prod_p \frac{p+1}{p} \leq h(n) \leq \prod_p \frac{p}{p-1}$$

Proof. The left hand side is for single prime factors, the right hand side for the full geometric series.

Problem 5.1. The function h takes values arbitrarily close to 1.

Problem 5.2. There are infinitely many n with $h(n) > 2$.

Moral: We only have to understand one function ...

Even perfect numbers

Theorem (Euclid) A number $(2^n-1)2^{n-1}$ is a perfect prime if 2^n-1 is prime.

Proof. $\sigma(2^n-1)2^{n-1}) = \sigma(2^n-1)\sigma(2^{n-1}) = 2^n(2^n-1) = 22^n(2^n-1)$.
The perfect numbers form **Sloane sequence** A000396.

$$6, 28, 496, 8128, 33550336, 8589869056, 137438691328,$$
$$2305843008139952128, 2658455991569831744654692615953842176,$$
$$191561942608236107294793378084303638130997321548169216...$$

Theorem (Euler) All even perfect numbers n are of the form $(2^p-1)2^{p-1}$, where 2^p-1 is prime.

Proof. Write $n = 2^k m$ where m is odd. From the multiplicativity:

$$\sigma(n) = (2^{k+1}-1)\sigma(m) = 2^{k+1}m$$

where the second equality uses the perfectness $\sigma(n) = 2n$. The equation shows that $2^{k+1}-1$ is a factor of m and so $m = (2^{k+1}-1)M$ implying

$$\sigma(m) = 2^{k+1}\sigma(M) \ .$$

We show now $M = 1$ and m is prime. Assume it is false. Then, because m and M are divisors of M and because there is an other divisor, we have $\sigma(m) > m + M$ and $m + M < \sigma(m) = 2^{k+1}M$ and $m < (2^{k+1}-1)M$ which contradicts $m = (2^{k+1}-1)M$. This contradiction establishs the truth of $M = 1$ and that m is prime.

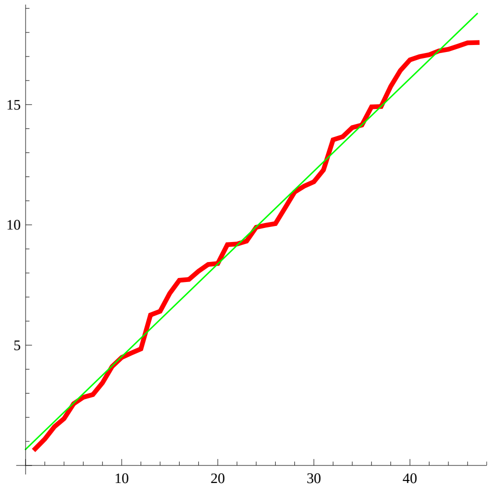
Primes of the form $a^n - 1$ are called **Mersenne primes**. You will verify that necessarily $a = 2$. The first 39 Mersenne primes are n = 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521, 607, 1279, 2203, 2281, 3217, 4253, 4423, 9689, 9941, 11213, 19937, 21701, 23209, 44497, 86243, 110503, 132049, 216091, 756839, 859433, 1257787, 1398269, 2976221, 3021377, 6972593, 13466917. There are 5 more known, but it is not clear whether there is one between the previous 39 and those: n = 20996011, 24036583, 25964951, 30402457, 32582657. **Update 2008:** two more have been added 37156667, 43112609. **Update 2009:** one more is known 242643801. One does not know whether there are infinitely many even perfect numbers.

Problem 6.1. Verify that every Mersenne prime a^n-1 is of the form 2^p-1 where p is a prime.

Problem 6.2. Show that every even perfect number is a triangular number $n(n+1)/2$.

Moral: Beating records can motivate.

Predicting Mersenne primes



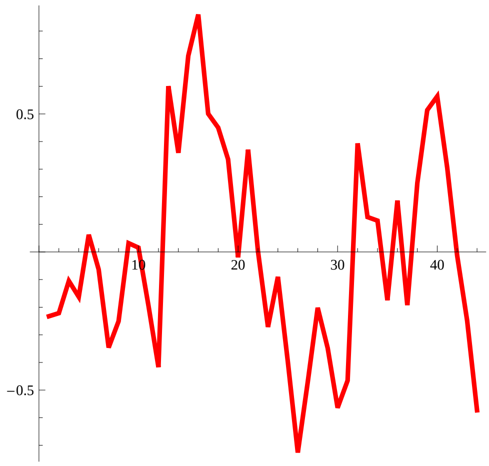


Figure. The growth of the prime exponents n for which $2^n - 1$ is prime. It is about $g(n) = 1.70212e^{0.394108n}$. The pictures shows the growth of the known Mersenne prime exponents in a logarithmic scale fitted by the best linear fit. The right picture shows the deviation from the predicted growth.

Experimentally, one observes that $2^p - 1$ never has a square factor. In other words, $\mu(2^p - 1) \neq 0$. Nobody knows whether this is true or not. [13]

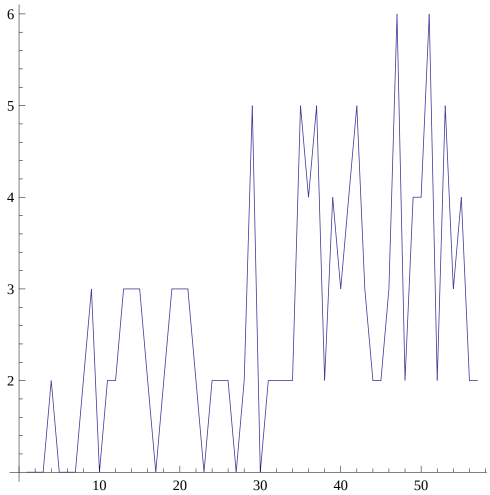


Figure. The number of prime factors of $2^p - 1$ for the first odd 54 primes p. No case has been observed where $2^p - 1$ has a square factor different from 1. The currently largest known prime number has 9,808,358 digits. The Electronic Frontier Foundation EEF offers a 100'000 award for the first 10 million digit prime. The GIMPS project is working on that.

The electronic frontier foundation (EFF) has promised 100'000 dollars for the first prime with more than 10 million digits. Lets try to estimate with a statistical prediction, where the next prime will be. If we use the above linear approximation of the exponential growth, we would predict that the 45'th Mersenne number has 25.8 Million digits Why not start searching near $p = 85731913$, the prime closest to the prediction? The reason is that we are still searching a needle in a hay stack and that the linear prediction gives not much information for an exponentially growing sequence. **Added: Jun 13, 2009** In the mean time, 47 Mersenne Primes are known: $2^{242643801} - 1$ is the largest known.

Problem 7.1. Estimate the size of the second next perfect number.

Moral: Even perfect numbers appear to come with statistical regularities.

Quotes

- **Leonard Euler**: Whether ... there are any odd perfect numbers is a most difficult question.
- **Chris Caldwell** in [2]": "This is probably the oldest unsolved problem in all of mathematics."
- **Jay Goldman** in [10]: "Probably the oldes unsolved problem in number theory and possibly in mathematics."
- **Bressoud and Wagon** say in their book on page 75: "It is one of the oldest unsolved mysteries of mathematics, as it goes back to the ancient Greeks." [1]
- **T.M. Putnam**. "It is a problem of much historic interest."
- **J. J. Sylvester**, who worked on the problem for a long time came to the conclusion: "The existence of an odd perfect number – its escape, so to say, from the complex web of conditions which hem it in on all sides – would be little short of a miracle."
- **Conway and Guy** write: "There probably aren't any!" [4]
- **Stan Wagon** writes in [32] "Maybe some simple combination of a dozen or so primes in fact yield an odd perfect number!"
- **P. Ribenboim**: "This is a question which has been extensively searched, but its answer is still unknown." [27] and
- **P. Ribenboim**: "Yet, I believe the problem stands like a unconquerable fortress. For all that is known, it would be almost by luck that an odd perfect number would be found. On the other hand, nothing that has been proved is promising to show that odd perfect numbers do not exist. New ideas are required." [27]
- **A. Beiler**: "Man ever seeks perfection but inevitably it eludes him. He has sought perfect numbers through the ages and has found a very few - twenty-three up to 1964". (cited in [24]).
- **C. Pickover"** "They may remain forever shrouded in mystery". "Throughout both ancient and modern history the feverish hunt for perfect numbers became a religion."

It is not clear who first explicitly asked about the existence of odd perfect numbers. Descartes wrote in a letter to Mersenne that every even perfect number is of Euclid's form and that he sees no reason why an odd perfect number could not exist. Before Descartes, people often seem have assumed that all perfect numbers are even [7] p 6-12.

Problem 8.1. How would YOU summarize the problem of odd perfect numbers in one centence?

Moral: While its unlikely that odd perfect numbers exist, there are no reasons, why they would not.

Experimental mathematics

An odd perfect number has

- more than 300 digits
- at least 75 prime factors
- at least 9 distinct prime factors
- the largest prime factor must have at least 20 digits

Dickson proved in 1911 that there are only finitely many perfect numbers with k prime factors [14].

There are explicit bounds on the size:

10^6	Turcaninov	1908	[9]
10^{20}	Kanold	1957	[20]
10^{36}	Tuckerman	1973	[20]
10^{50}	Hagis	1973	[20]
10^{160}	Brent/Cohen	1989	[20]
10^{300}	Brent et al	1991	[20]

Here are the known bounds of the least number of distinct prime factors:

3	Nocco	1863	[32]
4	B. Peirce	1830	[9]
5	Sylvester	1888	[9]
6	Gradshtein	1925	[20]
7	Robbins/Pomerance	1972	[32]
8	Chein	1979	[9]
8	Hagis	1980	[20]
9	Nielsen	2006	[20]

Here are the known lower bounds on the number of prime factors

47	Hare	2004	
75	Hare	2005	[20]

The opinions of whether there are odd perfect numbers vary. The search for odd perfect numbers is called "OPN research".

Much work has been done in estimating the size of odd perfect numbers. This research has shown that they must be huge. It looks impossible to search for it randomly.

The area of computational number theory reassembles very much the work of an experimental physicist. Instead of particle accelerators, which search for an elementary particle like the Higgs particle, or telescopes which search for planets in other solar systems, there are computers which look for numbers like odd perfect numbers. In both cases, one does not know whether the search will be successful.

Problem 9.1. How would you estimate the chance to get an odd perfect number by guessing?

Moral: The computer is the particle accelerator of experimental mathematicians.

Nocco Numbers

In an earlier exercice you have proven a result of Nocco in 1863:

An odd perfect number has more than two prime divisors.

Proof. $\sigma(p^a p^b)/(p^a p^b) = (1+p+p^2+\ldots+p^a)(1+q+q^2+\ldots+q^b)/(p^a p^b) \leq (1+1/p+1/p^2+ \ldots 1/p^a)(1+1/q+1/q^2+\ldots 1/q^b) \leq (1-1/p)^{-1}(1-1/q)^{-1} \leq (3/2)(5/4) < 2$.
In general, if $\prod_{p|n}(1-\frac{1}{p}) > 1/2$, we can not be a perfect number. Note that $\prod_p(1-\frac{1}{p}) = 1/(\sum_{k=1}^n 1/n) = 0$. How close can we get? Let P_k be the set of natural numbers which have exactly k different prime factors. Define the**Nocco numbers** as **Nocco**$(k) = \inf_{n \in P_k} |h(n)-2|$.
Examples:
Nocco$(1) = 1/2$ because $n = 3^k$ leads to the asymptotic value $1/(1-1/3) = 3/2$.
Nocco$(2) = 1/8$ because $n = 3^k 5^l$ gives us the closest asymptotic value $(3/2)(5/4) = 15/8$. We can not make this larger because these are the smallest primes.
Nocco$(3) = 2/6075$ for $n = 3^5 5^2 13$ Proof. We have $\sigma(n)/n > 2 - 1/8$ only for prime triples $(3,5,7), (3,5,11), (3,5,13)$ now check all powers. There are only finitely many to look at.
To find **Nocco**(4), we first find all quadruples (p_1,p_2,p_3,p_4) for which $f(p_1,p_2,p_3,p_4) = (1/(1-1/p_1))(1/(1-1/p_2))(1/(1-1/p_3))(1/(1-1/p_4)) > 2$. Without $p = 3$, it can not work since $f(5,7,11,13) = 1001/576$; So, we have to look at all p_2,p_3,p_4 such that $f(p_2,p_3,p_4) > 2/(1/(1-1/3)) = 4/3$.

Is Nocco$(k) > 0$ for all nonperfect numbers k?

It is not yet excluded that 2 as a limiting case with a finite number of primes. $Rp/(p-1) = 2$ implies $\prod \frac{p_i^{k_i+1}-1}{(p_i-1)p_i^{k_i}} = 2(p-1)/p$ which means $\prod_i p(p_i^{k_i+1}-1) = 2(p-1)\prod_i p_i^{k_i}(p_i-1)$. Estimates for **Nocco**(n) with $n = 6,7,8$ are cited in [31], The **Nocco**(5) estimates have been established by [16]:

$$Nocco(5) \leq h(3^{25} * 5^5 * 17^7 * 251 * 570407) - 2$$

$$Nocco(6) \leq h(3^2 * 7^10 * 11^8 * 13^1 * 541^4 * 291457^2) - 2$$

$$Nocco(7) \leq h(3^4 * 5^1 * 11^18 * 73^{10} * 2647^6 * 348031^4 * 24133257823^2) - 2$$

$$Nocco(8) \leq h(3^2 * 7^32 * 11^26 * 13^1 * 541^{10} * 291619^6 * 4475618963^2 * 399988521576257^2)$$

Almost perfect numbers are numbers for which $|\sigma(n) - 2n| = 1$. They produce close hits because $|\sigma(n)/n - 2| < 1/n$ in that case.

Problem 10.1. Find an upper bound for Nocco(8).

Problem 10.2. Show that Nocco$(k) \to 0$ for $k \to \infty$.

Moral: By refining a problem, one can continue to work at an impossibly hard problem.

Hunting for odd perfect numbers

Let p_i denote the i'th prime number. For fixed n, consider an integer vector $x = (x_1, x_2, ..., x_n)$. We want to find x so that

$$f(x) = h(p_1^{x_1} \cdots p_n^{x_n}) - 2 = 0 .$$

For smooth functions f, the **Newton method** or **gradient method** to find roots of functions. Replace x with $T(x) = x - f(x)f'(x)/|f'(x)|^2$ to get closer to a root. The idea of this method is very simple. The gradient gives the direction in which to go to decrease the function best. Can we adopt this to find perfect numbers? Here are some ideas:

- Do the Newton iteration step with LLL [3]
- Find for every $\{p_1, ..., p_n\}$ the best approximation.
- Change $x \to y$ on one coordinate until we have the best fit. Then change two coordinates until we have the best fit etc. This corresponds to a random walk with increasing jump sizes.
- Possibly add new primes to have a good fit.
- Use some theory to find good initial starting distribution. Find for each p, the k such that $\log((p^{k+1} - 1)/(p - 1) \sim \log(2)^{1/n} p$ where n are the number of primes.

What is known about OPN?

Theorem (Euler) Every OPN is the product of a prime power p^{4k+1} and a square.

Proof. We compare the number of factors 2 in the equation $\sigma(n) = 2n$. The right hand side has only one factor 2. On the left hand side we have the sum of the number of 2 factors $\sigma(p^k)$ for each prime k which appears in n. For odd k we have at least one factor. So, only one factor p^k can be an odd power. The number of 2 factors of $\sigma(p^k)$ is larger than 1 if k is of the form $4k + 3$. See the project [29].

Problem 11.1. Show that $\sigma(p^m)$ is divisible by 4 if $m = 4k - 1$.

Problem 11.2. Any perfect odd perfect number with 8 prime factors must be divisible by 3 or 5.

Moral: Some modular considerations allow to narrow down the search.

Pseudo perfect numbers

Descartes found an example of an odd perfect number which almost worked. Can you find a flaw in the following proof that odd perfect numbers exist?
Claim: The number $n = 3^2 \cdot 7^2 \cdot 11^2 \cdot 13^2 \cdot 22021$ satisfies

$$\begin{aligned} \sigma(n) &= (1+3+3^2)(1+7+7^2)(1+11+11^2)(1+13+13^2)(1+22021) \\ &= 397171152378 = 2n \; . \end{aligned}$$

Did we find an odd perfect number? This exercise appears both in the book of Klee and Wagon [17] as well as the book of Bressoud and Wagon [1]. The origin of the exercise is a a letter of Descartes to Mersenne in 1638 in which he writes:

... I think I am able to prove that there are no even numbers which are perfect apart from those of Euclid; and that there are no odd perfect numbers, unless they are composed of a single prime number, multiplied by a square whose root is composed of several other prime number. But I can see nothing which would prevent one from finding numbers of this sort. For example, if 22021 were prime, in multiplying it by 9018009 which is a square whose root is composed of the prime numbers 3, 7, 11, 13, one would have 198585576189, which would be a perfect number. But, whatever method one might use, it would require a great deal of time to look for these numbers...
Source: M Crubellier and J Sip, Looking for perfect numbers, History of Mathematics: History of Problems (Paris, 1997), 389-410
This riddle inspires the following question: given a random set P of odd numbers which has the same asymptotic as the prime numbers. Look for a sequence of such numbers $p_1^{k_1}, .., p_l^{k_l}$ and define the index as before. What is the probability to get such a **pseudo perfect number**?

To have pseudo perfect numbers, we need (not necessarily primes) p_i and integers n_i such that

$$\prod_i (1 + \frac{1}{p_i} + \frac{1}{p_i^2} + \ldots + \frac{1}{p_i}^{n_i}) = 2 \; .$$

Can we realize such numbers with $n_i = 1$? We call them "simple". In other words, are there integers p_i such that $\prod(1 + 1/p_i) = 2$ which means $\prod_i (p_i + 1) = \prod_i 2p_i$. Because

$$(1 + 1/3)(1 + 1/4)(1 + 1/5) = 2 \; .$$

the integer $3 * 4 * 5 = 60$ is a simple pseudo perfect number. It is even.

Problem 12.1. Solve Descartes riddle.

Problem 12.2. Show that there are no simple odd pseudo perfect numbers.

Moral: Its easy how to disguise an implicit assumption in a proof.

Ore Harmonic numbers

The **harmonic mean** of k numbres $n_1, ..., n_k$ is $\frac{k}{\frac{1}{n_1}+\frac{1}{n_k}}$. The harmonic mean of 1 and 2 for example is $2/(1+1/2) = 4/3$. An integer is called an **Ore harmonic number** or **harmonic divisor numbers** if the harmonic mean of its divisors is an integer. (The word **harmonic number** has been used for real numbers $1+1/2+1/3+...+1/n$ already and should not be used.) For example, the number 140 is an Ore harmonic number because the harmonic mean of its divisors is 5. Oystein Ore (1899-1968), a Norwegian mathematician introduced harmonic numbers in 1948 [23, 22].

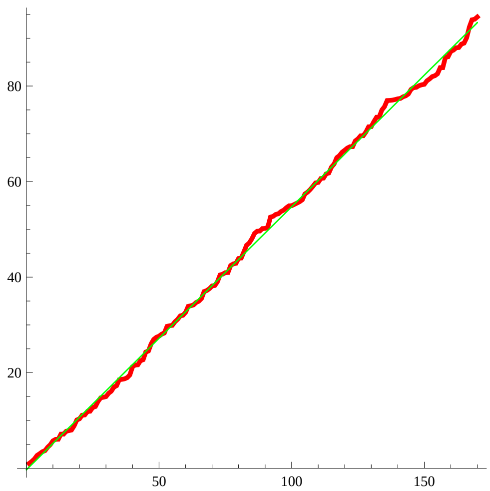
Theorem of Ore: Every perfect number is harmonic.

Proof. Assume the harmonic mean of the divisors n_j of n is equal to H and the arithmetic mean of the divisors is A. Then $AH = n$ because every divisor d also n/d is a divisor implies that $A = \frac{1}{k}(\frac{1}{n_1} + \cdots + \frac{1}{n_k})$. If n is perfect then

$$2n = (\frac{1}{n_1} + \cdots + \frac{1}{n_k}) = kA = \frac{kn}{H}$$

so that for a perfect number $H = k/2$, where k is the number of divisors. The number of divisors of a non-square number is even so that H is an integer. No perfect number can be a square number because the sum of the divisors of a square number n is odd and can not be $2n$.
There are more harmonic numbers than perfect numbers but it is also not known whether there is an odd harmonic number.

The list of harmonic numbers 1,6,28,140,270,496,672,1638,2970,6200,8128,8190, 18600,18620,27846,30240,32760,55860,105664,117800, 167400,173600,237510,242060,332640,360360,539400, 695520,726180,753480,950976,1089270,1421280, 1539720 is called the Sloane sequence A001599. The growth of Ore Harmonic numbers seems polynomial. The picture shows the list of the first known 170 harmonic numbers with x^5. it grows like $-0.317795 + 0.550454x^5$.



Evenso much more harmonic numbers than perfect numbers seem to exist: (harmonic numbers grow polynomially, while perfect numbers grow superexponentially) one does not know whether there are infinitely many, nor whether there is an odd harmonic divisor function beside $n = 1$.

Problem 13.1 A harmonic integer can not be a square.

Problem 13.2. Is the function $H(n)$ multiplicative?

Moral: Even after simplification, a problems can stay hard. One does not know of odd harmonic numbers, nor whether there are infinitely many.

Multi, Super and Quasi perfect numbers

Numbers for which the index is an integer k are called k-perfect. One also calles them **multiply perfect** or **multiperfect numbers of class** k. Perfect numbers are 2-perfect numbers. Numbers which are 3-perfect are called **triperfect**.
Only 6 triperfect numbers are known.

The list of smallest k-perfect numbers is called Sloane sequence A007539:

k	smallest k perfect number	
1	1	the only 1 perfect number
2	6	smallest perfect number
3	find it	smallest triperfect number
4	30240	Rene Descartes 1638
5	14182439040	Rene Descartes 1638
6	154345556085770649600	R.D. Carmichael 1907
7	141310897947438348259849402738485523264343544818565120000	T.E. Mason, 1911

A number n is called **superperfect**, if $\sigma\sigma(n) = 2n$. For example: 4 is a superperfect number: $\sigma(4) = 1+2+4 = 7, \sigma(7) = 8$. Here are the first superperfect numbers: $1, 2, 4, 16, 64, 4096, 65536, 262144$. They are all powers of 2 and form Sloan's sequence $A019279$.

Every even superperfect number is of the form 2^{p-1} where $2^p - 1$ is a Mersenne prime.

Proof. If $n = 2^{p-1}$, then $\sigma(n) = (2^p - 1)$. If this is prime, then $\sigma(\sigma(n)) = 2^p$. If it is not prime, it is larger.
Assume now that a superperfect number n has an odd prime factor p^k. Then $\sigma(n)$ contains the factor $1 + p + p^2 + ... + p^k$.

A number is called **quasiperfect** if $\sigma(n) = 2n + 1$. [8]

Problem 14.1 Find the smallest number which is 3-perfect.

Problem 14.2 If n is quasiperfect, then its index is

$$2 + \frac{1}{\sigma(n)}$$

Moral: Quasiperfect numbers are useful to explore close hits to perfectness.

The Catalan Dickson dynamical system

Define $T(x) = \sigma(x) - x$, the sum of all proper divisors of x and $T(1) = 1$. Because $T^{-1}(1)$ is a prime or 1, the set of prime numbers is maped to 1. What is the fate of an integer when applying the map T again and again? **Eugene Catalan** studied the dynamical system T in the year 1888. Fixed points of T are **perfect numbers**, periodic orbits of period 2 are called **amicable pairs**. Orbits of T in general are called an **aliquot sequences**, periodic cycles **aliquot cycles**. If the period is larger than 2, they are called **sociable numbers**. Periodic cycles of length 1,2,4,5,6,8,9 and 28 are known. If the orbit of n contains a prime p, one says it belongs to the **prime family** of p. It is in the basin of attraction of the prime p. In 1918 Poulet found a sociable chain of length 5 starting with 12496. An other 28 cycle starts with 14316.

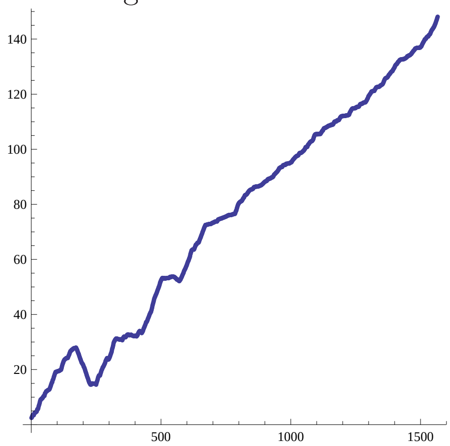


Figure. The orbit starting at 276. This orbit of T has length 1566 was obtained in [5]. Because the sequence has not terminated yet, it is called an **open-end-sequence**.

It is not known whether there exists an unbounded orbit of T. In $\{1 \ldots, 1000\}$, 5 open-end-sequences 276,552,564,660 and 966 called **Lehmer 5** are known. This stability problem looks very similar to other stability problems in dynamical system theory. Here it is called the **Catalan-Dickson problem**:

Are there unbounded orbits of T?

Experimens indicate that about 1 percent of all numbers start open ended sequences. [5]. One knows that for every n, there are orbits of T which are longer than n. [30] Guy and Selfridge [11] give heuristic probabilistic reasonings why the Catalan-Dickson conjecture of having ony bounded orbits is false. These arguments explain why the orbit of T does not look like a random walk. There are "drivers", factors like even perfect numbers. For example, if $n = 6 \cdot 5 = 30$. The perfect number 6 is a driver. It drives the sequence until 259, where it is lost as a factor. The sequence then decays: $30, 42, 54, 66, 78, 90, 144, 259, 45, 33, 15, 9, 4, 3, 1$. Why does a perfect number drive the orbit up? Because $\sigma(65) = \sigma(6)\sigma(5) = 2 \cdot 6\sigma(5) = 72$ we have $T(30) = 72 - 30 > 30$. Paul Erdos calls a number n **untouchable** if it is not in the image of the map T. [8]. One can decide in finitely many steps, whether a number is untouchable or not:

Problem 15.1 Verify that $\sigma(n) > n + \sqrt{n} > 2\sqrt{n}$.

Problem 15.2 Verify that if n is not in the image of $\{1, \ldots, m^2/4\}$ then it is untouchable.

Moral: Number theory reaches in other fields of mathematics.

Amicable numbers

Amicable numbers are period 2 orbits of the Catalan-Dickson dynamical system T. **Thabit's rule** tells that if $p = 3 \cdot 2^{n-1} - 1, q = 3 \cdot 2^n - 1, r = 9 \cdot 2^{2n-1} - 1$ are prime then $(2^n pq, 2^n r)$ is an amicable pair.
Proof: This is a direct computation. Show that $\sigma(p) = \sigma(q) = p + q$.
More generally, **Eulers rule** tells that if $p = 2^{k-l}f - 1, q = 2^k f - 1$ and $r = 2^{2k-l}f^2 - 1$ are all primes with $rf = 2^l + 1$, then $(2^k pq, 2^k r)$ forms an amicable pair.

Example: For $n = 2$, Thabits rule produces the cycle $220, 284$.
For more ways to compute amicable pairs, see [19]. To see how many numbers belonging to amicable pairs are there below one million and one billion?
We looked at the first 500 or so amicable numbers. Plotting them in a log-log coordinate system shows that they seem to grow cubically. We then fitted the sequence $a_n^{1/3}$ with a linear function.

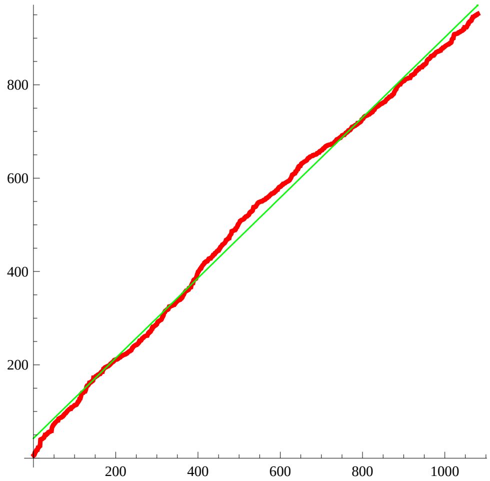


Figure. Experimentally, we measure that the n'th amicable number grows in a cubic manner. Pomerance has upper bounds. No lower bounds are known because we do not even know whether there are infinitely many amicable pairs.

Conjecture: There is a periodic cycle of period 3 of T.

Not much is known even about modular properties of amicable pairs:

Conjecture: Is there an amicable pair with an even and an odd number?

Problem 16.1. Verify directly that $220, 284$ is an amicable pair. It had been found by Pythagoras.

Problem 16.2. Verify that Eulers rule produces amicable pairs.

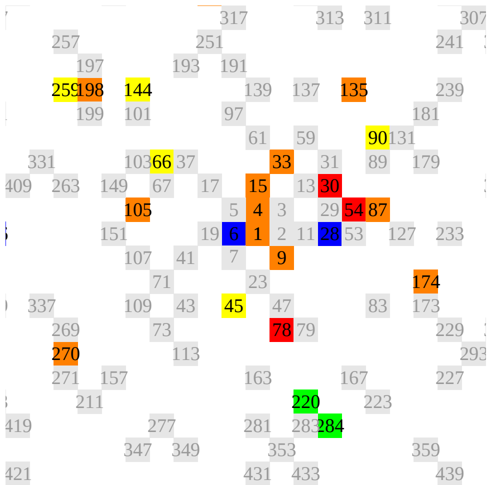
Moral: Period 2 seems to be the most frequent periodic points of T.

The Ulam spiral

The **Ulam spiral** maps the positive integers into squares on the plane.

	99	98	97	96	95	94	93	92	91
65	64	63	62	61	60	59	58	57	90
66	37	36	35	34	33	32	31	56	89
67	38	17	16	15	14	13	30	55	88
68	39	18	5	4	3	12	29	54	87
69	40	19	6	1	2	11	28	53	86
70	41	20	7	8	9	10	27	52	85
71	42	21	22	23	24	25	26	51	84
72	43	44	45	46	47	48	49	50	83
73	74	75	76	77	78	79	80	81	82

The Catalan-Dickson dynamical system is defined on the set of positive integers if we also define $T(1) = 1$. To visualize this better, we can find a map on the plane using the Ulam spiral.



The set of primes are characterized as the set of points which are maped to 1 by T.

The Ulam prime spiral has been used to visualize the set of primes. But unlike displayed in the movie "Code conspiracy", there are no big scale patterns in the distribution of the primes. Indeed, it looks as if the radial distribution of the primes is as if it would be random. We can also look at the statistics of the radial distribution of orbits of the Catalan-Dickson dynamical system.

Problem 17.1. Draw all multiples of 3 on the Ulam spiral. Do you see patterns?

Problem 17.2. It is not possible to march to infinity on prime numbers in the Ulam spiral.

Moral: Who would have guessed that there are conspiracies in math.

The law of small numbers

One of the key lessons in this area is Guy's **strong law of small number** [12]:

There are not enough small numbers to satisfy all the demands placed on them.

This means that things can happen with small numbers which fail for large numbers. Here are examples:

- Fermat numbers $2^{2^n}+1$ are prime for $n = 0, 1, 2, 3, 4$ but false for $n = 5$ as Euler first saw. Fermat was fooled by the strong law of small numbers.
- $\gcd(n^{17}+9, (n+1)^{17}+9) = 1$ which fails first for 8424432925592889329288197322308900672459420460792433.
- 31,331,3331,33331,333331,3333331 are all prime.
- All even perfect numbers have a decimal expansion ending with 6 or 8 $6, 28, 496, 8128, 33550336, \ldots$.

Guy restates in [12] the law in the following way:

You can't tell by looking at a few examples.
Superficial similarities spawn spurious statements.
Capricious coincidences cause careless conjectures.
Early exceptions eclipse eventual essentials.
Initial irregularities inhibit incisive intuition.

Examples:
We see experimentally that every number $2^p - 1$ with prime p is square free.
We see that every amicable pair has a common divisor > 2. Is no amicable pair relatively prime?
All known amicable pairs are either both even or both odd.

Problem 19.1. Disprove the statement that $n^2 + n + 41$ is prime for all n.

Problem 19.2. Every integer can be written as $T(x)+T(y)$, where $T(x) = \sigma(x) - x$ is the sum of proper divisors of x. Is this true?

Moral: One might need large numbers until the law of small numbers kicks in.

Sums of cubes

Every even perfect number $n > 6$ is the sum of k successive odd cubes $f(k) = \sum_{j=1}^{k}(2j+1)^3$.

Pickover [24] mentioned this in his book without proof. Examples are $28 = 1 + 3^3 = f(1)$ or $496 = 1 + 3^3 + 5^3 + 7^3 = f(3)$ or $8128 = 1 + 3^3 + 5^3 + 7^3 + 9^3 + 11^3 + 13^3 + 15^3 = f(7)$. If you want to have fun, stop reading and prove it by yourself. The main obstacle I experienced myself while searching and finding the proof was that I had assumed it is a deep result. It is not.

Proof: for an odd p, define $k = 2^{(p-1)/2}+1$. Then $(1+k)^2 = 2^{p-1}$ and $(1+4k+2k^2) = 2(1+k)^2-1 = 2^p - 1$ so that $2^{p-1}(2^p - 1) = (1+k)^2(1+4k+2k^2) = 2k^4 + 8k^3 + 11k^2 + 6k + 1 = \sum_{j=1}^{k}(2j+1)^3$. The argument does not work for the first perfect number 6 because the prime $p_0 = 2$ is even in that case and because 2 is the only even prime. (This result seems well known: I found mentioned a reference to Kahan. April Mathematics Magazine 1998).

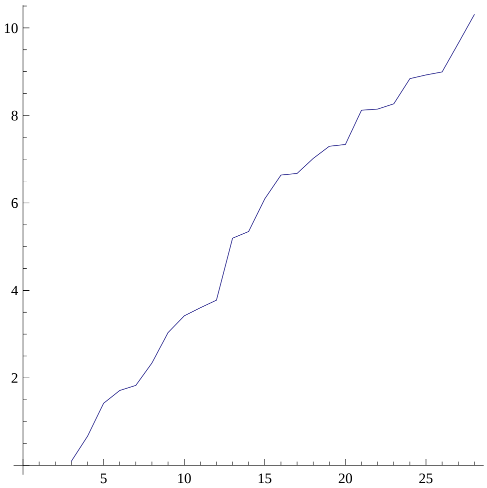


Figure. The Pickover sequence visualized. We plot $\log(\log(b(n))$, where $g(b(n))$ is the n'th Mersenne number. The Pickover sequence $1, 3, 7, ...$ grows super exponentially too.

Problem 20.1. Verify that the sum of the first k odd cubes is always a triangular number.

Problem 20.2. Verify by induction that the sum of the first k odd cubes is $2k^4 + 8k^3 + 11k^2 + 6k + 1$.

Moral: Even easy problems can look scary.

Why look for perfect numbers?

As many problems in number theory, mathematics or science in general, one can question the use and merit of its investigation, because immediate applications are sparse. History has shown so many times that even the seemingly esoteric mathematical topics can become relevant. For example, the abstract theory of operators became the foundation of **quantum mechanics** and the study of primes became the core of some of the best **cryptology** we know.

Here is a quote of Littlewood which looks nihilistic: [33]

Citation: "Perfect numbers certainly never did any good, but then they never did any particular harm."

On the other hand, this mathematical problem has a cultural importance because

Citation: "This is probably the oldest unsolved problem in all of mathematics."

(Chris Caldwell). It has been tackled since many hundreds of years. The best mathematicians in the last millennia like Euclid, Euler, Fermat, Descartes, Pierce have worked on perfect numbers and the quest has gone on to modern times. Sylvester, who worked heavily on the problem, writes:

Citation: "Whoever shall succeed in demonstrating the absolute nonexistence of odd perfect numbers will have solved a problem of the ages comparable in difficulty to that which previously to the labors of Hermite and Lindemann environed the subject of the quadrature of the circle".

The problem has now become a task in experimental mathematics. Like finding an **elementary particle** in physics or a **habitable planet** in space, it is a quest which inspires the push of technology, in this case algorithms and computers. Similar than with the quest for fundamental particles or other life forms, there is no guarantee that the search is successful.

T.M. Putnam writes in 1910 [26]: "There is always the possibility, too, that the pursuit of solutions of even these elusive problems may lead to the discovery of mathematical relations, or processes that are new and of much more general application than to the immediate problem to be solved."

Also the problem of finding large even perfect numbers through Mersenne primes has become a computational project in which many people are involved. The problems makes headlines from time to time.

A warning to the end: elementary problems in mathematics are a dangerous field to get into research. Most low hanging fruits have been picked and it is relatively unlikely to make progress. It is likely that future progress in this field too will need heavy machinery.

Moral: Finally, the topic is accessible to a large group of people, many of them being non mathematicians. The problem has so become a bridge to communicate and motivate mathematics for nonmathematicians.

Answers to some problems

Answer: 1.1 The proper divisors of 28 are 1,2,4,7,14 which sum up to 28.

Answer: 1.2. The divisors of 18 including 18 are $1,2,3,6,9,18$. The divisors of 2 are $1,2$ the divisor of 9 are $1,3,9$. A divisor of 18 is the product of a divisor of 2 and a divisor of 9. These are 2×3 possibilities.

Answer: 2.1. $\sigma(6) = 1+2+3+6 = 12, \sigma(35) = 1+5+7+35 = 48$. $\sigma(210) = 1+2+3+5+6+7+10+14+15+21+30+35+42+70+105+210 = 12*48$.

Answer: 2.2. Multiply both sides by $p-1$:

$$(p-1)(1+p+p^2+\cdots+p^k) = (p+p^2+p^3+\ldots+p^{k+1})-(1+p+p^2+p^3+\ldots+p^k) = p^{k+1}-1 .$$

Answer: 3.1. No, $\sigma(n)$ would be even. More generally, all primes p_i must occur with even multiplicity and one with odd multiplicity.

Answer: 3.2. Just look at the condition modulo 2. For a square number $\sigma(n)$ is odd.

Answer: 4.1.

$$\sigma(p^n)/p^n = (1+\frac{1}{p}+\frac{1}{p^2}+\cdots+\frac{1}{p^k}) < \frac{1}{1-1/p} \leq \frac{1}{1-1/3} = \frac{3}{2} .$$

Answer: 4.2. If $p=q$ we are in the same situation as in the previous exercice. Otherwise:

$$\begin{aligned} \frac{\sigma(p^n q^n)}{p^n q^n} &= (1+\frac{1}{p}+\frac{1}{p^2}+\cdots+\frac{1}{p^k})(1+\frac{1}{q}+\frac{1}{q^2}+\cdots+\frac{1}{q^k}) \\ &\leq \frac{1}{1-1/p}\frac{1}{1-1/q} \leq \frac{1}{1-1/3}\frac{1}{1-1/5} \\ &= \frac{3}{2}\frac{5}{4} = \frac{15}{8} < 2 \end{aligned}$$

Answer: 5.1. For prime numbers p and $p \to \infty$ we have

$$h(p) = \frac{p+1}{p} \to 1 .$$

Answer: 5.2. We have $h(30) = (1+2+3+5+6+10+12+15+30)/30 = 72/30 = 12/5 > 2$ and

$$h(na) = h(n)h(a)$$

if a and n have no common divisor. So, for any p not dividing n, we have

$$h(np) > h(n) .$$

So, there are also infinitely many superabundant numbers.

Answer: 6.1. $a = 2$ because $a^n - 1$ is divisible by $a - 1$.

Answer: 6.2. This follows directly from the formula. If $n = 2^p$ then a perfect number is of the form $n(n-1)/2$.

Answer: 8.1. Perfect numbers are fun.

Answer: 8.2. Even asking questions is hard.

Answer: 10.1. Charles Greathouse suggests in [25] to define a sequence of primes

$$3, 5, 7, 11, 389, 29959, 128194589$$

where we always pick the prime so that we get closest to 2 but still are below 2. For example, the prime $p = 128194589$ has the property that

$$\begin{aligned} h(3*5*7*11*389*29959*p) &= (1+\frac{1}{3})(1+\frac{1}{5})(1+\frac{1}{7}) \\ &+ \frac{1}{11})(1+\frac{1}{389})(1+\frac{1}{29959})(1+\frac{1}{p}) \end{aligned}$$

is < 2 but closest to 2 among all primes. Suppose that the Greathouse prime sequence p_k produce a limit $2 - \epsilon < 2$. Then the sequence p_k has to grow so fast that $\prod_k (1 + 1/p_k)$ converges. Because with all primes, the product diverges we can slow down any given sequence p_k for which the sum converges to a value < 2 and make it larger. [Let $p(m)$ be a prime of the form $4k + 1$ greater than m. Then $gcd(m, p(m)) = 1$. and $h(p(m)m^2) = h(p(m))h(m^2) = (1 + 1/p(m))$ Pomerance claims that the sequence $h(p(m)m^2)$ is dense in $[1, \infty)$. [25]]

Answer: 11.2. Check the primality of all factors involved.

Answer: 12.1. $\sigma(p^{4l-1}) = (p^{4m-1})/(p-1)$. Write $p^m = q$ so that this is $(q^2+1)(q+1)$ and both factors are even.

Answer: 12.2. We have to find odd integers p_i such that $\prod_{i=1}^k (p_i + 1) = \prod_i 2p_i$. If $k > 1$, then the left hand side has two factors 2 while the right hand side has only 1. Solving $p + 1 = 2p$ gives $p = 1$. So: the only odd simple pseudo perfect number is $p = 1$.

Answer: 13.1. $H = k/(\sum_{d|n,d\leq\sqrt{n}} 2/d)$ **and** k **is odd for a square, we know that**

$$H(\sum_{d|n,d\leq\sqrt{n}} \frac{1}{d} = k/2$$

Answer: 13.2.

$$((1-1/11)(1-1/13)(1-1/17)(1-1/19)(1-1/23)(1-1/29)(1-1/31)(1-1/37))^(-1) < 2$$

$$((1-1/7)(1-1/11)(1-1/13)(1-1/17)(1-1/19)(1-1/23)(1-1/29)(1-1/31))^(-1) < 2$$

$$((1-1/5)(1-1/11)(1-1/13)(1-1/17)(1-1/19)(1-1/23)(1-1/29)(1-1/31))^(-1) > 2$$

$$((1-1/5)(1-1/7)(1-1/11)(1-1/13)(1-1/17)(1-1/19)(1-1/23)(1-1/29)(1-1/31))^(-1) > 2$$

Answer: 14.1 120 **is** 3 **perfect.**

Answer: 14.2 Directly from the definition $\sigma(n) = 2n + 1$**.**

Answer: 15.1 If n **is prime then** $\sigma(n) = n+1$**. But if** n **is not prime, then it contains a factor different from** 1**. Either this factor** a **or** n/a **are larger than** $\sqrt{n}$**.**

Answer: 15.2 All primes are touchable. For a nonprime, the estimate in 15.1 applies. For every n **not in** $\{1, ..., m^2/4\}$**, the image** $T(n)$ **is larger than** m**.**

Answer: 19.1

Answer: 19.2 The first counter example is 221**.**

Answer: 20.1. This follows from the result proven in this section and the fact that every even perfect number is a triangular number.

Answer: 20.2. Use induction

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