Heuristic Search (Informed Search)

Heuristic Search

- In **uninformed search**, we don't try to evaluate which of the nodes on the frontier are most promising. We never "look-ahead" to the goal.

  E.g., in uniform cost search we always expand the cheapest path. We don't consider the cost of getting to the goal.

- Often we have some other knowledge about the merit of nodes, e.g., going the wrong direction in Romania.

Heuristic Search

Merit of a frontier node: different notions of merit.

- If we are concerned about the cost of the solution, we might want a notion of merit of how costly it is to get to the goal from that search node.
- If we are concerned about minimizing computation in search we might want a notion of ease in finding the goal from that search node.
- We will focus on the "cost of solution" notion of merit.

Heuristic Search

- The idea is to develop a domain specific heuristic function h(n).
- h(n) guesses the cost of getting to the goal from node n.
- There are different ways of guessing this cost in different domains. I.e., heuristics are domain specific.

Heuristic Search

- Convention: If $h(n_1) < h(n_2)$ this means that we guess that it is cheaper to get to the goal from n_1 than from n_2.

- We require that
  - h(n) = 0 for every node n that satisfies the goal.
    - Zero cost of getting to a goal node from a goal node.

Using only h(n): Greedy best-first search

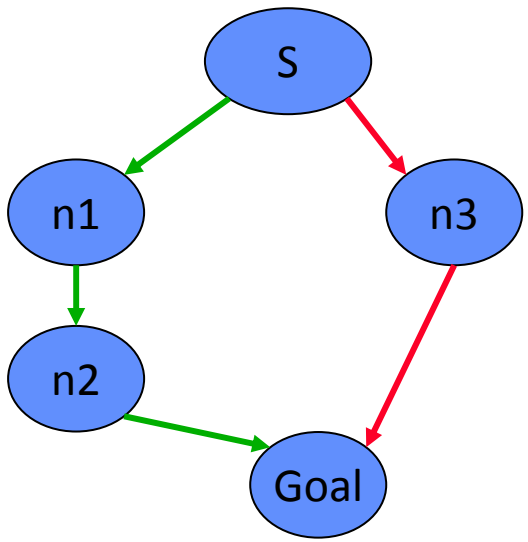
- We use h(n) to rank the nodes on the frontier.
  - Always expand node with lowest h-value.
- We are greedily trying to achieve a low cost solution.

- However, this method **ignores the cost of getting to n**, so it can be lead astray exploring nodes that cost a lot to get to but seem to be close to the goal:

→ step cost = 10

→ step cost = 100

h(n1) = 70



h(n3) = 50

(Incomplete)

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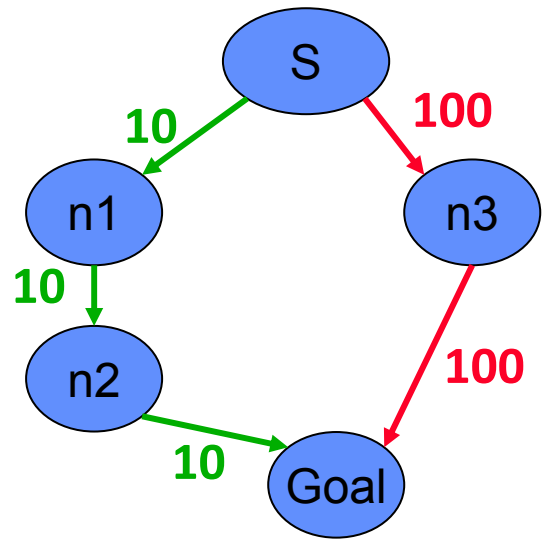

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A* search

- Take into account the cost of getting to the node as well as our estimate of the cost of getting to the goal from n.

- Define an evaluation function f(n)

  f(n) = g(n) + h(n)

  - g(n) is the cost of the path to node n
  - h(n) is the heuristic estimate of the cost of getting to a goal node from n.

- Always expand the node with lowest f-value on the frontier.

- The f-value is an estimate of the cost of getting to the goal via this node (path).

Conditions on h(n)

- We want to analyze the behavior of the resultant search.
- Completeness, time and space, optimality?
- To obtain such results we must put some further conditions on the heuristic function h(n) and the search space.

Conditions on h(n): Admissible

- We always assume that c(n1→ n2) ≥ ε > 0. The cost of any transition is greater than zero and can't be arbitrarily small.

- Let h*(n) be the **cost of an optimal path** from n to a goal node (∞ if there is no path). Then an admissible heuristic satisfies the condition

  $h(n) \leq h^*(n)$

  i.e. *h* always underestimates of the true cost.

- Hence
  - h(g) = 0, for any goal note, g
  - h(n) = ∞ if there is not path from n to a goal node

Consistency (aka monotonicity)

- Is a stronger condition than $h(n) \leq h^*(n)$.

- A monotone/consistent heuristic satisfies the triangle inequality (for all nodes n1,n2):

$$h(n1) \leq c(n1 \rightarrow n2) + h(n2)$$

- Note that there might be more than one transition (action) between n1 and n2, the inequality must hold for all of them.

- Note that monotonicity implies admissibility. Why?

Intuition behind admissibility

$h(n) \leq h^*(n)$ means that the search won't miss any promising paths.

- If it really is cheap to get to a goal via n (i.e., both g(n) and $h^*(n)$ are low), then f(n) = g(n) + h(n) will also be low, and the search won't ignore n in favor of more expensive options.

- This can be formalized to show that admissibility implies optimality.

Intuition behind monotonicity

$h(n1) \leq c(n1 \rightarrow n2) + h(n2)$

- This says something similar, but in addition one won't be "locally" mislead. See next example.

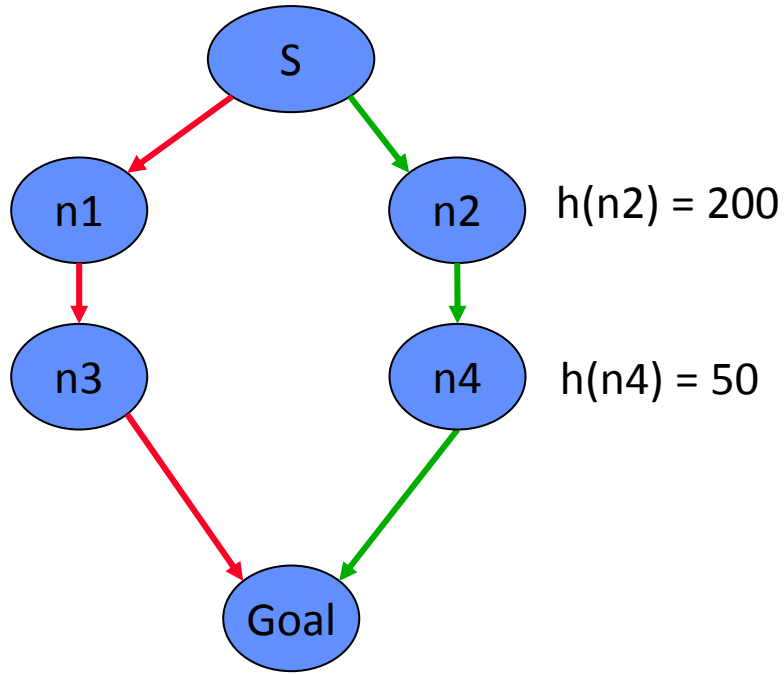
Example: admissible but nonmonotonic

The following *h* is not consistent since *h(n2)>c(n2→n4)+h(n4).*
But it is admissible.

→ step cost = 200
→ step cost = 100

h(n1) =50

h(n3) =50



g(n)+h(n)=f(n)

{S} → {n1 [200+50=**250**], n2 [200+100=**300**]}
→ {n2 [100+200=**300**], n3 [400+50=**450**]}
→ {n4 [200+50=**250**], n3 [400+50=**450**]}
→ {goal [300+0=**300**], n3 [400+50=**450**]}

We **do find** the optimal path as the heuristic is still admissible. **But** we are mislead into ignoring n2 until after we expand n1.

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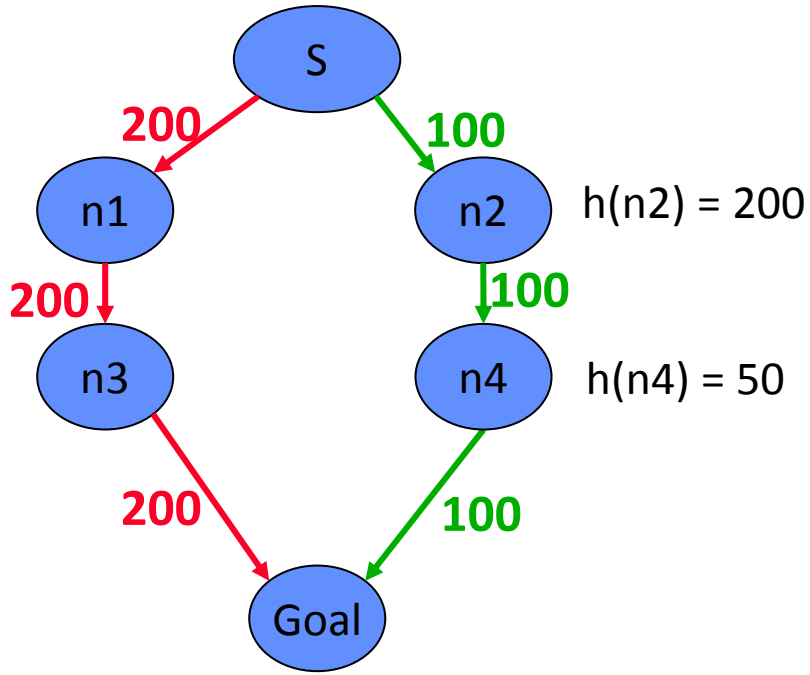

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Consequences of monotonicity

1. The f-values of nodes along a path must be non-decreasing.

- Let <Start→ n1→ n2…→ nk> be a path. We claim that
  $f(ni) \leq f(ni+1)$

- Proof:

$f(ni) = c(Start \rightarrow \ldots \rightarrow ni) + h(ni)$
$\leq c(Start \rightarrow \ldots \rightarrow ni) + c(ni \rightarrow ni+1) + h(ni+1)$
$= c(Start \rightarrow \ldots \rightarrow ni \rightarrow ni+1) + h(ni+1)$
$= g(ni+1) + h(ni+1)$
$= f(ni+1).$

Consequences of monotonicity

2. If n2 is expanded after n1, then f(n1) ≤ f(n2)
   (the f-value increases monotonically)

Proof:

- If n2 was on the frontier when n1 was expanded,
  then f(n1) ≤ f(n2) otherwise we would have expanded n2.

- If n2 was added to the frontier after n1's expansion, then let n be an ancestor of n2 that was present when n1 was being expanded (this could be n1 itself). We have f(n1) ≤ f(n) since A* chose n1 while n was present in the frontier. Also, since n is along the path to n2, by property (1) we have f(n)≤f(n2). So, we have f(n1) ≤ f(n2).

Consequences of monotonicity

3. When n is expanded every path with lower f-value has already been expanded.

- **Proof:** Assume by contradiction that there exists a path <Start, n0, n1, ni-1, ni, ni+1, …, nk> with f(nk) < f(n) and ni is its last expanded node.
  - ni+1 must be on the frontier while n is expanded, so

  a) by (1) f(ni+1) ≤ f(nk) since they lie along the same path.

  b) since f(nk) < f(n) so we have f(ni+1) < f(n)

  c) by (2) f(n) ≤ f(ni+1) because n is expanded before ni+1.
  - Contradiction from b&c!

Consequences of monotonicity

4. With a monotone heuristic, the first time A* expands a state, it has found the minimum cost path to that state.

Proof:

- Let **PATH1 = <Start, n0, n1, …, nk, n>** be **the first** path to n found. We have f(path1) = c(PATH1) + h(n).
- Let **PATH2 = <Start, m0,m1, …, mj, n>** be another path to n found later. we have f(path2) = c(PATH2) + h(n).
- By property (3), f(path1) ≤ f(path2)
- hence: c(PATH1) ≤ c(PATH2)

Consequences of monotonicity

Complete.

- Yes, consider a least cost path to a goal node
  - SolutionPath = <Start→ n1→ …→ G> with cost c(SolutionPath)
  - Since each action has a cost ≥ ε > 0, there are only a finite number of paths that have cost ≤ c(SolutionPath).
  - All of these paths must be explored before any path of cost > c(SolutionPath).
  - So eventually SolutionPath, or some equal cost path to a goal must be expanded.

Time and Space complexity.

- When h(n) = 0, for all n h is monotone.
  - A* becomes uniform-cost search!
- It can be shown that when h(n) > 0 for some n, the number of nodes expanded can be no larger than uniform-cost.
- Hence the same bounds as uniform-cost apply. (These are worst case bounds). Still exponential unless we have a very very good *h*!
- In real world problems, we run out of time and memory! IDA* used to address memory issues.

Consequences of monotonicity

Optimality

- Yes, by (4) the first path to a goal node must be optimal.

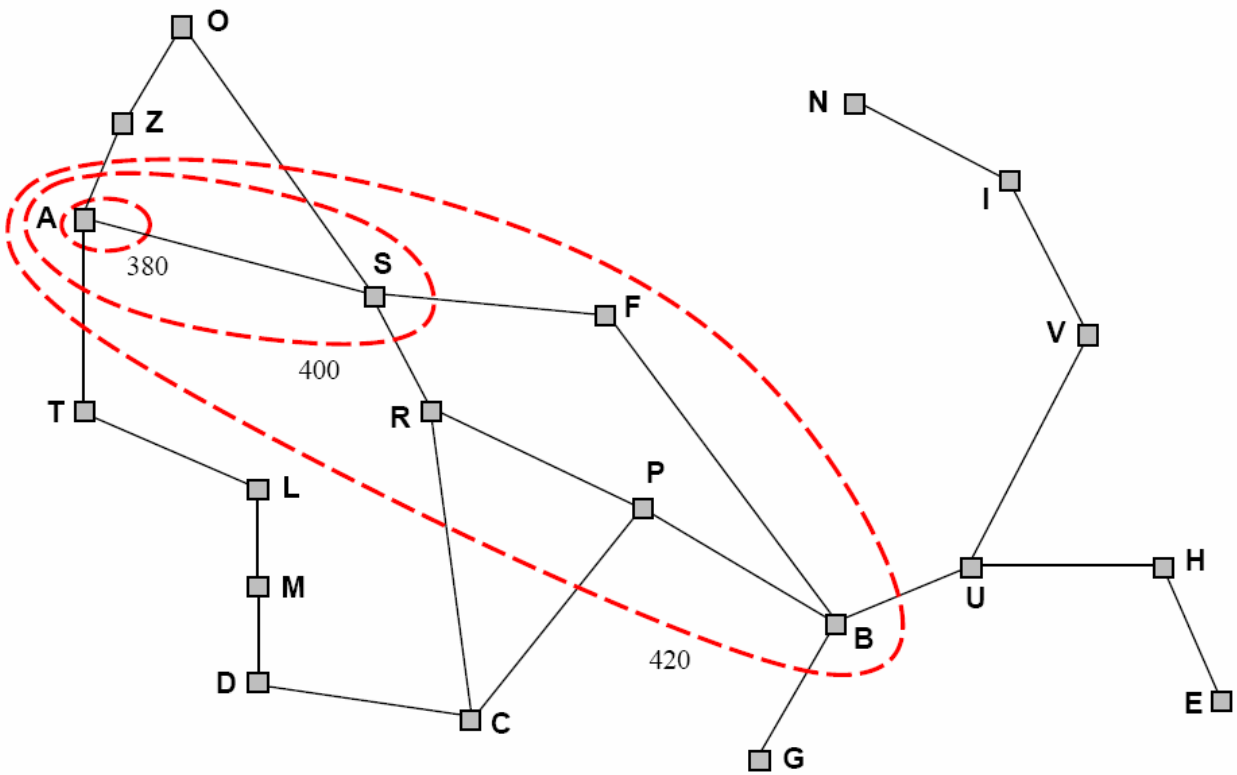
Cycle Checking

- If we do cycle checking (multiple path checking) e.g. using GraphSearch instead of TreeSearch, it is still optimal. Because by property (4) we need keep only the first path to a node, rejecting all subsequent paths.

Search generated by monotonicity

Gradually adds "f-contours" of nodes (cf. breadth-first adds layers)

Inside each counter, the f values are less than or equal to counter value!



- **For uniform cost search, bands are *"circular".***
- **With more accurate heuristics, bands stretch out more toward the goal.**

Admissibility without monotonicity

When "h" is admissible but not monotonic.

- Time and Space complexity remain the same. Completeness holds.
- Optimality still holds (without cycle checking), but need a different argument: don't know that paths are explored in order of cost.

Proof of optimality (without cycle checking):

- Assume the goal path <S,…,G> found by A* has cost bigger than the optimal cost: i.e. $C^*(G) < f(G)$.
- There must exists a node n in the optimal path that is still in the frontier.
- We have: $f(n)=g(n)+h(n) \leq g(n)+h^*(n)=C^*(G) < f(G)$

  why?
- Therefore, f(n) must have been selected before G by A*. contradiction!

Admissibility without monotonicity

What about Cycle Checking?

- No longer guaranteed we have found an optimal path to a node ***the first time*** we visit it.

- So, cycle checking might not preserve optimality.
  - To fix this: for previously visited nodes, must remember cost of previous path. If new path is cheaper must explore again.

- contours of monotonic heuristics don't hold.

Space Problems with A*

- A* has the same potential space problems as BFS

- IDA* - Iterative Deepening A* is similar to Iterative Lengthening Search and similarly addresses space issues.

IDA* - Iterative Deepening A*

Objective: reduce memory requirements for A*

- Like iterative deepening, but now the "cutoff" is the f-value (g+h) rather than the depth
- At each iteration, the cutoff value is the smallest f-value of any node that exceeded the cutoff on the previous iteration
- Avoids overhead associated with keeping a sorted queue of nodes
- Two new parameters:
  - curBound (any node with a bigger f-value is discarded)
  - smallestNotExplored (the smallest f-value for discarded nodes in a round) when frontier becomes empty, the search starts a new round with this bound.

Building Heuristics: Relaxed Problem

- One useful technique is to consider an easier problem, and let h(n) be the cost of reaching the goal in the easier problem.

- 8-Puzzle moves.
  - Can move a tile from square A to B if
    - A is adjacent (left, right, above, below) to B
    - and B is blank

- Can relax some of these conditions
  1. can move from A to B if A is adjacent to B (ignore whether or not position is blank)
  2. can move from A to B if B is blank (ignore adjacency)
  3. can move from A to B (ignore both conditions).

Building Heuristics: Relaxed Problem

- **#3** *"can move from A to B (ignore both conditions)".*

  leads to the misplaced tiles heuristic.
  - To solve the puzzle, we need to move each tile into its final position.
  - Number of moves = number of misplaced tiles.
  - Clearly h(n) = number of misplaced tiles ≤ the h*(n) the cost of an optimal sequence of moves from n.

- **#1** *"can move from A to B if A is adjacent to B (ignore whether or not position is blank)"*

  leads to the manhattan distance heuristic.
  - To solve the puzzle we need to slide each tile into its final position.
  - We can move vertically or horizontally.
  - Number of moves = sum over all of the tiles of the number of vertical and horizontal slides we need to move that tile into place.
  - Again h(n) = sum of the manhattan distances ≤ h*(n)
    - in a real solution we need to move each tile at least that that far and we can only move one tile at a time.

Building Heuristics: Relaxed Problem

The optimal cost to nodes in the relaxed problem is an admissible heuristic for the original problem!

Proof: the optimal solution in the original problem is a (*not necessarily optimal*) solution for relaxed problem, therefore it must be at least as expensive as the optimal solution in the relaxed problem.

Building Heuristics: Relaxed Problem

Comparison of IDS and A* (average total nodes expanded):

Depth	IDS	A*(Misplaced) **h1**	A*(Manhattan) **h2**
10	47,127	93	39
14	3,473,941	539	113
24	---	39,135	1,641

Let h1=Misplaced, h2=Manhattan

- Does h2 **always** expand fewer nodes than h1?
  - Yes! Note that **h2 dominates h1**, i.e. for all n: $h1(n) \leq h2(n)$. From this you can prove h2 is faster than h1.
  - Therefore, among several admissible heuristic the one with highest value is the fastest.

Building Heuristics: Pattern databases.

- Admissible heuristics can also be derived from solution to **subproblems: Each state is mapped into a partial specification, e.g. in 15-puzzle only *position of specific tiles matters.***

• Here are goals for two sub-problems (called Corner and Fringe) of 15puzzle.

• Note the location of BLANK!

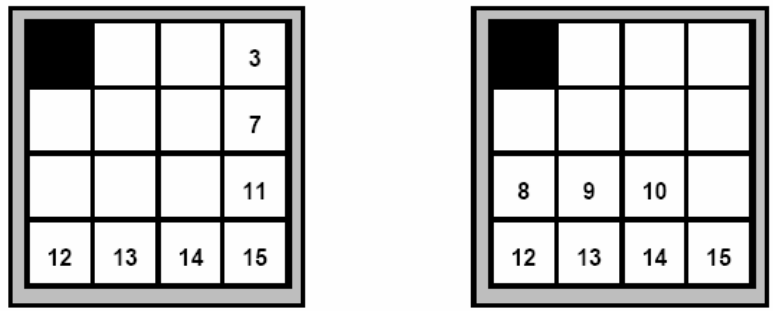


Fig. 2. The Fringe and Corner Target Patterns.

· By searching backwards from these goal states, we can compute the distance of any configuration of these tiles to their goal locations. We are ignoring the identity of the other tiles.

· For any state n, the number of moves required to get these tiles into place form a lower bound on the cost of getting to the goal from n.

Building Heuristics: Pattern databases.

- These configurations are stored in a database, along with the number of moves required to move the tiles into place.
- The maximum number of moves taken over all of the databases can be used as a heuristic.
- On the 15-puzzle
  - The fringe data base yields about a 345 fold decrease in the search tree size.
  - The corner data base yields about 437 fold decrease.
- Sometimes disjoint patterns can be found, then the number of moves can be added rather than taking the max (if we only count moves of the target tiles).

Local Search

- So far, we keep the paths to the goal.
- For some problems (like 8-queens) we don't care about the path, we **only care about the solution.** Many real problem like Scheduling, IC design, and network optimizations are of this form.
- Local search algorithms operate using a single current state and generally move to neighbors of that state.
- There is an objective function that tells the value of each state. The goal has the highest value (global maximum).
- Algorithms like Hill Climbing try to move to a neighbour with the highest value.
- Danger of being stuck in a local maximum. So some randomness is added to "shake" out of local maxima.
- Simulated Annealing: Instead of the best move, take a random move and if it improves the situation then always accept, otherwise accept with a probability <1.
- [If interested read these two algorithms from the R&N book].