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First-order logic

CMSC 421: Chapter 8 and Section 10.3

Pros and cons of propositional logic

$+$ Propositional logic is **declarative**: pieces of syntax correspond to facts

$+$ Propositional logic allows partial/disjunctive/negated information
(unlike most data structures and databases)

$+$ Propositional logic is **compositional**:
meaning of $B_{1,1} \wedge P_{1,2}$ is derived from meaning of $B_{1,1}$ and of $P_{1,2}$

$+$ Meaning in propositional logic is **context-independent**
(unlike natural language, where meaning depends on context)

$-$ Propositional logic has very limited expressive power
E.g., cannot say "pits cause breezes in adjacent squares"
except by writing one sentence for each square

Need a logic that's more expressive
$\Rightarrow$ First Order Logic (FOL)

Outline

- ♢ Syntax and semantics of FOL
- ♢ Examples of sentences
- ♢ Wumpus world in FOL

Basic entities in FOL

Whereas propositional logic assumes world contains **facts**,
first-order logic (like natural language) assumes the world contains

- *Objects*: people, houses, numbers, theories, Ronald McDonald, colors, baseball games, wars, centuries ...
- *Relations*: red, round, bogus, prime, multistoried ...,
  is the brother of, is bigger than, is inside, is part of, has color, occurred after, owns, comes between, ...
- *Functions*: father of, best friend, third inning of, one more than, end of ...

Syntax of FOL: Basic elements

Constant symbols	$KingJohn,\ 2,\ UniversityofMaryland, \ldots$
Predicate symbols	$Brother,\ >, \ldots$
Function symbols	$Sqrt,\ LeftLegOf, \ldots$
Variable symbols	$x,\ y,\ a,\ b, \ldots$
Connectives	$\land \quad \lor \quad \neg \quad \Rightarrow \quad \Leftrightarrow$
Equality	$=$
Quantifiers	$\forall\ \exists$
Punctuation	$(\ )\ ,$

Atomic sentences

Atomic sentence $= predicate(term_1, \ldots, term_n)$
or $term_1 = term_2$

Term $= function(term_1, \ldots, term_n)$
or $constant$ or $variable$

E.g.,

$Brother(KingJohn, RichardTheLionheart)$

$> (Length(LeftLegOf(Richard)), Length(LeftLegOf(KingJohn)))$

Complex sentences

Complex sentences are made from atomic sentences using connectives

$$\neg S, \quad S_1 \wedge S_2, \quad S_1 \vee S_2, \quad S_1 \;\Rightarrow\; S_2, \quad S_1 \;\Leftrightarrow\; S_2$$

E.g.,

$$Sibling(KingJohn, Richard) \;\Rightarrow\; Sibling(Richard, KingJohn)$$

$$>(1,2) \;\vee\; \leq(1,2)$$

$$>(1,2) \;\wedge\; \neg{>}(1,2)$$

Truth in first-order logic

In FOL, a *model* is a pair $M = (D, I)$, where D is a *domain* and I is an *interpretation*

D contains ≥ 1 objects (*domain elements*)
and relations among them

I specifies referents for
constant symbols $\rightarrow$ objects in the domain
predicate symbols $\rightarrow$ relations over objects in the domain
function symbols $\rightarrow$ functional relations over objects in the domain

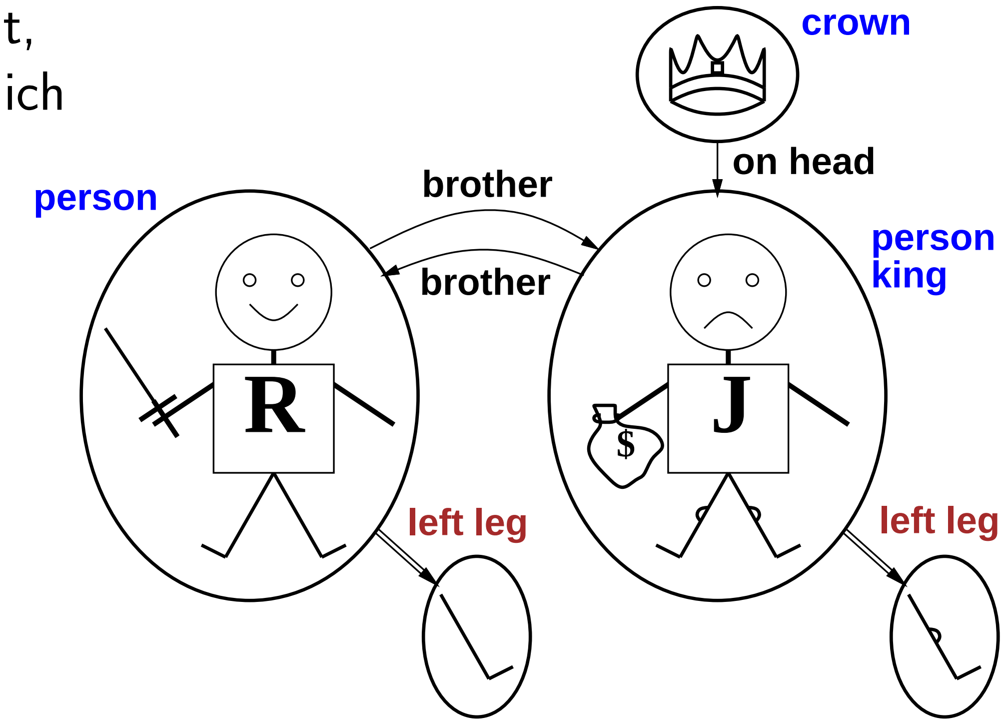
Recall that mathematically, a *relation* is a set of ordered n-tuples

An atomic sentence $predicate(term_1, \ldots, term_n)$ is true in M
iff the objects referred to by $term_1, \ldots, term_n$
are in the relation referred to by $predicate$

Like before, we say M *is a model of* a sentence α if α is true in M

Truth example

Suppose $M = (D, I)$, where
D is the domain shown at right,
and I is an interpretation in which



$Richard \rightarrow$
Richard the Lionheart

$John \rightarrow$
the evil King John

$Brother \rightarrow$
the brotherhood relation

$Brother(Richard, John)$ is true in M iff the pair consisting of
Richard the Lionheart and the evil King John
is in the brotherhood relation

Models for FOL: Lots!

Entailment in propositional logic can be computed by enumerating the possible worlds (i.e., model checking)

How to enumerate possible worlds in FOL?

For each number of domain elements n from 1 to ∞
    For each k-ary predicate P_k in the vocabulary
        For each possible k-ary relation on n objects
            For each constant symbol C in the vocabulary
                For each choice of referent for C from n objects . . .

Computing entailment in this way is not easy!

Universal quantification

$\forall \langle variables \rangle \;\; \langle sentence \rangle$

Everyone at the University of Maryland is smart:
$\forall x \;\; At(x, UMD) \;\Rightarrow\; Smart(x)$

$\forall x \;\; P$ is true in a model m iff P is true with x being **each** possible object in the model

Roughly equivalent to the conjunction of instantiations of P

$$\begin{aligned}
& (At(KingJohn, UMD) \;\Rightarrow\; Smart(KingJohn)) \\
\wedge\; & (At(Richard, UMD) \;\Rightarrow\; Smart(Richard)) \\
\wedge\; & (At(UMD, UMD) \;\Rightarrow\; Smart(UMD)) \\
\wedge\; & \ldots
\end{aligned}$$

A common mistake to avoid

Common mistake with $\forall$:
using $\wedge$ when you meant to use $\Rightarrow$

$$\forall\, x \;\; At(x, UMD) \wedge Smart(x)$$

means "Everyone is at UMD and everyone is smart"

Probably you meant to say

$$\forall\, x \;\; At(x, UMD) \;\Rightarrow\; Smart(x)$$

Everyone at UMD is smart.

Existential quantification

$\exists \langle variables \rangle \ \ \langle sentence \rangle$

Someone at UMD is smart:
$\exists\, x \ \ At(x, UMD) \land Smart(x)$

$\exists\, x \ \ P$ is true in a model m iff P is true with x being **some** possible object in the model

Roughly equivalent to the disjunction of instantiations of P

$$
\begin{aligned}
& (At(KingJohn, UMD) \land Smart(KingJohn)) \\
\lor\ & (At(Richard, UMD) \land Smart(Richard)) \\
\lor\ & (At(UMD, UMD) \land Smart(UMD)) \\
\lor\ & \ldots
\end{aligned}
$$

Another common mistake to avoid

A common mistake with $\exists$:
using $\Rightarrow$ when you meant to use $\wedge$:

$$\exists x \;\; At(x, UMD) \;\Rightarrow\; Smart(x)$$

This is equivalent to

$$\exists x \;\; \neg At(x, UMD) \vee Smart(x)$$

There's someone who either is smart or isn't at UMD.

That's true if there's anyone who is not at UMD.

Probably you meant to say this instead:

$$\exists x \;\; At(x, UMD) \wedge Smart(x)$$

There's someone who is at UMD and is smart.

Properties of quantifiers

$\forall x \ \forall y$ is the same as $\forall y \ \forall x$

$\exists x \ \exists y$ is the same as $\exists y \ \exists x$

$\exists x \ \forall y$ is **not** the same as $\forall y \ \exists x$

$\exists x \ \forall y \ Loves(x, y)$
"There is a person who loves everyone in the world"

$\forall y \ \exists x \ Loves(x, y)$
"Everyone in the world is loved by at least one person"

Quantifier duality: each can be expressed using the other

$\forall x \ Likes(x, IceCream) \qquad \neg\exists x \ \neg Likes(x, IceCream)$

$\exists x \ Likes(x, Broccoli) \qquad \neg\forall x \ \neg Likes(x, Broccoli)$

Examples of sentences

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$\forall\, x, y \;\; Brother(x, y) \;\Rightarrow\; Sibling(x, y)$

"Sibling" is symmetric

$\forall\, x, y \;\; Sibling(x, y) \;\Leftrightarrow\; Sibling(y, x)$

One's mother is one's female parent

Examples of sentences

Brothers are siblings

$\forall\, x, y \;\; Brother(x, y) \;\Rightarrow\; Sibling(x, y)$

"Sibling" is symmetric

$\forall\, x, y \;\; Sibling(x, y) \;\Leftrightarrow\; Sibling(y, x)$

One's mother is one's female parent

$\forall\, x, y \;\; Mother(x, y) \;\Leftrightarrow\; (Female(x) \wedge Parent(x, y))$

A first cousin is a child of a parent's sibling

Examples of sentences

Brothers are siblings

$\forall\, x, y \;\; Brother(x, y) \;\Rightarrow\; Sibling(x, y)$

"Sibling" is symmetric

$\forall\, x, y \;\; Sibling(x, y) \;\Leftrightarrow\; Sibling(y, x)$

One's mother is one's female parent

$\forall\, x, y \;\; Mother(x, y) \;\Leftrightarrow\; (Female(x) \wedge Parent(x, y))$

A first cousin is a child of a parent's sibling

$\forall\, x, y \;\; FirstCousin(x, y) \;\Leftrightarrow$
$\qquad \exists\, px, py \;\; Parent(px, x) \wedge Sibling(px, py) \wedge Parent(py, y)$

Equality

$term_1 = term_2$ is true under a given interpretation
if and only if $term_1$ and $term_2$ refer to the same object

E.g., $1 = 2$ and $\forall\, x \;\; \times(Sqrt(x), Sqrt(x)) = x$ are satisfiable
(true under at least one interpretation)

$2 = 2$ is valid (true in every interpretation)

E.g., definition of $Sibling$ in terms of $Parent$:

$$\forall\, x, y \;\; Sibling(x, y) \;\Leftrightarrow\; [\neg(x = y) \wedge \exists\, m, f \;\; \neg(m = f) \wedge Parent(m, x) \wedge Parent(f, x) \wedge Parent(m, y) \wedge Parent(f, y)]$$

Substitutions

Substitution: a set of variable bindings

Consider a substitution σ that assigns $x = 1$, $y = f(z)$

A logician would write $\sigma = \{1/x, f(z)/y\}$

Russell and Norvig write $\sigma = \{x/1, y/f(z)\}$

To try to avoid ambiguity, I'll try to write $\sigma = \{x \leftarrow 1, y \leftarrow f(z)\}$

Given a sentence S and a substitution σ,
$S\sigma$ (postfix notation) is the result of applying σ to S

$$S = \ GreaterThan(x, y)$$
$$\sigma = \{x \leftarrow 1, y \leftarrow f(z)\}$$
$$S\sigma = \ GreaterThan(1, f(z))$$

The substitutions are performed simultaneously like `let`,
not sequentially like `let*`

$$S = \ GreaterThan(x, y)$$
$$\sigma = \{x \leftarrow 2, y \leftarrow g(x)\}$$
$$S\sigma = \ GreaterThan(2, g(x))$$

Interacting with FOL KBs

Suppose an agent has an FOL KB of
axioms for how the Wumpus world works

A model of the KB consists of a domain and interpretation (e.g., an actual Wumpus World) that makes every sentence in the KB true

Suppose we have a way to do inference in the FOL KB (see next chapter)

Suppose the agent perceives a smell and a breeze (but no glitter) at $t = 5$:

$Tell(KB, Percept([Smell, Breeze, None], 5))$
$Ask(KB, \exists\, a \;\; Action(a, 5))$

I.e., does KB entail any particular actions at $t = 5$?

Answer: $Yes,\; \{a \leftarrow Shoot\}$ $\leftarrow$ *substitution*

$Ask(KB, S)$ returns some/all σ such that $KB \models S\sigma$

Knowledge base for the wumpus world

"Perception"
$\forall\, b, g, t \;\; Percept([Smell, b, g], t) \;\Rightarrow\; Smelled(t)$
$\forall\, s, b, t \;\; Percept([s, b, Glitter], t) \;\Rightarrow\; AtGold(t)$

Reflex: $\forall\, t \;\; AtGold(t) \;\Rightarrow\; Action(Grab, t)$

Reflex with internal state: do we have the gold already?
$\forall\, t \;\; AtGold(t) \land \neg Holding(Gold, t) \;\Rightarrow\; Action(Grab, t)$

$Holding(Gold, t)$ cannot be observed
$\Rightarrow$ keeping track of change is essential

Deducing hidden properties

Properties of locations:
$\forall\, x, t \;\; At(Agent, x, t) \land Smelled(t) \;\Rightarrow\; Smelly(x)$
$\forall\, x, t \;\; At(Agent, x, t) \land Breeze(t) \;\Rightarrow\; Breezy(x)$

Squares are breezy near a pit:

Diagnostic rule—infer cause from effect

$$\forall\, y \;\; Breezy(y) \;\Rightarrow\; \exists\, x \;\; Pit(x) \land Adjacent(x, y)$$

Causal rule—infer effect from cause

$$\forall\, x, y \;\; Pit(x) \land Adjacent(x, y) \;\Rightarrow\; Breezy(y)$$

Neither of these is complete—e.g., the causal rule doesn't say whether squares far away from pits can be breezy

Definition for the $Breezy$ predicate:

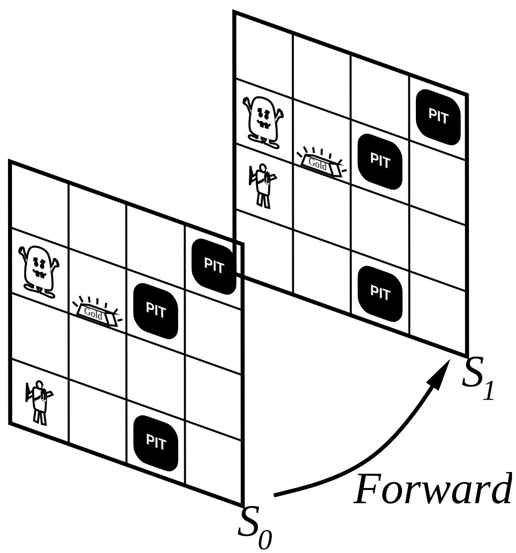
$$\forall\, y \;\; Breezy(y) \;\Leftrightarrow\; [\exists\, x \;\; Pit(x) \land Adjacent(x, y)]$$

Keeping track of change

Facts hold in *situations*, rather than eternally
E.g., truth of $Holding(Gold)$ depends on whether we've grabbed the gold

Situation calculus is one way to represent change in FOL:
Adds a situation argument to each non-eternal predicate
e.g., $Holding(Gold, s)$ to mean we're holding the gold in situation s

Situations are connected by the $Result$ function
$Result(a, s)$ is the situation that results from doing a in s



$= Result(Forward, S_0)$

Describing actions I

"Effect" axiom—describe changes due to action
$\forall\, s \;\; AtGold(s) \;\Rightarrow\; Holding(Gold, Result(Grab, s))$

Is this the only effect?

Describing actions I

"Effect" axiom—describe changes due to action
$\forall s \;\; AtGold(s) \;\Rightarrow\; Holding(Gold, Result(Grab, s))$

"Frame" axiom—describe **non-changes** due to action
$\forall s \;\; HaveArrow(s) \;\Rightarrow\; HaveArrow(Result(Grab, s))$

Frame problem: find a way to handle non-change

Qualification problem: true descriptions of real actions require endless caveats—what if gold is slippery or nailed down or ...

Ramification problem: real actions have many secondary consequences—what about the dust on the gold, wear and tear on gloves, ...

The Wumpus world is so simple that it's easy to specify all the qualifications and ramifications of each action

But we still need to handle the frame problem

Summary

First-order logic:
– objects and relations are semantic primitives
– syntax: constants, functions, predicates, equality, quantifiers

Increased expressive power: sufficient to define wumpus world

Situation calculus:
– conventions for describing actions and change in FOL
– can formulate and solve planning problems (inefficiently)
  as inference on a situation calculus KB