

Neural Networks

Assignment 4

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November 11, 2017

1 Summary

1.1 Single-Layer Perceptrons

- Pioneering contributions in neural network
 - McCulloch and Pitts introduced idea of neural networks as computing machines
 - Hebb introduced first rule of self-organized learning
 - Rosenblatt proposed model of perceptron for supervised learning
- **Rosenblatt's perceptron**
 - Perceptron is simplest form of neural network used for classification of linearly separable patterns
 - Consists of single neuron with adjustable weights and bias
 - Rosenblatt introduced perceptron convergence theorem that proved, if patterns/vectors used for training the network are linearly separable, then perceptron always converges and hyperplane can easily separate these patterns
 - Single neuron corresponds to the adaptive filter such as ADALINE (adaptive linear neuron)

1.2 Adaptive Filtering Problem

- Adaptive filtering operation consists of two continuous processes
 - Filtering process : Involves the computation of two signals
 - * Output is produced based on input vectors
 - * Error signal obtained by comparing current output to the desired output
 - Adaptive process : Involves weights adjustment based on error signal
- Combination of these two processes produce feedback loop acting around the neuron

1.3 Unconstrained Optimization Techniques

- It is a way of adjusting weight vector w of an adaptive filter algorithm to minimize the cost function with respect of weight vector
- Class of unconstrained optimization algorithms is based on idea of local iterative descent
 - Start with a initial guess denoted by $w(0)$
 - Generate a sequence of weight vectors $w(1), w(2), \dots$, such that cost function is reduced at each iteration of the algorithm

Three unconstrained optimization methods that are based on idea of local iterative descent are:

1.3.1 Steepest Descent

- Successive adjustments of the weight vector w are in the direction of steepest descent; in a direction opposite to the gradient vector $\Delta\epsilon(w)$
- Steepest descent algorithm is formally described by:

$$w(n+1) = w(n) - \eta g(n)$$

where η is positive constant called step-size or learning parameter and $g(n)$ is the gradient vector evaluated at the point $w(n)$

- Weight is adjusted as :

$$\Delta w(n) = w(n+1) - w(n) = -\eta g(n)$$

- Learning-rate parameter has good influence on convergence behavior
 - When η is small, transient response is over-damped, and trajectory traced by $w(n)$ follows a smooth path in the W -plane
 - When η is large, transient response is over-damped, and trajectory traced by $w(n)$ follows a zigzag(oscillatory) path in the W -plane
 - When it exceeds than certain threshold, algorithm becomes unstable(and it diverges)

1.3.2 Newton's Method

- Basic idea of this method is to minimize the quadratic approximation of the cost function $\epsilon(w)$ around the current point $w(n)$
- second-order Taylor series expansion of the cost function is used around the point $w(n)$
- Generally, this method converges quickly asymptotically and does not exhibit the zigzagging behavior, unlike steepest descent method
- However, for newton's method to work
 - Hessian $H(n)$ should be positive definite matrix for all iteration n
 - Unfortunately, in general there is no guarantee that $H(n)$ is positive definite
 - If the Hessian $H(n)$ is not positive definite, modifications of Newton's method is necessary

1.3.3 Gauss-Newton Method

- This method is applicable to a cost function that is expressed as sum of error squares(i.e. Special case of Newton) :

$$\epsilon(w) = \frac{1}{2} \sum_{i=1}^n e^2(i)$$

- Weight update is given as:

$$w(n+1) = w(n) - (J^T(n)J(n))^{-1}J^T(n)e(n)$$

- Gauss-Newton method only requires the Jacobian matrix of the error vector $e(n)$, rather than knowledge of Hessian matrix of the cost function as in Gauss method
- However, for this method, iteration should be computable and the matrix product $J^T(n)J(n)$ must be nonsingular. In other words, n rows of $J(n)$ must be linearly independent
- Unfortunately, rows of $J(n)$ are always linearly independent, therefore, to get rid of this deficiency we add diagonal matrix δI to the matrix $J^T(n)J(n)$

- The parameter ρ is a small positive constant chosen to ensure that

$$J^T(n)J(n) + \rho I$$

is positive definite for all n

- On this basis, Gauss-Newton method is slightly modified as :

$$w(n+1) = w(n) - (J^T(n)J(n) + \delta I)^{-1} J^T(n)e(n)$$

1.4 Linear Least-Squares Filter

- Linear least-squares filter has two distinctive characteristics
 - This is one of application of Gauss-Newton method
 - Single neuron is linear
 - Cost function $\epsilon(w)$ used to design the filter consists of the sum of error squares
 - Error vector $e(n)$ is given as:

$$e(n) = d(n) - X(n)w(n)$$

where $d(n)$ is desired vector of n -by-1 and $X(n)$ is input matrix of n -by- m

- Differentiating above equation, we get:

$$\Delta e(n) = -X^T(n)$$

- Jacobian of $e(n)$ is:

$$J(n) = -X(n)$$

- Apply Gauss Newton, we get:

$$w(n+1) = X^+(n)d(n)$$

where X^+ is pseudo-inverse

1.5 Least-Mean-Square Algorithm

- Least-mean-square(LMS) algorithm is based on use of instantaneous values for cost function:

$$\epsilon(w) = \frac{1}{2}e^2(n)$$

- LMS algorithm operates with a linear neuron, error signal is expressed as :

$$e(n) = d(n) - x^T(n)w(n)$$

- Weight update is given as :

$$\hat{w}(n+1) = \hat{w}(n) + \eta x(n)e(n)$$

where η is learning-rate parameter and $\hat{w}(n)$ is estimate of the weight vector

- In steepest descent algorithm weight vector $w(n)$ follows a well-defined trajectory in weight space for prescribed n . While, in LMS algorithm, weight vector $\hat{w}(n)$ traces a random trajectory. Due to this reason, LMS algorithm is also called as Stochastic gradient algorithm

1.5.1 Convergence Considerations of the LMS Algorithm

- From practical point of view, first criterion for convergence of the LMS algorithm is convergence in the mean square :

$$E[e^2(n)] \longrightarrow \text{constant} \text{ as } n \longrightarrow \infty$$

- LMS is convergent in the mean square provided that η satisfies the condition

$$0 < \eta < \frac{2}{\lambda_{max}}$$

where λ_{max} is the largest eigenvalue of correlation matrix R

- In typical applications, knowledge of λ_{max} is not available. Therefore, we use trace of R_x as a conservative estimate for λ_{max} . Equation can be re-written as:

$$0 < \eta < \frac{2}{tr[R_x]}$$

1.5.2 Virtues and Limitations of the LMS Algorithm

- One of important virtue of the LMS algorithm is its simplicity, and since it is model independent and therefore robust; that means small model uncertainty and small disturbances can only result in small estimation errors
- Primary limitations of LMS algorithm are its slow rate of convergence and sensitivity to variations in the eigen structure of the input
- Mostly these algorithms require number of iterations equal to about 10 times the dimensionality of the input space for it to reach a steady-state condition
- Hessian matrix H is defined as the second derivative of the cost function $\epsilon(w)$ with respect to w , which is equal to R_x

2 Exercise 3.8

2.1 (a)

Cost function is given as :

$$J(w) = \frac{1}{2}E[(d(n) - x^T(n)w)^2]$$

Expanding square formula, we have:

$$J(w) = \frac{1}{2}E[d^2(n) - 2d(n)x^T(n)w + w^T x^T(n)x(n)w]$$

Above equation can be simplified as :

$$J(w) = \frac{1}{2}E[d^2(n)] - E[d(n)x^T(n)w] + \frac{1}{2}E[w^T x^T(n)x(n)w]$$

Since $R_x = E[x(n)x^T(n)]$, $r_x d^T = E[x(n)^T d(n)]$ and $\sigma^2 = E[d^2(n)]$
Therefore, above equation can be written as:

$$J(w) = \frac{1}{2}\sigma^2 - r_{xd}^T w + \frac{1}{2}w^T R_x w$$

2.2 (b)

Since cost function is given as:

$$J(w) = \frac{1}{2}\sigma^2 - r_{xd}^T w + \frac{1}{2}w^T R_x w$$

Gradient vector is first partial derivative of cost function with respect to w , therefore, it can be written as:

$$\frac{\partial J(w)}{\partial w} = 0 - r_{xd} + \frac{1}{2}2R_x w$$

Simplify it, we have:

$$g = -r_{xd} + R_x w$$

Hessian matrix is defined as second partial derivative of cost function with respect to w , therefore, it can be computed as :

$$\frac{\partial J(w)}{\partial w} = -r_{xd} + R_x w$$

$$\frac{\partial J^2(w)}{\partial w^2} = 0 + R_x$$

Simplify it, we have

$$H = -R_x$$

2.3 (c)

In the newton method, update weight is given as :

$$w(n+1) = w(n) - H^{-1}(n)g(n) \quad (1)$$

Since Hessian matrix is equal to correlation matrix R_x . Therefore, above equation can be re-written as:

$$w(n+1) = w(n) - R^{-1}g(n) \quad (2)$$

When we replace g with instantaneous values:

$$g = -r_{xd} + R_x w$$

Substituting values of r_{xd} and R_x , we have

$$g = -x(n)d(n) + x(n)x^T(n)w(n)$$

Substitute g in equation(2) or LMS algorithm , we get :

$$w(n+1) = \hat{w}(n) + \eta R^{-1}(-x(n)d(n) + x(n)x^T(n)w(n)) \quad (3)$$

Simplifying it, we get:

$$w(n+1) = \hat{w}(n) + \eta R^{-1}(x(n)d(n) - x(n)x^T(n)w(n)) \quad (4)$$