Design and Analysis of Algorithms CSE2012

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Asymptotic Notations

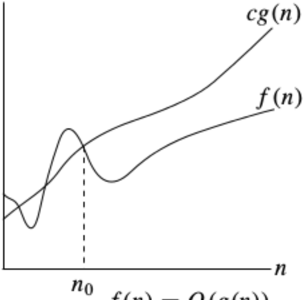
- The step count for estimating Time Complexity is inexact.
- Reason: The step count is one for either instruction x=y or x = y+(x*y)
- The running time for latter instruction is high
- So this step count is not very useful comparative purpose of algorithms.
- But Step Count method can be used to predict the running time growth of the algorithm for larger instance size accurately.
- Larger Instance Size → asymptote as n approaches infinity.
- So, We can use this to predict the relative performance of two programs when the instance size becomes large.
- Notations are used to Characterise main factors, affecting the Running time of algorithm
  - These Notations are not considering detail examination of primitive operations in algorithm.
- Asymptotic Notations are used to formalized.

- Simply, Asymptotic Notations are used to formalize " An algorithm has running time or storage requirement that are NEVER MORE THAN, ALWAYS GREATER THAN, or EXACTLY some amount"
- There are five asymptotic notations are available
  - Big - Oh ($\bigcirc$) Notation.
  - Big - Omega (Ω) Notation.
  - Theta (θ) Notation.
  - Little -Oh ($\circ$) Notation.
  - Little - Omega (ω) Notation.

- Definition: " Let *f(n)* and *g(n)* are two positive functions, the function *f(n)* = *O(g(n))* [read as "f of n is big Oh of g of n "] if and only if, ∃ positive constants c and n_0 such that f(n) ≤ c g(n) for all n, where n ≥ n_0".

- The Big-Oh notation allows
  - f(n) is less that or equal to another function, up to a constant factor-c and in the asymptotic sense as n grows towards infinity [n $\geq$ no].
  - The growth rate of f(n) is always less than or equal to the growth rate of g(n) for large n-values.
  - The Big - Oh notation is used widely to characterize running times and space bounds in terms of some parameters - n.
  - f(n)=O(g(n)) states only that g(n) is an upper bound on the value of f(n) for all n, where n $\geq$ n_0.
  - Big -Oh does not say about how good the bound is.

- Big -Oh Graph Representation:



$f(n) = O(g(n))$

- Rules of Big - Oh Notation:
  - The constants are ignored. Example: O(100n) is write as O(n). Here 100 is ignored.
  - Lower order terms are also ignored. Example: $O(2n^2+26n+39)$ is written as $O(n^2)$.

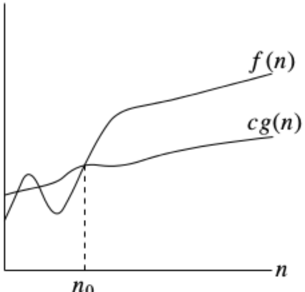
- Find the Big - Oh Representation of f(n) = 3n+2.
- According to the definition of Big Oh Notation, $f(n) \leq c\ g(n)$ for $n \geq n_0$.

$$3n + 2 \leq cn \quad \text{here g(n) is n}$$
$$2 \leq c.n - 3n$$
$$2 \leq n(c - 3)$$
$$2/(c - 3) \leq n \quad \text{here we take n as } n_0$$
$$n_0 \geq 2/(c - 3)$$

- According to the definition of Big - Oh, c and n_0 are positive constants. $\rightarrow$ So we should choose c-value that leads n_0 should to positive.
- So choose c has 4, that implies n_0 has 2.
- So f(n) = 3n+2 = O(4n). here we can neglect c-value in Big - Oh representation.
- So f(n) = 3n+2 = O(n), where $n \geq 2$.

- Definition: " Let *f(n)* and *g(n)* are two positive functions, the function $f(n) = \Omega(g(n))$ [read as "f of n is big Omega of g of n "] if and only if, $\exists$ positive constants c and n_0 such that $f(n) \geq c\, g(n)$ for all n, where $n \geq n_0$".
- $f(n)=\Omega(g(n))$ states only that g(n) is an Lower bound on the value of f(n) for all n, where $n \geq n_0$.

- Big -Omega Graph Representation:



$f(n) = \Omega(g(n))$

Big-Omega Notation Example

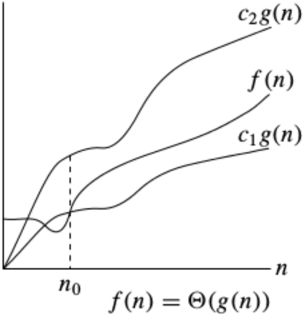
- Find Big-Omega Representation of f(n) = 3n+2
- According to the deffinition of Big-Omega notation, f(n) $\geq$ c g(n) for n $\geq n_0$.

$$3n + 3 \geq cn \quad \text{here g(n) is n}$$
$$2 \geq c.n - 3n$$
$$2 \geq n(c - 3)$$
$$2/(c - 3) \geq n \text{here we take n as } n_0$$
$$n_0 \leq 2/(c - 3)$$

- According to the definition of Big - Omega, c and n_0 are positive constants. $\rightarrow$ So we should choose c-value that leads n_0 should to positive.
- So choose c has 3, that implies n_0 starts from 0.
- So f(n) =3n+2 = Ω(3n). here we can neglect c-value in Big-Omega representation.
- So f(n) = 3n+2 = Ω(n), where n $\geq$ 0.

- Definition: " Let *f(n)* and *g(n)* are two positive functions, the function $f(n) = \theta(g(n))$ [read as "f of n is Theta of g of n "] if and only if, $\exists$ positive constants c_1, c_2 and n_0 such that $c_1 g(n) \leq f(n) \leq c_2\ g(n)$ for all n, where $n \geq n_0$".
- g(n) is the an asymptotically tight bound for f(n).

- Theta Graph Representation:



Theta Notation Example

- Find the Theta Representation of $f(n) = 3n+2$?
- Here we have calculate lower bound and upper bound of $f(n) = 3n+2$.
- In previous Slide we did estimate for the same function.
- So the lower Bound is $3n+2 = 3n$, $n \geq 0$, where $c_1 = 3$ and n_0 starts from 0.
- The Upper Bound is $3n+2 = 4n$, $n \geq 2$, where $c_2 = 4$ and n_0 starts from 2.
- Here we have to find the same n_0 for lower and upper bound.
- Assume n_0 is starting from 2.
- According to definition, $c_1 g(n) \leq f(n) \leq c_2\, g(n)$ for all n, where $n \geq n_0$.
- $3n \leq 3n+2 \leq 3n+4$ for all n, where $n \geq 2$
  - Here $c_1 = 3$ and $c_2 = 4$ and $n_0 \geq 2$
  - So $f(n) = 3n+2 = \theta(n)$ where $c_1 = 3$ and $c_2 = 4$ and $n_0 \geq 2$

Theorem

For any two functions f(n) and g(n), f(n) $= \theta(g(n))$ if and only if f(n) $=$ O(g(n)) and f(n)$=$ Ω(g(n))

- Definition: " Let *f(n)* and *g(n)* are two positive functions, the function $f(n) = o(g(n))$ [read as "f of n is little-oh of g of n "] if and only if, $\exists$ positive constants c and n_0 such that $f(n) < c\, g(n)$ for all n, where $n > n_0$".

- Definition: " Let *f(n)* and *g(n)* are two positive functions, the function *f(n)* $=$ *o(g(n))* [read as "f of n is little-omega of g of n "] if and only if, $\exists$ positive constants c and n_0 such that f(n) $>$ c g(n) for all n, where n $>$ n_0".

- Let f(n) and g(n) be asymptotically positive functions.
  - $f(n) = O(g(n))$ implies $g(n) = O(f(n))$
  - $f(n) + g(n) = \Theta(\min(f(n), g(n)))$
  - $f(n) + o(g(n)) = \Theta(f(n))$
  - $f(n) = O(g(n))$ implies $g(n) = \Omega(f(n))$
- Many of the relational properties of real numbers apply to asymptotic comparisons as well.
  - **Transitivity:**
    - $f(n) = \Theta(g(n))$ and $g(n) = \Theta(h(n))$ imply $f(n) = \Theta(h(n))$
    - $f(n) = O(g(n))$ and $g(n) = O(h(n))$ imply $f(n) = O(h(n))$
    - $f(n) = \Omega(g(n))$ and $g(n) = \Omega(h(n))$ imply $f(n) = \Omega(h(n))$
    - $f(n) = o(g(n))$ and $g(n) = o(h(n))$ imply $f(n) = o(h(n))$
    - $f(n) = \omega(g(n))$ and $g(n) = \omega(h(n))$ imply $f(n) = \omega(h(n))$
  - **Reflexivity:**
    - $f(n) = \Theta(f(n))$
    - $f(n) = \Omega(f(n))$
    - $f(n) = O(f(n))$

- **Symmetry:**
  - $f(n) = \Theta(g(n))$ if and only if $g(n) = \Theta(f(n))$
- **Transpose symmetry:**
  - $f(n) = O(g(n))$ if and only if $g(n) = \Omega(f(n))$
  - $f(n) = o(g(n))$ if and only if $g(n) = \omega(f(n))$
- Let f(n) and g(n) be asymptotically nonnegative functions. Using the basic definition of Θ notation, it can be proved that $\max(f(n), g(n)) = \Theta\ (f(n)+g(n))$.