Moore-Penrose Inverses of Block Circulant and Block k-Circulant Matrices

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ABSTRACT

The Moore-Penrose inverse A^+ of a block circulant matrix whose blocks are arbitrary square matrices is obtained. An explicit form is given for A^+ in terms of the blocks of A. The eigenvalues of A are determined in terms of the eigenvalues of the blocks where the blocks themselves are circulants.

1. INTRODUCTION

In recent papers Cline, Plemmons, and Worm [2] have determined the spectral and Moore-Penrose inverses of certain k-circulants, and Pye, Boullion, and Atchison [3] and Trapp [4] have determined the Moore-Penrose inverse of square block circulant matrices whose blocks are square circulants. Block circulant matrices have also been investigated by Chao [1]. Here we define a block k-circulant and generalize some of the results of [2] to block k-circulants. Then, we obtain the result that A is a block circulant if and only if A^+ is a block circulant. Lastly, we determine the Moore-Penrose inverse of square block circulant matrices in the case where the blocks are arbitrary square matrices and in the case where the blocks are $[r]$-circulants.

2. PROPERTIES OF k-CIRCULANTS

The results listed in this section all come from [2] and are used in later sections.

Let k be a complex number, and let $Q=(q_{ij})$ be the $n\times n$ matrix defined by $q_{i,i+1}=1$ for $i=1,2,\dots,n-1$, $q_{n1}=k$, and $q_{ij}=0$ otherwise. An $n\times n$ complex matrix $A=(a_{ij})$ is a k-circulant if A is defined by $a_{ij}=a_{j-1}$ for $i\leqslant j$ and $a_{ij}=ka_{n+j-1}$ for $i>j$.

THEOREM A (Cline-Plemmons-Worm). *An $n\times n$ matrix A is a k-circulant if and only if $AQ=QA$. In this case, $A=\sum_{i=0}^{n-1}a_iQ^i$, where $(a_0,a_1,\dots,a_{n-1})$ is the first row of A.*

COROLLARY A. *The Moore-Penrose inverse of a circulant matrix is a circulant matrix.*

3. BLOCK CIRCULANTS AND BLOCK k-CIRCULANTS

A complex matrix $A=(A_{ij})$, $1\leqslant i,\ j\leqslant n$, is a *block k-circulant* if A is defined by $A_{ij}=A_{j-1}$ for $i\leqslant j$ and $A_{ij}=kA_{n+j-1}$ for $i>j$, where each A_i is $m\times m$. If $k=1$, then A is a *block circulant.*

Let A be an $m\times n$ complex matrix and B a $p\times q$ complex matrix. The *Kronecker product* of A and B, denoted $A\otimes B$, is defined by $A\otimes B=(C_{ij})$ $(1\leqslant i\leqslant m;\ 1\leqslant j\leqslant n)$, where $C_{ij}=a_{ij}B$. It is well known that $(A\otimes B)(C\otimes D)=AC\otimes BD$ if all multiplications are defined.

We have the following generalization of Theorem B below, where $\hat{Q}=Q\otimes I_m$. The proof is quite similar to the proof of Theorem B.

LEMMA 1. *Let $A=(A_{ij})$, $1\leqslant i,\ j\leqslant n$, where each A_{ij} is $m\times m$. Then A is a block k-circulant if and only if $A\hat{Q}=\hat{Q}A$. In this case, $A=\sum_{i=0}^{n-1}Q^i\otimes A_i$, where $(A_0,A_1,\dots,A_{n-1})$ is the first "block" row of A.*

Proof. Define $A_{j-1}=A_{1j}$ for all $1\leqslant j\leqslant n$. Throughout the proof the subscripts are assumed to be reduced modulo n. Let $A\hat{Q}=(S_{ij})$ and $A\hat{Q}=(T_{ij})$, where S_{ij} and T_{ij} are $m\times m$ blocks for all $1\leqslant i,\ j<n$. Since $Q_{ij}=0$ for $i\neq j-1 \bmod n$, we have $S_{i,j+1}=A_{i,j}Q_{j,j+1}$ and $T_{i,j+1}=Q_{i,i+1}A_{i+1,j+1}$. Now, $A\hat{Q}=\hat{Q}A$ if and only if $A_{i,j}Q_{j,j+1}=Q_{i,i+1}A_{i+1,j+1}$. The last condition is equivalent to

$$A_{ij}=\begin{cases} A_{i+1,j+1} & \text{if } 1\leqslant i,\ j<n \text{ or } i=j=n,\\ kA_{1,j+1} & \text{if } i=n \text{ and } 1\leqslant j<n.\end{cases}$$

But this means A is a block k-circulant, i.e.,

$$A_{ij}=\begin{cases} A_{j-1} & \text{if } i\leqslant j,\\ kA_{n+j-1} & \text{if } i>j.\end{cases}$$

It is then obvious that $A=\sum_{i=0}^{n-1}Q^i\otimes A_i$. ■

THEOREM 1. *Let A be a block k-circulant, where k has unit modulus. Then A^+ is a block k-circulant. In particular, the Moore-Penrose inverse of a block circulant is a block circulant.*

Proof. If k has unit modulus, then $\hat{Q}$ is unitary. Hence, $A^+ = (\hat{Q}A\hat{Q}^*)^+ = \hat{Q}A^+\hat{Q}^*$. ∎

We shall now introduce a construction made in [2]. Let ω be any primitive nth root of unity and λ any nth root of k. Let $\Omega_n = (m_{ij})$, where

$$m_{ij} = \frac{1}{\sqrt{n}}\omega^{(i-1)(j-1)}, \qquad 1 \leqslant i,j \leqslant n,$$

for each positive integer n. Let $\Lambda = \operatorname{diag}(1,\lambda,\lambda^2,\dots,\lambda^{n-1})$, and let $T = \Lambda\Omega_n$. We have the following properties.

(1) Ω_n is unitary.

(2) C is a circulant in C_n if and only if $\Omega_n^* C\Omega_n = \operatorname{diag}(\lambda_0,\lambda_1,\dots,\lambda_{n-1})$, where the λ_i are the eigenvalues of C.

(3) $T^{-1}QT = \operatorname{diag}(\lambda,\lambda\omega,\lambda\omega^2,\dots,\lambda\omega^{n-1})$.

THEOREM B (C-P-W). *A is a k-circulant if and only if $A = \Lambda C\Lambda^{-1}$ for some circulant matrix C.*

We have the following characterization of block k-circulants as a generalization of Theorem B. Since the proof is analogous to the proof of Theorem B, it is omitted.

LEMMA 2. *Let $\hat{\Lambda} = \Lambda \otimes I$. A is a block k-circulant if and only if $A = \hat{\Lambda}C\hat{\Lambda}^{-1}$ for some block circulant matrix C.*

In [3] it was noted that if A is a nonsingular k-circulant, then A^{-1} is a k-circulant. On the other hand, the Moore-Penrose inverse A^+ of a singular k-circulant need not be a k-circulant. The following result gave necessary and sufficient conditions for A^+ to be k-circulant.

THEOREM C (C-P-W). *Let A be a k-circulant and singular. Then A^+ is a k-circulant if and only if k lies on the unit circle.*

This result is generalized to the case of block k-circulants. It is noted that if A is a nonsingular block k-circulant, then A^{-1} is a block k-circulant, but as in the case of k-circulants, the Moore-Penrose inverse A^+ of a singular block k-circulant need not be a block k-circulant.

THEOREM 2. *Let A be a block k-circulant matrix such that $AA^+ = (AA^+)^*$ is not a diagonal matrix. Then A^+ is a block k-circulant if and only if k lies on the unit circle.*

Proof. Let $R = (r_{ij})$ be the n-square matrix defined by $r_{i+1,i} = 1$ for $i = 1, 2, \ldots, n-1$, $r_{1n} = 1/k$, and $r_{ij} = 0$ otherwise. Then $R = Q^{-1}$, and thus $\hat{R} = R \otimes I = \hat{Q}^{-1}$. Observe that if $C = \hat{R}^* C (\hat{R}^*)^{-1}$, then C is a block $1/\bar{k}$-circulant matrix. If A^+ is a block k-circulant, then $A^+ = \hat{Q} A^+ \hat{Q}^{-1}$. Thus, $AA^+ = \hat{Q} AA^+ \hat{Q}^{-1}$, which implies AA^+ is a block k-circulant by Lemma 1. Since $AA^+ = (AA^+)^*$, we have $AA^+ = (\hat{Q}^{-1})^* AA^+ \hat{Q}^* = \hat{R}^* AA^+ (\hat{R}^*)^{-1}$, which implies AA^+ is a block $1/\bar{k}$-circulant. Therefore, $k = 1/\bar{k}$, which implies $|k| = 1$. The converse follows from Theorem 1.

Let C_i, $1 \leqslant i \leqslant p$, be a set of matrices, each of order m_i. The direct sum of this set, denoted $\sum_{i=1}^{p}{}^{\bullet}\, C_i$, is defined by

$$\sum_{i=1}^{p}{}^{\bullet}\, C_i = \operatorname{diag}(C_1, C_2, \ldots, C_p).$$

In the following let $\sum_{i=0}^{n-1} (\lambda\omega^j)^i A_i = B_j$, $T \otimes I_m = \hat{T}$, and $\Omega_n \otimes I_m = \hat{\Omega}_n$.

THEOREM 3. *A is a block k-circulant if and only if*

$$A = \hat{T}\left(\sum_{j=0}^{n-1}{}^{\bullet}\, B_j\right)\hat{T}^{-1}.$$

Proof. Assuming A is a block k-circulant, we have

$$\begin{aligned}
\hat{T}^{-1} A \hat{T} &= (T^{-1} \otimes I)\left(\sum_{i=0}^{n-1} Q^i \otimes A_i\right)(T \otimes I) \\
&= \sum_{i=0}^{n-1} \left(T^{-1} Q^i T \otimes A_i\right) \\
&= \sum_{i=0}^{n-1} \operatorname{diag}\left(\lambda^i, (\lambda\omega)^i, (\lambda\omega^2)^i, \ldots, (\lambda\omega^{n-1})^i\right) \otimes A_i \\
&= \sum_{j=0}^{n-1}{}^{\bullet}\left(\sum_{i=0}^{n-1} (\lambda\omega^j)^i A_i\right) = \sum_{j=0}^{n-1}{}^{\bullet}\, B_j.
\end{aligned}$$

Since the steps are reversible, the converse holds. ∎

COROLLARY 1. *If k has unit modulus, then A is a block k-circulant if and only if*

$$A^{+}=\hat{T}\left(\sum_{j=0}^{n-1}{}^{\bullet} B_j^{+}\right)\hat{T}^{*}.$$

Proof. If k has unit modulus, T is unitary and thus T is unitary. The result follows from these two well-known properties of the Moore-Penrose inverse:

(1) if $A=UBV$ where U and V are unitary, then $A^{+}=V^{*}B^{+}U^{*}$, and
(2) if $A=\sum_{j=0}^{k}{}^{\bullet} B_j$, then $A^{+}=\sum_{j=0}^{k}{}^{\bullet} \mathrm{B}_{\mathrm{j}}^{+}$.

COROLLARY 2. *A is a block circulant if and only if*

$$A^{+}=\hat{\Omega}_n\left(\sum_{j=0}^{n-1}{}^{\bullet} B_j^{+}\right)\hat{\Omega}_n^{*}.$$

Proof. Take $\lambda=1$ and thus $\hat{T}=\hat{\Omega}_n$. Now apply Corollary 1. ■

THEOREM 4. *Suppose k has unit modulus. Then the following are equivalent*:

(1) *A is a block k-circulant.*
(2) *A^{+} is a block k-circulant.*
(3) $A^{+}=\hat{T}\left(\sum_{j=0}^{n-1}{}^{\bullet} B_j^{+}\right)\hat{T}^{*}$.

Proof. (1) is equivalent to (2) by Theorem 1. (1) is equivalent to (3) by Corollary 1. ■

Note that $k=1$ gives the result that A is a block circulant if and only if A^{+} is a block circulant, and in either case

$$A^{+}=\hat{\Omega}_n\left[\sum_{j=0}^{n-1}{}^{\bullet}\left(\sum_{i=0}^{n-1} w^{ji}A_i\right)^{+}\right]\hat{\Omega}_n^{*}.$$

In the single block case with $|k|=1$, A is a k-circulant if and only if A^+ is a k-circulant, and in either case

$$A^+ = T\mathrm{diag}\left[\left(\sum_{i=0}^{n-1} a_i\right)^+, \left(\sum_{i=0}^{n-1} \omega^i a_i\right)^+, \ldots, \left(\sum_{i=0}^{n-1} \omega^{(n-1)i} a_i\right)^+\right] T^*.$$

4. SPECIAL BLOCK AND BLOCK k-CIRCULANTS

The notion of r-circulants was introduced in [5]; we refer to them as $[r]$-circulants here to avoid confusion.

We say that a matrix $C=(a_{ij})$ is a $[r]$-*circulant* if $a_{ij}=a_{(j-1)-r(i-1)}$, $1 \leqslant i, j \leqslant n$, where r is a nonnegative integer and each of the subscripts is understood to be reduced modulo n.

We now consider the case where A is a block circulant and the blocks themselves are $[r]$-circulants. We shall need the following results from [5].

THEOREM C. *If C is a $[r]$-circulant, then $C^+ = C^*\Omega_n D^+\Omega_n^*$ where D is the diagonal matrix such that $\Omega_n^* CC^* \Omega_n = D$.*

THEOREM D. *If C and D are $[r]$-circulants, then CD^* is a $[1]$-circulant.*

We mention that the necessity part of Corollary 3 was shown in [3] and both parts were shown in [4] under a different construction.

COROLLARY 3. *Suppose the blocks of A are circulants. Then A is a block circulant if and only if*

$$A^+ = \hat{\Omega}_n\left(\sum_{j=0}^{n-1}{}^{\bullet}\, \Omega_m D_j^{\,+}\Omega_m^*\right)\hat{\Omega}_n^*,$$

where D_j is the diagonal matrix such that

$$\sum_{i=0}^{n-1} (\lambda\omega^j)^i A_i = \Omega_m D_j \Omega_m^*.$$

Proof. Since each block is a circulant, $\Sigma_{i=0}^{n-1}(\lambda\omega^j)^i A_i$ is a circulant and thus

$$D_j = \Omega_m^* \left[\sum_{i=0}^{n-1} (\lambda\omega^j)^i A_i \right] \Omega_m$$

is diagonal. Now apply Corollary 2. ■

COROLLARY 4. *Suppose the blocks of A are $[r]$-circulants. Then A is a block circulant if and only if*

$$A^+ = \hat{\Omega}_n \left(\sum_{j=0}^{n-1}{}^{\bullet} B_j^* \Omega_m D_j^+ \Omega_m^* \right) \hat{\Omega}_n^*,$$

where D_j is the diagonal matrix such that $\Omega_m^* B_j B_j^* \Omega_m = D_j$.

Proof. Since each A_i is a $[r]$-circulant, B_j is a $[r]$-circulant. By Theorem D, $B_j B_j^*$ is a $[1]$-circulant and thus $\Omega_m^* B_j B_j^* \Omega_m = D_j^+$, where D_j is a diagonal matrix. By Theorem C, $C_j^* \Omega_m D_j^+ \Omega_m^* = C_j$. Now apply Corollary 2. ■

We note that the computation of the Moore-Penrose inverse of a block circulant requires only matrix multiplication when the blocks are $[r]$-circulants. In the general case where the blocks are arbitrary, the computation has been reduced to the computation of n Moore-Penrose inverses of order m. Further, we mention that all this applies to block k-circulants where k has unit modulus.

We now state an explicit correspondence between eigenvalues of A and eigenvalues of the blocks comprising A in the case where the blocks are circulants.

COROLLARY 5. *Suppose A is a block k-circulant with each A_i a circulant. For each i, $0 \leqslant i \leqslant n-1$, let $\mu_{i1}, \mu_{i2}, \ldots, \mu_{in}$ denote the eigenvalues of A_i. Then α is an eigenvalue of A if and only if $\alpha = \Sigma_{i=0}^{n-1}(\lambda\omega^j)^i \mu_{ik}$ for some j and k, $1 \leqslant j, k \leqslant n$.*

Proof. Let $\tilde{\Omega}_m = \sum_{j=0}^{n-1}{}^{\bullet} \Omega_m$. By Theorem 3,

$$A = \hat{T}\left[\sum_{j=0}^{n-1}{}^{\bullet}\left(\sum_{i=0}^{n-1}(\lambda\omega^{j})^{i}A_{i}\right)\right]T^{-1}$$

$$= \hat{T}\left[\sum_{j=0}^{n-1}{}^{\bullet}\left(\sum_{i=0}^{n-1}(\lambda\omega^{j})^{i}\Omega_{m}D_{i}\Omega_{m}^{*}\right)\right]\hat{T}^{-1}$$

$$= \hat{T}\tilde{\Omega}_{m}\left[\sum_{j=0}^{n-1}{}^{\bullet}\left(\sum_{i=0}^{n-1}(\lambda\omega^{j})^{i}D_{i}\right)\right]\tilde{\Omega}_{m}^{*}\hat{T}^{-1}.$$

Since $[\hat{T}\tilde{\Omega}_{m}]^{-1}=\tilde{\Omega}_{m}^{*}\hat{T}^{-1}$, we are done. ■

We now consider certain block k-circulants which either are idempotent or else are equal to their Moore-Penrose inverse.

THEOREM 5. *Suppose that A is a block k-circulant, where k has unit modules, and the blocks A_i are circulants. Then*

(i) *A is idempotent if and only if A is unitarily similar to a diagonal matrix with eigenvalues* 0 *or* 1.

(ii) *$A=A^{+}$ if and only if A is unitarily similar to a diagonal matrix with eigenvalues* 0, 1, *or* -1.

Proof. If A is block k-circulant where k has unit modulus, then

$$A = \hat{T}\left[\sum_{j=0}^{n-1}{}^{\bullet}\left(\sum_{i=0}^{n-1}(\lambda\omega^{j})^{i}A_{i}\right)\right]\hat{T}^{*}$$

by Corollary 1.

Since each A_i is a circulant, $\sum_{i=0}^{n-1}(\lambda\omega^{j})^{i}A_{i}$ is a circulant, and thus there exists a diagonal matrix D_j such that $\sum_{i=0}^{n-1}(\lambda\omega^{j})^{i}A_{i}=\Omega_{m}D_{j}\Omega_{m}^{*}$ for $0\leqslant j\leqslant n-1$. Thus, $A=\hat{T}\tilde{\Omega}_{m}D\tilde{\Omega}_{m}^{*}\hat{T}^{*}$, where $D=\sum_{j=0}^{n-1}{}^{\bullet}D_{j}$ is diagonal.

Proof of (i). It is well known that the n-square matrix B is idempotent if and only if B is similar to a diagonal matrix with roots 0 or 1. Now $(\tilde{\Omega}_{m}\hat{T})^{-1}=\hat{T}^{*}\tilde{\Omega}_{m}^{*}$; hence if A is idempotent, D is idempotent, which implies the diagonal elements are either 0 or 1. The converse is obvious.

Proof of (ii). If $A^{+}=A$, then $A^{3}=A$, which implies $D^{3}=D$. Thus, each diagonal element is either 0, 1, or -1. Conversely, if $A=U^{*}DU$, where U is unitary and D is diagonal with roots 0, 1, or -1, then $A^{+}=U^{*}D^{+}U=U^{*}DU=A$. ■

It is to be noted that these results hold for block circulants where the blocks are circulant and for k-circulants where k has unit modulus.

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