ELSEVIER Linear Algebra and its Applications 283 (1998) 35–60

LINEAR ALGEBRA AND ITS APPLICATIONS

The Moore–Penrose inverses of $m \times n$ block matrices and their applications

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Received 22 September 1997; accepted 23 March 1998

Submitted by L. Rodman

Abstract

This article gives the expressions for the Moore–Penrose inverses of $m \times n$ block matrices when they satisfy rank additivity conditions, and presents some of their special cases and applications.

AMS classification: 15A09

Keywords: Block matrix; Moore–Penrose inverse; Inversion formula

1. Introduction

Let

$$M = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{pmatrix} \tag{1.1}$$

be an $m \times n$ block matrix over the complex number field $\mathscr{C}$, where A_{ij} is an $s_i \times t_j$ matrix $(1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n)$, and suppose that M satisfies the following rank additivity condition

$$r(M) = r(W_1) + r(W_2) + \cdots + r(W_m) = r(V_1) + r(V_2) + \cdots + r(V_n), \tag{1.2}$$

where

0024-3795/98/$19.00
PII: S0024-3795(98)10049-6

$$W_i = (A_{i1}, A_{i2}, \dots, A_{in}), \quad V_j = \begin{pmatrix} A_{1j} \\ A_{2j} \\ \vdots \\ A_{mj} \end{pmatrix}, \quad 1 \leqslant i \leqslant m, \ 1 \leqslant j \leqslant n. \tag{1.3}$$

The aim of this article is to consider the expressions of the Moore–Penrose inverse of the block matrix M in Eq. (1.1) when it satisfies Eq. (1.2) and to give some applications of the corresponding results in the theory of generalized inverses of matrices.

The work in this article is organized as follows. In Section 2, we first present the expressions of the Moore–Penrose inverse of the 2×2 block matrix

$$M_1 = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{1.4}$$

under the following rank additivity condition

$$r\begin{pmatrix} A & B \\ C & D \end{pmatrix} = r\begin{pmatrix} A \\ C \end{pmatrix} + r\begin{pmatrix} B \\ D \end{pmatrix} = r(A, \ B) + r(C, \ D), \tag{1.5}$$

where A, B, C and D are $m \times n$, $m \times k$, $l \times n$ and $l \times k$ matrices, respectively, and then list a group of consequences of the corresponding results. In Section 3, we extend the results in Section 2 to the general block matrix M in Eq. (1.1) when it satisfies the rank additivity condition (1.2). In Section 4, we give a set of formulas for representing the Moore–Penrose inverses of matrix sums, and illustrate the duality between the Moore–Penrose inverses of block matrices and matrix sums. In Section 5, we list a group of inversion formulas for the Moore–Penrose inverses of matrices derived from the results in Section 2.

Throughout this article, all our matrices will be over the complex number field $\mathscr{C}$. For a matrix A over $\mathscr{C}$, A^* and $A^\dagger$ stand for the conjugate transpose and the Moore–Penrose inverse of A, respectively, $r(A)$, $R(A)$ and $N(A)$ stand for the rank, the range, and the null space of A, respectively, the abbreviated symbols E_A and F_A stand for the two projectors $E_A = I - AA^\dagger$ and $F_A = I - A^\dagger A$. For the 2×2 block matrix M_1 in Eq. (1.4), $S_A = D - CA^\dagger B$ stands for the generalized Schur complement of A in M_1. Similarly, $S_B = C - DB^\dagger A$, $S_C = B - AC^\dagger D$ and $S_D = A - BD^\dagger C$ stand for the generalized Schur complements of B, C and D in M_1, respectively.

Next we list some well-known results on ranks and the Moore–Penrose inverses of matrices.

Lemma 1.1 [1]. *Let A, B, C and D be $m \times n$, $m \times k$, $l \times n$ and $l \times k$ matrices, respectively. Then*

$$r(A,\ B) = r(A) + r(E_AB) = r(E_BA) + r(B), \tag{1.6}$$

$$r\begin{pmatrix} A \\ C \end{pmatrix} = r(A) + r(CF_A) = r(AF_C) + r(C), \tag{1.7}$$

$$r\begin{pmatrix} A & B \\ C & O \end{pmatrix} = r(B) + r(C) + r(E_BAF_C), \tag{1.8}$$

$$r\begin{pmatrix} A & B \\ C & D \end{pmatrix} = r\begin{pmatrix} A & E_AB \\ CF_A & S_A \end{pmatrix}, \tag{1.9}$$

$$r\begin{pmatrix} A & B \\ C & D \end{pmatrix} = r(A) + r\begin{pmatrix} O & E_AB \\ CF_A & S_A \end{pmatrix}, \tag{1.10}$$

$$r\begin{pmatrix} A & B \\ C & D \end{pmatrix} = r\begin{pmatrix} A \\ C \end{pmatrix} + r(\ A,\ B) - r(A) + r(E_{C_1}S_AF_{B_1}), \tag{1.11}$$

where $S_A = D - CA^{\dagger}B,\ B_1 = E_AB$ *and* $C_1 = CF_A$.

For convenience of statement, we call the term $E_{C_1}S_AF_{B_1}$ in Eq. (1.11) the *rank complement* of D in $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, and denote it by $J(D) = E_{C_1}S_AF_{B_1}$, which will be widely used in the sequel.

Lemma 1.2 [2]. *Let* $A,\ B$ *and* C *be given as in Eqs.* (1.6) *and* (1.7).

(i) *If* $r(\ A,\ B) = r(A) + r(B)$, *i.e.*, $R(A) \cap R(B) = \{0\}$, *then*

$$(E_AB)^{\dagger}(E_AB) = B^{\dagger}B, \quad (\ A,\ B)^{\dagger}(\ A,\ B) = \begin{pmatrix} A^{\dagger}A & O \\ O & B^{\dagger}B \end{pmatrix}. \tag{1.12}$$

(ii) *If* $r\begin{pmatrix} A \\ C \end{pmatrix} = r(A) + r(C)$, *i.e.*, $R(A^*) \cap R(C^*) = \{0\}$, *then*

$$(CF_A)(CF_A)^{\dagger} = CC^{\dagger}, \quad \begin{pmatrix} A \\ C \end{pmatrix}\begin{pmatrix} A \\ C \end{pmatrix}^{\dagger} = \begin{pmatrix} AA^{\dagger} & O \\ O & CC^{\dagger} \end{pmatrix}. \tag{1.13}$$

Lemma 1.3 [1]. *Let* $A,\ B \in \mathscr{C}^{m\times n}$. *Then*

$$R(A) \cap R(B) = \{0\} \quad \Rightarrow \quad r(A+B) = r\begin{pmatrix} A \\ B \end{pmatrix}, \tag{1.14}$$

$$R(A^*) \cap R(B^*) = \{0\} \quad \Rightarrow \quad r(A+B) = r(A,\ B). \tag{1.15}$$

Lemma 1.4 [2,4]. *Let* $M_2 = \begin{pmatrix} A & B \\ C & O \end{pmatrix}$ *be a bordered matrix over* $\mathscr{C}$.

(i) *If M_2 satisfies the following rank additivity condition*

$$r(M_2) = r\begin{pmatrix} A \\ C \end{pmatrix} + r(B) = r(\ A,\ B) + r(C), \tag{1.16}$$

then

$$M_2^\dagger = \begin{pmatrix} L & C^\dagger - LAC^\dagger \\ B^\dagger - B^\dagger AL & B^\dagger(ALA - A)C^\dagger \end{pmatrix}, \tag{1.17}$$

where $L = (E_BAF_C)^\dagger$.

(ii) *If M_2 satisfies the following rank additivity condition*

$$r(M_2) = r(A) + r(B) + r(C), \tag{1.18}$$

or equivalently $R(A) \cap R(B) = \{0\}$ *and* $R(A^*) \cap R(C^*) = \{0\}$, *then*

$$M_2^\dagger = \begin{pmatrix} L & C^\dagger - LAC^\dagger \\ B^\dagger - B^\dagger AL & O \end{pmatrix}, \tag{1.19}$$

where $L = (E_BAF_C)^\dagger$ *satisfies* $A(E_BAF_C)^\dagger A = A$.

Finally we give an equivalent statement for the rank additivity condition (1.5), which will directly be used for determining the Moore–Penrose inverse of M_1 in Eq. (1.4).

Lemma 1.5. *The rank additivity condition* (1.5) *is equivalent to the following four conditions*

$$R\begin{pmatrix} A \\ O \end{pmatrix} \subseteq R(M_1), \quad R\begin{pmatrix} A^* \\ O \end{pmatrix} \subseteq R(M_1^*), \tag{1.20}$$

$$r\begin{pmatrix} O & E_AB \\ CF_A & S_A \end{pmatrix} = r\begin{pmatrix} E_AB \\ S_A \end{pmatrix} + r(CF_A) = r(\ CF_A,\ S_A) + r(E_AB), \tag{1.21}$$

where $S_A = D - CA^\dagger B$.

Proof. Let

$$V_1 = \begin{pmatrix} A \\ C \end{pmatrix}, \quad V_2 = \begin{pmatrix} B \\ D \end{pmatrix}, \quad W_1 = (\ A,\ B), \quad W_2 = (\ C,\ D). \tag{1.22}$$

Then Eq. (1.5) is equivalent to

$$R(V_1) \cap R(V_2) = \{0\} \quad \text{and} \quad R(W_1^*) \cap R(W_2^*) = \{0\}. \tag{1.23}$$

In that case, we easily find

$$r\begin{pmatrix} W_1 & A \\ W_2 & O \end{pmatrix} = r(\ W_1,\ A) + r(\ W_2,\ O) = r(W_1) + r(W_2) = r(M_1),$$

$$r\begin{pmatrix} V_1 & V_2 \\ A & O \end{pmatrix} = r\begin{pmatrix} V_1 \\ A \end{pmatrix} + r\begin{pmatrix} V_2 \\ O \end{pmatrix} = r(V_1) + r(V_2) = r(M_1),$$

both of which are equivalent to the two inclusions in Eq. (1.20). On the other hand, observe that

$$\begin{pmatrix} E_AB \\ S_A \end{pmatrix} = \begin{pmatrix} B - AA^\dagger B \\ D - CA^\dagger B \end{pmatrix} = \begin{pmatrix} B \\ D \end{pmatrix} - \begin{pmatrix} A \\ C \end{pmatrix} A^\dagger B = V_2 - V_1 A^\dagger B,$$

$$(CF_A,\ S_A) = (\ C - CAA^\dagger,\ D - CA^\dagger B) = (C,\ D) - CA^\dagger(A,\ B)$$

$$= W_2 - CA^\dagger W_1.$$

Hence according to Eq. (1.23) and Lemma 1.3, we find the following equalities

$$r\begin{pmatrix} E_AB \\ S_A \end{pmatrix} = r(V_2 - V_1A^\dagger B) = r\begin{pmatrix} V_2 \\ V_1A^\dagger B \end{pmatrix} = r(V_2),$$

$$r(\ CF_A,\ S_A) = r(W_2 - CA^\dagger W_1) = r(\ W_2,\ CA^\dagger W_1) = r(W_2).$$

From both of them and the rank formulas in Eqs. (1.6), (1.7) and (1.10), we can derive the following two equalities

$$r\begin{pmatrix} O & E_AB \\ CF_A & S_A \end{pmatrix} = r(M_1) - r(A) = r(V_1) + r(V_2) - r(A)$$
$$= r\begin{pmatrix} E_AB \\ S_A \end{pmatrix} + r(CF_A),$$

$$r\begin{pmatrix} O & E_AB \\ CF_A & S_A \end{pmatrix} = r(M_1) - r(A) = r(W_1) + r(W_2) - r(A)$$
$$= r(\ CF_A,\ S_A) + r(E_AB).$$

Both of them are exactly the rank additivity condition (1.21). Conversely, adding $r(A)$ to the three sides of Eq. (1.21) and then applying Eqs. (1.6)–(1.10) to the corresponding result we first obtain

$$r\begin{pmatrix} A & E_AB \\ CF_A & S_A \end{pmatrix} = r\begin{pmatrix} A \\ CF_A \end{pmatrix} + r\begin{pmatrix} E_AB \\ S_A \end{pmatrix} = r(\ A,\ E_AB) + r(\ CF_A,\ S_A). \tag{1.24}$$

On the other hand, the two inclusions in Eq. (1.20) are also equivalent to

$$r(M_1) = r\begin{pmatrix} A & B & A \\ C & D & O \end{pmatrix} = r\begin{pmatrix} A & B & O \\ C & D & C \end{pmatrix}, \quad r(M_1) = r\begin{pmatrix} A & B \\ C & D \\ A & O \end{pmatrix} = r\begin{pmatrix} A & B \\ C & D \\ O & B \end{pmatrix}.$$

Applying Eq. (1.9) to the right-hand sides of the above two equalities and then combining them with Eq. (1.24), we find

$$r(M_1) = r\begin{pmatrix} A & E_AB & O \\ CF_A & S_A & C \end{pmatrix} = r(A,\ E_AB) + r(CF_A,\ S_A,\ C)$$

$$= r(A,\ B) + r(C,\ D),$$

$$r(M_1) = r\begin{pmatrix} A & E_AB \\ CF_A & S_A \\ O & B \end{pmatrix} = r\begin{pmatrix} A \\ CF_A \end{pmatrix} + r\begin{pmatrix} E_AB \\ S_A \\ B \end{pmatrix} = r\begin{pmatrix} A \\ C \end{pmatrix} + r\begin{pmatrix} B \\ D \end{pmatrix}.$$

Both of them are exactly Eq. (1.5). □

Similarly we have the following.

Lemma 1.6. *The rank additivity condition* (1.5) *is equivalent to the following four conditions*

$$R\begin{pmatrix} A \\ O \end{pmatrix} \subseteq R(M_1), \quad R\begin{pmatrix} A^* \\ O \end{pmatrix} \subseteq R(M_1^*), \tag{1.25}$$

$$r\begin{pmatrix} S_D & BF_D \\ E_DC & O \end{pmatrix} = r\begin{pmatrix} S_D \\ E_DC \end{pmatrix} + r(BF_D) = r(\ S_D,\ BF_D) + r(E_DC), \tag{1.26}$$

where $S_D = A - BD^{\dagger}C$.

2. The Moore–Penrose inverses of 2 × 2 block matrices under rank additivity conditions

It is well known that the 2×2 block matrix M_1 in Eq. (1.4) can be factored as the following product

$$M_1 = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = PNQ = \begin{pmatrix} I_m & O \\ CA^{\dagger} & I_l \end{pmatrix}\begin{pmatrix} A & E_AB \\ CF_A & S_A \end{pmatrix}\begin{pmatrix} I_n & A^{\dagger}B \\ O & I_k \end{pmatrix}, \tag{2.1}$$

where $S_A = D - CA^{\dagger}B$ is the Schur complement of A in M_1, P and Q are two nonsingular matrices. Our examination to the Moore–Penrose inverse of M_1 under Eq. (1.5) will base on this decomposition of M_1. Following this decomposition of M_1, a quite simple question can first be asked that under what conditions the reverse product $Q^{-1}N^{\dagger}P^{-1}$ is the Moore–Penrose inverse of M_1. To answer this question, we only need a known result (see Ref. [3]) that $(PNQ)^{\dagger} = Q^{-1}N^{\dagger}P^{-1}$ holds if and only if

$$r(N,\ P^*PN) = r(N), \quad r\begin{pmatrix} N \\ NQQ^* \end{pmatrix} = r(N), \tag{2.2}$$

which is also equivalent to

$$r(PNQ,\ PP^*(PNQ)) = r(PNQ), \quad r\begin{pmatrix} PNQ \\ (PNQ)Q^*Q \end{pmatrix} = r(PNQ). \tag{2.2'}$$

Now substituting $M_1 = PNQ$, P and Q into Eq. (2.2′) and then simplifying the corresponding two expressions, we obtain the following simple result.

Lemma 2.1. *The Moore–Penrose inverse of the block matrix M_1 in Eq.* (2.1) *can be expressed as*

$$M_1^\dagger = Q^{-1}N^\dagger P^{-1} = \begin{pmatrix} I_n & -A^\dagger B \\ O & I_k \end{pmatrix}\begin{pmatrix} A & E_AB \\ CF_A & S_A \end{pmatrix}^\dagger\begin{pmatrix} I_m & O \\ -CA^\dagger & I_l \end{pmatrix}, \tag{2.3}$$

if and only if M_1 satisfies the following two conditions

$$R\begin{pmatrix} A \\ O \end{pmatrix} \subseteq R(M_1), \quad R\begin{pmatrix} A^* \\ O \end{pmatrix} \subseteq R(M_1^*). \tag{2.4}$$

Clearly Eq. (2.4) is exactly the two inclusions in Eq. (1.20), which is implied by the general rank additivity condition (1.5). Thus the Moore–Penrose inverse of M_1 under Eq. (1.5) can naturally be written as Eq. (2.3). The subsequent question on Eq. (2.3) is how to give the expression of $N^\dagger$ in Eq. (2.3). Now we write N as

$$N = N_1 + N_2 = \begin{pmatrix} A & O \\ O & O \end{pmatrix} + \begin{pmatrix} O & E_AB \\ CF_A & S_A \end{pmatrix}. \tag{2.5}$$

It is easy to see that $N_1^*N_2 = O$ and $N_2N_1^* = O$. Thus the Moore–Penrose inverse of $N = N_1 + N_2$ can be written as

$$N^\dagger = N_1^\dagger + N_2^\dagger = \begin{pmatrix} A & O \\ O & O \end{pmatrix}^\dagger + \begin{pmatrix} O & E_AB \\ CF_A & S_A \end{pmatrix}^\dagger. \tag{2.6}$$

If we know the block expression of $N_2^\dagger$ in Eq. (2.6), we also know the block expression of $N^\dagger$. Consequently we can give the block expression of $M_1^\dagger$ in Eq. (2.3). In fact, the block expressions of $N_2^\dagger$ in Eq. (2.6) can be derived from various known results on bordered matrices (see, e.g., Refs. [6–11] But in this article we only consider one of such cases – the Moore–Penrose inverse of N_2 when it satisfies the following rank additivity condition

$$r\begin{pmatrix} O & E_AB \\ CF_A & S_A \end{pmatrix} = r\begin{pmatrix} E_AB \\ S_A \end{pmatrix} + r(CF_A) = r(CF_A,\ S_A) + r(E_AB).$$

Combining the above discussion with Lemma 1.4 (i), we find the following natural result.

Theorem 2.2. *If the block matrix M_1 in Eq.* (2.1) *satisfies the rank additivity condition* (1.5), *then the Moore–Penrose inverse of M_1 can be expressed in the following two forms*

$$M_1^\dagger = \begin{pmatrix} H_1 - H_2CA^\dagger - A^\dagger BH_3 + A^\dagger BJ^\dagger(D)CA^\dagger & H_2 - A^\dagger BJ^\dagger(D) \\ H_3 - J^\dagger(D)CA^\dagger & J^\dagger(D) \end{pmatrix}, \tag{2.7}$$

$$M_1^\dagger = \begin{pmatrix} J^\dagger(A) & J^\dagger(C) \\ J^\dagger(B) & J^\dagger(D) \end{pmatrix} = \begin{pmatrix} (E_{B_2}S_DF_{C_2})^\dagger & (E_{D_2}S_BF_{A_1})^\dagger \\ (E_{A_2}S_CF_{D_1})^\dagger & (E_{C_1}S_AF_{B_1})^\dagger \end{pmatrix}, \tag{2.8}$$

where

$$S_A = D - CA^\dagger B, \quad S_B = C - DB^\dagger A, \quad S_C = B - AC^\dagger D, \quad S_D = A - BD^\dagger C,$$

$$A_1 = E_BA, \quad A_2 = AF_C, \quad B_1 = E_AB, \quad B_2 = BF_D,$$

$$C_1 = CF_A, \quad C_2 = E_DC, \quad D_1 = E_CD, \quad D_2 = DF_B,$$

$$H_1 = A^\dagger + C_1^\dagger[S_AJ^\dagger(D)S_A - S_A]B_1^\dagger,$$

$$H_2 = C_1^\dagger[I - S_AJ^\dagger(D)], \quad H_3 = [I - J^\dagger(D)S_A]B_1^\dagger.$$

Proof. According to the above discussion we know that if M_1 in Eq. (2.1) satisfies the rank additivity condition (1.5), then the Moore–Penrose inverse of M_1 can be expressed as Eq. (2.3) and $N^\dagger$ in it can be written as Eq. (2.6). On the other hand, Lemma 1.5 also shows that if M_1 in Eq. (2.1) satisfies Eq. (1.5), then the bordered matrix N_2 in Eq. (2.5) satisfies the rank additivity condition in Eq. (1.21). Hence the Moore–Penrose inverse of N_2 by Lemma 1.4 (i) can be written as

$$N_2^\dagger = \begin{pmatrix} C_1^\dagger[S_AJ^\dagger(D)S_A - S_A]B_1^\dagger & C_1^\dagger - C_1^\dagger S_AJ^\dagger(D) \\ B_1^\dagger - J^\dagger(D)S_AB_1^\dagger & J^\dagger(D) \end{pmatrix}, \tag{2.9}$$

where $B_1 = E_AB$, $C_1 = CF_A$ and $J(D) = E_{C_1}S_AF_{B_1}$. Now substituting Eq. (2.9) into Eq. (2.6) and then Eq. (2.6) into Eq. (2.3), we have the following

$$M_1^\dagger = Q^{-1}\begin{pmatrix} A^\dagger + C_1^\dagger[S_AJ^\dagger(D)S_A - S_A]B_1^\dagger & C_1^\dagger - C_1^\dagger S_AJ^\dagger(D) \\ B_1^\dagger - J^\dagger(D)S_AB_1^\dagger & J^\dagger(D) \end{pmatrix}P^{-1}. \tag{2.10}$$

Written in a 2×2 block matrix, Eq. (2.10) is the first expression of $M_1^\dagger$ in Eq. (2.7). In the same way, we can also decompose M_1 in Eq. (2.1) into the other three forms analogous to Eq. (2.1), in which the Schur complements S_B, S_C and S_D of B, C and D in M_1 are the lower left, upper right and upper left submatrices of N, respectively. Based on these three decompositions of M_1, we know that the Moore–Penrose inverse of M_1 can also be expressed as

$$M_1^\dagger = \begin{pmatrix} * & J^\dagger(C) \\ * & * \end{pmatrix} = \begin{pmatrix} * & * \\ J^\dagger(B) & * \end{pmatrix} = \begin{pmatrix} J^\dagger(A) & * \\ * & * \end{pmatrix}.$$

Finally from the uniqueness of the Moore–Penrose inverse of a matrix and the expressions in Eq. (2.7) and the above, we obtain Eq. (2.8). □

From the structure of $M_1^\dagger$ given in Eqs. (2.7) and (2.8), we can directly derive some fundamental properties on the Moore–Penrose inverse of M_1 when it satisfies Eq. (1.5).

Theorem 2.3. *Denote the Moore–Penrose inverse of M_1 in Eq. (2.1) by*

$$M_1^\dagger = \begin{pmatrix} G_1 & G_2 \\ G_3 & G_4 \end{pmatrix}, \tag{2.11}$$

where G_1, G_2, G_3 and G_4 are $n \times m$, $n \times l$, $k \times m$ and $k \times l$ matrices, respectively. If M_1 satisfies the rank additivity condition (1.5), then the submatrices in M_1 and $M_1^\dagger$ satisfy the following rank equalities

$$r(G_1) = r(V_1) + r(W_1) - r(M_1) + r(D), \tag{2.12}$$

$$r(G_2) = r(V_1) + r(W_2) - r(M_1) + r(B), \tag{2.13}$$

$$r(G_3) = r(V_2) + r(W_1) - r(M_1) + r(C), \tag{2.14}$$

$$r(G_4) = r(V_2) + r(W_2) - r(M_1) + r(A), \tag{2.15}$$

$$r(G_1) + r(G_4) = r(A) + r(D), \quad r(G_2) + r(G_3) = r(B) + r(C), \tag{2.16}$$

where V_1, V_2, W_1 and W_2 are defined in Eq. (1.22); the products $M_1M_1^\dagger$ and $M_1^\dagger M_1$ have the following forms

$$M_1M_1^\dagger = \begin{pmatrix} W_1W_1^\dagger & O \\ O & W_2W_2^\dagger \end{pmatrix}, \quad M_1^\dagger M_1 = \begin{pmatrix} V_1^\dagger V_1 & O \\ O & V_2^\dagger V_2 \end{pmatrix}. \tag{2.17}$$

Proof. The four rank equalities in Eqs. (2.12)–(2.15) can directly be derived from the expression in Eq. (2.8) for $M_1^\dagger$ and the rank formula in Eq. (1.11). The two equalities in Eq. (2.16) come from the sums of (2.12) and (2.15), (2.13) and (2.14), respectively. The two results in Eq. (2.17) are derived from Eq. (1.5) and Lemma 1.2(i) and (ii). □

The rank additivity condition (1.5) is in fact a quite weak restriction to a 2×2 block matrix. Hence from the above two theorems we can derive a variety of consequences. The following are some of them.

Corollary 2.4. *If the block matrix M_1 in Eq.* (2.1) *satisfies Eq.* (2.4) *and the following two conditions*

$$R(C_1) \cap R(S_A) = \{0\} \quad \text{and} \quad R(B_1^*) \cap R(S_A^*) = \{0\}, \tag{2.18}$$

then the Moore–Penrose inverse of M_1 can be expressed as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{\dagger} = Q^{-1} \begin{pmatrix} A^{\dagger} & C_1^{\dagger} - C_1^{\dagger} S_A J^{\dagger}(D) \\ B_1^{\dagger} - J^{\dagger}(D) S_A B_1^{\dagger} & J^{\dagger}(D) \end{pmatrix} P^{-1}$$

$$= \begin{pmatrix} A^{\dagger} - H_2 C A^{\dagger} - A^{\dagger} B H_3 + A^{\dagger} B J^{\dagger}(D) C A^{\dagger} & H_2 - A^{\dagger} B J^{\dagger}(D) \\ H_3 - J^{\dagger}(D) C A^{\dagger} & J^{\dagger}(D) \end{pmatrix},$$

where C_1, B_1, H_2, H_3 and $J(D)$ are as in Eq. (2.7), *P and Q are as in Eq.* (2.1).

Proof. The conditions in Eq. (2.18) imply that the block matrix N_2 in Eq. (2.6) satisfies the following rank additivity condition

$$r(N_2) = r(E_A B) + r(C F_A) + r(S_A),$$

which is a special case of Eq. (1.21). Hence according to Lemma 1.4(ii), the Moore–Penrose inverse of N_2 has a simpler expression as in Eq. (1.19), correspondingly Eq. (2.10) can be simplified to the result in this corollary. □

Corollary 2.5. *If the block matrix M_1 in Eq.* (2.1) *satisfies Eq.* (2.4) *and the following two conditions*

$$R(B S_A^*) \subseteq R(A) \quad \text{and} \quad R(C^* S_A) \subseteq R(A^*), \tag{2.19}$$

then the Moore–Penrose inverse of M_1 can be expressed as

$$M_1^{\dagger} = \begin{pmatrix} I_n & -A^{\dagger} B \\ O & I_k \end{pmatrix} \begin{pmatrix} A^{\dagger} & (C F_A)^{\dagger} \\ (E_A B)^{\dagger} & S_A^{\dagger} \end{pmatrix} \begin{pmatrix} I_m & O \\ -C A^{\dagger} & I_l \end{pmatrix}$$

$$= \begin{pmatrix} A^{\dagger} - A^{\dagger} B (E_A B)^{\dagger} - (C F_A)^{\dagger} C A^{\dagger} + A^{\dagger} B S_A^{\dagger} C A^{\dagger} & (C F_A)^{\dagger} - A^{\dagger} B S_A^{\dagger} \\ (E_A B)^{\dagger} - S_A^{\dagger} C A^{\dagger} & S_A^{\dagger} \end{pmatrix},$$

where $S_A = D - C A^{\dagger} B$.

Proof. Clearly Eq. (2.19) is equivalent to $(E_A B) S_A^* = O$ and $S_A^* (C F_A) = O$. In that case,

$$N_2^\dagger = \begin{pmatrix} O & (CF_A)^\dagger \\ (E_AB)^\dagger & S_A^\dagger \end{pmatrix}$$

in Eq. (2.6). Hence Eq. (2.10) can be simplified to the result in this corollary. □

Corollary 2.6 [4]. *If the block matrix M_1 in Eq. (2.1) satisfies the following four conditions*

$$R(B) \subseteq R(A), \quad R(C^*) \subseteq R(A^*), \quad R(C) \subseteq R(S_A), \quad R(B^*) \subseteq R(S_A^*), \tag{2.20}$$

then the Moore–Penrose inverse of M_1 can be expressed as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^\dagger = \begin{pmatrix} I_n & -A^\dagger B \\ O & I_k \end{pmatrix} \begin{pmatrix} A^\dagger & O \\ O & S_A^\dagger \end{pmatrix} \begin{pmatrix} I_m & O \\ -CA^\dagger & I_l \end{pmatrix}$$

$$= \begin{pmatrix} A^\dagger + A^\dagger B S_A^\dagger C A^\dagger & -A^\dagger B S_A^\dagger \\ -S_A^\dagger C A^\dagger & S_A^\dagger \end{pmatrix},$$

where $S_A = D - CA^\dagger B$.

Proof. It is easy to verify that under the conditions in Eq. (2.20), the rank of M_1 satisfies the rank additivity condition (1.5). In that case, $N_2^\dagger = \begin{pmatrix} O & O \\ O & S_A^\dagger \end{pmatrix}$ in Eq. (2.6). Correspondingly Eq. (2.10) is simplified to the result in this corollary. □

Corollary 2.7. *If the block matrix M_1 in Eq. (2.1) satisfies the four conditions*

$$R(A) \cap R(B) = \{0\}, \quad R(A^*) \cap R(C^*) = \{0\}, \tag{2.21}$$

$$R(D) \subseteq R(C), \quad R(D^*) \subseteq R(B^*), \tag{2.22}$$

then the Moore–Penrose inverse of M_1 can be expressed as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^\dagger = \begin{pmatrix} I_n & -A^\dagger B \\ O & I_k \end{pmatrix} \begin{pmatrix} A^\dagger - C_1^\dagger S_A B_1^\dagger & C_1^\dagger \\ B_1^\dagger & O \end{pmatrix} \begin{pmatrix} I_m & O \\ -CA^\dagger & I_l \end{pmatrix}$$

$$= \begin{pmatrix} A^\dagger - A^\dagger B B_1^\dagger - C_1^\dagger C A^\dagger - C_1^\dagger S_A B_1^\dagger & C_1^\dagger \\ B_1^\dagger & O \end{pmatrix},$$

where $S_A = D - CA^\dagger B$, $B_1 = E_A B$ and $C_1 = CF_A$.

Proof. It is not difficult to verify by Eq. (1.10) that under Eqs. (2.21) and (2.22) the rank of M_1 satisfies Eq. (1.5). In that case, $J(D) = O$ and

$$N_2^{\dagger} = \begin{pmatrix} -C_1^{\dagger}S_AB_1^{\dagger} & C_1^{\dagger} \\ B_1^{\dagger} & O \end{pmatrix}$$

in Eq. (2.6). Hence Eq. (2.10) is simplified to the result in this corollary. □

Corollary 2.8. *If the block matrix M_1 in Eq.* (2.1) *satisfies the four conditions*

$$R(A) \cap R(B) = \{0\}, \quad R(A^*) \cap R(C^*) = \{0\}, \tag{2.23}$$

$$R(S_A) \subseteq N(C^*), \quad R(S_A^*) \subseteq N(B), \tag{2.24}$$

then the Moore–Penrose inverse of M_1 can be expressed as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{\dagger} = \begin{pmatrix} I_n & -A^{\dagger}B \\ O & I_k \end{pmatrix} \begin{pmatrix} A^{\dagger} & (CF_A)^{\dagger} \\ (E_AB)^{\dagger} & S_A^{\dagger} \end{pmatrix} \begin{pmatrix} I_m & O \\ -CA^{\dagger} & I_l \end{pmatrix}$$
$$= \begin{pmatrix} A^{\dagger} - A^{\dagger}B(E_AB)^{\dagger} - (CF_A)^{\dagger}CA^{\dagger} & (CF_A)^{\dagger} \\ (E_AB)^{\dagger} & S_A^{\dagger} \end{pmatrix},$$

where $S_A = D - CA^{\dagger}B$.

Proof. Clearly Eq. (2.24) is equivalent to $C^{\dagger}S_A = O$ and $S_AB^{\dagger} = O$, as well as $S_A^{\dagger}C = O$ and $BS_A^{\dagger} = O$. From them and Eq. (2.23), as well as Lemma 1.2 (i) and (ii), we also have

$$(CF_A)^{\dagger}S_A = O \quad \text{and} \quad S_A(E_AB)^{\dagger} = O. \tag{2.25}$$

Combining Eqs. (2.23) and (2.25) shows that M_1 satisfies Eqs. (1.20) and (1.21). In that case, Eq. (2.10) can be simplified to the result in this corollary. □

Corollary 2.9. *If the block matrix M_1 in Eq.* (2.1) *satisfies the following rank additivity condition*

$$r(M_1) = r(A) + r(B) + r(C) + r(D), \tag{2.26}$$

then the Moore–Penrose inverse of M_1 can be expressed as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{\dagger} = \begin{pmatrix} (E_BAF_C)^{\dagger} & (E_DCF_A)^{\dagger} \\ (E_ABF_D)^{\dagger} & (E_CDF_B)^{\dagger} \end{pmatrix}. \tag{2.27}$$

Proof. Obviously Eq. (2.26) is a special case of Eq. (1.5). Besides, Eq. (2.26) is also equivalent to the following four conditions

$$R(A) \cap R(B) = \{0\}, \quad R(C) \cap R(D) = \{0\}, \quad R(A^*) \cap R(C^*) = \{0\},$$
$$R(B^*) \cap R(D^*) = \{0\}.$$

In that case, applying Eqs. (1.12) and (1.13) to Eq. (2.8) leads to Eq. (2.27). □

Corollary 2.10. *If the block matrix M_1 in Eq.* (2.1) *satisfies* $r(M_1) = r(A) + r(D)$ *and*

$$R(B) \subseteq R(A), \quad R(C) \subseteq R(D), \quad R(C^*) \subseteq R(A^*), \quad R(B^*) \subseteq R(D^*),$$

then the Moore–Penrose inverse of M_1 can be expressed as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{\dagger} = \begin{pmatrix} (A - BD^{\dagger}C)^{\dagger} & -A^{\dagger}B(D - CA^{\dagger}B)^{\dagger} \\ -(D - CA^{\dagger}B)^{\dagger}CA^{\dagger} & (D - CA^{\dagger}B)^{\dagger} \end{pmatrix}.$$

Corollary 2.11. *If the block matrix M_1 in Eq.* (2.1) *satisfies* $r(M_1) = r(A) + r(D)$ *and the following four conditions*

$$R(A) = R(B), \quad R(C) = R(D), \quad R(A^*) = R(C^*), \quad R(B^*) = R(D^*),$$

then the Moore–Penrose inverse of M_1 can be expressed as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{\dagger} = \begin{pmatrix} S_D^{\dagger} & S_B^{\dagger} \\ S_C^{\dagger} & S_A^{\dagger} \end{pmatrix} = \begin{pmatrix} (A - BD^{\dagger}C)^{\dagger} & (C - DB^{\dagger}A)^{\dagger} \\ (B - AC^{\dagger}D)^{\dagger} & (D - CA^{\dagger}B)^{\dagger} \end{pmatrix}.$$

The above two corollaries can directly be derived from Eqs. (2.7) and (2.8), the proofs are omitted here.

3. The Moore–Penrose inverses of general block matrices under rank additivity conditions

For convenience of representation, we adopt the following notation. Let $M = (A_{ij})$ be given in Eq. (1.1), where $A_{ij} \in \mathscr{C}^{s_i \times t_j}$, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$, and $\sum_{i=1}^{m} s_i = s$, $\sum_{i=1}^{n} t_i = t$. For each A_{ij} in M we associate three block matrices as follows

$$B_{ij} = (A_{i1}, \ldots, A_{i,j-1}, A_{i,j+1}, \ldots, A_{in}), \tag{3.1}$$

$$C_{ij}^* = (A_{1j}^*, \ldots, A_{i-1,j}^*, A_{i+1,j}^*, \ldots, A_{mj}^*), \tag{3.2}$$

$$D_{ij} = \begin{pmatrix} A_{11} & \cdots & A_{1,j-1} & A_{1,j+1} & \cdots & A_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ A_{i-1,1} & \cdots & A_{i-1,j-1} & A_{i-1,j+1} & \cdots & A_{i-1,n} \\ A_{i+1,1} & \cdots & A_{i+1,j-1} & A_{i+1,j+1} & \cdots & A_{i+1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ A_{m1} & \cdots & A_{m,j-1} & A_{m,j+1} & \cdots & A_{mn} \end{pmatrix}, \tag{3.3}$$

and the symbol $J(A_{ij})$ stands for

$$J(A_{ij}) = E_{\alpha_{ij}} S_{D_{ij}} F_{\beta_{ij}}, \quad 1 \leqslant i \leqslant m, \quad 1 \leqslant j \leqslant n, \tag{3.4}$$

where $\alpha_{ij} = B_{ij}F_{D_{ij}}$, $\beta_{ij} = E_{D_{ij}}C_{ij}$ and $S_{D_{ij}} = A_{ij} - B_{ij}D_{ij}^{\dagger}C_{ij}$ is the Schur complement of D_{ij} in M. We call the matrix $J(A_{ij})$ the *rank complement* of A_{ij} in M. Besides we partition the Moore–Penrose inverse of M in Eq. (1.1) into the following form

$$M^{\dagger} = \begin{pmatrix} G_{11} & G_{12} & \cdots & G_{1m} \\ G_{21} & G_{22} & \cdots & G_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ G_{n1} & G_{n2} & \cdots & G_{nm} \end{pmatrix}, \tag{3.5}$$

where G_{ij} is a $t_i \times s_j$ matrix, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$.

Next we build two groups of block permutation matrices as follows.

$$P_1 = I_s, \quad P_i = \begin{pmatrix} O & & & I_{s_i} & & & \\ I_{s_1} & \ddots & & & & & \\ & \ddots & \ddots & & & & \\ & & I_{s_{i-1}} & O & & & \\ & & & & I_{s_{i+1}} & & \\ & & & & & \ddots & \\ & & & & & & I_{s_m} \end{pmatrix}, \tag{3.6}$$

$$Q_1 = I_s, \quad Q_j = \begin{pmatrix} O & I_{t_1} & & & & & \\ & \ddots & \ddots & & & & \\ & & \ddots & I_{t_{j-1}} & & & \\ I_{t_j} & & & O & & & \\ & & & & I_{t_{j+1}} & & \\ & & & & & \ddots & \\ & & & & & & I_{t_n} \end{pmatrix}, \tag{3.7}$$

where $2 \leqslant i \leqslant m$, $2 \leqslant j \leqslant n$. Applying Eqs. (3.6) and (3.7) to M in Eq. (1.1) and $M^{\dagger}$ in Eq. (3.5) we have the following two groups of expressions

$$P_iMQ_j = \begin{pmatrix} A_{ij} & B_{ij} \\ C_{ij} & D_{ij} \end{pmatrix}, \quad 1 \leqslant i \leqslant m, \quad 1 \leqslant j \leqslant n, \tag{3.8}$$

$$Q_j^{\mathrm{T}} M^\dagger P_i^{\mathrm{T}} = \begin{pmatrix} G_{ji} & * \\ * & * \end{pmatrix}, \quad 1 \leqslant i \leqslant m, \quad 1 \leqslant j \leqslant n. \tag{3.9}$$

These two equalities show that we can use two block permutation matrices to permute A_{ij} in M and the corresponding block G_{ji} in $M^\dagger$ to the upper left corners of M and $M^\dagger$, respectively. Observe that P_i and Q_j in Eqs. (3.5) and (3.6) are all orthogonal matrices. The Moore–Penrose inverse of P_iMQ_j in Eq. (3.8) can be expressed as $(P_iMQ_j)^\dagger = Q_j^{\mathrm{T}} M^\dagger P_i^{\mathrm{T}}$. Combining Eq. (3.8) with Eq. (3.9), we have the following simple result

$$\begin{pmatrix} A_{ij} & B_{ij} \\ C_{ij} & D_{ij} \end{pmatrix}^\dagger = \begin{pmatrix} G_{ji} & * \\ * & * \end{pmatrix}, \quad 1 \leqslant i \leqslant m, \quad 1 \leqslant j \leqslant n. \tag{3.10}$$

If the block matrix M in Eq. (1.1) satisfies the rank additivity condition Eq. (1.2), then the 2×2 block matrix on the right-hand side of Eq. (3.8) naturally satisfies the following rank additivity condition

$$r\begin{pmatrix} A_{ij} & B_{ij} \\ C_{ij} & D_{ij} \end{pmatrix} = r\begin{pmatrix} A_{ij} \\ C_{ij} \end{pmatrix} + r\begin{pmatrix} B_{ij} \\ D_{ij} \end{pmatrix} = r(A_{ij}, \ B_{ij}) + r(C_{ij}, \ D_{ij}), \tag{3.11}$$

where $1 \leqslant i \leqslant m, \ 1 \leqslant j \leqslant n$. Hence combining Eqs. (3.10) and (3.11) with Theorems 2.2 and 2.3, we obtain the following general result.

Theorem 3.1. *Suppose that the $m \times n$ block matrix M in Eq.* (1.1) *satisfies the rank additivity condition* (1.2). *Then*

(i) *the Moore–Penrose inverse of M can be expressed as*

$$M^\dagger = \begin{pmatrix} J^\dagger(A_{11}) & J^\dagger(A_{21}) & \cdots & J^\dagger(A_{m1}) \\ J^\dagger(A_{12}) & J^\dagger(A_{22}) & \cdots & J^\dagger(A_{m2}) \\ \cdots & \cdots & \cdots & \cdots \\ J^\dagger(A_{1n}) & J^\dagger(A_{2n}) & \cdots & J^\dagger(A_{mn}) \end{pmatrix}, \tag{3.12}$$

where $J(A_{ij})$ is defined in Eq. (3.4);

(ii) *the rank of the block entry $G_{ji} = J^\dagger(A_{ij})$ in $M^\dagger$ is*

$$r(G_{ji}) = r[J(A_{ij})] = r(W_i) + r(V_j) - r(M) + r(D_{ij}), \tag{3.13}$$

where $1 \leqslant i \leqslant m, \ 1 \leqslant j \leqslant n$, W_i and V_j are defined in Eq. (1.3);

(iii) *$MM^\dagger$ and $M^\dagger M$ are two block diagonal matrices*

$$MM^\dagger = \mathrm{diag}(\ W_1W_1^\dagger, \ W_2W_2^\dagger, \ldots, \ W_mW_m^\dagger), \tag{3.14}$$

$$M^\dagger M = \mathrm{diag}(\ V_1^\dagger V_1, \ V_2^\dagger V_2, \ldots, \ V_n^\dagger V_n), \tag{3.15}$$

written in explicit forms, Eqs. (3.14) *and* (3.15) *are equivalent to*

$$A_{i1}G_{1j} + A_{i2}G_{2j} + \cdots + A_{in}G_{nj} = \begin{cases} W_iW_i^\dagger & i = j, \\ O & i \neq j, \end{cases} \quad i,j = 1, 2, \cdots, m,$$

$$G_{i1}A_{1j} + G_{i2}A_{2j} + \cdots + G_{im}A_{mj} = \begin{cases} V_i^\dagger V_i & i = j, \\ O & i \neq j, \end{cases} \quad i,j = 1, 2, \ldots, n.$$

In addition to the expression given in Eq. (3.12) for $M^\dagger$, we can also derive some other expressions for G_{ij} in $M^\dagger$ from Eq. (2.7) when M satisfies Eq. (1.2). But they are quite complicated in form, so we omit them here.

Just as Theorems 2.2 and 2.3 for 2×2 block matrices, Theorem 3.1 is also a general result, from which we can derive a variety of consequences when the submatrices in M satisfies some additional conditions or M has some particular patterns, such as triangular forms, circulant forms and tridiagonal forms. Here we only give two direct consequences.

Corollary 3.2. *If the block matrix M in Eq. (1.1) satisfies the following rank additivity condition*

$$r(M) = \sum_{i=1}^{m}\sum_{j=1}^{n} r(A_{ij}), \tag{3.16}$$

then the Moore–Penrose inverse of M can be expressed as

$$M^\dagger = \begin{pmatrix} (E_{B_{11}}A_{11}F_{C_{11}})^\dagger & \cdots & (E_{B_{m1}}A_{m1}F_{C_{m1}})^\dagger \\ \vdots & & \vdots \\ (E_{B_{1n}}A_{1n}F_{C_{1n}})^\dagger & \cdots & (E_{B_{mn}}A_{mn}F_{C_{mn}})^\dagger \end{pmatrix}, \tag{3.17}$$

where B_{ij} and C_{ij} are defined in Eqs. (3.1) and (3.2).

Proof. In fact, Eq. (3.16) is equivalent to

$$R(A_{ij}) \cap R(B_{ij}) = \{0\}, \quad R(A_{ij}^*) \cap R(C_{ij}^*) = \{0\}, \quad 1 \leqslant i \leqslant m, \ 1 \leqslant j \leqslant n,$$

$$R(C_{ij}) \cap R(D_{ij}) = \{0\}, \quad R(B_{ij}^*) \cap R(D_{ij}^*) = \{0\}, \quad 1 \leqslant i \leqslant m, \ 1 \leqslant j \leqslant n.$$

From them we can get $J(A_{ij}) = E_{B_{ij}}A_{ij}F_{C_{ij}}$. Putting them in Eq. (3.12) yields Eq. (3.17). □

Corollary 3.3. *Let $A_i \in \mathscr{C}^{s \times t_i}$ and $B_i \in \mathscr{C}^{s_i \times t}$, $i = 1, 2, \ldots, k$. If they satisfy*

$$r(\, A_1, \, A_2, \, \ldots, \, A_k) = \sum_{i=1}^{k} r(A_i), \quad r(\, B_1^*, \, B_2^*, \, \ldots, \, B_k^*) = \sum_{i=1}^{k} r(B_i),$$

then

$$(A_1, A_2, \ldots, A_k)^\dagger = \begin{pmatrix} (E_{\alpha_1}A_1)^\dagger \\ (E_{\alpha_2}A_2)^\dagger \\ \vdots \\ (E_{\alpha_k}A_k)^\dagger \end{pmatrix},$$

$$\begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ B_k \end{pmatrix}^\dagger = ((B_1F_{\beta_1})^\dagger, (B_2F_{\beta_2})^\dagger, \ldots, (B_kF_{\beta_k})^\dagger),$$

where

$$\alpha_i = (A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_k), \quad 1 \leqslant i \leqslant k,$$

$$\beta_i^* = (B_1^*, \ldots, B_{i-1}^*, B_{i+1}^*, \ldots, B_k^*), \quad 1 \leqslant i \leqslant k.$$

4. The Moore–Penrose inverses of matrix sums

As one of the important applications of the results in Sections 2 and 3, we intend to present in this section some formulas for expressing the Moore–Penrose inverses of matrix sums. For doing so, we first make some preparations.

Let $A_1, A_2, \ldots, A_k \in \mathscr{C}^{m\times n}$, and denote

$$\alpha = (A_1, A_2, \ldots, A_k), \quad \alpha_i = (A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_k), \tag{4.1}$$

$$\beta = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_k \end{pmatrix}, \quad \beta_i = \begin{pmatrix} A_1 \\ \vdots \\ A_{i-1} \\ A_{i+1} \\ \vdots \\ A_k \end{pmatrix}, \quad A = \begin{pmatrix} A_1 & A_2 & \cdots & A_k \\ A_k & A_1 & \cdots & A_{k-1} \\ \cdots & \cdots & \cdots & \cdots \\ A_2 & A_3 & \cdots & A_1 \end{pmatrix}, \tag{4.2}$$

where $1 \leqslant i \leqslant k$ and we use D_i to denote the $(k-1)\times(k-1)$ block matrix resulting from the deletion of the first block row and the ith block column of A in Eq. (4.2). In addition, we need the following result.

Lemma 4.1 [3]. *Let P, N and Q are three complex matrices such that the product PNQ is defined. If $PP^* = I$, $Q^*Q = I$, $P^*PN = NQQ^*$ and $QQ^*N^\dagger = N^\dagger P^*P$, then $(PNQ)^\dagger = Q^*N^\dagger P^*$.*

Next we present a known equality on the Moore–Penrose inverses of matrix sums and block matrices and reprove it here.

Theorem 4.2 [5]. *Let $A_1, A_2, \ldots, A_k \in \mathscr{C}^{m\times n}$. Then the Moore–Penrose inverse of their sum $\sum_{i=1}^k A_i$ satisfies the following identity*

$$\left(\sum_{i=1}^k A_i\right)^\dagger = \frac{1}{k}(I_n, I_n, \ldots, I_n)A^\dagger \begin{pmatrix} I_m \\ I_m \\ \vdots \\ I_m \end{pmatrix}, \tag{4.3}$$

where A is the circulant block matrix composed by $A_1, A_2, \ldots, A_k$ shown in Eq. (4.2).

Proof. It is easy to see that the sum $\sum_{i=1}^k A_i$ can be expressed as a product $\sum_{i=1}^k A_i = PAQ$, where A is the circulant block matrix defined in Eq. (4.2), and

$$P = \frac{1}{\sqrt{k}}(I_m, I_m, \ldots, I_m), \quad Q^{\mathrm{T}} = \frac{1}{\sqrt{k}}(I_n, I_n, \ldots, I_n). \tag{4.4}$$

If we can prove $(PAQ)^\dagger = Q^{\mathrm{T}}A^\dagger P^{\mathrm{T}}$, then Eq. (4.3) is true. To do so,we first show that the Moore–Penrose inverse of any circulant block matrix is also a circulant block matrix. Note that $UAV = A$, where U and V are the following two permutation block matrices

$$U = \begin{pmatrix} O & I_m & & \\ & O & \ddots & \\ & & \ddots & I_m \\ I_m & & & O \end{pmatrix}, \quad V = \begin{pmatrix} O & & & I_n \\ I_n & O & & \\ & \ddots & \ddots & \\ & & I_n & O \end{pmatrix}. \tag{4.5}$$

By Lemma 4.1, we easily know that $A^\dagger = (UAV)^\dagger = V^{\mathrm{T}}A^\dagger U^{\mathrm{T}}$. Note that U^{T} and V^{T} are also two permutation block matrices, hence this equality implies that $A^\dagger$ is also a circulant block matrix with the form

$$A^\dagger = \begin{pmatrix} G_1 & G_2 & \cdots & G_k \\ G_k & G_1 & \cdots & G_{k-1} \\ \cdots & \cdots & \cdots & \cdots \\ G_2 & G_3 & \cdots & G_1 \end{pmatrix}, \tag{4.6}$$

where $G_1, G_2, \ldots, G_k \in \mathscr{C}^{n\times m}$. According to the structure of P, Q, A and $A^\dagger$ in Eqs. (4.2), (4.4) and (4.6), we easily see that

$$PP^{\mathrm{T}} = I_m, \quad Q^{\mathrm{T}}Q = I_n, \quad P^{\mathrm{T}}PA = AQQ^{\mathrm{T}}, \quad A^\dagger P^{\mathrm{T}}P = QQ^{\mathrm{T}}A^\dagger.$$

Combining these four equalities with Lemma 4.1, we know that $(PAQ)^{\dagger} = Q^{T}A^{\dagger}P^{T}$ is true. Hence Eq. (4.3) holds. □

If the matrices $A_1, A_2, \ldots, A_n$ in Eq. (4.3) are square and the circulant block matrix A composed by them is nonsingular, then the sum $\sum_{i=1}^{k} A_i$ is also nonsingular and Eq. (4.3) becomes an equality on the standard inverse of matrix sum

$$(A_1 + A_2 + \cdots + A_k)^{-1} = PA^{-1}P^{T}. \tag{4.7}$$

It is easily seen that combining Eq. (4.3) with the results in Sections 2 and 3 may produce lots of formulas for the Moore–Penrose inverses of matrix sums. We start with the simplest case – the Moore–Penrose inverse of sum of two matrices.

Let A and B be two $m \times n$ matrices. Then according to Eq. (4.3) we have

$$(A+B)^{\dagger} = \frac{1}{2}(\, I_n,\ I_n) \begin{pmatrix} A & B \\ B & A \end{pmatrix}^{\dagger} \begin{pmatrix} I_m \\ I_m \end{pmatrix}. \tag{4.8}$$

As a special case of Eq. (4.8), if we replace $A + B$ in Eq. (4.1) by a complex matrix $A + iB$, where A and B are two real matrices, then Eq. (4.8) becomes the following equality

$$(A+iB)^{\dagger} = \frac{1}{2}(I_n,\ I_n) \begin{pmatrix} A & iB \\ iB & A \end{pmatrix}^{\dagger} \begin{pmatrix} I_m \\ I_m \end{pmatrix} = \frac{1}{2}(I_n,\ iI_n) \begin{pmatrix} A & -B \\ B & A \end{pmatrix}^{\dagger} \begin{pmatrix} I_m \\ -iI_m \end{pmatrix}. \tag{4.9}$$

Now applying Theorems 2.2 and 2.3 to Eqs. (4.8) and (4.9) we find the following two results.

Theorem 4.3. *Let A and B be two $m \times n$ complex matrices. If they satisfy*

$$r\begin{pmatrix} A & B \\ B & A \end{pmatrix} = r\begin{pmatrix} A \\ B \end{pmatrix} + r\begin{pmatrix} B \\ A \end{pmatrix} = r(\, A,\ B) + r(\, B,\ A), \tag{4.10}$$

which is equivalent to $R(A) \subseteq R(A \pm B)$ and $R(A^*) \subseteq R(A^* \pm B^*)$, then

(i) *the Moore–Penrose inverse of $A + B$ can be expressed as*

$$(A+B)^{\dagger} = J^{\dagger}(A) + J^{\dagger}(B) = (E_{B_2}S_AF_{B_1})^{\dagger} + (E_{A_2}S_BF_{A_1})^{\dagger}, \tag{4.11}$$

where $J(A)$ and $J(B)$ are, respectively, the rank complements of A and B in $\begin{pmatrix} A & B \\ A & B \end{pmatrix}$, $S_A = A - BA^{\dagger}B$, $S_B = B - AB^{\dagger}A$, $A_1 = E_BA$, $A_2 = A\,F_B$, $B_1 = E_AB$, $B_2 = BF_A$;

(ii) *A, B and the two terms $G_1 = J^\dagger(A)$, $G_2 = J^\dagger(B)$ in the right-hand side of Eq.* (4.11) *satisfy the following several equalities*

$$r(G_1) = r(A), \quad r(G_2) = r(B),$$

$$(A+B)(A+B)^\dagger = AG_1 + BG_2, \quad (A+B)^\dagger(A+B) = G_1A + G_2B,$$

$$AG_2 + BG_1 = O, \quad G_2A + G_1B = O.$$

Proof. From Theorem 2.2 we know that under the condition (4.10), the Moore–Penrose inverse of

$$\begin{pmatrix} A & B \\ B & A \end{pmatrix}$$

can be expressed as

$$\begin{pmatrix} A & B \\ B & A \end{pmatrix}^\dagger = \begin{pmatrix} J^\dagger(A) & J^\dagger(B) \\ J^\dagger(B) & J^\dagger(A) \end{pmatrix}.$$

Then putting it in Eq. (4.8) immediately yields Eq. (4.11). The results in (ii) come from Theorem 2.3. □

Theorem 4.4. *Let $A + \mathrm{i}B$ be an $m \times n$ complex matrix, where A and B are two real matrices. If A and B satisfy*

$$r\begin{pmatrix} A & -B \\ B & A \end{pmatrix} = r\begin{pmatrix} A \\ B \end{pmatrix} + r\begin{pmatrix} -B \\ A \end{pmatrix} = r(\,A,\ -B) + r(\,B,\ A), \tag{4.12}$$

which is equivalent to $R(A) \subseteq R(A \pm \mathrm{i}B)$ and $R(A^*) \subseteq R(A^* \pm \mathrm{i}B^*)$, then the Moore–Penrose inverse of $A + \mathrm{i}B$ can be expressed as

$$(A+\mathrm{i}B)^\dagger = G_1 - iG_2 = [E_{B_2}(A + BA^\dagger B)F_{B_1}]^\dagger - \mathrm{i}[E_{A_2}(B + AB^\dagger A)F_{A_1}]^\dagger, \tag{4.13}$$

where $A_1 = E_BA$, $A_2 = AF_B$, $B_1 = E_AB$ and $B_2 = BF_A$.

Corollary 4.5. *Suppose that $A + \mathrm{i}B$ is a nonsingular complex matrix, where A and B are real.*

(i) *If A and B are also nonsingular, then*

$$(A+\mathrm{i}B)^{-1} = (A + BA^{-1}B)^{-1} - \mathrm{i}(B + AB^{-1}A)^{-1}.$$

(ii) *If $R(A) \cap R(B) = \{0\}$ and $R(A^*) \cap R(B^*) = \{0\}$, then*

$$(A+\mathrm{i}B)^{-1} = (E_BAF_B)^\dagger - \mathrm{i}(E_ABF_A)^\dagger.$$

(iii) *Let $A = \lambda I_m$, where λ is a real number such that $\lambda I_m + \mathrm{i}B$ is nonsingular,*

then

$$(A+\mathrm{i}B)^{-1} = \lambda(\lambda^2 I_m + B^2)^{-1} - \mathrm{i}(\lambda^2 B + B^\dagger B^3 B^\dagger)^\dagger.$$

Next we turn our attention to the Moore–Penrose inverse of the sum of k matrices, and give some general formulas.

Theorem 4.6. *Let $A_1, A_2, \ldots, A_k \in \mathscr{C}^{m\times n}$. If they satisfy the following rank additivity condition*

$$r(A) = kr(A_1, \ldots, A_k) = kr(A_1^*, \ldots, A_k^*), \tag{4.14}$$

where A is the circulant block matrix defined in Eq. (4.2), *then*

(i) *the Moore–Penrose inverse of the sum $\sum_{i=1}^k A_i$ can be expressed as*

$$(A_1 + A_2 + \cdots + A_k)^\dagger = J^\dagger(A_1) + J^\dagger(A_2) + \cdots + J^\dagger(A_k), \tag{4.15}$$

where $J(A_i)$ is the rank complement of $A_i (1 \leqslant i \leqslant k)$ in A;

(ii) *the rank of $J(A_i)$ is*

$$r(J(A_i)) = r(A_1, \ldots, A_k) + r(A_1^*, \ldots, A_k^*) - r(A) + r(D_i), \tag{4.16}$$

where $1 \leqslant i \leqslant k$, D_i is the $(k-1)\times(k-1)$ block matrix resulting from the deletion of the first block row and ith block column of A;

(iii) *$A_1, A_2, \ldots, A_k$ and $J^\dagger(A_1), J^\dagger(A_2), \ldots, J^\dagger(A_k)$ satisfy the following two equalities*

$$(A_1 + \cdots + A_k)(A_1 + \cdots + A_k)^\dagger = A_1 J^\dagger(A_1) + \cdots + A_k J^\dagger(A_k),$$

$$(A_1 + \cdots + A_k)^\dagger(A_1 + \cdots + A_k) = J^\dagger(A_1)A_1 + \cdots + J^\dagger(A_k)A_k.$$

Proof. Follow from the combination of Theorem 3.1 with the equality in (4.3). □

Corollary 4.7. *Let $A_1, A_2, \ldots, A_k \in \mathscr{C}^{m\times n}$. If they satisfy the following rank additivity condition*

$$r(A_1 + A_2 + \cdots + A_k) = r(A_1) + r(A_2) + \cdots + r(A_k), \tag{4.17}$$

then

$$\begin{aligned}(A_1 + A_2 + \cdots + A_k)^\dagger \\ = (E_{\alpha_1}A_1F_{\beta_1})^\dagger + (E_{\alpha_2}A_2F_{\beta_2})^\dagger + \cdots + (E_{\alpha_k}A_kF_{\beta_k})^\dagger,\end{aligned} \tag{4.18}$$

where α_i and β_i are defined in Eqs. (4.1) *and* (4.2), $1 \leqslant i \leqslant k$.

Proof. We first show that under the condition (4.17) the rank of the circulant matrix A in Eq. (4.2) is

$$r(A) = k[r(A_1) + r(A_2) + \cdots + r(A_k)]. \tag{4.19}$$

In fact, A in Eq. (4.2) can be expressed as

$$A = A_1' + U_m A_2' + U_m^2 A_3' + \cdots + U_m^{k-1} A_k', \tag{4.20}$$

$$A = A_1' + A_2' U_n + A_3' U_n^2 + \cdots + A_k' U_n^{k-1}, \tag{4.21}$$

where $A_i' = \mathrm{diag}(A_i, A_i, \ldots, A_i)$, $1 \leqslant i \leqslant k$, and

$$U_t = \begin{pmatrix} O & I_t & & \\ & O & \ddots & \\ & & \ddots & I_t \\ I_t & & & O \end{pmatrix}, \quad t = m, n.$$

Under Eq. (4.17) we know that

$$r(A_1' + A_2' + \cdots + A_k') = r(A_1') + r(A_2') + \cdots + r(A_k'). \tag{4.22}$$

On the other hand, applying the known rank inequality

$$r(N_1, \ldots, N_k) \geqslant r\begin{pmatrix} N_1 \\ \vdots \\ N_k \end{pmatrix} + r(N_1, \ldots, N_k) - [r(N_1) + \cdots + r(N_k)]$$

to Eq. (4.20) and combining it with Eqs. (2.21) and (2.22), we can find

$$\begin{aligned} r(A) &\geqslant r\begin{pmatrix} A_1' \\ \vdots \\ U_m^{k-1} A_k' \end{pmatrix} + r(A_1', \ldots, U_m^{k-1} A_k') - r(A_1') - \cdots - r(U_m^{k-1} A_k') \\ &= r\begin{pmatrix} A_1' \\ \vdots \\ A_k' \end{pmatrix} + r(A_1', \ldots, A_k') - r(A_1') - \cdots - r(A_k') \\ &= r(A_1') + \cdots + r(A_k') = k[r(A_1) + \cdots + r(A_k)]. \end{aligned}$$

Note that the rank of A naturally satisfies $r(A) \leqslant k[r(A_1) + \cdots + r(A_k)]$. Thus Eq. (4.19) holds under Eq. (4.17). In that case, applying the result in Corollary 3.2 to the circulant block matrix A in Eq. (4.3) produces the formula (4.18). □

At the end of this section, we should point out that the formulas on the Moore–Penrose inverses of matrix sums given in this section and those on the Moore–Penrose inverses of block matrices given in Sections 2 and 3 are, in fact, a group of dual results. That is to say, not only can we derive the Moore–Penrose inverses of matrix sums from the Moore–Penrose inverses of block matrices, but also we can make the contrary derivation. For simplicity, here we illustrate this assertion by a 2×2 block matrix. In fact, for any 2×2 block matrix in Eq. (1.4) we can decompose it as

$$M_1 = \begin{pmatrix} A & O \\ O & D \end{pmatrix} + \begin{pmatrix} O & B \\ C & O \end{pmatrix} = N_1 + N_2.$$

If M_1 satisfies the rank additivity condition (1.5), then N_1 and N_2 satisfy

$$r\begin{pmatrix} N_1 & N_2 \\ N_2 & N_1 \end{pmatrix} = r\begin{pmatrix} A & O & O & B \\ O & D & C & O \\ O & B & A & O \\ C & O & O & D \end{pmatrix} = 2r\begin{pmatrix} A & B \\ C & D \end{pmatrix} = r\begin{pmatrix} N_1 \\ N_2 \end{pmatrix} + r(N_1, \; N_2).$$

Hence by Theorem 4.3(i), we have

$$M_1^\dagger = (N_1 + N_2)^\dagger = J^\dagger(N_1) + J^\dagger(N_2), \tag{4.23}$$

where $J(N_1)$ and $J(N_2)$ are, respectively, the rank complements of N_1 and N_2 in

$$\begin{pmatrix} N_1 & N_2 \\ N_2 & N_1 \end{pmatrix}.$$

Written in an explicit form, Eq. (4.23) is exactly the formula (2.8). For a general $m \times m$ block matrix with the rank additivity condition (1.2), we can also derive the expression of its Moore–Penrose inverse from the formula (4.15). But the process is too tedious, we omit it here.

5. The inversion formulas for the Moore–Penrose inverses of matrices

Inversion formulas for the Moore–Penrose inverses of matrices is also one of the fundamental topics in the theory of generalized inverses of matrices. Various results on this topic and their applications can easily be found in the literature. In general cases, such kind of formulas are derived from the comparison of the different expressions of the Moore–Penrose inverses of block matrices. Along with the presentation of the Moore–Penrose inverse of 2×2 block matrix under the rank additivity condition (1.5), we now can find a group of new results on this topics.

We first present a general equality derived from Eqs. (2.7) and (2.8).

Theorem 5.1. *Let A, B, C and D are given by Eq.* (1.4). *If they satisfy the rank additivity condition* (1.5), then the following inversion formula holds

$$\begin{aligned}(E_{B_2}S_DF_{C_2})^\dagger = {} & A^\dagger + A^\dagger BJ^\dagger(D)CA^\dagger + C_1^\dagger[S_AJ^\dagger(D)S_A - S_A]B_1^\dagger \\ & - A^\dagger B[I - J^\dagger(D)S_A]B_1^\dagger - C_1^\dagger[I - S_AJ^\dagger(D)]CA^\dagger,\end{aligned} \tag{5.1}$$

where

$$S_A = D - CA^\dagger B,\quad S_D = A - BD^\dagger C,\quad J(D) = E_{C_1}S_AF_{B_1},$$
$$B_1 = E_AB,\quad B_2 = BF_D,\quad C_1 = CF_A,\quad C_2 = E_DC.$$

The results given below are all the special cases of the general formula (5.1).

Corollary 5.2. *If A, B, C and D satisfy*

$$R\begin{pmatrix} A \\ O \end{pmatrix} \subseteq R\begin{pmatrix} A & B \\ C & D \end{pmatrix},\quad R\begin{pmatrix} A^* \\ O \end{pmatrix} \subseteq R\begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} \tag{5.2}$$

and the following two conditions

$$R(CS_D^*) \subseteq R(D),\quad R(B^*S_D) \subseteq R(D^*), \tag{5.3}$$

or more specifically satisfy the following four conditions

$$R(C) \subseteq R(D),\quad R(B^*) \subseteq R(D^*),\quad R(B) \subseteq R(S_D),\quad R(C^*) \subseteq R(S_D^*), \tag{5.4}$$

then the Moore–Penrose inverse of $S_D = A - BD^\dagger C$ satisfies the inversion formula

$$\begin{aligned}(A - BD^\dagger C)^\dagger = {} & A^\dagger + A^\dagger BJ^\dagger(D)CA^\dagger + C_1^\dagger[S_AJ^\dagger(D)S_A - S_A]B_1^\dagger \\ & - A^\dagger B[I - J^\dagger(D)S_A]B_1^\dagger - C_1^\dagger[I - S_AJ^\dagger(D)]CA^\dagger,\end{aligned} \tag{5.5}$$

where S_A, B_1, C_1 and $J(D)$ are defined in Eq. (5.1).

Proof. It is obvious that Eq. (5.3) is equivalent to $(E_DC)S_D^* = O$ and $S_D^*(BF_D) = O$, or equivalently

$$S_D(E_DC)^\dagger = O \quad \text{and} \quad (BF_D)^\dagger S_D = O. \tag{5.6}$$

These two equalities clearly imply that S_D, E_DC and BF_D satisfy Eq. (1.26). Hence by Lemma 1.6, we know that under Eqs. (5.2) and (5.3), A, B, C and D naturally satisfy Eq. (1.5). Now substituting Eq. (5.6) into the left-hand side of Eq. (5.1) yields $J^\dagger(A) = (A - BD^\dagger C)^\dagger$. Hence Eq. (5.1) becomes Eq. (5.5). Observe that Eq. (5.4) is a special case of Eq. (5.3), hence Eq. (5.5) is also true under Eq. (5.4). □

Corollary 5.3. *If A, B, C and D satisfy Eqs.* (5.2) *and* (5.4) *and the following two conditions*

$$R(CF_A) \cap R(S_A) = \{0\} \quad \textit{and} \quad R[(E_AB)^*] \cap R(S_A^*) = \{0\}, \tag{5.7}$$

then

$$(A - BD^{\dagger}C)^{\dagger} = A^{\dagger} + A^{\dagger}BJ^{\dagger}(D)CA^{\dagger} - A^{\dagger}B[I - J^{\dagger}(D)S_A]B_1^{\dagger} - C_1^{\dagger}[I - S_AJ^{\dagger}(D)]CA^{\dagger}, \tag{5.8}$$

where S_A, B_1, C_1 *and* $J(D)$ *are defined in Eq.* (5.1).

Proof. According to Lemma 1.4(ii), the two conditions in Eq. (5.7) implies that $S_AJ^{\dagger}(D)S_A = S_A$. Hence Eq. (5.5) is simplified to Eq. (5.8). □

Corollary 5.4. *If* A, B, C *and* D *satisfy Eqs.* (5.2) *and* (5.4) *and the following two conditions*

$$R(BS_A^*) \subseteq R(A) \quad \textit{and} \quad R(C^*S_A) \subseteq R(A^*), \tag{5.9}$$

then

$$(A - BD^{\dagger}C)^{\dagger} = A^{\dagger} + A^{\dagger}BS_A^{\dagger}CA^{\dagger} - A^{\dagger}B(E_AB)^{\dagger} - (CF_A)^{\dagger}CA^{\dagger}. \tag{5.10}$$

where $S_A = D - CA^{\dagger}B$.

Proof. Clearly, Eq. (5.9) is equivalent to $(E_AB)S_A^* = O$ and $S_A^*(CF_A) = O$, which can also equivalently be expressed as $S_A(E_AB)^{\dagger} = O$ and $(CF_A)^{\dagger}S_A = O$. In that case, $J(D) = E_{C_1}S_AF_{B_1} = S_A$. Hence Eq. (5.5) is simplified to Eq. (5.10). □

Corollary 5.5. *If* A, B, C *and* D *satisfy Eqs.* (5.2) *and* (5.4) *and the following two conditions*

$$R(B) \subseteq R(A) \quad \textit{and} \quad R(C^*) \subseteq R(A^*), \tag{5.11}$$

then

$$(A - BD^{\dagger}C)^{\dagger} = A^{\dagger} + A^{\dagger}B(D - CA^{\dagger}B)^{\dagger}CA^{\dagger}. \tag{5.12}$$

Proof. The two inclusions in Eq. (5.11) are equivalent to $E_AB = O$ and $CF_A = O$. Substituting them into Eq. (5.5) yields Eq. (5.12). □

Corollary 5.6. *If* A, B, C *and* D *satisfy the following four conditions*

$$R(A) \cap R(B) = \{0\}, \quad R(A^*) \cap R(C^*) = \{0\}, \tag{5.13}$$

$$R(C) = R(D), \quad R(B^*) = R(D^*), \tag{5.14}$$

then

$$(A - BD^{\dagger}C)^{\dagger} = A^{\dagger} - A^{\dagger}B(E_AB)^{\dagger} - (CF_A)^{\dagger}CA^{\dagger} + (CF_A)^{\dagger}S_A(E_AB)^{\dagger}. \tag{5.15}$$

Proof. Under Eqs. (5.13) and (5.14), A, B, C and D naturally satisfy the rank additivity condition in Eq. (1.5). Besides, from Eqs. (5.13) and (5.14) and Lemma 1.2 we can derive

$$B_1^\dagger B_1 = B^\dagger B, \quad C_1 C_1^\dagger = CC^\dagger, \quad B_2 = O, \quad C_2 = O, \quad J(D) = O.$$

Substituting them into Eq. (5.1) yields Eq. (5.15). □

If D is invertible, or $D = I$, or $B = C = -D$, then the inversion formula (5.1) can be reduced to some other simpler forms. The reader could easily list the corresponding results and their various special cases.

Ref. [12] is a general reference book for the topic of this paper.

Acknowledgment

The author wishes to thank the referee for helpful comments and valuable suggestions.

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