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On the Moore–Penrose generalized inverse matrix

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Abstract

In this paper we exhibit some different methods for computing the Moore–Penrose inverse of different type of matrices. We discuss the Moore–Penrose inverse of block matrices, full-rank factorization. We give some numerical computations relative to this theory.

Keywords: Moore–Penrose inverse; Generalized inverse matrices; Pseudoinverses for non-invertible matrices

1. Introduction

Moore [2] (1862–1932) introduced and studied the *general reciprocal* during the decade 1910–1920. He stated the objective as follows: "*The effectiveness of the reciprocal of a nonsingular finite matrix in the study of properties of such matrices makes it desirable to define if possible in an analogous matrix to be associated with each finite matrix* κ^{12} *even if* κ^{12} *is not square or, if square, is not necessarily nonsingular*" [14].

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[2] Eliakim Hastings Moore (1862–1932) was a "true forefather of modern American mathematics" [15]. Several comprehensive accounts of Moore's life and work are available in Archibald [16], Bliss [17,18], Parshall [19], Parshall and Rowe [20], Siegmund [15], and others see [20] [see p. 281].

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Moore constructed the general reciprocal, established its uniqueness and main properties, and justified its application to linear equations [13,14].

The general reciprocal was rediscovered by Penrose [3] [1] in 1955, and is nowadays called the *Moore–Penrose inverse*. It had to be rediscovered because Moore's work was sinking into oblivion even during his lifetime: it was much to idiosyncratic, and used unnecessarily complicated notation, making it illegible for all but very dedicated readers.

Generalized inverses are matrices satisfying some of the characteristic properties of inverses. They may be thought of as pseudoinverses for non-invertible matrices. A matrix X is called a generalized inverse for a (not necessarily square) matrix A, if $XAX = X$, $AXA = A$ and the matrices AX and XA are Hermitian [6].

In fact, the Moore–Penrose inverse, denoted by $A^{\dagger}$ (read "A dagger"), is a special type of a generalized inverse which is some times referred to as the pseudoinverse of A.

2. Block matrices and full-rank factorization

Let $\mathbb{C}^{m}$ be the set of $m \times n$ complex matrices, and $\mathbb{C}^{m}_{m\times n} = \{X \in \mathbb{C}^{m,n} : \mathrm{rank}(X) = r\}$. With $A^{[r}$ and $A_{[r}$ we denote the submatrix of A which contains the first r columns of A and the first r rows of A, respectively. Similarly $A^{r]}$ and $A_{r]}$ denote the last r columns and last r rows of A, respectively. Finally, $A^{[r}_{r]}$ denotes the submatrix of A generated by the first r columns and the last r rows of A. The identity matrix of the order k is denoted by I_k, and $\mathbb{O}$ denotes the zero matrix of a convenient size.

Penrose [1,2] has shown the existence and uniqueness of a solution $X \in \mathbb{C}^{m\times n}$ of the following equations, that is for any $A \in \mathbb{C}^{m\times n}$, we will have

$$AXA = A, \tag{2.1}$$

$$XAX = X, \tag{2.2}$$

$$(AX)^{*} = AX, \tag{2.3}$$

$$(XA)^{*} = XA. \tag{2.4}$$

[3] Sir Roger Penrose made profound contributions to Physics, Mathematics, Geometry, Philosophy of Science, Artificial Intelligence, and theories of Mind and Consciousness [21–24]. The 1988 Wolf Prize in Physics, shared with Stephen of general relativity, in which W. Hawking, cites their brilliant development of the theory of general relativity, in which they have shown the necessary for cosmological singularities and have elucidated the physics of black holes. Penrose was awarded many other prizes and honors, and was knighted in 1994. Sir Roger discovered the Moore-Penrose inverse while a student at Cambridge, and his seminal papers [1,2] started the field of generalized inverses (AMS subject class 15A09).

For a sequence S of elements from the set $\{1,2,3,4\}$, the set of matrices which satisfy the equations represented in S is denoted by $A(S)$. A matrix form $\{S\}$ is called as S-inverse of A and denoted by $A^{(S)}$. We use the following useful expansion for the Moore–Penrose generalized inverse $A^\dagger$ of A, based on the full rank factorization $A = PQ$ of A [7,8]:

$$A^\dagger = Q^\dagger P^\dagger = Q^*(QQ^*)^{-1}(P^*P)^{-1}P^* = Q^*(P^*AQ^*)^{-1}P^*.$$

The *weighted Moore–Penrose inverse* is investigated in [9,10]. The main results of these papers are

1. If M, N are given positive definite matrices such that $M \in \mathbb{C}^{m\times m}$ and $N \in \mathbb{C}^{n\times n}$, then for any matrix $A \in \mathbb{C}^{m\times n}$ there exists the unique solution $X = A^\dagger_{M\circ,\circ N} \in A\{1,2\}$ which satisfy

$$(MAX)^* = MAX, \tag{2.5}$$

$$(XAN)^* = XAN. \tag{2.6}$$

Similarly, we use the following notations:

- $A^\dagger_{M\circ,N\circ}$ denotes the unique solution of Eqs. (2.1) and (2.2), and

$$(MAX)^* = MAX \quad \text{and} \quad (NXA)^* = NXA; \tag{2.7}$$

- $A^\dagger_{\circ M,N\circ}$ is the unique solution of Eqs. (2.1) and (2.2), and

$$(AXM)^* = AXM \quad \text{and} \quad (NXA)^* = NXA; \tag{2.8}$$

- $A^\dagger_{\circ M,\circ N}$ is the unique solution of Eqs. (2.1) and (2.2), and

$$(AXM)^* = AXM \quad \text{and} \quad (XAN)^* = XAN; \tag{2.9}$$

2. Eq. (2.5) is equivalent to $(AXM^{-1})^* = AXM^{-1}$, and (2.6) can be expressed in the form $(N^{-1}XA)^* = N^{-1}XA$.
3. If $A = PQ$ is a full rank factorization of A, then

$$A^\dagger_{M\circ,\circ N} = (QN)^*(Q(QN)^*)^{-1}((MP)^*P)^{-1}(MP)^* = NQ^*(P^*MANQ^*)^{-1}P^*M. \tag{2.10}$$

One can easily use the above results to verify the following facts:

$$A^\dagger_{M\circ,\circ N} = A^\dagger_{\circ M^{-1},\circ N} = A^\dagger_{M\circ,N^{-1}\circ} = A^\dagger_{\circ M^{-1},N^{-1}\circ},$$

$$A^\dagger_{M\circ,N\circ} = (QN^{-1})^*(Q(QN^{-1})^*)^{-1}((MP)^*P)^{-1}(MP)^* = A_{M\circ,\circ N^{-1}},$$

$$A^\dagger_{\circ M,N\circ} = (QN^{-1})^*(Q(QN^{-1})^*)^{-1}((M^{-1}P)^*P)^{-1}(M^{-1}P)^* = A_{M^{-1}\circ,\circ N^{-1}},$$

$$A^\dagger_{\circ M,\circ N} = (QN)^*(Q(QN)^*)^{-1}((M^{-1}P)^*P)^{-1}(M^{-1}P)^*.$$

Now, we restate the main block decomposition [11,12] as well as the block decompositions of the Moore–Penrose inverse [12].

• Main block decompositions:

For a given matrix $A \in \mathbb{C}_r^{m\times n}$ there exists the regular matrices R, G, the permutation matrices E, F and the unitary matrices U, V, such that

$$RAG = \begin{bmatrix} I_r & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{bmatrix} = N_1; \tag{2.11}$$

$$RAG = \begin{bmatrix} B & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{bmatrix} = N_2; \tag{2.12}$$

$$RAF = \begin{bmatrix} I_r & K \\ \mathbb{O} & \mathbb{O} \end{bmatrix} = N_3; \tag{2.13}$$

$$EAG = \begin{bmatrix} I_r & \mathbb{O} \\ K & \mathbb{O} \end{bmatrix} = N_4; \tag{2.14}$$

$$UAG = \begin{bmatrix} I_r & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{bmatrix} = N_1; \tag{2.15}$$

$$RAV = \begin{bmatrix} I_r & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{bmatrix} = N_1; \tag{2.16}$$

$$UAV = \begin{bmatrix} B & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{bmatrix} = N_2; \tag{2.17}$$

$$UAF = \begin{bmatrix} B & K \\ \mathbb{O} & \mathbb{O} \end{bmatrix} = N_5; \tag{2.18}$$

and

$$EAV = \begin{bmatrix} B & \mathbb{O} \\ K & \mathbb{O} \end{bmatrix} = N_6; \tag{2.19}$$

and with the multipliers S, T satisfy $T = A_{11}^{-1}A_{12}$, $S = A_{21}A_{11}^{-1}$, we have

$$EAF = \begin{bmatrix} A_{11} & A_{11}T \\ SA_{11} & SA_{11}T \end{bmatrix} = N_7; \tag{2.20}$$

and

$$EAF = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{21}A_{11}^{-1}A_{12} \end{bmatrix} = N_7; \tag{2.21}$$

while for similarity for square matrices we have the transformation:

$$RAR^{-1} = RAFF^*R^{-1} = \begin{bmatrix} I_r & K \\ \mathbb{O} & \mathbb{O} \end{bmatrix} F^*R^{-1} = \begin{bmatrix} T_1 & T_2 \\ \mathbb{O} & \mathbb{O} \end{bmatrix}. \tag{2.22}$$

For $A_r^{m\times n}$, let

$$R = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}, \quad G = [\,G_1 \quad G_2\,], \quad U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}, \quad V = [\,V_1 \quad V_2\,],$$

where R_1, U_1 are the first r rows of R and U, respectively, and G_1, V_1 denote the first r columns of G and V, respectively. Then the Moore–Penrose inverse can be represented in the following way, where the block representations (M_i) correspond to the block decompositions (2.11)–(2.19).

$$\begin{aligned} A^\dagger &= G\begin{bmatrix} I_r & -R_1R_2^\dagger \\ -G_2^\dagger G_1 & G_2^\dagger G_1 R_1 R_2^\dagger \end{bmatrix} R = G\begin{bmatrix} I_r \\ -G_2^\dagger G_1 \end{bmatrix}[\,I_r \quad -R_1R_2^\dagger\,]R \\ &= (G_1 - G_2G_2^\dagger G_1)(R_1 - R_1R_2^\dagger R_2), \end{aligned} \tag{2.23}$$

$$\begin{aligned} A^\dagger &= G\begin{bmatrix} B^{-1} & -B^{-1}R_1R_2^\dagger \\ -G_2^\dagger G_1 B^{-1} & G_2^\dagger G_1 B^{-1} R_1 R_2^\dagger \end{bmatrix} R \\ &= G\begin{bmatrix} I_r \\ -G_2^\dagger G_1 \end{bmatrix} B^{-1}[\,I_r \quad -R_1R_2^\dagger\,]R \\ &= (G_1 - G_2G_2^\dagger G_1)B^{-1}(R_1 - R_1R_2^\dagger R_2), \end{aligned} \tag{2.24}$$

$$\begin{aligned} A^\dagger &= F\begin{bmatrix} (I_r + KK^*)^{-1} & -(I_r + KK^*)^{-1}R_1R_2^\dagger \\ K^*(I_r + KK^*)^{-1} & -K^*(I_r + KK^*)^{-1}R_1R_2^\dagger \end{bmatrix} R \\ &= F\begin{bmatrix} I_r \\ K^* \end{bmatrix}(I_r + KK^*)^{-1}[\,I_r \quad -R_1R_2^\dagger\,]R, \end{aligned} \tag{2.25}$$

$$\begin{aligned} A^\dagger &= G\begin{bmatrix} (I_r + K^*K)^{-1} & (I_r + K^*K)^{-1}K^* \\ -G_2^\dagger G_1(I_r + K^*K)^{-1} & -G_2^\dagger G_1(I_r + K^*K)^{-1}K^* \end{bmatrix} R \\ &= G\begin{bmatrix} I_r \\ -G_2^\dagger G_1 \end{bmatrix}(I_r + K^*K)^{-1}[\,I_r \quad K^*\,]R, \end{aligned} \tag{2.26}$$

$$\begin{aligned} A^\dagger &= G\begin{bmatrix} I_r & \mathbb{O} \\ -G_2^\dagger G_1 & \mathbb{O} \end{bmatrix} U = G\begin{bmatrix} I_r \\ -G_2^\dagger G_1 \end{bmatrix}[\,I_r \quad \mathbb{O}\,]U \\ &= (G_1 - G_2G_2^\dagger G_1)U_1, \end{aligned} \tag{2.27}$$

$$\begin{aligned} A^\dagger &= V\begin{bmatrix} I_r & -R_1R_2^\dagger \\ \mathbb{O} & \mathbb{O} \end{bmatrix} R = V\begin{bmatrix} I_r \\ \mathbb{O} \end{bmatrix}[\,I_r \quad -R_1R_2^\dagger\,]R \\ &= V_1(R_1 - R_1R_2^\dagger R_2), \end{aligned} \tag{2.28}$$

$$A^\dagger = V\begin{bmatrix} B^{-1} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{bmatrix} U = V\begin{bmatrix} I_r \\ \mathbb{O} \end{bmatrix}[\,B^{-1} \quad \mathbb{O}\,]U = V_1B^{-1}U_1, \tag{2.29}$$

$$
\begin{aligned}
A^{\dagger} &= F\begin{bmatrix} B^*(BB^* + KK^*)^{-1} & \mathbb{O} \\ K^*(BB^* + KK^*)^{-1} & \mathbb{O} \end{bmatrix} U \\
&= F\begin{bmatrix} B^* \\ K^* \end{bmatrix}(BB^* + KK^*)^{-1}[\, I_r \quad \mathbb{O}\,]U,
\end{aligned} \tag{2.30}
$$

$$
\begin{aligned}
A^{\dagger} &= V\begin{bmatrix} (B^*B + K^*K)^{-1}B^* & (B^*B + K^*K)^{-1}K^* \\ \mathbb{O} & \mathbb{O} \end{bmatrix} E \\
&= V\begin{bmatrix} I_r \\ \mathbb{O} \end{bmatrix}(BB^* + K^*K)^{-1}[\, B^* \quad K^*\,]E.
\end{aligned} \tag{2.31}
$$

Block decomposition (2.20) is investigated in [26], but in two different ways. Moore–Penrose inverse is represented by solving the corresponding set of matrix equations, while in [26] the results obtained using a full rank factorization, implied by the block decomposition (2.20). The corresponding representation of the Moore–Penrose inverse is

$$
\begin{aligned}
A^{\dagger} &= \begin{bmatrix} I_r \\ T^* \end{bmatrix}\left(A_{11}^*[\, I_r \quad S^*\,]EAF\begin{bmatrix} I_r \\ T^* \end{bmatrix}\right)^{-1} A_{11}^*[\, I_r \quad S^*\,] \\
&= F\begin{bmatrix} I_r \\ T^* \end{bmatrix}(I_r + TT^*)^{-1}A_{11}^{-1}(I_r + S^*S)^{-1}[\, I_r \quad S^*\,].
\end{aligned} \tag{2.32}
$$

In [27], the block representation of the Moore–Penrose inverse was obtained for the block decomposition (2.21) as

$$
\begin{aligned}
A^{\dagger} = {} & \begin{bmatrix} I_r \\ (A_{11}^{-1}A_{12})^* \end{bmatrix}\left(I_r + A_{11}^{-1}A_{12}(A_{11}^{-1}A_{12})^*\right)^{-1} \\
& \times A_{11}^{-1}(I_r + (A_{21}A_{11}^{-1})^*A_{21}A_{11}^{-1})^{-1}[\, I_r \quad (A_{21}A_{11}^{-1})^*\,].
\end{aligned} \tag{2.33}
$$

Milovanovic and Stainmirovic in [25], continue the papers [26,27], from the presented block decompositions of matrices they find the corresponding full rank factorizations $A = PQ$, and then applying the described general representations for $A^{\dagger}$ and $A^{\dagger}_{\varphi(M,N)}$. The most important result obtained in [5], is the Moore–Penrose inverse of a given matrix $A \in \mathbb{C}_r^{m\times n}$ which can be presented as follows, where each block representation (2.34)–(2.45) is derived from the block decomposition (2.11)–(2.19) as well as (2.20), (2.21) and (2.22):

$$
\begin{aligned}
A^{\dagger} &= (G_{[r}^{-1})^*\left((R^{-1[r})^*A(G_{[r}^{-1})^*\right)^{-1}(R^{-1[r})^* \\
&= (G_{[r}^{-1})^*\left((RR^*)_{[r}^{-1[r}(G^*G)_{[r}^{-1[r}\right)^{-1}(R^{-1[r}),
\end{aligned} \tag{2.34}
$$

$$A^\dagger = (G^{-1}_{[r})^*\left((R^{-1[r}B)^*A(G^{-1}_{[r})^*\right)^{-1}(R^{-1[r}B)^*$$
$$= (G^{-1}_{[r})^*\left(B^*(RR^*)^{-1[r}_{[r}B(G^*G)^{-1r]}_{r]}\right)^{-1}B^*(R^{-1[r})^*, \tag{2.35}$$

$$A^\dagger = F\begin{bmatrix}I_r\\K^*\end{bmatrix}\left((R^{-1[r})^*AF\begin{bmatrix}I_r\\K^*\end{bmatrix}\right)^{-1}(R^{-1[r})^*$$
$$= (F^{[r}+F^{n-r]}K^*)\left((RR^*)^{-1[r}_{[r}(I_r+KK^*)\right)^{-1}(R^{-1[r})^*, \tag{2.36}$$

$$A^\dagger = (G^{-1}_{r]})^*\left([\,I_r\quad K^*\,]EA(G^{-1}_{[r})^*\right)^{-1}[\,I_r\quad K^*\,]E$$
$$= (G^{-1}_{r]})^*\left((I_r+K^*K)(G^*G)^{-1[r}_{[r}\right)^{-1}(E_{[r}+K^*_{n-r]}), \tag{2.37}$$

$$A^\dagger = (G^{-1}_{[r})^*\left(U_{(r}A(G^{-1}_{[r})^*\right)^{-1}U_{[r} = (G^{-1}_{[r})^*\left((G^*G)^{-1[r}_{[r}\right)^{-1}U_{[r}, \tag{2.38}$$

$$A^\dagger = V^{[r}\left((R^{-1[r})^*AV^{[r}\right)^{-1}(R^{-1[r})^* = V^{[r}\left((RR^*)^{-1[r}_{[r}\right)^{-1}(R^{-1[r})^*, \tag{2.39}$$

$$A^\dagger = V^{[r}(B^*U_{[r}AV^{[r})^{-1}B^*U_{[r} = V^{[r}B^{-1}U_{[r}, \tag{2.40}$$

$$A^\dagger = F\begin{bmatrix}B^*\\K^*\end{bmatrix}\left(U_{[r}AF\begin{bmatrix}B^*\\K^*\end{bmatrix}\right)^{-1}U_{[r} = F\begin{bmatrix}B^*\\K^*\end{bmatrix}(BB^*+KK^*)^{-1}U_{[r}, \tag{2.41}$$

$$A^\dagger = V^{[r}\left([\,B^*\quad K^*\,]EAV^{[r}\right)^{-1}[\,B^*\quad K^*\,]E$$
$$= V^{[r}(B^*B+K^*K)^{-1}[\,B^*\quad K^*\,]E, \tag{2.42}$$

$$A^\dagger = F\begin{bmatrix}I_r\\T^*\end{bmatrix}\left(A^*_{11}[\,I_r\quad S^*\,]EAF\begin{bmatrix}I_r\\T^*\end{bmatrix}\right)^{-1}A^*_{11}[\,I_r\quad S^*\,]E$$
$$= (F^{[r}+F^{n-r]}T^*)((I_r+S^*S)A_{11}(I_r+TT^*))^{-1}(E_{[r}+S^*E_{n-r]}), \tag{2.43}$$

$$A^\dagger = F\begin{bmatrix}A^*_{11}\\A^*_{12}\end{bmatrix}\left((A^*_{11})^{-1}[\,A^*_{11}\quad A^*_{21}\,]EAF\begin{bmatrix}A^*_{11}\\A^*_{12}\end{bmatrix}\right)^{-1}(A^*_{11})^{-1}[\,A^*_{11}\quad A^*_{21}\,]$$
$$E = F\begin{bmatrix}A^*_{11}\\A^*_{12}\end{bmatrix}(A_{11}A^*_{11}+A_{12}A^*_{12})^{-1}A_{11}(A^*_{11}A_{11}+A^*_{21}A_{21})^{-1}[\,A^*_{11}\quad A^*_{21}\,]E, \tag{2.44}$$

$$A^\dagger = R^*\begin{bmatrix}I_r\\(T_1^{-1}T_2)^*\end{bmatrix}\left(T_1^*(RR^*)^{-1[r}_{[r}[\,T_1\quad T_2\,]RR^*\begin{bmatrix}I_r\\(T_1^{-1}T_2)^*\end{bmatrix}\right)^{-1}(R^{-1[r}T_1)^*. \tag{2.45}$$

Now, using the general presentation of the weighted Moore–Penrose presented in (2.10), and with the block representations (2.34)–(2.45), we can describe the following block representation of the weighted Moore–Penrose inverse $A^{\dagger}_{M\circ,\circ N}$.

The weighted Moore–Penrose inverse $A^{\dagger}_{M\circ,\circ N}$ of $A \in \mathbb{C}_r^{m\times n}$ possesses the following block representations (Z_i), which correspond to the block decompositions (2.11)–(2.19), as well as (2.20), (2.21) and (2.22) are

$$N(G_{[r}^{-1})^*\left((R^{-1^{[r}})^*MAN(G_{[r}^{-1})^*\right)^{-1}(R^{-1^{[r}})^*M, \tag{2.46}$$

$$N(G_{[r}^{-1})^*\left((R^{-1^{[r}}B)^*MAN(G_{[r}^{-1})^*\right)^{-1}(R^{-1^{[r}}B)^*M, \tag{2.47}$$

$$NF\begin{bmatrix} I_r \\ K^* \end{bmatrix}\left((R^{-1^{[r}})^*MANF\begin{bmatrix} I_r \\ K^* \end{bmatrix}\right)^{-1}(R^{-1^{[r}})^*M, \tag{2.48}$$

$$N(G_{[r}^{-1})^*\left([I_r \quad K^*]EMAN(G_{[r}^{-1})^*\right)^{-1}[I_r \quad K^*]EM, \tag{2.49}$$

$$N(G_{[r}^{-1})^*\left(U_{(r}MAN(G_{[r}^{-1})^*\right)^{-1}U_{[r}M, \tag{2.50}$$

$$NV^{[r}\left((R^{-1^{[r}})^*MANV^{[r}\right)^{-1}(R^{-1^{[r}})^*M, \tag{2.51}$$

$$NV^{[r}\left(B^*U_{[r}MANV^{[r}\right)^{-1}B^*U_{[r}M, \tag{2.52}$$

$$NF\begin{bmatrix} B^* \\ K^* \end{bmatrix}\left(U_{[r}MANF\begin{bmatrix} B^* \\ K^* \end{bmatrix}\right)^{-1}U_{[r}M, \tag{2.53}$$

$$NV^{[r}\left([B^* \quad K^*]EMANV^{[r}\right)^{-1}[B^* \quad K^*]EM, \tag{2.54}$$

$$NF\begin{bmatrix} I_r \\ T^* \end{bmatrix}\left(A_{11}^*[I_r \quad S^*]EMANF\begin{bmatrix} I_r \\ T^* \end{bmatrix}\right)^{-1}A_{11}^*[I_r \quad S^*]EM, \tag{2.55}$$

$$NF\begin{bmatrix} A_{11}^* \\ A_{12}^* \end{bmatrix}\left((A_{11}^*)^{-1}[A_{11}^* \quad A_{21}^*]EMANF\begin{bmatrix} A_{11}^* \\ A_{12}^* \end{bmatrix}\right)^{-1}(A_{11}^*)^{-1}[A_{11}^* \quad A_{21}^*]EM, \tag{2.56}$$

$$NR^*\begin{bmatrix} I_r \\ (T_1^{-1}T_2)^* \end{bmatrix}\left((R^{-1^{[r}}T_1)^*MANR^*\begin{bmatrix} I_r \\ (T_1^{-1}T_2)^* \end{bmatrix}\right)^{-1}(R^{-1^{[r}}T_1)^*M. \tag{2.57}$$

3. Reverse order laws for a matrix product

Now, we will establish necessary and sufficient conditions for mixed-type reveres order laws, and so on to hold the Moore–Penrose inverse of a matrix product. Revisiting

$$(AB)^\dagger = B^\dagger(A^\dagger ABB^\dagger)^\dagger A^\dagger \tag{3.1}$$

and then consider the following four mixed type reverse order laws

$$(AB)^\dagger = B^*(A^*ABB^*)^\dagger A^*, \tag{3.2}$$

$$(AB)^\dagger = (B^*B)^\dagger[(B^\dagger A^\dagger)^\dagger]^*(AA^*)^\dagger, \tag{3.3}$$

$$(AB)^\dagger = B^*B(AA^*ABB^*B)^\dagger AA^*, \tag{3.4}$$

$$(AB)^\dagger = B^\dagger A^\dagger - B^\dagger[(I_n - BB^\dagger)(I_n - A^\dagger A)]^\dagger A^\dagger. \tag{3.5}$$

The following, the five dual expressions which are the above reverse order laws, are equivalent to (3.1)–(3.5) respectively:

$$(B^\dagger A^\dagger)^\dagger = A(BB^\dagger A^\dagger A)^\dagger B, \tag{3.6}$$

$$(B^\dagger A^\dagger)^\dagger = (A^\dagger)^*[(BB^*)^\dagger(A^*A)^\dagger]^\dagger(B^\dagger), \tag{3.7}$$

$$(B^\dagger A^\dagger)^\dagger = AA^*[(AB)^\dagger]^*B^*B, \tag{3.8}$$

$$(B^\dagger A^\dagger)^\dagger = (AA^*)^\dagger[(BB^*B)^\dagger(AA^*A)^\dagger]^\dagger(B^*B)^\dagger, \tag{3.9}$$

$$(B^\dagger A^\dagger)^\dagger = AB - A[(I_n - A^\dagger A)(I_n - BB^\dagger)]^\dagger B. \tag{3.10}$$

A direct motivation to write $(AB)^\dagger$ in the four forms in (3.1)–(3.5) arises from the different expressions of AB through Moore–Penrose inverses of A and B. Recall that

$$A = AA^\dagger A = AA^*(A^\dagger)^* = (A^\dagger)^*A^*A \tag{3.11}$$

then

$$AB = AA^\dagger ABB^\dagger B = A(A^\dagger ABB^\dagger)B := P_1N_1Q_1,$$

hence (3.1) comes from considering the reverse order law $(P_1N_1Q_1)^\dagger = Q_1^\dagger\, N_1^\dagger P_1^\dagger$. Also note that

$$AB = AA^\dagger ABB^\dagger B = (A^\dagger)^*A^*ABB^*(B^\dagger)^* = (A^\dagger)^*A^*ABB^*(B^\dagger)^* := P_2N_2Q_2.$$

Hence (3.2) comes from considering the reverse order law $(P_2N_2Q_2)^{\text{“}-1\text{”}} = Q_2^{\text{“}-1\text{”}}N_2^{\text{“}-1\text{”}}P_2^{\text{“}-1\text{”}}$. Moreover

$$\begin{aligned} AB &= AA^{\text{“}-1\text{”}}ABB^{\text{“}-1\text{”}}B = AA^*(A^{\text{“}-1\text{”}})^*(B^{\text{“}-1\text{”}})^*B^*B \\ &= AA^*[(A^{\text{“}-1\text{”}})^*(B^{\text{“}-1\text{”}})^*]BB^* := P_3N_3Q_3. \end{aligned}$$

Thus (3.3) comes from considering the reverse order law $(P_3N_3Q_3)^\dagger = Q_3^\dagger$ $N_3^\dagger P_3^\dagger$. Furthermore, we can rewrite A as

$$A = (AA^*A)(A^*A)^\dagger = (AA^*)^\dagger(AA^*A).$$

Hence AB can also be rewritten as

$$AB = (A^*A)^\dagger(AA^*A)(BB^*B)(B^*B)^\dagger = (AA^*)^\dagger(AA^*ABB^*B)(B^*B)^\dagger := P_4N_4Q_4,$$

and (3.4) comes from considering the reverse order law $(P_4N_4Q_4)^\dagger = Q_4^\dagger N_4^\dagger P_4^\dagger$.

Finally, and before we start our numerical examples, let us have the following theorem.

Theorem 1. *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times p}$ be given, for any A^{-1} and B^{-1} the product $B^{-1}(A^\dagger ABB^\dagger)^\dagger A^{-1}$ is an inner inverse of AB, that is, the set inclusion*

$$\{B^{-1}(A^\dagger ABB^\dagger)^\dagger A^{-1}\} \subseteq \{(AB)^{-1}\}$$

holds.

Proof. It is well known that the inverse order law $(AB)^{-1} = B^{-1}A^{-1}$ does not hold in general for an inner inverse of the matrix product AB. Let P_1 and P_2 be orthogonal projectors, i.e., $P_i^2 = P_i^*$, $i = 1, 2$. Then

$$(P_1P_2)^\dagger = P_2(P_1P_2)^\dagger P_1, \tag{3.12}$$

which follows straightforwardly from the definition of the Moore–Penrose inverse by verifying that

$$P_1P_2^2(P_1P_2)^\dagger P_1^2P_2P_1P_2(P_1P_2)^\dagger P_1P_2 = P_1P_2,$$

$$P_2(P_1P_2)^\dagger P_1^2P_2^2(P_1P_2)^\dagger P_1 = P_2(P_1P_2)^\dagger P_1P_2(P_1P_2)^\dagger P_1 = P_2(P_1P_2)^\dagger P_1$$

and noting that the products

$$P_1P_2^2(P_1P_2)^\dagger P_1 = (P_1P_2(P_1P_2)^\dagger)^*P_1 = (P_1^2P_2(P_1P_2)^\dagger)^* = P_1P_2(P_1P_2)^\dagger$$

and

$$P_2(P_1P_2)^\dagger P_1^2P_2 = P_2((P_1P_2)^\dagger P_1P_2)^* = ((P_1P_2)^\dagger P_1P_2^2)^* = (P_1P_2)^\dagger P_1P_2$$

are both Hermitian. Since $A^\dagger$ and $BB^\dagger$ are the orthogonal projectors on the column spaces $C(A^*)$ and $C(B)$, respectively, we adopt the notation $A^\dagger A = P_{A^*}$ and $BB^\dagger = P_{B^*}$. Then on account of (3.12), it follows that

$$\begin{aligned} ABB^{-1}(P_{A^*}P_B)^\dagger A^{-1}AB &= ABB^{-1}P_B(P_{A^*}P_B)^\dagger P_{A^*}A^\dagger AB = AP_B(P_{A^*}P_B)^\dagger P_{A^*}B \\ &= AP_{A^*}P_B(P_{A^*}P_B)^\dagger P_{A^*}P_BB = AP_{A^*}P_BB = AB \end{aligned}$$

and so $B^{-1}(P_{A^*}P_B)^\dagger A^{-1}$ is a generalized inverse of AB irrespective of the choice of generalized inverses A^{-1} and B^{-1}. $\square$

4. Numerical examples

Let us now have some computational examples, all computations are verified with the use of *Mathematica* one of the most interested CAS, Computer Algebra System.

Example. Consider

$$A = \begin{bmatrix} -1 & 0 & 1 & 2 \\ -1 & 1 & 0 & -1 \\ 0 & -1 & 1 & 3 \\ 0 & 1 & -1 & -3 \\ 1 & -1 & 0 & 1 \\ 1 & 0 & -1 & -2 \end{bmatrix}.$$

Using the Gauss–Jordan transformation, we get the following reduced row-echelon form of the matrix A:

$$RAF = R_A = \begin{bmatrix} I_r & K \\ \mathbb{O} & \mathbb{O} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

the matrix R_A is obtained using the permutation matrix $F = I_4$, and the following regular matrix:

$$R = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Using the method (2.36) we obtain

$$R^{-1|^r} = \begin{bmatrix} -1 & 0 \\ -1 & 1 \\ 0 & -1 \\ 0 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}, \quad F\begin{bmatrix} I_r \\ K^* \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \\ -2 & -3 \end{bmatrix},$$

and the following Moore–Penrose inverse of A:

$$A^{\dagger} = \begin{bmatrix} -\frac{5}{34} & -\frac{3}{17} & \frac{1}{34} & -\frac{1}{34} & \frac{3}{17} & \frac{5}{34} \\ \frac{4}{51} & \frac{13}{102} & -\frac{5}{102} & \frac{5}{102} & -\frac{13}{102} & -\frac{4}{51} \\ \frac{7}{102} & \frac{5}{102} & \frac{1}{51} & -\frac{1}{51} & -\frac{5}{102} & -\frac{7}{102} \\ \frac{1}{17} & -\frac{1}{34} & \frac{3}{34} & -\frac{3}{34} & \frac{1}{34} & -\frac{1}{17} \end{bmatrix}.$$

Example. For a matrix

$$A = \begin{bmatrix} 4 & -1 & 1 & 2 \\ -2 & 2 & 0 & -1 \\ 6 & -3 & 1 & 3 \\ -10 & 4 & -2 & -5 \end{bmatrix}$$

we obtain

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 2 & -1 & 0 & 1 \end{bmatrix}, \quad T_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad T_2 = \begin{bmatrix} -1 & -2 \\ -1 & -3 \end{bmatrix}, \quad F = I_4.$$

Then, easily we can verify the following:

$$(R^{-1}|^{r} T_1)^{*} = \begin{bmatrix} -1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 1 \end{bmatrix}, \quad R^{*}\begin{bmatrix} I_r \\ (T_1^{-1}T_2)^{*} \end{bmatrix} = \begin{bmatrix} -4 & -6 \\ 1 & 3 \\ -1 & -1 \\ -2 & -3 \end{bmatrix}.$$

Finally, using (2.45), we get

$$A^{\dagger} = \begin{bmatrix} \frac{8}{81} & \frac{10}{81} & -\frac{2}{81} & -\frac{2}{27} \\ \frac{47}{162} & \frac{79}{162} & -\frac{16}{81} & -\frac{5}{54} \\ \frac{7}{54} & \frac{11}{54} & -\frac{2}{27} & -\frac{1}{18} \\ \frac{4}{81} & \frac{5}{81} & -\frac{1}{81} & -\frac{1}{27} \end{bmatrix}.$$

Example. Consider the matrix

$$A = \begin{bmatrix} 0 & a_3 & -a_2 & a_5 & -a_4 & a_7 & -a_6 \\ -a_3 & 0 & a_1 & a_6 & -a_7 & -a_4 & a_5 \\ a_2 & -a_1 & 0 & -a_7 & -a_6 & a_5 & a_4 \\ -a_5 & -a_6 & a_7 & 0 & a_1 & a_2 & -a_3 \\ a_4 & a_7 & a_6 & -a_1 & 0 & -a_3 & -a_2 \\ -a_7 & a_4 & -a_5 & -a_2 & a_3 & 0 & a_1 \\ a_6 & -a_5 & -a_4 & a_3 & a_2 & -a_1 & 0 \end{bmatrix},$$

to find the Moore–Penrose inverse $A^\dagger$ of A, let a denote the 7×1 vector whose successive components are a_i, $i = 1,\dots,7$. Since the transpose of A satisfies $A^t = -A$, the well-know formula $A^\dagger = A^t(A^tA)^\dagger$ takes the form

$$A^\dagger = A(A^2)^\dagger.$$

The structure of A ensures that the entries of A^2 are

$$(A^2)_{ii} = -\alpha + a_i^2, \quad i = 1,\dots,7 \quad \text{and} \quad (A^2)_{ij} = a_ia_j; \\ i,j = 1,\dots,7; \ i \neq j,$$

where $\alpha = a^ta = \sum_{i=1}^{7} a_i^2$. This means that

$$A^2 = -\alpha I_7 + aa^t = -\alpha(I_7 - \alpha^{\text{“}-1\text{”}}aa^t).$$

The matrix $I_7 - \alpha^\dagger aa^t$ is idempotent and Hermitian (i.e., an orthogonal projectors), and therefore has the Moore–Penrose inverse equal to itself. Consequently, $A^\dagger = A(A^2)^\dagger$ takes the form

$$A^\dagger = -\alpha^\dagger A(I_7 - \alpha^\dagger aa^t).$$

But the structure of A ensures also that $Aa = 0$, and the solution is

$$A^\dagger = -\alpha^\dagger A.$$

The following remark present another method of solution.

Remark. Clearly, if each a_i is zero then $A^\dagger$ is the zero matrix. Below let $a_i \neq 0$ for some i. Then we show that

$$A^\dagger = -(aa^t)^{-1}A,$$

where $a = [a_1, a_2, a_3, a_4, a_5, a_6, a_7]$. It is easily verified that $Aa^t = 0$ and that $AA^t = (aa^t)I - a^ta$. Also, $AA^t = -A^2 = A^tA$ since A is skew-symmetric. Put $X := -(aa^t)^{-1}A$. Then it follows from these observations that $AXA = A$, $XAX = X$ and that AX (which equals XA) is symmetric. Hence the result is proved.

Example. Let

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}; \quad A_{11} \in \mathbb{C}^{m\times n},\ A_{12} \in \mathbb{C}^{m\times k},\ A_{21} \in \mathbb{C}^{l\times n},\ A_{22} \in \mathbb{C}^{l\times k}$$

then

1. The rank of the upper-right $m \times l$ block of $P_A = AA^{\dagger}$ is

   $$\operatorname{rank}((P_A)_{12}) = \operatorname{rank}(A_{11} : A_{12}) + \operatorname{rank}(A_{21} : A_{22}) - \operatorname{rank}(A),$$

   where $(E : F)$ denotes the partitioned block matrix with E placed next to F.
2. If A is an orthogonal projection, Hermitian and idempotent, then

   $$\operatorname{rank}(A) = \operatorname{rank}(A_{11}) + \operatorname{rank}(A_{22}) - \operatorname{rank}(A_{12}).$$

Solution: We solve the parts (1) and (2) of the problem in the reverse order. First observe that without the additional assumption that $n = m$, $k = l$ the result in (2) is invalid. This is can be easily seen from the example in which $m = 2$, $n = 1$, $k = 2$, $l = 1$ and

$$A_{11} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \quad A_{21} = [\,0\,] \quad \text{and} \quad A_{22} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

Then $\operatorname{rank}(A) = 2$, whereas $\operatorname{rank}(A_{11}) + \operatorname{rank}(A_{22}) - \operatorname{rank}(A_{12}) = 1$. Since every orthogonal projector is a non-negative definite matrix, it follows from [3] that if $n = m$ and $k = l$, then the blocks A_{ij} $(i, j = 1, 2)$ satisfy

$$\begin{aligned} &A_{11}A_{11}^{\dagger}A_{12} = A_{12}, \quad A_{12}A_{22}^{\dagger}A_{22} = A_{12}, \\ &A_{21}A_{11}^{\dagger}A_{11} = A_{21} \quad \text{and} \quad A_{22}A_{22}^{\dagger}A_{21} = A_{21} \end{aligned} \tag{4.1}$$

using [4] assures that condition (4.1) are sufficient for the rank of A to be additive on the Schur complement, i.e.,

$$\operatorname{rank}(A) = \operatorname{rank}(A_{11}) + \operatorname{rank}(A_{22} - A_{21}A_{11}^{\dagger}A_{12}). \tag{4.2}$$

An immediate consequence of (4.1) is that

$$A_{22}A_{22}^{\dagger}(A_{21}A_{11}^{\dagger}A_{12}) = A_{21}A_{11}^{\dagger}A_{12} \quad \text{and} \quad (A_{21}A_{11}^{\dagger}A_{12})A_{22}^{\dagger}A_{22} = A_{21}A_{11}^{\dagger}A_{12}. \tag{4.3}$$

On the other hand, from the idempotency of A it follows that $A_{21} = A_{21}A_{11} + A_{22}A_{21}$, $A_{12}A_{21} = (I - A_{11})A_{11}$, and $A_{21}A_{12} = A_{22}(I - A_{22})$. Hence in view of (4.1),

$$\begin{aligned} (A_{21}A_{11}^{\dagger}A_{12})A_{22}^{\dagger}(A_{21}A_{11}^{\dagger}A_{12}) &= A_{21}A_{11}^{\dagger}A_{12}A_{22}^{\dagger}(A_{21}A_{11} + A_{22}A_{21})A_{11}^{\dagger}A_{12} \\ &= A_{21}A_{11}^{\dagger}(A_{12}A_{22}^{\dagger}A_{21}A_{12} + A_{12}A_{21}A_{11}^{\dagger}A_{12}) \\ &= A_{21}A_{11}^{\dagger}[A_{12}(I - A_{22}) + (I - A_{11})A_{12}] \\ &= A_{21}A_{11}^{\dagger}(2A_{12} - A_{21}^{*}) = A_{21}A_{11}^{\dagger}A. \end{aligned} \tag{4.4}$$

According to [4], the condition (4.3) and (4.4) are necessary and sufficient for rank subtractivity of A_{22} and $A_{21}A_{11}^{\dagger}A_{12}$, i.e.,

$$\mathrm{rank}(A_{22} - A_{21}A_{11}^{\dagger}A_{12}) = \mathrm{rank}(A_{22}) - \mathrm{rank}(A_{21}A_{11}^{\dagger}A_{12}). \tag{4.5}$$

Moreover, since A_{11} is a non-negative definite matrix, it follows that

$$rank(A_{21}A_{11}^{\dagger}A_{12}) = \mathrm{rank}(A_{12}^{*}A_{11}^{\dagger}A_{11}A_{11}^{\dagger}A_{12}) = \mathrm{rank}(A_{11}A_{11}^{\dagger}A_{12}) = \mathrm{rank}(A_{12}). \tag{4.6}$$

Substituting (4.5) and (4.6) into (4.2) yields the required equality

$$\mathrm{rank}(A) = \mathrm{rank}(A_{11}) + \mathrm{rank}(A_{22}) - \mathrm{rank}(A_{12}). \tag{4.7}$$

The result in part (1) can be obtained as a corollary to (4.7). Denoting $(A_{11} : A_{12})$ and $(A_{21} : A_{22})$ by A_1 and A_2 respectively, and assuming that $A^{\dagger} = (G_1 : G_2)$, the equality (4.7) applied to

$$P_A = AA^{``-1"} = \begin{bmatrix} A_1G_1 & A_1G_2 \\ A_2G_1 & A_2G_2 \end{bmatrix}$$

takes the form

$$\mathrm{rank}(AA^{\dagger}) = \mathrm{rank}(A_1G_1) + \mathrm{rank}(A_2G_2) - \mathrm{rank}(A_1G_2). \tag{4.8}$$

It is clear now that $\mathrm{rank}(AA^{\dagger}) = \mathrm{rank}(A)$. Moreover, since the non-negative definiteness of $AA^{\dagger}$ implies the range inclusions

$$R(A_1G_2) \subseteq R(A_1G_1) \quad \text{and} \quad R(A_2G_1) \subseteq R(A_2G_2),$$

it follows that

$$\begin{aligned} \mathrm{rank}(A_i) &\geqslant \mathrm{rank}(A_iG_i) = \mathrm{rank}(A_i(G_1 : G_2)) \\ &\geqslant \mathrm{rank}(A_iA^{\dagger}A) = \mathrm{rank}(A_i), \quad i = 1, 2. \end{aligned}$$

Consequently, the equality (4.8) can be expressed in the

$$\mathrm{rank}(A) = \mathrm{rank}(A_1) + \mathrm{rank}(A_2) - \mathrm{rank}((P_A)_{12})$$

as required.

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