

# United-atom, Mie $\lambda$ -6 force fields cannot simultaneously predict vapor-liquid equilibria and high pressure properties

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Molecular simulation results at extreme temperatures and pressures can supplement experimental data when developing fundamental equations of state. Since most force fields are optimized to agree with vapor-liquid equilibria (VLE) properties, however, the reliability of the molecular simulation results depends on the validity/transferability of the force field at higher temperatures and pressures. As demonstrated in this study, although state-of-the-art united-atom Mie  $\lambda$ -6 potentials for normal and branched alkanes provide accurate estimates for VLE, they tend to over-predict pressures for dense supercritical fluids and compressed liquids. The physical explanation for this observation is that the repulsive barrier is too steep for the “optimal” united-atom Mie  $\lambda$ -6 potential parameterized with VLE properties. Bayesian inference confirms that no feasible combination of non-bonded parameters ( $\epsilon$ ,  $\sigma$ , and  $\lambda$ ) is capable of simultaneously predicting saturated vapor pressures, saturated liquid densities, and pressures at high temperatures and densities. This conclusion has both practical and theoretical ramifications, as more realistic non-bonded potentials may be required for accurate extrapolation to high pressures of industrial interest.

Keywords: Transferability, Molecular Dynamics, Molecular Simulation, Monte Carlo, Markov Chain, Bayesian Inference

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## I. INTRODUCTION

An accurate understanding of the relationship between pressure ( $P$ ), volume ( $V$ , or density,  $\rho$ ), and temperature ( $T$ ) and caloric properties (such as internal energy,  $U$ , and heat capacity) for a given compound is essential for designing industrial chemical processes. Fundamental equations of state (FEOS), such as those based on the Helmholtz free energy, are a powerful approach for estimating  $PVT$  behavior and caloric properties. For example, the National Institute of Standards and Technology (NIST) Reference Fluid Properties (REFPROP) currently provides FEOS for around one hundred chemical species.<sup>1</sup> Unfortunately, most compounds do not have sufficient (reliable) experimental data covering a wide range of pressures, densities, and temperatures to develop a highly-accurate FEOS. Since FEOS are semi-empirical and have 15-30 fitting parameters, the FEOS predictions can result in large errors at temperatures and pressures that are significantly higher than those used in parameterizing the FEOS. Therefore, improvement in an FEOS at high temperatures and pressures necessitates additional data for those conditions.

The lack of experimental data at high temperatures and pressures, especially, is likely attributed to the inherent safety, cost, and complexity of such experiments. By contrast, molecular simulation (i.e. Monte Carlo, MC, and molecular dynamics, MD) methods at high temperatures and pressures do not suffer from any of these limitations. Therefore, in principle, molecular simulation can aid in developing FEOS.<sup>2-6</sup> Although it is possible to fit an FEOS to just molecular simulation results, most studies implement hybrid data sets, i.e. from both experiment and molecular simulation.<sup>7</sup>

For example, several recent studies supplement experimental data with molecular simulation results at temperatures and pressures beyond the range of available experimental temperatures and pressures.<sup>8-11</sup> Specifically, experimental data were available for temperatures and pressures up to 580 K and 130 MPa, 590 K and 180 MPa, 450 K and 2 MPa, and 560 K and 100 MPa for hexamethyldisiloxane,<sup>8</sup> octamethylcyclotetrasiloxane,<sup>9</sup> ethylene

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oxide,<sup>11</sup> and 1,2-dichloroethane,<sup>10</sup> respectively. Molecular simulations were performed for these compounds at temperatures and pressures up to 1200 K and 600 MPa, 1200 K and 520 MPa, 1000 K and 700 MPa, 1000 K and 1200 MPa, respectively. The inclusion of these simulation results improved the performance of the FEOS at extreme temperatures and pressures.

While previous studies have focused on small/hazardous compounds, the present study investigates normal and branched alkanes. Hydrocarbons are a fundamental feed-stock for many petrochemical processes and, therefore, large amounts of experimental data exist covering a wide range of  $PVT$  phase space for some alkanes. For these reasons, REFPROP provides highly-accurate FEOS for several hydrocarbons, most of which are shorter-chains (less than 20 carbons) with limited branching (i.e. only methyl branches). The use of hybrid data sets is an appealing approach to develop FEOS for industrially relevant hydrocarbons with less experimental data, i.e. those with longer chain-lengths or a higher degree of branching.

The primary limitation for implementing molecular simulation at extreme temperatures and pressures is whether or not the force field, which is typically parameterized using VLE data, is reliable at those conditions. For example, it was demonstrated that VLE-optimized force fields for small compounds, such as noble gases, hydrogen sulfide, and hydrogen chloride, do adequately represent the homogeneous fluid region.<sup>7</sup> In this study, we investigate how well the traditional force fields for predicting VLE of normal and branched alkanes extrapolate to higher temperatures and pressures, i.e dense supercritical fluids and compressed liquids. This analysis is performed for four normal and four branched alkanes by comparing the simulated compressibility factor ( $Z$ ) with the REFPROP correlations. Note that the simulation conditions do not go beyond the range of validity for the respective compounds, so that we can assume the REFPROP correlations are reliable.

The most accurate force fields for estimating hydrocarbon VLE properties, such as  $\rho_1^{\text{sat}}$  and  $P_v^{\text{sat}}$ , are the Transferable Potentials for Phase Equilibria (TraPPE)<sup>12,13</sup> (and, especially, the recent TraPPE-2<sup>14</sup>), Errington,<sup>15</sup> fourth generation anisotropic-United-atom (AUA4),<sup>16,17</sup> Potoff,<sup>18,19</sup> and Transferable anisotropic Mie potential (TAMie).<sup>20,21</sup> The

TraPPE and Potoff force fields use a united-atom (UA) model while the TraPPE-2, Errington, AUA4, and TAMie force fields use an anisotropic-united-atom (AUA) model. Both a UA and AUA model group the hydrogen interactions with their neighboring carbon atom. However, the UA model assumes that the UA interaction site is that of the carbon atom, while an AUA model assumes that the AUA interaction site is shifted away from the carbon atom and towards the hydrogen atom(s). Although, in theory, an all-atom (AA) force field should yield more accurate results, from a parameterization standpoint, it is much easier to ensure that a global minimum is obtained when parameterizing UA and AUA force fields since fewer (highly correlated) parameters are optimized simultaneously. Furthermore, the reduced computational cost is an additional benefit of the UA and AUA approach.

In addition to the division between UA and AUA force fields, the existing force fields differ in the non-bonded functional form and corresponding parameters. The TraPPE, TraPPE-2, and AUA4 force fields use a Lennard-Jones (LJ) 12-6 potential, while the Potoff and TAMie force fields use the Mie  $\lambda$ -6 (or generalized Lennard-Jones) potential, and the Errington force field uses the Buckingham exponential-6 (Exp-6) potential.

The Mie  $\lambda$ -6 potential is:<sup>22</sup>

$$u^{\text{vdw}}(\epsilon, \sigma, \lambda; r) = \left( \frac{\lambda}{\lambda - 6} \right) \left( \frac{\lambda}{6} \right)^{\frac{6}{\lambda - 6}} \epsilon \left[ \left( \frac{\sigma}{r} \right)^\lambda - \left( \frac{\sigma}{r} \right)^6 \right] \quad (1)$$

where  $u^{\text{vdw}}$  is the van der Waals interaction,  $\sigma$  is the distance ( $r$ ) where  $u^{\text{vdw}} = 0$ ,  $-\epsilon$  is the energy of the potential at the minimum (i.e.  $u^{\text{vdw}} = -\epsilon$  and  $\frac{\partial u^{\text{vdw}}}{\partial r} = 0$  for  $r = r_{\min}$ ), and  $\lambda$  is the repulsive exponent.

The Buckingham exponential-6 model is:<sup>15</sup>

$$u^{\text{vdw}}(\epsilon, r_{\min}, \alpha; r) = \begin{cases} \frac{\epsilon}{1 - \frac{6}{\alpha}} \left[ \frac{6}{\alpha} \exp \left( \alpha \left[ 1 - \frac{r}{r_{\min}} \right] \right) - \left( \frac{r_{\min}}{r} \right)^6 \right] & \text{for } r > r_{\max} \\ \infty & \text{for } r < r_{\max} \end{cases} \quad (2)$$

where  $u^{\text{vdw}}$ ,  $\epsilon$ , and  $r$  are the same as in Equation 1,  $r_{\min}$  is the distance that corresponds to the minimum in the potential (i.e.  $u^{\text{vdw}}(r_{\min}) = -\epsilon$ ),  $\alpha$  is a Buckingham exponential-6 parameter, and  $r_{\max}$  is the smallest positive value for which  $\frac{du^{\text{vdw}}}{dr} = 0$ .

The three-parameter Mie  $\lambda$ -6 and Exp-6 potentials are more flexible than the two-parameter LJ 12-6 potential as an additional adjustable parameter controls the steepness of the repulsive barrier. Note that the Mie  $\lambda$ -6 potential reduces to the LJ 12-6 potential for  $\lambda = 12$ . Therefore, the LJ 12-6 potential can be considered a special subclass of the Mie  $\lambda$ -6 potential.

Previous work demonstrated that the UA LJ 12-6 potential cannot adequately estimate both  $\rho_1^{\text{sat}}$  and  $P_v^{\text{sat}}$  for  $n$ -alkanes.<sup>23,24</sup> For this reason, the TraPPE-UA force field was primarily developed to agree with  $\rho_1^{\text{sat}}$  (and the critical temperature,  $T_c$ ).<sup>12</sup> By contrast, accurate prediction of both  $\rho_1^{\text{sat}}$  and  $P_v^{\text{sat}}$  over a wide temperature range is possible by varying the repulsive exponent of the LJ potential (i.e. the Mie  $\lambda$ -6 potential). It is important to mention that, although an attractive exponent of 6 has a strong theoretical basis,  $\lambda = 12$  (LJ 12-6) is a historical artifact that was chosen primarily for computational purposes.<sup>25</sup>

Typically, when parameterized to VLE data, the optimal value of  $\lambda$  is greater than 12 with a corresponding increase in the well depth ( $\epsilon$ ). Specifically, for most hydrocarbons, the Potoff UA force field<sup>18,19</sup> uses  $\lambda = 16$  while the TAMie force field<sup>20</sup> uses  $\lambda = 14$ . Gordon also demonstrated that reliable viscosities can be obtained from a UA Mie  $\lambda$ -6 model for  $n$ -alkanes by using  $\lambda = 14$  and  $\lambda = 20$  for the  $\text{CH}_3$  and  $\text{CH}_2$  sites, respectively (note the subtle difference in how Gordon defines the Mie  $\lambda$ -6 potential, a.k.a. “mod- $n$ -6”).<sup>26</sup> However, it is important to note that Gordon and Galliéro et al. report  $\lambda$  values of 11 and 10, respectively, for UA methane when optimized with viscosity data.<sup>26,27</sup>

There are some theoretical concerns that increasing the repulsive exponent might have some undesirable consequences, especially at high pressures, where particles will spend more time with very short pairwise distances than at VLE conditions. For example, References 28–30 demonstrate that neither an all-atom LJ 12-6 or an all-atom LJ 9-6 is adequate to reproduce high-level *ab initio* calculations of  $n$ -alkanes ranging from methane to  $n$ -butane. The studies of Rowley et al. suggest a modified-Morse potential is necessary for accurate representation of *ab initio* dimer energies.<sup>28,29</sup> Hayes et al. confirms these results while also emphasizing that the short-range repulsive forces, which are most important when computing high pressures in molecular simulation, are poorly represented with an AA LJ 12-6 or an AA LJ 9-6 model.<sup>30</sup> Specifically, the LJ 12-6 potential is too steep, and

only slight improvement in the repulsive region is observed for the LJ 9-6 potential. It is interesting that Hayes et al. also highlights deficiencies in the repulsive region for the Buckingham exponential-6 potential.

Recently, Kulakova et al. used Bayesian inference to conclude that experimental data for argon, specifically the liquid and vapor radial distribution functions at varying temperatures and densities, support  $\lambda$  values between 6 and 10, while argon dimer *ab initio* energies support  $\lambda$  values between 12 and 14.<sup>31</sup> They suggest that these larger values of  $\lambda$  should not be used for liquid phase simulations. By contrast, two other studies of noble gases, including argon, support  $\lambda \geq 12$ .<sup>27,32</sup> Specifically, Mick et al. reports a 13-6 potential for argon, while Galliéro et al. states that the 12-6 potential is superior for argon than the 10-6, 14-6, 16-6, 18-6, and 20-6 potentials. The likely explanation for this discrepancy is the choice of experimental data. The conclusions from Kulakova et al. are based on the radial distribution function, while Galliéro et al. used viscosity and pressure, and Mick et al. utilized VLE data.

Structural properties, such as the radial distribution function, and *ab initio* calculations provide considerable insight into the true repulsive barrier.<sup>27,31</sup> However, the “correct” value of  $\lambda$  does not guarantee adequate prediction of VLE and/or *PVT* behavior. This is primarily because the Mie  $\lambda$ -6 potential is only an approximation to the real potential and, thus, it is not flexible enough to agree with both the repulsive and attractive regions. Instead, only the region that is most sensitive to the target experimental data will be adequately represented, which for VLE data is typically the effective-size, depth of the energy minimum, and the dispersive tail. Furthermore, the “optimal”  $\lambda$  is not necessarily the “true”  $\lambda$  because force fields use effective parameter sets that account for assumptions, such as pair-wise additivity (i.e. excluding three-, four-, etc. body interactions) or the lack of explicit hydrogens. For example, a UA Mie  $\lambda$ -6 potential is simply not capable of predicting VLE properties of ethane for  $\lambda < 12$  (see Figures 1-2 of Reference 18), despite theoretical evidence that the repulsive barrier should be softer than  $\lambda = 12$ .

The purpose of this study is to determine whether or not the UA Mie  $\lambda$ -6 model is adequate for predicting *both* VLE and *PVT* at high temperatures and pressures for alkanes. Although it is difficult to know if the theoretical studies discussed previously for

noble gases and all-atom  $n$ -alkane models are applicable to UA models for normal and branched alkanes, the working hypothesis based on the literature is that a UA Mie  $\lambda$ -6 potential parameterized with VLE data is too repulsive and, thus, performs poorly at high pressures. This assessment is of practical engineering importance for deciding whether or not UA Mie  $\lambda$ -6 force fields should be used when developing fundamental equations of state for alkanes based on hybrid data sets.

The outline for this study is the following. Section II discusses the simulation and force field details. Section III is a case study for normal and branched alkanes using the existing force fields developed based on VLE properties. Section IV explains how Bayesian inference is employed to investigate the adequacy of the UA Mie  $\lambda$ -6 potential. Section V presents the results from the Bayesian analysis with recommendations and limitations in Section VI. Section VII reports the primary conclusions of this study.

## II. METHODS I

### A. Simulation Details

Four normal and four branched alkanes of varying chain-length and degree of branching are simulated in this study. Specifically, we simulate ethane, propane,  $n$ -butane,  $n$ -octane, isobutane (2-methylpropane), isohexane (2-methylpentane), isooctane (2,2,4-trimethylpentane), and neopentane (2,2-dimethylpropane). These compounds were chosen to represent a diverse set of the normal and branched alkanes available in REFPROP.<sup>1</sup>

Simulations for this study are performed in the  $NVT$  ensemble (constant number of molecules,  $N$ , constant volume,  $V$ , and constant temperature,  $T$ ) using GROMACS version 2018.<sup>33</sup> Each simulation uses the Velocity Verlet integrator with a 2 fs time-step, 1.4 nm cut-off for non-bonded interactions with tail corrections for energy and pressure, Nosé-Hoover thermostat with a time constant of 1 ps, and fixed bond-lengths are constrained using LINCS with a LINCS-order of eight. Coulombic interactions are not computed as none of the force fields require partial charges for the compounds studied. The equilibration time is 0.1 ns for ethane and propane, 0.2 ns for  $n$ -butane, and 0.5 ns for all

other compounds. The production time is 1 ns for ethane, 2 ns for propane and *n*-butane, and 4 ns for all other compounds. Replicate simulations are performed for *n*-octane to validate that a single MD run of this length agrees with the average of several replicates, to within the combined uncertainty. A system size of 400 molecules is used for ethane, propane, and *n*-butane, while all other compounds use 800 molecules. Example input files are provided in Section [SII](#) of Supporting Information.

The specific state points for each compound studied are depicted in Figure 1 (for tabulated values, see Section [SII](#) of Supporting Information). These state points correspond to the recommended conditions for the isothermal isochoric integration (ITIC) algorithm discussed in Section [IV](#).<sup>34–36</sup> Simulations are performed along a supercritical isotherm (with a reduced temperature,  $T_r \approx 1.2$ ) and five saturated liquid density isochores. Nine densities are simulated along the supercritical isotherm ( $T^{\text{IT}}$ ) with five densities being the same as the isochore densities ( $\rho^{\text{IC}}$ ). Two additional temperatures are simulated along each isochore, with one being the REFPROP saturation temperature ( $T^{\text{sat}}$ ) while the inverse of the second isochore temperature is the average of  $1/T^{\text{IT}}$  and  $1/T^{\text{sat}}$ . Thus, a total of 19 simulations are performed for each compound and force field.

## B. Force field

A united-atom (UA) or anisotropic-united-atom (AUA) representation is used for each compound studied. The UA and AUA groups required for normal and branched alkanes are  $\text{sp}^3$  hybridized  $\text{CH}_3$ ,  $\text{CH}_2$ ,  $\text{CH}$ , and  $\text{C}$  sites. For most literature models, a single (transferable) parameter set is assigned for each interaction site. However, two exceptions exist for the force fields studied. First, TAMie implements a different set of  $\text{CH}_3$  parameters for ethane. Second, Potoff reports a “generalized” and “short/long” (S/L)  $\text{CH}$  and  $\text{C}$  parameter set. The Potoff “generalized”  $\text{CH}$  and  $\text{C}$  parameter set is an attempt at a completely transferable set. However, since the “generalized” parameters performed poorly for some compounds, the S/L parameter set was proposed, where the “short” and “long” parameters are implemented when the number of carbons in the backbone is  $\leq 4$  and  $> 4$ , respectively.



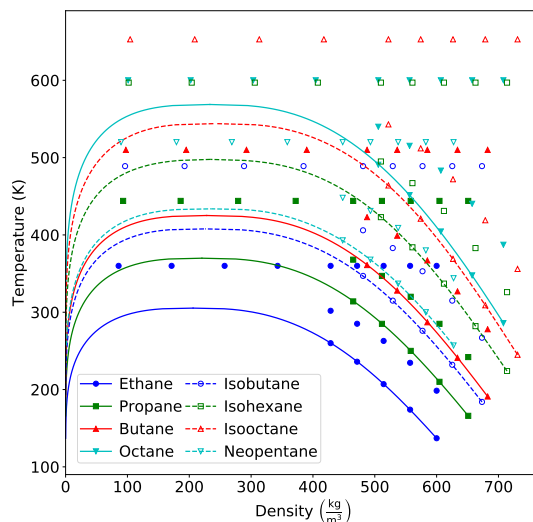


FIG. 1. State points simulated for each compound studied. A total of 19 simulations are performed: nine densities along the supercritical isotherm and two temperatures along five liquid density isochores. Filled symbols and solid lines correspond to *n*-alkanes, while empty symbols and dashed lines correspond to branched alkanes. The REFPROP saturation curve for each compound is included as a reference.

A fixed bond-length is used for each bond between UA or AUA sites. Although TAMie is an AUA force field, only the terminal  $\text{CH}_3$  sites have a displacement in the interaction site. For example, Figure 2 depicts both the UA and AUA representations of isooctane when only terminal  $\text{CH}_3$  interaction sites are shifted from the carbon center. This convention is much simpler to implement than other AUA approaches (such as AUA4) where non-terminal (i.e.  $\text{CH}_2$  and  $\text{CH}$ ) interaction sites also have a displacement distance. For this reason, we do not attempt to simulate the AUA4 force field for any compounds containing  $\text{CH}_2$  and  $\text{CH}$  interaction sites. For the compounds and force fields simulated, the anisotropic shift in a terminal interaction site (i.e.  $\text{CH}_3$ ) is treated simply as a longer effective bond-length (see Table I). The bond-length for all non-terminal sites is 0.154 nm, except for the Errington Exp-6 force field which uses 0.1535 nm for  $\text{CH}_2$ - $\text{CH}_2$  bonds.

The angle and dihedral energies are computed using the same functional forms for

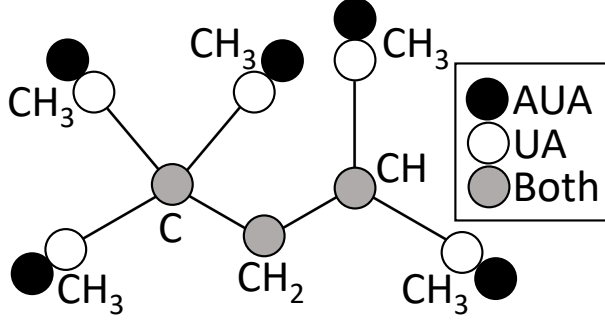


FIG. 2. If only the terminal sites have an anisotropic displacement (e.g. TAMie), AUA force fields have the same complexity as UA force fields since the only difference is a longer effective bond-length for the  $\text{CH}_3$  sites. Note that the AUA4 representation of isooctane requires a more complicated shifting of  $\text{CH}_2$  and  $\text{CH}$  sites than that depicted here.

TABLE I. Effective bond-lengths in units of nm for terminal ( $\text{CH}_3$ ) UA or AUA interaction sites. Empty table entries for Exp-6 and TraPPE-2 denote that the force field does not contain the corresponding interaction site type. Empty table entries in AUA4 arise because this force field uses a more complicated construct than the simple effective bond-length approach. Specifically, AUA4 requires  $\text{CH}_2$  and  $\text{CH}$  interaction sites that are not along the C-C bond axis.

Bond	TraPPE, Potoff	TAMie	Exp-6	AUA4	TraPPE-2
$\text{CH}_3\text{-CH}_3$	0.154	0.194	0.1839	0.1967	0.230
$\text{CH}_3\text{-CH}_2$	0.154	0.174	0.1687	–	–
$\text{CH}_3\text{-CH}$	0.154	0.174	–	–	–
$\text{CH}_3\text{-C}$	0.154	0.174	–	0.1751	–

each force field. Angular bending interactions are evaluated using a harmonic potential:

$$u^{\text{bend}} = \frac{k_\theta}{2} (\theta - \theta_0)^2 \quad (3)$$

where  $u^{\text{bend}}$  is the bending energy,  $\theta$  is the instantaneous bond angle,  $\theta_0$  is the equilibrium bond angle, and  $k_\theta$  is the harmonic force constant which is equal to 62500 K/rad<sup>2</sup> for all

bonding angles. Dihedral torsional interactions are determined using a cosine series:

$$u^{\text{tors}} = c_1[1 + \cos \phi] + c_2[1 - \cos 2\phi] + c_3[1 + \cos 3\phi] \quad (4)$$

where  $u^{\text{tors}}$  is the torsional energy,  $\phi$  is the dihedral angle and  $c_i$  are the Fourier constants. The equilibrium bond angles and torsional parameters are found in Tables II-III, respectively. Note that the Errington  $c_i$  values for  $\text{CH}_x\text{-CH}_2\text{-CH}_2\text{-CH}_y$  are a factor of two less than those reported in Table III.

TABLE II. Equilibrium bond angles ( $\theta_0$ ).  $x$  and  $y$  are values between 0-3.

Bending sites	$\theta_0$ (degrees)
$\text{CH}_x\text{-CH}_2\text{-CH}_y$	114.0
$\text{CH}_x\text{-CH-CH}_y$	112.0
$\text{CH}_x\text{-C-CH}_y$	109.5

TABLE III. Fourier constants ( $c_i$ ) in units of K.  $x$  and  $y$  are values between 0-3.

Torsion sites	$c_0$	$c_1$	$c_2$	$c_3$
$\text{CH}_x\text{-CH}_2\text{-CH}_2\text{-CH}_y$	0.0	355.03	-68.19	791.32
$\text{CH}_x\text{-CH}_2\text{-CH-CH}_y$	-251.06	428.73	-111.85	441.27
$\text{CH}_x\text{-CH}_2\text{-C-CH}_y$	0.0	0.0	0.0	461.29
$\text{CH}_x\text{-CH-CH-CH}_y$	-251.06	428.73	-111.85	441.27

Non-bonded interaction energies and forces between sites located in two different molecules or separated by more than three bonds are calculated using either a Lennard-Jones 12-6, Mie  $\lambda$ -6, or Buckingham Exponential-6 potential (see Equations 1-2). The non-bonded LJ 12-6 or Mie  $\lambda$ -6 force field parameters for TraPPE, TraPPE-2, Potoff, AUA4, and TAMie are provided in Table IV. Note that, for computational purposes, a common practice to date is to use integer values of  $\lambda$  in Equation 1.

TABLE IV. Non-bonded (intermolecular) parameters for TraPPE<sup>12,13</sup> (and TraPPE-2<sup>14</sup>), Potoff,<sup>18,19</sup> AUA4,<sup>16,37</sup> and TAMie<sup>20,21</sup> force fields. The “short/long” Potoff CH and C parameters are included in parenthesis. The ethane specific parameters for TAMie are included in parenthesis.

	TraPPE (TraPPE-2 )			Potoff (S/L)		
United-atom	$\epsilon$ (K)	$\sigma$ (nm)	$\lambda$	$\epsilon$ (K)	$\sigma$ (nm)	$\lambda$
CH <sub>3</sub>	98 (134.5)	0.375 (0.352)	12	121.25	0.3783	16
CH <sub>2</sub>	46	0.395	12	61	0.399	16
CH	10	0.468	12	15 (15/14)	0.46 (0.47/0.47)	16
C	0.5	0.640	12	1.2 (1.45/1.2)	0.61 (0.61/0.62)	16
	AUA4			TAMie		
CH <sub>3</sub>	120.15	0.3607	12	136.318 (130.780)	0.36034 (0.36463)	14
CH <sub>2</sub>	86.29	0.3461	12	52.9133	0.40400	14
CH	50.98	0.3363	12	14.5392	0.43656	14
C	15.04	0.244	12	–	–	–

The Errington Exp-6 non-bonded parameters are found in Table V. Note that Errington reported values for  $\epsilon$ ,  $\sigma$ , and  $\alpha$ . We computed  $r_{\min}$  and  $r_{\max}$  to facilitate compatibility with Equation 2 and future validation of our results.

TABLE V. Non-bonded (intermolecular) parameters for Errington Exp-6 force field.

United-atom	$\epsilon$ (K)	$\sigma$ (nm)	$\alpha$	$r_{\min}$ (nm)	$r_{\max}$ (nm)
CH <sub>3</sub>	129.6	0.3679	16	0.4094	0.0574
CH <sub>2</sub>	73.5	0.400	22	0.436	0.0221

Non-bonded interactions between two different site types (i.e. cross-interactions) are determined using Lorentz-Berthelot combining rules<sup>25</sup> for  $\epsilon$  and  $\sigma$ , an arithmetic mean for the repulsive exponent  $\lambda$  (as recommended in Reference 18), and a geometric mean for  $\alpha$

(as recommended in Reference 15):

$$\epsilon_{ij} = \sqrt{\epsilon_{ii}\epsilon_{jj}} \quad (5)$$

$$\sigma_{ij} = \frac{\sigma_{ii} + \sigma_{jj}}{2} \quad (6)$$

$$\lambda_{ij} = \frac{\lambda_{ii} + \lambda_{jj}}{2} \quad (7)$$

$$\alpha_{ij} = \sqrt{\alpha_{ii}\alpha_{jj}} \quad (8)$$

where the  $ij$  subscript refers to cross-interactions and the subscripts  $ii$  and  $jj$  refer to same-site interactions.

### III. CASE STUDY

The purpose of this case study is to demonstrate that the existing UA and AUA force fields for normal and branched alkanes that were parameterized with VLE properties do not predict the proper  $PVT$  behavior at higher temperatures and pressures (with the exception of ethane for the TraPPE-2 potential). Figures 3–4 plot the compressibility factor with respect to inverse temperature for  $n$ -alkanes and branched alkanes, respectively. Note that saturation corresponds to  $Z \approx 0$  for each isochore. The “Potoff” results in Figure 4 are only for the “short/long” model, since the “short/long” model is more accurate than the “generalized” model (available in Section SI.II of Supporting Information).

Figure 3 demonstrates that the existing literature force fields for  $n$ -alkanes, while accurate for VLE, do not capture the correct  $PVT$  behavior at high pressures ( $P^{\text{high}}$ ), i.e. the higher temperatures ( $T > T^{\text{sat}}$ ) and highest isochore densities ( $\rho_3$  and  $\rho_4$ ). Figure 4 shows the same erroneous trend in  $Z$  for branched alkanes. Note that the error in  $Z$  at high temperatures is less obvious because these force fields are typically not as reliable at predicting VLE for branched alkanes as for  $n$ -alkanes, i.e. notice the large deviations at  $Z \approx 0$ . That being said, it is clear in both Figures 3–4 that none of the force fields adequately reproduces  $Z$  over the entire temperature range, or the slope of  $Z$  with respect to inverse  $T$ .

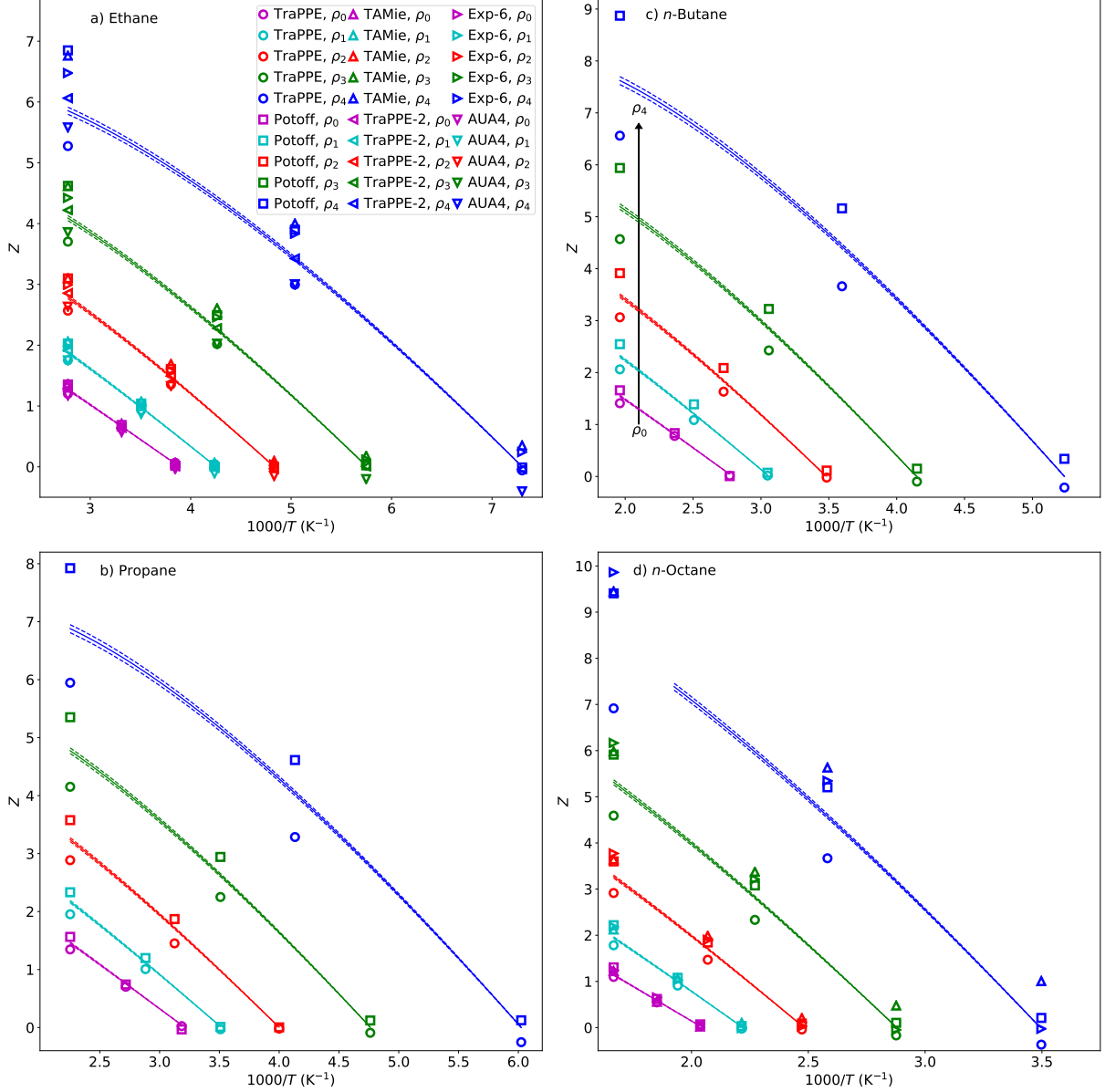


FIG. 3. Compressibility factors ( $Z$ ) along isochores agree at saturation ( $Z \approx 0$ ) but deviate strongly at higher pressures. Densities are distinguished by color, increase vertically, and are labeled such that  $\rho_0 < \rho_1 < \rho_2 < \rho_3 < \rho_4$ . Panels a)-d) correspond to ethane, propane, *n*-butane, and *n*-octane, respectively. Solid lines represent REFPROP correlations, with dashed lines representing a 1% uncertainty in REFPROP values. Simulation error bars are approximately one symbol size.

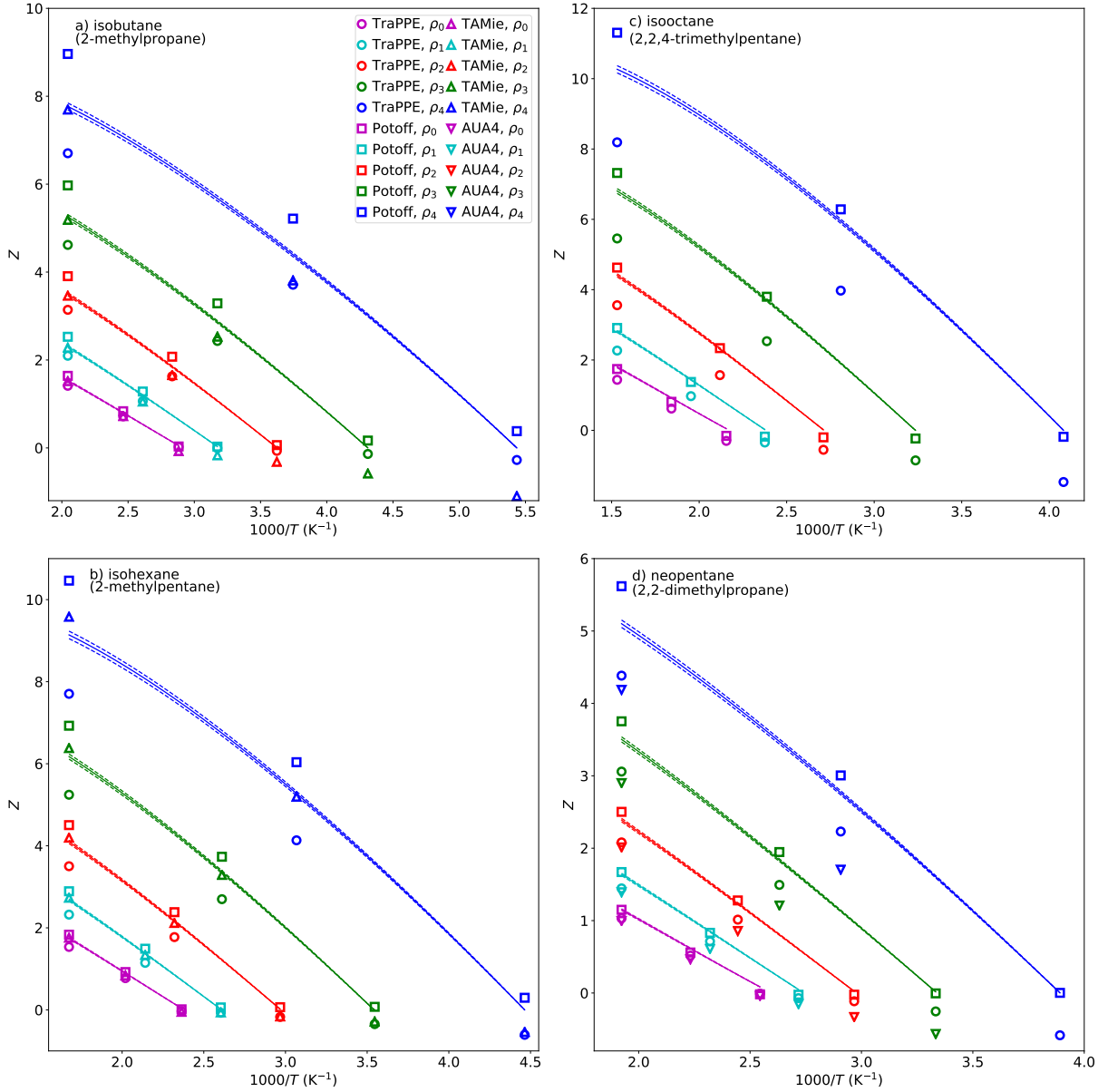


FIG. 4. Compressibility factors ( $Z$ ) along isochores for branched alkanes are not as accurate as normal alkanes at saturation ( $Z \approx 0$ ) and deviate strongly at higher pressures. Panels a)-d) correspond to isobutane, isohexane, isooctane, and neopentane, respectively. Symbols, lines, uncertainties, and formatting are the same as those in Figure 3. The Potoff results for isobutane and neopentane use the “short” parameters, while isohexane and isooctane use the “long” parameters (see Table IV).

A surprising trend is that the Errington (AUA Exp-6) model has a positive bias at high pressures. This appears to suggest that the repulsive barrier is too steep, despite the fact that the Exp-6 model is typically considered softer than the LJ 12-6. However, for  $\alpha_{\text{CH}_3} = 16$ , the Exp-6 is less repulsive than the LJ 12-6 only at very short distances, i.e.  $r < 0.7r_{\text{min}}$ , while it is actually somewhat more repulsive for the closest-range distances sampled in molecular dynamics at these conditions, i.e.  $0.7r_{\text{min}} < r < r_{\text{min}}$ . More definitive and straight-forward conclusions regarding the shape of the Mie  $\lambda$ -6 repulsive barrier are possible by directly comparing different values of  $\lambda$ .

In general, clear systematic biases are observed for the LJ 12-6 potentials (TraPPE-UA and AUA4) and the Mie  $\lambda$ -6 potentials (Potoff and TAMie). Specifically, the LJ 12-6 and Mie  $\lambda$ -6 potentials under- and over-predict  $Z$  at high pressures, respectively. These results are intuitive as the repulsive barriers are steeper for the respective Mie 16-6 and 14-6 potentials of the Potoff and TAMie force fields.

The one exception to this trend is the TraPPE-2 model for ethane, which has the most accurate prediction of the entire  $PVT$  phase space simulated. Specifically, TraPPE-2 reproduces  $Z$  to within the REFPROP uncertainties for all state points except at  $P^{\text{high}}$ , where the average percent deviation (AD%) relative to the REFPROP correlations is still only 3%.

The performance of TraPPE-2 is somewhat surprising considering this force field has only three fitting parameters ( $\epsilon$ ,  $\sigma$ , and the effective bond-length) while the TAMie model has these three parameters and an additional fitting parameter ( $\lambda$ ). It is possible that a four parameter optimization, such as that used by TAMie, is overfit to the VLE data and would perform better if high pressure  $PVT$  data were included in the parameterization. Furthermore, it is important to note that TraPPE-2 uses a much longer effective bond-length of 0.230 nm while TAMie did not consider bond-lengths larger than 0.194 nm. Therefore, the fact that the TraPPE-2 force field extrapolates to high pressures better than TAMie suggests that, at high pressures, it is important to account for hydrogens with a longer effective bond-length than that typically used for AUA models (see Table I). It seems logical that at even higher pressures an AA model may become necessary.

Unfortunately, a direct comparison of the non-bonded interactions for AUA force fields is difficult because each model has a different anisotropic displacement, i.e. effective



bond-length. By contrast, comparing TraPPE-UA and Potoff is straightforward because they use the same bond-lengths and the same non-bonded Mie  $\lambda$ -6 potential (Equation 1). For example, since the TraPPE-UA (LJ 12-6) potential under-predicts  $Z$  and the Potoff (UA Mie 16-6) potential over-predicts  $Z$ , it seems reasonable that a UA Mie 13-6, 14-6, or 15-6 model could demonstrate the proper trend (although it is important to remember that certain values of  $\lambda$  cannot reproduce both  $\rho_1^{\text{sat}}$  and  $P_v^{\text{sat}}$ ).

To investigate this hypothesis, the remainder of this document focuses on the UA Mie  $\lambda$ -6 potential, where all bond-lengths are 0.154 nm. Specifically, we perform a Bayesian uncertainty quantification analysis to determine if there exists a set of  $\epsilon$ ,  $\sigma$ , and  $\lambda$  that reasonably predicts  $\rho_1^{\text{sat}}$ ,  $P_v^{\text{sat}}$ , and  $P^{\text{high}}$ . The results in Section V demonstrate that the optimal value of  $\lambda$  for predicting  $PVT$  of supercritical fluids and compressed liquids is not capable of predicting VLE properties accurately, and vice-versa.

## IV. METHODS II

The results presented in Section III demonstrate that none of the literature UA or AUA force fields, parameterized with VLE data, can reproduce the  $PVT$  behavior for supercritical fluids and compressed liquids. However, there is uncertainty in the non-bonded parameters inherited from the VLE data. Therefore, by considering the inherent uncertainty, it is possible that a feasible parameter set exists that adequately predicts VLE and  $P^{\text{high}}$ . By contrast, if none of the  $\epsilon$ ,  $\sigma$ , and  $\lambda$  sets is capable of simultaneously predicting VLE properties and  $Z$  at high pressures, we can conclude that the UA Mie  $\lambda$ -6 potential (and Lennard-Jones 12-6 as a special case) is inadequate for this purpose and, therefore, should not be used when developing FEOS with molecular simulation results.

Bayesian inference is a rigorous approach to quantify if the UA Mie  $\lambda$ -6 potential is “adequate”. We refer the reader to the literature for a thorough discussion of Bayesian statistics.<sup>31,38–41</sup> In Section IV A, we review some basic concepts of Bayes theorem, define the posterior, likelihood, and prior distributions, and discuss the Markov Chain Monte Carlo (MCMC) approach for sampling from the posterior joint distribution of the parameters. MCMC can be computationally burdensome, especially when molecular simula-

tion is required to compute the likelihood. For this reason, we utilize surrogate models to reduce the computational cost of MCMC by several orders of magnitude. Section IV B demonstrates how these surrogate models estimate  $\rho_1^{\text{sat}}$ ,  $P_v^{\text{sat}}$ , and  $Z$  for a given set of  $\epsilon$ ,  $\sigma$ , and  $\lambda$ . We implement this analysis for  $n$ -alkanes to generate joint distributions of  $\epsilon_{\text{CH}_3}-\sigma_{\text{CH}_3}$  and  $\epsilon_{\text{CH}_2}-\sigma_{\text{CH}_2}$  for different values of  $\lambda_{\text{CH}_3}$  and  $\lambda_{\text{CH}_2}$ , respectively.

## A. Bayesian Analysis

### 1. Theory

Bayes theorem states

$$Pr(\theta|D, M) = \frac{Pr(D|\theta, M)Pr(\theta|M)}{Pr(D|M)} \quad (9)$$

where  $Pr$  denotes a probability distribution function,  $\theta$  is the parameter set,  $M$  is the model, and  $D$  are the data.  $Pr(\theta|D, M)$  is commonly referred to as the “posterior”,  $Pr(D|\theta, M)$  is the “likelihood” (alternatively expressed as  $L(\theta|D, M)$ ),  $Pr(\theta|M)$  is the “prior”, and  $Pr(D|M)$  is a normalization constant which is also the “model evidence”.

The “model evidence” is used in model selection, by computing the probability of different models given the data:

$$Pr(M|D) = \frac{Pr(D|M)Pr(M)}{Pr(D)} \quad (10)$$

where  $Pr(M)$  is the “model prior”,  $Pr(D)$  is a normalization constant, and  $Pr(M|D)$  is the “model posterior”. The ratio of  $Pr(M|D)$ , known as the Bayes factor ( $K$ ), provides the relative probability of two models.

The parameter uncertainty propagates when estimating another quantity of interest ( $QoI$ ), which may or may not be included in  $D$ , according to:<sup>31</sup>

$$Pr(QoI|D, M) = \int Pr(QoI|\theta, M)Pr(\theta|D, M)d\theta \quad (11)$$

This expression is commonly referred to as “robust posterior prediction.” Note that the uncertainty in  $QoI$ , obtained from  $Pr(QoI|D, M)$ , does not account for deficiencies in the model itself, only the uncertainty in the model parameters.

## 2. Application

Bayesian inference is used to quantify the uncertainty in the non-bonded parameters ( $\epsilon$  and  $\sigma$ ) and to determine the evidence for different values of  $\lambda$  based on VLE data. For clarity, we rewrite Equations 9-10 for the specific case studied by substituting  $\epsilon$  and  $\sigma$  for  $\theta$ ,  $\lambda$  for  $M$ , and  $\rho_1^{\text{sat}}$  and  $P_v^{\text{sat}}$  for  $D$ :

$$Pr(\epsilon, \sigma | \rho_1^{\text{sat}}, P_v^{\text{sat}}, \lambda) = \frac{L(\epsilon, \sigma | \rho_1^{\text{sat}}, P_v^{\text{sat}}, \lambda) Pr(\epsilon, \sigma | \lambda)}{Pr(\rho_1^{\text{sat}}, P_v^{\text{sat}} | \lambda)} \quad (12)$$

$$Pr(\lambda | \rho_1^{\text{sat}}, P_v^{\text{sat}}) = \frac{Pr(\rho_1^{\text{sat}}, P_v^{\text{sat}} | \lambda) Pr(\lambda)}{Pr(\rho_1^{\text{sat}}, P_v^{\text{sat}})} \quad (13)$$

where in this context  $\rho_1^{\text{sat}}$  and  $P_v^{\text{sat}}$  should each be interpreted as an array of experimental data values. Notice that  $\theta$  does not include  $\lambda$ , since we use  $\lambda$  to distinguish between models. The “model evidence”,  $Pr(\rho_1^{\text{sat}}, P_v^{\text{sat}} | \lambda)$  in Equation 13, for different values of  $\lambda$  is determined by integrating the numerator of Equation 12 for all values of  $\epsilon$  and  $\sigma$ .

To compute the Bayes factor between two values of  $\lambda$  (i.e. two different models,  $M$ ), we assume that the prior evidence is equal for all positive values of  $\epsilon$ ,  $\sigma$ , and  $\lambda$  (within a feasible range). Specifically, we use bounded uniform prior distributions for  $Pr(\epsilon, \sigma | \lambda)$  in Equation 12 and  $Pr(\lambda)$  in Equation 13, where the lower bound is 0 and the upper bound is an order of magnitude greater than the literature values for  $\epsilon$ ,  $\sigma$ , and  $\lambda$ . Due to the large amount of information contained in the data,  $D$ , the use of a uniform prior does not impact our results, i.e. the data “overwhelms” the prior. The advantage of this assumption is that the Bayes factor,  $K$ , depends completely on the likelihood:

$$K = \frac{Pr(\lambda_j | \rho_1^{\text{sat}}, P_v^{\text{sat}})}{Pr(\lambda_i | \rho_1^{\text{sat}}, P_v^{\text{sat}})} = \frac{\int L(\epsilon, \sigma | \rho_1^{\text{sat}}, P_v^{\text{sat}}, \lambda_j) \partial \epsilon \partial \sigma}{\int L(\epsilon, \sigma | \rho_1^{\text{sat}}, P_v^{\text{sat}}, \lambda_i) \partial \epsilon \partial \sigma} \quad (14)$$

where  $\lambda_i$  and  $\lambda_j$  are the different (fixed) values of  $\lambda$  being compared.

We utilize robust posterior prediction (Equation 11) to propagate the  $\epsilon$ - $\sigma$  parameter uncertainty (for a given  $\lambda$ ) to three different  $QoI$ , specifically,  $\rho_1^{\text{sat}}$ ,  $P_v^{\text{sat}}$ , and  $Z$ . For example, the uncertainty in predicting  $Z$  is obtained from:

$$Pr(Z | \rho_1^{\text{sat}}, P_v^{\text{sat}}, \lambda) = \int Pr(Z | \epsilon, \sigma, \lambda) Pr(\epsilon, \sigma | \rho_1^{\text{sat}}, P_v^{\text{sat}}, \lambda) \partial \epsilon \partial \sigma \quad (15)$$

Similar expressions exist for  $Pr(\rho_1^{\text{sat}}|\rho_1^{\text{sat}}, P_v^{\text{sat}}, \lambda)$  and  $Pr(P_v^{\text{sat}}|\rho_1^{\text{sat}}, P_v^{\text{sat}}, \lambda)$ , where posterior prediction allows for uncertainty estimates in  $\rho_1^{\text{sat}}$  and  $P_v^{\text{sat}}$  at any temperature, not just those included in  $D$ .

### 3. Implementation

Markov Chain Monte Carlo (MCMC) is the traditional approach for numerically sampling from the probability distribution  $Pr(\epsilon, \sigma|\rho_1^{\text{sat}}, P_v^{\text{sat}}, \lambda)$ . We use the Metropolis-Hastings algorithm to create a Markov Chain by proposing new  $\epsilon$ - $\sigma$  values and accepting those moves based on the criterion:

$$\alpha = \min \left( 1, \frac{Pr(\epsilon_{i+1}, \sigma_{i+1}|\rho_1^{\text{sat}}, P_v^{\text{sat}}, \lambda)Q(\epsilon_i, \sigma_i|\epsilon_{i+1}, \sigma_{i+1}, s_\epsilon^2, s_\sigma^2)}{Pr(\epsilon_i, \sigma_i|\rho_1^{\text{sat}}, P_v^{\text{sat}}, \lambda)Q(\epsilon_{i+1}, \sigma_{i+1}|\epsilon_i, \sigma_i, s_\epsilon^2, s_\sigma^2)} \right) \quad (16)$$

where  $\alpha$  is the acceptance probability,  $\epsilon_i$  and  $\sigma_i$  are the previous parameter set,  $\epsilon_{i+1}$  and  $\sigma_{i+1}$  are the proposed parameter set, and  $Q$  is the proposal distribution from which  $\epsilon_{i+1}$  and  $\sigma_{i+1}$  are sampled. In this study,  $Q$  is a bi-variate normal distribution with mean equal to  $\epsilon_i$  and  $\sigma_i$ , variance of  $s_\epsilon^2$  and  $s_\sigma^2$ , and a covariance of 0. The amount to which  $\epsilon$  or  $\sigma$  is varied for each MCMC step ( $\epsilon_{i+1} - \epsilon_i$  or  $\sigma_{i+1} - \sigma_i$ ) depends on  $s_\epsilon^2$  and  $s_\sigma^2$ . These parameters are tuned such that approximately  $\frac{1}{3}$  of the moves are accepted. This “tuning” period (also referred to as a “burn-in” period) is followed by a production period where  $s_\epsilon^2$  and  $s_\sigma^2$  do not change. Since  $\epsilon_{i+1}$  and  $\sigma_{i+1}$  are highly correlated with  $\epsilon_i$  and  $\sigma_i$ , it is important to “thin” the MCMC  $\epsilon$  and  $\sigma$  parameter sets ( $\epsilon_{\text{MCMC}}$  and  $\sigma_{\text{MCMC}}$ ), i.e. every  $j^{\text{th}}$  parameter set is stored. Section [SI.III](#) of Supporting Information provides an MCMC example with some details (i.e. number of steps for burn-in and production, frequency that  $s_\epsilon^2$  and  $s_\sigma^2$  are updated, resulting acceptance rates, etc.).

The joint distribution of the parameter sets sampled from MCMC ( $\theta_{\text{MCMC}}$ , or specifically,  $\epsilon_{\text{MCMC}}$  and  $\sigma_{\text{MCMC}}$ ) provide an estimate of the uncertainty in the  $\epsilon$  and  $\sigma$  (see Figure 6 and 11 in Section V). The integration required for robust posterior prediction (see Equations 11 and 15) is approximated from the distribution of  $QoI$  values evaluated for each MCMC parameter set, i.e.  $QoI(\theta_{\text{MCMC}}|D, M)$  or the more succinct notation  $QoI_{\text{MCMC}}$ . From  $QoI_{\text{MCMC}}$ , standard statistical methods are used to approximate the  $QoI$  uncertainty

at a desired credible level. For example, the histograms and uncertainties reported in Section V are obtained from distributions of  $\rho_{1,\text{MCMC}}^{\text{sat}}$ ,  $P_{\text{v},\text{MCMC}}^{\text{sat}}$ , and  $Z_{\text{MCMC}}$ .

Because MCMC moves are accepted based on Equation 16 and the denominator in Equation 12 (i.e.  $Pr(\rho_1^{\text{sat}}, P_{\text{v}}^{\text{sat}}|\lambda)$ ) does not depend on  $\epsilon$  and  $\sigma$ , the acceptance probability is independent of  $Pr(\rho_1^{\text{sat}}, P_{\text{v}}^{\text{sat}}|\lambda)$ . Also, as mentioned previously, we use a “weakly informative prior” such that the acceptance probability is independent of  $Pr(\epsilon, \sigma|\lambda)$ . Furthermore,  $Q$  is chosen to be symmetric such that the  $Q$  terms in the numerator and denominator of Equation 16 cancel. Therefore, the probability of accepting  $\epsilon_{i+1}$  and  $\sigma_{i+1}$  is based completely on the likelihood:

$$\alpha = \min \left( 1, \frac{L(\epsilon_{i+1}, \sigma_{i+1} | \rho_1^{\text{sat}}, P_{\text{v}}^{\text{sat}}, \lambda)}{L(\epsilon_i, \sigma_i | \rho_1^{\text{sat}}, P_{\text{v}}^{\text{sat}}, \lambda)} \right) \quad (17)$$

For this reason, we discuss in some detail how we calculate  $L(\epsilon, \sigma, |\rho_1^{\text{sat}}, P_{\text{v}}^{\text{sat}}, \lambda)$ .

The likelihood is calculated using a normal distribution,  $N(D, s_{\text{D,SM}}^2)$ , where  $s_{\text{D,SM}}^2$  is the combined variance of the experimental data and the surrogate model used to estimate  $\rho_1^{\text{sat}}$  and  $P_{\text{v}}^{\text{sat}}$  for a given  $\epsilon$ ,  $\sigma$ , and  $\lambda$  (see Section IV B). The variances are independent, meaning that the combined variance is the sum of the experimental and surrogate model variances, i.e.  $s_{\text{D}}^2 + s_{\text{SM}}^2$ .<sup>39</sup> The experimental ( $u_{\text{D}}$  and surrogate model  $u_{\text{SM}}$  uncertainties are computed at the 95% confidence level using the respective  $s_{\text{D}}^2$  and  $s_{\text{SM}}^2$  values. Figure 5 compares the experimental data for  $\rho_1^{\text{sat}}$  and  $P_{\text{v}}^{\text{sat}}$  with the corresponding values of  $u_{\text{D}}$  and  $u_{\text{SM}}$ . The experimental data and uncertainties are obtained through the Thermodynamics Research Center (TRC) ThermoData Engine (TDE).<sup>42</sup>

As shown in Figure 5, the surrogate model uncertainty for  $\rho_1^{\text{sat}}$  is 0.3% up to  $0.75T_c$  and increases linearly to 1.5% at the maximum  $T^{\text{sat}}$ . The surrogate model uncertainty for  $P_{\text{v}}^{\text{sat}}$  is 20% at the minimum  $T_{\text{sat}}$  and decreases linearly to 7% at  $0.6T_c$ , where it remains constant for higher temperatures. Note that these are conservative estimates of  $u_{\text{SM}}$ , where other studies suggest smaller uncertainties.<sup>34–36</sup> In fact, for the compounds investigated in this study, these uncertainties are much larger than the experimental uncertainties ( $u_{\text{D}}$ ) and, therefore, the size of the parameter space sampled by MCMC depends almost entirely on  $u_{\text{SM}}$ . The use of a conservative  $u_{\text{SM}}$  model is intentional in this regard, namely, so that the  $\theta_{\text{MCMC}}$  sampled points represent practically all of the feasible  $\epsilon$ – $\sigma$  parameter sets

(for a given  $\lambda$ ) optimized with  $\rho_l^{\text{sat}}$  and  $P_v^{\text{sat}}$ .

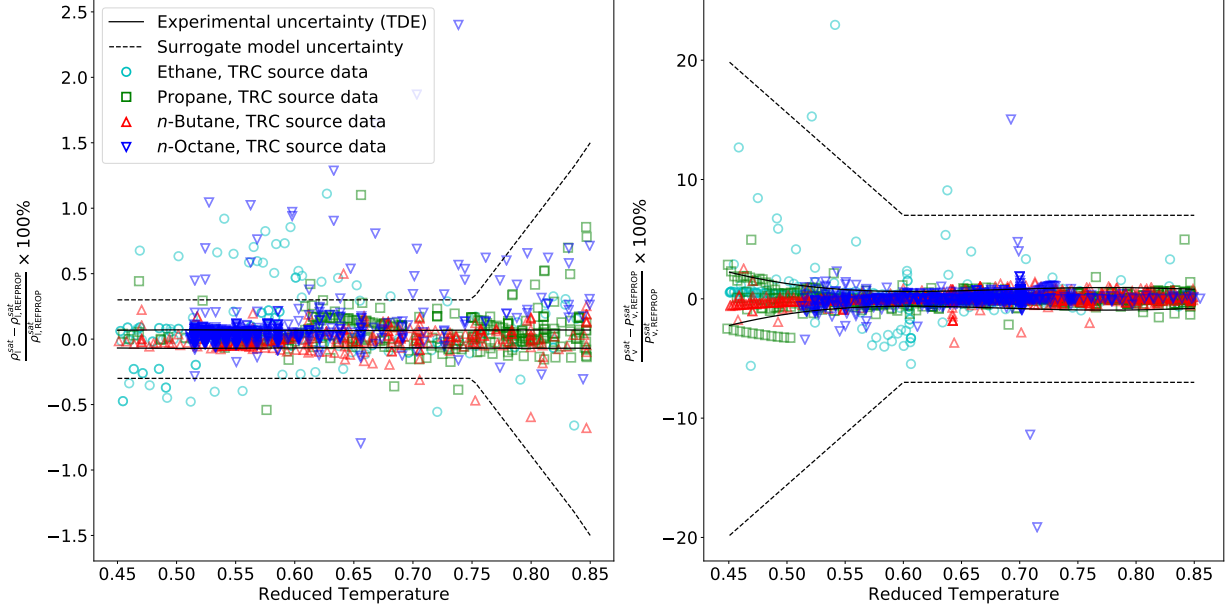


FIG. 5. Experimental (TDE) uncertainties are negligible compared to surrogate model uncertainties. Panels a)–b) plot the uncertainties for  $\rho_l^{\text{sat}}$  and  $P_v^{\text{sat}}$  with respect to reduced temperature (absolute temperature divided by the REFPROP  $T_c$ ). Also included are percent deviations between the experimental data for ethane, propane, *n*-butane, and *n*-octane relative to the REFPROP correlations.

## B. Surrogate Model

A typical Markov Chain requires  $O(10^4\text{--}10^5)$  Monte Carlo steps, where the likelihood function must be evaluated at each step. Since  $L(\theta|D, M)$  depends on the force field parameters ( $\epsilon$ ,  $\sigma$ , and  $\lambda$ ), an MCMC approach is computationally infeasible if computing  $L(\theta|D, M)$  requires performing direct molecular simulations for every proposed parameter set. Furthermore, propagation of uncertainty with robust posterior prediction may require  $O(10^2\text{--}10^3)$   $\theta_{\text{MCMC}}$  parameter sets for adequate representations of  $Pr(QoI|D, M)$  (see Equations 11 and 15). For these reasons, surrogate models to estimate  $\rho_{l,\text{MCMC}}^{\text{sat}}$ ,  $P_{v,\text{MCMC}}^{\text{sat}}$  and  $Z_{\text{MCMC}}$  are essential for this study.

We use a configuration-sampling-based surrogate model, where configurations are sampled using a small group of reference parameter sets ( $\epsilon_{\text{ref}}$ ,  $\sigma_{\text{ref}}$ , and  $\lambda_{\text{ref}}$ ).<sup>35</sup> As discussed in our previous study, we use a single value of  $\epsilon_{\text{ref}}$  with nine evenly spaced  $\sigma_{\text{ref}}$  values for a fixed value of  $\lambda_{\text{ref}} = \lambda$ . Ensemble averages for the MCMC parameter sets ( $\theta_{\text{MCMC}}$ ) are estimated by reweighting the sampled reference configurations using Multi-state Bennett Acceptance Ratio (MBAR).<sup>43</sup> MBAR is a nearly exact surrogate model when there are a sufficient number of effective samples.<sup>35</sup>

The properties that are estimated using MBAR are the departure internal energy ( $U^{\text{dep}}$ ) and the compressibility factor ( $Z$ ). Isothermal isochoric integration (ITIC) converts the MBAR estimated  $U^{\text{dep}}$  and  $Z$  values at the 19 ITIC state points to saturation temperatures ( $T^{\text{sat}}$ ), saturated liquid densities ( $\rho_1^{\text{sat}}$ ), saturated vapor densities ( $\rho_v^{\text{sat}}$ ), and saturated vapor pressures ( $P_v^{\text{sat}}$ ). This is important since  $\rho_1^{\text{sat}}$  and  $P_v^{\text{sat}}$  are the data ( $D$ ) included in  $L(\theta|D)$ . Details for the combined implementation of MBAR and ITIC (MBAR-ITIC) is discussed elsewhere.<sup>35</sup>

The ITIC equations are:

$$\frac{A^{\text{dep}}}{R_g T^{\text{sat}}} = \int_0^{\rho_1^{\text{sat}}} \frac{Z-1}{\rho} \partial \rho|_{T=T^{\text{IT}}} + \int_{T^{\text{IT}}}^{T^{\text{sat}}} U^{\text{dep}} \partial \left( \frac{1}{R_g T} \right) |_{\rho=\rho_1^{\text{sat}}} \quad (18)$$

$$\rho_v^{\text{sat}} \approx \rho_1^{\text{sat}} \exp \left( \frac{A^{\text{dep}}}{R_g T^{\text{sat}}} + Z_1^{\text{sat}} - 1 - 2B_2 \rho_v^{\text{sat}} - 1.5B_3 (\rho_v^{\text{sat}})^2 \right) \quad (19)$$

$$P_v^{\text{sat}} \approx (1 + B_2 \rho_v^{\text{sat}} + B_3 (\rho_v^{\text{sat}})^2) \rho_v^{\text{sat}} R_g T^{\text{sat}} \quad (20)$$

$$Z_1^{\text{sat}} = \frac{P_v^{\text{sat}}}{\rho_1^{\text{sat}} R_g T^{\text{sat}}} \quad (21)$$

where  $A^{\text{dep}} \equiv A - A^{\text{ig}}$  is the Helmholtz free energy departure from ideal gas for  $T = T^{\text{sat}}$  and  $\rho = \rho_1^{\text{sat}}$ ,  $U^{\text{dep}} \equiv U - U^{\text{ig}}$  is the internal energy departure,  $Z_1^{\text{sat}}$  is the saturated liquid compressibility factor,  $B_2$  is the second virial coefficient,  $B_3$  is the third virial coefficient,  $T^{\text{IT}}$  is the isothermal temperature, and  $R_g$  is the universal gas constant. For details regarding the implementation of ITIC, see References 34–36. As discussed in our previous work,<sup>35</sup> the  $B_2$  and  $B_3$  values found in Equations 19–20 are calculated using REFPROP correlations.<sup>1</sup> The use of REFPROP correlations introduces a small bias in the resulting  $\rho_1^{\text{sat}}$  and  $P_v^{\text{sat}}$ , which is accounted for in the surrogate model uncertainty.

The ITIC analysis provides VLE properties at only 5 saturation temperature values ( $T_{\text{ITIC}}^{\text{sat}}$ ), while the experimental data set may have hundreds of saturation temperatures ( $T_{\text{D}}^{\text{sat}}$ ). Although it is possible to use computed values from an empirical correlation fit to experimental data (i.e. REFPROP, ThermoData Engine (TDE)) as the data set, it is considered best practice for Bayesian inference that  $D$  consist of raw experimental data. For this reason, we instead use empirical model fits to interpolate the ITIC VLE properties ( $T_{\text{ITIC}}^{\text{sat}}$ ,  $\rho_{\text{l,ITIC}}^{\text{sat}}$ , and  $(P_{\text{v,ITIC}}^{\text{sat}})$  so that  $\rho_1^{\text{sat}}$  and  $P_{\text{v}}^{\text{sat}}$  can be estimated at any value of  $T^{\text{sat}}$ . Specifically, we fit  $P_{\text{v,ITIC}}^{\text{sat}}$  and  $T_{\text{ITIC}}^{\text{sat}}$  to the Antoine equation:

$$\log_{10}(P_{\text{v}}^{\text{sat}}) = a_0 + \frac{a_1}{T^{\text{sat}} + a_2} \quad (22)$$

where  $a_i$  are fitting parameters. We fit  $\rho_{\text{l,ITIC}}^{\text{sat}}$  and  $T_{\text{ITIC}}^{\text{sat}}$  to a combined rectilinear and density scaling law expression:<sup>24</sup>

$$\rho_1^{\text{sat}} = b_0 + b_1(b_2 - T^{\text{sat}}) + b_3(b_2 - T^{\text{sat}})^{\beta} \quad (23)$$

where  $b_i$  are fitting parameters, and  $\beta = 0.326$ .  $b_0$  and  $b_2$  only provide rough estimates of the critical density ( $\rho_c$ ) and critical temperature ( $T_c$ ). More reliable estimates of the critical point require simultaneous fitting of  $\rho_{\text{v,ITIC}}^{\text{sat}}$  to a similar expression, but this is unnecessary for our purposes since  $D$  does not include the critical constants. Note that Equations 22-23 are only used to interpolate ITIC values, and not to extrapolate to higher or lower  $T^{\text{sat}}$ . These equations are reliable over the limited temperature range studied ( $0.45 < T_{\text{r}} < 0.85$ ), whereas a wider temperature range would require more flexible models.<sup>44,45</sup>

In summary, MBAR, ITIC, and Equations 22-23 enable prediction of  $\rho_1^{\text{sat}}$  and  $P_{\text{v}}^{\text{sat}}$  over a range of  $T^{\text{sat}}$  for any  $\epsilon$ ,  $\sigma$ , and  $\lambda$  by performing a small number of direct NVT simulations with only a few reference parameter sets. In addition, since the Mie  $\lambda$ -6 potential is linear with respect to  $r^{-6}$  and  $r^{-\lambda}$  (see Equation 1), we implement basis functions to efficiently recompute the energies and forces that are required for MBAR and ITIC (for details see Reference 46 and Section SI.IV of Supporting Information in Reference 35). In total, this methodology reduces the computational cost for computing  $L(\theta|D)$  by several orders of magnitude compared to direct simulation of VLE, using Gibbs Ensemble Monte Carlo (GEMC) or Grand Canonical Monte Carlo (GCMC) histogram reweighting (HR).



Quantifying the surrogate model variance ( $s_{\text{SM}}^2$ ) is essential for evaluating  $L(\theta|D)$ . While only a brief description is provided here, details are found in Section SI.IV of Supporting Information. Rather than performing a rigorous statistical assessment of MBAR, ITIC, and Equations 22-23, we use an empirical approach for estimating  $s_{\text{SM}}^2$ . Specifically, we compute the deviation between the surrogate model estimates of  $\rho_1^{\text{sat}}$  and  $P_v^{\text{sat}}$  for TraPPE-UA and Potoff with those reported in the literature for the respective force fields obtained using Gibbs Ensemble Monte Carlo (GEMC) or Grand Canonical Monte Carlo (GCMC) histogram reweighting (HR).<sup>12,18</sup> Although this is a rough approximation for estimating  $s_{\text{SM}}^2$ , this inter-laboratory comparison has the added benefit that it incorporates possible “dark uncertainty”,<sup>47</sup> which can be significant.<sup>48</sup> These non-statistical deviations are typically associated with different simulation packages, MD instead of MC, finite-size effects, and post-simulation analysis (e.g. ITIC rather than HR).

## V. RESULTS

In this section, we use MCMC and the aforementioned surrogate models to determine the parameter uncertainty in  $\text{CH}_3$  and  $\text{CH}_2$  interaction sites of  $n$ -alkanes. Since the common practice is to limit  $\lambda$  to integer values (see Section II B), we perform several independent MCMC runs using a single, fixed, integer value of  $\lambda$ . The Bayesian inference analysis for  $\text{CH}_3$  and  $\text{CH}_2$  sites is performed sequentially. Specifically, rather than sampling from a four-dimensional parameter space (i.e.  $\epsilon_{\text{CH}_3}$ ,  $\epsilon_{\text{CH}_2}$ ,  $\sigma_{\text{CH}_3}$ , and  $\sigma_{\text{CH}_2}$  for a given value of  $\lambda_{\text{CH}_3}$  and  $\lambda_{\text{CH}_2}$ ), we implement a pair of two-dimensional MCMC runs by assuming the  $\text{CH}_3$  parameters from ethane are transferable to propane,  $n$ -butane, and  $n$ -octane.

### A. Ethane

Figures 6-9 present the MCMC results for ethane with  $\lambda_{\text{CH}_3} = 13$ -18. Figure 6 demonstrates that the feasible region of  $\epsilon_{\text{CH}_3}$  depends strongly on  $\lambda_{\text{CH}_3}$ , namely, larger values of  $\lambda_{\text{CH}_3}$  require larger values of  $\epsilon_{\text{CH}_3}$ . By contrast, we observe a much smaller shift towards larger values of  $\sigma_{\text{CH}_3}$  with increasing  $\lambda_{\text{CH}_3}$ . This observation is consistent with Reference

18.

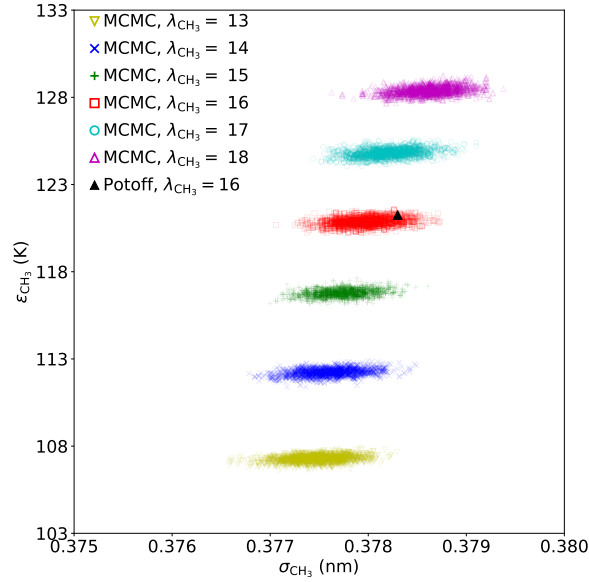


FIG. 6. MCMC sampled parameter sets for different values of  $\lambda_{\text{CH}_3}$  ( $\epsilon_{\text{CH}_3, \text{MCMC}}$  and  $\sigma_{\text{CH}_3, \text{MCMC}}$ ) quantify the uncertainty in  $\epsilon_{\text{CH}_3}-\sigma_{\text{CH}_3}$ . The Potoff parameter set is included as a reference for  $\lambda_{\text{CH}_3} = 16$ .

The results depicted in Figures 7-8 are useful for comparing the performance of different values of  $\lambda$  for  $\rho_1^{\text{sat}}$ ,  $P_v^{\text{sat}}$ , and  $Z$ . Notice that the insets in Figure 7 plot the mean absolute percent deviation (MAPD%) to quantify the goodness of fit to VLE data, while the inset in Figure 8 plots the average deviation (AD%) to demonstrate the positive bias in  $P^{\text{high}}$ . Note that because MAPD% and AD% are percent deviations they are not directly related to the squared deviations of the normal distribution used to compute the likelihood. We plot MAPD% and AD% as these are easier to conceptualize and quantify.

Figure 7 Panel a) with the corresponding inset demonstrates that the best prediction of  $\rho_1^{\text{sat}}$  is obtained for higher values of  $\lambda_{\text{CH}_3}$ . However, while the  $\rho_1^{\text{sat}}$  MAPD% for  $\lambda_{\text{CH}_3} = 15-18$  are similar,  $\lambda_{\text{CH}_3} = 13-14$  have significantly higher  $\rho_1^{\text{sat}}$  MAPD%. Figure 7 Panel b) demonstrates that  $\lambda_{\text{CH}_3} = 13-14$  and  $\lambda_{\text{CH}_3} = 17-18$  over- and under-predict  $P_v^{\text{sat}}$  at low temperatures, respectively, while  $\lambda_{\text{CH}_3} = 15-16$  have the best trend for  $P_v^{\text{sat}}$ . The inset for Panel b) shows that  $\lambda_{\text{CH}_3} = 15$  has the lowest MAPD% in  $P_v^{\text{sat}}$ .

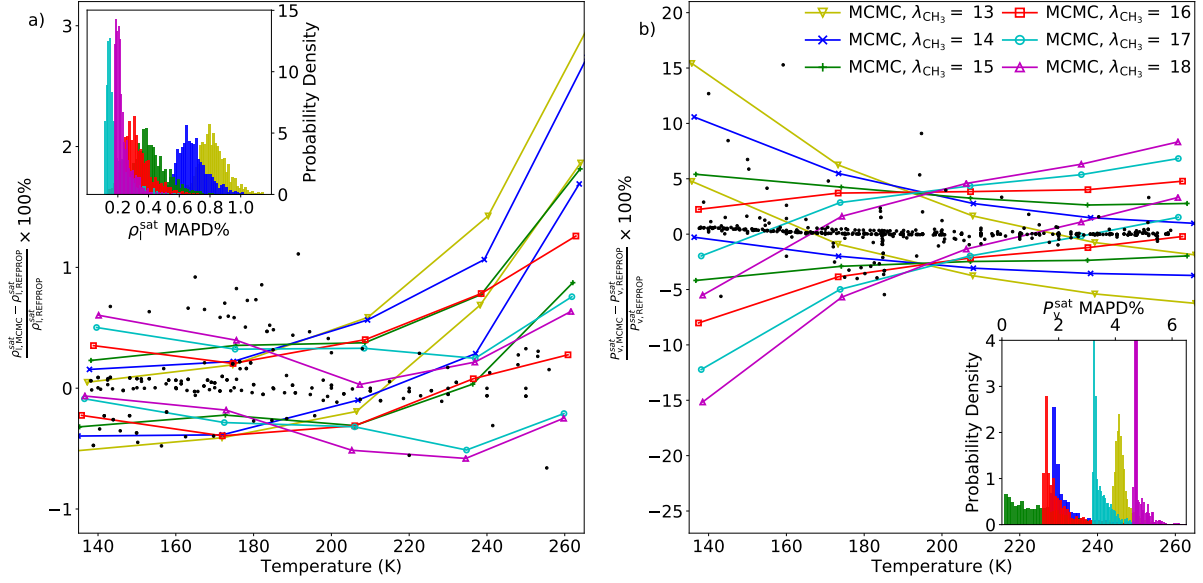


FIG. 7. Uncertainties in  $\epsilon_{\text{CH}_3}-\sigma_{\text{CH}_3}$  propagate to  $\rho_1^{\text{sat}}$  and  $P_v^{\text{sat}}$ . Panels a)–b) plot the percent deviation from REFPROP correlations for  $\rho_{1,\text{MCMC}}^{\text{sat}}$  and  $P_{v,\text{MCMC}}^{\text{sat}}$ , respectively. The upper and lower lines for each  $\lambda$  correspond to the 95% credible interval obtained from  $QoI_{\text{MCMC}}$ . Insets of Panels a)–b) are histograms of the MAPD% in  $\rho_{1,\text{MCMC}}^{\text{sat}}$  and  $P_{v,\text{MCMC}}^{\text{sat}}$ , respectively. Experimental data used to compute the likelihood are included as black dots.

Finally, Figure 8 demonstrates that all of the sampled  $\epsilon_{\text{CH}_3,\text{MCMC}}$  and  $\sigma_{\text{CH}_3,\text{MCMC}}$  parameter sets for  $\lambda_{\text{CH}_3} \geq 14$  over-predict  $Z$  at high temperatures and densities ( $P^{\text{high}}$ ). As expected, the larger the value of  $\lambda_{\text{CH}_3}$ , the more the force field over-predicts  $P^{\text{high}}$ .

While Figures 7–8 plot the results for  $\rho_1^{\text{sat}}$ ,  $P_v^{\text{sat}}$ , and  $Z$  individually, Figure 9 helps to visualize the overall performance of different values of  $\lambda_{\text{CH}_3}$  for simultaneously predicting all three quantities of interest. In Panel a), notice the trade-off between the MAPD% of  $\rho_1^{\text{sat}}$  and  $P_v^{\text{sat}}$ . This compromise between two competing properties included in the objective function, namely,  $\rho_1^{\text{sat}}$  and  $P_v^{\text{sat}}$ , is known as a Pareto front.<sup>23,49,50</sup> The optimal location for a Pareto front is the bottom left region of the plot (low MAPD% for both  $\rho_1^{\text{sat}}$  and  $P_v^{\text{sat}}$ ) while the worst location is the top right region (high MAPD% for both  $\rho_1^{\text{sat}}$  and  $P_v^{\text{sat}}$ ). Note that the inset of Panel a) includes an approximate “overall” Pareto front that combines the results for all values of  $\lambda_{\text{CH}_3}$ . Although not depicted for visual clarity, the “L” shaped

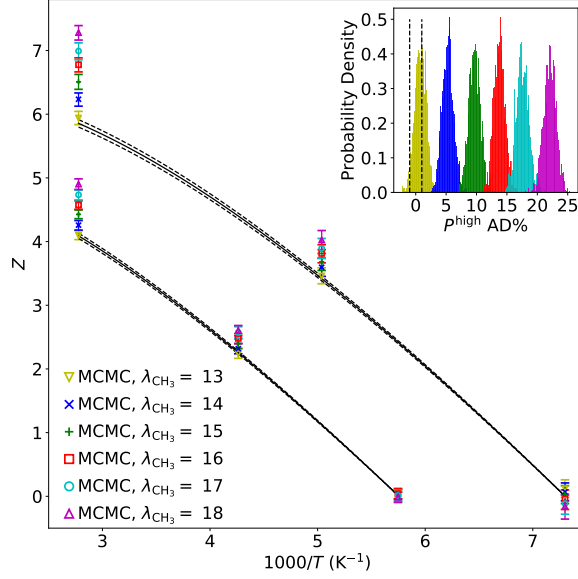


FIG. 8. Inadequacies of the UA Mie  $\lambda$ -6 potential are observed in  $Z$  for the two highest isochore densities ( $\rho_3$  and  $\rho_4$ ) by propagating the uncertainties in  $\epsilon_{\text{CH}_3}-\sigma_{\text{CH}_3}$  for different values of  $\lambda_{\text{CH}_3}$ . The inset plots the distribution of average deviation (AD%) in  $P^{\text{high}}$ , i.e.  $P_{\text{MCMC}}$  for  $\rho = \rho_3-\rho_4$  and  $T = T^{\text{IT}}$ . REFPROP uncertainty in  $P^{\text{high}}$  is  $\pm 1\%$ .

frontier for different colors/symbols demonstrates that each  $\lambda_{\text{CH}_3}$  value also has its own Pareto front. Because the overall Pareto front consists of points from the  $\lambda_{\text{CH}_3} = 15-17$  Pareto fronts, the Pareto optimal  $\lambda_{\text{CH}_3}$  value is either 15, 16, or 17, depending on the relative weight assigned to  $\rho_1^{\text{sat}}$  and  $P_v^{\text{sat}}$ . By contrast, since the  $\lambda_{\text{CH}_3} = 13, 14$ , and 18 Pareto fronts are completely inside the overall Pareto front, these  $\lambda_{\text{CH}_3}$  values are not optimal, regardless of the weighting.

Finally, and most importantly for our purposes, Figure 9 Panels b)-c) demonstrate the depreciation in  $\rho_1^{\text{sat}}$  and  $P_v^{\text{sat}}$  that is required to achieve more accurate prediction of  $P^{\text{high}}$ . For example, although  $\lambda_{\text{CH}_3} = 15, 16$ , and 17 are the best values based on VLE data, they over-predict  $P^{\text{high}}$  by around 10, 14, and 18%, respectively. By contrast, while  $\lambda_{\text{CH}_3} = 13$  is the most accurate for  $P^{\text{high}}$ , the MAPD% for  $\rho_1^{\text{sat}}$  and  $P_v^{\text{sat}}$  are 4 and 40 factors larger than the respective minimum MAPD%. These results support the fundamental claim of this work, namely, that the UA Mie  $\lambda$ -6 potential cannot adequately predict both VLE and

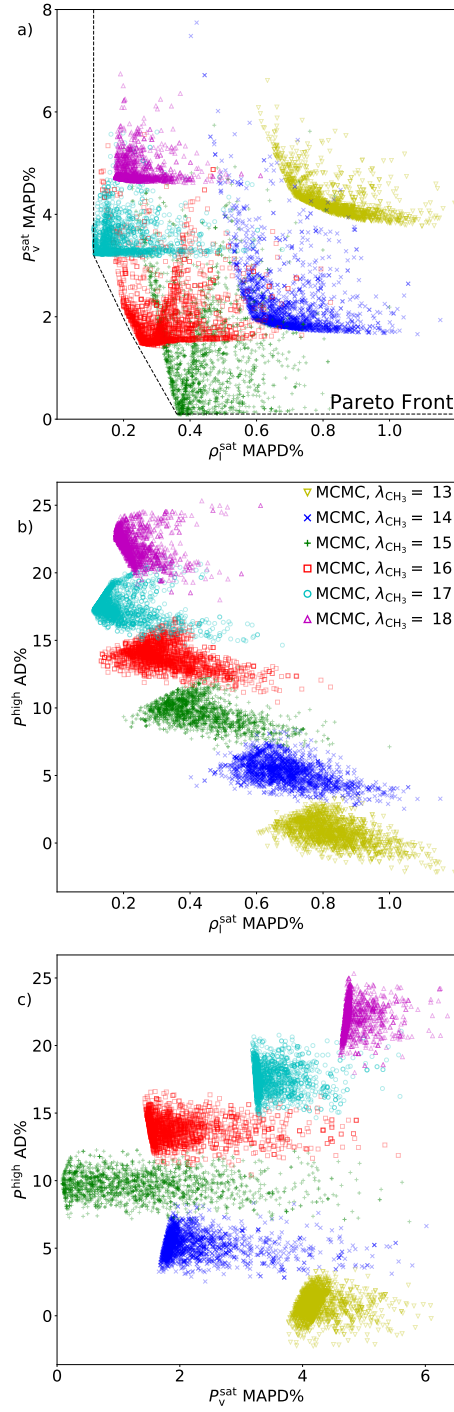


FIG. 9. MCMC results confirm that the UA Mie  $\lambda$ -6 potential cannot adequately predict both VLE and high pressures for supercritical fluids and compressed liquids. Panel a) plots the mean absolute percent deviation (MAPD%) of  $\rho_{l,MCMC}^{sat}$  and  $P_{v,MCMC}^{sat}$ . Panels b)-c) plot the average deviation (AD%) in  $P^{high}$  with respect to MAPD% of  $\rho_{l,MCMC}^{sat}$  and  $P_{v,MCMC}^{sat}$ , respectively.

high pressures for supercritical fluids and compressed liquids.

To statistically verify this statement, Figure 10 provides the Bayes factor for each value of  $\lambda_{\text{CH}_3}$  (normalized with respect to  $\lambda_{\text{CH}_3} = 14$ ) based solely on  $\rho_1^{\text{sat}}$  and  $P_v^{\text{sat}}$ . Bayes Factors between 1–3.2, 3.2–10, 10–32, 32–100, and  $> 100$  are typically classified as “not substantial”, “substantial”, “strong”, “very strong”, and “decisive” evidence, respectively.<sup>51</sup> With  $\frac{3.6}{0.02} = 180$ , there is “decisive” evidence against the use of  $\lambda_{\text{CH}_3} = 13$  for predicting  $\rho_1^{\text{sat}}$  and  $P_v^{\text{sat}}$ . As  $\lambda_{\text{CH}_3} = 13$  is the only value that predicts  $P^{\text{high}}$  within the REFPROP uncertainty, we conclude that no set of  $\epsilon_{\text{CH}_3}$ ,  $\sigma_{\text{CH}_3}$ , and  $\lambda_{\text{CH}_3}$  can predict *both* VLE and  $P^{\text{high}}$ .

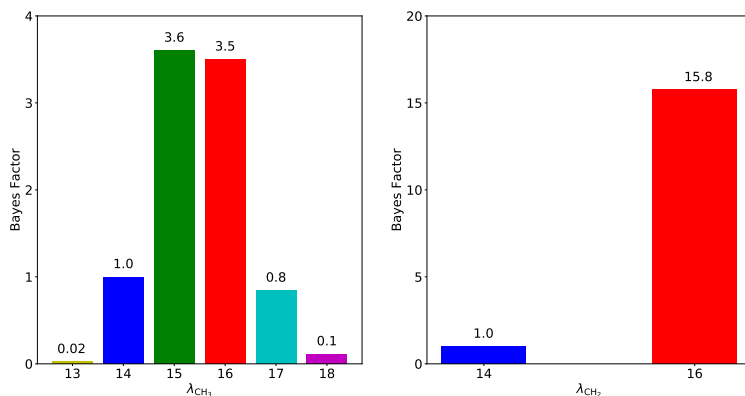


FIG. 10. Evidence (Bayes factor) in Panel a) supports  $\lambda_{\text{CH}_3} = 15$  or 16, while evidence in Panel b) strongly supports  $\lambda_{\text{CH}_2} = 16$  over  $\lambda_{\text{CH}_2} = 14$ .  $\text{CH}_3$  values depend only on ethane while  $\text{CH}_2$  values are based on propane, *n*-butane, and *n*-octane. Note that all values are normalized with respect to  $\lambda = 14$ .

In addition, there is “very strong” evidence that the 18-6 potential is not justified by VLE data ( $\frac{3.6}{0.1} = 36$ ). The evidence in favor of the 15-6 or 16-6 potentials over the 14-6 and 17-6 potentials is not as definitive, although it is still considered “substantial” ( $\frac{3.6 \text{ or } 3.5}{1.0 \text{ or } 0.8} > 3.5$ ). By contrast, the evidence for  $\lambda_{\text{CH}_3} = 15$  instead of  $\lambda_{\text{CH}_3} = 16$  is “not substantial” ( $\frac{3.6}{3.5} \approx 1.03$ ).

It is important to mention that these Bayes factors depend primarily on the VLE data and the  $s_{\text{D,SM}}^2$  model used to compute  $L(\theta|D)$ . We use a very conservative error model

for  $\rho_1^{\text{sat}}$  and  $P_v^{\text{sat}}$  (see Figure 5) so that our MCMC samples cover a large region of parameter space. This is done primarily to demonstrate that the UA Mie  $\lambda$ -6 is inadequate for predicting VLE and  $P^{\text{high}}$ . However, a less conservative error model would provide more convincing evidence regarding the optimal  $\lambda$  value based solely on VLE data.

Also, ITIC is limited to  $T^{\text{sat}} < 0.85T_c$ . Therefore, it is possible that the optimal value of  $\lambda_{\text{CH}_3}$  could be deduced (i.e. larger Bayes factors) if higher temperature VLE data were included (say from 260–290 K). Based on the observed bias in  $\rho_1^{\text{sat}}$  at higher temperatures (240–260 K) for  $\lambda_{\text{CH}_3} = 14$ , it appears that higher temperature VLE data would strengthen the counter evidence against the 14-6 potential. It is unclear whether higher temperature data would support the 15-6 or 16-6 potential, although the optimal  $\lambda_{\text{CH}_3}$  is likely a non-integer value between 15 and 16. Implementing MBAR with GCMC may be necessary to include VLE data from  $0.85 < T_r^{\text{sat}} < 0.95$ .

## B. Larger $n$ -alkanes

This section demonstrates that the conclusions regarding the UA Mie  $\lambda$ -6 potential for ethane are general for larger  $n$ -alkanes. Specifically, we observe that improved accuracy in predicting VLE requires a larger value of  $\lambda_{\text{CH}_2}$ . However, this improvement comes at the cost of significantly over-predicting  $P^{\text{high}}$ . Figure 11 presents the MCMC sampled  $\epsilon_{\text{CH}_2}$  and  $\sigma_{\text{CH}_2}$  parameter sets with Panels a) and b) corresponding to  $\lambda_{\text{CH}_2} = 16$  and  $\lambda_{\text{CH}_2} = 14$ , respectively. Note that these results were obtained using fixed values of  $\epsilon_{\text{CH}_3}$ ,  $\sigma_{\text{CH}_3}$ , and  $\lambda_{\text{CH}_3}$ , where  $\lambda_{\text{CH}_3} = \lambda_{\text{CH}_2}$ . The values of  $\epsilon_{\text{CH}_3}$  and  $\sigma_{\text{CH}_3}$  are the maximum likelihood parameter set from ethane for the corresponding  $\lambda_{\text{CH}_3}$  value.

Notice in Figure 11 that the MCMC sampled  $\epsilon_{\text{CH}_2}$  and  $\sigma_{\text{CH}_2}$  parameter sets, for a given value of  $\lambda_{\text{CH}_2}$ , overlap considerably for propane,  $n$ -butane, and  $n$ -octane. These joint-distributions provide statistical evidence in favor of the common assumption that  $\text{CH}_2$  parameters are transferable between different  $n$ -alkanes. To further demonstrate this point, Figure 11 includes the MCMC results when the posterior is based on the combined likelihoods from all three compounds, referred to as “MCMC transferable.”

Panel a) shows that the Potoff  $\text{CH}_2$  parameter set is within the MCMC sample regions

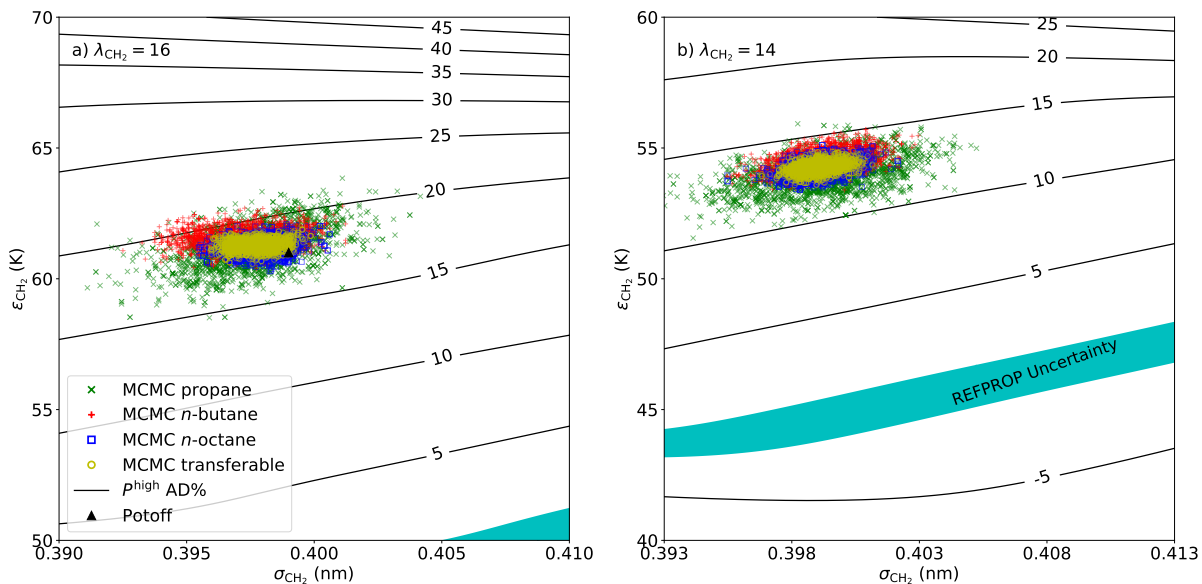


FIG. 11. MCMC sampled  $\epsilon_{\text{CH}_2}$  and  $\sigma_{\text{CH}_2}$  parameter sets result in large AD% for  $P^{\text{high}}$ . Panels a) and b) correspond to  $\lambda_{\text{CH}_2} = 16$  and  $\lambda_{\text{CH}_2} = 14$ , respectively. Contours are the AD% in  $P^{\text{high}}$  relative to the REFPROP correlations, where the “REFPROP uncertainty” region represents  $\pm 1\%$  deviation. Panel a) includes the Potoff parameter set as a reference for  $\lambda_{\text{CH}_2} = 16$ .

for  $\lambda_{\text{CH}_2} = 16$ . The same result was also observed for ethane (see Figure 6). This suggests that the Potoff  $\text{CH}_3$  and  $\text{CH}_2$  parameters are supported by the VLE data used in this study, even though the Potoff force field was parameterized using VLE data in a higher temperature range ( $0.6 < T_r^{\text{sat}} < 0.95$ ).

Also, note that the uncertainty in the parameters is largest for propane and smallest for  $n$ -octane. Therefore, the sensitivity of  $\rho_1^{\text{sat}}$  and  $P_v^{\text{sat}}$ , with respect to the  $\text{CH}_2$  parameters, increases with increasing number of  $\text{CH}_2$  interaction sites. Although this result is fairly intuitive, it is a valuable insight when selecting a training set of molecules for force field development. For example, notice that the MCMC transferable region is almost identical to that of  $n$ -octane, which shows that propane and  $n$ -butane data provide relatively little additional information that is not contained in the  $n$ -octane data.

For the purposes of this study, it is important to focus on the contours in Figure 11, which correspond to AD% in  $P^{\text{high}}$ . Notice that the MCMC sampled  $\epsilon_{\text{CH}_2}$  and  $\sigma_{\text{CH}_2}$  pa-



parameter sets have a strong positive bias (i.e. large AD%) in  $P^{\text{high}}$ . Specifically,  $\lambda_{\text{CH}_2} = 16$  and  $\lambda_{\text{CH}_2} = 14$  have, respectively, AD% of  $\approx 16$ -21 and  $\approx 10$ -15, much greater than the REFPROP uncertainty of around 1%. Furthermore, because the “REFPROP uncertainty” contours are roughly parallel to the MCMC region and found at much lower  $\epsilon_{\text{CH}_2}$  values, it is necessary to sacrifice considerable accuracy in  $\rho_1^{\text{sat}}$  and  $P_v^{\text{sat}}$  in order to accurately predict  $P^{\text{high}}$ . It is interesting that, for corresponding values of  $\lambda$ , the AD% for these larger  $n$ -alkanes is higher than that of ethane. This suggests that longer chain-lengths, with a UA Mie  $\lambda$ -6 force field, exacerbate the erroneous  $Z$  trend at high pressures.

Although the AD% in  $P^{\text{high}}$  is slightly lower for  $\lambda_{\text{CH}_2} = 14$  than for  $\lambda_{\text{CH}_2} = 16$ , the UA Mie 14-6 potential is significantly less reliable for VLE. Figure 10 demonstrates that there is “strong” evidence for  $\lambda_{\text{CH}_2} = 16$  over  $\lambda_{\text{CH}_2} = 14$ , based on VLE data. Note that the evidence in Figure 10 for  $\text{CH}_2$  is stronger than that for  $\text{CH}_3$ . This suggests that the ethane  $\rho_1^{\text{sat}}$  and  $P_v^{\text{sat}}$  results are less sensitive to  $\lambda$  than the larger  $n$ -alkanes and/or that the ethane VLE data contains less information than the combined data of propane, *n*-butane, and *n*-octane. In conclusion, these results suggest that neither UA Mie 16-6 or 14-6 force fields are capable of predicting VLE and  $PVT$  for supercritical fluids and compressed liquids of  $n$ -alkanes.

## VI. DISCUSSION

### A. Recommendations

Although the UA Mie  $\lambda$ -6 potential is not quantitatively reliable at high pressures, it may still be of use for FEOS parameterization when considering the insight gained in this study. For example, since the Potoff force field consistently over-predicts high pressures, a non-linear FEOS optimization could utilize the simulation results as an upper constraint for the FEOS pressure.<sup>3</sup> Furthermore, the primary purpose to include molecular simulation data for FEOS development is to increase the range of validity by ensuring good behavior of the FEOS at high temperatures and pressures. While FEOS are based on empirical equations with 15–30 fitting parameters, even an inaccurate force field has

a more sound theoretical basis. Therefore, the UA Mie  $\lambda$ -6 simulation output for a given property should not demonstrate non-physical oscillations, inflection points, derivative sign-changes, etc., which can plague a poorly-fit FEOS.

Essentially, whether or not a FEOS should be developed using a hybrid data set consisting of UA Mie  $\lambda$ -6 simulation results depends on the quality and quantity of available experimental data. If the data cover a wide range of state points and properties, it is possible that the UA Mie  $\lambda$ -6 potential may still be useful, despite the systematic deviations at high pressures. By contrast, if the experimental data are limited such that the FEOS depends almost entirely on the molecular simulation results, the UA Mie  $\lambda$ -6 force field will lead to large deviations at high pressures. Therefore, in this scenario, we advise against the use of UA Mie  $\lambda$ -6 force fields when developing a FEOS for normal and branched alkanes. For this purpose, we recommend further investigation of alternative potentials with a softer repulsive barrier and a more sound theoretical basis, e.g. Buckingham exponential-6, modified-Morse,<sup>28–30</sup> or an extended Lennard-Jones.<sup>34,52,53</sup>

## B. Limitations

This section discusses limitations to the primary conclusion from this study, namely, that UA Mie  $\lambda$ -6 force fields parameterized with VLE data should not be used to develop fundamental equations of state for normal and branched alkanes. The main limitation is that the poor extrapolation at high pressures is based solely on the trend of  $Z$  with respect to inverse temperature. By contrast, the simulation values that are typically included in hybrid data sets are derivatives of the residual (or departure) Helmholtz free energy with respect to inverse temperature and/or density:<sup>7–11</sup>

$$A_{xy}^{\text{dep}} R_g T \equiv (1/T)^x \rho^y \frac{\partial^{x+y} A^{\text{dep}}}{\partial (1/T)^x \partial \rho^y} \quad (24)$$

We would like to emphasize the advantage of using  $A_{xy}^{\text{dep}}$  for developing FEOS, as this approach eliminates redundant information found in traditional macroscopic properties.<sup>2,4–7</sup> For example, the following expressions demonstrate the interdependency of the properties we computed, namely,  $Z$  and  $U^{\text{dep}}$  with their derivatives along isochores and

isotherms:<sup>6</sup>

$$Z = 1 + A_{01}^{\text{dep}} \quad (25)$$

$$\frac{1}{T} \left( \frac{-\partial Z}{\partial(1/T)} \right)_{\rho} = 1 + A_{01}^{\text{dep}} - A_{11}^{\text{dep}} \quad (26)$$

$$\frac{U^{\text{dep}}}{R_g T} = A_{10}^{\text{dep}} \quad (27)$$

$$\frac{1}{R_g} \left( \frac{\partial U^{\text{dep}}}{\partial T} \right)_{\rho} = -A_{20}^{\text{dep}} \quad (28)$$

$$\rho \left( \frac{\partial Z}{\partial \rho} \right)_T = 1 + 2A_{01}^{\text{dep}} + A_{02}^{\text{dep}} \quad (29)$$

Unfortunately, with the exception of *ms2*,<sup>54</sup> we are not aware of any open-source simulation package that readily provides  $A_{xy}^{\text{dep}}$ .

In addition, macroscopic properties, such as  $Z$  and  $U^{\text{dep}}$  (with their respective derivatives), are more readily understood and visualized than  $A_{xy}^{\text{dep}}$ . Also, it is easier to quantify their impact on process design than derivatives in the residual Helmholtz free energy. For example, as demonstrated in Reference,<sup>8</sup> an inaccurate prediction of some  $A_{xy}^{\text{dep}}$  does not necessarily result in poor prediction of  $PVT$  behavior or heat capacities. For these reasons, we did not base our conclusions on the adequacy of the UA Mie  $\lambda$ -6 potential to predict  $A_{xy}^{\text{dep}}$ .

Instead, we have presented  $Z$  and, by inspection, the slope of  $Z$  with respect to  $1/T$  at constant  $\rho$ . Since these properties are equivalent to Equations 25-26, respectively, we have indirectly focused on only two of the Helmholtz derivatives, namely,  $A_{01}^{\text{dep}}$  and  $A_{11}^{\text{dep}}$ . However, we also observed some deviations in  $U^{\text{dep}}$ , the slope of  $U^{\text{dep}}$  with respect to  $T$  at constant  $\rho$ , and the slope of  $Z$  with respect to  $\rho$  at constant  $T$ , which are equivalent to Equations 27-29. Unfortunately, deficiencies are not as obvious for these properties due to the relatively large uncertainties in the REFPROP correlations, ca. 5% and 10% for  $U^{\text{dep}}$  and  $\left( \frac{\partial U^{\text{dep}}}{R_g \partial T} \right)_{\rho}$ . Although these additional properties provide insight regarding  $A_{10}^{\text{dep}}$ ,  $A_{20}^{\text{dep}}$ ,  $A_{01}^{\text{dep}}$ , and  $A_{02}^{\text{dep}}$ , because the results are less conclusive they are provided only in Section SI.V of Supporting Information. Furthermore, the relationship between Equations 27-29 and the repulsive barrier,  $\lambda$ , is not obvious. By contrast, the observation that larger

values of  $\lambda$  over-predict  $Z$  with increasing temperature is clear evidence that  $\lambda > 12$  is not theoretically justified but, rather, is an empirical remedy that performs well for VLE.

Future work should investigate more thoroughly the adequacy of UA Mie  $\lambda$ -6 (or other) force fields to predict  $U^{\text{dep}}$  and isochoric/isobaric heat capacities at high temperatures and pressures. For example, rigorous studies of this type have been performed on the single-site Lennard-Jones fluid.<sup>2-4</sup>

Another limitation is that we utilize a single layer Bayes model as opposed to a hierarchical model.<sup>31,41</sup> Wu et al. demonstrated that it is important to use hierarchical models when there are discrepancies in the data set,  $D$ . Fortunately, since we use a conservative estimate for the surrogate model uncertainty, i.e.  $u_{\text{SM}} \gg u_{\text{D}}$ , any discrepancies in the VLE data should not affect the parameter uncertainties.

The hierarchical approach is also useful when accounting for model inadequacies, i.e. when the force field is not capable of representing multiple data types. A hierarchical method should thus be favored if determining the parameter uncertainty when simultaneously considering  $\rho_1^{\text{sat}}$ ,  $P_{\text{v}}^{\text{sat}}$  and  $P^{\text{high}}$ . Furthermore, a hierarchical model should be used if the parameters are not transferable between molecules, i.e. if the  $\text{CH}_2$  parameters were different for propane,  $n$ -butane, and  $n$ -octane. However, the hierarchical approach is unnecessary for our purposes, since the transferable UA Mie  $\lambda$ -6 force field for  $n$ -alkanes is capable of reproducing  $\rho_1^{\text{sat}}$  and  $P_{\text{v}}^{\text{sat}}$ , which are the only properties included in  $D$ .

## VII. CONCLUSIONS

Recently, molecular simulation results at extreme temperatures and pressures have supplemented experimental data when developing fundamental equations of state for compounds with limited experimental data. For this hybrid data set approach to work, it is imperative that the force field be reliable and transferable over different  $PVT$  conditions. Unfortunately, literature united-atom force fields that are highly accurate for estimating VLE properties of normal and branched alkanes have systematic deviations in  $Z$  at non-VLE conditions. Bayesian inference suggests that the UA Mie  $\lambda$ -6 model type is not adequate for simultaneously predicting  $\rho_1^{\text{sat}}$ ,  $P_{\text{v}}^{\text{sat}}$ , and  $P^{\text{high}}$ . In the case of ethane, ev-

idence from VLE data supports  $14 \leq \lambda \leq 17$ , while  $Z$  at high pressures requires  $\lambda = 13$ . A similar trend is observed for larger  $n$ -alkanes. Specifically, evidence from VLE data supports  $\lambda = 16$ , while we observe only slight improvement in  $Z$  at high pressures for  $\lambda = 14$ . Therefore, while considerable improvement in VLE is observed for  $\lambda > 12$ , the use of UA Mie  $\lambda$ -6 potentials over the traditional UA Lennard-Jones 12-6 potential does not appear to have physical/theoretical justification. For these reasons, we recommend that alternative models be considered for developing FEOS of normal and branched alkanes, such as force fields that use anisotropic-united-atom, all-atom, and/or alternative non-bonded potentials.

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