DEVELOPMENTS IN HIGHER-DIMENSIONAL AUTOMATA THEORY

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Higher-dimensional automata (HDAs) were introduced by Pratt [11] and van Glabbeek [12] as a geometrical model for non-interleaving concurrency. HDAs are general models of concurrency that subsume, for example, event structures and safe Petri nets [13]. Asynchronous transition systems [5,15] are two-dimensional HDAs; HDAs without concurrency, the 1-dimensional case, are standard automata.

Some years ago we have set out to create a proper automata theory for HDAs. Starting with a paper which defined languages of HDAs [6] and an accompanying paper on the elements of their languages, partially ordered multisets with interfaces (ipomsets) [8], we have now been able to show a Kleene theorem [7], a Myhill-Nerode theorem [9], and a Büchi-Elgot-Trakhtenbrot theorem [3].

Languages of HDAs need to account for both the sequential and the concurrent nature of computations. Their elements are thus finite *pomsets* or *partial words*. As an example, Figure 1 displays an HDA consisting of two squares, with three events labeled a, c, and d. Here the a-labeled event is executed concurrently to the sequence $c.d$, so that the language of this HDA will contain the pomset

$$\begin{bmatrix} & a & \\ c & \longrightarrow & d \end{bmatrix}.$$

Higher-dimensional automata consist of a collection of cells in which events are running concurrently, connected by face maps which model termination of some events or which map to cells in which some of the currently active events have not yet been started. More precisely, HDAs are presheaves over a category of labeled precubical sets equipped with sets of initial and accepting cells. Their languages are sets of interval-order pomsets [10] equipped with interfaces (ipomsets) which are closed under refinement of partial orders (subsumption).

Subsumption-closed sets of interval ipomsets can be combined using union $\cup$, sequential composition $*$, parallel composition $\|$, and the (sequential) Kleene plus $^+$. The languages generated from singletons using these operations yield a natural notion of rational language. In [7] we show a *Kleene* type theorem that the rational languages are precisely the regular languages, *i.e.*, languages of finite HDAs.

The prefix quotient of a set L of ipomsets by an ipomset P is the set $P\backslash L = \{Q \in \mathsf{iiPoms} \mid PQ \in L\}$. Defining $\mathrm{suff}(L) = \{P\backslash L \mid P \in \mathsf{iiPoms}\}$, we show in [9] a *Myhill-Nerode* type theorem that a language L is regular iff $\mathrm{suff}(L)$ is finite. In order to turn a suffix-finite language into an HDA we use a refinement of the standard Nerode equivalence which takes interfaces into account.

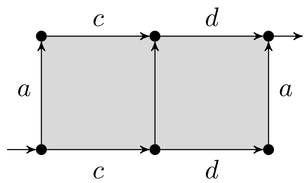

FIGURE 1. HDA which executes a in parallel with $c.d$. Initial and accepting cells marked with incoming and outgoing arrows.

Finally, in [3] we define an interpretation of monadic second-order logic (MSO) over ipomsets and show a theorem à la *Büchi-Elgot-Trakhtenbrot* stating that any regular language is MSO-definable and that if L is MSO-definable, then for any $k \geq 0$, $L_{\leq k}\downarrow$, the subsumption closure of the width-k restriction of L, is regular.

Taken together, these works provide a solid basis for a higher-dimensional automata theory. They have already led to further work on closure and decision properties of HDAs [4], on the representation theory of interval ipomsets [2], and on languages of higher-dimensional *timed* automata [1], and much work is ongoing.

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My proposed talk will start with a gentle introduction to HDAs. Afterwards I will give an overview of the three main results mentioned above (Kleene; Myhill-Nerode; Büchi-Elgot-Trakhtenbrot), mention differences with the equivalent theorems for finite automata, and hint at applications and future work. If possible, I could also have a poster.

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