Varieties of BL-Algebras With the Amalgamation Property: An Exhaustive Classification

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BL-algebras are the models of the equational theory of continuous triangular norms. On a logical level, they make up the equivalent algebraic semantics of Petr Hájek's basic fuzzy logic. Since their introduction in [6], BL-algebras have become the subject of a sophisticated theory that has attracted contributions from specialists in computer science, algebra, logic, and engineering.

One long-standing open problem in this theory concerns equational classes of BL-algebras with the amalgamation property. Montagna was the first to study this problem, and proved in [7] that several of the most important equational classes of BL-algebras have the amalgamation property but that there are uncountably many equational classes of BL-algebras that do not. Montagna left the characterization of equational classes of BL-algebras with amalgamation as an open problem, and in particular posed the question of whether there are uncountably many of these or only countably many.

Considerable effort has been expended over the years in efforts to solve Montagna's problem. Tools from first-order model theory were used to study amalgamation in BL-algebras in [3], and [1] gave a partial classification of equational classes of BL-algebras with amalgamation using algebraic structure theory. Then, using general results on amalgamation in equational classes from [4], the analysis of [1] was sharpened in [2]. However, a complete description of equational classes with amalgamation still proved elusive.

In this work, we completely characterize the equational classes of BL-algebras with the amalgamation property. In particular, we show that the subposet of the subvariety lattice of BL-algebras consisting of those equational classes with the amalgamation property may be partitioned into countably infinitely many finite intervals, and give a tangible description of these. This entails an answer to the question posed by Montagna: There are only countably many equational classes of BL-algebras with the amalgamation property.

The analysis enabling us to resolve Montagna's problem rests on two ingredients. First, we provide a novel characterization of the amalgamation property for certain equational classes in terms of essential extensions. Second, we offer a systematic way of naming equational classes of BL-algebras with amalgamation that is inspired by the theory of regular expressions, and use our nomenclature scheme to execute the necessary combinatorial arguments.

Further information may be found in our preprint [5].

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