Right-orders on Free Groups

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Let $\mathbf{G}$ be a group (or monoid). A *right-order* on $\mathbf{G}$ is a total order $\leq$ on G such that for all $a, b, c \in G$, $a \leq b$ implies $ac \leq bc$. For a subset $S \subseteq G$ of a group $\mathbf{G}$ we say that S *extends to a right-order* on $\mathbf{G}$ if there exists a right-order $\leq$ on $\mathbf{G}$ such that for all $g \in S$, $\mathrm{e} < g$, where e is the neutral element of $\mathbf{G}$. Free groups are well-known to be right-orderable and in [1] it is shown that given a free group $\mathbf{F}_g(X)$ over a set X the problem of checking for a finite subset $S \subseteq \mathrm{F}_g(X)$ if it extends to a right-order on $\mathbf{F}_g(X)$ is decidable.

In this talk, which is based on results from [5], we show that this problem is NP-complete for $|X| \geq 2$. For the proof we exploit the following connection between right-orders on free groups and the validity of certain equations in the class LG of ℓ-groups[1] obtained in [3]:

Proposition. *Let $\mathbf{F}_g(X)$ be the free group over a set X. Then a finite subset $\{s_1, \ldots, s_n\} \subseteq \mathrm{F}_g(X)$ extends to a right-order on $\mathbf{F}_g(X)$ if and only if $\mathsf{LG} \not\models \mathrm{e} \leq s_1 \vee \ldots \vee s_n$.*

Since the equational theory of LG is co-NP-complete (see [4]), it follows that the problem of checking if a finite set of elements of a free group extends to a right-order is in NP. To show that the problem is NP-hard we use a method of eliminating meets from ℓ-group equations. This method allows us to reduce the validity of certain equations, the set of which is known to be co-NP-hard, to equations of the form $\mathrm{e} \leq s_1 \vee \ldots \vee s_n$ with group terms $s_1, \ldots, s_n$, yielding:

Theorem. *Let $\mathbf{F}_g(X)$ be the free group over a set X with $|X| \geq 2$. Then the problem of checking for $\{s_1, \ldots, s_n\} \subseteq \mathrm{F}(X)$ if it extends to a right-order is NP-complete.*

Using a method of eliminating inverses in ℓ-group equations first considered in [2] we also obtain an analogue of the result for free monoids in the case where the generating set is infinite:

Corollary. *Let $\mathbf{F}_m(X)$ be the free monoid over an infinite set X. Then the problem of checking for $s_1, t_1, \ldots, s_n, t_n \in \mathrm{F}_m(X)$ if there exists a right-order $\leq$ on $\mathbf{F}_m(X)$ with $s_1 < t_1, \ldots, s_n < t_n$ is NP-complete.*

References

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[1] An ℓ-group is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \cdot, ^{-1}, \mathrm{e}\rangle$ such that $\langle L, \wedge, \vee\rangle$ is a lattice, $\langle L, \cdot, ^{-1}, \mathrm{e}\rangle$ is a group, and for all $a, b, c, d \in L$, $a \leq b \implies cad \leq cbd$.