TALK 6: L-PARAMETERS

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First of all, I should apologize that I still do not prepare so well for this talk(so I also LaTeX them after the talk).

This note records contents in the talk. Feel free to ask me questions and give me typos and suggestions!

1. Prelude

During the last five talks, we've discussed a lot about the representation theory of a reductive group G over an non-Archimedean local field F, which is viewed as the one side of the local Langlands correspondence (LLC). Today we will focus on the other side of the local Langlands correspondence: Langlands paremeters (L-parameters). Roughly speaking, it encodes 1-cocycles of the Weil group W_F over the dual group $\hat{G}$.

$$\mathrm{Irr}_\Lambda\big(G(F)\big) \longrightarrow Z^1\big(W_F, \hat{G}(\Lambda)\big)$$

We will not focus on the correspondence in this talk. Instead, we only care about L-parameters themselves. After introducing the concepts of Weil group, dual group and 1-cocycles, we will define a functor of L-parameters and show it's represented by a scheme with nice properties. After the break, we will also see the properties of GIT[1] quotients of L-parameters, or, to be precise, we will consider the points and functions of the GIT quotients.

Conventions and Notations. *Throughout this talk, F is a non-Archimedean local field with residue field $\kappa = \mathbb{F}_q$, $q = p^k$, G/F is a reductive group. l is any prime not equal to p, Λ is a $\mathbb{Z}_l$-algebra, and F_n denotes the free group generated by n elements.*

[1] GIT=geometric invariant theory.

2. Weil group

The Weil group is defined as a special subgroup of the absolute Galois group $G_F := \mathrm{Gal}(F^{sep}/F)$, whose structure is already carefully studied and understood well (see [1]). Information relevant to this talk is summarized below:

$$\begin{tikzcd}
F^{sep} \arrow[d, dash, "P_F"] \arrow[dd, phantom, "I_F" left, xshift=-3em] \\
F^{tr} \arrow[d, dash, "\hat{\mathbb{Z}}^{(p)}"] \arrow[dd, phantom, "\hat{\mathbb{Z}}^{(p)} \rtimes \hat{\mathbb{Z}}" right, xshift=3em] \\
F^{un} \arrow[d, dash, "\hat{\mathbb{Z}}"] \\
F
\end{tikzcd}$$

where

F^{un}	is the maximal unramified extension,
F^{tr}	is the maximal tame ramified extension,
$I_F = \mathrm{Gal}(F^{sep}/F^{un})$	is the inertia group,
$P_F = \mathrm{Gal}(F^{sep}/F^{tr})$	is the sylow p-subgroup of I_F, called the wild inertia group,

$$\hat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n\mathbb{Z} = \prod_l \mathbb{Z}_l, \qquad \hat{\mathbb{Z}}^{(p)} = \varprojlim_{(n,p)=1} \mathbb{Z}/n\mathbb{Z} = \prod_{l \neq p} \mathbb{Z}_l.$$

We take the geometric Frobenius $\sigma \in \mathrm{Gal}(F^{un}/F)$ as well as a choice of the geometrical generator $\tau \in \mathrm{Gal}(F^{tr}/F^{un})$, and then lift them to the absolute Galois group G_F.[2] We get

$$\mathrm{Gal}(F^{tr}/F) \cong \hat{\mathbb{Z}}^{(p)} \rtimes \hat{\mathbb{Z}} \qquad \sigma\tau\sigma^{-1} = \tau^{\frac{1}{q}}.$$

Until now we have not talked about the Weil group, and any group we mentioned in the tower of fields is given by the usual Krull topology. The Weil group W_F and the related smaller group W_F^0 [3] can be viewed as the discretization of the absolute Galois group G_F:

$$\begin{aligned}
&G_F \\
&\cup\!\mid \\
&W_F := \langle I_F, \sigma \rangle = \textstyle\bigsqcup_{g \in \langle \sigma \rangle} gI_F \\
&\cup\!\mid \\
&W_F^0 := \langle P_F, \tau, \sigma \rangle = \textstyle\bigsqcup_{g \in \langle \tau, \sigma \rangle} gP_F
\end{aligned}$$

To be exact, the topology of W_F is defined such that $I_F \subseteq W_F$ is open and closed, and has the same subspace topology as $I_F \subseteq G_F$; similarly for W_F^0. The word "discretization"

[2] We fix this lift during the whole talk. Also, see [stackexchange] for a proof of the equation $\sigma\tau\sigma^{-1} = \tau^{\frac{1}{q}}$.

[3] I would like to call it the 0-Weil group, or the skeloton Weil group.

can be seen clearer once we quotient P_F:

$$\begin{aligned} G_F/P_F &= \hat{\mathbb{Z}}^{(p)} \rtimes \hat{\mathbb{Z}} \\ &\cup\!| \\ W_F/P_F &= \hat{\mathbb{Z}}^{(p)} \rtimes \mathbb{Z} \\ &\cup\!| \\ W_F^0/P_F &= \mathbb{Z}\left[\frac{1}{p}\right] \rtimes \mathbb{Z} \end{aligned}$$

3. (Langlands) Dual group

4. 1-cocycle

5. $\mathcal{Z}^1(W_F, \hat{G}(\Lambda))$ is a good scheme

6. GIT and 1-parameter groups

7. Geometrical points of GIT quotient

8. functions of GIT quotient

References

[1] Alex Youcis. Galois groups of local and global fields.

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