Thanks for coming. ~~In this talk I will explain the L-parameters~~

Recall that ~~we~~ we've discussed a lot about the rep theory of

$$\mathrm{Irr}_{\Lambda}(G(F)) \xrightarrow{\text{L-parameters}} Z^1(W_F, \hat{G}(\Lambda))$$

a red gp over an non-archi

Notation. F: NA local field $k = \mathbb{F}_q$

G/F reductive gp

$q = p^k$ $\quad l \neq p$ (any prime)

Λ: $\mathbb{Z}_l$-alg. $\quad F_n$: free group gen by n elements.

---

1. Weil gp. $W_F \subset G_F := \mathrm{Gal}(F^{sep}/F)$

$$I_F\left\{\begin{array}{l} F^{sep} \\ \mid P_F \\ F^{tr.} \\ \mid \hat{\mathbb{Z}}^{(p)} \quad \tau \\ F^{un} \qquad \hat{\mathbb{Z}}^{(p)} \rtimes \hat{\mathbb{Z}} \\ \mid \hat{\mathbb{Z}} \quad \sigma \\ F \end{array}\right.$$

$$\begin{array}{l} G \\ \cup \\ W_F := \langle I_F, \sigma \rangle = \bigsqcup_{g \in \langle \sigma \rangle} g I_F \\ \cup \\ W_F^\circ := \langle P_F, \tau, \sigma \rangle = \bigsqcup_{g \in \langle \tau, \sigma \rangle} g P_F \end{array}$$

$$\rightsquigarrow \begin{array}{l} G/P_F = \hat{\mathbb{Z}}^{(p)} \rtimes \hat{\mathbb{Z}} \\ \cup \\ W_F/P_F = \hat{\mathbb{Z}}^{(p)} \rtimes \mathbb{Z} \\ \cup \\ W_F^\circ/P_F = \mathbb{Z}[\tfrac{1}{p}] \rtimes \mathbb{Z} \end{array}$$

$\tau \in \mathrm{Gal}(F^{tr}/F^{un})$

$\sigma \in \mathrm{Gal}(F^{un}/F)$ $\rightsquigarrow \tau, \sigma \in G_F$. $\quad \sigma \tau \sigma^{-1} = \tau^{\frac{1}{q}}$

Krull topology

363234ES.

$$\hat{\mathbb{Z}}^{(p)} = \varprojlim_{(n,p)=1} \mathbb{Z}/n\mathbb{Z} = \prod_{l \neq p} \mathbb{Z}_l$$

$$F^{un} = \bigcup_{(n,p)=1} F(\zeta_n)$$

$$F^{tr} = F(\zeta_n, \pi_F^{\frac{1}{n}} \mid p \nmid n)$$

2. ~~Pa~~ (Langlands) dual gp.

Def/Construction

$$G/F \overset{(T,B)}{\rightsquigarrow} (X^*(T), \Delta(B), X_*(T), \check{\Delta}(B)) \circlearrowleft G_F$$

$$\overset{\text{dual}}{\rightsquigarrow} (X_*(T), \check{\Delta}(B), X^*(T), \Delta(B)) \circlearrowleft G_F$$

$$\underset{\text{attach}}{\overset{\text{forget}}{\rightsquigarrow}} \hat{G} \circlearrowleft G_F \leftarrow W_F$$

$$\rightsquigarrow {}^LG := \hat{G} \rtimes W_F.$$

E.g. $G/F \rightsquigarrow \hat{G}/\mathbb{Z}_l \rightsquigarrow {}^LG/\mathbb{Z}_l$

$GL_n \rightsquigarrow GL_n \rightsquigarrow GL_n \times W_F$

$U(n, E/F) \rightsquigarrow GL_n \rightsquigarrow GL_n \rtimes W_F$

$W_F \twoheadrightarrow Gal(E/F) = \{1, \gamma\} \circlearrowleft GL_n$

$\gamma: A \mapsto w_0 (A^T)^{-1} w_0$ ← may figure it out. Not clear yet.

Rmk. (E/F ⇝ split ⇝ trivial action)

$\exists W \leq W_F \quad [W_F : W] < +\infty$ s.t. $W \circlearrowleft \hat{G}$ is trivial

Rmk. This dual gp has the God-given stardard borel & torus,

E/F Galois ext of deg 2

~~W_F~~

$$U(n, E/F)(R) = \left\{ A := (a_{ij})_{i,j=1}^n \;\middle|\; \begin{array}{l} a_{ij} \in E \otimes_F R \\ A w A^H = w \end{array} \right\}$$

$w = Id_{n \times n}$ or $\begin{bmatrix} & & & 1 \\ & & -1 & \\ & \iddots & & \\ \pm 1 & & & \end{bmatrix}$ $\quad A^H := \gamma(A^T)$

${}^LG$ encodes the information of W_F-action.

semidirect product ~~will~~ makes our life harder.

3. 1-cocycle.

W, A : topo gp $\quad W \curvearrowright A$ by $\phi: W \to \mathrm{Aut}(A)$

$\gamma \mapsto \gamma(-)$

$$Z^1(W,A) := \{ {}^L\varphi : W \to A \rtimes W \mid \text{cont gp homo section} \} \circlearrowleft A$$

$$\gamma \mapsto (\gamma_0, \gamma) =: {}^L\gamma$$

$$= \{ \varphi : W \to A \mid \varphi(\gamma\gamma') = \varphi(\gamma)\, {}^{\gamma}(\varphi(\gamma')) \}$$

E.p. For W: subquot of W_F acting on $\hat{G}$, we define a fctor.

$$\underline{Z}^1(W, \hat{G}) : \mathbb{Z}_\ell\text{-alg} \subset (Sch/_{\mathbb{Z}_\ell})^{op} \longrightarrow Set$$

$$\Lambda \longmapsto Z^1(W, \hat{G}(\Lambda))$$

e.g. ~~when~~ $\underline{Z}^1(W, GL_n)(\Lambda) = \{ \rho : W \to GL_n(\Lambda) \}$

$H^0(W,A) = A^W$

$H^1(W,A) = Z^1(W,A)/A$

$G \rightsquigarrow P, M$ in [VIII].

4. $\underline{Z}'(W_F,\hat{G})$

Thm $\underline{Z}'(W_F,\hat{G})/\mathbb{Z}_\ell$ is repped by a disjoint union of affine schemes of f.t over $\mathbb{Z}_\ell$.

It is flat, complete intersection of dim $Z'(W_F,\hat{G})$.

↖ regular embedding

Proof. Representability $\underline{Z}'(-,\hat{G})$.

W_F

$\Downarrow$

$W_F/P_{F,0}$

$\Downarrow$ field ext.

W_F/P_F

$\Downarrow$ "skeloton"

W_F^0/P_F

$\Downarrow$

F_n

Step 1. $\underline{Z}'(W_F,\hat{G}) = \bigcup_{\substack{P_{F,0}\leq P_F \text{ open} \\ \text{trivial action}}} \underline{Z}'(W_F/P_{F,0},\hat{G})$

Step 2. $\exists\ E/F$ Galois $\begin{cases} W_E \curvearrowright \hat{G} \text{ trivial} \\ P_{F,0}\cap W_E = P_E \end{cases}$

Step 3. $\underline{Z}'(W_F/P_F,\hat{G}) = \underline{Z}'(W_F^0/P_F,\hat{G})$

Step 4.
$$\begin{tikzcd} 0 \arrow[r] & F_1 \arrow[r] & F_2 \arrow[r] & W_F^0/P_F \arrow[r] & 0 \end{tikzcd}$$
$\sigma^{-1}\tau\sigma\tau^{-q}$ $\qquad$ τ,σ

Step 5
$$\begin{tikzcd} \rightsquigarrow \underline{Z}'(F_1,\hat{G}) \arrow[d, "\cong"] & \underline{Z}'(F_2,\hat{G}) \arrow[l] \arrow[d, "\cong"] & \underline{Z}'(W_F^0/P_F,\hat{G}) \arrow[l] \arrow[d, equal] & 0 \arrow[l] \\ \hat{G} & \hat{G}\times\hat{G} \arrow[l] & \underline{Z}'(W_F^0/P_F,\hat{G}) \arrow[l] & 0 \arrow[l] \end{tikzcd}$$

$\varphi \mapsto (\varphi(\tau),\varphi(\sigma))$

$\varphi(\sigma^{-1}\tau\sigma\tau^{-q}) \longleftarrow\!\!\shortmid (\varphi(\tau),\varphi(\sigma))$

Step 1 $\quad \underline{Z}'(W_F/P_{F,0},\hat{G})$ is an open and closed subfct of $Z'(W_F,\hat{G})$

reduced to: $Z'(W_F/P'_{F,0},\hat{G})$

$W: W_F/P'_{F,0}$

Claim. When $\gamma\in P_{F,0}$, $\underline{Z}'(W/\langle\gamma\rangle,\hat{G})$ is open and closed,

then $\underline{Z}'(W/\bar{P}_{F,0},\hat{G}) = \bigcap_{\gamma\in P_{F,0}} \underline{Z}'(W/\langle\gamma\rangle,\hat{G})$ is open & closed ~~open~~

$\underline{Z}'(W_F/P_{F,0},\hat{G})$ ------ $\underline{Z}'(W_F,\hat{G})$

$\underline{Z}'(W/\bar{P}_{F,0},\hat{G})$ ------ $\underline{Z}'(W,\hat{G})$

$\underline{Z}'(W/\langle\gamma\rangle,\hat{G})$ ------ $\underline{Z}'(W,\hat{G})$

$W := W_F/P'_{F,0}$: ~~finite p-gp~~.

$\gamma\in P_{F,0}$

$\gamma \curvearrowright W$ trivial.

~~$\gamma^{p^r}=1$ for some r in W~~

$\langle\gamma\rangle = \mathbb{Z}/p^r\mathbb{Z}$.

Step 1. $\underline{Z}'(W_F/P_{F,0}, \hat{G}) \subseteq \underline{Z}'(W_F, \hat{G})$ is open and closed

$W := W_F/P'_{F,0}$

$\overline{P}_{F,0} := P_{F,0}/P'_{F,0}$

$\Leftarrow \underline{Z}'(W/\overline{P}_{F,0}, \hat{G}) \subseteq \underline{Z}'(W, \hat{G})$ ————

$\Leftarrow \underline{Z}'(W/\langle\gamma\rangle, \hat{G}) \subseteq \underline{Z}'(W, \hat{G})$ ————

$\forall \gamma \in \overline{P}_{F,0}$

$\langle\gamma\rangle = \mathbb{Z}/p^r\mathbb{Z}$

$$0 \to \langle\gamma\rangle \to W \to W/\langle\gamma\rangle \to 0$$

$$\underline{Z}'(\langle\gamma\rangle, \hat{G}) \leftarrow \underline{Z}'(W, \hat{G}) \leftarrow \underline{Z}'(W/\langle\gamma\rangle, \hat{G}) \leftarrow 0$$

$\underline{Z}'(\langle\gamma\rangle, \hat{G}) = \hat{G} \cap \mu_{p^r}$ discrete components.

Step 2.

$$\begin{tikzcd} & & F_n \arrow[d, "\text{take}"] \arrow[dr, two heads] & & \\ 0 \arrow[r] & W_E/P_E \arrow[r] & W_F/P_{F,0} \arrow[r] & Gal(E/F) \arrow[r] & 0 \end{tikzcd} \quad (\star)$$

$\rightsquigarrow W_E/P_E * F_n \twoheadrightarrow W_F/P_{F,0}$

$\rightsquigarrow \underline{Z}'(W_E/P_E, \hat{G}) \times \underline{Z}'(F_n, \hat{G}) \hookleftarrow \underline{Z}'(W_F/P_{F,0}, \hat{G})$

$\rightsquigarrow \underline{Z}'(W_F/P_{F,0}, \hat{G})$ is rep'ble if $\underline{Z}'(W_E/P_E, \hat{G})$ is.

$(\star) \rightsquigarrow \underline{Z}'(W_E/P_E, \hat{G}) \leftarrow \underline{Z}'(W_F/P_{F,0}, \hat{G}) \leftarrow \underline{Z}'(Gal(E/F), \hat{G}) \leftarrow 0$

finite morphism.
same dim.

may be not true

$\underline{Z}'(Gal(E/F), \hat{G})$ can be complicated.

~~E.x. 证明 集合~~

~~是 Cantor 集中的离散稠密子集 (dense di discrete dense subset)~~

~~注意~~

Step 3. Suppose $W_F \curvearrowright \hat{G}$ trivial, $\hat{G} \hookrightarrow GL_n$.

$$\begin{array}{ccc} Z^1(W_F/P_F, \hat{G}(\Lambda)) & \longrightarrow & Z^1(W_F^0/P_F, \hat{G}(\Lambda)) \\ \| & & \| \\ \mathrm{Hom}_{cont}(\hat{\mathbb{Z}}^{(p)} \rtimes \mathbb{Z}, \hat{G}(\Lambda)) & \xhookrightarrow{\ *\ } & \mathrm{Hom}(\mathbb{Z}[\tfrac{1}{p}] \rtimes \mathbb{Z}, \hat{G}(\Lambda)) \end{array}$$

$$\varphi: \tau \mapsto \tau_0, \quad \sigma \mapsto \sigma_0$$

Task: extend $\mathbb{Z} \longrightarrow \hat{G}(\Lambda)$, $m \mapsto \tau_0^m$ to $\hat{\mathbb{Z}}^{(p)} \longrightarrow \hat{G}(\Lambda)$

$\sigma_0^{-1}\tau_0\sigma_0 = \tau_0^q \Rightarrow$ eigenvalues of τ_0 are the roots of unity

$\Rightarrow \tau_0^N$ is unipotent for $N = N(\hat{G})$

$\Rightarrow$ extends to $N\mathbb{Z}_l \longrightarrow \hat{G}(\Lambda)$

$$N.m \mapsto \tau_0^{N\cdot m} = (1 + (\tau_0^N - 1))^m$$

$$= \sum_{i \geq 0} \binom{m}{i} (\tau_0^N - 1)^i$$

$$\binom{m}{i} \in \mathbb{Z}_l$$

image commutes

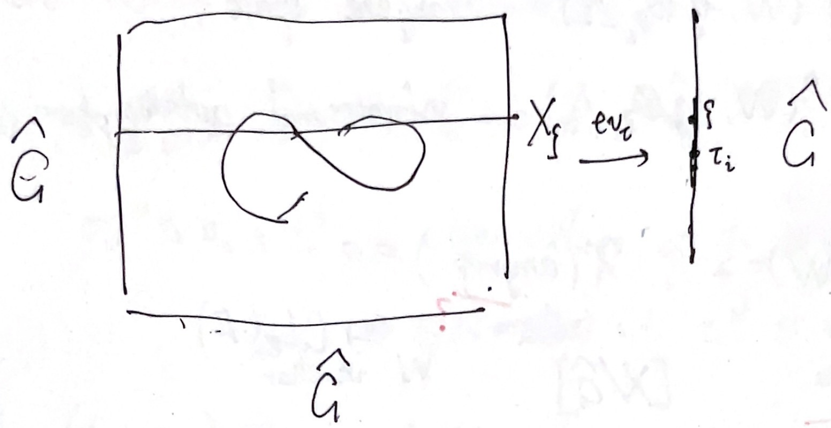
$\Rightarrow$ extends to $\mathbb{Z}_l = N\mathbb{Z}_l + \mathbb{Z}$

$\Rightarrow$ extends to $\hat{\mathbb{Z}}^{(p)} \twoheadrightarrow \mathbb{Z}_l$ $\square$

So. $\dim Z'(W_F^\circ/P_F, \hat{G})_{\hat{G}\cdot\tau_i} \leq \dim \hat{G}\cdot\tau_i + \dim X_\xi$

$= \dim \hat{G}\cdot\xi + \dim \hat{G}_\xi$

$= \dim \hat{G}$.



For the rest of the proof, we reduce to the case of

$$Z'(W_F^\circ/P_F, \hat{G})(\Lambda) \cong \{(\tau_0, \sigma_0) \in \hat{G}(\Lambda) \times \hat{G}(\Lambda) \mid \sigma_0^{-1}\tau_0\sigma_0 = \tau_0^q\}$$

$$\varphi \mapsto (\varphi(\tau), \varphi(\sigma))$$

where W_F°/P_F acts on $\hat{G}$ trivially.

w.l.o.g. $\Lambda = \bar{\Lambda}$. $\qquad a \rightsquigarrow a^{q^n-1}$

$\dim \leq \dim \hat{G}$: fix $\hat{G} \hookrightarrow GL_n$.

$\sigma_0^{-1}\tau_0\sigma_0 = \tau_0^q \Rightarrow$ eigenvalues of τ_0 are roots of unity $\quad \cancel{N = q^n - 1}$.

$\Rightarrow \exists N$ s.t. τ_0^N is unipotent.

Thm. (Lusztig) $\exists$ only finite many unipotent conj class in $\hat{G}(\Lambda)$

Cor $\exists$ __________ possibilities of τ_0 (up to conj)

$\{\tau_1, \dots \tau_M\}$

~~Denote X~~

So. $\quad Z'(W_F^\circ/P_F, \hat{G}) = \bigsqcup_i Z'(W_F^\circ/P_F, \hat{G})_{\hat{G}\cdot\tau_i}$

For $\xi \in \hat{G}\cdot\tau_i$, denote (choose τ_0) $\qquad ev_\tau: Z'(W_F^\circ/P_F, \hat{G}) \to \hat{G}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \varphi \mapsto \varphi(\tau)$

$X_\xi := ev_\tau^{-1}(\xi)$

$\cong \{\sigma_0 \in \hat{G}(\Lambda) \mid \sigma_0^{-1}\xi\sigma_0 = \xi^q\}$

(choose σ_0)

This is a $\hat{G}_\xi$ - torsor by

$$\hat{G}_\xi \times X_\xi \to X_\xi$$
$$(\gamma, \sigma_0) \mapsto \gamma\sigma_0\gamma^{-1}$$

$(\gamma\sigma_0\gamma^{-1})^{-1}\xi(\gamma\sigma_0\gamma^{-1})$
$= \gamma\sigma_0^{-1}(\gamma^{-1}\xi\gamma)\sigma_0\gamma^{-1}$
$= \gamma\sigma_0^{-1}\xi\sigma_0\gamma^{-1}$
$= \gamma\xi^q\gamma^{-1}$
$= \xi^q$.

If $\sigma_0^{-1}\xi\sigma_0 = \xi^q = \sigma_0'^{-1}\xi\sigma_0'$

$\Rightarrow$ $(\sigma_0\sigma_0'^{-1})^{-1}\xi\sigma_0\sigma_0'^{-1} = \xi$

$\Rightarrow \sigma_0\sigma_0'^{-1} \in \hat{G}_\xi$

$\dim \geq \dim \hat{G}$:

$$\begin{tikzcd} \underline{Z}^1(W_F^0/P_F, \hat{G}) \arrow[r, two heads, "\pi"] \arrow[d] & \{*\} = \mathrm{Spec}\, \mathbb{Z}_\ell \arrow[d] \\ \hat{G} \times \hat{G} \arrow[r, two heads] & \hat{G} \\[-20pt] (\tau_0, \sigma_0) \arrow[r, mapsto] & \sigma_0^{-1}\tau_0\sigma_0\tau_0^{q} \end{tikzcd}$$

Others.

~~Im~~

$\dim = \dim \hat{G}$

$\sigma_0^{-1}\tau_0\sigma_0\tau_0^{q} = 1$ ~~has~~ $\dim \hat{G}$-many equations

$\Big\} \Rightarrow$ complete intersection

$\left(\underline{Z}^1(W_F^0/P_F, \hat{G}) \subset \hat{G} \times \hat{G}\right.$ is complete intersection$\left.\right)$

$\Rightarrow \underline{Z}^1(W_F^0/P_F, \hat{G})$ is ~~Cohen-Macaulay~~ CM

$\mathrm{Spec}\, \mathbb{Z}_\ell$ is regular

fibers of π have $\dim = \dim \hat{G}$

$\Bigg\} \xRightarrow[\text{flatness}]{\text{Miracle}} \pi$ is flat.

~~Miracle flatness~~

~~$\Rightarrow \underline{Z}^1(W_F^0/P_F, \hat{G})$ is flat~~

Rmk. For more properties of $\underline{Z}^1(W_F^0/P_F, \hat{G})$.

$\begin{cases} \text{reduced, generical smooth [Prop 2.7]} \\ \ell\text{-adic separated, i.e. } \bigcap_n \ell^n \mathcal{O}(\underline{Z}^1(\ )) = 0 \text{ [Cor 2.10]} \end{cases}$

5. GIT & 1-parameter gps.

Before we discuss the GIT quotient

$$Z^1(W_F,\widehat{G})/\!/\widehat{G}$$

let us recall some preliminary facts about GIT & 1-parameter gps.

Notation. G: red. gp $\curvearrowright X = \operatorname{Spec} A$, $\varphi \in X$ | specify to $G \rightsquigarrow \widehat{G}_L$

A f.t./L $\Rightarrow A^G$ f.t./L | $X \rightsquigarrow Z^1(W_F,\widehat{G})_L$

$L = \bar{L}$ (everything is over L)

Thm. [2.9. Jose_S16]

$$\pi: X \longrightarrow X/\!/G \text{ is surj, and induces}$$

$$\{\text{closed } G\text{-orbits in } X\} \xleftrightarrow{1:1} X/\!/G\,(L)$$

We also need the Hilbert-Mumford theorem/Fundamental thm in GIT, ~~so that we can have~~ a which gives us a useful criterion of closed G-orbits.

But before that, let me introduce the notion of 1-parameter gp.

Def (1-Parameter subgp.)

$$PS(G) := \{\lambda : \mathbb{G}_m \to G \mid \text{homo}\}$$

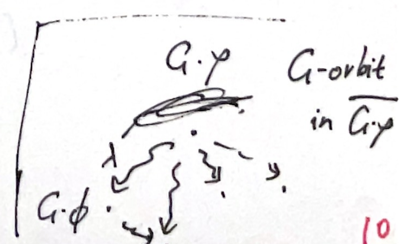
$$|X,\varphi| := \{\lambda \in PS(G) \mid \lim_{t\to 0} \lambda(t)\cdot\varphi \text{ exists}\}$$

$$\text{“} \lambda(t)\varphi\lambda(t)^{-1} \text{”}$$

Thm [14, Thm 1.4].

$$\forall \phi \in \overline{G\cdot\varphi},\ \exists \lambda \in |X,\varphi| \text{ s.t. } \lim_{t\to 0}\lambda(t)\cdot\varphi \in G\cdot\phi.$$

Cor. $G\cdot\varphi \subset X$ is closed $\Leftrightarrow \forall \lambda \in |X,\varphi|,\ \lim_{t\to 0}\lambda(t)\cdot\varphi \in G\cdot\varphi$



For every $\lambda \in PS(G)$, we can attach ~~corr~~ some corr. parabolic / levi / unipotent subgps.

Def. $(P_\lambda, M_\lambda, U_\lambda)$

$$P_\lambda = \{ g \in G \mid \lim_{t\to 0} \lambda(t) g \lambda(t)^{-1} \text{ exists} \}$$

$$M_\lambda = \{ g \in G \mid \lambda(t) g = g \lambda(t) \}$$

$$U_\lambda = \{ g \in G \mid \lim_{t\to 0} \lambda(t) g \lambda(t)^{-1} = 1 \}$$

E.g. $G = \cancel{SL_n}\ GL_3$ $\quad \lambda : \mathbb{G}_m \to SL_n \quad t \mapsto \mathrm{diag}(t^{k_1}, t^{k_2}, t^{k_3})$

$$\lambda(t) (a_{ij})_{i,j} \lambda(t)^{-1} = (t^{k_i - k_j} a_{ij})_{i,j}$$

when $k_1 = k_2 > k_3$.

$$P_\lambda = \left(\begin{array}{cc|c} * & * & * \\ * & * & * \\ \hline & & * \end{array}\right) \quad M_\lambda = \left(\begin{array}{cc|c} * & * & \\ * & * & \\ \hline & & * \end{array}\right) \quad U_\lambda = \left(\begin{array}{cc|c} 1 & & * \\ & 1 & * \\ \hline & & 1 \end{array}\right)$$

Fact. 1. $pr_\lambda : P_\lambda \to M_\lambda \quad g \mapsto \lim_{t\to 0} \lambda(t) g \lambda(t)^{-1}$ is a projection as in Levi decompo.

2. $\{(P_\lambda, M_\lambda)\}_{\lambda \in PS(G)}$ give all parabolic/Levi subgps.

Suppose W ~~acts on~~ G, ~~then~~ $\rightsquigarrow W \curvearrowright PS(G)$
$\qquad W \curvearrowright G$

$\gamma : \lambda \mapsto {}^\gamma\lambda := \gamma\lambda(-)$

Claim. 1. If γ fix λ, then γ fix P_λ $\qquad \gamma P_\lambda = P_{\gamma\lambda}$. $\quad \gamma M_\lambda = M_{\gamma\lambda}$ ..

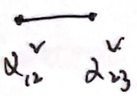
2. If W fix P, then $\exists \lambda \in PS(G)$ s.t $\begin{cases} P_\lambda = P_\lambda \\ W \text{ fix } \lambda. \end{cases}$

W fix $T \le P$ (max torus)

$\Rightarrow \lambda$ fix M_λ

$n=3 \qquad \lambda = k_1\varepsilon_1^* + k_2\varepsilon_2^* + k_3\varepsilon_3^* : t \longmapsto \begin{pmatrix} t^{k_1} & & \\ & t^{k_2} & \\ & & t^{k_3} \end{pmatrix}$

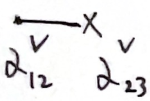
$$\lambda(t)\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\lambda(t)^{-1} = \begin{pmatrix} a_{11} & t^{k_1-k_2}a_{12} & t^{k_1-k_3}a_{13} \\ t^{k_2-k_1}a_{21} & a_{22} & t^{k_2-k_3}a_{23} \\ t^{k_3-k_1}a_{31} & t^{k_3-k_2}a_{32} & a_{33} \end{pmatrix}$$



$k_1 = k_2 = k_3 :\quad P_\lambda = GL_3$

$U_\lambda = Id$

$M_\lambda = GL_2,$



$k_1 = k_2 > k_3 :\quad P_\lambda = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix}$

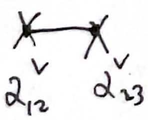
$U_\lambda = \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$

$M_\lambda = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix}$

$k_1 > k_2 = k_3 :\quad P_\lambda = \left(\begin{array}{c|cc} * & * & * \\ 0 & * & * \\ 0 & * & * \end{array}\right)$

$U_\lambda = \left(\begin{array}{c|cc} 1 & * & * \\ & 1 & \\ & & 1 \end{array}\right)$

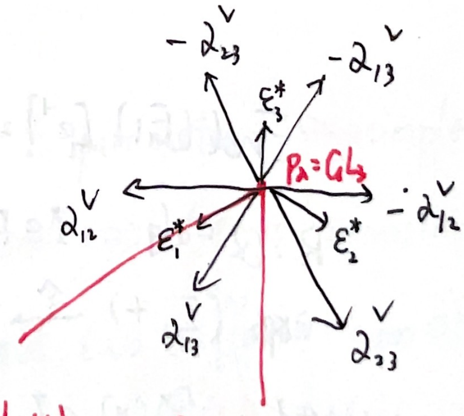
$M_\lambda = \left(\begin{array}{c|cc} * & & \\ \hline & * & * \\ & * & * \end{array}\right)$



$k_1 > k_2 > k_3 :\quad P_\lambda = \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix}$

$U_\lambda = \begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix}$

$M_\lambda = \begin{pmatrix} * & & \\ & * & * \\ & * & * \end{pmatrix}$



$P_\lambda = \left(\begin{array}{c|cc} * & * & * \\ \hline & * & * \\ & * & * \end{array}\right) \qquad P_\lambda = B \qquad P_\lambda = \left(\begin{array}{cc|c} * & & * \\ \hline & & * \end{array}\right)$

6. Geometrical pt in $\underline{Z}'(W_F, \hat{G})_L / \hat{G}_L$ $\qquad L = \bar{L}/\mathbb{Z}_l$

∇ $\lambda(t)\cdot\varphi = \lambda(t)\varphi^{\gamma}(\lambda(t)^{-1}) : \gamma \longmapsto \lambda(t)\varphi(\gamma)\,{}^{\gamma}(\lambda(t)^{-1}) \in \underline{Z}'(W_F, \hat{G})_L$

$\lambda(t)\varphi\lambda(t)^{-1} : \gamma \longmapsto \lambda(t)\varphi(\gamma)(\lambda(t))^{-1} \notin \underline{Z}'$

Def. ${}^L\varphi : W_F \longrightarrow \hat{G}(L) \rtimes W_F$ is called semisimple if

$$\left.\begin{array}{l} \operatorname{Im} {}^L\varphi \overset{\text{conj}}{\subseteq} \underset{\text{standard, i.e. contains } B}{P} \rtimes W_F \\ W_F \text{ fix } P \end{array}\right\} \Rightarrow \operatorname{Im} {}^L\varphi \overset{\text{conj}}{\subseteq} \underset{\text{standard, i.e. contains } T}{M} \rtimes W_F$$

I believe comes from Claim 2

i.e.

$$\left.\begin{array}{l} \operatorname{Im} {}^L\varphi \overset{\text{conj}}{\subseteq} P_\lambda \rtimes W_F \\ W_F \text{ fix } \lambda \end{array}\right\} \Rightarrow \operatorname{Im} {}^L\varphi \overset{\text{conj}}{\subseteq} M_\lambda \rtimes W_F$$

i.e. ${}^L\varphi \sim {}^L\varphi' : W_F \longrightarrow P_\lambda(L) \rtimes W_F$ + W_F fix λ

$\Rightarrow {}^L\varphi \sim {}^L\varphi'' : W_F \xrightarrow{\varphi'} P_\lambda(L) \rtimes W_F \xrightarrow{pr_\lambda} M_\lambda(L) \rtimes W_F$

$$\varphi'' = \lim_{t\to 0} \lambda(t)\varphi'\lambda(t)^{-1}$$

i.e.

$$\left.\begin{array}{l} \varphi' \in \hat{G}\cdot\varphi \\ \lim\limits_{t\to 0} \lambda(t)\varphi'\lambda(t)^{-1} \ \exists \\ W_F \text{ fix } \lambda \end{array}\right\} \Rightarrow \lim_{t\to 0} \lambda(t)\varphi'\lambda(t)^{-1} \in \hat{G}\cdot\varphi$$

E.g. $G = GL_n$. then semisimple L-parameter = semisimple rep of W_F

Prop. $\hat{G}\cdot\varphi \subseteq Z'(W_F, \hat{G})_L$ is closed $\Leftrightarrow$ ${}^L\varphi$ is semisimple

We need a lemma to connect two different ~~no~~ actions.

Lemma. For $\lambda \in PS(\hat{G}_L)$, $g \rtimes \gamma \in \hat{G}(L) \rtimes W_F$.

$$\lim_{t\to 0} \lambda(t)\, g\, {}^{\gamma}(\lambda(t)^{-1}) \ \exists \iff \begin{cases} \lim\limits_{t\to 0} \lambda(t)\, g\, (\lambda(t))^{-1} \ \exists \\ {}^{\gamma}\lambda = \lambda \end{cases}$$

Proof of Prop.

"~~LHS~~" $\Rightarrow$ "~~RHS~~": ~~Suppose~~

~~$\varphi' \in \hat{G}\,\varphi$~~

$\varphi' \in \hat{G} \cdot \varphi$

$\lim_{t\to 0} \lambda(t)\varphi'\lambda(t)^{-1}$ exists

W_F fixes λ

$\Big\} \overset{\text{Lemma}}{\Longleftrightarrow} \lim_{t\to 0} \lambda(t)\varphi'(\,{}^{\gamma}\lambda(t)^{-1})$ exists

$\overset{Z' \text{ closed}}{\Longrightarrow} \lim_{t\to 0} \lambda(t)\cdot\varphi' \in \hat{G}\cdot\varphi' \overset{\varphi' \in \hat{G}\cdot\varphi}{=} \hat{G}\cdot\varphi$

"~~RHS~~ $\Leftarrow$": $\lambda \in |X,\varphi| \Rightarrow \lim_{t\to 0} \lambda(t)\,\varphi\,({}^{\gamma}\lambda(t)^{-1})$ exists

$\overset{\text{Lemma}}{\Longleftrightarrow} \begin{cases} \lim_{t\to 0} \lambda(t)\varphi\lambda(t)^{-1} \text{ exists} \\ W_F \text{ fixes } \lambda. \end{cases}$

$\overset{{}^{L}\varphi \text{ semi}}{\Longrightarrow} \lim_{t\to 0} \lambda(t)\varphi\lambda(t)^{-1} \in \hat{G}\cdot\varphi$

$\overset{{}^{\gamma}\lambda=\lambda}{\Longrightarrow} \lim_{t\to 0} \lambda(t)\cdot\varphi \in \hat{G}\cdot\varphi.$

$\square$

8. Excursion operators

Idea. understand fcts on GIT quotient detect it by free gps

Fix $W = W_F^{\circ}/P_F \subset W_F/P_F$.

$$\begin{array}{ccc} F_n & & \\ \downarrow & \searrow & \\ & & W G \hat{G} \\ F_m & \nearrow & \end{array} \Rightarrow \begin{tikzcd} \mathcal{O}(\hat{G}^{\oplus n}) = \mathcal{O}(\underline{Z}'(F_n,\hat{G})) \arrow[d] \arrow[r] & \mathcal{O}(\underline{Z}'(W,\hat{G})) = \mathcal{O}(\underline{Z}'(W_F/P_F,\hat{G})) \\ \mathcal{O}(\underline{Z}'(F_m,\hat{G})) \arrow[ur] & \end{tikzcd}$$

$$\rightsquigarrow \Phi : \varinjlim_{(n, F_n \to W)} \mathcal{O}(\underline{Z}'(F_n,\hat{G})) \longrightarrow \mathcal{O}(\underline{Z}'(W,\hat{G}))$$

$(n, F_n\to W)$: sifted index category

i.e. $\varinjlim$ commutes with finite $\prod$

$$\Phi^{\hat{G}} : \varinjlim_{(n, F_n \to W)} \mathcal{O}(\underline{Z}'(F_n,\hat{G}))^{\hat{G}} \longrightarrow \underbrace{\mathcal{O}(\underline{Z}'(W,\hat{G}))^{\hat{G}}}_{=\mathrm{Exc}(W,\hat{G})}$$

spectral Bernstein Center.

Claim. Φ is iso of $\hat{G}$-algs.

$\Phi^{\hat{G}}$ is a universal homo of f.t. $\mathbb{Z}_\ell$-algs,

~~and~~ $\Phi^{\hat{G}}[\frac{1}{\ell}]$ is an iso. ~~after inve~~

Thm VIII 3.6. Assume that ℓ is a very good prime for $\hat{G}$. Then

Φ is iso in (the presentable stable ∞-category) $\mathrm{Ind\,Perf}(B\hat{G})$.

$\Rightarrow \Phi^{\hat{G}}$ is ~~an~~ iso.

~~Proof~~ (Difficulty of) Proof. Need. ∞-category

Perfect cplx good filtrations

Modular rep. theory

~~good~~ discussion one-by-one.

......

Rmk. $\mathrm{Exc}(W,\hat{G})$ Will be used in

- Construction of LLC.
- categoricalization of LLC

Task. Section 2).