TALK 6: L-PARAMETERS

XIAOXIANG ZHOU

Contents

First of all, I should apologize that I still do not prepare so well for this talk(so I also LaTeX them after the talk).

This note records contents in the talk. Feel free to ask me questions and give me typos and suggestions!

1. Prelude

During the last five talks, we've discussed a lot about the representation theory of a reductive group G over an non-Archimedean local field F, which is viewed as the one side of the local Langlands correspondence (LLC). Today we will focus on the other side of the local Langlands correspondence: Langlands paremeters (L-parameters). Roughly speaking, it encodes 1-cocycles of the Weil group W_F over the dual group $\hat{G}$[1].

$$\mathrm{Irr}_\Lambda\big(G(F)\big) \longrightarrow Z^1\big(W_F, \hat{G}(\Lambda)\big)$$

We will not focus on the correspondence in this talk. Instead, we only care about L-parameters themselves. After introducing the concepts of Weil group, dual group and 1-cocycles, we will define a functor of L-parameters and show it's represented by a scheme with nice properties. After the break, we will also see the properties of GIT[2] quotients of L-parameters, or, to be precise, we will consider the points and functions of the GIT quotients.

[1]Usually people denote $\hat{G}$ for the characters of G. Luckily, we won't discuss anything about the representation theory of G in this note, so there won't be any confusion. (Replace $\hat{G}$ by $G^\vee$ if you don't like it.)

[2]GIT=geometric invariant theory.

Conventions and Notations. *Throughout this talk, F is a non-Archimedean local field with residue field $\kappa = \mathbb{F}_q$, $q = p^k$, G/F is a reductive group. l is any prime not equal to p, Λ is a $\mathbb{Z}_l$-algebra, and F_n denotes the free group generated by n elements.*

2. Weil group

The Weil group is defined as a special subgroup of the absolute Galois group $G_F := \mathrm{Gal}(F^{sep}/F)$, whose structure is already carefully studied and understood well (see [4]). Information relevant to this talk is summarized below:

$$\begin{tikzcd}
F^{sep} \arrow[d, dash, "P_F"] \arrow[dd, dash, bend right=60, swap, "I_F"] \\
F^{tr} \arrow[d, dash, "\hat{\mathbb{Z}}^{(p)}"] \arrow[dd, dash, bend left=60, "\hat{\mathbb{Z}}^{(p)} \rtimes \hat{\mathbb{Z}}"] \\
F^{un} \arrow[d, dash, "\hat{\mathbb{Z}}"] \\
F
\end{tikzcd}$$

where

F^{un}	is the maximal unramified extension,
F^{tr}	is the maximal tame ramified extension,
$I_F = \mathrm{Gal}(F^{sep}/F^{un})$	is the inertia group,
$P_F = \mathrm{Gal}(F^{sep}/F^{tr})$	is the sylow p-subgroup of I_F, called the wild inertia group,

$$\hat{\mathbb{Z}} = \varprojlim_{n} \mathbb{Z}/n\mathbb{Z} = \prod_{l} \mathbb{Z}_l, \qquad \hat{\mathbb{Z}}^{(p)} = \varprojlim_{(n,p)=1} \mathbb{Z}/n\mathbb{Z} = \prod_{l \neq p} \mathbb{Z}_l.$$

We take the geometric Frobenius $\sigma \in \mathrm{Gal}(F^{un}/F)$ as well as a choice of the geometrical generator $\tau \in \mathrm{Gal}(F^{tr}/F^{un})$, and then lift them to the absolute Galois group G_F.[3] We get

$$\mathrm{Gal}(F^{tr}/F) \cong \hat{\mathbb{Z}}^{(p)} \rtimes \hat{\mathbb{Z}} \qquad \sigma\tau\sigma^{-1} = \tau^{\frac{1}{q}}.$$

Until now we have not talked about the Weil group, and any group we mentioned in the tower of fields is given by the usual Krull topology. The Weil group W_F and the related smaller group W_F^0 [4] can be viewed as the discretization of the absolute Galois group G_F:

$$\begin{array}{l}
G_F \\
\cup\!\mathsf{I} \\
W_F := \langle I_F, \sigma \rangle = \bigsqcup_{g \in \langle \sigma \rangle} g I_F \\
\cup\!\mathsf{I} \\
W_F^0 := \langle P_F, \tau, \sigma \rangle = \bigsqcup_{g \in \langle \tau, \sigma \rangle} g P_F
\end{array}$$

[3]We fix this lift during the whole talk. Also, see [stackexchange] for a proof of the equation $\sigma\tau\sigma^{-1} = \tau^{\frac{1}{q}}$.

[4]I would like to call it the 0-Weil group, or the skeloton Weil group.

To be exact, the topology of W_F is defined such that $I_F \subseteq W_F$ is open and closed, and has the same subspace topology as $I_F \subseteq G_F$; similarly for W_F^0. The word "discretization" can be seen clearer once we quotient P_F:

$$\begin{gathered} G_F/P_F = \hat{\mathbb{Z}}^{(p)} \rtimes \hat{\mathbb{Z}} \\ \cup\!\mathrm{I} \\ W_F/P_F = \hat{\mathbb{Z}}^{(p)} \rtimes \mathbb{Z} \\ \cup\!\mathrm{I} \\ W_F^0/P_F = \mathbb{Z}\left[\frac{1}{p}\right] \rtimes \mathbb{Z} \end{gathered}$$

3. (Langlands) Dual group

The second ingredient of L-parameters is the dual group. The main reference are [2, Section 4-5] and [1]. If you have never seen the structure theory of reductive groups, this expository paper is highly recommanded.

Definition 3.1 (Dual group, Langlands dual group)**.** *Fix a quasi-split reductive connected group*[5] *G over F, a Borel subgroup $B \le G$ as well as a maximal torus $T \le B$. We get a root system*

$$(X^*(T), \Delta(B), X_*(T), \Delta^\vee(B))$$

with a G_F-action. Temporarily forget the G_F-action, there is a unique split group $\hat{G}$ (with standard Borel $\hat{B}$ and standard tori $\hat{T}$) whose root system

$$\left(X^*(\hat{T}), \Delta(\hat{B}), X_*(\hat{T}), \Delta^\vee(\hat{B})\right) \cong (X_*(T), \Delta^\vee(B), X^*(T), \Delta(B))$$

*dual to the root system of G. $\hat{G}$ is called the **dual group** of G. We attach $\hat{G}$ with a G_F-action such that the induced G_F-action on the root system is the same as the original one. Finally, the **Langlands dual group** ${}^L G$ is defined as the semidirect product of $\hat{G}$ and W_F, where W_F acts on $\hat{G}$ as a subgroup of G_F.*

The whole process can be encapsulated in the following diagram:

$$\begin{aligned} G/F &\overset{(T,B)}{\rightsquigarrow} (X^*(T), \Delta(B), X_*(T), \Delta^\vee(B)) \circlearrowleft G_F \\ &\overset{\text{dual}}{\rightsquigarrow} (X_*(T), \Delta^\vee(B), X^*(T), \Delta(B)) \circlearrowleft G_F \\ &\underset{\text{attach}}{\overset{\text{forget}}{\rightsquigarrow}} \hat{G} \circlearrowleft G_F \longleftarrow W_F \\ &\rightsquigarrow {}^L G := \hat{G} \rtimes W_F \end{aligned}$$

Example 3.2.

$$\begin{aligned} G/F &\rightsquigarrow \hat{G}/\mathbb{Z}_l \rightsquigarrow {}^L G/\mathbb{Z}_l \\ \mathrm{GL}_n &\rightsquigarrow \mathrm{GL}_n \rightsquigarrow \mathrm{GL}_n \times W_F \\ \mathrm{U}(n, E/F) &\rightsquigarrow \mathrm{GL}_n \rightsquigarrow \mathrm{GL}_n \rtimes W_F \end{aligned}$$

[5]Later, when (quasi-)split group is mentioned, it's always assumed to be a reductive connected group.

Here, E/F is a degree 2 Galois extension (of NA local field), $\mathrm{U}(n, E/F)$ is an quasi-split group defined by

$$\mathrm{U}(n, E/F)(R) = \left\{ A = (a_{ij})_{i,j=1}^n \,\middle|\, \begin{array}{l} a_{ij} \in E \otimes_F R \\ A\omega A^H = \omega \end{array} \right\}$$

where

$$\omega := \begin{bmatrix} & & 1 \\ & -1 & \\ \iddots & 1 & \end{bmatrix} \qquad \mathrm{Gal}(E/F) := \{1, \gamma\} \qquad A^H := \gamma(A^T).$$

In this case, the Weil group W_F acts on GL_n by

$$W_F \twoheadrightarrow \mathrm{Gal}(E/F) = \{1, \gamma\} \circlearrowleft \mathrm{GL}_n \qquad \gamma : A \longmapsto (\omega^{-1} A^{-1} \omega)^T$$

The details can be found in [1, 2]*, or my calculation for* $\mathrm{U}(3, E/F)$*.*

Remark 3.3. Quasi-split group always becomes split after some finite extension. Therefore, the action of the Weil group W_F always factors through a finite quotient Q. In many articles the Langlands dual group is defined as $\hat{G} \rtimes Q$, to obtain the structure of finite type scheme.[6]

As you see, the Langlands dual group encodes information of W_F-action, and the semidirect product inside twists everything to be less understandable. But in this talk, the real difficulty happens even when you consider $G = \mathrm{GL}_n$ case, so feel free to make your life easier, and replace $\rtimes$ by $\times$ whenever you get troubles.

4. 1-cocycle

The third ingredient is easier to define.[7] Let us begin in a little more general.

Definition 4.1. *Let W, A be topological groups, and let W acts on A by*

$$\phi : W \longrightarrow \mathrm{Aut}(A) \qquad \gamma \longrightarrow {}^{\gamma}(-).$$

The 1-cocycle is defined as

$$\begin{aligned} Z^1(W, A) &:= \left\{ \begin{array}{r} {}^L\varphi : W \longrightarrow A \rtimes W \\ \gamma \longmapsto (\gamma_0, \gamma) := {}^L\gamma \end{array} \,\middle|\, \begin{array}{l} {}^L\varphi : \textit{continuous group homo} \\ {}^L\varphi \textit{ is a section} \end{array} \right\} \\ &= \left\{ \varphi : W \longrightarrow A \,\middle|\, \varphi(\gamma\gamma') = \varphi(\gamma)\, {}^{\gamma}(\varphi(\gamma)) \right\} \end{aligned}$$

Finally we can define L-parameters.[8]

---

[6]The main problem is, we have topology on both $\hat{G}$ and W_F, and the "correct" semidirect product should be compatible with those topologies. It's also not perfect ot consider them as abstract groups, since $\hat{G}$ is a scheme and there's no nature group structure even on the closed points of $\hat{G}$.

[7]We omit everything about Galois cohomology in [3]. Here we list two formulas for comparison:

$$H^0(W, A) = A^W, \qquad H^0(W, A) = Z^1(W, A)/A.$$

[8]Strictly speaking, we replace topological space by condensed things in the following definition. In Section 7, the same definition applies for $\hat{P}$, $\hat{B}$, or $\hat{T}$.

Definition 4.2. *Denote W as any subquotient group of W_F which can (naturally) act on $\hat{G}$, we define a functor*

$$\mathcal{Z}^1(W, \hat{G}) : \mathbb{Z}_l\text{-Alg} \longrightarrow \mathrm{Set} \qquad \Lambda \longmapsto Z^1\big(W_F, \hat{G}(\Lambda)\big),$$

and elements in $Z^1\big(W_F, \hat{G}(\Lambda)\big)$ are called L-parameters.

Example 4.3. *The Λ-points of $\mathcal{Z}^1(W, \mathrm{GL}_n)$ are just n-dim (continuous?) representations of W (with coefficient in Λ), i.e.,*

$$\mathcal{Z}^1(W, \mathrm{GL}_n)(\Lambda) = \big\{\rho : W \longrightarrow \mathrm{GL}_n(\Lambda)\big\}.$$

In the next section, we will prove the representability of $\mathcal{Z}^1(W_F, \hat{G})$.

5. $\mathcal{Z}^1(W_F, \hat{G})$ is a good scheme

6. Interlude: GIT and 1-parameter groups

7. Geometrical points of GIT quotient

8. functions of GIT quotient

References

[1] Kevin Buzzard. Unitary groups: basic definitions. `https://www.ma.imperial.ac.uk/~buzzard/maths/research/notes/unitary_groups_basic_definitions.pdf`, 2012.

[2] Bill Casselman. The l-group. `https://personal.math.ubc.ca/~cass/research/pdf/miyake.pdf`, 2001.

[3] Jean-Pierre Serre. Galois cohomology. In *Local Fields*, pages 150–163. Springer, 1979.

[4] Alex Youcis. Galois groups of local and global fields. `https://alex-youcis.github.io/localglobalgalois.pdf`.

School of Mathematical Sciences, University of Bonn, Bonn, 53115, Germany,

Email address: `email:xx352229@mail.ustc.edu.cn`