AFFINE PAVING OF PARTIAL FLAG QUIVER VARIETY

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Abstract. In this article, we establish an affine paving for partial flag quiver varieties when the quiver is of Dynkin type. By copying results in [1, section 6] word by word, the same problem for affine quiver reduced to the case where the representation is regular quasi-simple. The idea of the proof mainly comes from [1], and the result is a natural continuation of [2].

Contents

1. Introduction

Let Q be a quiver of Dynkin or affine type(without loops), $X \in \mathrm{Rep}(Q)$ be an quiver representation.[1] We are interested in three objects related to $X \in \mathrm{Rep}(Q)$:

quiver Grassmannian	$\mathrm{Gr}^{KQ}(X) := \{M_1 \mid M_1 \subseteq X\}$
partial flag variety $d \geqslant 1$	$\mathrm{Flag}_{\mathbf{d}}(X) := \{0 \subseteq M_1 \subseteq \cdots M_d \subseteq X\}$
strict partial flag variety $d \geqslant 2$	$\mathrm{Flag}_{\mathbf{d},\mathrm{str}}(X) := \{0 \subseteq M_1 \subseteq \cdots M_d \subseteq X \mid x.M_{i+1} \subseteq M_i\}$ [2]

It's easy to see that $\mathrm{Flag}_{\mathbf{1}}(X) = \mathrm{Gr}^{KQ}(X)$. These geometrical objects can be divided into different pieces according to the dimension vectors of $M_1, \ldots, M_d$, and each piece have its own natural (complex/Zarisky) topology. It was proved in [1] that $\mathrm{Gr}^{KQ}(X)$ have an affine paving, and in [2] that $\mathrm{Flag}_{\mathbf{d}}(X)$ have the same property when Q is Dynkin quiver of type A/E. Here we go one step further, the results are concluded in the ???.

???Here is one table

The idea of proof is very simple: first, we view the partial flag quiver variety as the quiver Grassmannian of the more complicated quiver; then we establish the decomposition so that

[1] We fix the base field $K = \mathbb{C}$ for convinience.

[2] for any $x \in Q_1, i \in \{2, \ldots, d\}$.

one may solve the problem by induction; finally we set a special way of decomposition for each indecomposable module so that we can avoid meeting the bad decomposition. These contents are in Section ???, accordingly.

Before the end of this section, let us see in one example how partial flag variety is viewed as quiver Grassmannian.

Example 1.1. *Let* $Q\colon x \longrightarrow y \longleftarrow z \longrightarrow w$ *be a quiver, and let* $X\colon X_x \longrightarrow X_y \longleftarrow X_z \longrightarrow X_w$ *be a quiver representation, then* $\mathrm{Flag}_{\mathbf{3}}(X), \mathrm{Flag}_{\mathbf{3},str}$ *can be viewed as quiver Grassmannian in ???:*

???[pictures here]

2. Preliminary Facts

Fix the quiver Q ant the integer $d \geqslant 2$, we define the new bigger quiver $Q_d, Q_{d,str}$ as follows:

- The vertex set of Q_d(resp. $Q_{d,str}$) is defined as the Cartesian product of the vertex set of Q and $\{1, \dots, d\}$, i.e.

$$v(Q_d) = v(Q_{d,str}) = v(Q) \times \{1, \dots, d\}$$

- The arrows of Q_d(resp. $Q_{d,str}$) is defined as follows:
  - for each $(i, r) \in v(Q) \times \{1, \dots, d-1\}$, there is one arrow from (i, r) to $(i, r+1)$;
  - for each arrow $i \longrightarrow j$ in quiver Q:
    * Q_d case: there is one arrow from (i, r) to (j, r); (for any $r \in \{1, \dots, d\}$)
    * $Q_{d,str}$ case: there is one arrow from (i, r) to $(j, r-1)$; (for any $r \in \{1, \dots, d-1\}$)

Example 2.1. *Here is the picture of new quiver:*

???[pictures here]

We define the ring

$$R = ???or???$$

and define the canonical map $\Phi : \mathrm{Rep}(Q) \longrightarrow \mathrm{Mod}(R) \subseteq \mathrm{Rep}(Q_d \text{ or } Q_{d,str})$ as follows:

- $(\Phi(X))_{(i,r)} := X_i$;
- $(\Phi(X))_{(i,r)\to(i,r+1)} := \mathrm{Id}_{X_i}$;
- $(\Phi(X))_{(i,r)\to(j,-)} := X_{i\to j} \quad (i \to j)$;

For the module $T \in \mathrm{Mod}(R)$, we define $\mathrm{Gr}^R(T)\colon = \{T' \subseteq T \text{ as the submodule}\}$.

Proposition 2.2. *Fix the representation* $X \in \mathrm{Rep}(Q)$*, we have the isomorphism*

$$\mathrm{Flag}(X) \cong \mathrm{Gr}^R(\Phi(X)).$$

Proof. The two side give us same amount of informations. Or, you can easily construct the bijection from $\mathrm{Flag}(X)$ to $\mathrm{Gr}^R(\Phi(X))$. □

2.1. **Ext-vanishing properties.** We would like to show some higher rank Extension group to be 0, which would be a key ingredient in the proof of the next section.

Lemma 2.3. *Let* $M, N, X, S \in \mathrm{Rep}(Q)$*,* $V, W, T \in \mathrm{Mod}(R)$*.*

(1) $\mathrm{gl.dim}\, R \leqslant 2$*;*
(2) The functor $\Phi : \mathrm{Rep}(Q) \longrightarrow \mathrm{Mod}(R)$ *is exact;*
(3) Φ *maps projective module to projective module;*
(4) $\mathrm{Hom}_{KQ}(M, N) \cong \mathrm{Hom}_R(\Phi(M), \Phi(N)), \quad \mathrm{Ext}^i_{KQ}(M, N) \cong \mathrm{Ext}^i_R(\Phi(M), \Phi(N))$*;*
(5) $\mathrm{proj.dim}\, \Phi(M) \leqslant 1, \mathrm{inj.dim}\, \Phi(M) \leqslant 1$*;*
(6) Suppose $V \subseteq \Phi(X), W \subseteq \Phi(S)$*, then* $\mathrm{Ext}^2_R(W, T) = 0, \mathrm{Ext}^2_R(T, \Phi(X)/V) = 0$*.*

Proof. For (1), we just need to check minimal projective resolution of $S(i)$ in $\mathrm{Mod}(R)$; for (3), we reduced to the case of indecomposable projective modules. The rests are easy exercises of homological algebra. □

2.2. **How much do we understand the quiver representation?** To understand the category $\mathrm{Rep}(Q)$, one should understand indecomposable modules(as well as their relations). This has almost been done in the Auslander-Reiten theory. For example, when the quiver Q is of Dynkin type, then there are only finite ind rep(up to isomorphism) and each ind rep corresponds to the positive root of Dynkin diagram. One can compute the Auslander-Reiten quiver by knitting algorithm and get the structure of ind rep. Moreover, one can directly get Hom space between M and N by looking at nontrivial paths from M to N[3].

We will use the Auslander-Reiten quiver to find ”good monomorphisms” in Section ???. For more informations about Auslander-Reiten theory, one can see ???.

3. Main Theorem

In the following sections, we always suppose that

- $X, Y, S, M, N \in \mathrm{Rep}(Q), V, U, W, T, T' \in \mathrm{Mod}(R)$;
- we have the short exact sequence $\eta : 0 \longrightarrow X \longrightarrow Y \longrightarrow S \longrightarrow 0$, and $V \subseteq \Phi(X), U \subseteq \Phi(Y), W \subseteq \Phi(S)$;
- $\mathbf{f}, \mathbf{g}$ are dimension vectors of quiver Q_d or $Q_{d,str}$. $\mathrm{Gr}^R_{\mathbf{f}}(T) := \{T' \subseteq T \mid \underline{\mathbf{dim}}\, T' = \mathbf{f}\}$ is defined as the set of subrepresentations with dimension vector $\mathbf{f}$.

We use

$$[M, N]^i := \dim_K \mathrm{Ext}^i_{KQ}(M, N), \qquad [M, N] := \dim_K \mathrm{Hom}_{KQ}(M, N)$$
$$[T, T']^i := \dim_K \mathrm{Ext}^i_R(T, T'), \qquad [T, T'] := \dim_K \mathrm{Hom}_R(T, T')$$
$$\langle T, T' \rangle_R := \sum_{i=0}^{\infty} (-1)^i [T, T']^i = [T, T'] - [T, T']^1 + [T, T']^2$$
$$\langle \mathbf{f}, \mathbf{g} \rangle_R := \sum_{i \in v(R)} f_i g_i - \sum_{b \in a(R)} f_{s(b)} g_{t(b)} + \sum_{c \in va(R)} f_{s(c)} g_{t(c)}$$

[3]These paths may be linear dependent, so it’s not too easy.

for the shorthand notation, where

$$\begin{aligned}
v(R) &:= \{\text{vertices in quiver } KQ_d \text{ or } KQ_{d,str}\}\\
a(R) &:= \{\text{arrows in quiver } KQ_d \text{ or } KQ_{d,str}\}\\
va(R) &:= \{\text{"virtual arrows" in quiver } KQ_d \text{ or } KQ_{d,str}\}
\end{aligned}$$

???[pictures with caption "virtual arrow: can ne thought as the "face" of the quiver"]

Let $\eta : 0 \longrightarrow X \xrightarrow{\iota} Y \xrightarrow{\pi} S \longrightarrow 0$ be a short exact sequence in $Rep(Q)$. Consider the canonical **non-continuous** map

$$\Psi : \mathrm{Gr}^R(\Phi(Y)) \longrightarrow \mathrm{Gr}^R(\Phi(X)) \times \mathrm{Gr}^R(\Phi(S)) \qquad U \longmapsto \big([\Phi(\iota)]^{-1}(U), [\Phi(\pi)](U)\big)$$

and $\Psi_{\mathbf{f},\mathbf{g}}$ is the map Ψ restricted to the preimage of $\mathrm{Gr}^R_{\mathbf{f}}(\Phi(X)) \times \mathrm{Gr}^R_{\mathbf{g}}(\Phi(S))$.

The goal of this section is to prove the following theorems:

Theorem 3.1. *When η splits, Ψ is surjective. Moreover, $\Psi_{\mathbf{f},\mathbf{g}}$ is a Zarisky-locally trivial affine bundle of rank $\langle \mathbf{g}, \underline{\mathbf{dim}}\,\Phi(X) - \mathbf{f}\rangle_R$.*

Theorem 3.2 (follows [1, Theorem 32])**.** *When η does not split and $[S,X]^1 = 1$,*

$$\mathrm{Im}\,\Psi_{\mathbf{f},\mathbf{g}} = \Big(\mathrm{Gr}^R_{\mathbf{f}}(\Phi(X)) \times \mathrm{Gr}^R_{\mathbf{g}}(\Phi(S))\Big) \setminus \Big(\mathrm{Gr}^R_{\mathbf{f}}(\Phi(X_S)) \times \mathrm{Gr}^R_{\mathbf{g}-\underline{\mathbf{dim}}\,\Phi(S^X)}\big(\Phi(S/S^X)\big)\Big)$$

where

$$\begin{aligned}
X_S &:= \max\big\{M \subseteq X \,\big|\, [S, X/M]^1 = 1\big\} \subseteq X\\
S^X &:= \max\big\{M \subseteq S \,\big|\, [M, X]^1 = 1\big\} \subseteq S
\end{aligned}$$

Moreover, $\Psi_{\mathbf{f},\mathbf{g}}$ is a Zarisky-locally trivial affine bundle of rank $\langle \mathbf{g}, \underline{\mathbf{dim}}\,\Phi(X) - \mathbf{f}\rangle_R$ over $\mathrm{Im}\,\Psi_{\mathbf{f},\mathbf{g}}$.

We will spend the rest of the section proving this theorem. We decide the image as well as the fiber of Ψ respectively.

Lemma 3.3 (follows [1, Lemma 21])**.** *The element $(V, W) \in \mathrm{Gr}^R(\Phi(X)) \times \mathrm{Gr}^R(\Phi(S))$ lies in the image of $/\Psi$ iff the canonical map $\mathrm{Ext}^1(\Phi(S), \Phi(X)) \longrightarrow \mathrm{Ext}^1(W, \Phi(X)/V)$ maps η to 0.*

Proof. This follows from the definition of the canonical map.

[??? Picture π'] ☐

Corollary 3.4. *When η splits, Ψ is surjective.*

Lemma 3.5. *the canonical map $\mathrm{Ext}^1(\Phi(S), \Phi(X)) \longrightarrow \mathrm{Ext}^1(W, \Phi(X)/V)$ is surjective.*

Proof. By using the LES of Extension groups and the Lemma 2.3(6), the maps

$$\mathrm{Ext}^1(\Phi(S), \Phi(X)) \longrightarrow \mathrm{Ext}^1(W, \Phi(X)) \qquad \mathrm{Ext}^1(W, \Phi(X)) \longrightarrow \mathrm{Ext}^1(W, \Phi(X)/V)$$

are both surjective. Thus the composition is also surjective. ☐

Corollary 3.6. *When $[S,X]^1 = 1$, then $[W, \Phi(X)/V]^1 = 0$ or 1.*

Suppose η generates $\mathrm{Ext}^1(S, X)$, then

$$(V, W) \in \mathrm{Im}\,\Psi \iff [W, \Phi(X)/V]^1 = 0.$$

In the case where η generates $\mathrm{Ext}^1(S,X)$, we want to describe $\mathrm{Im}\,\Psi$ more precisely. For this reason we need to introduce two new R-modules:

$$\widetilde{X_S} := \max\{V \subseteq \Phi(X) \mid [\Phi(S), \Phi(X)/V]^1 = 1\} \subseteq \Phi(X)$$
$$\widetilde{S^X} := \max\{W \subseteq \Phi(S) \mid [W, \Phi(X)]^1 = 1\} \subseteq \Phi(S)$$

$\widetilde{X_S}$ and $\widetilde{S^X}$ are well-defined because of the following lemma:

Lemma 3.7 (follows [1, Lemma 27])**.**
(i) Let $V, V' \subset \Phi(X)$ such that $[\Phi(S), \Phi(X)/V]^1 = [\Phi(S), \Phi(X)/V']^1 = 1$. Then $[\Phi(S), \Phi(X)/V+V']^1 = 1$.
(ii) Let $W, W' \subset \Phi(S)$ such that $[W, \Phi(X)]^1 = [W', \Phi(X)]^1 = 1$. Then $[W\cap W', \Phi(X)]^1 = 1$.

Proof. We only prove (i). (ii) is similar.

From the SES

$$0 \longrightarrow \Phi(X)/V\cap V' \longrightarrow \Phi(X)/V \oplus \Phi(X)/V' \longrightarrow \Phi(X)/V+V' \longrightarrow 0,$$

we get the LES ???change the way of format

$$\cdots \longrightarrow \mathrm{Ext}^1(\Phi(S), \Phi(X)/V\cap V') \longrightarrow \mathrm{Ext}^1(\Phi(S), \Phi(X)/V) \oplus \mathrm{Ext}^1(\Phi(S), \Phi(X)/V') \longrightarrow \mathrm{Ext}^1(\Phi(S), \Phi(X)/V$$

By Corollary 3.6, $[\Phi(S), \Phi(X)/V\cap V']^1 \leqslant 1$, $[\Phi(S), \Phi(X)/V+V']^1 \leqslant 1$, and this forces that $[\Phi(S), \Phi(X)/V+V']^1 = 1$. □

Lemma 3.8 (follows [1, Lemma 31(1)(2)], and the proof is same)**.** *Let $f : X \longrightarrow \tau S$ be a non-zero morphism, then $X_S = \ker(f)$; also, $\Phi(f) : \Phi(X) \longrightarrow \Phi(\tau S)$ is a non-zero morphism, $\widetilde{X_S} = \ker(\Phi(f))$.*

Corollary 3.9. $\widetilde{X_S} = \Phi(X_S)$.*(since $\widetilde{X_S} = \ker(\Phi(f)) = \Phi(\ker(f)) = \Phi(X_S)$)*

By the similar argument, one can show that $\widetilde{S^X} = \Phi(S^X)$.

Lemma 3.10 (follows [1, Lemma 31(6)])**.** *Given $V \subseteq \Phi(X)$ and $W \subseteq \Phi(S)$, we have*

$$[W, \Phi(X)/V]^1 = 0 \iff V \nsubseteq \Phi(X_S) \text{ or } W \nsupseteq \Phi(S^X).$$

Proof. $\Leftarrow$: without lose of generation suppose $V \nsubseteq \Phi(X_S)$, then

$$V \nsubseteq \Phi(X_S) \iff [\Phi(S), \Phi(X)/V]^1 = 0 \Rightarrow [W, \Phi(X)/V]^1 = 0.$$

$\Rightarrow$: If not, then $V \subseteq \Phi(X_S)$ and $W \supseteq \Phi(S^X)$, and[4]

$$[W, \Phi(X)/V]^1 \geqslant [\Phi(S^X), \Phi(X)/\Phi(X_S)]^1 = [S^X, X/X_S]^1 = 1.$$

We get the contradiction! □

Corollary 3.11. *When η generates $\mathrm{Ext}^1(S,X)$, we have*

$$\mathrm{Im}\,\Psi_{\mathbf{f},\mathbf{g}} = \left(\mathrm{Gr}^R_{\mathbf{f}}(\Phi(X)) \times \mathrm{Gr}^R_{\mathbf{g}}(\Phi(S))\right) \setminus \left(\mathrm{Gr}^R_{\mathbf{f}}(\Phi(X_S)) \times \mathrm{Gr}^R_{\mathbf{g}-\underline{\mathbf{dim}}\,\Phi(S^X)}\left(\Phi(S/S^X)\right)\right)$$

[4] $[S^X, X/X_S]^1 = 1$ follows from [1, Lemma 31(5)]

Lemma 3.12. *For $(V, W) \in \operatorname{Im} \Psi$, the preimage of (V, W) is a torsor of* $\operatorname{Hom}_R(W, \Phi(X)/V)$. *Or we could say, there is one non-canonical isomorphism*

$$\Psi^{-1}((V, W)) \cong \operatorname{Hom}_R(W, \Phi(X)/V).$$

Proof. ???[pictures]

When $(V, W) \in \operatorname{Im} \Psi$, $\bar{\eta}$ is split, and each split morphism θ give us an element in $\Psi^{-1}((V, W))$. If we fix one split morphism θ_0, then the other split morphisms are all of the form $\theta_0 + \iota \circ f$ where $f \in \operatorname{Hom}_R(W, \Phi(X)/V)$(and this form is unique). So

$$\Psi^{-1}((V, W)) \cong \{\theta : \text{ split morphism } \} \cong \operatorname{Hom}_R(W, \Phi(X)/V).$$

$\square$

Lemma 3.13. *For two R-modules T, T', we have*

$$\left\langle T, T' \right\rangle_R \cong \left\langle \underline{\mathbf{dim}}\, T, \underline{\mathbf{dim}}\, T' \right\rangle_R.$$

Proof. We construct one canonical resolution of T,

[??? Pictures]

by applying the functor $\operatorname{Hom}_R(-, T')$ to this resolution, we get

$$\left\langle T, T' \right\rangle_R \cong \left\langle \underline{\mathbf{dim}}\, T, \underline{\mathbf{dim}}\, T' \right\rangle_R.$$

$\square$

Proof of Theorem 3.1 and 3.2. We have already computed $\operatorname{Im} \Psi$ in Corollary 3.4 and 3.11. For the rank of the affine bundle, we have

$$\begin{aligned} (V, W) \in \operatorname{Im} \Psi_{\mathbf{f},\mathbf{g}} &\Longrightarrow [W, \Phi(X)/V]^1 = 0 \\ &\Longrightarrow [W, \Phi(X)/V] = \langle W, \Phi(X)/V \rangle_R = \langle \mathbf{f}, \underline{\mathbf{dim}}\, \Phi(X) - \mathbf{g} \rangle_R \end{aligned}$$

$\square$

4. Application: Dynkin Case

5. Application: Affine Case

Task:
fillin the ???
change ind rep to indecomposable representations
change iff to if and only if
change SES LES

References

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