

Geometry of Quiver Flag Varieties

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January 10, 2023

Introduction

Year	People	Cohomology theory	Algebra
1980	Kazhdan–Lusztig	$\mathbb{C}[W]$	$H_*^{\text{BM}}(\mathcal{Z})$
1985	Lusztig	$\mathcal{H}_q(W)$	$K^{G \times \mathbb{C}^\times}(\mathcal{Z})$
2011	Varagnolo–Vasserot	KLR algebra	$H_{G_d}^*(\mathcal{Z}_d)$

In the first part, we compute the G -equivariant K -theory of the Steinberg variety in the quiver version.

Variety structure

$$V_1 \longrightarrow V_2 \quad \mathbf{d} = (\dim_{\mathbb{C}} V_1, \dim_{\mathbb{C}} V_2) \hat{=} (\mathbf{d}_1, \mathbf{d}_2)$$

$$\mathcal{F}_{\mathbf{d}} = \{ \text{complete flags of } V_1 \oplus V_2 \text{ respect to the index} \}$$

$$\text{Rep}_{\mathbf{d}}(Q) = \text{Hom}_{\mathbb{C}}(V_1, V_2)$$

$$\begin{array}{ccc} \widetilde{\text{Rep}}_{\mathbf{d}}(Q) \subseteq \text{Rep}_{\mathbf{d}}(Q) \times \mathcal{F}_{\mathbf{d}} & & Z_{\mathbf{d}} \subseteq \text{Rep}_{\mathbf{d}}(Q) \times \mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}} \\ \mu_{\mathbf{d}} \swarrow & \searrow \pi_{\mathbf{d}} & \mu_{\mathbf{d}, \mathbf{d}} \swarrow \quad \searrow \pi_{\mathbf{d}, \mathbf{d}} \\ \text{Rep}_{\mathbf{d}}(Q) & \mathcal{F}_{\mathbf{d}} & \text{Rep}_{\mathbf{d}}(Q) \quad \mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}} \end{array}$$

$$\text{Flag}(M) := \mu_{\mathbf{d}}^{-1}(M) = \{ (\text{complete}) \text{ flags of } M \}$$

Stratification structure

$$\varpi \in \mathbb{W}_{|\mathbf{d}|} := S_{|\mathbf{d}|}, \quad G_{\mathbf{d}} := \mathrm{GL}_{\mathbf{d}_1} \times \mathrm{GL}_{\mathbf{d}_2}, \quad B_{\mathbf{d}}, T_{\mathbf{d}}.$$

$$\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q) = \bigsqcup_{\varpi} \tilde{\Omega}_{\varpi}$$



$$\mathcal{F}_{\mathbf{d}} = \bigsqcup_{\varpi} \Omega_{\varpi}$$

$$\mathcal{Z}_{\mathbf{d}} = \bigsqcup_{\varpi} \tilde{\mathcal{O}}_{\varpi}$$



$$\mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}} = \bigsqcup_{\varpi} \mathcal{O}_{\varpi}$$

$$K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}}) \cong \bigoplus_{\varpi} K^{G_{\mathbf{d}}}(\tilde{\mathcal{O}}_{\varpi})$$

Cellular fibration theorem

$$\cong \bigoplus_{\varpi} K^{G_{\mathbf{d}}}(\mathcal{O}_{\varpi})$$

Thom isomorphism

$$\cong \bigoplus_{\varpi} K^{B_{\mathbf{d}}}(\Omega_{\varpi})$$

Induction isomorphism

$$\cong \bigoplus_{\varpi} K^{T_{\mathbf{d}}}(\Omega_{\varpi})$$

Reduction isomorphism

$$\cong \bigoplus_{\varpi} R(T_{\mathbf{d}})$$

Thom isomorphism

Main theorem

Theorem

Under the convolution product, $K^{G_d}(\mathcal{Z}_d)$ has a $R(T_d)$ -algebra structure. Moreover,

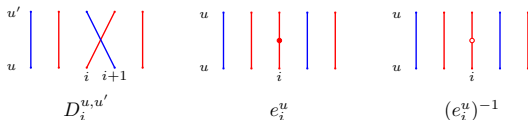
- (1) As an $R(T_d)$ -module, $K^{G_d}(\mathcal{Z}_d)$ is free of rank $|d|!$;*
- (2) As an $R(T_d)$ -algebra, we can write down generators and relations explicitly.*

For proving (2), we mainly use the localization formula and the excess intersection formula.

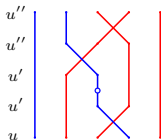
Generators

$$\bullet \longrightarrow \bullet, \quad \mathbf{d} = (3, 2), \quad u = \bullet \cdots \bullet \cdots \bullet, \quad u \in \text{Min}(\mathbb{W}_{|\mathbf{d}|}, W_{\mathbf{d}}).$$

Generators:



A typical element in $K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}})$ is a \mathbb{Z} -linear combination of diagrams shown below:



Compositions and trivial relations

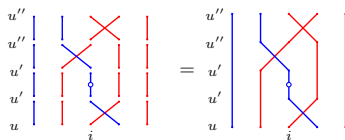


Diagram illustrating a relation between compositions of operators D and e . The left side shows a sequence of four diagrams: $D_3^{u,u'}$, $(e_3^{u'})^{-1}$, $D_2^{u',u''}$, and $D_3^{u'',u''}$. The right side shows a single diagram $D_3^{u,u'}$. The diagrams consist of vertical lines (blue and red) and crossings, with labels u, u', u'', u''' indicating the states at the top and bottom.

$$D_3^{u,u'} * (e_3^{u'})^{-1} * D_2^{u',u''} * D_3^{u'',u''} = D_3^{u,u'}$$

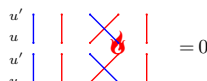


Diagram illustrating a relation between compositions of operators D . The left side shows a sequence of two diagrams: $D_3^{u,u'}$ and $D_3^{u,u'}$. The right side is zero. The diagrams consist of vertical lines (blue and red) and crossings, with labels u, u', u'' indicating the states at the top and bottom.

$$D_3^{u,u'} * D_3^{u,u'} = 0$$

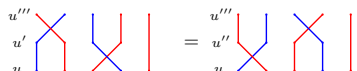


Diagram illustrating a relation between compositions of operators D and e . The left side shows a sequence of two diagrams: $D_3^{u,u'}$ and $D_1^{u',u''}$. The right side shows a sequence of two diagrams: $D_1^{u,u''}$ and $D_3^{u'',u''}$. The diagrams consist of vertical lines (blue and red) and crossings, with labels u, u', u'', u''' indicating the states at the top and bottom.

$$D_3^{u,u'} * D_1^{u',u''} = D_1^{u,u''} * D_3^{u'',u''}$$

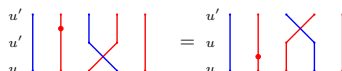


Diagram illustrating a relation between compositions of operators D and e . The left side shows a sequence of two diagrams: $D_3^{u,u'}$ and $e_2^{u'}$. The right side shows a sequence of two diagrams: e_2^u and $D_3^{u,u'}$. The diagrams consist of vertical lines (blue and red) and crossings, with labels u, u', u'' indicating the states at the top and bottom.

$$D_3^{u,u'} * e_2^{u'} = e_2^u * D_3^{u,u'}$$

Nontrivial relations I

Same color: $(D_i^{u,u} \hat{=} D_i, e_i^u \hat{=} e_i)$

$$\begin{array}{c} \text{Diagram: } \begin{array}{|c|} \hline \diagup \bullet \diagdown \\ \hline \end{array} = \begin{array}{|c|} \hline \diagdown \bullet \diagup \\ \hline \end{array} - \begin{array}{|c|} \hline | \bullet | \\ \hline \end{array} \\ D_i e_i = e_{i+1} D_i - e_{i+1} \end{array}$$

$$\begin{array}{c} \text{Diagram: } \begin{array}{|c|} \hline \diagdown \bullet \diagup \\ \hline \end{array} = \begin{array}{|c|} \hline \diagup \bullet \diagdown \\ \hline \end{array} + \begin{array}{|c|} \hline | \bullet | \\ \hline \end{array} \\ D_i e_{i+1} = e_i D_i + e_{i+1} \end{array}$$

$$\begin{array}{c} \text{Diagram: } \begin{array}{|c|} \hline \diagup \circ \diagdown \\ \hline \end{array} = \begin{array}{|c|} \hline \diagdown \circ \diagup \\ \hline \end{array} + \begin{array}{|c|} \hline | \circ | \\ \hline \end{array} \\ D_i e_i^{-1} = e_{i+1}^{-1} D_i + e_i^{-1} \end{array}$$

$$\begin{array}{c} \text{Diagram: } \begin{array}{|c|} \hline \diagdown \circ \diagup \\ \hline \end{array} = \begin{array}{|c|} \hline \diagup \circ \diagdown \\ \hline \end{array} - \begin{array}{|c|} \hline | \circ | \\ \hline \end{array} \\ D_i e_{i+1}^{-1} = e_i^{-1} D_i - e_{i+1}^{-1} \end{array}$$

$$\begin{array}{c} \text{Diagram: } \begin{array}{|c|} \hline \diagup \diagdown \diagup \diagdown \\ \hline \end{array} = \begin{array}{|c|} \hline \diagdown \diagup \diagdown \diagup \\ \hline \end{array} \\ D_i D_{i+1} D_i = D_{i+1} D_i D_{i+1} \end{array}$$

$$\begin{array}{c} \text{Diagram: } \begin{array}{|c|} \hline \diagup \diagdown \diagup \diagdown \\ \hline \end{array} = \begin{array}{|c|} \hline \diagup \diagdown \\ \hline \end{array} \\ D_i^2 = D_i \end{array}$$

Nontrivial relations II

Different color:

Diagrammatic equation: A crossing of a blue line (top-left to bottom-right) and a red line (top-right to bottom-left). The blue line has a solid dot at the crossing. This is equal to the same crossing, but the dot is on the red line. Both diagrams are labeled with a circled 3.

$$D_i^{u,u'} e_i^{u'} = e_{i+1}^u D_i^{u,u'}$$

Diagrammatic equation: A crossing of a blue line (top-left to bottom-right) and a red line (top-right to bottom-left). The blue line has a solid dot at the crossing. This is equal to the same crossing, but the dot is on the red line. Both diagrams are labeled with a circled 2 and a circled 3.

$$D_i^{u,u'} D_i^{u',u} = 1^u - \left(\frac{e_i}{e_{i+1}}\right)^u$$

Diagrammatic equation: A crossing of a blue line (top-left to bottom-right) and a red line (top-right to bottom-left). The red line has a solid dot at the crossing. This is equal to the same crossing, but the dot is on the blue line. Both diagrams are labeled with a circled 3.

$$D_i^{u,u'} (e_i^{u'})^{-1} = (e_{i+1}^u)^{-1} D_i^{u,u'}$$

Diagrammatic equation: A crossing of a blue line (top-left to bottom-right) and a red line (top-right to bottom-left). The red line has a solid dot at the crossing. This is equal to the same crossing, but the dot is on the blue line. Both diagrams are labeled with a circled 3 and a circled 2.

$$D_i^{u,u'} D_i^{u',u} = 1^u - \left(\frac{e_{i+1}}{e_i}\right)^u$$

Nontrivial relations III

Different color:

$$\begin{array}{c} u \\ u' \\ u' \\ u \end{array} \begin{array}{c} \text{Diagram 1} \end{array} \begin{array}{c} \textcircled{2} \\ \textcircled{1} \\ \textcircled{3} \end{array} = \begin{array}{c} u \\ u'' \\ u'' \\ u \end{array} \begin{array}{c} \text{Diagram 2} \end{array} \begin{array}{c} \textcircled{3} \\ \textcircled{1} \\ \textcircled{2} \end{array} + \begin{array}{c} u \\ u \\ u \end{array} \begin{array}{c} \text{Diagram 3} \end{array}$$

$$D_i^{u,u'} D_{i+1}^{u',u'} D_i^{u',u} = D_{i+1}^{u,u''} D_i^{u'',u''} D_{i+1}^{u'',u} + \left(\frac{e_{i+2}}{e_{i+1}} \right)^u$$

$$\begin{array}{c} u \\ u' \\ u' \\ u \end{array} \begin{array}{c} \text{Diagram 4} \end{array} \begin{array}{c} \textcircled{3} \\ \textcircled{1} \\ \textcircled{2} \end{array} = \begin{array}{c} u \\ u'' \\ u'' \\ u \end{array} \begin{array}{c} \text{Diagram 5} \end{array} \begin{array}{c} \textcircled{2} \\ \textcircled{1} \\ \textcircled{3} \end{array} - \begin{array}{c} u \\ u \\ u \end{array} \begin{array}{c} \text{Diagram 6} \end{array}$$

$$D_i^{u,u'} D_{i+1}^{u',u'} D_i^{u',u} = D_{i+1}^{u,u''} D_i^{u'',u''} D_{i+1}^{u'',u} - \left(\frac{e_{i+1}}{e_i} \right)^u$$

$$\begin{array}{c} u'' \\ u'' \\ u' \\ u \end{array} \begin{array}{c} \text{Diagram 7} \end{array} \begin{array}{c} \textcircled{1} \\ \textcircled{2} \\ \textcircled{2} \\ \textcircled{2} \end{array} = \begin{array}{c} u'' \\ u' \\ u \\ u \end{array} \begin{array}{c} \text{Diagram 8} \end{array} \begin{array}{c} \textcircled{2} \\ \textcircled{2} \\ \textcircled{2} \\ \textcircled{1} \end{array}$$

$$D_i^{u,u'} D_{i+1}^{u',u''} D_i^{u'',u''} = D_{i+1}^{u,u} D_i^{u,u'} D_{i+1}^{u',u''}$$

Affine pavings

Theorem

For a Dynkin quiver Q and $M \in \text{Rep}(Q)$, the **partial flag variety** of length d

$$\text{Flag}_d(X) := \{F: 0 \subseteq N_1 \subseteq \cdots \subseteq N_d \subseteq M \mid N_i \in \text{Rep}(Q)\}$$

has an affine paving.

Roughly speaking, we decompose $\text{Flag}_d(M)$ into several pieces, and each piece is an affine space.

Idea of affine pavings

Find a nice short exact sequence

$$0 \longrightarrow X \xrightarrow{\iota} M \xrightarrow{\pi} S \longrightarrow 0$$

which induces a nice morphism

$$\begin{aligned} \Psi : \operatorname{Flag}_d(M) &\longrightarrow \operatorname{Flag}_d(X) \times \operatorname{Flag}_d(S) \\ F &\longmapsto (\iota^{-1}(F), \pi(F)) \end{aligned}$$

We construct the affine paving of $\operatorname{Flag}_d(M)$ from the affine paving of $\operatorname{Flag}_d(X)$ and $\operatorname{Flag}_d(S)$. Then, we use mathematical induction.

Further discussion

- Discuss the representation theory of $K^{G_d}(\mathcal{Z}_d)$, and connect that with the geometry of $\text{Flag}_d(M)$;
- Generalize the result of first part to partial flag varieties (by introducing merge and split);
- Understand Kazhdan-Lusztig isomorphism and its categorifications;
- Understand “KLR-algebra categorifies quantum groups”.