

Affine pavings of partial flag varieties

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Process

- 1 Setting and Statement
- 2 Case study
- 3 Auslander–Reiten theory
- 4 Tackle the type E case

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Affine paving

Setting

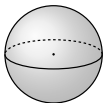
$K = \mathbb{C}$, X : algebraic variety over K .

Definition

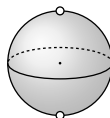
An **affine paving** of X is a filtration

$$0 = X_0 \subset X_1 \subset \cdots \subset X_d = X$$

with X_i closed and $X_{i+1} \setminus X_i \cong \mathbb{A}_K^k$.



$$\mathbb{P}^1 = \{\infty\} \sqcup \mathbb{A}^1$$



$\mathbb{P}^1 \setminus \{0, \infty\}$ has no affine paving

Quiver and quiver representation

Quiver is a graph. It has some vertices & arrows.
In this talk, all the quivers are finite and connected.

Quiver and quiver representation

We focus on the Dynkin quiver.

That means, the graph of the Dynkin diagrams in the ADE series.

Partial flag variety

Definition

Fix a quiver Q and $M \in \text{rep}(Q)$,

$$\text{Flag}_d(M) := \{F: 0 \subseteq N_1 \subseteq \cdots \subseteq N_d \subseteq M\}$$

$$\text{Flag}_{\underline{\mathbf{f}}}(M) := \{F: 0 \subseteq N_1 \subseteq \cdots \subseteq N_d \subseteq M \mid \underline{\dim} M_i = \underline{\mathbf{f}}_i\}$$

Example

$$Q = \bullet, M = \mathbb{C}^n, \underline{\mathbf{f}} := \begin{pmatrix} n \\ \vdots \\ 1 \end{pmatrix}$$

$$\text{Flag}_d(\mathbb{C}^n) = \{F: 0 \subseteq N_1 \subseteq \cdots \subseteq N_d \subseteq \mathbb{C}^n\}$$

$$\text{Flag}_1(\mathbb{C}^n) = \{F: 0 \subseteq N_1 \subseteq \mathbb{C}^n\} = \sqcup_{k=0}^n \text{Gr}(n, k)$$

$$\text{Flag}_{\underline{\mathbf{f}}}(\mathbb{C}^n) = \text{complete flags of } \mathbb{C}^n$$

$$\text{Flag}_{(k)}(\mathbb{C}^n) = \text{Gr}(n, k)$$

Statement

Theorem

For a Dynkin quiver Q and $M \in \text{rep}(Q)$,

$\text{Flag}_d(M)$ has an affine paving.

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Task 1. $Q = \bullet$, $M = \mathbb{C}^n$

In this case,

$$\begin{array}{ccc} \mathrm{GL}_n(\mathbb{C}) \curvearrowright \mathbb{C}^n & \rightsquigarrow & \mathrm{GL}_n(\mathbb{C}) \curvearrowright \mathrm{Flag}_d(\mathbb{C}^n) \\ & \rightsquigarrow & B \curvearrowright \mathrm{Flag}_d(\mathbb{C}^n) \end{array}$$

$\mathrm{Flag}_d(\mathbb{C}^n)$ has an affine paving given by Schubert cells (i.e., B -orbits).

Note

When $Q = \bullet \longrightarrow \bullet$, $\mathrm{Flag}_{\mathbf{f}}(M)$ have no natural group actions.

Task 2a. $Q = \bullet \rightarrow \bullet$, $M = \left[\mathbb{C}^2 \xrightarrow{\text{Id}} \mathbb{C}^2 \right]$, $d = 1$

$$\underline{\mathbf{f}} = (1, 0) : \quad \text{Flag}_{\underline{\mathbf{f}}}(M) = \emptyset$$

$$\underline{\mathbf{f}} = (0, 0) : \quad \text{Flag}_{\underline{\mathbf{f}}}(M) = \{*\}$$

$$\underline{\mathbf{f}} = (1, 1) : \quad \text{Flag}_{\underline{\mathbf{f}}}(M) = \mathbb{P}^1$$

In this case, $\text{Flag}_{\underline{\mathbf{f}}}(M)$ is Grassmannian or empty, so it has an affine paving.

Task 2b. $Q = \bullet \rightarrow \bullet$, $M = \left[\mathbb{C}^2 \xrightarrow{0} \mathbb{C}^2 \right]$, $d = 1$

$$\underline{\mathbf{f}} = (1, 0) : \quad \text{Flag}_{\underline{\mathbf{f}}}(M) = \mathbb{P}^1$$

$$\underline{\mathbf{f}} = (0, 0) : \quad \text{Flag}_{\underline{\mathbf{f}}}(M) = \{*\}$$

$$\underline{\mathbf{f}} = (1, 1) : \quad \text{Flag}_{\underline{\mathbf{f}}}(M) = \mathbb{P}^1 \times \mathbb{P}^1$$

In this case, $\text{Flag}_{\underline{\mathbf{f}}}(M) \cong \text{Flag}_{\underline{\mathbf{f}}_1}(M) \times \text{Flag}_{\underline{\mathbf{f}}_2}(M)$ has an affine paving.

Task 2c. $Q = \bullet \rightarrow \bullet$, $M = \left[\mathbb{C}^2 \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} \mathbb{C}^2 \right]$, $d = 1$

$$\underline{\mathbf{f}} = (1, 0) : \quad \text{Flag}_{\underline{\mathbf{f}}}(M) = \{*\}$$

$$\underline{\mathbf{f}} = (0, 0) : \quad \text{Flag}_{\underline{\mathbf{f}}}(M) = \{*\}$$

$$\underline{\mathbf{f}} = (1, 1) : \quad \text{Flag}_{\underline{\mathbf{f}}}(M) = \mathbb{P}^1 \vee \mathbb{P}^1$$

$$\underline{\mathbf{f}} = (0, 1) : \quad \dots$$

To construct affine pavings systematically, we need to construct an uniform method.

Task 2c. $Q = \bullet \rightarrow \bullet$, $M = \left[\mathbb{C}^2 \xrightarrow{\begin{pmatrix} 10 \\ 00 \end{pmatrix}} \mathbb{C}^2 \right]$, $d = 1$

First try

Let $X = [0 \rightarrow \mathbb{C}]$, $S = \left[\mathbb{C}^2 \xrightarrow{\begin{pmatrix} 10 \\ 00 \end{pmatrix}} \mathbb{C} \right]$, then $M = X \oplus S$, and the short exact sequence

$$0 \longrightarrow X \xrightarrow{\iota} M \xrightarrow{\pi} S \longrightarrow 0$$

induces

$$\begin{aligned} \Psi : \operatorname{Flag}_d(M) &\longrightarrow \operatorname{Flag}_d(X) \times \operatorname{Flag}_d(S) \\ F &\longmapsto (\iota^{-1}(F), \pi(F)) \end{aligned}$$

Idea of affine pavings

Find a nice short exact sequence

$$0 \longrightarrow X \xrightarrow{\iota} M \xrightarrow{\pi} S \longrightarrow 0$$

which induces a nice morphism

$$\begin{aligned} \Psi : \operatorname{Flag}_d(M) &\longrightarrow \operatorname{Flag}_d(X) \times \operatorname{Flag}_d(S) \\ F &\longmapsto (\iota^{-1}(F), \pi(F)) \end{aligned}$$

We construct the affine paving of $\operatorname{Flag}_d(M)$ from the affine paving of $\operatorname{Flag}_d(X)$ and $\operatorname{Flag}_d(S)$. Then, we use mathematical induction.

Example. $Q = \bullet$, $M = \mathbb{C}^2$

$$0 \longrightarrow \mathbb{C} \xrightarrow{\iota} \mathbb{C}^2 \xrightarrow{\pi} \mathbb{C} \longrightarrow 0$$

$$\Psi_1 : \text{Flag}_1(\mathbb{C}^2) \longrightarrow \text{Flag}_1(\mathbb{C}) \times \text{Flag}_1(\mathbb{C})$$

$$\begin{array}{ccccc} \Psi_{(1)} : \text{Flag}_{(1)}(\mathbb{C}) & \longrightarrow & \text{Flag}_{(1)}(\mathbb{C}) \times \text{Flag}_{(0)}(\mathbb{C}) & \sqcup & \text{Flag}_{(0)}(\mathbb{C}) \times \text{Flag}_{(1)}(\mathbb{C}) \\ \mathbb{P}^1 & \longrightarrow & \{*\} & \sqcup & \{*\} \end{array}$$

Question

How does $\Psi_{(1)}$ give an affine paving of $\text{Flag}_{(1)}(\mathbb{C})$?

$$\begin{array}{c} \mathbb{P}^1 = \{\infty\} \sqcup \mathbb{C} \\ \downarrow \Psi_{(1)} \\ \{*\} \sqcup \{*\} \end{array}$$

Example. $Q = \bullet$, $M = \mathbb{C}^8 = \bigoplus_{i=1}^8 \mathbb{C}v_i$

$$0 \longrightarrow \mathbb{C}^3 \xrightarrow{\iota} \mathbb{C}^8 \xrightarrow{\pi} \mathbb{C}^5 \longrightarrow 0$$

$$\begin{aligned} \Psi^{-1}\left(\langle v_1 \rangle, \langle v_4, v_5 \rangle\right) &= \left\{ \langle v_1, v_4 + av_2 + bv_3, v_5 + cv_2 + dv_3 \rangle \mid a, b, c, d \in \mathbb{C} \right\} \\ &\cong \mathbb{C}^4 \end{aligned}$$

In general,

$$\mathrm{Flag}_{(3)}(\mathbb{C}^8) \dashrightarrow \mathrm{Flag}_{(1)}(\mathbb{C}^3) \times \mathrm{Flag}_{(2)}(\mathbb{C}^5)$$

is a Zarisky-locally trivial affine bundle of rank $2 \cdot (3 - 1) = 4$.

Consider the short exact sequence of representations

$$\eta : 0 \longrightarrow X \xrightarrow{\iota} Y \xrightarrow{\pi} S \longrightarrow 0$$

which induce maps

$$\begin{aligned} \Psi : \text{Flag}_d(Y) &\longrightarrow \text{Flag}_d(X) \times \text{Flag}_d(S) \\ \Psi_{\underline{f}, \underline{g}} : \text{Flag}(Y)_{\underline{f}, \underline{g}} &\longrightarrow \text{Flag}_{\underline{f}}(X) \times \text{Flag}_{\underline{g}}(S) \end{aligned}$$

Theorem A

When η splits, then Ψ is surjective.

Moreover, if $\text{Ext}^1(S, X) = 0$, then

$\Psi_{\underline{f}, \underline{g}}$ is a Zarisky-locally trivial affine bundle.

By this theorem,

$\text{Flag}_d(Y)$ has an affine paving $\Leftarrow \text{Flag}_d(X), \text{Flag}_d(S)$ have.

Warming

η splits and $\text{Ext}^1(S, X) = 0$ are necessary for Theorem A.

Example

Consider the quiver $Q : \bullet \rightarrow \bullet \leftarrow \bullet$ and the short exact sequence

$$0 \longrightarrow [\mathbb{C}e_1 \rightarrow \mathbb{C}^2 \leftarrow \mathbb{C}e_2] \longrightarrow [\mathbb{C}^2 \xrightarrow{\text{Id}} \mathbb{C}^2 \xleftarrow{\text{Id}} \mathbb{C}^2] \longrightarrow [\mathbb{C}e_2 \rightarrow 0 \leftarrow \mathbb{C}e_1] \longrightarrow 0$$

we get

$$\text{Im } \Psi_{(0,1,0),(1,0,1)} \cong (\mathbb{P}^1 \setminus \{0, \infty\}) \times \{*\} \cong \mathbb{C}^*,$$

so Ψ is not surjective.

In this way, we get a bad stratification

$$\text{Flag}_{(1,1,1)}(Y) \cong \mathbb{P}^1 = \{0\} \sqcup \{\infty\} \sqcup \mathbb{C}^\times.$$

Task 3. $Q = \begin{array}{c} \bullet \\ \downarrow \\ \bullet \rightarrow \bullet \leftarrow \bullet \end{array}, M = \begin{smallmatrix} 1 \\ 121 \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 111 \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 111 \end{smallmatrix}$

We use following short exact sequences

$$0 \longrightarrow \begin{smallmatrix} 1 \\ 111 \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 111 \end{smallmatrix} \longrightarrow M \longrightarrow \begin{smallmatrix} 1 \\ 121 \end{smallmatrix} \longrightarrow 0$$

$$0 \longrightarrow \begin{smallmatrix} 1 \\ 111 \end{smallmatrix} \longrightarrow \begin{smallmatrix} 1 \\ 111 \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 111 \end{smallmatrix} \longrightarrow \begin{smallmatrix} 1 \\ 111 \end{smallmatrix} \longrightarrow 0$$

to reduced the problem to indecomposable representations.

Notice that we use the result

$$\mathrm{Ext}^1\left(\begin{smallmatrix} 1 \\ 121 \end{smallmatrix}, \begin{smallmatrix} 1 \\ 111 \end{smallmatrix}\right) = 0, \quad \mathrm{Ext}^1\left(\begin{smallmatrix} 1 \\ 111 \end{smallmatrix}, \begin{smallmatrix} 1 \\ 111 \end{smallmatrix}\right) = 0.$$

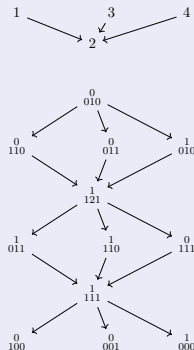
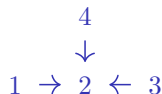
$\mathrm{Flag}_d\left(\begin{smallmatrix} 1 \\ 111 \end{smallmatrix}\right)$ has an affine paving: obvious.

$\mathrm{Flag}_d\left(\begin{smallmatrix} 1 \\ 121 \end{smallmatrix}\right)$ has an affine paving: it is \mathbb{P}^1 , $\{*\}$ or empty.

Need: more informations of indecomposable representations!

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Auslander–Reiten quiver: D_4 

For other examples, see [here](#).

First application

Corollary

For a Dynkin quiver, we can give an total order to the set of all indecomposable representations, such that

$$i \leq j \implies \operatorname{Ext}^1(M_i, M_j) = 0.$$

By Theorem A, the problem reduced to

For a Dynkin quiver Q and $M \in \operatorname{ind}(Q)$,

$\operatorname{Flag}_d(M)$ has an affine paving.

Indecomposable representations of low order are easy!

Lemma

Suppose Q is a tree. For $M \in \text{ind}(Q)$, $\text{ord}(M) \leq 2$,

$$\text{Flag}_{\underline{\mathbf{f}}}(M) \cong \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \quad \text{or} \quad \emptyset.$$

Example

$$Q = \begin{array}{ccccc} & & \bullet & & \\ & & \downarrow & & \\ \bullet & \rightarrow & \bullet & \leftarrow & \bullet \rightarrow \bullet \end{array} \quad M = \begin{array}{ccccccc} & & & & \mathbb{C} & & \\ & & & & \downarrow & & \\ \mathbb{C} & \hookrightarrow & \mathbb{C}^2 & \xleftarrow{\sim} & \mathbb{C}^2 & \twoheadrightarrow & \mathbb{C} \end{array} \quad \underline{\mathbf{f}} = \begin{pmatrix} 0 & & & \\ 12 & 11 & & \\ & 0 & & \\ 11 & 0 & 1 & \end{pmatrix}$$

$$\begin{aligned} \text{Flag}_{\underline{\mathbf{f}}}(M) &\hookrightarrow \text{Flag}_{\binom{1}{1}}(\mathbb{C}) \times \text{Flag}_{\binom{2}{1}}(\mathbb{C}^2) \times \text{Flag}_{\binom{1}{0}}(\mathbb{C}^2) \\ &\quad \times \text{Flag}_{\binom{1}{1}}(\mathbb{C}) \times \text{Flag}_{\binom{0}{0}}(\mathbb{C}) \\ &\cong \mathbb{P}^1 \times \mathbb{P}^1 \end{aligned}$$

Continue

	$\mathbb{C} \hookrightarrow \mathbb{C}^2$		$\mathbb{C}^2 \twoheadrightarrow \mathbb{C}$		$\mathbb{C}^2 \rightarrow \mathbb{C}^2$	
No restriction	—	2	0	—	—	1
	0	—	1	2	0	0
Reduce	1	1	1	1	1	0
Impossible	2	1	1	0	2	0
	2	0				
	1	0				

Corollary

The main theorem is true for quivers of type A , D .

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Theorem B

When η does not split and generates $\text{Ext}^1(S, X)$,

$\Psi_{\underline{\mathbf{f}}, \underline{\mathbf{g}}}$ is a Zarisky-locally trivial affine bundle over $\text{Im } \Psi_{\underline{\mathbf{f}}, \underline{\mathbf{g}}}$.

In this case, we have a clear description of $\text{Im } \Psi_{\underline{\mathbf{f}}, \underline{\mathbf{g}}}$.

Statement

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How to find nice η ?

Proposition

For $X \hookrightarrow Y$ minimal sectional mono, the induced SES

$$\eta : 0 \longrightarrow X \xrightarrow{\iota} Y \xrightarrow{\pi} S \longrightarrow 0$$

satisfies the condition of Theorem B. Moreover,

$$\operatorname{Im} \Psi_{\underline{\mathbf{f}}, \underline{\mathbf{g}}} = \begin{cases} (\operatorname{Flag}_{\underline{\mathbf{f}}}(X) \setminus \operatorname{Flag}_{\underline{\mathbf{f}}}(X_S)) \times \operatorname{Flag}_{\underline{\mathbf{g}}}(S), & \underline{\mathbf{g}}_i = \underline{\dim} S \\ \operatorname{Flag}_{\underline{\mathbf{f}}}(X) \times \operatorname{Flag}_{\underline{\mathbf{g}}}(S), & \text{otherwise} \end{cases}$$

where

$$X_S := \max \{ M \subseteq X \mid \operatorname{Ext}^1(S, X/M) \cong \mathbb{C} \} \subseteq X.$$

How to find nice η ?

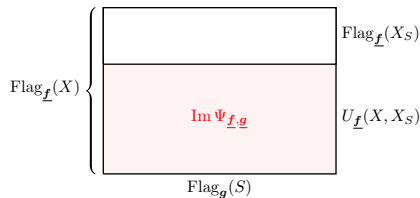
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minimal sectional mono case

How to find nice η ?

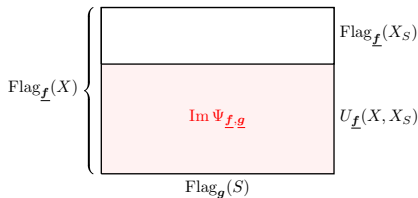
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$$\operatorname{Im} \Psi_{\underline{f}, \underline{g}} = \begin{cases} U_{\underline{f}}(X, X_S), & \underline{g}_i = \underline{\dim} S \\ \operatorname{Flag}_{\underline{f}}(X) \times \operatorname{Flag}_{\underline{g}}(S), & \text{otherwise} \end{cases}$$



minimal sectional mono case

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Proposition

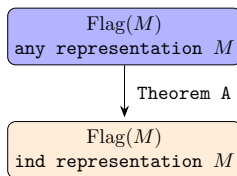
In addition, if $X \hookrightarrow Y$ is irreducible mono, then

$$X_S = 0 \quad \text{or} \quad X_S \hookrightarrow X \text{ is irreducible mono.}$$

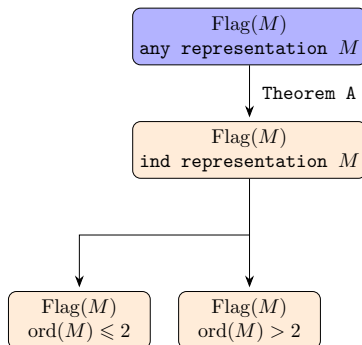
Corollary

For $M \in \mathrm{ind}(Q)$, if exist irreducible mono $X \hookrightarrow M$, then $\mathrm{Flag}_d(M)$ has an affine paving.

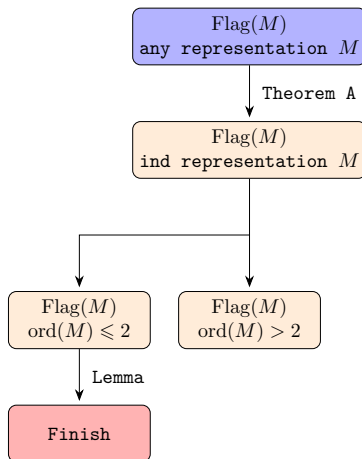
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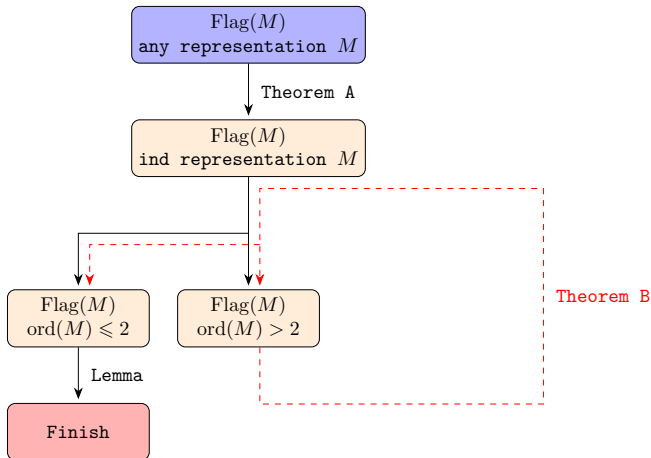
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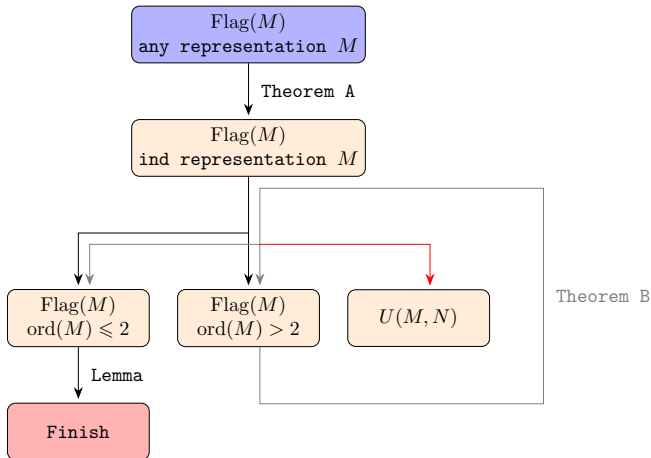
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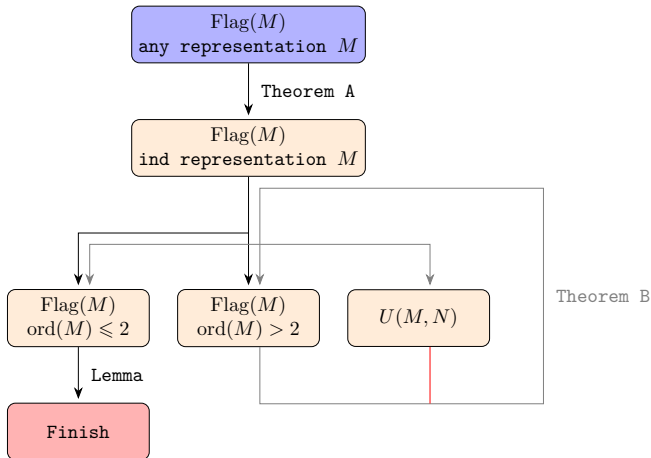
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Process



Process



What is remaining?

$$E_6 : \begin{array}{cccccccccc} & & 1 & & 1 & & 1 & & 1 & & \\ & 1 & & 2 & & 2 & & 2 & & 1 & \\ 1 & 1 & 2 & 1 & 3 & 2 & 3 & 1 & 2 & 1 & 1 \\ & 1 & & 2 & & 2 & & 2 & & 1 & \\ & & 1 & & 1 & & 1 & & 1 & & \end{array}$$

$$E_7 : \begin{array}{cccccccccccccc} & & 1 & & 1 & & 1 & & 2 & & 1 & & 1 & & 1 & \\ & 1 & & 2 & & 2 & & 3 & & 3 & & 2 & & 2 & & 1 & \\ 1 & 1 & 2 & 1 & 3 & 2 & 4 & 2 & 4 & 2 & 4 & 2 & 3 & 1 & 2 & 1 & 1 \\ & 1 & & 2 & & 3 & & 3 & & 3 & & 3 & & 2 & & 1 & \\ & & 1 & & 2 & & 2 & & 2 & & 2 & & 2 & & 1 & \\ & & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \end{array}$$

$$E_8 : \begin{array}{cccccccccccccccccccc} & & & & 1 & & 1 & & 1 & & 2 & & 2 & & 2 & & 2 & & 2 & & 2 & & 2 & & 1 & & 1 & & 1 & \\ & & 1 & & 2 & & 2 & & 3 & & 4 & & 4 & & 4 & & 4 & & 4 & & 4 & & 3 & & 2 & & 2 & & 1 & \\ 1 & 1 & 2 & 1 & 3 & 2 & 4 & 2 & 5 & 3 & 6 & 3 & 6 & 3 & 6 & 3 & 6 & 3 & 6 & 3 & 5 & 2 & 4 & 2 & 3 & 1 & 2 & 1 & 1 & \\ & 1 & & 2 & & 3 & & 4 & & 4 & & 5 & & 5 & & 5 & & 5 & & 4 & & 4 & & 3 & & 2 & & 1 & \\ & & 1 & & 2 & & 3 & & 3 & & 3 & & 4 & & 4 & & 4 & & 3 & & 3 & & 3 & & 2 & & 1 & \\ & & & 1 & & 2 & & 2 & & 2 & & 2 & & 3 & & 3 & & 2 & & 2 & & 2 & & 2 & & 2 & & 1 & \\ & & & & 1 & & 1 & & 1 & & 1 & & 1 & & 2 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \end{array}$$

What is remaining?

$$E_6 :$$

		1		1		1		1		
	1		2		2		2		1	
1	1	2	1	3	2	3	1	2	1	1
	1		2		2		2		1	
		1		1		1		1		

$$E_7 :$$

		1		1		1		2		1		1		1
	1		2		2		3		3		2		2	1
1	1	2	1	3	2	4	2	4	2	4	2	3	1	2
	1		2		3		3		3		3		2	1
		1		2		2		2		2		2		1
			1		1		1		1		1		1	

$$E_8 :$$

			1		1		1		2		2		2		2		2		2		2		1		1		1	
		1		2		2		3		4		4		4		4		4		4		4		3		2		2
1	1	2	1	3	2	4	2	5	3	6	3	6	3	6	3	6	3	6	3	5	2	4	2	3	1	2	1	1
		1		2		3		4		4		5		5		5		5		4		4		3		2		1
			1		2		3		3		3		4		4		4		3		3		3		2		1	
				1		2		2		2		2		3		3		2		2		2		2		2		1
					1		1		1		1		1		2		1		1		1		1		1		1	

What is remaining?

$$E_6 : \begin{array}{ccccccccc} & & 1 & & 1 & & 1 & & 1 \\ & 1 & & 2 & & 2 & & 2 & & 1 \\ 1 & 1 & 2 & 1 \rightarrow 3 & 2 \rightarrow 3 & 1 & 2 & 1 & 1 \\ & 1 & & 2 & & 2 & & 2 & & 1 \\ & & 1 & & 1 & & 1 & & 1 \end{array}$$

$$E_7 : \begin{array}{ccccccccccccccc} & & 1 & & 1 & & 1 & \searrow & 2 & \searrow & 1 & & 1 & & 1 \\ & 1 & & 2 & & 2 & & 3 & \searrow & 3 & & 2 & & 2 & & 1 \\ 1 & 1 & 2 & 1 \rightarrow 3 & 2 \rightarrow 4 & 2 \rightarrow 4 & 2 \rightarrow 4 & 2 \rightarrow 3 & 1 & 2 & 1 & 1 \\ & 1 & & 2 & & 3 & \searrow & 3 & \searrow & 3 & \searrow & 3 & & 2 & & 1 \\ & & 1 & & 2 & \searrow & 2 & \searrow & 2 & \searrow & 2 & \searrow & 2 & & 1 \\ & & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 \end{array}$$

$$E_8 : \begin{array}{ccccccccccccccccccccccccccc} & & & & 1 & & 1 & & 2 & & 2 & & 2 & & 2 & & 2 & & 2 & & 2 & & 1 & & 1 & & 1 \\ & 1 & & 2 & & 2 & & 3 & \searrow & 4 & \searrow & 4 & \searrow & 4 & \searrow & 4 & \searrow & 4 & \searrow & 4 & \searrow & 3 & & 2 & & 2 & & 1 \\ 1 & 1 & 2 & 1 \rightarrow 3 & 2 \rightarrow 4 & 2 \rightarrow 5 & 3 \rightarrow 6 & 3 \rightarrow 6 & 3 \rightarrow 6 & 3 \rightarrow 6 & 3 \rightarrow 6 & 3 \rightarrow 5 & 2 \rightarrow 4 & 2 \rightarrow 3 & 1 & 2 & 1 & 1 \\ & 1 & & 2 & & 3 & \searrow & 4 & \searrow & 4 & \searrow & 4 & \searrow & 4 & \searrow & 3 & & 2 & & 1 \\ & & 1 & & 2 & \searrow & 3 & \searrow & 3 & \searrow & 3 & \searrow & 3 & \searrow & 3 & \searrow & 3 & & 2 & & 1 \\ & & & 1 & & 2 & \searrow & 2 & \searrow & 2 & \searrow & 2 & \searrow & 2 & \searrow & 2 & \searrow & 2 & & 1 \\ & & & & 1 & & 1 & & 1 & & 1 & & 2 & \searrow & 3 & \searrow & 3 & & 1 & & 1 & & 1 & & 1 & & 1 \end{array}$$