Master thesis

Contents

Warning 0.0.1. *I made some assumptions during the writing. To avoid confusing readers, these assumptions are listed here:*

- *For quivers, all the quivers we considered (except Auslander–Reiten quivers) are connected and finite (Remark 1.2.2). For simplicity, From ??? to ???, all the quivers have no loops or cycles.*
- *For any $\varpi \in \mathbb{W}_{|\mathbf{d}|}$, we always write $\varpi = wu$, where $w \in W_{\mathbf{d}}$ and u is the shortest element in the coset $W_{\mathbf{d}}\varpi$. The flag-type dimension vector $\underline{\mathbf{d}} \in W_{\mathbf{d}} \backslash \mathbb{W}_{|\mathbf{d}|}$ corresponds to u, i.e., $\underline{\mathbf{d}} = W_{\mathbf{d}} u$. Whenever $\tilde{w}$ and $\tilde{u}$ emerge, they are always defined by $uw'u' = \tilde{w}\tilde{u}$.*
- *For the diagram, we always read from top to bottom.*

Chapter 1

Variety and stratification

1.1 Initial case: $\mathcal{F}$ and $\mathcal{F} \times \mathcal{F}$

We introduce the complete flag variety to give a bird's eye view on the whole section. Actually, the entire difficulty is bundled in this example.

Setting 1.1.1. *Fix $n \geqslant 1$, we denote $\mathrm{GL}_n := \mathrm{GL}_n(\mathbb{C})$, B, T, N, W be the standard Borel subgroup, standard torus, unipotent subgroup and Weyl group, respectively, i.e.,*

$$\mathrm{GL}_n = \begin{pmatrix} * & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & * \end{pmatrix} \quad B = \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix} \quad T = \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \quad N = \begin{pmatrix} 1 & \cdots & * \\ & \ddots & \vdots \\ 0 & & 1 \end{pmatrix}$$

$$W := N_{\mathrm{GL}_n}(T)/T \cong S_n$$

1.1.1 $\mathcal{F}$

Definition 1.1.2 (Flag)**.** *For a finite dimensional $\mathbb{C}$-vector space V, a flag of V is an increasing sequence of subspaces of V:*

$$F : 0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_k = V.$$

F is called a complete flag if $\dim M_j = j$ for all j, otherwise F is called a partial flag.

Definition 1.1.3 (Complete flag variety)**.** *The complete flag variety $\mathcal{F}$ is defined as*

$$\begin{aligned} \mathcal{F} &:= \mathrm{GL}_n / B \\ &\cong \{\textit{complete flags of } \mathbb{C}^n\} \\ &= \{0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n = \mathbb{C}^n \mid \dim M_j = j\} \\ &\cong \{\textit{Borel subgroups of } \mathrm{GL}_n\} \\ &= \left\{gBg^{-1} \;\middle|\; g \in \mathrm{GL}_n\right\} \end{aligned}$$

Remark 1.1.4.

1. $\mathcal{F}$ is a smooth projective variety of dimension $\frac{n(n+1)}{2}$, which can be seen from the embedding
$$\mathcal{F} \hookrightarrow \mathrm{Gr}(1,n) \times \cdots \times \mathrm{Gr}(n-1,n)$$

2. We implicitly give the base point of $\mathcal{F}$, which is not considered as the data of $\mathcal{F}$. Fix a standard basis of $\mathbb{C}^n$ by $\{v_1, \ldots, v_n\}$, we define the standard flag
$$F_{\mathrm{Id}} : 0 \subseteq \langle v_1 \rangle \subseteq \langle v_1, v_2 \rangle \subseteq \cdots \subseteq \langle v_1, \ldots, v_n \rangle = \mathbb{C}^n.$$

3. We have the natural GL_n-action on $\mathcal{F}$, which is considered as the data of $\mathcal{F}$.

   For $g \in \mathrm{GL}_n$, we define the flag attached to g:
$$F_g \triangleq gF_{\mathrm{Id}} : 0 \subseteq \langle gv_1 \rangle \subseteq \langle gv_1, gv_2 \rangle \subseteq \cdots \subseteq \langle gv_1, \ldots, gv_n \rangle = \mathbb{C}^n.$$
   Especially, for $w \in W = N_{\mathrm{GL}_n}(T)/T \cong S_n$, the flag attached to w
$$\begin{aligned} F_w : 0 &\subseteq \langle \tilde{w}v_1 \rangle \subseteq \langle \tilde{w}v_1, \tilde{w}v_2 \rangle \subseteq \cdots \subseteq \langle \tilde{w}v_1, \ldots, \tilde{w}v_n \rangle = \mathbb{C}^n \\ 0 &\subseteq \left\langle v_{w(1)} \right\rangle \subseteq \left\langle v_{w(1)}, v_{w(2)} \right\rangle \subseteq \cdots \subseteq \left\langle v_{w(1)}, \ldots, v_{w(n)} \right\rangle = \mathbb{C}^n \end{aligned}$$
   does not depend on the choice of the lift $\tilde{w} \in N_{\mathrm{GL}_n}(T)$ of w.

   Readers can verify that $\{F_w \,|\, w \in W\}$ are all T-fixed points of $\mathcal{F}$, while the set $\left\{wBw^{-1} \,\middle|\, w \in W\right\}$ consists of all Borel subgroups of G containing the standard torus T.

4. For $n = 2$, $\mathcal{F} \cong \mathbb{P}^1$. We encourage readers to use $\mathbb{P}^1$ as a toy example for the whole theory.

interpretation	GL_n/B	flags	Borel subgroups
base point	Id	F_{Id}	B
GL_n-action	left multiplication	$\{V_i\} \mapsto \{gV_i\}$	conjugation
general point	g	F_g	gBg^{-1}

$\mathcal{F}$ is a well-studied variety, and has many combinatorical properties. For example, from the well-known Bruhat decomposition, [1]
$$\mathrm{GL}_n = \bigsqcup_{w \in W} BwB,$$
we get a stratification of $\mathcal{F}$ by B-orbits:
$$\mathcal{F} = \mathrm{GL}_n/B \cong \bigsqcup_{w \in W} BwB/B$$
The B-orbit BwB/B is called the **Schubert cell**, denoted by Ω_w. Since
$$\Omega_w = BwB/B \cong B/\left(B \cap wBw^{-1}\right) \cong \mathbb{A}^{l(w)},$$

$i \backslash n$	0	2	4	6	8	10	12
1	1						
2	1	1					
3	1	2	2	1			
4	1	3	5	6	5	3	1
5	1	4	9	15	20	22	20

Table 1.1: The singular cohomology $H^i(\mathcal{F};\mathbb{C})$ for $\mathcal{F} = \mathrm{GL}_n /B$

G	Orbit	G-fixed points
GL_n	$\mathcal{F} \cong \mathrm{GL}_n /B$	$\varnothing$
B	$\Omega_w \cong B/(B \cap wBw^{-1})$	$\{F_{\mathrm{Id}}\}$
T	$-$	$\{F_w \mid w \in W\}$

Table 1.2: Orbit and fixed points

the Schubert cell is an affine space of dimension $l(w)$.

As a result, we know a lot of information of $\mathcal{F}$:

$\overline{\Omega}_w \subseteq \mathcal{F}$ is called the **Schubert variety**. It is well-known that

$$\overline{\Omega}_w = \bigsqcup_{w' \leq w} \Omega_w$$

as a set. Especially, for any $s \in W$ with $l(s) = 1$, denote $P_s = B \sqcup BsB$,

$$\overline{\Omega}_s = \Omega_{\mathrm{Id}} \sqcup \Omega_s = B/B \sqcup BsB/B = P_s/B \cong \mathbb{P}^1.$$

For other Schubert varieties, the structures are quite dedicate and far away from the scope of this master thesis. For example, most Schubert varieties are not smooth.

1.1.2 $\mathcal{F} \times \mathcal{F}$

As a more complicated geometrical object, $\mathcal{F} \times \mathcal{F}$ works as the base space for the Steinberg variety, which turns out to be the central focus in the thesis. $\mathcal{F} \times \mathcal{F}$ has naturally a diagonal GL_n-action:

$$\mathrm{GL}_n \times \mathcal{F} \times \mathcal{F} \longrightarrow \mathcal{F} \times \mathcal{F} \qquad (g, F_1, F_2) \longmapsto (gF_1, gF_2).$$

Under this action, $\mathcal{F} \times \mathcal{F}$ has a stratification consisting of GL_n-orbits, indexed by the Weyl group:

$$\mathrm{GL}_n \backslash (\mathcal{F} \times \mathcal{F}) \cong \mathrm{GL}_n \backslash (\mathrm{GL}_n /B \times \mathrm{GL}_n /B) \cong B \backslash \mathrm{GL}_n /B \cong W \quad \text{as sets.}$$

[1]For the most time the formula does not depend on the lift of w, so we abuse the notation of $w \in N_{\mathrm{GL}_n}(T)/T$ and $\tilde{w} \in N_{\mathrm{GL}_n}(T)$.

For $w' \in W$, denote $\mathbf{\Omega}_{w'} := \mathrm{GL}_n \cdot (F_{\mathrm{Id}}, F_{w'})$, then $\mathcal{F} \times \mathcal{F} = \sqcup_{w'} \mathbf{\Omega}_{w'}$. Moreover, by the orbit-stabilizer theorem, we get

$$\mathbf{\Omega}_{w'} \cong \mathrm{GL}_n / \left(B \cap w' B (w')^{-1}\right)$$

which is an $\mathbb{A}^{l(w')}$-bundle over $\mathcal{F}$, as shown below:

$$\begin{array}{ccc} \mathbb{A}^{l(w')} \cong B/\left(B \cap w' B (w')^{-1}\right) & \longrightarrow & \mathrm{GL}_n / \left(B \cap w' B (w')^{-1}\right) \\ & & \big\downarrow \\ & & \mathcal{F} = \mathrm{GL}_n / B \end{array}$$

Different from $\mathcal{F}$, the GL_n-action on $\mathcal{F} \times \mathcal{F}$ is not transitive. To facilitate the stratification of $\mathcal{F} \times \mathcal{F}$, we introduce the twisted $\mathrm{GL}_n \times \mathrm{GL}_n$-action:

$$\mathrm{GL}_n \times \mathrm{GL}_n \times \mathcal{F} \times \mathcal{F} \longrightarrow \mathcal{F} \times \mathcal{F} \qquad (g_1, g_2, F_g, F_{g'}) \longmapsto (F_{g_1 g}, F_{g_1 (g g_2 g^{-1}) g'}).$$

If we write $\underline{F}_{g,g'} := (F_g, F_{gg'}) \in \mathcal{F} \times \mathcal{F}$, then

$$(g_1, g_2) \cdot \underline{F}_{g,g'} = \underline{F}_{g_1 g, g_2 g'}.$$

This $\mathrm{GL}_n \times \mathrm{GL}_n$-action is now transitive, and decomposes $\mathcal{F} \times \mathcal{F}$ as disjoint union of finite many $B \times B$-orbits, which are compatible with G-orbits:

$$\begin{aligned} \mathbf{\Omega}_{w,w'} &:= (B \times B) \cdot \underline{F}_{w,w'} \subseteq \mathcal{F} \times \mathcal{F} \\ \mathcal{F} \times \mathcal{F} &= \bigsqcup_{w,w' \in W} \mathbf{\Omega}_{w,w'} \qquad \mathbf{\Omega}_{w'} = \bigsqcup_{w \in W} \mathbf{\Omega}_{w,w'} \end{aligned}$$

$$\begin{aligned} \mathbf{\Omega}_{w,w'} &\cong (B \times B) / \{(g_1, g_2) \in B \times B \mid (g_1, g_2) \cdot (F_w, F_{ww'}) = (F_w, F_{ww'})\} \\ &= (B \times B) / \left\{(g_1, g_2) \in B \times B \;\middle|\; g_1 w B = wB, g_1 w g_2 w' B = ww'B\right\} \\ &= (B \times B) / \left\{(g_1, g_2) \in B \times B \;\middle|\; g_1 w B = wB, (w^{-1} g_1 w) g_2 w' B = w'B\right\} \\ &= (B \times B) / \left\{(g_1, g_2) \in B \times B \;\middle|\; g_1 \in wBw^{-1}, g_2 \in (w^{-1} g_1^{-1} w)\left(w' B w'^{-1}\right)\right\} \\ &= (B \times B) / \left\{(g_1, g_2) \in \left(B \cap wBw^{-1}\right) \times (w^{-1} g_1^{-1} w)\left(B \cap w' B w'^{-1}\right)\right\} \\ &\cong B/(B \cap wBw^{-1}) \times B/(B \cap w' B w'^{-1}) \cong \mathbb{A}^{l(w) + l(w')} \end{aligned} \tag{1.1.1}$$

We conclude the information of orbits and fixed points of $\mathcal{F} \times \mathcal{F}$ in Table 1.3:

Like $\mathcal{F}$, we also study the closure of $\mathbf{\Omega}_{w'}$ and $\mathbf{\Omega}_{w,w'}$ in special case. It can be shown that

$$\overline{\mathbf{\Omega}}_{w'} = \bigsqcup_{x' \leq w'} \mathbf{\Omega}_{x'} \qquad \overline{\mathbf{\Omega}}_{w,w'} = \bigsqcup_{x \leq w, x' \leq w'} \mathbf{\Omega}_{x,x'}$$

as a set. Especially, for any $s \in W$ with $l(s) = 1$, [2]

[2]Here, $\mathrm{GL}_n \times^B X$ is called contracted product. Roughly, it is defined by

$$\mathrm{GL}_n \times^B X := \mathrm{GL}_n \times X / \big((gb, x) \sim (g, bx)\big)$$

We will discuss contracted product in Subsection 2.4.1 thoroughly.

G	Orbit	G-fixed points
$\mathrm{GL}_n \times \mathrm{GL}_n$	$\mathcal{F} \times \mathcal{F}$	$\varnothing$
GL_n	$\mathbf{\Omega}_{w'}$	$\varnothing$
$B \times B$	$\mathbf{\Omega}_{w,w'}$	$\{F_{\mathrm{Id},\mathrm{Id}}\}$
T	$-$	$\{\underline{F}_{w,w'} \mid w, w' \in W\}$

Table 1.3: Orbit and fixed points of $\mathcal{F} \times \mathcal{F}$

$$\begin{aligned}\overline{\mathbf{\Omega}}_s = \mathbf{\Omega}_{\mathrm{Id}} \sqcup \mathbf{\Omega}_s &\cong \mathrm{GL}_n /(B \cap sBs^{-1}) && \sqcup \mathrm{GL}_n /B \\ &\cong \mathrm{GL}_n \times^B \big(B/(B \cap sBs^{-1})\big) && \sqcup \mathrm{GL}_n \times^B (B/B) \\ &\cong \mathrm{GL}_n \times^B (BsB/B) && \sqcup \mathrm{GL}_n \times^B (B/B) \\ &\cong \mathrm{GL}_n \times^B (P_s/B)\end{aligned}$$

is an $\mathcal{F}$-bundle over $\mathbb{P}^1$. Also,

$$\begin{aligned}\overline{\mathbf{\Omega}}_{\mathrm{Id},s} = \mathbf{\Omega}_{\mathrm{Id},s} \sqcup \mathbf{\Omega}_{\mathrm{Id},\mathrm{Id}} &\cong \big(B/B \times BsB/B\big) \sqcup \big(B/B \times B/B\big) \\ &\cong P_s/B \cong \mathbb{P}^1\end{aligned}$$

Other closure can be highly singular.

Example 1.1.5. *In the Table* **??**, *$n = 3$, $t = (12)$, $s = (23)$. In this case, $\mathcal{F} \times \mathcal{F}$ has* 6 GL_3*-orbits, and each* GL_3*-orbits decompose as* 6 *$B \times B$-orbits, with dimensions equal to $l(w) + l(w')$.*

We can also see the GL_3*-orbit (and its closure) from the table, for example,*

$$\begin{aligned}\mathbf{\Omega}_s &= \mathbf{\Omega}_{\mathrm{Id},s} \sqcup \mathbf{\Omega}_{t,s} \sqcup \mathbf{\Omega}_{s,s} \sqcup \mathbf{\Omega}_{ts,s} \sqcup \mathbf{\Omega}_{st,s} \sqcup \mathbf{\Omega}_{sts,s} \\ \overline{\mathbf{\Omega}}_s &= \mathbf{\Omega}_s \sqcup \mathbf{\Omega}_{\mathrm{Id}} = \bigsqcup_w (\mathbf{\Omega}_{w,s} \sqcup \mathbf{\Omega}_{w,\mathrm{Id}})\end{aligned}$$

We color pieces of $\mathbf{\Omega}_s$ by blue, and $\mathbf{\Omega}_{ts,s}$ by ink blue.

Now we understand the structure a lot about $\mathcal{F}$ and $\mathcal{F} \times \mathcal{F}$, and the whole process will be applied repeatedly in Section 1.5 and 1.6.

1.2 Quiver

To introduce more complicated spaces and discuss their stratifications, we fix notations related to quiver and algebraic group in the following sections.

Roughly speaking, a quiver is a directed multigraph permitting loops.

Definition 1.2.1 (Quiver). *A quiver is a quadruple*

$$Q = (Q_0, Q_1, s, t)$$

where

dim; $(B\times B)\cdot \underline{F}_{w,w'}$; $B\cdot F_{ww'}$ / $B\cdot F_w$	0 Ω_{Id}	1 Ω_t	1 Ω_s	2 Ω_{ts}	2 Ω_{st}	3 Ω_{sts}
0 Ω_{Id}	0 $\boldsymbol{\Omega}_{\mathrm{Id},\mathrm{Id}}$	1 $\boldsymbol{\Omega}_{\mathrm{Id},t}$	1 $\boldsymbol{\Omega}_{\mathrm{Id},s}$	2 $\boldsymbol{\Omega}_{\mathrm{Id},ts}$	2 $\boldsymbol{\Omega}_{\mathrm{Id},st}$	3 $\boldsymbol{\Omega}_{\mathrm{Id},sts}$
1 Ω_t	2 $\boldsymbol{\Omega}_{t,t}$	1 $\boldsymbol{\Omega}_{t,\mathrm{Id}}$	3 $\boldsymbol{\Omega}_{t,ts}$	2 $\boldsymbol{\Omega}_{t,s}$	4 $\boldsymbol{\Omega}_{t,sts}$	3 $\boldsymbol{\Omega}_{t,st}$
1 Ω_s	2 $\boldsymbol{\Omega}_{s,s}$	3 $\boldsymbol{\Omega}_{s,st}$	1 $\boldsymbol{\Omega}_{s,\mathrm{Id}}$	4 $\boldsymbol{\Omega}_{s,sts}$	2 $\boldsymbol{\Omega}_{s,t}$	3 $\boldsymbol{\Omega}_{s,ts}$
2 Ω_{ts}	4 $\boldsymbol{\Omega}_{ts,st}$	3 $\boldsymbol{\Omega}_{ts,s}$	5 $\boldsymbol{\Omega}_{ts,sts}$	2 $\boldsymbol{\Omega}_{ts,\mathrm{Id}}$	4 $\boldsymbol{\Omega}_{ts,ts}$	3 $\boldsymbol{\Omega}_{ts,t}$
2 Ω_{st}	4 $\boldsymbol{\Omega}_{st,ts}$	5 $\boldsymbol{\Omega}_{st,sts}$	3 $\boldsymbol{\Omega}_{st,t}$	4 $\boldsymbol{\Omega}_{st,st}$	2 $\boldsymbol{\Omega}_{st,\mathrm{Id}}$	3 $\boldsymbol{\Omega}_{st,s}$
3 Ω_{sts}	6 $\boldsymbol{\Omega}_{sts,sts}$	5 $\boldsymbol{\Omega}_{sts,ts}$	5 $\boldsymbol{\Omega}_{sts,st}$	4 $\boldsymbol{\Omega}_{sts,t}$	4 $\boldsymbol{\Omega}_{sts,s}$	3 $\boldsymbol{\Omega}_{sts,\mathrm{Id}}$

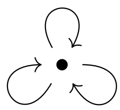
Table 1.4: stratifications of $\mathcal{F}\times\mathcal{F}$

- *Q_0 is a non-empty set consisting of vertices of Q,*
- *Q_1 is a set consisting of arrows of Q,*
- *$s : Q_1 \longrightarrow Q_0$ is a map indicating the start vertex of arrows,*
- *$t : Q_1 \longrightarrow Q_0$ is a map indicating the terminal vertex of arrows.*

Remark 1.2.2. In the first part of our master thesis, all the quivers are supposed to be connected and finite (i.e., Q_0, Q_1 are finite sets). We will only encounter disconnected and infinite quiver as Auslander–Reiten quiver later on.

Example 1.2.3. *The following graphs are quivers.*

$$\bullet \longrightarrow \bullet \longleftarrow \bullet \longleftarrow \bullet$$



$$\begin{tikzcd} 1 \arrow[r, bend left, "a"] \arrow[r, bend right, "b"'] & 2 \end{tikzcd}$$

quiver of type A_3 3-loop quiver $L(3)$ 2-Kronecker quiver $K(2)$

The reader can easily read the quadruple of quivers from the graphs. Take $Q = K(2)$ as an example, we have

$$Q_0 = \{1, 2\}, \qquad Q_1 = \{a, b\} \qquad s, t : \{a, b\} \longrightarrow \{1, 2\}$$

by $s(a) = s(b) = 1$, $t(a) = t(b) = 2$.

For convenience, we mainly use simpler quivers as examples:

$\bullet$	$\bullet \longrightarrow \bullet$	$\bullet \circlearrowleft$
trivial quiver	quiver of type A_1	1-loop quiver $L(1)$

From those quivers we are able to construct relatively complicated algebraic and geometrical objects.

Definition 1.2.4 (Quiver representation)**.** *Fix a quiver Q.*

A representation of Q consists of the following data:

- *A finite dimensional $\mathbb{C}$-vector space V_i for each vertex $i \in Q_0$;*
- *A $\mathbb{C}$-linear map $V_a : V_{s(a)} \longrightarrow V_{t(a)}$ for each arrow $a \in Q_1$.*

A morphism $f : V \longrightarrow W$ is a collection of morphisms $f_i : V_i \longrightarrow W_i$ (for every $i \in Q_0$) which make the following diagram commute:

$$\begin{tikzcd} V_{s(a)} \arrow[r, "V_a"] \arrow[d, "f_{s(a)}"'] & V_{t(a)} \arrow[d, "f_{t(a)}"] \\ W_{s(a)} \arrow[r, "W_a"] & W_{t(a)} \end{tikzcd}$$

We denote $\mathrm{rep}(Q)$ as the category of representations of Q.

Example 1.2.5. *A representation of 1-loop quiver $L(1)$ is a 2-tuple*

$$(V,\ \alpha : V \longrightarrow V)$$

which is equivalent to a (finite dimensional) $\mathbb{C}[t]$-module.

Remark 1.2.6. The equivalence appeared in the example can actually be generalized to arbitrary quivers. For a quiver Q, we can define the path algebra $\mathbb{C}Q$, and view any Q-representation as $\mathbb{C}Q$-module, and vice versa.

For many constructions, we only care about the data of vector space.

Definition 1.2.7 (Q-vector space/Vector space with quiver partition)**.** *Fix a quiver Q, a Q-vector space is a finite dimensional $\mathbb{C}$-vector space with a direct sum decomposition*

$$V = \bigoplus_{i \in Q_0} V_i.$$

The dimension vector of a Q-vector space is defined as

$$\underline{\mathbf{dim}}\, V := (\dim_{\mathbb{C}} V_i)_{i \in Q_0} \subseteq \prod_{i \in Q_0} \mathbb{Z}.$$

On the contrary, given $\mathbf{d} \in \prod_{i \in Q_0} \mathbb{N}_{\geqslant 0}$, we can construct a canonical Q-vector space of dimension vector $\mathbf{d}$, as follows:

$$V = \bigoplus_{i \in Q_0} V_i \qquad \text{with } V_i = \mathbb{C}^{\mathbf{d}_i}.$$

Definition 1.2.8. *The total dimension vector of a Q-vector space V is defined as*

$$|\underline{\mathbf{dim}}\, V| := \dim_{\mathbb{C}} V.$$

For $\mathbf{d} \in \prod_{i \in Q_0} \mathbb{N}_{\geqslant 0}$, denote $|\mathbf{d}| := \sum_{i \in Q_0} \mathbf{d}_i$.

Definition 1.2.9 (Space of representations with given dimension vector)**.** *For any quiver Q, dimension vector $\mathbf{d}$, fix the canonical Q-vector space $V = \bigoplus_{i \in Q_0} V_i$, the space of representations with dimension vector $\mathbf{d}$ is defined as*

$$\begin{aligned} \mathrm{Rep}_{\mathbf{d}}(Q) &:= \left\{(V_i, V_a : V_{s(a)} \longrightarrow V_{t(a)}) \textit{ as a representation of } Q\right\} \\ &= \bigoplus_{a \in Q_1} \mathrm{Hom}\left(\mathbb{C}^{\mathbf{d}_{s(a)}}, \mathbb{C}^{\mathbf{d}_{t(a)}}\right) \end{aligned}$$

Since we encode the information of vector space in $\mathbf{d}$, $\mathrm{Rep}_{\mathbf{d}}(Q)$ only records the information of linear maps.

For both Q-vector spaces and Q-representations, we can define (complete) flags.

Definition 1.2.10 (Flag with quiver)**.** *For a quiver representation $V \in \mathrm{rep}(Q)$, a flag of V is defined as an increasing sequence of subrepresentation of V, i.e.,*

$$F : 0 \subseteq M_1 \subseteq \ldots \subseteq M_k = V \qquad M_j \in \mathrm{rep}(Q).$$

For a Q-vector space $V = \oplus_{i \in Q_0} V_i$, a (quiver-graded) flag of V is defined as an increasing sequence of Q-subspace of V, i.e.,

$$F : 0 \subseteq M_1 \subseteq \ldots \subseteq M_k = V \qquad M_j = \bigoplus_{i \in Q_0} M_{j,i}.$$

For both Q-vector space and Q-representation, F is called a complete flag if $k = \dim_{\mathbb{C}} V$ and

$$\dim_{\mathbb{C}} M_j = j \qquad \textit{for any } j \in \{1, \ldots, |\mathbf{d}|\}$$

For the flag we also have the notation of dimension vector.

Definition 1.2.11 (flag-type dimension vector)**.** *For any flag $F : 0 \subseteq M_1 \subseteq \ldots \subseteq M_k = V$, the dimension vector of F is defined as*

$$\underline{\mathbf{d}} = \left(\underline{\mathbf{dim}}\, M_j\right)_{j \in \{1, \ldots, k\}} \subseteq \prod_{\substack{i \in Q_0 \\ j \in \{1, \ldots, k\}}} \mathbb{Z}.$$

$\underline{\mathbf{d}}$ is called a flag-type dimension vector if $\underline{\mathbf{d}}$ is the dimension vector of some complete flag F, i.e., [3]

$$|\underline{\mathbf{dim}}\, M_{j+1}/M_j| = 1 \qquad \textit{for any } j \in \{0, \ldots, |\mathbf{d}| - 1\}.$$

[3] For convenience, we denote M_0 by 0.

Example 1.2.12. *For quiver $Q : i \longrightarrow i'$, $\mathbf{d} = (3, 2)$, the canonical Q-vector space of dimension vector $\mathbf{d}$ is*

$$\begin{aligned} V &= V_i \oplus V_{i'} \\ &= \langle v_1, v_2, v_3\rangle_{\mathbb{C}} \oplus \langle v_4, v_5\rangle_{\mathbb{C}} \end{aligned}$$

The flag

$$F : 0 \subseteq \langle v_4\rangle \subseteq \langle v_4, v_1\rangle \subseteq \langle v_4, v_1, v_2\rangle \subseteq \langle v_4, v_1, v_2, v_5\rangle \subseteq \langle v_4, v_1, v_2, v_5, v_3\rangle = V$$

is a complete flag of V, with dimension vector

$$\underline{\mathbf{d}} = \begin{pmatrix} 3, 2 \\ 2, 2 \\ 2, 1 \\ 1, 1 \\ 0, 1 \end{pmatrix}.$$

Remark 1.2.13. The flag-type dimension vector $\underline{\mathbf{d}}$ can be viewed as a partition of set $\{1, \dots, |\mathbf{d}|\}$, i.e., a map

$$\mathrm{par} : \{1, \dots, |\mathbf{d}|\} \longrightarrow Q_0$$

such that $\#\,\mathrm{par}^{-1}(i) = \mathbf{d}_i$.[4] . As an example,

$$\underline{\mathbf{d}} = \begin{pmatrix} 3, 2 \\ 2, 2 \\ 2, 1 \\ 1, 1 \\ 0, 1 \end{pmatrix} \qquad \text{corresponds to} \qquad \{1, 2, 3, 4, 5\} = \{2, 3, 5\} \sqcup \{1, 4\}.$$

1.3 Symmetric group calculus

As a reminder, we recall some basic diagrams referring to the elements in S_n, and do some calculations by these diagrams. We will also relate cosets with flag-type dimension vectors.

Fix a quiver Q and dimension vector $\mathbf{d}$. Later (Definition 1.4.2, 1.4.3) we will define

$$\mathbb{W}_{|\mathbf{d}|} = S_{|\mathbf{d}|} \qquad W_{\mathbf{d}} = \prod_{i \in Q_0} S_{\mathbf{d}_i} \leqslant \mathbb{W}_{|\mathbf{d}|}$$

For simplicity, we take $Q_0 = \{1, \dots, k\}$, then $W_{\mathbf{d}} = S_{\mathbf{d}_1} \times \cdots \times S_{\mathbf{d}_k}$ embed in $S_{|\mathbf{d}|}$ in the most natural way.

Remark 1.3.1. We have different ways to express $\varpi \in \mathbb{W}_{|\mathbf{d}|} = S_{|\mathbf{d}|}$. For example, take $|\mathbf{d}| = 5$, $\varpi \in S_5$ by

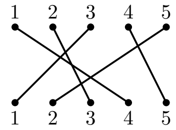
$$\varpi(1) = 4, \quad \varpi(2) = 3, \quad \varpi(3) = 1, \quad \varpi(4) = 5, \quad \varpi(5) = 2,$$

[4]The partition corresponding to par is

$$\{1, \dots, |\mathbf{d}|\} = \bigsqcup_{i \in Q_0} \mathrm{par}^{-1}(i).$$

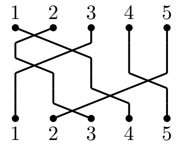
then

$$\varpi = (14523) = \binom{1\,2\,3\,4\,5}{4\,3\,1\,5\,2} =$$



$$= \begin{bmatrix} & & 1 & & \\ & & & & 1 \\ & 1 & & & \\ 1 & & & & \\ & & & 1 & \end{bmatrix}$$

$$= (23)(34)(45)(12)(23)(12) =$$



Even though all expressions give us the same amount of information, the diagram presents them more vividly. For example, each intersection of strands corresponds to a simple reflection, so we read from the diagram that $l(\varpi) = 6$. Readers can also check that

$$\begin{aligned} &l(\varpi s_1) = 5, \quad l(\varpi s_2) = 5, \quad l(\varpi s_3) = 7, \quad l(\varpi s_4) = 5, \\ &l(s_1 \varpi) = 7, \quad l(s_2 \varpi) = 5, \quad l(s_3 \varpi) = 5, \quad l(s_4 \varpi) = 7, \end{aligned}$$

where $s_i := (i, i+1) \in S_5$ are simple reflections.

Definition 1.3.2 (Simple reflections in the Weyl group)**.** *For $i \in \{1, \ldots, |\mathbf{d}| - 1\}$, the simple reflection is defined as*

$$s_i := (i, i+1) \in S_{|\mathbf{d}|}.$$

We denote

$$\begin{aligned} \Pi &= \left\{ s_i \in S_{|\mathbf{d}|} \;\middle|\; i \in \{1, \ldots, |\mathbf{d}| - 1\} \right\} \\ \Pi_{\mathbf{d}} &= \left\{ s_i \in S_{\mathbf{d}_1} \times \cdots \times S_{\mathbf{d}_k} \;\middle|\; i \in \{1, \ldots, |\mathbf{d}| - 1\} \right\} \\ &= \left\{ s_1, \ldots, s_{|\mathbf{d}|-1} \right\} \setminus \left\{ s_{\mathbf{d}_1}, s_{\mathbf{d}_1 + \mathbf{d}_2}, \ldots, s_{\mathbf{d}_1 + \cdots + \mathbf{d}_{k-1}} \right\} \end{aligned}$$

to be the set of simple reflections in $\mathbb{W}_{|\mathbf{d}|}$ and $W_{\mathbf{d}}$, respectively.

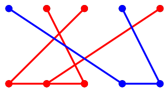
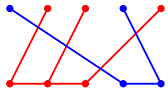
We also denote $\varpi_{\max} \in \mathbb{W}_{|\mathbf{d}|}$, $w_{\max} \in W_{\mathbf{d}}$ to be the longest elements in $\mathbb{W}_{|\mathbf{d}|}$ and $W_{\mathbf{d}}$, respectively. See Table 1.5 for a picture of $\varpi_{\max}$ and $w_{\max}$.

We discuss about right cosets $W_{\mathbf{d}} \setminus \mathbb{W}_{|\mathbf{d}|}$ and minimal length coset representatives now.

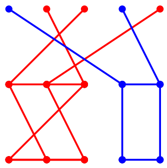
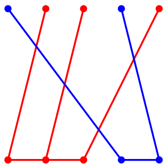
Multiplying on left by $w \in W_{\mathbf{d}}$ is equivalent to plugging in a diagram representing $w \in W_{\mathbf{d}}$ underneath the original diagram. Therefore, we connect some bottom points by lines, indicating that switching them will cause no trouble. Furthermore, we color different parts to make the following fact more explicitly.

Fact 1.3.3. *Every element $W_{\mathbf{d}} \varpi \in W_{\mathbf{d}} \setminus \mathbb{W}_{|\mathbf{d}|}$ corresponds to a partition on set $\{1, \ldots, |\mathbf{d}|\}$ (of a given number partition $\mathbf{d}$), which corresponds to a flag-type dimension vector $\underline{\mathbf{d}}$.*

Example 1.3.4.

 $=$  *in* $W_{\mathbf{d}} \setminus \mathbb{W}_{|\mathbf{d}|}$

since

 $=$  *in* $\mathbb{W}_{|\mathbf{d}|}$

This coset corresponds to the partition $\{1, 2, 3, 4, 5\} = \{2, 3, 5\} \sqcup \{1, 4\}$.

It is easy to see from the diagram that in every coset, there exists an unique element $u \in \mathbb{W}_{|\mathbf{d}|}$ of minimal length. We collect these minimal length coset representatives as a set, and denote it by $\mathrm{Min}(\mathbb{W}_{|\mathbf{d}|}, \mathrm{W}_{\mathbf{d}})$.[5]

Proposition 1.3.5. *For any* $\varpi \in \mathbb{W}_{|\mathbf{d}|}$*, exists unique* $w \in W_{\mathbf{d}}$*,* $u \in \mathrm{Min}(\mathbb{W}_{|\mathbf{d}|}, \mathrm{W}_{\mathbf{d}})$ *such that* $\varpi = wu$.

Exercise 1.3.6. *For* $u \in \mathrm{Min}(\mathbb{W}_{|\mathbf{d}|}, \mathrm{W}_{\mathbf{d}})$*,* $s_i \in \Pi$*, show that*

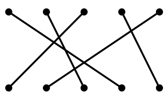
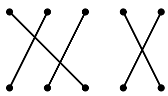
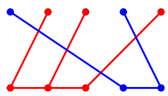
$$\begin{aligned} us_iu^{-1} \in W_{\mathbf{d}} \quad &\Longrightarrow \quad us_iu^{-1} = s_{u(i)} \in \Pi_{\mathbf{d}}, \\ us_iu^{-1} \notin W_{\mathbf{d}} \quad &\Longrightarrow \quad us_i \in \mathrm{Min}(\mathbb{W}_{|\mathbf{d}|}, \mathrm{W}_{\mathbf{d}})\,. \end{aligned}$$

The picture of us_i *is shown in Table 1.7.*

We finish this section with figures and examples.

$$\begin{tikzcd} & & & \mathrm{Min}(\mathbb{W}_{|\mathbf{d}|}, \mathrm{W}_{\mathbf{d}}) \arrow[dl, hook'] \arrow[d, "\cong"] & & & u \arrow[d, mapsto] \\ 0 \arrow[r] & W_{\mathbf{d}} \arrow[r] & \mathbb{W}_{|\mathbf{d}|} \arrow[r] & W_{\mathbf{d}} \setminus \mathbb{W}_{|\mathbf{d}|} \arrow[r] & 0 & \varpi = wu \arrow[r, mapsto] & \underline{\mathbf{d}} \end{tikzcd}$$

Example 1.3.7. *In this table,* $|\mathbf{d}| = 5$, $\mathbf{d} = (3, 2)$*, typical elements would be*

$\varpi =$  $w =$  $u =$ 

Example 1.3.8. *In this table,*

$$|\mathbf{d}| = 3, \quad \mathbf{d} = (1, 2), \qquad \mathbb{W}_{|\mathbf{d}|} = S_3, \quad W_{\mathbf{d}} = S_1 \times S_2, \qquad s = (12), \quad t = (23).$$

The columns "order of basis" and "Borel subgroups" will be introduced in Definition 1.5.5 and Remark 1.4.4.

[5] In some references $\mathrm{Min}(\mathbb{W}_{|\mathbf{d}|}, \mathrm{W}_{\mathbf{d}})$ is also denoted by $\mathrm{Shuffle}_{\mathbf{d}}$, since those elements can be thought as ways off riffle shuffling several words together.

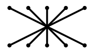
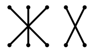
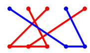
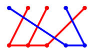
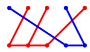
set	element	special element	others/alias
$\mathbb{W}_{\lvert\mathbf{d}\rvert} = S_5$	ϖ, x	$\varpi_{\max} =$ 	$\Pi \;= \{s_1, s_2, s_3, s_4\}$
$W_{\mathbf{d}} = S_3 \times S_2$	w	$w_{\max} =$ 	$\Pi_{\mathbf{d}} = \{s_1, s_2, \quad s_4\}$
$W_{\mathbf{d}} \setminus \mathbb{W}_{\lvert\mathbf{d}\rvert} \cong (S_3 \times S_2) \setminus S_5$	$\varpi, \underline{\mathbf{d}}$		$\mathrm{Comp}_{\mathbf{d}}$
$\mathrm{Min}(\mathbb{W}_{\lvert\mathbf{d}\rvert}, \mathrm{W}_{\mathbf{d}}) = \{$  $, \dots \}$	u		$\mathrm{Shuffle}_{\mathbf{d}}$

Table 1.5: Collected notations in $(3,2)$-case

$\varpi = wu$					w	$\underline{\mathbf{d}}, u$		order of basis	$l(\varpi)$	$l(w)$	$\mathbb{B}_{\varpi}$	B_{ϖ}	$\varpi B_{\mathbf{d}} \varpi^{-1}$
Id	Id	$\begin{pmatrix}1\,2\,3\\1\,2\,3\end{pmatrix}$		$\begin{bmatrix}1&&\\&1&\\&&1\end{bmatrix}$		*abb*		$\{v_1, v_2, v_3\}$	0	0	$\begin{bmatrix}*&*&*\\&*&*\\&&*\end{bmatrix}$	$\begin{bmatrix}*&&\\&*&*\\&&*\end{bmatrix}$	$\begin{bmatrix}*&&\\&*&*\\&&*\end{bmatrix}$
t	(23)	$\begin{pmatrix}1\,2\,3\\1\,3\,2\end{pmatrix}$		$\begin{bmatrix}1&&\\&&1\\&1&\end{bmatrix}$		*abb*		$\{v_1, v_3, v_2\}$	1	1	$\begin{bmatrix}*&*&*\\&*&\\&*&*\end{bmatrix}$	$\begin{bmatrix}*&&\\&*&\\&*&*\end{bmatrix}$	$\begin{bmatrix}*&&\\&*&\\&*&*\end{bmatrix}$
s	(12)	$\begin{pmatrix}1\,2\,3\\2\,1\,3\end{pmatrix}$		$\begin{bmatrix}&1&\\1&&\\&&1\end{bmatrix}$		*bab*		$\{v_2, v_1, v_3\}$	1	0	$\begin{bmatrix}*&&*\\*&*&*\\&&*\end{bmatrix}$	$\begin{bmatrix}*&&\\&*&*\\&&*\end{bmatrix}$	$\begin{bmatrix}*&&*\\&*&\\&&*\end{bmatrix}$
ts	(132)	$\begin{pmatrix}1\,2\,3\\3\,1\,2\end{pmatrix}$		$\begin{bmatrix}&1&\\&&1\\1&&\end{bmatrix}$		*bab*		$\{v_3, v_1, v_2\}$	2	1	$\begin{bmatrix}*&*&\\&*&\\*&*&*\end{bmatrix}$	$\begin{bmatrix}*&&\\&*&\\&*&*\end{bmatrix}$	$\begin{bmatrix}*&*&\\&*&\\&&*\end{bmatrix}$
st	(123)	$\begin{pmatrix}1\,2\,3\\2\,3\,1\end{pmatrix}$		$\begin{bmatrix}&&1\\1&&\\&1&\end{bmatrix}$		*bba*		$\{v_2, v_3, v_1\}$	2	0	$\begin{bmatrix}*&&\\*&*&*\\*&&*\end{bmatrix}$	$\begin{bmatrix}*&&\\&*&*\\&&*\end{bmatrix}$	$\begin{bmatrix}*&&\\&*&\\*&&*\end{bmatrix}$
sts	(13)	$\begin{pmatrix}1\,2\,3\\3\,2\,1\end{pmatrix}$		$\begin{bmatrix}&&1\\&1&\\1&&\end{bmatrix}$		*bba*		$\{v_3, v_2, v_1\}$	3	1	$\begin{bmatrix}*&&\\*&*&\\*&*&*\end{bmatrix}$	$\begin{bmatrix}*&&\\&*&\\&*&*\end{bmatrix}$	$\begin{bmatrix}*&&\\*&*&\\&&*\end{bmatrix}$

Table 1.6: basic information of $(1,2)$-case

1.4 Algebraic group and Lie algebra

In this section we fix notations of algebraic group and Lie algebras. Later, the algebraic group will act on varieties, and some Lie algebra will serve as tangent spaces.

Setting 1.4.1. *Fix a quiver Q, a dimension vector $\mathbf{d}$ and a Q-vector space*

$$V = \bigoplus_{i \in Q_0} V_i \qquad \text{with } V_i = \mathbb{C}^{\mathbf{d}_i}.$$

When a basis of V is needed, we fix a total order on Q_0, and denote

$$V = \langle v_1, \dots v_{\lvert\mathbf{d}\rvert} \rangle$$

where

$$V_i = \langle v_{f_i+1}, \dots v_{f_i+\mathbf{d}_i} \rangle \qquad f_i = \sum_{i' < i} \mathbf{d}_{i'}.$$

1.4.1 Algebraic group

Definition 1.4.2 (absolute algebraic groups)**.** *We set*

$$\mathbb{G}_{|\mathbf{d}|} := \mathrm{GL}(V) = \mathrm{GL}_{|\mathbf{d}|}(\mathbb{C}),$$

and $\mathbb{B}_{|\mathbf{d}|}$, $\mathbb{T}_{|\mathbf{d}|}$, $\mathbb{N}_{|\mathbf{d}|}$ *are corresponding standard Borel, torus and unipotent subgroups, respectively.*

The Weyl group is

$$\mathbb{W}_{|\mathbf{d}|} := N_{\mathbb{G}_{|\mathbf{d}|}}(\mathbb{T}_{|\mathbf{d}|})/\mathbb{T}_{|\mathbf{d}|} \cong S_{|\mathbf{d}|}.$$

For $\varpi \in \mathbb{W}_{|\mathbf{d}|}$, *we define*[6]

$$\mathbb{B}_{\varpi} := \varpi \mathbb{B}_{|\mathbf{d}|}\varpi^{-1}.$$

We will view $\mathbb{B}_{\varpi}$ *as the stabilizer of the flag* F_{ϖ} *with* $\mathbb{G}_{|\mathbf{d}|}$*-action.*

We also have a series of algebraic groups compatible with the quiver partition of V, and they're more common in this thesis.

Definition 1.4.3 (relative algebraic groups)**.** *We set*

$$G_{\mathbf{d}} := \bigoplus_{i\in Q_0} \mathrm{GL}(V_i) = \bigoplus_{i\in Q_0} \mathrm{GL}_{\mathbf{d}_i}(\mathbb{C}) \subseteq \mathbb{G}_{|\mathbf{d}|},$$

and $B_{\mathbf{d}}$, $T_{\mathbf{d}}$, $N_{\mathbf{d}}$ *are corresponding standard Borel, torus and unipotent subgroups.*

The Weyl group is

$$W_{\mathbf{d}} := N_{G_{\mathbf{d}}}(T_{\mathbf{d}})/T_{\mathbf{d}} \cong \prod_{i\in Q_0} S_{\mathbf{d}_i}.$$

For $\varpi = wu \in W_{\mathbf{d}}$, *we define*

$$B_{\varpi} := wB_{\mathbf{d}}w^{-1}.$$

We will view B_{ϖ} *as the stabilizer of the flag* F_{ϖ} *with* $G_{\mathbf{d}}$*-action.*

Remark 1.4.4. Be careful that $B_{\varpi} \neq \varpi B_{\mathbf{d}}\varpi^{-1}$. Actually,

$$B_{\varpi} = \varpi \mathbb{B}_{|\mathbf{d}|}\varpi^{-1} \cap B_{\mathbf{d}} = wB_{\mathbf{d}}w^{-1}$$

The difference is clearly shown in Table 1.6.

We also have a series of algebraic groups indexed by elements in the Weyl group:

Definition 1.4.5 (more algebraic groups)**.** *For* $\varpi, \varpi'' \in \mathbb{W}_{|\mathbf{d}|}$, *define*

$$\begin{aligned} N_{\varpi} &:= R_u(B_{\varpi}), \\ N_{\varpi,\varpi''} &:= N_{\varpi} \cap N_{\varpi''}, \\ M_{\varpi,\varpi''} &:= N_{\varpi}/N_{\varpi,\varpi''}, \end{aligned}$$

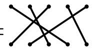
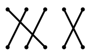
[6] As usual, we abuse the notation of ϖ and its lift.

where R_u denotes the unipotent radical.

For $s \in \Pi$ such that $\varpi s \varpi^{-1} \in W_d$ (i.e., $W_{\mathbf{d}}\varpi = W_{\mathbf{d}}\varpi s$), define

$$\begin{aligned} P_{\varpi,\varpi s} &:\xlongequal{\varpi = wu} w\left(B_{\mathbf{d}} usu^{-1} B_{\mathbf{d}} \cup B_{\mathbf{d}}\right) w^{-1} \\ &\xlongequal{\quad} B_{\varpi} \varpi s \varpi^{-1} B_{\varpi} \cup B_{\varpi} \end{aligned}$$

Remark 1.4.6. One can easily show that $N_{\varpi,\varpi s} = R_u(P_{\varpi,\varpi s})$.

Example 1.4.7 (Follows Example 1.3.7)**.** *For $|\mathbf{d}| = 5$, $\mathbf{d} = (3,2)$, $\varpi =$* *, $w =$* *, $s = s_2$, we compute all the algebraic groups we mentioned:*

$$\mathbb{G}_{|\mathbf{d}|} = \begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{pmatrix} \qquad \mathbb{B}_{|\mathbf{d}|} = \begin{pmatrix} * & * & * & * & * \\ & * & * & * & * \\ & & * & * & * \\ & & & * & * \\ & & & & * \end{pmatrix} \qquad \mathbb{T}_{|\mathbf{d}|} = \begin{pmatrix} * & & & & \\ & * & & & \\ & & * & & \\ & & & * & \\ & & & & * \end{pmatrix} \qquad \mathbb{N}_{|\mathbf{d}|} = \begin{pmatrix} 1 & * & * & * & * \\ & 1 & * & * & * \\ & & 1 & * & * \\ & & & 1 & * \\ & & & & 1 \end{pmatrix}$$

$$\mathbb{W}_{|\mathbf{d}|} \cong S_5 \qquad \mathbb{B}_{\varpi} = \begin{pmatrix} * & * & & & * \\ & * & & & \\ * & * & * & & * \\ * & * & * & * & * \\ & * & & & * \end{pmatrix} \qquad \mathbb{B}_{\varpi s} = \begin{pmatrix} * & * & * & & * \\ & * & & & \\ & * & * & & * \\ * & * & * & * & * \\ & * & & & * \end{pmatrix}$$

$$G_{\mathbf{d}} = \left(\begin{array}{ccc|cc} * & * & * & & \\ * & * & * & & \\ * & * & * & & \\ \hline & & & * & * \\ & & & * & * \end{array}\right) \qquad B_{\mathbf{d}} = \left(\begin{array}{ccc|cc} * & * & * & & \\ & * & * & & \\ & & * & & \\ \hline & & & * & * \\ & & & & * \end{array}\right) \qquad T_{\mathbf{d}} = \left(\begin{array}{ccc|cc} * & & & & \\ & * & & & \\ & & * & & \\ \hline & & & * & \\ & & & & * \end{array}\right) \qquad N_{\mathbf{d}} = \left(\begin{array}{ccc|cc} 1 & * & * & & \\ & 1 & * & & \\ & & 1 & & \\ \hline & & & 1 & * \\ & & & & 1 \end{array}\right)$$

$$W_{\mathbf{d}} \cong S_3 \times S_2 \qquad B_{\varpi} = \left(\begin{array}{ccc|cc} * & * & & & \\ & * & & & \\ * & * & * & & \\ \hline & & & * & \\ & & & * & * \end{array}\right) \qquad B_{\varpi s} = \left(\begin{array}{ccc|cc} * & & & & \\ * & * & & & \\ * & * & * & & \\ \hline & & & * & \\ & & & * & * \end{array}\right)$$

$$N_{\varpi} = \left(\begin{array}{ccc|cc} 1 & * & & & \\ & 1 & & & \\ * & * & 1 & & \\ \hline & & & 1 & \\ & & & * & 1 \end{array}\right) \qquad N_{\varpi,\varpi s} = \left(\begin{array}{ccc|cc} 1 & & & & \\ & 1 & & & \\ * & * & 1 & & \\ \hline & & & 1 & \\ & & & * & 1 \end{array}\right) \qquad M_{\varpi,\varpi s} = \left(\begin{array}{ccc|cc} 1 & * & & & \\ & 1 & & & \\ - & - & 1 & & \\ \hline & & & 1 & \\ & & & - & 1 \end{array}\right) \qquad P_{\varpi,\varpi s} = \left(\begin{array}{ccc|cc} * & * & & & \\ * & * & & & \\ * & * & * & & \\ \hline & & & * & \\ & & & * & * \end{array}\right)$$

1.4.2 Lie algebra

We use Fraktur-font symbols to represent the Lie algebras of the corresponding algebraic groups introduced in the last section:

$$\begin{array}{lllll} \mathfrak{g}_{|\mathbf{d}|}, & \mathfrak{b}_{|\mathbf{d}|}, & \mathfrak{t}_{|\mathbf{d}|}, & \mathfrak{n}_{|\mathbf{d}|}, & \mathfrak{b}_{\varpi} \\ \mathfrak{g}_{\mathbf{d}}, & \mathfrak{b}_{\mathbf{d}}, & \mathfrak{t}_{\mathbf{d}}, & \mathfrak{n}_{\mathbf{d}}, & \mathfrak{b}_{\varpi}, \\ \mathfrak{n}_{\varpi}, & \mathfrak{n}_{\varpi,\varpi''}, & \mathfrak{m}_{\varpi,\varpi''}, & \mathfrak{p}_{\varpi,\varpi s}, & \end{array}$$

We also have to encode the information of representations as Lie algebra. Notice that

$$\mathrm{Hom}(V_{s(a)}, V_{t(a)}) \hookrightarrow \mathrm{Hom}(V,V) \cong \mathfrak{g}_{|\mathbf{d}|} \qquad f \longmapsto \iota_{t(a)} \circ f \circ \pi_{s(a)}$$

realizes $\mathrm{Hom}(V_{s(a)}, V_{t(a)})$ as a Lie subalgebra of $\mathfrak{g}_{|\mathbf{d}|}$, so

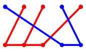
$$\mathrm{Rep}_{\mathbf{d}}(Q) = \bigoplus_{a \in Q_1} \mathrm{Hom}\left(\mathbb{C}^{\mathbf{d}_{s(a)}}, \mathbb{C}^{\mathbf{d}_{t(a)}}\right) \subseteq \bigoplus_{a \in Q_1} \mathfrak{g}_{|\mathbf{d}|}.$$

Definition 1.4.8 (Lie algebras connected with representations)**.** *For $\varpi \in \mathbb{W}_{|\mathbf{d}|}$, denote temperately*

$$V_{\varpi,j} := \left\langle e_{\varpi(1)}, \dots e_{\varpi(j)} \right\rangle \subseteq V.$$

We define Lie subalgebras of $\mathrm{Rep}_{\mathbf{d}}(Q)$ *as follows.*

$$\mathfrak{r}_{\varpi} := \left\{ (f_a)_{a\in Q_1} \in \mathrm{Rep}_{\mathbf{d}}(Q) \;\middle|\; f_a(V_{\varpi,j} \cap V_{s(a)}) \subseteq V_{\varpi,j} \text{ for any } j \right\},$$
$$\mathfrak{r}_{\varpi,\varpi''} := \mathfrak{r}_{\varpi} \cap \mathfrak{r}_{\varpi''},$$
$$\mathfrak{d}_{\varpi,\varpi''} := \mathfrak{r}_{\varpi}/\mathfrak{r}_{\varpi,\varpi''},$$

Example 1.4.9 (Follows Example 1.4.7)**.** *Consider the quiver* $\bullet \longrightarrow \bullet$ *, and* $u =$ *. Table 1.7 gives us an example of the shape of these Lie algebras. Symbols like* $\frac{e_1}{e_2}$ *will be explained in Example 2.1.4.*

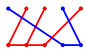
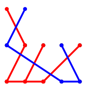
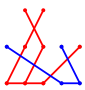
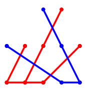
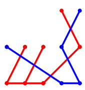
		$\mathfrak{n}_u$	$\mathfrak{m}_{u,u}$		$\mathfrak{r}_u$	$\mathfrak{d}_{u,u}$	
	$u =$ 	$\begin{bmatrix} \begin{smallmatrix} *\,* \\ \;\;* \end{smallmatrix} & \\ & * \end{bmatrix}$	$\begin{bmatrix} & \\ & \end{bmatrix}$		$\begin{bmatrix} & \\ \begin{smallmatrix} *\,*\,* \\ \;\;\;\;* \end{smallmatrix} & \end{bmatrix}$	$\begin{bmatrix} & \\ & \end{bmatrix}$	
s	cases	$\mathfrak{n}_{us}$	$\mathfrak{m}_{u,us}$		$\mathfrak{r}_{us}$	$\mathfrak{d}_{u,us}$	
$s = s_1$	$us_1 =$  $\notin W_{\mathbf{d}}u$	$\begin{bmatrix} \begin{smallmatrix} *\,* \\ \;\;* \end{smallmatrix} & \\ & * \end{bmatrix}$	$\begin{bmatrix} & \\ & \end{bmatrix}$		$\begin{bmatrix} & \\ \begin{smallmatrix} *\,* \\ \;\;* \end{smallmatrix} & \end{bmatrix}$	$\begin{bmatrix} & \\ * & \end{bmatrix}$	$\frac{e_4}{e_1}$
$s = s_2$	$us_2 =$  $\in W_{\mathbf{d}}u$	$\begin{bmatrix} \begin{smallmatrix} \;\;\;\;* \\ *\;\;* \end{smallmatrix} & \\ & * \end{bmatrix}$	$\begin{bmatrix} * & \\ & \end{bmatrix}$	$\frac{e_1}{e_2}$	$\begin{bmatrix} & \\ \begin{smallmatrix} *\,*\,* \\ \;\;\;\;* \end{smallmatrix} & \end{bmatrix}$	$\begin{bmatrix} & \\ & \end{bmatrix}$	
$s = s_3$	$us_3 =$  $\notin W_{\mathbf{d}}u$	$\begin{bmatrix} \begin{smallmatrix} *\,* \\ \;\;* \end{smallmatrix} & \\ & * \end{bmatrix}$	$\begin{bmatrix} & \\ & \end{bmatrix}$		$\begin{bmatrix} & \\ \begin{smallmatrix} *\,*\,* \\ \;\;*\,* \end{smallmatrix} & \end{bmatrix}$	$\begin{bmatrix} & \\ & \end{bmatrix}$	
$s = s_4$	$us_4 =$  $\notin W_{\mathbf{d}}u$	$\begin{bmatrix} \begin{smallmatrix} *\,* \\ \;\;* \end{smallmatrix} & \\ & * \end{bmatrix}$	$\begin{bmatrix} & \\ & \end{bmatrix}$		$\begin{bmatrix} & \\ *\,*\,* & \end{bmatrix}$	$\begin{bmatrix} & \\ * & \end{bmatrix}$	$\frac{e_5}{e_3}$

Table 1.7: examples of Lie algebras

Remark 1.4.10. We also have twisted notations for Lie algebras. For example,

$$\underline{\mathfrak{n}}_{\varpi,\varpi'} = \mathfrak{n}_{\varpi,\varpi\varpi'}, \quad \underline{\mathfrak{m}}_{\varpi,\varpi'} = \mathfrak{m}_{\varpi,\varpi\varpi'}, \quad \underline{\mathfrak{p}}_{\varpi,s} = \mathfrak{p}_{\varpi,\varpi s},$$
$$\underline{\mathfrak{r}}_{\varpi,\varpi'} = \mathfrak{r}_{\varpi,\varpi\varpi'}, \quad \underline{\mathfrak{d}}_{\varpi,\varpi'} = \mathfrak{d}_{\varpi,\varpi\varpi'}.$$

Another twist happens when we add minus sign as the superscript:

$$\mathfrak{b}^-_{\varpi} = \mathfrak{b}_{\varpi_{\max}\varpi},$$
$$\mathfrak{b}^-_{\varpi} = \mathfrak{b}_{w_{\max}\varpi}, \qquad \mathfrak{n}^-_{\varpi} = \mathfrak{n}_{w_{\max}\varpi},$$
$$\mathfrak{n}^-_{\varpi,\varpi''} = \mathfrak{n}_{w_{\max}\varpi, w_{\max}\varpi''}, \quad \mathfrak{m}^-_{\varpi,\varpi''} = \mathfrak{m}_{w_{\max}\varpi, w_{\max}\varpi''}.$$

1.5 Typical variety

In this section, we define nearly all the varieties we care about in the same spirit as Section 1.1. Their stratifications and related "Schubert" varieties will be defined in Section 1.6.

Recall Setting 1.1 and Definition 1.2.10.

1.5.1 Flag variety

Definition 1.5.1 (Absolute complete flag variety). *The absolute complete flag variety $\mathcal{F}_{|\mathbf{d}|}$ is defined as*

$$\begin{aligned}\mathcal{F}_{|\mathbf{d}|} &= \mathbb{G}_{|\mathbf{d}|}/\mathbb{B}_{|\mathbf{d}|} \\ &\cong \left\{\textit{complete flags of } \mathbb{C}^{|\mathbf{d}|}\right\} \\ &= \left\{0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_{|\mathbf{d}|} = \mathbb{C}^{|\mathbf{d}|} \;\middle|\; \dim M_j = j\right\} \\ &\cong \left\{\textit{Borel subgroups of } \mathbb{G}_{|\mathbf{d}|}\right\} \\ &= \left\{g\mathbb{B}_{|\mathbf{d}|}g^{-1} \;\middle|\; g \in \mathbb{G}_{|\mathbf{d}|}\right\}\end{aligned}$$

Here, M_i can have no Q-vector space structure.

Definition 1.5.2 (complete flag variety with flag-type dimension vector). *For a flag-type dimension vector $\underline{\mathbf{d}}$, the flag variety $\mathcal{F}_{\underline{\mathbf{d}}}$ is defined as*

$$\begin{aligned}\mathcal{F}_{\underline{\mathbf{d}}} &= \left\{\textit{complete flags of } V = \bigoplus_{i\in Q_0} V_i \textit{ with dimension vector } \underline{\mathbf{d}}\right\} \\ &= \left\{F : 0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_{|\mathbf{d}|} = V \;\middle|\; \underline{\mathbf{dim}}\, F = \underline{\mathbf{d}}\right\}\end{aligned}$$

Definition 1.5.3 (Relative complete flag variety). *The relative complete flag variety $\mathcal{F}_{\mathbf{d}}$ is defined as*

$$\begin{aligned}\mathcal{F}_{\mathbf{d}} &= \left\{\textit{complete flags of } V = \bigoplus_{i\in Q_0} V_i\right\} \\ &= \left\{0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_{|\mathbf{d}|} = V \;\middle|\; |\underline{\mathbf{dim}}\, M_j| = j\right\} \\ &= \bigsqcup_{\underline{\mathbf{d}}} \mathcal{F}_{\underline{\mathbf{d}}}\end{aligned}$$

Here, M_i are Q-vector spaces.

Remark 1.5.4.

1. $\mathcal{F}_{|\mathbf{d}|}$, $\mathcal{F}_{\underline{\mathbf{d}}}$ and $\mathcal{F}_{\mathbf{d}}$ are smooth varieties, since

$$\mathcal{F}_{|\mathbf{d}|} \cong \mathrm{GL}_{|\mathbf{d}|}/B \qquad \mathcal{F}_{\underline{\mathbf{d}}} \cong \prod_{i \in Q_0} \mathrm{GL}_{\mathbf{d}_i}/B$$

are products of usual flag varieties.

2. $\mathcal{F}_{|\mathbf{d}|}$ is an $\mathrm{GL}_{|\mathbf{d}|}$-variety, while $\mathcal{F}_{\underline{\mathbf{d}}}$, $\mathcal{F}_{\mathbf{d}}$ are $G_{\mathbf{d}}$-varieties. The actions are induced by the actions on the vector space V.

We need to simplify our notations of flags.

Definition 1.5.5 (Coordinate flags and related flags)**.** *For a basis* $\{x_1, \ldots, x_{|\mathbf{d}|}\}$*, denote the flag*

$$F_{\{x_1,\ldots,x_{|\mathbf{d}|}\}} : 0 \subseteq \langle x_1 \rangle \subseteq \langle x_1, x_2 \rangle \subseteq \cdots \subseteq \left\langle x_1, \ldots x_{|\mathbf{d}|} \right\rangle = V.$$

For $g \in \mathbb{G}_{|\mathbf{d}|}$, $\varpi \in \mathbb{W}_{|\mathbf{d}|}$, define

$$\begin{aligned} F_{\mathrm{Id}} &= F_{\{v_1,\ldots,v_{|\mathbf{d}|}\}} && \in \mathcal{F}_{\mathbf{d}} \\ F_g &= gF_{\mathrm{Id}} = F_{\{gv_1,\ldots,gv_{|\mathbf{d}|}\}} && \in \mathcal{F}_{|\mathbf{d}|} \\ F_{\varpi} &= \varpi F_{\mathrm{Id}} = F_{\{v_{\varpi(1)},\ldots,v_{\varpi(|\mathbf{d}|)}\}} && \in \mathcal{F}_{\mathbf{d}} \end{aligned}$$

F_{Id} is called the **standard flag** of V.

Now we can define flag varieties attached to $\varpi \in \mathbb{W}_{|\mathbf{d}|}$.

Definition 1.5.6. *For $\varpi = wu \in \mathbb{W}_{|\mathbf{d}|}$, define $\mathcal{F}_{\varpi}$ as the $G_{\mathbf{d}}$-orbit of F_{ϖ}. By the orbit-stabilizer theorem,*

$$\mathcal{F}_{\varpi} \cong G_{\mathbf{d}}/B_{\varpi}.$$

We can generalize it a little bit: for $g \in G_{\mathbf{d}}$, $F_{g\varpi} \in \mathcal{F}_{\mathbf{d}}$,

$$\mathcal{F}_{g\varpi} := G_{\mathbf{d}} \cdot F_{g\varpi} \cong G_{\mathbf{d}}/B_{g\varpi} = G_{\mathbf{d}}/gB_{\varpi}g^{-1}.$$

Remark 1.5.7. F_{ϖ} is the preferred base point of $\mathcal{F}_{\varpi}$. Ignoring the base point,

$$\mathcal{F}_{\varpi} = \mathcal{F}_u = \mathcal{F}_{\underline{\mathbf{d}}} \quad \text{for } \varpi = wu \quad \underline{\mathbf{d}} = W_{\mathbf{d}}\varpi.$$

In fact, we are not defining new varieties; we give old varieties new names, so that we can manipulate them more freely.

Like Section 1.1, we also consider the product of two flag varieties. For $g, g', g'' \in \mathbb{G}_{|\mathbf{d}|}$, $\varpi, \varpi', \varpi'' \in \mathbb{W}_{|\mathbf{d}|}$, denote

$$\begin{aligned} F_{\mathrm{Id},\mathrm{Id}} &= (F_{\mathrm{Id}}, F_{\mathrm{Id}}) \\ F_{g,g''} &= (F_g, F_{g''}) && \underline{F}_{g,g'} = F_{g,gg'} = (F_g, F_{gg'}) \\ F_{\varpi,\varpi''} &= (F_{\varpi}, F_{\varpi''}) && \underline{F}_{\varpi,\varpi'} = F_{\varpi,\varpi\varpi'} = (F_{\varpi}, F_{\varpi\varpi'}) \end{aligned}$$

Table 1.8 concludes all varieties we get until now.

	base point		base point
$\mathcal{F}_{\lvert\mathbf{d}\rvert} \cong \mathbb{G}_{\lvert\mathbf{d}\rvert}/\mathbb{B}_{\lvert\mathbf{d}\rvert}$	F_{Id}	$\mathcal{F}_{\lvert\mathbf{d}\rvert} \times \mathcal{F}_{\lvert\mathbf{d}\rvert}$	$F_{\mathrm{Id},\mathrm{Id}}$
$\mathcal{F}_{\underline{\mathbf{d}}} \cong G_{\mathbf{d}}/B_{\mathbf{d}}$	F_u	$\mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}}'}$	$F_{u,u'}$
$\mathcal{F}_{\varpi} \cong G_{\mathbf{d}}/B_{\varpi}$	F_{ϖ}	$\mathcal{F}_{\varpi} \times \mathcal{F}_{\varpi'}$	$F_{\varpi,\varpi'}$
$\mathcal{F}_{\mathbf{d}} = \bigsqcup_{\underline{\mathbf{d}}} \mathcal{F}_{\underline{\mathbf{d}}}$	$-$	$\mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}} = \bigsqcup_{\underline{\mathbf{d}},\underline{\mathbf{d}}'} \mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}}'}$	$-$

Table 1.8: Base varieties and their preferred base point

1.5.2 Incidence variety

Now it is time to conclude information about arrows, and construct spaces over varieties in Table 1.8.

Definition 1.5.8 (Incidence variety). *For a quiver Q with flag-type dimension vector $\underline{\mathbf{d}}$, define*

$$\begin{aligned}
\widetilde{\mathrm{Rep}}_{\underline{\mathbf{d}}}(Q) &:= \left\{(\rho, F) \in \mathrm{Rep}_{\mathbf{d}}(Q) \times \mathcal{F}_{\underline{\mathbf{d}}} \,\middle|\, \rho(M_j) \subseteq M_j \textit{ for any } j\right\} \\
\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q) &:= \{(\rho, F) \in \mathrm{Rep}_{\mathbf{d}}(Q) \times \mathcal{F}_{\mathbf{d}} \mid \rho(M_j) \subseteq M_j \textit{ for any } j\} \\
&= \bigsqcup_{\underline{\mathbf{d}}} \widetilde{\mathrm{Rep}}_{\underline{\mathbf{d}}}(Q)
\end{aligned}$$

and $\mu_{\underline{\mathbf{d}}}$, $\pi_{\underline{\mathbf{d}}}$, $\mu_{\mathbf{d}}$, $\pi_{\mathbf{d}}$ to be the natural morphisms from the incidence varieties to $\mathrm{Rep}_{\mathbf{d}}(Q)$ *or flag varieties, as follows:*

$$\begin{tikzcd}
 & \widetilde{\mathrm{Rep}}_{\underline{\mathbf{d}}}(Q) \subseteq \mathrm{Rep}_{\mathbf{d}}(Q) \times \mathcal{F}_{\underline{\mathbf{d}}} \arrow[dl, bend left, "\mu_{\underline{\mathbf{d}}}"'] \arrow[dr, bend left, "\pi_{\underline{\mathbf{d}}}"] & \\
\mathrm{Rep}_{\mathbf{d}}(Q) & & \mathcal{F}_{\underline{\mathbf{d}}}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q) \subseteq \mathrm{Rep}_{\mathbf{d}}(Q) \times \mathcal{F}_{\mathbf{d}} \arrow[dl, bend left, "\mu_{\mathbf{d}}"'] \arrow[dr, bend left, "\pi_{\mathbf{d}}"] & \\
\mathrm{Rep}_{\mathbf{d}}(Q) & & \mathcal{F}_{\mathbf{d}}
\end{tikzcd}$$

Remark 1.5.9. For $M \in \mathrm{Rep}_{\mathbf{d}}(Q)$, the **Springer fiber**

$$\mathrm{Flag}_{\underline{\mathbf{d}}}(M) := \mu_{\underline{\mathbf{d}}}^{-1}(M) \cong \pi_{\underline{\mathbf{d}}}\big(\mu_{\underline{\mathbf{d}}}^{-1}(M)\big) \subseteq \mathcal{F}_{\underline{\mathbf{d}}}$$

records the complete flags of subrepresentations of M. The partial flag variety version of $\mathrm{Flag}_{\underline{\mathbf{d}}}(M)$ will become the key object in the second part.

Definition 1.5.10 (Steinberg variety). *For quiver Q with flag-type dimension vectors $\underline{\mathbf{d}}$, $\underline{\mathbf{d}}'$, define*

$$\begin{aligned}
\mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}'} &:= \widetilde{\mathrm{Rep}}_{\underline{\mathbf{d}}}(Q) \times_{\mathrm{Rep}_{\mathbf{d}}(Q)} \widetilde{\mathrm{Rep}}_{\underline{\mathbf{d}}'}(Q) \\
\mathcal{Z}_{\mathbf{d}} &:= \widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q) \times_{\mathrm{Rep}_{\mathbf{d}}(Q)} \widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q) \\
&= \bigsqcup_{\underline{\mathbf{d}},\underline{\mathbf{d}}'} \mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}'}
\end{aligned}$$

*$\mathcal{Z}_{\mathbf{d}}$ is called the **Steinberg variety**.*

$\mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}'}$ can actually be realized as the incidence variety between $\mathrm{Rep}_{\mathbf{d}}(Q)$ and $\mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}}'}$, since

$$\begin{aligned}
\mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}'} &= \widetilde{\mathrm{Rep}}_{\underline{\mathbf{d}}}(Q) \times_{\mathrm{Rep}_{\mathbf{d}}(Q)} \widetilde{\mathrm{Rep}}_{\underline{\mathbf{d}}'}(Q) \\
&\subseteq \left(\mathrm{Rep}_{\mathbf{d}}(Q) \times \mathcal{F}_{\underline{\mathbf{d}}}\right) \times_{\mathrm{Rep}_{\mathbf{d}}(Q)} \left(\mathrm{Rep}_{\mathbf{d}}(Q) \times \mathcal{F}_{\underline{\mathbf{d}}'}\right) \\
&\cong \mathrm{Rep}_{\mathbf{d}}(Q) \times \mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}}'}
\end{aligned}$$

For that reason, we denote $\mu_{\underline{\mathbf{d}},\underline{\mathbf{d}}'}$, $\pi_{\underline{\mathbf{d}},\underline{\mathbf{d}}'}$, $\mu_{\mathbf{d},\mathbf{d}}$, $\pi_{\mathbf{d},\mathbf{d}}$ as natural morphisms from $\mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}'}$, $\mathcal{Z}_{\mathbf{d}}$ to $\mathrm{Rep}_{\mathbf{d}}(Q)$ or product of flag varieties, as follows:

$$\begin{tikzcd}
\mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}'} \subseteq \mathrm{Rep}_{\mathbf{d}}(Q) \times \mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}}'} \arrow[d, bend left, "\mu_{\underline{\mathbf{d}},\underline{\mathbf{d}}'}"'] \arrow[dr, bend left, "\pi_{\underline{\mathbf{d}},\underline{\mathbf{d}}'}"] & \\
\mathrm{Rep}_{\mathbf{d}}(Q) & \mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}}'}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{Z}_{\mathbf{d},\mathbf{d}'} \subseteq \mathrm{Rep}_{\mathbf{d}}(Q) \times \mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}'} \arrow[d, bend left, "\mu_{\mathbf{d},\mathbf{d}'}"'] \arrow[dr, bend left, "\pi_{\mathbf{d},\mathbf{d}'}"] & \\
\mathrm{Rep}_{\mathbf{d}}(Q) & \mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}'}
\end{tikzcd}$$

Remark 1.5.11 (Group actions).

1. $\mathrm{Rep}_{\mathbf{d}}(Q) \subseteq \oplus_{a \in Q_1} \mathfrak{g}_{|\mathbf{d}|}$ has a natural $G_{\mathbf{d}}$-action, which is induced by the conjugation action of $G_{\mathbf{d}}$ on $\mathfrak{g}_{|\mathbf{d}|}$. We have already mentioned the $G_{\mathbf{d}}$-action on $\mathcal{F}_{\underline{\mathbf{d}}}$ and $\mathcal{F}_{\mathbf{d}}$ in Remark 1.5.4. Therefore, by restriction we automatically get $G_{\mathbf{d}}$-actions on $\widetilde{\mathrm{Rep}}_{\underline{\mathbf{d}}}(Q)$, $\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)$, $\mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}'}$ and $\mathcal{Z}_{\mathbf{d}}$. All the maps we mentioned in Definition 1.5.8 are $G_{\mathbf{d}}$-equivariant.

2. In Section 6.2 we will also view all the varieties as $G_{\mathbf{d}} \times \mathbb{C}^\times$-varieties, so we also shortly introduce $\mathbb{C}^\times$-action here. View $\mathrm{Rep}_{\mathbf{d}}(Q)$ as a $\mathbb{C}$-vector space, $\mathbb{C}^\times$ acts on $\mathrm{Rep}_{\mathbf{d}}(Q)$ by scalar multiplication. For $\mathcal{F}_{\underline{\mathbf{d}}}$ and $\mathcal{F}_{\mathbf{d}}$, $\mathbb{C}^\times$ acts trivially, and by restriction we get $\mathbb{C}^\times$-actions on $\widetilde{\mathrm{Rep}}_{\underline{\mathbf{d}}}(Q)$, $\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)$, $\mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}'}$ and $\mathcal{Z}_{\mathbf{d}}$. Also, all the maps we mentioned above are $\mathbb{C}^\times$-equivariant.

3. It may worth mentioning that $\mathcal{F}_{\mathbf{d}}$ has an $\mathbb{W}_{|\mathbf{d}|}$-action which can be extended neither to $\mathbb{G}_{|\mathbf{d}|}$-action on $\mathcal{F}_{\mathbf{d}}$ nor to $\mathbb{W}_{|\mathbf{d}|}$-action on $\widetilde{\mathrm{Rep}}_{\underline{\mathbf{d}}}(Q)$.

1.6 Stratification and T-fixed points

Natural defined varieties resemble burr puzzles, they have delicate structures and can be decomposed as relatively easy pieces. In this subsection, we will find stratifications of varieties introduced in Section 1.5, and fix notations of orbits. We will also mention about their T-fixed points. These stratifications will give us a basis for the K-theory and cohomology theory in Chapter 2, while those T-fixed points will give us another "basis" in Chapter 4.

1.6.1 Stratification: flag variety

We begin with $\mathcal{F}_{|\mathbf{d}|}$ and $\mathcal{F}_{|\mathbf{d}|} \times \mathcal{F}_{|\mathbf{d}|}$, which is roughly a repetition of Section 1.1.

Definition 1.6.1 (Twisted action)**.** *We define the twisted $\mathbb{G}_{|\mathbf{d}|} \times \mathbb{G}_{|\mathbf{d}|}$-action on $\mathcal{F}_{|\mathbf{d}|} \times \mathcal{F}_{|\mathbf{d}|}$:*

$$\mathbb{G}_{|\mathbf{d}|} \times \mathbb{G}_{|\mathbf{d}|} \times \mathcal{F}_{|\mathbf{d}|} \times \mathcal{F}_{|\mathbf{d}|} \longrightarrow \mathcal{F}_{|\mathbf{d}|} \times \mathcal{F}_{|\mathbf{d}|} \qquad \left(g_1, g_2, \underline{F}_{g,g'}\right) \longmapsto \underline{F}_{g_1 g, g_2 g'}$$

which is the same as original $\mathbb{G}_{|\mathbf{d}|}$-action when we restrict to $\mathbb{G}_{|\mathbf{d}|} \times \{\mathrm{Id}\}$-action. Other $G \times G$-actions on $\mathcal{F} \times \mathcal{F}$ are defined in a similar way.

Definition 1.6.2 (Stratifications of $\mathcal{F}_{|\mathbf{d}|}$ and $\mathcal{F}_{|\mathbf{d}|} \times \mathcal{F}_{|\mathbf{d}|}$)**.** *For $\varpi, \varpi' \in \mathbb{W}_{|\mathbf{d}|}$, we define*

$$\begin{aligned} \mathcal{V}_{\varpi} &= \mathbb{B}_{|\mathbf{d}|} \cdot F_{\varpi} && \subseteq \mathcal{F}_{|\mathbf{d}|} \\ \boldsymbol{\mathcal{V}}_{\varpi,\varpi'} &= \left(\mathbb{B}_{|\mathbf{d}|} \times \mathbb{B}_{|\mathbf{d}|}\right) \cdot \underline{F}_{\varpi,\varpi'} && \subseteq \mathcal{F}_{|\mathbf{d}|} \times \mathcal{F}_{|\mathbf{d}|} \\ \boldsymbol{\mathcal{V}}_{\varpi'} &= \mathbb{G}_{|\mathbf{d}|} \cdot \underline{F}_{\mathrm{Id},\varpi'} && \subseteq \mathcal{F}_{|\mathbf{d}|} \times \mathcal{F}_{|\mathbf{d}|} \end{aligned}$$

as $\mathbb{B}_{|\mathbf{d}|}$-orbit, $\mathbb{B}_{|\mathbf{d}|} \times \mathbb{B}_{|\mathbf{d}|}$-orbit, $\mathbb{G}_{|\mathbf{d}|}$-orbit of $\mathcal{F}_{|\mathbf{d}|}$, $\mathcal{F}_{|\mathbf{d}|} \times \mathcal{F}_{|\mathbf{d}|}$, $\mathcal{F}_{|\mathbf{d}|} \times \mathcal{F}_{|\mathbf{d}|}$, respectively.

By Bruhat-decomposition, we are able to show

$$\mathcal{F}_{|\mathbf{d}|} = \bigsqcup_{\varpi} \mathcal{V}_{\varpi} \qquad \mathcal{F}_{|\mathbf{d}|} \times \mathcal{F}_{|\mathbf{d}|} = \bigsqcup_{\varpi'} \boldsymbol{\mathcal{V}}_{\varpi'} = \bigsqcup_{\varpi,\varpi'} \boldsymbol{\mathcal{V}}_{\varpi,\varpi'}.$$

We also realize these orbits as quotients of algebraic groups by the orbit-stabilizer theorem, as follows:

$$\begin{aligned} \mathcal{V}_{\varpi} &\cong \mathbb{B}_{|\mathbf{d}|} / \left(\mathbb{B}_{|\mathbf{d}|} \cap \mathbb{B}_{\varpi}\right) && \cong \mathbb{A}^{l(\varpi)} \\ \boldsymbol{\mathcal{V}}_{\varpi,\varpi'} &\cong \left(\mathbb{B}_{|\mathbf{d}|} \times \mathbb{B}_{|\mathbf{d}|}\right) / \left(\mathbb{B}_{|\mathbf{d}|} \cap \mathbb{B}_{\varpi} \times \mathbb{B}_{|\mathbf{d}|} \cap \mathbb{B}_{\varpi'}\right) && \cong \mathbb{A}^{l(\varpi)+l(\varpi')} \\ \boldsymbol{\mathcal{V}}_{\varpi'} &\cong \mathbb{G}_{|\mathbf{d}|} / \left(\mathbb{B}_{|\mathbf{d}|} \cap \mathbb{B}_{\varpi'}\right) && \cong \mathbb{A}^{l(\varpi')}\text{-bundle over } \mathcal{F}_{|\mathbf{d}|} \end{aligned}$$

Similar stratifications happen for $\mathcal{F}_u$ and $\mathcal{F}_{\mathbf{d}}$.

Definition 1.6.3 (Stratifications of $\mathcal{F}_u$ and $\mathcal{F}_u \times \mathcal{F}_{u'}$)**.** *For $u, u' \in \mathrm{Min}(\mathbb{W}_{|\mathbf{d}|}, \mathrm{W}_{\mathbf{d}})$, $w, w' \in W_{\mathbf{d}}$, we define*

$$\begin{aligned} \Omega_w^u &= B_{\mathbf{d}} \cdot F_{wu} && \subseteq \mathcal{F}_u \\ \boldsymbol{\Omega}_{w,w'}^{u,u'} &= \left(B_{\mathbf{d}} \times B_{\mathbf{d}}\right) \cdot \left(F_{wu}, F_{ww'u'}\right) && \subseteq \mathcal{F}_u \times \mathcal{F}_{u'} \\ \boldsymbol{\Omega}_{w'}^{u,u'} &= G_{\mathbf{d}} \cdot \left(F_u, F_{w'u'}\right) && \subseteq \mathcal{F}_u \times \mathcal{F}_{u'} \end{aligned}$$

as $B_{\mathbf{d}}$-orbit, $B_{\mathbf{d}} \times B_{\mathbf{d}}$-orbit, $G_{\mathbf{d}}$-orbit of $\mathcal{F}_u$, $\mathcal{F}_u \times \mathcal{F}_{u'}$, $\mathcal{F}_u \times \mathcal{F}_{u'}$, respectively.

By Bruhat decomposition, we are again able to show

$$\mathcal{F}_u = \bigsqcup_{w} \Omega_w^u \qquad \mathcal{F}_u \times \mathcal{F}_{u'} = \bigsqcup_{w'} \boldsymbol{\Omega}_{w'}^{u,u'} = \bigsqcup_{w,w'} \boldsymbol{\Omega}_{w,w'}^{u,u'}$$

and

$$\begin{aligned} \Omega_w^u &\cong B_{\mathbf{d}} / \left(B_{\mathbf{d}} \cap B_w\right) && \cong \mathbb{A}^{l(w)} \\ \boldsymbol{\Omega}_{w,w'}^{u,u'} &\cong \left(B_{\mathbf{d}} \times B_{\mathbf{d}}\right) / \left(B_{\mathbf{d}} \cap B_w \times B_{\mathbf{d}} \cap B_{w'}\right) && \cong \mathbb{A}^{l(w)+l(w')} \\ \boldsymbol{\Omega}_{w'}^{u,u'} &\cong G_{\mathbf{d}} / \left(B_{\mathbf{d}} \cap B_{w'}\right) && \cong \mathbb{A}^{l(w')}\text{-bundle over } \mathcal{F}_u \end{aligned}$$

Definition 1.6.4 (Stratifications of $\mathcal{F}_{\mathbf{d}}$ and $\mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}}$)**.** *For $\varpi, \varpi' \in \mathbb{W}_{|\mathbf{d}|}$, we define*

$$\begin{array}{rlll} \mathcal{O}_{\varpi} = & B_{\mathbf{d}} \cdot F_{\varpi} & \subseteq \mathcal{F}_{\varpi} & \subseteq \mathcal{F}_{\mathbf{d}} \\ \boldsymbol{\mathcal{O}}_{\varpi,\varpi'} = & (B_{\mathbf{d}} \times B_{\mathbf{d}}) \cdot \underline{F}_{\varpi,\varpi'} & \subseteq \mathcal{F}_{\varpi} \times \mathcal{F}_{\varpi\varpi'} & \subseteq \mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}} \\ \boldsymbol{\mathcal{O}}_{\varpi'} = & \bigsqcup_u G_{\mathbf{d}} \cdot \underline{F}_{u,\varpi'} & \subseteq \bigsqcup_u \mathcal{F}_u \times \mathcal{F}_{u\varpi'} & \subseteq \mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}} \end{array}$$

as $B_{\mathbf{d}}$-orbit, $B_{\mathbf{d}} \times B_{\mathbf{d}}$-orbit, (union of) $G_{\mathbf{d}}$-orbit of $\mathcal{F}_{\mathbf{d}}$, $\mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}}$, $\mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}}$, respectively.

Notice that $\mathcal{O}_{\varpi}$, $\boldsymbol{\mathcal{O}}_{\varpi,\varpi'}$, $\boldsymbol{\mathcal{O}}_{\varpi'}$ are preimages of $\mathcal{V}_{\varpi}$, $\boldsymbol{\mathcal{V}}_{\varpi,\varpi'}$, $\boldsymbol{\mathcal{V}}_{\varpi'}$ under the maps

$$\mathcal{F}_{\mathbf{d}} \hookrightarrow \mathcal{F}_{|\mathbf{d}|} \qquad \mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}} \hookrightarrow \mathcal{F}_{|\mathbf{d}|} \times \mathcal{F}_{|\mathbf{d}|}.$$

Therefore,

$$\mathcal{F}_{\mathbf{d}} = \bigsqcup_{\varpi} \mathcal{O}_{\varpi} \qquad \mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}} = \bigsqcup_{\varpi'} \boldsymbol{\mathcal{O}}_{\varpi'} = \bigsqcup_{\varpi,\varpi'} \boldsymbol{\mathcal{O}}_{\varpi,\varpi'}.$$

We still need to care about symbols. For $\varpi = wu$, $\varpi' = w'u'$, denote $uw'u' = \tilde{w}\tilde{u}$ for $\tilde{w}' \in W_{\mathbf{d}}$, $\tilde{u} \in \mathrm{Min}(\mathbb{W}_{|\mathbf{d}|}, \mathrm{W}_{\mathbf{d}})$, then

$$\underline{F}_{\varpi,\varpi'} = (F_{\varpi}, F_{\varpi\varpi'}) = (F_{wu}, F_{wuw'u'}) = (F_{wu}, F_{w\tilde{w}\tilde{u}}) \in \mathcal{F}_u \times \mathcal{F}_{\tilde{u}}.$$

This incompatibility comes from our twisted $G_{\mathbf{d}} \times G_{\mathbf{d}}$-actions. In particular, denote

$$\boldsymbol{\mathcal{O}}^u_{\varpi'} := G_{\mathbf{d}} \cdot \underline{F}_{u,\varpi'} \subseteq \mathcal{F}_u \times \mathcal{F}_{\tilde{u}},$$

we have $\boldsymbol{\mathcal{O}}_{\varpi'} = \sqcup_u \boldsymbol{\mathcal{O}}^u_{\varpi'}$ and identifications

$$\mathcal{O}_{\varpi} = \Omega^u_w \qquad \boldsymbol{\mathcal{O}}_{\varpi,\varpi'} = \boldsymbol{\Omega}^{u,\tilde{u}}_{w,\tilde{w}} \qquad \boldsymbol{\mathcal{O}}^u_{\varpi'} = \boldsymbol{\Omega}^{u,\tilde{u}}_{\tilde{w}}. \tag{$\star$}$$

We can also describe the closure of orbits, for example,

$$\overline{\Omega}^u_w = \bigsqcup_{x \leq w} \Omega^u_x \qquad \overline{\boldsymbol{\Omega}}^{u,u'}_{w,w'} = \bigsqcup_{x \leq w, x' \leq w'} \boldsymbol{\Omega}^{u,u'}_{x,x'} \qquad \overline{\boldsymbol{\Omega}}^{u,u'}_{w'} = \bigsqcup_{x' \leq w'} \boldsymbol{\Omega}^{u,u'}_{x'}$$

Especially, for any $s \in \Pi_{\mathbf{d}}$, $u, u' \in \mathrm{Min}(\mathbb{W}_{|\mathbf{d}|}, \mathrm{W}_{\mathbf{d}})$, we have

$$\overline{\boldsymbol{\Omega}}^{u,u'}_s = \boldsymbol{\Omega}^{u,u'}_s \sqcup \boldsymbol{\Omega}^{u,u'}_{\mathrm{Id}} \cong G_{\mathbf{d}} \times^{B_{\mathbf{d}}} (P_{\mathrm{Id},s}/B_{\mathbf{d}})$$

when we work over base point $F_{u,u'}$. If we work over different base points, we will get different isomorphisms, as follows:

$$\begin{aligned} \overline{\boldsymbol{\Omega}}^{u,u'}_s = \boldsymbol{\Omega}^{u,u'}_{\mathrm{Id}} \sqcup \boldsymbol{\Omega}^{u,u'}_s &\cong G_{\mathbf{d}}/(B_w \cap B_{ws}) && \sqcup G_{\mathbf{d}}/B_w \\ &\cong G_{\mathbf{d}} \times^{B_w} \big(B_w/(B_w \cap B_{ws})\big) && \sqcup G_{\mathbf{d}} \times^{B_w} (B_w/B_w) \\ &\cong G_{\mathbf{d}} \times^{B_w} (B_w s B_w/B_w) && \sqcup G_{\mathbf{d}} \times^{B_w} (B_w/B_w) \\ &\cong G_{\mathbf{d}} \times^{B_w} (\underline{P}_{w,s}/B_w) && \text{base point } F_{wu,wu'} \\ &\cong G_{\mathbf{d}} \times^{B_w} (\underline{P}_{w,s}/B_{ws}) && \text{base point } F_{wu,wsu'} \end{aligned}$$

Closures of $\boldsymbol{\mathcal{O}}$-cells are obtained by identifications ($\star$). To illustrate it, we compute $\overline{\boldsymbol{\mathcal{O}}}_s$ by hand. Let $\varpi' = s, us = \tilde{w}\tilde{u}$,

$$\begin{aligned}
\overline{\boldsymbol{\mathcal{O}}}_s &= \bigsqcup_u \overline{\boldsymbol{\mathcal{O}}}_s^u = \bigsqcup_u \overline{\boldsymbol{\Omega}}_{\tilde{w}}^{u,\tilde{u}} \\
&= \left(\bigsqcup_{u:usu^{-1}\in W_{\mathbf{d}}} \overline{\boldsymbol{\Omega}}_{usu^{-1}}^{u,u} \right) \sqcup \left(\bigsqcup_{u:usu^{-1}\notin W_{\mathbf{d}}} \overline{\boldsymbol{\Omega}}_{\mathrm{Id}}^{u,us} \right) \\
&= \left(\bigsqcup_{u:usu^{-1}\in W_{\mathbf{d}}} \boldsymbol{\Omega}_{usu^{-1}}^{u,u} \right) \sqcup \left(\bigsqcup_{u:usu^{-1}\notin W_{\mathbf{d}}} \boldsymbol{\Omega}_{\mathrm{Id}}^{u,us} \right) \sqcup \left(\bigsqcup_{u:usu^{-1}\in W_{\mathbf{d}}} \boldsymbol{\Omega}_{\mathrm{Id}}^{u,u} \right) \\
&= \boldsymbol{\mathcal{O}}_s \sqcup \left(\bigsqcup_{u:usu^{-1}\in W_{\mathbf{d}}} \boldsymbol{\mathcal{O}}_{\mathrm{Id}}^{u} \right)
\end{aligned}$$

We restrict the result of $\overline{\boldsymbol{\Omega}}_s^{u,u'}$ to $\overline{\boldsymbol{\mathcal{O}}}_s^u$ in Lemma 1.6.5.

Lemma 1.6.5. *For $\varpi = wu \in \mathbb{W}_{|\mathbf{d}|}$, $s \in \Pi$ such that $\varpi s \varpi^{-1} \in W_{\mathbf{d}}$, we have isomorphisms of $G_{\mathbf{d}}$-varieties*

$$\begin{aligned}
G_{\mathbf{d}} \times^{B_\varpi} (\underline{P}_{\varpi,s}/B_\varpi) &\longrightarrow \overline{\boldsymbol{\mathcal{O}}}_s^u & (g,p) &\longmapsto (g\cdot F_\varpi, gp\cdot F_\varpi) \\
G_{\mathbf{d}} \times^{B_\varpi} (\underline{P}_{\varpi,s}/B_{\varpi s}) &\longrightarrow \overline{\boldsymbol{\mathcal{O}}}_s^u & (g,p) &\longmapsto (g\cdot F_\varpi, gp\cdot F_{\varpi s})
\end{aligned}$$

Proof. Notice that when $\varpi s \varpi^{-1} \in W_{\mathbf{d}}$, $\boldsymbol{\mathcal{O}}_s^u = \boldsymbol{\Omega}_{usu^{-1}}^{u,u}$. Therefore,

$$\begin{aligned}
\overline{\boldsymbol{\mathcal{O}}}_s^u = \overline{\boldsymbol{\Omega}}_{usu^{-1}}^{u,u} &\cong \begin{cases} G_{\mathbf{d}} \times^{B_w} (\underline{P}_{w,usu^{-1}}/B_w) & \text{base point } F_{wu,wu} \\ G_{\mathbf{d}} \times^{B_w} (\underline{P}_{w,usu^{-1}}/B_{wusu^{-1}}) & \text{base point } F_{wu,wus} \end{cases} \\
&\cong \begin{cases} G_{\mathbf{d}} \times^{B_\varpi} (\underline{P}_{\varpi,s}/B_\varpi) & \text{base point } F_{\varpi,\varpi} \\ G_{\mathbf{d}} \times^{B_\varpi} (\underline{P}_{\varpi,s}/B_{\varpi s}) & \text{base point } F_{\varpi,\varpi s} \end{cases}
\end{aligned}$$

□

After so many notations are introduced rapidly, an enlightening example is needed here.

Example 1.6.6 (Follows Example 1.3.8)**.** *Here, $\mathbb{W}_{|\mathbf{d}|} = S_3$, $W_{\mathbf{d}} = S_1 \times S_2$,*

$$\varpi = ts = t\cdot s, \qquad \varpi' = s = \mathrm{Id}\cdot s, \qquad \varpi\varpi' = t = t\cdot \mathrm{Id}\,.$$

$\mathcal{F}_{\mathbf{d}}$ has 3 connected components, each of them has 2 $B_{\mathbf{d}}$-orbits;

$\mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}}$ has 9 connected components, each of them has 4 $B_{\mathbf{d}} \times B_{\mathbf{d}}$-orbits. We have

shape $(B_{\mathbf{d}}\times B_{\mathbf{d}})\cdot \underline{F}_{\varpi,\varpi'}$ $B_{\mathbf{d}}\cdot F_{\varpi\varpi'}$ $B_{\mathbf{d}}\cdot F_{\varpi}$		$\mathcal{F}_{\mathrm{Id}}$		$\mathcal{F}_s$		$\mathcal{F}_{st}$	
		$\mathcal{O}_{\mathrm{Id}}$	$\mathcal{O}_t$	$\mathcal{O}_s$	$\mathcal{O}_{ts}$	$\mathcal{O}_{st}$	$\mathcal{O}_{sts}$
$\mathcal{F}_{\mathrm{Id}}$	$\mathcal{O}_{\mathrm{Id}} = \Omega^{\mathrm{Id}}_{\mathrm{Id}}$	$\boldsymbol{\Omega}^{\mathrm{Id},\mathrm{Id}}_{\mathrm{Id},\mathrm{Id}}$	$\boldsymbol{\Omega}^{\mathrm{Id},\mathrm{Id}}_{\mathrm{Id},t}$	$\boldsymbol{\Omega}^{\mathrm{Id},s}_{\mathrm{Id},\mathrm{Id}}$	$\boldsymbol{\Omega}^{\mathrm{Id},s}_{\mathrm{Id},t}$	$\boldsymbol{\Omega}^{\mathrm{Id},st}_{\mathrm{Id},\mathrm{Id}}$	$\boldsymbol{\Omega}^{\mathrm{Id},st}_{\mathrm{Id},t}$
	$\mathcal{O}_t = \Omega^{\mathrm{Id}}_t$	$\boldsymbol{\Omega}^{\mathrm{Id},\mathrm{Id}}_{t,t}$	$\boldsymbol{\Omega}^{\mathrm{Id},\mathrm{Id}}_{t,\mathrm{Id}}$	$\boldsymbol{\Omega}^{\mathrm{Id},s}_{t,t}$	$\boldsymbol{\Omega}^{\mathrm{Id},s}_{t,\mathrm{Id}}$	$\boldsymbol{\Omega}^{\mathrm{Id},st}_{t,t}$	$\boldsymbol{\Omega}^{\mathrm{Id},st}_{t,\mathrm{Id}}$
$\mathcal{F}_s$	$\mathcal{O}_s = \Omega^s_{\mathrm{Id}}$	$\boldsymbol{\Omega}^{s,\mathrm{Id}}_{\mathrm{Id},\mathrm{Id}}$	$\boldsymbol{\Omega}^{s,\mathrm{Id}}_{\mathrm{Id},t}$	$\boldsymbol{\Omega}^{s,s}_{\mathrm{Id},\mathrm{Id}}$	$\boldsymbol{\Omega}^{s,s}_{\mathrm{Id},t}$	$\boldsymbol{\Omega}^{s,st}_{\mathrm{Id},\mathrm{Id}}$	$\boldsymbol{\Omega}^{s,st}_{\mathrm{Id},t}$
	$\mathcal{O}_{ts} = \Omega^s_t$	$\boldsymbol{\Omega}^{s,\mathrm{Id}}_{t,t}$	$\boldsymbol{\Omega}^{s,\mathrm{Id}}_{t,\mathrm{Id}}$	$\boldsymbol{\Omega}^{s,s}_{t,t}$	$\boldsymbol{\Omega}^{s,s}_{t,\mathrm{Id}}$	$\boldsymbol{\Omega}^{s,st}_{t,t}$	$\boldsymbol{\Omega}^{s,st}_{t,\mathrm{Id}}$
$\mathcal{F}_{st}$	$\mathcal{O}_{ts} = \Omega^{st}_{\mathrm{Id}}$	$\boldsymbol{\Omega}^{st,\mathrm{Id}}_{\mathrm{Id},\mathrm{Id}}$	$\boldsymbol{\Omega}^{st,\mathrm{Id}}_{\mathrm{Id},t}$	$\boldsymbol{\Omega}^{st,s}_{\mathrm{Id},\mathrm{Id}}$	$\boldsymbol{\Omega}^{st,s}_{\mathrm{Id},t}$	$\boldsymbol{\Omega}^{st,st}_{\mathrm{Id},\mathrm{Id}}$	$\boldsymbol{\Omega}^{st,st}_{\mathrm{Id},t}$
	$\mathcal{O}_{sts} = \Omega^{st}_t$	$\boldsymbol{\Omega}^{st,\mathrm{Id}}_{t,t}$	$\boldsymbol{\Omega}^{st,\mathrm{Id}}_{t,\mathrm{Id}}$	$\boldsymbol{\Omega}^{st,s}_{t,t}$	$\boldsymbol{\Omega}^{st,s}_{t,\mathrm{Id}}$	$\boldsymbol{\Omega}^{st,st}_{t,t}$	$\boldsymbol{\Omega}^{st,st}_{t,\mathrm{Id}}$

Table 1.9: stratifications of $\mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}}$

given every orbit a name, and other spaces are finite union of these orbits. For example,

$$\begin{aligned}
\boldsymbol{\mathcal{O}}_{ts,s} &= \boldsymbol{\Omega}^{s,\mathrm{Id}}_{t,\mathrm{Id}} \\
\boldsymbol{\mathcal{O}}^s_s &= \boldsymbol{\Omega}^{s,\mathrm{Id}}_{\mathrm{Id}} = \boldsymbol{\Omega}^{s,\mathrm{Id}}_{\mathrm{Id},\mathrm{Id}} \sqcup \boldsymbol{\Omega}^{s,\mathrm{Id}}_{t,\mathrm{Id}} \\
\boldsymbol{\mathcal{O}}_s &= \boldsymbol{\mathcal{O}}^s_s \sqcup \boldsymbol{\mathcal{O}}^{\mathrm{Id}}_s \sqcup \boldsymbol{\mathcal{O}}^{st}_s \\
&= \boldsymbol{\mathcal{O}}^s_s \sqcup \boldsymbol{\mathcal{O}}^{\mathrm{Id}}_s \sqcup \boldsymbol{\mathcal{O}}^{st}_s \\
&= \boldsymbol{\Omega}^{s,\mathrm{Id}}_{\mathrm{Id}} \sqcup \boldsymbol{\Omega}^{\mathrm{Id},s}_{\mathrm{Id}} \sqcup \boldsymbol{\Omega}^{st,st}_{\mathrm{Id}} \\
&= \boldsymbol{\Omega}^{s,\mathrm{Id}}_{\mathrm{Id},\mathrm{Id}} \sqcup \boldsymbol{\Omega}^{s,\mathrm{Id}}_{t,\mathrm{Id}} \sqcup \boldsymbol{\Omega}^{\mathrm{Id},s}_{\mathrm{Id},\mathrm{Id}} \sqcup \boldsymbol{\Omega}^{\mathrm{Id},s}_{t,\mathrm{Id}} \sqcup \boldsymbol{\Omega}^{st,st}_{\mathrm{Id},\mathrm{Id}} \sqcup \boldsymbol{\Omega}^{st,st}_{t,\mathrm{Id}}
\end{aligned}$$

Their closures are also clear from the table, for example,

$$\overline{\boldsymbol{\mathcal{O}}}_s = \boldsymbol{\mathcal{O}}_s \sqcup \boldsymbol{\Omega}^{st,st}_{\mathrm{Id}}$$

contains 8 $B_{\mathbf{d}} \times B_{\mathbf{d}}$-orbits.

1.6.2 Stratification: incidence variety

Now comes the stratifications of incidence varieties. Those stratifications are produced by taking the preimage of stratifications on base spaces. They are relatively easy to obtain, while their closures are quite difficult to analyze.

Definition 1.6.7 (Stratifications of incidence varieties). *For $\varpi = wu$, $\varpi' = w'u' \in \mathbb{W}_{|\mathbf{d}|}$, denote $uwu' = \tilde{w}\tilde{u}$, $\underline{\mathbf{d}} = W_{\mathbf{d}}u$, $\underline{\mathbf{d}}' = W_{\mathbf{d}}u'$, $\widetilde{\underline{\mathbf{d}}} = W_{\mathbf{d}}\tilde{u}$, we define*

$$\begin{aligned}
\widetilde{\Omega}^u_w &:= \pi^{-1}_{\underline{\mathbf{d}}}(\Omega^u_w) && \subseteq \widetilde{\mathrm{Rep}}_{\underline{\mathbf{d}}}(Q)\\
\widetilde{\boldsymbol{\Omega}}^{u,u'}_{w,w'} &:= \pi^{-1}_{\underline{\mathbf{d}},\underline{\mathbf{d}}'}(\boldsymbol{\Omega}^{u,u'}_{w,w'}) && \subseteq \mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}'}\\
\widetilde{\boldsymbol{\Omega}}^{u,u'}_{w'} &:= \pi^{-1}_{\underline{\mathbf{d}},\underline{\mathbf{d}}'}(\boldsymbol{\Omega}^{u,u'}_{w'}) && \subseteq \mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}'}\\
\widetilde{\boldsymbol{\mathcal{O}}}^u_{\varpi'} &:= \pi^{-1}_{\underline{\mathbf{d}},\widetilde{\underline{\mathbf{d}}}}(\boldsymbol{\mathcal{O}}^u_{\varpi'}) = \widetilde{\boldsymbol{\Omega}}^{u,\tilde{u}}_{\tilde{w}} && \subseteq \mathcal{Z}_{\underline{\mathbf{d}},\widetilde{\underline{\mathbf{d}}}}\\
\widetilde{\mathcal{O}}_\varpi &:= \pi^{-1}_{\mathbf{d}}(\mathcal{O}_\varpi) && \subseteq \widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)\\
\widetilde{\boldsymbol{\mathcal{O}}}_{\varpi,\varpi'} &:= \pi^{-1}_{\mathbf{d},\mathbf{d}'}(\boldsymbol{\mathcal{O}}_{\varpi,\varpi'}) && \subseteq \mathcal{Z}_{\mathbf{d}}\\
\widetilde{\boldsymbol{\mathcal{O}}}_{\varpi'} &:= \pi^{-1}_{\mathbf{d},\mathbf{d}'}(\boldsymbol{\mathcal{O}}_{\varpi'}) && \subseteq \mathcal{Z}_{\mathbf{d}}
\end{aligned}$$

It is not hard to see that they are stratifications:

$$\begin{aligned}
\widetilde{\mathrm{Rep}}_{\underline{\mathbf{d}}}(Q) &= \bigsqcup_{\varpi} \widetilde{\Omega}^u_w & \mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}'} &= \bigsqcup_{w} \widetilde{\boldsymbol{\Omega}}^{u,u'}_{w'} = \bigsqcup_{w,w'} \widetilde{\boldsymbol{\Omega}}^{u,u'}_{w,w'}\\
\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q) &= \bigsqcup_{\varpi} \widetilde{\mathcal{O}}_\varpi & \mathcal{Z}_{\mathbf{d}} &= \bigsqcup_{\varpi'} \widetilde{\boldsymbol{\mathcal{O}}}_{\varpi'} = \bigsqcup_{\varpi,\varpi'} \widetilde{\boldsymbol{\mathcal{O}}}_{\varpi,\varpi'}
\end{aligned}$$

Proposition 1.6.8. *Those stratifications are affine spaces over corresponding base spaces. To be precise,*

$$\begin{aligned}
\widetilde{\Omega}^u_w &= \mathfrak{r}_{wu}\text{-}\textit{bundle over } \Omega^u_w\\
\widetilde{\boldsymbol{\Omega}}^{u,u'}_{w,w'} &= \mathfrak{r}_{wu,ww'u'}\text{-}\textit{bundle over } \boldsymbol{\Omega}^{u,u'}_{w,w'}\\
\widetilde{\boldsymbol{\Omega}}^{u,u'}_{w'} &= \mathfrak{r}_{u,w'u'}\text{-}\textit{bundle over } \boldsymbol{\Omega}^{u,u'}_{w'}\\
\widetilde{\boldsymbol{\mathcal{O}}}^u_{\varpi'} &= \underline{\mathfrak{r}}_{u,\varpi'}\text{-}\textit{bundle over } \boldsymbol{\mathcal{O}}^u_{\varpi'}\\
\widetilde{\mathcal{O}}_\varpi &= \mathfrak{r}_{\varpi}\text{-}\textit{bundle over } \mathcal{O}_\varpi\\
\widetilde{\boldsymbol{\mathcal{O}}}_{\varpi,\varpi'} &= \underline{\mathfrak{r}}_{\varpi,\varpi'}\text{-}\textit{bundle over } \boldsymbol{\mathcal{O}}_{\varpi,\varpi'}\\
\widetilde{\boldsymbol{\mathcal{O}}}_{\varpi'} &= \underline{\mathfrak{r}}_{\mathrm{Id},\varpi'}\text{-}\textit{bundle over } \boldsymbol{\mathcal{O}}_{\varpi'}
\end{aligned}$$

Proof. The fibers are all computed over the preferred base point. The group action induces the isomorphism between different fibers, and lift affine local charts on base space (viewed as group quotient) to the local charts of fiber bundles. □

We will frequently use closures of some stratifications, so we give them names.

Definition 1.6.9. *We define*

$$\begin{aligned} \mathcal{Z}^{u,u'}_{w'} &:= \overline{\widetilde{\boldsymbol{\Omega}}}^{u,u'}_{w'} &\subseteq \qquad \mathcal{Z}^{u,u'} &:= \mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}'}, \\ \mathcal{Z}_{\varpi'} &:= \overline{\widetilde{\boldsymbol{\mathcal{O}}}}_{\varpi'} &\subseteq \qquad & \mathcal{Z}_{\mathbf{d}}\,. \end{aligned}$$

Proposition 1.6.10 (Properties of the closure)**.** *$\mathcal{Z}_{\varpi'}$ is a Zarisky-locally trivial cone bundle over $\mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}}$. To be precise, under the map*

$$\pi_{\mathbf{d},\mathbf{d},\varpi'} : \mathcal{Z}_{\varpi'} \longrightarrow \mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}},$$

for any $x, x' \in \mathbb{W}_{|\mathbf{d}|}$, $\pi^{-1}_{\mathbf{d},\mathbf{d},\varpi'}(\boldsymbol{\mathcal{O}}_{x,x'})$ is a trivial fiber bundle over $\boldsymbol{\mathcal{O}}_{x,x'}$, whose fibers are cones.

Notice that

$$\begin{aligned} \mathcal{Z}^{u,u'}_{w'} &:= \overline{\widetilde{\boldsymbol{\Omega}}}^{u,u'}_{w'} &\subseteq \qquad \widetilde{\overline{\boldsymbol{\Omega}}}^{u,u'}_{w'} &:= \pi^{-1}_{\underline{\mathbf{d}},\underline{\mathbf{d}}'}\left(\overline{\boldsymbol{\Omega}}^{u,u'}_{w'}\right), \\ \mathcal{Z}_{\varpi'} &:= \overline{\widetilde{\boldsymbol{\mathcal{O}}}}_{\varpi'} &\subseteq \qquad \widetilde{\overline{\boldsymbol{\mathcal{O}}}}_{\varpi'} &:= \pi^{-1}_{\mathbf{d},\mathbf{d}}\left(\overline{\boldsymbol{\mathcal{O}}}_{\varpi'}\right). \end{aligned}$$

Even though these inclusions are usually not equalities, we can still say something when the length of w' or ϖ' is small. For example,

$$\begin{aligned} \mathcal{Z}^{u,u'}_{\mathrm{Id}} &= \widetilde{\boldsymbol{\Omega}}^{u,u'}_{\mathrm{Id}} \\ \mathcal{Z}_{\mathrm{Id}} &= \widetilde{\boldsymbol{\mathcal{O}}}_{\mathrm{Id}} \\ \widetilde{\boldsymbol{\Omega}}^{u,u'}_{s} \sqcup \boldsymbol{\Omega}^{u,u'}_{\mathrm{Id}} \subseteq \mathcal{Z}^{u,u'}_{s} &\subseteq \widetilde{\boldsymbol{\Omega}}^{u,u'}_{s} \sqcup \widetilde{\boldsymbol{\Omega}}^{u,u'}_{\mathrm{Id}} && (s \in \Pi_{\mathbf{d}}) \\ \widetilde{\boldsymbol{\mathcal{O}}}_{s} \sqcup \left(\bigsqcup_{u:usu^{-1} \in W_{\mathbf{d}}} \boldsymbol{\mathcal{O}}^{u}_{\mathrm{Id}} \right) \subseteq \mathcal{Z}_{s} &\subseteq \widetilde{\boldsymbol{\mathcal{O}}}_{s} \sqcup \left(\bigsqcup_{u:usu^{-1} \in W_{\mathbf{d}}} \widetilde{\boldsymbol{\mathcal{O}}}^{u}_{\mathrm{Id}} \right) && (s \in \Pi\ ) \end{aligned}$$

Proposition 1.6.11. *$\mathcal{Z}_s$ is a Zarisky-locally trivial vector bundle over $\overline{\boldsymbol{\mathcal{O}}}_s$, with fiber $\mathfrak{r}_{u,us}$ at point $\underline{F}_{u,s}$.*

Proof. This is claimed in [6, 2.20(c)]. In fact, we have a $G_{\mathbf{d}}$-equivariant morphism

$$\phi : G_{\mathbf{d}} \times^{B_u} \left(\underline{P}_{u,s}/B_{us} \times \underline{\mathfrak{r}}_{u,s}\right) \hookrightarrow \mathrm{Rep}_{\mathbf{d}}(Q) \times \mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}} \qquad (g,p,x) \longmapsto (gx, g\cdot F_u, gp\cdot F_{us})$$

which realized $G_{\mathbf{d}} \times^{B_u} \left(\underline{P}_{u,s}/B_{us} \times \underline{\mathfrak{r}}_{u,s}\right)$ as a closed subset of $\mathrm{Rep}_{\mathbf{d}}(Q) \times \mathcal{F}_{\mathbf{d}}$. In the meantime, the open dense subset

$$G_{\mathbf{d}} \times^{B_u} \left(B_{us} s B_{us}/B_{us} \times \underline{\mathfrak{r}}_{u,s}\right) \subseteq G_{\mathbf{d}} \times^{B_u} \left(\underline{P}_{u,s}/B_{us} \times \underline{\mathfrak{r}}_{u,s}\right)$$

is identified with $\widetilde{\boldsymbol{\mathcal{O}}}^{u}_{s}$ by ϕ. Therefore, ϕ identifies $\mathcal{Z}^{u,\tilde{u}}_{s}$ with the vector bundle $G_{\mathbf{d}} \times^{B_u} \left(\underline{P}_{u,s}/B_{us} \times \underline{\mathfrak{r}}_{u,s}\right)$ over $\overline{\boldsymbol{\mathcal{O}}}^{u}_{s}$, with fiber $\underline{\mathfrak{r}}_{u,s} = \mathfrak{r}_{u,us}$. ☐

Remark 1.6.12. By the same method, one can show that $\overline{\widetilde{\mathcal{O}}}_s$ is a Zarisky-locally trivial vector bundle over $\overline{\mathcal{O}}_s$, with fiber $\mathfrak{r}_s$ at point F_s.

We end this subsection by Table 1.10:

stratification *stabilizer* *variety* *base point*	*type*	B-orbit	$B \times B$-orbit twisted stabilizer	G-orbit	Remark
$\mathcal{F}$ F_g	$\mathcal{F} \times \mathcal{F}$ $(F_g, F_{gg'})$	Ω_g $B \cap gBg^{-1}$	$\boldsymbol{\Omega}_{g,g'}$ $\left(B \cap gBg^{-1}\right) \times \left(B \cap g'Bg'^{-1}\right)$	$\boldsymbol{\Omega}_{g'}$ $gBg^{-1} \cap gg'B(gg')^{-1}$	
$\mathcal{F}_{\lvert\mathbf{d}\rvert}$ F_ϖ	$\mathcal{F}_{\lvert\mathbf{d}\rvert} \times \mathcal{F}_{\lvert\mathbf{d}\rvert}$ $(F_\varpi, F_{\varpi\varpi'})$	$\mathcal{V}_\varpi$ $\mathbb{B}_{\lvert\mathbf{d}\rvert} \cap \mathbb{B}_\varpi$	$\boldsymbol{\mathcal{V}}_{\varpi,\varpi'}$ $\left(\mathbb{B}_{\lvert\mathbf{d}\rvert} \cap \mathbb{B}_\varpi\right) \times \left(\mathbb{B}_{\lvert\mathbf{d}\rvert} \cap \mathbb{B}_{\varpi'}\right)$	$\boldsymbol{\mathcal{V}}_{\varpi'}$ $\mathbb{B}_\varpi \cap \mathbb{B}_{\varpi\varpi'}$	
$\mathcal{F}_u$ F_{wu}	$\mathcal{F}_u \times \mathcal{F}_{u'}$ $(F_{wu}, F_{ww'u'})$	Ω^u_w $B_\mathbf{d} \cap B_w$	$\boldsymbol{\Omega}^{u,u'}_{w,w'}$ $(B_\mathbf{d} \cap B_w) \times (B_\mathbf{d} \cap B_{w'})$	$\boldsymbol{\Omega}^{u,u'}_{w'}$ $B_w \cap B_{ww'}$	
$\mathcal{F}_\mathbf{d}$ F_ϖ F_{wu}	$\mathcal{F}_\mathbf{d} \times \mathcal{F}_\mathbf{d}$ $(F_\varpi, F_{\varpi\varpi'})$ $(F_{wu}, F_{w\tilde{w}\tilde{u}})$	Ω^u_w $B_\mathbf{d} \cap B_w$	$\boldsymbol{\Omega}^{u,\tilde{u}}_{w,\tilde{w}}$ $(B_\mathbf{d} \cap B_w) \times (B_\mathbf{d} \cap B_{\tilde{w}})$	$\boldsymbol{\mathcal{O}}^u_{\varpi'} = \boldsymbol{\Omega}^{u,\tilde{u}}_{\tilde{w}}$ $B_w \cap B_{w\tilde{w}}$	
The following may not be single orbit, but derived from the above definition.					
$\mathcal{F}_\mathbf{d}$ F_ϖ	$\mathcal{F}_\mathbf{d} \times \mathcal{F}_\mathbf{d}$ $(F_\varpi, F_{\varpi\varpi'})$	$\mathcal{O}_\varpi$ Ω^u_w	$\boldsymbol{\mathcal{O}}_{\varpi,\varpi'}$ $\boldsymbol{\Omega}^{u,\tilde{u}}_{w,\tilde{w}}$	$\boldsymbol{\mathcal{O}}_{\varpi'}$ $\sqcup_u \boldsymbol{\mathcal{O}}^u_{\varpi'}$	preimage of $\mathcal{F}_\mathbf{d} \times \mathcal{F}_\mathbf{d} \hookrightarrow \mathcal{F}_{\lvert\mathbf{d}\rvert} \times \mathcal{F}_{\lvert\mathbf{d}\rvert}$
$\widetilde{\mathrm{Rep}}_{\underline{\mathbf{d}}}(Q)$ F_{wu}	$\mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}'}$ $(F_{wu}, F_{ww'u'})$	$\widetilde{\Omega}^u_w$	$\widetilde{\boldsymbol{\Omega}}^{u,u'}_{w,w'}$	$\widetilde{\boldsymbol{\Omega}}^{u,u'}_{w'}$	preimage of $\mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}'} \hookrightarrow \mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}}'}$
$\widetilde{\mathrm{Rep}}_\mathbf{d}(Q)$ F_ϖ	$\mathcal{Z}_\mathbf{d}$ $(F_\varpi, F_{\varpi\varpi'})$	$\widetilde{\Omega}^u_w$	$\widetilde{\boldsymbol{\Omega}}^{u,\tilde{u}}_{w,\tilde{w}}$	$\widetilde{\boldsymbol{\mathcal{O}}}^u_{\varpi'} = \widetilde{\boldsymbol{\Omega}}^{u,\tilde{u}}_{\tilde{w}}$	preimage of $\mathcal{Z}_\mathbf{d} \hookrightarrow \mathcal{F}_\mathbf{d} \times \mathcal{F}_\mathbf{d}$
$\widetilde{\mathrm{Rep}}_\mathbf{d}(Q)$ F_ϖ	$\mathcal{Z}_\mathbf{d}$ $(F_\varpi, F_{\varpi\varpi'})$	$\widetilde{\mathcal{O}}_\varpi$ $\widetilde{\Omega}^u_w$	$\widetilde{\boldsymbol{\mathcal{O}}}_{\varpi,\varpi'}$ $\widetilde{\boldsymbol{\Omega}}^{u,\tilde{u}}_{w,\tilde{w}}$	$\widetilde{\boldsymbol{\mathcal{O}}}_{\varpi'}$ $\sqcup_u \widetilde{\boldsymbol{\mathcal{O}}}^u_{\varpi'}$	preimage of $\mathcal{Z}_\mathbf{d} \hookrightarrow \mathcal{F}_\mathbf{d} \times \mathcal{F}_\mathbf{d}$

Table 1.10: stratifications of typical varieties

1.6.3 T-fixed points

Compare with stratifications, T-fixed points are easy to compute and have clear structures. Somewhat surprisingly, these T-fixed points encode most information of varieties.

Recall that the T-fixed points of a complete flag variety $\mathcal{F}$ are exactly those coordinate flags $\{F_w \,|\, w \in W\}$. For absolute or relative flag varieties, we have similar results:

$$\mathcal{F}_{\lvert\mathbf{d}\rvert}^{\mathbb{T}_{\lvert\mathbf{d}\rvert}} = \mathcal{F}_\mathbf{d}^{T_\mathbf{d}} = \left\{F_\varpi \,\middle|\, \varpi \in \mathbb{W}_{\lvert\mathbf{d}\rvert}\right\} \qquad \mathcal{F}_u^{T_\mathbf{d}} = \{F_{wu} \,|\, w \in W_\mathbf{d}\}$$

For $\mathrm{Rep}_\mathbf{d}(Q)$, we get

$$\left(\mathrm{Rep}_\mathbf{d}(Q)\right)^{T_\mathbf{d}} = \bigoplus_{a \in Q_1} \left(\mathrm{Hom}\left(\mathbb{C}^{\mathbf{d}_{s(a)}}, \mathbb{C}^{\mathbf{d}_{t(a)}}\right)\right)^{T_\mathbf{d}} = \{\rho_0\}$$

where ρ_0 is the zero representation in $\mathrm{Rep}_\mathbf{d}(Q)$.

Combining these two results, one can easily describe T-fixed points of varieties constructed over them:

$$\begin{aligned}
\big(\mathcal{F}_{|\mathbf{d}|}\times\mathcal{F}_{|\mathbf{d}|}\big)^{\mathbb{T}_{|\mathbf{d}|}} &= \big\{(F_\varpi,F_{\varpi'})\,\big|\,\varpi,\varpi'\in\mathbb{W}_{|\mathbf{d}|}\big\}\\
(\mathcal{F}_u\times\mathcal{F}_{u'})^{T_\mathbf{d}} &= \big\{(F_{wu},F_{w'u'})\,\big|\,w,w'\in W_\mathbf{d}\big\}\\
(\mathcal{F}_\mathbf{d}\times\mathcal{F}_\mathbf{d})^{T_\mathbf{d}} &= \big\{(F_\varpi,F_{\varpi'})\,\big|\,\varpi,\varpi'\in\mathbb{W}_{|\mathbf{d}|}\big\}\\
\Big(\widetilde{\mathrm{Rep}}_{\underline{\mathbf{d}}}(Q)\Big)^{T_\mathbf{d}} &= \{(\rho_0,F_{wu})\,|\,w\in W_\mathbf{d}\}\\
\Big(\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)\Big)^{T_\mathbf{d}} &= \big\{(\rho_0,F_\varpi)\,\big|\,\varpi\in\mathbb{W}_{|\mathbf{d}|}\big\}\\
\big(\mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}'}\big)^{T_\mathbf{d}} &= \big\{(\rho_0,F_{wu},F_{w'u'})\,\big|\,w,w'\in W_\mathbf{d}\big\}\\
(\mathcal{Z}_\mathbf{d})^{T_\mathbf{d}} &= \big\{(\rho_0,F_\varpi,F_{\varpi'})\,\big|\,\varpi,\varpi'\in\mathbb{W}_{|\mathbf{d}|}\big\}
\end{aligned}$$

Notice that, each $B_\mathbf{d}\times B_\mathbf{d}$-orbit of $\mathcal{F}_\mathbf{d}\times\mathcal{F}_\mathbf{d}$ contains exactly one $T_\mathbf{d}$-fixed point. Also, all the T-fixed points lie in the zero sections. By this reason, we can compute more:

$$\begin{aligned}
\Big(\mathcal{Z}_{\mathrm{Id}}^{u,u'}\Big)^{T_\mathbf{d}} &= \{(\rho_0,F_{wu},F_{wu'})\,|\,w\in W_\mathbf{d}\}\\
(\mathcal{Z}_{\mathrm{Id}})^{T_\mathbf{d}} &= \big\{(\rho_0,F_\varpi,F_\varpi)\,\big|\,\varpi\in\mathbb{W}_{|\mathbf{d}|}\big\}\\
\Big(\mathcal{Z}_{s}^{u,u'}\Big)^{T_\mathbf{d}} &= \{(\rho_0,F_{wu},F_{wsu'})\,|\,w\in W_\mathbf{d}\}\sqcup\{(\rho_0,F_{wu},F_{wu'})\,|\,w\in W_\mathbf{d}\}\\
(\mathcal{Z}_s)^{T_\mathbf{d}} &= \big\{(\rho_0,F_\varpi,F_{\varpi s})\,\big|\,\varpi\in\mathbb{W}_{|\mathbf{d}|}\big\}\sqcup\big\{(\rho_0,F_\varpi,F_\varpi)\,\big|\,\varpi\in\mathbb{W}_{|\mathbf{d}|},\varpi s\varpi^{-1}\in W_\mathbf{d}\big\}
\end{aligned}$$

1.6.4 Tangent spaces of T-fixed points

The tangent space of T-fixed points will be used in Chapter 4, so we fix symbols of them and compute some of them as Lie algebras.[7]

Definition 1.6.13 (Tangent space of T-fixed points)**.** *For $\varpi,\varpi',x\in\mathbb{W}_{|\mathbf{d}|}$, we denote the following tangent spaces:*

$$\begin{array}{lll}
\mathcal{T}_\varpi := T_{F_\varpi}\mathcal{F}_\mathbf{d} & \mathcal{T}^x_\varpi := T_{F_\varpi}\overline{\mathcal{O}}_x & \mathcal{T}^x_{\varpi,\varpi'} := T_{F_{\varpi,\varpi'}}\overline{\boldsymbol{\mathcal{O}}}_x\\
\widetilde{\mathcal{T}}_\varpi := T_{(\rho_0,F_\varpi)}\,\widetilde{\mathrm{Rep}}_\mathbf{d}(Q) & \widetilde{\mathcal{T}}^x_\varpi := T_{(\rho_0,F_\varpi)}\overline{\widetilde{\mathcal{O}}}_x & \widetilde{\mathcal{T}}^x_{\varpi,\varpi'} := T_{(\rho_0,F_\varpi,F_{\varpi'})}\,\mathcal{Z}_x
\end{array}$$

For completeness, denote

$$\mathcal{T}_{\varpi,\varpi'} := T_{F_{\varpi,\varpi'}}\big(\mathcal{F}_\mathbf{d}\times\mathcal{F}_\mathbf{d}\big)\qquad \widetilde{\mathcal{T}}_{\varpi,\varpi'} := T_{(\rho_0,F_\varpi,F_{\varpi'})}\,\mathcal{Z}_\mathbf{d}\,.$$

When we underline, the subscripts are twisted. For example,

$$\underline{\mathcal{T}}^x_{\varpi,\varpi'} := \mathcal{T}^x_{\varpi,\varpi\varpi'} = T_{F_{\varpi,\varpi\varpi'}}\overline{\boldsymbol{\mathcal{O}}}_x.$$

[7] In algebraic geometry, we can define the tangent space at even singular points, see [5, 12.1].

From the description of $\mathcal{F}_{\mathbf{d}}$ and $\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)$, we know that

$$\begin{aligned}
\mathcal{T}_{\varpi} &= T_{F_{\varpi}}\mathcal{F}_{\mathbf{d}} \cong T_{\mathrm{Id}}(G_{\mathbf{d}}/B_{\varpi}) \cong \mathfrak{g}_{\mathbf{d}}/\mathfrak{b}_{\varpi} && \cong \mathfrak{n}_{\varpi}^{-} \\
\widetilde{\mathcal{T}}_{\varpi} &= T_{(\rho_0,F_{\varpi})}\,\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q) \cong T_{\rho_0}\mathfrak{r}_{\varpi} \oplus T_{F_{\varpi}}\mathcal{F}_{\mathbf{d}} && \cong \mathfrak{r}_{\varpi} \oplus \mathfrak{n}_{\varpi}^{-}
\end{aligned}$$

For the rest, we can only compute special cases.

Proposition 1.6.14. *For $s \in \Pi$, We have identifications*

$$\begin{aligned}
\mathcal{T}_{\mathrm{Id}}^{s} &\cong \mathfrak{m}_{s,\mathrm{Id}} & \widetilde{\mathcal{T}}_{\mathrm{Id}}^{s} &\cong \mathfrak{r}_{s} \oplus \mathfrak{m}_{s,\mathrm{Id}} \\
\mathcal{T}_{s}^{s} &\cong \mathfrak{m}_{\mathrm{Id},s} & \widetilde{\mathcal{T}}_{s}^{s} &\cong \mathfrak{r}_{s} \oplus \mathfrak{m}_{\mathrm{Id},s}.
\end{aligned}$$

Proof. We know from Remark 1.6.12 that

$$\begin{aligned}
\mathcal{T}_{\mathrm{Id}}^{s} &\cong T_{\mathrm{Id}}(P_{\mathrm{Id},s}/B_{\mathbf{d}}) \cong \mathfrak{p}_{\mathrm{Id},s}/\mathfrak{b}_{\mathbf{d}} \cong \mathfrak{b}_{s}/\left(\mathfrak{b}_{s} \cap \mathfrak{b}_{\mathbf{d}}\right) && \cong \mathfrak{m}_{s,\mathrm{Id}} \\
\widetilde{\mathcal{T}}_{\mathrm{Id}}^{s} &\cong T_{\rho_0}\mathfrak{r}_{s} \oplus \mathcal{T}_{\mathrm{Id}}^{s} && \cong \mathfrak{r}_{s} \oplus \mathfrak{m}_{s,\mathrm{Id}}
\end{aligned}$$

Other proofs are the same. □

Proposition 1.6.15. *For $\varpi \in \mathbb{W}_{|\mathbf{d}|}$, $s \in \mathrm{Min}(\mathbb{W}_{|\mathbf{d}|}, \mathrm{W}_{\mathbf{d}})$, We have identifications*

$$\begin{aligned}
\mathcal{T}_{\varpi,\varpi}^{s} &\cong \mathfrak{n}_{\varpi}^{-} \oplus \mathfrak{m}_{\varpi s,\varpi} & \widetilde{\mathcal{T}}_{\varpi,\varpi}^{s} &\cong \mathfrak{r}_{\varpi,\varpi s} \oplus \mathfrak{n}_{\varpi}^{-} \oplus \mathfrak{m}_{\varpi s,\varpi} \\
\mathcal{T}_{\varpi,\varpi s}^{s} &\cong \mathfrak{n}_{\varpi}^{-} \oplus \mathfrak{m}_{\varpi,\varpi s} & \widetilde{\mathcal{T}}_{\varpi,\varpi s}^{s} &\cong \mathfrak{r}_{\varpi,\varpi s} \oplus \mathfrak{n}_{\varpi}^{-} \oplus \mathfrak{m}_{\varpi,\varpi s}
\end{aligned}$$

Proof. We know from Lemma 1.6.5 and Proposition 1.6.11 that

$$\begin{aligned}
\mathcal{T}_{\varpi,\varpi}^{s} &\cong T_{(\mathrm{Id},\mathrm{Id})}\left(G_{\mathbf{d}} \times^{B_{\varpi}} \left(\underline{P}_{\varpi,s}/B_{\varpi}\right)\right) \cong \mathfrak{g}_{\mathbf{d}}/\mathfrak{b}_{\varpi} \oplus \mathfrak{p}_{\varpi,\varpi s}/\mathfrak{b}_{\varpi} && \cong \mathfrak{n}_{\varpi}^{-} \oplus \mathfrak{m}_{\varpi s,\varpi} \\
\widetilde{\mathcal{T}}_{\varpi,\varpi}^{s} &\cong T_{\rho_0}\mathfrak{r}_{\varpi,\varpi s} \oplus \mathcal{T}_{\varpi,\varpi}^{s} && \cong \mathfrak{r}_{\varpi,\varpi s} \oplus \mathfrak{n}_{\varpi}^{-} \oplus \mathfrak{m}_{\varpi s,\varpi}
\end{aligned}$$

Other proofs are the same. □

Remark 1.6.16. We know a little more on the biggest cells. Here is an example. When $\varpi' = \varpi x$, $F_{\varpi,\varpi x} \in \mathcal{O}_x$, so

$$\begin{aligned}
\mathcal{T}_{\varpi,\varpi x}^{x} &= T_{F_{\varpi,\varpi x}}\overline{\mathcal{O}}_{x} = T_{F_{\varpi,\varpi x}}\mathcal{O}_{x} = T_{F_{\varpi,\varpi x}}\mathcal{O}_{x}^{u} && \cong \mathfrak{n}_{\varpi}^{-} \oplus \mathfrak{m}_{\varpi,\varpi x} \\
\widetilde{\mathcal{T}}_{\varpi,\varpi x}^{x} &= T_{(\rho_0,F_{\varpi},F_{\varpi x})}\,\mathcal{Z}_{x} = T_{(\rho_0,F_{\varpi},F_{\varpi x})}\widetilde{\mathcal{O}}_{x} \cong T_{\rho_0}\mathfrak{r}_{\varpi,\varpi x} \oplus \mathcal{T}_{\varpi,\varpi x}^{x} && \cong \mathfrak{r}_{\varpi,\varpi x} \oplus \mathfrak{n}_{\varpi}^{-} \oplus \mathfrak{m}_{\varpi,\varpi x}
\end{aligned}$$

In particular,

$$\begin{aligned}
\mathcal{T}_{\varpi,\varpi}^{\mathrm{Id}} &\cong \mathfrak{n}_{\varpi}^{-} & \widetilde{\mathcal{T}}_{\varpi,\varpi}^{s} &\cong \mathfrak{r}_{\varpi,\varpi} \oplus \mathfrak{n}_{\varpi}^{-} \\
\mathcal{T}_{\varpi,\varpi s}^{s} &\cong \mathfrak{n}_{\varpi}^{-} \oplus \mathfrak{m}_{\varpi,\varpi s} & \widetilde{\mathcal{T}}_{\varpi,\varpi s}^{s} &\cong \mathfrak{r}_{\varpi,\varpi s} \oplus \mathfrak{n}_{\varpi}^{-} \oplus \mathfrak{m}_{\varpi,\varpi s}
\end{aligned}$$

With huge effort, finally we fixed all the symbols and understand those typical varieties in detail.

Chapter 2

K-theory and cohomology theory

From my humble point of view, there is no easy cohomology theory, in a sense that key properties are usually hard to prove. On the other hand, plenty of examples can be quickly computed once we grasp some properties and use them in black boxes. Therefore, we won't prove any properties we stated. We have no choice but to do so, for the restricted space and time.

The main reference for the K-theory is [1, Chapter 5].

Setting 2.0.1. *Throughout abstract results of K-theory, we use the following notations:*

- *G stands for a linear algebraic group, i.e., a closed subgroup of* $\mathrm{GL}_n(\mathbb{C})$.[1] *Denote $m : G \times G \longrightarrow G$ as the multiplication map of G.*
- *X is a variety over $\mathbb{C}$, i.e., a reduced, separated scheme of finite type over $\mathbb{C}$. We assume X to be quasi-projective.*
- *Usually, X is equipped with an algebraic G-action (which is compatible with the variety structure of G and X), then we say that X is a G-variety. In that case, we will denote $\alpha : G \times X \longrightarrow X$ as the G-action map.*
- *$\mathcal{F}$ is usually a sheaf on X, which is not flag variety* GL_n / B.

2.1 Definitions and initial examples

2.1.1 G-equivariant sheaf and $K_0^G(X)$

We give definition for K-theory, which is lengthy already. Roughly speaking, a G-equivariant coherent sheaf over X is a sheaf $\mathcal{F} \in \mathrm{Coh}(X)$ equipped with G-action which is compatible with the G-action on X, and K-theory is the Grothendieck group of G-equivariant coherent sheaves over X.

Definition 2.1.1 (G-equivariant sheaf, [1, Definition 5.1.6])**.** *For a G-variety X, denote*

[1] The closed embedding $G \hookrightarrow \mathrm{GL}_n(\mathbb{C})$ is not considered as the data of G.

$p_i^{jk}, p_i := p_i^{123}, p_{ij} := p_{ij}^{123}$ *as projections onto some factors, as follows.*[2]

$$\begin{tikzcd}
 & & & G\times G\times X \arrow[dll, "p_{12}"'] \arrow[dr, "p_{23}"] \arrow[drrr, bend left, "p_{13}"] & & & \\
 & G\times G \arrow[dl, "p_1^{12}"'] \arrow[dr, "p_2^{12}"] & & & G\times X \arrow[dll, "p_2^{23}"'] \arrow[dr, "p_3^{23}"] & & G\times X \arrow[dllllll, dashed, "p_1^{13}"'] \arrow[dl, dashed, "p_3^{13}"] \\
G & & G & & & X &
\end{tikzcd}$$

We have morphisms

$$\begin{tikzcd}
G\times G\times X \arrow[r, shift left=2ex, "m\times \mathrm{Id}_X"] \arrow[r, "p_{23}"] \arrow[r, shift right=2ex, "\mathrm{Id}_G\times\alpha"'] & G\times X \arrow[r, shift left=1ex, "p_3^{23}=p_3^{13}"] \arrow[r, shift right=1ex, "\alpha"'] & X
\end{tikzcd}$$

which satisfies the "coequalizer conditions":

$$\begin{aligned}
p_3^{23}\circ(m\times \mathrm{Id}_X) &= p_3^{23}\circ p_{23} & \qquad (g_1,g_2,x)&\longmapsto x\\
p_3^{23}\circ(\mathrm{Id}_G\times\alpha) &= \alpha\circ p_{23} & (g_1,g_2,x)&\longmapsto g_2x\\
\alpha\circ(m\times \mathrm{Id}_X) &= \alpha\circ(\mathrm{Id}_G\times\alpha) & (g_1,g_2,x)&\longmapsto g_1g_2x
\end{aligned}$$

A ***G-equivariant (coherent) sheaf***[3] *on X is a sheaf $\mathcal{F}\in \mathrm{Coh}(X)$ equipped with an isomorphism*

$$\phi_{\mathcal{F}} : p_3^{23,*}\mathcal{F} \longrightarrow \alpha^*\mathcal{F}$$

such that the following diagram commutes:

$$\begin{tikzcd}
 & (m\times \mathrm{Id}_X)^* p_3^{23,*}\mathcal{F} \arrow[r, "(m\times \mathrm{Id}_X)^*\phi_{\mathcal{F}}"] \arrow[dl, equal] & (m\times \mathrm{Id}_X)^*\alpha^*\mathcal{F} \arrow[dr, equal] & \\
p_{23}^* p_3^{23,*}\mathcal{F} \arrow[dr, "p_{23}^*\phi_{\mathcal{F}}"] & & & (\mathrm{Id}_G\times\alpha)^*\alpha^*\mathcal{F} \\
 & p_{23}^*\alpha^*\mathcal{F} \arrow[r, equal] & (\mathrm{Id}_G\times\alpha)^* p_3^{23,*}\mathcal{F} \arrow[ur, "(\mathrm{Id}_G\times\alpha)^*\phi_{\mathcal{F}}"] &
\end{tikzcd} \tag{2.1.1}$$

A ***(G-equivariant) morphism*** *$f : (\mathcal{F},\phi_{\mathcal{F}}) \longrightarrow (\mathcal{G},\phi_{\mathcal{G}})$ between two G-equivariant sheaves is a morphism $f : \mathcal{F}\longrightarrow\mathcal{G}$ in $\mathrm{Coh}(X)$ such that the diagram*

$$\begin{tikzcd}
p_3^{23,*}\mathcal{F} \arrow[r, "\phi_{\mathcal{F}}"] \arrow[d, "p_3^{23,*}f"'] & \alpha^*\mathcal{F} \arrow[d, "\alpha^* f"] \\
p_3^{23,*}\mathcal{G} \arrow[r, "\phi_{\mathcal{G}}"] & \alpha^*\mathcal{G}
\end{tikzcd} \tag{2.1.2}$$

[2] Be careful, under this convention, the projection map $p_3^{23} = p_3^{13} : G\times X \longrightarrow X$ has subscription 3, and p_2 means the projection from $G\times G\times X$ to the second G. This convention is different with notations in [1, 5.1].

[3] we will omit the word "coherent" for shorter notation.

commutes.

We denote $\mathrm{Coh}^G(X)$ *as the category of* G*-equivariant sheaves.*

Definition 2.1.2 (G-equivariant K-theory)**.** *For a* G*-variety* X*, the* G*-equivariant* K*-theory is defined as the Grothendieck group of* G*-equivariant coherent sheaves over* X*, i.e.,*

$$K_0^G(X) := K_0(\mathrm{Coh}^G(X)).$$

Specifically, for a point $\mathrm{pt} = \operatorname{Spec}\mathbb{C}$ *with trivial* G*-action, denote*

$$\mathrm{R}(G) := K_0^G(\mathrm{pt}) = K_0(\mathrm{Rep}(G))$$

as the representation ring of group G*.*

We may omit 0 *for the convenience of writing and typing.*

Let us unravel this construction a little bit. For (geometrical) points $g, g_1, g_2 \in G$, denote that

$$\begin{aligned} \iota_g &: X \longrightarrow G \times X & x &\longmapsto (g, x) \\ \iota_{g_1,g_2} &: X \longrightarrow G \times G \times X & x &\longmapsto (g_1, g_2, x) \\ \alpha_g &: X \xhookrightarrow{\iota_g} G \times X \xrightarrow{\alpha} X & x &\longmapsto gx \end{aligned}$$

By pulling back along ι_g and ι_{g_1,g_2}, we can see geometrical meanings in the expressions. Apply ι_g^* to $\phi_{\mathcal{F}}$, one get

$$\iota_g^*\phi_{\mathcal{F}} : \mathcal{F} \longrightarrow \alpha_g^*\mathcal{F} \qquad \rightsquigarrow \qquad \phi^{\mathcal{F}}_{g,x} \hat{=} \left(\iota_g^*\phi_{\mathcal{F}}\right)_x : \mathcal{F}_x \longrightarrow \mathcal{F}_{gx}$$

Therefore, $\phi_{\mathcal{F}}$ encodes information of G-action on $\mathcal{F}$, which is G-equivariant.

Now we apply $\iota^*_{g_1,g_2}$ to (2.1.1):

$$\begin{tikzcd} \mathcal{F} \arrow[rr, "\iota^*_{g_1,g_2}\phi_{\mathcal{F}}"] \arrow[dr, "\iota^*_{g_2}\phi_{\mathcal{F}}"'] & & \alpha^*_{g_1,g_2}\mathcal{F} = \alpha^*_{g_1}\alpha^*_{g_2}\mathcal{F} \\ & \alpha^*_{g_2}\mathcal{F} \arrow[ur, "\iota^*_{g_1}\phi_{\alpha^*_{g_2}\mathcal{F}}"'] & \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \mathcal{F}_x \arrow[rr, "\phi^{\mathcal{F}}_{g_1g_2,x}"] \arrow[dr, "\phi^{\mathcal{F}}_{g_2,x}"'] & & \mathcal{F}_{g_1g_2x} \\ & \mathcal{F}_{g_2x} \arrow[ur, "\phi^{\mathcal{F}}_{g_1,g_2x} = \phi^{\alpha^*_{g_2}\mathcal{F}}_{g_1,x}"'] & \end{tikzcd}$$

So (2.1.1) is just the associative constraint of the G-structure on $\mathcal{F}$.

Similarly, apply ι_g^* to (2.1.2), we get

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\iota_g^*\phi_{\mathcal{F}}"] \arrow[d, "f"'] & \alpha_g^*\mathcal{F} \arrow[d, "\alpha_g^* f"] \\ \mathcal{G} \arrow[r, "\iota_g^*\phi_{\mathcal{G}}"] & \alpha_g^*\mathcal{G} \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \mathcal{F}_x \arrow[r, "\phi^{\mathcal{F}}_{g,x}"] \arrow[d, "f_x"'] & \mathcal{F}_{gx} \arrow[d, "f_{gx}"] \\ \mathcal{G}_x \arrow[r, "\phi^{\mathcal{G}}_{g,x}"] & \mathcal{G}_{gx} \end{tikzcd}$$

So (2.1.2) is just the condition for f to be G-equivariant.

There are two extreme situations worth mentioning about. When $G = \mathrm{Id}$, there is no G-action structure constrain on varieties and sheaves. Therefore,

$$\mathrm{Coh}^{\mathrm{Id}}(X) = \mathrm{Coh}(X) \qquad K_0^{\mathrm{Id}}(X) = K_0(X) \mathrel{\hat{=}} K_0(\mathrm{Coh}(X)).$$

When G acts on $X = \operatorname{Spec} A$ trivially, any sheaf $\mathcal{F} \in \mathrm{Coh}^G(X)$ can be viewed as an (finitely generated)[4] A-module M with G-action, so

$$\mathrm{Coh}^G(X) = \mathrm{rep}_A(G) \xlongequal{\text{when } G \text{ is finite}} \mathrm{Mod}(A[G]).$$

In particular, any sheaf $\mathcal{F} \in \mathrm{Coh}^G(\mathrm{pt})$ can be viewed as a finite dimensional complex G-representation, so

$$\mathrm{Coh}^G(\mathrm{pt}) = \mathrm{rep}_{\mathbb{C}}(G) \xlongequal{\text{when } G \text{ is finite}} \mathrm{Mod}(\mathbb{C}[G]).$$

???(If I have time I will compute $K_0(\mathbb{P}^1)$ here.)

2.1.2 Representation ring $\mathrm{R}(G)$

Now let us try to figure out some examples.

Recall that any coherent sheaf over pt is equivalent to a finite dimensional $\mathbb{C}$-vector space, and any G-equivariant coherent sheaf over pt is equivalent to a finite dimensional complex G-representation. Moreover, by Jordan-Hölder theorem, every finite dimensional complex G-representation can be written as a composition series such that each quotient object is irreducible. Therefore,

$$\mathrm{R}(G) = \bigoplus_{\rho \in \mathrm{Irr}(G)} \mathbb{Z}$$

as a free $\mathbb{Z}$-module.

For $\mathrm{R}(G)$, we have the multiplication structure induced by tensor products on complex G-representations. Let us see some examples now. We use Setting 1.1.1 in these examples.

Example 2.1.3. *For trivial group* Id*, every* Id*-representation is just a* $\mathbb{C}$*-vector space, which can be written as the direct sum of* 1*-dimensional vector spaces. Therefore,*

$$\mathrm{R}(\mathrm{Id}) = \mathbb{Z}.$$

Example 2.1.4. *For group T, since T is abelian, every T-representation can be written as direct sum of* 1*-dimensional vector spaces. Furthermore,*

$$\begin{aligned}\mathrm{Irr}(T) &= \left\{\rho : T \longrightarrow \mathbb{C}^\times \;\middle|\; \rho \textit{ is an (algebraic) group homomorphism}\right\} \\ &= \mathrm{Hom}_{\mathbb{C}\text{-}Alg\ gp}(T, \mathbb{C}^\times) := X^*(T)\end{aligned}$$

[4] We already assume X to be of finite type, so coherent condition is equivalent to finitely generated condition.

We get

$$\mathrm{R}(T) = \bigoplus_{\rho \in \mathrm{Irr}(T)} \mathbb{Z} = \mathbb{Z}\left[X^*(T)\right].$$

The group structure in $X^*(T)$ is given by tensor product, so the multiplication structure is induced by the group structure in $X^*(T)$. Denote

$$\varepsilon_i : T \longrightarrow \mathbb{C}^\times \qquad \begin{pmatrix} t_1 & & & & \\ & \ddots & & & \\ & & t_i & & \\ & & & \ddots & \\ & & & & t_n \end{pmatrix} \longmapsto t_i$$

as a $\mathbb{Z}$-basis of $X^*(T)$, then $X^*(T) \cong \oplus_{i=1}^n \mathbb{Z}\varepsilon_i$.

To distinguish the addition in $X^*(T)$ and $\mathbb{Z}\left[X^*(T)\right]$, we rewrite ε_i as e_i. In that case, $\sum_{i=1}^n k_i\varepsilon_i$ is sent to $\prod_{i=1}^n e_i^{k_i}$, and

$$\mathrm{R}(T) \cong \mathbb{Z}\big[e_1^{\pm 1}, \dots, e_n^{\pm 1}\big]$$

as a $\mathbb{Z}$-algebra.

By forgetting T-actions, we get a morphism of $\mathbb{Z}$-algebra

$$\mathrm{R}(T) \longrightarrow \mathrm{R}(\mathrm{Id}) \qquad f(e_1, \dots, e_n) \longmapsto f(1, \dots, 1).$$

Example 2.1.5. *After we state the reduction isomorphism 2.5.1, we can show that*

$$\mathrm{R}(N) \cong \mathrm{R}(\mathrm{Id}) \cong \mathbb{Z} \qquad \mathrm{R}(B) \cong \mathrm{R}(T) \cong \mathbb{Z}\big[e_1^{\pm 1}, \dots, e_n^{\pm 1}\big]$$

Example 2.1.6. *By [1, Theorem 6.1.4],*

$$\mathrm{R}(\mathrm{GL}_n) \cong \mathrm{R}(T)^W \cong \mathbb{Z}\big[e_1^{\pm 1}, \dots, e_n^{\pm 1}\big]^{S_n}.$$

This can be viewed as a "group" analogue of Chevalley restriction theorem. Notice that we have clear description of finite dimensional irreducible representations of GL_n*, and the forget map*

$$\mathrm{rep}(\mathrm{GL}_n) \longrightarrow \mathrm{rep}(T) \qquad \rightsquigarrow \qquad \mathrm{R}(\mathrm{GL}_n) \longrightarrow \mathrm{R}(T)$$

views GL_n*-representations as special W-invariant T-representations.*

From these examples we already see the difficulty of computing K-theories. Therefore, a series of properties of K-theories are definitely needed for computations. To state these properties, we need to define some tools (or weapons???) in K-theory.

2.2 Three functors: pullback, proper pushforward and tensor product

In this section, we will construct three basic functors of equivariant K-theory: pullback, proper pushforward and tensor product.

2.2.1 Non-derived three functors in $\mathrm{Coh}^G(X)$

We assume that readers know the non-derived pullback, pushforward and tensor product of normal coherent sheaves. (See [5, Chapter 16])

As a special reminder, the pushforward of coherent sheaves may be not coherent. This problem can be remedied by Grothendieck's coherence theorem [5, Theorem 18.9.1], once we impose morphisms to be proper (and Noetherian hypotheses on varieties). That is why we only consider about proper pushforward.

Now let us consider the effect of G-equivariance. Somewhat surprising, these three functors behave quite well with group actions.

Definition 2.2.1 (Group action on pullback, proper pushforward and tensor product)**.** *Let X, Y be G-varieties, $f : Y \longrightarrow X$ be a G-equivariant morphism. For $(\mathcal{G}, \phi_{\mathcal{G}}) \in \mathrm{Coh}^G(Y)$, we define group actions on $f^*\mathcal{F}$, $f_*\mathcal{G}$ and $\mathcal{F} \otimes \mathcal{G}$, as follows.*

$$\begin{tikzcd}
 & & \mathcal{G} \arrow[dl, dash] \\
G \times Y \arrow[r, shift left, "p^{23}_{3,Y}"] \arrow[r, shift right, "\alpha_Y"'] \arrow[d, "\mathrm{Id}_G \times f"'] \arrow[dr, phantom, very near start, "\lrcorner"] & Y \arrow[d, "f"] & \\
G \times X \arrow[r, shift left, "p^{23}_{3,X}"] \arrow[r, shift right, "\alpha_X"'] & X & \mathcal{F} \arrow[l, dash]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathcal{F} \arrow[d, dash] & \mathcal{F}' \arrow[dl, dash] \\
G \times X \arrow[r, shift left, "p^{23}_{3,X}"] \arrow[r, shift right, "\alpha_X"'] & X &
\end{tikzcd}$$

By definition, we get

$$p^{23}_{3,X} \circ (\mathrm{Id}_G \times f) = f \circ p^{23}_{3,Y}.$$

Since f is G-equivariant,

$$\alpha_X \circ (\mathrm{Id}_G \times f) = f \circ \alpha_Y.$$

These two diagrams are Cartesian, and $p^{23}_{3,X}, \alpha_X$ are flat.

The pullback $(f^*\mathcal{F}, \phi_{f^*\mathcal{F}}) \in \mathrm{Coh}^G(Y)$ is defined by

$$\phi_{f^*\mathcal{F}} : p^{23,*}_{3,Y} f^*\mathcal{F} = (\mathrm{Id}_G \times f)^* \, p^{23,*}_{3,X} \mathcal{F} \xrightarrow{(\mathrm{Id}_G \times f)^* \phi_{\mathcal{F}}} (\mathrm{Id}_G \times f)^* \, \alpha_X^* \mathcal{F} = \alpha_Y^* f^* \mathcal{F}$$

By flat base change [5, Theorem 24.2.8], assuming f is proper, the proper pushforward $(f_*\mathcal{G}, \phi_{f_*\mathcal{G}}) \in \mathrm{Coh}^G(X)$ is defined by

$$\phi_{f_*\mathcal{G}} : p^{23,*}_{3,X} f_*\mathcal{G} \cong (\mathrm{Id}_G \times f)_* \, p^{23,*}_{3,Y} \mathcal{G} \xrightarrow{(\mathrm{Id}_G \times f)_* \phi_{\mathcal{G}}} (\mathrm{Id}_G \times f)_* \, \alpha_Y^* \mathcal{G} \cong \alpha_X^* f_* \mathcal{G}$$

In general, we can also define $(\mathrm{R}^i f_*\mathcal{G}, \phi_{\mathrm{R}^i f_*\mathcal{G}}) \in \mathrm{Coh}^G(X)$ by

$$\phi_{\mathrm{R}^i f_*\mathcal{G}} : p^{23,*}_{3,X} \, \mathrm{R}^i f_*\mathcal{G} \cong \mathrm{R}^i (\mathrm{Id}_G \times f)_* \, p^{23,*}_{3,Y} \mathcal{G} \xrightarrow{\mathrm{R}^i (\mathrm{Id}_G \times f)_* \phi_{\mathcal{G}}} \mathrm{R}^i (\mathrm{Id}_G \times f)_* \, \alpha_Y^* \mathcal{G} \cong \alpha_X^* \, \mathrm{R}^i f_* \mathcal{G}$$

Similarly, the tensor product $(\mathcal{F} \otimes \mathcal{F}', \phi_{\mathcal{F} \otimes \mathcal{F}'}) \in \mathrm{Coh}^G(X)$ is defined by

$$\phi_{\mathcal{F} \otimes \mathcal{F}'} : p^{23,*}_{3,X} \left(\mathcal{F} \otimes \mathcal{F}'\right) \cong p^{23,*}_{3,X} \mathcal{F} \otimes p^{23,*}_{3,X} \mathcal{F}' \xrightarrow{\phi_{\mathcal{F}} \otimes \phi_{\mathcal{F}'}} \alpha_X^* \mathcal{F} \otimes \alpha_X^* \mathcal{F}' \cong \alpha_X^* \left(\mathcal{F} \otimes \mathcal{F}'\right).$$

The following definition will be useful in redefining tensor products.

Definition 2.2.2 (External tensor product)**.** *For two G-varieties X and Y, define a functor*

$$\boxtimes : \mathrm{Coh}^G(X) \times \mathrm{Coh}^G(Y) \longrightarrow \mathrm{Coh}^G(X \times Y) \qquad (\mathcal{F}, \mathcal{G}) \longmapsto \mathcal{F} \boxtimes \mathcal{G}$$

where

$$\mathcal{F} \boxtimes \mathcal{G} := p_X^* \mathcal{F} \otimes p_X^* \mathcal{G}.$$

*$\boxtimes$ is called the **external tensor product**.*

Remark 2.2.3. For G-variety X and $\mathcal{F}$, $\mathcal{F}' \in \mathrm{Coh}^G(X)$, denote $\Delta : X \hookrightarrow X \times X$ to be the diagonal embedding, we have

$$\mathcal{F} \otimes \mathcal{F}' \cong \Delta^*(\mathcal{F} \boxtimes \mathcal{F}')$$

Unlike $\otimes$, $\boxtimes$ is always an exact functor. This feature let us redefine tensor product in K-theory later on.

2.2.2 Smooth case

We would like to extend functors in $\mathrm{Coh}^G(X)$ to $K^G(X)$. However, these (non-derived) functors are usually not exact, so we have to work over (G-equivariant) derived category of cohrent sheaves $\mathcal{D}^G_{\mathrm{Coh}}(X)$ and replace every functor by its derived version.

Still, we can not extend functors from $\mathcal{D}^G_{\mathrm{Coh}}(X)$ to $K^G(X)$. The chain complex in $\mathcal{D}^G_{\mathrm{Coh}}(X)$ can have infinite many non-zero terms, which can not be viewed as an element in $K^G(X)$. Therefore, we consider the bounded (G-equivariant) derived category $\mathcal{D}^{b,G}_{\mathrm{Coh}}(X)$ as a full subcategory of $\mathcal{D}^G_{\mathrm{Coh}}(X)$.

The last problem comes when we restrict functors to $\mathcal{D}^{b,G}_{\mathrm{Coh}}(X)$:

$$\begin{aligned}
f^* &: \mathcal{D}^{b,G}_{\mathrm{Coh}}(X) \longrightarrow \mathcal{D}^G_{\mathrm{Coh}}(Y) \\
f_* &: \mathcal{D}^{b,G}_{\mathrm{Coh}}(Y) \longrightarrow \mathcal{D}^{b,G}_{\mathrm{Coh}}(X) \\
\otimes &: \mathcal{D}^{b,G}_{\mathrm{Coh}}(X) \times \mathcal{D}^{b,G}_{\mathrm{Coh}}(X) \longrightarrow \mathcal{D}^G_{\mathrm{Coh}}(X)
\end{aligned}$$

Other than proper pushforward, [5] pullback and tensor product may not preserve boundness.

For pullback, boundness preserving is equivalent to the following condition:

$$f : Y \longrightarrow X \text{ is } G\text{-equivariant of globally finite Tor-dimension.} \tag{2.2.1}$$

When X, Y are smooth, the condition (2.2.1) is automatically satisfied. (See [1, 5.2.5(ii)]). The condition is concluded as follows:

$$X,\ Y \text{ are smooth } G\text{-varieties, and } f : Y \longrightarrow X \text{ is } G\text{-equivariant.} \tag{2.2.2}$$

[5]See [1, 5.2.13] for proper pushforward preserving boundness, and it essentially use the higher cohomology vanishing theorem [5, Theorem 18.8.5].

Tensor product also preserve boundness when X is smooth. By Remark 2.2.3, $\boxtimes$ is exact, and Δ^* preserves boundness when X is smooth, so $\otimes$ also preserves boundness. In particular, one can define tensor product on $K^G(X)$ for X smooth:

$$\otimes : K^G(X) \times K^G(X) \xrightarrow{\boxtimes} K^G(X \times X) \xrightarrow{\Delta^*} K^G(X) \qquad \mathcal{F} \otimes \mathcal{F}' = \Delta^*\left(\mathcal{F} \boxtimes \mathcal{F}'\right)$$

Remark 2.2.4. When $f : Y \longrightarrow X$ is open embedding, the non-derived pullback f^* is exact, so we can define pullback on K-theory automatically.

2.2.3 Restriction with supports

In practice, the varieties we consider are not smooth. Luckily, these varieties are always embedded in some ambiance spaces which are smooth.

Definition 2.2.5 (Restriction with supports)**.** *For a triple (X, Y, f) satisfying assumption* (2.2.2)*, and a G-equivariant closed subvariety Z of X, the triple $\left(Z, f^{-1}(X), f\big|_{f^{-1}(X)}\right)$ is called a restriction with supports of (X, Y, f).*

We can now define pullback of f in the following assumption:

$$\begin{gathered} f : Y \longrightarrow X \text{ is } G\text{-equivariant, and } f \text{ is a restriction with supports} \\ \text{by some } f' : Y' \longrightarrow X', \text{ where } X', Y' \text{ are smooth.} \end{gathered} \tag{2.2.3}$$

Definition 2.2.6 (Pullback with supports)**.** *Let Z, Z' be G-varieties, $h : Z' \longrightarrow Z$ be a G-equivariant closed embedding. Suppose that h is a restriction with support of some (X, Y, f) satisfying the assumption* (2.2.2)*, i.e., we have a G-equivariant closed embedding $\iota_Z : Z \longrightarrow X$ such that $Z' \cong f^{-1}(Z)$ and $h = f\big|_{Z'}$. Denote $\iota_{Z'} : Z' \longrightarrow Y$ as the induced G-equivariant closed embedding, we would like to construct the pullback $h^* : K^G(Z) \longrightarrow K^G(Z')$.*

$$\begin{tikzcd} Z' \arrow[r, hook, "h"] \arrow[d, hook, "\iota_{Z'}"'] & Z \arrow[d, hook, "\iota_Z"] \\ Y \arrow[r, "f"] & X \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} K^G(Z') \arrow[d, "\iota_{Z',*}"'] & K^G(Z) \arrow[l, dashed, "h^*"'] \arrow[d, "\iota_{Z,*}"] \\ K^G(Y) \arrow[u, dotted, bend right, "\mathrm{gr}"'] & K^G(X) \arrow[l, "f^*"] \end{tikzcd} \tag{2.2.4}$$

Follows [1, 5.2.7(ii)], one can construct a morphism

$$\mathrm{gr} : \mathrm{Im}\,(f^* \circ \iota_{Z,*}) \longrightarrow K^G(Z'),$$

and the pullback is defined as

$$h^* : K^G(Z) \xrightarrow{\iota_{Z,*}} K^G(X) \xrightarrow{f^*} K^G(Y) \overset{\mathrm{gr}}{\dashrightarrow} K^G(Z').$$

Warning 2.2.7. *The diagram* (2.2.4) *of K-group is usually not commutative. In fact, we will state the excess base change in Section 4.2, in which the Euler class measures the failure of diagram to be commutative.*

Definition 2.2.8 (Tensor product with supports/Intersection product)**.** *Let X be a smooth G-variety, and Z, $Z' \subseteq X$ be two closed G-subvarieties. The tensor product with supports is defined as*

$$\otimes : K^G(Z) \times K^G(Z') \xrightarrow{\boxtimes} K^G(Z \times Z') \xrightarrow{\Delta^*} K^G(Z \cap Z')$$

i.e., $\mathcal{F} \otimes \mathcal{F}' := \Delta^*(\mathcal{F} \boxtimes \mathcal{F}')$.

The following diagram explains the word "restriction with supports":

$$\begin{tikzcd} K^G(Z) \times K^G(Z') \arrow[r, "\boxtimes"] \arrow[d] & K^G(Z \times Z') \arrow[r, dashed, "\Delta^*"] \arrow[d] & K^G(Z \cap Z') \arrow[d] \\ K^G(X) \times K^G(X) \arrow[r, "\boxtimes"] & K^G(X \times X) \arrow[r, "\Delta^*"] & K^G(X) \end{tikzcd}$$

Lemma 2.2.9. *Let X be a smooth variety, $Z \subseteq X$ be a closed G-subvariety, $\pi_Z : Z \longrightarrow \mathrm{pt}$ be the projection map. For any $\alpha \in K^G(Z)$, $\alpha \otimes \pi_Z^* 1_{\mathrm{R}(G)} = \alpha$.*

Proof. This comes from the definition of the tensor product. □

2.2.4 Algebraic structures of K-theory

With enough tools in hand, we can define some extra structures on $K^G(X)$. (By priority $K^G(X)$ is an abelian group)

Proposition 2.2.10 ($\mathrm{R}(G)$-module)**.** *For any G-variety X, $K^G(X)$ is a $\mathrm{R}(G)$-module by*

$$\mathrm{R}(G) \times K^G(X) \cong K^G(\mathrm{pt}) \times K^G(X) \xrightarrow{\boxtimes} K^G(\mathrm{pt} \times X) \cong K^G(X).$$

Under this proposition, these three functors become $\mathrm{R}(G)$-homomorphisms.

Proposition 2.2.11 ($\otimes$ as multiplication)**.** *For any smooth G-variety X, $K^G(X)$ is a unital commutative associative $\mathrm{R}(G)$-algebra, where the multiplication (call the $\otimes$-product on $K^G(X)$) is defined by*

$$K^G(X) \times K^G(X) \xrightarrow{\otimes} K^G(X).$$

Under this proposition, for any morphism $f : Y \longrightarrow X$ of smooth G-varieties, f^* is a ring homomorphism.

Warning 2.2.12. *We will define another product (called the convolution product) on some K-theories in Section 5.1. These two products are essentially different products, and people have to specify which one they are using, when they discuss the "algebra structures on K-theories". The final task is to compute the convolution product of $K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}})$, not the $\otimes$-product.*

After that, whenever we see an isomorphism of K-theories, we need to specify which structures this isomorphism preserve.

2.3 Thom isomorphism

In this section we state Thom isomorphism theorem, which is an analogy of Poincaré lemma in K-theory.

Proposition 2.3.1 (Thom isomorphism, [1, Theorem 5.4.17])**.** *Let X be a G-variety, $\pi : E \longrightarrow X$ be a G-equivariant affine bundle on X. The pullback*

$$\pi^* : K^G(X) \longrightarrow K^G(E)$$

is an isomorphism of K-theories as $\mathrm{R}(G)$-modules.

For a proof, see [1, Theorem 5.4.17].

With Thom isomorphism, we can compute K-theory of affine bundles by the K-theory of the base spaces. Proposition 1.6.8 offers plenty of cases to apply Thom isomorphism. Also, for any $k \in \mathbb{N}_{>0}$,

$$K^G(\mathbb{A}^k) \cong K^G(\mathrm{pt}) \cong \mathrm{R}(G).$$

as an $\mathrm{R}(G)$-module. This can be applied to $\mathbf{\Omega}_w^u$ and $\mathbf{\Omega}_{w,w'}^{u,u'}$.

2.4 Induction

2.4.1 Contracted product

Before we state the induction isomorphism, let us recall one basic construction of spaces: the contracted product.

Definition 2.4.1 (Contracted product)**.** *Let $H \subseteq G$ be a closed algebraic subgroup and X be an H-variety. The contracted product of G and X over H is defined as*

$$G \times^H X := (G \times X)/\sim$$

where

$$(gh, x) \sim (g, hx) \qquad \text{for any } g \in G,\ h \in H,\ x \in X.$$

$G \times^H X$ has a natural variety structure, which is not easy to construct. G acts on $G \times^H X$ by multiplying from the left side. We have a G-equivariant flat morphism

$$G \times^H X \longrightarrow G/H \qquad (g, x) \longrightarrow gH$$

which realize $G \times^H X$ as an X-bundle over G/H. In particular, for $X = \mathrm{pt}$, we get an isomorphism of G-varieties

$$G \times^H \mathrm{pt} \xrightarrow{\sim} G/H.$$

The contracted product is not only used for the induction isomorphism, but also used in the definition of equivariant cohomology theory (see Definition 2.6.1) and description of some typical varieties (see the description of $\overline{\mathbf{\Omega}}_s$ in 1.1.2).

Example 2.4.2. *In the setting 1.1.1, the* GL_n*-equivariant map*

$$\mathrm{GL}_n \times^B \mathcal{F} \xrightarrow{\sim} \mathrm{GL}_n / B \times \mathcal{F} = \mathcal{F} \times \mathcal{F} \qquad (g, g'B) \longmapsto (gB, gg'B)$$

realizes $\mathcal{F} \times \mathcal{F}$ *as a contracted product, and*

$$\boldsymbol{\Omega}_{w'} \cong \mathrm{GL}_n \times^B \Omega_{w'}$$

under this isomorphism.

2.4.2 Statement

Proposition 2.4.3 (Induction isomorphism, [1, 5.2.16])**.** *Let* $H \subseteq G$ *be a closed algebraic subgroup and* X *be an* H*-variety, we have a Cartesian diagram of* H*-varieties*

$$\begin{tikzcd} X = H \times^H X \arrow[r, "\iota_X"] \arrow[d] & G \times^H X \arrow[d, "\pi"] \\ \mathrm{pt} = H/H \arrow[r, "\iota_{\mathrm{pt}}"] & G/H \end{tikzcd}$$

The functor

$$\mathrm{Res}^G_H : \mathrm{Coh}^G(G \times^H X) \xrightarrow{\textit{forget}} \mathrm{Coh}^H(G \times^H X) \xrightarrow{\iota_X^*} \mathrm{Coh}^H(X)$$

is an equivalence of categories, and descend to an $\mathrm{R}(H)$*-module homomorphism of* K*-groups:*

$$\mathrm{Res}^G_H : K^G(G \times^H X) \xrightarrow{\textit{forget}} K^H(G \times^H X) \xrightarrow{\iota_X^*} K^H(X)$$

When X *is smooth,* Res^G_H *is an isomorphism as algebras (for* $\otimes$*-product).*

We denote the inverse functor of Res^G_H by Ind^G_H, called the induction, which is also explicitly constructed by pulling back and descent argument in [1, 5.2.16].

(???Present the construction of Ind^G_H and example of $K^{\mathrm{GL}_2}(\mathbb{P}^1)$, if time permits.)

Remark 2.4.4. The isomorphism Res^G_H also gives $K^G(G \times^H X)$ a $\mathrm{R}(H)$-module structure.

2.4.3 Applications

This induction formula is usually used for computing G-equivariant K-theory of G-orbits. For example, in Setting 1.1.1,

$$K^{\mathrm{GL}_n}(\mathcal{F}) = K^{\mathrm{GL}_n}(\mathrm{GL}_n / B) \cong K^B(\mathrm{pt}) = \mathrm{R}(B)$$

is an isomorphism as $\mathrm{R}(\mathrm{GL}_n)$-modules. Notice that $K^{\mathrm{GL}_n}(\mathcal{F})$ is a free $\mathrm{R}(\mathrm{GL}_n)$-module of rank $\#W = n!$.

Also, the isomorphism

$$K^{\mathrm{GL}_n}(\mathcal{F} \times \mathcal{F}) \cong K^{\mathrm{GL}_n}(\mathrm{GL}_n \times^B \mathcal{F}) \cong K^B(\mathcal{F})$$

gives $K^{\mathrm{GL}_n}(\mathcal{F} \times \mathcal{F})$ a $\mathrm{R}(B)$-module structure.

In the next section we will explore how to reduce B-equivariant K-theory to T-equivariant K-theory.

2.5 Reduction

Let $P = M \ltimes U$ be a linear algebraic group in this section, where M is reductive and $U = R_u(M)$ is the unipotent radical of P.

Proposition 2.5.1 (Reduction isomorphism, [1, 5.2.18]). *For any P-variety X, the forgetful map*

$$K^P(X) \longrightarrow K^M(X)$$

is an isomorphism as $\mathrm{R}(M)$*-modules. (and as algebras for* $\otimes$*-product, when X is smooth)*

In the proof of reduction isomorphism, induction isomorphism and Thom isomorphism are used in an essential way.

This isomorphism allows us to identify B-equivariant K-theory and T-equivariant K-theory. In particular, $\mathrm{R}(B) \cong \mathrm{R}(T)$ as $\mathbb{Z}$-algebras.

2.6 Equivariant cohomology theory

The theory of equivariant cohomology theory is completely parallel with the theory of equivariant K-theory. We shortly sketch the definition and refer readers to see [3, Chapter 2] for details (like the definition of universal principle bundle $\mathrm{E}G \longrightarrow \mathrm{B}G$)

Nearly all the abstract results for K-theory have a corresponding cohomology theory version in [3]. We will mention about the difference of Euler class in Section 4.1, compute some examples in Chapter 7, and compare these two theories in Chapter 8.

2.6.1 G-equivariant cohomology $H^*_G(X;\mathbb{Q})$

Definition 2.6.1 (G-equivariant cohomology, [3, Definition 2.7]). *For a G-variety X, the G-equivariant cohomology theory is defined as the cohomology ring of the contracted product space* $\mathrm{E}G \times^G X$*, i.e.,*

$$H^*_G(X;\mathbb{Q}) := H^*\big(\mathrm{E}G \times^G X;\mathbb{Q}\big)\,.$$

Specifically, for a point $\{\mathrm{pt}\} = \operatorname{Spec}\mathbb{C}$ *with trivial G-action, denote*

$$\mathrm{S}(G) := H^*_G\big(\{\mathrm{pt}\};\mathbb{Q}\big) = H^*(\mathrm{B}G;\mathbb{Q})$$

as the cohomology ring of classifying space $\mathrm{B}G$*.*

We work with coefficient $\mathbb{Q}$ for simplicity, and we may omit $\mathbb{Q}$ for the convenience of writing and typing.

Parallelly, there are two extreme situations worth mentioning about. When $G = \mathrm{Id}$, $\mathrm{E}G = \{\mathrm{pt}\}$. Therefore,

$$H^*_{\mathrm{Id}}(X;\mathbb{Q}) = H^*\Big(\{\mathrm{pt}\} \times^{\mathrm{Id}} X;\mathbb{Q}\Big) \cong H^*(X;\mathbb{Q}).$$

When G acts on X trivially, we get

$$H^*_G(X;\mathbb{Q}) = H^*(\mathrm{B}G \times X;\mathbb{Q}) \cong H^*(\mathrm{B}G;\mathbb{Q}) \otimes_{\mathbb{Q}} H^*(X;\mathbb{Q}).$$

2.6.2 Cohomology ring $\mathrm{S}(G)$

We also list examples in parallel with subsection 2.1.2. Everything is much more sketchy though. We use Setting 1.1.1.

Example 2.6.2. *For trivial group* Id, $\mathrm{BId} = \{\mathrm{pt}\}$*, so*

$$\mathrm{S}(\mathrm{Id}) = H^*\big(\{\mathrm{pt}\};\mathbb{Q}\big) \cong \mathbb{Q}.$$

Example 2.6.3 ([3, Example 2.9(i)])**.** *For group T, $\mathrm{B}T = \prod_{j=i}^{n} \mathbb{CP}^\infty$, so*

$$\mathrm{S}(T) = H^*\left(\prod_{j=1}^{n} \mathbb{CP}^\infty;\mathbb{Q}\right) \cong \bigotimes_{j=1}^{n} H^*\left(\mathbb{CP}^\infty\ ;\mathbb{Q}\right) \cong \bigotimes_{j=1}^{n} \mathbb{Q}[t_j] = \mathbb{Q}[t_1,\dots,t_n]$$

where $\deg t_j = 2$ *for any* j*.*

By forgetting T-actions, we get a morphism of $\mathbb{Q}$-algebra

$$\mathrm{S}(T) \longrightarrow \mathrm{S}(\mathrm{Id}) \qquad f(t_1,\dots,t_n) \longmapsto f(0,\dots,0).$$

Example 2.6.4. *By using the reduction isomorphism 2.5.1 in the version of cohomology theory, we can show that*

$$\mathrm{S}(N) \cong \mathrm{S}(\mathrm{Id}) \cong \mathbb{Q} \qquad \mathrm{S}(B) \cong \mathrm{S}(T) \cong \mathbb{Q}[t_1,\dots,t_n]$$

Example 2.6.5 ([3, Example 2.9(ii)])**.** *For group* GL_n*,* $\mathrm{BGL}_n = \mathrm{Gr}(n,\infty)$*, so*

$$\mathrm{S}(T) = H^*(\mathrm{Gr}(n,\infty);\mathbb{Q}) \cong \mathbb{Q}[t_1,\dots,t_n]^{S_n}$$

We also have the Chevalley restriction theorem in the version of cohomology theory. In this case, it says

$$\mathrm{S}(\mathrm{GL}_n) \cong \mathrm{S}(T)^W \cong \mathbb{Q}[t_1,\dots,t_n]^{S_n}.$$

The three functors on cohomology theory are defined in a different way, which are induced from the three functors in normal cohomology theory, see [3, 2.3.2]. Thom isomorphism, induction isomorphism and reduction isomorphism are still true in the equivariant cohomology theory case. In particular, we have

$$H^*_{\mathrm{GL}_n}(\mathcal{F}) \cong H^*_B(\mathrm{pt}) \cong H^*_T(\mathrm{pt}) \cong \mathbb{Q}[t_1,\dots,t_n]$$

as an $\mathrm{S}(\mathrm{GL}_n)$-module. $H^*_{\mathrm{GL}_n}(\mathcal{F})$ is a free $\mathrm{S}(\mathrm{GL}_n)$-module with rank $\#W = n!$.

Also, the isomorphism

$$H^*_{\mathrm{GL}_n}(\mathcal{F}\times\mathcal{F}) \cong H^*_{\mathrm{GL}_n}(\mathrm{GL}_n \times^B \mathcal{F}) \cong H^*_B(\mathcal{F})$$

gives $H^*_{\mathrm{GL}_n}(\mathcal{F}\times\mathcal{F})$ an $\mathrm{S}(B)$-module structure.

Chapter 3

Cellular fibration theorem

3.1 Statement

We first state one general theorem, and then apply it repeatly to get the cellular fibration theorem.

Theorem 3.1.1 (Glueing theorem). *Suppose the triple (X, Y, π) satisfies assumption* (2.2.3). *For a G-equivariant closed embedding $i : Z \hookrightarrow Y$, denote $U := Y \smallsetminus Z$, and $j : U \hookrightarrow Y$ as the open immersion, as follows.*

$$\begin{tikzcd} Z \arrow[r, hook, "i"] & Y \arrow[d, "\pi"] & U \arrow[l, hook', "j"'] \\ & X & \end{tikzcd}$$

Suppose that $\pi|_U = \pi \circ j : U \longrightarrow X$ realizes U as a G-equivariant affine bundle on X, so

$$\pi\big|_U^* : K^G(X) \xrightarrow{\cong} K^G(U)$$

as $\mathrm{R}(G)$*-modules.*

1. *We have a canonical short exact sequence*

$$\begin{tikzcd} 0 \arrow[r] & K^G(Z) \arrow[r, "i_*"] & K^G(Y) \arrow[r, "j^*"] & K^G(U) \arrow[l, dashed, bend left, "s"] \arrow[r] & 0 \end{tikzcd} \tag{3.1.1}$$

2. *If $K^G(X)$ is a free* $\mathrm{R}(G)$*-module with basis $\{y_1, \dots y_m\}$, then the short exact sequence* (3.1.1) *(non-naturally) splits, and*

$$K^G(Y) \cong K^G(Z) \oplus K^G(U)$$

as $\mathrm{R}(G)$*-modules. The splitting s is defined on basis of $K^G(U)$:*

$$s : K^G(U) \longrightarrow K^G(Y) \qquad \pi\big|_U^*(y_l) \longmapsto \iota_{\overline{U},*}\pi\big|_{\overline{U}}^*(y_l)$$

where $\iota_{\overline{U}}$, $\pi\big|_{\overline{U}}$ *are defined in the following diagram:*

$$\begin{tikzcd} U \arrow[r, hook] \arrow[drr, "\pi|_U"'] & \overline{U} \arrow[r, hook, "\iota_{\overline{U}}"] \arrow[dr, "\pi|_{\overline{U}}"] & Y \arrow[d, "\pi"] \\ & & X \end{tikzcd}$$

In practice, we will use Theorem 3.1.1 by repetition.

Definition 3.1.2 (Cellular fibration). *Let* $\pi : E \longrightarrow X$ *be a* G*-equivariant morphism satisfying the assumption* (2.2.3)*. A (*G*-equivariant)* ***cellular fibration structure*** *of* E *is a fibration of closed* G*-equivariant subvarieties*

$$\varnothing = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_k = E$$

such that $\pi_j := \pi\big|_{E_j \smallsetminus E_{j-1}} : E_j \smallsetminus E_{j-1} \longrightarrow X$ *is a* G*-equivariant affine bundle over* X*, for any* $j \in \{1, \ldots, k\}$*.*

When $X = \mathrm{pt}$*, this filtration is called a* ***cellular decomposition*** *of* E*.*

Theorem 3.1.3 (Cellular fibration). *Suppose a* G*-equivariant morphism* $\pi : E \longrightarrow X$ *has a cellular fibration structure*

$$\varnothing = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_k = E$$

and $K^G(X)$ *is a free* $\mathrm{R}(G)$*-module with basis* $\{y_1, \ldots, y_m\}$*.*

For $j \in \{1, \ldots, k\}$*, denote* $U_j := E_j \setminus E_{j-1}$*,* $\overline{U}_j$ *as the closure of* U_j *in* E_j*,* $\iota_{\overline{U}_j}$ *as* $\overline{U}_j$ *embedded in* E*,* $\pi_{\overline{U}_j} := \pi\big|_{\overline{U}_j} = \pi \circ \iota$*, as follows.*

$$\begin{tikzcd} \overline{U}_j \arrow[r, hook, "\iota_{\overline{U}_j}"] \arrow[d, "\pi_{\overline{U}_j}"'] & E \arrow[dl, dashed, "\pi"] \\ X & \end{tikzcd}$$

- $K^G(E)$ *is a free* $\mathrm{R}(G)$*-module with basis*

$$\left\{ \iota_{\overline{U}_j,*}\pi^*_{\overline{U}_j}(y_l) \,\middle|\, 1 \leq l \leq m, 1 \leq j \leq k \right\}$$

- *In particular, when* $X = \mathrm{pt}$ *is a point,*

$$K^G(E) \cong \bigoplus_j \mathrm{R}(G)\iota_{\overline{U}_j,*}\pi^*_{\overline{U}_j}(1_{\mathrm{R}(G)}).$$

When $\overline{U}_j$ *is smooth,* $\pi^*_{\overline{U}_j}(1_{\mathrm{R}(G)}) = 1_{K^G\left(\pi_{\overline{U}_j}\right)}$*.*

This theorem is powerful. Most stratifications can be (non-canonically) viewed as cellular decompositions, and the theorem gives us the $\mathrm{R}(G)$-module structure of the total space. Readers can compare this theorem with the cellular cohomology of CW-complexs with no cell in odd dimension.

3.2 Application: module structure

Before we really start working, let us make a shorthand for the basis.

Definition 3.2.1. *Let $\iota_Y : Y \longrightarrow X$ be a closed G-equivariant embedding, $\pi_Y : Y \longrightarrow \mathrm{pt}$ be the projection map. Denote*

$$[Y]^G := \iota_{Y,*}\pi_Y^* 1_{\mathrm{R}(G)} \in K^G(X).$$

$$\begin{tikzcd} Y \arrow[r, hook, "\iota_Y"] \arrow[d, "\pi_Y"'] & X \\ \mathrm{pt} & \end{tikzcd}$$

Warning 3.2.2. *The symbol $[Y]^G$ (weakly) depends on X, and we don't want to mention X all the time. In practice, Y will be the closure of some U_i for the stratification $X = \sqcup_i U_i$, so we can read X from the symbol in the bracket. In case X is not clear from the context, we write $[Y]^G_X$ to emphasize X.*

Table 3.1 to 3.3 conclude the results in this section.

	pt	$\mathcal{F}$	$\mathcal{F} \otimes \mathcal{F}$
GL_n	$\mathrm{R}(T)^W$	$\mathrm{R}(T)$	$\bigoplus_{w'} \mathrm{R}(T) \left[\boldsymbol{\Omega}_{w'}\right]^{\mathrm{GL}_n}$
B	$\mathrm{R}(T)$	$\bigoplus_{w} \mathrm{R}(T) \left[\overline{\Omega}_{w}\right]^{B}$	$\bigoplus_{w,w'} \mathrm{R}(T) \left[\boldsymbol{\Omega}_{w,w'}\right]^{B}$
Id	$\mathbb{Z}$	$\bigoplus_{w} \mathbb{Z} \left[\overline{\Omega}_{w}\right]$	$\bigoplus_{w,w'} \mathbb{Z} \left[\boldsymbol{\Omega}_{w,w'}\right]$

Table 3.1: Initial case

	pt	$\mathcal{F}_{\underline{\mathbf{d}}}$	$\mathcal{F}_{\underline{\mathbf{d}}} \otimes \mathcal{F}_{\underline{\mathbf{d}}'}$	$\widetilde{\mathrm{Rep}}_{\underline{\mathbf{d}}}(Q)$	$\mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}'}$
$G_{\mathbf{d}}$	$\mathrm{R}(T_{\mathbf{d}})^{W_{\mathbf{d}}}$	$\mathrm{R}(T_{\mathbf{d}})$	$\bigoplus_{w'} \mathrm{R}(T_{\mathbf{d}}) \left[\overline{\boldsymbol{\Omega}}^{u,u'}_{w'}\right]^{G_{\mathbf{d}}}$	$\mathrm{R}(T_{\mathbf{d}})$	$\bigoplus_{w'} \mathrm{R}(T_{\mathbf{d}}) \left[\mathcal{Z}^{u,u'}_{w'}\right]^{G_{\mathbf{d}}}$
$B_{\mathbf{d}}$	$\mathrm{R}(T_{\mathbf{d}})$	$\bigoplus_{w} \mathrm{R}(T_{\mathbf{d}}) \left[\overline{\Omega}^{u}_{w}\right]^{B_{\mathbf{d}}}$	$\bigoplus_{w,w'} \mathrm{R}(T_{\mathbf{d}}) \left[\overline{\boldsymbol{\Omega}}^{u,u'}_{w,w'}\right]^{B_{\mathbf{d}}}$	$\bigoplus_{w} \mathrm{R}(T_{\mathbf{d}}) \left[\overline{\widetilde{\Omega}}^{u}_{w}\right]^{B_{\mathbf{d}}}$	$\bigoplus_{w,w'} \mathrm{R}(T_{\mathbf{d}}) \left[\overline{\widetilde{\boldsymbol{\Omega}}}^{u,u'}_{w,w'}\right]^{B_{\mathbf{d}}}$
Id	$\mathbb{Z}$	$\bigoplus_{w} \mathbb{Z} \left[\overline{\Omega}^{u}_{w}\right]$	$\bigoplus_{w,w'} \mathbb{Z} \left[\overline{\boldsymbol{\Omega}}^{u,u'}_{w,w'}\right]$	$\bigoplus_{w} \mathbb{Z} \left[\overline{\widetilde{\Omega}}^{u}_{w}\right]$	$\bigoplus_{w,w'} \mathbb{Z} \left[\overline{\widetilde{\boldsymbol{\Omega}}}^{u,u'}_{w,w'}\right]$

Table 3.2: Relative case

First, we work over Setting 1.1.1.

Example 3.2.3. *The complete flag variety $\mathcal{F}$ has a stratification $\mathcal{F} = \sqcup_w \Omega_w$. By extending the Bruhat order on W to a total order $\preccurlyeq$, we get a cellular decomposition of $\mathcal{F}$:*

$$0 \subseteq \Omega_{\mathrm{Id}} \subseteq \quad \cdots \quad \subseteq \bigsqcup\nolimits_{x \preccurlyeq w} \Omega_x \subseteq \quad \cdots \quad \subseteq \bigsqcup\nolimits_{x} \Omega_x = \mathcal{F}$$

By Theorem 3.1.3,

$$K^B(\mathcal{F}) \cong \bigoplus_w \mathrm{R}(B) \left[\overline{\Omega}_w\right]^B \qquad K(\mathcal{F}) \cong \bigoplus_w \mathbb{Z} \left[\overline{\Omega}_w\right].$$

	pt	$\mathcal{F}_{\mathbf{d}}$	$\mathcal{F}_{\mathbf{d}} \otimes \mathcal{F}_{\mathbf{d}}$	$\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)$	$\mathcal{Z}_{\mathbf{d},\mathbf{d}}$
$G_{\mathbf{d}}$	$\mathrm{R}(T_{\mathbf{d}})^{W_{\mathbf{d}}}$	$\bigoplus_{\underline{\mathbf{d}}} \mathrm{R}(T_{\mathbf{d}}) \left[\mathcal{F}_{\underline{\mathbf{d}}}\right]^{G_{\mathbf{d}}}$	$\bigoplus_{\varpi'} \mathrm{R}(T_{\mathbf{d}}) \left[\overline{\boldsymbol{\mathcal{O}}}_{\varpi'}\right]^{G_{\mathbf{d}}}$	$\bigoplus_{\underline{\mathbf{d}}} \mathrm{R}(T_{\mathbf{d}}) \left[\widetilde{\mathrm{Rep}}_{\underline{\mathbf{d}}}(Q)\right]^{G_{\mathbf{d}}}$	$\bigoplus_{\varpi'} \mathrm{R}(T_{\mathbf{d}}) \left[\mathcal{Z}_{\varpi'}\right]^{G_{\mathbf{d}}}$
$B_{\mathbf{d}}$	$\mathrm{R}(T_{\mathbf{d}})$	$\bigoplus_{\varpi} \mathrm{R}(T_{\mathbf{d}}) \left[\overline{\mathcal{O}}_{\varpi}\right]^{B_{\mathbf{d}}}$	$\bigoplus_{\varpi,\varpi'} \mathrm{R}(T_{\mathbf{d}}) \left[\overline{\boldsymbol{\mathcal{O}}}_{\varpi,\varpi'}\right]^{B_{\mathbf{d}}}$	$\bigoplus_{\varpi} \mathrm{R}(T_{\mathbf{d}}) \left[\widetilde{\overline{\mathcal{O}}}_{\varpi}\right]^{B_{\mathbf{d}}}$	$\bigoplus_{\varpi,\varpi'} \mathrm{R}(T_{\mathbf{d}}) \left[\widetilde{\overline{\boldsymbol{\mathcal{O}}}}_{\varpi,\varpi'}\right]^{B_{\mathbf{d}}}$
Id	$\mathbb{Z}$	$\bigoplus_{\varpi} \mathbb{Z} \left[\overline{\mathcal{O}}_{\varpi}\right]$	$\bigoplus_{\varpi,\varpi'} \mathbb{Z} \left[\overline{\boldsymbol{\mathcal{O}}}_{\varpi,\varpi'}\right]$	$\bigoplus_{\varpi} \mathbb{Z} \left[\widetilde{\overline{\mathcal{O}}}_{\varpi}\right]$	$\bigoplus_{\varpi,\varpi'} \mathbb{Z} \left[\widetilde{\overline{\boldsymbol{\mathcal{O}}}}_{\varpi,\varpi'}\right]$

Table 3.3: Absolute case

In particular,

$$\begin{aligned} K^{\mathrm{GL}_n}(\mathcal{F} \times \mathcal{F}) \cong K^B(\mathcal{F}) &\cong \bigoplus_{w} \mathrm{R}(B) \cdot \mathrm{Ind}_B^{\mathrm{GL}_n} \left(\left[\overline{\Omega}_w\right]^B\right) \\ &\cong \bigoplus_{w'} \mathrm{R}(B) \left[\overline{\boldsymbol{\Omega}}_{w'}\right]^{\mathrm{GL}_n} \end{aligned}$$

Example 3.2.4. *$\mathcal{F} \times \mathcal{F}$ has many stratifications. Consider the stratification $\mathcal{F} \times \mathcal{F} = \sqcup_{w,w' \in W} \boldsymbol{\Omega}_{w,w'}$. By extending the Bruhat order on $W \times W$ (i.e., $(x, x') \leq (w, w')$ if and only if $x \leq w$ and $x' \leq w'$) to a total order $\preccurlyeq$, we get a cellular decomposition of $\mathcal{F} \times \mathcal{F}$:*

$$0 \subseteq \boldsymbol{\Omega}_{\mathrm{Id},\mathrm{Id}} \subseteq \quad \cdots \quad \subseteq \bigsqcup_{(x,x') \preccurlyeq (w,w')} \boldsymbol{\Omega}_{x,x'} \subseteq \quad \cdots \quad \subseteq \bigsqcup_{x,x'} \boldsymbol{\Omega}_{x,x'} = \mathcal{F} \times \mathcal{F}$$

By Theorem 3.1.3,

$$K^B(\mathcal{F} \times \mathcal{F}) \cong \bigoplus_{w,w'} \mathrm{R}(B) \left[\overline{\boldsymbol{\Omega}}_{w,w'}\right]^B \qquad K(\mathcal{F} \times \mathcal{F}) \cong \bigoplus_{w,w'} \mathbb{Z} \left[\overline{\boldsymbol{\Omega}}_{w,w'}\right].$$

Example 3.2.5. *For computing $K^{\mathrm{GL}_n}(\mathcal{F} \times \mathcal{F})$, consider the ($GL_n$-equivariant) stratification $\mathcal{F} \times \mathcal{F} = \sqcup_{w'} \boldsymbol{\Omega}_{w'}$. Again, we get a cellular decomposition of $\pi_2 : \mathcal{F} \times \mathcal{F} \longrightarrow \mathcal{F}$:*

$$0 \subseteq \boldsymbol{\Omega}_{\mathrm{Id}} \subseteq \quad \cdots \quad \subseteq \bigsqcup_{x' \preccurlyeq w'} \boldsymbol{\Omega}_{x'} \subseteq \quad \cdots \quad \subseteq \bigsqcup_{x'} \boldsymbol{\Omega}_{x'} = \mathcal{F} \times \mathcal{F}$$

By Theorem 3.1.3 and Example 2.4.2, we get

$$\begin{aligned} K^{\mathrm{GL}_n}(\mathcal{F} \times \mathcal{F}) &\cong \bigoplus_{w'} K^{\mathrm{GL}_n}(\boldsymbol{\Omega}_{w'}) \\ &\cong \bigoplus_{w'} K^B(\Omega_{w'}) \\ &\cong \bigoplus_{w'} \mathrm{R}(B) \left[\overline{\boldsymbol{\Omega}}_{w'}\right]^{\mathrm{GL}_n} \end{aligned}$$

The general case can be solved by the same method.

Example 3.2.6. *By repeating Example 2.1.3 to 2.1.6, we get*

$$\mathrm{R}(N_{\mathbf{d}}) \cong \mathrm{R}(\mathrm{Id}) \cong \mathbb{Z} \qquad \mathrm{R}(B_{\mathbf{d}}) \cong \mathrm{R}(T_{\mathbf{d}}) \cong \mathbb{Z}\Big[e_1^{\pm 1}, \dots, e_{|\mathbf{d}|}^{\pm 1}\Big]$$

$$\mathrm{R}(G_{\mathbf{d}}) \cong \mathrm{R}(T_{\mathbf{d}})^{W_{\mathbf{d}}} \cong \mathbb{Z}\Big[e_1^{\pm 1}, \dots, e_{|\mathbf{d}|}^{\pm 1}\Big]^{W_{\mathbf{d}}}$$

The induction formula tells us

$$K^{G_{\mathbf{d}}}(\mathcal{F}_{\underline{\mathbf{d}}}) \cong K^{B_{\mathbf{d}}}(\mathrm{pt}) = \mathrm{R}(B_{\mathbf{d}}) \cong \mathbb{Z}\Big[e_1^{\pm 1}, \dots, e_{|\mathbf{d}|}^{\pm 1}\Big].$$

By repeating Example 3.2.3 to 3.2.4, we get

$$K^{B_{\mathbf{d}}}(\mathcal{F}_{\underline{\mathbf{d}}}) \cong \bigoplus_{w} \mathrm{R}(B_{\mathbf{d}}) \left[\overline{\Omega}_w^u\right]^{B_{\mathbf{d}}} \qquad K(\mathcal{F}_{\underline{\mathbf{d}}}) \cong \bigoplus_{w} \mathbb{Z} \left[\overline{\Omega}_w^u\right]$$

$$K^{G_{\mathbf{d}}}(\mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}}'}) \cong \bigoplus_{w'} \mathrm{R}(B_{\mathbf{d}}) \left[\overline{\boldsymbol{\Omega}}_{w'}^{u,u'}\right]^{G_{\mathbf{d}}}$$

$$K^{B_{\mathbf{d}}}(\mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}}'}) \cong \bigoplus_{w,w'} \mathrm{R}(B_{\mathbf{d}}) \left[\overline{\boldsymbol{\Omega}}_{w,w'}^{u,u'}\right]^{B_{\mathbf{d}}} \qquad K(\mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}}'}) \cong \bigoplus_{w,w'} \mathbb{Z} \left[\overline{\boldsymbol{\Omega}}_{w,w'}^{u,u'}\right]$$

Since

$$\mathcal{F}_{\mathbf{d}} = \bigsqcup_{\underline{\mathbf{d}}} \mathcal{F}_{\underline{\mathbf{d}}} \qquad \mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}} = \bigsqcup_{\underline{\mathbf{d}},\underline{\mathbf{d}}'} \mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}}'}$$

as topological spaces, we get K-theory of $\mathcal{F}_{\mathbf{d}}$ and $\mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}}$ for free. (See Table???)

The calculations of incidence spaces use the same method we introduced in Example 3.2.5.

Example 3.2.7. *We compute $G_{\mathbf{d}}$-equivariant K-theory of the Steinberg variety in this example.*

$$\begin{aligned}
K^{G_{\mathbf{d}}}(\mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}'}) &\cong \bigoplus_{w'} K^{G_{\mathbf{d}}}(\widetilde{\boldsymbol{\Omega}}_{w'}^{u,u'}) \\
&\cong \bigoplus_{w'} K^{G_{\mathbf{d}}}(\boldsymbol{\Omega}_{w'}^{u,u'}) \\
&\cong \bigoplus_{w'} K^{B_{\mathbf{d}}}(\Omega_{w'}^{u,u'}) \\
&\cong \bigoplus_{w'} \mathrm{R}(B_{\mathbf{d}}) \left[\overline{\widetilde{\boldsymbol{\Omega}}}_{w'}^{u,u'}\right]^{G_{\mathbf{d}}}
\end{aligned}$$

Now we've done with the module structure. The equivariant cohomology theory can be also computed in the exact same way, see [3, Chapter 7].

Chapter 4

Localization theorem

We have already gotten the module structure of K-theories. However, this basis behaves badly with the convolution product (will be introduced in Section 5.1), because "the information is not concentrated enough". In this chapter we will introduce another basis, which "concentrates information in the T-fixed points". The localization formula describes the transition matrix of two basis. Readers with topological background can compare the localization theorem with the Poincaré-Hopf theorem.

4.1 Euler class

In the category of coherent sheaf, the "proper base change" is usually not true. In order to describe the defect of the diagram, we introduce the Euler class.

Definition 4.1.1 (Euler class, for K-group)**.** *Let X be a G-variety, and $\mathcal{T}$ be a G-equivariant vector bundle over X. The Euler class is defined by*

$$\mathrm{eu}(\mathcal{T}) := \sum_{k=0}^{\infty} (-1)^k \left[\Lambda^k \mathcal{T}^*\right] \in K^G(X)$$

In practice, X are points and G is a torus. In that case, since we know the representation of a torus (see Example 2.1.4), the Euler class can be explicitly written down. For example, ($X = \mathrm{pt}$)

$$\begin{aligned}
&\mathrm{eu}\,(1) = 1 \\
&\mathrm{eu}\left(\frac{e_1}{e_2}\right) = 1 - \frac{e_2}{e_1} \\
&\mathrm{eu}\left(\frac{e_1}{e_2} + \frac{e_2}{e_3} + \frac{e_3}{e_1}\right) = \left(1 - \frac{e_2}{e_1}\right)\left(1 - \frac{e_3}{e_2}\right)\left(1 - \frac{e_1}{e_3}\right)
\end{aligned}$$

Here we confuse the notation of $\mathrm{R}(T)$ and $\mathrm{Rep}(T)$: the elements inside the bracket of Euler class should be viewed as a vector bundle rather than a $\mathbb{Z}$-linear combination of coherent sheaves.

Warning 4.1.2. *Compared with usual Euler class, some properties are kept in K-theory version, while some are not. For example, for line bundles $\mathcal{L}$, $\mathcal{L}_1$, $\mathcal{L}_2$ over X,*

$$\begin{aligned}
&\mathrm{eu}(\mathcal{T} \oplus \mathcal{T}') \cong \mathrm{eu}(\mathcal{T}) \cdot \mathrm{eu}(\mathcal{T}'),\\
&\mathrm{eu}(\mathcal{L}_1 \otimes \mathcal{L}_2) \neq \mathrm{eu}(\mathcal{L}_1) + \mathrm{eu}(\mathcal{L}_2) \qquad \mathrm{eu}(\mathcal{L}^*) \neq -\mathrm{eu}(\mathcal{L}).
\end{aligned}$$

Remark 4.1.3. We also have equivariant Euler class for cohomology theory, see [3, Chapter 9], ??? for more details. In particular, for any T-representation $\mathcal{T}$ with weight space decomposition $\mathcal{T}^* = \oplus \mathcal{T}^*_\lambda$, the Euler class of $\mathcal{T}$ (for cohomology theory) is defined by

$$\mathrm{eu}'(\mathcal{T}) := \prod_{\lambda \in X^*(T)} \lambda^{\dim \mathcal{T}^*_\lambda} \in \mathrm{S}(T)$$

where $X^*(T)$ embeds in $\mathrm{S}(T)$ by

$$X^*(T) \longrightarrow \mathrm{S}(T) \qquad \sum_i k_i \varepsilon_i \longmapsto \sum_i k_i \lambda_i.$$

For example,

$$\begin{aligned}
&\mathrm{eu}'(1) = 1\\
&\mathrm{eu}'\left(\frac{e_1}{e_2}\right) = \lambda_2 - \lambda_1\\
&\mathrm{eu}'\left(\frac{e_1}{e_2} + \frac{e_2}{e_3} + \frac{e_3}{e_1}\right) = (\lambda_2 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_1 - \lambda_3)
\end{aligned}$$

4.2 Statement

We first state one general theorem, which will be connected with both localization formula and excess intersection formula.

Theorem 4.2.1 (Excess base change, [4, Théorème 3.1])**.** *Let* (4.2.1) *be a Cartesian square of G-varieties, ϕ, φ are regular embeddings and f, g are of globally finite* Tor*-dimension. Denote $\mathcal{N}_\phi$ and $\mathcal{N}_\varphi$ as the normal cone of ϕ, φ respectively, and $\mathcal{T} := (g^*\mathcal{N}_\varphi)/\mathcal{N}_\phi$ as a vector bundle over W.*

$$\begin{tikzcd}
\mathcal{N}_\phi \arrow[dr, dash] & g^*\mathcal{N}_\varphi \arrow[d, dash] & \mathcal{N}_\varphi \arrow[d, dash] \\
& W \arrow[r, "g"] \arrow[d, "\phi"'] & Z \arrow[d, "\varphi"] \\
& Y \arrow[r, "f"] & X
\end{tikzcd} \tag{4.2.1}$$

*For any $\alpha \in K^G(Z)$, we have the **excess base change formula**:*

$$f^* \circ \varphi_*(\alpha) = \phi_*\Big(\mathrm{eu}(\mathcal{T}) \cdot g^*(\alpha)\Big) \qquad \textit{in } K^G(Y)$$

where the dot product of $\mathrm{eu}(\mathcal{T})$ *is given by the tensor product in $K^G(W)$.*

By applying Theorem 4.2.1 to the Cartesian square (4.2.2), we get the (fake) localization formula:

$$\begin{array}{ccc} X^T & \xrightarrow{\mathrm{Id}} & X^T \\ {\scriptstyle \mathrm{Id}}\big\downarrow & & \big\downarrow{\scriptstyle i} \\ X^T & \xrightarrow{i} & X \end{array} \tag{4.2.2}$$

Proposition 4.2.2 (Fake localization formula)**.** *For a smooth T-variety X with finite fixed points $\{x_1, \dots, x_m\}$, denote $i : X^T \longrightarrow X$ and $i_k : \{x_k\} \longrightarrow X$ as embeddings. For any $\beta \in K^T(X^T)$, $\beta_k \in K^T(\{x_k\})$, we have formulas*

$$i^* i_* \beta = \mathrm{eu}\left(\bigoplus_k T_{x_k} X\right) \cdot \beta \qquad i_k^* i_{k,*} \beta = \mathrm{eu}\,(T_{x_k} X) \cdot \beta_k.$$

This proposition is not as powerful as it is supposed to be, but it explains some technical details in the localization theorem and localization formula. First, we would like to work on a base ring where Euler classes are invertible, so we denote the curly font as everything in the fraction field.

$$\mathcal{R}(T) := \mathrm{Frac}\big(\mathrm{R}(T)\big) \qquad \mathcal{K}^T(X) := K^T(X) \otimes_{\mathrm{R}(T)} \mathcal{R}(T)$$
$$\mathcal{S}(T) := \mathrm{Frac}\big(\mathrm{S}(T)\big) \qquad \mathcal{H}_T^*(X) := H_T^*(X) \otimes_{\mathrm{S}(T)} \mathcal{S}(T)$$

Now we can do linear algebras and discuss about the actual basis:

Theorem 4.2.3 (Localization theorem, [3, Theorem 10.1] or [1, Corollary 5.11.3])**.** *Let X be a smooth T-variety, $i : X^T \longrightarrow X$ be the embedding. The morphisms i_*, i^* are isomorphism after tensored over the fraction field, i.e.,*

$$\mathcal{K}^T(X^T) \xrightarrow{i_*} \mathcal{K}^T(X) \xrightarrow{i^*} \mathcal{K}^T(X^T)$$
$$\mathcal{H}_T^*(X^T) \xrightarrow{i_*} \mathcal{H}_T^*(X) \xrightarrow{i^*} \mathcal{H}_T^*(X^T)$$

are isomorphism as $\mathcal{R}(T)$ or $\mathcal{S}(T)$-modules.

The genuine localization formula is stated as follows.

Theorem 4.2.4 (Localization formula, [3, Theorem 10.2] or [2, Proposition 6])**.** *For a smooth T-variety X with finite fixed points $\{x_1, \dots, x_m\}$, denote $i_k : \{x_k\} \longrightarrow X$ as embeddings. For any $\alpha \in \mathcal{K}^T(X)$, we have formula*

$$\alpha = \sum_{k=1}^{m} \eta_k \cdot i_{k,*} i_k^* \alpha$$

where $\eta_k := \big(\mathrm{eu}(T_{x_k} X)\big)^{-1} \in \mathcal{R}(T)$.

More generally, suppose $f : Y \hookrightarrow X$ is a T-equivariant closed subvariety with finite fixed points $\{x_1, \dots, x_{m'}\}$, denote $i'_k : \{x_k\} \longrightarrow Y$ as embeddings. For any $\beta \in \mathcal{K}^T(Y)$, we have formula

$$\beta = \sum_{k=1}^{m} \eta_k \cdot i'_{k,*} i_k^* f_* \beta.$$

Let us unravel Theorem 4.2.4 a little bit. For the closed T-equivariant subset Z of Y, denote $[Z]^T_X \in K^T(X)$, $[Z]^T_Y \in K^T(Y)$, $[x_k]^T_Y \in K^T(Y)$. Substitute the localization formula, we get

$$\begin{aligned}
[Z]^T_Y &= \sum_{k=1}^{m} \eta_k \cdot i'_{k,*} i^*_k f_* [Z]^T_Y \\
&= \sum_{k=1}^{m} \eta_k \cdot i'_{k,*} \left(i^*_k [Z]^T_X \cdot 1_{\mathrm{R}(T)} \right) && \text{definition of } [Z]^T_X \\
&= \sum_{k=1}^{m} \eta_k \cdot \left(i^*_k [Z]^T_X \right) \cdot \left(i'_{k,*} 1_{\mathrm{R}(T)} \right) && i'_{k,*} \text{ is a } \mathrm{R}(T)\text{-module homomorphism} \\
&= \sum_{k=1}^{m} \eta_k \cdot \left(i^*_k [Z]^T_X \right) \cdot [x_k]^T_Y && \text{definition of } [x_k]^T_Y
\end{aligned}$$

When Z is smooth at x_k,[1] denote $g : Z \hookrightarrow X$ and $j_k : \{x_k\} \longrightarrow Z$,

$$\begin{aligned}
i^*_k [Z]^T_X &= i^*_k g_* \left(\pi^*_Z 1_{\mathrm{R}(T)} \right) \\
&= \mathrm{eu}\,(j^*_k N_Z X) \cdot j^*_k \left(\pi^*_Z 1_{\mathrm{R}(T)} \right) && \text{excess base change} \\
&= \mathrm{eu} \left(\frac{T_{x_k} X}{T_{x_k} Z} \right) \cdot 1_{\mathrm{R}(T)} && \pi_Z \circ j_k = \mathrm{Id}_{\mathrm{pt}} \\
&= \frac{\mathrm{eu}\,(T_{x_k} X)}{\mathrm{eu}\,(T_{x_k} Z)} && \mathrm{Rep}(T) \text{ is semisimple}
\end{aligned}$$

Therefore, the coefficient before $[x_k]^T_Y$ is

$$\eta_k \cdot \left(i^*_k [Z]^T_X \right) = \frac{1}{\mathrm{eu}\,(T_{x_k} X)} \cdot \frac{\mathrm{eu}\,(T_{x_k} X)}{\mathrm{eu}\,(T_{x_k} Z)} = \frac{1}{\mathrm{eu}\,(T_{x_k} Z)}.$$

In other word, we computed the transition matrix between two basis, where the matrix coefficient is roughly the inverse of the Euler class. Keep this is mind, and let us see applications now.

4.3 Application: change of basis

Before we really start working, let us make a shorthand for the basis and the Euler class.

Definition 4.3.1 (Another basis). *For ϖ, ϖ', $x \in \mathbb{W}_{|\mathbf{d}|}$, denote*

$$\begin{aligned}
\psi_\varpi &:= [\{F_\varpi\}]^{T_\mathbf{d}} = (i_\varpi)_* 1_{\mathrm{R}(T_\mathbf{d})} && \in K^{T_\mathbf{d}}(\mathcal{F}_\mathbf{d}) \\
\psi^x_\varpi &:= [\{F_\varpi\}]^{T_\mathbf{d}} = (i^x_\varpi)_* 1_{\mathrm{R}(T_\mathbf{d})} && \in K^{T_\mathbf{d}}(\overline{\mathcal{O}}_x) \\
\psi_{\varpi,\varpi'} &:= \left[\{F_{\varpi,\varpi'}\}\right]^{T_\mathbf{d}} = (i_{\varpi,\varpi'})_* 1_{\mathrm{R}(T_\mathbf{d})} && \in K^{T_\mathbf{d}}(\mathcal{F}_\mathbf{d} \times \mathcal{F}_\mathbf{d}) \\
\psi^x_{\varpi,\varpi'} &:= \left[\{F_{\varpi,\varpi'}\}\right]^{T_\mathbf{d}} = (i^x_{\varpi,\varpi'})_* 1_{\mathrm{R}(T_\mathbf{d})} && \in K^{T_\mathbf{d}}(\overline{\boldsymbol{\mathcal{O}}}_x)
\end{aligned}$$

[1]The smoothness guarantees the regular embedding condition in Theorem 4.2.1.

The same symbols are used for

$$\widetilde{\psi}_\varpi \in K^{T_{\mathbf{d}}}\left(\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)\right) \quad \widetilde{\psi}^x_\varpi \in K^{T_{\mathbf{d}}}\left(\widetilde{\overline{\mathcal{O}}}_x\right) \quad \widetilde{\psi}_{\varpi,\varpi'} \in K^{T_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}}) \quad \widetilde{\psi}^x_{\varpi,\varpi'} \in K^{T_{\mathbf{d}}}(\mathcal{Z}_x).$$

Also, we use underline to twist subscripts, like $\underline{\psi}_{\varpi,\varpi'} := \psi_{\varpi,\varpi\varpi'}$.

By Theorem 4.2.3,

$$\mathcal{K}^{T_{\mathbf{d}}}(\mathcal{F}_{\mathbf{d}}) \cong \bigoplus_{\varpi} \mathcal{R}(T_{\mathbf{d}})\psi_\varpi \qquad \mathcal{K}^{T_{\mathbf{d}}}(\mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}}) \cong \bigoplus_{\varpi,\varpi'} \mathcal{R}(T_{\mathbf{d}})\psi_{\varpi,\varpi'}$$
$$\mathcal{K}^{T_{\mathbf{d}}}\left(\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)\right) \cong \bigoplus_{\varpi} \mathcal{R}(T_{\mathbf{d}})\widetilde{\psi}_\varpi \qquad \mathcal{K}^{T_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}}) \cong \bigoplus_{\varpi,\varpi'} \mathcal{R}(T_{\mathbf{d}})\widetilde{\psi}_{\varpi,\varpi'}.$$

Definition 4.3.2 (Shorthand for Euler class)**.** *For* ϖ*,* ϖ'*,* $x \in \mathbb{W}_{|\mathbf{d}|}$*, denote the Euler class in* $\mathrm{R}(T_{\mathbf{d}})$*:*

$$\begin{aligned} \Lambda_\varpi &:= \mathrm{eu}\left(\mathcal{T}_\varpi\right) & \Lambda^x_\varpi &:= \mathrm{eu}\left(\mathcal{T}^x_\varpi\right) & \Lambda^x_{\varpi,\varpi'} &:= \mathrm{eu}\left(\mathcal{T}^x_{\varpi,\varpi'}\right) \\ \widetilde{\Lambda}_\varpi &:= \mathrm{eu}\left(\widetilde{\mathcal{T}}_\varpi\right) & \widetilde{\Lambda}^x_\varpi &:= \mathrm{eu}\left(\widetilde{\mathcal{T}}^x_\varpi\right) & \widetilde{\Lambda}^x_{\varpi,\varpi'} &:= \mathrm{eu}\left(\widetilde{\mathcal{T}}^x_{\varpi,\varpi'}\right) \end{aligned}$$

For completeness, denote

$$\Lambda_{\varpi,\varpi'} := \mathrm{eu}\left(\mathcal{T}_{\varpi,\varpi'}\right) \qquad \widetilde{\Lambda}_{\varpi,\varpi'} := \mathrm{eu}\left(\widetilde{\mathcal{T}}_{\varpi,\varpi'}\right).$$

Also, we use underline to twist subscripts.

Now we can compute the transition matrix of two basis.

Example 4.3.3. *Let* $X = Y = \mathcal{F}_{\mathbf{d}}$*,* $T = T_{\mathbf{d}}$*,* $i_\varpi : \{F_\varpi\} \hookrightarrow \mathcal{F}_{\mathbf{d}}$ *be the embedding,* $y \in W_{\mathbf{d}}$*, we get*

$$\left[\overline{\Omega}^u_y\right]^{T_{\mathbf{d}}} = \sum_{w \le y} \Lambda^{-1}_{wu}\left(i^*_{wu}\left[\overline{\Omega}^u_y\right]^{T_{\mathbf{d}}}\right) \cdot \psi_{wu}.$$

When $\overline{\Omega}^u_y$ *is smooth at* F_{wu}*,* $\Lambda^{-1}_{wu}\left(i^*_{wu}\left[\overline{\Omega}^u_y\right]^{T_{\mathbf{d}}}\right) = \left(\mathrm{eu}\left(T_{F_{wu}}\overline{\Omega}^u_y\right)\right)^{-1} = \left(\Lambda^{yu}_{wu}\right)^{-1}$*. Especially, for* $s \in \Pi_{\mathbf{d}}$*,*

$$\begin{aligned} \left[\overline{\Omega}^u_{\mathrm{Id}}\right]^{T_{\mathbf{d}}} &= \left(\Lambda^u_u\right)^{-1}\psi_u = \psi_u \\ \left[\overline{\Omega}^u_s\right]^{T_{\mathbf{d}}} &= \left(\Lambda^{su}_u\right)^{-1}\psi_u + \left(\Lambda^{su}_{su}\right)^{-1}\psi_{su} \\ \left[\mathcal{F}_u\right]^{T_{\mathbf{d}}} &= \sum_w \Lambda^{-1}_{wu}\psi_{wu} \\ \left[\mathcal{F}_{\mathbf{d}}\right]^{T_{\mathbf{d}}} &= \sum_\varpi \Lambda^{-1}_\varpi\psi_\varpi \end{aligned}$$

Also, for $s \in \Pi$*,*

$$\left[\overline{\mathcal{O}}_s\right]^{T_{\mathbf{d}}} = \begin{cases} \left(\Lambda^s_{\mathrm{Id}}\right)^{-1}\psi_{\mathrm{Id}} + \left(\Lambda^s_s\right)^{-1}\psi_s, & s \in \Pi_{\mathbf{d}} \\ \psi_s, & s \notin \Pi_{\mathbf{d}} \end{cases}$$

Example 4.3.4. *Let $X = Y = \widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)$, $T = T_{\mathbf{d}}$, $i_\varpi : \{(\rho_0, F_\varpi)\} \hookrightarrow \widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)$ be the embedding, $y \in W_{\mathbf{d}}$, we get*

$$\left[\overline{\widetilde{\Omega}}{}^u_y\right]^{T_{\mathbf{d}}} = \sum_{w \le y} \widetilde{\Lambda}^{-1}_{wu}\left(i^*_{wu}\left[\overline{\widetilde{\Omega}}{}^u_y\right]^{T_{\mathbf{d}}}\right) \cdot \widetilde{\psi}_{wu}.$$

When $\overline{\widetilde{\Omega}}{}^u_y$ is smooth at F_{wu}, $\widetilde{\Lambda}^{-1}_{wu}\left(i^*_{wu}\left[\overline{\widetilde{\Omega}}{}^u_y\right]^{T_{\mathbf{d}}}\right) = \left(\mathrm{eu}\left(T_{F_{wu}}\overline{\widetilde{\Omega}}{}^u_y\right)\right)^{-1} = \left(\widetilde{\Lambda}^{yu}_{wu}\right)^{-1}$. Especially, for $s \in \Pi_{\mathbf{d}}$,

$$\begin{aligned}
\left[\overline{\widetilde{\Omega}}{}^u_{\mathrm{Id}}\right]^{T_{\mathbf{d}}} &= \left(\widetilde{\Lambda}^u_u\right)^{-1}\widetilde{\psi}_u = \widetilde{\psi}_u \\
\left[\overline{\widetilde{\Omega}}{}^u_s\right]^{T_{\mathbf{d}}} &= \left(\widetilde{\Lambda}^{su}_u\right)^{-1}\widetilde{\psi}_u + \left(\widetilde{\Lambda}^{su}_{su}\right)^{-1}\widetilde{\psi}_{su} \\
\left[\widetilde{\mathrm{Rep}}_{\underline{\mathbf{d}}}(Q)\right]^{T_{\mathbf{d}}} &= \sum_w \widetilde{\Lambda}^{-1}_{wu}\widetilde{\psi}_{wu} \\
\left[\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)\right]^{T_{\mathbf{d}}} &= \sum_\varpi \widetilde{\Lambda}^{-1}_\varpi\widetilde{\psi}_\varpi
\end{aligned}$$

Also, for $s \in \Pi$,

$$\left[\widetilde{\mathcal{O}}_s\right]^{T_{\mathbf{d}}} = \begin{cases} \left(\widetilde{\Lambda}^s_{\mathrm{Id}}\right)^{-1}\widetilde{\psi}_{\mathrm{Id}} + \left(\widetilde{\Lambda}^s_s\right)^{-1}\widetilde{\psi}_s, & s \in \Pi_{\mathbf{d}} \\ \widetilde{\psi}_s, & s \notin \Pi_{\mathbf{d}} \end{cases}$$

Example 4.3.5. *Let $X = Y = \mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}}$, $T = T_{\mathbf{d}}$, $s \in \Pi$. Since $\overline{\boldsymbol{\mathcal{O}}}_s$ is smooth, we get*

$$[\overline{\boldsymbol{\mathcal{O}}}_s]^{T_{\mathbf{d}}} = \sum_{\varpi \in \mathbb{W}_{|\mathbf{d}|}} \left(\Lambda^s_{\varpi,\varpi s}\right)^{-1}\psi_{\varpi,\varpi s} + \sum_{\substack{\varpi \in \mathbb{W}_{|\mathbf{d}|} \\ \varpi s \varpi^{-1} \in W_{\mathbf{d}}}} \left(\Lambda^s_{\varpi,\varpi}\right)^{-1}\psi_{\varpi,\varpi}.$$

One can also write $\left[\overline{\boldsymbol{\mathcal{O}}}_\varpi\right]$ in terms of $\mathcal{R}(T_{\mathbf{d}})$-linear combination of those $\psi_{\varpi,\varpi'}$.

Example 4.3.6. *Let $X = \mathrm{Rep}_{\mathbf{d}}(Q) \times \mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}}$, $Y = \mathcal{Z}_{\mathbf{d}}$, $T = T_{\mathbf{d}}$, $s \in \Pi$. Since $\mathcal{Z}_s$ is smooth, we get*

$$[\mathcal{Z}_s]^{T_{\mathbf{d}}} = \sum_{\varpi \in \mathbb{W}_{|\mathbf{d}|}} \left(\widetilde{\Lambda}^s_{\varpi,\varpi s}\right)^{-1}\widetilde{\psi}_{\varpi,\varpi s} + \sum_{\substack{\varpi \in \mathbb{W}_{|\mathbf{d}|} \\ \varpi s \varpi^{-1} \in W_{\mathbf{d}}}} \left(\widetilde{\Lambda}^s_{\varpi,\varpi}\right)^{-1}\widetilde{\psi}_{\varpi,\varpi}.$$

One can also write $\left[\overline{\boldsymbol{\mathcal{O}}}_\varpi\right]$ in terms of $\mathcal{R}(T_{\mathbf{d}})$-linear combination of those $\widetilde{\psi}_{\varpi,\varpi'}$.

Chapter 5

Excess intersection formula

Finally, we are able to compute the convolution structure of the Steinberg variety in this Chapter. We first introduce the convolution product, then give an explicit intersection formula, and finally apply theorems to our settings.

5.1 Convolution

The construction of the convolution product has a similar flavor with Fourier-Mukai transformation, which is the composition of pullback, tensor product and proper pushforward.

Definition 5.1.1 (Convolution product)**.** *For the convenience of reading, we divide the whole process into three steps.*

Step1. *Setting.*

Let M_1, M_2, M_3 be smooth quasi-projective G-varieties. For convenience, denote

$$M_{ij} := M_i \times M_j \qquad M_{123} = M_1 \times M_2 \times M_3$$

and $p_i^{jk}, p_i := p_i^{123}, p_{ij} := p_{ij}^{123}$ as projections onto some factors, as follows.

$$\begin{tikzcd}
& & M_{123} \arrow[dl, "p_{12}"'] \arrow[dr, "p_{23}"] \arrow[rrrd, bend left, "p_{13}"] & & & \\
& M_{12} \arrow[dl, "p_1^{12}"'] \arrow[dr, "p_2^{12}"] & & M_{23} \arrow[dl, "p_2^{23}"'] \arrow[dr, "p_3^{23}"] & & M_{13} \arrow[llllld, dashed, "p_1^{13}"'] \arrow[dl, dashed, "p_3^{13}"] \\
M_1 & & M_2 & & M_3 &
\end{tikzcd}$$

(Check that $p_i = p_i^{jk} \circ p_{jk}$ for $1 \leqslant j < k \leqslant 3$, $i = j$ or $i = k$)

Step2. *Convolution product on the level of varieties.*

For closed G-subvarieties $Z_{12} \subseteq M_{12}$, $Z_{23} \subseteq M_{23}$, denote

$$Z_{123} := p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23}) \subseteq M_{123}$$

as the intersection of two preimages. The ***convolution product*** *of Z_{12} and Z_{23} is defined as*

$$Z_{12} \circ Z_{23} := p_{13}(Z_{123}) \subseteq M_{13}$$

which is a closed G-subvariety of M_{13}.

Step3. *Convolution product on the level of K-theories.*

Denote

$$\pi_{12} := p_{12}\big|_{p_{12}^{-1}(Z_{12})} \qquad \pi_{23} := p_{23}\big|_{p_{23}^{-1}(Z_{23})} \qquad \pi_{13} := p_{13}\big|_{Z_{123}}$$

as corresponding morphisms restricted to $p_{12}^{-1}(Z_{12})$, $p_{23}^{-1}(Z_{23})$ and Z_{123}, respectively. We assume that π_{13} is proper, so that we can use proper pushforward in K-theory.

We define the convolution product by

$$* : K_0^G(Z_{12}) \times K_0^G(Z_{23}) \longrightarrow K_0^G(Z_{12} \circ Z_{23}) \qquad (\mathcal{F}_{12}, \mathcal{F}_{23}) \longmapsto \mathcal{F}_{12} * \mathcal{F}_{23}$$

$$\mathcal{F}_{12} * \mathcal{F}_{23} = \pi_{13,*}\left(\pi_{12}^* \mathcal{F}_{12} \otimes \pi_{23}^* \mathcal{F}_{23}\right) \in K_0^G(Z_{12} \circ Z_{23})$$

Remark 5.1.2. Those "Z-varieties" (Z_{12}, $p_{12}^{-1}(Z_{12})$, Z_{123}, etc.) are often singular in practice, so π_{12}^*, π_{23}^* and $\otimes$ are defined in the sense of "restriction with supports", under the "M-varieties" which are smooth. The following diagram best illustrates the "actual" definition.

$$\begin{tikzcd}
K_0^G(Z_{12}) \times K_0^G(Z_{23}) \arrow[r, dashed, "\pi_{12}^* \times \pi_{23}^*"] \arrow[d, hook] & K_0^G\big(p_{12}^{-1}(Z_{12})\big) \times K_0^G\big(p_{12}^{-1}(Z_{23})\big) \arrow[r, dashed, "\otimes"] \arrow[d, hook] & K_0^G(Z_{123}) \arrow[r, "\pi_{13,*}"] \arrow[d, hook] & K_0^G(Z_{12} \circ Z_{23}) \arrow[d, hook] \\
K_0^G(M_{12}) \times K_0^G(M_{23}) \arrow[r, "p_{12}^* \times p_{23}^*"] & K_0^G(M_{123}) \times K_0^G(M_{123}) \arrow[r, "\otimes"] & K_0^G(M_{123}) \arrow[r, "p_{13,*}"] & K_0^G(M_{13})
\end{tikzcd}$$

The whole process can be concluded in Figure 5.1.

5.2 Statement

To facilitate the computation of intersection (i.e., tensor product in the construction of convolution product), we state the excess intersection formula.

Theorem 5.2.1 (Excess intersection formula, [3, Corollary 9.4])**.** *Let X' be a smooth G-variety, $X \subseteq X'$ be a (maybe singular) closed G-subvariety, and $Y_1, Y_2 \subseteq X$ be closed G-equivariant embeddings (of globally finite* Tor*-dimension). Denote*

$$Y := Y_1 \cap Y_2 \qquad \iota_Y : Y \hookrightarrow X$$
$$\mathcal{T} := TX\big|_Y / \left(TY_1\big|_Y + TY_2\big|_Y\right)$$

$$
\begin{tikzcd}[column sep=small]
 & & \pi_{12}^*\mathcal{F}_{12}\big|_{Z_{123}} \arrow[d, dash] & \pi_{23}^*\mathcal{F}_{23}\big|_{Z_{123}} \arrow[dl, dash] & & & & & \\
 & & Z_{123} \arrow[r, hook] & M_{123} \arrow[ddl, "p_{12}"'] \arrow[ddrr, "p_{23}"] \arrow[ddrrrrr, bend left=15, "p_{13}"] & & & & & \\
 & \mathcal{F}_{12} \arrow[d, dash] & & & \mathcal{F}_{23} \arrow[d, dash] & & & \mathcal{F}_{13} \arrow[d, dash] & \\
 & Z_{12} \arrow[r, hook] & M_{12} \arrow[dll] \arrow[dr] & & Z_{23} \arrow[r, hook] & M_{23} \arrow[dll] \arrow[dr] & & Z_{13} \arrow[r, hook] & M_{13} \\
M_1 & & & M_2 & & & M_3 & &
\end{tikzcd}
$$

Figure 5.1: Convolution Product

$$
\begin{tikzcd}
N_Y Y_2 \arrow[dr, dash] & \frac{N_Y X}{N_Y Y_1} \arrow[d, dash] & N_{Y_1} X \arrow[d, dash] \\
 & Y \arrow[r, "g"] \arrow[d, "\phi"'] & Y_1 \arrow[d, "\varphi"] \\
 & Y_2 \arrow[r, "f"] & X
\end{tikzcd}
\tag{5.2.1}
$$

Assume that $TY_1\big|_Y \cap TY_2\big|_Y = TY$*, we get excess intersection formula:*

$$[Y_1]_X^G \otimes [Y_2]_X^G = \iota_{Y,*}\left(\mathrm{eu}(\mathcal{T}) \cdot [Y]_Y^G\right).$$

In particular, when $Y = \mathrm{pt}$ *is a point, we get simplified formula in* $K^G(X)$*:*

$$[Y_1]^G \otimes [Y_2]^G = \mathrm{eu}(\mathcal{T}) \cdot [Y]^G$$

where $\mathrm{eu}(\mathcal{T}) \in \mathrm{R}(G)$ *acts by scalar multiplication.*

Readers may find Theorem 5.2.1 as a special case of excess base change theorem. In fact,

$$
\begin{aligned}
[Y_1]_X^G \otimes [Y_2]_X^G &= [Y_1]_X^G \otimes f_*[Y_2]_{Y_2}^G && \text{definition of } [Y_2]_X^G \\
&= f_*\left(f^*[Y_1]_X^G \otimes [Y_2]_{Y_2}^G\right) && \text{proper projection formula} \\
&= f_*\left(f^*[Y_1]_X^G\right) && \text{Lemma 2.2.9} \\
&= f_*\left(f^*\varphi_*[Y_1]_{Y_1}^G\right) && \text{definition of } [Y_1]_X^G \\
&= f_*\left(\phi_*\left(\mathrm{eu}(\mathcal{T}) \cdot g^*[Y_1]_{Y_1}^G\right)\right) && \text{excess base change to (5.2.1)} \\
&= \iota_{Y,*}\left(\mathrm{eu}(\mathcal{T}) \cdot [Y]_Y^G\right)
\end{aligned}
$$

5.3 Application: convolution structure

In this section, we will apply Definition 5.1.1 and Theorem 5.2.1 to our typical varieties. In particular, we will get the convolution product formula in terms of basis elements $\widetilde{\phi}_{\varpi}$ and $\widetilde{\phi}_{\varpi,\varpi'}$.

5.3.1 Algebraic structures induced by convolution product

Definition 5.3.1 (Convolution product sturcture on $K^{G_\mathbf{d}}(\mathcal{Z}_\mathbf{d})$). *Following notations in 5.1.1, We take $G = G_\mathbf{d}$,*

$$\begin{aligned} M_1 &= M_2 = M_3 = \widetilde{\mathrm{Rep}}_\mathbf{d}(Q) \\ Z_{12} &= Z_{23} = \mathcal{Z}_\mathbf{d} \\ \mathcal{Z}_\mathbf{d} &= \widetilde{\mathrm{Rep}}_\mathbf{d}(Q) \times_{\mathrm{Rep}_\mathbf{d}(Q)} \widetilde{\mathrm{Rep}}_\mathbf{d}(Q) \subseteq \widetilde{\mathrm{Rep}}_\mathbf{d}(Q) \times \widetilde{\mathrm{Rep}}_\mathbf{d}(Q) \end{aligned}$$

By definition, we see that $\mathcal{Z}_\mathbf{d} \circ \mathcal{Z}_\mathbf{d} = \mathcal{Z}_\mathbf{d}$. Therefore, we define a ring structure on $K^{G_\mathbf{d}}(\mathcal{Z}_\mathbf{d})$:

$$* : K^{G_\mathbf{d}}(\mathcal{Z}_\mathbf{d}) \times K^{G_\mathbf{d}}(\mathcal{Z}_\mathbf{d}) \longrightarrow K^{G_\mathbf{d}}(\mathcal{Z}_\mathbf{d}).$$

Definition 5.3.2 ($K^{G_\mathbf{d}}(\mathcal{Z}_\mathbf{d})$-module sturcture on $K^{G_\mathbf{d}}(\widetilde{\mathrm{Rep}}_\mathbf{d}(Q))$). *Following notations in 5.1.1, We take $G = G_\mathbf{d}$,*

$$\begin{aligned} M_1 &= M_2 = \widetilde{\mathrm{Rep}}_\mathbf{d}(Q) & M_3 &= \mathrm{pt} \\ Z_{12} &= \mathcal{Z}_\mathbf{d} & Z_{23} &= \widetilde{\mathrm{Rep}}_\mathbf{d}(Q) \end{aligned}$$

By definition, we see that $\mathcal{Z}_\mathbf{d} \circ \widetilde{\mathrm{Rep}}_\mathbf{d}(Q) = \widetilde{\mathrm{Rep}}_\mathbf{d}(Q)$. Therefore, we define a $K^{G_\mathbf{d}}(\mathcal{Z}_\mathbf{d})$-module sturcture on $K^{G_\mathbf{d}}\left(\widetilde{\mathrm{Rep}}_\mathbf{d}(Q)\right)$:

$$\divideontimes : K^{G_\mathbf{d}}(\mathcal{Z}_\mathbf{d}) \times K^{G_\mathbf{d}}\left(\widetilde{\mathrm{Rep}}_\mathbf{d}(Q)\right) \longrightarrow K^{G_\mathbf{d}}\left(\widetilde{\mathrm{Rep}}_\mathbf{d}(Q)\right).$$

Remark 5.3.3. Notice that in the construction of the convolution product, pullback, tensor product and proper pushforward are compatible with the forgetful map of groups. Therefore, the following diagrams commute:

$$\begin{tikzcd} K^{G_\mathbf{d}}(\mathcal{Z}_\mathbf{d}) \times K^{G_\mathbf{d}}(\mathcal{Z}_\mathbf{d}) \arrow[r, "*"] \arrow[d] & K^{G_\mathbf{d}}(\mathcal{Z}_\mathbf{d}) \arrow[d] \\ K^{T_\mathbf{d}}(\mathcal{Z}_\mathbf{d}) \times K^{T_\mathbf{d}}(\mathcal{Z}_\mathbf{d}) \arrow[r, "*"] \arrow[d] & K^{T_\mathbf{d}}(\mathcal{Z}_\mathbf{d}) \arrow[d] \\ \mathcal{K}^{T_\mathbf{d}}(\mathcal{Z}_\mathbf{d}) \times \mathcal{K}^{T_\mathbf{d}}(\mathcal{Z}_\mathbf{d}) \arrow[r, "*"] & \mathcal{K}^{T_\mathbf{d}}(\mathcal{Z}_\mathbf{d}) \end{tikzcd}$$

$$\begin{tikzcd} K^{G_\mathbf{d}}(\mathcal{Z}_\mathbf{d}) \times K^{G_\mathbf{d}}\left(\widetilde{\mathrm{Rep}}_\mathbf{d}(Q)\right) \arrow[r, "\divideontimes"] \arrow[d] & K^{G_\mathbf{d}}\left(\widetilde{\mathrm{Rep}}_\mathbf{d}(Q)\right) \arrow[d] \\ K^{T_\mathbf{d}}(\mathcal{Z}_\mathbf{d}) \times K^{T_\mathbf{d}}\left(\widetilde{\mathrm{Rep}}_\mathbf{d}(Q)\right) \arrow[r, "\divideontimes"] \arrow[d] & K^{T_\mathbf{d}}\left(\widetilde{\mathrm{Rep}}_\mathbf{d}(Q)\right) \arrow[d] \\ \mathcal{K}^{T_\mathbf{d}}(\mathcal{Z}_\mathbf{d}) \times \mathcal{K}^{T_\mathbf{d}}\left(\widetilde{\mathrm{Rep}}_\mathbf{d}(Q)\right) \arrow[r, "\divideontimes"] & \mathcal{K}^{T_\mathbf{d}}\left(\widetilde{\mathrm{Rep}}_\mathbf{d}(Q)\right) \end{tikzcd}$$

Definition 5.3.4 ($K^{G_{\mathbf{d}}}(\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q))$-module sturcture on $K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}})$)**.** *We know that*

$$\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q) \cong \mathcal{Z}_{\mathrm{Id}} \subseteq \mathcal{Z}_{\mathbf{d}}, \qquad \mathcal{Z}_{\mathrm{Id}} \circ \mathcal{Z}_{\mathrm{Id}} = \mathcal{Z}_{\mathrm{Id}},$$

so $K^{G_{\mathbf{d}}}\left(\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)\right)$ *can be realized as a* $R(G_{\mathbf{d}})$*-subalgebra of* $K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}})$*, and* $K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}})$ *has the* $K^{G_{\mathbf{d}}}\left(\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)\right)$*-module structure induced by the convolution product:*

$$* : K^{G_{\mathbf{d}}}\left(\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)\right) \times K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}}) \longrightarrow K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}}).$$

5.3.2 Convolution product formula

???(add preimage before psi)

In this subsection, we prove the convolution product formula.

Proposition 5.3.5 (Convolution product formula)**.** *For* ϖ*,* ϖ'*,* ϖ''*,* $\varpi''' \in \mathbb{W}_{|\mathbf{d}|}$*, we have*

$$\psi_{\varpi''',\varpi''} * \psi_{\varpi',\varpi} = \delta_{\varpi'',\varpi'}\widetilde{\Lambda}_{\varpi'}\psi_{\varpi''',\varpi}$$
$$\psi_{\varpi'',\varpi'} \divideontimes \psi_{\varpi} = \delta_{\varpi',\varpi}\widetilde{\Lambda}_{\varpi}\psi_{\varpi''}.$$

Proof. Follow the Definition 5.1.1 and Theorem 5.2.1 if needed.

For clearance, we divide the proof into 4 cases.

<u>**Case 1.**</u> Assume $\varpi'' \neq \varpi'$, need to show $\psi_{\varpi''',\varpi''} * \psi_{\varpi',\varpi} = 0$.

<u>**Case 2.**</u> Assume $\varpi' \neq \varpi'$, need to show $\psi_{\varpi'',\varpi'} \divideontimes \psi_{\varpi} = 0$.

<u>**Case 3.**</u> For ϖ, ϖ', $\varpi'' \in \mathbb{W}_{|\mathbf{d}|}$, need to show that

$$\psi_{\varpi'',\varpi'} * \psi_{\varpi',\varpi} = \widetilde{\Lambda}_{\varpi'}\psi_{\varpi'',\varpi}.$$

<u>**Case 4.**</u> For ϖ, $\varpi' \in \mathbb{W}_{|\mathbf{d}|}$, need to show that

$$\psi_{\varpi',\varpi} * \psi_{\varpi} = \widetilde{\Lambda}_{\varpi}\psi_{\varpi'}.$$

□

5.3.3 Demazure operator

5.3.4 Miscellaneous

Chapter 6

Generalization

6.1 quiver with loops

6.2 $G \times \mathbb{C}^{\times}$-action

Chapter 7

From formula to diagram

7.1 One point quiver

7.2 A_2-quiver

7.3 1-loop quiver

Chapter 8

Atiyah-Segal completion theorem

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