Affine pavings of partial flag varieties

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Process

1. Setting and Statement
2. Case study
3. Auslander–Reiten theory
4. Tackle the type E case

Process

Affine paving

Setting

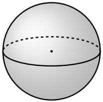
$K = \mathbb{C}$, X: *algebraic variety over* K.

Definition

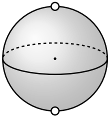
An **affine paving** of X is a filtration

$$\varnothing = X_0 \subset X_1 \subset \cdots \subset X_d = X$$

with X_i closed and $X_{i+1} \setminus X_i \cong \mathbb{A}^k_K$.



$\mathbb{P}^1 = \{\infty\} \sqcup \mathbb{A}^1$



$\mathbb{P}^1 \setminus \{0, \infty\}$ has no affine paving

Quiver and quiver representation

$$\bullet \longrightarrow \bullet \longleftarrow \bullet \longleftarrow \bullet \qquad \begin{tikzcd} \bullet \arrow[loop right] \end{tikzcd} \qquad \bullet \rightrightarrows \bullet$$

Quiver is a graph. It has some vertices & arrows.
In this talk, all the quivers are finite and connected.

Quiver and quiver representation

$$\bullet \longrightarrow \bullet \longleftarrow \bullet \longleftarrow \bullet \qquad \begin{tikzcd} \bullet \arrow[loop right] \end{tikzcd} \qquad \bullet \rightrightarrows \bullet$$

We focus on the Dynkin quiver.

That means, the graph of the Dynkin diagrams in the ADE series.

$$\begin{tikzcd} & \bullet \arrow[d] & & \\ \bullet \arrow[r] & \bullet \arrow[r] & \bullet & \bullet \arrow[l] \end{tikzcd} \qquad \begin{tikzcd} & & \bullet \arrow[d] & & \\ \bullet & \bullet \arrow[l] & \bullet \arrow[l] \arrow[r] & \bullet \arrow[r] & \bullet \end{tikzcd} \qquad \bullet \rightrightarrows \bullet \longrightarrow \bullet \longleftarrow \bullet$$

$$D_5 \qquad\qquad E_6 \qquad\qquad B_4$$

Quiver representation

$$\begin{tikzcd} & \bullet \arrow[d] & & \\ \bullet \arrow[r] & \bullet \arrow[r] & \bullet & \bullet \arrow[l] \end{tikzcd} \qquad \begin{tikzcd} & V_5 \arrow[d, "\delta"] & & \\ V_1 \arrow[r, "\alpha"] & V_2 \arrow[r, "\beta"] & V_3 & V_4 \arrow[l, "\gamma"'] \end{tikzcd}$$

$$Q \qquad\qquad V \in \operatorname{rep}(Q)$$

Quiver representation

$$
\begin{tikzcd}
 & \bullet \arrow[d] & & \\
\bullet \arrow[r] & \bullet \arrow[r] & \bullet & \bullet \arrow[l]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & 0 \arrow[d] & & \\
\mathbb{C}^2 \arrow[r, "\mathrm{Id}"] & \mathbb{C}^2 \arrow[r, "\mathrm{Id}"] & \mathbb{C}^2 & \mathbb{C} \arrow[l, "(1\,0)"']
\end{tikzcd}
$$

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Quiver representation

$$
\begin{tikzcd}
 & \bullet \arrow[d] & & \\
\bullet \arrow[r] & \bullet \arrow[r] & \bullet & \bullet \arrow[l]
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\end{tikzcd}
$$

Q $\qquad\qquad$ $V \in \mathrm{rep}(Q)$

Dimension vector: $\underline{\mathbf{dim}}\, V = {}^{\;\;0}_{2221}$

Partial flag variety

Definition

Fix a quiver Q and $M \in \mathrm{rep}(Q)$,

$$\mathrm{Flag}_d(M) := \{F\colon 0 \subseteq N_1 \subseteq \cdots \subseteq N_d \subseteq M\}$$
$$\mathrm{Flag}_{\underline{\mathbf{f}}}(M) := \left\{F\colon 0 \subseteq N_1 \subseteq \cdots \subseteq N_d \subseteq M \;\middle|\; \underline{\mathbf{dim}}\, N_i = \underline{\mathbf{f}}_i\right\}$$

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Example

$Q = \bullet$, $M = \mathbb{C}^n$,

$$\mathrm{Flag}_1(\mathbb{C}^n) = \{F \colon 0 \subseteq N_1 \subseteq \mathbb{C}^n\} \quad = \bigsqcup_{k=0}^{n} \mathrm{Gr}(n,k)$$
$$\mathrm{Flag}_{(k)}(\mathbb{C}^n) = \mathrm{Gr}(n,k)$$

Statement

Theorem

For a Dynkin quiver Q and $M \in \mathrm{rep}(Q)$,

$$\mathrm{Flag}_d(M) \text{ has an affine paving.}$$

Process

Task 1. $Q = \bullet$, $M = \mathbb{C}^n$

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In this case,

$$\begin{aligned} \mathrm{GL}_n(\mathbb{C}) \circlearrowright \mathbb{C}^n \quad &\rightsquigarrow \quad \mathrm{GL}_n(\mathbb{C}) \circlearrowright \mathrm{Flag}_d(\mathbb{C}^n) \\ &\rightsquigarrow \quad B \circlearrowright \mathrm{Flag}_d(\mathbb{C}^n) \end{aligned}$$

$\mathrm{Flag}_d(\mathbb{C}^n)$ has an affine paving given by Schubert cells (i.e., B-orbits).

Task 1. $Q = \bullet$, $M = \mathbb{C}^n$

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$\mathrm{Flag}_d(\mathbb{C}^n)$ has an affine paving given by Schubert cells (i.e., B-orbits).

Note

When $Q = \bullet \longrightarrow \bullet$, $\mathrm{Flag}_{\mathbf{f}}(M)$ have no natural group actions.

Task 2a. $Q = \bullet \to \bullet$, $M = \left[\mathbb{C}^2 \xrightarrow{0} \mathbb{C}^2\right]$, $d = 1$

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$$
\begin{aligned}
\underline{\mathbf{f}} = (1,0): &\qquad \mathrm{Flag}_{\underline{\mathbf{f}}}(M) = \mathbb{P}^1 \\
\underline{\mathbf{f}} = (0,0): &\qquad \mathrm{Flag}_{\underline{\mathbf{f}}}(M) = \{*\} \\
\underline{\mathbf{f}} = (1,1): &\qquad \mathrm{Flag}_{\underline{\mathbf{f}}}(M) = \mathbb{P}^1 \times \mathbb{P}^1
\end{aligned}
$$

Task 2a. $Q = \bullet \to \bullet$, $M = \left[\mathbb{C}^2 \xrightarrow{0} \mathbb{C}^2\right]$, $d = 1$

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\underline{\mathbf{f}} = (1,1): &\qquad \mathrm{Flag}_{\underline{\mathbf{f}}}(M) = \mathbb{P}^1 \times \mathbb{P}^1
\end{aligned}
$$

In this case,

$$\mathrm{Flag}_{(a,b)}(M) \cong \mathrm{Flag}_{(a)}(\mathbb{C}^2) \times \mathrm{Flag}_{(b)}(\mathbb{C}^2)$$

has an affine paving.

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$$\begin{aligned}
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\underline{\mathbf{f}} = (0,0): &\qquad \mathrm{Flag}_{\underline{\mathbf{f}}}(M) = \{*\} \\
\underline{\mathbf{f}} = (1,1): &\qquad \mathrm{Flag}_{\underline{\mathbf{f}}}(M) = \mathbb{P}^1 \\
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$$
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\underline{\mathbf{f}} = (1,1): &\qquad \mathrm{Flag}_{\underline{\mathbf{f}}}(M) = \mathbb{P}^1 \\
\underline{\mathbf{f}} = (0,1): &\qquad \mathrm{Flag}_{\underline{\mathbf{f}}}(M) = \mathbb{P}^1
\end{aligned}
$$

In this case,

$$\mathrm{Flag}_{(a,b)}(M) \cong \mathrm{Flag}_{\binom{b}{a}}(\mathbb{C}^2)$$

has an affine paving.

Task 2c. $Q = \bullet \to \bullet$, $M = \left[\mathbb{C}^2 \xrightarrow{\begin{pmatrix}1&0\\0&0\end{pmatrix}} \mathbb{C}^2\right]$, $d = 1$

Task 2c. $Q = \bullet \to \bullet$, $M = \left[\mathbb{C}^2 \xrightarrow{\left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right)} \mathbb{C}^2\right]$, $d = 1$

$$\underline{\mathbf{f}} = (1,0): \qquad \mathrm{Flag}_{\underline{\mathbf{f}}}(M) = \{*\}$$

$$\underline{\mathbf{f}} = (0,0): \qquad \mathrm{Flag}_{\underline{\mathbf{f}}}(M) = \{*\}$$

$$\underline{\mathbf{f}} = (1,1): \qquad \mathrm{Flag}_{\underline{\mathbf{f}}}(M) = \mathbb{P}^1 \vee \mathbb{P}^1$$

$$\underline{\mathbf{f}} = (0,1): \qquad \ldots$$

Task 2c. $Q = \bullet \to \bullet$, $M = \left[\mathbb{C}^2 \xrightarrow{\binom{1\,0}{0\,0}} \mathbb{C}^2\right]$, $d = 1$

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\underline{\mathbf{f}} &= (0,1): && \ldots
\end{aligned}
$$

To construct affine pavings systematically, we need to construct an uniform method.

Task 2c. $Q = \bullet \to \bullet$, $M = \left[\mathbb{C}^2 \xrightarrow{\binom{1\,0}{0\,0}} \mathbb{C}^2\right]$, $d = 1$

First try

Let $X = \left[0 \longrightarrow \mathbb{C}\right]$, $S = \left[\mathbb{C}^2 \xrightarrow{(1\,0)} \mathbb{C}\right]$, then $M = X \oplus S$, and the short exact sequence

$$0 \longrightarrow X \xrightarrow{\iota} M \xrightarrow{\pi} S \longrightarrow 0$$

induces

$$\begin{aligned} \Psi : \mathrm{Flag}_d(M) &\longrightarrow \mathrm{Flag}_d(X) \times \mathrm{Flag}_d(S) \\ F &\longmapsto \left(\iota^{-1}(F), \pi(F)\right) \end{aligned}$$

Idea of affine pavings

Find a nice short exact sequence

$$0 \longrightarrow X \xrightarrow{\iota} M \xrightarrow{\pi} S \longrightarrow 0$$

which induces a nice morphism

$$\begin{aligned} \Psi : \mathrm{Flag}_d(M) &\longrightarrow \mathrm{Flag}_d(X) \times \mathrm{Flag}_d(S) \\ F &\longmapsto \big(\iota^{-1}(F), \pi(F)\big) \end{aligned}$$

We construct the affine paving of $\mathrm{Flag}_d(M)$ from the affine paving of $\mathrm{Flag}_d(X)$ and $\mathrm{Flag}_d(S)$. Then, we use mathematical induction.

Example. $Q = \bullet$, $M = \mathbb{C}^2$

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$$0 \longrightarrow \mathbb{C} \xrightarrow{\iota} \mathbb{C}^2 \xrightarrow{\pi} \mathbb{C} \longrightarrow 0$$

$$\Psi_1 : \quad \mathrm{Flag}_1(\mathbb{C}^2) \longrightarrow \quad \mathrm{Flag}_1(\mathbb{C}) \times \mathrm{Flag}_1(\mathbb{C})$$

$$\begin{array}{cccccc} \Psi_{(1)} : \mathrm{Flag}_{(1)}(\mathbb{C}^2) & \longrightarrow & \mathrm{Flag}_{(1)}(\mathbb{C}) \times \mathrm{Flag}_{(0)}(\mathbb{C}) & \bigsqcup & \mathrm{Flag}_{(0)}(\mathbb{C}) \times \mathrm{Flag}_{(1)}(\mathbb{C}) \\ \mathbb{P}^1 & \longrightarrow & \{*\} & \bigsqcup & \{*\} \end{array}$$

Example. $Q = \bullet$, $M = \mathbb{C}^2$

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$$\begin{array}{c} \mathbb{P}^1 = \{\infty\} \sqcup \mathbb{C} \\ \Big\downarrow \Psi_{(1)} \\ \{*\} \sqcup \{*\} \end{array}$$

Example. $Q = \bullet$, $M = \mathbb{C}^8 = \bigoplus_{i=1}^{8} \mathbb{C}v_i$

$$0 \longrightarrow \mathbb{C}^3 \xrightarrow{\iota} \mathbb{C}^8 \xrightarrow{\pi} \mathbb{C}^5 \longrightarrow 0$$

$$\Psi : \mathrm{Flag}_{(3)}(\mathbb{C}^8) \longrightarrow \mathrm{Flag}_{(1)}(\mathbb{C}^3) \times \mathrm{Flag}_{(2)}(\mathbb{C}^5) \bigsqcup \cdots$$

$$\begin{aligned}\Psi^{-1}\Big(\langle v_1\rangle, \langle v_4, v_5\rangle\Big) &= \Big\{\langle v_1, v_4 + av_2 + bv_3, v_5 + cv_2 + dv_3\rangle \,\Big|\, a, b, c, d \in \mathbb{C}\Big\} \\ &\cong \mathbb{C}^4\end{aligned}$$

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Ψ is a Zarisky-locally trivial affine bundle of rank $2 \cdot (3-1) = 4$.

Consider the short exact sequence of representations

$$\eta : 0 \longrightarrow X \xrightarrow{\iota} Y \xrightarrow{\pi} S \longrightarrow 0$$

which induce maps

$$\begin{array}{rccc} \Psi : & \mathrm{Flag}_d(Y) & \longrightarrow & \mathrm{Flag}_d(X) \times \mathrm{Flag}_d(S) \\ & \cup & & \cup \\ \Psi_{\underline{\mathbf{f}},\underline{\mathbf{g}}} : & \mathrm{Flag}(Y)_{\underline{\mathbf{f}},\underline{\mathbf{g}}} & \longrightarrow & \mathrm{Flag}_{\underline{\mathbf{f}}}(X) \times \mathrm{Flag}_{\underline{\mathbf{g}}}(S) \end{array}$$

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Theorem A

When η splits, then Ψ is surjective.
Moreover, if $\mathrm{Ext}^1(S, X) = 0$, then

$$\Psi_{\underline{\mathbf{f}},\underline{\mathbf{g}}} \text{ is a Zarisky-locally trivial affine bundle.}$$

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$\Psi_{\underline{\mathbf{f}},\underline{\mathbf{g}}}$ is a Zarisky-locally trivial affine bundle.

By this theorem,

$\boxed{\mathrm{Flag}_d(Y) \text{ has an affine paving} \Longleftarrow \mathrm{Flag}_d(X),\ \mathrm{Flag}_d(S) \text{ have.}}$

Warming

η splits and $\mathrm{Ext}^1(S, X) = 0$ are necessary for Theorem A.

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Example

Consider the quiver $Q : \bullet \to \bullet \leftarrow \bullet$ and the short exact sequence

$$0 \longrightarrow \left[\mathbb{C}e_1 \to \mathbb{C}^2 \leftarrow \mathbb{C}e_2\right] \longrightarrow \left[\mathbb{C}^2 \overset{\mathrm{Id}}{\to} \mathbb{C}^2 \overset{\mathrm{Id}}{\leftarrow} \mathbb{C}^2\right] \longrightarrow \left[\mathbb{C}e_2 \to 0 \leftarrow \mathbb{C}e_1\right] \longrightarrow 0$$

Warming

η splits and $\mathrm{Ext}^1(S, X) = 0$ are necessary for Theorem A.

Example

Consider the quiver $Q : \bullet \to \bullet \leftarrow \bullet$ *and the short exact sequence*

$$0 \longrightarrow \left[\mathbb{C}e_1 \to \mathbb{C}^2 \leftarrow \mathbb{C}e_2\right] \longrightarrow \left[\mathbb{C}^2 \xrightarrow{\mathrm{Id}} \mathbb{C}^2 \xleftarrow{\mathrm{Id}} \mathbb{C}^2\right] \longrightarrow \left[\mathbb{C}e_2 \to 0 \leftarrow \mathbb{C}e_1\right] \longrightarrow 0$$

we get

$$\mathrm{Im}\,\Psi_{(0,1,0),(1,0,1)} \cong \left(\mathbb{P}^1 \setminus \{0, \infty\}\right) \times \{*\} \cong \mathbb{C}^*,$$

so Ψ *is not surjective.*

Warming

η splits and $\mathrm{Ext}^1(S,X)=0$ are necessary for Theorem A.

Example

Consider the quiver $Q : \bullet \to \bullet \leftarrow \bullet$ *and the short exact sequence*

$$0 \longrightarrow \left[\mathbb{C}e_1 \to \mathbb{C}^2 \leftarrow \mathbb{C}e_2\right] \longrightarrow \left[\mathbb{C}^2 \xrightarrow{\mathrm{Id}} \mathbb{C}^2 \xleftarrow{\mathrm{Id}} \mathbb{C}^2\right] \longrightarrow \left[\mathbb{C}e_2 \to 0 \leftarrow \mathbb{C}e_1\right] \longrightarrow 0$$

we get

$$\mathrm{Im}\,\Psi_{(0,1,0),(1,0,1)} \cong \left(\mathbb{P}^1 \setminus \{0,\infty\}\right) \times \{*\} \cong \mathbb{C}^*,$$

so Ψ *is not surjective.*

In this way, we get a bad stratification

$$\mathrm{Flag}_{(1,1,1)}(Y) \cong \mathbb{P}^1 = \{0\} \sqcup \{\infty\} \sqcup \mathbb{C}^*.$$

Task 3. $Q = \begin{smallmatrix} & \bullet & \\ & \downarrow & \\ \bullet \to & \bullet & \leftarrow \bullet \end{smallmatrix}$, $M = {}_{1\,2\,1}^{\;1} \oplus {}_{1\,1\,1}^{\;1} \oplus {}_{1\,1\,1}^{\;1}$

We use following short exact sequences

$$0 \longrightarrow {}_{111}^{\ 1} \oplus {}_{111}^{\ 1} \longrightarrow M \longrightarrow {}_{121}^{\ 1} \longrightarrow 0$$

$$0 \longrightarrow {}_{111}^{\ 1} \longrightarrow {}_{111}^{\ 1} \oplus {}_{111}^{\ 1} \longrightarrow {}_{111}^{\ 1} \longrightarrow 0$$

to reduced the problem to indecomposable representations.

Task 3. $Q = \begin{tikzcd}[row sep=small, column sep=small] & \bullet \arrow[d] & \\ \bullet \arrow[r] & \bullet & \bullet \arrow[l] \end{tikzcd}$, $M = \begin{smallmatrix} & 1 & \\ 1 & 2 & 1 \end{smallmatrix} \oplus \begin{smallmatrix} & 1 & \\ 1 & 1 & 1 \end{smallmatrix} \oplus \begin{smallmatrix} & 1 & \\ 1 & 1 & 1 \end{smallmatrix}$

We use following short exact sequences

$$0 \longrightarrow \begin{smallmatrix} & 1 & \\ 1 & 1 & 1 \end{smallmatrix} \oplus \begin{smallmatrix} & 1 & \\ 1 & 1 & 1 \end{smallmatrix} \longrightarrow M \longrightarrow \begin{smallmatrix} & 1 & \\ 1 & 2 & 1 \end{smallmatrix} \longrightarrow 0$$

$$0 \longrightarrow \begin{smallmatrix} & 1 & \\ 1 & 1 & 1 \end{smallmatrix} \longrightarrow \begin{smallmatrix} & 1 & \\ 1 & 1 & 1 \end{smallmatrix} \oplus \begin{smallmatrix} & 1 & \\ 1 & 1 & 1 \end{smallmatrix} \longrightarrow \begin{smallmatrix} & 1 & \\ 1 & 1 & 1 \end{smallmatrix} \longrightarrow 0$$

to reduced the problem to indecomposable representations.

Notice that we use the result

$$\operatorname{Ext}^1\left(\begin{smallmatrix} & 1 & \\ 1 & 2 & 1 \end{smallmatrix}, \begin{smallmatrix} & 1 & \\ 1 & 1 & 1 \end{smallmatrix}\right) = 0, \qquad \operatorname{Ext}^1\left(\begin{smallmatrix} & 1 & \\ 1 & 1 & 1 \end{smallmatrix}, \begin{smallmatrix} & 1 & \\ 1 & 1 & 1 \end{smallmatrix}\right) = 0.$$

Task 3. $Q = \begin{smallmatrix} & \bullet & \\ & \downarrow & \\ \bullet \rightarrow & \bullet & \leftarrow \bullet \end{smallmatrix}$, $M = {}^{\;1}_{121} \oplus {}^{\;1}_{111} \oplus {}^{\;1}_{111}$

We use following short exact sequences

$$0 \longrightarrow {}^{\;1}_{111} \oplus {}^{\;1}_{111} \longrightarrow M \longrightarrow {}^{\;1}_{121} \longrightarrow 0$$

$$0 \longrightarrow {}^{\;1}_{111} \longrightarrow {}^{\;1}_{111} \oplus {}^{\;1}_{111} \longrightarrow {}^{\;1}_{111} \longrightarrow 0$$

to reduced the problem to indecomposable representations.

Notice that we use the result

$$\mathrm{Ext}^1\left({}^{\;1}_{121}, {}^{\;1}_{111}\right) = 0, \qquad \mathrm{Ext}^1\left({}^{\;1}_{111}, {}^{\;1}_{111}\right) = 0.$$

We can't put ${}^{\;1}_{121}$ on the left, since

$$\mathrm{Ext}^1\left({}^{\;1}_{111}, {}^{\;1}_{121}\right) \cong \mathbb{C} \neq 0.$$

Task 3. $Q = \begin{smallmatrix} & \bullet & \\ & \downarrow & \\ \bullet \rightarrow & \bullet & \leftarrow \bullet \end{smallmatrix}$, $M = {}_{1\,2\,1}^{\;1} \oplus {}_{1\,1\,1}^{\;1} \oplus {}_{1\,1\,1}^{\;1}$

We use following short exact sequences

$$0 \longrightarrow {}_{111}^{\;1} \oplus {}_{111}^{\;1} \longrightarrow M \longrightarrow {}_{121}^{\;1} \longrightarrow 0$$

$$0 \longrightarrow {}_{111}^{\;1} \longrightarrow {}_{111}^{\;1} \oplus {}_{111}^{\;1} \longrightarrow {}_{111}^{\;1} \longrightarrow 0$$

to reduced the problem to indecomposable representations.

$\mathrm{Flag}_d\left({}_{111}^{\;1}\right)$ has an affine paving: obvious.
$\mathrm{Flag}_d\left({}_{121}^{\;1}\right)$ has an affine paving: it is $\mathbb{P}^1$, $\{*\}$ or empty.

Task 3. $Q = \begin{array}{ccccc} & & \bullet & & \\ & & \downarrow & & \\ \bullet & \rightarrow & \bullet & \leftarrow & \bullet \end{array}$, $M = {}^{\;1}_{121} \oplus {}^{\;1}_{111} \oplus {}^{\;1}_{111}$

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to reduced the problem to indecomposable representations.

$\mathrm{Flag}_d\,({}^{\;1}_{111})$ has an affine paving: obvious.
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Need: more informations of indecomposable representations!

Process

1. Setting and Statement
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Auslander–Reiten quiver: D_4

$$
\begin{tikzcd}
 & 4 \arrow[d] & \\
2 \arrow[r] & 1 & 3 \arrow[l]
\end{tikzcd}
$$

$$
\begin{tikzcd}
2 \arrow[dr] & \\
4 \arrow[r] & 1 \\
3 \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \substack{0\\110} \arrow[dr] & & \substack{1\\011} \arrow[dr] & & \substack{0\\100} \\
\substack{0\\010} \arrow[ur] \arrow[r] \arrow[dr] & \substack{1\\010} \arrow[r] & \substack{1\\121} \arrow[ur] \arrow[r] \arrow[dr] & \substack{0\\111} \arrow[r] & \substack{1\\111} \arrow[ur] \arrow[r] \arrow[dr] & \substack{1\\000} \\
 & \substack{0\\011} \arrow[ur] & & \substack{1\\110} \arrow[ur] & & \substack{0\\001}
\end{tikzcd}
$$

Vertices	$\Longleftrightarrow$	Indecomposable representations
Arrows	$\Longleftrightarrow$	Irreducible morphisms
Paths	$\Longleftrightarrow$	Morphisms
Shift cards	$\Longleftrightarrow$	Switch arrows in Q

Auslander–Reiten quiver: D_4

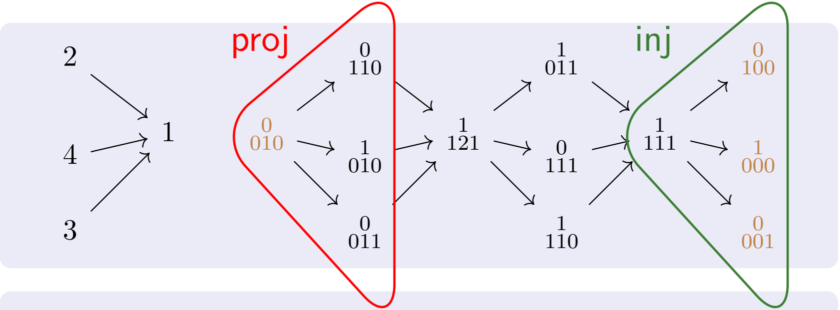
$$\begin{tikzcd} & 4 \arrow[d] & \\ 2 \arrow[r] & 1 & 3 \arrow[l] \end{tikzcd}$$

$$\begin{tikzcd} 2 \arrow[dr] & \\ 4 \arrow[r] & 1 \\ 3 \arrow[ur] & \end{tikzcd}
\qquad
\begin{tikzcd}
 & \begin{smallmatrix}0\\110\end{smallmatrix} \arrow[dr] & & \begin{smallmatrix}1\\011\end{smallmatrix} \arrow[dr] & & \begin{smallmatrix}0\\100\end{smallmatrix} \\
\begin{smallmatrix}0\\010\end{smallmatrix} \arrow[ur] \arrow[r] \arrow[dr] & \begin{smallmatrix}1\\010\end{smallmatrix} \arrow[r] & \begin{smallmatrix}1\\121\end{smallmatrix} \arrow[ur] \arrow[r] \arrow[dr] & \begin{smallmatrix}0\\111\end{smallmatrix} \arrow[r] & \begin{smallmatrix}1\\111\end{smallmatrix} \arrow[ur] \arrow[r] \arrow[dr] & \begin{smallmatrix}1\\000\end{smallmatrix} \\
 & \begin{smallmatrix}0\\011\end{smallmatrix} \arrow[ur] & & \begin{smallmatrix}1\\110\end{smallmatrix} \arrow[ur] & & \begin{smallmatrix}0\\001\end{smallmatrix}
\end{tikzcd}$$

Vertices $\iff$ Indecomposable representations

Auslander–Reiten quiver: D_4

$$\begin{array}{ccccc} & & 4 & & \\ & & \downarrow & & \\ 2 & \rightarrow & 1 & \leftarrow & 3 \end{array}$$



Vertices $\Longleftrightarrow$ Indecomposable representations

irreducible rep, projective rep, injective rep

Auslander–Reiten quiver: D_4

$$\begin{tikzcd} & 4 \arrow[d] & \\ 2 \arrow[r] & 1 & 3 \arrow[l] \end{tikzcd}$$

$$\begin{tikzcd} 2 \arrow[dr] & \\ 4 \arrow[r] & 1 \\ 3 \arrow[ur] & \end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
 & \substack{0\\110} \arrow[dr] & & \substack{1\\011} \arrow[dr] & & \substack{0\\100} \\
\substack{0\\010} \arrow[ur] \arrow[r] \arrow[dr] & \substack{1\\010} \arrow[r] & \substack{1\\121} \arrow[ur] \arrow[r] \arrow[dr] & \substack{0\\111} \arrow[r] & \substack{1\\111} \arrow[ur] \arrow[r] \arrow[dr] & \substack{1\\000} \\
 & \substack{0\\011} \arrow[ur] & & \substack{1\\110} \arrow[ur] & & \substack{0\\001}
\end{tikzcd}$$

Arrows $\Longleftrightarrow$ Irreducible morphisms

Instead, I will show you how to construct AR-quiver.

Auslander–Reiten quiver: D_4

$$\begin{tikzcd} & 4 \arrow[d] & \\ 2 \arrow[r] & 1 & 3 \arrow[l] \end{tikzcd}$$

$$\begin{tikzcd} 2 \arrow[dr] & & & & P(2) \\ 4 \arrow[r] & 1 & & P(1) \arrow[ur] \arrow[r] \arrow[dr] & P(4) \\ 3 \arrow[ur] & & & & P(3) \end{tikzcd}$$

<u>Construction</u>:

Auslander–Reiten quiver: D_4

$$\begin{tikzcd} & 4 \arrow[d] & \\ 2 \arrow[r] & 1 & 3 \arrow[l] \end{tikzcd}$$

$$\begin{tikzcd}[column sep=small] 2 \arrow[ddr] & & & & P(2) \\ & & & & \\ & 1 & & P(1) \arrow[uur] \arrow[r] \arrow[ddr] & P(4) \\ 4 \arrow[ur] & & & & \\ 3 \arrow[uur] & & & & P(3) \end{tikzcd}$$

<u>Construction</u>:

$$0 \longrightarrow P(1) \longrightarrow P(2) \oplus P(4) \oplus P(3) \longrightarrow \substack{1\\121} \longrightarrow 0$$

Auslander–Reiten quiver: D_4

$$\begin{tikzcd}
 & 4 \arrow[d] & \\
2 \arrow[r] & 1 & 3 \arrow[l]
\end{tikzcd}$$

$$\begin{tikzcd}
2 \arrow[dr] & \\
4 \arrow[r] & 1 \\
3 \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & P(2) \arrow[dr] & \\
P(1) \arrow[ur] \arrow[r] \arrow[dr] & P(4) \arrow[r] & \begin{smallmatrix}1\\121\end{smallmatrix} \\
 & P(3) \arrow[ur] &
\end{tikzcd}$$

<u>Construction</u>:

$$0 \longrightarrow P(1) \longrightarrow P(2) \oplus P(4) \oplus P(3) \longrightarrow \begin{smallmatrix}1\\121\end{smallmatrix} \longrightarrow 0$$

Auslander–Reiten quiver: D_4

$$\begin{tikzcd} & 4 \arrow[d] & \\ 2 \arrow[r] & 1 & 3 \arrow[l] \end{tikzcd}$$

$$\begin{tikzcd} 2 \arrow[dr] & \\ 4 \arrow[r] & 1 \\ 3 \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} & P(2) \arrow[dr] & \\ P(1) \arrow[ur] \arrow[r] \arrow[dr] & P(4) \arrow[r] & \begin{smallmatrix} & 1 & \\ 1 & 2 & 1 \end{smallmatrix} \\ & P(3) \arrow[ur] & \end{tikzcd}$$

<u>Construction</u>:

$$0 \longrightarrow P(2) \longrightarrow \begin{smallmatrix} & 1 & \\ 1 & 2 & 1 \end{smallmatrix} \longrightarrow \begin{smallmatrix} & 1 & \\ 0 & 1 & 1 \end{smallmatrix} \longrightarrow 0$$

Auslander–Reiten quiver: D_4

$$\begin{tikzcd} & 4 \arrow[d] & \\ 2 \arrow[r] & 1 & 3 \arrow[l] \end{tikzcd}$$

$$\begin{tikzcd} 2 \arrow[dr] & \\ 4 \arrow[r] & 1 \\ 3 \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} & P(2) \arrow[dr] & & \substack{1\\011} \\ P(1) \arrow[ur] \arrow[r] \arrow[dr] & P(4) \arrow[r] & \substack{1\\121} \arrow[ur] & \\ & P(3) \arrow[ur] & & \end{tikzcd}$$

Construction:

$$0 \longrightarrow P(2) \longrightarrow \substack{1\\121} \longrightarrow \substack{1\\011} \longrightarrow 0$$

Auslander–Reiten quiver: D_4

$$
\begin{tikzcd}
 & 4 \arrow[d] & \\
2 \arrow[r] & 1 & 3 \arrow[l]
\end{tikzcd}
$$

$$
\begin{tikzcd}
2 \arrow[dr] & \\
4 \arrow[r] & 1 \\
3 \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & P(2) \arrow[dr] & & \begin{smallmatrix}1\\011\end{smallmatrix} \arrow[dr] & & \begin{smallmatrix}0\\100\end{smallmatrix} \\
P(1) \arrow[ur] \arrow[r] \arrow[dr] & P(4) \arrow[r] & \begin{smallmatrix}1\\121\end{smallmatrix} \arrow[ur] \arrow[r] \arrow[dr] & \begin{smallmatrix}0\\111\end{smallmatrix} \arrow[r] & \begin{smallmatrix}1\\111\end{smallmatrix} \arrow[ur] \arrow[r] \arrow[dr] & \begin{smallmatrix}1\\000\end{smallmatrix} \\
 & P(3) \arrow[ur] & & \begin{smallmatrix}1\\110\end{smallmatrix} \arrow[ur] & & \begin{smallmatrix}0\\001\end{smallmatrix}
\end{tikzcd}
$$

Construction:

Auslander–Reiten quiver: D_4

$$
\begin{tikzcd}
 & 4 \arrow[d] & \\
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\end{tikzcd}
$$

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\begin{tikzcd}
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\end{tikzcd}
\qquad
\begin{tikzcd}
 & P(2) \arrow[dr] & & \begin{smallmatrix}1\\011\end{smallmatrix} \arrow[dr] & & \begin{smallmatrix}0\\100\end{smallmatrix} \\
P(1) \arrow[ur] \arrow[r] \arrow[dr] & P(4) \arrow[r] & \begin{smallmatrix}1\\121\end{smallmatrix} \arrow[ur] \arrow[r] \arrow[dr] & \begin{smallmatrix}0\\111\end{smallmatrix} \arrow[r] & \begin{smallmatrix}1\\111\end{smallmatrix} \arrow[ur] \arrow[r] \arrow[dr] & \begin{smallmatrix}1\\000\end{smallmatrix} \\
 & P(3) \arrow[ur] & & \begin{smallmatrix}1\\110\end{smallmatrix} \arrow[ur] & & \begin{smallmatrix}0\\001\end{smallmatrix}
\end{tikzcd}
$$

Construction:

AR-quiver, AR-sequence, AR-translation

Auslander–Reiten quiver: D_4

$$\begin{tikzcd} & 4 \arrow[d] & \\ 2 \arrow[r] & 1 & 3 \arrow[l] \end{tikzcd}$$

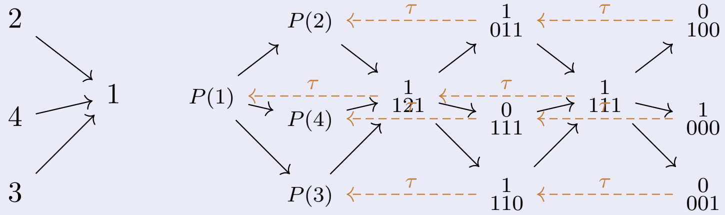
$$\begin{tikzcd} 2 \arrow[dr] & \\ 4 \arrow[r] & 1 \\ 3 \arrow[ur] & \end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & P(2) \arrow[dr] & & \substack{1\\011} \arrow[dr] & & \substack{0\\100} \\
P(1) \arrow[ur] \arrow[r] \arrow[dr] & P(4) \arrow[r] & \substack{1\\121} \arrow[ur] \arrow[r] \arrow[dr] & \substack{0\\111} \arrow[r] & \substack{1\\111} \arrow[ur] \arrow[r] \arrow[dr] & \substack{1\\000} \\
 & P(3) \arrow[ur] & & \substack{1\\110} \arrow[ur] & & \substack{0\\001}
\end{tikzcd}$$

Construction:

AR-quiver, AR-sequence, AR-translation

Auslander–Reiten quiver: D_4

$$\begin{tikzcd} & 4 \arrow[d] & \\ 2 \arrow[r] & 1 & 3 \arrow[l] \end{tikzcd}$$



Construction:

AR-quiver, AR-sequence, AR-translation

Auslander–Reiten quiver: D_4

$$\begin{tikzcd} & 4 \arrow[d] & \\ 2 \arrow[r] & 1 & 3 \arrow[l] \end{tikzcd}$$

$$\begin{tikzcd} 2 \arrow[dr] & \\ 4 \arrow[r] & 1 \\ 3 \arrow[ur] & \end{tikzcd}
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 & \substack{0\\110} \arrow[dr] & & \substack{1\\011} \arrow[dr] & & \substack{0\\100} \\
\substack{0\\010} \arrow[ur] \arrow[r] \arrow[dr] & \substack{1\\010} \arrow[r] & \substack{1\\121} \arrow[ur] \arrow[r] \arrow[dr] & \substack{0\\111} \arrow[r] & \substack{1\\111} \arrow[ur] \arrow[r] \arrow[dr] & \substack{1\\000} \\
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\end{tikzcd}$$

Paths $\Longleftrightarrow$ Morphisms

Auslander–Reiten quiver: D_4

$$\begin{tikzcd} & 4 \arrow[d] & \\ 2 \arrow[r] & 1 & 3 \arrow[l] \end{tikzcd}$$

$$\begin{tikzcd} 2 \arrow[dr] & \\ 4 \arrow[r] & 1 \\ 3 \arrow[ur] & \end{tikzcd}$$

$$\begin{tikzcd}
 & \substack{0\\110} \arrow[dr] & & \substack{1\\011} \arrow[dr] & & \substack{0\\100} \\
\substack{0\\010} \arrow[ur] \arrow[r] \arrow[dr] & \substack{1\\010} \arrow[r] & \substack{1\\121} \arrow[ur] \arrow[r] \arrow[dr] & \substack{0\\111} \arrow[r] & \substack{1\\111} \arrow[ur] \arrow[r] \arrow[dr] & \substack{1\\000} \\
 & \substack{0\\011} \arrow[ur] & & \substack{1\\110} \arrow[ur] & & \substack{0\\001}
\end{tikzcd}$$

In the Dynkin quiver case,

$$\mathrm{Hom}(T,T') \cong \langle \text{ paths from } T \text{ to } T' \ \rangle /_{\text{AR-seq}}$$

For example,

$$\mathrm{Hom}(\substack{1\\010}, \substack{1\\011}) \cong \mathbb{C}, \quad \mathrm{Hom}(\substack{1\\010}, \substack{0\\111}) \cong 0, \quad \mathrm{Hom}(\substack{1\\010}, \substack{1\\111}) \cong \mathbb{C}$$

Auslander–Reiten quiver: D_4

$$\begin{tikzcd} & 4 \arrow[d] & \\ 2 \arrow[r] & 1 & 3 \arrow[l] \end{tikzcd}$$

$$\begin{tikzcd}
2 \arrow[dr] & & & \substack{0\\110} \arrow[dr] & & \substack{1\\011} \arrow[dr] & & \substack{0\\100} \\
4 \arrow[r] & 1 & \substack{0\\010} \arrow[ur] \arrow[r] \arrow[dr] & \substack{1\\010} \arrow[r] & \substack{1\\121} \arrow[ur] \arrow[r] \arrow[dr] & \substack{0\\111} \arrow[r] & \substack{1\\111} \arrow[ur] \arrow[r] \arrow[dr] & \substack{1\\000} \\
3 \arrow[ur] & & & \substack{0\\011} \arrow[ur] & & \substack{1\\110} \arrow[ur] & & \substack{0\\001}
\end{tikzcd}$$

$$\mathrm{Ext}^1(T, T') \cong \overline{\mathrm{Hom}}(T', \tau T)^\vee$$

For example, $$\mathrm{Ext}^1\left(\substack{1\\121}, \substack{0\\111}\right) \cong \mathrm{Hom}\left(\substack{0\\111}, \substack{0\\010}\right)^\vee \cong 0$$

Auslander–Reiten quiver: D_4

$$\begin{tikzcd}
 & 4 \arrow[d] & \\
2 \arrow[r] & 1 & 3 \arrow[l]
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
2 \arrow[dr] & & & \begin{smallmatrix}0\\110\end{smallmatrix} \arrow[dr] & & \begin{smallmatrix}1\\011\end{smallmatrix} \arrow[dr] & & \begin{smallmatrix}0\\100\end{smallmatrix} \\
4 \arrow[r] & 1 & \begin{smallmatrix}0\\010\end{smallmatrix} \arrow[ur] \arrow[r] \arrow[dr] & \begin{smallmatrix}1\\010\end{smallmatrix} \arrow[r] & \begin{smallmatrix}1\\121\end{smallmatrix} \arrow[ur] \arrow[r] \arrow[dr] & \begin{smallmatrix}0\\111\end{smallmatrix} \arrow[r] & \begin{smallmatrix}1\\111\end{smallmatrix} \arrow[ur] \arrow[r] \arrow[dr] & \begin{smallmatrix}1\\000\end{smallmatrix} \\
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\end{tikzcd}$$

$$\mathrm{Ext}^1(T, T') \cong \overline{\mathrm{Hom}}(T', \tau T)^\vee$$

For example,

$$\mathrm{Ext}^1\left(\begin{smallmatrix}1\\121\end{smallmatrix}, \begin{smallmatrix}0\\111\end{smallmatrix}\right) \cong \mathrm{Hom}\left(\begin{smallmatrix}0\\111\end{smallmatrix}, \begin{smallmatrix}0\\010\end{smallmatrix}\right)^\vee \cong 0$$

First application

Corollary

For a Dynkin quiver, we can give an total order to the set of all indecomposable representations, such that

$$i \leqslant j \quad \Longrightarrow \quad \mathrm{Ext}^1(M_i, M_j) = 0.$$

First application

Corollary

For a Dynkin quiver, we can give an total order to the set of all indecomposable representations, such that

$$i \leqslant j \quad \Longrightarrow \quad \mathrm{Ext}^1(M_i, M_j) = 0.$$

By Theorem A, the problem reduced to

For a Dynkin quiver Q and $M \in \mathrm{ind}(Q)$,

$$\mathrm{Flag}_d(M) \text{ has an affine paving.}$$

Auslander–Reiten quiver: D_4

$$\begin{tikzcd} & 4 \arrow[d] & \\ 2 \arrow[r] & 1 & 3 \arrow[l] \end{tikzcd}$$

$$\begin{tikzcd} 2 \arrow[dr] & \\ 4 \arrow[r] & 1 \\ 3 \arrow[ur] & \end{tikzcd}
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\begin{tikzcd}[column sep=small]
 & \begin{smallmatrix}0\\110\end{smallmatrix} \arrow[dr] & & \begin{smallmatrix}1\\011\end{smallmatrix} \arrow[dr] & & \begin{smallmatrix}0\\100\end{smallmatrix} \\
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\end{tikzcd}$$

Shift cards $\iff$ Switch arrows in Q

Auslander–Reiten quiver: D_4

$$
\begin{tikzcd}
 & 4 \arrow[d] & \\
2 \arrow[r] & 1 & 3 \arrow[l]
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & \substack{0\\110} \arrow[dr] & & \substack{1\\011} \arrow[dr] & & \substack{0\\100} \\
\substack{0\\010} \arrow[r] \arrow[ur] \arrow[dr] & \substack{1\\010} \arrow[r] & \substack{1\\121} \arrow[r] \arrow[ur] \arrow[dr] & \substack{0\\111} \arrow[r] & \substack{1\\111} \arrow[r] \arrow[ur] \arrow[dr] & \substack{1\\000} \\
 & \substack{0\\011} \arrow[ur] & & \substack{1\\110} \arrow[ur] & & \substack{0\\001}
\end{tikzcd}
$$

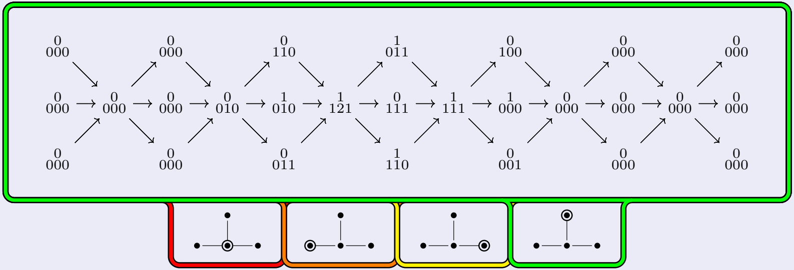
Auslander–Reiten quiver: D_4

$$
\begin{tikzcd}
 & 4 \arrow[d] & \\
2 \arrow[r] & 1 & 3 \arrow[l]
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=small]
\substack{0\\000} \arrow[dr] & & \substack{0\\000} \arrow[dr] & & \substack{0\\110} \arrow[dr] & & \substack{1\\011} \arrow[dr] & & \substack{0\\100} \arrow[dr] & & \substack{0\\000} \arrow[dr] & & \substack{0\\000} \\
\substack{0\\000} \arrow[r] & \substack{0\\000} \arrow[r] \arrow[ur] \arrow[dr] & \substack{0\\000} \arrow[r] & \substack{0\\010} \arrow[r] \arrow[ur] \arrow[dr] & \substack{1\\010} \arrow[r] & \substack{1\\121} \arrow[r] \arrow[ur] \arrow[dr] & \substack{0\\111} \arrow[r] & \substack{1\\111} \arrow[r] \arrow[ur] \arrow[dr] & \substack{1\\000} \arrow[r] & \substack{0\\000} \arrow[r] \arrow[ur] \arrow[dr] & \substack{0\\000} \arrow[r] & \substack{0\\000} \arrow[r] \arrow[ur] \arrow[dr] & \substack{0\\000} \\
\substack{0\\000} \arrow[ur] & & \substack{0\\000} \arrow[ur] & & \substack{0\\011} \arrow[ur] & & \substack{1\\110} \arrow[ur] & & \substack{0\\001} \arrow[ur] & & \substack{0\\000} \arrow[ur] & & \substack{0\\000}
\end{tikzcd}
$$

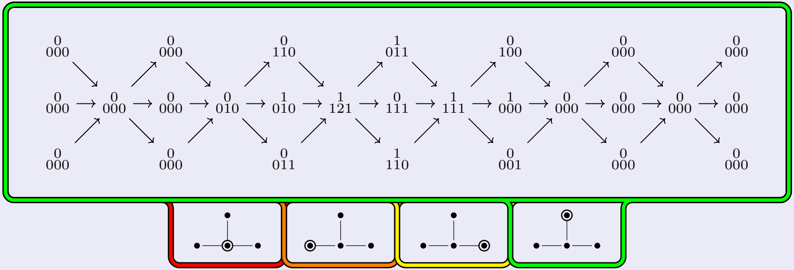
Auslander–Reiten quiver: D_4

$$\begin{tikzcd} & 4 \arrow[d] & \\ 2 \arrow[r] & 1 & 3 \arrow[l] \end{tikzcd}$$



Auslander–Reiten quiver: D_4

$$\begin{tikzcd} & 4 \arrow[d] & \\ 2 \arrow[r] & 1 & 3 \arrow[l] \end{tikzcd}$$



Interactive webversion

Indecomposable representations of low order are easy!

Lemma

Suppose Q is a tree. For $M \in \operatorname{ind}(Q)$, $\operatorname{ord}(M) \leqslant 2$,

$$\operatorname{Flag}_{\underline{\mathbf{f}}}(M) \;\cong\; \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \quad \text{or} \quad \varnothing.$$

Example

$$Q = \begin{tikzcd} & & \bullet \arrow[d] & \\ \bullet \arrow[r] & \bullet & \bullet \arrow[l] \arrow[r] & \bullet \end{tikzcd} \qquad M = \begin{tikzcd} & & \mathbb{C} \arrow[d, hook] & \\ \mathbb{C} \arrow[r, hook] & \mathbb{C}^2 & \mathbb{C}^2 \arrow[l, "\sim"'] \arrow[r, two heads] & \mathbb{C} \end{tikzcd} \qquad \underline{\mathbf{f}} = \begin{pmatrix} & & 0 & \\ 1 & 2 & 1 & 1 \\ & & 0 & \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned} \operatorname{Flag}_{\underline{\mathbf{f}}}(M) \hookrightarrow\; & \operatorname{Flag}_{\binom{1}{1}}(\mathbb{C}) \times \operatorname{Flag}_{\binom{2}{1}}(\mathbb{C}^2) \times \operatorname{Flag}_{\binom{1}{0}}(\mathbb{C}^2) \\ & \times \operatorname{Flag}_{\binom{1}{1}}(\mathbb{C}) \times \operatorname{Flag}_{\binom{0}{0}}(\mathbb{C}) \\ & \cong \mathbb{P}^1 \times \mathbb{P}^1 \end{aligned}$$

Continue

<table>
<tr><th></th><th colspan="2">$\mathbb{C} \hookrightarrow \mathbb{C}^2$</th><th colspan="2">$\mathbb{C}^2 \twoheadrightarrow \mathbb{C}$</th><th colspan="2">$\mathbb{C}^2 \to \mathbb{C}^2$</th></tr>
<tr><td rowspan="2">No restriction</td><td>–</td><td>2</td><td>0</td><td>–</td><td>–</td><td>1</td></tr>
<tr><td>0</td><td>–</td><td>1</td><td>2</td><td>0</td><td>0</td></tr>
<tr><td>Reduce</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>0</td></tr>
<tr><td rowspan="3">Impossible</td><td>2</td><td>1</td><td rowspan="3">1</td><td rowspan="3">0</td><td rowspan="3">2</td><td rowspan="3">0</td></tr>
<tr><td>2</td><td>0</td></tr>
<tr><td>1</td><td>0</td></tr>
</table>

Corollary

The main theorem is true for quivers of type A, D.

Process

1. Setting and Statement
2. Case study
3. Auslander–Reiten theory
4. Tackle the type E case

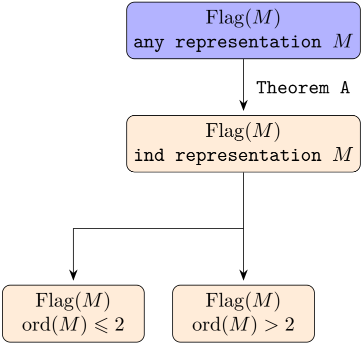
Process

$\mathrm{Flag}(M)$
`any representation` M

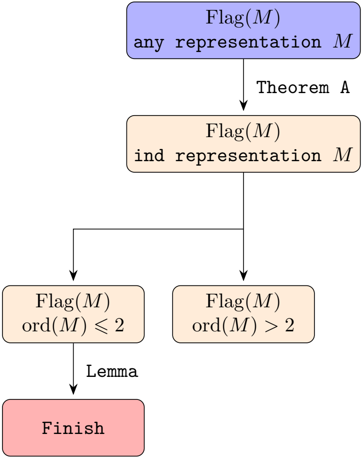
Process

$$\begin{tikzcd} \boxed{\begin{array}{c}\mathrm{Flag}(M)\\ \texttt{any representation } M\end{array}} \arrow[d, "\texttt{Theorem A}"] \\ \boxed{\begin{array}{c}\mathrm{Flag}(M)\\ \texttt{ind representation } M\end{array}} \end{tikzcd}$$

Process


Flag(M)
any representation M
Theorem A
Flag(M)
ind representation M
Flag(M)
ord(M) $\leqslant 2$
Flag(M)
ord(M) > 2

Process


$\mathrm{Flag}(M)$
any representation M
Theorem A
$\mathrm{Flag}(M)$
ind representation M
$\mathrm{Flag}(M)$
$\mathrm{ord}(M) \leqslant 2$
$\mathrm{Flag}(M)$
$\mathrm{ord}(M) > 2$
Lemma
Finish

What is remaining?

$$E_6 : \begin{array}{ccccccccccc}
& & 1 & & 1 & & 1 & & 1 & & \\
& 1 & & 2 & & 2 & & 2 & & 1 & \\
1 & 1 & 2 & 1 & 3 & 2 & 3 & 1 & 2 & 1 & 1 \\
& 1 & & 2 & & 2 & & 2 & & 1 & \\
& & 1 & & 1 & & 1 & & 1 & &
\end{array}$$

$$E_7 : \begin{array}{ccccccccccccccccc}
& & 1 & & 1 & & 1 & & 2 & & 1 & & 1 & & 1 & & \\
& 1 & & 2 & & 2 & & 3 & & 3 & & 2 & & 2 & & 1 & \\
1 & 1 & 2 & 1 & 3 & 2 & 4 & 2 & 4 & 2 & 4 & 2 & 3 & 1 & 2 & 1 & 1 \\
& 1 & & 2 & & 3 & & 3 & & 3 & & 3 & & 2 & & 1 & \\
& & 1 & & 2 & & 2 & & 2 & & 2 & & 2 & & 1 & & \\
& & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & &
\end{array}$$

$$E_8 : \begin{array}{ccccccccccccccccccccccccccccc}
& & 1 & & 1 & & 1 & & 2 & & 2 & & 2 & & 2 & & 2 & & 2 & & 2 & & 1 & & 1 & & 1 & & \\
& 1 & & 2 & & 2 & & 3 & & 4 & & 4 & & 4 & & 4 & & 4 & & 4 & & 3 & & 2 & & 2 & & 1 & \\
1 & 1 & 2 & 1 & 3 & 2 & 4 & 2 & 5 & 3 & 6 & 3 & 6 & 3 & 6 & 3 & 6 & 3 & 6 & 3 & 5 & 2 & 4 & 2 & 3 & 1 & 2 & 1 & 1 \\
& 1 & & 2 & & 3 & & 4 & & 4 & & 5 & & 5 & & 5 & & 5 & & 4 & & 4 & & 3 & & 2 & & 1 & \\
& & 1 & & 2 & & 3 & & 3 & & 3 & & 4 & & 4 & & 4 & & 3 & & 3 & & 3 & & 2 & & 1 & & \\
& & & 1 & & 2 & & 2 & & 2 & & 2 & & 3 & & 3 & & 2 & & 2 & & 2 & & 2 & & 1 & & & \\
& & & & 1 & & 1 & & 1 & & 1 & & 1 & & 2 & & 1 & & 1 & & 1 & & 1 & & 1 & & & &
\end{array}$$

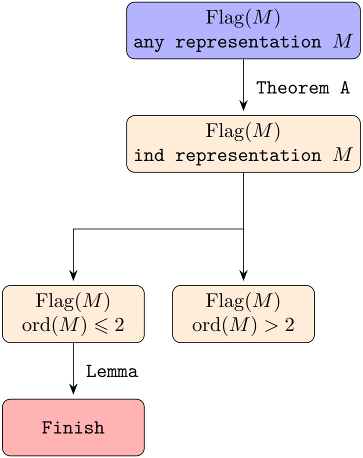
What is remaining?

$$E_6: \quad \begin{array}{ccccccccccc}
 & & 1 & & 1 & & 1 & & 1 & & \\
 & 1 & & 2 & & 2 & & 2 & & 1 & \\
1 & 1 & 2 & 1 & {\color{red}3} & 2 & {\color{red}3} & 1 & 2 & 1 & 1 \\
 & 1 & & 2 & & 2 & & 2 & & 1 & \\
 & & 1 & & 1 & & 1 & & 1 & &
\end{array}$$

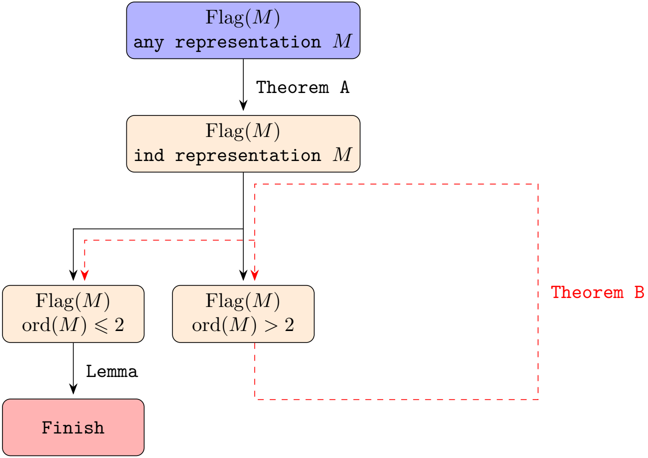
$$E_7: \quad \begin{array}{ccccccccccccccccc}
 & & 1 & & 1 & & 1 & & 2 & & 1 & & 1 & & 1 & & \\
 & 1 & & 2 & & 2 & & {\color{red}3} & & {\color{red}3} & & 2 & & 2 & & 1 & \\
1 & 1 & 2 & 1 & {\color{red}3} & 2 & {\color{red}4} & 2 & {\color{red}4} & 2 & {\color{red}4} & 2 & {\color{red}3} & 1 & 2 & 1 & 1 \\
 & 1 & & 2 & & {\color{red}3} & & {\color{red}3} & & {\color{red}3} & & {\color{red}3} & & 2 & & 1 & \\
 & & 1 & & 2 & & 2 & & 2 & & 2 & & 2 & & 1 & & \\
 & & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & &
\end{array}$$

$$E_8: \quad \begin{array}{ccccccccccccccccccccccccccccc}
 & & 1 & & 1 & & 1 & & 2 & & 2 & & 2 & & 2 & & 2 & & 2 & & 2 & & 1 & & 1 & & 1 & & \\
 & 1 & & 2 & & 2 & & {\color{red}3} & & {\color{red}4} & & {\color{red}4} & & {\color{red}4} & & {\color{red}4} & & {\color{red}4} & & {\color{red}4} & & {\color{red}3} & & 2 & & 2 & & 1 & \\
1 & 1 & 2 & 1 & {\color{red}3} & 2 & {\color{red}4} & 2 & {\color{red}5} & {\color{red}3} & {\color{red}6} & {\color{red}3} & {\color{red}6} & {\color{red}3} & {\color{red}6} & {\color{red}3} & {\color{red}6} & {\color{red}3} & {\color{red}6} & {\color{red}3} & {\color{red}5} & 2 & {\color{red}4} & 2 & {\color{red}3} & 1 & 2 & 1 & 1 \\
 & 1 & & 2 & & {\color{red}3} & & {\color{red}4} & & {\color{red}4} & & {\color{red}5} & & {\color{red}5} & & {\color{red}5} & & {\color{red}5} & & {\color{red}4} & & {\color{red}4} & & {\color{red}3} & & 2 & & 1 & \\
 & & 1 & & 2 & & {\color{red}3} & & {\color{red}3} & & {\color{red}3} & & {\color{red}4} & & {\color{red}4} & & {\color{red}4} & & {\color{red}3} & & {\color{red}3} & & {\color{red}3} & & 2 & & 1 & & \\
 & & & 1 & & 2 & & 2 & & 2 & & 2 & & {\color{red}3} & & {\color{red}3} & & 2 & & 2 & & 2 & & 2 & & 1 & & & \\
 & & & & 1 & & 1 & & 1 & & 1 & & 1 & & 2 & & 1 & & 1 & & 1 & & 1 & & 1 & & & &
\end{array}$$

Process


Flag(M)
any representation M
Theorem A
Flag(M)
ind representation M
Flag(M)
ord(M) $\leqslant 2$
Flag(M)
ord(M) > 2
Lemma
Finish

Process


Flag(M)
any representation M
Theorem A
Flag(M)
ind representation M
Flag(M)
ord(M) $\leqslant 2$
Flag(M)
ord(M) > 2
Theorem B
Lemma
Finish

Consider the short exact sequence of representations

$$\eta : 0 \longrightarrow X \xrightarrow{\iota} Y \xrightarrow{\pi} S \longrightarrow 0$$

which induce maps

$$\begin{array}{rccc} \Psi : & \mathrm{Flag}_d(Y) & \longrightarrow & \mathrm{Flag}_d(X) \times \mathrm{Flag}_d(S) \\ & \cup & & \cup \\ \Psi_{\underline{\mathbf{f}},\underline{\mathbf{g}}} : & \mathrm{Flag}(Y)_{\underline{\mathbf{f}},\underline{\mathbf{g}}} & \longrightarrow & \mathrm{Flag}_{\underline{\mathbf{f}}}(X) \times \mathrm{Flag}_{\underline{\mathbf{g}}}(S) \end{array}$$

Theorem B

When η does not split and generates $\mathrm{Ext}^1(S, X)$,

$\Psi_{\underline{\mathbf{f}},\underline{\mathbf{g}}}$ is a Zarisky-locally trivial affine bundle over $\mathrm{Im}\,\Psi_{\underline{\mathbf{f}},\underline{\mathbf{g}}}$.

In this case, we have a clear description of $\mathrm{Im}\,\Psi_{\underline{\mathbf{f}},\underline{\mathbf{g}}}$.

How to find nice η?

Proposition

For $X \hookrightarrow Y$ irreducible mono, the induced SES

$$\eta : 0 \longrightarrow X \xrightarrow{\iota} Y \xrightarrow{\pi} S \longrightarrow 0$$

satisfies the condition of Theorem B. Moreover,

$$\operatorname{Im} \Psi_{\underline{\mathbf{f}},\underline{\mathbf{g}}} = \begin{cases} \big(\mathrm{Flag}_{\underline{\mathbf{f}}}(X) \smallsetminus \mathrm{Flag}_{\underline{\mathbf{f}}}(X_S)\big) \times \mathrm{Flag}_{\underline{\mathbf{g}}}(S), & \underline{\mathbf{g}}_i = \underline{\mathbf{dim}}\, S \\ \mathrm{Flag}_{\underline{\mathbf{f}}}(X) \times \mathrm{Flag}_{\underline{\mathbf{g}}}(S), & \text{otherwise} \end{cases}$$

where

$$X_S := \max\left\{M \subseteq X \,\middle|\, \mathrm{Ext}^1(S, X/M) \cong \mathbb{C}\right\} \subseteq X.$$

How to find nice η?

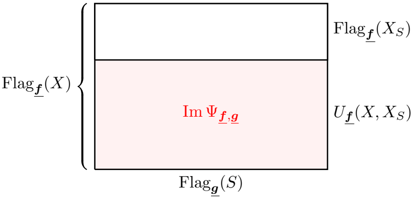
Proposition

For $X \hookrightarrow Y$ irreducible mono, the induced SES

$$\eta : 0 \longrightarrow X \xrightarrow{\iota} Y \xrightarrow{\pi} S \longrightarrow 0$$

satisfies the condition of Theorem B. Moreover,

$$\operatorname{Im} \Psi_{\underline{\mathbf{f}},\underline{\mathbf{g}}} = \begin{cases} \big(\mathrm{Flag}_{\underline{\mathbf{f}}}(X) \setminus \mathrm{Flag}_{\underline{\mathbf{f}}}(X_S)\big) \times \mathrm{Flag}_{\underline{\mathbf{g}}}(S), & \underline{\mathbf{g}}_i = \underline{\mathbf{dim}}\, S \\ \mathrm{Flag}_{\underline{\mathbf{f}}}(X) \times \mathrm{Flag}_{\underline{\mathbf{g}}}(S), & \textit{otherwise} \end{cases}$$



How to find nice η?

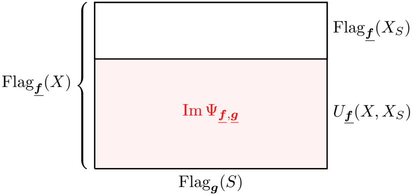
Proposition

For $X \hookrightarrow Y$ irreducible mono, the induced SES

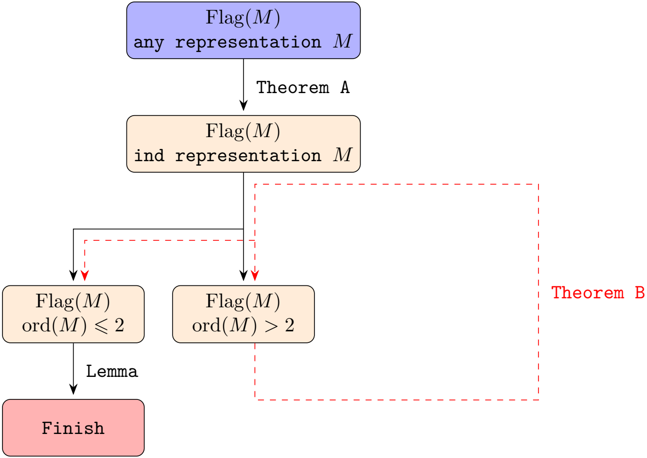
$$\eta : 0 \longrightarrow X \xrightarrow{\iota} Y \xrightarrow{\pi} S \longrightarrow 0$$

satisfies the condition of Theorem B. Moreover,

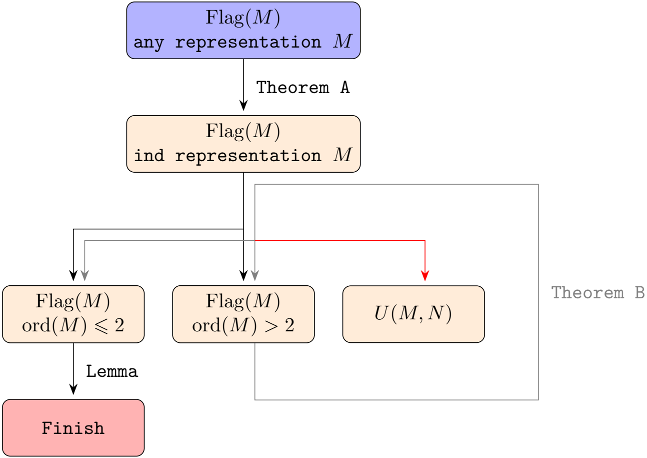
$$\operatorname{Im} \Psi_{\underline{\mathbf{f}},\underline{\mathbf{g}}} = \begin{cases} U_{\underline{\mathbf{f}}}(X, X_S), & \underline{\mathbf{g}}_i = \underline{\mathbf{dim}}\, S \\ \mathrm{Flag}_{\underline{\mathbf{f}}}(X) \times \mathrm{Flag}_{\underline{\mathbf{g}}}(S), & \textit{otherwise} \end{cases}$$



Process


Flag(M)
any representation M
Theorem A
Flag(M)
ind representation M
Flag(M)
ord(M) $\leqslant 2$
Flag(M)
ord(M) > 2
Theorem B
Lemma
Finish

Process


Flag(M)
any representation M
Theorem A
Flag(M)
ind representation M
Flag(M)
ord(M) $\leqslant 2$
Flag(M)
ord(M) > 2
$U(M, N)$
Theorem B
Lemma
Finish

Induction?

Proposition

For $X \hookrightarrow Y$ irreducible mono, the induced SES

$$\eta : 0 \longrightarrow X \xrightarrow{\iota} Y \xrightarrow{\pi} S \longrightarrow 0$$

satisfies the condition of Theorem B. Moreover,

$$\operatorname{Im} \Psi_{\underline{\mathbf{f}},\underline{\mathbf{g}}} = \begin{cases} U_{\underline{\mathbf{f}}}(X, X_S), & \underline{\mathbf{g}}_i = \underline{\mathbf{dim}}\, S \\ \operatorname{Flag}_{\underline{\mathbf{f}}}(X) \times \operatorname{Flag}_{\underline{\mathbf{g}}}(S), & \text{otherwise} \end{cases}$$

Proposition

In addition,

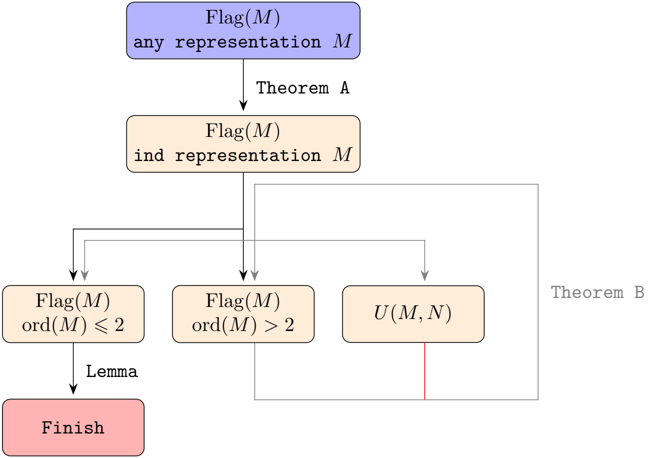
$$X_S = 0 \quad \text{or} \quad X_S \hookrightarrow X \text{ is irreducible mono.}$$

Corollary

For $M \in \operatorname{ind}(Q)$, if exist irreducible mono $X \hookrightarrow M$, then

$$\operatorname{Flag}_d(M) \text{ has an affine paving.}$$

Process



Flag(M)
any representation M
Theorem A
Flag(M)
ind representation M
Flag(M)
ord(M) $\leqslant 2$
Flag(M)
ord(M) > 2
$U(M, N)$
Theorem B
Lemma
Finish

What is remaining?

$$E_6: \quad \begin{array}{*{11}{c}}
& & 1 & & 1 & & 1 & & 1 & & \\
& 1 & & 2 & & 2 & & 2 & & 1 & \\
1 & 1 & 2 & 1 & 3 & 2 & 3 & 1 & 2 & 1 & 1 \\
& 1 & & 2 & & 2 & & 2 & & 1 & \\
& & 1 & & 1 & & 1 & & 1 & &
\end{array}$$

$$E_7: \quad \begin{array}{*{17}{c}}
& & 1 & & 1 & & 1 & & 2 & & 1 & & 1 & & 1 & & \\
& 1 & & 2 & & 2 & & 3 & & 3 & & 2 & & 2 & & 1 & \\
1 & 1 & 2 & 1 & 3 & 2 & 4 & 2 & 4 & 2 & 4 & 2 & 3 & 1 & 2 & 1 & 1 \\
& 1 & & 2 & & 3 & & 3 & & 3 & & 3 & & 2 & & 1 & \\
& & 1 & & 2 & & 2 & & 2 & & 2 & & 2 & & 1 & & \\
& & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & &
\end{array}$$

$$E_8: \quad \begin{array}{*{29}{c}}
& & 1 & & 1 & & 1 & & 2 & & 2 & & 2 & & 2 & & 2 & & 2 & & 2 & & 1 & & 1 & & 1 & & \\
& 1 & & 2 & & 2 & & 3 & & 4 & & 4 & & 4 & & 4 & & 4 & & 4 & & 3 & & 2 & & 2 & & 1 & \\
1 & 1 & 2 & 1 & 3 & 2 & 4 & 2 & 5 & 3 & 6 & 3 & 6 & 3 & 6 & 3 & 6 & 3 & 6 & 3 & 5 & 2 & 4 & 2 & 3 & 1 & 2 & 1 & 1 \\
& 1 & & 2 & & 3 & & 4 & & 4 & & 5 & & 5 & & 5 & & 5 & & 4 & & 4 & & 3 & & 2 & & 1 & \\
& & 1 & & 2 & & 3 & & 3 & & 3 & & 4 & & 4 & & 4 & & 3 & & 3 & & 3 & & 2 & & 1 & & \\
& & & 1 & & 2 & & 2 & & 2 & & 2 & & 3 & & 3 & & 2 & & 2 & & 2 & & 2 & & 1 & & & \\
& & & & 1 & & 1 & & 1 & & 1 & & 1 & & 2 & & 1 & & 1 & & 1 & & 1 & & 1 & & & &
\end{array}$$

What is remaining?

$$E_6 : \quad \begin{array}{ccccccccccc}
 & & 1 & & 1 & & 1 & & 1 & & \\
 & 1 & & 2 & & 2 & & 2 & & 1 & \\
1 & 1 & 2 & 1 & {\color{red}\to 3} & 2 & {\color{red}\to 3} & 1 & 2 & 1 & 1 \\
 & 1 & & 2 & & 2 & & 2 & & 1 & \\
 & & 1 & & 1 & & 1 & & 1 & &
\end{array}$$

$$E_7 : \quad \begin{array}{ccccccccccccccccc}
 & & 1 & & 1 & & 1 & & 2 & & 1 & & 1 & & 1 & & \\
 & 1 & & 2 & & 2 & & {\color{red}\searrow 3} & & {\color{red}\searrow 3} & & 2 & & 2 & & 1 & \\
1 & 1 & 2 & 1 & {\color{red}\to 3} & 2 & {\color{red}\to 4} & 2 & {\color{red}\to 4} & 2 & {\color{red}\to 4} & 2 & {\color{red}\to 3} & 1 & 2 & 1 & 1 \\
 & 1 & & 2 & & {\color{red}\nearrow 3} & & {\color{red}\nearrow 3} & & {\color{red}\nearrow 3} & & {\color{red}\nearrow 3} & & 2 & & 1 & \\
 & & 1 & & 2 & & 2 & & 2 & & 2 & & 2 & & 1 & & \\
 & & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & &
\end{array}$$

$$E_8 : \quad \begin{array}{ccccccccccccccccccccccccccccc}
 & & 1 & & 1 & & 1 & & 2 & & 2 & & 2 & & 2 & & 2 & & 2 & & 2 & & 1 & & 1 & & 1 & & \\
 & 1 & & 2 & & 2 & & {\color{red}\searrow 3} & & {\color{red}\searrow 4} & & {\color{red}\searrow 4} & & {\color{red}\searrow 4} & & {\color{red}\searrow 4} & & {\color{red}\searrow 4} & & {\color{red}\searrow 4} & & {\color{red}\searrow 3} & & 2 & & 2 & & 1 & \\
1 & 1 & 2 & 1 & {\color{red}\nearrow 3} & 2 & {\color{red}\nearrow 4} & 2 & {\color{red}\nearrow 5} & {\color{blue}3} & {\color{red}\nearrow 6} & {\color{blue}3} & {\color{red}\nearrow 6} & {\color{blue}3} & {\color{red}\nearrow 6} & {\color{blue}3} & {\color{red}\nearrow 6} & {\color{blue}3} & {\color{red}\nearrow 6} & {\color{blue}3} & {\color{red}\nearrow 5} & 2 & {\color{red}\to 4} & 2 & {\color{red}\to 3} & 1 & 2 & 1 & 1 \\
 & 1 & & 2 & & {\color{red}\nearrow 3} & & {\color{red}\nearrow 4} & & {\color{red}\nearrow 4} & & {\color{red}\nearrow 5} & & {\color{red}\nearrow 5} & & {\color{red}\nearrow 5} & & {\color{red}\nearrow 5} & & {\color{red}\nearrow 4} & & {\color{red}\nearrow 4} & & {\color{blue}3} & & 2 & & 1 & \\
 & & 1 & & 2 & & {\color{red}\nearrow 3} & & {\color{red}\nearrow 3} & & {\color{red}\nearrow 3} & & {\color{red}\nearrow 4} & & {\color{red}\nearrow 4} & & {\color{red}\nearrow 4} & & {\color{red}\nearrow 3} & & {\color{red}\nearrow 3} & & {\color{red}\nearrow 3} & & 2 & & 1 & & \\
 & & & 1 & & 2 & & 2 & & 2 & & 2 & & {\color{red}\nearrow 3} & & {\color{red}\nearrow 3} & & 2 & & 2 & & 2 & & 2 & & 1 & & & \\
 & & & & 1 & & 1 & & 1 & & 1 & & 1 & & 2 & & 1 & & 1 & & 1 & & 1 & & 1 & & & &
\end{array}$$

What is remaining?

E_6 :

1 1 1 1
1 2 2 2 1
1 1 2 1 3 2 3 1 2 1 1
1 2 2 2 1
1 1 1 1

E_7 :

1 1 1 2 1 1 1
1 2 2 3 3 2 2 1
1 1 2 1 3 2 4 2 4 2 4 2 3 1 2 1 1
1 2 3 3 3 3 2 1
1 2 2 2 2 2 1
1 1 1 1 1 1

E_8 :

1 1 1 2 2 2 2 2 2 2 1 1 1
1 2 2 3 4 4 4 4 4 4 3 2 2 1
1 1 2 1 3 2 4 2 5 3 6 3 6 3 6 3 6 3 6 3 5 2 4 2 3 1 2 1 1
1 2 3 4 4 5 5 5 5 4 4 3 2 1
1 2 3 3 3 4 4 4 3 3 3 2 1
1 2 2 2 2 3 3 2 2 2 2 1
1 1 1 1 1 2 1 1 1 1 1

Q & A

Thank you! Any questions?