Geometry of Quiver Flag Varieties

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Introduction

Year	People	Cohomology theory	Algebra
1980	Kazhdan–Lusztig	$\mathbb{C}[W]$	$H_*^{\mathrm{BM}}(\mathcal{Z})$
1985	Lusztig	$\mathcal{H}_q(W)$	$K^{G\times\mathbb{C}^\times}(\mathcal{Z})$
2011	Varagnolo-Vasserot	KLR algebra	$H^*_{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}})$

In the first part, we compute the G-equivariant K-theory of the Steinberg variety in the quiver version.

Variety structure

$$\begin{tikzcd} \widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q) \subseteq \mathrm{Rep}_{\mathbf{d}}(Q) \times \mathcal{F}_{\mathbf{d}} \arrow[d, bend left, "\mu_{\mathbf{d}}"'] \arrow[dr, bend left, "\pi_{\mathbf{d}}"] & \\ \mathrm{Rep}_{\mathbf{d}}(Q) & \mathcal{F}_{\mathbf{d}} \end{tikzcd}
\qquad
\begin{tikzcd} \mathcal{Z}_{\mathbf{d}} \subseteq \mathrm{Rep}_{\mathbf{d}}(Q) \times \mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}} \arrow[d, bend left, "\mu_{\mathbf{d},\mathbf{d}}"'] \arrow[dr, bend left, "\pi_{\mathbf{d},\mathbf{d}}"] & \\ \mathrm{Rep}_{\mathbf{d}}(Q) & \mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}} \end{tikzcd}$$

Stratification structure

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Main theorem

Theorem

Under the convolution product, $K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}})$ has a $\mathrm{R}(T_{\mathbf{d}})$-algebra structure. Moreover,

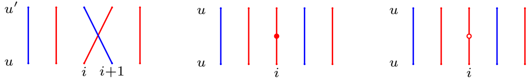
(1) As an $\mathrm{R}(T_{\mathbf{d}})$-module, $K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}})$ is free of rank $|\mathbf{d}|!$;

(2) As an $\mathrm{R}(T_{\mathbf{d}})$-algebra, we can write down generators and relations explicitly.

For proving (2), we mainly use the localization formula and the excess intersection formula.

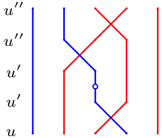
Generators

- $\bullet \longrightarrow \bullet$, $\mathbf{d} = (3,2)$, $\mathrm{Min}(\mathbb{W}_{|\mathbf{d}|}, W_{\mathbf{d}}) = \{, \dots\}$

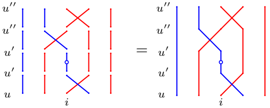


$D_i^{u,u'}$ e_i^u $(e_i^u)^{-1}$

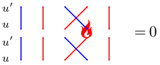
A typical element in $K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}})$ is a $\mathbb{Z}$-linear combination of diagrams shown below:



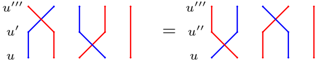
Compositions and trivial relations



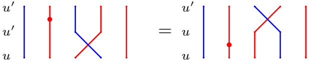
$$D_3^{u,u'} * (e_3^{u'})^{-1} * D_2^{u',u''} * D_3^{u'',u''}$$



$$D_3^{u,u'} * D_3^{u,u'} = 0$$



$$D_3^{u,u'} * D_1^{u',u'''} \quad = \quad D_1^{u,u''} * D_3^{u'',u'''}$$



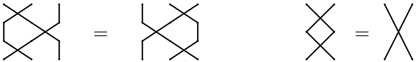
$$D_3^{u,u'} * e_2^{u'} \quad = \quad e_2^{u} * D_3^{u,u'}$$

Nontrivial relations I

Same color: $(D_i^{u,u} \hat{=} D_i, e_i^u \hat{=} e_i)$

$D_i e_i = e_{i+1} D_i - e_{i+1}$ $\qquad$ $D_i e_{i+1} = e_i D_i + e_{i+1}$

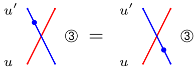
$D_i e_i^{-1} = e_{i+1}^{-1} D_i + e_i^{-1}$ $\qquad$ $D_i e_{i+1}^{-1} = e_i^{-1} D_i - e_i^{-1}$



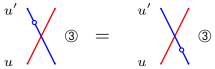
$D_i D_{i+1} D_i = D_{i+1} D_i D_{i+1}$ $\qquad$ $D_i^2 = D_i$

Nontrivial relations II

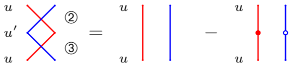
Different color:



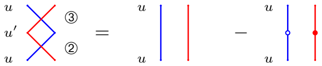
$$D_i^{u,u'} e_i^{u'} = e_{i+1}^u D_i^{u,u'}$$



$$D_i^{u,u'} (e_i^{u'})^{-1} = (e_{i+1}^u)^{-1} D_i^{u,u'}$$



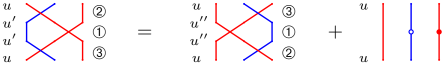
$$D_i^{u,u'} D_i^{u',u} = 1^u - \left(\frac{e_i}{e_{i+1}}\right)^u$$



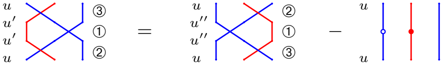
$$D_i^{u,u'} D_i^{u',u} = 1^u - \left(\frac{e_{i+1}}{e_i}\right)^u$$

Nontrivial relations III

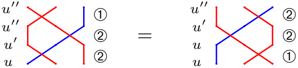
Different color:



$$D_i^{u,u'} D_{i+1}^{u',u'} D_i^{u',u} = D_{i+1}^{u,u''} D_i^{u'',u''} D_{i+1}^{u'',u} + \left(\frac{e_{i+2}}{e_{i+1}}\right)^u$$



$$D_i^{u,u'} D_{i+1}^{u',u'} D_i^{u',u} = D_{i+1}^{u,u''} D_i^{u'',u''} D_{i+1}^{u'',u} - \left(\frac{e_{i+1}}{e_i}\right)^u$$



$$D_i^{u,u'} D_{i+1}^{u',u''} D_i^{u'',u''} = D_{i+1}^{u,u} D_i^{u,u'} D_{i+1}^{u',u''}$$

Affine pavings

$\dots$

Idea of affine pavings

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Outlook

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