Affine pavings of partial flag varieties

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Process

1. Setting and Statement
2. Case study
3. Auslander–Reiten theory
4. Sketch of proof

Process

Affine paving

Setting

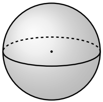
$K = \mathbb{C}$, X: *algebraic variety over* K.

Definition

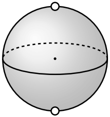
An **affine paving** of X is a filtration

$$0 = X_0 \subset X_1 \subset \cdots \subset X_d = X$$

with X_i closed and $X_{i+1} \setminus X_i \cong \mathbb{A}^k_K$.



$\mathbb{P}^1 = \{\infty\} \sqcup \mathbb{A}^1$



$\mathbb{P}^1 \setminus \{0, \infty\}$ has no affine paving

Quiver and quiver representation

Quiver is a graph. It has some vertices & arrows.
In this talk, all the quivers are finite and connected.

Quiver and quiver representation

We focus on the Dynkin quiver.

That means, the graph of the Dynkin diagrams in the ADE series.

Partial flag variety

Definition

Fix a quiver Q and $M \in \mathrm{rep}(Q)$,

$$\begin{aligned}\mathrm{Flag}_d(M) &:= \{F\colon 0 \subseteq N_1 \subseteq \cdots \subseteq N_d \subseteq M\}\\ \mathrm{Flag}_{\underline{\mathbf{f}}}(M) &:= \left\{F\colon 0 \subseteq N_1 \subseteq \cdots \subseteq N_d \subseteq M \;\middle|\; \underline{\mathbf{dim}}\, M_i = \underline{\mathbf{f}}_i\right\}\end{aligned}$$

Example

$Q = \bullet$, $M = \mathbb{C}^n$, $\underline{\mathbf{f}} := \begin{pmatrix} n \\ \vdots \\ 1 \end{pmatrix}$

$$\begin{aligned}\mathrm{Flag}_d(\mathbb{C}^n) &= \{F\colon 0 \subseteq N_1 \subseteq \cdots \subseteq N_d \subseteq \mathbb{C}^n\}\\ \mathrm{Flag}_1(\mathbb{C}^n) &= \{F\colon 0 \subseteq N_1 \subseteq \mathbb{C}^n\} \qquad = \sqcup_{k=0}^{n} \mathrm{Gr}(n,k)\\ \mathrm{Flag}_{\underline{\mathbf{f}}}(\mathbb{C}^n) &= \textit{complete flags of } \mathbb{C}^n\\ \mathrm{Flag}_{(k)}(\mathbb{C}^n) &= \mathrm{Gr}(n,k)\end{aligned}$$

Statement

Theorem

For a Dynkin quiver Q and $M \in \mathrm{rep}(Q)$,

$$\mathrm{Flag}_d(M) \text{ has an affine paving.}$$

Process

Task 1. $Q = \bullet$, $M = \mathbb{C}^n$

In this case,

$$\begin{array}{ccc} \mathrm{GL}_n(\mathbb{C}) \circlearrowright \mathbb{C}^n & \rightsquigarrow & \mathrm{GL}_n(\mathbb{C}) \circlearrowright \mathrm{Flag}_d(\mathbb{C}^n) \\ & \rightsquigarrow & B \quad \circlearrowright \mathrm{Flag}_d(\mathbb{C}^n) \end{array}$$

$\mathrm{Flag}_d(\mathbb{C}^n)$ has an affine paving given by Schubert cells (i.e., B-orbits).

Note

When $Q = \bullet \longrightarrow \bullet$, $\mathrm{Flag}_{\mathbf{f}}(M)$ have no natural group actions.

Task 2a. $Q = \bullet \to \bullet$, $M = \left[\mathbb{C}^2 \xrightarrow{\mathrm{Id}} \mathbb{C}^2\right]$, $d = 1$

$$\begin{aligned} \underline{\mathbf{f}} = (1,0): &\qquad \mathrm{Flag}_{\underline{\mathbf{f}}}(M) = \varnothing \\ \underline{\mathbf{f}} = (0,0): &\qquad \mathrm{Flag}_{\underline{\mathbf{f}}}(M) = \{*\} \\ \underline{\mathbf{f}} = (1,1): &\qquad \mathrm{Flag}_{\underline{\mathbf{f}}}(M) = \mathbb{P}^1 \end{aligned}$$

In this case, $\mathrm{Flag}_{\underline{\mathbf{f}}}(M)$ is Grassmannian or empty, so it has an affine paving.

Task 2b. $Q = \bullet \to \bullet$, $M = \left[\mathbb{C}^2 \xrightarrow{0} \mathbb{C}^2\right]$, $d = 1$

$$\begin{aligned}
\underline{\mathbf{f}} = (1,0): &\qquad \mathrm{Flag}_{\underline{\mathbf{f}}}(M) = \mathbb{P}^1 \\
\underline{\mathbf{f}} = (0,0): &\qquad \mathrm{Flag}_{\underline{\mathbf{f}}}(M) = \{*\} \\
\underline{\mathbf{f}} = (1,1): &\qquad \mathrm{Flag}_{\underline{\mathbf{f}}}(M) = \mathbb{P}^1 \times \mathbb{P}^1
\end{aligned}$$

In this case, $\mathrm{Flag}_{\underline{\mathbf{f}}}(M) \cong \mathrm{Flag}_{\underline{\mathbf{f}}_1}(M) \times \mathrm{Flag}_{\underline{\mathbf{f}}_2}(M)$ has an affine paving.

Task 2c. $Q = \bullet \to \bullet$, $M = \left[\mathbb{C}^2 \xrightarrow{\binom{1\,0}{0\,0}} \mathbb{C}^2\right]$, $d = 1$

$$\begin{aligned}
\underline{\mathbf{f}} = (1,0):& \qquad \mathrm{Flag}_{\underline{\mathbf{f}}}(M) = \{*\} \\
\underline{\mathbf{f}} = (0,0):& \qquad \mathrm{Flag}_{\underline{\mathbf{f}}}(M) = \{*\} \\
\underline{\mathbf{f}} = (1,1):& \qquad \mathrm{Flag}_{\underline{\mathbf{f}}}(M) = \mathbb{P}^1 \vee \mathbb{P}^1 \\
\underline{\mathbf{f}} = (0,1):& \qquad \dots
\end{aligned}$$

To construct affine pavings systematically, we need to construct an uniform method.

Task 2c. $Q = \bullet \to \bullet$, $M = \left[\mathbb{C}^2 \xrightarrow{\binom{1\,0}{0\,0}} \mathbb{C}^2\right]$, $d = 1$

First try

Let $X = [0 \to \mathbb{C}]$, $S = \left[\mathbb{C}^2 \xrightarrow{(1\,0)} \mathbb{C}\right]$, then $M = X \oplus S$, and the short exact sequence

$$0 \longrightarrow X \xrightarrow{\ \iota\ } M \xrightarrow{\ \pi\ } S \longrightarrow 0$$

induces

$$\begin{aligned} \Psi : \mathrm{Flag}_d(M) &\longrightarrow \mathrm{Flag}_d(X) \times \mathrm{Flag}_d(S) \\ F &\longmapsto \big(\iota^{-1}(F), \pi(F)\big) \end{aligned}$$

Idea of affine pavings

Find a nice short exact sequence

$$0 \longrightarrow X \xrightarrow{\iota} M \xrightarrow{\pi} S \longrightarrow 0$$

which induces a nice morphism

$$\begin{aligned} \Psi : \mathrm{Flag}_d(M) &\longrightarrow \mathrm{Flag}_d(X) \times \mathrm{Flag}_d(S) \\ F &\longmapsto \big(\iota^{-1}(F), \pi(F)\big) \end{aligned}$$

We construct the affine paving of $\mathrm{Flag}_d(M)$ from the affine paving of $\mathrm{Flag}_d(X)$ and $\mathrm{Flag}_d(S)$. Then, we use mathematical induction.

Example. $Q = \bullet$, $M = \mathbb{C}^2$

$$0 \longrightarrow \mathbb{C} \xrightarrow{\iota} \mathbb{C}^2 \xrightarrow{\pi} \mathbb{C} \longrightarrow 0$$

$$\Psi_1 : \ \mathrm{Flag}_1(\mathbb{C}^2) \longrightarrow \mathrm{Flag}_1(\mathbb{C}) \times \mathrm{Flag}_1(\mathbb{C})$$

$$\begin{array}{ccccccc} \Psi_{(1)} : \mathrm{Flag}_{(1)}(\mathbb{C}) & \longrightarrow & \mathrm{Flag}_{(1)}(\mathbb{C}) \times \mathrm{Flag}_{(0)}(\mathbb{C}) & \bigsqcup & \mathrm{Flag}_{(0)}(\mathbb{C}) \times \mathrm{Flag}_{(1)}(\mathbb{C}) \\ \mathbb{P}^1 & \longrightarrow & \{*\} & \bigsqcup & \{*\} \end{array}$$

Question

How does $\Psi_{(1)}$ give an affine paving of $\mathrm{Flag}_{(1)}(\mathbb{C})$?

$$\begin{array}{c} \mathbb{P}^1 = \{\infty\} \sqcup \mathbb{C} \\ \Big\downarrow {\scriptstyle \Psi_{(1)}} \\ \{*\} \sqcup \{*\} \end{array}$$

Example. $Q = \bullet$, $M = \mathbb{C}^8 = \bigoplus_{i=1}^8 \mathbb{C}v_i$

$$0 \longrightarrow \mathbb{C}^3 \xrightarrow{\iota} \mathbb{C}^8 \xrightarrow{\pi} \mathbb{C}^5 \longrightarrow 0$$

$$\begin{aligned}\Psi^{-1}\Big(\langle v_1\rangle, \langle v_4, v_5\rangle\Big) &= \Big\{\langle v_1, v_4+av_2+bv_3, v_5+cv_2+dv_3\rangle \,\Big|\, a,b,c,d \in \mathbb{C}\Big\} \\ &\cong \mathbb{C}^4\end{aligned}$$

In general,

$$\mathrm{Flag}_{(3)}(\mathbb{C}^8) \dashrightarrow \mathrm{Flag}_{(1)}(\mathbb{C}^3) \times \mathrm{Flag}_{(2)}(\mathbb{C}^5)$$

is a Zarisky-locally trivial affine bundle of rank $2 \cdot (3-1) = 4$.

Consider the short exact sequence of representations

$$\eta : 0 \longrightarrow X \xrightarrow{\iota} Y \xrightarrow{\pi} S \longrightarrow 0$$

which induce maps

$$\begin{array}{rccc} \Psi : & \mathrm{Flag}_d(Y) & \longrightarrow & \mathrm{Flag}_d(X) \times \mathrm{Flag}_d(S) \\ & \cup & & \cup \\ \Psi_{\underline{\mathbf{f}},\underline{\mathbf{g}}} : & \mathrm{Flag}(Y)_{\underline{\mathbf{f}},\underline{\mathbf{g}}} & \longrightarrow & \mathrm{Flag}_{\underline{\mathbf{f}}}(X) \times \mathrm{Flag}_{\underline{\mathbf{g}}}(S) \end{array}$$

Theorem A

When η splits, then Ψ is surjective.
Moreover, if $\mathrm{Ext}^1(S, X) = 0$, then

$\Psi_{\underline{\mathbf{f}},\underline{\mathbf{g}}}$ is a Zarisky-locally trivial affine bundle.

By this theorem,

$$\boxed{\mathrm{Flag}_d(Y) \text{ has an affine paving} \Longleftarrow \mathrm{Flag}_d(X),\ \mathrm{Flag}_d(S) \text{ have.}}$$

Warming

η splits and $\mathrm{Ext}^1(S, X) = 0$ are necessary for Theorem A.

Example

Consider the quiver $Q : \bullet \to \bullet \leftarrow \bullet$ *and the short exact sequence*

$$0 \longrightarrow \left[\mathbb{C}e_1 \to \mathbb{C}^2 \leftarrow \mathbb{C}e_2\right] \longrightarrow \left[\mathbb{C}^2 \xrightarrow{\mathrm{Id}} \mathbb{C}^2 \xleftarrow{\mathrm{Id}} \mathbb{C}^2\right] \longrightarrow \left[\mathbb{C}e_2 \to 0 \leftarrow \mathbb{C}e_1\right] \longrightarrow 0$$

we get

$$\mathrm{Im}\,\Psi_{(0,1,0),(1,0,1)} \cong \left(\mathbb{P}^1 \setminus \{0, \infty\}\right) \times \{*\} \cong \mathbb{C}^*,$$

so Ψ *is not surjective.*

In this way, we get a bad stratification

$$\mathrm{Flag}_{(1,1,1)}(Y) \cong \mathbb{P}^1 = \{0\} \sqcup \{\infty\} \sqcup \mathbb{C}^\times.$$

Task 3. $Q = \begin{tikzcd}[column sep=small, row sep=small] & \bullet \arrow[d] & \\ \bullet \arrow[r] & \bullet & \bullet \arrow[l] \end{tikzcd}$, $M = \begin{smallmatrix} & 1 & \\ 1 & 2 & 1 \end{smallmatrix} \oplus \begin{smallmatrix} & 1 & \\ 1 & 1 & 1 \end{smallmatrix} \oplus \begin{smallmatrix} & 1 & \\ 1 & 1 & 1 \end{smallmatrix}$

We use following short exact sequences

$$0 \longrightarrow \begin{smallmatrix} & 1 & \\ 1 & 1 & 1 \end{smallmatrix} \oplus \begin{smallmatrix} & 1 & \\ 1 & 1 & 1 \end{smallmatrix} \longrightarrow M \longrightarrow \begin{smallmatrix} & 1 & \\ 1 & 2 & 1 \end{smallmatrix} \longrightarrow 0$$

$$0 \longrightarrow \begin{smallmatrix} & 1 & \\ 1 & 1 & 1 \end{smallmatrix} \longrightarrow \begin{smallmatrix} & 1 & \\ 1 & 1 & 1 \end{smallmatrix} \oplus \begin{smallmatrix} & 1 & \\ 1 & 1 & 1 \end{smallmatrix} \longrightarrow \begin{smallmatrix} & 1 & \\ 1 & 1 & 1 \end{smallmatrix} \longrightarrow 0$$

to reduced the problem to indecomposable representations.

Notice that we use the result

$$\mathrm{Ext}^1\left(\begin{smallmatrix} & 1 & \\ 1 & 2 & 1 \end{smallmatrix}, \begin{smallmatrix} & 1 & \\ 1 & 1 & 1 \end{smallmatrix}\right) = 0, \qquad \mathrm{Ext}^1\left(\begin{smallmatrix} & 1 & \\ 1 & 1 & 1 \end{smallmatrix}, \begin{smallmatrix} & 1 & \\ 1 & 1 & 1 \end{smallmatrix}\right) = 0.$$

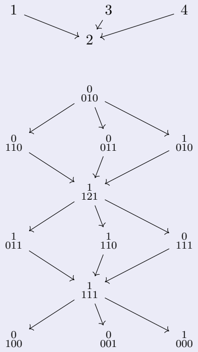
$\mathrm{Flag}_d\left(\begin{smallmatrix} & 1 & \\ 1 & 1 & 1 \end{smallmatrix}\right)$ has an affine paving: obvious.
$\mathrm{Flag}_d\left(\begin{smallmatrix} & 1 & \\ 1 & 2 & 1 \end{smallmatrix}\right)$ has an affine paving: it is $\mathbb{P}^1$, $\{*\}$ or empty.

Need: more informations of indecomposable representations!

Process

Another example: D_4

$$\begin{tikzcd} & 4 \arrow[d] & \\ 1 \arrow[r] & 2 & 3 \arrow[l] \end{tikzcd}$$



For other examples, see here.

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