Geometry of Quiver Flag Varieties

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Introduction

Year	People	Cohomology theory	Algebra
1980	Kazhdan–Lusztig	$\mathbb{C}[W]$	$H_*^{\mathrm{BM}}(\mathcal{Z})$
1985	Lusztig	$\mathcal{H}_q(W)$	$K^{G\times\mathbb{C}^\times}(\mathcal{Z})$
2011	Varagnolo-Vasserot	KLR algebra	$H^*_{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}})$

In the first part, we compute the G-equivariant K-theory of the Steinberg variety in the quiver version.

Variety structure

$$V_1 \longrightarrow V_2 \qquad \mathbf{d} = (\dim_{\mathbb{C}} V_1, \dim_{\mathbb{C}} V_2) \hat{=} (\mathbf{d}_1, \mathbf{d}_2)$$

$$\mathcal{F}_{\mathbf{d}} = \left\{\text{complete flags of } V_1 \oplus V_2 \text{ respect to the index}\right\}$$

$$\mathrm{Rep}_{\mathbf{d}}(Q) = \mathrm{Hom}_{\mathbb{C}}(V_1, V_2)$$

$$\begin{tikzcd}
 & \widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q) \subseteq \mathrm{Rep}_{\mathbf{d}}(Q) \times \mathcal{F}_{\mathbf{d}} \arrow[dl, bend left, "\mu_{\mathbf{d}}"'] \arrow[dr, bend left, "\pi_{\mathbf{d}}"] & \\
\mathrm{Rep}_{\mathbf{d}}(Q) & & \mathcal{F}_{\mathbf{d}}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathcal{Z}_{\mathbf{d}} \subseteq \mathrm{Rep}_{\mathbf{d}}(Q) \times \mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}} \arrow[dl, bend left, "\mu_{\mathbf{d},\mathbf{d}}"'] \arrow[dr, bend left, "\pi_{\mathbf{d},\mathbf{d}}"] & \\
\mathrm{Rep}_{\mathbf{d}}(Q) & & \mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}}
\end{tikzcd}$$

$$\mathrm{Flag}(M) := \mu_{\mathbf{d}}^{-1}(M) = \left\{\text{(complete) flags of } M\right\}$$

Stratification structure

$$\varpi \in \mathbb{W}_{|\mathbf{d}|} := S_{|\mathbf{d}|}, \quad G_{\mathbf{d}} := \mathrm{GL}_{\mathbf{d}_1} \times \mathrm{GL}_{\mathbf{d}_2}, \quad B_{\mathbf{d}}, T_{\mathbf{d}}.$$

$$\begin{array}{ccc} \widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q) = \bigsqcup_{\varpi} \widetilde{\Omega}_{\varpi} & & \mathcal{Z}_{\mathbf{d}} \quad = \bigsqcup_{\varpi} \widetilde{\mathcal{O}}_{\varpi} \\ \downarrow & & \downarrow \\ \mathcal{F}_{\mathbf{d}} \quad = \bigsqcup_{\varpi} \Omega_{\varpi} & & \mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}} = \bigsqcup_{\varpi} \mathcal{O}_{\varpi} \end{array}$$

$K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}}) \cong \bigoplus_{\varpi} K^{G_{\mathbf{d}}}\left(\widetilde{\mathcal{O}}_{\varpi}\right)$	Cellular fibration theorem
$\cong \bigoplus_{\varpi} K^{G_{\mathbf{d}}}\left(\mathcal{O}_{\varpi}\right)$	Thom isomorphism
$\cong \bigoplus_{\varpi} K^{B_{\mathbf{d}}}\left(\Omega_{\varpi}\right)$	Induction isomorphism
$\cong \bigoplus_{\varpi} K^{T_{\mathbf{d}}}\left(\Omega_{\varpi}\right)$	Reduction isomorphism
$\cong \bigoplus_{\varpi} \mathrm{R}(T_d)$	Thom isomorphism

Main theorem

Theorem

Under the convolution product, $K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}})$ has a $\mathrm{R}(T_{\mathbf{d}})$-algebra structure. Moreover,

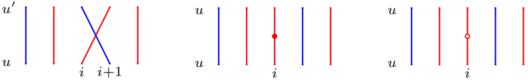
1. *As an $\mathrm{R}(T_{\mathbf{d}})$-module, $K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}})$ is free of rank $|\mathbf{d}|!$;*
2. *As an $\mathrm{R}(T_{\mathbf{d}})$-algebra, we can write down generators and relations explicitly.*

For proving (2), we mainly use the localization formula and the excess intersection formula.

Generators

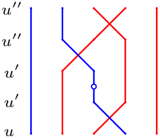
$$\bullet \longrightarrow \bullet\ , \quad \mathbf{d} = (3,2), \quad u = \cdot\ \cdot\ \cdot\ \cdot\ \cdot, \quad u \in \mathrm{Min}(\mathbb{W}_{|\mathbf{d}|}, W_{\mathbf{d}}).$$

Generators:

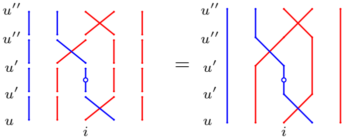


$D_i^{u,u'}$ $\quad$ e_i^u $\quad$ $(e_i^u)^{-1}$

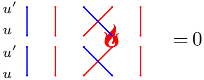
A typical element in $K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}})$ is a $\mathbb{Z}$-linear combination of diagrams shown below:



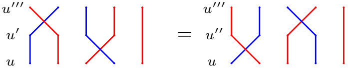
Compositions and trivial relations



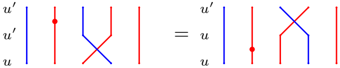
$$D_3^{u,u'} * (e_3^{u'})^{-1} * D_2^{u',u''} * D_3^{u'',u''}$$



$$D_3^{u,u'} * D_3^{u,u'} = 0$$



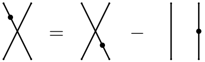
$$D_3^{u,u'} * D_1^{u',u'''} = D_1^{u,u''} * D_3^{u'',u'''}$$



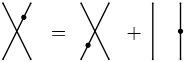
$$D_3^{u,u'} * e_2^{u'} = e_2^{u} * D_3^{u,u'}$$

Nontrivial relations I

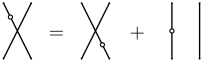
Same color: $(D_i^{u,u} \hat{=} D_i, e_i^u \hat{=} e_i)$



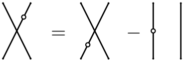
$$D_i e_i = e_{i+1} D_i - e_{i+1}$$



$$D_i e_{i+1} = e_i D_i + e_{i+1}$$

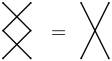


$$D_i e_i^{-1} = e_{i+1}^{-1} D_i + e_i^{-1}$$



$$D_i e_{i+1}^{-1} = e_i^{-1} D_i - e_i^{-1}$$

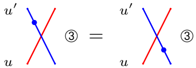
$$D_i D_{i+1} D_i = D_{i+1} D_i D_{i+1}$$



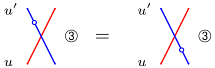
$$D_i^2 = D_i$$

Nontrivial relations II

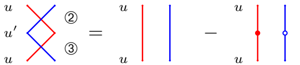
Different color:



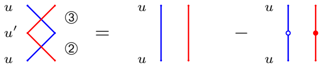
$$D_i^{u,u'} e_i^{u'} = e_{i+1}^u D_i^{u,u'}$$



$$D_i^{u,u'} (e_i^{u'})^{-1} = (e_{i+1}^u)^{-1} D_i^{u,u'}$$



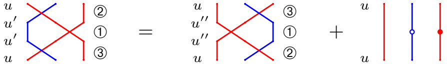
$$D_i^{u,u'} D_i^{u',u} = 1^u - \left(\frac{e_i}{e_{i+1}}\right)^u$$



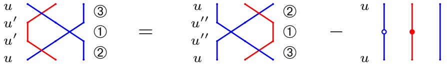
$$D_i^{u,u'} D_i^{u',u} = 1^u - \left(\frac{e_{i+1}}{e_i}\right)^u$$

Nontrivial relations III

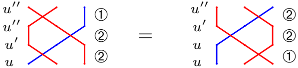
Different color:



$$D_i^{u,u'} D_{i+1}^{u',u'} D_i^{u',u} = D_{i+1}^{u,u''} D_i^{u'',u''} D_{i+1}^{u'',u} + \left(\frac{e_{i+2}}{e_{i+1}}\right)^u$$



$$D_i^{u,u'} D_{i+1}^{u',u'} D_i^{u',u} = D_{i+1}^{u,u''} D_i^{u'',u''} D_{i+1}^{u'',u} - \left(\frac{e_{i+1}}{e_i}\right)^u$$



$$D_i^{u,u'} D_{i+1}^{u',u''} D_i^{u'',u''} = D_{i+1}^{u,u} D_i^{u,u'} D_{i+1}^{u',u''}$$

Affine pavings

Theorem

For a Dynkin quiver Q and $M \in \mathrm{Rep}(Q)$, the ***partial flag variety*** *of length d*

$$\mathrm{Flag}_d(M) := \left\{ F : 0 \subseteq N_1 \subseteq \cdots \subseteq N_d \subseteq M \;\middle|\; N_i \in \mathrm{Rep}(Q) \right\}$$

has an affine paving.

Roughly speaking, we decompose $\mathrm{Flag}_d(M)$ into several pieces, and each piece is an affine space.

Idea of affine pavings

Find a nice short exact sequence

$$0 \longrightarrow X \xrightarrow{\iota} M \xrightarrow{\pi} S \longrightarrow 0$$

which induces a nice morphism

$$\begin{aligned} \Psi : \mathrm{Flag}_d(M) &\longrightarrow \mathrm{Flag}_d(X) \times \mathrm{Flag}_d(S) \\ F &\longmapsto \big(\iota^{-1}(F), \pi(F)\big) \end{aligned}$$

We construct the affine paving of $\mathrm{Flag}_d(M)$ from the affine paving of $\mathrm{Flag}_d(X)$ and $\mathrm{Flag}_d(S)$. Then, we use mathematical induction.

Further discussion

- Discuss the representation theory of $K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}})$, and connect that with the geometry of $\mathrm{Flag}_d(M)$;
- Generalize the result of first part to partial flag varieties (by introducing merge and split);
- Understand Kazhdan-Lusztig isomorphism and its categorifications;
- Understand "KLR-algebra categorifies quantum groups".