Affine pavings of partial flag varieties

Xiaoxiang Zhou

Advisor: Prof. Dr. Catharina Stroppel
Second Advisor: Dr. Jens Niklas Eberhardt

Universität Bonn

March 11, 2023

Process

Process

Affine paving

Setting

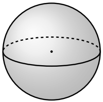
$K = \mathbb{C}$, X: *algebraic variety over* K.

Definition

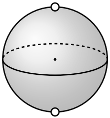
An **affine paving** of X is a filtration

$$0 = X_0 \subset X_1 \subset \cdots \subset X_d = X$$

with X_i closed and $X_{i+1} \setminus X_i \cong \mathbb{A}^k_K$.



$\mathbb{P}^1 = \{\infty\} \sqcup \mathbb{A}^1$



$\mathbb{P}^1 \setminus \{0, \infty\}$ has no affine paving

Quiver and quiver representation

Quiver is a graph. It has some vertices & arrows.
In this talk, all the quivers are finite and connected.

Quiver and quiver representation

We focus on the Dynkin quiver.

That means, the graph of the Dynkin diagrams in the ADE series.

Partial flag variety

Definition

Fix a quiver Q and $M \in \mathrm{rep}(Q)$,

$$\mathrm{Flag}_d(M) :\ = \{F : 0 \subseteq N_1 \subseteq \cdots \subseteq N_d \subseteq M\}$$
$$\mathrm{Flag}_{\underline{\mathbf{f}}}(M) :\ = \left\{F : 0 \subseteq N_1 \subseteq \cdots \subseteq N_d \subseteq M \;\middle|\; \underline{\mathbf{dim}}\, M_i = \underline{\mathbf{f}}_i\right\}$$

Example

$Q = \bullet$, $M = \mathbb{C}^n$, $\underline{\mathbf{f}} := \begin{pmatrix} n \\ \vdots \\ 1 \end{pmatrix}$

$$\mathrm{Flag}_d(\mathbb{C}^n) = \{F : 0 \subseteq N_1 \subseteq \cdots \subseteq N_d \subseteq \mathbb{C}^n\}$$
$$\mathrm{Flag}_1(\mathbb{C}^n) = \{F : 0 \subseteq N_1 \subseteq \mathbb{C}^n\} \qquad = \sqcup_{k=0}^{n} \mathrm{Gr}(n,k)$$
$$\mathrm{Flag}_{\underline{\mathbf{f}}}(\mathbb{C}^n) = \text{ \emph{complete flags of} } \mathbb{C}^n$$
$$\mathrm{Flag}_{(k)}(\mathbb{C}^n) = \mathrm{Gr}(n,k)$$

Statement

Theorem

For a Dynkin quiver Q *and* $M \in \mathrm{rep}(Q)$,

$$\mathrm{Flag}_d(M) \text{ has an affine paving.}$$

Process

Task 1. $Q = \bullet$, $M = \mathbb{C}^n$

In this case,

$$\begin{array}{ccc} \mathrm{GL}_n(\mathbb{C}) \circlearrowright \mathbb{C}^n & \rightsquigarrow & \mathrm{GL}_n(\mathbb{C}) \circlearrowright \mathrm{Flag}_d(\mathbb{C}^n) \\ & \rightsquigarrow & B \circlearrowright \mathrm{Flag}_d(\mathbb{C}^n) \end{array}$$

$\mathrm{Flag}_d(\mathbb{C}^n)$ has an affine paving given by Schubert cells (i.e., B-orbits).

Note

When $Q = \bullet \longrightarrow \bullet$, $\mathrm{Flag}_{\mathbf{f}}(M)$ have no natural group actions.

Task 2a. $Q = \bullet \to \bullet$, $M = \left[\mathbb{C}^2 \xrightarrow{\mathrm{Id}} \mathbb{C}^2\right]$, $d = 1$

$$
\begin{aligned}
\underline{\mathbf{f}} = (1,0): &\qquad \mathrm{Flag}_{\underline{\mathbf{f}}}(M) = \varnothing \\
\underline{\mathbf{f}} = (0,0): &\qquad \mathrm{Flag}_{\underline{\mathbf{f}}}(M) = \{*\} \\
\underline{\mathbf{f}} = (1,1): &\qquad \mathrm{Flag}_{\underline{\mathbf{f}}}(M) = \mathbb{P}^1
\end{aligned}
$$

In this case, $\mathrm{Flag}_{\underline{\mathbf{f}}}(M)$ is Grassmannian or empty, so it has an affine paving.

Task 2b. $Q = \bullet \to \bullet$, $M = \left[\mathbb{C}^2 \xrightarrow{0} \mathbb{C}^2\right]$, $d = 1$

$$\begin{aligned}
\underline{\mathbf{f}} = (1,0) : &\qquad \mathrm{Flag}_{\underline{\mathbf{f}}}(M) = \mathbb{P}^1 \\
\underline{\mathbf{f}} = (0,0) : &\qquad \mathrm{Flag}_{\underline{\mathbf{f}}}(M) = \{*\} \\
\underline{\mathbf{f}} = (1,1) : &\qquad \mathrm{Flag}_{\underline{\mathbf{f}}}(M) = \mathbb{P}^1 \times \mathbb{P}^1
\end{aligned}$$

In this case, $\mathrm{Flag}_{\underline{\mathbf{f}}}(M) \cong \mathrm{Flag}_{\underline{\mathbf{f}}_1}(M) \times \mathrm{Flag}_{\underline{\mathbf{f}}_2}(M)$ has an affine paving.

Task 2c. $Q = \bullet \to \bullet$, $M = \left[\mathbb{C}^2 \xrightarrow{\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)} \mathbb{C}^2\right]$, $d = 1$

$$\begin{aligned}
\underline{\mathbf{f}} &= (1,0): && \mathrm{Flag}_{\underline{\mathbf{f}}}(M) = \{*\}\\
\underline{\mathbf{f}} &= (0,0): && \mathrm{Flag}_{\underline{\mathbf{f}}}(M) = \{*\}\\
\underline{\mathbf{f}} &= (1,1): && \mathrm{Flag}_{\underline{\mathbf{f}}}(M) = \mathbb{P}^1 \vee \mathbb{P}^1\\
\underline{\mathbf{f}} &= (0,1): && \dots
\end{aligned}$$

To construct affine pavings systematically, we need to construct an uniform method.

Task 2c. $Q = \bullet \to \bullet$, $M = \left[\mathbb{C}^2 \xrightarrow{\binom{1\,0}{0\,0}} \mathbb{C}^2\right]$, $d = 1$

First try

Let $X = [0 \to \mathbb{C}]$, $S = \left[\mathbb{C}^2 \xrightarrow{(1\,0)} \mathbb{C}\right]$, then $M = X \oplus S$, and the short exact sequence

$$0 \longrightarrow X \xrightarrow{\iota} M \xrightarrow{\pi} S \longrightarrow 0$$

induces

$$\begin{aligned} \Psi : \mathrm{Flag}_d(M) &\longrightarrow \mathrm{Flag}_d(X) \times \mathrm{Flag}_d(S) \\ F &\longmapsto \left(\iota^{-1}(F), \pi(F)\right) \end{aligned}$$

Idea of affine pavings

Find a nice short exact sequence

$$0 \longrightarrow X \xrightarrow{\iota} M \xrightarrow{\pi} S \longrightarrow 0$$

which induces a nice morphism

$$\begin{aligned} \Psi : \mathrm{Flag}_d(M) &\longrightarrow \mathrm{Flag}_d(X) \times \mathrm{Flag}_d(S) \\ F &\longmapsto \big(\iota^{-1}(F), \pi(F)\big) \end{aligned}$$

We construct the affine paving of $\mathrm{Flag}_d(M)$ from the affine paving of $\mathrm{Flag}_d(X)$ and $\mathrm{Flag}_d(S)$. Then, we use mathematical induction.

Example. $Q = \bullet$, $M = \mathbb{C}^2$

$$0 \longrightarrow \mathbb{C} \xrightarrow{\iota} \mathbb{C}^2 \xrightarrow{\pi} \mathbb{C} \longrightarrow 0$$

$$\Psi_1 : \ \mathrm{Flag}_1(\mathbb{C}^2) \longrightarrow \mathrm{Flag}_1(\mathbb{C}) \times \mathrm{Flag}_1(\mathbb{C})$$

$$\begin{array}{ccccccc} \Psi_{(1)} : \mathrm{Flag}_{(1)}(\mathbb{C}) & \longrightarrow & \mathrm{Flag}_{(1)}(\mathbb{C}) \times \mathrm{Flag}_{(0)}(\mathbb{C}) & \bigsqcup & \mathrm{Flag}_{(0)}(\mathbb{C}) \times \mathrm{Flag}_{(1)}(\mathbb{C}) \\ \mathbb{P}^1 & \longrightarrow & \{*\} & \bigsqcup & \{*\} \end{array}$$

Question

How does $\Psi_{(1)}$ give an affine paving of $\mathrm{Flag}_{(1)}(\mathbb{C})$?

$$\begin{array}{c} \mathbb{P}^1 = \{\infty\} \sqcup \mathbb{C} \\ \big\downarrow \Psi_{(1)} \\ \{*\} \sqcup \{*\} \end{array}$$

Example. $Q = \bullet$, $M = \mathbb{C}^8 = \bigoplus_{i=1}^8 \mathbb{C} v_i$

$$0 \longrightarrow \mathbb{C}^3 \xrightarrow{\iota} \mathbb{C}^8 \xrightarrow{\pi} \mathbb{C}^5 \longrightarrow 0$$

$$\begin{aligned}\Psi^{-1}\Big(\langle v_1\rangle, \langle v_4, v_5\rangle\Big) &= \Big\{\langle v_1, v_4+av_2+bv_3, v_5+cv_2+dv_3\rangle \,\Big|\, a,b,c,d \in \mathbb{C}\Big\} \\ &\cong \mathbb{C}^4\end{aligned}$$

In general,

$$\mathrm{Flag}_{(3)}(\mathbb{C}^8) \dashrightarrow \mathrm{Flag}_{(1)}(\mathbb{C}^3) \times \mathrm{Flag}_{(2)}(\mathbb{C}^5)$$

is a Zarisky-locally trivial affine bundle of rank $2 \cdot (3-1) = 4$.

Consider the short exact sequence of representations

$$\eta : 0 \longrightarrow X \xrightarrow{\iota} Y \xrightarrow{\pi} S \longrightarrow 0$$

which induce maps

$$\begin{array}{rccc} \Psi : & \mathrm{Flag}_d(Y) & \longrightarrow & \mathrm{Flag}_d(X) \times \mathrm{Flag}_d(S) \\ & \cup & & \cup \\ \Psi_{\underline{\mathbf{f}},\underline{\mathbf{g}}} : & \mathrm{Flag}(Y)_{\underline{\mathbf{f}},\underline{\mathbf{g}}} & \longrightarrow & \mathrm{Flag}_{\underline{\mathbf{f}}}(X) \times \mathrm{Flag}_{\underline{\mathbf{g}}}(S) \end{array}$$

Theorem A

When η splits, then Ψ is surjective.
Moreover, if $\mathrm{Ext}^1(S, X) = 0$, then

$\Psi_{\underline{\mathbf{f}},\underline{\mathbf{g}}}$ is a Zarisky-locally trivial affine bundle.

By this theorem,

$\boxed{\mathrm{Flag}_d(Y) \text{ has an affine paving } \Longleftarrow \mathrm{Flag}_d(X),\ \mathrm{Flag}_d(S) \text{ have.}}$

Warming

η splits and $\mathrm{Ext}^1(S, X) = 0$ are necessary for Theorem A.

Example

Consider the quiver $Q : \bullet \to \bullet \leftarrow \bullet$ *and the short exact sequence*

$$0 \longrightarrow \left[\mathbb{C}e_1 \to \mathbb{C}^2 \leftarrow \mathbb{C}e_2\right] \longrightarrow \left[\mathbb{C}^2 \xrightarrow{\mathrm{Id}} \mathbb{C}^2 \xleftarrow{\mathrm{Id}} \mathbb{C}^2\right] \longrightarrow \left[\mathbb{C}e_2 \to 0 \leftarrow \mathbb{C}e_1\right] \longrightarrow 0$$

we get

$$\mathrm{Im}\,\Psi_{(0,1,0),(1,0,1)} \cong \left(\mathbb{P}^1 \setminus \{0, \infty\}\right) \times \{*\} \cong \mathbb{C}^*,$$

so Ψ *is not surjective.*

In this way, we get a bad stratification

$$\mathrm{Flag}_{(1,1,1)}(Y) \cong \mathbb{P}^1 = \{0\} \sqcup \{\infty\} \sqcup \mathbb{C}^\times.$$

Task 3. $Q = \begin{smallmatrix} & \bullet & \\ & \downarrow & \\ \bullet \to & \bullet & \leftarrow \bullet \end{smallmatrix}$, $M = {}^{\;1}_{121} \oplus {}^{\;1}_{111} \oplus {}^{\;1}_{111}$

We use following short exact sequences

$$0 \longrightarrow {}^{\;1}_{111} \oplus {}^{\;1}_{111} \longrightarrow M \longrightarrow {}^{\;1}_{121} \longrightarrow 0$$

$$0 \longrightarrow {}^{\;1}_{111} \longrightarrow {}^{\;1}_{111} \oplus {}^{\;1}_{111} \longrightarrow {}^{\;1}_{111} \longrightarrow 0$$

to reduced the problem to indecomposable representations.

Notice that we use the result

$$\mathrm{Ext}^1\left({}^{\;1}_{121}, {}^{\;1}_{111}\right) = 0, \qquad \mathrm{Ext}^1\left({}^{\;1}_{111}, {}^{\;1}_{111}\right) = 0.$$

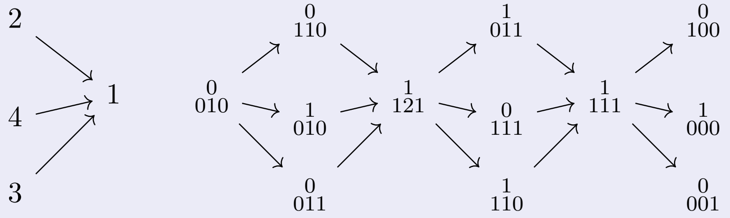
$\mathrm{Flag}_d\left({}^{\;1}_{111}\right)$ has an affine paving: obvious.
$\mathrm{Flag}_d\left({}^{\;1}_{121}\right)$ has an affine paving: it is $\mathbb{P}^1$, $\{*\}$ or empty.

Need: more informations of indecomposable representations!

Process

Auslander–Reiten quiver: D_4

$$\begin{tikzcd} & 4 \arrow[d] & \\ 2 \arrow[r] & 1 & 3 \arrow[l] \end{tikzcd}$$



Vertices $\Longleftrightarrow$ Indecomposable representations

Arrows $\Longleftrightarrow$ Irreducible morphisms

Paths $\Longleftrightarrow$ Morphisms

Shift of cards $\Longleftrightarrow$ Switch arrows in Q

Auslander–Reiten quiver: D_4

$$\begin{tikzcd} & 4 \arrow[d] & \\ 2 \arrow[r] & 1 & 3 \arrow[l] \end{tikzcd}$$

$$\begin{tikzcd} 2 \arrow[dr] & \\ 4 \arrow[r] & 1 \\ 3 \arrow[ur] & \end{tikzcd}$$

$$\begin{tikzcd}
 & \substack{0\\110} \arrow[dr] & & \substack{1\\011} \arrow[dr] & & \substack{0\\100} \\
\substack{0\\010} \arrow[ur] \arrow[r] \arrow[dr] & \substack{1\\010} \arrow[r] & \substack{1\\121} \arrow[ur] \arrow[r] \arrow[dr] & \substack{0\\111} \arrow[r] & \substack{1\\111} \arrow[ur] \arrow[r] \arrow[dr] & \substack{1\\000} \\
 & \substack{0\\011} \arrow[ur] & & \substack{1\\110} \arrow[ur] & & \substack{0\\001}
\end{tikzcd}$$

Vertices $\Longleftrightarrow$ Indecomposable representations

Auslander–Reiten quiver: D_4

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\end{tikzcd}$$

Vertices $\iff$ Indecomposable representations

irreducible rep, projective rep, injective rep

Auslander–Reiten quiver: D_4

$$
\begin{tikzcd}
 & 4 \arrow[d] & \\
2 \arrow[r] & 1 & 3 \arrow[l]
\end{tikzcd}
$$

$$
\begin{tikzcd}
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 & \substack{0\\011} \arrow[ur] & & \substack{1\\110} \arrow[ur] & & \substack{0\\001}
\end{tikzcd}
$$

Arrows $\Longleftrightarrow$ Irreducible morphisms

Instead, I will show you how to construct AR-quiver.

Auslander–Reiten quiver: D_4

$$\begin{tikzcd} & 4 \arrow[d] & \\ 2 \arrow[r] & 1 & 3 \arrow[l] \end{tikzcd}$$

$$\begin{tikzcd} 2 \arrow[dr] & & & & P(2) \\ & 1 & P(1) \arrow[urr] \arrow[rr] \arrow[drr] & & P(4) \\ 4 \arrow[ur] & & & & \\ 3 \arrow[uur] & & & & P(3) \end{tikzcd}$$

<u>Construction</u>:

Auslander–Reiten quiver: D_4

$$\begin{array}{ccccc} & & 4 & & \\ & & \downarrow & & \\ 2 & \rightarrow & 1 & \leftarrow & 3 \end{array}$$

$$\begin{tikzcd}[column sep=small] 2 \arrow[rd] & & & & P(2) \\ 4 \arrow[r] & 1 & & P(1) \arrow[ru] \arrow[r] \arrow[rd] & P(4) \\ 3 \arrow[ru] & & & & P(3) \end{tikzcd}$$

Construction:

$$0 \longrightarrow P(1) \longrightarrow P(2) \oplus P(4) \oplus P(3) \longrightarrow \begin{smallmatrix} & 1 & \\ 1 & 2 & 1 \end{smallmatrix} \longrightarrow 0$$

Auslander–Reiten quiver: D_4

$$\begin{tikzcd}
& 4 \arrow[d] & \\
2 \arrow[r] & 1 & 3 \arrow[l]
\end{tikzcd}$$

$$\begin{tikzcd}
2 \arrow[dr] & \\
4 \arrow[r] & 1 \\
3 \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& P(2) \arrow[dr] & \\
P(1) \arrow[ur] \arrow[r] \arrow[dr] & P(4) \arrow[r] & \begin{smallmatrix}1\\121\end{smallmatrix} \\
& P(3) \arrow[ur] &
\end{tikzcd}$$

<u>Construction</u>:

$$0 \longrightarrow P(1) \longrightarrow P(2) \oplus P(4) \oplus P(3) \longrightarrow \begin{smallmatrix}1\\121\end{smallmatrix} \longrightarrow 0$$

Auslander–Reiten quiver: D_4

$$\begin{tikzcd} & 4 \arrow[d] & \\ 2 \arrow[r] & 1 & 3 \arrow[l] \end{tikzcd}$$

$$\begin{tikzcd} 2 \arrow[dr] & \\ 4 \arrow[r] & 1 \\ 3 \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} & P(2) \arrow[dr] & \\ P(1) \arrow[ur] \arrow[r] \arrow[dr] & P(4) \arrow[r] & {\begin{smallmatrix} & 1 & \\ 1 & 2 & 1 \end{smallmatrix}} \\ & P(3) \arrow[ur] & \end{tikzcd}$$

<u>Construction</u>:

$$0 \longrightarrow P(2) \longrightarrow \begin{smallmatrix} & 1 & \\ 1 & 2 & 1 \end{smallmatrix} \longrightarrow \begin{smallmatrix} & 1 & \\ 0 & 1 & 1 \end{smallmatrix} \longrightarrow 0$$

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$$\begin{tikzcd} & 4 \arrow[d] & \\ 2 \arrow[r] & 1 & 3 \arrow[l] \end{tikzcd}$$

$$\begin{tikzcd}
2 \arrow[dr] & & & P(2) \arrow[dr] & & \substack{1\\011} \\
4 \arrow[r] & 1 & P(1) \arrow[ur] \arrow[r] \arrow[dr] & P(4) \arrow[r] & \substack{1\\121} \arrow[ur] & \\
3 \arrow[ur] & & & P(3) \arrow[ur] & &
\end{tikzcd}$$

Construction:

$$0 \longrightarrow P(2) \longrightarrow \substack{1\\121} \longrightarrow \substack{1\\011} \longrightarrow 0$$

Auslander–Reiten quiver: D_4

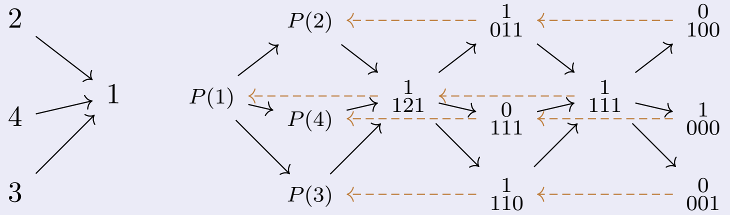
$$\begin{array}{ccccc} & & 4 & & \\ & & \downarrow & & \\ 2 & \rightarrow & 1 & \leftarrow & 3 \end{array}$$

$$\begin{tikzcd}
2 \arrow[dr] & & & P(2) \arrow[dr] & & \begin{smallmatrix}1\\011\end{smallmatrix} \arrow[dr] & & \begin{smallmatrix}0\\100\end{smallmatrix} \\
4 \arrow[r] & 1 & P(1) \arrow[ur] \arrow[r] \arrow[dr] & P(4) \arrow[r] & \begin{smallmatrix}1\\121\end{smallmatrix} \arrow[ur] \arrow[r] \arrow[dr] & \begin{smallmatrix}0\\111\end{smallmatrix} \arrow[r] & \begin{smallmatrix}1\\111\end{smallmatrix} \arrow[ur] \arrow[r] \arrow[dr] & \begin{smallmatrix}1\\000\end{smallmatrix} \\
3 \arrow[ur] & & & P(3) \arrow[ur] & & \begin{smallmatrix}1\\110\end{smallmatrix} \arrow[ur] & & \begin{smallmatrix}0\\001\end{smallmatrix}
\end{tikzcd}$$

<u>Construction</u>:

Auslander–Reiten quiver: D_4

$$\begin{array}{ccccc} & & 4 & & \\ & & \downarrow & & \\ 2 & \rightarrow & 1 & \leftarrow & 3 \end{array}$$



Construction:

AR-quiver, AR-sequence, AR-translation

Auslander–Reiten quiver: D_4

$$\begin{array}{ccccc} & & 4 & & \\ & & \downarrow & & \\ 2 & \rightarrow & 1 & \leftarrow & 3 \end{array}$$

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Paths $\Longleftrightarrow$ Morphisms

Auslander–Reiten quiver: D_4

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\end{tikzcd}$$

In the Dynkin quiver case,

$$\mathrm{Hom}(T, T') \cong \langle \text{ paths from } T \text{ to } T' \rangle /_{\text{AR-seq}}$$

For example,

$$\mathrm{Hom}(\substack{1\\010}, \substack{1\\011}) \cong \mathbb{C}, \quad \mathrm{Hom}(\substack{1\\010}, \substack{0\\111}) \cong 0, \quad \mathrm{Hom}(\substack{1\\010}, \substack{1\\111}) \cong \mathbb{C}$$

Auslander–Reiten quiver: D_4

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Auslander–Reiten quiver: D_4

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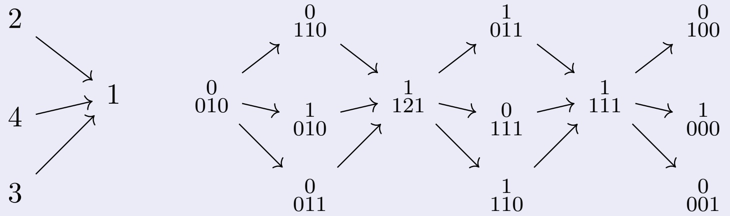

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Auslander–Reiten quiver: D_4

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$$\begin{tikzcd} 2 \arrow[dr] & \\ 4 \arrow[r] & 1 \\ 3 \arrow[ur] & \end{tikzcd}
\qquad
\begin{tikzcd}
 & \substack{0\\110} \arrow[dr] & & \substack{1\\011} \arrow[dr] & & \substack{0\\100} \\
\substack{0\\010} \arrow[ur] \arrow[r] \arrow[dr] & \substack{1\\010} \arrow[r] & \substack{1\\121} \arrow[ur] \arrow[r] \arrow[dr] & \substack{0\\111} \arrow[r] & \substack{1\\111} \arrow[ur] \arrow[r] \arrow[dr] & \substack{1\\000} \\
 & \substack{0\\011} \arrow[ur] & & \substack{1\\110} \arrow[ur] & & \substack{0\\001}
\end{tikzcd}$$

$$\mathrm{Ext}^1(T, T') \cong \overline{\mathrm{Hom}}(T', \tau T)^\vee$$

For example,
$$\mathrm{Ext}^1\left(\substack{1\\121}, \substack{0\\111}\right) \cong \mathrm{Hom}\left(\substack{0\\111}, \substack{0\\010}\right)^\vee \cong 0$$

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2 \arrow[dr] & & & \substack{0\\110} \arrow[dr] & & \substack{1\\011} \arrow[dr] & & \substack{0\\100} \\
4 \arrow[r] & 1 & \substack{0\\010} \arrow[ur] \arrow[r] \arrow[dr] & \substack{1\\010} \arrow[r] & \substack{1\\121} \arrow[ur] \arrow[r] \arrow[dr] & \substack{0\\111} \arrow[r] & \substack{1\\111} \arrow[ur] \arrow[r] \arrow[dr] & \substack{1\\000} \\
3 \arrow[ur] & & & \substack{0\\011} \arrow[ur] & & \substack{1\\110} \arrow[ur] & & \substack{0\\001}
\end{tikzcd}$$

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\end{tikzcd}
$$

$$
\begin{tikzcd}
2 \arrow[dr] & \\
4 \arrow[r] & 1 \\
3 \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \genfrac{}{}{0pt}{}{0}{110} \arrow[dr] & & \genfrac{}{}{0pt}{}{1}{011} \arrow[dr] & & \genfrac{}{}{0pt}{}{0}{100} \\
\genfrac{}{}{0pt}{}{0}{010} \arrow[ur] \arrow[r] \arrow[dr] & \genfrac{}{}{0pt}{}{1}{010} \arrow[r] & \genfrac{}{}{0pt}{}{1}{121} \arrow[ur] \arrow[r] \arrow[dr] & \genfrac{}{}{0pt}{}{0}{111} \arrow[r] & \genfrac{}{}{0pt}{}{1}{111} \arrow[ur] \arrow[r] \arrow[dr] & \genfrac{}{}{0pt}{}{1}{000} \\
& \genfrac{}{}{0pt}{}{0}{011} \arrow[ur] & & \genfrac{}{}{0pt}{}{1}{110} \arrow[ur] & & \genfrac{}{}{0pt}{}{0}{001}
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$$\mathrm{Ext}^1(T, T') \cong \overline{\mathrm{Hom}}(T', \tau T)^\vee$$

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First application

Corollary

For a Dynkin quiver, we can give an total order to the set of all indecomposable representations, such that

$$i \leqslant j \quad \Longrightarrow \quad \mathrm{Ext}^1(M_i, M_j) = 0.$$

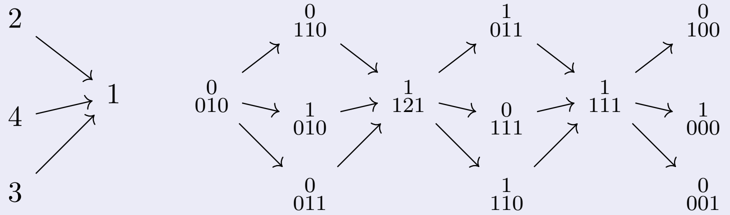
By Theorem A, the problem reduced to

For a Dynkin quiver Q and $M \in \mathrm{ind}(Q)$,

$$\mathrm{Flag}_d(M) \text{ has an affine paving.}$$

Auslander–Reiten quiver: D_4

$$\begin{array}{ccccc} & & 4 & & \\ & & \downarrow & & \\ 2 & \rightarrow & 1 & \leftarrow & 3 \end{array}$$



Shift of cards $\iff$ Switch arrows in Q

Auslander–Reiten quiver: D_4

$$\begin{tikzcd} & 4 \arrow[d] & \\ 2 \arrow[r] & 1 & 3 \arrow[l] \end{tikzcd}$$

$$\begin{tikzcd}[column sep=small, row sep=small]
& \substack{0\\110} \arrow[dr] & & \substack{1\\011} \arrow[dr] & & \substack{0\\100} \\
\substack{0\\010} \arrow[ur] \arrow[r] \arrow[dr] & \substack{1\\010} \arrow[r] & \substack{1\\121} \arrow[ur] \arrow[r] \arrow[dr] & \substack{0\\111} \arrow[r] & \substack{1\\111} \arrow[ur] \arrow[r] \arrow[dr] & \substack{1\\000} \\
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\end{tikzcd}$$

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\begin{tikzcd}[column sep=small]
\overset{0}{000} \arrow[dr] & & \overset{0}{000} \arrow[dr] & & \overset{0}{110} \arrow[dr] & & \overset{1}{011} \arrow[dr] & & \overset{0}{100} \arrow[dr] & & \overset{0}{000} \arrow[dr] & & \overset{0}{000} \\
\overset{0}{000} \arrow[r] & \overset{0}{000} \arrow[r] \arrow[ur] \arrow[dr] & \overset{0}{000} \arrow[r] & \overset{0}{010} \arrow[r] \arrow[ur] \arrow[dr] & \overset{1}{010} \arrow[r] & \overset{1}{121} \arrow[r] \arrow[ur] \arrow[dr] & \overset{0}{111} \arrow[r] & \overset{1}{111} \arrow[r] \arrow[ur] \arrow[dr] & \overset{1}{000} \arrow[r] & \overset{0}{000} \arrow[r] \arrow[ur] \arrow[dr] & \overset{0}{000} \arrow[r] & \overset{0}{000} \arrow[r] \arrow[ur] \arrow[dr] & \overset{0}{000} \\
\overset{0}{000} \arrow[ur] & & \overset{0}{000} \arrow[ur] & & \overset{0}{011} \arrow[ur] & & \overset{1}{110} \arrow[ur] & & \overset{0}{001} \arrow[ur] & & \overset{0}{000} \arrow[ur] & & \overset{0}{000}
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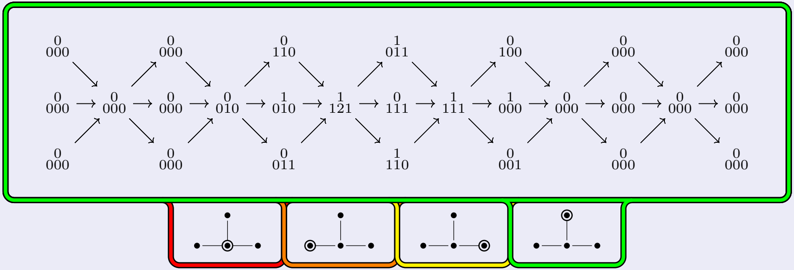

Auslander–Reiten quiver: D_4

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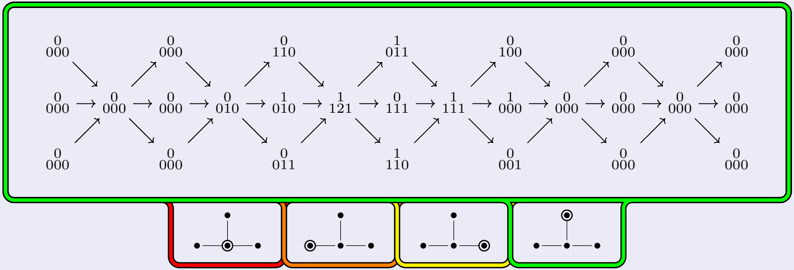

Interactive webversion

Indecomposable representations of low order are easy!

Lemma

Suppose Q is a tree. For $M \in \mathrm{ind}(Q)$, $\mathrm{ord}(M) \leqslant 2$,

$$\mathrm{Flag}_{\underline{\mathbf{f}}}(M) \;\cong\; \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \quad \text{or} \quad \varnothing.$$

Example

$$Q = \begin{tikzcd} & & \bullet \arrow[d] & \\ \bullet \arrow[r] & \bullet & \bullet \arrow[l] \arrow[r] & \bullet \end{tikzcd} \quad M = \begin{tikzcd} & & \mathbb{C} \arrow[d, hook] & \\ \mathbb{C} \arrow[r, hook] & \mathbb{C}^2 & \mathbb{C}^2 \arrow[l, "\sim"'] \arrow[r, two heads] & \mathbb{C} \end{tikzcd} \quad \underline{\mathbf{f}} = \begin{pmatrix} \;\;0\;\; \\ 1211 \\ \;\;0\;\; \\ 1101 \end{pmatrix}$$

$$\begin{aligned} \mathrm{Flag}_{\underline{\mathbf{f}}}(M) \hookrightarrow\; & \mathrm{Flag}_{\binom{1}{1}}(\mathbb{C}) \times \mathrm{Flag}_{\binom{2}{1}}(\mathbb{C}^2) \times \mathrm{Flag}_{\binom{1}{0}}(\mathbb{C}^2) \\ & \times \mathrm{Flag}_{\binom{1}{1}}(\mathbb{C}) \times \mathrm{Flag}_{\binom{0}{0}}(\mathbb{C}) \\ & \cong \mathbb{P}^1 \times \mathbb{P}^1 \end{aligned}$$

Continue

<table>
<tr><th></th><th colspan="2">$\mathbb{C} \hookrightarrow \mathbb{C}^2$</th><th colspan="2">$\mathbb{C}^2 \twoheadrightarrow \mathbb{C}$</th><th colspan="2">$\mathbb{C}^2 \to \mathbb{C}^2$</th></tr>
<tr><td rowspan="2">No restriction</td><td>–</td><td>2</td><td>0</td><td>–</td><td>–</td><td>1</td></tr>
<tr><td>0</td><td>–</td><td>1</td><td>2</td><td>0</td><td>0</td></tr>
<tr><td>Reduce</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>0</td></tr>
<tr><td rowspan="3">Impossible</td><td>2</td><td>1</td><td rowspan="3">1</td><td rowspan="3">0</td><td rowspan="3">2</td><td rowspan="3">0</td></tr>
<tr><td>2</td><td>0</td></tr>
<tr><td>1</td><td>0</td></tr>
</table>

Corollary

The main theorem is true for quivers of type A, D.

Process

1. Setting and Statement
2. Case study
3. Auslander–Reiten theory
4. Tackle the type E case

Consider the short exact sequence of representations

$$\eta : 0 \longrightarrow X \xrightarrow{\iota} Y \xrightarrow{\pi} S \longrightarrow 0$$

which induce maps

$$\begin{array}{rccc} \Psi : & \mathrm{Flag}_d(Y) & \longrightarrow & \mathrm{Flag}_d(X) \times \mathrm{Flag}_d(S) \\ & \cup & & \cup \\ \Psi_{\underline{\mathbf{f}},\underline{\mathbf{g}}} : & \mathrm{Flag}(Y)_{\underline{\mathbf{f}},\underline{\mathbf{g}}} & \longrightarrow & \mathrm{Flag}_{\underline{\mathbf{f}}}(X) \times \mathrm{Flag}_{\underline{\mathbf{g}}}(S) \end{array}$$

Theorem B

When η does not split and generates $\mathrm{Ext}^1(S, X)$,

$\Psi_{\underline{\mathbf{f}},\underline{\mathbf{g}}}$ is a Zarisky-locally trivial affine bundle over $\mathrm{Im}\,\Psi_{\underline{\mathbf{f}},\underline{\mathbf{g}}}$.

In this case, we have a clear description of $\mathrm{Im}\,\Psi_{\underline{\mathbf{f}},\underline{\mathbf{g}}}$.

Statement

233

How to find nice η?

Proposition

For $X \hookrightarrow Y$ minimal sectional mono, the induced SES

$$\eta : 0 \longrightarrow X \xrightarrow{\iota} Y \xrightarrow{\pi} S \longrightarrow 0$$

satisfies the condition of Theorem B. Moreover,

$$\operatorname{Im} \Psi_{\underline{\mathbf{f}},\underline{\mathbf{g}}} = \begin{cases} \big(\mathrm{Flag}_{\underline{\mathbf{f}}}(X) \smallsetminus \mathrm{Flag}_{\underline{\mathbf{f}}}(X_S)\big) \times \mathrm{Flag}_{\underline{\mathbf{g}}}(S), & \underline{\mathbf{g}}_i = \underline{\mathbf{dim}}\, S \\ \mathrm{Flag}_{\underline{\mathbf{f}}}(X) \times \mathrm{Flag}_{\underline{\mathbf{g}}}(S), & \textit{otherwise} \end{cases}$$

where

$$X_S := \max\left\{ M \subseteq X \;\middle|\; \mathrm{Ext}^1(S, X/M) \cong \mathbb{C} \right\} \subseteq X.$$

How to find nice η?

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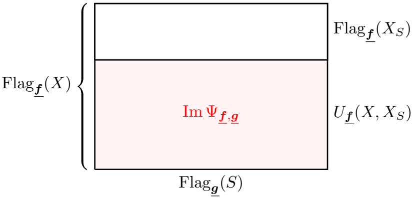

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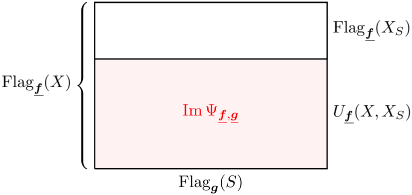

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Proposition

In addition, if $X \hookrightarrow Y$ is irreducible mono, then

$$X_S = 0 \quad \text{or} \quad X_S \hookrightarrow X \text{ is irreducible mono.}$$

Corollary

For $M \in \operatorname{ind}(Q)$, if exist irreducible mono $X \hookrightarrow M$, then

$$\operatorname{Flag}_d(M) \text{ has an affine paving.}$$

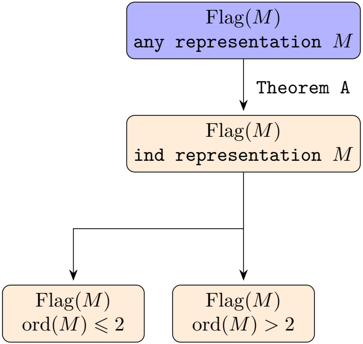
Process

$\mathrm{Flag}(M)$
`any representation` M

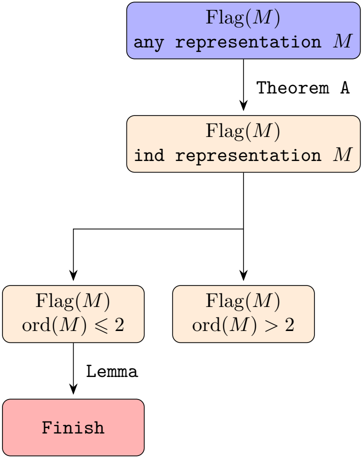
Process

$$\begin{tikzcd}
\boxed{\begin{array}{c}\mathrm{Flag}(M)\\ \texttt{any representation } M\end{array}} \arrow[d, "\texttt{Theorem A}"] \\
\boxed{\begin{array}{c}\mathrm{Flag}(M)\\ \texttt{ind representation } M\end{array}}
\end{tikzcd}$$

Process


Flag(M)
any representation M
Theorem A
Flag(M)
ind representation M
Flag(M)
ord(M) $\leqslant 2$
Flag(M)
ord(M) > 2

Process


Flag(M)
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Theorem A
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Flag(M)
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Flag(M)
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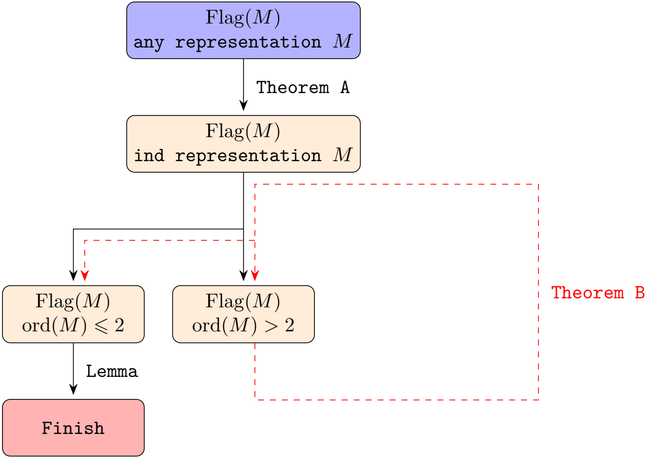
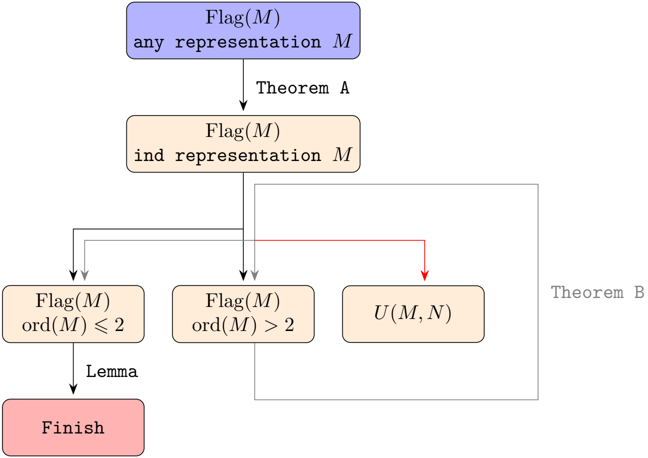
Lemma
Finish

Process



$\mathrm{Flag}(M)$
any representation M
Theorem A
$\mathrm{Flag}(M)$
ind representation M
Theorem B
$\mathrm{Flag}(M)$
$\mathrm{ord}(M) \leqslant 2$
$\mathrm{Flag}(M)$
$\mathrm{ord}(M) > 2$
Lemma
Finish

Process


Flag(M)
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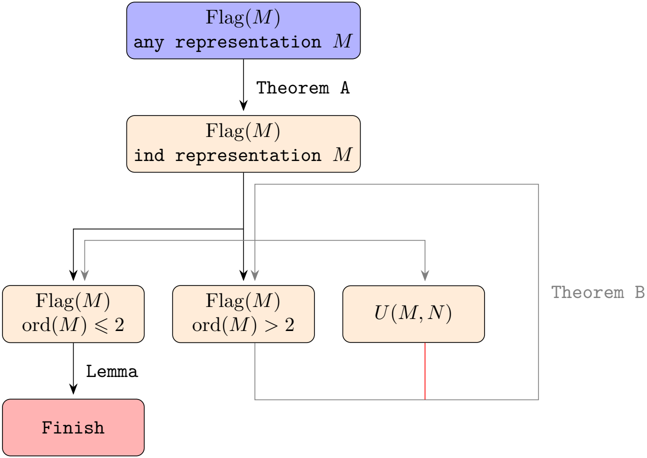
$U(M, N)$
Theorem B
Lemma
Finish

Process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Flag(M)
ord(M) > 2
$U(M, N)$
Theorem B
Lemma
Finish

What is remaining?

$$E_6 : \quad \begin{array}{ccccccccccc}
 & & 1 & & 1 & & 1 & & 1 & & \\
 & 1 & & 2 & & 2 & & 2 & & 1 & \\
1 & 1 & 2 & 1 & 3 & 2 & 3 & 1 & 2 & 1 & 1 \\
 & 1 & & 2 & & 2 & & 2 & & 1 & \\
 & & 1 & & 1 & & 1 & & 1 & &
\end{array}$$

$$E_7 : \quad \begin{array}{ccccccccccccccccc}
 & & 1 & & 1 & & 1 & & 2 & & 1 & & 1 & & 1 & & \\
 & 1 & & 2 & & 2 & & 3 & & 3 & & 2 & & 2 & & 1 & \\
1 & 1 & 2 & 1 & 3 & 2 & 4 & 2 & 4 & 2 & 4 & 2 & 3 & 1 & 2 & 1 & 1 \\
 & 1 & & 2 & & 3 & & 3 & & 3 & & 3 & & 2 & & 1 & \\
 & & 1 & & 2 & & 2 & & 2 & & 2 & & 2 & & 1 & & \\
 & & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & &
\end{array}$$

$$E_8 : \quad \begin{array}{ccccccccccccccccccccccccccccc}
 & & 1 & & 1 & & 1 & & 2 & & 2 & & 2 & & 2 & & 2 & & 2 & & 2 & & 1 & & 1 & & 1 & & \\
 & 1 & & 2 & & 2 & & 3 & & 4 & & 4 & & 4 & & 4 & & 4 & & 4 & & 3 & & 2 & & 2 & & 1 & \\
1 & 1 & 2 & 1 & 3 & 2 & 4 & 2 & 5 & 3 & 6 & 3 & 6 & 3 & 6 & 3 & 6 & 3 & 6 & 3 & 5 & 2 & 4 & 2 & 3 & 1 & 2 & 1 & 1 \\
 & 1 & & 2 & & 3 & & 4 & & 4 & & 5 & & 5 & & 5 & & 5 & & 4 & & 4 & & 3 & & 2 & & 1 & \\
 & & 1 & & 2 & & 3 & & 3 & & 3 & & 4 & & 4 & & 4 & & 3 & & 3 & & 3 & & 2 & & 1 & & \\
 & & & 1 & & 2 & & 2 & & 2 & & 2 & & 3 & & 3 & & 2 & & 2 & & 2 & & 2 & & 1 & & & \\
 & & & & 1 & & 1 & & 1 & & 1 & & 1 & & 2 & & 1 & & 1 & & 1 & & 1 & & 1 & & & &
\end{array}$$

What is remaining?

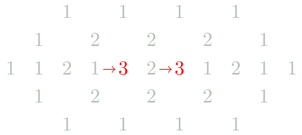
$$E_6 : \quad \begin{array}{c}
1 \quad 1 \quad 1 \quad 1 \\
1 \quad 2 \quad 2 \quad 2 \quad 1 \\
1 \;\; 1 \;\; 2 \;\; 1 \;\; 3 \;\; 2 \;\; 3 \;\; 1 \;\; 2 \;\; 1 \;\; 1 \\
1 \quad 2 \quad 2 \quad 2 \quad 1 \\
1 \quad 1 \quad 1 \quad 1
\end{array}$$

$$E_7 : \quad \begin{array}{c}
1 \quad 1 \quad 1 \quad 2 \quad 1 \quad 1 \quad 1 \\
1 \quad 2 \quad 2 \quad 3 \quad 3 \quad 2 \quad 2 \quad 1 \\
1 \;\; 1 \;\; 2 \;\; 1 \;\; 3 \;\; 2 \;\; 4 \;\; 2 \;\; 4 \;\; 2 \;\; 4 \;\; 2 \;\; 3 \;\; 1 \;\; 2 \;\; 1 \;\; 1 \\
1 \quad 2 \quad 3 \quad 3 \quad 3 \quad 3 \quad 2 \quad 1 \\
1 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 1 \\
1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1
\end{array}$$

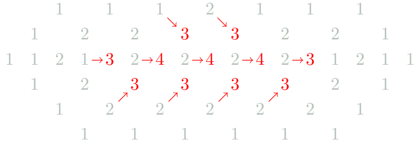
$$E_8 : \quad \begin{array}{c}
1 \quad 1 \quad 1 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 1 \quad 1 \quad 1 \\
1 \quad 2 \quad 2 \quad 3 \quad 4 \quad 4 \quad 4 \quad 4 \quad 4 \quad 4 \quad 3 \quad 2 \quad 2 \quad 1 \\
1 \;\; 1 \;\; 2 \;\; 1 \;\; 3 \;\; 2 \;\; 4 \;\; 2 \;\; 5 \;\; 3 \;\; 6 \;\; 3 \;\; 6 \;\; 3 \;\; 6 \;\; 3 \;\; 6 \;\; 3 \;\; 6 \;\; 3 \;\; 5 \;\; 2 \;\; 4 \;\; 2 \;\; 3 \;\; 1 \;\; 2 \;\; 1 \;\; 1 \\
1 \quad 2 \quad 3 \quad 4 \quad 4 \quad 5 \quad 5 \quad 5 \quad 5 \quad 4 \quad 4 \quad 3 \quad 2 \quad 1 \\
1 \quad 2 \quad 3 \quad 3 \quad 3 \quad 4 \quad 4 \quad 4 \quad 3 \quad 3 \quad 3 \quad 2 \quad 1 \\
1 \quad 2 \quad 2 \quad 2 \quad 2 \quad 3 \quad 3 \quad 2 \quad 2 \quad 2 \quad 2 \quad 1 \\
1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 2 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1
\end{array}$$

What is remaining?

E_6 :



E_7 :



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