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Affine pavings of partial flag varieties

Xiaoxiang Zhou

Advisor: Prof. Dr. Catharina Stroppel
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Universität Bonn

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Process

Process

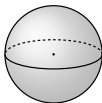# Affine paving

Setting

$K = \mathbb{C}$, X: *algebraic variety over* K.
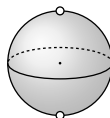
Definition

An **affine paving** of X is a filtration

$$\varnothing = X_0 \subset X_1 \subset \cdots \subset X_d = X$$

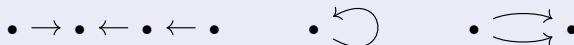with X_i closed and $X_{i+1} \setminus X_i \cong \mathbb{A}_K^k$.

$$\mathbb{P}^1 = \{\infty\} \sqcup \mathbb{A}^1$$

$\mathbb{P}^1 \setminus \{0, \infty\}$ has no affine paving

Quiver and quiver representation

Quiver is a graph. It has some vertices & arrows.
In this talk, all the quivers are finite and connected.

Quiver and quiver representation

$$\bullet \rightarrow \bullet \leftarrow \bullet \leftarrow \bullet \qquad \bullet \circlearrowleft \qquad \bullet \rightrightarrows \bullet$$

We focus on the Dynkin quiver.

That means, the graph of the Dynkin diagrams in the ADE series.

$$D_5 \qquad\qquad E_6 \qquad\qquad B_4$$

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Quiver representation

$$
\begin{array}{cc}
\bullet & V_5 \\
\downarrow & \downarrow{\scriptstyle \delta} \\
\bullet \longrightarrow \bullet \longrightarrow \bullet \longleftarrow \bullet \qquad & V_1 \xrightarrow{\alpha} V_2 \xrightarrow{\beta} V_3 \xleftarrow{\gamma} V_4 \\
Q & V \in \mathrm{rep}(Q)
\end{array}
$$

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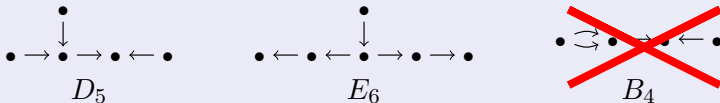
Quiver representation

$$
\begin{array}{ccc}
\bullet & \quad & 0 \\
\downarrow & \quad & \downarrow \\
\bullet \longrightarrow \bullet \longrightarrow \bullet \longleftarrow \bullet & \quad & \mathbb{C}^2 \xrightarrow{\mathrm{Id}} \mathbb{C}^2 \xrightarrow{\mathrm{Id}} \mathbb{C}^2 \xleftarrow{(1\,0)} \mathbb{C} \\
\\
Q & \quad & V \in \mathrm{rep}(Q)
\end{array}
$$

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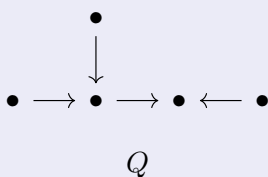
Quiver representation
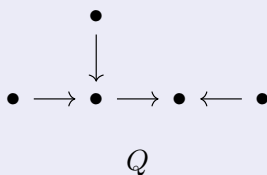
$$
\begin{array}{c}
\bullet \\
\downarrow \\
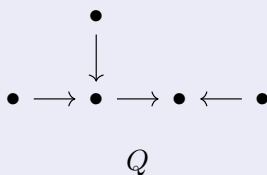\bullet \longrightarrow \bullet \longrightarrow \bullet \longleftarrow \bullet
\end{array}
\qquad\qquad
\begin{array}{c}
0 \\
\downarrow \\
\mathbb{C}^2 \xrightarrow{\mathrm{Id}} \mathbb{C}^2 \xrightarrow{\mathrm{Id}} \mathbb{C}^2 \xleftarrow{(10)} \mathbb{C}
\end{array}
$$

$$Q \qquad\qquad\qquad V \in \mathrm{rep}(Q)$$

Dimension vector: $\qquad \underline{\dim}\, V = {}^{\ 0}_{2221}$

Partial flag variety

Definition

Fix a quiver Q and $M \in \mathrm{rep}(Q)$,

$$\mathrm{Flag}_d(M) \colon = \{F \colon 0 \subseteq N_1 \subseteq \cdots \subseteq N_d \subseteq M\}$$

$$\mathrm{Flag}_{\underline{\mathbf{f}}}(M) \colon = \left\{F \colon 0 \subseteq N_1 \subseteq \cdots \subseteq N_d \subseteq M \mid \underline{\mathbf{dim}}\, N_i = \underline{\mathbf{f}}_i\right\}$$

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Partial flag variety

Definition

Fix a quiver Q and $M \in \operatorname{rep}(Q)$,

$$\operatorname{Flag}_d(M) \colon = \{F \colon 0 \subseteq N_1 \subseteq \cdots \subseteq N_d \subseteq M\}$$
$$\operatorname{Flag}_{\underline{\mathbf{f}}}(M) \colon = \left\{F \colon 0 \subseteq N_1 \subseteq \cdots \subseteq N_d \subseteq M \mid \underline{\dim} N_i = \underline{\mathbf{f}}_i\right\}$$

Example

$Q = \bullet$, $M = \mathbb{C}^n$,

$$\operatorname{Flag}_1(\mathbb{C}^n) = \{F \colon 0 \subseteq N_1 \subseteq \mathbb{C}^n\} \ = \bigsqcup_{k=0}^{n} \operatorname{Gr}(n,k)$$
$$\operatorname{Flag}_{(k)}(\mathbb{C}^n) = \operatorname{Gr}(n,k)$$

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Statement

Theorem [Cerulli-Irelli–Esposito–Franzen–Reineke, Zhou]

For a Dynkin quiver Q and $M \in \mathrm{rep}(Q)$,

$$\mathrm{Flag}_d(M) \text{ has an affine paving.}$$

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Task 1. $Q = \bullet$, $M = \mathbb{C}^n$

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In this case,

$$\mathrm{GL}_n(\mathbb{C}) \circlearrowright \mathbb{C}^n \quad \rightsquigarrow \quad \mathrm{GL}_n(\mathbb{C}) \circlearrowright \mathrm{Flag}_d(\mathbb{C}^n)$$

$$\rightsquigarrow \quad B \quad \circlearrowright \mathrm{Flag}_d(\mathbb{C}^n)$$

$\mathrm{Flag}_d(\mathbb{C}^n)$ has an affine paving given by Schubert cells (i.e., B-orbits).

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In this case,

$$\mathrm{GL}_n(\mathbb{C}) \circlearrowleft \mathbb{C}^n \quad \rightsquigarrow \quad \mathrm{GL}_n(\mathbb{C}) \circlearrowleft \mathrm{Flag}_d(\mathbb{C}^n)$$

$$\rightsquigarrow \quad B \quad \circlearrowleft \mathrm{Flag}_d(\mathbb{C}^n)$$

$\mathrm{Flag}_d(\mathbb{C}^n)$ has an affine paving given by Schubert cells (i.e., B-orbits).

Note

When $Q = \bullet \longrightarrow \bullet$, $\mathrm{Flag}_{\mathbf{f}}(M)$ have no natural group actions.

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Task 2a. $Q = \bullet \to \bullet$, $M = \left[\mathbb{C}^2 \xrightarrow{0} \mathbb{C}^2 \right]$, $d = 1$

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Task 2a. $Q = \bullet \rightarrow \bullet$, $M = \left[\mathbb{C}^2 \xrightarrow{0} \mathbb{C}^2 \right]$, $d = 1$

$$
\begin{aligned}
\underline{\mathbf{f}} = (1, 0): &\qquad \mathrm{Flag}_{\underline{\mathbf{f}}}(M) = \mathbb{P}^1 \\
\underline{\mathbf{f}} = (0, 0): &\qquad \mathrm{Flag}_{\underline{\mathbf{f}}}(M) = \{*\} \\
\underline{\mathbf{f}} = (1, 1): &\qquad \mathrm{Flag}_{\underline{\mathbf{f}}}(M) = \mathbb{P}^1 \times \mathbb{P}^1
\end{aligned}
$$

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Task 2a. $Q = \bullet \to \bullet$, $M = \left[\mathbb{C}^2 \xrightarrow{0} \mathbb{C}^2\right]$, $d = 1$

$$
\begin{aligned}
\underline{\mathbf{f}} = (1,0): &\qquad \mathrm{Flag}_{\underline{\mathbf{f}}}(M) = \mathbb{P}^1 \\
\underline{\mathbf{f}} = (0,0): &\qquad \mathrm{Flag}_{\underline{\mathbf{f}}}(M) = \{*\} \\
\underline{\mathbf{f}} = (1,1): &\qquad \mathrm{Flag}_{\underline{\mathbf{f}}}(M) = \mathbb{P}^1 \times \mathbb{P}^1
\end{aligned}
$$

In this case,

$$
\mathrm{Flag}_{(a,b)}(M) \cong \mathrm{Flag}_{(a)}(\mathbb{C}^2) \times \mathrm{Flag}_{(b)}(\mathbb{C}^2)
$$

has an affine paving.

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Task 2b. $Q = \bullet \rightarrow \bullet$, $M = \left[\mathbb{C}^2 \xrightarrow{\mathrm{Id}} \mathbb{C}^2 \right]$, $d = 1$

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Task 2b. $Q = \bullet \to \bullet$, $M = \left[\mathbb{C}^2 \overset{\mathrm{Id}}{\to} \mathbb{C}^2\right]$, $d = 1$

$$
\begin{aligned}
\underline{\mathbf{f}} = (1, 0): && \mathrm{Flag}_{\underline{\mathbf{f}}}(M) &= \varnothing \\
\underline{\mathbf{f}} = (0, 0): && \mathrm{Flag}_{\underline{\mathbf{f}}}(M) &= \{*\} \\
\underline{\mathbf{f}} = (1, 1): && \mathrm{Flag}_{\underline{\mathbf{f}}}(M) &= \mathbb{P}^1 \\
\underline{\mathbf{f}} = (0, 1): && \mathrm{Flag}_{\underline{\mathbf{f}}}(M) &= \mathbb{P}^1
\end{aligned}
$$

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Task 2b. $Q = \bullet \to \bullet$, $M = \left[\mathbb{C}^2 \xrightarrow{\mathrm{Id}} \mathbb{C}^2 \right]$, $d = 1$

$$\begin{aligned}
\underline{\mathbf{f}} = (1,0): & \qquad \mathrm{Flag}_{\underline{\mathbf{f}}}(M) = \varnothing \\
\underline{\mathbf{f}} = (0,0): & \qquad \mathrm{Flag}_{\underline{\mathbf{f}}}(M) = \{*\} \\
\underline{\mathbf{f}} = (1,1): & \qquad \mathrm{Flag}_{\underline{\mathbf{f}}}(M) = \mathbb{P}^1 \\
\underline{\mathbf{f}} = (0,1): & \qquad \mathrm{Flag}_{\underline{\mathbf{f}}}(M) = \mathbb{P}^1
\end{aligned}$$

In this case,

$$\mathrm{Flag}_{(a,b)}(M) \cong \mathrm{Flag}_{\binom{b}{a}}(\mathbb{C}^2)$$

has an affine paving.

Task 2c. $Q = \bullet \to \bullet$, $M = \left[\mathbb{C}^2 \xrightarrow{\binom{10}{00}} \mathbb{C}^2 \right]$, $d = 1$

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Task 2c. $Q = \bullet \to \bullet$, $M = \left[\mathbb{C}^2 \xrightarrow{\binom{1\,0}{0\,0}} \mathbb{C}^2 \right]$, $d = 1$

$$
\begin{aligned}
\underline{\mathbf{f}} = (1,0): && \mathrm{Flag}_{\underline{\mathbf{f}}}(M) &= \{*\} \\
\underline{\mathbf{f}} = (0,0): && \mathrm{Flag}_{\underline{\mathbf{f}}}(M) &= \{*\} \\
\underline{\mathbf{f}} = (1,1): && \mathrm{Flag}_{\underline{\mathbf{f}}}(M) &= \mathbb{P}^1 \vee \mathbb{P}^1 \\
\underline{\mathbf{f}} = (0,1): && \ldots
\end{aligned}
$$

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Task 2c. $Q = \bullet \to \bullet$, $M = \left[\mathbb{C}^2 \xrightarrow{\binom{10}{00}} \mathbb{C}^2 \right]$, $d = 1$

$$\begin{aligned}
\underline{\mathbf{f}} = (1,0): && \mathrm{Flag}_{\underline{\mathbf{f}}}(M) &= \{*\} \\
\underline{\mathbf{f}} = (0,0): && \mathrm{Flag}_{\underline{\mathbf{f}}}(M) &= \{*\} \\
\underline{\mathbf{f}} = (1,1): && \mathrm{Flag}_{\underline{\mathbf{f}}}(M) &= \mathbb{P}^1 \vee \mathbb{P}^1 \\
\underline{\mathbf{f}} = (0,1): && \ldots
\end{aligned}$$

To construct affine pavings systematically, we need to construct an uniform method.

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Task 2c. $Q = \bullet \rightarrow \bullet$, $M = \left[\mathbb{C}^2 \xrightarrow{\binom{1\,0}{0\,0}} \mathbb{C}^2 \right]$, $d = 1$

First try

Let $X = \left[0 \longrightarrow \mathbb{C} \right]$, $S = \left[\mathbb{C}^2 \xrightarrow{(1\,0)} \mathbb{C} \right]$, then $M = X \oplus S$,
and the short exact sequence

$$0 \longrightarrow X \xrightarrow{\ \iota\ } M \xrightarrow{\ \pi\ } S \longrightarrow 0$$

induces

$$\Psi : \mathrm{Flag}_d(M) \longrightarrow \mathrm{Flag}_d(X) \times \mathrm{Flag}_d(S)$$
$$F \longmapsto \left(\iota^{-1}(F), \pi(F) \right)$$

Idea of affine pavings

Find a nice short exact sequence

$$0 \longrightarrow X \stackrel{\iota}{\longrightarrow} M \stackrel{\pi}{\longrightarrow} S \longrightarrow 0$$

which induces a nice morphism

$$\Psi : \mathrm{Flag}_d(M) \longrightarrow \mathrm{Flag}_d(X) \times \mathrm{Flag}_d(S)$$
$$F \longmapsto \left(\iota^{-1}(F), \pi(F) \right)$$

We construct the affine paving of $\mathrm{Flag}_d(M)$ from the affine paving of $\mathrm{Flag}_d(X)$ and $\mathrm{Flag}_d(S)$. Then, we use mathematical induction.

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Example. $Q = \bullet$, $M = \mathbb{C}^2$

$$0 \longrightarrow \mathbb{C} \xrightarrow{\ \iota\ } \mathbb{C}^2 \xrightarrow{\ \pi\ } \mathbb{C} \longrightarrow 0$$

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Example. $Q = \bullet$, $M = \mathbb{C}^2$

$$0 \longrightarrow \mathbb{C} \overset{\iota}{\longrightarrow} \mathbb{C}^2 \overset{\pi}{\longrightarrow} \mathbb{C} \longrightarrow 0$$

$$\Psi_1 : \quad \mathrm{Flag}_1(\mathbb{C}^2) \longrightarrow \quad \mathrm{Flag}_1(\mathbb{C}) \times \mathrm{Flag}_1(\mathbb{C})$$

$$\Psi_{(1)} : \mathrm{Flag}_{(1)}(\mathbb{C}^2) \longrightarrow \mathrm{Flag}_{(1)}(\mathbb{C}) \times \mathrm{Flag}_{(0)}(\mathbb{C}) \bigsqcup \mathrm{Flag}_{(0)}(\mathbb{C}) \times \mathrm{Flag}_{(1)}(\mathbb{C})$$

$$\mathbb{P}^1 \qquad \longrightarrow \qquad \{*\} \qquad \bigsqcup \qquad \{*\}$$

Example. $Q = \bullet$, $M = \mathbb{C}^2$

$$0 \longrightarrow \mathbb{C} \overset{\iota}{\longrightarrow} \mathbb{C}^2 \overset{\pi}{\longrightarrow} \mathbb{C} \longrightarrow 0$$

$$\Psi_1 : \ \mathrm{Flag}_1(\mathbb{C}^2) \longrightarrow \ \mathrm{Flag}_1(\mathbb{C}) \times \mathrm{Flag}_1(\mathbb{C})$$

$$\Psi_{(1)} : \mathrm{Flag}_{(1)}(\mathbb{C}^2) \longrightarrow \mathrm{Flag}_{(1)}(\mathbb{C}) \times \mathrm{Flag}_{(0)}(\mathbb{C}) \bigsqcup \mathrm{Flag}_{(0)}(\mathbb{C}) \times \mathrm{Flag}_{(1)}(\mathbb{C})$$
$$\mathbb{P}^1 \quad \longrightarrow \quad \{*\} \qquad \bigsqcup \qquad \{*\}$$

$$\mathbb{P}^1 = \{\infty\} \sqcup \mathbb{C}$$

$$\Big\downarrow \Psi_{(1)}$$

$$\{*\} \sqcup \{*\}$$

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Example. $Q = \bullet$, $M = \mathbb{C}^8 = \bigoplus_{i=1}^{8} \mathbb{C}v_i$

$$0 \longrightarrow \mathbb{C}^3 \xrightarrow{\iota} \mathbb{C}^8 \xrightarrow{\pi} \mathbb{C}^5 \longrightarrow 0$$

$$\Psi : \mathrm{Flag}_{(3)}(\mathbb{C}^8) \longrightarrow \mathrm{Flag}_{(1)}(\mathbb{C}^3) \times \mathrm{Flag}_{(2)}(\mathbb{C}^5) \bigsqcup \cdots$$

$$\Psi^{-1}\Big(\langle v_1 \rangle, \langle v_4, v_5 \rangle\Big) = \Big\{ \langle v_1, v_4 + av_2 + bv_3, v_5 + cv_2 + dv_3 \rangle \,\Big|\, a, b, c, d \in \mathbb{C} \Big\}$$
$$\cong \mathbb{C}^4$$

Example. $Q = \bullet$, $M = \mathbb{C}^8 = \bigoplus_{i=1}^{8} \mathbb{C}v_i$

$$0 \longrightarrow \mathbb{C}^3 \stackrel{\iota}{\longrightarrow} \mathbb{C}^8 \stackrel{\pi}{\longrightarrow} \mathbb{C}^5 \longrightarrow 0$$

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$$\cong \ \mathbb{C}^4$$

Ψ is a Zarisky-locally trivial affine bundle of rank $2 \cdot (3-1) = 4$.

Consider the short exact sequence of representations

$$\eta : 0 \longrightarrow X \xrightarrow{\iota} Y \xrightarrow{\pi} S \longrightarrow 0$$

which induce maps

$$\Psi : \quad \mathrm{Flag}_d(Y) \longrightarrow \mathrm{Flag}_d(X) \times \mathrm{Flag}_d(S)$$
$$\underset{\cup}{} \qquad\qquad\qquad\qquad \underset{\cup}{}$$
$$\Psi_{\underline{\mathbf{f}},\underline{\mathbf{g}}} : \quad \mathrm{Flag}(Y)_{\underline{\mathbf{f}},\underline{\mathbf{g}}} \longrightarrow \mathrm{Flag}_{\underline{\mathbf{f}}}(X) \times \mathrm{Flag}_{\underline{\mathbf{g}}}(S)$$

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Theorem A [Cerulli-Irelli–Esposito–Franzen–Reineke, Zhou]

When η splits, then Ψ is surjective. ▸ skip
Moreover, if $\mathrm{Ext}^1(S, X) = 0$, then

$$\Psi_{\underline{\mathbf{f}},\underline{\mathbf{g}}} \text{ is a Zarisky-locally trivial affine bundle.}$$

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Consider the short exact sequence of representations

$$\eta : 0 \longrightarrow X \xrightarrow{\iota} Y \xrightarrow{\pi} S \longrightarrow 0$$

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$$\cup \qquad\qquad\qquad \cup$$
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Theorem A [Cerulli-Irelli–Esposito–Franzen–Reineke, Zhou]

When η splits, then Ψ is surjective. ▸ skip
Moreover, if $\mathrm{Ext}^1(S, X) = 0$, then

$$\Psi_{\underline{\mathbf{f}},\underline{\mathbf{g}}} \text{ is a Zarisky-locally trivial affine bundle.}$$

By this theorem,

$\mathrm{Flag}_d(Y)$ has an affine paving $\Longleftarrow \mathrm{Flag}_d(X), \mathrm{Flag}_d(S)$ have.

Warning

η splits and $\mathrm{Ext}^1(S, X) = 0$ are necessary for Theorem A.

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Example

Consider the quiver $Q : \bullet \to \bullet \leftarrow \bullet$ and the short exact sequence

$$0 \longrightarrow \left[\mathbb{C}e_1 \to \mathbb{C}^2 \leftarrow \mathbb{C}e_2\right] \longrightarrow \left[\mathbb{C}^2 \overset{\mathrm{Id}}{\to} \mathbb{C}^2 \overset{\mathrm{Id}}{\leftarrow} \mathbb{C}^2\right] \longrightarrow \left[\mathbb{C}e_2 \to 0 \leftarrow \mathbb{C}e_1\right] \longrightarrow 0$$

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Warning

η splits and $\mathrm{Ext}^1(S, X) = 0$ are necessary for Theorem A.

Example

Consider the quiver $Q : \bullet \to \bullet \leftarrow \bullet$ and the short exact sequence

$$0 \longrightarrow \left[\mathbb{C}e_1 \to \mathbb{C}^2 \leftarrow \mathbb{C}e_2\right] \longrightarrow \left[\mathbb{C}^2 \overset{\mathrm{Id}}{\to} \mathbb{C}^2 \overset{\mathrm{Id}}{\leftarrow} \mathbb{C}^2\right] \longrightarrow \left[\mathbb{C}e_2 \to 0 \leftarrow \mathbb{C}e_1\right] \longrightarrow 0$$

we get

$$\mathrm{Im}\, \Psi_{(0,1,0),(1,0,1)} \cong \left(\mathbb{P}^1 \smallsetminus \{0, \infty\}\right) \times \{*\} \cong \mathbb{C}^*,$$

so Ψ is not surjective.

Warning

η splits and $\operatorname{Ext}^1(S, X) = 0$ are necessary for Theorem A.

Example

Consider the quiver $Q : \bullet \to \bullet \leftarrow \bullet$ and the short exact sequence

$$0 \longrightarrow \left[\mathbb{C}e_1 \to \mathbb{C}^2 \leftarrow \mathbb{C}e_2\right] \longrightarrow \left[\mathbb{C}^2 \overset{\mathrm{Id}}{\to} \mathbb{C}^2 \overset{\mathrm{Id}}{\leftarrow} \mathbb{C}^2\right] \longrightarrow \left[\mathbb{C}e_2 \to 0 \leftarrow \mathbb{C}e_1\right] \longrightarrow 0$$

we get

$$\operatorname{Im} \Psi_{(0,1,0),(1,0,1)} \cong \left(\mathbb{P}^1 \smallsetminus \{0, \infty\}\right) \times \{*\} \cong \mathbb{C}^*,$$

so Ψ is not surjective.

In this way, we get a bad stratification

$$\operatorname{Flag}_{(1,1,1)}(Y) \cong \mathbb{P}^1 = \{0\} \sqcup \{\infty\} \sqcup \mathbb{C}^*.$$

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Task 3. $Q = \begin{smallmatrix} & & \bullet & & \\ & & \downarrow & & \\ \bullet & \to & \bullet & \leftarrow & \bullet \end{smallmatrix}$, $M = {}_{121}^{\ 1} \oplus {}_{111}^{\ 1} \oplus {}_{111}^{\ 1}$

We use following short exact sequences

$$0 \longrightarrow {}_{111}^{\ 1} \oplus {}_{111}^{\ 1} \longrightarrow M \longrightarrow {}_{121}^{\ 1} \longrightarrow 0$$

$$0 \longrightarrow {}_{111}^{\ 1} \longrightarrow {}_{111}^{\ 1} \oplus {}_{111}^{\ 1} \longrightarrow {}_{111}^{\ 1} \longrightarrow 0$$

to reduced the problem to indecomposable representations.

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Task 3. $Q = \begin{smallmatrix} & & \bullet & & \\ & & \downarrow & & \\ \bullet & \to & \bullet & \leftarrow & \bullet \end{smallmatrix}$, $M = {}_{1}1{}_{2}1 \oplus {}_{1}1{}_{1}1 \oplus {}_{1}1{}_{1}1$

We use following short exact sequences

$$0 \longrightarrow {}_{1}1{}_{1}1 \oplus {}_{1}1{}_{1}1 \longrightarrow M \longrightarrow {}_{1}1{}_{2}1 \longrightarrow 0$$

$$0 \longrightarrow {}_{1}1{}_{1}1 \longrightarrow {}_{1}1{}_{1}1 \oplus {}_{1}1{}_{1}1 \longrightarrow {}_{1}1{}_{1}1 \longrightarrow 0$$

to reduced the problem to indecomposable representations.

_____

Notice that we use the result

$$\operatorname{Ext}^1\left({}_{1}1{}_{2}1, {}_{1}1{}_{1}1\right) = 0, \qquad \operatorname{Ext}^1\left({}_{1}1{}_{1}1, {}_{1}1{}_{1}1\right) = 0.$$

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Task 3. $Q = \begin{smallmatrix} & & \cdot \\ & & \downarrow \\ \cdot & \to & \cdot & \leftarrow & \cdot \end{smallmatrix}$, $M = {}_{1}^{1}2{}_{1} \oplus {}_{1}^{1}1{}_{1} \oplus {}_{1}^{1}1{}_{1}$

We use following short exact sequences

$$0 \longrightarrow {}_{1}^{1}1{}_{1} \oplus {}_{1}^{1}1{}_{1} \longrightarrow M \longrightarrow {}_{1}^{1}2{}_{1} \longrightarrow 0$$

$$0 \longrightarrow {}_{1}^{1}1{}_{1} \longrightarrow {}_{1}^{1}1{}_{1} \oplus {}_{1}^{1}1{}_{1} \longrightarrow {}_{1}^{1}1{}_{1} \longrightarrow 0$$

to reduced the problem to indecomposable representations.

Notice that we use the result

$$\mathrm{Ext}^1\big({}_{1}^{1}2{}_{1}, {}_{1}^{1}1{}_{1}\big) = 0, \qquad \mathrm{Ext}^1\big({}_{1}^{1}1{}_{1}, {}_{1}^{1}1{}_{1}\big) = 0.$$

We can't put ${}_{1}^{1}2{}_{1}$ on the left, since

$$\mathrm{Ext}^1\big({}_{1}^{1}1{}_{1}, {}_{1}^{1}2{}_{1}\big) \cong \mathbb{C} \neq 0.$$

Task 3. $Q = \begin{smallmatrix} & & \bullet \\ & & \downarrow \\ \bullet & \to & \bullet & \leftarrow & \bullet \end{smallmatrix}$, $M = {}_{1}^{1}2_{1} \oplus {}_{1}^{1}1_{1} \oplus {}_{1}^{1}1_{1}$

We use following short exact sequences

$$0 \longrightarrow {}_{1}^{1}1_{1} \oplus {}_{1}^{1}1_{1} \longrightarrow M \longrightarrow {}_{1}^{1}2_{1} \longrightarrow 0$$

$$0 \longrightarrow {}_{1}^{1}1_{1} \longrightarrow {}_{1}^{1}1_{1} \oplus {}_{1}^{1}1_{1} \longrightarrow {}_{1}^{1}1_{1} \longrightarrow 0$$

to reduced the problem to indecomposable representations.

$\mathrm{Flag}_d \left({}_{1}^{1}1_{1} \right)$ has an affine paving: obvious.
$\mathrm{Flag}_d \left({}_{1}^{1}2_{1} \right)$ has an affine paving: it is $\mathbb{P}^1$, $\{*\}$ or empty.

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Task 3. $Q = \underset{\bullet \ \to \ \bullet \ \leftarrow \ \bullet}{\overset{\bullet}{\downarrow}}$, $M = {}_1^1{}_2{}_1^1 \oplus {}_1^1{}_1{}_1^1 \oplus {}_1^1{}_1{}_1^1$

We use following short exact sequences

$$0 \longrightarrow {}_1^1{}_1{}_1^1 \oplus {}_1^1{}_1{}_1^1 \longrightarrow M \longrightarrow {}_1^1{}_2{}_1^1 \longrightarrow 0$$

$$0 \longrightarrow {}_1^1{}_1{}_1^1 \longrightarrow {}_1^1{}_1{}_1^1 \oplus {}_1^1{}_1{}_1^1 \longrightarrow {}_1^1{}_1{}_1^1 \longrightarrow 0$$

to reduced the problem to indecomposable representations.

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$\mathrm{Flag}_d\left({}_1^1{}_1{}_1^1\right)$ has an affine paving: obvious.
$\mathrm{Flag}_d\left({}_1^1{}_2{}_1^1\right)$ has an affine paving: it is $\mathbb{P}^1$, $\{*\}$ or empty.

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Need: more informations of indecomposable representations!

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Process

Auslander–Reiten quiver: D_4

$$\begin{matrix} & 4 & \\ & \downarrow & \\ 2 \to & 1 & \leftarrow 3 \end{matrix}$$

Vertices	$\Longleftrightarrow$	Indecomposable representations
Arrows	$\Longleftrightarrow$	Irreducible morphisms
Paths	$\Longleftrightarrow$	Morphisms
Shift cards	$\Longleftrightarrow$	Switch arrows in Q
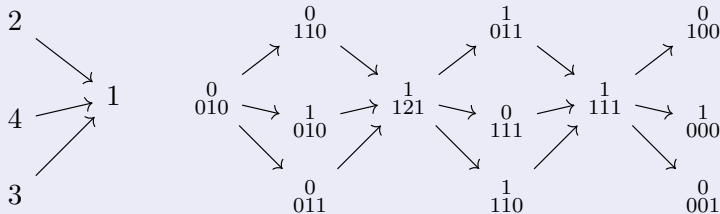
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Auslander–Reiten quiver: D_4

$$
\begin{array}{c}
4 \\
\downarrow \\
2 \to 1 \leftarrow 3
\end{array}
$$

Vertices $\iff$ Indecomposable representations

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Auslander–Reiten quiver: D_4

$$4$$
$$\downarrow$$
$$2 \to 1 \leftarrow 3$$

Vertices $\iff$ Indecomposable representations

irreducible rep, projective rep, injective rep

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Auslander–Reiten quiver: D_4

$$\begin{array}{c} 4 \\ \downarrow \\ 2 \rightarrow 1 \leftarrow 3 \end{array}$$

$$\text{Vertices} \quad \Longleftrightarrow \quad \text{Indecomposable representations}$$

irreducible rep,   projective rep,   injective rep

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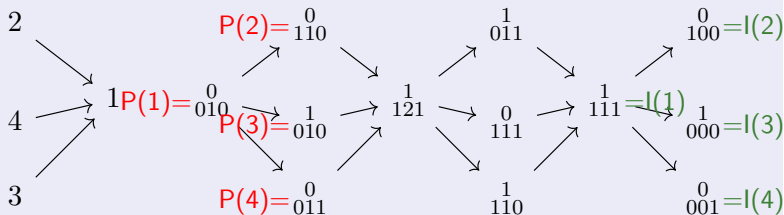
Auslander–Reiten quiver: D_4

$$\begin{array}{c} 4 \\ \downarrow \\ 2 \rightarrow 1 \leftarrow 3 \end{array}$$

Arrows $\iff$ Irreducible morphisms

I will show you how to construct AR-quiver.

Auslander–Reiten quiver: D_4
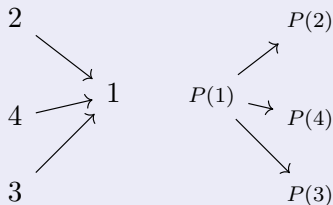
$$
\begin{array}{c}
4 \\
\downarrow \\
2 \rightarrow 1 \leftarrow 3
\end{array}
$$

Construction:

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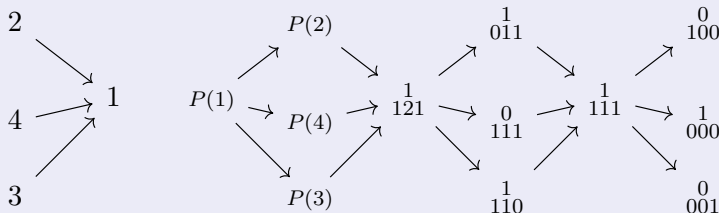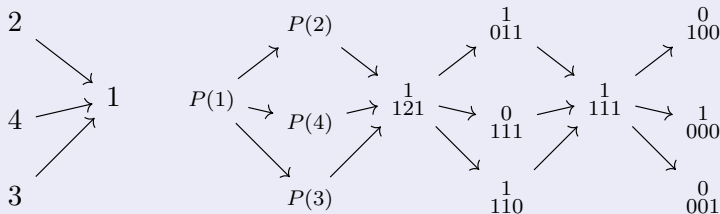000000000000

Auslander–Reiten quiver: D_4

$$
\begin{array}{c}
4 \\
\downarrow \\
2 \rightarrow 1 \leftarrow 3
\end{array}
$$
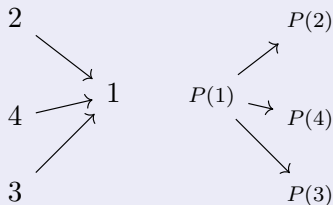
Construction:

$$0 \longrightarrow P(1) \longrightarrow P(2) \oplus P(4) \oplus P(3) \longrightarrow \begin{smallmatrix} 1 \\ 121 \end{smallmatrix} \longrightarrow 0$$

Auslander–Reiten quiver: D_4

$$\begin{array}{c} 4 \\ \downarrow \\ 2 \rightarrow 1 \leftarrow 3 \end{array}$$
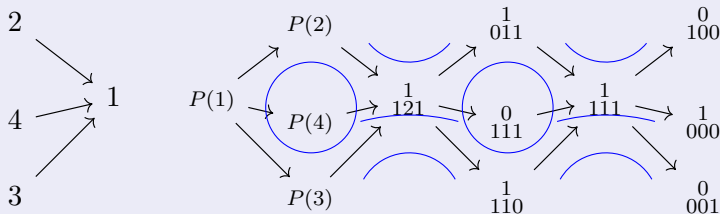

Construction:

$$0 \longrightarrow P(1) \longrightarrow P(2) \oplus P(4) \oplus P(3) \longrightarrow \begin{smallmatrix} 1 \\ 121 \end{smallmatrix} \longrightarrow 0$$

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Auslander–Reiten quiver: D_4

$$\begin{array}{c} 4 \\ \downarrow \\ 2 \to 1 \leftarrow 3 \end{array}$$

Construction:

$$0 \longrightarrow P(2) \longrightarrow {\textstyle\frac{1}{121}} \longrightarrow {\textstyle\frac{1}{011}} \longrightarrow 0$$

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Auslander–Reiten quiver: D_4

$$\begin{array}{c} 4 \\ \downarrow \\ 2 \rightarrow 1 \leftarrow 3 \end{array}$$

Construction:

$$0 \longrightarrow P(2) \longrightarrow {}^{1}_{121} \longrightarrow {}^{1}_{011} \longrightarrow 0$$

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Auslander–Reiten quiver: D_4

$$\begin{array}{c} 4 \\ \downarrow \\ 2 \to 1 \leftarrow 3 \end{array}$$

Construction:

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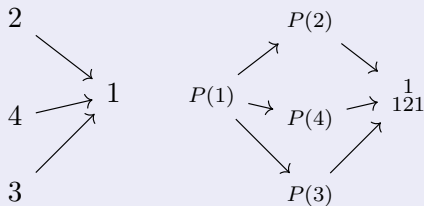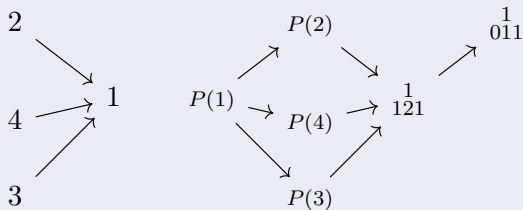
Auslander–Reiten quiver: D_4

$$\begin{array}{c} 4 \\ \downarrow \\ 2 \rightarrow 1 \leftarrow 3 \end{array}$$

Construction:

AR-quiver, AR-sequence, AR-translation

Auslander–Reiten quiver: D_4

$$\begin{array}{c} 4 \\ \downarrow \\ 2 \to 1 \leftarrow 3 \end{array}$$

Construction:

AR-quiver,   AR-sequence,   AR-translation

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Auslander–Reiten quiver: D_4

$$\begin{matrix} & & 4 & & \\ & & \downarrow & & \\ 2 & \to & 1 & \leftarrow & 3 \end{matrix}$$

Construction:

AR-quiver,   AR-sequence,   AR-translation

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Auslander–Reiten quiver: D_4

$$\begin{matrix} & 4 & \\ & \downarrow & \\ 2 \to & 1 & \leftarrow 3 \end{matrix}$$

Paths $\iff$ Morphisms

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Auslander–Reiten quiver: D_4

$$\begin{matrix} & 4 & \\ & \downarrow & \\ 2 \to & 1 & \leftarrow 3 \end{matrix}$$
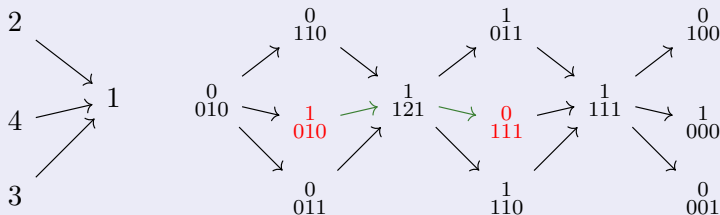

In the Dynkin quiver case,

$$\mathrm{Hom}(T, T') \cong \big\langle \text{ paths from } T \text{ to } T' \big\rangle / \text{AR-seq}$$

For example,

$$\mathrm{Hom}(\tfrac{1}{010}, \tfrac{1}{011}) \cong \mathbb{C}, \quad \mathrm{Hom}(\tfrac{1}{010}, \tfrac{0}{111}) \cong 0, \quad \mathrm{Hom}(\tfrac{1}{010}, \tfrac{1}{111}) \cong \mathbb{C}$$

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Auslander–Reiten quiver: D_4

$$\begin{array}{c} 4 \\ \downarrow \\ 2 \to 1 \leftarrow 3 \end{array}$$

In the Dynkin quiver case,

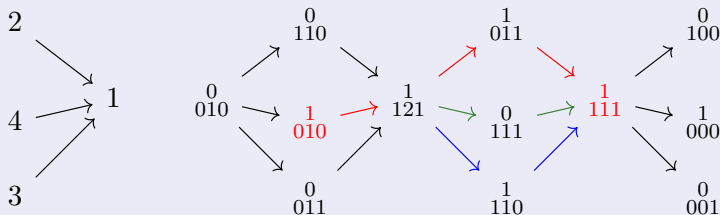$$\operatorname{Hom}(T, T') \cong \big\langle \text{ paths from } T \text{ to } T' \big\rangle /_{\text{AR-seq}}$$

For example,

$$\operatorname{Hom}(^1_{010}, ^1_{011}) \cong \mathbb{C}, \quad \operatorname{Hom}(^1_{010}, ^0_{111}) \cong 0, \quad \operatorname{Hom}(^1_{010}, ^1_{111}) \cong \mathbb{C}$$

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Auslander–Reiten quiver: D_4

$$\begin{array}{c} 4 \\ \downarrow \\ 2 \to 1 \leftarrow 3 \end{array}$$

In the Dynkin quiver case,

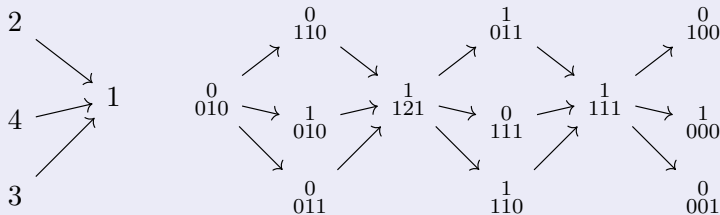$$\mathrm{Hom}(T, T') \cong \big\langle \text{ paths from } T \text{ to } T' \big\rangle /_{\mathsf{AR\text{-}seq}}$$

For example,

$$\mathrm{Hom}(^1_{010}, \, ^1_{011}) \cong \mathbb{C}, \quad \mathrm{Hom}(^1_{010}, \, ^0_{111}) \cong 0, \quad \mathrm{Hom}(^1_{010}, \, ^1_{111}) \cong \mathbb{C}$$

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Auslander–Reiten quiver: D_4

$$\begin{array}{c} 4 \\ \downarrow \\ 2 \to 1 \leftarrow 3 \end{array}$$

In the Dynkin quiver case,

$$\operatorname{Hom}(T, T') \cong \big\langle \text{ paths from } T \text{ to } T' \big\rangle /_{\textsf{AR-seq}}$$

For example,

$$\operatorname{Hom}(\tfrac{1}{010}, \tfrac{1}{011}) \cong \mathbb{C}, \quad \operatorname{Hom}(\tfrac{1}{010}, \tfrac{0}{111}) \cong 0, \quad \operatorname{Hom}(\tfrac{1}{010}, \tfrac{1}{111}) \cong \mathbb{C}$$

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Auslander–Reiten quiver: D_4

$$\begin{array}{c} 4 \\ \downarrow \\ 2 \to 1 \leftarrow 3 \end{array}$$
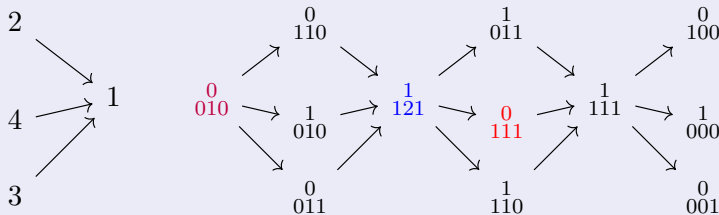

$$\mathrm{Ext}^1(T, T') \cong \overline{\mathrm{Hom}}(T', \tau T)^\vee$$

For example, $\quad \mathrm{Ext}^1\left(\begin{smallmatrix}1\\121\end{smallmatrix}, \begin{smallmatrix}0\\111\end{smallmatrix}\right) \cong \mathrm{Hom}\left(\begin{smallmatrix}0\\111\end{smallmatrix}, \begin{smallmatrix}0\\010\end{smallmatrix}\right)^\vee \cong 0$

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Auslander–Reiten quiver: D_4

$$\begin{array}{c} 4 \\ \downarrow \\ 2 \to 1 \leftarrow 3 \end{array}$$

$$\mathrm{Ext}^1(T, T') \cong \overline{\mathrm{Hom}}(T', \tau T)^\vee$$

For example, $\quad \mathrm{Ext}^1\left(\begin{smallmatrix} 1 \\ 121 \end{smallmatrix}, \begin{smallmatrix} 0 \\ 111 \end{smallmatrix}\right) \cong \mathrm{Hom}\left(\begin{smallmatrix} 0 \\ 111 \end{smallmatrix}, \begin{smallmatrix} 0 \\ 010 \end{smallmatrix}\right)^\vee \cong 0$

First application

Corollary

For a Dynkin quiver, we can give an total order to the set of all indecomposable representations, such that
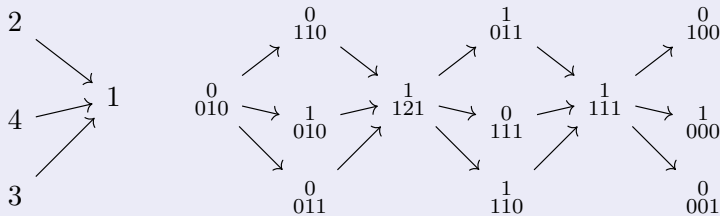
$$i \leqslant j \implies \mathrm{Ext}^1(M_i, M_j) = 0.$$

First application

Corollary

For a Dynkin quiver, we can give an total order to the set of all indecomposable representations, such that

$$i \leqslant j \quad \implies \quad \mathrm{Ext}^1(M_i, M_j) = 0.$$

By Theorem A, the problem reduced to

For a Dynkin quiver Q and $M \in \mathrm{ind}(Q)$,

$$\mathrm{Flag}_d(M) \text{ has an affine paving.}$$

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Auslander–Reiten quiver: D_4

$$\begin{array}{c} 4 \\ \downarrow \\ 2 \to 1 \leftarrow 3 \end{array}$$

Shift cards $\iff$ Switch arrows in Q
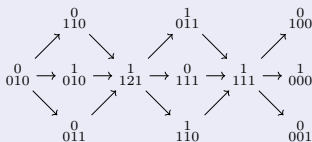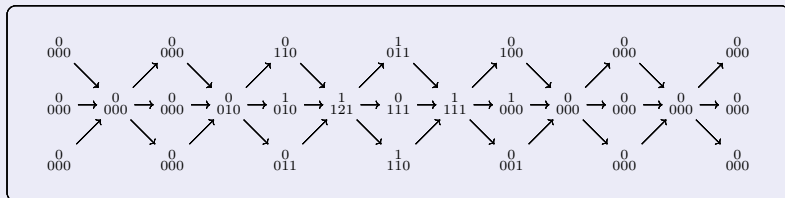
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Auslander–Reiten quiver: D_4

$$\begin{array}{c} 4 \\ \downarrow \\ 2 \to 1 \leftarrow 3 \end{array}$$

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Auslander–Reiten quiver: D_4

$$\begin{matrix} & 4 & \\ & \downarrow & \\ 2 \rightarrow & 1 & \leftarrow 3 \end{matrix}$$

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Auslander–Reiten quiver: D_4

$$\begin{array}{c} 4 \\ \downarrow \\ 2 \to 1 \leftarrow 3 \end{array}$$

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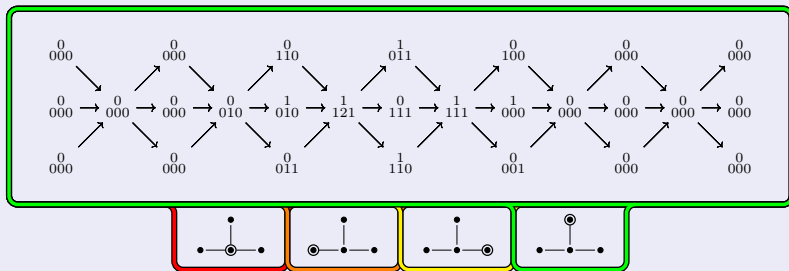
Auslander–Reiten quiver: D_4

$$\begin{array}{c} 4 \\ \downarrow \\ 2 \to 1 \leftarrow 3 \end{array}$$
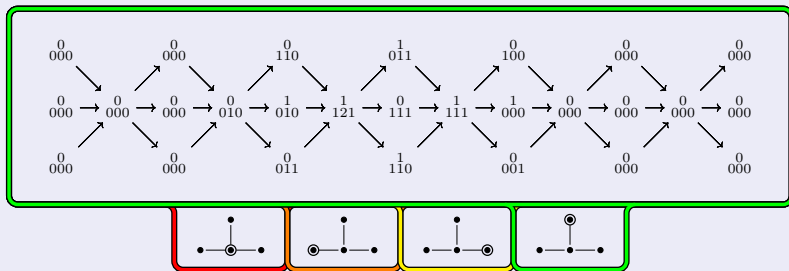
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Indecomposable representations of low order are easy!

lemma [Maksimau]

Suppose Q is a tree. For $M \in \mathrm{ind}(Q)$, $\mathrm{ord}(M) \leqslant 2$,

$$\mathrm{Flag}_{\underline{\mathbf{f}}}(M) \ \cong \ \ \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \quad \text{or} \quad \varnothing.$$

Example

$$Q = \ \begin{matrix} & & \bullet & & \\ & & \downarrow & & \\ \bullet \to \bullet \leftarrow \bullet \to \bullet \end{matrix} \quad M = \ \begin{matrix} & & \mathbb{C} & & \\ & & \downarrow & & \\ \mathbb{C} \hookrightarrow \mathbb{C}^2 \xleftarrow{\sim} \mathbb{C}^2 \twoheadrightarrow \mathbb{C} \end{matrix} \quad \underline{\mathbf{f}} = \begin{pmatrix} 0 \\ 1\,2\,1\,1 \\ 0 \\ 1\,1\,0\,1 \end{pmatrix}$$

$$\begin{aligned}
\mathrm{Flag}_{\underline{\mathbf{f}}}(M) &\hookrightarrow \mathrm{Flag}_{\binom{1}{1}}(\mathbb{C}) \times \mathrm{Flag}_{\binom{2}{1}}(\mathbb{C}^2) \times \mathrm{Flag}_{\binom{1}{0}}(\mathbb{C}^2) \\
&\quad \times \mathrm{Flag}_{\binom{1}{1}}(\mathbb{C}) \times \mathrm{Flag}_{\binom{0}{0}}(\mathbb{C}) \\
&\cong \mathbb{P}^1 \times \mathbb{P}^1
\end{aligned}$$

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Continue

	$\mathbb{C}^2 \to \mathbb{C}^2$		$\mathbb{C} \hookrightarrow \mathbb{C}^2$		$\mathbb{C}^2 \twoheadrightarrow \mathbb{C}$	
No restriction	–	2	0	–	–	1
	0	–	1	2	0	0
Reduce	1	1	1	1	1	0
Impossible	2	1	1	0	2	0
	2	0				
	1	0				

Corollary [Maksimau]

The main theorem is true for quivers of type A, D.

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Process

> Flag(M)
> any representation M

Process

Flag(M)
any representation M

Theorem A

Flag(M)
ind representation M
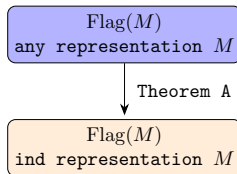
Process
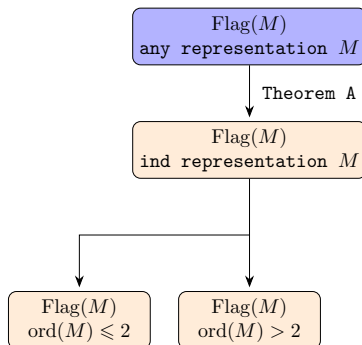
Process
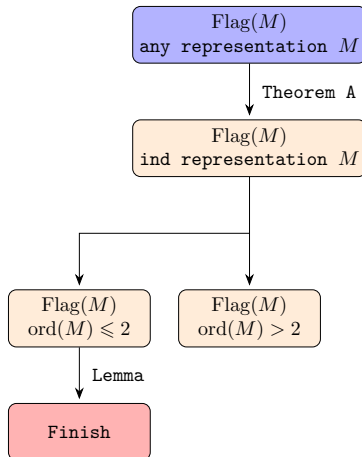
Process

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2. Case study

3. Auslander–Reiten theory

4. Tackle the type E case

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What is remaining?

$$
E_6 : \quad
\begin{array}{ccccccccccc}
 & & 1 & & 1 & & 1 & & 1 & & \\
 & 1 & & 2 & & 2 & & 2 & & 1 & \\
1 & 1 & 2 & 1 & 3 & 2 & 3 & 1 & 2 & 1 & 1 \\
 & 1 & & 2 & & 2 & & 2 & & 1 & \\
 & 1 & & 1 & & 1 & & 1 & & &
\end{array}
$$

$$
E_7 : \quad
\begin{array}{ccccccccccccccccc}
 & & 1 & & 1 & & 1 & & 2 & & 1 & & 1 & & 1 & & \\
 & 1 & & 2 & & 2 & & 3 & & 3 & & 2 & & 2 & & 1 & \\
1 & 1 & 2 & 1 & 3 & 2 & 4 & 2 & 4 & 2 & 4 & 2 & 3 & 1 & 2 & 1 & 1 \\
 & 1 & & 2 & & 3 & & 3 & & 3 & & 3 & & 2 & & 1 & \\
 & 1 & & 2 & & 2 & & 2 & & 2 & & 2 & & 1 & & & \\
 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & & & &
\end{array}
$$

$$
E_8 : \quad
\begin{array}{ccccccccccccccccccccccccccccc}
 & & 1 & & 1 & & 1 & & 2 & & 2 & & 2 & & 2 & & 2 & & 2 & & 2 & & 1 & & 1 & & 1 & & \\
 & 1 & & 2 & & 2 & & 3 & & 4 & & 4 & & 4 & & 4 & & 4 & & 4 & & 3 & & 2 & & 2 & & 1 & \\
1 & 1 & 2 & 1 & 3 & 2 & 4 & 2 & 5 & 3 & 6 & 3 & 6 & 3 & 6 & 3 & 6 & 3 & 6 & 3 & 5 & 2 & 4 & 2 & 3 & 1 & 2 & 1 & 1 \\
 & 1 & & 2 & & 3 & & 4 & & 4 & & 5 & & 5 & & 5 & & 5 & & 4 & & 4 & & 3 & & 2 & & 1 & \\
 & 1 & & 2 & & 3 & & 3 & & 3 & & 4 & & 4 & & 4 & & 3 & & 3 & & 3 & & 2 & & 1 & & \\
 & 1 & & 2 & & 2 & & 2 & & 2 & & 3 & & 3 & & 2 & & 2 & & 2 & & 2 & & 1 & & & \\
 & 1 & & 1 & & 1 & & 1 & & 1 & & 2 & & 1 & & 1 & & 1 & & 1 & & 1 & & & & &
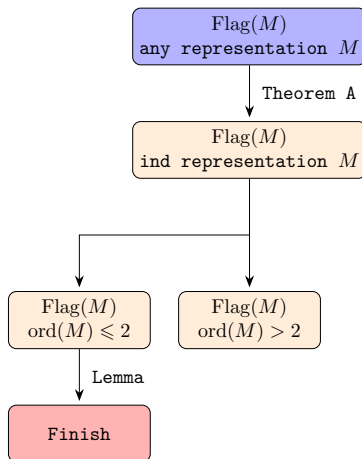\end{array}
$$

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What is remaining?

E_6 :

$$
\begin{array}{ccccccccccc}
 & 1 & & 1 & & 1 & & 1 & & & \\
 & 1 & & 2 & & 2 & & 2 & & 1 & \\
1 & 1 & 2 & 1 & \textcolor{red}{3} & 2 & \textcolor{red}{3} & 1 & 2 & 1 & 1 \\
 & 1 & & 2 & & 2 & & 2 & & 1 & \\
 & 1 & & 1 & & 1 & & 1 & & &
\end{array}
$$

E_7 :

$$
\begin{array}{ccccccccccccccc}
 & 1 & & 1 & & 1 & & 2 & & 1 & & 1 & & 1 & \\
 & 1 & & 2 & & 2 & & \textcolor{red}{3} & & \textcolor{red}{3} & & 2 & & 2 & & 1 \\
1 & 1 & 2 & 1 & \textcolor{red}{3} & 2 & 4 & 2 & 4 & 2 & 4 & 2 & \textcolor{red}{3} & 1 & 2 & 1 & 1 \\
 & 1 & & 2 & & \textcolor{red}{3} & & \textcolor{red}{3} & & \textcolor{red}{3} & & \textcolor{red}{3} & & 2 & & 1 \\
 & 1 & & 2 & & 2 & & 2 & & 2 & & 2 & & 2 & & 1 \\
 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & &
\end{array}
$$

E_8 :

$$
\begin{array}{ccccccccccccccccccccc}
 & 1 & & 1 & & 1 & & 2 & & 2 & & 2 & & 2 & & 2 & & 2 & & 2 & & 1 & & 1 & & 1 \\
 & 1 & & 2 & & 2 & & \textcolor{red}{3} & & 4 & & 4 & & 4 & & 4 & & 4 & & 4 & & \textcolor{red}{3} & & 2 & & 2 & & 1 \\
1 & 1 & 2 & 1 & \textcolor{red}{3} & 2 & 4 & 2 & 5 & \textcolor{red}{3} & 6 & \textcolor{red}{3} & 6 & \textcolor{red}{3} & 6 & \textcolor{red}{3} & 6 & \textcolor{red}{3} & 6 & \textcolor{red}{3} & 5 & 2 & 4 & 2 & \textcolor{red}{3} & 1 & 2 & 1 & 1 \\
 & 1 & & 2 & & \textcolor{red}{3} & & 4 & & 4 & & 5 & & 5 & & 5 & & 5 & & 4 & & 4 & & \textcolor{red}{3} & & 2 & & 1 \\
 & 1 & & 2 & & \textcolor{red}{3} & & \textcolor{red}{3} & & \textcolor{red}{3} & & 4 & & 4 & & 4 & & \textcolor{red}{3} & & \textcolor{red}{3} & & \textcolor{red}{3} & & 2 & & 1 \\
 & 1 & & 1 & & 1 & & 1 & & 1 & & 2 & & 1 & & 1 & & 1 & & 1 & & 1
\end{array}
$$

Process

Process
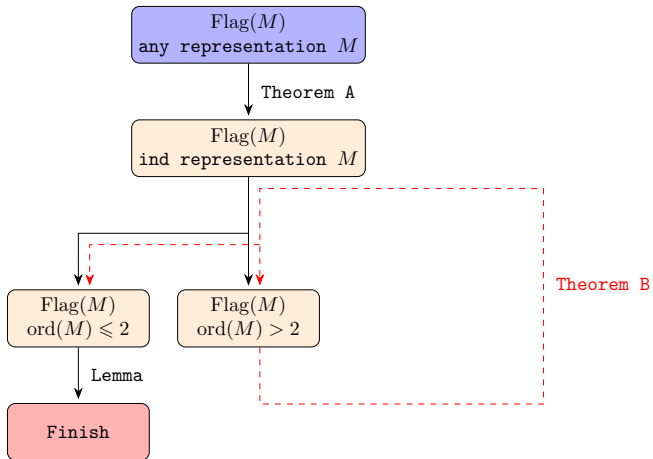
Consider the short exact sequence of representations

$$\eta : 0 \longrightarrow X \xrightarrow{\iota} Y \xrightarrow{\pi} S \longrightarrow 0$$

which induce maps

$$\Psi : \quad \mathrm{Flag}_d(Y) \longrightarrow \mathrm{Flag}_d(X) \times \mathrm{Flag}_d(S)$$
$$\Psi_{\underline{\mathbf{f}},\underline{\mathbf{g}}} : \quad \mathrm{Flag}(Y)_{\underline{\mathbf{f}},\underline{\mathbf{g}}} \longrightarrow \mathrm{Flag}_{\underline{\mathbf{f}}}(X) \times \mathrm{Flag}_{\underline{\mathbf{g}}}(S)$$

Theorem B [Cerulli-Irelli–Esposito–Franzen–Reineke, Zhou]

When η does not split and generates $\mathrm{Ext}^1(S, X)$,

$\Psi_{\underline{\mathbf{f}},\underline{\mathbf{g}}}$ is a Zarisky-locally trivial affine bundle over $\mathrm{Im}\,\Psi_{\underline{\mathbf{f}},\underline{\mathbf{g}}}$.

In this case, we have a clear description of $\mathrm{Im}\,\Psi_{\underline{\mathbf{f}},\underline{\mathbf{g}}}$.

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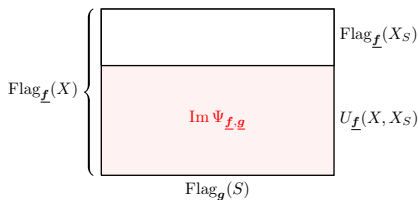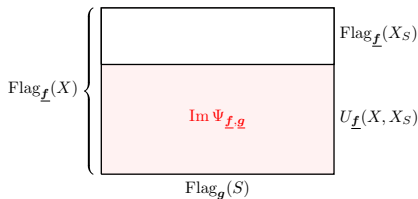000000●0000000

How to find nice η?

Proposition

For $X \hookrightarrow Y$ irreducible mono, the induced SES
$$\eta : 0 \longrightarrow X \xrightarrow{\iota} Y \xrightarrow{\pi} S \longrightarrow 0$$
satisfies the condition of Theorem B. Moreover,
$$\operatorname{Im} \Psi_{\mathbf{f},\mathbf{g}} = \begin{cases} \left(\operatorname{Flag}_{\underline{\mathbf{f}}}(X) \smallsetminus \operatorname{Flag}_{\underline{\mathbf{f}}}(X_S)\right) \times \operatorname{Flag}_{\underline{\mathbf{g}}}(S), & \underline{\mathbf{g}}_i = \underline{\dim} S \\ \operatorname{Flag}_{\underline{\mathbf{f}}}(X) \times \operatorname{Flag}_{\underline{\mathbf{g}}}(S), & \text{otherwise} \end{cases}$$
where
$$X_S := \max \left\{ M \subseteq X \mid \operatorname{Ext}^1(S, X/M) \cong \mathbb{C} \right\} \subseteq X.$$

How to find nice η?

Proposition

For $X \hookrightarrow Y$ *irreducible mono*, the induced SES
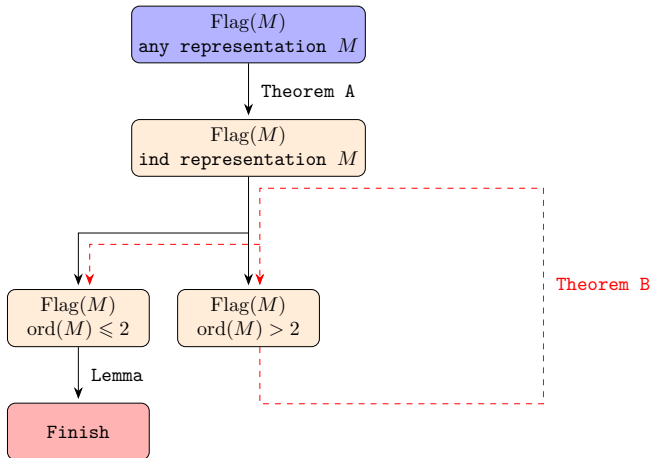$$\eta : 0 \longrightarrow X \xrightarrow{\iota} Y \xrightarrow{\pi} S \longrightarrow 0$$
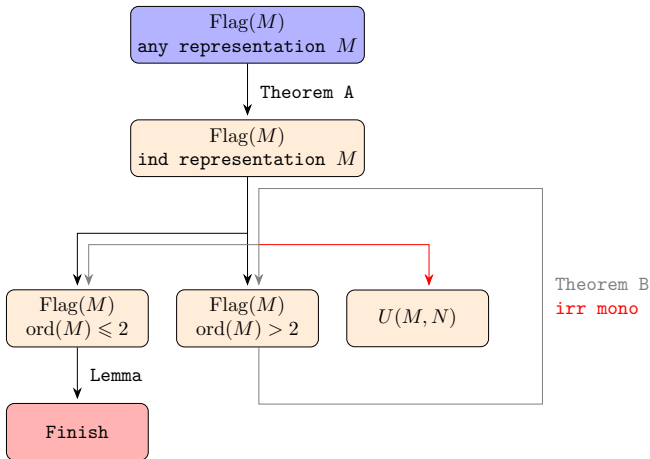satisfies the condition of Theorem B. Moreover,
$$\operatorname{Im} \Psi_{\underline{\mathbf{f}},\underline{\mathbf{g}}} = \begin{cases} \left(\operatorname{Flag}_{\underline{\mathbf{f}}}(X) \smallsetminus \operatorname{Flag}_{\underline{\mathbf{f}}}(X_S)\right) \times \operatorname{Flag}_{\underline{\mathbf{g}}}(S), & \underline{\mathbf{g}}_i = \underline{\dim} S \\ \operatorname{Flag}_{\underline{\mathbf{f}}}(X) \times \operatorname{Flag}_{\underline{\mathbf{g}}}(S), & \text{otherwise} \end{cases}$$

How to find nice η?

Proposition

For $X \hookrightarrow Y$ *irreducible mono*, the induced SES
$$\eta : 0 \longrightarrow X \xrightarrow{\iota} Y \xrightarrow{\pi} S \longrightarrow 0$$
satisfies the condition of Theorem B. Moreover,
$$\operatorname{Im} \Psi_{\mathbf{f},\mathbf{g}} = \begin{cases} U_{\underline{\mathbf{f}}}(X, X_S), & \underline{\mathbf{g}}_i = \underline{\dim} S \\ \operatorname{Flag}_{\underline{\mathbf{f}}}(X) \times \operatorname{Flag}_{\underline{\mathbf{g}}}(S), & \text{otherwise} \end{cases}$$
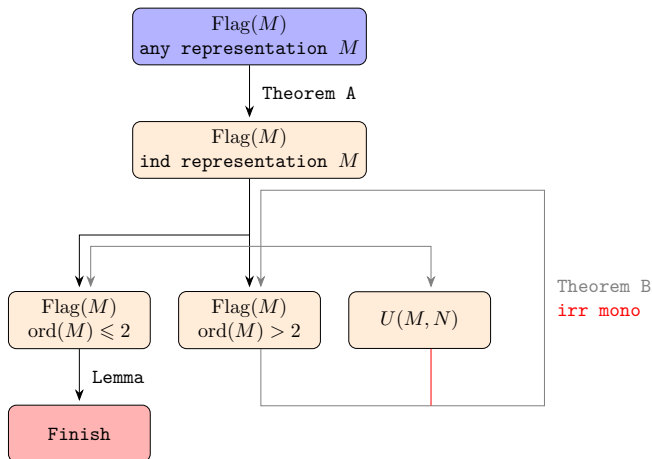
Setting and Statement
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Induction?

Proposition

For $X \hookrightarrow Y$ irreducible mono, the induced SES
$$\eta : 0 \longrightarrow X \xrightarrow{\iota} Y \xrightarrow{\pi} S \longrightarrow 0$$
satisfies the condition of Theorem B. Moreover,
$$\operatorname{Im} \Psi_{\underline{\mathbf{f}},\underline{\mathbf{g}}} = \begin{cases} U_{\underline{\mathbf{f}}}(X, X_S), & \underline{\mathbf{g}}_i = \underline{\dim} S \\ \operatorname{Flag}_{\underline{\mathbf{f}}}(X) \times \operatorname{Flag}_{\underline{\mathbf{g}}}(S), & \text{otherwise} \end{cases}$$

Proposition

In addition,
$$X_S = 0 \quad \text{or} \quad X_S \hookrightarrow X \text{ is irreducible mono.}$$

Corollary

For $M \in \operatorname{ind}(Q)$, if exist irreducible mono $X \hookrightarrow M$, then
$$\operatorname{Flag}_d(M) \text{ has an affine paving.}$$

Process

What is remaining?

E_6 :

$$
\begin{array}{ccccccccccc}
 & & 1 & & 1 & & 1 & & 1 & & \\
 & 1 & & 2 & & 2 & & 2 & & 1 & \\
1 & 1 & 2 & 1 & 3 & 2 & 3 & 1 & 2 & 1 & 1 \\
 & 1 & & 2 & & 2 & & 2 & & 1 & \\
 & 1 & & 1 & & 1 & & 1 & &
\end{array}
$$

E_7 :

$$
\begin{array}{ccccccccccccccc}
 & 1 & & 1 & & 1 & & 2 & & 1 & & 1 & & 1 & \\
1 & & 2 & & 2 & & 3 & & 3 & & 2 & & 2 & & 1 \\
1 & 1 & 2 & 1 & 3 & 2 & 4 & 2 & 4 & 2 & 4 & 2 & 3 & 1 & 2 & 1 & 1 \\
1 & & 2 & & 3 & & 3 & & 3 & & 3 & & 2 & & 1 \\
 & 1 & & 2 & & 2 & & 2 & & 2 & & 2 & & 1 & \\
 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 &
\end{array}
$$

E_8 :

$$
\begin{array}{c}
1 \quad 1 \quad 1 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 1 \quad 1 \quad 1 \\
1 \quad 2 \quad 2 \quad 3 \quad 4 \quad 4 \quad 4 \quad 4 \quad 4 \quad 4 \quad 3 \quad 2 \quad 2 \quad 1 \\
1 \ 1 \ 2 \ 1 \ 3 \ 2 \ 4 \ 2 \ 5 \ 3 \ 6 \ 3 \ 6 \ 3 \ 6 \ 3 \ 6 \ 3 \ 6 \ 3 \ 5 \ 2 \ 4 \ 2 \ 3 \ 1 \ 2 \ 1 \ 1 \\
1 \quad 2 \quad 3 \quad 4 \quad 4 \quad 5 \quad 5 \quad 5 \quad 5 \quad 4 \quad 4 \quad 3 \quad 2 \quad 1 \\
1 \quad 2 \quad 3 \quad 3 \quad 3 \quad 4 \quad 4 \quad 4 \quad 3 \quad 3 \quad 3 \quad 2 \quad 1 \\
1 \quad 2 \quad 2 \quad 2 \quad 2 \quad 3 \quad 3 \quad 2 \quad 2 \quad 2 \quad 2 \quad 1 \\
1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 2 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1
\end{array}
$$

What is remaining?

What is remaining?

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Q & A

Thank you for your listening!

Special thanks to Yilong Zhang, Luozi Shi, Chenji Fu, Zheng Huang and Junyan Liu for their valuable feedback.

Any questions?