Master thesis

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Chapter 1

Variety and stratification

1.1 Initial case: $\mathcal{F}$ and $\mathcal{F} \times \mathcal{F}$

We introduce the complete flag variety to give a bird's eye view on the whole section. Actually, the entire difficulty is bundled in this example.

Fix $n \geqslant 1$, we denote $\mathrm{GL}_n := \mathrm{GL}_n(\mathbb{C})$, B, T, N, W be the standard Borel subgroup, standard torus, unipotent subgroup, Weyl group respectively, i.e.,

$$\mathrm{GL}_n = \begin{pmatrix} * & \dots & * \\ \vdots & \ddots & \vdots \\ * & \dots & * \end{pmatrix} \quad B = \begin{pmatrix} * & \dots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix} \quad T = \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \quad B = \begin{pmatrix} 1 & \dots & * \\ & \ddots & \vdots \\ 0 & & 1 \end{pmatrix}$$

$$W := N_{\mathrm{GL}_n}(T)/T \cong S_n$$

Definition 1.1.1 (flag)**.** *For a finite dimensional $\mathbb{C}$-vector space V, a flag of V is an increasing sequence of subspaces of V:*

$$F : 0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_k = \mathbb{C}^n.$$

F is called a complete flag if $\dim V_i = i$ for all i, otherwise F is called a partial flag.

Definition 1.1.2 (complete flag variety)**.** *The complete flag variety $\mathcal{F}$ is defined as*

$$\begin{aligned} \mathcal{F} &= \mathrm{GL}_n / B \\ &\cong \{\textit{complete flags of } \mathbb{C}^n\} \\ &= \{0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n = \mathbb{C}^n \mid \dim V_i = i\} \\ &\cong \{\textit{Borel subgroups of } \mathrm{GL}_n\} \\ &= \left\{gBg^{-1} \mid g \in \mathrm{GL}_n\right\} \end{aligned}$$

Remark 1.1.3.

1. We implicitly give the base point of $\mathcal{F}$, which is not considered as the data of $\mathcal{F}$. Fix a standard basis of $\mathbb{C}^n$ by $\{v_1, \ldots, v_n\}$, we define the standard flag

$$F_{\mathrm{Id}} : 0 \subseteq \langle v_1 \rangle \subseteq \langle v_1, v_2 \rangle \subseteq \cdots \subseteq \langle v_1, \ldots, v_n \rangle = \mathbb{C}^n.$$

2. We have the natural GL_n-action on $\mathcal{F}$, which is considered as the data of $\mathcal{F}$.

   For $g \in \mathrm{GL}_n$, we define the flag attached to g:

$$F_g \triangleq gF_{\mathrm{Id}} : 0 \subseteq \langle gv_1 \rangle \subseteq \langle gv_1, gv_2 \rangle \subseteq \cdots \subseteq \langle gv_1, \ldots, gv_n \rangle = \mathbb{C}^n.$$

   Especially, for $w \in W = N_{\mathrm{GL}_n}(T)/T \cong S_n$, the flag attached to w

$$\begin{aligned} F_w : 0 &\subseteq \langle \tilde{w}v_1 \rangle \subseteq \langle \tilde{w}v_1, \tilde{w}v_2 \rangle \subseteq \cdots \subseteq \langle \tilde{w}v_1, \ldots, \tilde{w}v_n \rangle = \mathbb{C}^n \\ 0 &\subseteq \left\langle v_{w(1)} \right\rangle \subseteq \left\langle v_{w(1)}, v_{w(2)} \right\rangle \subseteq \cdots \subseteq \left\langle v_{w(1)}, \ldots, v_{w(n)} \right\rangle = \mathbb{C}^n \end{aligned}$$

   does not depend on the choice of the lift $\tilde{w} \in N_{\mathrm{GL}_n}(T)$ of w.

   Readers can verify that $\{F_w | w \in W\}$ are all T-fixed points of $\mathcal{F}$, while $\left\{wBw^{-1} \middle| w \in W\right\}$ are all Borel subgroups of G containing the standard torus T.

3. For $n = 2$, $\mathcal{F} \cong \mathbb{P}^1$. We encourage readers to use $\mathbb{P}^1$ as a toy example for the whole theory.

interpretation	GL_n / B	flags	Borel subgroups
base point	Id	F_{Id}	B
GL_n-action	left multiplication	$\{V_i\} \mapsto \{gV_i\}$	conjugation
general point	g	F_g	gBg^{-1}

$\mathcal{F}$ is a well-studied variety, and has many combinatorical properties. For example, from the well-known Bruhat decomposition, [1]

$$\mathrm{GL}_n \cong \bigsqcup_{w \in W} BwB$$

We get a stratification of $\mathcal{F}$ by B-orbits:

$$\mathcal{F} = \mathrm{GL}_n / B \cong \bigsqcup_{w \in W} BwB/B$$

The B-orbit BwB/B is called the Schubert cell, denoted by $\mathcal{V}_w$. Since

$$\mathcal{V}_w = BwB/B \cong B/\left(B \cap wBw^{-1}\right) \cong \mathbb{A}^{l(w)},$$

the Schubert cell is an affine space of dimension $l(w)$.

As a result, we know a lot of information of $\mathcal{F}$:

[1] For the most time the formula doees not depend on the lift of w, so we abuse the notation of $w \in N_{\mathrm{GL}_n}(T)/T$ and $\tilde{w} \in N_{\mathrm{GL}_n}(T)$.

$H^i(\mathcal{F};\mathbb{C})$	0	2	4	6	8	10	12
1	1						
2	1	1					
3	1	2	2	1			
4	1	3	5	6	5	3	1
5	1	4	9	15	20	22	20

G	Orbit	G-fixed points
GL_n	$\mathcal{F} \cong \mathrm{GL}_n / B$	$\varnothing$
B	$\mathcal{V}_w \cong B/(B \cap wBw^{-1})$	$\{F_{\mathrm{Id}}\}$
T	$-$	$\{F_w \mid w \in W\}$

1.2 quiver and Weyl group

1.3 algebraic group and Lie algebra

1.4 typical variety

1.5 stratification and T-fixed points

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