

Affine pavings of partial flag varieties

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Process

- 1 Setting and Statement
- 2 Case study
- 3 Auslander–Reiten theory
- 4 Tackle the type E case

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Affine paving

Setting

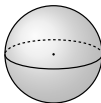
$K = \mathbb{C}$, X : algebraic variety over K .

Definition

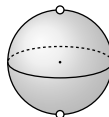
An **affine paving** of X is a filtration

$$\emptyset = X_0 \subset X_1 \subset \cdots \subset X_d = X$$

with X_i closed and $X_{i+1} \setminus X_i \cong \mathbb{A}_K^k$.



$$\mathbb{P}^1 = \{\infty\} \sqcup \mathbb{A}^1$$



$\mathbb{P}^1 \setminus \{0, \infty\}$ has no affine paving

Quiver and quiver representation



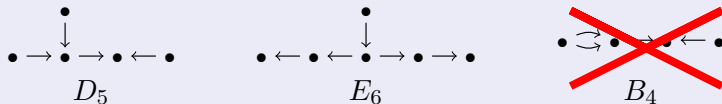
Quiver is a graph. It has some vertices & arrows.
In this talk, all the quivers are finite and connected.

Quiver and quiver representation

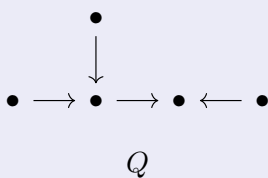


We focus on the Dynkin quiver.

That means, the graph of the Dynkin diagrams in the ADE series.

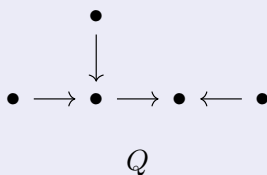


Quiver representation



$$\begin{array}{c} V_5 \\ \downarrow \delta \\ V_1 \xrightarrow{\alpha} V_2 \xrightarrow{\beta} V_3 \xleftarrow{\gamma} V_4 \\ V \in \text{rep}(Q) \end{array}$$

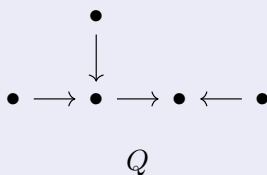
Quiver representation



$$\begin{array}{c}
 0 \\
 \downarrow \\
 \mathbb{C}^2 \xrightarrow{\text{Id}} \mathbb{C}^2 \xrightarrow{\text{Id}} \mathbb{C}^2 \xleftarrow{(10)} \mathbb{C}
 \end{array}$$

$V \in \text{rep}(Q)$

Quiver representation



Dimension vector:

$$\begin{array}{c} 0 \\ \downarrow \\ \mathbb{C}^2 \xrightarrow{\text{Id}} \mathbb{C}^2 \xrightarrow{\text{Id}} \mathbb{C}^2 \xleftarrow{(10)} \mathbb{C} \end{array}$$

$V \in \text{rep}(Q)$

$$\underline{\dim} V = \begin{smallmatrix} 0 \\ 2 \\ 2 \\ 2 \\ 1 \end{smallmatrix}$$

Partial flag variety

Definition

Fix a quiver Q and $M \in \text{rep}(Q)$,

$$\text{Flag}_d(M) := \{F: 0 \subseteq N_1 \subseteq \cdots \subseteq N_d \subseteq M\}$$

$$\text{Flag}_{\underline{\mathbf{f}}}(M) := \{F: 0 \subseteq N_1 \subseteq \cdots \subseteq N_d \subseteq M \mid \underline{\dim} N_i = \underline{\mathbf{f}}_i\}$$

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Example

$$Q = \bullet, M = \mathbb{C}^n,$$

$$\text{Flag}_1(\mathbb{C}^n) = \{F: 0 \subseteq N_1 \subseteq \mathbb{C}^n\} = \bigsqcup_{k=0}^n \text{Gr}(n, k)$$

$$\text{Flag}_{(k)}(\mathbb{C}^n) = \text{Gr}(n, k)$$

Statement

Theorem [Cerulli-Irelli–Esposito–Franzen–Reineke, Zhou]

For a Dynkin quiver Q and $M \in \text{rep}(Q)$,

$\text{Flag}_d(M)$ has an affine paving.

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Task 1. $Q = \bullet$, $M = \mathbb{C}^n$

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In this case,

$$\begin{array}{ccc} \mathrm{GL}_n(\mathbb{C}) \curvearrowright \mathbb{C}^n & \rightsquigarrow & \mathrm{GL}_n(\mathbb{C}) \curvearrowright \mathrm{Flag}_d(\mathbb{C}^n) \\ & \rightsquigarrow & B \curvearrowright \mathrm{Flag}_d(\mathbb{C}^n) \end{array}$$

$\mathrm{Flag}_d(\mathbb{C}^n)$ has an affine paving given by Schubert cells (i.e., B -orbits).

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$\mathrm{Flag}_d(\mathbb{C}^n)$ has an affine paving given by Schubert cells (i.e., B -orbits).

Note

When $Q = \bullet \longrightarrow \bullet$, $\mathrm{Flag}_{\mathbf{f}}(M)$ have no natural group actions.

Task 2a. $Q = \bullet \rightarrow \bullet$, $M = \begin{bmatrix} \mathbb{C}^2 & \xrightarrow{0} & \mathbb{C}^2 \end{bmatrix}$, $d = 1$

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$$\underline{\mathbf{f}} = (1, 0) : \quad \text{Flag}_{\underline{\mathbf{f}}}(M) = \mathbb{P}^1$$

$$\underline{\mathbf{f}} = (0, 0) : \quad \text{Flag}_{\underline{\mathbf{f}}}(M) = \{*\}$$

$$\underline{\mathbf{f}} = (1, 1) : \quad \text{Flag}_{\underline{\mathbf{f}}}(M) = \mathbb{P}^1 \times \mathbb{P}^1$$

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In this case,

$$\text{Flag}_{(a,b)}(M) \cong \text{Flag}_{(a)}(\mathbb{C}^2) \times \text{Flag}_{(b)}(\mathbb{C}^2)$$

has an affine paving.

Task 2b. $Q = \bullet \rightarrow \bullet$, $M = \left[\mathbb{C}^2 \xrightarrow{\text{Id}} \mathbb{C}^2 \right]$, $d = 1$

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In this case,

$$\text{Flag}_{(a,b)}(M) \cong \text{Flag}_{(b)}^{(a)}(\mathbb{C}^2)$$

has an affine paving.

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To construct affine pavings systematically, we need to construct an uniform method.

Task 2c. $Q = \bullet \rightarrow \bullet$, $M = \left[\mathbb{C}^2 \xrightarrow{\begin{pmatrix} 10 \\ 00 \end{pmatrix}} \mathbb{C}^2 \right]$, $d = 1$

First try

Let $X = \left[0 \longrightarrow \mathbb{C} \right]$, $S = \left[\mathbb{C}^2 \xrightarrow{\begin{pmatrix} 10 \\ 00 \end{pmatrix}} \mathbb{C} \right]$, then $M = X \oplus S$,
and the short exact sequence

$$0 \longrightarrow X \xrightarrow{\iota} M \xrightarrow{\pi} S \longrightarrow 0$$

induces

$$\begin{aligned} \Psi : \text{Flag}_d(M) &\longrightarrow \text{Flag}_d(X) \times \text{Flag}_d(S) \\ F &\longmapsto (\iota^{-1}(F), \pi(F)) \end{aligned}$$

Idea of affine pavings

Find a nice short exact sequence

$$0 \longrightarrow X \xrightarrow{\iota} M \xrightarrow{\pi} S \longrightarrow 0$$

which induces a nice morphism

$$\begin{aligned} \Psi : \operatorname{Flag}_d(M) &\longrightarrow \operatorname{Flag}_d(X) \times \operatorname{Flag}_d(S) \\ F &\longmapsto (\iota^{-1}(F), \pi(F)) \end{aligned}$$

We construct the affine paving of $\operatorname{Flag}_d(M)$ from the affine paving of $\operatorname{Flag}_d(X)$ and $\operatorname{Flag}_d(S)$. Then, we use mathematical induction.

Example. $Q = \bullet$, $M = \mathbb{C}^2$

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$$\Psi_1 : \text{Flag}_1(\mathbb{C}^2) \longrightarrow \text{Flag}_1(\mathbb{C}) \times \text{Flag}_1(\mathbb{C})$$

$$\begin{array}{ccccc} \Psi_{(1)} : \text{Flag}_{(1)}(\mathbb{C}^2) & \longrightarrow & \text{Flag}_{(1)}(\mathbb{C}) \times \text{Flag}_{(0)}(\mathbb{C}) & \sqcup & \text{Flag}_{(0)}(\mathbb{C}) \times \text{Flag}_{(1)}(\mathbb{C}) \\ \mathbb{P}^1 & \longrightarrow & \{*\} & \sqcup & \{*\} \end{array}$$

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$$\mathbb{P}^1 = \{\infty\} \sqcup \mathbb{C}$$

$$\begin{array}{c} \downarrow \Psi_{(1)} \\ \{*\} \sqcup \{*\} \end{array}$$

Example. $Q = \bullet$, $M = \mathbb{C}^8 = \bigoplus_{i=1}^8 \mathbb{C}v_i$

$$0 \longrightarrow \mathbb{C}^3 \xrightarrow{\iota} \mathbb{C}^8 \xrightarrow{\pi} \mathbb{C}^5 \longrightarrow 0$$

$$\Psi : \text{Flag}_{(3)}(\mathbb{C}^8) \longrightarrow \text{Flag}_{(1)}(\mathbb{C}^3) \times \text{Flag}_{(2)}(\mathbb{C}^5) \sqcup \dots$$

$$\begin{aligned} \Psi^{-1}\left(\langle v_1 \rangle, \langle v_4, v_5 \rangle\right) &= \left\{ \langle v_1, v_4 + av_2 + bv_3, v_5 + cv_2 + dv_3 \rangle \mid a, b, c, d \in \mathbb{C} \right\} \\ &\cong \mathbb{C}^4 \end{aligned}$$

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Ψ is a Zarisky-locally trivial affine bundle of rank $2 \cdot (3 - 1) = 4$.

Consider the short exact sequence of representations

$$\eta : 0 \longrightarrow X \xrightarrow{\iota} Y \xrightarrow{\pi} S \longrightarrow 0$$

which induce maps

$$\begin{aligned} \Psi : \text{Flag}_d(Y) &\longrightarrow \text{Flag}_d(X) \times \text{Flag}_d(S) \\ \Psi_{\underline{\mathbf{f}}, \underline{\mathbf{g}}} : \text{Flag}(Y)_{\underline{\mathbf{f}}, \underline{\mathbf{g}}} &\longrightarrow \text{Flag}_{\underline{\mathbf{f}}}(X) \times \text{Flag}_{\underline{\mathbf{g}}}(S) \end{aligned}$$

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Theorem A [Cerulli-Irelli–Esposito–Franzen–Reineke, Zhou]

When η splits, then Ψ is surjective.

► skip

Moreover, if $\text{Ext}^1(S, X) = 0$, then

$\Psi_{\underline{\mathbf{f}}, \underline{\mathbf{g}}}$ is a Zarisky-locally trivial affine bundle.

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By this theorem,

$\text{Flag}_d(Y)$ has an affine paving $\Leftarrow \text{Flag}_d(X), \text{Flag}_d(S)$ have.

Warning

η splits and $\text{Ext}^1(S, X) = 0$ are necessary for Theorem A.

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Example

Consider the quiver $Q : \bullet \rightarrow \bullet \leftarrow \bullet$ and the short exact sequence

$$0 \longrightarrow [\mathbb{C}e_1 \rightarrow \mathbb{C}^2 \leftarrow \mathbb{C}e_2] \longrightarrow [\mathbb{C}^2 \xrightarrow{\text{Id}} \mathbb{C}^2 \xleftarrow{\text{Id}} \mathbb{C}^2] \longrightarrow [\mathbb{C}e_2 \rightarrow 0 \leftarrow \mathbb{C}e_1] \longrightarrow 0$$

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we get

$$\text{Im } \Psi_{(0,1,0),(1,0,1)} \cong (\mathbb{P}^1 \setminus \{0, \infty\}) \times \{*\} \cong \mathbb{C}^*,$$

so Ψ is not surjective.

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In this way, we get a bad stratification

$$\text{Flag}_{(1,1,1)}(Y) \cong \mathbb{P}^1 = \{0\} \sqcup \{\infty\} \sqcup \mathbb{C}^*.$$

Task 3. $Q = \begin{array}{c} \bullet \\ \downarrow \\ \bullet \rightarrow \bullet \leftarrow \bullet \end{array}, M = \begin{smallmatrix} 1 \\ 121 \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 111 \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 111 \end{smallmatrix}$

We use following short exact sequences

$$0 \longrightarrow \begin{smallmatrix} 1 \\ 111 \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 111 \end{smallmatrix} \longrightarrow M \longrightarrow \begin{smallmatrix} 1 \\ 121 \end{smallmatrix} \longrightarrow 0$$

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to reduced the problem to indecomposable representations.

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Notice that we use the result

$$\mathrm{Ext}^1\left(\begin{smallmatrix} 1 \\ 121 \end{smallmatrix}, \begin{smallmatrix} 1 \\ 111 \end{smallmatrix}\right) = 0, \quad \mathrm{Ext}^1\left(\begin{smallmatrix} 1 \\ 111 \end{smallmatrix}, \begin{smallmatrix} 1 \\ 111 \end{smallmatrix}\right) = 0.$$

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We can't put $\begin{smallmatrix} 1 \\ 121 \end{smallmatrix}$ on the left, since

$$\mathrm{Ext}^1\left(\begin{smallmatrix} 1 \\ 111 \end{smallmatrix}, \begin{smallmatrix} 1 \\ 121 \end{smallmatrix}\right) \cong \mathbb{C} \neq 0.$$

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to reduced the problem to indecomposable representations.

$\text{Flag}_d \left(\begin{smallmatrix} 1 \\ 111 \end{smallmatrix} \right)$ has an affine paving: obvious.

$\text{Flag}_d \left(\begin{smallmatrix} 1 \\ 121 \end{smallmatrix} \right)$ has an affine paving: it is \mathbb{P}^1 , $\{*\}$ or empty.

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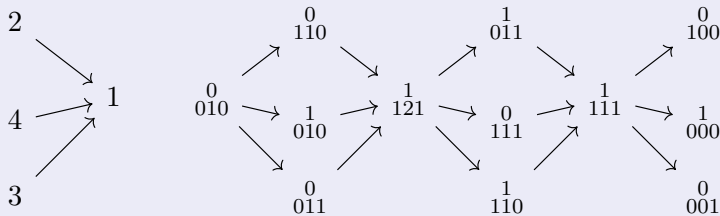
Need: more informations of indecomposable representations!

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Auslander–Reiten quiver: D_4

$$\begin{array}{c}
 4 \\
 \downarrow \\
 2 \rightarrow 1 \leftarrow 3
 \end{array}$$



Vertices \iff Indecomposable representations

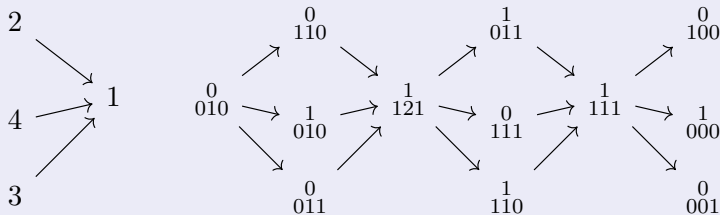
Arrows \iff Irreducible morphisms

Paths \iff Morphisms

Shift cards \iff Switch arrows in Q

Auslander–Reiten quiver: D_4

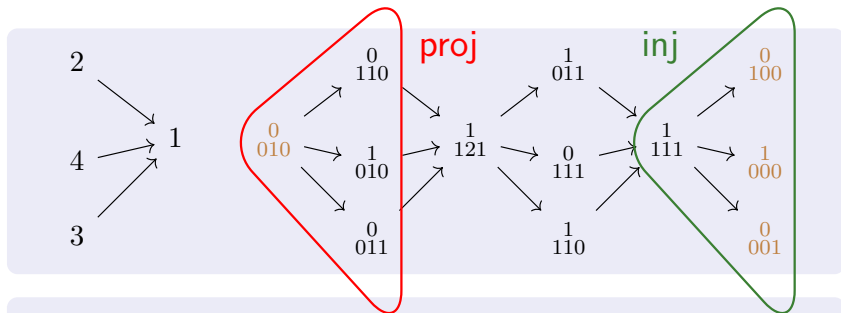
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Vertices \iff Indecomposable representations

Auslander–Reiten quiver: D_4

$$\begin{array}{c} 4 \\ \downarrow \\ 2 \rightarrow 1 \leftarrow 3 \end{array}$$

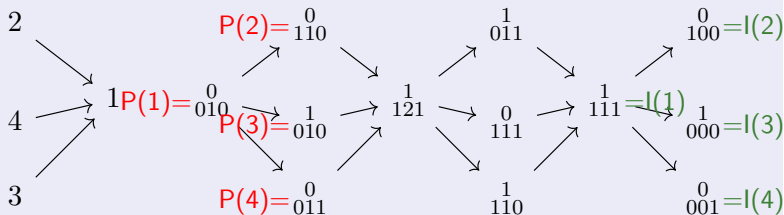


Vertices \iff Indecomposable representations

irreducible rep, projective rep, injective rep

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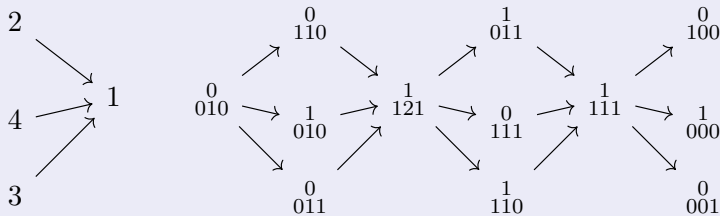


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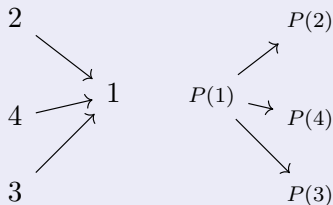


Arrows \iff Irreducible morphisms

I will show you how to construct AR-quiver.

Auslander–Reiten quiver: D_4

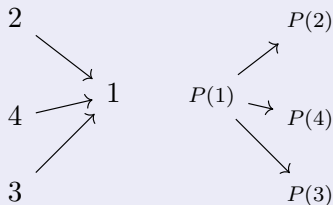
$$\begin{array}{c} 4 \\ \downarrow \\ 2 \rightarrow 1 \leftarrow 3 \end{array}$$



Construction:

Auslander–Reiten quiver: D_4

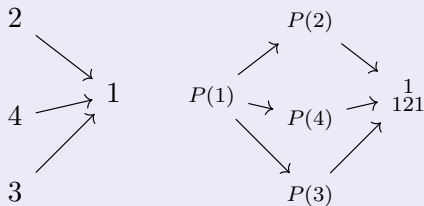
$$\begin{array}{c} 4 \\ \downarrow \\ 2 \rightarrow 1 \leftarrow 3 \end{array}$$

Construction:

$$0 \rightarrow P(1) \rightarrow P(2) \oplus P(4) \oplus P(3) \rightarrow \begin{smallmatrix} 1 \\ 121 \end{smallmatrix} \rightarrow 0$$

Auslander–Reiten quiver: D_4

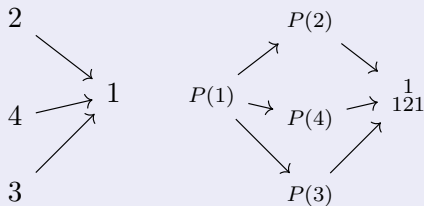
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Auslander–Reiten quiver: D_4

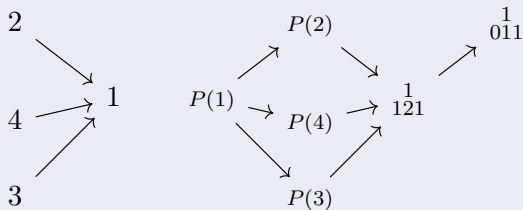
$$\begin{array}{c} 4 \\ \downarrow \\ 2 \rightarrow 1 \leftarrow 3 \end{array}$$

Construction:

$$0 \rightarrow P(2) \rightarrow 1_{121} \rightarrow 0_{11} \rightarrow 0$$

Auslander–Reiten quiver: D_4

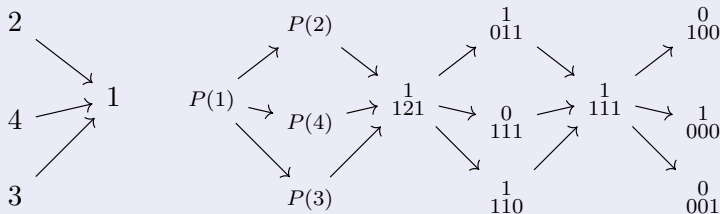
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Construction:

$$0 \rightarrow P(2) \rightarrow \begin{smallmatrix} 1 \\ 121 \end{smallmatrix} \rightarrow \begin{smallmatrix} 1 \\ 011 \end{smallmatrix} \rightarrow 0$$

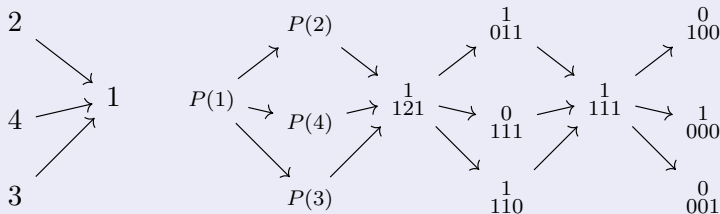
Auslander–Reiten quiver: D_4

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Auslander–Reiten quiver: D_4

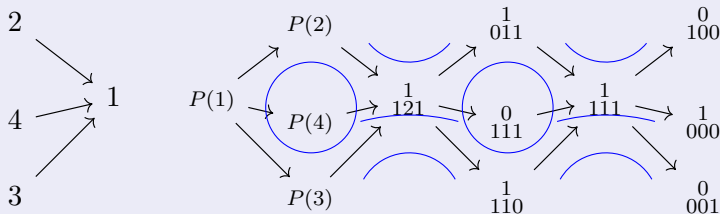
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Construction:

AR-quiver, AR-sequence, AR-translation

Auslander–Reiten quiver: D_4

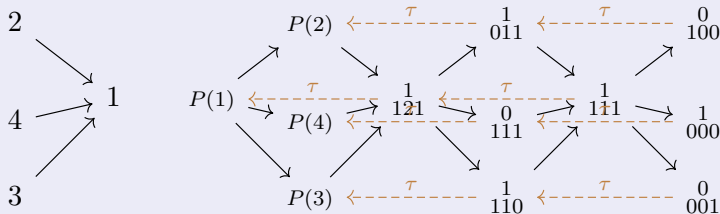
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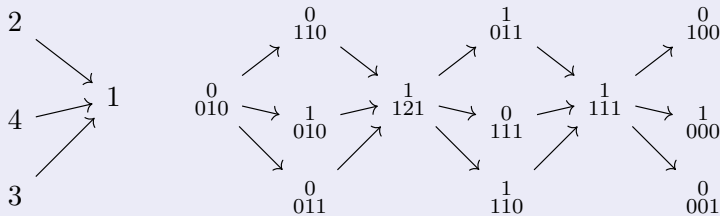
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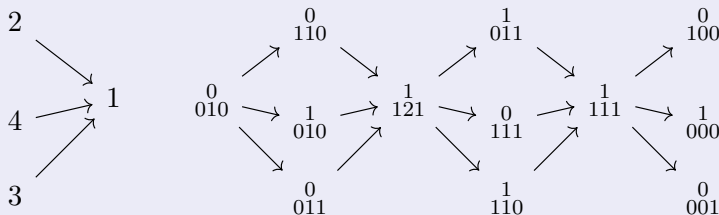
$$\begin{array}{c}
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 2 \rightarrow 1 \leftarrow 3
 \end{array}$$



Paths \iff Morphisms

Auslander–Reiten quiver: D_4

$$\begin{array}{c} 4 \\ \downarrow \\ 2 \rightarrow 1 \leftarrow 3 \end{array}$$



In the Dynkin quiver case,

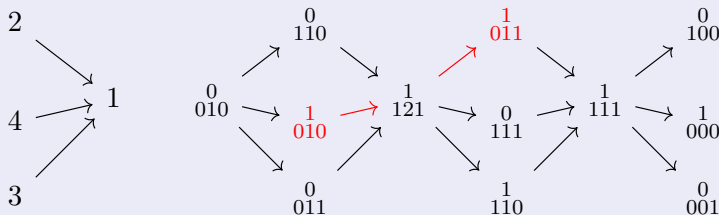
$$\text{Hom}(T, T') \cong \langle \text{paths from } T \text{ to } T' \rangle / \text{AR-seq}$$

For example,

$$\text{Hom}(\begin{smallmatrix} 1 \\ 010 \end{smallmatrix}, \begin{smallmatrix} 1 \\ 011 \end{smallmatrix}) \cong \mathbb{C}, \quad \text{Hom}(\begin{smallmatrix} 1 \\ 010 \end{smallmatrix}, \begin{smallmatrix} 0 \\ 111 \end{smallmatrix}) \cong 0, \quad \text{Hom}(\begin{smallmatrix} 1 \\ 010 \end{smallmatrix}, \begin{smallmatrix} 1 \\ 111 \end{smallmatrix}) \cong \mathbb{C}$$

Auslander–Reiten quiver: D_4

$$\begin{array}{c} 4 \\ \downarrow \\ 2 \rightarrow 1 \leftarrow 3 \end{array}$$



In the Dynkin quiver case,

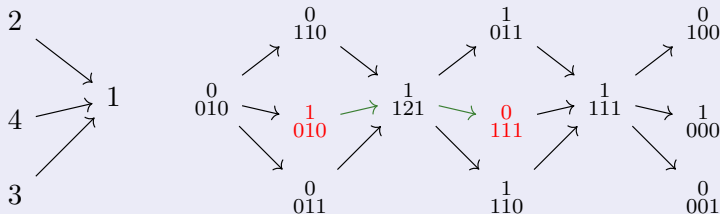
$$\text{Hom}(T, T') \cong \langle \text{paths from } T \text{ to } T' \rangle / \text{AR-seq}$$

For example,

$$\text{Hom}(\overset{1}{010}, \overset{1}{011}) \cong \mathbb{C}, \quad \text{Hom}(\overset{1}{010}, \overset{0}{111}) \cong 0, \quad \text{Hom}(\overset{1}{010}, \overset{1}{111}) \cong \mathbb{C}$$

Auslander–Reiten quiver: D_4

$$\begin{array}{c} 4 \\ \downarrow \\ 2 \rightarrow 1 \leftarrow 3 \end{array}$$



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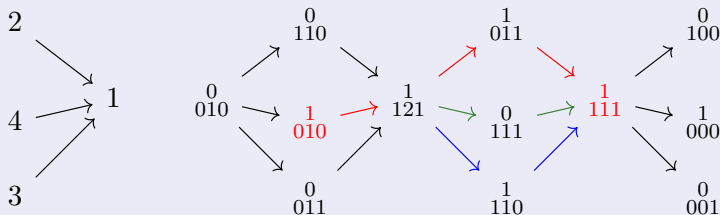
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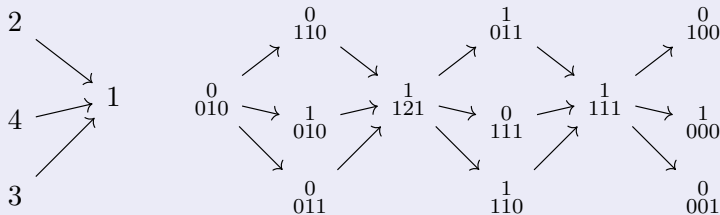
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Auslander–Reiten quiver: D_4

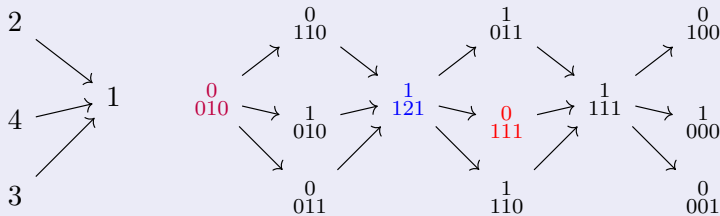
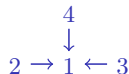
$$\begin{array}{c} 4 \\ \downarrow \\ 2 \rightarrow 1 \leftarrow 3 \end{array}$$



$$\mathrm{Ext}^1(T, T') \cong \overline{\mathrm{Hom}}(T', \tau T)^\vee$$

For example,

$$\mathrm{Ext}^1\left(\begin{smallmatrix} 1 \\ 121 \end{smallmatrix}, \begin{smallmatrix} 0 \\ 111 \end{smallmatrix}\right) \cong \mathrm{Hom}\left(\begin{smallmatrix} 0 \\ 111 \end{smallmatrix}, \begin{smallmatrix} 0 \\ 010 \end{smallmatrix}\right)^\vee \cong 0$$

Auslander–Reiten quiver: D_4 

$$\mathrm{Ext}^1(\textcolor{blue}{T}, \textcolor{red}{T}') \cong \overline{\mathrm{Hom}}(\textcolor{red}{T}', \tau \textcolor{violet}{T})^\vee$$

For example,

$$\mathrm{Ext}^1 \left(\begin{smallmatrix} 1 \\ 121 \end{smallmatrix}, \begin{smallmatrix} 0 \\ 111 \end{smallmatrix} \right) \cong \mathrm{Hom} \left(\begin{smallmatrix} 0 \\ 111 \end{smallmatrix}, \begin{smallmatrix} 0 \\ 010 \end{smallmatrix} \right)^\vee \cong 0$$

First application

Corollary

For a Dynkin quiver, we can give an total order to the set of all indecomposable representations, such that

$$i \leq j \implies \operatorname{Ext}^1(M_i, M_j) = 0.$$

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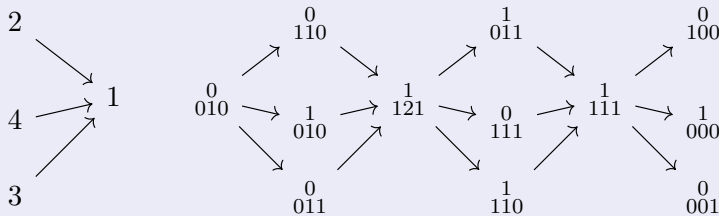
By Theorem A, the problem reduced to

For a Dynkin quiver Q and $M \in \operatorname{ind}(Q)$,

$\operatorname{Flag}_d(M)$ has an affine paving.

Auslander–Reiten quiver: D_4

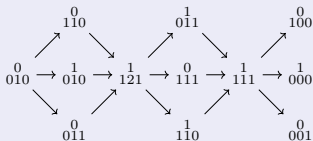
$$\begin{array}{c} 4 \\ \downarrow \\ 2 \rightarrow 1 \leftarrow 3 \end{array}$$



Shift cards \iff Switch arrows in Q

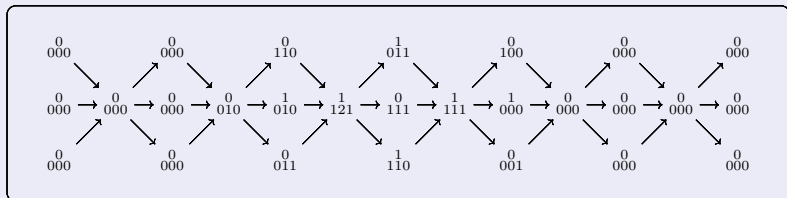
Auslander–Reiten quiver: D_4

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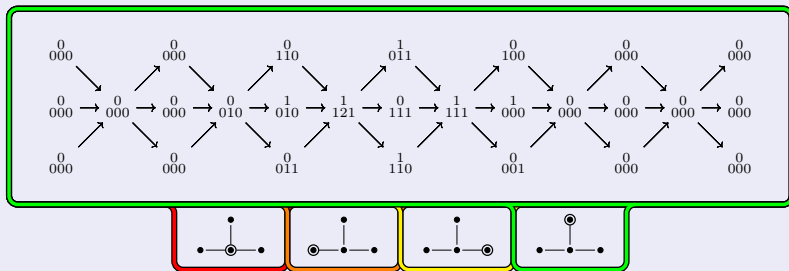
Auslander–Reiten quiver: D_4

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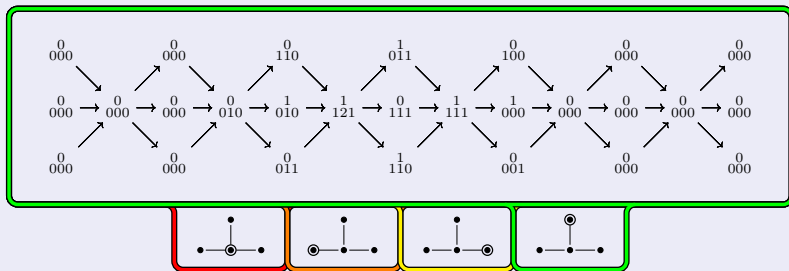
Auslander–Reiten quiver: D_4

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Auslander–Reiten quiver: D_4

$$\begin{array}{c}
 4 \\
 \downarrow \\
 2 \rightarrow 1 \leftarrow 3
 \end{array}$$



Interactive webversion

Indecomposable representations of low order are easy!

lemma [Maksimau]

Suppose Q is a tree. For $M \in \text{ind}(Q)$, $\text{ord}(M) \leq 2$,

$$\text{Flag}_{\underline{\mathbf{f}}}(M) \cong \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \quad \text{or} \quad \emptyset.$$

Example

$$Q = \begin{array}{ccccc} & & \bullet & & \\ & & \downarrow & & \\ \bullet & \rightarrow & \bullet & \leftarrow & \bullet \end{array} \quad M = \begin{array}{ccccccc} & & & & \mathbb{C} & & \\ & & & & \downarrow & & \\ \mathbb{C} & \hookrightarrow & \mathbb{C}^2 & \xleftarrow{\sim} & \mathbb{C}^2 & \twoheadrightarrow & \mathbb{C} \end{array} \quad \underline{\mathbf{f}} = \begin{pmatrix} 0 & & \\ 12 & 1 & 1 \\ 0 & & \\ 11 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned} \text{Flag}_{\underline{\mathbf{f}}}(M) &\hookrightarrow \text{Flag}_{(1)}(\mathbb{C}) \times \text{Flag}_{(2)}(\mathbb{C}^2) \times \text{Flag}_{(1)}(\mathbb{C}^2) \\ &\quad \times \text{Flag}_{(1)}(\mathbb{C}) \times \text{Flag}_{(0)}(\mathbb{C}) \\ &\cong \mathbb{P}^1 \times \mathbb{P}^1 \end{aligned}$$

Continue

	$\mathbb{C} \hookrightarrow \mathbb{C}^2$		$\mathbb{C}^2 \twoheadrightarrow \mathbb{C}$		$\mathbb{C}^2 \rightarrow \mathbb{C}^2$	
No restriction	—	2	0	—	—	1
	0	—	1	2	0	0
Reduce	1	1	1	1	1	0
Impossible	2	1	1	0	2	0
	2	0				
	1	0				

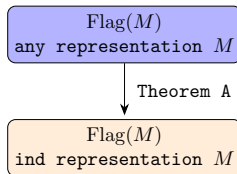
Corollary [Maksimau]

The main theorem is true for quivers of type A , D .

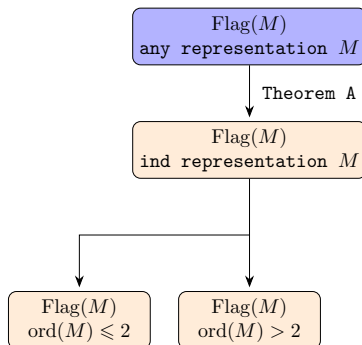
Process

$\text{Flag}(M)$
any representation M

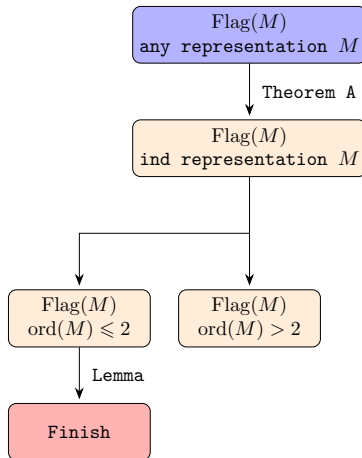
Process



Process



Process



Process

- 1 Setting and Statement
- 2 Case study
- 3 Auslander–Reiten theory
- 4 Tackle the type E case

What is remaining?

$$E_6 : \begin{array}{cccccccccc} & & 1 & & 1 & & 1 & & 1 & & \\ & 1 & & 2 & & 2 & & 2 & & 1 & \\ 1 & 1 & 2 & 1 & 3 & 2 & 3 & 1 & 2 & 1 & 1 \\ & 1 & & 2 & & 2 & & 2 & & 1 & \\ & & 1 & & 1 & & 1 & & 1 & & \end{array}$$

$$E_7 : \begin{array}{cccccccccccccccc} & & 1 & & 1 & & 1 & & 2 & & 1 & & 1 & & 1 & \\ & 1 & & 2 & & 2 & & 3 & & 3 & & 2 & & 2 & & 1 & \\ 1 & 1 & 2 & 1 & 3 & 2 & 4 & 2 & 4 & 2 & 4 & 2 & 3 & 1 & 2 & 1 & 1 \\ & 1 & & 2 & & 3 & & 3 & & 3 & & 3 & & 2 & & 1 & \\ & & 1 & & 2 & & 2 & & 2 & & 2 & & 2 & & 1 & \\ & & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \end{array}$$

$$E_8 : \begin{array}{cccccccccccccccccccc} & & & & 1 & & 1 & & 1 & & 2 & & 2 & & 2 & & 2 & & 2 & & 2 & & 2 & & 1 & & 1 & & 1 & \\ & & 1 & & 2 & & 2 & & 3 & & 4 & & 4 & & 4 & & 4 & & 4 & & 4 & & 3 & & 2 & & 2 & & 1 & \\ 1 & 1 & 2 & 1 & 3 & 2 & 4 & 2 & 5 & 3 & 6 & 3 & 6 & 3 & 6 & 3 & 6 & 3 & 6 & 3 & 5 & 2 & 4 & 2 & 3 & 1 & 2 & 1 & 1 & \\ & & 1 & & 2 & & 3 & & 4 & & 4 & & 5 & & 5 & & 5 & & 5 & & 4 & & 4 & & 3 & & 2 & & 1 & \\ & & & 1 & & 2 & & 3 & & 3 & & 3 & & 4 & & 4 & & 4 & & 3 & & 3 & & 3 & & 2 & & 1 & \\ & & & & 1 & & 2 & & 2 & & 2 & & 2 & & 3 & & 3 & & 2 & & 2 & & 2 & & 2 & & 2 & & 1 & \\ & & & & & & 1 & & 1 & & 1 & & 1 & & 1 & & 2 & & 1 & & 1 & & 1 & & 1 & & 1 & & \end{array}$$

What is remaining?

$$E_6 :$$

		1		1		1		1		
	1		2		2		2		1	
1	1	2	1	3	2	3	1	2	1	1
	1		2		2		2		1	
		1		1		1		1		

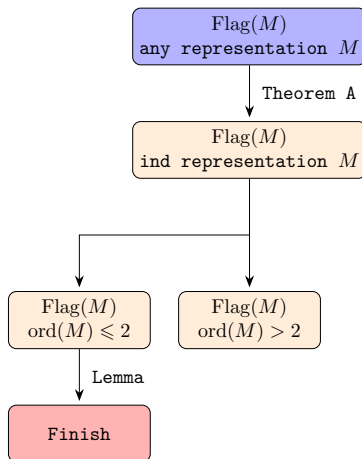
$$E_7 :$$

		1		1		1		2		1		1		1	
	1		2		2		3		3		2		2		1
1	1	2	1	3	2	4	2	4	2	4	2	3	1	2	1
	1		2		3		3		3		3		2		1
		1		2		2		2		2		2		1	
			1		1		1		1		1		1		

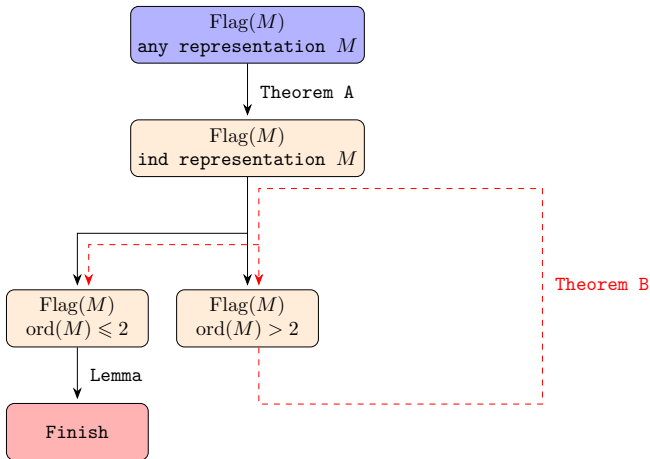
$$E_8 :$$

			1		1		1		2		2		2		2		2		2		2		1		1		1			
		1		2		2		3		4		4		4		4		4		4		4		3		2		2		1
1	1	2	1	3	2	4	2	5	3	6	3	6	3	6	3	6	3	6	3	5	2	4	2	3	1	2	1	1		
	1		2		3		4		4		5		5		5		5		4		4		3		2		1			
		1		2		3		3		3		4		4		4		3		3		3		2		1				
			1		2		2		2		2		3		3		2		2		2		2		2		1			
				1		1		1		1		1		2		1		1		1		1		1		1				

Process



Process



Consider the short exact sequence of representations

$$\eta : 0 \longrightarrow X \xrightarrow{\iota} Y \xrightarrow{\pi} S \longrightarrow 0$$

which induce maps

$$\begin{aligned} \Psi : \text{Flag}_d(Y) &\longrightarrow \text{Flag}_d(X) \times \text{Flag}_d(S) \\ \cup &\qquad \qquad \cup \\ \Psi_{\underline{\mathbf{f}}, \underline{\mathbf{g}}} : \text{Flag}(Y)_{\underline{\mathbf{f}}, \underline{\mathbf{g}}} &\longrightarrow \text{Flag}_{\underline{\mathbf{f}}}(X) \times \text{Flag}_{\underline{\mathbf{g}}}(S) \end{aligned}$$

Theorem B [Cerulli-Irelli–Esposito–Franzen–Reineke, Zhou]

When η does not split and generates $\text{Ext}^1(S, X)$,

$\Psi_{\underline{\mathbf{f}}, \underline{\mathbf{g}}}$ is a Zarisky-locally trivial affine bundle over $\text{Im } \Psi_{\underline{\mathbf{f}}, \underline{\mathbf{g}}}$.

In this case, we have a clear description of $\text{Im } \Psi_{\underline{\mathbf{f}}, \underline{\mathbf{g}}}$.

How to find nice η ?

Proposition

For $X \hookrightarrow Y$ *irreducible mono*, the induced SES

$$\eta : 0 \longrightarrow X \xrightarrow{\iota} Y \xrightarrow{\pi} S \longrightarrow 0$$

satisfies the condition of Theorem B. Moreover,

$$\operatorname{Im} \Psi_{\underline{\mathbf{f}}, \underline{\mathbf{g}}} = \begin{cases} (\operatorname{Flag}_{\underline{\mathbf{f}}}(X) \setminus \operatorname{Flag}_{\underline{\mathbf{f}}}(X_S)) \times \operatorname{Flag}_{\underline{\mathbf{g}}}(S), & \underline{\mathbf{g}}_i = \underline{\dim} S \\ \operatorname{Flag}_{\underline{\mathbf{f}}}(X) \times \operatorname{Flag}_{\underline{\mathbf{g}}}(S), & \text{otherwise} \end{cases}$$

where

$$X_S := \max \{ M \subseteq X \mid \operatorname{Ext}^1(S, X/M) \cong \mathbb{C} \} \subseteq X.$$

How to find nice η ?

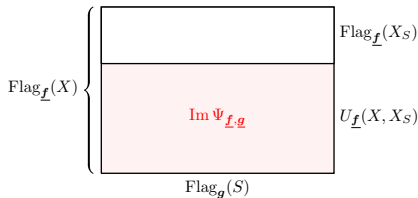
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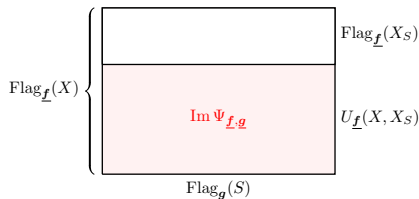
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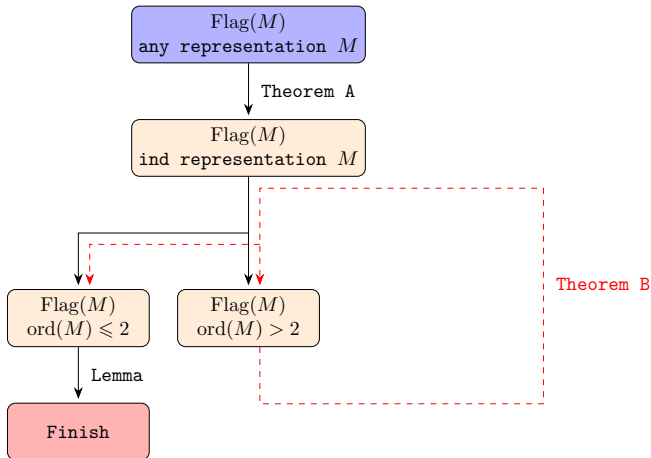
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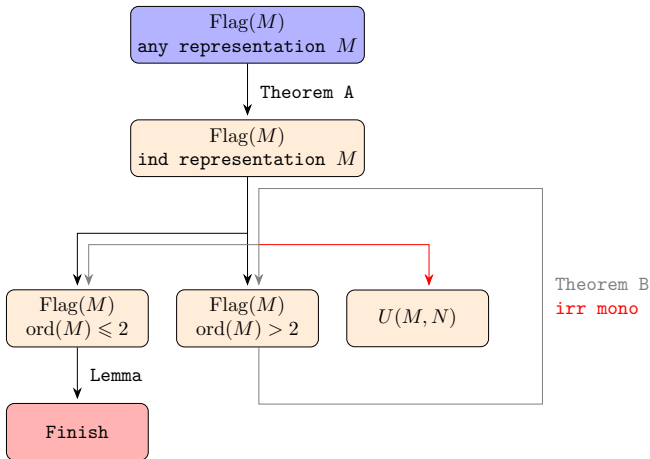
$$\mathrm{Im} \Psi_{\underline{f}, \underline{g}} = \begin{cases} U_{\underline{f}}(X, X_S), & \underline{g}_i = \underline{\dim} S \\ \mathrm{Flag}_{\underline{f}}(X) \times \mathrm{Flag}_{\underline{g}}(S), & \text{otherwise} \end{cases}$$



Process



Process



Induction?

Proposition

For $X \hookrightarrow Y$ irreducible mono, the induced SES

$$\eta : 0 \longrightarrow X \xrightarrow{\iota} Y \xrightarrow{\pi} S \longrightarrow 0$$

satisfies the condition of Theorem B. Moreover,

$$\operatorname{Im} \Psi_{\underline{\mathbf{f}}, \underline{\mathbf{g}}} = \begin{cases} U_{\underline{\mathbf{f}}}(X, X_S), & \underline{\mathbf{g}}_i = \underline{\dim} S \\ \operatorname{Flag}_{\underline{\mathbf{f}}}(X) \times \operatorname{Flag}_{\underline{\mathbf{g}}}(S), & \text{otherwise} \end{cases}$$

Proposition

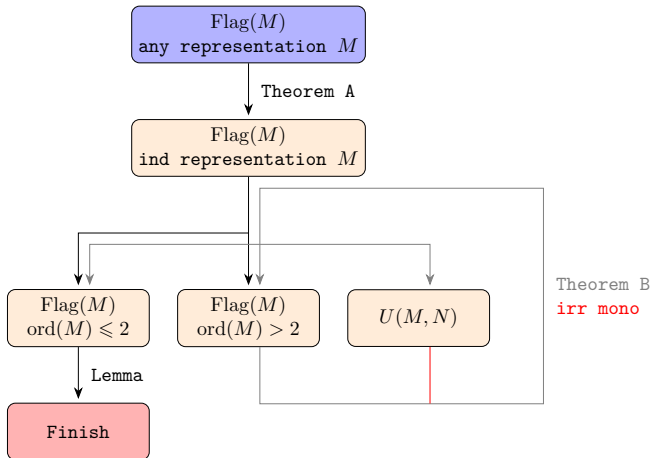
In addition,

$$X_S = 0 \quad \text{or} \quad X_S \hookrightarrow X \text{ is irreducible mono.}$$

Corollary

For $M \in \operatorname{ind}(Q)$, if exist irreducible mono $X \hookrightarrow M$, then $\operatorname{Flag}_d(M)$ has an affine paving.

Process



What is remaining?

$$E_6 :$$

		1		1		1		1		
	1		2		2		2		1	
1	1	2	1	3	2	3	1	2	1	1
	1		2		2		2		1	
		1		1		1		1		

$$E_7 :$$

		1		1		1		2		1		1		1	
	1		2		2		3		3		2		2		1
1	1	2	1	3	2	4	2	4	2	4	2	3	1	2	1
	1		2		3		3		3		3		2		1
		1		2		2		2		2		2		1	
			1		1		1		1		1		1		

$$E_8 :$$

			1		1		1		2		2		2		2		2		2		2		1		1		1			
		1		2		2		3		4		4		4		4		4		4		4		3		2		2		1
1	1	2	1	3	2	4	2	5	3	6	3	6	3	6	3	6	3	6	3	5	2	4	2	3	1	2	1	1		
		1		2		3		4		4		5		5		5		5		4		4		3		2		1		
			1		2		3		3		3		4		4		4		3		3		3		2		1			
				1		2		2		2		2		3		3		2		2		2		2		2		1		
					1		1		1		1		1		2		1		1		1		1		1		1			

What is remaining?

$$E_6 :$$

		1		1		1		1	
	1		2		2		2		1
1	1	2	1→3	2→3	1	2	1	1	
	1		2		2		2		1
		1		1		1		1	

$$E_7 :$$

		1		1		2		1		1		1	
	1		2		2	↓3	↓3		2		2		1
1	1	2	1→3	2→4	2→4	2→4	2→3	1	2	1	1		
	1		2	↗3	↗3	↗3	↗3		2		1		
		1		2		2		2		2		1	
			1		1		1		1		1		

$$E_8 :$$

			1		1		2		2		2		2		2		2		2		1		1		1
		1		2		2	↓3	↓4	↓4	↓4	↓4	↓4	↓4	↓4	↓4	↓3					2		2		1
1	1	2	1	↗3	↗4	↗5	↗6	↗3	↗6	↗3	↗6	↗3	↗6	↗3	↗6	↗3	↗5	2→4	2→3	1	2	1	1		
		1		2	↗3	↗4	↗4	↗4	↗5	↗5	↗5	↗5	↗5	↗5	↗4	↗4				3		2		1	
			1		2	↗3	↗3	↗3	↗4	↗4	↗4	↗4	↗3	↗3	↗3	↗3					2		1		
				1		2	↗3	↗2	↗2	↗3	↗3	↗2	↗2	↗2	↗2	↗3					2		1		
					1		1		1		1		2	↗3	↗1	↗1					1		1		

What is remaining?

$$E_6 :$$

		1		1		1		1		
	1		2		2		2		1	
1	1	2	1	3	2	3	1	2	1	1
	1		2		2		2		1	
		1		1		1		1		

$$E_7 :$$

		1		1		1		2		1		1		1	
	1		2		2		3		3		2		2		1
1	1	2	1	3	2	4	2	4	2	4	2	3	1	2	1
	1		2		3		3		3		3		2		1
		1		2		2		2		2		2		1	
			1		1		1		1		1		1		

$$E_8 :$$

			1		1		1		2		2		2		2		2		2		1		1		1			
		1		2		2		3		4		4		4		4		4		4		3		2		2		1
1	1	2	1	3	2	4	2	5	3	6	3	6	3	6	3	6	3	6	3	5	2	4	2	3	1	2	1	1
	1		2		3		4		4		5		5		5		5		4		4		3		2		1	
		1		2		3		3		3		4		4		4		3		3		3		3		2		1
			1		2		2		2		2		3		3		2		2		2		2		1			
				1		1		1		1		1		2		1		1		1		1		1				

Q & A

Thank you for your listening!

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Any questions?