Master thesis

Contents

Warning 0.0.1. *I made some assumptions during the writing. To avoid confusing readers, these assumptions are listed here:*

- *For quivers, all the quivers we considered (except Auslander–Reiten quivers) are connected and finite (Remark 1.2.2). For simplicity, From ??? to ???, all the quivers have no loops or cycles.*

- *For any $\varpi \in \mathbb{W}_{|\mathbf{d}|}$, we always write $\varpi = wu$, where $w \in W_{\mathbf{d}}$ and u is the shortest element in the coset $W_{\mathbf{d}}\varpi$. The flag-type dimension vector $\underline{\mathbf{d}} \in W_{\mathbf{d}} \backslash \mathbb{W}_{|\mathbf{d}|}$ corresponds to u, i.e., $\underline{\mathbf{d}} = W_{\mathbf{d}}u$. Whenever $\tilde{w}$ and $\tilde{u}$ emerge, they are always defined by $uw'u' = \tilde{w}\tilde{u}$.*

- *For the diagram, we always read from top to bottom.*

Chapter 1

Variety and stratification

1.1 Initial case: $\mathcal{F}$ and $\mathcal{F} \times \mathcal{F}$

We introduce the complete flag variety to give a bird's eye view on the whole section. Actually, the entire difficulty is bundled in this example.

Setting 1.1.1. *Fix $n \geqslant 1$, we denote $\mathrm{GL}_n := \mathrm{GL}_n(\mathbb{C})$, B, T, N, W be the standard Borel subgroup, standard torus, unipotent subgroup, Weyl group respectively, i.e.,*

$$\mathrm{GL}_n = \begin{pmatrix} * & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & * \end{pmatrix} \quad B = \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix} \quad T = \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \quad N = \begin{pmatrix} 1 & \cdots & * \\ & \ddots & \vdots \\ 0 & & 1 \end{pmatrix}$$

$$W := N_{\mathrm{GL}_n}(T)/T \cong S_n$$

1.1.1 $\mathcal{F}$

Definition 1.1.2 (Flag). *For a finite dimensional $\mathbb{C}$-vector space V, a flag of V is an increasing sequence of subspaces of V:*

$$F : 0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_k = V.$$

F is called a complete flag if $\dim V_j = j$ for all j, otherwise F is called a partial flag.

Definition 1.1.3 (Complete flag variety). *The complete flag variety $\mathcal{F}$ is defined as*

$$
\begin{aligned}
\mathcal{F} &= \mathrm{GL}_n / B \\
&\cong \{ \textit{complete flags of } \mathbb{C}^n \} \\
&= \{ 0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n = \mathbb{C}^n \mid \dim M_j = j \} \\
&\cong \{ \textit{Borel subgroups of } \mathrm{GL}_n \} \\
&= \{ gBg^{-1} \mid g \in \mathrm{GL}_n \}
\end{aligned}
$$

Remark 1.1.4.

1. $\mathcal{F}$ is a smooth projective variety of dimension $\frac{n(n+1)}{2}$, which can be seen from the embedding

$$\mathcal{F} \hookrightarrow \mathrm{Gr}(1, n) \times \cdots \times \mathrm{Gr}(n-1, n)$$

2. We implicitly give the base point of $\mathcal{F}$, which is not considered as the data of $\mathcal{F}$. Fix a standard basis of $\mathbb{C}^n$ by $\{v_1, \ldots, v_n\}$, we define the standard flag

$$F_{\mathrm{Id}} : 0 \subseteq \langle v_1 \rangle \subseteq \langle v_1, v_2 \rangle \subseteq \cdots \subseteq \langle v_1, \ldots, v_n \rangle = \mathbb{C}^n.$$

3. We have the natural GL_n-action on $\mathcal{F}$, which is considered as the data of $\mathcal{F}$.

   For $g \in \mathrm{GL}_n$, we define the flag attached to g:

$$F_g \triangleq g F_{\mathrm{Id}} : 0 \subseteq \langle gv_1 \rangle \subseteq \langle gv_1, gv_2 \rangle \subseteq \cdots \subseteq \langle gv_1, \ldots, gv_n \rangle = \mathbb{C}^n.$$

   Especially, for $w \in W = N_{\mathrm{GL}_n}(T)/T \cong S_n$, the flag attached to w

$$F_w : 0 \subseteq \langle \tilde{w}v_1 \rangle \subseteq \langle \tilde{w}v_1, \tilde{w}v_2 \rangle \subseteq \cdots \subseteq \langle \tilde{w}v_1, \ldots, \tilde{w}v_n \rangle = \mathbb{C}^n$$
$$0 \subseteq \langle v_{w(1)} \rangle \subseteq \langle v_{w(1)}, v_{w(2)} \rangle \subseteq \cdots \subseteq \langle v_{w(1)}, \ldots, v_{w(n)} \rangle = \mathbb{C}^n$$

   does not depend on the choice of the lift $\tilde{w} \in N_{\mathrm{GL}_n}(T)$ of w.

   Readers can verify that $\{ F_w \mid w \in W \}$ are all T-fixed points of $\mathcal{F}$, while $\{ wBw^{-1} \mid w \in W \}$ are all Borel subgroups of G containing the standard torus T.

4. For $n = 2$, $\mathcal{F} \cong \mathbb{P}^1$. We encourage readers to use $\mathbb{P}^1$ as a toy example for the whole theory.

interpretation	GL_n / B	flags	Borel subgroups
base point	Id	F_{Id}	B
GL_n-action	left multiplication	$\{V_i\} \mapsto \{gV_i\}$	conjugation
general point	g	F_g	gBg^{-1}

$\mathcal{F}$ is a well-studied variety, and has many combinatorical properties. For example, from the well-known Bruhat decomposition, [1]

$$\mathrm{GL}_n = \bigsqcup_{w \in W} BwB$$

We get a stratification of $\mathcal{F}$ by B-orbits:

$$\mathcal{F} = \mathrm{GL}_n / B \cong \bigsqcup_{w \in W} BwB/B$$

The B-orbit BwB/B is called the **Schubert cell**, denoted by Ω_w. Since

$$\Omega_w = BwB/B \cong B / \left(B \cap wBw^{-1} \right) \cong \mathbb{A}^{l(w)},$$

the Schubert cell is an affine space of dimension $l(w)$.

$H^i(\mathcal{F};\mathbb{C})$	0	2	4	6	8	10	12
1	1						
2	1	1					
3	1	2	2	1			
4	1	3	5	6	5	3	1
5	1	4	9	15	20	22	20

G	Orbit	G-fixed points
GL_n	$\mathcal{F} \cong \mathrm{GL}_n /B$	$\varnothing$
B	$\Omega_w \cong B/(B \cap wBw^{-1})$	$\{F_{\mathrm{Id}}\}$
T	$-$	$\{F_w \mid w \in W\}$

As a result, we know a lot of information of $\mathcal{F}$:

$\overline{\Omega}_w \subseteq \mathcal{F}$ is called the **Schubert variety**. It is well-known that

$$\overline{\Omega}_w = \bigsqcup_{w' \leq w} \Omega_w$$

as a set. Especially, for any $s \in W$ with $l(s) = 1$, denote $P_s = B \sqcup BsB$,

$$\overline{\Omega}_s = \Omega_{\mathrm{Id}} \sqcup \Omega_s = B/B \sqcup BsB/B = P_s/B \cong \mathbb{P}^1.$$

For other Schubert variety, the structures are quite dedicate and far away from the scope of this master thesis. For example, most Schubert varieties are not smooth.

1.1.2 $\mathcal{F} \times \mathcal{F}$

As a more complicated geometrical object, $\mathcal{F} \times \mathcal{F}$ works as the base space for the Steinberg variety, which turns out to be the central focus in the thesis. $\mathcal{F} \times \mathcal{F}$ has naturally a diagonal GL_n-action:

$$\mathrm{GL}_n \times \mathcal{F} \times \mathcal{F} \longrightarrow \mathcal{F} \times \mathcal{F} \qquad (g, F_1, F_2) \longmapsto (gF_1, gF_2).$$

Under this action, $\mathcal{F} \times \mathcal{F}$ has a stratification consisting of GL_n-orbits, indexed by the Weyl group:

$$\mathrm{GL}_n \backslash (\mathcal{F} \times \mathcal{F}) \cong \mathrm{GL}_n \backslash (\mathrm{GL}_n /B \times \mathrm{GL}_n /B) \cong B \backslash \mathrm{GL}_n /B \cong W \qquad \text{as sets.}$$

Denote $\boldsymbol{\Omega}_{w'} := \mathrm{GL}_n \cdot (F_{\mathrm{Id}}, F_{w'})$, then $\mathcal{F} \times \mathcal{F} = \sqcup_{w'} \boldsymbol{\Omega}_{w'}$. Moreover, by the orbit-stabilizer theorem, we get

$$\boldsymbol{\Omega}_{w'} \cong \mathrm{GL}_n / \left(B \cap w'B(w')^{-1} \right).$$

Different from $\mathcal{F}$, the GL_n-action on $\mathcal{F} \times \mathcal{F}$ is not transitive. To facilitate the stratification of $\mathcal{F} \times \mathcal{F}$, we introduce the twisted $\mathrm{GL}_n \times \mathrm{GL}_n$-action:

$$\mathrm{GL}_n \times \mathrm{GL}_n \times \mathcal{F} \times \mathcal{F} \longrightarrow \mathcal{F} \times \mathcal{F} \qquad (g_1, g_2, F_g, F_{g'}) \longmapsto (F_{g_1 g}, F_{g_1(g g_2 g^{-1})g'}).$$

---

[1]For the most time the formula does not depend on the lift of w, so we abuse the notation of $w \in N_{\mathrm{GL}_n}(T)/T$ and $\tilde{w} \in N_{\mathrm{GL}_n}(T)$.

If we write $\underline{F}_{g,g'} := (F_g, F_{gg'}) \in \mathcal{F} \times \mathcal{F}$, then

$$(g_1, g_2) \cdot \underline{F}_{g,g'} = \underline{F}_{g_1 g, g_2 g'}.$$

This $\mathrm{GL}_n \times \mathrm{GL}_n$-action is now transitive, and decompose $\mathcal{F} \times \mathcal{F}$ as disjoint union of finite many $B \times B$-orbits, which are compatible with G-orbits:

$$\boldsymbol{\Omega}_{w,w'} := (B \times B) \cdot \underline{F}_{w,w'} \subseteq \mathcal{F} \times \mathcal{F}$$

$$\mathcal{F} \times \mathcal{F} = \bigsqcup_{w,w' \in W} \boldsymbol{\Omega}_{w,w'} \qquad \boldsymbol{\Omega}_{w'} = \bigsqcup_{w \in W} \boldsymbol{\Omega}_{w,w'}$$

$$
\begin{aligned}
\boldsymbol{\Omega}_{w,w'} &\cong (B \times B)/\left\{(g_1, g_2) \in B \times B \mid (g_1, g_2) \cdot (F_w, F_{ww'}) = (F_w, F_{ww'})\right\} \\
&= (B \times B)/\left\{(g_1, g_2) \in B \times B \mid g_1 wB = wB, g_1 w g_2 w' B = ww' B\right\} \\
&= (B \times B)/\left\{(g_1, g_2) \in B \times B \mid g_1 wB = wB, (w^{-1} g_1 w) g_2 w' B = w' B\right\} \\
&= (B \times B)/\left\{(g_1, g_2) \in B \times B \mid g_1 \in wBw^{-1}, g_2 \in (w^{-1} g_1^{-1} w)\left(w' B w'^{-1}\right)\right\} \quad (1.1.1) \\
&= (B \times B)/\left\{(g_1, g_2) \in \left(B \cap wBw^{-1}\right) \times (w^{-1} g_1^{-1} w)\left(B \cap w' B w'^{-1}\right)\right\} \\
&\cong B/(B \cap wBw^{-1}) \times B/(B \cap w' B w'^{-1}) \cong \mathbb{A}^{l(w)+l(w')}
\end{aligned}
$$

We conclude the information of orbits and fixed points of $\mathcal{F} \times \mathcal{F}$ in Table 1.1:

G	Orbit	G-fixed points
$\mathrm{GL}_n \times \mathrm{GL}_n$	$\mathcal{F} \times \mathcal{F}$	$\varnothing$
GL_n	$\boldsymbol{\Omega}_{w'}$	$\varnothing$
$B \times B$	$\boldsymbol{\Omega}_{w,w'}$	$\{F_{\mathrm{Id},\mathrm{Id}}\}$
T	$-$	$\{\underline{F}_{w,w'} \mid w, w' \in W\}$

Table 1.1: Orbit and fixed points of $\mathcal{F} \times \mathcal{F}$

Like $\mathcal{F}$, we also study the closure of $\boldsymbol{\Omega}_{w'}$ and $\boldsymbol{\Omega}_{w,w'}$ in special case. It can be shown that

$$\overline{\boldsymbol{\Omega}}_{w'} = \bigsqcup_{x' \leq w'} \boldsymbol{\Omega}_{x'} \qquad \overline{\boldsymbol{\Omega}}_{w,w'} = \bigsqcup_{x \leq w, x' \leq w'} \boldsymbol{\Omega}_{x,x'}$$

as a set. Especially, for any $s \in W$ with $l(s) = 1$, [2]

$$
\begin{aligned}
\overline{\boldsymbol{\Omega}}_s = \boldsymbol{\Omega}_{\mathrm{Id}} \sqcup \boldsymbol{\Omega}_s &\cong \mathrm{GL}_n /(B \cap sBs^{-1}) \qquad\qquad \sqcup \mathrm{GL}_n /B \\
&\cong \mathrm{GL}_n \times^B \left(B/(B \cap sBs^{-1})\right) \sqcup \mathrm{GL}_n \times^B (B/B) \\
&\cong \mathrm{GL}_n \times^B (BsB/B) \qquad\qquad \sqcup \mathrm{GL}_n \times^B (B/B) \\
&\cong \mathrm{GL}_n \times^B (P_s/B)
\end{aligned}
$$

---

[2]Here, $\mathrm{GL}_n \times^B X$ is called contracted product. Roughly, it is defined by

$$\mathrm{GL}_n \times^B X := \mathrm{GL}_n \times X/\big((gb, x) \sim (g, bx)\big)$$

We will discuss contracted product in Subsection 2.4.1 thoroughly.

is an $\mathcal{F}$-bundle over $\mathbb{P}^1$. Also,

$$\overline{\mathbf{\Omega}}_{\mathrm{Id},s} = \mathbf{\Omega}_{\mathrm{Id},s} \sqcup \mathbf{\Omega}_{\mathrm{Id},\mathrm{Id}} \cong \left(B/B \times BsB/B\right) \sqcup \left(B/B \times B/B\right)$$
$$\cong P_s/B \cong \mathbb{P}^1$$

Other closure can be highly singular.

Example 1.1.5. *In the table, $n = 3$, $t = (12)$, $s = (23)$. In this case, $\mathcal{F} \times \mathcal{F}$ has 6 GL_3-orbits, and each GL_3-orbits decompose as 6 $B \times B$-orbits, with dimensions equal to $l(w) + l(w')$.*

dim $(B\times B)\cdot \underline{F}_{w,w'}$ / $B\cdot F_{ww'}$ / $B\cdot F_w$	0 Ω_{Id}	1 Ω_t	1 Ω_s	2 Ω_{ts}	2 Ω_{st}	3 Ω_{sts}
0 Ω_{Id}	0 $\mathbf{\Omega}_{\mathrm{Id},\mathrm{Id}}$	1 $\mathbf{\Omega}_{\mathrm{Id},t}$	1 $\mathbf{\Omega}_{\mathrm{Id},s}$	2 $\mathbf{\Omega}_{\mathrm{Id},ts}$	2 $\mathbf{\Omega}_{\mathrm{Id},st}$	3 $\mathbf{\Omega}_{\mathrm{Id},sts}$
1 Ω_t	2 $\mathbf{\Omega}_{t,t}$	1 $\mathbf{\Omega}_{t,\mathrm{Id}}$	3 $\mathbf{\Omega}_{t,ts}$	2 $\mathbf{\Omega}_{t,s}$	4 $\mathbf{\Omega}_{t,sts}$	3 $\mathbf{\Omega}_{t,st}$
1 Ω_s	2 $\mathbf{\Omega}_{s,s}$	3 $\mathbf{\Omega}_{s,st}$	1 $\mathbf{\Omega}_{s,\mathrm{Id}}$	4 $\mathbf{\Omega}_{s,sts}$	2 $\mathbf{\Omega}_{s,t}$	3 $\mathbf{\Omega}_{s,ts}$
2 Ω_{ts}	4 $\mathbf{\Omega}_{ts,st}$	3 $\mathbf{\Omega}_{ts,s}$	5 $\mathbf{\Omega}_{ts,sts}$	2 $\mathbf{\Omega}_{ts,\mathrm{Id}}$	4 $\mathbf{\Omega}_{ts,ts}$	3 $\mathbf{\Omega}_{ts,t}$
2 Ω_{st}	4 $\mathbf{\Omega}_{st,ts}$	5 $\mathbf{\Omega}_{st,sts}$	3 $\mathbf{\Omega}_{st,t}$	4 $\mathbf{\Omega}_{st,st}$	2 $\mathbf{\Omega}_{st,\mathrm{Id}}$	3 $\mathbf{\Omega}_{st,s}$
3 Ω_{sts}	6 $\mathbf{\Omega}_{sts,sts}$	5 $\mathbf{\Omega}_{sts,ts}$	5 $\mathbf{\Omega}_{sts,st}$	4 $\mathbf{\Omega}_{sts,t}$	4 $\mathbf{\Omega}_{sts,s}$	3 $\mathbf{\Omega}_{sts,\mathrm{Id}}$

Table 1.2: stratifications of $\mathcal{F} \times \mathcal{F}$

We can also see the GL_3-orbit (and its closure) from the table, for example,

$$\mathbf{\Omega}_s = \mathbf{\Omega}_{\mathrm{Id},s} \sqcup \mathbf{\Omega}_{t,s} \sqcup \mathbf{\Omega}_{s,s} \sqcup \mathbf{\Omega}_{ts,s} \sqcup \mathbf{\Omega}_{st,s} \sqcup \mathbf{\Omega}_{sts,s}$$
$$\overline{\mathbf{\Omega}}_s = \mathbf{\Omega}_s \sqcup \mathbf{\Omega}_{\mathrm{Id}} = \bigsqcup_w (\mathbf{\Omega}_{w,s} \sqcup \mathbf{\Omega}_{w,\mathrm{Id}})$$

Now wee understand a lot about $\mathcal{F}$ and $\mathcal{F} \times \mathcal{F}$, and the whole process of analysis(investigations?) will be applied repeatedly in Section 1.5 and 1.6.

1.2   Quiver

To introduce more complicated spaces and discuss their stratifications, we fix notations related to quiver and algebraic group in the following sections.

Roughly speaking, a quiver is a directed multigraph permitting loops.

**Definition 1.2.1** (Quiver)**.** *A quiver is a quadruple*

$$Q = (Q_0, Q_1, s, t)$$

where
- Q_0 *is a non-empty set consisting of vertices of Q,*
- Q_1 *is a set consisting of arrows of Q,*
- $s : Q_1 \longrightarrow Q_0$ *is a map indicating the start vertex of arrows,*
- $t : Q_1 \longrightarrow Q_0$ *is a map indicating the terminal vertex of arrows.*

Remark 1.2.2. In the first part of our master thesis, all the quivers are supposed to be connected and finite (i.e., Q_0, Q_1 are finite sets). We will only encounter disconnected and infinite quiver as Auslander–Reiten quiver later on.
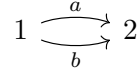
Example 1.2.3. *The following graphs are quivers.*
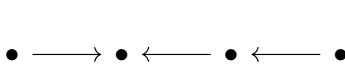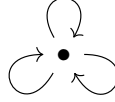
quiver of type A_3 3-loop quiver $L(3)$ 2-Kronecker quiver $K(2)$

The reader can easily read the quadruple of quivers from the graphs. Take $Q = K(2)$ as an example, we have

$$Q_0 = \{1, 2\}, \qquad Q_1 = \{a, b\} \qquad s, t : \{a, b\} \longrightarrow \{1, 2\}$$

by $s(a) = s(b) = 1$, $t(a) = t(b) = 2$.

For convenience, we mainly use simpler quivers as examples:

trivial quiver   quiver of type A_1 1-loop quiver $L(1)$

From those quivers we are able to construct relatively complicated algebraic and geometrical objects.

Definition 1.2.4 (Quiver representation)**.** *Fix a quiver Q. A representation of Q consists of the following data:*
- *A finite dimensional $\mathbb{C}$-vector space V_i for each vertex $i \in Q_0$;*
- *A $\mathbb{C}$-linear map $V_a : V_{s(a)} \longrightarrow V_{t(a)}$ for each arrow $a \in Q_1$.*

Example 1.2.5. *A representation of* 1*-loop quiver* $L(1)$ *is a 2-tuple*

$$(V, \; \alpha : V \longrightarrow V)$$

which is equivalent to a (finite dimensional) $\mathbb{C}[t]$*-module.*

Remark 1.2.6. The equivalence appeared in the example can actually be generalized to arbitrary quivers. For a quiver Q, we can define the path algebra $\mathbb{C}Q$, and view any Q-representation as $\mathbb{C}Q$-module, and vice versa.

For many constructions, we only care about the data of vector space.

Definition 1.2.7 (Q-vector space/Vector space with quiver partition)**.** *Fix a quiver Q, a Q-vector space is a finite dimensional $\mathbb{C}$-vector space with the direct sum decomposition*

$$V = \bigoplus_{i \in Q_0} V_i.$$

The dimension vector of a Q-vector space is defined as

$$\underline{\mathbf{dim}}\, V = (\dim_{\mathbb{C}} V_i)_{i \in Q_0} \subseteq \prod_{i \in Q_0} \mathbb{Z}.$$

On the country, given $\mathbf{d} \in \prod_{i \in Q_0} \mathbb{N}_{\geqslant 0}$, we can construct a canonical Q-vector space of dimension vector $\mathbf{d}$, as follows:

$$V = \bigoplus_{i \in Q_0} V_i \qquad \text{with } V_i = \mathbb{C}^{\mathbf{d}_i}.$$

Definition 1.2.8. *The total dimension vector of a Q-vector space V is defined as*

$$|\underline{\mathbf{dim}}\, V| := \dim_{\mathbb{C}} V.$$

For $\mathbf{d} \in \prod_{i \in Q_0} \mathbb{N}_{\geqslant 0}$, denote $|\mathbf{d}| := \sum_{i \in Q_0} \mathbf{d}_i$.

Definition 1.2.9 (Space of representations with given dimension vector)**.** *For any quiver Q, dimension vector $\mathbf{d}$, fix the canonical Q-vector space $V = \bigoplus_{i \in Q_0} V_i$, the space of representations with dimension vector $\mathbf{d}$ is defined as*

$$\begin{aligned}
\operatorname{Rep}_{\mathbf{d}}(Q) &= \left\{ (V_i, V_a : V_{s(a)} \longrightarrow V_{t(a)}) \text{ as a representation of } Q \right\} \\
&= \bigoplus_{a \in Q_1} \operatorname{Hom}\left(\mathbb{C}^{\mathbf{d}_{s(a)}}, \mathbb{C}^{\mathbf{d}_{t(a)}} \right)
\end{aligned}$$

Since we encode the information of vector space in $\mathbf{d}$, $\operatorname{Rep}_{\mathbf{d}}(Q)$ only records the information of linear maps.

For both Q-vector space and Q-representations, we can define (complete) flags.

Definition 1.2.10 (Flag with quiver)**.** *For a quiver representation $V \in \text{rep}(Q)$, a flag of V is defined as an increasing sequence of subrepresentation of V, i.e.,*

$$F : 0 \subseteq M_1 \subseteq \ldots \subseteq M_k = V \qquad M_j \in \text{rep}(Q).$$

For a Q-vector space $V = \oplus_{i \in Q_0} V_i$, a (quiver-graded) flag of V is defined as an increasing sequence of Q-subspace of V, i.e.,

$$F : 0 \subseteq M_1 \subseteq \ldots \subseteq M_k = V \qquad M_j = \bigoplus_{i \in Q_0} M_{j,i}.$$

For both Q-vector space and Q-representation, F is called a complete flag if $k = \dim_{\mathbb{C}} V$ and

$$\dim_{\mathbb{C}} M_j = j \qquad \textit{for any } j \in \{1, \ldots, |\mathbf{d}|\}$$

For the flag we also have the notation of dimension vector.

Definition 1.2.11 (flag-type dimension vector)**.** *For any flag $F : 0 \subseteq M_1 \subseteq \ldots \subseteq M_k = V$, the dimension vector of F is defined as*

$$\underline{\mathbf{d}} = \left(\underline{\dim} M_j \right)_{j \in \{1, \ldots, k\}} \subseteq \prod_{\substack{i \in Q_0 \\ j \in \{1, \ldots, k\}}} \mathbb{Z}.$$

$\underline{\mathbf{d}}$ is called a flag-type dimension vector if $\underline{\mathbf{d}}$ is the dimension vector of some complete flag F, i.e., [3]

$$|\underline{\dim} M_{j+1}/M_j| = 1 \qquad \textit{for any } j \in \{0, \ldots, |\mathbf{d}| - 1\}.$$

Example 1.2.12. *For quiver $Q : i \longrightarrow i'$, $\mathbf{d} = (3, 2)$, the canonical Q-vector space of dimension vector $\mathbf{d}$ is*

$$\begin{aligned}
V &= V_i \oplus V_{i'} \\
&= \langle v_1, v_2, v_3 \rangle_{\mathbb{C}} \oplus \langle v_4, v_5 \rangle_{\mathbb{C}}
\end{aligned}$$

The flag

$$F : 0 \subseteq \langle v_4 \rangle \subseteq \langle v_4, v_1 \rangle \subseteq \langle v_4, v_1, v_2 \rangle \subseteq \langle v_4, v_1, v_2, v_5 \rangle \subseteq \langle v_4, v_1, v_2, v_5, v_3 \rangle = V$$

is a complete flag of V, with dimension vector

$$\underline{\mathbf{d}} = \begin{pmatrix} 3,2 \\ 2,2 \\ 2,1 \\ 1,1 \\ 0,1 \end{pmatrix}.$$

Remark 1.2.13. The flag-type dimension vector $\underline{\mathbf{d}}$ can be viewed as a partition on set $\{1, \ldots, |\mathbf{d}|\}$, i.e., a map

$$\text{par} : \{1, \ldots, |\mathbf{d}|\} \longrightarrow Q_0$$

---

[3]For convenience, we denote M_0 by 0.

such that $\# \operatorname{par}^{-1}(i) = \mathbf{d}_i.$[4] . As an example,

$$\underline{\mathbf{d}} = \begin{pmatrix} 3,2 \\ 2,2 \\ 2,1 \\ 1,1 \\ 0,1 \end{pmatrix} \quad \text{corresponds to} \quad \{1,2,3,4,5\} = \{2,3,5\} \sqcup \{1,4\}.$$

1.3 Symmetric group calculus

As a reminder, we recall some basic diagrams referring to the elements in S_n, and do some calculations by these diagrams. We will also relate cosets with flag-type dimension vectors.

Fix a quiver Q and dimension vector $\mathbf{d}$. Later (Definition 1.4.2, 1.4.3) we will define

$$\mathbb{W}_{|\mathbf{d}|} = S_{|\mathbf{d}|} \qquad W_{\mathbf{d}} = \prod_{i \in Q_0} S_{\mathbf{d}_i} \leqslant \mathbb{W}_{|\mathbf{d}|}$$

For simplicity, we take $Q_0 = \{1,\ldots,k\}$, then $W_{\mathbf{d}} = S_{\mathbf{d}_1} \times \cdots \times S_{\mathbf{d}_k}$ embed in $S_{|\mathbf{d}|}$ in the most natural way.

Remark 1.3.1. We have different ways to express $\varpi \in \mathbb{W}_{|\mathbf{d}|} = S_{|\mathbf{d}|}$. For example, take $|\mathbf{d}| = 5$, $\varpi \in S_5$ by

$$\varpi(1) = 4, \quad \varpi(2) = 3, \quad \varpi(3) = 1, \quad \varpi(4) = 5, \quad \varpi(5) = 2,$$

then

$$\varpi = (14523) = \begin{pmatrix} 1\,2\,3\,4\,5 \\ 4\,3\,1\,5\,2 \end{pmatrix} = \underset{\substack{1\ 2\ 3\ 4\ 5}}{\overset{\substack{1\ 2\ 3\ 4\ 5}}{\times}} = \begin{bmatrix} & & & 1 & \\ & & & & 1 \\ & 1 & & & \\ 1 & & & & \\ & & 1 & & \end{bmatrix}$$

$$= (23)(34)(45)(12)(23)(12) = \underset{\substack{1\ 2\ 3\ 4\ 5}}{\overset{\substack{1\ 2\ 3\ 4\ 5}}{\diagdown}}$$

Even though all expressions give us the same amount of information, the diagram presents them more vividly. For example, each intersection of strands corresponds to a simple reflection, so we read from the diagram that $l(\varpi) = 6$. Readers can also check that

$$l(\varpi s_1) = 5, \quad l(\varpi s_2) = 5, \quad l(\varpi s_3) = 7, \quad l(\varpi s_4) = 5,$$
$$l(s_1 \varpi) = 7, \quad l(s_2 \varpi) = 5, \quad l(s_3 \varpi) = 5, \quad l(s_4 \varpi) = 7,$$

where $s_i := (i, i+1) \in S_5$ are simple reflections.

---

[4]The partition corresponding to par is

$$\{1,\ldots,|\mathbf{d}|\} = \bigsqcup_{i \in Q_0} \operatorname{par}^{-1}(i).$$

Definition 1.3.2 (Special elements in the Weyl group)**.** *For $i \in \{1, \ldots, |\mathbf{d}| - 1\}$, the simple reflection is defined as*

$$s_i := (i, i+1) \in S_{|\mathbf{d}|}.$$

We denote

$$\Pi = \left\{ s_i \in S_{|\mathbf{d}|} \ \middle| \ i \in \{1, \ldots, |\mathbf{d}| - 1\} \right\}$$

$$\Pi_{\mathbf{d}} = \left\{ s_i \in S_{\mathbf{d}_1} \times \cdots \times S_{\mathbf{d}_k} \ \middle| \ i \in \{1, \ldots, |\mathbf{d}| - 1\} \right\}$$

$$= \left\{ s_1, \ldots, s_{|\mathbf{d}|-1} \right\} \smallsetminus \left\{ s_{\mathbf{d}_1}, s_{\mathbf{d}_1 + \mathbf{d}_2}, \ldots, s_{\mathbf{d}_1 + \cdots + \mathbf{d}_{k-1}} \right\}$$

to be the set of simple reflections in $\mathbb{W}_{|\mathbf{d}|}$ and $W_{\mathbf{d}}$, respectively.

We also denote $\varpi_{\max} \in \mathbb{W}_{|\mathbf{d}|}$, $w_{\max} \in W_{\mathbf{d}}$ to be the longest elements in $\mathbb{W}_{|\mathbf{d}|}$, $W_{\mathbf{d}}$, respectively.

We discuss about right cosets $W_{\mathbf{d}} \backslash \mathbb{W}_{|\mathbf{d}|}$ and minimal length coset representatives now.

Multiplying on left by $w \in W_{\mathbf{d}}$ is equivalent to plugging in a diagram representing $w \in W_{\mathbf{d}}$ underneath the original diagram. Therefore, we connect some bottom points by lines, indicating that switching them will cause no trouble. Furthermore, we color different parts to make the following fact more explicitly.

Fact 1.3.3. *Every element $\varpi_{\max} \in W_{\mathbf{d}} \backslash \mathbb{W}_{|\mathbf{d}|}$ corresponds to a partition on set $\{1, \ldots, |\mathbf{d}|\}$ (of a given number partition $\mathbf{d}$), which corresponds to a flag-type dimension vector $\underline{\mathbf{d}}$.*

Example 1.3.4.
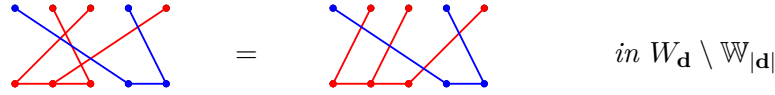
since
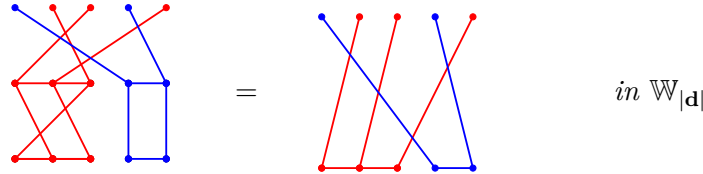
This coset corresponds to the partition $\{1, 2, 3, 4, 5\} = \{2, 3, 5\} \sqcup \{1, 4\}$.

It is easy to see from the diagram that in every coset, there exists an unique element $u \in \mathbb{W}_{|\mathbf{d}|}$ of minimal length. We collect these minimal length coset representatives as a set, and denote it by $\mathrm{Min}(\mathbb{W}_{|\mathbf{d}|}, W_{\mathbf{d}})$.[5]

Proposition 1.3.5. *For any $\varpi \in \mathbb{W}_{|\mathbf{d}|}$, exists unique $w \in W_{\mathbf{d}}$, $u \in \mathrm{Min}(\mathbb{W}_{|\mathbf{d}|}, W_{\mathbf{d}})$ such that $\varpi = wu$.*

---

[5]In some references $\mathrm{Min}(\mathbb{W}_{|\mathbf{d}|}, W_{\mathbf{d}})$ is also denoted by $\mathrm{Schuffle}_{\mathbf{d}}$, since those elements can be thought as ways off riffle shuffling several words together.

Exercise 1.3.6. *For* $u \in \mathrm{Min}(\mathbb{W}_{|\mathbf{d}|}, \mathrm{W}_{\mathbf{d}})$, $s_i \in \Pi$, *show that*

$$us_i u^{-1} \in W_{\mathbf{d}} \implies us_i u^{-1} = s_{u(i)} \in \Pi_{\mathbf{d}},$$
$$us_i u^{-1} \notin W_{\mathbf{d}} \implies us_i \in \mathrm{Min}(\mathbb{W}_{|\mathbf{d}|}, \mathrm{W}_{\mathbf{d}}).$$

An example is shown in Table 1.4.

We finish this section with figures and examples.

Example 1.3.7. *In this table,* $|\mathbf{d}| = 5$, $\mathbf{d} = (3, 2)$, *typical elements would be*

set	element	special element		others
$\mathbb{W}_{|\mathbf{d}|} = S_5$	ϖ, x	$\varpi_{\max} = $		$\Pi = \{s_1, s_2, s_3, s_4\}$
$W_{\mathbf{d}} = S_3 \times S_2$	w	$w_{\max} = $		$\Pi_{\mathbf{d}} = \{s_1, s_2, \quad s_4\}$
$W_{\mathbf{d}} \backslash \mathbb{W}_{|\mathbf{d}|} \cong (S_3 \times S_2) \backslash S_5$	$\varpi, \underline{\mathbf{d}}$			$\mathrm{Comp}_{\mathbf{d}}$
$\mathrm{Min}(\mathbb{W}_{|\mathbf{d}|}, \mathrm{W}_{\mathbf{d}}) = \left\{ \text{}, \ldots \right\}$	u			$\mathrm{Schuffle}_{\mathbf{d}}$

Example 1.3.8. *In this table,*

$$|\mathbf{d}| = 3, \quad \mathbf{d} = (1, 2), \qquad \mathbb{W}_{|\mathbf{d}|} = S_3, \quad W_{\mathbf{d}} = S_1 \times S_2, \qquad s = (12), \quad t = (23).$$

The columns "order of basis" and "Borel subgroups" will be introduced in Definition 1.5.5 and Remark 1.4.4.

1.4 Algebraic group and Lie algebra

In this section we fix notations of algebraic group and Lie algebras. Later, the algebraic group will act on varieties, and some Lie algebra will serve as tangent spaces.

|		$\varpi = wu$			w	$\underline{\mathbf{d}}, u$		order of basis	$l(\varpi)$	$l(w)$	$\mathbb{B}_\varpi$	B_ϖ	$\varpi B_{\mathbf{d}}\varpi^{-1}$
Id	Id	$\binom{1\,2\,3}{1\,2\,3}$	(diagram)	$\left[\begin{smallmatrix}1&&\\&1&\\&&1\end{smallmatrix}\right]$	(diagram)	abb	(diagram)	$\{v_1,v_2,v_3\}$	0	0	$\left[\begin{smallmatrix}*&*&*\\&*&*\\&&*\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}*&&\\&*&*\\&&*\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}*&&\\&*&*\\&&*\end{smallmatrix}\right]$
t	(23)	$\binom{1\,2\,3}{1\,3\,2}$	(diagram)	$\left[\begin{smallmatrix}1&&\\&&1\\&1&\end{smallmatrix}\right]$	(diagram)	abb	(diagram)	$\{v_1,v_3,v_2\}$	1	1	$\left[\begin{smallmatrix}*&*&*\\&*&\\&*&*\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}*&&\\&*&\\&*&*\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}*&&\\&*&\\&*&*\end{smallmatrix}\right]$
s	(12)	$\binom{1\,2\,3}{2\,1\,3}$	(diagram)	$\left[\begin{smallmatrix}&1&\\1&&\\&&1\end{smallmatrix}\right]$	(diagram)	bab	(diagram)	$\{v_2,v_1,v_3\}$	1	0	$\left[\begin{smallmatrix}*&&*\\ *&*&*\\&&*\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}*&&\\&*&*\\&&*\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}*&&*\\&*&\\&&*\end{smallmatrix}\right]$
ts	(132)	$\binom{1\,2\,3}{3\,1\,2}$	(diagram)	$\left[\begin{smallmatrix}&1&\\&&1\\1&&\end{smallmatrix}\right]$	(diagram)	bab	(diagram)	$\{v_3,v_1,v_2\}$	2	1	$\left[\begin{smallmatrix}*&*&\\&*&\\ *&*&*\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}*&&\\&*&\\&*&*\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}*&*&\\&*&\\&&*\end{smallmatrix}\right]$
st	(123)	$\binom{1\,2\,3}{2\,3\,1}$	(diagram)	$\left[\begin{smallmatrix}&&1\\1&&\\&1&\end{smallmatrix}\right]$	(diagram)	bba	(diagram)	$\{v_2,v_3,v_1\}$	2	0	$\left[\begin{smallmatrix}*&&\\ *&*&*\\&*&*\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}*&&\\&*&*\\&&*\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}*&&\\&*&\\ *&&*\end{smallmatrix}\right]$
sts	(13)	$\binom{1\,2\,3}{3\,2\,1}$	(diagram)	$\left[\begin{smallmatrix}&&1\\&1&\\1&&\end{smallmatrix}\right]$	(diagram)	bba	(diagram)	$\{v_3,v_2,v_1\}$	3	1	$\left[\begin{smallmatrix}*&&\\ *&*&\\ *&*&*\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}*&&\\&*&\\ *&*&*\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}*&&\\ *&*&\\ *&&*\end{smallmatrix}\right]$

Table 1.3: basic information of $(1,2)$-case

Setting 1.4.1. *We fix a quiver Q, a dimension vector $\mathbf{d}$ and a Q-vector space*

$$V = \bigoplus_{i\in Q_0} V_i \qquad with\ V_i = \mathbb{C}^{\mathbf{d}_i}.$$

When a basis of V is needed, we fix a total order on Q_0, and denote

$$V = \left\langle v_1, \dots v_{|\mathbf{d}|} \right\rangle$$

where

$$V_i = \left\langle v_{f_i+1}, \dots v_{f_i+\mathbf{d}_i} \right\rangle \qquad f_i = \sum_{i'<i} \mathbf{d}_{i'}.$$

1.4.1 Algebraic group

Definition 1.4.2 (absolute algebraic groups)**.** *We set*

$$\mathbb{G}_{|\mathbf{d}|} := \mathrm{GL}(V) = \mathrm{GL}_{|\mathbf{d}|}(\mathbb{C}),$$

and $\mathbb{B}_{|\mathbf{d}|}$, $\mathbb{T}_{|\mathbf{d}|}$, $\mathbb{N}_{|\mathbf{d}|}$ are corresponding standard Borel, torus and unipotent subgroups.
  The Weyl group is
$$\mathbb{W}_{|\mathbf{d}|} := N_{\mathbb{G}_{|\mathbf{d}|}}(\mathbb{T}_{|\mathbf{d}|})/\mathbb{T}_{|\mathbf{d}|} \cong S_{|\mathbf{d}|}.$$

  For $\varpi \in \mathbb{W}_{|\mathbf{d}|}$, we define[6]
$$\mathbb{B}_\varpi := \varpi\mathbb{B}_{|\mathbf{d}|}\varpi^{-1}.$$

We will view $\mathbb{B}_\varpi$ as the stabilizer of the flag F_ϖ with $\mathbb{G}_{|\mathbf{d}|}$-action.

  We also have a series of algebraic groups compatible with the quiver partition of V, and they're more common in this thesis.

---

[6]As usual, we abuse the notation of ϖ and its lift.

Definition 1.4.3 (relative algebraic groups)**.** *We set*

$$G_{\mathbf{d}} := \bigoplus_{i \in Q_0} \mathrm{GL}(V_i) = \mathrm{GL}_{\mathbf{d}_i}(\mathbb{C}) \subseteq \mathbb{G}_{|\mathbf{d}|},$$

and $B_{\mathbf{d}}$, $T_{\mathbf{d}}$, $N_{\mathbf{d}}$ are corresponding standard Borel, torus and unipotent subgroups.
The Weyl group is

$$W_{\mathbf{d}} := N_{G_{\mathbf{d}}}(T_{\mathbf{d}})/T_{\mathbf{d}} \cong \prod_{i \in Q_0} S_{\mathbf{d}_i}.$$

For $\varpi = wu \in W_{\mathbf{d}}$, we define

$$B_{\varpi} := wB_{\mathbf{d}}w^{-1}.$$

We will view B_{ϖ} as the stabilizer of the flag F_{ϖ} with $G_{\mathbf{d}}$-action.

Remark 1.4.4. Be careful that $B_{\varpi} \neq \varpi B_{\mathbf{d}} \varpi^{-1}$. Actually,

$$B_{\varpi} = \varpi \mathbb{B}_{|\mathbf{d}|} \varpi^{-1} \cap B_{\mathbf{d}} = wB_{\mathbf{d}}w^{-1}$$

The difference is clearly shown in Table 1.3.

We also have a series of algebraic groups with subscription as elements in the Weyl group:

Definition 1.4.5 (more algebraic groups)**.** *For $\varpi, \varpi'' \in \mathbb{W}_{|\mathbf{d}|}$, define*

$$\begin{aligned} N_{\varpi} &:= R_u(B_{\varpi}), \\ N_{\varpi,\varpi''} &:= N_{\varpi} \cap N_{\varpi''}, \\ M_{\varpi,\varpi''} &:= N_{\varpi}/N_{\varpi,\varpi''}, \end{aligned}$$

where R_u denotes for the unipotent radical.
For $s \in \Pi$ such that $\varpi s \varpi^{-1} \in W_d$ (i.e., $W_{\mathbf{d}}\varpi = W_{\mathbf{d}}\varpi s$), define

$$\begin{aligned} P_{\varpi,\varpi s} :&\overset{\varpi=wu}{=\!=\!=\!=} w\left(B_{\mathbf{d}}usu^{-1}B_{\mathbf{d}} \cup B_{\mathbf{d}}\right)w^{-1} \\ &=\!=\!=\!= B_{\varpi}\varpi s\varpi^{-1}B_{\varpi} \cup B_{\varpi} \end{aligned}$$

Remark 1.4.6. One can easily show that $N_{\varpi,\varpi s} = R_u(P_{\varpi,\varpi s})$.

Example 1.4.7 (Follows Example 1.3.7)**.** *For $|\mathbf{d}| = 5$, $\mathbf{d} = (3,2)$, $\varpi =$, $w =$*

$\diagup\!\!\!\diagup \; \diagup\!\!\!\diagdown$, $s = s_2$, we compute all the algebraic groups we mentioned:

$$\mathbb{G}_{|\mathbf{d}|} = \begin{pmatrix} * * * * * \\ * * * * * \\ * * * * * \\ * * * * * \\ * * * * * \end{pmatrix} \qquad \mathbb{B}_{|\mathbf{d}|} = \begin{pmatrix} * * * * * \\ * * * * \\ * * * \\ * * \\ * \end{pmatrix} \qquad \mathbb{T}_{|\mathbf{d}|} = \begin{pmatrix} * \\ \quad * \\ \quad\quad * \\ \quad\quad\quad * \\ \quad\quad\quad\quad * \end{pmatrix} \qquad \mathbb{N}_{|\mathbf{d}|} = \begin{pmatrix} 1 * * * * \\ 1 * * * \\ 1 * * \\ 1 * \\ 1 \end{pmatrix}$$

$$\mathbb{W}_{|\mathbf{d}|} \cong S_5 \qquad \mathbb{B}_\varpi = \begin{pmatrix} * * & \quad * \\ * & \\ * * * & \quad * \\ * * * * * \\ * & \quad * \end{pmatrix} \qquad \mathbb{B}_{\varpi s} = \begin{pmatrix} * * * & \quad * \\ * & \\ * * & \quad * \\ * * * * * \\ * & \quad * \end{pmatrix}$$

$$G_{\mathbf{d}} = \begin{pmatrix} * * * & | \\ * * * & | \\ * * * & | \\ \hline & | * * \\ & | * * \end{pmatrix} \qquad B_{\mathbf{d}} = \begin{pmatrix} * * * & | \\ * * & | \\ * & | \\ \hline & | * * \\ & | * \end{pmatrix} \qquad T_{\mathbf{d}} = \begin{pmatrix} * & | \\ \; * & | \\ \;\; * & | \\ \hline & | * \\ & | \; * \end{pmatrix} \qquad N_{\mathbf{d}} = \begin{pmatrix} 1 * * & | \\ 1 * & | \\ 1 & | \\ \hline & | 1 * \\ & | 1 \end{pmatrix}$$

$$W_{\mathbf{d}} \cong S_3 \times S_2 \qquad B_\varpi = \begin{pmatrix} * * & | \\ * & | \\ * * * & | \\ \hline & | * \\ & | * * \end{pmatrix} \qquad B_{\varpi s} = \begin{pmatrix} * & | \\ * * & | \\ * * * & | \\ \hline & | * \\ & | * * \end{pmatrix}$$

$$N_\varpi = \begin{pmatrix} 1 * & | \\ 1 & | \\ * * 1 & | \\ \hline & | 1 \\ & | * 1 \end{pmatrix} \qquad N_{\varpi,\varpi s} = \begin{pmatrix} 1 & | \\ 1 & | \\ * * 1 & | \\ \hline & | 1 \\ & | * 1 \end{pmatrix} \qquad M_{\varpi,\varpi s} = \begin{pmatrix} 1 * & | \\ 1 & | \\ - - 1 & | \\ \hline & | 1 \\ & | - 1 \end{pmatrix} \qquad P_{\varpi,\varpi s} = \begin{pmatrix} * * & | \\ * * & | \\ * * * & | \\ \hline & | * \\ & | * * \end{pmatrix}$$

1.4.2   Lie algebra

For the Lie algebra, we use the corresponding Fraktur-font symbols:

$$\mathfrak{g}_{|\mathbf{d}|}, \qquad \mathfrak{b}_{|\mathbf{d}|}, \qquad \mathfrak{t}_{|\mathbf{d}|}, \qquad \mathfrak{n}_{|\mathbf{d}|}, \qquad \mathfrak{b}_\varpi$$
$$\mathfrak{g}_{\mathbf{d}}, \qquad \mathfrak{b}_{\mathbf{d}}, \qquad \mathfrak{t}_{\mathbf{d}}, \qquad \mathfrak{n}_{\mathbf{d}}, \qquad \mathfrak{b}_{\varpi},$$
$$\mathfrak{n}_\varpi, \qquad \mathfrak{n}_{\varpi,\varpi''}, \qquad \mathfrak{m}_{\varpi,\varpi''}, \qquad \mathfrak{p}_{\varpi,\varpi s},$$

We also have to encode the information of representations as Lie algebra. Notice that

$$\operatorname{Hom}(V_{s(a)}, V_{t(a)}) \hookrightarrow \operatorname{Hom}(V, V) \cong \mathfrak{g}_{|\mathbf{d}|} \qquad f \longmapsto \iota_{t(a)} \circ f \circ \pi_{s(a)}$$

realizes $\operatorname{Hom}(V_{s(a)}, V_{t(a)})$ as a Lie subalgebra of $\mathfrak{g}_{|\mathbf{d}|}$, so

$$\operatorname{Rep}_{\mathbf{d}}(Q) = \bigoplus_{a \in Q_1} \operatorname{Hom}\left(\mathbb{C}^{\mathbf{d}_{s(a)}}, \mathbb{C}^{\mathbf{d}_{t(a)}} \right) \subseteq \bigoplus_{a \in Q_1} \mathfrak{g}_{|\mathbf{d}|}.$$

Definition 1.4.8 (Lie algebras connected with representations)**.** *For $\varpi \in \mathbb{W}_{|\mathbf{d}|}$, denote temperately*

$$V_{\varpi,j} := \left\langle e_{\varpi(1)}, \ldots e_{\varpi(j)} \right\rangle \subseteq V.$$

We define Lie subalgebras of $\operatorname{Rep}_{\mathbf{d}}(Q)$ as follows.

$$\mathfrak{r}_\varpi := \left\{ (f_a)_{a \in Q_1} \in \operatorname{Rep}_{\mathbf{d}}(Q) \mid f_a(V_{\varpi,j} \cap V_{s(a)}) \subseteq V_{\varpi,j} \text{ for any } j \right\},$$

$$\mathfrak{r}_{\varpi,\varpi''} := \mathfrak{r}_\varpi \cap \mathfrak{r}_{\varpi''},$$

$$\mathfrak{d}_{\varpi,\varpi''} := \mathfrak{r}_\varpi / \mathfrak{r}_{\varpi,\varpi''},$$

Example 1.4.9 (Follows Example 1.4.7)**.** *Consider the quiver* • ⟶ • *, and* $u =$ *. Table 1.4 gives us an example of the shape of these Lie algebras.*

|		$\mathfrak{n}_u$	$\mathfrak{m}_{u,u}$	$\mathfrak{r}_u$	$\mathfrak{d}_{u,u}$	
	$u \;\; = $	$\begin{bmatrix} * \,* & \\ \;\; * & \\ \hline & * \end{bmatrix}$	$\begin{bmatrix} & \\ \hline & \end{bmatrix}$	$\begin{bmatrix} & \\ \hline * \,* \,* & \\ \;\;\; * & \end{bmatrix}$	$\begin{bmatrix} & \\ \hline & \end{bmatrix}$	
s	cases	$\mathfrak{n}_{us}$	$\mathfrak{m}_{u,us}$	$\mathfrak{r}_{us}$	$\mathfrak{d}_{u,us}$	
$s = s_1$	$us_1 = $ $\notin W_{\mathbf{d}}u$	$\begin{bmatrix} * \,* & \\ \;\; * & \\ \hline & * \end{bmatrix}$	$\begin{bmatrix} & \\ \hline & \end{bmatrix}$	$\begin{bmatrix} & \\ \hline * \,* & \\ \;\; * & \end{bmatrix}$	$\begin{bmatrix} & \\ \hline * & \end{bmatrix}$	$\dfrac{e_4}{e_1}$
$s = s_2$	$us_2 = $ $\in W_{\mathbf{d}}u$	$\begin{bmatrix} * & \;* & \\ \;\; * & \\ \hline & * \end{bmatrix}$	$\begin{bmatrix} * & \\ \hline & \end{bmatrix} \;\dfrac{e_1}{e_2}$	$\begin{bmatrix} & \\ \hline * \,* \,* & \\ \;\;\; * & \end{bmatrix}$	$\begin{bmatrix} & \\ \hline & \end{bmatrix}$	
$s = s_3$	$us_3 = $ $\notin W_{\mathbf{d}}u$	$\begin{bmatrix} * \,* & \\ \;\; * & \\ \hline & * \end{bmatrix}$	$\begin{bmatrix} & \\ \hline & \end{bmatrix}$	$\begin{bmatrix} & \\ \hline * \,* \,* & \\ \;\; * \,* & \end{bmatrix}$	$\begin{bmatrix} & \\ \hline & \end{bmatrix}$	
$s = s_4$	$us_4 = $ $\notin W_{\mathbf{d}}u$	$\begin{bmatrix} * \,* & \\ \;\; * & \\ \hline & * \end{bmatrix}$	$\begin{bmatrix} & \\ \hline & \end{bmatrix}$	$\begin{bmatrix} & \\ \hline * \,* \,* & \end{bmatrix}$	$\begin{bmatrix} & \\ \hline & * \end{bmatrix}$	$\dfrac{e_5}{e_3}$

Table 1.4: examples of Lie algebras

Remark 1.4.10. We also have twisted notations for Lie algebras. For example,

$$\underline{\mathfrak{n}}_{\varpi,\varpi'} = \mathfrak{n}_{\varpi,\varpi\varpi'}, \quad \underline{\mathfrak{m}}_{\varpi,\varpi'} = \mathfrak{m}_{\varpi,\varpi\varpi'}, \quad \underline{\mathfrak{p}}_{\varpi,s} = \mathfrak{p}_{\varpi,\varpi s},$$

$$\underline{\mathfrak{r}}_{\varpi,\varpi'} = \mathfrak{r}_{\varpi,\varpi\varpi'}, \quad \underline{\mathfrak{d}}_{\varpi,\varpi'} = \mathfrak{d}_{\varpi,\varpi\varpi'}.$$

Another twist happens when we add minus sign as the superscript:

$$\mathfrak{b}_{\varpi}^- = \mathfrak{b}_{\varpi_{\max}\varpi},$$
$$\mathfrak{b}_{\varpi}^- = \mathfrak{b}_{w_{\max}\varpi}, \qquad\qquad \mathfrak{n}_{\varpi}^- = \mathfrak{b}_{w_{\max}\varpi},$$
$$\mathfrak{n}_{\varpi,\varpi''}^- = \mathfrak{n}_{w_{\max}\varpi,w_{\max}\varpi''}, \quad \mathfrak{m}_{\varpi,\varpi''}^- = \mathfrak{m}_{w_{\max}\varpi,w_{\max}\varpi''}.$$

1.5 Typical variety

In this section, we define nearly all the varieties we care about in the same spirit as Section 1.1. Their stratifications and related "Schubert" varieties will be defined in Section 1.6.

Recall Setting 1.1 and Definition 1.2.10.

1.5.1 Flag variety

Definition 1.5.1 (Absolute complete flag variety)**.** *The absolute complete flag variety* $\mathcal{F}_{|\mathbf{d}|}$ *is defined as*

$$
\begin{aligned}
\mathcal{F}_{|\mathbf{d}|} &= \mathbb{G}_{|\mathbf{d}|}/\mathbb{B}_{|\mathbf{d}|}\\
&\cong \left\{ complete\ flags\ of\ \mathbb{C}^{|\mathbf{d}|} \right\}\\
&= \left\{ 0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_{|\mathbf{d}|} = \mathbb{C}^{|\mathbf{d}|} \;\middle|\; \dim M_j = j \right\}\\
&\cong \left\{ Borel\ subgroups\ of\ \mathbb{G}_{|\mathbf{d}|} \right\}\\
&= \left\{ g\mathbb{B}_{|\mathbf{d}|}g^{-1} \;\middle|\; g \in \mathbb{G}_{|\mathbf{d}|} \right\}
\end{aligned}
$$

Here, M_i can have no Q-vector space structure.

Definition 1.5.2 (complete flag variety with flag-type dimension vector)**.** *For a flag-type dimension vector* $\underline{\mathbf{d}}$, *the flag variety* $\mathcal{F}_{\underline{\mathbf{d}}}$ *is defined as*

$$
\begin{aligned}
\mathcal{F}_{\underline{\mathbf{d}}} &= \left\{ complete\ flags\ of\ V = \bigoplus_{i \in Q_0} V_i\ with\ dimension\ vector\ \underline{\mathbf{d}} \right\}\\
&= \left\{ F : 0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_{|\mathbf{d}|} = V \;\middle|\; \underline{\dim}\,F = \underline{\mathbf{d}} \right\}
\end{aligned}
$$

Definition 1.5.3 (Relative complete flag variety)**.** *The relative complete flag variety* $\mathcal{F}_{\mathbf{d}}$ *is defined as*

$$
\begin{aligned}
\mathcal{F}_{\mathbf{d}} &= \left\{ complete\ flags\ of\ V = \bigoplus_{i \in Q_0} V_i \right\}\\
&= \left\{ 0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_{|\mathbf{d}|} = V \;\middle|\; |\underline{\dim}\,M_j| = j \right\}\\
&= \bigsqcup_{\underline{\mathbf{d}}} \mathcal{F}_{\underline{\mathbf{d}}}
\end{aligned}
$$

Here, M_i are Q-vector spaces.

Remark 1.5.4.

1. $\mathcal{F}_{|\mathbf{d}|}$, $\mathcal{F}_{\underline{\mathbf{d}}}$ and $\mathcal{F}_{\mathbf{d}}$ are smooth varieties, since

    $$
    \mathcal{F}_{|\mathbf{d}|} \cong \mathrm{GL}_{|\mathbf{d}|}/B \qquad \mathcal{F}_{\underline{\mathbf{d}}} \cong \prod_{i \in Q_0} \mathrm{GL}_{\mathbf{d}_i}/B
    $$

    are products of usual flag varieties.

2. $\mathcal{F}_{|\mathbf{d}|}$ is an $\mathrm{GL}_{|\mathbf{d}|}$-variety, while $\mathcal{F}_{\underline{\mathbf{d}}}$, $\mathcal{F}_{\mathbf{d}}$ are $G_{\mathbf{d}}$-varieties. The actions are induced by the actions on the vector space V.

We need to simplify our notations of flags.

Definition 1.5.5 (Special flags)**.** *For a basis $\{x_1, \ldots, x_{|\mathbf{d}|}\}$, denote the flag*

$$F_{\{x_1, \ldots, x_{|\mathbf{d}|}\}} : 0 \subseteq \langle x_1 \rangle \subseteq \langle x_1, x_2 \rangle \subseteq \cdots \subseteq \langle x_1, \ldots x_{|\mathbf{d}|} \rangle = V.$$

For $g \in \mathbb{G}_{|\mathbf{d}|}$, $\varpi \in \mathbb{W}_{|\mathbf{d}|}$, define

$$
\begin{aligned}
F_{\mathrm{Id}} &= F_{\{v_1, \ldots, v_{|\mathbf{d}|}\}} & &\in \mathcal{F}_{\mathbf{d}} \\
F_g &= g F_{\mathrm{Id}} = F_{\{gv_1, \ldots, gv_{|\mathbf{d}|}\}} & &\in \mathcal{F}_{|\mathbf{d}|} \\
F_{\varpi} &= \varpi F_{\mathrm{Id}} = F_{\{v_{\varpi(1)}, \ldots, v_{\varpi(|\mathbf{d}|)}\}} & &\in \mathcal{F}_{\mathbf{d}}
\end{aligned}
$$

F_{Id} is called the **standard flag** of V.

Now we can define flag varieties attached to $\varpi \in \mathbb{W}_{|\mathbf{d}|}$.

Definition 1.5.6. *For $\varpi = wu \in \mathbb{W}_{|\mathbf{d}|}$, define $\mathcal{F}_{\varpi}$ as the $G_{\mathbf{d}}$-orbit of F_{ϖ}. By the orbit-stabilizer theorem,*

$$\mathcal{F}_{\varpi} \cong G_{\mathbf{d}} / B_{\varpi}.$$

We can generalize it a little bit: for $g \in G_{\mathbf{d}}$, $F_{g\varpi} \in \mathcal{F}_{\mathbf{d}}$,

$$\mathcal{F}_{g\varpi} := G_{\mathbf{d}} \cdot F_{g\varpi} \cong G_{\mathbf{d}} / B_{g\varpi} = G_{\mathbf{d}} / g B_{\varpi} g^{-1}.$$

Remark 1.5.7. F_{ϖ} is the preferred base point of $\mathcal{F}_{\varpi}$. Ignoring the base point,

$$\mathcal{F}_{\varpi} = \mathcal{F}_u = \mathcal{F}_{\underline{\mathbf{d}}} \quad \text{for } \varpi = wu \quad \underline{\mathbf{d}} = W_{\mathbf{d}} \varpi.$$

In fact, we are not defining new varieties; we give old varieties new names, so that we can manipulate them more freely.

Like Section 1.1, we also consider the product of two flag varieties. For $g, g', g'' \in \mathbb{G}_{|\mathbf{d}|}$, $\varpi, \varpi', \varpi'' \in \mathbb{W}_{|\mathbf{d}|}$, denote

$$
\begin{aligned}
F_{\mathrm{Id},\mathrm{Id}} &= (F_{\mathrm{Id}}, F_{\mathrm{Id}}) \\
F_{g,g''} &= (F_g, F_{g''}) & \underline{F}_{g,g'} &= F_{g,gg'} = (F_g, F_{gg'}) \\
F_{\varpi,\varpi''} &= (F_{\varpi}, F_{\varpi''}) & \underline{F}_{\varpi,\varpi'} &= F_{\varpi,\varpi\varpi'} = (F_{\varpi}, F_{\varpi\varpi'})
\end{aligned}
$$

Table 1.5 concludes all varieties we get until now.

1.5.2 Incidence variety

Now it is time to conclude information about arrows, and construct spaces over varieties in Table 1.5.

	base point		base point
$\mathcal{F}_{|\mathbf{d}|} \cong \mathbb{G}_{|\mathbf{d}|}/\mathbb{B}_{|\mathbf{d}|}$	F_{Id}	$\mathcal{F}_{|\mathbf{d}|} \times \mathcal{F}_{|\mathbf{d}|}$	$F_{\mathrm{Id,Id}}$
$\mathcal{F}_{\underline{\mathbf{d}}} \cong G_{\mathbf{d}}/B_{\mathbf{d}}$	F_u	$\mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}}'}$	$F_{u,u'}$
$\mathcal{F}_{\varpi} \cong G_{\mathbf{d}}/B_{\varpi}$	F_{ϖ}	$\mathcal{F}_{\varpi} \times \mathcal{F}_{\varpi'}$	$F_{\varpi,\varpi'}$
$\mathcal{F}_{\mathbf{d}} = \bigsqcup\limits_{\underline{\mathbf{d}}} \mathcal{F}_{\underline{\mathbf{d}}}$	–	$\mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}} = \bigsqcup\limits_{\underline{\mathbf{d}},\underline{\mathbf{d}}'} \mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}}'}$	–

Table 1.5: Base varieties and their preferred base point

Definition 1.5.8 (Incidence variety)**.** *For a quiver Q with flag-type dimension vector $\underline{\mathbf{d}}$, define*

$$\widetilde{\mathrm{Rep}}_{\underline{\mathbf{d}}}(Q) := \big\{ (\rho, F) \in \mathrm{Rep}_{\mathbf{d}}(Q) \times \mathcal{F}_{\underline{\mathbf{d}}} \,\big|\, \rho(M_j) \subseteq M_j \text{ for any } j \big\}$$
$$\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q) := \big\{ (\rho, F) \in \mathrm{Rep}_{\mathbf{d}}(Q) \times \mathcal{F}_{\mathbf{d}} \,|\, \rho(M_j) \subseteq M_j \text{ for any } j \big\}$$
$$= \bigsqcup_{\underline{\mathbf{d}}} \widetilde{\mathrm{Rep}}_{\underline{\mathbf{d}}}(Q)$$

and $\mu_{\underline{\mathbf{d}}}$, $\pi_{\underline{\mathbf{d}}}$, $\mu_{\mathbf{d}}$, $\pi_{\mathbf{d}}$ to be the natural morphisms from the incidence varieties to $\mathrm{Rep}_{\mathbf{d}}(Q)$ or flag varieties, as follows:

$$\widetilde{\mathrm{Rep}}_{\underline{\mathbf{d}}}(Q) \subseteq \mathrm{Rep}_{\mathbf{d}}(Q) \times \mathcal{F}_{\underline{\mathbf{d}}} \qquad\qquad \widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q) \subseteq \mathrm{Rep}_{\mathbf{d}}(Q) \times \mathcal{F}_{\mathbf{d}}$$

$$\mu_{\underline{\mathbf{d}}} \qquad\qquad \pi_{\underline{\mathbf{d}}} \qquad\qquad\qquad \mu_{\mathbf{d}} \qquad\qquad \pi_{\mathbf{d}}$$

$$\mathrm{Rep}_{\mathbf{d}}(Q) \qquad\qquad \mathcal{F}_{\underline{\mathbf{d}}} \qquad\qquad \mathrm{Rep}_{\mathbf{d}}(Q) \qquad\qquad \mathcal{F}_{\mathbf{d}}$$

Remark 1.5.9. For $M \in \mathrm{Rep}_{\mathbf{d}}(Q)$, the **Springer fiber**

$$\mathrm{Flag}_{\underline{\mathbf{d}}}(M) := \mu_{\underline{\mathbf{d}}}^{-1}(M) \cong \pi_{\underline{\mathbf{d}}}\big(\mu_{\underline{\mathbf{d}}}^{-1}(M)\big) \subseteq \mathcal{F}_{\underline{\mathbf{d}}}$$

records the complete flags of subrepresentations of M. The partial flag variety version of $\mathrm{Flag}_{\underline{\mathbf{d}}}(M)$ will become the key object in the second part.

Definition 1.5.10 (Steinberg variety)**.** *For quiver Q with flag-type dimension vectors $\underline{\mathbf{d}}$, $\underline{\mathbf{d}}'$, define*

$$\mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}'} := \widetilde{\mathrm{Rep}}_{\underline{\mathbf{d}}}(Q) \times_{\mathrm{Rep}_{\mathbf{d}}(Q)} \widetilde{\mathrm{Rep}}_{\underline{\mathbf{d}}'}(Q)$$
$$\mathcal{Z}_{\mathbf{d}} := \widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q) \times_{\mathrm{Rep}_{\mathbf{d}}(Q)} \widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)$$
$$= \bigsqcup_{\underline{\mathbf{d}},\underline{\mathbf{d}}'} \mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}'}$$

*$\mathcal{Z}_{\mathbf{d}}$ is called the **Steinberg variety**.*

$\mathcal{Z}_{\mathbf{d},\mathbf{d}'}$ can actually be realized as the incidence variety between $\mathrm{Rep}_{\mathbf{d}}(Q)$ and $\mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}}'}$, since

$$\mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}'} = \widetilde{\mathrm{Rep}_{\underline{\mathbf{d}}}}(Q) \times_{\mathrm{Rep}_{\mathbf{d}}(Q)} \widetilde{\mathrm{Rep}_{\underline{\mathbf{d}}'}}(Q)$$
$$\subseteq \left(\mathrm{Rep}_{\mathbf{d}}(Q) \times \mathcal{F}_{\underline{\mathbf{d}}} \right) \times_{\mathrm{Rep}_{\mathbf{d}}(Q)} \left(\mathrm{Rep}_{\mathbf{d}}(Q) \times \mathcal{F}_{\underline{\mathbf{d}}'} \right)$$
$$\cong \mathrm{Rep}_{\mathbf{d}}(Q) \times \mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}}'}$$

For that reason, we denote $\mu_{\underline{\mathbf{d}},\underline{\mathbf{d}}'}$, $\pi_{\underline{\mathbf{d}},\underline{\mathbf{d}}'}$, $\mu_{\mathbf{d},\mathbf{d}}$, $\pi_{\mathbf{d},\mathbf{d}}$ as natural morphisms from $\mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}'}$, $\mathcal{Z}_{\mathbf{d}}$ to $\mathrm{Rep}_{\mathbf{d}}(Q)$ or product of flag varieties, as follows:

Remark 1.5.11 (Group actions).

1. $\mathrm{Rep}_{\mathbf{d}}(Q) \subseteq \oplus_{a \in Q_1} \mathfrak{g}_{|\mathbf{d}|}$ has a natural $G_{\mathbf{d}}$-action, which is induced by the conjugation action of $G_{\mathbf{d}}$ on $\mathfrak{g}_{|\mathbf{d}|}$. We have already mentioned the $G_{\mathbf{d}}$-action on $\mathcal{F}_{\underline{\mathbf{d}}}$ and $\mathcal{F}_{\mathbf{d}}$ in Remark 1.5.4. Therefore, by restriction we automatically get $G_{\mathbf{d}}$-actions on $\widetilde{\mathrm{Rep}_{\underline{\mathbf{d}}}}(Q)$, $\widetilde{\mathrm{Rep}_{\mathbf{d}}}(Q)$, $\mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}'}$ and $\mathcal{Z}_{\mathbf{d}}$. All the maps we mentioned in Definition 1.5.8 are $G_{\mathbf{d}}$-equivariant.

2. In Section 6.2 we will also view all the varieties as $G_{\mathbf{d}} \times \mathbb{C}^{\times}$-varieties, so we also shortly introduce $\mathbb{C}^{\times}$-action here. View $\mathrm{Rep}_{\mathbf{d}}(Q)$ as a $\mathbb{C}$-vector space, $\mathbb{C}^{\times}$ acts on $\mathrm{Rep}_{\mathbf{d}}(Q)$ by scalar multiplication. For $\mathcal{F}_{\underline{\mathbf{d}}}$ and $\mathcal{F}_{\mathbf{d}}$, $\mathbb{C}^{\times}$ acts trivially, and by restriction we get $\mathbb{C}^{\times}$-actions on $\widetilde{\mathrm{Rep}_{\underline{\mathbf{d}}}}(Q)$, $\widetilde{\mathrm{Rep}_{\mathbf{d}}}(Q)$, $\mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}'}$ and $\mathcal{Z}_{\mathbf{d}}$. Also, all the maps we mentioned above are $\mathbb{C}^{\times}$-equivariant.

3. It may worth mentioning that $\mathcal{F}_{\mathbf{d}}$ has an $\mathbb{W}_{|\mathbf{d}|}$-action which can be extended neither to $\mathbb{G}_{|\mathbf{d}|}$-action on $\mathcal{F}_{\mathbf{d}}$ nor to $\mathbb{W}_{|\mathbf{d}|}$-action on $\widetilde{\mathrm{Rep}_{\underline{\mathbf{d}}}}(Q)$.

1.6 Stratification and T-fixed points

Natural defined varieties resemble burr puzzles, they have delicate structures and can be decomposed as relatively easy pieces. In this subsection, we will find stratifications of varieties introduced in Section 1.5, and fix notations of orbits. We will also mention about their T-fixed points. These stratifications will give us a basis for the K-theory and cohomology theory in Chapter 2, while those T-fixed points will give us another "basis" in Chapter 4.

1.6.1 Stratification: flag variety

We begin with $\mathcal{F}_{|\mathbf{d}|}$ and $\mathcal{F}_{|\mathbf{d}|} \times \mathcal{F}_{|\mathbf{d}|}$, which is roughly a repetition of Section 1.1.

Definition 1.6.1 (Twisted action). *We define the twisted $\mathbb{G}_{|\mathbf{d}|} \times \mathbb{G}_{|\mathbf{d}|}$-action on $\mathcal{F}_{|\mathbf{d}|} \times \mathcal{F}_{|\mathbf{d}|}$:*

$$\mathbb{G}_{|\mathbf{d}|} \times \mathbb{G}_{|\mathbf{d}|} \times \mathcal{F}_{|\mathbf{d}|} \times \mathcal{F}_{|\mathbf{d}|} \longrightarrow \mathcal{F}_{|\mathbf{d}|} \times \mathcal{F}_{|\mathbf{d}|} \qquad \left(g_1, g_2, \underline{F}_{g,g'}\right) \longmapsto \underline{F}_{g_1 g, g_2 g'}$$

which is the same as original $\mathbb{G}_{|\mathbf{d}|}$-action when we restrict to $\mathbb{G}_{|\mathbf{d}|} \times \{\mathrm{Id}\}$-action. Other $G \times G$-actions on $\mathcal{F} \times \mathcal{F}$ are defined in a similar way.

Definition 1.6.2 (Stratifications of $\mathcal{F}_{|\mathbf{d}|}$ and $\mathcal{F}_{|\mathbf{d}|} \times \mathcal{F}_{|\mathbf{d}|}$). *For $\varpi, \varpi' \in \mathbb{W}_{|\mathbf{d}|}$, we define*

$$\begin{aligned}
\mathcal{V}_{\varpi} &= \mathbb{B}_{|\mathbf{d}|} \cdot F_{\varpi} & &\subseteq \mathcal{F}_{|\mathbf{d}|} \\
\boldsymbol{\mathcal{V}}_{\varpi,\varpi'} &= \left(\mathbb{B}_{|\mathbf{d}|} \times \mathbb{B}_{|\mathbf{d}|}\right) \cdot \underline{F}_{\varpi,\varpi'} & &\subseteq \mathcal{F}_{|\mathbf{d}|} \times \mathcal{F}_{|\mathbf{d}|} \\
\boldsymbol{\mathcal{V}}_{\varpi'} &= \mathbb{G}_{|\mathbf{d}|} \cdot \underline{F}_{\mathrm{Id},\varpi'} & &\subseteq \mathcal{F}_{|\mathbf{d}|} \times \mathcal{F}_{|\mathbf{d}|}
\end{aligned}$$

as $\mathbb{B}_{|\mathbf{d}|}$-orbit, $\mathbb{B}_{|\mathbf{d}|} \times \mathbb{B}_{|\mathbf{d}|}$-orbit, $\mathbb{G}_{|\mathbf{d}|}$-orbit of $\mathcal{F}_{|\mathbf{d}|}$, $\mathcal{F}_{|\mathbf{d}|} \times \mathcal{F}_{|\mathbf{d}|}$, $\mathcal{F}_{|\mathbf{d}|} \times \mathcal{F}_{|\mathbf{d}|}$, respectively.

By Bruhat-decomposition, we are able to show

$$\mathcal{F}_{|\mathbf{d}|} = \bigsqcup_{\varpi} \mathcal{V}_{\varpi} \qquad \mathcal{F}_{|\mathbf{d}|} \times \mathcal{F}_{|\mathbf{d}|} = \bigsqcup_{\varpi'} \boldsymbol{\mathcal{V}}_{\varpi'} = \bigsqcup_{\varpi,\varpi'} \boldsymbol{\mathcal{V}}_{\varpi,\varpi'}.$$

We also realize these orbits as quotients of algebraic groups by the orbit-stabilizer theorem, as follows:

$$\begin{aligned}
\mathcal{V}_{\varpi} &= \mathbb{B}_{|\mathbf{d}|} / \left(\mathbb{B}_{|\mathbf{d}|} \cap \mathbb{B}_{\varpi}\right) & &\cong \mathbb{A}^{l(\varpi)} \\
\boldsymbol{\mathcal{V}}_{\varpi,\varpi'} &= \left(\mathbb{B}_{|\mathbf{d}|} \times \mathbb{B}_{|\mathbf{d}|}\right) / \left(\mathbb{B}_{|\mathbf{d}|} \cap \mathbb{B}_{\varpi} \times \mathbb{B}_{|\mathbf{d}|} \cap \mathbb{B}_{\varpi'}\right) & &\cong \mathbb{A}^{l(\varpi)+l(\varpi')} \\
\boldsymbol{\mathcal{V}}_{\varpi'} &= \mathbb{G}_{|\mathbf{d}|} / \left(\mathbb{B}_{|\mathbf{d}|} \cap \mathbb{B}_{\varpi'}\right) & &\cong \mathbb{A}^{l(\varpi')}\text{-bundle over } \mathcal{F}_{|\mathbf{d}|}
\end{aligned}$$

Similar stratifications happen for $\mathcal{F}_u$ and $\mathcal{F}_{\mathbf{d}}$.

Definition 1.6.3 (Stratifications of $\mathcal{F}_u$ and $\mathcal{F}_u \times \mathcal{F}_{u'}$). *For $u, u' \in \mathrm{Min}(\mathbb{W}_{|\mathbf{d}|}, \mathrm{W}_{\mathbf{d}})$, $w, w' \in \mathrm{W}_{\mathbf{d}}$, we define*

$$\begin{aligned}
\Omega_w^u &= B_{\mathbf{d}} \cdot F_{wu} & &\subseteq \mathcal{F}_u \\
\boldsymbol{\Omega}_{w,w'}^{u,u'} &= (B_{\mathbf{d}} \times B_{\mathbf{d}}) \cdot (F_{wu}, F_{ww'u'}) & &\subseteq \mathcal{F}_u \times \mathcal{F}_{u'} \\
\boldsymbol{\Omega}_{w'}^{u,u'} &= G_{\mathbf{d}} \cdot (F_u, F_{w'u'}) & &\subseteq \mathcal{F}_u \times \mathcal{F}_{u'}
\end{aligned}$$

as $B_{\mathbf{d}}$-orbit, $B_{\mathbf{d}} \times B_{\mathbf{d}}$-orbit, $G_{\mathbf{d}}$-orbit of $\mathcal{F}_u$, $\mathcal{F}_u \times \mathcal{F}_{u'}$, $\mathcal{F}_u \times \mathcal{F}_{u'}$, respectively.

By Bruhat decomposition, we are again able to show

$$\mathcal{F}_u = \bigsqcup_w \Omega_w^u \qquad \mathcal{F}_u \times \mathcal{F}_{u'} = \bigsqcup_{w'} \boldsymbol{\Omega}_{w'}^{u,u'} = \bigsqcup_{w,w'} \boldsymbol{\Omega}_{w,w'}^{u,u'}$$

and

$$\begin{aligned}
\Omega_w^u &= B_{\mathbf{d}} / \left(B_{\mathbf{d}} \cap B_w\right) & &\cong \mathbb{A}^{l(w)} \\
\boldsymbol{\Omega}_{w,w'}^{u,u'} &= (B_{\mathbf{d}} \times B_{\mathbf{d}}) / \left(B_{\mathbf{d}} \cap B_w \times B_{\mathbf{d}} \cap B_{w'}\right) & &\cong \mathbb{A}^{l(w)+l(w')} \\
\boldsymbol{\Omega}_{w'}^{u,u'} &= G_{\mathbf{d}} / \left(B_{\mathbf{d}} \cap B_{w'}\right) & &\cong \mathbb{A}^{l(w')}\text{-bundle over } \mathcal{F}_u
\end{aligned}$$

Definition 1.6.4 (Stratifications of $\mathcal{F}_{\mathbf{d}}$ and $\mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}}$). *For* $\varpi, \varpi' \in \mathbb{W}_{|\mathbf{d}|}$, *we define*

$$
\begin{aligned}
\mathcal{O}_{\varpi} &= B_{\mathbf{d}} \cdot F_{\varpi} & &\subseteq \mathcal{F}_{\varpi} & &\subseteq \mathcal{F}_{\mathbf{d}} \\
\boldsymbol{\mathcal{O}}_{\varpi,\varpi'} &= (B_{\mathbf{d}} \times B_{\mathbf{d}}) \cdot \underline{F}_{\varpi,\varpi'} & &\subseteq \mathcal{F}_{\varpi} \times \mathcal{F}_{\varpi\varpi'} & &\subseteq \mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}} \\
\boldsymbol{\mathcal{O}}_{\varpi'} &= \bigsqcup_{u} G_{\mathbf{d}} \cdot \underline{F}_{u,\varpi'} & &\subseteq \bigsqcup_{u} \mathcal{F}_{u} \times \mathcal{F}_{u\varpi'} \subseteq \mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}}
\end{aligned}
$$

as $B_{\mathbf{d}}$-orbit, $B_{\mathbf{d}} \times B_{\mathbf{d}}$-orbit, (union of) $G_{\mathbf{d}}$-orbit of $\mathcal{F}_{\mathbf{d}}$, $\mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}}$, $\mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}}$, respectively.

Notice that $\mathcal{O}_{\varpi}$, $\boldsymbol{\mathcal{O}}_{\varpi,\varpi'}$, $\boldsymbol{\mathcal{O}}_{\varpi'}$ are preimages of $\mathcal{V}_{\varpi}$, $\boldsymbol{\mathcal{V}}_{\varpi,\varpi'}$, $\boldsymbol{\mathcal{V}}_{\varpi'}$ under the maps

$$
\mathcal{F}_{\mathbf{d}} \hookrightarrow \mathcal{F}_{|\mathbf{d}|} \qquad \mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}} \hookrightarrow \mathcal{F}_{|\mathbf{d}|} \times \mathcal{F}_{|\mathbf{d}|}.
$$

Therefore,

$$
\mathcal{F}_{\mathbf{d}} = \bigsqcup_{\varpi} \mathcal{O}_{\varpi} \qquad \mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}} = \bigsqcup_{\varpi'} \boldsymbol{\mathcal{O}}_{\varpi'} = \bigsqcup_{\varpi,\varpi'} \boldsymbol{\mathcal{O}}_{\varpi,\varpi'}.
$$

We still need to care about symbols. For $\varpi = wu$, $\varpi' = w'u'$, denote $uw'u' = \tilde{w}\tilde{u}$ for $\tilde{w}' \in W_{\mathbf{d}}$, $\tilde{u} \in \mathrm{Min}(\mathbb{W}_{|\mathbf{d}|}, W_{\mathbf{d}})$, then

$$
\underline{F}_{\varpi,\varpi'} = (F_{\varpi}, F_{\varpi\varpi'}) = (F_{wu}, F_{wuw'u'}) = (F_{wu}, F_{w\tilde{w}\tilde{u}}) \in \mathcal{F}_{u} \times \mathcal{F}_{\tilde{u}}.
$$

This incompatibility comes from our twisted $G_{\mathbf{d}} \times G_{\mathbf{d}}$-actions. In particular, denote

$$
\boldsymbol{\mathcal{O}}_{\varpi'}^{u} := G_{\mathbf{d}} \cdot \underline{F}_{u,\varpi'} \subseteq \mathcal{F}_{u} \times \mathcal{F}_{\tilde{u}},
$$

we have $\boldsymbol{\mathcal{O}}_{\varpi'} = \sqcup_{u} \boldsymbol{\mathcal{O}}_{\varpi'}^{u}$ and identifications

$$
\mathcal{O}_{\varpi} = \Omega_{w}^{u} \qquad \boldsymbol{\mathcal{O}}_{\varpi,\varpi'} = \boldsymbol{\Omega}_{w,\tilde{w}}^{u,\tilde{u}} \qquad \boldsymbol{\mathcal{O}}_{\varpi'}^{u} = \boldsymbol{\Omega}_{\tilde{w}}^{u,\tilde{u}}. \tag{$\star$}
$$

We can also describe the closure of orbits, for example,

$$
\overline{\Omega}_{w}^{u} = \bigsqcup_{x \leq w} \Omega_{x}^{u} \qquad \overline{\boldsymbol{\Omega}}_{w,w'}^{u,u'} = \bigsqcup_{x \leq w, x' \leq w'} \boldsymbol{\Omega}_{x,x'}^{u,u'} \qquad \overline{\boldsymbol{\Omega}}_{w'}^{u,u'} = \bigsqcup_{x' \leq w'} \boldsymbol{\Omega}_{x'}^{u,u'}
$$

Especially, for any $s \in \Pi_{\mathbf{d}}$, $u, u' \in \mathrm{Min}(\mathbb{W}_{|\mathbf{d}|}, W_{\mathbf{d}})$, we have

$$
\overline{\boldsymbol{\Omega}}_{s}^{u,u'} = \boldsymbol{\Omega}_{s}^{u,u'} \sqcup \boldsymbol{\Omega}_{\mathrm{Id}}^{u,u'} \cong G_{\mathbf{d}} \times^{B_{\mathbf{d}}} (P_{\mathrm{Id},s}/B_{\mathbf{d}})
$$

when we work over base point $F_{u,u'}$. If we work over different base points, we will get different isomorphisms, as follows:

$$
\begin{aligned}
\overline{\boldsymbol{\Omega}}_{s}^{u,u'} = \boldsymbol{\Omega}_{\mathrm{Id}}^{u,u'} \sqcup \boldsymbol{\Omega}_{s}^{u,u'} &\cong G_{\mathbf{d}}/(B_{w} \cap B_{ws}) & &\sqcup G_{\mathbf{d}}/B_{w} \\
&\cong G_{\mathbf{d}} \times^{B_{w}} (B_{w}/(B_{w} \cap B_{ws})) \sqcup G_{\mathbf{d}} \times^{B_{w}} (B_{w}/B_{w}) \\
&\cong G_{\mathbf{d}} \times^{B_{w}} (B_{w}sB_{w}/B_{w}) & &\sqcup G_{\mathbf{d}} \times^{B_{w}} (B_{w}/B_{w}) \\
&\cong G_{\mathbf{d}} \times^{B_{w}} (\underline{P}_{w,s}/B_{w}) & &\text{base point } F_{wu,wu'} \\
&\cong G_{\mathbf{d}} \times^{B_{w}} (\underline{P}_{w,s}/B_{ws}) & &\text{base point } F_{wu,wsu'}
\end{aligned}
$$

Closures of $\mathcal{O}$-cells are obtained by identifications $(\star)$. To illustrate it, we compute $\overline{\mathcal{O}}_s$ by hand. Let $\varpi' = s, us = \tilde{w}\tilde{u}$,

$$
\overline{\mathcal{O}}_s = \bigsqcup_u \overline{\mathcal{O}}_s^u = \bigsqcup_u \overline{\Omega}_{\tilde{w}}^{u,\tilde{u}}
$$

$$
= \left(\bigsqcup_{u:usu^{-1}\in W_{\mathbf{d}}} \overline{\Omega}_{usu^{-1}}^{u,u} \right) \sqcup \left(\bigsqcup_{u:usu^{-1}\notin W_{\mathbf{d}}} \overline{\Omega}_{\mathrm{Id}}^{u,us} \right)
$$

$$
= \left(\bigsqcup_{u:usu^{-1}\in W_{\mathbf{d}}} \Omega_{usu^{-1}}^{u,u} \right) \sqcup \left(\bigsqcup_{u:usu^{-1}\notin W_{\mathbf{d}}} \Omega_{\mathrm{Id}}^{u,us} \right) \sqcup \left(\bigsqcup_{u:usu^{-1}\in W_{\mathbf{d}}} \Omega_{\mathrm{Id}}^{u,u} \right)
$$

$$
= \mathcal{O}_s \sqcup \left(\bigsqcup_{u:usu^{-1}\in W_{\mathbf{d}}} \mathcal{O}_{\mathrm{Id}}^u \right)
$$

We restrict the result of $\overline{\Omega}_s^{u,u'}$ to $\overline{\mathcal{O}}_s^u$ in Lemma 1.6.5.

Lemma 1.6.5. *For $\varpi = wu \in \mathbb{W}_{|\mathbf{d}|}$, $s \in \Pi$ such that $\varpi s \varpi^{-1} \in W_{\mathbf{d}}$, we have isomorphisms of $G_{\mathbf{d}}$-varieties*

$$
G_{\mathbf{d}} \times^{B_\varpi} (\underline{P}_{\varpi,s}/B_\varpi) \longrightarrow \overline{\mathcal{O}}_s^u \qquad (g,p) \longmapsto (g \cdot F_\varpi, gp \cdot F_\varpi)
$$
$$
G_{\mathbf{d}} \times^{B_\varpi} (\underline{P}_{\varpi,s}/B_{\varpi s}) \longrightarrow \overline{\mathcal{O}}_s^u \qquad (g,p) \longmapsto (g \cdot F_\varpi, gp \cdot F_{\varpi s})
$$

Proof. Notice that when $\varpi s \varpi^{-1} \in W_{\mathbf{d}}$, $\mathcal{O}_s^u = \Omega_{usu^{-1}}^{u,u}$. Therefore,

$$
\overline{\mathcal{O}}_s^u = \overline{\Omega}_{usu^{-1}}^{u,u} \cong \begin{cases} G_{\mathbf{d}} \times^{B_w} (\underline{P}_{w,usu^{-1}}/B_w) & \text{base point } F_{wu,wu} \\ G_{\mathbf{d}} \times^{B_w} (\underline{P}_{w,usu^{-1}}/B_{wusu^{-1}}) & \text{base point } F_{wu,wus} \end{cases}
$$

$$
\cong \begin{cases} G_{\mathbf{d}} \times^{B_\varpi} (\underline{P}_{\varpi,s}/B_\varpi) & \text{base point } F_{\varpi,\varpi} \\ G_{\mathbf{d}} \times^{B_\varpi} (\underline{P}_{\varpi,s}/B_{\varpi s}) & \text{base point } F_{\varpi,\varpi s} \end{cases} \qquad \square
$$

After so many notations are introduced rapidly, an enlightening example is needed here.

Example 1.6.6 (Follows Example 1.3.8)**.** *Here,* $\mathbb{W}_{|\mathbf{d}|} = S_3$, $W_{\mathbf{d}} = S_1 \times S_2$,

$$
\varpi = ts = t \cdot s, \qquad \varpi' = s = \mathrm{Id} \cdot s, \qquad \varpi\varpi' = t = t \cdot \mathrm{Id}.
$$

$\mathcal{F}_{\mathbf{d}}$ *has 3 connected components, each of them has 2 $B_{\mathbf{d}}$-orbits;*
$\mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}}$ *has 9 connected components, each of them has 4 $B_{\mathbf{d}} \times B_{\mathbf{d}}$-orbits. We have*

shape $(B_\mathbf{d}\times B_\mathbf{d})\cdot F_{\varpi,\varpi'}$ $\quad B_\mathbf{d}\cdot F_{\varpi\varpi'}$ $B_\mathbf{d}\cdot F_{\varpi}$		$\mathcal{F}_\mathrm{Id}$		$\mathcal{F}_s$		$\mathcal{F}_{st}$	
		$\bullet\ \mathcal{O}_\mathrm{Id}$	$-\ \mathcal{O}_t$	$\bullet\ \mathcal{O}_s$	$-\ \mathcal{O}_{ts}$	$\bullet\ \mathcal{O}_{st}$	$-\ \mathcal{O}_{sts}$
$\mathcal{F}_\mathrm{Id}$	$\bullet\ \mathcal{O}_\mathrm{Id}=\Omega_\mathrm{Id}^\mathrm{Id}$	$\bullet\ \Omega_{\mathrm{Id},\mathrm{Id}}^{\mathrm{Id},\mathrm{Id}}$	$-\ \Omega_{\mathrm{Id},t}^{\mathrm{Id},\mathrm{Id}}$	$\bullet\ \Omega_{\mathrm{Id},\mathrm{Id}}^{\mathrm{Id},s}$	$-\ \Omega_{\mathrm{Id},t}^{\mathrm{Id},s}$	$\bullet\ \Omega_{\mathrm{Id},\mathrm{Id}}^{\mathrm{Id},st}$	$-\ \Omega_{\mathrm{Id},t}^{\mathrm{Id},st}$
	$\mathrm{I}\ \mathcal{O}_t=\Omega_t^\mathrm{Id}$	$\Omega_{t,t}^{\mathrm{Id},\mathrm{Id}}$	$\Omega_{t,\mathrm{Id}}^{\mathrm{Id},\mathrm{Id}}$	$\Omega_{t,t}^{\mathrm{Id},s}$	$\Omega_{t,\mathrm{Id}}^{\mathrm{Id},s}$	$\Omega_{t,t}^{\mathrm{Id},st}$	$\Omega_{t,\mathrm{Id}}^{\mathrm{Id},st}$
$\mathcal{F}_s$	$\bullet\ \mathcal{O}_s=\Omega_\mathrm{Id}^s$	$\bullet\ \Omega_{\mathrm{Id},\mathrm{Id}}^{s,\mathrm{Id}}$	$-\ \Omega_{\mathrm{Id},t}^{s,\mathrm{Id}}$	$\bullet\ \Omega_{\mathrm{Id},\mathrm{Id}}^{s,s}$	$-\ \Omega_{\mathrm{Id},t}^{s,s}$	$\bullet\ \Omega_{\mathrm{Id},\mathrm{Id}}^{s,st}$	$-\ \Omega_{\mathrm{Id},t}^{s,st}$
	$\mathrm{I}\ \mathcal{O}_{ts}=\Omega_t^s$	$\Omega_{t,t}^{s,\mathrm{Id}}$	$\Omega_{t,\mathrm{Id}}^{s,\mathrm{Id}}$	$\Omega_{t,t}^{s,s}$	$\Omega_{t,\mathrm{Id}}^{s,s}$	$\Omega_{t,t}^{s,st}$	$\Omega_{t,\mathrm{Id}}^{s,st}$
$\mathcal{F}_{st}$	$\bullet\ \mathcal{O}_{ts}=\Omega_\mathrm{Id}^{st}$	$\bullet\ \Omega_{\mathrm{Id},\mathrm{Id}}^{st,\mathrm{Id}}$	$-\ \Omega_{\mathrm{Id},t}^{st,\mathrm{Id}}$	$\bullet\ \Omega_{\mathrm{Id},\mathrm{Id}}^{st,s}$	$-\ \Omega_{\mathrm{Id},t}^{st,s}$	$\bullet\ \Omega_{\mathrm{Id},\mathrm{Id}}^{st,st}$	$-\ \Omega_{\mathrm{Id},t}^{st,st}$
	$\mathrm{I}\ \mathcal{O}_{sts}=\Omega_t^{st}$	$\Omega_{t,t}^{st,\mathrm{Id}}$	$\Omega_{t,\mathrm{Id}}^{st,\mathrm{Id}}$	$\Omega_{t,t}^{st,s}$	$\Omega_{t,\mathrm{Id}}^{st,s}$	$\Omega_{t,t}^{st,st}$	$\Omega_{t,\mathrm{Id}}^{st,st}$

Table 1.6: stratifications of $\mathcal{F}_\mathbf{d}\times\mathcal{F}_\mathbf{d}$

given every orbit a name, and other spaces are finite union of these orbits. For example,

$$\mathcal{O}_{ts,s}=\Omega_{t,\mathrm{Id}}^{s,\mathrm{Id}}$$
$$\mathcal{O}_s^s=\Omega_\mathrm{Id}^{s,\mathrm{Id}}=\Omega_{\mathrm{Id},\mathrm{Id}}^{s,\mathrm{Id}}\sqcup\Omega_{t,\mathrm{Id}}^{s,\mathrm{Id}}$$
$$\mathcal{O}_s=\mathcal{O}_s^s\sqcup\mathcal{O}_s^\mathrm{Id}\sqcup\mathcal{O}_s^{st}$$
$$=\mathcal{O}_s^s\sqcup\mathcal{O}_s^\mathrm{Id}\sqcup\mathcal{O}_s^{st}$$
$$=\Omega_\mathrm{Id}^{s,\mathrm{Id}}\sqcup\Omega_\mathrm{Id}^{\mathrm{Id},s}\sqcup\Omega_\mathrm{Id}^{st,st}$$
$$=\Omega_{\mathrm{Id},\mathrm{Id}}^{s,\mathrm{Id}}\sqcup\Omega_{t,\mathrm{Id}}^{s,\mathrm{Id}}\sqcup\Omega_{\mathrm{Id},\mathrm{Id}}^{\mathrm{Id},s}\sqcup\Omega_{t,\mathrm{Id}}^{\mathrm{Id},s}\sqcup\Omega_{\mathrm{Id},\mathrm{Id}}^{st,st}\sqcup\Omega_{t,\mathrm{Id}}^{st,st}$$

Their closures are also clear from the table, for example,

$$\overline{\mathcal{O}}_s=\mathcal{O}_s\sqcup\Omega_\mathrm{Id}^{st,st}$$

contains 8 $B_\mathbf{d}\times B_\mathbf{d}$-orbits.

1.6.2   Stratification: incidence variety

Now comes the stratifications of incidence varieties. Those stratifications are produced by taking the preimage of stratifications on base spaces. They are relatively easy to obtain,

while their closures are quite difficult to analyze.

Definition 1.6.7 (Stratifications of incidence varieties)**.** *For $\varpi = wu$, $\varpi' = w'u' \in \mathbb{W}_{|\mathbf{d}|}$, denote $uwu' = \tilde{w}\tilde{u}$, $\underline{\mathbf{d}} = W_{\mathbf{d}}u$, $\underline{\mathbf{d}}' = W_{\mathbf{d}}u'$, $\widetilde{\underline{\mathbf{d}}} = W_{\mathbf{d}}\tilde{u}$, we define*

$$
\begin{aligned}
\widetilde{\Omega}^u_w &:= \pi^{-1}_{\underline{\mathbf{d}}}(\Omega^u_w) & &\subseteq \widetilde{\mathrm{Rep}}_{\underline{\mathbf{d}}}(Q) \\[4pt]
\widetilde{\boldsymbol{\Omega}}^{u,u'}_{w,w'} &:= \pi^{-1}_{\underline{\mathbf{d}},\underline{\mathbf{d}}'}(\boldsymbol{\Omega}^{u,u'}_{w,w'}) & &\subseteq \mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}'} \\[4pt]
\widetilde{\boldsymbol{\Omega}}^{u,u'}_{w'} &:= \pi^{-1}_{\underline{\mathbf{d}},\underline{\mathbf{d}}'}(\boldsymbol{\Omega}^{u,u'}_{w'}) & &\subseteq \mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}'} \\[10pt]
\widetilde{\mathcal{O}}^u_{\varpi'} &:= \pi^{-1}_{\underline{\mathbf{d}},\widetilde{\underline{\mathbf{d}}}}(\mathcal{O}^u_{\varpi'}) = \widetilde{\boldsymbol{\Omega}}^{u,\tilde{u}}_{\tilde{w}} & &\subseteq \mathcal{Z}_{\underline{\mathbf{d}},\widetilde{\underline{\mathbf{d}}}} \\[10pt]
\widetilde{\mathcal{O}}_{\varpi} &:= \pi^{-1}_{\mathbf{d}}(\mathcal{O}_{\varpi}) & &\subseteq \widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q) \\[4pt]
\widetilde{\mathcal{O}}_{\varpi,\varpi'} &:= \pi^{-1}_{\mathbf{d},\mathbf{d}'}(\mathcal{O}_{\varpi,\varpi'}) & &\subseteq \mathcal{Z}_{\mathbf{d}} \\[4pt]
\widetilde{\mathcal{O}}_{\varpi'} &:= \pi^{-1}_{\mathbf{d},\mathbf{d}'}(\mathcal{O}_{\varpi'}) & &\subseteq \mathcal{Z}_{\mathbf{d}}
\end{aligned}
$$

It is not hard to see that they are stratifications:

$$
\begin{aligned}
\widetilde{\mathrm{Rep}}_{\underline{\mathbf{d}}}(Q) = \bigsqcup_{\varpi} \widetilde{\Omega}^u_w \qquad & \mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}'} = \bigsqcup_{w} \widetilde{\boldsymbol{\Omega}}^{u,u'}_{w'} = \bigsqcup_{w,w'} \widetilde{\boldsymbol{\Omega}}^{u,u'}_{w,w'} \\[6pt]
\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q) = \bigsqcup_{\varpi} \widetilde{\mathcal{O}}_{\varpi} \qquad & \mathcal{Z}_{\mathbf{d}} = \bigsqcup_{\varpi'} \widetilde{\mathcal{O}}_{\varpi'} = \bigsqcup_{\varpi,\varpi'} \widetilde{\mathcal{O}}_{\varpi,\varpi'}
\end{aligned}
$$

Proposition 1.6.8. *Those stratifications are affine spaces over corresponding base spaces. To be precise,*

$$
\begin{aligned}
\widetilde{\Omega}^u_w &= \mathfrak{r}_{wu}\text{-}bundle \ over \ \Omega^u_w \\[4pt]
\widetilde{\boldsymbol{\Omega}}^{u,u'}_{w,w'} &= \mathfrak{r}_{wu,ww'u'}\text{-}bundle \ over \ \boldsymbol{\Omega}^{u,u'}_{w,w'} \\[4pt]
\widetilde{\boldsymbol{\Omega}}^{u,u'}_{w'} &= \mathfrak{r}_{u,w'u'}\text{-}bundle \ over \ \boldsymbol{\Omega}^{u,u'}_{w'} \\[10pt]
\widetilde{\mathcal{O}}^u_{\varpi'} &= \underline{\mathfrak{r}}_{u,\varpi'}\text{-}bundle \ over \ \mathcal{O}^u_{\varpi'} \\[10pt]
\widetilde{\mathcal{O}}_{\varpi} &= \mathfrak{r}_{\varpi}\text{-}bundle \ over \ \mathcal{O}_{\varpi} \\[4pt]
\widetilde{\mathcal{O}}_{\varpi,\varpi'} &= \underline{\mathfrak{r}}_{\varpi,\varpi'}\text{-}bundle \ over \ \mathcal{O}_{\varpi,\varpi'} \\[4pt]
\widetilde{\mathcal{O}}_{\varpi'} &= \underline{\mathfrak{r}}_{\mathrm{Id},\varpi'}\text{-}bundle \ over \ \mathcal{O}_{\varpi'}
\end{aligned}
$$

Proof. The fibers are all computed over the preferred base point. The group action induces the isomorphism between different fibers, and lift affine local charts on base space (viewed as group quotient) to the local charts of fiber bundles. □

We will frequently use closures of some stratifications, so we give them names.

Definition 1.6.9. *We define*

$$\begin{aligned}
\mathcal{Z}_{w'}^{u,u'} &:= \overline{\widetilde{\Omega}}_{w'}^{u,u'} &\subseteq &\qquad \mathcal{Z}^{u,u'} := \mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}'}, \\
\mathcal{Z}_{\varpi'} &:= \overline{\widetilde{\mathcal{O}}}_{\varpi'} &\subseteq &\qquad \mathcal{Z}_{\mathbf{d}}.
\end{aligned}$$

Proposition 1.6.10 (Properties of the closure)**.** $\mathcal{Z}_{\varpi'}$ *is a Zarisky-locally trivial cone bundle over $\mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}}$. To be precise, under the map*

$$\pi_{\mathbf{d},\mathbf{d},\varpi'} : \mathcal{Z}_{\varpi'} \longrightarrow \mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}},$$

for any $x, x' \in \mathbb{W}_{|\mathbf{d}|}$, $\pi_{\mathbf{d},\mathbf{d},\varpi'}^{-1}(\mathcal{O}_{x,x'})$ is a trivial fiber bundle over $\mathcal{O}_{x,x'}$, whose fibers are cones.

I have no idea when $\mathcal{Z}_{\varpi'}$ is a Zarisky-locally trivial vector bundle. Let us see when we need this fact.

Remark 1.6.11. Notice that

$$\begin{aligned}
\mathcal{Z}_{w'}^{u,u'} &:= \overline{\widetilde{\Omega}}_{w'}^{u,u'} &\subseteq &\qquad \widetilde{\Omega}_{w'}^{u,u'} := \pi_{\underline{\mathbf{d}},\underline{\mathbf{d}}'}^{-1}\left(\overline{\Omega}_{w'}^{u,u'}\right), \\
\mathcal{Z}_{\varpi'} &:= \overline{\widetilde{\mathcal{O}}}_{\varpi'} &\subseteq &\qquad \widetilde{\mathcal{O}}_{\varpi'} := \pi_{\mathbf{d},\mathbf{d}}^{-1}\left(\overline{\mathcal{O}}_{\varpi'}\right).
\end{aligned}$$

Even though these inclusions are usually not equalities, we can still say something when the length of w' or ϖ' is small. For example,

$$\mathcal{Z}_{\mathrm{Id}}^{u,u'} = \widetilde{\Omega}_{\mathrm{Id}}^{u,u'}$$

$$\mathcal{Z}_{\mathrm{Id}} = \widetilde{\mathcal{O}}_{\mathrm{Id}}$$

$$\widetilde{\Omega}_{s}^{u,u'} \sqcup \Omega_{\mathrm{Id}}^{u,u'} \subseteq \mathcal{Z}_{s}^{u,u'} \subseteq \widetilde{\Omega}_{s}^{u,u'} \sqcup \widetilde{\Omega}_{\mathrm{Id}}^{u,u'} \qquad\qquad (s \in \Pi_{\mathbf{d}})$$

$$\widetilde{\mathcal{O}}_{s} \sqcup \left(\bigsqcup_{u:usu^{-1}\in W_{\mathbf{d}}} \mathcal{O}_{\mathrm{Id}}^{u}\right) \subseteq \mathcal{Z}_{s} \subseteq \widetilde{\mathcal{O}}_{s} \sqcup \left(\bigsqcup_{u:usu^{-1}\in W_{\mathbf{d}}} \widetilde{\mathcal{O}}_{\mathrm{Id}}^{u}\right) \qquad\qquad (s \in \Pi)$$

I have no idea if we have any accurate description for $\mathcal{Z}_{s}^{u,u'}$ and $\mathcal{Z}_{s}$.

We end this subsection by Table 1.7:

1.6.3 T-fixed points

Compare with stratifications, T-fixed points are easy to compute and have clear structures. Somewhat surprisingly, these T-fixed points encode most information of varieties.

Recall that the T-fixed points of a complete flag variety $\mathcal{F}$ are exactly those coordinate flags $\{F_w \mid w \in W\}$. For absolute or relative flag varieties, we have similar results:

$$\mathcal{F}_{|\mathbf{d}|}^{\mathbb{T}_{|\mathbf{d}|}} = \mathcal{F}_{\mathbf{d}}^{T_{\mathbf{d}}} = \left\{F_{\varpi} \mid \varpi \in \mathbb{W}_{|\mathbf{d}|}\right\} \qquad \mathcal{F}_{u}^{T_{\mathbf{d}}} = \{F_{wu} \mid w \in W_{\mathbf{d}}\}$$

stratification / stabilizer — type; variety / base point		B-orbit	$B \times B$-orbit — twisted stabilizer	G-orbit	Remark
$\mathcal{F}$	$\mathcal{F} \times \mathcal{F}$	Ω_g	$\boldsymbol{\Omega}_{g,g'}$	$\boldsymbol{\Omega}_{g'}$	
F_g	$(F_g, F_{gg'})$	$B \cap gBg^{-1}$	$\left(B \cap gBg^{-1}\right) \times \left(B \cap g'Bg'^{-1}\right)$	$gBg^{-1} \cap gg'B(gg')^{-1}$	
$\mathcal{F}_{|\mathbf{d}|}$	$\mathcal{F}_{|\mathbf{d}|} \times \mathcal{F}_{|\mathbf{d}|}$	$\mathcal{V}_\varpi$	$\boldsymbol{\mathcal{V}}_{\varpi,\varpi'}$	$\boldsymbol{\mathcal{V}}_{\varpi'}$	
F_ϖ	$(F_\varpi, F_{\varpi\varpi'})$	$\mathbb{B}_{|\mathbf{d}|} \cap \mathbb{B}_\varpi$	$\left(\mathbb{B}_{|\mathbf{d}|} \cap \mathbb{B}_\varpi\right) \times \left(\mathbb{B}_{|\mathbf{d}|} \cap \mathbb{B}_{\varpi\varpi'}\right)$	$\mathbb{B}_\varpi \cap \mathbb{B}_{\varpi\varpi'}$	
$\mathcal{F}_u$	$\mathcal{F}_u \times \mathcal{F}_{u'}$	Ω_w^u	$\boldsymbol{\Omega}_{w,w'}^{u,u'}$	$\boldsymbol{\Omega}_{w'}^{u,u'}$	
F_{wu}	$(F_{wu}, F_{ww'u'})$	$B_\mathbf{d} \cap B_w$	$(B_\mathbf{d} \cap B_w) \times (B_\mathbf{d} \cap B_{w'})$	$B_w \cap B_{ww'}$	
$\mathcal{F}_\mathbf{d}$	$\mathcal{F}_\mathbf{d} \times \mathcal{F}_\mathbf{d}$	Ω_w^u	$\boldsymbol{\Omega}_{w,\tilde{w}}^{u,\tilde{u}}$	$\boldsymbol{\mathcal{O}}_{\varpi'}^u = \boldsymbol{\Omega}_{\tilde{w}}^{u,\tilde{u}}$	
F_ϖ	$(F_\varpi, F_{\varpi\varpi'})$	$B_\mathbf{d} \cap B_w$	$(B_\mathbf{d} \cap B_w) \times (B_\mathbf{d} \cap B_{\tilde{w}})$	$B_w \cap B_{w\tilde{w}}$	
F_{wu}	$(F_{wu}, F_{w\tilde{w}\tilde{u}})$				
The following may not be single orbit, but derived from the above definition.					
$\mathcal{F}_\mathbf{d}$	$\mathcal{F}_\mathbf{d} \times \mathcal{F}_\mathbf{d}$	$\mathcal{O}_\varpi$	$\boldsymbol{\mathcal{O}}_{\varpi,\varpi'}$	$\boldsymbol{\mathcal{O}}_{\varpi'}$	preimage of
F_ϖ	$(F_\varpi, F_{\varpi\varpi'})$	Ω_w^u	$\boldsymbol{\Omega}_{w,\tilde{w}}^{u,\tilde{u}}$	$\sqcup_u \boldsymbol{\mathcal{O}}_{\varpi'}^u$	$\mathcal{F}_\mathbf{d} \times \mathcal{F}_\mathbf{d} \hookrightarrow \mathcal{F}_{|\mathbf{d}|} \times \mathcal{F}_{|\mathbf{d}|}$
$\widetilde{\mathrm{Rep}}_{\underline{\mathbf{d}}}(Q)$	$\mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}'}$	$\widetilde{\Omega}_w^u$	$\widetilde{\boldsymbol{\Omega}}_{w,w'}^{u,u'}$	$\widetilde{\boldsymbol{\Omega}}_{w'}^{u,u'}$	preimage of
F_{wu}	$(F_{wu}, F_{ww'u'})$				$\mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}'} \hookrightarrow \mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}}'}$
$\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)$	$\mathcal{Z}_{\mathbf{d}}$	$\widetilde{\Omega}_w^u$	$\widetilde{\boldsymbol{\Omega}}_{w,\tilde{w}}^{u,\tilde{u}}$	$\widetilde{\boldsymbol{\mathcal{O}}}_{\varpi'}^u = \widetilde{\boldsymbol{\Omega}}_{\tilde{w}}^{u,\tilde{u}}$	preimage of
F_ϖ	$(F_\varpi, F_{\varpi\varpi'})$				$\mathcal{Z}_{\mathbf{d}} \hookrightarrow \mathcal{F}_\mathbf{d} \times \mathcal{F}_\mathbf{d}$
$\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)$	$\mathcal{Z}_{\mathbf{d}}$	$\widetilde{\mathcal{O}}_\varpi$	$\widetilde{\boldsymbol{\mathcal{O}}}_{\varpi,\varpi'}$	$\widetilde{\boldsymbol{\mathcal{O}}}_{\varpi'}$	preimage of
F_ϖ	$(F_\varpi, F_{\varpi\varpi'})$	$\widetilde{\Omega}_w^u$	$\widetilde{\boldsymbol{\Omega}}_{w,\tilde{w}}^{u,\tilde{u}}$	$\sqcup_u \widetilde{\boldsymbol{\mathcal{O}}}_{\varpi'}^u$	$\mathcal{Z}_{\mathbf{d}} \hookrightarrow \mathcal{F}_\mathbf{d} \times \mathcal{F}_\mathbf{d}$

Table 1.7: stratifications of typical varieties

For $\mathrm{Rep}_\mathbf{d}(Q)$, we get

$$\left(\mathrm{Rep}_\mathbf{d}(Q)\right)^{T_\mathbf{d}} = \bigoplus_{a \in Q_1} \left(\mathrm{Hom}\left(\mathbb{C}^{\mathbf{d}_{s(a)}}, \mathbb{C}^{\mathbf{d}_{t(a)}}\right) \right)^{T_\mathbf{d}} = \{\rho_0\}$$

where ρ_0 is the zero representation in $\mathrm{Rep}_\mathbf{d}(Q)$.

Combining these two results, one can easily describe T-fixed points of varieties constructed over them:

$$\left(\mathcal{F}_{|\mathbf{d}|} \times \mathcal{F}_{|\mathbf{d}|}\right)^{\mathbb{T}_{|\mathbf{d}|}} = \left\{(F_\varpi, F_{\varpi'}) \,\middle|\, \varpi, \varpi' \in \mathbb{W}_{|\mathbf{d}|}\right\}$$

$$\left(\mathcal{F}_u \times \mathcal{F}_{u'}\right)^{T_\mathbf{d}} = \left\{(F_{wu}, F_{w'u'}) \,\middle|\, w, w' \in W_\mathbf{d}\right\}$$

$$\left(\mathcal{F}_\mathbf{d} \times \mathcal{F}_\mathbf{d}\right)^{T_\mathbf{d}} = \left\{(F_\varpi, F_{\varpi'}) \,\middle|\, \varpi, \varpi' \in \mathbb{W}_{|\mathbf{d}|}\right\}$$

$$\left(\widetilde{\mathrm{Rep}}_{\underline{\mathbf{d}}}(Q)\right)^{T_\mathbf{d}} = \left\{(\rho_0, F_{wu}) \,|\, w \in W_\mathbf{d}\right\}$$

$$\left(\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)\right)^{T_\mathbf{d}} = \left\{(\rho_0, F_\varpi) \,\middle|\, \varpi \in \mathbb{W}_{|\mathbf{d}|}\right\}$$

$$\left(\mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}'}\right)^{T_\mathbf{d}} = \left\{(\rho_0, F_{wu}, F_{w'u'}) \,\middle|\, w, w' \in W_\mathbf{d}\right\}$$

$$\left(\mathcal{Z}_\mathbf{d}\right)^{T_\mathbf{d}} = \left\{(\rho_0, F_\varpi, F_{\varpi'}) \,\middle|\, \varpi, \varpi' \in \mathbb{W}_{|\mathbf{d}|}\right\}$$

Notice that, each $B_{\mathbf{d}} \times B_{\mathbf{d}}$-orbit of $\mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}}$ contains exactly one $T_{\mathbf{d}}$-fixed point. Also, all the T-fixed points lie in the zero sections. By this reason, we can compute more:

$$
\left(\mathcal{Z}_{\mathrm{Id}}^{u,u'}\right)^{T_{\mathbf{d}}} = \{(\rho_0, F_{wu}, F_{wu'}) \mid w \in W_{\mathbf{d}}\}
$$

$$
(\mathcal{Z}_{\mathrm{Id}})^{T_{\mathbf{d}}} = \left\{(\rho_0, F_{\varpi}, F_{\varpi}) \mid \varpi \in \mathbb{W}_{|\mathbf{d}|}\right\}
$$

$$
\left(\mathcal{Z}_{s}^{u,u'}\right)^{T_{\mathbf{d}}} = \{(\rho_0, F_{wu}, F_{wsu'}) \mid w \in W_{\mathbf{d}}\} \sqcup \{(\rho_0, F_{wu}, F_{wu'}) \mid w \in W_{\mathbf{d}}\}
$$

$$
(\mathcal{Z}_{s})^{T_{\mathbf{d}}} = \left\{(\rho_0, F_{\varpi}, F_{\varpi s}) \mid \varpi \in \mathbb{W}_{|\mathbf{d}|}\right\} \sqcup \left\{(\rho_0, F_{\varpi}, F_{\varpi}) \mid \varpi \in \mathbb{W}_{|\mathbf{d}|}, \varpi s \varpi^{-1} \in W_{\mathbf{d}}\right\}
$$

With huge effort, finally we fixed all the symbols and understand those typical varieties in detail.

Chapter 2

K-theory and cohomology theory

From my humble point of view, there is no easy cohomology theory, in a sense that key properties are usually hard to prove. On the other hand, plenty of examples can be quickly computed once we grasp some properties and use them in black boxes. Therefore, we won't prove any properties we stated. We have no choice but to do so, for the restricted space and time.

The main reference for the K-theory is ???.

Setting 2.0.1. *Throughout abstract results of K-theory, we use the following notations:*

- *G stands for a linear algebraic group, i.e., a closed subgroup of $\mathrm{GL}_n(\mathbb{C})$.* [1] *Denote $m : G \times G \longrightarrow G$ as the multiplication map of G.*
- *X is a variety over $\mathbb{C}$, i.e., a reduced, separated scheme of finite type over $\mathbb{C}$. We assume X to be quasi-projective.*
- *Usually, X is equipped with an algebraic G-action (which is compatible with the variety structure of G and X), then we say that X is a G-variety. In that case, we will denote $\alpha : G \times X \longrightarrow X$ as the G-action map.*
- *$\mathcal{F}$ is usually a sheaf on X, which is not flag variety GL_n / B.*
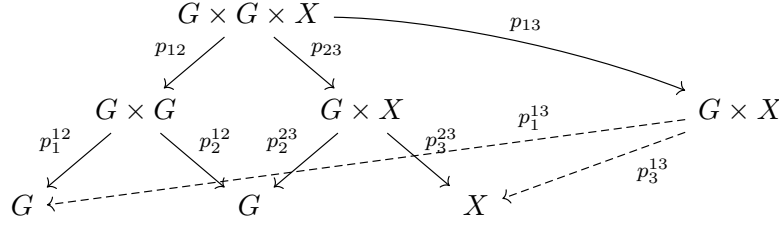
2.1 Definitions and initial examples

We give definition for K-theory, which is lengthy already. Roughly speaking, a G-equivariant coherent sheaf over X is a sheaf $\mathcal{F} \in \mathrm{Coh}(X)$ equipped with G-action which is compatible with the G-action on X, and K-theory is the Grothendieck group of G-equivariant coherent sheaves over X.

Definition 2.1.1 (G-equivariant sheaf, ???)**.** *For a G-variety X, denote $p_i^{jk}, p_i := p_i^{123}, p_{ij} :=$*

---

[1] The closed embedding $G \hookrightarrow \mathrm{GL}_n(\mathbb{C})$ is not considered as the data of G.

p_{ij}^{123} *as projections onto some factors, as follows.*[2]

We have morphisms

$$G \times G \times X \xrightarrow[\substack{\text{Id}_G \times \alpha}]{\substack{m \times \text{Id}_X \\ p_{23}}} G \times X \xrightarrow[\alpha]{p_3^{23}=p_3^{13}} X$$

which satisfies the "coequalizer conditions":

$$\begin{aligned}
p_3^{23} \circ (m \times \text{Id}_X) &= p_3^{23} \circ p_{23} & (g_1, g_2, x) &\longmapsto x \\
p_3^{23} \circ (\text{Id}_G \times \alpha) &= \alpha \circ p_{23} & (g_1, g_2, x) &\longmapsto g_2 x \\
\alpha \circ (m \times \text{Id}_X) &= \alpha \circ (\text{Id}_G \times \alpha) & (g_1, g_2, x) &\longmapsto g_1 g_2 x
\end{aligned}$$

A **G-equivariant (coherent) sheaf**[3] *on X is a sheaf $\mathcal{F} \in \text{Coh}(X)$ equipped with an isomorphism*

$$\phi_{\mathcal{F}} : p_3^{23,*} \mathcal{F} \longrightarrow \alpha^* \mathcal{F}$$

such that the following diagram commutes:

$$(2.1.1)$$

A **(G-equivariant) morphism** $f : (\mathcal{F}, \phi_{\mathcal{F}}) \longrightarrow (\mathcal{G}, \phi_{\mathcal{G}})$ *between two G-equivariant sheaves is a morphism $f : \mathcal{F} \longrightarrow \mathcal{G}$ in $\text{Coh}(X)$ such that the diagram*

$$(2.1.2)$$

---

[2]Be careful, under this convention, the projection map $p_3^{23} = p_3^{13} : G \times X \longrightarrow X$ has subscription 3, and p_2 means the projection from $G \times G \times X$ to the second G. This convention is different with notations in ???.

[3]we will omit the word "coherent" for shorter notation.

commutes.

We denote $\mathrm{Coh}^G(X)$ as the category of G-equivariant sheaves.

Definition 2.1.2 (G-equivariant K-theory)**.** *For a G-variety X, the G-equivariant K-theory is defined as the Grothendieck group of G-equivariant coherent sheaves over X, i.e.,*

$$K_0^G(X) := K_0(\mathrm{Coh}^G(X)).$$

Specifically, for a point $\mathrm{pt} = \mathrm{Spec}\,\mathbb{C}$ *with trivial G-action, denote*

$$\mathrm{R}(G) := K_0^G(\mathrm{pt}) = K_0(\mathrm{Rep}(G))$$

as the representation ring of group G.

Let us unravel this construction a little bit. For (geometrical) points $g, g_1, g_2 \in G$, denote that

$$
\begin{aligned}
\iota_g &: X \longrightarrow G \times X & x &\longmapsto (g, x) \\
\iota_{g_1,g_2} &: X \longrightarrow G \times G \times X & x &\longmapsto (g_1, g_2, x) \\
\alpha_g &: X \xhookrightarrow{\iota_g} G \times X \xrightarrow{\alpha} X & x &\longmapsto gx
\end{aligned}
$$

By pulling back along ι_g and ι_{g_1,g_2}, we can see geometrical meanings in the expressions. Apply ι_g^* to $\phi_{\mathcal{F}}$, one get

$$\iota_g^*\phi_{\mathcal{F}} : \mathcal{F} \longrightarrow \alpha_g^*\mathcal{F} \qquad \rightsquigarrow \qquad \phi_{g,x}^{\mathcal{F}} \hat{=} \left(\iota_g^*\phi_{\mathcal{F}}\right)_x : \mathcal{F}_x \longrightarrow \mathcal{F}_{gx}$$

Therefore, $\phi_{\mathcal{F}}$ encodes information of G-action on $\mathcal{F}$, which is G-equivariant.

Now we apply ι_{g_1,g_2}^* to (2.1.1):

So (2.1.1) is just the associative constraint of the G-structure on $\mathcal{F}$.

Similarly, apply ι_g^* to (2.1.2), we get

So (2.1.2) is just the condition for f to be G-equivariant.

There are two extreme situations worth mentioning about. When $G = \mathrm{Id}$, there is no G-action structure constrain on varieties and sheaves. Therefore,

$$\mathrm{Coh}^{\mathrm{Id}}(X) = \mathrm{Coh}(X) \qquad K_0^{\mathrm{Id}}(X) = K_0(X) \hat{=} K_0(\mathrm{Coh}(X)).$$

When G acts on $X = \mathrm{Spec}\, A$ trivially, any sheaf $\mathcal{F} \in \mathrm{Coh}^G(X)$ can be viewed as an (finitely generated)[4] A-module M with G-action, so

$$\mathrm{Coh}^G(X) = \mathrm{rep}_A(G) \xrightarrow{\text{when } G \text{ is finite}} \mathrm{Mod}(A[G]).$$

In particular, any sheaf $\mathcal{F} \in \mathrm{Coh}^G(\mathrm{pt})$ can be viewed as a finite dimensional complex G-representation, so

$$\mathrm{Coh}^G(\mathrm{pt}) = \mathrm{rep}_{\mathbb{C}}(G) \xrightarrow{\text{when } G \text{ is finite}} \mathrm{Mod}(\mathbb{C}[G]).$$

???(If I have time I will compute $K_0(\mathbb{P}^1)$ here.)

2.1.1 Representation ring $\mathrm{R}(G)$

Now let us try to figure out some examples.

Recall that any coherent sheaf over pt is equivalent to a finite dimensional $\mathbb{C}$-vector space, and any G-equivariant coherent sheaf over pt is equivalent to a finite dimensional complex G-representation. Moreover, by Jordan-Hölder theorem, every finite dimensional complex G-representation can be written as a composition series such that each quotient object is irreducible. Therefore,

$$\mathrm{R}(G) = \bigoplus_{\rho \in \mathrm{Irr}(G)} \mathbb{Z}$$

as a free $\mathbb{Z}$-module.

For $\mathrm{R}(G)$, we have the multiplication structure induced by tensor products on complex G-representations. Let us see some examples now. We use Setting 1.1.1 in these examples.

Example 2.1.3. *For trivial group* Id, *every* Id*-representation is just a* $\mathbb{C}$*-vector space, which can be written as the direct sum of* 1*-dimensional vector spaces. Therefore,*

$$\mathrm{R}(\mathrm{Id}) = \mathbb{Z}.$$

Example 2.1.4. *For group* T, *since* T *is abelian, every* T*-representation can be written as direct sum of* 1*-dimensional vector spaces. Furthermore,*

$$\begin{aligned}
\mathrm{Irr}(T) &= \big\{ \rho : T \longrightarrow \mathbb{C}^{\times} \,\big|\, \rho \text{ is an (algebraic) group homomorphism} \big\} \\
&= \mathrm{Hom}_{\mathbb{C}\text{-}Alg\ gp}(T, \mathbb{C}^{\times}) := X^*(T)
\end{aligned}$$

---

[4]We already assume X to be of finite type, so coherent condition is equivalent to finitely generated condition.

We get

$$\mathrm{R}(T) = \bigoplus_{\rho \in \mathrm{Irr}(T)} \mathbb{Z} = \mathbb{Z}\left[X^*(T)\right].$$

The group structure in $X^*(T)$ is given by tensor product, so the multiplication structure is induced by the group structure in $X^*(T)$. Denote

$$\varepsilon_i : T \longrightarrow \mathbb{C}^\times \qquad \begin{pmatrix} t_1 & & & & \\ & \ddots & & & \\ & & t_i & & \\ & & & \ddots & \\ & & & & t_n \end{pmatrix} \longmapsto t_i$$

as a $\mathbb{Z}$-basis of $X^*(T)$, then

$$\mathrm{R}(T) \cong \mathbb{Z}\left[\varepsilon_1^{\pm 1}, \ldots, \varepsilon_n^{\pm 1}\right]$$

as a $\mathbb{Z}$-algebra.

By forgetting T-actions, we get a morphism of $\mathbb{Z}$-algebra

$$\mathrm{R}(T) \longrightarrow \mathrm{R}(\mathrm{Id}) \qquad f(\varepsilon_1, \ldots, \varepsilon_n) \longmapsto f(1, \ldots, 1).$$

Example 2.1.5. *After we prove the reduction theorem ???(ref), we can show that*

$$\mathrm{R}(N) \cong \mathrm{R}(\mathrm{Id}) \cong \mathbb{Z} \qquad \mathrm{R}(B) \cong \mathrm{R}(T) \cong \mathbb{Z}\left[\varepsilon_1^{\pm 1}, \ldots, \varepsilon_n^{\pm 1}\right]$$

Example 2.1.6. *By ???(ref Thm 6.1.4),*

$$\mathrm{R}(\mathrm{GL}_n) \cong \mathrm{R}(T)^W \cong \mathbb{Z}\left[\varepsilon_1^{\pm 1}, \ldots, \varepsilon_n^{\pm 1}\right]^{S_n}.$$

This can be viewed as a "group" analogue of Chevalley restriction theorem. Notice that we have clear description of finite dimensional irreducible representations of GL_n, and the forget map

$$\mathrm{rep}(\mathrm{GL}_n) \longrightarrow \mathrm{rep}(T) \qquad \rightsquigarrow \qquad \mathrm{R}(\mathrm{GL}_n) \longrightarrow \mathrm{R}(T)$$

views GL_n-representations as special W-invariant T-representations.

From these examples we already see the difficulty of computing K-theories. Therefore, a series of properties of K-theories are definitely needed for computations. To state these properties, we need to define some factors of K-theory.

2.2   Basic constructions: pullback, proper pushforward and tensor product

2.2.1   pullback

2.2.2   proper pushforward

2.2.3   tensor product

2.3   Thom isomorphism

2.4   Induction

2.4.1   Contracted product

Definition 2.4.1 (Contracted product). *Let $H \subseteq G$ be a closed algebraic subgroup and X be an H-variety. The contracted product of G and X over H is defined as*

$$G \times^H X := (G \times X)/\sim$$

where

$$(gh, x) \sim (g, hx) \qquad \text{for any } g \in G,\ h \in H,\ x \in X.$$

  $G \times^H X$ has a natural variety structure, which is not easy to construct. G acts on $G \times^H X$ by multiplying from the left side. We have a G-equivariant flat morphism

$$G \times^H X \longrightarrow G/H \qquad (g, x) \longrightarrow gH$$

which realize $G \times^H X$ as an X-bundle over G/H. In particular, for $X = \mathrm{pt}$, we get an isomorphism of G-varieties

$$G \times^H \mathrm{pt} \xrightarrow{\sim} G/H.$$

2.5   Reduction

2.6   Equivariant cohomology theory

Chapter 3

Cellular fibration theorem

3.1 Statement

3.2 Application: module structure

Chapter 4

Localization theorem

Chapter 5

Excess intersection formula

5.1 Convolution

The construction of the convolution product has a similar flavor with Fourier-Mukai transformation, which is the composition of pullback, tensor product and proper pushforward.
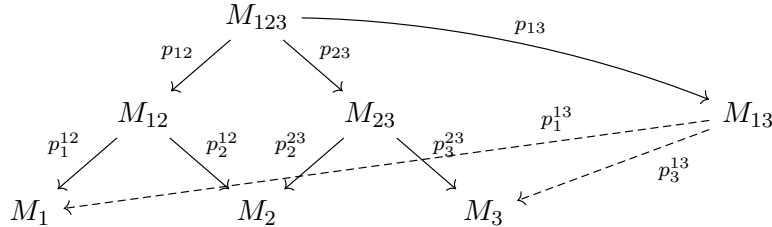
Definition 5.1.1 (Convolution product). *For the convenience of reading, we divide the whole process into three steps.*

Step1 *Setting.*
    Let M_1, M_2, M_3 be smooth quasi-projective G-varieties. For convenience, denote

$$M_{ij} := M_i \times M_j \qquad M_{123} = M_1 \times M_2 \times M_3$$

and $p_i^{jk}, p_i := p_i^{123}, p_{ij} := p_{ij}^{123}$ as projections onto some factors, as follows.

(Check that $p_i = p_i^{jk} \circ p_{jk}$ for $1 \leqslant j < k \leqslant 3$, $i = j$ or $i = k$)

Step2 *Convolution product on the level of varieties.*
    For closed G-subvarieties $Z_{12} \subseteq M_{12}$, $Z_{23} \subseteq M_{23}$, denote

$$Z_{123} := p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23}) \subseteq M_1 23$$

*as the intersection of two preimages. The **convolution product** of Z_{12} and Z_{23} is defined as*

$$Z_{12} \circ Z_{23} := p_{13}(Z_{123}) \subseteq M_{13}$$

which is a closed G-subvariety of M_{13}.

Step3   *Convolution product on the level of K-theories.*

5.2   Statement

5.3   Application: convolution formula

5.4   Demazure operator

Chapter 6

Generalization

6.1   quiver with loops

6.2 $G \times \mathbb{C}^{\times}$-action

Chapter 7

From formula to diagram

7.1 One point quiver

7.2 A_2-quiver

7.3 1-loop quiver

Chapter 8

Atiyah-Segal completion theorem