Affine pavings of partial flag varieties

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Process

Process

Affine paving

Setting

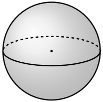
$K = \mathbb{C}$, X: *algebraic variety over* K.

Definition

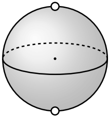
An **affine paving** of X is a filtration

$$0 = X_0 \subset X_1 \subset \cdots \subset X_d = X$$

with X_i closed and $X_{i+1} \setminus X_i \cong \mathbb{A}_K^k$.



$\mathbb{P}^1 = \{\infty\} \sqcup \mathbb{A}^1$



$\mathbb{P}^1 \setminus \{0, \infty\}$ has no affine paving

Quiver and quiver representation

Quiver is a graph. It has some vertices & arrows.
In this talk, all the quivers are finite and connected.

Quiver and quiver representation

We focus on the Dynkin quiver.

That means, the graph of the Dynkin diagrams in the ADE series.

Partial flag variety

Definition

Fix a quiver Q and $M \in \mathrm{rep}(Q)$,

$$\mathrm{Flag}_d(M) \colon = \{F \colon 0 \subseteq N_1 \subseteq \cdots \subseteq N_d \subseteq M\}$$
$$\mathrm{Flag}_{\underline{\mathbf{f}}}(M) \colon = \left\{F \colon 0 \subseteq N_1 \subseteq \cdots \subseteq N_d \subseteq M \;\middle|\; \underline{\mathbf{dim}}\, M_i = \underline{\mathbf{f}}_i\right\}$$

Example

$Q = \bullet$, $M = \mathbb{C}^n$, $\underline{\mathbf{f}} := \begin{pmatrix} n \\ \vdots \\ 1 \end{pmatrix}$

$$\mathrm{Flag}_d(\mathbb{C}^n) = \{F \colon 0 \subseteq N_1 \subseteq \cdots \subseteq N_d \subseteq \mathbb{C}^n\}$$
$$\mathrm{Flag}_1(\mathbb{C}^n) = \{F \colon 0 \subseteq N_1 \subseteq \mathbb{C}^n\} \qquad = \sqcup_{k=0}^{n} \mathrm{Gr}(n,k)$$
$$\mathrm{Flag}_{\underline{\mathbf{f}}}(\mathbb{C}^n) = \textit{ complete flags of } \mathbb{C}^n$$
$$\mathrm{Flag}_{(k)}(\mathbb{C}^n) = \mathrm{Gr}(n,k)$$

Statement

Theorem

For a Dynkin quiver Q and $M \in \mathrm{rep}(Q)$,

$$\mathrm{Flag}_d(M) \text{ has an affine paving.}$$

Process

Task 1. $Q = \bullet$, $M = \mathbb{C}^n$

In this case,

$$\begin{array}{ccc} \mathrm{GL}_n(\mathbb{C}) \circlearrowright \mathbb{C}^n & \rightsquigarrow & \mathrm{GL}_n(\mathbb{C}) \circlearrowright \mathrm{Flag}_d(\mathbb{C}^n) \\ & \rightsquigarrow & B \circlearrowright \mathrm{Flag}_d(\mathbb{C}^n) \end{array}$$

$\mathrm{Flag}_d(\mathbb{C}^n)$ has an affine paving given by Schubert cells (i.e., B-orbits).

Note

When $Q = \bullet \longrightarrow \bullet$, $\mathrm{Flag}_{\mathbf{f}}(M)$ have no natural group actions.

Idea of affine pavings

Find a nice short exact sequence

$$0 \longrightarrow X \xrightarrow{\iota} M \xrightarrow{\pi} S \longrightarrow 0$$

which induces a nice morphism

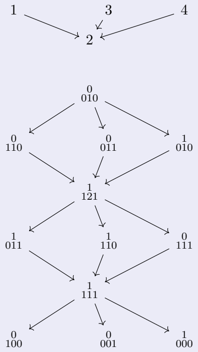
$$\begin{aligned} \Psi : \mathrm{Flag}_d(M) &\longrightarrow \mathrm{Flag}_d(X) \times \mathrm{Flag}_d(S) \\ F &\longmapsto \big(\iota^{-1}(F), \pi(F)\big) \end{aligned}$$

We construct the affine paving of $\mathrm{Flag}_d(M)$ from the affine paving of $\mathrm{Flag}_d(X)$ and $\mathrm{Flag}_d(S)$. Then, we use mathematical induction.

Process

Another example: D_4

$$\begin{tikzcd} & 4 \arrow[d] & \\ 1 \arrow[r] & 2 & 3 \arrow[l] \end{tikzcd}$$



For other examples, see here.

Process