

Geometry of Quiver Flag Varieties

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Introduction

Year	People	Cohomology theory	Algebra
1980	Kazhdan–Lusztig	$\mathbb{C}[W]$	$H_*^{\text{BM}}(\mathcal{Z})$
1985	Lusztig	$\mathcal{H}_q(W)$	$K^{G \times \mathbb{C}^\times}(\mathcal{Z})$
2011	Varagnolo–Vasserot	KLR algebra	$H_{G_d}^*(\mathcal{Z}_d)$

In the first part, we compute the G -equivariant K -theory of the Steinberg variety in the quiver version.

Variety structure

$$\begin{array}{ccc} \widetilde{\text{Rep}}_{\mathbf{d}}(Q) \subseteq \text{Rep}_{\mathbf{d}}(Q) \times \mathcal{F}_{\mathbf{d}} & & \\ \mu_{\mathbf{d}} \swarrow & \searrow \pi_{\mathbf{d}} & \\ \text{Rep}_{\mathbf{d}}(Q) & & \mathcal{F}_{\mathbf{d}} \end{array}$$

$$\begin{array}{ccc} \mathcal{Z}_{\mathbf{d}} \subseteq \text{Rep}_{\mathbf{d}}(Q) \times \mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}} & & \\ \mu_{\mathbf{d},\mathbf{d}} \swarrow & \searrow \pi_{\mathbf{d},\mathbf{d}} & \\ \text{Rep}_{\mathbf{d}}(Q) & & \mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}} \end{array}$$

Stratification structure

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Main theorem

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Generators

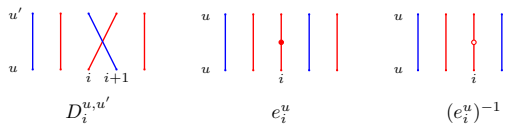


Figure: Generators

Compositions and trivial relations

$$\begin{array}{c} u'' \\ u'' \\ u' \\ u' \\ u \end{array} \left| \begin{array}{c} \text{diagram} \end{array} \right| \begin{array}{c} u'' \\ u'' \\ u' \\ u' \\ u \end{array} = \begin{array}{c} u'' \\ u'' \\ u' \\ u' \\ u \end{array} \left| \begin{array}{c} \text{diagram} \end{array} \right| \begin{array}{c} u'' \\ u'' \\ u' \\ u' \\ u \end{array}$$

$$D_3^{u,u'} * (e_3^{u'})^{-1} * D_2^{u',u''} * D_3^{u'',u''}$$

$$\begin{array}{c} u' \\ u \\ u' \\ u \end{array} \left| \begin{array}{c} \text{diagram} \end{array} \right| \begin{array}{c} u' \\ u \\ u' \\ u \end{array} = 0$$

$$D_3^{u,u'} * D_3^{u,u'} = 0$$

$$\begin{array}{c} u''' \\ u' \\ u \end{array} \left| \begin{array}{c} \text{diagram} \end{array} \right| \begin{array}{c} u''' \\ u' \\ u \end{array} = \begin{array}{c} u''' \\ u'' \\ u \end{array} \left| \begin{array}{c} \text{diagram} \end{array} \right| \begin{array}{c} u''' \\ u'' \\ u \end{array}$$

$$D_3^{u,u'} * D_1^{u',u'''} = D_1^{u,u''} * D_3^{u'',u''}$$

$$\begin{array}{c} u' \\ u' \\ u \end{array} \left| \begin{array}{c} \text{diagram} \end{array} \right| \begin{array}{c} u' \\ u' \\ u \end{array} = \begin{array}{c} u' \\ u \\ u \end{array} \left| \begin{array}{c} \text{diagram} \end{array} \right| \begin{array}{c} u' \\ u \\ u \end{array}$$

$$D_3^{u,u'} * e_2^{u'} = e_2^u * D_3^{u,u'}$$

Nontrivial relations I

Same color: $(D_i^{u,u} \hat{=} D_i, e_i^u \hat{=} e_i)$

$$\begin{array}{c} \text{Diagram: } \begin{array}{|c|} \hline \diagup \text{ (black dot)} \diagdown \\ \hline \end{array} = \begin{array}{|c|} \hline \diagdown \text{ (black dot)} \diagup \\ \hline \end{array} - \begin{array}{|c|} \hline | \text{ (black dot)} \\ \hline \end{array} \\ D_i e_i = e_{i+1} D_i - e_{i+1} \end{array}$$

$$\begin{array}{c} \text{Diagram: } \begin{array}{|c|} \hline \diagdown \text{ (black dot)} \diagup \\ \hline \end{array} = \begin{array}{|c|} \hline \diagup \text{ (black dot)} \diagdown \\ \hline \end{array} + \begin{array}{|c|} \hline | \text{ (black dot)} \\ \hline \end{array} \\ D_i e_{i+1} = e_i D_i + e_{i+1} \end{array}$$

$$\begin{array}{c} \text{Diagram: } \begin{array}{|c|} \hline \diagup \text{ (white dot)} \diagdown \\ \hline \end{array} = \begin{array}{|c|} \hline \diagdown \text{ (white dot)} \diagup \\ \hline \end{array} + \begin{array}{|c|} \hline | \text{ (white dot)} \\ \hline \end{array} \\ D_i e_i^{-1} = e_{i+1}^{-1} D_i + e_i^{-1} \end{array}$$

$$\begin{array}{c} \text{Diagram: } \begin{array}{|c|} \hline \diagdown \text{ (white dot)} \diagup \\ \hline \end{array} = \begin{array}{|c|} \hline \diagup \text{ (white dot)} \diagdown \\ \hline \end{array} - \begin{array}{|c|} \hline | \text{ (white dot)} \\ \hline \end{array} \\ D_i e_{i+1}^{-1} = e_i^{-1} D_i - e_i^{-1} \end{array}$$

$$\begin{array}{c} \text{Diagram: } \begin{array}{|c|} \hline \diagup \diagdown \diagup \diagdown \\ \hline \end{array} = \begin{array}{|c|} \hline \diagdown \diagup \diagdown \diagup \\ \hline \end{array} \\ D_i D_{i+1} D_i = D_{i+1} D_i D_{i+1} \end{array}$$

$$\begin{array}{c} \text{Diagram: } \begin{array}{|c|} \hline \diagup \diagdown \diagup \diagdown \\ \hline \end{array} = \begin{array}{|c|} \hline \diagdown \diagup \diagdown \diagup \\ \hline \end{array} \\ D_i^2 = D_i \end{array}$$

Nontrivial relations II

Different color:

Diagrammatic equation: A crossing of a blue line (top-left to bottom-right) and a red line (top-right to bottom-left) with a blue dot on the blue line is equal to the same crossing with a blue dot on the red line. Both diagrams are labeled with a circled 3.

$$D_i^{u,u'} e_i^{u'} = e_{i+1}^u D_i^{u,u'}$$

Diagrammatic equation: A crossing of a blue line (top-left to bottom-right) and a red line (top-right to bottom-left) with a blue dot on the blue line is equal to the same crossing with a blue dot on the red line. Both diagrams are labeled with a circled 2 and a circled 3.

$$D_i^{u,u'} D_i^{u',u} = 1^u - \left(\frac{e_i}{e_{i+1}}\right)^u$$

Diagrammatic equation: A crossing of a blue line (top-left to bottom-right) and a red line (top-right to bottom-left) with a red dot on the red line is equal to the same crossing with a red dot on the blue line. Both diagrams are labeled with a circled 3.

$$D_i^{u,u'} (e_i^{u'})^{-1} = (e_{i+1}^u)^{-1} D_i^{u,u'}$$

Diagrammatic equation: A crossing of a blue line (top-left to bottom-right) and a red line (top-right to bottom-left) with a red dot on the red line is equal to the same crossing with a red dot on the blue line. Both diagrams are labeled with a circled 3 and a circled 2.

$$D_i^{u,u'} D_i^{u',u} = 1^u - \left(\frac{e_{i+1}}{e_i}\right)^u$$

Nontrivial relations III

Different color:

$$\begin{array}{c} u \\ u' \\ u' \\ u \end{array} \begin{array}{c} \text{Diagram 1} \end{array} \begin{array}{c} \textcircled{2} \\ \textcircled{1} \\ \textcircled{3} \end{array} = \begin{array}{c} u \\ u'' \\ u'' \\ u \end{array} \begin{array}{c} \text{Diagram 2} \end{array} \begin{array}{c} \textcircled{3} \\ \textcircled{1} \\ \textcircled{2} \end{array} + \begin{array}{c} u \\ u \\ u \end{array} \begin{array}{c} \text{Diagram 3} \end{array}$$

$$D_i^{u,u'} D_{i+1}^{u',u'} D_i^{u',u} = D_{i+1}^{u,u''} D_i^{u'',u''} D_{i+1}^{u'',u} + \left(\frac{e_{i+2}}{e_{i+1}} \right)^u$$

$$\begin{array}{c} u \\ u' \\ u' \\ u \end{array} \begin{array}{c} \text{Diagram 4} \end{array} \begin{array}{c} \textcircled{3} \\ \textcircled{1} \\ \textcircled{2} \end{array} = \begin{array}{c} u \\ u'' \\ u'' \\ u \end{array} \begin{array}{c} \text{Diagram 5} \end{array} \begin{array}{c} \textcircled{2} \\ \textcircled{1} \\ \textcircled{3} \end{array} - \begin{array}{c} u \\ u \\ u \end{array} \begin{array}{c} \text{Diagram 6} \end{array}$$

$$D_i^{u,u'} D_{i+1}^{u',u'} D_i^{u',u} = D_{i+1}^{u,u''} D_i^{u'',u''} D_{i+1}^{u'',u} - \left(\frac{e_{i+1}}{e_i} \right)^u$$

$$\begin{array}{c} u'' \\ u'' \\ u' \\ u \end{array} \begin{array}{c} \text{Diagram 7} \end{array} \begin{array}{c} \textcircled{1} \\ \textcircled{2} \\ \textcircled{2} \\ \textcircled{2} \end{array} = \begin{array}{c} u'' \\ u' \\ u \\ u \end{array} \begin{array}{c} \text{Diagram 8} \end{array} \begin{array}{c} \textcircled{2} \\ \textcircled{2} \\ \textcircled{2} \\ \textcircled{1} \end{array}$$

$$D_i^{u,u'} D_{i+1}^{u',u''} D_i^{u'',u''} = D_{i+1}^{u,u} D_i^{u,u'} D_{i+1}^{u',u''}$$

Affine pavings

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Idea of affine pavings

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Outlook

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