

Master thesis

Contents

1	Variety and stratification	5
1.1	Initial case: \mathcal{F} and $\mathcal{F} \times \mathcal{F}$	5
1.2	quiver and Weyl group	7
1.3	algebraic group and Lie algebra	7
1.4	typical variety	8
1.5	stratification and T -fixed points	8
2	K-theory and cohomology theory	9
3	Localization theorem	11
4	Excess intersection formula	13
5	From formula to diagram	15
6	Generalization	17
7	Atiyah-Segal completion theorem	19

Chapter 1

Variety and stratification

1.1 Initial case: \mathcal{F} and $\mathcal{F} \times \mathcal{F}$

We introduce the complete flag variety to give a bird's eye view on the whole section. Actually, the entire difficulty is bundled in this example.

Fix $n \geq 1$, we denote $\mathrm{GL}_n := \mathrm{GL}_n(\mathbb{C})$, B , T , N , W be the standard Borel subgroup, standard torus, unipotent subgroup, Weyl group respectively, i.e.,

$$\mathrm{GL}_n = \begin{pmatrix} * & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & * \end{pmatrix} \quad B = \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix} \quad T = \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \quad B = \begin{pmatrix} 1 & \cdots & * \\ & \ddots & \vdots \\ 0 & & 1 \end{pmatrix}$$

$$W := N_{\mathrm{GL}_n}(T)/T \cong S_n$$

Definition 1.1.1 (flag). *For a finite dimensional \mathbb{C} -vector space V , a flag of V is an increasing sequence of subspaces of V :*

$$F : 0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_k = \mathbb{C}^n.$$

F is called a complete flag if $\dim V_i = i$ for all i , otherwise F is called a partial flag.

Definition 1.1.2 (complete flag variety). *The complete flag variety \mathcal{F} is defined as*

$$\begin{aligned} \mathcal{F} &= \mathrm{GL}_n / B \\ &\cong \{\text{complete flags of } \mathbb{C}^n\} \\ &= \{0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n = \mathbb{C}^n \mid \dim V_i = i\} \\ &\cong \{\text{Borel subgroups of } \mathrm{GL}_n\} \\ &= \{gBg^{-1} \mid g \in \mathrm{GL}_n\} \end{aligned}$$

Remark 1.1.3.

1. We implicitly give the base point of \mathcal{F} , which is not considered as the data of \mathcal{F} . Fix a standard basis of \mathbb{C}^n by $\{v_1, \dots, v_n\}$, we define the standard flag

$$F_{\text{Id}} : 0 \subseteq \langle v_1 \rangle \subseteq \langle v_1, v_2 \rangle \subseteq \dots \subseteq \langle v_1, \dots, v_n \rangle = \mathbb{C}^n.$$

2. We have the natural GL_n -action on \mathcal{F} , which is considered as the data of \mathcal{F} . For $g \in \text{GL}_n$, we define the flag attached to g :

$$F_g \triangleq gF_{\text{Id}} : 0 \subseteq \langle gv_1 \rangle \subseteq \langle gv_1, gv_2 \rangle \subseteq \dots \subseteq \langle gv_1, \dots, gv_n \rangle = \mathbb{C}^n.$$

Especially, for $w \in W = N_{\text{GL}_n}(T)/T \cong S_n$, the flag attached to w

$$\begin{aligned} F_w : 0 \subseteq \langle \tilde{w}v_1 \rangle \subseteq \langle \tilde{w}v_1, \tilde{w}v_2 \rangle \subseteq \dots \subseteq \langle \tilde{w}v_1, \dots, \tilde{w}v_n \rangle &= \mathbb{C}^n \\ 0 \subseteq \langle v_{w(1)} \rangle \subseteq \langle v_{w(1)}, v_{w(2)} \rangle \subseteq \dots \subseteq \langle v_{w(1)}, \dots, v_{w(n)} \rangle &= \mathbb{C}^n \end{aligned}$$

does not depend on the choice of the lift $\tilde{w} \in N_{\text{GL}_n}(T)$ of w .

Readers can verify that $\{F_w | w \in W\}$ are all T -fixed points of \mathcal{F} , while $\{wBw^{-1} | w \in W\}$ are all Borel subgroups of G containing the standard torus T .

3. For $n = 2$, $\mathcal{F} \cong \mathbb{P}^1$. We encourage readers to use \mathbb{P}^1 as a toy example for the whole theory.

interpretation	GL_n/B	flags	Borel subgroups
base point	Id	F_{Id}	B
GL_n -action	left multiplication	$\{V_i\} \mapsto \{gV_i\}$	conjugation
general point	g	F_g	gBg^{-1}

\mathcal{F} is a well-studied variety, and has many combinatorial properties. For example, from the well-known Bruhat decomposition,¹

$$\text{GL}_n \cong \bigsqcup_{w \in W} BwB$$

We get a stratification of \mathcal{F} by B -orbits:

$$\mathcal{F} = \text{GL}_n/B \cong \bigsqcup_{w \in W} BwB/B$$

The B -orbit BwB/B is called the Schubert cell, denoted by \mathcal{V}_w . Since

$$\mathcal{V}_w = BwB/B \cong B/(B \cap wBw^{-1}) \cong \mathbb{A}^{l(w)},$$

the Schubert cell is an affine space of dimension $l(w)$.

As a result, we know a lot of information of \mathcal{F} :

¹For the most time the formula does not depend on the lift of w , so we abuse the notation of $w \in N_{\text{GL}_n}(T)/T$ and $\tilde{w} \in N_{\text{GL}_n}(T)$.

$H^i(\mathcal{F}; \mathbb{C})$	0	2	4	6	8	10	12
1	1						
2	1	1					
3	1	2	2	1			
4	1	3	5	6	5	3	1
5	1	4	9	15	20	22	20

G	Orbit	G -fixed points
GL_n	$\mathcal{F} \cong GL_n/B$	\emptyset
B	$\mathcal{V}_w \cong B/(B \cap wBw^{-1})$	$\{F_{\text{Id}}\}$
T	–	$\{F_w w \in W\}$

1.2 quiver and Weyl group

1.3 algebraic group and Lie algebra

In this subsection we fix notations of algebraic group and Lie algebras. Later, the algebraic group will act on varieties, and some Lie algebra will serve as tangent spaces.

We fix a quiver Q , a dimension vector \mathbf{d} and a \mathbb{C} -vector space with quiver partition

$$V = \bigoplus_{i \in Q_0} V_i \quad \text{with } V_i = \mathbb{C}^{\mathbf{d}_i}.$$

Definition 1.3.1 (absolute algebraic groups). *We set*

$$\mathbb{G}_{|\mathbf{d}|} := GL(V) = GL_{|\mathbf{d}|}(\mathbb{C}),$$

and $\mathbb{B}_{|\mathbf{d}|}$, $\mathbb{T}_{|\mathbf{d}|}$, $\mathbb{N}_{|\mathbf{d}|}$ are corresponding standard Borel, torus and unipotent subgroups.

The Weyl group is

$$\mathbb{W}_{|\mathbf{d}|} := N_{\mathbb{G}_{|\mathbf{d}|}}(\mathbb{T}_{|\mathbf{d}|})/\mathbb{T}_{|\mathbf{d}|} \cong S_{|\mathbf{d}|}.$$

For $\varpi \in \mathbb{W}_{|\mathbf{d}|}$, we define²

$$\mathbb{B}_{\varpi} := \varpi \mathbb{B}_{|\mathbf{d}|} \varpi^{-1}.$$

We will view \mathbb{B}_{ϖ} as the stabilizer of the flag F_{ϖ} with $\mathbb{G}_{|\mathbf{d}|}$ -action.

We also have a series of algebraic groups compatible with the quiver partition of V , and they're more common in this thesis.

Definition 1.3.2 (relative algebraic groups). *We set*

$$G_{\mathbf{d}} := \bigoplus_{i \in Q_0} GL(V_i) = GL_{\mathbf{d}_i}(\mathbb{C}) \subseteq \mathbb{G}_{|\mathbf{d}|},$$

and $B_{\mathbf{d}}$, $T_{\mathbf{d}}$, $N_{\mathbf{d}}$ are corresponding standard Borel, torus and unipotent subgroups.

²As usual, we abuse the notation of ϖ and its lift.

The Weyl group is

$$W_{\mathbf{d}} := N_{G_{\mathbf{d}}}(T_{\mathbf{d}})/T_{\mathbf{d}} \cong \prod_{i \in Q_0} S_{\mathbf{d}_i}.$$

For $\varpi = wu \in W_{\mathbf{d}}$, we define

$$B_{\varpi} := wB_{\mathbf{d}}w^{-1}.$$

We will view B_{ϖ} as the stabilizer of the flag F_{ϖ} with $G_{\mathbf{d}}$ -action.

We also have a series of algebraic groups with subscription as elements in the Weyl group:

Definition 1.3.3 (more algebraic groups). For $\varpi, \varpi'' \in \mathbb{W}_{|\mathbf{d}|}$, define

$$\begin{aligned} N_{\varpi} &:= R_u(B_{\varpi}), \\ N_{\varpi, \varpi''} &:= N_{\varpi} \cap N_{\varpi''}, \\ M_{\varpi, \varpi''} &:= N_{\varpi}/N_{\varpi, \varpi''}, \end{aligned}$$

where R_u denotes for the unipotent radical.

For $s \in \Pi$ such that $\varpi s \varpi^{-1} \in W_{\mathbf{d}}$ (i.e., $W_{\mathbf{d}}\varpi = W_{\mathbf{d}}\varpi s$), define

$$\begin{aligned} P_{\varpi, \varpi s} &:= \overline{\overline{\overline{\varpi = wu}}} w (B_{\mathbf{d}} u s u^{-1} B_{\mathbf{d}} \cup B_{\mathbf{d}}) w^{-1} \\ &= B_{\varpi} \varpi s \varpi^{-1} B_{\varpi} \cup B_{\varpi} \end{aligned}$$

Remark 1.3.4. One can easily show that $N_{\varpi, \varpi s} = R_u(P_{\varpi, \varpi s})$.

Example 1.3.5. For $|\mathbf{d}| = 5$, $\mathbf{d} = (3, 2)$, ???,

$$\begin{array}{llll} \mathbb{G}_{|\mathbf{d}|} = \begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{pmatrix} & \mathbb{B}_{|\mathbf{d}|} = \begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{pmatrix} & \mathbb{T}_{|\mathbf{d}|} = \begin{pmatrix} * & & & & \\ * & & & & \\ & * & & & \\ & & * & & \\ & & & * & \end{pmatrix} & \mathbb{N}_{|\mathbf{d}|} = \begin{pmatrix} 1 & * & * & * & * \\ & 1 & * & * & * \\ & & 1 & * & * \\ & & & 1 & * \\ & & & & 1 \end{pmatrix} \\ \mathbb{W}_{|\mathbf{d}|} \cong S_5 & \mathbb{B}_{\varpi} = \begin{pmatrix} * & * & & & * \\ * & * & & & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{pmatrix} & \mathbb{B}_{\varpi s} = \begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{pmatrix} & \\ G_{\mathbf{d}} = \left(\begin{array}{ccc|cc} * & * & * & & \\ * & * & * & & \\ * & * & * & & \\ \hline & & & * & * \\ & & & * & * \end{array} \right) & B_{\mathbf{d}} = \left(\begin{array}{ccc|cc} * & * & * & & \\ * & * & * & & \\ * & * & * & & \\ \hline & & & * & * \\ & & & * & * \end{array} \right) & T_{\mathbf{d}} = \left(\begin{array}{ccc|cc} * & & & & \\ * & & & & \\ & * & & & \\ \hline & & & * & * \\ & & & * & * \end{array} \right) & N_{\mathbf{d}} = \left(\begin{array}{ccc|cc} 1 & * & * & & \\ & 1 & * & & \\ & & 1 & & \\ \hline & & & 1 & * \\ & & & * & 1 \end{array} \right) \\ W_{\mathbf{d}} \cong S_3 \times S_2 & B_{\varpi} = \left(\begin{array}{ccc|cc} * & * & & & * \\ * & * & & & * \\ * & * & * & * & * \\ \hline & & & * & * \\ & & & * & * \end{array} \right) & B_{\varpi s} = \left(\begin{array}{ccc|cc} * & & & & \\ * & * & & & \\ * & * & * & * & * \\ \hline & & & * & * \\ & & & * & * \end{array} \right) & \\ N_{\varpi} = \left(\begin{array}{ccc|cc} 1 & * & & & \\ & 1 & & & \\ * & * & 1 & & \\ \hline & & & 1 & * \\ & & & * & 1 \end{array} \right) & N_{\varpi, \varpi s} = \left(\begin{array}{ccc|cc} 1 & & & & \\ & 1 & & & \\ * & * & 1 & & \\ \hline & & & 1 & * \\ & & & * & 1 \end{array} \right) & M_{\varpi, \varpi s} = \left(\begin{array}{ccc|cc} 1 & * & & & \\ & 1 & & & \\ - & - & 1 & & \\ \hline & & & 1 & -1 \end{array} \right) & P_{\varpi, \varpi s} = \left(\begin{array}{ccc|cc} * & * & & & \\ * & * & & & \\ * & * & * & * & * \\ \hline & & & * & * \\ & & & * & * \end{array} \right) \end{array}$$

1.4 typical variety

1.5 stratification and T -fixed points

Chapter 2

K-theory and cohomology theory

Chapter 3

Localization theorem

Chapter 4

Excess intersection formula

Chapter 5

From formula to diagram

Chapter 6

Generalization

Chapter 7

Atiyah-Segal completion theorem