Master thesis

Contents

Chapter 1

Variety and stratification

1.1 Initial case: $\mathcal{F}$ and $\mathcal{F} \times \mathcal{F}$

We introduce the complete flag variety to give a bird's eye view on the whole section. Actually, the entire difficulty is bundled in this example.

Fix $n \geqslant 1$, we denote $\mathrm{GL}_n := \mathrm{GL}_n(\mathbb{C})$, B, T, N, W be the standard Borel subgroup, standard torus, unipotent subgroup, Weyl group respectively, i.e.,

$$\mathrm{GL}_n = \begin{pmatrix} * & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & * \end{pmatrix} \quad B = \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix} \quad T = \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \quad N = \begin{pmatrix} 1 & \cdots & * \\ & \ddots & \vdots \\ 0 & & 1 \end{pmatrix}$$

$$W := N_{\mathrm{GL}_n}(T)/T \cong S_n$$

Definition 1.1.1 (Flag)**.** *For a finite dimensional $\mathbb{C}$-vector space V, a flag of V is an increasing sequence of subspaces of V:*

$$F : 0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_k = \mathbb{C}^n.$$

F is called a complete flag if $\dim V_i = i$ for all i, otherwise F is called a partial flag.

Definition 1.1.2 (Complete flag variety)**.** *The complete flag variety $\mathcal{F}$ is defined as*

$$\begin{aligned} \mathcal{F} &= \mathrm{GL}_n / B \\ &\cong \{\textit{complete flags of } \mathbb{C}^n\} \\ &= \{0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n = \mathbb{C}^n \mid \dim V_i = i\} \\ &\cong \{\textit{Borel subgroups of } \mathrm{GL}_n\} \\ &= \left\{gBg^{-1} \;\middle|\; g \in \mathrm{GL}_n\right\} \end{aligned}$$

Remark 1.1.3.

1. We implicitly give the base point of $\mathcal{F}$, which is not considered as the data of $\mathcal{F}$. Fix a standard basis of $\mathbb{C}^n$ by $\{v_1, \ldots, v_n\}$, we define the standard flag

$$F_{\mathrm{Id}} : 0 \subseteq \langle v_1 \rangle \subseteq \langle v_1, v_2 \rangle \subseteq \cdots \subseteq \langle v_1, \ldots, v_n \rangle = \mathbb{C}^n.$$

2. We have the natural GL_n-action on $\mathcal{F}$, which is considered as the data of $\mathcal{F}$.

   For $g \in \mathrm{GL}_n$, we define the flag attached to g:

$$F_g \triangleq gF_{\mathrm{Id}} : 0 \subseteq \langle gv_1 \rangle \subseteq \langle gv_1, gv_2 \rangle \subseteq \cdots \subseteq \langle gv_1, \ldots, gv_n \rangle = \mathbb{C}^n.$$

   Especially, for $w \in W = N_{\mathrm{GL}_n}(T)/T \cong S_n$, the flag attached to w

$$\begin{aligned} F_w : 0 &\subseteq \langle \tilde{w}v_1 \rangle \subseteq \langle \tilde{w}v_1, \tilde{w}v_2 \rangle \subseteq \cdots \subseteq \langle \tilde{w}v_1, \ldots, \tilde{w}v_n \rangle = \mathbb{C}^n \\ 0 &\subseteq \left\langle v_{w(1)} \right\rangle \subseteq \left\langle v_{w(1)}, v_{w(2)} \right\rangle \subseteq \cdots \subseteq \left\langle v_{w(1)}, \ldots, v_{w(n)} \right\rangle = \mathbb{C}^n \end{aligned}$$

   does not depend on the choice of the lift $\tilde{w} \in N_{\mathrm{GL}_n}(T)$ of w.

   Readers can verify that $\{F_w | w \in W\}$ are all T-fixed points of $\mathcal{F}$, while $\left\{wBw^{-1} \middle| w \in W\right\}$ are all Borel subgroups of G containing the standard torus T.

3. For $n = 2$, $\mathcal{F} \cong \mathbb{P}^1$. We encourage readers to use $\mathbb{P}^1$ as a toy example for the whole theory.

interpretation	GL_n / B	flags	Borel subgroups
base point	Id	F_{Id}	B
GL_n-action	left multiplication	$\{V_i\} \mapsto \{gV_i\}$	conjugation
general point	g	F_g	gBg^{-1}

$\mathcal{F}$ is a well-studied variety, and has many combinatorical properties. For example, from the well-known Bruhat decomposition, [1]

$$\mathrm{GL}_n \cong \bigsqcup_{w \in W} BwB$$

We get a stratification of $\mathcal{F}$ by B-orbits:

$$\mathcal{F} = \mathrm{GL}_n / B \cong \bigsqcup_{w \in W} BwB/B$$

The B-orbit BwB/B is called the Schubert cell, denoted by $\mathcal{V}_w$. Since

$$\mathcal{V}_w = BwB/B \cong B/\left(B \cap wBw^{-1}\right) \cong \mathbb{A}^{l(w)},$$

the Schubert cell is an affine space of dimension $l(w)$.

As a result, we know a lot of information of $\mathcal{F}$:

[1] For the most time the formula does not depend on the lift of w, so we abuse the notation of $w \in N_{\mathrm{GL}_n}(T)/T$ and $\tilde{w} \in N_{\mathrm{GL}_n}(T)$.

$H^i(\mathcal{F};\mathbb{C})$	0	2	4	6	8	10	12
1	1						
2	1	1					
3	1	2	2	1			
4	1	3	5	6	5	3	1
5	1	4	9	15	20	22	20

G	Orbit	G-fixed points
GL_n	$\mathcal{F} \cong \mathrm{GL}_n / B$	$\varnothing$
B	$\mathcal{V}_w \cong B/(B \cap wBw^{-1})$	$\{F_{\mathrm{Id}}\}$
T	$-$	$\{F_w \mid w \in W\}$

1.2 quiver and Weyl group

To introduce more complicated spaces and discuss their stratifications, we fix notations related to quiver and algebraic group in the following subsections.

Roughly speaking, a quiver is a directed multigraph permitting loops.

Definition 1.2.1 (Quiver)**.** *???*

Remark 1.2.2. In the first part of our master thesis, all the quivers are supposed to be connected and finite (i.e., Q_0, Q_1 are finite sets). We will only encounter disconnected and infinite quiver as Auslander–Reiten quiver later on.

Example 1.2.3. *The following graphs are quivers.*

???

The reader can easily writen down the quadruple of these quivers. ???

For convenience, we mainly use simpler quivers as examples:

???

From those quivers we are able to construct relatively complicated algebraic and geometrical objects.

Definition 1.2.4 (Quiver representation)**.** *???*

Example 1.2.5. *A representation of* 1*-loop quiver* $L(1)$ *is a* 2*-tuple*

$$(V,\ \alpha : V \longrightarrow V)$$

which is equivalent to a (finite dimensional) $\mathbb{C}[t]$*-module.*

Remark 1.2.6. The equivalence appeared in the example can actually be generalized to arbitrary quivers. For a quiver Q, we can define the path algebra $\mathbb{C}Q$, and view any Q-representation as $\mathbb{C}Q$-module, and vice versa.

One important information of the quiver representation is the dimension vector.

Definition 1.2.7 (Dimension vector)**.** *???*

1.3 algebraic group and Lie algebra

In this subsection we fix notations of algebraic group and Lie algebras. Later, the algebraic group will act on varieties, and some Lie algebra will serve as tangent spaces.

We fix a quiver Q, a dimension vector $\mathbf{d}$ and a $\mathbb{C}$-vector space with quiver partition

$$V = \bigoplus_{i \in Q_0} V_i \qquad \text{with } V_i = \mathbb{C}^{\mathbf{d}_i}.$$

Definition 1.3.1 (absolute algebraic groups)**.** *We set*

$$\mathbb{G}_{|\mathbf{d}|} := \mathrm{GL}(V) = \mathrm{GL}_{|\mathbf{d}|}(\mathbb{C}),$$

and $\mathbb{B}_{|\mathbf{d}|}$, $\mathbb{T}_{|\mathbf{d}|}$, $\mathbb{N}_{|\mathbf{d}|}$ *are corresponding standard Borel, torus and unipotent subgroups.*

The Weyl group is

$$\mathbb{W}_{|\mathbf{d}|} := N_{\mathbb{G}_{|\mathbf{d}|}}(\mathbb{T}_{|\mathbf{d}|})/\mathbb{T}_{|\mathbf{d}|} \cong S_{|\mathbf{d}|}.$$

For $\varpi \in \mathbb{W}_{|\mathbf{d}|}$, *we define*[2]

$$\mathbb{B}_{\varpi} := \varpi \mathbb{B}_{|\mathbf{d}|} \varpi^{-1}.$$

We will view $\mathbb{B}_{\varpi}$ *as the stabilizer of the flag* F_{ϖ} *with* $\mathbb{G}_{|\mathbf{d}|}$*-action.*

We also have a series of algebraic groups compatible with the quiver partition of V, and they're more common in this thesis.

Definition 1.3.2 (relative algebraic groups)**.** *We set*

$$G_{\mathbf{d}} := \bigoplus_{i \in Q_0} \mathrm{GL}(V_i) = \mathrm{GL}_{\mathbf{d}_i}(\mathbb{C}) \subseteq \mathbb{G}_{|\mathbf{d}|},$$

and $B_{\mathbf{d}}$, $T_{\mathbf{d}}$, $N_{\mathbf{d}}$ *are corresponding standard Borel, torus and unipotent subgroups.*

The Weyl group is

$$W_{\mathbf{d}} := N_{G_{\mathbf{d}}}(T_{\mathbf{d}})/T_{\mathbf{d}} \cong \prod_{i \in Q_0} S_{\mathbf{d}_i}.$$

For $\varpi = wu \in W_{\mathbf{d}}$, *we define*

$$B_{\varpi} := w B_{\mathbf{d}} w^{-1}.$$

We will view B_{ϖ} *as the stabilizer of the flag* F_{ϖ} *with* $G_{\mathbf{d}}$*-action.*

We also have a series of algebraic groups with subscription as elements in the Weyl group:

[2] As usual, we abuse the notation of ϖ and its lift.

Definition 1.3.3 (more algebraic groups)**.** *For $\varpi, \varpi'' \in \mathbb{W}_{|\mathbf{d}|}$, define*

$$
\begin{aligned}
N_\varpi &:= R_u(B_\varpi), \\
N_{\varpi,\varpi''} &:= N_\varpi \cap N_{\varpi''}, \\
M_{\varpi,\varpi''} &:= N_\varpi / N_{\varpi,\varpi''},
\end{aligned}
$$

where R_u denotes for the unipotent radical.

For $s \in \Pi$ such that $\varpi s \varpi^{-1} \in W_d$ (i.e., $W_{\mathbf{d}}\varpi = W_{\mathbf{d}}\varpi s$), define

$$
\begin{aligned}
P_{\varpi,\varpi s} &:\xlongequal{\varpi = wu} w\left(B_{\mathbf{d}} u s u^{-1} B_{\mathbf{d}} \cup B_{\mathbf{d}}\right) w^{-1} \\
&\xlongequal{\quad} B_\varpi \varpi s \varpi^{-1} B_\varpi \cup B_\varpi
\end{aligned}
$$

Remark 1.3.4. One can easily show that $N_{\varpi,\varpi s} = R_u(P_{\varpi,\varpi s})$.

Example 1.3.5. *For $|\mathbf{d}| = 5$, $\mathbf{d} = (3,2)$, ???,*

$$
\mathbb{G}_{|\mathbf{d}|} = \begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{pmatrix} \quad
\mathbb{B}_{|\mathbf{d}|} = \begin{pmatrix} * & * & * & * & * \\ & * & * & * & * \\ & & * & * & * \\ & & & * & * \\ & & & & * \end{pmatrix} \quad
\mathbb{T}_{|\mathbf{d}|} = \begin{pmatrix} * & & & & \\ & * & & & \\ & & * & & \\ & & & * & \\ & & & & * \end{pmatrix} \quad
\mathbb{N}_{|\mathbf{d}|} = \begin{pmatrix} 1 & * & * & * & * \\ & 1 & * & * & * \\ & & 1 & * & * \\ & & & 1 & * \\ & & & & 1 \end{pmatrix}
$$

$$
\mathbb{W}_{|\mathbf{d}|} \cong S_5 \quad
\mathbb{B}_\varpi = \begin{pmatrix} * & * & & & * \\ & * & & & \\ * & * & * & & * \\ * & * & * & * & * \\ & * & & & * \end{pmatrix} \quad
\mathbb{B}_{\varpi s} = \begin{pmatrix} * & * & * & & * \\ & * & & & \\ & * & * & & * \\ * & * & * & * & * \\ & * & & & * \end{pmatrix}
$$

$$
G_{\mathbf{d}} = \left(\begin{array}{ccc|cc} * & * & * & & \\ * & * & * & & \\ * & * & * & & \\ \hline & & & * & * \\ & & & * & * \end{array}\right) \quad
B_{\mathbf{d}} = \left(\begin{array}{ccc|cc} * & * & * & & \\ & * & * & & \\ & & * & & \\ \hline & & & * & * \\ & & & & * \end{array}\right) \quad
T_{\mathbf{d}} = \left(\begin{array}{ccc|cc} * & & & & \\ & * & & & \\ & & * & & \\ \hline & & & * & \\ & & & & * \end{array}\right) \quad
N_{\mathbf{d}} = \left(\begin{array}{ccc|cc} 1 & * & * & & \\ & 1 & * & & \\ & & 1 & & \\ \hline & & & 1 & * \\ & & & & 1 \end{array}\right)
$$

$$
W_{\mathbf{d}} \cong S_3 \times S_2 \quad
B_\varpi = \left(\begin{array}{ccc|cc} * & * & & & \\ & * & & & \\ * & * & * & & \\ \hline & & & * & \\ & & & * & * \end{array}\right) \quad
B_{\varpi s} = \left(\begin{array}{ccc|cc} * & & & & \\ * & * & & & \\ * & * & * & & \\ \hline & & & * & \\ & & & * & * \end{array}\right)
$$

$$
N_\varpi = \left(\begin{array}{ccc|cc} 1 & * & & & \\ & 1 & & & \\ * & * & 1 & & \\ \hline & & & 1 & \\ & & & * & 1 \end{array}\right) \quad
N_{\varpi,\varpi s} = \left(\begin{array}{ccc|cc} 1 & & & & \\ & 1 & & & \\ * & * & 1 & & \\ \hline & & & 1 & \\ & & & * & 1 \end{array}\right) \quad
M_{\varpi,\varpi s} = \left(\begin{array}{ccc|cc} 1 & * & & & \\ & 1 & & & \\ - & - & 1 & & \\ \hline & & & 1 & \\ & & & - & 1 \end{array}\right) \quad
P_{\varpi,\varpi s} = \left(\begin{array}{ccc|cc} * & * & & & \\ * & * & & & \\ * & * & * & & \\ \hline & & & * & \\ & & & * & * \end{array}\right)
$$

For the Lie algebra, we use the corresponding Fraktur-font symbols:

$$
\begin{array}{lllll}
\mathfrak{g}_{|\mathbf{d}|}, & \mathfrak{b}_{|\mathbf{d}|}, & \mathfrak{t}_{|\mathbf{d}|}, & \mathfrak{n}_{|\mathbf{d}|}, & \mathfrak{b}_\varpi \\
\mathfrak{g}_{\mathbf{d}}, & \mathfrak{b}_{\mathbf{d}}, & \mathfrak{t}_{\mathbf{d}}, & \mathfrak{n}_{\mathbf{d}}, & \mathfrak{b}_\varpi, \\
\mathfrak{n}_\varpi, & \mathfrak{n}_{\varpi,\varpi''}, & \mathfrak{m}_{\varpi,\varpi''}, & \mathfrak{p}_{\varpi,\varpi s}, &
\end{array}
$$

1.4 typical variety

1.5 stratification and T-fixed points

Chapter 2

K-theory and cohomology theory

Chapter 3

Localization theorem

Chapter 4

Excess intersection formula

Chapter 5

From formula to diagram

Chapter 6

Generalization

Chapter 7

Atiyah-Segal completion theorem