Master thesis

Contents

Warning 0.0.1. *I made some assumptions during the writing. To avoid confusing readers, these assumptions are listed here:*

- *For quivers, all the quivers we considered (except Auslander–Reiten quivers) are connected and finite (Remark 1.2.2). For simplicity, From ??? to ???, all the quivers have no loops or cycles.*
- *For any $\varpi \in \mathbb{W}_{|\mathbf{d}|}$, we always write $\varpi = wu$, where $w \in W_{\mathbf{d}}$ and u is the shortest element in the coset $W_{\mathbf{d}}\varpi$. The flag-type dimension vector $\underline{\mathbf{d}} \in W_{\mathbf{d}} \backslash \mathbb{W}_{|\mathbf{d}|}$ corresponds to u, i.e., $\underline{\mathbf{d}} = W_{\mathbf{d}}u$.*
- *For the diagram, we always read from top to bottom.*

Chapter 1

Variety and stratification

1.1 Initial case: $\mathcal{F}$ and $\mathcal{F} \times \mathcal{F}$

We introduce the complete flag variety to give a bird's eye view on the whole section. Actually, the entire difficulty is bundled in this example.

Fix $n \geqslant 1$, we denote $\mathrm{GL}_n := \mathrm{GL}_n(\mathbb{C})$, B, T, N, W be the standard Borel subgroup, standard torus, unipotent subgroup, Weyl group respectively, i.e.,

$$\mathrm{GL}_n = \begin{pmatrix} * & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & * \end{pmatrix} \quad B = \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix} \quad T = \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \quad N = \begin{pmatrix} 1 & \cdots & * \\ & \ddots & \vdots \\ 0 & & 1 \end{pmatrix}$$

$$W := N_{\mathrm{GL}_n}(T)/T \cong S_n$$

1.1.1 $\mathcal{F}$

Definition 1.1.1 (Flag)**.** *For a finite dimensional $\mathbb{C}$-vector space V, a flag of V is an increasing sequence of subspaces of V:*

$$F : 0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_k = V.$$

F is called a complete flag if $\dim V_j = j$ for all j, otherwise F is called a partial flag.

Definition 1.1.2 (Complete flag variety)**.** *The complete flag variety $\mathcal{F}$ is defined as*

$$\begin{aligned} \mathcal{F} &= \mathrm{GL}_n / B \\ &\cong \{\textit{complete flags of } \mathbb{C}^n\} \\ &= \{0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n = \mathbb{C}^n \mid \dim M_j = j\} \\ &\cong \{\textit{Borel subgroups of } \mathrm{GL}_n\} \\ &= \left\{gBg^{-1} \;\middle|\; g \in \mathrm{GL}_n\right\} \end{aligned}$$

Remark 1.1.3.

1. $\mathcal{F}$ is a smooth projective variety of dimension $\frac{n(n+1)}{2}$, which can be seen from the embedding
$$\mathcal{F} \hookrightarrow \mathrm{Gr}(1,n) \times \cdots \times \mathrm{Gr}(n-1,n)$$

2. We implicitly give the base point of $\mathcal{F}$, which is not considered as the data of $\mathcal{F}$. Fix a standard basis of $\mathbb{C}^n$ by $\{v_1,\ldots,v_n\}$, we define the standard flag
$$F_{\mathrm{Id}} : 0 \subseteq \langle v_1 \rangle \subseteq \langle v_1, v_2 \rangle \subseteq \cdots \subseteq \langle v_1, \ldots, v_n \rangle = \mathbb{C}^n.$$

3. We have the natural GL_n-action on $\mathcal{F}$, which is considered as the data of $\mathcal{F}$.

   For $g \in \mathrm{GL}_n$, we define the flag attached to g:
$$F_g \triangleq gF_{\mathrm{Id}} : 0 \subseteq \langle gv_1 \rangle \subseteq \langle gv_1, gv_2 \rangle \subseteq \cdots \subseteq \langle gv_1, \ldots, gv_n \rangle = \mathbb{C}^n.$$
   Especially, for $w \in W = N_{\mathrm{GL}_n}(T)/T \cong S_n$, the flag attached to w
$$\begin{aligned} F_w : 0 &\subseteq \langle \tilde{w}v_1 \rangle \subseteq \langle \tilde{w}v_1, \tilde{w}v_2 \rangle \subseteq \cdots \subseteq \langle \tilde{w}v_1, \ldots, \tilde{w}v_n \rangle = \mathbb{C}^n \\ 0 &\subseteq \left\langle v_{w(1)} \right\rangle \subseteq \left\langle v_{w(1)}, v_{w(2)} \right\rangle \subseteq \cdots \subseteq \left\langle v_{w(1)}, \ldots, v_{w(n)} \right\rangle = \mathbb{C}^n \end{aligned}$$
   does not depend on the choice of the lift $\tilde{w} \in N_{\mathrm{GL}_n}(T)$ of w.

   Readers can verify that $\{F_w | w \in W\}$ are all T-fixed points of $\mathcal{F}$, while $\left\{wBw^{-1} \middle| w \in W\right\}$ are all Borel subgroups of G containing the standard torus T.

4. For $n = 2$, $\mathcal{F} \cong \mathbb{P}^1$. We encourage readers to use $\mathbb{P}^1$ as a toy example for the whole theory.

interpretation	GL_n / B	flags	Borel subgroups
base point	Id	F_{Id}	B
GL_n-action	left multiplication	$\{V_i\} \mapsto \{gV_i\}$	conjugation
general point	g	F_g	gBg^{-1}

$\mathcal{F}$ is a well-studied variety, and has many combinatorical properties. For example, from the well-known Bruhat decomposition, [1]
$$\mathrm{GL}_n \cong \bigsqcup_{w \in W} BwB$$
We get a stratification of $\mathcal{F}$ by B-orbits:
$$\mathcal{F} = \mathrm{GL}_n / B \cong \bigsqcup_{w \in W} BwB/B$$
The B-orbit BwB/B is called the **Schubert cell**, denoted by Ω_w. Since
$$\Omega_w = BwB/B \cong B/\left(B \cap wBw^{-1}\right) \cong \mathbb{A}^{l(w)},$$
the Schubert cell is an affine space of dimension $l(w)$.

$H^i(\mathcal{F};\mathbb{C})$	0	2	4	6	8	10	12
1	1						
2	1	1					
3	1	2	2	1			
4	1	3	5	6	5	3	1
5	1	4	9	15	20	22	20

G	Orbit	G-fixed points
GL_n	$\mathcal{F}\cong \mathrm{GL}_n/B$	$\varnothing$
B	$\Omega_w\cong B/(B\cap wBw^{-1})$	$\{F_{\mathrm{Id}}\}$
T	$-$	$\{F_w \mid w\in W\}$

As a result, we know a lot of information of $\mathcal{F}$:
$\overline{\Omega}_w\subseteq\mathcal{F}$ is called the **Schubert variety**. It is well-known that

$$\overline{\Omega}_w=\bigsqcup_{w'\leq w}\Omega_w$$

as a set. Especially, for any $s\in W$ with $l(s)=1$, denote $P_s=B\sqcup BsB$,

$$\overline{\Omega}_s=\Omega_{\mathrm{Id}}\sqcup\Omega_s=B/B\sqcup BsB/B=P_s/B\cong\mathbb{P}^1.$$

For other Schubert variety, the structures are quite dedicate and far away from the scope of this master thesis. For example, most Schubert variety are not smooth.

1.1.2 $\mathcal{F}\times\mathcal{F}$

As a more complicated geometrical object, $\mathcal{F}\times\mathcal{F}$ works as the base space for the Steinberg variety, which turns out to be the central focus in the thesis. $\mathcal{F}\times\mathcal{F}$ has naturally a diagonal GL_n-action:

$$\mathrm{GL}_n\times\mathcal{F}\times\mathcal{F}\longrightarrow\mathcal{F}\times\mathcal{F}\qquad (g,F_1,F_2)\longmapsto(gF_1,gF_2).$$

Under this action, $\mathcal{F}\times\mathcal{F}$ has a stratification consisting of GL_n-orbits, indexed by the Weyl group:

$$\mathrm{GL}_n\backslash(\mathcal{F}\times\mathcal{F})\cong\mathrm{GL}_n\backslash(\mathrm{GL}_n/B\times\mathrm{GL}_n/B)\cong B\backslash\mathrm{GL}_n/B\cong W\qquad\text{as sets.}$$

Denote $\boldsymbol{\Omega}_{w'}:=\mathrm{GL}_n\cdot(F_{\mathrm{Id}},F_{w'})$, then $\mathcal{F}\times\mathcal{F}=\sqcup_{w'}\boldsymbol{\Omega}_{w'}$. Moreover, by the orbit-stabilizer theorem, we get

$$\boldsymbol{\Omega}_{w'}\cong\mathrm{GL}_n/\left(B\cap w'B(w')^{-1}\right)$$

Different from $\mathcal{F}$, the GL_n-action on $\mathcal{F}\times\mathcal{F}$ is not transitive. To facilitate the stratification of $\mathcal{F}\times\mathcal{F}$, we introduce the twisted $\mathrm{GL}_n\times\mathrm{GL}_n$-action:

$$\mathrm{GL}_n\times\mathrm{GL}_n\times\mathcal{F}\times\mathcal{F}\longrightarrow\mathcal{F}\times\mathcal{F}\qquad(g_1,g_2,F_g,F_{g'})\longmapsto(F_{g_1g},F_{g_1(gg_2g^{-1})g'}).$$

[1] For the most time the formula does not depend on the lift of w, so we abuse the notation of $w\in N_{\mathrm{GL}_n}(T)/T$ and $\tilde{w}\in N_{\mathrm{GL}_n}(T)$.

If we write $\underline{F}_{g,g'} := (F_g, F_{gg'}) \in \mathcal{F} \times \mathcal{F}$, then

$$(g_1, g_2) \cdot \underline{F}_{g,g'} = \underline{F}_{g_1 g, g_2 g'}.$$

This $\mathrm{GL}_n \times \mathrm{GL}_n$-action is now transitive, and decompose $\mathcal{F} \times \mathcal{F}$ as disjoint union of finite many $B \times B$-orbits, which are compatible with G-orbits:

$$\begin{aligned}
&\boldsymbol{\Omega}_{w,w'} := (B \times B) \cdot \underline{F}_{w,w'} \subseteq \mathcal{F} \times \mathcal{F} \\
&\mathcal{F} \times \mathcal{F} = \bigsqcup_{w,w' \in W} \boldsymbol{\Omega}_{w,w'} \qquad \boldsymbol{\Omega}_{w'} = \bigsqcup_{w \in W} \boldsymbol{\Omega}_{w,w'} \\
&\boldsymbol{\Omega}_{w,w'} \cong B/(B \cap wBw^{-1}) \times B/(B \cap w'Bw'^{-1}) \cong \mathbb{A}^{l(w)+l(w')}
\end{aligned}$$

We conclude the information of orbits and fixed points of $\mathcal{F} \times \mathcal{F}$ in Table 1.1:

G	Orbit	G-fixed points
$\mathrm{GL}_n \times \mathrm{GL}_n$	$\mathcal{F} \times \mathcal{F}$	$\varnothing$
GL_n	$\boldsymbol{\Omega}_{w'}$	$\varnothing$
$B \times B$	$\boldsymbol{\Omega}_{w,w'}$	$\{F_{\mathrm{Id},\mathrm{Id}}\}$
T	$-$	$\{\underline{F}_{w,w'} \mid w, w' \in W\}$

Table 1.1: Orbit and fixed points of $\mathcal{F} \times \mathcal{F}$

Like $\mathcal{F}$, we also study the closure of $\boldsymbol{\Omega}_{w'}$ and $\boldsymbol{\Omega}_{w,w'}$ in special case. It can be shown that

$$\overline{\boldsymbol{\Omega}}_{w'} = \bigsqcup_{x' \leq w'} \boldsymbol{\Omega}_{x'} \qquad \overline{\boldsymbol{\Omega}}_{w,w'} = \bigsqcup_{x \leq w, x' \leq w'} \boldsymbol{\Omega}_{x,x'}$$

as a set. Especially, for any $s \in W$ with $l(s) = 1$,

$$\begin{aligned}
\overline{\boldsymbol{\Omega}}_s = \boldsymbol{\Omega}_{\mathrm{Id}} \sqcup \boldsymbol{\Omega}_s &\cong \mathrm{GL}_n /B \sqcup \mathrm{GL}_n /(B \cap sBs^{-1}) \\
&\cong \mathrm{GL}_n \times^B (B/B) \sqcup GL_n \times^B (B \cap sBs^{-1}) \\
&\cong \mathrm{GL}_n \times^B (B/B) \sqcup GL_n \times^B (BsB/B) \\
&\cong \mathrm{GL}_n \times^B (P_s/B)
\end{aligned}$$

is an $\mathcal{F}$-bundle over $\mathbb{P}^1$.[2] Also,

$$\begin{aligned}
\overline{\boldsymbol{\Omega}}_{\mathrm{Id},s} = \boldsymbol{\Omega}_{\mathrm{Id},\mathrm{Id}} \sqcup \boldsymbol{\Omega}_{\mathrm{Id},s} &\cong \big(B/B \times B/B\big) \sqcup \big(B/B \times BsB/B\big) \\
&\cong P_s/B \cong \mathbb{P}^1
\end{aligned}$$

Other closure can be highly singular.

Example 1.1.4. *In the table, $n = 3$, $t = (12)$, $s = (23)$. In this case, $\mathcal{F} \times \mathcal{F}$ has 6 GL_3-orbits, and each GL_3-orbits decompose as 6 $B \times B$-orbits, with dimensions equal to $l(w) + l(w')$.*

Now wee understand a lot about $\mathcal{F}$ and $\mathcal{F} \times \mathcal{F}$, and the whole process of analysis(investigations?) will be applied repeatedly in section ???.

[2] ??? need to explain $\times^B$

dim; $\mathbb{B}_{\|\omega\|}\cdot F_{\omega\omega'}$; $\mathbb{B}_{\|\omega\|}\times\mathbb{B}_{\|\omega\|}\cdot(F_\omega, F_{\omega\omega'})$ $\omega\omega'$; $\mathbb{B}_{\|\omega\|}\cdot F_\omega$ ω	0 ϑ_{Id}	1 ϑ_t	1 ϑ_s	2 ϑ_{ts}	2 ϑ_{st}	3 ϑ_{sts}	$pr_1^{-1}(\vartheta_{ts})$
0 ϑ_{Id}	0 $\vartheta_{Id,Id}$	1 $\vartheta_{Id,t}$	1 $\vartheta_{Id,s}$	2 $\vartheta_{Id,ts}$	2 $\vartheta_{Id,st}$	3 $\vartheta_{Id,sts}$	ϑ_s
1 ϑ_t	2 $\vartheta_{t,t}$	1 $\vartheta_{t,Id}$	3 $\vartheta_{t,ts}$	2 $\vartheta_{t,s}$	4 $\vartheta_{t,sts}$	3 $\vartheta_{t,st}$	
1 ϑ_s	2 $\vartheta_{s,s}$	3 $\vartheta_{s,st}$	1 $\vartheta_{s,Id}$	4 $\vartheta_{s,sts}$	2 $\vartheta_{s,t}$	3 $\vartheta_{s,ts}$	
2 ϑ_{ts}	4 $\vartheta_{ts,st}$	3 $\vartheta_{ts,s}$	5 $\vartheta_{ts,sts}$	2 $\vartheta_{ts,Id}$	4 $\vartheta_{ts,ts}$	3 $\vartheta_{ts,t}$	
2 ϑ_{st}	4 $\vartheta_{st,ts}$	5 $\vartheta_{st,sts}$	3 $\vartheta_{st,t}$	4 $\vartheta_{st,st}$	2 $\vartheta_{st,Id}$	3 $\vartheta_{st,s}$	
3 ϑ_{sts}	6 $\vartheta_{sts,sts}$	5 $\vartheta_{sts,ts}$	5 $\vartheta_{sts,st}$	4 $\vartheta_{sts,t}$	4 $\vartheta_{sts,s}$	3 $\vartheta_{sts,Id}$	

Figure 1.1: stratifications of $\mathcal{F} \times \mathcal{F}$

1.2 Quiver

To introduce more complicated spaces and discuss their stratifications, we fix notations related to quiver and algebraic group in the following sections.

Roughly speaking, a quiver is a directed multigraph permitting loops.

Definition 1.2.1 (Quiver)**.** *???*

Remark 1.2.2. In the first part of our master thesis, all the quivers are supposed to be connected and finite (i.e., Q_0, Q_1 are finite sets). We will only encounter disconnected and infinite quiver as Auslander–Reiten quiver later on.

Example 1.2.3. *The following graphs are quivers.*

???

The reader can easily writen down the quadruple of these quivers. ???

For convenience, we mainly use simpler quivers as examples:

???

From those quivers we are able to construct relatively complicated algebraic and geometrical objects.

Definition 1.2.4 (Quiver representation)**.** *Fix a quiver Q. A representation of Q consists of the following data:*

- *A finite dimensional $\mathbb{C}$-vector space V_i for each vertex $i \in Q_0$;*

- *A $\mathbb{C}$-linear map $V_a : V_{s(a)} \longrightarrow V_{t(a)}$ for each arrow $a \in Q_1$.*

Example 1.2.5. *A representation of 1-loop quiver $L(1)$ is a 2-tuple*

$$(V,\ \alpha : V \longrightarrow V)$$

which is equivalent to a (finite dimensional) $\mathbb{C}[t]$-module.

Remark 1.2.6. The equivalence appeared in the example can actually be generalized to arbitrary quivers. For a quiver Q, we can define the path algebra $\mathbb{C}Q$, and view any Q-representation as $\mathbb{C}Q$-module, and vice versa.

For many constructions, we only care about the data of vector space.

Definition 1.2.7 (Q-vector space/Vector space with quiver partition)**.** *Fix a quiver Q, a Q-vector space is a finite dimensional $\mathbb{C}$-vector space with the direct sum decomposition*

$$V = \bigoplus_{i \in Q_0} V_i.$$

The dimension vector of a Q-vector space is defined as

$$\underline{\mathbf{dim}}\, V = (\dim_{\mathbb{C}} V_i)_{i \in Q_0} \subseteq \prod_{i \in Q_0} \mathbb{Z}.$$

On the country, given $\mathbf{d} \in \prod_{i \in Q_0} \mathbb{N}_{\geqslant 0}$, we can construct a canonical Q-vector space of dimension vector $\mathbf{d}$, as follows:

$$V = \bigoplus_{i \in Q_0} V_i \qquad \text{with } V_i = \mathbb{C}^{\mathbf{d}_i}.$$

Definition 1.2.8. *The total dimension vector of a Q-vector space V is defined as*

$$|\underline{\mathbf{dim}}\, V| := \dim_{\mathbb{C}} V.$$

For $\mathbf{d} \in \prod_{i \in Q_0} \mathbb{N}_{\geqslant 0}$, denote $|\mathbf{d}| := \sum_{i \in Q_0} \mathbf{d}_i$.

Definition 1.2.9 (Space of representations with given dimension vector)**.** *For any quiver Q, dimension vector $\mathbf{d}$, fix the canonical Q-vector space $V = \bigoplus_{i \in Q_0} V_i$, the space of representations with dimension vector $\mathbf{d}$ is defined as*

$$\begin{aligned}\mathrm{Rep}_{\mathbf{d}}(Q) &= \left\{(V_i, V_a : V_{s(a)} \longrightarrow V_{t(a)}) \text{ as a representation of } Q\right\} \\ &= \bigoplus_{a \in Q_1} \mathrm{Hom}\left(\mathbb{C}^{\mathbf{d}_{s(a)}}, \mathbb{C}^{\mathbf{d}_{t(a)}}\right)\end{aligned}$$

Since we encode the information of vector space in $\mathbf{d}$, $\mathrm{Rep}_{\mathbf{d}}(Q)$ only records the information of linear maps.

For both Q-vector space and Q-representations, we can define (complete) flags.

Definition 1.2.10 (Flag with quiver). *For a quiver representation $V \in \mathrm{rep}(Q)$, a flag of V is defined as an increasing sequence of subrepresentation of V, i.e.,*

$$F : 0 \subseteq M_1 \subseteq \ldots \subseteq M_k = V \qquad M_j \in \mathrm{rep}(Q).$$

For a Q-vector space $V = \oplus_{i \in Q_0} V_i$, a (quiver-graded) flag of V is defined as an increasing sequence of Q-subspace of V, i.e.,

$$F : 0 \subseteq M_1 \subseteq \ldots \subseteq M_k = V \qquad M_j = \bigoplus_{i \in Q_0} M_{j,i}.$$

For both Q-vector space and Q-representation, F is called a complete flag if $k = \dim_{\mathbb{C}} V$ and

$$\dim_{\mathbb{C}} M_j = j \qquad \text{for any } j \in \{1, \ldots, |\mathbf{d}|\}$$

For the flag we also have the notation of dimension vector.

Definition 1.2.11 (flag-type dimension vector). *For any flag $F : 0 \subseteq M_1 \subseteq \ldots \subseteq M_k = V$, the dimension vector of F is defined as*

$$\underline{\mathbf{d}} = \left(\underline{\mathbf{dim}}\, M_j\right)_{j \in \{1,\ldots,k\}} \subseteq \prod_{\substack{i \in Q_0 \\ j \in \{1,\ldots,k\}}} \mathbb{Z}.$$

$\underline{\mathbf{d}}$ is called a flag-type dimension vector if $\underline{\mathbf{d}}$ is the dimension vector of some complete flag F, i.e., [3]

$$|\underline{\mathbf{dim}}\, M_{j+1}/M_j| = 1 \qquad \text{for any } j \in \{0, \ldots, |\mathbf{d}| - 1\}.$$

Example 1.2.12. *For quiver $Q : i \longrightarrow i'$, $\mathbf{d} = (3, 2)$, the canonical Q-vector space of dimension vector $\mathbf{d}$ is*

$$\begin{aligned} V &= V_i \oplus V_{i'} \\ &= \langle v_1, v_2, v_3 \rangle_{\mathbb{C}} \oplus \langle v_4, v_5 \rangle_{\mathbb{C}} \end{aligned}$$

The flag

$$F : 0 \subseteq \langle v_4 \rangle \subseteq \langle v_4, v_1 \rangle \subseteq \langle v_4, v_1, v_2 \rangle \subseteq \langle v_4, v_1, v_2, v_5 \rangle \subseteq \langle v_4, v_1, v_2, v_5, v_3 \rangle = V$$

is a complete flag of V, with dimension vector

$$\underline{\mathbf{d}} = \begin{pmatrix} 3,2 \\ 2,2 \\ 2,1 \\ 1,1 \\ 0,1 \end{pmatrix}.$$

[3]For convenience, we denote M_0 by 0.

Remark 1.2.13. The flag-type dimension vector $\underline{\mathbf{d}}$ can be viewed as a partition on set $\{1, \dots, |\mathbf{d}|\}$, i.e., a map

$$\mathrm{par} : \{1, \dots, |\mathbf{d}|\} \longrightarrow Q_0$$

such that $\#\,\mathrm{par}^{-1}(i) = \mathbf{d}_i$.[4] . As an example,

$$\underline{\mathbf{d}} = \begin{pmatrix} 3,2 \\ 2,2 \\ 2,1 \\ 1,1 \\ 0,1 \end{pmatrix} \qquad \text{corresponds to} \qquad \{1,2,3,4,5\} = \{2,3,5\} \sqcup \{1,4\}.$$

1.3 Symmetric group calculus

As a reminder, we recall some basic diagrams referring to the elements in S_n, and do some calculations by these diagrams. We will also relate cosets with flag-type dimension vectors.

Fix a quiver Q and dimension vector $\mathbf{d}$. Later (Definition 1.4.2, 1.4.3) we will define

$$\mathbb{W}_{|\mathbf{d}|} = S_{|\mathbf{d}|} \qquad W_{\mathbf{d}} = \prod_{i \in Q_0} S_{\mathbf{d}_i} \leqslant \mathbb{W}_{|\mathbf{d}|}$$

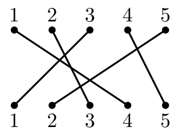
For simplicity, we take $Q_0 = \{1, \dots, k\}$, then $W_{\mathbf{d}} = S_{\mathbf{d}_1} \times \cdots \times S_{\mathbf{d}_k}$ embed in $S_{|\mathbf{d}|}$ in the most natural way.

Remark 1.3.1. We have different ways to express $\varpi \in \mathbb{W}_{|\mathbf{d}|} = S_{|\mathbf{d}|}$. For example, take $|\mathbf{d}| = 5$, $\varpi \in S_5$ by

$$\varpi(1) = 4, \quad \varpi(2) = 3, \quad \varpi(3) = 1, \quad \varpi(4) = 5, \quad \varpi(5) = 2,$$

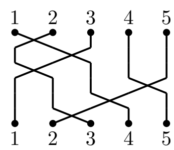
then

$$\varpi = (14523) = \begin{pmatrix} 1\,2\,3\,4\,5 \\ 4\,3\,1\,5\,2 \end{pmatrix} =$$



$$= \begin{bmatrix} & & 1 & & \\ & & & & 1 \\ & 1 & & & \\ 1 & & & & \\ & & & 1 & \end{bmatrix}$$

$$= (23)(34)(45)(12)(23)(12) =$$



[4]The partition corresponding map par is

$$\{1, \dots, |\mathbf{d}|\} = \mathrm{par}^{-1}(i).$$

Even though all expressions give us the same amount of information, the diagram presents them more vividly. For example, each intersection of strands corresponds to a simple reflection, so we read from the diagram that $l(\varpi) = 6$. Readers can also check that

$$\begin{aligned} &l(\varpi s_1) = 5, \quad l(\varpi s_2) = 5, \quad l(\varpi s_3) = 7, \quad l(\varpi s_4) = 5, \\ &l(s_1\varpi) = 7, \quad l(s_2\varpi) = 5, \quad l(s_3\varpi) = 5, \quad l(s_4\varpi) = 7, \end{aligned}$$

where $s_i := (i, i+1) \in S_5$ are simple reflections.

Definition 1.3.2 (Special elements in the Weyl group)**.** *For $i \in \{1, \ldots, |\mathbf{d}| - 1\}$, the simple reflection is defined as*

$$s_i := (i, i+1) \in S_{|\mathbf{d}|}.$$

We denote

$$\begin{aligned} \Pi &= \left\{ s_i \in S_{|\mathbf{d}|} \;\middle|\; i \in \{1, \ldots, |\mathbf{d}| - 1\} \right\} \\ \Pi_{\mathbf{d}} &= \left\{ s_i \in S_{\mathbf{d}_1} \times \cdots \times S_{\mathbf{d}_k} \;\middle|\; i \in \{1, \ldots, |\mathbf{d}| - 1\} \right\} \\ &= \left\{ s_1, \ldots, s_{|\mathbf{d}|-1} \right\} \smallsetminus \left\{ s_{\mathbf{d}_1}, s_{\mathbf{d}_1+\mathbf{d}_2}, \ldots, s_{\mathbf{d}_1+\cdots+\mathbf{d}_{k-1}} \right\} \end{aligned}$$

to be the set of simple reflections in $\mathbb{W}_{|\mathbf{d}|}$ and $W_{\mathbf{d}}$, respectively.

We also denote $\varpi_{\max} \in \mathbb{W}_{|\mathbf{d}|}$, $w_{\max} \in W_{\mathbf{d}}$ to be the longest elements in $\mathbb{W}_{|\mathbf{d}|}$, $W_{\mathbf{d}}$, respectively.

We discuss about right cosets $W_{\mathbf{d}} \backslash \mathbb{W}_{|\mathbf{d}|}$ and minimal length coset representatives now.

Multiplying on left by $w \in W_{\mathbf{d}}$ is equivalent to plugging in a diagram representing $w \in W_{\mathbf{d}}$ underneath the original diagram. Therefore, we connect some bottom points by lines, indicating that switching them will cause no trouble. Furthermore, we color different parts to make the following fact more explicitly.

Fact 1.3.3. *Every element $\varpi_{\max} \in W_{\mathbf{d}} \backslash \mathbb{W}_{|\mathbf{d}|}$ corresponds to a partition on set $\{1, \ldots, |\mathbf{d}|\}$ (of a given number partition $\mathbf{d}$), which corresponds to a flag-type dimension vector $\underline{\mathbf{d}}$.*

Example 1.3.4. *???*

This coset corresponds to the partition $\{1, 2, 3, 4, 5\} = \{2, 3, 5\} \sqcup \{1, 4\}$.

???

It is easy to see from the diagram that in every coset, there exists an unique element $u \in \mathbb{W}_{|\mathbf{d}|}$ of minimal length. We collect these minimal length coset representatives as a set, and denote it by $\mathrm{Min}(\mathbb{W}_{|\mathbf{d}|}, \mathrm{W}_{\mathbf{d}})$.[5]

Proposition 1.3.5. *For any $\varpi \in \mathbb{W}_{|\mathbf{d}|}$, exists unique $w \in W_{\mathbf{d}}$, $u \in \mathrm{Min}(\mathbb{W}_{|\mathbf{d}|}, \mathrm{W}_{\mathbf{d}})$ such that $\varpi = wu$.*

Exercise 1.3.6. *For $u \in \mathrm{Min}(\mathbb{W}_{|\mathbf{d}|}, \mathrm{W}_{\mathbf{d}})$, $s_i \in \Pi$, show that*

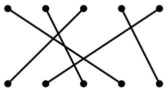
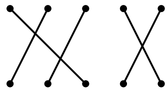
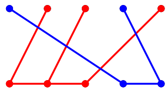
$$us_iu^{-1} \in W_{\mathbf{d}} \quad \Longrightarrow \quad us_iu^{-1} = s_{u(i)} \in \Pi_{\mathbf{d}}.$$

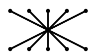
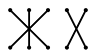
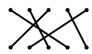
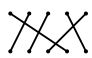
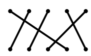
[5]In some references $\mathrm{Min}(\mathbb{W}_{|\mathbf{d}|}, \mathrm{W}_{\mathbf{d}})$ is also denoted by $\mathrm{Schuffle}_{\mathbf{d}}$, since those elements can be thought as ways off riffle shuffling several words together.

We finish this section with figures and examples.

$$\begin{tikzcd}
 & & & \mathrm{Min}(\mathbb{W}_{|\mathbf{d}|}, \mathrm{W}_{\mathbf{d}}) \arrow[dl, hook] \arrow[d, "\cong"] & & & u \arrow[d, mapsto] \\
0 \arrow[r] & W_{\mathbf{d}} \arrow[r] & \mathbb{W}_{|\mathbf{d}|} \arrow[r] & W_{\mathbf{d}} \setminus \mathbb{W}_{|\mathbf{d}|} \arrow[r] & 0 & \varpi = wu \arrow[r, mapsto] & \underline{\mathbf{d}}
\end{tikzcd}$$

Example 1.3.7. *In this table,* $|\mathbf{d}| = 5$, $\mathbf{d} = (3, 2)$, *typical elements would be*

$\varpi =$  $w =$  $u =$ 

set	element	special element	others
$\mathbb{W}_{\lvert\mathbf{d}\rvert} = S_5$	ϖ, x	$\varpi_{\max} =$ 	$\Pi \ = \{s_1, s_2, s_3, s_4\}$
$W_{\mathbf{d}} = S_3 \times S_2$	w	$w_{\max} =$ 	$\Pi_d = \{s_1, s_2, \quad s_4\}$
$W_{\mathbf{d}} \setminus \mathbb{W}_{\lvert\mathbf{d}\rvert} \cong (S_3 \times S_2) \setminus S_5$	$\varpi, \underline{\mathbf{d}}$		$\mathrm{Comp}_{\mathbf{d}}$
$\mathrm{Min}(\mathbb{W}_{\lvert\mathbf{d}\rvert}, \mathrm{W}_{\mathbf{d}}) = \{$  $, \ldots\}$	u		$\mathrm{Schuffle}_{\mathbf{d}}$

Example 1.3.8. *In this table,* $|\mathbf{d}| = 3$, $\mathbf{d} = (1, 2)$, $s = (12)$, $t = (23)$. *The columns "order of basis" and Borelsubgroups have not been introduced yet, and they are here for the future usage.*

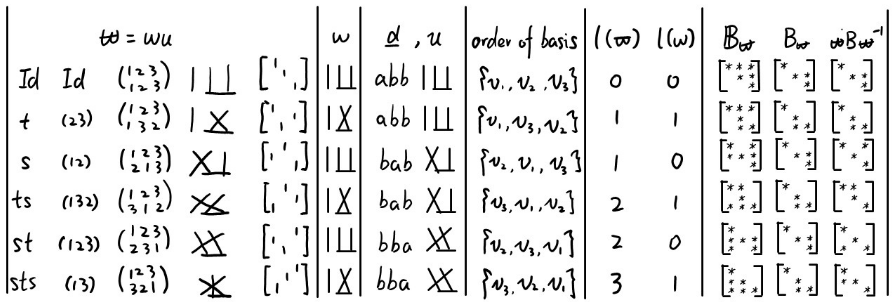


Figure 1.2: desired picture

1.4 Algebraic group and Lie algebra

In this section we fix notations of algebraic group and Lie algebras. Later, the algebraic group will act on varieties, and some Lie algebra will serve as tangent spaces.

Setting 1.4.1. *We fix a quiver Q, a dimension vector $\mathbf{d}$ and a Q-vector space*

$$V = \bigoplus_{i \in Q_0} V_i \qquad \text{with } V_i = \mathbb{C}^{\mathbf{d}_i}.$$

When a basis of V is needed, we fix a total order on Q_0, and denote

$$V = \langle v_1, \dots v_{|\mathbf{d}|} \rangle$$

where

$$V_i = \langle v_{f_i+1}, \dots v_{f_i+\mathbf{d}_i} \rangle \qquad f_i = \sum_{i'<i} \mathbf{d}_{i'}.$$

1.4.1 Algebraic group

Definition 1.4.2 (absolute algebraic groups)**.** *We set*

$$\mathbb{G}_{|\mathbf{d}|} := \mathrm{GL}(V) = \mathrm{GL}_{|\mathbf{d}|}(\mathbb{C}),$$

and $\mathbb{B}_{|\mathbf{d}|}$, $\mathbb{T}_{|\mathbf{d}|}$, $\mathbb{N}_{|\mathbf{d}|}$ are corresponding standard Borel, torus and unipotent subgroups.

The Weyl group is

$$\mathbb{W}_{|\mathbf{d}|} := N_{\mathbb{G}_{|\mathbf{d}|}}(\mathbb{T}_{|\mathbf{d}|})/\mathbb{T}_{|\mathbf{d}|} \cong S_{|\mathbf{d}|}.$$

For $\varpi \in \mathbb{W}_{|\mathbf{d}|}$, we define[6]

$$\mathbb{B}_\varpi := \varpi \mathbb{B}_{|\mathbf{d}|} \varpi^{-1}.$$

We will view $\mathbb{B}_\varpi$ as the stabilizer of the flag F_ϖ with $\mathbb{G}_{|\mathbf{d}|}$-action.

We also have a series of algebraic groups compatible with the quiver partition of V, and they're more common in this thesis.

Definition 1.4.3 (relative algebraic groups)**.** *We set*

$$G_{\mathbf{d}} := \bigoplus_{i \in Q_0} \mathrm{GL}(V_i) = \mathrm{GL}_{\mathbf{d}_i}(\mathbb{C}) \subseteq \mathbb{G}_{|\mathbf{d}|},$$

and $B_{\mathbf{d}}$, $T_{\mathbf{d}}$, $N_{\mathbf{d}}$ are corresponding standard Borel, torus and unipotent subgroups.

The Weyl group is

$$W_{\mathbf{d}} := N_{G_{\mathbf{d}}}(T_{\mathbf{d}})/T_{\mathbf{d}} \cong \prod_{i \in Q_0} S_{\mathbf{d}_i}.$$

For $\varpi = wu \in W_{\mathbf{d}}$, we define

$$B_\varpi := w B_{\mathbf{d}} w^{-1}.$$

We will view B_ϖ as the stabilizer of the flag F_ϖ with $G_{\mathbf{d}}$-action.

[6] As usual, we abuse the notation of ϖ and its lift.

We also have a series of algebraic groups with subscription as elements in the Weyl group:

Definition 1.4.4 (more algebraic groups)**.** *For $\varpi, \varpi'' \in \mathbb{W}_{|\mathbf{d}|}$, define*

$$\begin{aligned} N_{\varpi} &:= R_u(B_{\varpi}), \\ N_{\varpi,\varpi''} &:= N_{\varpi} \cap N_{\varpi''}, \\ M_{\varpi,\varpi''} &:= N_{\varpi}/N_{\varpi,\varpi''}, \end{aligned}$$

where R_u denotes for the unipotent radical.

For $s \in \Pi$ such that $\varpi s \varpi^{-1} \in W_d$ (i.e., $W_{\mathbf{d}}\varpi = W_{\mathbf{d}}\varpi s$), define

$$\begin{aligned} P_{\varpi,\varpi s} &:\xlongequal{\varpi = wu} w\left(B_{\mathbf{d}} u s u^{-1} B_{\mathbf{d}} \cup B_{\mathbf{d}}\right) w^{-1} \\ &\xlongequal{\quad\quad} B_{\varpi} \varpi s \varpi^{-1} B_{\varpi} \cup B_{\varpi} \end{aligned}$$

Remark 1.4.5. One can easily show that $N_{\varpi,\varpi s} = R_u(P_{\varpi,\varpi s})$.

Example 1.4.6. *For $|\mathbf{d}| = 5$, $\mathbf{d} = (3,2)$, ???,*

$$\mathbb{G}_{|\mathbf{d}|} = \begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{pmatrix} \qquad \mathbb{B}_{|\mathbf{d}|} = \begin{pmatrix} * & * & * & * & * \\ & * & * & * & * \\ & & * & * & * \\ & & & * & * \\ & & & & * \end{pmatrix} \qquad \mathbb{T}_{|\mathbf{d}|} = \begin{pmatrix} * & & & & \\ & * & & & \\ & & * & & \\ & & & * & \\ & & & & * \end{pmatrix} \qquad \mathbb{N}_{|\mathbf{d}|} = \begin{pmatrix} 1 & * & * & * & * \\ & 1 & * & * & * \\ & & 1 & * & * \\ & & & 1 & * \\ & & & & 1 \end{pmatrix}$$

$$\mathbb{W}_{|\mathbf{d}|} \cong S_5 \qquad \mathbb{B}_{\varpi} = \begin{pmatrix} * & * & & & * \\ & * & & & \\ * & * & * & & * \\ * & * & * & * & * \\ & * & & & * \end{pmatrix} \qquad \mathbb{B}_{\varpi s} = \begin{pmatrix} * & * & * & & * \\ & * & & & \\ & * & * & & * \\ * & * & * & * & * \\ & * & & & * \end{pmatrix}$$

$$G_{\mathbf{d}} = \left(\begin{array}{ccc|cc} * & * & * & & \\ * & * & * & & \\ * & * & * & & \\ \hline & & & * & * \\ & & & * & * \end{array}\right) \qquad B_{\mathbf{d}} = \left(\begin{array}{ccc|cc} * & * & * & & \\ & * & * & & \\ & & * & & \\ \hline & & & * & * \\ & & & & * \end{array}\right) \qquad T_{\mathbf{d}} = \left(\begin{array}{ccc|cc} * & & & & \\ & * & & & \\ & & * & & \\ \hline & & & * & \\ & & & & * \end{array}\right) \qquad N_{\mathbf{d}} = \left(\begin{array}{ccc|cc} 1 & * & * & & \\ & 1 & * & & \\ & & 1 & & \\ \hline & & & 1 & * \\ & & & & 1 \end{array}\right)$$

$$W_{\mathbf{d}} \cong S_3 \times S_2 \qquad B_{\varpi} = \left(\begin{array}{ccc|cc} * & * & & & \\ & * & & & \\ * & * & * & & \\ \hline & & & * & \\ & & & * & * \end{array}\right) \qquad B_{\varpi s} = \left(\begin{array}{ccc|cc} * & & & & \\ * & * & & & \\ * & * & * & & \\ \hline & & & * & \\ & & & * & * \end{array}\right)$$

$$N_{\varpi} = \left(\begin{array}{ccc|cc} 1 & * & & & \\ & 1 & & & \\ * & * & 1 & & \\ \hline & & & 1 & \\ & & & * & 1 \end{array}\right) \qquad N_{\varpi,\varpi s} = \left(\begin{array}{ccc|cc} 1 & & & & \\ & 1 & & & \\ * & * & 1 & & \\ \hline & & & 1 & \\ & & & * & 1 \end{array}\right) \qquad M_{\varpi,\varpi s} = \left(\begin{array}{ccc|cc} 1 & * & & & \\ & 1 & & & \\ - & - & 1 & & \\ \hline & & & 1 & \\ & & & - & 1 \end{array}\right) \qquad P_{\varpi,\varpi s} = \left(\begin{array}{ccc|cc} * & * & & & \\ * & * & & & \\ * & * & * & & \\ \hline & & & * & \\ & & & * & * \end{array}\right)$$

1.4.2 Lie algebra

For the Lie algebra, we use the corresponding Fraktur-font symbols:

$$\begin{array}{lllll} \mathfrak{g}_{|\mathbf{d}|}, & \mathfrak{b}_{|\mathbf{d}|}, & \mathfrak{t}_{|\mathbf{d}|}, & \mathfrak{n}_{|\mathbf{d}|}, & \mathfrak{b}_{\varpi} \\ \mathfrak{g}_{\mathbf{d}}, & \mathfrak{b}_{\mathbf{d}}, & \mathfrak{t}_{\mathbf{d}}, & \mathfrak{n}_{\mathbf{d}}, & \mathfrak{b}_{\varpi}, \\ \mathfrak{n}_{\varpi}, & \mathfrak{n}_{\varpi,\varpi''}, & \mathfrak{m}_{\varpi,\varpi''}, & \mathfrak{p}_{\varpi,\varpi s}, & \end{array}$$

We also have to encode the information of representations as Lie algebra. Notice that

$$\mathrm{Hom}(V_{s(a)}, V_{t(a)}) \hookrightarrow \mathrm{Hom}(V, V) \cong \mathfrak{g}_{|\mathbf{d}|} \qquad f \longmapsto \iota_{t(a)} \circ f \circ \pi_{s(a)}$$

realizes $\mathrm{Hom}(V_{s(a)}, V_{t(a)})$ as a Lie subalgebra of $\mathfrak{g}_{|\mathbf{d}|}$, so

$$\mathrm{Rep}_{\mathbf{d}}(Q) = \bigoplus_{a\in Q_1} \mathrm{Hom}\left(\mathbb{C}^{\mathbf{d}_{s(a)}}, \mathbb{C}^{\mathbf{d}_{t(a)}}\right) \subseteq \bigoplus_{a\in Q_1} \mathfrak{g}_{|\mathbf{d}|}.$$

Definition 1.4.7 (Lie algebras connected with representations). *For $\varpi \in \mathbb{W}_{|\mathbf{d}|}$, denote temperately*

$$V_{\varpi,j} := \left\langle e_{\varpi(1)}, \dots e_{\varpi(j)} \right\rangle \subseteq V.$$

We define Lie subalgebras of $\mathrm{Rep}_{\mathbf{d}}(Q)$ *as follows.*

$$\begin{aligned}
\mathfrak{r}_{\varpi} &:= \left\{ (f_a)_{a\in Q_1} \in \mathrm{Rep}_{\mathbf{d}}(Q) \;\middle|\; f_a(V_{\varpi,j} \cap V_{s(a)}) \subseteq V_{\varpi,j} \right\}, \\
\mathfrak{r}_{\varpi,\varpi''} &:= \mathfrak{r}_{\varpi} \cap \mathfrak{r}_{\varpi''}, \\
\mathfrak{d}_{\varpi,\varpi''} &:= \mathfrak{r}_{\varpi} / \mathfrak{r}_{\varpi,\varpi''},
\end{aligned}$$

Remark 1.4.8. We also have twisted notations for Lie algebras. For example,

$$\begin{aligned}
&\underline{\mathfrak{n}}_{\varpi,\varpi'} = \mathfrak{n}_{\varpi,\varpi\varpi'}, \quad \underline{\mathfrak{m}}_{\varpi,\varpi'} = \mathfrak{m}_{\varpi,\varpi\varpi'}, \quad \underline{\mathfrak{p}}_{\varpi,s} = \mathfrak{p}_{\varpi,\varpi s}, \\
&\underline{\mathfrak{r}}_{\varpi,\varpi'} = \mathfrak{r}_{\varpi,\varpi\varpi'}, \quad \underline{\mathfrak{d}}_{\varpi,\varpi'} = \mathfrak{d}_{\varpi,\varpi\varpi'}.
\end{aligned}$$

Another twist happens when we add minus sign as the superscript:

$$\begin{aligned}
&\mathfrak{b}^-_{\varpi} = \mathfrak{b}_{\varpi_{\max}\varpi}, \\
&\mathfrak{b}^-_{\varpi} = \mathfrak{b}_{w_{\max}\varpi}, \qquad\qquad \mathfrak{n}^-_{\varpi} = \mathfrak{b}_{w_{\max}\varpi}, \\
&\mathfrak{n}^-_{\varpi,\varpi''} = \mathfrak{n}_{w_{\max}\varpi, w_{\max}\varpi''}, \quad \mathfrak{m}^-_{\varpi,\varpi''} = \mathfrak{m}_{w_{\max}\varpi, w_{\max}\varpi''}.
\end{aligned}$$

1.5 Typical variety

In this section, we define nearly all the varieties we care about in the same spirit as Section 1.1. Their stratifications and related "Schubert" varieties will be defined in Section 1.6.

Recall Setting 1.1 and Definition 1.2.10.

1.5.1 Flag variety

Definition 1.5.1 (Absolute complete flag variety). *The absolute complete flag variety $\mathcal{F}_{|\mathbf{d}|}$ is defined as*

$$\begin{aligned}
\mathcal{F}_{|\mathbf{d}|} &= \mathbb{G}_{|\mathbf{d}|}/\mathbb{B}_{|\mathbf{d}|} \\
&\cong \left\{ \textit{complete flags of } \mathbb{C}^{|\mathbf{d}|} \right\} \\
&= \left\{ 0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_{|\mathbf{d}|} = \mathbb{C}^{|\mathbf{d}|} \;\middle|\; \dim M_j = j \right\} \\
&\cong \left\{ \textit{Borel subgroups of } \mathbb{G}_{|\mathbf{d}|} \right\} \\
&= \left\{ g\mathbb{B}_{|\mathbf{d}|}g^{-1} \;\middle|\; g \in \mathbb{G}_{|\mathbf{d}|} \right\}
\end{aligned}$$

Here, M_i can have no Q-vector space structure.

Definition 1.5.2 (Relative complete flag variety)**.** *The relative complete flag variety $\mathcal{F}_{\mathbf{d}}$ is defined as*

$$\begin{aligned}\mathcal{F}_{\mathbf{d}} &= \left\{ \text{complete flags of } V = \bigoplus_{i\in Q_0} V_i \right\} \\ &= \left\{ 0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_{|\mathbf{d}|} = V \;\middle|\; |\underline{\mathbf{dim}}\, M_j| = j \right\}\end{aligned}$$

Here, M_i are Q-vector spaces.

Definition 1.5.3 (complete flag variety with flag-type dimension vector)**.** *For a flag-type dimension vector $\underline{\mathbf{d}}$, the flag variety $\mathcal{F}_{\underline{\mathbf{d}}}$ is defined as*

$$\begin{aligned}\mathcal{F}_{\underline{\mathbf{d}}} &= \left\{ \text{complete flags of } V = \bigoplus_{i\in Q_0} V_i \text{ with dimension vector } \underline{\mathbf{d}} \right\} \\ &= \left\{ F : 0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_{|\mathbf{d}|} = V \;\middle|\; \underline{\mathbf{dim}}\, F = \underline{\mathbf{d}} \right\}\end{aligned}$$

It is easy to see that

$$\mathcal{F}_{\mathbf{d}} = \bigsqcup_{\underline{\mathbf{d}}} \mathcal{F}_{\underline{\mathbf{d}}}.$$

Remark 1.5.4.

1. $\mathcal{F}_{|\mathbf{d}|}$, $\mathcal{F}_{\mathbf{d}}$ and $\mathcal{F}_{\underline{\mathbf{d}}}$ are smooth varieties, since

$$\mathcal{F}_{|\mathbf{d}|} \cong \mathrm{GL}_{|\mathbf{d}|}/B \qquad \mathcal{F}_{\underline{\mathbf{d}}} \cong \prod_{i\in Q_0} \mathrm{GL}_{\mathbf{d}_i}/B$$

   are products of usual flag varieties.

2. $\mathcal{F}_{|\mathbf{d}|}$ is an $\mathrm{GL}_{|\mathbf{d}|}$-variety, while $\mathcal{F}_{\mathbf{d}}$, $\mathcal{F}_{\underline{\mathbf{d}}}$ are $G_{\mathbf{d}}$-varieties. The actions are induced by the actions on the vector space V.

We need to simplify our notations of flags.

Definition 1.5.5 (Special flags)**.** *For a basis $\{x_1, \ldots, x_{|\mathbf{d}|}\}$, denote the flag*

$$F_{\{x_1,\ldots,x_{|\mathbf{d}|}\}} : 0 \subseteq \langle x_1\rangle \subseteq \langle x_1, x_2\rangle \subseteq \cdots \subseteq \left\langle x_1, \ldots x_{|\mathbf{d}|}\right\rangle = V.$$

For $g \in \mathbb{G}_{|\mathbf{d}|}$, $\varpi \in \mathbb{W}_{|\mathbf{d}|}$, define

$$\begin{aligned} F_{\mathrm{Id}} &= F_{\{v_1,\ldots,v_{|\mathbf{d}|}\}} && \in \mathcal{F}_{\mathbf{d}} \\ F_g &= gF_{\mathrm{Id}} = F_{\{gv_1,\ldots,gv_{|\mathbf{d}|}\}} && \in \mathcal{F}_{|\mathbf{d}|} \\ F_{\varpi} &= \varpi F_{\mathrm{Id}} = F_{\{v_{\varpi(1)},\ldots,v_{\varpi(|\mathbf{d}|)}\}} && \in \mathcal{F}_{\mathbf{d}} \end{aligned}$$

F_{Id} is called the **standard flag** of V.

Now we can define flag varieties attached to $\varpi \in \mathbb{W}_{|\mathbf{d}|}$.

Definition 1.5.6. *For $\varpi = wu \in \mathbb{W}_{|\mathbf{d}|}$, define $\mathcal{F}_\varpi$ as the $G_\mathbf{d}$-orbit of F_ϖ. By the orbit-stabilizer theorem,*

$$\mathcal{F}_\varpi \cong G_\mathbf{d}/B_\varpi.$$

We can generalize it a little bit: for $g \in G_\mathbf{d}$, $F_{g\varpi} \in \mathcal{F}_\mathbf{d}$,

$$\mathcal{F}_{g\varpi} := G_\mathbf{d} \cdot F_{g\varpi} \cong G_\mathbf{d}/B_{g\varpi} = G_\mathbf{d}/gB_\varpi g^{-1}.$$

Remark 1.5.7. F_ϖ is the preferred base point of $\mathcal{F}_\varpi$. Ignoring the base point,

$$\mathcal{F}_\varpi = \mathcal{F}_u = \mathcal{F}_{\underline{\mathbf{d}}} \quad \text{for } \varpi = wu \quad \underline{\mathbf{d}} = W_\mathbf{d}\varpi.$$

In fact, we are not defining new varieties; we give old varieties new names, so that we can manipulate them more freely.

Like Section 1.1, we also consider the product of two flag varieties. For $g, g', g'' \in \mathbb{G}_{|\mathbf{d}|}$, $\varpi, \varpi', \varpi'' \in \mathbb{W}_{|\mathbf{d}|}$, denote

$$\begin{aligned} F_{\mathrm{Id},\mathrm{Id}} &= (F_{\mathrm{Id}}, F_{\mathrm{Id}}) & & \\ F_{g,g''} &= (F_g, F_{g''}) & \underline{F}_{g,g'} &= F_{g,gg'} = (F_g, F_{gg'}) \\ F_{\varpi,\varpi''} &= (F_\varpi, F_{\varpi''}) & \underline{F}_{\varpi,\varpi'} &= F_{\varpi,\varpi\varpi'} = (F_\varpi, F_{\varpi\varpi'}) \end{aligned}$$

Table 1.2 concludes all varieties we get until now.

	base point		base point
$\mathcal{F}_{\|\mathbf{d}\|} \cong \mathbb{G}_{\|\mathbf{d}\|}/\mathbb{B}_{\|\mathbf{d}\|}$	F_{Id}	$\mathcal{F}_{\|\mathbf{d}\|} \times \mathcal{F}_{\|\mathbf{d}\|}$	$F_{\mathrm{Id},\mathrm{Id}}$
$\mathcal{F}_{\underline{\mathbf{d}}} \cong G_\mathbf{d}/B_\mathbf{d}$	F_u	$\mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}}'}$	$F_{u,u'}$
$\mathcal{F}_\varpi \cong G_\mathbf{d}/B_\varpi$	F_ϖ	$\mathcal{F}_\varpi \times \mathcal{F}_{\varpi'}$	$F_{\varpi,\varpi'}$
$\mathcal{F}_\mathbf{d} = \bigsqcup_{\underline{\mathbf{d}}} \mathcal{F}_{\underline{\mathbf{d}}}$	$-$	$\mathcal{F}_\mathbf{d} \times \mathcal{F}_\mathbf{d} = \bigsqcup_{\underline{\mathbf{d}},\underline{\mathbf{d}}'} \mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}}'}$	$-$

Table 1.2: Base varieties and their preferred base point

1.5.2 Incidence variety

Now it is time to conclude information about arrows, and construct spaces over varieties in Table 1.2.

Definition 1.5.8 (Incidence variety)**.** *For a quiver Q with flag-type dimension vector $\underline{\mathbf{d}}$, define*

$$\begin{aligned} \widetilde{\mathrm{Rep}}_{\underline{\mathbf{d}}}(Q) &:= \left\{(\rho, F) \in \mathrm{Rep}_\mathbf{d}(Q) \times \mathcal{F}_{\underline{\mathbf{d}}} \middle| \rho(M_j) \subseteq M_j\right\} \\ \widetilde{\mathrm{Rep}}_\mathbf{d}(Q) &:= \{(\rho, F) \in \mathrm{Rep}_\mathbf{d}(Q) \times \mathcal{F}_\mathbf{d} | \rho(M_j) \subseteq M_j\} \\ &= \bigsqcup_{\underline{\mathbf{d}}} \widetilde{\mathrm{Rep}}_{\underline{\mathbf{d}}}(Q) \end{aligned}$$

and $\mu_{\underline{\mathbf{d}}}$, $\pi_{\underline{\mathbf{d}}}$, $\mu_\mathbf{d}$, $\pi_\mathbf{d}$ to be the natural morphisms from the incidence varieties to $\mathrm{Rep}_\mathbf{d}(Q)$ *or flag varieties, as follows:*

$$\begin{tikzcd} & \widetilde{\mathrm{Rep}}_{\underline{\mathbf{d}}}(Q) \subseteq \mathrm{Rep}_{\mathbf{d}}(Q) \times \mathcal{F}_{\underline{\mathbf{d}}} \arrow[dl, bend left, "\mu_{\underline{\mathbf{d}}}"'] \arrow[dr, bend left, "\pi_{\underline{\mathbf{d}}}"] & \\ \mathrm{Rep}_{\mathbf{d}}(Q) & & \mathcal{F}_{\underline{\mathbf{d}}} \end{tikzcd} \qquad \begin{tikzcd} & \widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q) \subseteq \mathrm{Rep}_{\mathbf{d}}(Q) \times \mathcal{F}_{\mathbf{d}} \arrow[dl, bend left, "\mu_{\mathbf{d}}"'] \arrow[dr, bend left, "\pi_{\mathbf{d}}"] & \\ \mathrm{Rep}_{\mathbf{d}}(Q) & & \mathcal{F}_{\mathbf{d}} \end{tikzcd}$$

Remark 1.5.9. For $M \in \mathrm{Rep}_{\mathbf{d}}(Q)$, the **Springer fiber**

$$\mathrm{Flag}_{\underline{\mathbf{d}}}(M) := \mu_{\underline{\mathbf{d}}}^{-1}(M) \cong \pi_{\underline{\mathbf{d}}}\big(\mu_{\underline{\mathbf{d}}}^{-1}(M)\big) \subseteq \mathcal{F}_{\underline{\mathbf{d}}}$$

records the complete flags of subrepresentations of M. The partial flag variety version of $\mathrm{Flag}_{\underline{\mathbf{d}}}(M)$ will become the key object in the second part.

Definition 1.5.10 (Steinberg variety)**.** *For quiver Q with flag-type dimension vectors $\underline{\mathbf{d}}$, $\underline{\mathbf{d}}'$, define*

$$\begin{aligned} \mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}'} &:= \widetilde{\mathrm{Rep}}_{\underline{\mathbf{d}}}(Q) \times_{\mathrm{Rep}_{\mathbf{d}}(Q)} \widetilde{\mathrm{Rep}}_{\underline{\mathbf{d}}'}(Q) \\ \mathcal{Z}_{\mathbf{d}} &:= \widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q) \times_{\mathrm{Rep}_{\mathbf{d}}(Q)} \widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q) \\ &= \bigsqcup_{\underline{\mathbf{d}},\underline{\mathbf{d}}'} \mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}'} \end{aligned}$$

*$\mathcal{Z}_{\mathbf{d}}$ is called the **Steinberg variety**.*

$\mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}'}$ can actually be realized as the incidence variety between $\mathrm{Rep}_{\mathbf{d}}(Q)$ and $\mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}}'}$, since

$$\begin{aligned} \mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}'} &:= \widetilde{\mathrm{Rep}}_{\underline{\mathbf{d}}}(Q) \times_{\mathrm{Rep}_{\mathbf{d}}(Q)} \widetilde{\mathrm{Rep}}_{\underline{\mathbf{d}}'}(Q) \\ &\subseteq \big(\mathrm{Rep}_{\mathbf{d}}(Q) \times \mathcal{F}_{\underline{\mathbf{d}}}\big) \times_{\mathrm{Rep}_{\mathbf{d}}(Q)} \big(\mathrm{Rep}_{\mathbf{d}}(Q) \times \mathcal{F}_{\underline{\mathbf{d}}'}\big) \\ &\cong \mathrm{Rep}_{\mathbf{d}}(Q) \times \mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}}'} \end{aligned}$$

For that reason, we denote $\mu_{\underline{\mathbf{d}},\underline{\mathbf{d}}'}$, $\pi_{\underline{\mathbf{d}},\underline{\mathbf{d}}'}$, $\mu_{\mathbf{d},\mathbf{d}}$, $\pi_{\mathbf{d},\mathbf{d}}$ as natural morphisms from $\mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}'}$, $\mathcal{Z}_{\mathbf{d}}$ to $\mathrm{Rep}_{\mathbf{d}}(Q)$ or product of flag varieties, as follows:

$$\begin{tikzcd} & \mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}'} \subseteq \mathrm{Rep}_{\mathbf{d}}(Q) \times \mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}}'} \arrow[dl, bend left, "\mu_{\underline{\mathbf{d}},\underline{\mathbf{d}}'}"'] \arrow[dr, bend left, "\pi_{\underline{\mathbf{d}},\underline{\mathbf{d}}'}"] & \\ \mathrm{Rep}_{\mathbf{d}}(Q) & & \mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}}'} \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{Z}_{\mathbf{d},\mathbf{d}'} \subseteq \mathrm{Rep}_{\mathbf{d}}(Q) \times \mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}'} \arrow[dl, bend left, "\mu_{\mathbf{d},\mathbf{d}'}"'] \arrow[dr, bend left, "\pi_{\mathbf{d},\mathbf{d}'}"] & \\ \mathrm{Rep}_{\mathbf{d}}(Q) & & \mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}'} \end{tikzcd}$$

Remark 1.5.11 (Group actions).

1. $\mathrm{Rep}_{\mathbf{d}}(Q) \subseteq \oplus_{a \in Q_1} \mathfrak{g}_{|\mathbf{d}|}$ has a natural $G_{\mathbf{d}}$-action, which is induced by the conjugation action of $G_{\mathbf{d}}$ on $\mathfrak{g}_{|\mathbf{d}|}$. We have already mentioned the $G_{\mathbf{d}}$-action on $\mathcal{F}_{\underline{\mathbf{d}}}$ and $\mathcal{F}_{\mathbf{d}}$ in Remark 1.5.4. Therefore, by restriction we automatically get $G_{\mathbf{d}}$-actions on $\widetilde{\mathrm{Rep}}_{\underline{\mathbf{d}}}(Q)$, $\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)$, $\mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}'}$ and $\mathcal{Z}_{\mathbf{d}}$. All the maps we mentioned in Definition 1.5.8 are $G_{\mathbf{d}}$-equivariant.

2. In Section 6.2 we will also view all the varieties as $G_{\mathbf{d}} \times \mathbb{C}^{\times}$-varieties, so we also shortly introduce $\mathbb{C}^{\times}$-action here. View $\mathrm{Rep}_{\mathbf{d}}(Q)$ as a $\mathbb{C}$-vector space, $\mathbb{C}^{\times}$ acts on $\mathrm{Rep}_{\mathbf{d}}(Q)$ by scalar multiplication. For $\mathcal{F}_{\underline{\mathbf{d}}}$ and $\mathcal{F}_{\mathbf{d}}$, $\mathbb{C}^{\times}$ acts trivially, and by restriction we get $\mathbb{C}^{\times}$-actions on $\widetilde{\mathrm{Rep}}_{\underline{\mathbf{d}}}(Q)$, $\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)$, $\mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}'}$ and $\mathcal{Z}_{\mathbf{d}}$. Also, all the maps we mentioned above are $\mathbb{C}^{\times}$-equivariant.

3. It may worth mentioning that $\mathcal{F}_{\mathbf{d}}$ has an $\mathbb{W}_{|\mathbf{d}|}$-action which can be extended neither to $\mathbb{G}_{|\mathbf{d}|}$-action on $\mathcal{F}_{\mathbf{d}}$ nor to $\mathbb{W}_{|\mathbf{d}|}$-action on $\widetilde{\mathrm{Rep}}_{\underline{\mathbf{d}}}(Q)$.

1.6 Stratification and T-fixed points

Natural defined varieties resemble burr puzzles, they have delicate structures and can be decomposed as relatively easy pieces. In this subsection, we will find stratifications of varieties introduced in Section 1.5, and fix notations of orbits. We will also mention about their T-fixed points. These stratifications will give us a basis for the K-theory and cohomology theory in Chapter 2, while those T-fixed points will give us another "basis" in Chapter 4.

We begin with $\mathcal{F}_{|\mathbf{d}|}$ and $\mathcal{F}_{|\mathbf{d}|} \times \mathcal{F}_{|\mathbf{d}|}$, which is roughly a repetition of Section 1.1.

Definition 1.6.1 (Twisted action)**.** *We define the twisted $\mathbb{G}_{|\mathbf{d}|} \times \mathbb{G}_{|\mathbf{d}|}$-action on $\mathcal{F}_{|\mathbf{d}|} \times \mathcal{F}_{|\mathbf{d}|}$:*

$$\mathbb{G}_{|\mathbf{d}|} \times \mathbb{G}_{|\mathbf{d}|} \times \mathcal{F}_{|\mathbf{d}|} \times \mathcal{F}_{|\mathbf{d}|} \longrightarrow \mathcal{F}_{|\mathbf{d}|} \times \mathcal{F}_{|\mathbf{d}|} \qquad \left(g_1, g_2, \underline{F}_{g,g'}\right) \longmapsto \underline{F}_{g_1 g, g_2 g'}$$

which is the same as original $G_{\mathbf{d}}$-action when we restrict to $G_{\mathbf{d}} \times \{\mathrm{Id}\}$-action. Other $G \times G$-actions on $\mathcal{F} \times \mathcal{F}$ are defined in a similar way.

Definition 1.6.2 (Stratifications of $\mathcal{F}_{|\mathbf{d}|}$ and $\mathcal{F}_{|\mathbf{d}|} \times \mathcal{F}_{|\mathbf{d}|}$)**.** *For $\varpi, \varpi' \in \mathbb{W}_{|\mathbf{d}|}$, we define*

$$\begin{aligned} \mathcal{V}_{\varpi} &= \mathbb{B}_{|\mathbf{d}|} \cdot F_{\varpi} && \subseteq \mathcal{F}_{|\mathbf{d}|} \\ \boldsymbol{\mathcal{V}}_{\varpi,\varpi'} &= (\mathbb{B}_{|\mathbf{d}|} \times \mathbb{B}_{|\mathbf{d}|}) \cdot \underline{F}_{\varpi,\varpi'} && \subseteq \mathcal{F}_{|\mathbf{d}|} \times \mathcal{F}_{|\mathbf{d}|} \\ \boldsymbol{\mathcal{V}}_{\varpi'} &= \mathbb{G}_{|\mathbf{d}|} \cdot \underline{F}_{\varpi,\varpi'} && \subseteq \mathcal{F}_{|\mathbf{d}|} \times \mathcal{F}_{|\mathbf{d}|} \end{aligned}$$

as $\mathbb{B}_{|\mathbf{d}|}$-orbit, $\mathbb{B}_{|\mathbf{d}|} \times \mathbb{B}_{|\mathbf{d}|}$-orbit, $\mathbb{G}_{|\mathbf{d}|}$-orbit of $\mathcal{F}_{|\mathbf{d}|}$, $\mathcal{F}_{|\mathbf{d}|} \times \mathcal{F}_{|\mathbf{d}|}$, $\mathcal{F}_{|\mathbf{d}|} \times \mathcal{F}_{|\mathbf{d}|}$, respectively.

Chapter 2

K-theory and cohomology theory

From my humble point of view, there is no easy cohomology theory, in a sense that key properties are usually hard to prove. On the other hand, plenty of examples can be quickly computed once we grasp some properties and use them in black boxes. Therefore, we won't prove any properties we stated. We have no choice but to do so, for the restricted space and time.

The main reference for the K-theory is ???.

2.1 Definitions and initial examples

We give definitions for both K-theory and cohomology theory, which are lengthy already.

2.2 Basic constructions: pullbask, pushforward and tensor product

2.3 Induction

2.4 Reduction

2.5 Equivariant cohomology theory

Chapter 3

Cellular fibration theorem

3.1 Thom isomorphism

3.2 Statement

3.3 Application: module structure

Chapter 4

Localization theorem

4.1 Euler class

4.2 Statement

4.3 Application: change of basis

Chapter 5

Excess intersection formula

5.1 Convolution

5.2 Statement

5.3 Application: convolution formula

5.4 Demazure operator

Chapter 6

Generalization

6.1 quiver with loops

6.2 $G \times \mathbb{C}^\times$-action

Chapter 7

From formula to diagram

7.1 One point quiver

7.2 A_2-quiver

7.3 1-loop quiver

Chapter 8

Atiyah-Segal completion theorem