Geometry of Quiver Flag Varieties

Xiaoxiang Zhou

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Advisor: Prof. Dr. Catharina Stroppel

Second Advisor: Dr. Jens Niklas Eberhardt

Mathematical Institute

Mathematisch-Naturwissenschaftliche Fakultät der

Rheinischen Friedrich-Wilhelms-Universität Bonn

Contents

Warning 0.0.1. *I made some assumptions during the writing. To avoid confusing readers, these assumptions are listed here:*

- *We use* $\leqslant$ *to represent subgroups and Bruhat orders. For example,* $H \leqslant G$ *means* H *is a subgroup of* G.
- *For the diagram, we always read from top to bottom.*
- *For quivers, all the quivers we considered (except Auslander–Reiten quivers) are connected and finite (Remark 1.2.2). For simplicity, From Section 1.4 to Chapter 5, all the quivers have no loops or cycles.*
- *For any* $\varpi \in \mathbb{W}_{|\mathbf{d}|}$*, we always write* $\varpi = wu$*, where* $w \in W_{\mathbf{d}}$ *and* u *is the shortest element in the coset* $W_{\mathbf{d}}\varpi$*. The flag-type dimension vector* $\underline{\mathbf{d}} \in W_{\mathbf{d}} \backslash \mathbb{W}_{|\mathbf{d}|}$ *corresponds to* u*, i.e.,* $\underline{\mathbf{d}} = W_{\mathbf{d}} u$*. Whenever* $\tilde{w}$ *and* $\tilde{u}$ *emerge, they are always defined by* $uw'u' = \tilde{w}\tilde{u}$*. See Section 1.4.*
- *Usually, the symbol* $\underline{\mathbf{d}}$ *represents a complete dimension vector in Part I, while the symbols* $\underline{\mathbf{f}}, \underline{\mathbf{g}}$ *represent a (partial) dimension vector in Part II.*
- *All varieties are defined over* $\mathbb{C}$.
- *We relabel the coefficient ring before the basis* $\widetilde{\psi}_{\varpi}$*; see Remark 5.3.6.*

Preprint and electronic version

Over the course of this master project the second part of this thesis have appeared in the preprint [26] on the arXiv. The updated version of this thesis can be found here: https://github.com/ramified/master_thesis/raw/main/master_thesis_Xiaoxiang_Zhou.pdf

Introduction

With two different goals, this master thesis is naturally divided into two parts. The first part is dedicated to computing the equivariant K-theory of Steinberg varieties, while the second part is dedicated to constructing affine pavings for quiver partial flag varieties.

Part I

The Steinberg variety $\mathcal{Z}$ was introduced in [21] and, in type A, consists of triples of a nilpotent operator and two flags fixed by this operator. The (top) Borel–Moore homology of $\mathcal{Z}$ can be equipped with the structure of convolution algebra which yields the group algebra of Weyl group, see [13].

Similarly, Lusztig showed in [15] that the affine Hecke algebra can realized as the $G\times\mathbb{C}^\times$-equivariant K-theory of $\mathcal{Z}$—a fact that is often referred to as the Kazhdan–Lusztig isomorphism. Moreover, irreducible representations of these algebras can be constructed using the geometry of Springer fibers.

Khovanov and Lauda [14] and Rouquier [18] defined an algebra, called the quiver Hecke algebra or KLR algebra, in order to categorify quantum groups. In fact, Varagnolo–Vasserot [25] show that this algebra arises as the G-equivariant cohomology of a quiver version $\mathcal{Z}_{\mathbf{d}}$ of the Steinberg variety.

Motivated by this, we study a K-theoretic version of the KLR algebra in the first part of this thesis. We follow methods from Varagnolo–Vasserot [25] and Przezdziecki's master thesis [17]. We base our K-theoretic arguments on the exposition in [3, Chapter 5].

To state our results, we fix some notation. For a G-variety X, denote by $K^G(X) = K\big(\mathrm{Coh}^G(X)\big)$ the Grothendieck group of G-equivariant coherent sheaves on X, and by $\mathrm{R}(G) = K^G(\mathrm{pt})$ the representation ring. Fix a quiver Q (without loops and cycles) and dimension vector $\mathbf{d}$. Denote by $G_{\mathbf{d}} = \prod_{i\in v(Q)} \mathrm{GL}_{\mathbf{d}_i}(\mathbb{C})$ and by $T_{\mathbf{d}} \subset G_{\mathbf{d}}$ the maximal torus of diagonal matrices. Let $\mathcal{F}_{\mathbf{d}}$ be the complete flag variety and $\mathrm{Rep}_{\mathbf{d}}(Q)$ be the vector space of representations of Q with dimension vector $\mathbf{d}$.

We will study the geometry of the incidence varieties $\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q) \subseteq \mathrm{Rep}_{\mathbf{d}}(Q) \times \mathcal{F}_{\mathbf{d}}$ and $\mathcal{Z}_{\mathbf{d}} \subseteq \mathrm{Rep}_{\mathbf{d}}(Q) \times \mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}}$ consisting of pairs, respectively triples, of a representation of Q and one, respectively two, flags fixed by this representation. Now we state Theorem A as the main result in Part I.

Theorem A *Under the convolution product, $K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}})$ has a $K^{G_{\mathbf{d}}}\Big(\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)\Big) \cong \mathrm{R}(T_{\mathbf{d}})$-algebra structure. Moreover,*

(1) $K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}})$ is a free $\mathrm{R}(T_{\mathbf{d}})$-module of rank $|\mathbf{d}|!$, with a basis corresponding to the $\mathcal{O}$-cells of $\mathcal{Z}_{\mathbf{d}}$;

(2) After base change to fraction field, $\mathcal{K}^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}})$ is a free $\mathcal{R}(T_{\mathbf{d}})$-module of rank $|\mathbf{d}|!$, with a new basis corresponding to the $T_{\mathbf{d}}$-fixed points of $\mathcal{Z}_{\mathbf{d}}$;

(3) As an $\mathrm{R}(T_{\mathbf{d}})$-algebra, $K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}})$ is generated by the Demazure operators $\{D_i\}_{i=1}^{|\mathbf{d}|-1}$;

(4) $K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}})$ has a faithful action on $K^{G_{\mathbf{d}}}\left(\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)\right)$, which embeds $K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}})$ as a subalgebra of the endomorphism ring $\mathrm{End}_{\mathbb{Z}}\left(K^{G_{\mathbf{d}}}\left(\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)\right)\right)$. Also, we have an explicit formula for the Demazure operator action on $K^{G_{\mathbf{d}}}\left(\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)\right)$.

(5) Any element in $K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}})$ can be written as formal sum of certain planar diagrams, and the algebraic structure of $K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}})$ can be understood in a diagrammatic way.

We will show (1) in Chapter 3 (see Table 3.3), (2) in Chapter 4 (see Theorem 4.2.3, Definition 4.3.1), (3), (4) in Chapter 5 (see Proposition 5.3.10 for (3), Proposition 5.3.9, Theorem 5.3.8 for (4)). (5) will be explained in detail in Chapter 6 (see Section 6.2). Varieties as well as their stratifications and T-fixed points will be defined in Chapter 1, while the basic results on K-theory will be treated in Chapter 2.

Similar methods should apply to the case of partial flag varieties and yield a K-theoretic version of the quiver Schur algebra introduced by Stroppel–Webster [22]. We also note that similar K-theoretic analogs in the context of Soergel bimodules were studied by Eberhardt in [7, 8]. Using the work of Stroppel–Eberhardt [22], one should be able to show that the derived categories of modules of the K-theoretic KLR algebra can be realized in terms of certain K-motives on $\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)$.

We proceed as follows.

In Chapter 1, we fix notation and collect properties of quiver flag varieties. Especially, the Steinberg varieties are also defined as an incidence variety, and their properties are described for future use.

From Chapter 2 to Chapter 5, we introduce general results of K-theory, and then specify them to our cases. Both K-theory and cohomology are defined in Chapter 2, with examples and functorialities carefully discussed (in K-theoretical version). Three isomorphisms are also stated in K-theoretical version in Section 2.3-2.5. We compute the module structure of K-groups by the cellular fibration theorem 3.1.3, see Chapter 3. For the computation of convolution product, we introduce another basis of K-groups (in the field of fractions) and compute the transition matrix by the localization formula 4.2.4, see Chapter 4. Finally, we compute the convolution structure of K-theory (Proposition 5.3.5 for $T_{\mathbf{d}}$-equivariant, and Theorem 5.3.8 for $G_{\mathbf{d}}$-equivariant) by the excess intersection formula 5.2.1.

Different from the previous chapters, the three sections in Chapter 6 are quite independent, and can be read in any order. In Section 6.1, we slightly relax the conditions on quivers and group actions. Section 6.2 collect examples and present them by diagrams. In Section 6.3, K-theory and cohomology are connected by the Atiyah–Segal completion theorem 6.3.1, and the Chern class and the Todd class emerge explicitly in examples.

Part II

As explained in the introduction of Part I, irreducible representations of convolution algebras, such as (quiver) Hecke algebras, can be realized in terms of Springer fibers or generalizations thereof. While the geometry of Springer fibers can be very intricate, one often show that they admit affine pavings.

Affine pavings are an important concept in algebraic geometry similar to cellular decompositions in topology. A complex algebraic variety X has an affine paving if X has a filtration

$$0 = X_0 \subset X_1 \subset \cdots \subset X_d = X$$

with X_i closed and $X_{i+1} \setminus X_i$ isomorphic to some affine space $\mathbb{A}^k_{\mathbb{C}}$.

Affine pavings imply nice properties about the cohomology of varieties, for example the vanishing of cohomology in odd degrees. For other properties see [5, 1.7].

Affine pavings have been constructed in many cases, as for Grassmannians, flag varieties, as well as certain Springer fibers, quiver Grassmannians, and quiver flag varieties. Part II focuses on the case of (strict) partial flag varieties which parameterize subrepresentations of a fixed indecomposable representation of a quiver. In particular, we consider quivers of Dynkin type or affine type. In this case, affine pavings have been constructed in [12] for quiver Grassmannians in all types and in [16] for partial flag varieties of type A and D (see Table 1). Besides, affine pavings have been constructed in [9, Theorem 6.3] for strict partial flag varieties in type $\tilde{A}$ with cyclic orientation, which generalized the result in [19] for complete quiver flag varieties in nilpotent representations of an oriented cycle. In this part, we will tackle the remaining cases.

Theorem B *Fix a quiver Q and a representation M of Q.*

(1) If Q is Dynkin, then any (strict) partial flag variety $\mathrm{Flag}(M)$ has an affine paving;
(2) If Q is of type $\tilde{A}$ or $\tilde{D}$, then for any indecomposable representation M, the (strict) partial flag variety $\mathrm{Flag}(M)$ has an affine paving;
(3) If Q is of type $\tilde{E}$, assume that $\mathrm{Flag}(N)$ has an affine paving for any regular quasi-simple representation $N \in \mathrm{rep}(Q)$, then $\mathrm{Flag}(M)$ has an affine paving for any indecomposable representation M.

<table>
<tr><th></th><th>$\mathrm{Gr}^{KQ}(X)$</th><th>$\mathrm{Flag}_{\mathrm{d}}(X)$</th><th>$\mathrm{Flag}_{\mathrm{d,str}}(X)$</th></tr>
<tr><td>A</td><td rowspan="3">[12, Section 5]</td><td rowspan="2">[16, Theorem 2.20]</td><td rowspan="2">Theorem 8.2.1</td></tr>
<tr><td>D</td></tr>
<tr><td>E</td><td colspan="2">Theorem 8.2.1</td></tr>
<tr><td>$\tilde{A}$</td><td rowspan="3">[12, Section 6]</td><td colspan="2" rowspan="2">Theorem 8.4.3</td></tr>
<tr><td>$\tilde{D}$</td></tr>
<tr><td>$\tilde{E}$</td><td colspan="2">reduced to the regular quasi-finite case.</td></tr>
</table>

Table 1

We will show (1) in Theorem 8.2.1 and (2), (3) in Theorem 8.4.3.

We proceed as follows. In Chapter 7, we discuss basic definitions and properties of partial flags. In Section 8.1 we will prove key Theorems 8.1.2 and 8.1.3, which allow us to construct affine pavings for quiver partial flag varieties inductively. We apply these theorems to partial flag varieties of Dynkin type, see Section 8.2, and to partial flag varieties of affine type, see Section 8.4. We will combine and extend results from [12] and [16]. Following the arguments of [16] would require studying millions of cases when we consider the Dynkin quivers of type E. To avoid this, we extend the methods of [12] from quiver Grassmannian to quiver partial flag variety. This will reduce the case by case analysis to a feasible computation of (mostly) 8 critical cases, which we carry out in Section 8.2 and Section 8.3. The reduction uses Auslander–Reiten theory which we recall in Section 7.5.

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Part I

K-theory of the Steinberg varieties

Chapter 1

Varieties and stratifications

In this chapter we fix notation and state properties of various varieties. In particular, we will:

- investigate stratifications for each variety;
- describe the closure of some cells;
- list T-fixed points;
- describe the tangent space of some varieties.

1.1 Initial case: $\mathcal{F}$ and $\mathcal{F} \times \mathcal{F}$

In this section, we introduce the complete flag variety to give a bird's eye view on the whole chapter. We refer readers to the book [2] for detailed description of flag varieties, Bruhat order and Bruhat decomposition.

Setting 1.1.1. *Fix $n \geqslant 1$. Let $\mathrm{GL}_n := \mathrm{GL}_n(\mathbb{C})$, B, T, N, W be the standard Borel subgroup, standard torus, unipotent subgroup and Weyl group, respectively, i.e.,*

$$\mathrm{GL}_n = \begin{pmatrix} * & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & * \end{pmatrix} \quad B = \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix} \quad T = \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \quad N = \begin{pmatrix} 1 & \cdots & * \\ & \ddots & \vdots \\ 0 & & 1 \end{pmatrix}$$

$$W := N_{\mathrm{GL}_n}(T)/T \cong S_n$$

1.1.1 The flag variety $\mathcal{F}$

Definition 1.1.2 (Flag)**.** *For a finite dimensional $\mathbb{C}$-vector space V, a **(partial) flag** of V of length $d \in \mathbb{N}_{>0}$ is an increasing sequence of subspaces of V:*

$$F : 0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_d \subseteq V.$$

*F is called a **complete flag** if $d = \dim_{\mathbb{C}} V$ and $\dim M_k = k$ for all k.*

Definition 1.1.3 (Complete flag variety)**.** *The complete flag variety $\mathcal{F}$ is defined as*

$$\begin{aligned} \mathcal{F} &:= \mathrm{GL}_n / B \\ &\cong \{\textit{complete flags of } \mathbb{C}^n\} \\ &= \{0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n = \mathbb{C}^n \mid \dim M_k = k\} \\ &\cong \{\textit{Borel subgroups of } \mathrm{GL}_n\} \\ &= \left\{gBg^{-1} \mid g \in \mathrm{GL}_n\right\} \end{aligned}$$

Remark 1.1.4.

1. $\mathcal{F}$ is a smooth projective variety of dimension $\frac{n(n+1)}{2}$, which can be seen from the embedding
$$\mathcal{F}\hookrightarrow \mathrm{Gr}(1,n)\times\cdots\times\mathrm{Gr}(n-1,n)$$

2. We implicitly give the base point of $\mathcal{F}$, which is not considered as the data of $\mathcal{F}$. Fix a standard basis of $\mathbb{C}^n$ by $\{v_1,\ldots,v_n\}$, we define the standard flag
$$F_{\mathrm{Id}}:0\subseteq\langle v_1\rangle\subseteq\langle v_1,v_2\rangle\subseteq\cdots\subseteq\langle v_1,\ldots,v_n\rangle=\mathbb{C}^n.$$

3. There is a natural GL_n-action on $\mathcal{F}$.

   For $g\in\mathrm{GL}_n$, we define the flag attached to g:
$$F_g\triangleq gF_{\mathrm{Id}}:0\subseteq\langle gv_1\rangle\subseteq\langle gv_1,gv_2\rangle\subseteq\cdots\subseteq\langle gv_1,\ldots,gv_n\rangle=\mathbb{C}^n.$$
   Especially, for $w\in W=N_{\mathrm{GL}_n}(T)/T\cong S_n$, the flag attached to w
$$\begin{aligned}F_w:0&\subseteq\langle\tilde{w}v_1\rangle\subseteq\langle\tilde{w}v_1,\tilde{w}v_2\rangle\subseteq\cdots\subseteq\langle\tilde{w}v_1,\ldots,\tilde{w}v_n\rangle=\mathbb{C}^n\\ 0&\subseteq\left\langle v_{w(1)}\right\rangle\subseteq\left\langle v_{w(1)},v_{w(2)}\right\rangle\subseteq\cdots\subseteq\left\langle v_{w(1)},\ldots,v_{w(n)}\right\rangle=\mathbb{C}^n\end{aligned}$$
   does not depend on the choice of the lift $\tilde{w}\in N_{\mathrm{GL}_n}(T)$ of w.

   Readers can verify that $\{F_w\,|\,w\in W\}$ are all T-fixed points of $\mathcal{F}$, while the set $\left\{wBw^{-1}\,\middle|\,w\in W\right\}$ consists of all Borel subgroups of G containing the standard torus T.

4. For $n=2$, $\mathcal{F}\cong\mathbb{P}^1$, and we use $\mathbb{P}^1$ as a toy example for the whole theory.

interpretation	GL_n/B	flags	Borel subgroups
base point	Id	F_{Id}	B
GL_n-action	left multiplication	$\{V_i\}\mapsto\{gV_i\}$	conjugation
general point	g	F_g	gBg^{-1}

$\mathcal{F}$ is a well-studied variety, and has many combinatorical properties. For example, from the well-known Bruhat decomposition,[1]
$$\mathrm{GL}_n=\bigsqcup_{w\in W}BwB,$$
we get a stratification of $\mathcal{F}$ by B-orbits:
$$\mathcal{F}=\mathrm{GL}_n/B\cong\bigsqcup_{w\in W}BwB/B$$
The B-orbit BwB/B is called the **Schubert cell**, denoted by Ω_w. Since
$$\Omega_w=BwB/B\cong B/\left(B\cap wBw^{-1}\right)\cong\mathbb{A}^{l(w)},$$
the Schubert cell is an affine space of dimension $l(w)$.

[1]The formula does not depend on the lift of w, so we abuse the notation of $w\in N_{\mathrm{GL}_n}(T)/T$ and $\tilde{w}\in N_{\mathrm{GL}_n}(T)$.

$\overline{\Omega}_w \subseteq \mathcal{F}$ is called the **Schubert variety**. It is well-known that

$$\overline{\Omega}_w = \bigsqcup_{w' \leqslant w} \Omega_w$$

as a set. Especially, for any $s \in W$ with $l(s) = 1$, denote $P_s = B \sqcup BsB$,

$$\overline{\Omega}_s = \Omega_{\mathrm{Id}} \sqcup \Omega_s = B/B \sqcup BsB/B = P_s/B \cong \mathbb{P}^1.$$

Most Schubert varieties are not smooth.

1.1.2 $\mathcal{F} \times \mathcal{F}$

As a more complicated geometrical object, $\mathcal{F} \times \mathcal{F}$ works as the base space for the Steinberg variety, which turns out to be the central focus in the thesis. $\mathcal{F} \times \mathcal{F}$ has naturally a diagonal GL_n-action:

$$\mathrm{GL}_n \times \mathcal{F} \times \mathcal{F} \longrightarrow \mathcal{F} \times \mathcal{F} \qquad (g, F_1, F_2) \longmapsto (gF_1, gF_2).$$

Under this action, $\mathcal{F} \times \mathcal{F}$ has a stratification consisting of GL_n-orbits, indexed by the Weyl group:

$$\mathrm{GL}_n \backslash (\mathcal{F} \times \mathcal{F}) \cong \mathrm{GL}_n \backslash (\mathrm{GL}_n / B \times \mathrm{GL}_n / B) \cong B \backslash \mathrm{GL}_n / B \cong W \quad \text{as sets.}$$

For $w' \in W$, denote $\mathbf{\Omega}_{w'} := \mathrm{GL}_n \cdot (F_{\mathrm{Id}}, F_{w'})$. Then $\mathcal{F} \times \mathcal{F} = \sqcup_{w'} \mathbf{\Omega}_{w'}$. Moreover, by the orbit-stabilizer theorem, we get

$$\mathbf{\Omega}_{w'} \cong \mathrm{GL}_n / \left(B \cap w' B (w')^{-1}\right)$$

which is an $\mathbb{A}^{l(w')}$-bundle over $\mathcal{F}$.

Different from $\mathcal{F}$, the GL_n-action on $\mathcal{F} \times \mathcal{F}$ is not transitive. To facilitate the stratification of $\mathcal{F} \times \mathcal{F}$, we introduce the twisted $\mathrm{GL}_n \times \mathrm{GL}_n$-action:

$$\mathrm{GL}_n \times \mathrm{GL}_n \times \mathcal{F} \times \mathcal{F} \longrightarrow \mathcal{F} \times \mathcal{F} \qquad (g_1, g_2, F_g, F_{g'}) \longmapsto (F_{g_1 g}, F_{g_1 (g g_2 g^{-1}) g'}).$$

If we write $\underline{F}_{g,g'} := (F_g, F_{gg'}) \in \mathcal{F} \times \mathcal{F}$, then

$$(g_1, g_2) \cdot \underline{F}_{g,g'} = \underline{F}_{g_1 g, g_2 g'}.$$

This $\mathrm{GL}_n \times \mathrm{GL}_n$-action is now transitive, and decomposes $\mathcal{F} \times \mathcal{F}$ as disjoint union of finite many $B \times B$-orbits, which are compatible with G-orbits:

$$\begin{aligned} \mathbf{\Omega}_{w,w'} &:= \ (B \times B) \cdot \underline{F}_{w,w'} \subseteq \mathcal{F} \times \mathcal{F} \\ \mathcal{F} \times \mathcal{F} &= \bigsqcup_{w,w' \in W} \mathbf{\Omega}_{w,w'} \qquad \mathbf{\Omega}_{w'} = \bigsqcup_{w \in W} \mathbf{\Omega}_{w,w'} \end{aligned}$$

$$\begin{aligned} \mathbf{\Omega}_{w,w'} &\cong (B \times B) / \{(g_1, g_2) \in B \times B \mid (g_1, g_2) \cdot (F_w, F_{ww'}) = (F_w, F_{ww'})\} \\ &= (B \times B) / \left\{(g_1, g_2) \in B \times B \;\middle|\; g_1 w B = wB, g_1 w g_2 w' B = ww' B\right\} \\ &= (B \times B) / \left\{(g_1, g_2) \in B \times B \;\middle|\; g_1 \in wBw^{-1}, g_2 \in (w^{-1} g_1^{-1} w)\left(w' B w'^{-1}\right)\right\} \\ &= (B \times B) / \left\{(g_1, g_2) \in \left(B \cap wBw^{-1}\right) \times (w^{-1} g_1^{-1} w)\left(B \cap w' B w'^{-1}\right)\right\} \\ &\cong B/(B \cap wBw^{-1}) \times B/(B \cap w' B w'^{-1}) \cong \mathbb{A}^{l(w) + l(w')} \end{aligned} \tag{1.1.1}$$

G	Orbit	G-fixed points
GL_n	$\mathcal{F}$	$\varnothing$
B	Ω_w	
T	$-$	$\{F_w \mid w \in W\}$

(a) $\mathcal{F}$

G	Orbit	G-fixed points
$\mathrm{GL}_n \times \mathrm{GL}_n$	$\mathcal{F} \times \mathcal{F}$	$\varnothing$
GL_n	$\mathbf{\Omega}_{w'}$	$\varnothing$
$B \times B$	$\mathbf{\Omega}_{w,w'}$	$\{F_{\mathrm{Id},\mathrm{Id}}\}$
T	$-$	$\{\underline{F}_{w,w'} \mid w, w' \in W\}$

(b) $\mathcal{F} \times \mathcal{F}$

Table 1.1: Orbit and fixed points

We conclude the information of orbits and fixed points in Table 1.1.

Like $\mathcal{F}$, we also study the closure of $\mathbf{\Omega}_{w'}$ and $\mathbf{\Omega}_{w,w'}$ in special case. It can be shown that

$$\overline{\mathbf{\Omega}}_{w'} = \bigsqcup_{x' \leqslant w'} \mathbf{\Omega}_{x'} \qquad \overline{\mathbf{\Omega}}_{w,w'} = \bigsqcup_{x \leqslant w, x' \leqslant w'} \mathbf{\Omega}_{x,x'}$$

as a set. Especially, for any $s \in W$ with $l(s) = 1$, [2]

$$\begin{aligned} \overline{\mathbf{\Omega}}_s = \mathbf{\Omega}_{\mathrm{Id}} \sqcup \mathbf{\Omega}_s &\cong \mathrm{GL}_n /(B \cap sBs^{-1}) && \sqcup \mathrm{GL}_n /B \\ &\cong \mathrm{GL}_n \times^B \big(B/(B \cap sBs^{-1})\big) && \sqcup \mathrm{GL}_n \times^B (B/B) \\ &\cong \mathrm{GL}_n \times^B (BsB/B) && \sqcup \mathrm{GL}_n \times^B (B/B) \\ &\cong \mathrm{GL}_n \times^B (P_s/B) \end{aligned}$$

is an $\mathcal{F}$-bundle over $\mathbb{P}^1$. Also,

$$\begin{aligned} \overline{\mathbf{\Omega}}_{\mathrm{Id},s} = \mathbf{\Omega}_{\mathrm{Id},s} \sqcup \mathbf{\Omega}_{\mathrm{Id},\mathrm{Id}} &\cong \big(B/B \times BsB/B\big) \sqcup \big(B/B \times B/B\big) \\ &\cong P_s/B \cong \mathbb{P}^1 \end{aligned}$$

Other closure can be highly singular.

Example 1.1.5. *In the Table 1.6, $n = 3$, $t = (12)$, $s = (23)$. In this case, $\mathcal{F} \times \mathcal{F}$ has* 6 GL_3*-orbits, and each* GL_3*-orbits decompose as* 6 *$B \times B$-orbits, with dimensions equal to $l(w) + l(w')$.*

We can also see the GL_3*-orbit (and its closure) from the table, for example,*

$$\begin{aligned} \mathbf{\Omega}_s &= \mathbf{\Omega}_{\mathrm{Id},s} \sqcup \mathbf{\Omega}_{t,s} \sqcup \mathbf{\Omega}_{s,s} \sqcup \mathbf{\Omega}_{ts,s} \sqcup \mathbf{\Omega}_{st,s} \sqcup \mathbf{\Omega}_{sts,s} \\ \overline{\mathbf{\Omega}}_s &= \mathbf{\Omega}_s \sqcup \mathbf{\Omega}_{\mathrm{Id}} = \bigsqcup_w (\mathbf{\Omega}_{w,s} \sqcup \mathbf{\Omega}_{w,\mathrm{Id}}) \end{aligned}$$

We color pieces of $\mathbf{\Omega}_s$ by blue, and $\mathbf{\Omega}_{ts,s}$ by light blue.

Now we understand the structure a lot about $\mathcal{F}$ and $\mathcal{F} \times \mathcal{F}$, and the whole process will be applied repeatedly in Section 1.5 and 1.6.

[2]Here, $\mathrm{GL}_n \times^B X$ is called balanced product. Roughly, it is defined by

$$\mathrm{GL}_n \times^B X := \mathrm{GL}_n \times X/\big((gb, x) \sim (g, bx)\big)$$

We will discuss balanced product in Subsection 2.4.1 thoroughly.

1.2 Quivers

To introduce more complicated spaces and discuss their stratifications, we fix notation related to quiver and algebraic group in the following sections.

Roughly speaking, a quiver is a directed multigraph permitting loops.

Definition 1.2.1 (Quiver)**.** *A quiver is a quadruple*

$$Q = (v(Q), a(Q), s, t)$$

where

- *$v(Q)$ is a non-empty set consisting of vertices of Q,*
- *$a(Q)$ is a set consisting of arrows of Q,*
- *$s : a(Q) \longrightarrow v(Q)$ is a map indicating the start vertex of arrows,*
- *$t : a(Q) \longrightarrow v(Q)$ is a map indicating the terminal vertex of arrows.*

Remark 1.2.2. In the first part of our master thesis, all the quivers are supposed to be connected and finite (i.e., $v(Q)$, $a(Q)$ are finite sets). We will only encounter disconnected and infinite quiver as Auslander–Reiten quiver later on.

Example 1.2.3. *The following graphs are quivers.*

$$\bullet \longrightarrow \bullet \longleftarrow \bullet \longleftarrow \bullet \qquad \begin{tikzcd} \bullet \arrow[loop above] \arrow[loop left] \arrow[loop right] \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r, bend left, "a"] \arrow[r, bend right, "b"'] & 2 \end{tikzcd}$$

quiver of type A_3 3-loop quiver $L(3)$ 2-Kronecker quiver $K(2)$

The reader can easily read the quadruple of quivers from the graphs. Take $Q = K(2)$ as an example, we have

$$v(Q) = \{1, 2\}, \qquad a(Q) = \{a, b\} \qquad s, t : \{a, b\} \longrightarrow \{1, 2\}$$

by $s(a) = s(b) = 1$, $t(a) = t(b) = 2$.

For simplicity, we mainly use simpler quivers as examples:

$$\bullet \qquad\qquad \bullet \longrightarrow \bullet \qquad\qquad \begin{tikzcd} \bullet \arrow[loop right] \end{tikzcd}$$

trivial quiver quiver of type A_1 1-loop quiver $L(1)$

From those quivers we are able to construct relatively complicated algebraic and geometrical objects.

Definition 1.2.4 (Quiver representation)**.** *Fix a quiver Q.*

A representation of Q consists of the following data:

- *A finite dimensional $\mathbb{C}$-vector space V_i for each vertex $i \in v(Q)$;*
- *A $\mathbb{C}$-linear map $V_r : V_{s(r)} \longrightarrow V_{t(r)}$ for each arrow $r \in a(Q)$.*

A morphism $f : V \longrightarrow W$ is a collection of morphisms $f_i : V_i \longrightarrow W_i$ (for every $i \in v(Q)$) which makes the following diagram commute:

$$\begin{tikzcd} V_{s(r)} \arrow[r, "V_r"] \arrow[d, "f_{s(r)}"'] & V_{t(r)} \arrow[d, "f_{t(r)}"] \\ W_{s(r)} \arrow[r, "W_r"] & W_{t(r)} \end{tikzcd}$$

We denote the category of representations of Q by $\mathrm{rep}(Q)$*.*

Example 1.2.5. *A representation of the* 1*-loop quiver* $L(1)$ *is a* 2*-tuple*

$$(V,\ \alpha : V \longrightarrow V)$$

which is equivalent to a (finite dimensional) $\mathbb{C}[t]$*-module.*

Remark 1.2.6. The equivalence in Example 1.2.5 can be generalized to arbitrary quivers. For a quiver Q, we can define the path algebra $\mathbb{C}Q$, and view any Q-representation as $\mathbb{C}Q$-module, and vice versa.

Definition 1.2.7 (Q-vector space/Vector space with quiver partition)**.** *Fix a quiver Q, a Q-vector space is a finite dimensional $\mathbb{C}$-vector space with a direct sum decomposition*

$$V = \bigoplus_{i \in v(Q)} V_i.$$

The dimension vector of a Q-vector space is defined as

$$\underline{\mathbf{dim}}\, V := (\dim_{\mathbb{C}} V_i)_{i \in v(Q)} \in \prod_{i \in v(Q)} \mathbb{Z}.$$

On the contrary, given $\mathbf{d} \in \prod_{i \in v(Q)} \mathbb{N}_{\geqslant 0}$, we can construct a canonical Q-vector space of dimension vector $\mathbf{d}$, as follows:

$$V = \bigoplus_{i \in v(Q)} V_i \qquad \text{with } V_i = \mathbb{C}^{\mathbf{d}_i}.$$

Definition 1.2.8. *The total dimension vector of a Q-vector space V is defined as*

$$|\underline{\mathbf{dim}}\, V| := \dim_{\mathbb{C}} V.$$

For $\mathbf{d} \in \prod_{i \in v(Q)} \mathbb{N}_{\geqslant 0}$, denote $|\mathbf{d}| := \sum_{i \in v(Q)} \mathbf{d}_i$.

Definition 1.2.9 (Space of representations with given dimension vector)**.** *For any quiver Q, dimension vector $\mathbf{d}$, fix the canonical Q-vector space $V = \bigoplus_{i \in v(Q)} V_i$, the space of representations with dimension vector $\mathbf{d}$ is defined as*

$$\begin{aligned} \mathrm{Rep}_{\mathbf{d}}(Q) &:= \left\{ (V_i, V_a : V_{s(a)} \longrightarrow V_{t(a)}) \textit{ as a representation of } Q \right\} \\ &= \bigoplus_{r \in a(Q)} \mathrm{Hom}\left(\mathbb{C}^{\mathbf{d}_{s(a)}}, \mathbb{C}^{\mathbf{d}_{t(a)}} \right) \end{aligned}$$

Since we encode the information of vector space in $\mathbf{d}$, $\mathrm{Rep}_{\mathbf{d}}(Q)$ *only records the information of linear maps.*

For both Q-vector spaces and Q-representations, we can define (complete) flags.

Definition 1.2.10 (Flag with quiver)**.** *For a quiver representation $V \in \mathrm{rep}(Q)$, a (partial) flag of V is defined as an increasing sequence of subrepresentation of V, i.e.,*

$$F : 0 \subseteq M_1 \subseteq \ldots \subseteq M_d = V \qquad M_k \in \mathrm{rep}(Q).$$

For a Q-vector space $V = \oplus_{i \in v(Q)} V_i$, a (quiver-graded partial) flag of V is defined as an increasing sequence of Q-subspace of V, i.e.,

$$F : 0 \subseteq M_1 \subseteq \ldots \subseteq M_d = V \qquad M_k = \bigoplus_{i \in v(Q)} M_{k,i}.$$

For both Q-vector space and Q-representation, F is called a ***complete flag*** *if $d = \dim_{\mathbb{C}} V$ and*

$$\dim_{\mathbb{C}} M_k = k \qquad \textit{for any } k \in \{1, \ldots, d\}$$

For the flag we also have the notation of dimension vector.

Definition 1.2.11 (flag-type dimension vector)**.** *For any flag $F : 0 \subseteq M_1 \subseteq \ldots \subseteq M_d \subseteq V$, the (flag-type) dimension vector of F is defined as*

$$\underline{\mathbf{f}} = \left(\underline{\mathbf{dim}}\, M_k\right)_{k\in\{1,\ldots,d\}} \in \prod_{\substack{i\in v(Q)\\ k\in\{1,\ldots,d\}}} \mathbb{Z}.$$

$\underline{\mathbf{f}}$ is called a complete flag-type dimension vector if $\underline{\mathbf{f}}$ is the dimension vector of some complete flag F, i.e.,[3] *$d = |\mathbf{f}|$ and*

$$|\underline{\mathbf{dim}}\, M_{k+1}/M_k| = 1 \qquad \textit{for any } k \in \{0,\ldots,d-1\}.$$

From now on, we denote a complete dimension vector by $\underline{\mathbf{d}}$, and (partial) dimension vectors by $\underline{\mathbf{f}}$, $\underline{\mathbf{g}}$.

Example 1.2.12. *For the quiver $Q : i \longrightarrow i'$, $\mathbf{d} = (3, 2)$, the canonical Q-vector space of dimension vector $\mathbf{d}$ is*

$$\begin{aligned} V &= V_i \oplus V_{i'} \\ &= \langle v_1, v_2, v_3\rangle_{\mathbb{C}} \oplus \langle v_4, v_5\rangle_{\mathbb{C}} \end{aligned}$$

The flag

$$F : 0 \subseteq \langle v_4\rangle \subseteq \langle v_4, v_1\rangle \subseteq \langle v_4, v_1, v_2\rangle \subseteq \langle v_4, v_1, v_2, v_5\rangle \subseteq \langle v_4, v_1, v_2, v_5, v_3\rangle = V$$

is a complete flag of V, with complete dimension vector

$$\underline{\mathbf{d}} = \begin{pmatrix} 3,2\\ 2,2\\ 2,1\\ 1,1\\ 0,1 \end{pmatrix}.$$

Remark 1.2.13. Any complete flag-type dimension vector $\underline{\mathbf{d}}$ can be viewed as a partition of set $\{1,\ldots,|\mathbf{d}|\}$, i.e., a map

$$\mathrm{par} : \{1,\ldots,|\mathbf{d}|\} \longrightarrow v(Q)$$

such that $\#\,\mathrm{par}^{-1}(i) = \mathbf{d}_i$.[4] We color the set $\{1,\ldots,|\mathbf{d}|\}$ by the partition par. In the Example 1.2.12,

$$\underline{\mathbf{d}} = \begin{pmatrix} 3,2\\ 2,2\\ 2,1\\ 1,1\\ 0,1 \end{pmatrix} \quad \begin{array}{ll} \text{corresponds to} & \{1,2,3,4,5\} = \{2,3,5\} \sqcup \{1,4\} \\ \text{corresponds to} & \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \end{array}$$

[3] For convenience, we define $M_0 := 0$.

[4] The partition corresponding to par is

$$\{1,\ldots,|\mathbf{d}|\} = \bigsqcup_{i\in v(Q)} \mathrm{par}^{-1}(i).$$

1.3 Symmetric group calculus

As a reminder, we recall some basic diagrams referring to the elements in S_n, and do some calculations by these diagrams. We will also relate cosets with complete flag-type dimension vectors.

Fix a quiver Q and dimension vector $\mathbf{d}$. Later (Definition 1.4.2, 1.4.3) we will define

$$\mathbb{W}_{|\mathbf{d}|} = S_{|\mathbf{d}|} \qquad W_{\mathbf{d}} = \prod_{i \in v(Q)} S_{\mathbf{d}_i} \leqslant \mathbb{W}_{|\mathbf{d}|}$$

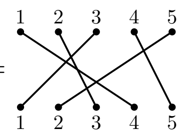
For simplicity, we take $v(Q) = \{1, \ldots, k\}$, then $W_{\mathbf{d}} = S_{\mathbf{d}_1} \times \cdots \times S_{\mathbf{d}_k}$ embeds in $S_{|\mathbf{d}|}$.

Remark 1.3.1. We have different ways to express $\varpi \in \mathbb{W}_{|\mathbf{d}|} = S_{|\mathbf{d}|}$. For example, take $|\mathbf{d}| = 5$, $\varpi \in S_5$ by

$$\varpi(1) = 4, \quad \varpi(2) = 3, \quad \varpi(3) = 1, \quad \varpi(4) = 5, \quad \varpi(5) = 2,$$

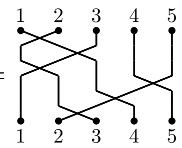
then one can represent ϖ via

$$\varpi \mathrel{\hat{=}} (14523) \mathrel{\hat{=}} \begin{pmatrix} 1\,2\,3\,4\,5 \\ 4\,3\,1\,5\,2 \end{pmatrix} \mathrel{\hat{=}}$$



$$\mathrel{\hat{=}} \begin{bmatrix} & & 1 & & \\ & & & & 1 \\ & 1 & & & \\ 1 & & & & \\ & & & 1 & \end{bmatrix}$$

$$\mathrel{\hat{=}} (23)(34)(45)(12)(23)(12) \mathrel{\hat{=}}$$



Even though all expressions give us the same amount of information, the diagram presents them more vividly. For example, each intersection of strands corresponds to a simple reflection, so we read from the diagram that $l(\varpi) = 6$. Readers can also check that

$$\begin{aligned} &l(\varpi s_1) = 5, \quad l(\varpi s_2) = 5, \quad l(\varpi s_3) = 7, \quad l(\varpi s_4) = 5, \\ &l(s_1 \varpi) = 7, \quad l(s_2 \varpi) = 5, \quad l(s_3 \varpi) = 5, \quad l(s_4 \varpi) = 7, \end{aligned}$$

where $s_i := (i, i+1) \in S_5$ are simple reflections.

Definition 1.3.2 (Simple reflections in the Weyl group)**.** *For $i \in \{1, \ldots, |\mathbf{d}| - 1\}$, the simple reflection is defined as*

$$s_i := (i, i+1) \in S_{|\mathbf{d}|}.$$

We define

$$\Pi = \left\{ s_i \in S_{|\mathbf{d}|} \,\middle|\, i \in \{1, \ldots, |\mathbf{d}| - 1\} \right\}$$

as the set of simple reflections in $\mathbb{W}_{|\mathbf{d}|}$, and

$$\begin{aligned} \Pi_{\mathbf{d}} &= \left\{ s_i \in S_{\mathbf{d}_1} \times \cdots \times S_{\mathbf{d}_k} \,\middle|\, i \in \{1, \ldots, |\mathbf{d}| - 1\} \right\} \\ &= \left\{ s_1, \ldots, s_{|\mathbf{d}|-1} \right\} \setminus \left\{ s_{\mathbf{d}_1}, s_{\mathbf{d}_1 + \mathbf{d}_2}, \ldots, s_{\mathbf{d}_1 + \cdots + \mathbf{d}_{k-1}} \right\} \end{aligned}$$

as the set of simple reflections in $W_{\mathbf{d}}$.

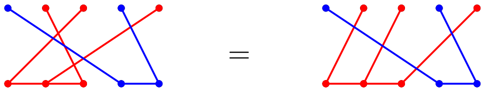
We also denote the longest elements in $\mathbb{W}_{|\mathbf{d}|}$ and $W_{\mathbf{d}}$ by $\varpi_{\max}$ and $w_{\max}$, respectively. See Table 1.2 for a picture of $\varpi_{\max}$ and $w_{\max}$.

We discuss right cosets $W_{\mathbf{d}} \setminus \mathbb{W}_{|\mathbf{d}|}$ and minimal length coset representatives now.

Multiplying on the left by $w \in W_{\mathbf{d}}$ is equivalent to plugging in a diagram representing $w \in W_{\mathbf{d}}$ underneath the original diagram. Therefore, we connect some bottom points by lines, indicating that switching them will cause no trouble. Furthermore, we color different parts to make the following more explicitly.

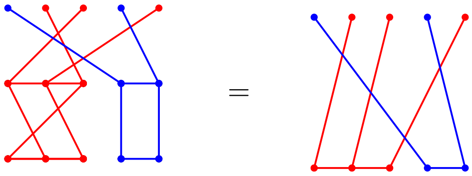
Proposition 1.3.3. *Every element $W_{\mathbf{d}}\varpi \in W_{\mathbf{d}} \setminus \mathbb{W}_{|\mathbf{d}|}$ corresponds to a partition on set* $\{1, \ldots, |\mathbf{d}|\}$ *(of a given number partition* $\mathbf{d}$*), which corresponds to a complete flag-type dimension vector* $\underline{\mathbf{d}}$*, i.e., an ordered set of points colored by the vertices of* Q*.*

Example 1.3.4.



in $W_{\mathbf{d}} \setminus \mathbb{W}_{|\mathbf{d}|}$

since



in $\mathbb{W}_{|\mathbf{d}|}$

This coset corresponds to the partition $\{1,2,3,4,5\} = \{2,3,5\} \sqcup \{1,4\}$*, and this corresponds to the ordered set of colored points:*

It is easy to see from the diagram that in every coset, there exists a unique element $u \in \mathbb{W}_{|\mathbf{d}|}$ of minimal length. Let $\mathrm{Min}(\mathbb{W}_{|\mathbf{d}|}, \mathrm{W}_{\mathbf{d}})$ be the collection of these minimal length coset representatives.[5]

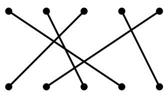
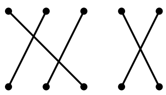
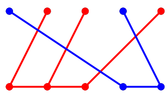
Proposition 1.3.5. *For any $\varpi \in \mathbb{W}_{|\mathbf{d}|}$, there exists unique $w \in W_{\mathbf{d}}$, $u \in \mathrm{Min}(\mathbb{W}_{|\mathbf{d}|}, \mathrm{W}_{\mathbf{d}})$ such that $\varpi = wu$.*

Proposition 1.3.6. *For $u \in \mathrm{Min}(\mathbb{W}_{|\mathbf{d}|}, \mathrm{W}_{\mathbf{d}})$, $s_i \in \Pi$,*

$$\begin{aligned} us_iu^{-1} \in W_{\mathbf{d}} \quad &\Longrightarrow \quad us_iu^{-1} = s_{u(i)} \in \Pi_{\mathbf{d}}, \\ us_iu^{-1} \notin W_{\mathbf{d}} \quad &\Longrightarrow \quad us_i \in \mathrm{Min}(\mathbb{W}_{|\mathbf{d}|}, \mathrm{W}_{\mathbf{d}})\,. \end{aligned}$$

We finish this section with figures and examples.

Example 1.3.7. *In Table 1.2,* $|\mathbf{d}| = 5$*,* $\mathbf{d} = (3,2)$*, typical elements would be*

$\varpi =$  $w =$  $u =$ 

Example 1.3.8. *In Table 1.3,*

$$|\mathbf{d}| = 3, \quad \mathbf{d} = (1,2), \qquad \mathbb{W}_{|\mathbf{d}|} = S_3, \quad W_{\mathbf{d}} = S_1 \times S_2, \qquad s = (12), \quad t = (23).$$

The columns "order of basis" and "Borel subgroups" will be introduced in Definition 1.5.5 and Remark 1.4.4.

[5] In some references $\mathrm{Min}(\mathbb{W}_{|\mathbf{d}|}, \mathrm{W}_{\mathbf{d}})$ is also denoted by $\mathrm{Shuffle}_{\mathbf{d}}$, since those elements can be thought as ways off riffle shuffling several words together.

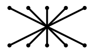
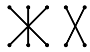
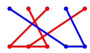
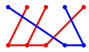
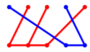
set	element	special element	others/alias
$\mathbb{W}_{\lvert\mathbf{d}\rvert} = S_5$	ϖ, x	$\varpi_{\max} =$ 	$\Pi \ = \{s_1, s_2, s_3, s_4\}$
$W_{\mathbf{d}} = S_3 \times S_2$	w	$w_{\max} =$ 	$\Pi_{\mathbf{d}} = \{s_1, s_2, \quad s_4\}$
$W_{\mathbf{d}} \setminus \mathbb{W}_{\lvert\mathbf{d}\rvert} \cong (S_3 \times S_2) \setminus S_5$	$\varpi, \underline{\mathbf{d}}$		$\mathrm{Comp}_{\mathbf{d}}$
$\mathrm{Min}(\mathbb{W}_{\lvert\mathbf{d}\rvert}, \mathrm{W}_{\mathbf{d}}) = \{$  $, \dots \}$	u		$\mathrm{Shuffle}_{\mathbf{d}}$

Table 1.2: Collected notation in $(3, 2)$-case

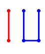
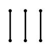
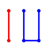
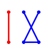
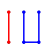
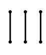
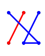
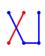
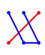
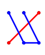
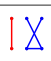
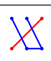
$\varpi = wu$					w	$\underline{\mathbf{d}}, u$		order of basis	$l(\varpi)$	$l(w)$	$\mathbb{B}_{\varpi}$	B_{ϖ}	$\varpi B_{\mathbf{d}} \varpi^{-1}$
Id	Id	$\begin{pmatrix}1\,2\,3\\1\,2\,3\end{pmatrix}$		$\begin{bmatrix}1&&\\&1&\\&&1\end{bmatrix}$		abb		$\{v_1, v_2, v_3\}$	0	0	$\begin{bmatrix}*&*&*\\&*&*\\&&*\end{bmatrix}$	$\begin{bmatrix}*&&\\&*&*\\&&*\end{bmatrix}$	$\begin{bmatrix}*&&\\&*&*\\&&*\end{bmatrix}$
t	(23)	$\begin{pmatrix}1\,2\,3\\1\,3\,2\end{pmatrix}$		$\begin{bmatrix}1&&\\&&1\\&1&\end{bmatrix}$		abb		$\{v_1, v_3, v_2\}$	1	1	$\begin{bmatrix}*&*&*\\&*&\\&*&*\end{bmatrix}$	$\begin{bmatrix}*&&\\&*&\\&*&*\end{bmatrix}$	$\begin{bmatrix}*&&\\&*&\\&*&*\end{bmatrix}$
s	(12)	$\begin{pmatrix}1\,2\,3\\2\,1\,3\end{pmatrix}$		$\begin{bmatrix}&1&\\1&&\\&&1\end{bmatrix}$		bab		$\{v_2, v_1, v_3\}$	1	0	$\begin{bmatrix}*&&*\\*&*&*\\&&*\end{bmatrix}$	$\begin{bmatrix}*&&\\&*&*\\&&*\end{bmatrix}$	$\begin{bmatrix}*&&*\\&*&\\&&*\end{bmatrix}$
ts	(132)	$\begin{pmatrix}1\,2\,3\\3\,1\,2\end{pmatrix}$		$\begin{bmatrix}&1&\\&&1\\1&&\end{bmatrix}$		bab		$\{v_3, v_1, v_2\}$	2	1	$\begin{bmatrix}*&*&\\&*&\\*&*&*\end{bmatrix}$	$\begin{bmatrix}*&&\\&*&\\&*&*\end{bmatrix}$	$\begin{bmatrix}*&*&\\&*&\\&&*\end{bmatrix}$
st	(123)	$\begin{pmatrix}1\,2\,3\\2\,3\,1\end{pmatrix}$		$\begin{bmatrix}&&1\\1&&\\&1&\end{bmatrix}$		bba		$\{v_2, v_3, v_1\}$	2	0	$\begin{bmatrix}*&&\\*&*&*\\*&&*\end{bmatrix}$	$\begin{bmatrix}*&&\\&*&*\\&&*\end{bmatrix}$	$\begin{bmatrix}*&&\\&*&\\*&&*\end{bmatrix}$
sts	(13)	$\begin{pmatrix}1\,2\,3\\3\,2\,1\end{pmatrix}$		$\begin{bmatrix}&&1\\&1&\\1&&\end{bmatrix}$		bba		$\{v_3, v_2, v_1\}$	3	1	$\begin{bmatrix}*&&\\*&*&\\*&*&*\end{bmatrix}$	$\begin{bmatrix}*&&\\&*&\\&*&*\end{bmatrix}$	$\begin{bmatrix}*&&\\*&*&\\&&*\end{bmatrix}$

Table 1.3: Basic information of $(1, 2)$-case

1.4 Algebraic groups and Lie algebras

In this section we fix notation for some algebraic groups and Lie algebras.

Setting 1.4.1. *Fix a quiver Q, a dimension vector $\mathbf{d}$ and a Q-vector space*

$$V = \bigoplus_{i \in v(Q)} V_i \qquad \textit{with } V_i = \mathbb{C}^{\mathbf{d}_i}.$$

When a basis of V is needed, we fix a total order on $v(Q)$, and denote

$$V = \langle v_1, \dots v_{\lvert\mathbf{d}\rvert} \rangle$$

where

$$V_i = \langle v_{f_i+1}, \dots v_{f_i+\mathbf{d}_i} \rangle \qquad f_i = \sum_{i' < i} \mathbf{d}_{i'}.$$

1.4.1 Algebraic groups

Definition 1.4.2 (Absolute algebraic groups)**.** *We set*

$$\mathbb{G}_{\lvert\mathbf{d}\rvert} := \mathrm{GL}(V) = \mathrm{GL}_{\lvert\mathbf{d}\rvert}(\mathbb{C}),$$

and $\mathbb{B}_{|\mathbf{d}|}$, $\mathbb{T}_{|\mathbf{d}|}$, $\mathbb{N}_{|\mathbf{d}|}$ are corresponding standard Borel, torus and unipotent subgroups, respectively.

The Weyl group is

$$\mathbb{W}_{|\mathbf{d}|} := N_{\mathbb{G}_{|\mathbf{d}|}}(\mathbb{T}_{|\mathbf{d}|})/\mathbb{T}_{|\mathbf{d}|} \cong S_{|\mathbf{d}|}.$$

For $\varpi \in \mathbb{W}_{|\mathbf{d}|}$, we define[6]

$$\mathbb{B}_{\varpi} := \varpi \mathbb{B}_{|\mathbf{d}|} \varpi^{-1}.$$

We will view $\mathbb{B}_{\varpi}$ as the stabilizer of the flag F_{ϖ} with $\mathbb{G}_{|\mathbf{d}|}$-action.

Definition 1.4.3 (Relative algebraic groups)**.** *We set*

$$G_{\mathbf{d}} := \prod_{i\in v(Q)} \mathrm{GL}(V_i) = \prod_{i\in v(Q)} \mathrm{GL}_{\mathbf{d}_i}(\mathbb{C}) \subseteq \mathbb{G}_{|\mathbf{d}|},$$

and $B_{\mathbf{d}}$, $T_{\mathbf{d}}$, $N_{\mathbf{d}}$ are corresponding standard Borel, torus and unipotent subgroups.

The Weyl group is

$$W_{\mathbf{d}} := N_{G_{\mathbf{d}}}(T_{\mathbf{d}})/T_{\mathbf{d}} \cong \prod_{i\in v(Q)} S_{\mathbf{d}_i}.$$

For $\varpi = wu \in W_{\mathbf{d}}$, we define

$$B_{\varpi} := w B_{\mathbf{d}} w^{-1}.$$

We will view B_{ϖ} as the stabilizer of the flag F_{ϖ} with $G_{\mathbf{d}}$-action.

Remark 1.4.4. Be careful that $B_{\varpi} \neq \varpi B_{\mathbf{d}} \varpi^{-1}$. Actually,

$$B_{\varpi} = \varpi \mathbb{B}_{|\mathbf{d}|} \varpi^{-1} \cap G_{\mathbf{d}} = w B_{\mathbf{d}} w^{-1}$$

The difference is clearly shown in Table 1.3.

We also have a series of algebraic groups indexed by elements in the Weyl group:

Definition 1.4.5 (More algebraic groups)**.** *For $\varpi, \varpi'' \in \mathbb{W}_{|\mathbf{d}|}$, define*

$$\begin{aligned} N_{\varpi} &:= R_u(B_{\varpi}), \\ N_{\varpi,\varpi''} &:= N_{\varpi} \cap N_{\varpi''}, \\ M_{\varpi,\varpi''} &:= N_{\varpi}/N_{\varpi,\varpi''}, \end{aligned}$$

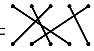
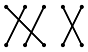
where R_u denotes the unipotent radical.

For $s \in \Pi$ such that $\varpi s \varpi^{-1} \in W_d$ (i.e., $W_{\mathbf{d}}\varpi = W_{\mathbf{d}}\varpi s$), define

$$\begin{aligned} P_{\varpi,\varpi s} &:\overset{\varpi = wu}{=\!=\!=} w\left(B_{\mathbf{d}} u s u^{-1} B_{\mathbf{d}} \cup B_{\mathbf{d}}\right) w^{-1} \\ &=\!=\!= B_{\varpi} \varpi s \varpi^{-1} B_{\varpi} \cup B_{\varpi} \end{aligned}$$

Remark 1.4.6. One can easily show that $N_{\varpi,\varpi s} = R_u(P_{\varpi,\varpi s})$.

[6]As usual, we abuse the notation of ϖ and its lift.

Example 1.4.7 (Follows Example 1.3.7)**.** *For* $|\mathbf{d}| = 5$*,* $\mathbf{d} = (3, 2)$*,* $\varpi =$ *,* $w =$ *,* $s = s_2$*, we compute all the algebraic groups we mentioned:*

$$\mathbb{G}_{|\mathbf{d}|} = \begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{pmatrix} \qquad \mathbb{B}_{|\mathbf{d}|} = \begin{pmatrix} * & * & * & * & * \\ & * & * & * & * \\ & & * & * & * \\ & & & * & * \\ & & & & * \end{pmatrix} \qquad \mathbb{T}_{|\mathbf{d}|} = \begin{pmatrix} * & & & & \\ & * & & & \\ & & * & & \\ & & & * & \\ & & & & * \end{pmatrix} \qquad \mathbb{N}_{|\mathbf{d}|} = \begin{pmatrix} 1 & * & * & * & * \\ & 1 & * & * & * \\ & & 1 & * & * \\ & & & 1 & * \\ & & & & 1 \end{pmatrix}$$

$$\mathbb{W}_{|\mathbf{d}|} \cong S_5 \qquad \mathbb{B}_{\varpi} = \begin{pmatrix} * & * & & & * \\ & * & & & \\ * & * & * & & * \\ * & * & * & * & * \\ & * & & & * \end{pmatrix} \qquad \mathbb{B}_{\varpi s} = \begin{pmatrix} * & * & * & & * \\ & * & & & \\ & * & * & & * \\ * & * & * & * & * \\ & * & & & * \end{pmatrix}$$

$$G_{\mathbf{d}} = \left(\begin{array}{ccc|cc} * & * & * & & \\ * & * & * & & \\ * & * & * & & \\ \hline & & & * & * \\ & & & * & * \end{array}\right) \qquad B_{\mathbf{d}} = \left(\begin{array}{ccc|cc} * & * & * & & \\ & * & * & & \\ & & * & & \\ \hline & & & * & * \\ & & & & * \end{array}\right) \qquad T_{\mathbf{d}} = \left(\begin{array}{ccc|cc} * & & & & \\ & * & & & \\ & & * & & \\ \hline & & & * & \\ & & & & * \end{array}\right) \qquad N_{\mathbf{d}} = \left(\begin{array}{ccc|cc} 1 & * & * & & \\ & 1 & * & & \\ & & 1 & & \\ \hline & & & 1 & * \\ & & & & 1 \end{array}\right)$$

$$W_{\mathbf{d}} \cong S_3 \times S_2 \qquad B_{\varpi} = \left(\begin{array}{ccc|cc} * & * & & & \\ & * & & & \\ * & * & * & & \\ \hline & & & * & \\ & & & * & * \end{array}\right) \qquad B_{\varpi s} = \left(\begin{array}{ccc|cc} * & & & & \\ * & * & & & \\ * & * & * & & \\ \hline & & & * & \\ & & & * & * \end{array}\right)$$

$$N_{\varpi} = \left(\begin{array}{ccc|cc} 1 & * & & & \\ & 1 & & & \\ * & * & 1 & & \\ \hline & & & 1 & \\ & & & * & 1 \end{array}\right) \qquad N_{\varpi,\varpi s} = \left(\begin{array}{ccc|cc} 1 & & & & \\ & 1 & & & \\ * & * & 1 & & \\ \hline & & & 1 & \\ & & & * & 1 \end{array}\right) \qquad M_{\varpi,\varpi s} = \left(\begin{array}{ccc|cc} 1 & * & & & \\ & 1 & & & \\ - & - & 1 & & \\ \hline & & & 1 & \\ & & & - & 1 \end{array}\right) \qquad P_{\varpi,\varpi s} = \left(\begin{array}{ccc|cc} * & * & & & \\ * & * & & & \\ * & * & * & & \\ \hline & & & * & \\ & & & * & * \end{array}\right)$$

1.4.2 Lie algebra

We use Fraktur-font symbols to represent the Lie algebras of the corresponding algebraic groups introduced in the last section:

$$\begin{array}{lllll} \mathfrak{g}_{|\mathbf{d}|}, & \mathfrak{b}_{|\mathbf{d}|}, & \mathfrak{t}_{|\mathbf{d}|}, & \mathfrak{n}_{|\mathbf{d}|}, & \mathfrak{b}_{\varpi} \\ \mathfrak{g}_{\mathbf{d}}, & \mathfrak{b}_{\mathbf{d}}, & \mathfrak{t}_{\mathbf{d}}, & \mathfrak{n}_{\mathbf{d}}, & \mathfrak{b}_{\varpi}, \\ \mathfrak{n}_{\varpi}, & \mathfrak{n}_{\varpi,\varpi''}, & \mathfrak{m}_{\varpi,\varpi''}, & \mathfrak{p}_{\varpi,\varpi s}, & \end{array}$$

We also have to encode the information of representations as Lie algebra. Notice that

$$\mathrm{Hom}(V_{s(a)}, V_{t(a)}) \hookrightarrow \mathrm{Hom}(V, V) \cong \mathfrak{g}_{|\mathbf{d}|} \qquad f \longmapsto \iota_{t(a)} \circ f \circ \pi_{s(a)}$$

realizes $\mathrm{Hom}(V_{s(a)}, V_{t(a)})$ as a subspace of $\mathfrak{g}_{|\mathbf{d}|}$, so

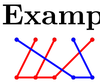
$$\mathrm{Rep}_{\mathbf{d}}(Q) = \bigoplus_{r \in a(Q)} \mathrm{Hom}\left(\mathbb{C}^{\mathbf{d}_{s(a)}}, \mathbb{C}^{\mathbf{d}_{t(a)}}\right) \subseteq \bigoplus_{r \in a(Q)} \mathfrak{g}_{|\mathbf{d}|}.$$

Definition 1.4.8 (Lie algebras connected with representations)**.** *For* $\varpi \in \mathbb{W}_{|\mathbf{d}|}$*, denote*

$$V_{\varpi,k} := \langle e_{\varpi(1)}, \dots e_{\varpi(k)} \rangle \subseteq V.$$

We define Lie subalgebras of $\mathrm{Rep}_{\mathbf{d}}(Q)$ *as follows.*

$$\mathfrak{r}_{\varpi} := \left\{ (f_a)_{a \in a(Q)} \in \mathrm{Rep}_{\mathbf{d}}(Q) \,\middle|\, f_a(V_{\varpi,k} \cap V_{s(a)}) \subseteq V_{\varpi,k} \text{ for any } k \right\},$$
$$\mathfrak{r}_{\varpi,\varpi''} := \mathfrak{r}_{\varpi} \cap \mathfrak{r}_{\varpi''},$$
$$\mathfrak{d}_{\varpi,\varpi''} := \mathfrak{r}_{\varpi} / \mathfrak{r}_{\varpi,\varpi''},$$

Example 1.4.9 (Follows Example 1.4.7)**.** *Consider the quiver* $\color{red}{\bullet} \longrightarrow \color{blue}{\bullet}$ *, and* $u =$ *. Table 1.4 gives us an example of the shape of these Lie algebras. Symbols like* $\frac{e_1}{e_2}$ *will be explained in Example 2.1.4.*

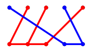
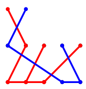
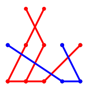
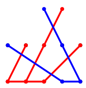
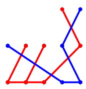
		$\mathfrak{n}_u$	$\mathfrak{m}_{u,u}$		$\mathfrak{r}_u$	$\mathfrak{d}_{u,u}$	
	$u =$ 	$\left[\begin{array}{ccc\|cc} & * & * & & \\ & & * & & \\ & & & & \\ \hline & & & & * \\ & & & & \end{array}\right]$	$\left[\begin{array}{ccc\|cc} & & & & \\ & & & & \\ & & & & \\ \hline & & & & \\ & & & & \end{array}\right]$		$\left[\begin{array}{ccc\|cc} & & & & \\ & & & & \\ & & & & \\ \hline * & * & * & & \\ & & * & & \end{array}\right]$	$\left[\begin{array}{ccc\|cc} & & & & \\ & & & & \\ & & & & \\ \hline & & & & \\ & & & & \end{array}\right]$	
s	cases	$\mathfrak{n}_{us}$	$\mathfrak{m}_{u,us}$		$\mathfrak{r}_{us}$	$\mathfrak{d}_{u,us}$	
$s = s_1$	$us_1 =$  $\notin W_{\mathbf{d}}u$	$\left[\begin{array}{ccc\|cc} & * & * & & \\ & & * & & \\ & & & & \\ \hline & & & & * \\ & & & & \end{array}\right]$	$\left[\begin{array}{ccc\|cc} & & & & \\ & & & & \\ & & & & \\ \hline & & & & \\ & & & & \end{array}\right]$		$\left[\begin{array}{ccc\|cc} & & & & \\ & & & & \\ & & & & \\ \hline & * & * & & \\ & & * & & \end{array}\right]$	$\left[\begin{array}{ccc\|cc} & & & & \\ & & & & \\ & & & & \\ \hline * & & & & \\ & & & & \end{array}\right]$	$\frac{e_4}{e_1}$
$s = s_2$	$us_2 =$  $\in W_{\mathbf{d}}u$	$\left[\begin{array}{ccc\|cc} & & * & & \\ * & & * & & \\ & & & & \\ \hline & & & & * \\ & & & & \end{array}\right]$	$\left[\begin{array}{ccc\|cc} & * & & & \\ & & & & \\ & & & & \\ \hline & & & & \\ & & & & \end{array}\right]$	$\frac{e_1}{e_2}$	$\left[\begin{array}{ccc\|cc} & & & & \\ & & & & \\ & & & & \\ \hline * & * & * & & \\ & & * & & \end{array}\right]$	$\left[\begin{array}{ccc\|cc} & & & & \\ & & & & \\ & & & & \\ \hline & & & & \\ & & & & \end{array}\right]$	
$s = s_3$	$us_3 =$  $\notin W_{\mathbf{d}}u$	$\left[\begin{array}{ccc\|cc} & * & * & & \\ & & * & & \\ & & & & \\ \hline & & & & * \\ & & & & \end{array}\right]$	$\left[\begin{array}{ccc\|cc} & & & & \\ & & & & \\ & & & & \\ \hline & & & & \\ & & & & \end{array}\right]$		$\left[\begin{array}{ccc\|cc} & & & & \\ & & & & \\ & & & & \\ \hline * & * & * & & \\ & * & * & & \end{array}\right]$	$\left[\begin{array}{ccc\|cc} & & & & \\ & & & & \\ & & & & \\ \hline & & & & \\ & & & & \end{array}\right]$	
$s = s_4$	$us_4 =$  $\notin W_{\mathbf{d}}u$	$\left[\begin{array}{ccc\|cc} & * & * & & \\ & & * & & \\ & & & & \\ \hline & & & & * \\ & & & & \end{array}\right]$	$\left[\begin{array}{ccc\|cc} & & & & \\ & & & & \\ & & & & \\ \hline & & & & \\ & & & & \end{array}\right]$		$\left[\begin{array}{ccc\|cc} & & & & \\ & & & & \\ & & & & \\ \hline * & * & * & & \\ & & & & \end{array}\right]$	$\left[\begin{array}{ccc\|cc} & & & & \\ & & & & \\ & & & & \\ \hline & & & & \\ & & * & & \end{array}\right]$	$\frac{e_5}{e_3}$

Table 1.4: examples of Lie algebras

Remark 1.4.10. We also have twisted notation for Lie algebras. For example,

$$\underline{\mathfrak{n}}_{\varpi,\varpi'} = \mathfrak{n}_{\varpi,\varpi\varpi'}, \quad \underline{\mathfrak{m}}_{\varpi,\varpi'} = \mathfrak{m}_{\varpi,\varpi\varpi'}, \quad \underline{\mathfrak{p}}_{\varpi,s} = \mathfrak{p}_{\varpi,\varpi s},$$
$$\underline{\mathfrak{r}}_{\varpi,\varpi'} = \mathfrak{r}_{\varpi,\varpi\varpi'}, \quad \underline{\mathfrak{d}}_{\varpi,\varpi'} = \mathfrak{d}_{\varpi,\varpi\varpi'}.$$

Another twist happens when we add minus sign as the superscript:

$$\begin{aligned} \mathfrak{b}^-_{\varpi} &= \mathfrak{b}_{\varpi_{\max}\varpi}, \\ \mathfrak{b}^-_{\varpi} &= \mathfrak{b}_{w_{\max}\varpi}, & \mathfrak{n}^-_{\varpi} &= \mathfrak{n}_{w_{\max}\varpi}, \\ \mathfrak{n}^-_{\varpi,\varpi''} &= \mathfrak{n}_{w_{\max}\varpi,w_{\max}\varpi''}, & \mathfrak{m}^-_{\varpi,\varpi''} &= \mathfrak{m}_{w_{\max}\varpi,w_{\max}\varpi''}. \end{aligned}$$

1.5 Quiver flag varieties

In this section, we define quiver flag varieties we care about in the same spirit as Section 1.1. Their stratifications and related "Schubert" varieties will be defined in Section 1.6.

Recall Setting 1.1 and Definition 1.2.10.

1.5.1 Flag variety

Definition 1.5.1 (Absolute complete flag variety)**.** *The absolute complete flag variety $\mathcal{F}_{|\mathbf{d}|}$ is defined as*

$$\begin{aligned} \mathcal{F}_{|\mathbf{d}|} &= \mathbb{G}_{|\mathbf{d}|}/\mathbb{B}_{|\mathbf{d}|} \\ &\cong \left\{ \textit{complete flags of } \mathbb{C}^{|\mathbf{d}|} \right\} \\ &= \left\{ 0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_{|\mathbf{d}|} = \mathbb{C}^{|\mathbf{d}|} \;\middle|\; \dim M_k = k \right\} \end{aligned}$$

$$\begin{aligned}&\cong \left\{\textit{Borel subgroups of } \mathbb{G}_{|\mathbf{d}|}\right\}\\&= \left\{g\mathbb{B}_{|\mathbf{d}|}g^{-1} \;\middle|\; g \in \mathbb{G}_{|\mathbf{d}|}\right\}\end{aligned}$$

Here, M_i can have no Q-vector space structure.

Definition 1.5.2 (complete flag variety with flag-type dimension vector)**.** *For a complete flag-type dimension vector $\underline{\mathbf{d}}$, the flag variety $\mathcal{F}_{\underline{\mathbf{d}}}$ is defined as*

$$\begin{aligned}\mathcal{F}_{\underline{\mathbf{d}}} &= \left\{\textit{complete flags of } V = \bigoplus_{i\in v(Q)} V_i \textit{ with dimension vector } \underline{\mathbf{d}}\right\}\\&= \left\{F : 0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_{|\mathbf{d}|} = V \;\middle|\; \underline{\mathbf{dim}}\, F = \underline{\mathbf{d}}\right\}\end{aligned}$$

Definition 1.5.3 (Relative complete flag variety)**.** *The relative complete flag variety $\mathcal{F}_{\mathbf{d}}$ is defined as*

$$\begin{aligned}\mathcal{F}_{\mathbf{d}} &= \left\{\textit{complete flags of } V = \bigoplus_{i\in v(Q)} V_i\right\}\\&= \left\{0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_{|\mathbf{d}|} = V \;\middle|\; |\underline{\mathbf{dim}}\, M_k| = k\right\}\\&= \bigsqcup_{\underline{\mathbf{d}}} \mathcal{F}_{\underline{\mathbf{d}}}\end{aligned}$$

Here, M_i are Q-vector spaces.

Remark 1.5.4.

1. $\mathcal{F}_{|\mathbf{d}|}$, $\mathcal{F}_{\underline{\mathbf{d}}}$ and $\mathcal{F}_{\mathbf{d}}$ are smooth varieties, since

$$\mathcal{F}_{|\mathbf{d}|} \cong \mathrm{GL}_{|\mathbf{d}|}/B \qquad \mathcal{F}_{\underline{\mathbf{d}}} \cong \prod_{i\in v(Q)} \mathrm{GL}_{\mathbf{d}_i}/B$$

   are products of usual flag varieties.

2. $\mathcal{F}_{|\mathbf{d}|}$ is an $\mathbb{G}_{|\mathbf{d}|}$-variety, while $\mathcal{F}_{\underline{\mathbf{d}}}$, $\mathcal{F}_{\mathbf{d}}$ are $G_{\mathbf{d}}$-varieties. The actions are induced by the actions on the vector space V.

We need to simplify our notation of flags.

Definition 1.5.5 (Coordinate flags and related flags)**.** *For a basis $\{x_1, \dots, x_{|\mathbf{d}|}\}$, define the flag*

$$F_{\{x_1,\dots,x_{|\mathbf{d}|}\}} : 0 \subseteq \langle x_1\rangle \subseteq \langle x_1, x_2\rangle \subseteq \cdots \subseteq \left\langle x_1, \dots x_{|\mathbf{d}|}\right\rangle = V.$$

For $g \in \mathbb{G}_{|\mathbf{d}|}$, $\varpi \in \mathbb{W}_{|\mathbf{d}|}$, define

$$\begin{aligned}F_{\mathrm{Id}} &= F_{\{v_1,\dots,v_{|\mathbf{d}|}\}} && \in \mathcal{F}_{\mathbf{d}}\\F_g &= gF_{\mathrm{Id}} = F_{\{gv_1,\dots,gv_{|\mathbf{d}|}\}} && \in \mathcal{F}_{|\mathbf{d}|}\\F_\varpi &= \varpi F_{\mathrm{Id}} = F_{\{v_{\varpi(1)},\dots,v_{\varpi(|\mathbf{d}|)}\}} && \in \mathcal{F}_{\mathbf{d}}\end{aligned}$$

F_{Id} is called the **standard flag** of V.

Now we can define flag varieties attached to $\varpi \in \mathbb{W}_{|\mathbf{d}|}$.

Definition 1.5.6. *For $\varpi = wu \in \mathbb{W}_{|\mathbf{d}|}$, define $\mathcal{F}_\varpi$ as the $G_\mathbf{d}$-orbit of F_ϖ. By the orbit-stabilizer theorem,*

$$\mathcal{F}_\varpi \cong G_\mathbf{d}/B_\varpi.$$

We can generalize it a little bit: for $g \in G_\mathbf{d}$, $F_{g\varpi} \in \mathcal{F}_\mathbf{d}$,

$$\mathcal{F}_{g\varpi} := G_\mathbf{d} \cdot F_{g\varpi} \cong G_\mathbf{d}/B_{g\varpi} = G_\mathbf{d}/gB_\varpi g^{-1}.$$

Remark 1.5.7. F_ϖ is the preferred base point of $\mathcal{F}_\varpi$. Ignoring the base point,

$$\mathcal{F}_\varpi = \mathcal{F}_u = \mathcal{F}_{\underline{\mathbf{d}}} \quad \text{for } \varpi = wu, \quad \underline{\mathbf{d}} = W_\mathbf{d}\varpi.$$

In fact, we are not defining new varieties; we give old varieties new names, so that we can manipulate them more freely.

Like Section 1.1, we also consider the product of two flag varieties. For $g, g', g'' \in \mathbb{G}_{|\mathbf{d}|}$, $\varpi, \varpi', \varpi'' \in \mathbb{W}_{|\mathbf{d}|}$, denote

$$\begin{aligned} F_{\mathrm{Id},\mathrm{Id}} &= (F_{\mathrm{Id}}, F_{\mathrm{Id}}) & & \\ F_{g,g''} &= (F_g, F_{g''}) & \underline{F}_{g,g'} &= F_{g,gg'} = (F_g, F_{gg'}) \\ F_{\varpi,\varpi''} &= (F_\varpi, F_{\varpi''}) & \underline{F}_{\varpi,\varpi'} &= F_{\varpi,\varpi\varpi'} = (F_\varpi, F_{\varpi\varpi'}) \end{aligned}$$

Table 1.5 concludes all varieties we get until now.

	base point		base point
$\mathcal{F}_{\lvert\mathbf{d}\rvert} \cong \mathbb{G}_{\lvert\mathbf{d}\rvert}/\mathbb{B}_{\lvert\mathbf{d}\rvert}$	F_{Id}	$\mathcal{F}_{\lvert\mathbf{d}\rvert} \times \mathcal{F}_{\lvert\mathbf{d}\rvert}$	$F_{\mathrm{Id},\mathrm{Id}}$
$\mathcal{F}_{\underline{\mathbf{d}}} \cong G_\mathbf{d}/B_\mathbf{d}$	F_u	$\mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}}'}$	$F_{u,u'}$
$\mathcal{F}_\varpi \cong G_\mathbf{d}/B_\varpi$	F_ϖ	$\mathcal{F}_\varpi \times \mathcal{F}_{\varpi'}$	$F_{\varpi,\varpi'}$
$\mathcal{F}_\mathbf{d} = \bigsqcup_{\underline{\mathbf{d}}} \mathcal{F}_{\underline{\mathbf{d}}}$	–	$\mathcal{F}_\mathbf{d} \times \mathcal{F}_\mathbf{d} = \bigsqcup_{\underline{\mathbf{d}},\underline{\mathbf{d}}'} \mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}}'}$	–

Table 1.5: Base varieties and their preferred base point

1.5.2 Incidence variety

Now it is time to include information about arrows.

Definition 1.5.8 (Incidence variety)**.** *For a quiver Q with complete flag-type dimension vector $\underline{\mathbf{d}}$, define*

$$\begin{aligned} \widetilde{\mathrm{Rep}}_{\underline{\mathbf{d}}}(Q) &:= \left\{(\rho, F) \in \mathrm{Rep}_\mathbf{d}(Q) \times \mathcal{F}_{\underline{\mathbf{d}}} \,\middle|\, \rho(M_k) \subseteq M_k \text{ for any } k\right\} \\ \widetilde{\mathrm{Rep}}_\mathbf{d}(Q) &:= \{(\rho, F) \in \mathrm{Rep}_\mathbf{d}(Q) \times \mathcal{F}_\mathbf{d} \mid \rho(M_k) \subseteq M_k \text{ for any } k\} = \bigsqcup_{\underline{\mathbf{d}}} \widetilde{\mathrm{Rep}}_{\underline{\mathbf{d}}}(Q) \end{aligned}$$

and $\mu_{\underline{\mathbf{d}}}$, $\pi_{\underline{\mathbf{d}}}$, $\mu_\mathbf{d}$, $\pi_\mathbf{d}$ to be the natural morphisms from the incidence varieties to $\mathrm{Rep}_\mathbf{d}(Q)$ *or flag varieties, as follows:*

$$\begin{tikzcd} \widetilde{\mathrm{Rep}}_{\underline{\mathbf{d}}}(Q) \subseteq \mathrm{Rep}_\mathbf{d}(Q) \times \mathcal{F}_{\underline{\mathbf{d}}} \arrow[d, bend left, "\mu_{\underline{\mathbf{d}}}"'] \arrow[dr, bend left, "\pi_{\underline{\mathbf{d}}}"] & \\ \mathrm{Rep}_\mathbf{d}(Q) & \mathcal{F}_{\underline{\mathbf{d}}} \end{tikzcd} \qquad \begin{tikzcd} \widetilde{\mathrm{Rep}}_\mathbf{d}(Q) \subseteq \mathrm{Rep}_\mathbf{d}(Q) \times \mathcal{F}_\mathbf{d} \arrow[d, bend left, "\mu_\mathbf{d}"'] \arrow[dr, bend left, "\pi_\mathbf{d}"] & \\ \mathrm{Rep}_\mathbf{d}(Q) & \mathcal{F}_\mathbf{d} \end{tikzcd}$$

Remark 1.5.9. Fix $X \in \mathrm{Rep}_{\mathbf{d}}(Q)$, the flag variety associated to X

$$\mathrm{Flag}_{\underline{\mathbf{d}}}(X) := \mu_{\underline{\mathbf{d}}}^{-1}(X) \cong \pi_{\underline{\mathbf{d}}}\big(\mu_{\underline{\mathbf{d}}}^{-1}(X)\big) \subseteq \mathcal{F}_{\underline{\mathbf{d}}}$$

records the complete flags of subrepresentations of X. The partial flag variety version of $\mathrm{Flag}_{\underline{\mathbf{d}}}(X)$ will become the key object in Part II.

Definition 1.5.10 (Steinberg variety)**.** *For a quiver Q with complete flag-type dimension vectors $\underline{\mathbf{d}}$, $\underline{\mathbf{d}}'$, define*

$$\begin{aligned}
\mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}'} &:= \widetilde{\mathrm{Rep}}_{\underline{\mathbf{d}}}(Q) \times_{\mathrm{Rep}_{\mathbf{d}}(Q)} \widetilde{\mathrm{Rep}}_{\underline{\mathbf{d}}'}(Q) \\
\mathcal{Z}_{\mathbf{d}} &:= \widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q) \times_{\mathrm{Rep}_{\mathbf{d}}(Q)} \widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q) \\
&= \bigsqcup_{\underline{\mathbf{d}},\underline{\mathbf{d}}'} \mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}'}
\end{aligned}$$

*$\mathcal{Z}_{\mathbf{d}}$ is called the **Steinberg variety**.*

$\mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}'}$ can actually be realized as the incidence variety between $\mathrm{Rep}_{\mathbf{d}}(Q)$ and $\mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}}'}$, since

$$\begin{aligned}
\mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}'} &= \widetilde{\mathrm{Rep}}_{\underline{\mathbf{d}}}(Q) \times_{\mathrm{Rep}_{\mathbf{d}}(Q)} \widetilde{\mathrm{Rep}}_{\underline{\mathbf{d}}'}(Q) \\
&\subseteq \big(\mathrm{Rep}_{\mathbf{d}}(Q) \times \mathcal{F}_{\underline{\mathbf{d}}}\big) \times_{\mathrm{Rep}_{\mathbf{d}}(Q)} \big(\mathrm{Rep}_{\mathbf{d}}(Q) \times \mathcal{F}_{\underline{\mathbf{d}}'}\big) \\
&\cong \mathrm{Rep}_{\mathbf{d}}(Q) \times \mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}}'}
\end{aligned}$$

For that reason, we define $\mu_{\underline{\mathbf{d}},\underline{\mathbf{d}}'}$, $\pi_{\underline{\mathbf{d}},\underline{\mathbf{d}}'}$, $\mu_{\mathbf{d},\mathbf{d}}$, $\pi_{\mathbf{d},\mathbf{d}}$ as natural morphisms from $\mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}'}$, $\mathcal{Z}_{\mathbf{d}}$ to $\mathrm{Rep}_{\mathbf{d}}(Q)$ or product of flag varieties, as follows:

$$\begin{tikzcd}
& \mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}'} \subseteq \mathrm{Rep}_{\mathbf{d}}(Q) \times \mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}}'} \arrow[dl, bend left, "\mu_{\underline{\mathbf{d}},\underline{\mathbf{d}}'}"'] \arrow[dr, bend left, "\pi_{\underline{\mathbf{d}},\underline{\mathbf{d}}'}"] & \\
\mathrm{Rep}_{\mathbf{d}}(Q) & & \mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}}'}
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathcal{Z}_{\mathbf{d},\mathbf{d}'} \subseteq \mathrm{Rep}_{\mathbf{d}}(Q) \times \mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}'} \arrow[dl, bend left, "\mu_{\mathbf{d},\mathbf{d}'}"'] \arrow[dr, bend left, "\pi_{\mathbf{d},\mathbf{d}'}"] & \\
\mathrm{Rep}_{\mathbf{d}}(Q) & & \mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}'}
\end{tikzcd}$$

Remark 1.5.11 (Group actions).

1. $\mathrm{Rep}_{\mathbf{d}}(Q) \subseteq \oplus_{r\in a(Q)} \mathfrak{g}_{|\mathbf{d}|}$ has a natural $G_{\mathbf{d}}$-action, which is induced by the conjugation action of $G_{\mathbf{d}}$ on $\mathfrak{g}_{|\mathbf{d}|}$. We have already mentioned the $G_{\mathbf{d}}$-action on $\mathcal{F}_{\underline{\mathbf{d}}}$ and $\mathcal{F}_{\mathbf{d}}$ in Remark 1.5.4. Therefore, by restriction we automatically get $G_{\mathbf{d}}$-actions on $\widetilde{\mathrm{Rep}}_{\underline{\mathbf{d}}}(Q)$, $\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)$, $\mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}'}$ and $\mathcal{Z}_{\mathbf{d}}$. All the maps we mentioned in Definition 1.5.8 are $G_{\mathbf{d}}$-equivariant.

2. In Subsection 6.1.2 we will also view all the varieties as $G_{\mathbf{d}} \times \mathbb{C}^{\times}$-varieties, so we also shortly introduce $\mathbb{C}^{\times}$-action here. View $\mathrm{Rep}_{\mathbf{d}}(Q)$ as a $\mathbb{C}$-vector space, $\mathbb{C}^{\times}$ acts on $\mathrm{Rep}_{\mathbf{d}}(Q)$ by scalar multiplication. For $\mathcal{F}_{\underline{\mathbf{d}}}$ and $\mathcal{F}_{\mathbf{d}}$, $\mathbb{C}^{\times}$ acts trivially, and by restriction we get $\mathbb{C}^{\times}$-actions on $\widetilde{\mathrm{Rep}}_{\underline{\mathbf{d}}}(Q)$, $\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)$, $\mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}'}$ and $\mathcal{Z}_{\mathbf{d}}$. Also, all the maps we mentioned above are $\mathbb{C}^{\times}$-equivariant.

3. It may be worth mentioning that $\mathcal{F}_{\mathbf{d}}$ has an $\mathbb{W}_{|\mathbf{d}|}$-action which can be extended neither to $\mathbb{G}_{|\mathbf{d}|}$-action on $\mathcal{F}_{\mathbf{d}}$ nor to $\mathbb{W}_{|\mathbf{d}|}$-action on $\widetilde{\mathrm{Rep}}_{\underline{\mathbf{d}}}(Q)$.

1.6 Stratifications and T-fixed points

In this subsection, we will find stratifications of varieties introduced in Section 1.5, and fix notation of orbits. We will also discuss their T-fixed points. These stratifications will give us a basis for the K-theory and cohomology in Chapter 2, while those T-fixed points will give us another "basis" in Chapter 4.

1.6.1 Stratifications: flag varieties

We begin with $\mathcal{F}_{|\mathbf{d}|}$ and $\mathcal{F}_{|\mathbf{d}|} \times \mathcal{F}_{|\mathbf{d}|}$, which is roughly a repetition of Section 1.1.

Definition 1.6.1 (Twisted action)**.** *We define the twisted $\mathbb{G}_{|\mathbf{d}|} \times \mathbb{G}_{|\mathbf{d}|}$-action on $\mathcal{F}_{|\mathbf{d}|} \times \mathcal{F}_{|\mathbf{d}|}$:*

$$\mathbb{G}_{|\mathbf{d}|} \times \mathbb{G}_{|\mathbf{d}|} \times \mathcal{F}_{|\mathbf{d}|} \times \mathcal{F}_{|\mathbf{d}|} \longrightarrow \mathcal{F}_{|\mathbf{d}|} \times \mathcal{F}_{|\mathbf{d}|} \qquad \left(g_1, g_2, \underline{F}_{g,g'}\right) \longmapsto \underline{F}_{g_1 g, g_2 g'}$$

which is the same as original $\mathbb{G}_{|\mathbf{d}|}$-action when we restrict to $\mathbb{G}_{|\mathbf{d}|} \times \{\mathrm{Id}\}$-action. Other $G \times G$-actions on $\mathcal{F} \times \mathcal{F}$ are defined in a similar way.

Definition 1.6.2 (Stratifications of $\mathcal{F}_{|\mathbf{d}|}$ and $\mathcal{F}_{|\mathbf{d}|} \times \mathcal{F}_{|\mathbf{d}|}$)**.** *For $\varpi, \varpi' \in \mathbb{W}_{|\mathbf{d}|}$, we define*

$$\begin{aligned} \mathcal{V}_{\varpi} &= \mathbb{B}_{|\mathbf{d}|} \cdot F_{\varpi} && \subseteq \mathcal{F}_{|\mathbf{d}|} \\ \boldsymbol{\mathcal{V}}_{\varpi,\varpi'} &= \left(\mathbb{B}_{|\mathbf{d}|} \times \mathbb{B}_{|\mathbf{d}|}\right) \cdot \underline{F}_{\varpi,\varpi'} && \subseteq \mathcal{F}_{|\mathbf{d}|} \times \mathcal{F}_{|\mathbf{d}|} \\ \boldsymbol{\mathcal{V}}_{\varpi'} &= \mathbb{G}_{|\mathbf{d}|} \cdot \underline{F}_{\mathrm{Id},\varpi'} && \subseteq \mathcal{F}_{|\mathbf{d}|} \times \mathcal{F}_{|\mathbf{d}|} \end{aligned}$$

as $\mathbb{B}_{|\mathbf{d}|}$-orbit, $\mathbb{B}_{|\mathbf{d}|} \times \mathbb{B}_{|\mathbf{d}|}$-orbit, $\mathbb{G}_{|\mathbf{d}|}$-orbit of $\mathcal{F}_{|\mathbf{d}|}$, $\mathcal{F}_{|\mathbf{d}|} \times \mathcal{F}_{|\mathbf{d}|}$ and $\mathcal{F}_{|\mathbf{d}|} \times \mathcal{F}_{|\mathbf{d}|}$, respectively.

By Bruhat-decomposition, we are able to show

$$\mathcal{F}_{|\mathbf{d}|} = \bigsqcup_{\varpi} \mathcal{V}_{\varpi} \qquad \mathcal{F}_{|\mathbf{d}|} \times \mathcal{F}_{|\mathbf{d}|} = \bigsqcup_{\varpi'} \boldsymbol{\mathcal{V}}_{\varpi'} = \bigsqcup_{\varpi,\varpi'} \boldsymbol{\mathcal{V}}_{\varpi,\varpi'}.$$

We also realize these orbits as quotients of algebraic groups by the orbit-stabilizer theorem, as follows:

$$\begin{aligned} \mathcal{V}_{\varpi} &\cong \mathbb{B}_{|\mathbf{d}|} / \left(\mathbb{B}_{|\mathbf{d}|} \cap \mathbb{B}_{\varpi}\right) && \cong \mathbb{A}^{l(\varpi)} \\ \boldsymbol{\mathcal{V}}_{\varpi,\varpi'} &\cong \left(\mathbb{B}_{|\mathbf{d}|} \times \mathbb{B}_{|\mathbf{d}|}\right) / \left(\mathbb{B}_{|\mathbf{d}|} \cap \mathbb{B}_{\varpi} \times \mathbb{B}_{|\mathbf{d}|} \cap \mathbb{B}_{\varpi'}\right) && \cong \mathbb{A}^{l(\varpi)+l(\varpi')} \\ \boldsymbol{\mathcal{V}}_{\varpi'} &\cong \mathbb{G}_{|\mathbf{d}|} / \left(\mathbb{B}_{|\mathbf{d}|} \cap \mathbb{B}_{\varpi'}\right) && \cong \mathbb{A}^{l(\varpi')}\text{-bundle over } \mathcal{F}_{|\mathbf{d}|} \end{aligned}$$

Similar stratifications happen for $\mathcal{F}_u$ and $\mathcal{F}_{\mathbf{d}}$.

Definition 1.6.3 (Stratifications of $\mathcal{F}_u$ and $\mathcal{F}_u \times \mathcal{F}_{u'}$)**.** *For $u, u' \in \mathrm{Min}(\mathbb{W}_{|\mathbf{d}|}, \mathrm{W}_{\mathbf{d}})$, $w, w' \in W_{\mathbf{d}}$, we define*

$$\begin{aligned} \Omega_w^u &= B_{\mathbf{d}} \cdot F_{wu} && \subseteq \mathcal{F}_u \\ \boldsymbol{\Omega}_{w,w'}^{u,u'} &= \left(B_{\mathbf{d}} \times B_{\mathbf{d}}\right) \cdot \left(F_{wu}, F_{ww'u'}\right) && \subseteq \mathcal{F}_u \times \mathcal{F}_{u'} \\ \boldsymbol{\Omega}_{w'}^{u,u'} &= G_{\mathbf{d}} \cdot \left(F_u, F_{w'u'}\right) && \subseteq \mathcal{F}_u \times \mathcal{F}_{u'} \end{aligned}$$

as $B_{\mathbf{d}}$-orbit, $B_{\mathbf{d}} \times B_{\mathbf{d}}$-orbit, $G_{\mathbf{d}}$-orbit of $\mathcal{F}_u$, $\mathcal{F}_u \times \mathcal{F}_{u'}$ and $\mathcal{F}_u \times \mathcal{F}_{u'}$, respectively.

By Bruhat decomposition, we are again able to show

$$\mathcal{F}_u = \bigsqcup_{w} \Omega_w^u \qquad \mathcal{F}_u \times \mathcal{F}_{u'} = \bigsqcup_{w'} \boldsymbol{\Omega}_{w'}^{u,u'} = \bigsqcup_{w,w'} \boldsymbol{\Omega}_{w,w'}^{u,u'}$$

and

$$\begin{aligned} \Omega_w^u &\cong B_{\mathbf{d}} / \left(B_{\mathbf{d}} \cap B_w\right) && \cong \mathbb{A}^{l(w)} \\ \boldsymbol{\Omega}_{w,w'}^{u,u'} &\cong \left(B_{\mathbf{d}} \times B_{\mathbf{d}}\right) / \left(B_{\mathbf{d}} \cap B_w \times B_{\mathbf{d}} \cap B_{w'}\right) && \cong \mathbb{A}^{l(w)+l(w')} \\ \boldsymbol{\Omega}_{w'}^{u,u'} &\cong G_{\mathbf{d}} / \left(B_{\mathbf{d}} \cap B_{w'}\right) && \cong \mathbb{A}^{l(w')}\text{-bundle over } \mathcal{F}_u \end{aligned}$$

Definition 1.6.4 (Stratifications of $\mathcal{F}_{\mathbf{d}}$ and $\mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}}$). *For $\varpi, \varpi' \in \mathbb{W}_{|\mathbf{d}|}$, we define*

$$\begin{aligned} \mathcal{O}_{\varpi} &= B_{\mathbf{d}} \cdot F_{\varpi} && \subseteq \mathcal{F}_{\varpi} && \subseteq \mathcal{F}_{\mathbf{d}} \\ \boldsymbol{\mathcal{O}}_{\varpi,\varpi'} &= (B_{\mathbf{d}} \times B_{\mathbf{d}}) \cdot \underline{F}_{\varpi,\varpi'} && \subseteq \mathcal{F}_{\varpi} \times \mathcal{F}_{\varpi\varpi'} && \subseteq \mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}} \\ \boldsymbol{\mathcal{O}}_{\varpi'} &= \bigsqcup_u G_{\mathbf{d}} \cdot \underline{F}_{u,\varpi'} && \subseteq \bigsqcup_u \mathcal{F}_u \times \mathcal{F}_{u\varpi'} && \subseteq \mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}} \end{aligned}$$

as $B_{\mathbf{d}}$-orbit, $B_{\mathbf{d}} \times B_{\mathbf{d}}$-orbit, (union of) $G_{\mathbf{d}}$-orbit of $\mathcal{F}_{\mathbf{d}}$, $\mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}}$ and $\mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}}$, respectively.

Notice that $\mathcal{O}_{\varpi}$, $\boldsymbol{\mathcal{O}}_{\varpi,\varpi'}$, $\boldsymbol{\mathcal{O}}_{\varpi'}$ are preimages of $\mathcal{V}_{\varpi}$, $\boldsymbol{\mathcal{V}}_{\varpi,\varpi'}$, $\boldsymbol{\mathcal{V}}_{\varpi'}$ under the maps

$$\mathcal{F}_{\mathbf{d}} \hookrightarrow \mathcal{F}_{|\mathbf{d}|} \qquad \mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}} \hookrightarrow \mathcal{F}_{|\mathbf{d}|} \times \mathcal{F}_{|\mathbf{d}|}.$$

Therefore,

$$\mathcal{F}_{\mathbf{d}} = \bigsqcup_{\varpi} \mathcal{O}_{\varpi} \qquad \mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}} = \bigsqcup_{\varpi'} \boldsymbol{\mathcal{O}}_{\varpi'} = \bigsqcup_{\varpi,\varpi'} \boldsymbol{\mathcal{O}}_{\varpi,\varpi'}.$$

We still need to care about symbols. For $\varpi = wu$, $\varpi' = w'u'$, denote $uw'u'$ by $\tilde{w}\tilde{u}$ for some unique $\tilde{w} \in W_{\mathbf{d}}$, $\tilde{u} \in \mathrm{Min}(\mathbb{W}_{|\mathbf{d}|}, \mathrm{W}_{\mathbf{d}})$, then

$$\underline{F}_{\varpi,\varpi'} = (F_{\varpi}, F_{\varpi\varpi'}) = (F_{wu}, F_{wuw'u'}) = (F_{wu}, F_{w\tilde{w}\tilde{u}}) \in \mathcal{F}_u \times \mathcal{F}_{\tilde{u}}.$$

This incompatibility comes from our twisted $G_{\mathbf{d}} \times G_{\mathbf{d}}$-actions. In particular, denote

$$\boldsymbol{\mathcal{O}}^u_{\varpi'} := G_{\mathbf{d}} \cdot \underline{F}_{u,\varpi'} \subseteq \mathcal{F}_u \times \mathcal{F}_{\tilde{u}},$$

we have $\boldsymbol{\mathcal{O}}_{\varpi'} = \sqcup_u \boldsymbol{\mathcal{O}}^u_{\varpi'}$ and identifications

$$\mathcal{O}_{\varpi} = \Omega^u_w \qquad \boldsymbol{\mathcal{O}}_{\varpi,\varpi'} = \boldsymbol{\Omega}^{u,\tilde{u}}_{w,\tilde{w}} \qquad \boldsymbol{\mathcal{O}}^u_{\varpi'} = \boldsymbol{\Omega}^{u,\tilde{u}}_{\tilde{w}}. \tag{1.6.1}$$

After so much notation is introduced rapidly, an enlightening example is needed here.

Example 1.6.5 (Follows Example 1.3.8). *In Table 1.7, $\mathbb{W}_{|\mathbf{d}|} = S_3$, $W_{\mathbf{d}} = S_1 \times S_2$,*

$$\varpi = ts = t \cdot s, \qquad \varpi' = s = \mathrm{Id} \cdot s, \qquad \varpi\varpi' = t = t \cdot \mathrm{Id}.$$

$\mathcal{F}_{\mathbf{d}}$ has 3 connected components, each of them has 2 $B_{\mathbf{d}}$-orbits;
$\mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}}$ has 9 connected components, each of them has 4 $B_{\mathbf{d}} \times B_{\mathbf{d}}$-orbits.

We have given every orbit a name, and other spaces are finite union of these orbits. For example,

$$\begin{aligned} \boldsymbol{\mathcal{O}}_{ts,s} &= \boldsymbol{\Omega}^{s,\mathrm{Id}}_{t,\mathrm{Id}} \\ \boldsymbol{\mathcal{O}}^s_s &= \boldsymbol{\Omega}^{s,\mathrm{Id}}_{\mathrm{Id}} = \boldsymbol{\Omega}^{s,\mathrm{Id}}_{\mathrm{Id},\mathrm{Id}} \sqcup \boldsymbol{\Omega}^{s,\mathrm{Id}}_{t,\mathrm{Id}} \\ \boldsymbol{\mathcal{O}}_s &= \boldsymbol{\mathcal{O}}^s_s \sqcup \boldsymbol{\mathcal{O}}^{\mathrm{Id}}_s \sqcup \boldsymbol{\mathcal{O}}^{st}_s \\ &= \boldsymbol{\mathcal{O}}^s_s \sqcup \boldsymbol{\mathcal{O}}^{\mathrm{Id}}_s \sqcup \boldsymbol{\mathcal{O}}^{st}_s \\ &= \boldsymbol{\Omega}^{s,\mathrm{Id}}_{\mathrm{Id}} \sqcup \boldsymbol{\Omega}^{\mathrm{Id},s}_{\mathrm{Id}} \sqcup \boldsymbol{\Omega}^{st,st}_{\mathrm{Id}} \\ &= \boldsymbol{\Omega}^{s,\mathrm{Id}}_{\mathrm{Id},\mathrm{Id}} \sqcup \boldsymbol{\Omega}^{s,\mathrm{Id}}_{t,\mathrm{Id}} \sqcup \boldsymbol{\Omega}^{\mathrm{Id},s}_{\mathrm{Id},\mathrm{Id}} \sqcup \boldsymbol{\Omega}^{\mathrm{Id},s}_{t,\mathrm{Id}} \sqcup \boldsymbol{\Omega}^{st,st}_{\mathrm{Id},\mathrm{Id}} \sqcup \boldsymbol{\Omega}^{st,st}_{t,\mathrm{Id}} \end{aligned}$$

dim; $(B\times B)\cdot\underline{F}_{w,w'}$; $B\cdot F_{ww'}$ / $B\cdot F_w$	0 Ω_{Id}	1 Ω_t	1 Ω_s	2 Ω_{ts}	2 Ω_{st}	3 Ω_{sts}
0 Ω_{Id}	0 $\boldsymbol{\Omega}_{\mathrm{Id},\mathrm{Id}}$	1 $\boldsymbol{\Omega}_{\mathrm{Id},t}$	1 $\boldsymbol{\Omega}_{\mathrm{Id},s}$	2 $\boldsymbol{\Omega}_{\mathrm{Id},ts}$	2 $\boldsymbol{\Omega}_{\mathrm{Id},st}$	3 $\boldsymbol{\Omega}_{\mathrm{Id},sts}$
1 Ω_t	2 $\boldsymbol{\Omega}_{t,t}$	1 $\boldsymbol{\Omega}_{t,\mathrm{Id}}$	3 $\boldsymbol{\Omega}_{t,ts}$	2 $\boldsymbol{\Omega}_{t,s}$	4 $\boldsymbol{\Omega}_{t,sts}$	3 $\boldsymbol{\Omega}_{t,st}$
1 Ω_s	2 $\boldsymbol{\Omega}_{s,s}$	3 $\boldsymbol{\Omega}_{s,st}$	1 $\boldsymbol{\Omega}_{s,\mathrm{Id}}$	4 $\boldsymbol{\Omega}_{s,sts}$	2 $\boldsymbol{\Omega}_{s,t}$	3 $\boldsymbol{\Omega}_{s,ts}$
2 Ω_{ts}	4 $\boldsymbol{\Omega}_{ts,st}$	3 $\boldsymbol{\Omega}_{ts,s}$	5 $\boldsymbol{\Omega}_{ts,sts}$	2 $\boldsymbol{\Omega}_{ts,\mathrm{Id}}$	4 $\boldsymbol{\Omega}_{ts,ts}$	3 $\boldsymbol{\Omega}_{ts,t}$
2 Ω_{st}	4 $\boldsymbol{\Omega}_{st,ts}$	5 $\boldsymbol{\Omega}_{st,sts}$	3 $\boldsymbol{\Omega}_{st,t}$	4 $\boldsymbol{\Omega}_{st,st}$	2 $\boldsymbol{\Omega}_{st,\mathrm{Id}}$	3 $\boldsymbol{\Omega}_{st,s}$
3 Ω_{sts}	6 $\boldsymbol{\Omega}_{sts,sts}$	5 $\boldsymbol{\Omega}_{sts,ts}$	5 $\boldsymbol{\Omega}_{sts,st}$	4 $\boldsymbol{\Omega}_{sts,t}$	4 $\boldsymbol{\Omega}_{sts,s}$	3 $\boldsymbol{\Omega}_{sts,\mathrm{Id}}$

Table 1.6: stratifications of $\mathcal{F}\times\mathcal{F}$

shape; $(B_{\mathbf{d}}\times B_{\mathbf{d}})\cdot\underline{F}_{\varpi,\varpi'}$; $B_{\mathbf{d}}\cdot F_{\varpi\varpi'}$ / $B_{\mathbf{d}}\cdot F_{\varpi}$		$\mathcal{F}_{\mathrm{Id}}$		$\mathcal{F}_s$		$\mathcal{F}_{st}$	
		$\mathcal{O}_{\mathrm{Id}}$	$\mathcal{O}_t$	$\mathcal{O}_s$	$\mathcal{O}_{ts}$	$\mathcal{O}_{st}$	$\mathcal{O}_{sts}$
$\mathcal{F}_{\mathrm{Id}}$	$\mathcal{O}_{\mathrm{Id}}=\Omega^{\mathrm{Id}}_{\mathrm{Id}}$	$\boldsymbol{\Omega}^{\mathrm{Id},\mathrm{Id}}_{\mathrm{Id},\mathrm{Id}}$	$\boldsymbol{\Omega}^{\mathrm{Id},\mathrm{Id}}_{\mathrm{Id},t}$	$\boldsymbol{\Omega}^{\mathrm{Id},s}_{\mathrm{Id},\mathrm{Id}}$	$\boldsymbol{\Omega}^{\mathrm{Id},s}_{\mathrm{Id},t}$	$\boldsymbol{\Omega}^{\mathrm{Id},st}_{\mathrm{Id},\mathrm{Id}}$	$\boldsymbol{\Omega}^{\mathrm{Id},st}_{\mathrm{Id},t}$
	$\mathcal{O}_t=\Omega^{\mathrm{Id}}_t$	$\boldsymbol{\Omega}^{\mathrm{Id},\mathrm{Id}}_{t,t}$	$\boldsymbol{\Omega}^{\mathrm{Id},\mathrm{Id}}_{t,\mathrm{Id}}$	$\boldsymbol{\Omega}^{\mathrm{Id},s}_{t,t}$	$\boldsymbol{\Omega}^{\mathrm{Id},s}_{t,\mathrm{Id}}$	$\boldsymbol{\Omega}^{\mathrm{Id},st}_{t,t}$	$\boldsymbol{\Omega}^{\mathrm{Id},st}_{t,\mathrm{Id}}$
$\mathcal{F}_s$	$\mathcal{O}_s=\Omega^s_{\mathrm{Id}}$	$\boldsymbol{\Omega}^{s,\mathrm{Id}}_{\mathrm{Id},\mathrm{Id}}$	$\boldsymbol{\Omega}^{s,\mathrm{Id}}_{\mathrm{Id},t}$	$\boldsymbol{\Omega}^{s,s}_{\mathrm{Id},\mathrm{Id}}$	$\boldsymbol{\Omega}^{s,s}_{\mathrm{Id},t}$	$\boldsymbol{\Omega}^{s,st}_{\mathrm{Id},\mathrm{Id}}$	$\boldsymbol{\Omega}^{s,st}_{\mathrm{Id},t}$
	$\mathcal{O}_{ts}=\Omega^s_t$	$\boldsymbol{\Omega}^{s,\mathrm{Id}}_{t,t}$	$\boldsymbol{\Omega}^{s,\mathrm{Id}}_{t,\mathrm{Id}}$	$\boldsymbol{\Omega}^{s,s}_{t,t}$	$\boldsymbol{\Omega}^{s,s}_{t,\mathrm{Id}}$	$\boldsymbol{\Omega}^{s,st}_{t,t}$	$\boldsymbol{\Omega}^{s,st}_{t,\mathrm{Id}}$
$\mathcal{F}_{st}$	$\mathcal{O}_{ts}=\Omega^{st}_{\mathrm{Id}}$	$\boldsymbol{\Omega}^{st,\mathrm{Id}}_{\mathrm{Id},\mathrm{Id}}$	$\boldsymbol{\Omega}^{st,\mathrm{Id}}_{\mathrm{Id},t}$	$\boldsymbol{\Omega}^{st,s}_{\mathrm{Id},\mathrm{Id}}$	$\boldsymbol{\Omega}^{st,s}_{\mathrm{Id},t}$	$\boldsymbol{\Omega}^{st,st}_{\mathrm{Id},\mathrm{Id}}$	$\boldsymbol{\Omega}^{st,st}_{\mathrm{Id},t}$
	$\mathcal{O}_{sts}=\Omega^{st}_t$	$\boldsymbol{\Omega}^{st,\mathrm{Id}}_{t,t}$	$\boldsymbol{\Omega}^{st,\mathrm{Id}}_{t,\mathrm{Id}}$	$\boldsymbol{\Omega}^{st,s}_{t,t}$	$\boldsymbol{\Omega}^{st,s}_{t,\mathrm{Id}}$	$\boldsymbol{\Omega}^{st,st}_{t,t}$	$\boldsymbol{\Omega}^{st,st}_{t,\mathrm{Id}}$

Table 1.7: stratifications of $\mathcal{F}_{\mathbf{d}}\times\mathcal{F}_{\mathbf{d}}$

1.6.2 Stratifications: incidence varieties

Now comes the stratifications of incidence varieties. Those stratifications are produced by taking the preimage of stratifications on base spaces. They are relatively easy to obtain, while their closures are quite difficult to analyze.

Definition 1.6.6 (Stratifications of incidence varieties)**.** *For $\varpi = wu$, $\varpi' = w'u' \in \mathbb{W}_{|\mathbf{d}|}$, denote $\tilde{w}\tilde{u} = uwu'$, $\underline{\mathbf{d}} = W_{\mathbf{d}}u$, $\underline{\mathbf{d}}' = W_{\mathbf{d}}u'$, $\underline{\widetilde{\mathbf{d}}} = W_{\mathbf{d}}\tilde{u}$. We define*

$$\begin{aligned}
\widetilde{\Omega}^u_w &:= \pi_{\underline{\mathbf{d}}}^{-1}(\Omega^u_w) && \subseteq \widetilde{\mathrm{Rep}}_{\underline{\mathbf{d}}}(Q) & \widetilde{\mathcal{O}}_\varpi &:= \pi_{\mathbf{d}}^{-1}(\mathcal{O}_\varpi) && \subseteq \widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)\\
\widetilde{\boldsymbol{\Omega}}^{u,u'}_{w,w'} &:= \pi^{-1}_{\underline{\mathbf{d}},\underline{\mathbf{d}}'}(\boldsymbol{\Omega}^{u,u'}_{w,w'}) && \subseteq \mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}'} & \widetilde{\boldsymbol{\mathcal{O}}}_{\varpi,\varpi'} &:= \pi^{-1}_{\mathbf{d},\mathbf{d}'}(\boldsymbol{\mathcal{O}}_{\varpi,\varpi'}) && \subseteq \mathcal{Z}_{\mathbf{d}}\\
\widetilde{\boldsymbol{\Omega}}^{u,u'}_{w'} &:= \pi^{-1}_{\underline{\mathbf{d}},\underline{\mathbf{d}}'}(\boldsymbol{\Omega}^{u,u'}_{w'}) && \subseteq \mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}'} & \widetilde{\boldsymbol{\mathcal{O}}}_{\varpi'} &:= \pi^{-1}_{\mathbf{d},\mathbf{d}'}(\boldsymbol{\mathcal{O}}_{\varpi'}) && \subseteq \mathcal{Z}_{\mathbf{d}}\\
\widetilde{\boldsymbol{\mathcal{O}}}^u_{\varpi'} &:= \pi^{-1}_{\underline{\mathbf{d}},\underline{\widetilde{\mathbf{d}}}}(\boldsymbol{\mathcal{O}}^u_{\varpi'}) = \widetilde{\boldsymbol{\Omega}}^{u,\tilde{u}}_{\tilde{w}} && \subseteq \mathcal{Z}_{\underline{\mathbf{d}},\underline{\widetilde{\mathbf{d}}}}
\end{aligned}$$

It is not hard to see that they are stratifications:

$$\begin{aligned}
\widetilde{\mathrm{Rep}}_{\underline{\mathbf{d}}}(Q) &= \bigsqcup_{\varpi} \widetilde{\Omega}^u_w & \mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}'} &= \bigsqcup_{w} \widetilde{\boldsymbol{\Omega}}^{u,u'}_{w'} = \bigsqcup_{w,w'} \widetilde{\boldsymbol{\Omega}}^{u,u'}_{w,w'}\\
\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q) &= \bigsqcup_{\varpi} \widetilde{\mathcal{O}}_\varpi & \mathcal{Z}_{\mathbf{d}} &= \bigsqcup_{\varpi'} \widetilde{\boldsymbol{\mathcal{O}}}_{\varpi'} = \bigsqcup_{\varpi,\varpi'} \widetilde{\boldsymbol{\mathcal{O}}}_{\varpi,\varpi'}
\end{aligned}$$

Proposition 1.6.7. *Those stratifications are affine spaces over corresponding base spaces. To be precise,*

$$\begin{aligned}
\widetilde{\Omega}^u_w &= \mathfrak{r}_{wu}\text{-bundle over } \Omega^u_w & \widetilde{\mathcal{O}}_\varpi &= \mathfrak{r}_\varpi\text{-bundle over } \mathcal{O}_\varpi\\
\widetilde{\boldsymbol{\Omega}}^{u,u'}_{w,w'} &= \mathfrak{r}_{wu,ww'u'}\text{-bundle over } \boldsymbol{\Omega}^{u,u'}_{w,w'} & \widetilde{\boldsymbol{\mathcal{O}}}_{\varpi,\varpi'} &= \underline{\mathfrak{r}}_{\varpi,\varpi'}\text{-bundle over } \boldsymbol{\mathcal{O}}_{\varpi,\varpi'}\\
\widetilde{\boldsymbol{\Omega}}^{u,u'}_{w'} &= \mathfrak{r}_{u,w'u'}\text{-bundle over } \boldsymbol{\Omega}^{u,u'}_{w'} & \widetilde{\boldsymbol{\mathcal{O}}}_{\varpi'} &= \underline{\mathfrak{r}}_{\mathrm{Id},\varpi'}\text{-bundle over } \boldsymbol{\mathcal{O}}_{\varpi'}\\
\widetilde{\boldsymbol{\mathcal{O}}}^u_{\varpi'} &= \underline{\mathfrak{r}}_{u,\varpi'}\text{-bundle over } \boldsymbol{\mathcal{O}}^u_{\varpi'}
\end{aligned}$$

Proof. The fibers are all computed over the preferred base point. The group action induces the isomorphism between different fibers, and lift affine local charts on base space (viewed as group quotient) to the local charts of fiber bundles. □

We end this subsection by Table 1.8:

1.6.3 Closure of cells

In this subsection we describe the closure of cells. We begin with $\boldsymbol{\Omega}$-cells:

$$\overline{\Omega}^u_w = \bigsqcup_{x \leqslant w} \Omega^u_x \qquad \overline{\boldsymbol{\Omega}}^{u,u'}_{w,w'} = \bigsqcup_{x \leqslant w, x' \leqslant w'} \boldsymbol{\Omega}^{u,u'}_{x,x'} \qquad \overline{\boldsymbol{\Omega}}^{u,u'}_{w'} = \bigsqcup_{x' \leqslant w'} \boldsymbol{\Omega}^{u,u'}_{x'}$$

Especially, for any $s \in \Pi_{\mathbf{d}}$, $u, u' \in \mathrm{Min}(\mathbb{W}_{|\mathbf{d}|}, \mathrm{W}_{\mathbf{d}})$, we have

$$\overline{\boldsymbol{\Omega}}^{u,u'}_s = \boldsymbol{\Omega}^{u,u'}_s \sqcup \boldsymbol{\Omega}^{u,u'}_{\mathrm{Id}} \cong G_{\mathbf{d}} \times^{B_{\mathbf{d}}} (P_{\mathrm{Id},s}/B_{\mathbf{d}})$$

stratification / *type* *stabilizer* *variety* *base point*		B-orbit	$B\times B$-orbit twisted stabilizer	G-orbit	Remark
$\mathcal{F}$ F_g	$\mathcal{F}\times\mathcal{F}$ $(F_g, F_{gg'})$	Ω_g $B\cap gBg^{-1}$	$\boldsymbol{\Omega}_{g,g'}$ $\left(B\cap gBg^{-1}\right)\times\left(B\cap g'Bg'^{-1}\right)$	$\boldsymbol{\Omega}_{g'}$ $gBg^{-1}\cap gg'B(gg')^{-1}$	
$\mathcal{F}_{\lvert\mathbf{d}\rvert}$ F_ϖ	$\mathcal{F}_{\lvert\mathbf{d}\rvert}\times\mathcal{F}_{\lvert\mathbf{d}\rvert}$ $(F_\varpi, F_{\varpi\varpi'})$	$\mathcal{V}_\varpi$ $\mathbb{B}_{\lvert\mathbf{d}\rvert}\cap\mathbb{B}_\varpi$	$\boldsymbol{\mathcal{V}}_{\varpi,\varpi'}$ $\left(\mathbb{B}_{\lvert\mathbf{d}\rvert}\cap\mathbb{B}_\varpi\right)\times\left(\mathbb{B}_{\lvert\mathbf{d}\rvert}\cap\mathbb{B}_{\varpi'}\right)$	$\boldsymbol{\mathcal{V}}_{\varpi'}$ $\mathbb{B}_\varpi\cap\mathbb{B}_{\varpi\varpi'}$	
$\mathcal{F}_u$ F_{wu}	$\mathcal{F}_u\times\mathcal{F}_{u'}$ $(F_{wu}, F_{ww'u'})$	Ω^u_w $B_\mathbf{d}\cap B_w$	$\boldsymbol{\Omega}^{u,u'}_{w,w'}$ $(B_\mathbf{d}\cap B_w)\times(B_\mathbf{d}\cap B_{w'})$	$\boldsymbol{\Omega}^{u,u'}_{w'}$ $B_w\cap B_{ww'}$	
$\mathcal{F}_\mathbf{d}$ F_ϖ F_{wu}	$\mathcal{F}_\mathbf{d}\times\mathcal{F}_\mathbf{d}$ $(F_\varpi, F_{\varpi\varpi'})$ $(F_{wu}, F_{w\tilde{w}\tilde{u}})$	Ω^u_w $B_\mathbf{d}\cap B_w$	$\boldsymbol{\Omega}^{u,\tilde{u}}_{w,\tilde{w}}$ $(B_\mathbf{d}\cap B_w)\times(B_\mathbf{d}\cap B_{\tilde{w}})$	$\boldsymbol{\mathcal{O}}^u_{\varpi'}=\boldsymbol{\Omega}^{u,\tilde{u}}_{\tilde{w}}$ $B_w\cap B_{w\tilde{w}}$	
The following may not be single orbit, but derived from the above definition.					
$\mathcal{F}_\mathbf{d}$ F_ϖ	$\mathcal{F}_\mathbf{d}\times\mathcal{F}_\mathbf{d}$ $(F_\varpi, F_{\varpi\varpi'})$	$\mathcal{O}_\varpi$ Ω^u_w	$\boldsymbol{\mathcal{O}}_{\varpi,\varpi'}$ $\boldsymbol{\Omega}^{u,\tilde{u}}_{w,\tilde{w}}$	$\boldsymbol{\mathcal{O}}_{\varpi'}$ $\sqcup_u\boldsymbol{\mathcal{O}}^u_{\varpi'}$	preimage of $\mathcal{F}_\mathbf{d}\times\mathcal{F}_\mathbf{d}\hookrightarrow\mathcal{F}_{\lvert\mathbf{d}\rvert}\times\mathcal{F}_{\lvert\mathbf{d}\rvert}$
$\widetilde{\mathrm{Rep}}_{\underline{\mathbf{d}}}(Q)$ F_{wu}	$\mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}'}$ $(F_{wu}, F_{ww'u'})$	$\widetilde{\Omega}^u_w$	$\widetilde{\boldsymbol{\Omega}}^{u,u'}_{w,w'}$	$\widetilde{\boldsymbol{\Omega}}^{u,u'}_{w'}$	preimage of $\mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}'}\hookrightarrow\mathcal{F}_{\underline{\mathbf{d}}}\times\mathcal{F}_{\underline{\mathbf{d}}'}$
$\widetilde{\mathrm{Rep}}_\mathbf{d}(Q)$ F_ϖ	$\mathcal{Z}_\mathbf{d}$ $(F_\varpi, F_{\varpi\varpi'})$	$\widetilde{\Omega}^u_w$	$\widetilde{\boldsymbol{\Omega}}^{u,\tilde{u}}_{w,\tilde{w}}$	$\widetilde{\boldsymbol{\mathcal{O}}}^u_{\varpi'}=\widetilde{\boldsymbol{\Omega}}^{u,\tilde{u}}_{\tilde{w}}$	preimage of $\mathcal{Z}_\mathbf{d}\hookrightarrow\mathcal{F}_\mathbf{d}\times\mathcal{F}_\mathbf{d}$
$\widetilde{\mathrm{Rep}}_\mathbf{d}(Q)$ F_ϖ	$\mathcal{Z}_\mathbf{d}$ $(F_\varpi, F_{\varpi\varpi'})$	$\widetilde{\mathcal{O}}_\varpi$ $\widetilde{\Omega}^u_w$	$\widetilde{\boldsymbol{\mathcal{O}}}_{\varpi,\varpi'}$ $\widetilde{\boldsymbol{\Omega}}^{u,\tilde{u}}_{w,\tilde{w}}$	$\widetilde{\boldsymbol{\mathcal{O}}}_{\varpi'}$ $\sqcup_u\widetilde{\boldsymbol{\mathcal{O}}}^u_{\varpi'}$	preimage of $\mathcal{Z}_\mathbf{d}\hookrightarrow\mathcal{F}_\mathbf{d}\times\mathcal{F}_\mathbf{d}$

Table 1.8: stratifications of typical varieties

when we work over base point $F_{u,u'}$. If we work over different base points, we will get different isomorphisms, as follows:

$$\begin{aligned}
\overline{\boldsymbol{\Omega}}^{u,u'}_s = \boldsymbol{\Omega}^{u,u'}_{\mathrm{Id}}\sqcup\boldsymbol{\Omega}^{u,u'}_s &\cong G_\mathbf{d}/(B_w\cap B_{ws}) && \sqcup\, G_\mathbf{d}/B_w\\
&\cong G_\mathbf{d}\times^{B_w}\big(B_w/(B_w\cap B_{ws})\big) && \sqcup\, G_\mathbf{d}\times^{B_w}(B_w/B_w)\\
&\cong G_\mathbf{d}\times^{B_w}(B_wsB_w/B_w) && \sqcup\, G_\mathbf{d}\times^{B_w}(B_w/B_w)\\
&\cong G_\mathbf{d}\times^{B_w}(\underline{P}_{w,s}/B_w) && \text{base point } F_{wu,wu'}\\
&\cong G_\mathbf{d}\times^{B_w}(\underline{P}_{w,s}/B_{ws}) && \text{base point } F_{wu,wsu'}
\end{aligned}$$

Closures of $\boldsymbol{\mathcal{O}}$-cells are obtained by identifications (1.6.1). To illustrate it, we compute $\overline{\boldsymbol{\mathcal{O}}}_s$ by hand. Let $\varpi'=s, us=\tilde{w}\tilde{u}$,

$$\begin{aligned}
\overline{\boldsymbol{\mathcal{O}}}_s &= \bigsqcup_u \overline{\boldsymbol{\mathcal{O}}}^u_s = \bigsqcup_u \overline{\boldsymbol{\Omega}}^{u,\tilde{u}}_{\tilde{w}}\\
&= \left(\bigsqcup_{u:usu^{-1}\in W_\mathbf{d}} \overline{\boldsymbol{\Omega}}^{u,u}_{usu^{-1}}\right)\sqcup\left(\bigsqcup_{u:usu^{-1}\notin W_\mathbf{d}} \overline{\boldsymbol{\Omega}}^{u,us}_{\mathrm{Id}}\right)\\
&= \left(\bigsqcup_{u:usu^{-1}\in W_\mathbf{d}} \boldsymbol{\Omega}^{u,u}_{usu^{-1}}\right)\sqcup\left(\bigsqcup_{u:usu^{-1}\notin W_\mathbf{d}} \boldsymbol{\Omega}^{u,us}_{\mathrm{Id}}\right)\sqcup\left(\bigsqcup_{u:usu^{-1}\in W_\mathbf{d}} \boldsymbol{\Omega}^{u,u}_{\mathrm{Id}}\right)\\
&= \boldsymbol{\mathcal{O}}_s\sqcup\left(\bigsqcup_{u:usu^{-1}\in W_\mathbf{d}} \boldsymbol{\mathcal{O}}^u_{\mathrm{Id}}\right)
\end{aligned}$$

We restrict the result of $\overline{\mathbf{\Omega}}_s^{u,u'}$ to $\overline{\boldsymbol{\mathcal{O}}}_s^u$ in Lemma 1.6.8.

Lemma 1.6.8. *For $\varpi = wu \in \mathbb{W}_{|\mathbf{d}|}$, $s \in \Pi$ such that $\varpi s \varpi^{-1} \in W_{\mathbf{d}}$, we have isomorphisms of $G_{\mathbf{d}}$-varieties*

$$
\begin{aligned}
G_{\mathbf{d}} \times^{B_\varpi} (\underline{P}_{\varpi,s}/B_\varpi) &\longrightarrow \overline{\boldsymbol{\mathcal{O}}}_s^u & (g,p) &\longmapsto (g\cdot F_\varpi, gp\cdot F_\varpi)\\
G_{\mathbf{d}} \times^{B_\varpi} (\underline{P}_{\varpi,s}/B_{\varpi s}) &\longrightarrow \overline{\boldsymbol{\mathcal{O}}}_s^u & (g,p) &\longmapsto (g\cdot F_\varpi, gp\cdot F_{\varpi s})
\end{aligned}
$$

Proof. Notice that when $\varpi s \varpi^{-1} \in W_{\mathbf{d}}$, $\boldsymbol{\mathcal{O}}_s^u = \mathbf{\Omega}_{usu^{-1}}^{u,u}$. Therefore,

$$
\begin{aligned}
\overline{\boldsymbol{\mathcal{O}}}_s^u = \overline{\mathbf{\Omega}}_{usu^{-1}}^{u,u} &\cong \begin{cases} G_{\mathbf{d}} \times^{B_w} (\underline{P}_{w,usu^{-1}}/B_w) & \text{base point } F_{wu,wu} \\ G_{\mathbf{d}} \times^{B_w} (\underline{P}_{w,usu^{-1}}/B_{wusu^{-1}}) & \text{base point } F_{wu,wus} \end{cases}\\
&\cong \begin{cases} G_{\mathbf{d}} \times^{B_\varpi} (\underline{P}_{\varpi,s}/B_\varpi) & \text{base point } F_{\varpi,\varpi} \\ G_{\mathbf{d}} \times^{B_\varpi} (\underline{P}_{\varpi,s}/B_{\varpi s}) & \text{base point } F_{\varpi,\varpi s} \end{cases}
\end{aligned}
$$

□

Definition 1.6.9. *We define*

$$
\begin{aligned}
\mathcal{Z}_{w'}^{u,u'} &:= \overline{\widetilde{\mathbf{\Omega}}}_{w'}^{u,u'} &\subseteq\quad & \mathcal{Z}^{u,u'} := \mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}'},\\
\mathcal{Z}_{\varpi'} &:= \overline{\widetilde{\mathcal{O}}}_{\varpi'} &\subseteq\quad & \mathcal{Z}_{\mathbf{d}}\,.
\end{aligned}
$$

Proposition 1.6.10. *$\mathcal{Z}_s$ is a Zarisky-locally trivial vector bundle over $\overline{\boldsymbol{\mathcal{O}}}_s$, with fiber $\mathfrak{r}_{u,us}$ at point $\underline{F}_{u,s}$.*

Proof. This is claimed in [25, 2.20(c)]. In fact, we have a $G_{\mathbf{d}}$-equivariant morphism

$$\phi : G_{\mathbf{d}} \times^{B_u} \left(\underline{P}_{u,s}/B_{us} \times \underline{\mathfrak{r}}_{u,s}\right) \hookrightarrow \mathrm{Rep}_{\mathbf{d}}(Q) \times \mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}} \qquad (g,p,x) \longmapsto (gx, g\cdot F_u, gp\cdot F_{us})$$

which realized $G_{\mathbf{d}} \times^{B_u} \left(\underline{P}_{u,s}/B_{us} \times \underline{\mathfrak{r}}_{u,s}\right)$ as a closed subset of $\mathrm{Rep}_{\mathbf{d}}(Q) \times \mathcal{F}_{\mathbf{d}}$. In the meantime, the open dense subset

$$G_{\mathbf{d}} \times^{B_u} \left(B_{us} s B_{us}/B_{us} \times \underline{\mathfrak{r}}_{u,s}\right) \subseteq G_{\mathbf{d}} \times^{B_u} \left(\underline{P}_{u,s}/B_{us} \times \underline{\mathfrak{r}}_{u,s}\right)$$

is identified with $\widetilde{\boldsymbol{\mathcal{O}}}_s^u$ by ϕ. Therefore, ϕ identifies $\mathcal{Z}_s^{u,\tilde{u}}$ with the vector bundle $G_{\mathbf{d}} \times^{B_u} \left(\underline{P}_{u,s}/B_{us} \times \underline{\mathfrak{r}}_{u,s}\right)$ over $\overline{\boldsymbol{\mathcal{O}}}_s^u$, with fiber $\underline{\mathfrak{r}}_{u,s} = \mathfrak{r}_{u,us}$. □

Remark 1.6.11. By the same method, one can show that $\overline{\widetilde{\mathcal{O}}}_s$ is a Zarisky-locally trivial vector bundle over $\overline{\mathcal{O}}_s$, with fiber $\mathfrak{r}_s$ at point F_s.

1.6.4 T-fixed points

Recall that the T-fixed points of a complete flag variety $\mathcal{F}$ are exactly the coordinate flags $\{F_w \mid w \in W\}$. For absolute or relative flag varieties, we have similar results:

$$\mathcal{F}_{|\mathbf{d}|}^{\mathbb{T}_{|\mathbf{d}|}} = \mathcal{F}_{\mathbf{d}}^{T_{\mathbf{d}}} = \left\{F_\varpi \,\middle|\, \varpi \in \mathbb{W}_{|\mathbf{d}|}\right\} \qquad \mathcal{F}_u^{T_{\mathbf{d}}} = \{F_{wu} \mid w \in W_{\mathbf{d}}\}$$

For $\mathrm{Rep}_{\mathbf{d}}(Q)$, we get

$$\left(\mathrm{Rep}_{\mathbf{d}}(Q)\right)^{T_{\mathbf{d}}} = \bigoplus_{r \in a(Q)} \left(\mathrm{Hom}\left(\mathbb{C}^{\mathbf{d}_{s(a)}}, \mathbb{C}^{\mathbf{d}_{t(a)}}\right) \right)^{T_{\mathbf{d}}} = \{\rho_0\}$$

where ρ_0 is the zero representation in $\mathrm{Rep}_{\mathbf{d}}(Q)$.

Combining these two results, one can easily describe T-fixed points of varieties constructed over them:

$$\begin{aligned}
\left(\mathcal{F}_{|\mathbf{d}|} \times \mathcal{F}_{|\mathbf{d}|}\right)^{\mathbb{T}_{|\mathbf{d}|}} &= \left\{(F_\varpi, F_{\varpi'}) \,\middle|\, \varpi, \varpi' \in \mathbb{W}_{|\mathbf{d}|}\right\} & \left(\widetilde{\mathrm{Rep}}_{\underline{\mathbf{d}}}(Q)\right)^{T_{\mathbf{d}}} &= \{(\rho_0, F_{wu}) \mid w \in W_{\mathbf{d}}\} \\
\left(\mathcal{F}_u \times \mathcal{F}_{u'}\right)^{T_{\mathbf{d}}} &= \left\{(F_{wu}, F_{w'u'}) \,\middle|\, w, w' \in W_{\mathbf{d}}\right\} & \left(\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)\right)^{T_{\mathbf{d}}} &= \left\{(\rho_0, F_\varpi) \,\middle|\, \varpi \in \mathbb{W}_{|\mathbf{d}|}\right\} \\
\left(\mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}}\right)^{T_{\mathbf{d}}} &= \left\{(F_\varpi, F_{\varpi'}) \,\middle|\, \varpi, \varpi' \in \mathbb{W}_{|\mathbf{d}|}\right\} & \left(\mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}'}\right)^{T_{\mathbf{d}}} &= \left\{(\rho_0, F_{wu}, F_{w'u'}) \,\middle|\, w, w' \in W_{\mathbf{d}}\right\} \\
& & \left(\mathcal{Z}_{\mathbf{d}}\right)^{T_{\mathbf{d}}} &= \left\{(\rho_0, F_\varpi, F_{\varpi'}) \,\middle|\, \varpi, \varpi' \in \mathbb{W}_{|\mathbf{d}|}\right\}
\end{aligned}$$

Notice that, each $B_{\mathbf{d}} \times B_{\mathbf{d}}$-orbit of $\mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}}$ contains exactly one $T_{\mathbf{d}}$-fixed point. Also, all the T-fixed points lie in the zero sections. By this reason, we can compute more:

$$\begin{aligned}
\left(\mathcal{Z}_{\mathrm{Id}}^{u,u'}\right)^{T_{\mathbf{d}}} &= \{(\rho_0, F_{wu}, F_{wu'}) \mid w \in W_{\mathbf{d}}\} \\
\left(\mathcal{Z}_{\mathrm{Id}}\right)^{T_{\mathbf{d}}} &= \left\{(\rho_0, F_\varpi, F_\varpi) \,\middle|\, \varpi \in \mathbb{W}_{|\mathbf{d}|}\right\} \\
\left(\mathcal{Z}_s^{u,u'}\right)^{T_{\mathbf{d}}} &= \{(\rho_0, F_{wu}, F_{wsu'}) \mid w \in W_{\mathbf{d}}\} \sqcup \{(\rho_0, F_{wu}, F_{wu'}) \mid w \in W_{\mathbf{d}}\} \\
\left(\mathcal{Z}_s\right)^{T_{\mathbf{d}}} &= \left\{(\rho_0, F_\varpi, F_{\varpi s}) \,\middle|\, \varpi \in \mathbb{W}_{|\mathbf{d}|}\right\} \sqcup \left\{(\rho_0, F_\varpi, F_\varpi) \,\middle|\, \varpi \in \mathbb{W}_{|\mathbf{d}|}, \varpi s \varpi^{-1} \in W_{\mathbf{d}}\right\}
\end{aligned}$$

1.6.5 Tangent spaces of T-fixed points

The tangent space of T-fixed points will be used in Chapter 4, so we fix symbols of them and compute some of them as Lie algebras.[7]

Definition 1.6.12 (Tangent space of T-fixed points)**.** *For $\varpi, \varpi', x \in \mathbb{W}_{|\mathbf{d}|}$, we define the following tangent spaces:*

$$\begin{aligned}
\mathcal{T}_\varpi &:= T_{F_\varpi}\mathcal{F}_{\mathbf{d}} & \mathcal{T}_\varpi^x &:= T_{F_\varpi}\overline{\mathcal{O}}_x & \mathcal{T}_{\varpi,\varpi'}^x &:= T_{F_{\varpi,\varpi'}}\overline{\boldsymbol{\mathcal{O}}}_x \\
\widetilde{\mathcal{T}}_\varpi &:= T_{(\rho_0, F_\varpi)}\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q) & \widetilde{\mathcal{T}}_\varpi^x &:= T_{(\rho_0, F_\varpi)}\overline{\widetilde{\mathcal{O}}}_x & \widetilde{\mathcal{T}}_{\varpi,\varpi'}^x &:= T_{(\rho_0, F_\varpi, F_{\varpi'})}\mathcal{Z}_x
\end{aligned}$$

For completeness, denote

$$\mathcal{T}_{\varpi,\varpi'} := T_{F_{\varpi,\varpi'}}\left(\mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}}\right) \qquad \widetilde{\mathcal{T}}_{\varpi,\varpi'} := T_{(\rho_0, F_\varpi, F_{\varpi'})}\mathcal{Z}_{\mathbf{d}}\,.$$

When we underline, the subscripts are twisted. For example,

$$\underline{\mathcal{T}}_{\varpi,\varpi'}^x := \mathcal{T}_{\varpi,\varpi\varpi'}^x = T_{F_{\varpi,\varpi\varpi'}}\overline{\boldsymbol{\mathcal{O}}}_x.$$

From the description of $\mathcal{F}_{\mathbf{d}}$ and $\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)$, we know that

$$\begin{aligned}
\mathcal{T}_\varpi &= T_{F_\varpi}\mathcal{F}_{\mathbf{d}} \cong T_{\mathrm{Id}}(G_{\mathbf{d}}/B_\varpi) \cong \mathfrak{g}_{\mathbf{d}}/\mathfrak{b}_\varpi & &\cong \mathfrak{n}_\varpi^- \\
\widetilde{\mathcal{T}}_\varpi &= T_{(\rho_0, F_\varpi)}\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q) \cong T_{\rho_0}\mathfrak{r}_\varpi \oplus T_{F_\varpi}\mathcal{F}_{\mathbf{d}} & &\cong \mathfrak{r}_\varpi \oplus \mathfrak{n}_\varpi^-
\end{aligned}$$

For the rest, we can only compute special cases.

[7] In algebraic geometry, we can define the tangent space even at singular points, see [24, 12.1].

Proposition 1.6.13. *For $s \in \Pi$, We have identifications*

$$\begin{aligned} \mathcal{T}^s_{\mathrm{Id}} &\cong \mathfrak{m}_{s,\mathrm{Id}} & \widetilde{\mathcal{T}}^s_{\mathrm{Id}} &\cong \mathfrak{r}_s \oplus \mathfrak{m}_{s,\mathrm{Id}} \\ \mathcal{T}^s_{s} &\cong \mathfrak{m}_{\mathrm{Id},s} & \widetilde{\mathcal{T}}^s_{s} &\cong \mathfrak{r}_s \oplus \mathfrak{m}_{\mathrm{Id},s}. \end{aligned}$$

Proof. We know from Remark 1.6.11 that

$$\begin{aligned} \mathcal{T}^s_{\mathrm{Id}} &\cong T_{\mathrm{Id}}(P_{\mathrm{Id},s}/B_{\mathbf{d}}) \cong \mathfrak{p}_{\mathrm{Id},s}/\mathfrak{b}_{\mathbf{d}} \cong \mathfrak{b}_s/\left(\mathfrak{b}_s \cap \mathfrak{b}_{\mathbf{d}}\right) &\cong& \quad \mathfrak{m}_{s,\mathrm{Id}} \\ \widetilde{\mathcal{T}}^s_{\mathrm{Id}} &\cong T_{\rho_0}\mathfrak{r}_s \oplus \mathcal{T}^s_{\mathrm{Id}} &\cong& \; \mathfrak{r}_s \oplus \mathfrak{m}_{s,\mathrm{Id}} \end{aligned}$$

Other proofs are the same. □

Proposition 1.6.14. *For $\varpi \in \mathbb{W}_{|\mathbf{d}|}$, $s \in \mathrm{Min}(\mathbb{W}_{|\mathbf{d}|}, \mathrm{W}_{\mathbf{d}})$, We have identifications*

$$\begin{aligned} \mathcal{T}^s_{\varpi,\varpi} &\cong \mathfrak{n}^-_{\varpi} \oplus \mathfrak{m}_{\varpi s,\varpi} & \widetilde{\mathcal{T}}^s_{\varpi,\varpi} &\cong \mathfrak{r}_{\varpi,\varpi s} \oplus \mathfrak{n}^-_{\varpi} \oplus \mathfrak{m}_{\varpi s,\varpi} \\ \mathcal{T}^s_{\varpi,\varpi s} &\cong \mathfrak{n}^-_{\varpi} \oplus \mathfrak{m}_{\varpi,\varpi s} & \widetilde{\mathcal{T}}^s_{\varpi,\varpi s} &\cong \mathfrak{r}_{\varpi,\varpi s} \oplus \mathfrak{n}^-_{\varpi} \oplus \mathfrak{m}_{\varpi,\varpi s} \end{aligned}$$

Proof. We know from Lemma 1.6.8 and Proposition 1.6.10 that

$$\begin{aligned} \mathcal{T}^s_{\varpi,\varpi} &\cong T_{(\mathrm{Id},\mathrm{Id})}\left(G_{\mathbf{d}} \times^{B_{\varpi}} \left(\underline{P}_{\varpi,s}/B_{\varpi}\right)\right) \cong \mathfrak{g}_{\mathbf{d}}/\mathfrak{b}_{\varpi} \oplus \mathfrak{p}_{\varpi,\varpi s}/\mathfrak{b}_{\varpi} &\cong& \quad \mathfrak{n}^-_{\varpi} \oplus \mathfrak{m}_{\varpi s,\varpi} \\ \widetilde{\mathcal{T}}^s_{\varpi,\varpi} &\cong T_{\rho_0}\mathfrak{r}_{\varpi,\varpi s} \oplus \mathcal{T}^s_{\varpi,\varpi} &\cong& \; \mathfrak{r}_{\varpi,\varpi s} \oplus \mathfrak{n}^-_{\varpi} \oplus \mathfrak{m}_{\varpi s,\varpi} \end{aligned}$$

Other proofs are the same. □

Remark 1.6.15. We know a little more on the biggest cells. Here is an example. When $\varpi' = \varpi x$, $F_{\varpi,\varpi x} \in \boldsymbol{\mathcal{O}}_x$, so

$$\begin{aligned} \mathcal{T}^x_{\varpi,\varpi x} &= T_{F_{\varpi,\varpi x}}\overline{\boldsymbol{\mathcal{O}}}_x = T_{F_{\varpi,\varpi x}}\boldsymbol{\mathcal{O}}_x = T_{F_{\varpi,\varpi x}}\boldsymbol{\mathcal{O}}^u_x &\cong& \quad \mathfrak{n}^-_{\varpi} \oplus \mathfrak{m}_{\varpi,\varpi x} \\ \widetilde{\mathcal{T}}^x_{\varpi,\varpi x} &= T_{(\rho_0,F_{\varpi},F_{\varpi x})}\mathcal{Z}_x = T_{(\rho_0,F_{\varpi},F_{\varpi x})}\widetilde{\boldsymbol{\mathcal{O}}}_x \cong T_{\rho_0}\mathfrak{r}_{\varpi,\varpi x} \oplus \mathcal{T}^x_{\varpi,\varpi x} &\cong& \; \mathfrak{r}_{\varpi,\varpi x} \oplus \mathfrak{n}^-_{\varpi} \oplus \mathfrak{m}_{\varpi,\varpi x} \end{aligned}$$

In particular,

$$\begin{aligned} \mathcal{T}^{\mathrm{Id}}_{\varpi,\varpi} &\cong \mathfrak{n}^-_{\varpi}, & \widetilde{\mathcal{T}}^s_{\varpi,\varpi} &\cong \mathfrak{r}_{\varpi,\varpi} \oplus \mathfrak{n}^-_{\varpi}, \\ \mathcal{T}^s_{\varpi,\varpi s} &\cong \mathfrak{n}^-_{\varpi} \oplus \mathfrak{m}_{\varpi,\varpi s}, & \widetilde{\mathcal{T}}^s_{\varpi,\varpi s} &\cong \mathfrak{r}_{\varpi,\varpi s} \oplus \mathfrak{n}^-_{\varpi} \oplus \mathfrak{m}_{\varpi,\varpi s}. \end{aligned}$$

Chapter 2

K-theory and cohomology

From my humble point of view, there is no easy cohomology theory, in a sense that key properties are usually hard to prove. On the other hand, plenty of examples can be quickly computed once we grasp some properties and use them in black boxes. Therefore, we will not prove any properties we stated, for the restricted space and time.

The main reference for the K-theory is [3, Chapter 5].

Setting 2.0.1. *Throughout abstract results of K-theory, we use the following notation:*

- *G stands for a linear algebraic group, i.e., a closed subgroup of* $\mathrm{GL}_n(\mathbb{C})$.[1] *Let $m : G \times G \longrightarrow G$ denote the multiplication map of G.*
- *X is a variety over $\mathbb{C}$, i.e., a reduced, separated scheme of finite type over $\mathbb{C}$. We assume X to be quasi-projective.*
- *Usually, X is equipped with an algebraic G-action (which is compatible with the variety structure of G and X), and we say that X is a G-variety. In that case, let $\alpha : G \times X \longrightarrow X$ denote the G-action map.*
- *$\mathcal{F}$ is usually a sheaf on X.*

2.1 Definitions and initial examples

2.1.1 G-equivariant sheaf and $K_0^G(X)$

We give definition of equivariant algebraic K-theory. Roughly speaking, a G-equivariant coherent sheaf over X is a sheaf $\mathcal{F} \in \mathrm{Coh}(X)$ equipped with G-action which is compatible with the G-action on X, and K-theory is the Grothendieck group of G-equivariant coherent sheaves over X.

Definition 2.1.1 (G-equivariant sheaf, [3, Definition 5.1.6])**.** *For a G-variety X, let $p_i^{jk}, p_i := p_i^{123}, p_{ij} := p_{ij}^{123}$ represent projections onto some factors, as follows.*

$$\begin{tikzcd}
 & & G \times G \times X \arrow[dl, "p_{12}"'] \arrow[dr, "p_{23}"] \arrow[drrr, bend left, "p_{13}"] & & & \\
 & G \times G \arrow[dl, "p_1^{12}"'] \arrow[dr, "p_2^{12}"] & & G \times X \arrow[dl, "p_2^{23}"'] \arrow[dr, "p_3^{23}"] & & G \times X \arrow[dlllll, dashed, "p_1^{13}"'] \arrow[dl, dashed, "p_3^{13}"] \\
G & & G & & X &
\end{tikzcd}$$

[1] The closed embedding $G \hookrightarrow \mathrm{GL}_n(\mathbb{C})$ is not considered as the data of G.

We have morphisms

$$
\begin{tikzcd}
G \times G \times X \arrow[r, shift left=2, "m\times \mathrm{Id}_X"] \arrow[r, "p_{23}"] \arrow[r, shift right=2, "\mathrm{Id}_G \times\alpha"'] & G \times X \arrow[r, shift left, "p_3^{23}=p_3^{13}"] \arrow[r, shift right, "\alpha"'] & X
\end{tikzcd}
$$

which satisfies the "coequalizer conditions":

$$
\begin{aligned}
p_3^{23} \circ (m \times \mathrm{Id}_X) &= p_3^{23} \circ p_{23} & \qquad (g_1, g_2, x) &\longmapsto x \\
p_3^{23} \circ (\mathrm{Id}_G \times\alpha) &= \alpha \circ p_{23} & (g_1, g_2, x) &\longmapsto g_2 x \\
\alpha \circ (m \times \mathrm{Id}_X) &= \alpha \circ (\mathrm{Id}_G \times\alpha) & (g_1, g_2, x) &\longmapsto g_1 g_2 x
\end{aligned}
$$

A ***G-equivariant (coherent) sheaf***[2] *on X is a sheaf $\mathcal{F} \in \mathrm{Coh}(X)$ equipped with an isomorphism*

$$\phi_{\mathcal{F}} : p_3^{23,*}\mathcal{F} \longrightarrow \alpha^*\mathcal{F}$$

such that the following diagram commutes:

$$
\begin{tikzcd}
& (m \times \mathrm{Id}_X)^* p_3^{23,*}\mathcal{F} \arrow[r, "(m\times \mathrm{Id}_X)^*\phi_{\mathcal{F}}"] \arrow[dl, equal] & (m \times \mathrm{Id}_X)^*\alpha^*\mathcal{F} \arrow[dr, equal] & \\
p_{23}^* p_3^{23,*}\mathcal{F} \arrow[dr, "p_{23}^*\phi_{\mathcal{F}}"] & & & (\mathrm{Id}_G \times\alpha)^*\alpha^*\mathcal{F} \\
& p_{23}^*\alpha^*\mathcal{F} \arrow[r, equal] & (\mathrm{Id}_G \times\alpha)^* p_3^{23,*}\mathcal{F} \arrow[ur, "(\mathrm{Id}_G \times\alpha)^*\phi_{\mathcal{F}}"] &
\end{tikzcd}
\tag{2.1.1}
$$

A ***(G-equivariant) morphism*** *$f : (\mathcal{F}, \phi_{\mathcal{F}}) \longrightarrow (\mathcal{G}, \phi_{\mathcal{G}})$ between two G-equivariant sheaves is a morphism $f : \mathcal{F} \longrightarrow \mathcal{G}$ in $\mathrm{Coh}(X)$ such that the diagram*

$$
\begin{tikzcd}
p_3^{23,*}\mathcal{F} \arrow[r, "\phi_{\mathcal{F}}"] \arrow[d, "p_3^{23,*}f"'] & \alpha^*\mathcal{F} \arrow[d, "\alpha^* f"] \\
p_3^{23,*}\mathcal{G} \arrow[r, "\phi_{\mathcal{G}}"] & \alpha^*\mathcal{G}
\end{tikzcd}
\tag{2.1.2}
$$

commutes.

We denote the category of G-equivariant sheaves by $\mathrm{Coh}^G(X)$.

Definition 2.1.2 (G-equivariant K-theory)**.** *For a G-variety X, the G-equivariant K-theory is defined as the Grothendieck group of G-equivariant coherent sheaves over X, i.e.,*

$$K_0^G(X) := K_0(\mathrm{Coh}^G(X)).$$

Specifically, for a point $\mathrm{pt} = \operatorname{Spec}\mathbb{C}$ with trivial G-action, we obtain

$$\mathrm{R}(G) := K_0^G(\mathrm{pt}) = K_0(\mathrm{Rep}(G))$$

the representation ring of group G.

We may omit 0 for convenience.

[2] we will omit the word "coherent" for shorter notation.

Let us unravel Definition 2.1.1 a little bit. For (geometric) points $g, g_1, g_2 \in G$, denote

$$\begin{aligned} \iota_g &: X \longrightarrow G \times X & x &\longmapsto (g, x) \\ \iota_{g_1,g_2} &: X \longrightarrow G \times G \times X & x &\longmapsto (g_1, g_2, x) \\ \alpha_g &: X \xhookrightarrow{\iota_g} G \times X \xrightarrow{\alpha} X & x &\longmapsto gx \end{aligned}$$

By pulling back along ι_g and ι_{g_1,g_2}, we can see geometrical meanings in the expressions. Apply ι_g^* to $\phi_{\mathcal{F}}$, one get

$$\iota_g^*\phi_{\mathcal{F}} : \mathcal{F} \longrightarrow \alpha_g^*\mathcal{F} \qquad \rightsquigarrow \qquad \phi^{\mathcal{F}}_{g,x} \hat{=} \left(\iota_g^*\phi_{\mathcal{F}}\right)_x : \mathcal{F}_x \longrightarrow \mathcal{F}_{gx}$$

Therefore, $\phi_{\mathcal{F}}$ encodes information of G-action on $\mathcal{F}$, which is G-equivariant.

Now we apply ι_{g_1,g_2}^* to (2.1.1):

$$\begin{tikzcd} \mathcal{F} \arrow[rr, "\iota^*_{g_1,g_2}\phi_{\mathcal{F}}"] \arrow[dr, "\iota^*_{g_2}\phi_{\mathcal{F}}"'] & & \alpha^*_{g_1,g_2}\mathcal{F} = \alpha^*_{g_1}\alpha^*_{g_2}\mathcal{F} \\ & \alpha^*_{g_2}\mathcal{F} \arrow[ur, "\iota^*_{g_1}\phi_{\alpha^*_{g_2}\mathcal{F}}"'] & \end{tikzcd} \qquad \rightsquigarrow \qquad \begin{tikzcd} \mathcal{F}_x \arrow[rr, "\phi^{\mathcal{F}}_{g_1g_2,x}"] \arrow[dr, "\phi^{\mathcal{F}}_{g_2,x}"'] & & \mathcal{F}_{g_1g_2x} \\ & \mathcal{F}_{g_2x} \arrow[ur, "\phi^{\mathcal{F}}_{g_1,g_2x} = \phi^{\alpha^*_{g_2}\mathcal{F}}_{g_1,x}"'] & \end{tikzcd}$$

So (2.1.1) is just the associative constraint of the G-structure on $\mathcal{F}$.

Similarly, apply ι_g^* to (2.1.2), we get

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\iota_g^*\phi_{\mathcal{F}}"] \arrow[d, "f"'] & \alpha_g^*\mathcal{F} \arrow[d, "\alpha_g^* f"] \\ \mathcal{G} \arrow[r, "\iota_g^*\phi_{\mathcal{G}}"] & \alpha_g^*\mathcal{G} \end{tikzcd} \qquad \rightsquigarrow \qquad \begin{tikzcd} \mathcal{F}_x \arrow[r, "\phi^{\mathcal{F}}_{g,x}"] \arrow[d, "f_x"'] & \mathcal{F}_{gx} \arrow[d, "f_{gx}"] \\ \mathcal{G}_x \arrow[r, "\phi^{\mathcal{G}}_{g,x}"] & \mathcal{G}_{gx} \end{tikzcd}$$

So (2.1.2) is just the condition for f to be G-equivariant.

There are two extreme situations worth mentioning. When $G = \mathrm{Id}$, there is no G-action structure constrain on varieties and sheaves. Therefore,

$$\mathrm{Coh}^{\mathrm{Id}}(X) = \mathrm{Coh}(X) \qquad K_0^{\mathrm{Id}}(X) = K_0(X) \hat{=} K_0(\mathrm{Coh}(X)).$$

When G acts on $X = \operatorname{Spec} A$ trivially, any sheaf $\mathcal{F} \in \mathrm{Coh}^G(X)$ can be viewed as an (finitely generated)[3] A-module M with G-action, so

$$\mathrm{Coh}^G(X) = \mathrm{rep}_A(G) \xlongequal{\text{when } G \text{ is finite}} \mathrm{Mod}(A[G]).$$

In particular, any sheaf $\mathcal{F} \in \mathrm{Coh}^G(\mathrm{pt})$ can be viewed as a finite dimensional complex G-representation, so

$$\mathrm{Coh}^G(\mathrm{pt}) = \mathrm{rep}_{\mathbb{C}}(G) \xlongequal{\text{when } G \text{ is finite}} \mathrm{Mod}(\mathbb{C}[G]).$$

[3]We already assume X to be of finite type, so coherent condition is equivalent to finitely generated condition.

2.1.2 Representation ring $\mathrm{R}(G)$

Recall that any coherent sheaf over a point pt is equivalent to a finite dimensional $\mathbb{C}$-vector space, and any G-equivariant coherent sheaf over pt is equivalent to a finite dimensional complex G-representation. Moreover, by the Jordan-Hölder theorem, every finite dimensional complex G-representation can be written as a composition series such that each quotient object is irreducible. Therefore,

$$\mathrm{R}(G) = \bigoplus_{\rho \in \mathrm{Irr}(G)} \mathbb{Z}$$

as a free $\mathbb{Z}$-module.

For $\mathrm{R}(G)$, we have the multiplication structure induced by tensor products on complex G-representations. Let us see some examples now. We use Setting 1.1.1 in these examples.

Example 2.1.3. *For trivial group* Id*, every* Id*-representation is just a* $\mathbb{C}$*-vector space, which can be written as the direct sum of* 1*-dimensional vector spaces. Therefore,*

$$\mathrm{R}(\mathrm{Id}) = \mathbb{Z}.$$

Example 2.1.4. *For a torus* T*, every* T*-representation can be written as direct sum of* 1*-dimensional vector spaces. Furthermore,*

$$\begin{aligned} \mathrm{Irr}(T) &= \left\{\rho : T \longrightarrow \mathbb{C}^\times \;\middle|\; \rho \text{ is an (algebraic) group homomorphism}\right\} \\ &= \mathrm{Hom}_{\mathbb{C}\text{-}Alg\ gp}(T, \mathbb{C}^\times) \hat{=} X^*(T) \end{aligned}$$

We get

$$\mathrm{R}(T) = \bigoplus_{\rho \in \mathrm{Irr}(T)} \mathbb{Z} = \mathbb{Z}\left[X^*(T)\right].$$

The group structure in $X^*(T)$ *is given by tensor product, so the multiplication structure is induced by the group structure in* $X^*(T)$*. Take the following* $\mathbb{Z}$*-basis of* $X^*(T)$*:*

$$\varepsilon_i : T \longrightarrow \mathbb{C}^\times \qquad \begin{pmatrix} t_1 & & & & \\ & \ddots & & & \\ & & t_i & & \\ & & & \ddots & \\ & & & & t_n \end{pmatrix} \longmapsto t_i.$$

Then $X^*(T) \cong \oplus_{i=1}^n \mathbb{Z}\varepsilon_i$*.*

To distinguish the addition in $X^*(T)$ *and* $\mathbb{Z}\left[X^*(T)\right]$*, we rewrite* ε_i *as* e_i*. In that case,* $\sum_{i=1}^n k_i\varepsilon_i$ *is sent to* $\prod_{i=1}^n e_i^{k_i}$*, and*

$$\mathrm{R}(T) \cong \mathbb{Z}\big[e_1^{\pm 1}, \ldots, e_n^{\pm 1}\big]$$

as a $\mathbb{Z}$*-algebra.*

By forgetting T*-actions, we get a morphism of* $\mathbb{Z}$*-algebras*

$$\mathrm{R}(T) \longrightarrow \mathrm{R}(\mathrm{Id}) \qquad f(e_1, \ldots, e_n) \longmapsto f(1, \ldots, 1).$$

Example 2.1.5. *After stating the reduction isomorphism 2.5.1, we can show that*

$$\mathrm{R}(N) \cong \mathrm{R}(\mathrm{Id}) \cong \mathbb{Z} \qquad \mathrm{R}(B) \cong \mathrm{R}(T) \cong \mathbb{Z}\big[e_1^{\pm 1}, \ldots, e_n^{\pm 1}\big]$$

Example 2.1.6. *By [3, Theorem 6.1.4],*

$$\mathrm{R}(\mathrm{GL}_n) \cong \mathrm{R}(T)^W \cong \mathbb{Z}\big[e_1^{\pm 1}, \ldots, e_n^{\pm 1}\big]^{S_n}.$$

This can be viewed as an analogue of the Chevalley restriction theorem.

From these examples we already see the difficulty of computing K-theories. Therefore, a series of properties of K-theories are definitely needed for computations. To state these properties, we need to define some tools in K-theory.

2.2 Three functors: pullback, proper pushforward and tensor product

In this section, we will construct three basic functors of equivariant K-theory: pullback, proper pushforward and tensor product.

2.2.1 Non-derived three functors in $\mathrm{Coh}^G(X)$

We assume that readers know the non-derived pullback, pushforward and tensor product of (ordinary) coherent sheaves. (See [24, Chapter 16])

As a special reminder, the pushforward of coherent sheaves may be not coherent. This problem can be remedied by Grothendieck's coherence theorem [24, Theorem 18.9.1], once we impose morphisms to be proper (and Noetherian hypotheses on varieties). That is why we only consider proper pushforwards.

Now let us consider the effect of G-equivariance. Somewhat surprising, these three functors behave quite well with group actions.

Definition 2.2.1 (Group action on pullback, proper pushforward and tensor product)**.** *Let X, Y be G-varieties, $f : Y \longrightarrow X$ be a G-equivariant morphism. For $(\mathcal{F}, \phi_{\mathcal{F}}), (\mathcal{F}', \phi_{\mathcal{F}'}) \in \mathrm{Coh}^G(X)$, $(\mathcal{G}, \phi_{\mathcal{G}}) \in \mathrm{Coh}^G(Y)$, we define group actions on $f^*\mathcal{F}$, $f_*\mathcal{G}$ and $\mathcal{F} \otimes \mathcal{F}'$, as follows.*

$$\begin{tikzcd}
 & & \mathcal{G} \arrow[dl, dash] \\
G \times Y \arrow[r, shift left, "p^{23}_{3,Y}"] \arrow[r, shift right, "\alpha_Y"'] \arrow[d, "\mathrm{Id}_G \times f"'] \arrow[dr, phantom, very near start, "\lrcorner"] & Y \arrow[d, "f"] & \\
 & & \mathcal{F} \arrow[dl, dash] \\
G \times X \arrow[r, shift left, "p^{23}_{3,X}"] \arrow[r, shift right, "\alpha_X"'] & X &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathcal{F} \arrow[d, dash] & \mathcal{F}' \arrow[dl, dash] \\
G \times X \arrow[r, shift left, "p^{23}_{3,X}"] \arrow[r, shift right, "\alpha_X"'] & X &
\end{tikzcd}$$

By definition, we get

$$p^{23}_{3,X} \circ (\mathrm{Id}_G \times f) = f \circ p^{23}_{3,Y}.$$

Since f is G-equivariant,

$$\alpha_X \circ (\mathrm{Id}_G \times f) = f \circ \alpha_Y.$$

These two diagrams are Cartesian, and $p^{23}_{3,X}, \alpha_X$ are flat.

The pullback $(f^*\mathcal{F}, \phi_{f^*\mathcal{F}}) \in \mathrm{Coh}^G(Y)$ is defined by

$$\phi_{f^*\mathcal{F}} : p^{23,*}_{3,Y} f^*\mathcal{F} = (\mathrm{Id}_G \times f)^* \, p^{23,*}_{3,X}\mathcal{F} \xrightarrow{(\mathrm{Id}_G \times f)^*\phi_{\mathcal{F}}} (\mathrm{Id}_G \times f)^* \, \alpha_X^*\mathcal{F} = \alpha_Y^* f^*\mathcal{F}$$

By flat base change [24, Theorem 24.2.8], assuming f is proper, the proper pushforward $(f_*\mathcal{G}, \phi_{f_*\mathcal{G}}) \in \mathrm{Coh}^G(X)$ is defined by

$$\phi_{f_*\mathcal{G}} : p_{3,X}^{23,*} f_*\mathcal{G} \cong (\mathrm{Id}_G \times f)_*\, p_{3,Y}^{23,*}\mathcal{G} \xrightarrow{(\mathrm{Id}_G \times f)_*\phi_{\mathcal{G}}} (\mathrm{Id}_G \times f)_*\, \alpha_Y^*\mathcal{G} \cong \alpha_X^* f_*\mathcal{G}$$

In general, we can also define $(\mathrm{R}^i f_*\mathcal{G}, \phi_{\mathrm{R}^i f_*\mathcal{G}}) \in \mathrm{Coh}^G(X)$ by

$$\phi_{\mathrm{R}^i f_*\mathcal{G}} : p_{3,X}^{23,*}\, \mathrm{R}^i f_*\mathcal{G} \cong \mathrm{R}^i(\mathrm{Id}_G \times f)_*\, p_{3,Y}^{23,*}\mathcal{G} \xrightarrow{\mathrm{R}^i(\mathrm{Id}_G \times f)_*\phi_{\mathcal{G}}} \mathrm{R}^i(\mathrm{Id}_G \times f)_*\, \alpha_Y^*\mathcal{G} \cong \alpha_X^*\, \mathrm{R}^i f_*\mathcal{G}$$

Similarly, the tensor product $(\mathcal{F} \otimes \mathcal{F}', \phi_{\mathcal{F}\otimes\mathcal{F}'}) \in \mathrm{Coh}^G(X)$ is defined by

$$\phi_{\mathcal{F}\otimes\mathcal{F}'} : p_{3,X}^{23,*}\left(\mathcal{F} \otimes \mathcal{F}'\right) \cong p_{3,X}^{23,*}\mathcal{F} \otimes p_{3,X}^{23,*}\mathcal{F}' \xrightarrow{\phi_{\mathcal{F}}\otimes\phi_{\mathcal{F}'}} \alpha_X^*\mathcal{F} \otimes \alpha_X^*\mathcal{F}' \cong \alpha_X^*\left(\mathcal{F} \otimes \mathcal{F}'\right).$$

The following definition will be useful in redefining tensor products.

Definition 2.2.2 (External tensor product)**.** *For two G-varieties X and Y, define a functor*

$$\boxtimes : \mathrm{Coh}^G(X) \times \mathrm{Coh}^G(Y) \longrightarrow \mathrm{Coh}^G(X \times Y) \qquad (\mathcal{F}, \mathcal{G}) \longmapsto \mathcal{F} \boxtimes \mathcal{G}$$

where

$$\mathcal{F} \boxtimes \mathcal{G} := p_X^*\mathcal{F} \otimes p_Y^*\mathcal{G}, \qquad p_X,\, p_Y \text{ are projections.}$$

*$\boxtimes$ is called the **external tensor product**.*

Remark 2.2.3. For G-variety X and $\mathcal{F}$, $\mathcal{F}' \in \mathrm{Coh}^G(X)$, let $\Delta : X \hookrightarrow X \times X$ be the diagonal embedding. Then

$$\mathcal{F} \otimes \mathcal{F}' \cong \Delta^*(\mathcal{F} \boxtimes \mathcal{F}').$$

Unlike $\otimes$, $\boxtimes$ is always an exact functor. This feature allows us to redefine tensor product in K-theory later on.

2.2.2 Smooth case

We would like to extend functors in $\mathrm{Coh}^G(X)$ to $K^G(X)$. However, these (non-derived) functors are usually not exact, so we have to work over (G-equivariant) derived category of coherent sheaves $\mathcal{D}^G_{\mathrm{Coh}}(X)$ and replace every functor by its derived version.

Still, we can not extend functors from $\mathcal{D}^G_{\mathrm{Coh}}(X)$ to $K^G(X)$. The chain complex in $\mathcal{D}^G_{\mathrm{Coh}}(X)$ can have infinite many non-zero terms, which can not be viewed as an element in $K^G(X)$. Therefore, we consider the bounded (G-equivariant) derived category $\mathcal{D}^{b,G}_{\mathrm{Coh}}(X)$ as a full subcategory of $\mathcal{D}^G_{\mathrm{Coh}}(X)$.

The last problem comes when we restrict functors to $\mathcal{D}^{b,G}_{\mathrm{Coh}}(X)$:

$$\begin{aligned} f^* &: \mathcal{D}^{b,G}_{\mathrm{Coh}}(X) \longrightarrow \mathcal{D}^G_{\mathrm{Coh}}(Y) \\ f_* &: \mathcal{D}^{b,G}_{\mathrm{Coh}}(Y) \longrightarrow \mathcal{D}^{b,G}_{\mathrm{Coh}}(X) \\ \otimes &: \mathcal{D}^{b,G}_{\mathrm{Coh}}(X) \times \mathcal{D}^{b,G}_{\mathrm{Coh}}(X) \longrightarrow \mathcal{D}^G_{\mathrm{Coh}}(X) \end{aligned}$$

Other than proper pushforward, [4] pullback and tensor product may not preserve boundedness.

[4] See [3, 5.2.13] for proper pushforward preserving boundedness, and it essentially uses the higher cohomology vanishing theorem [24, Theorem 18.8.5].

For pullback, preserving boundedness is equivalent to the following condition:

$$f : Y \longrightarrow X \text{ is } G\text{-equivariant of globally finite Tor-dimension.} \tag{2.2.1}$$

When X, Y are smooth, the condition (2.2.1) is automatically satisfied. (See [3, 5.2.5(ii)]). The condition is concluded as follows:

$$X,\ Y \text{ are smooth } G\text{-varieties, and } f : Y \longrightarrow X \text{ is } G\text{-equivariant.} \tag{2.2.2}$$

Tensor product also preserves boundedness when X is smooth. By Remark 2.2.3, $\boxtimes$ is exact, and Δ^* preserves boundedness when X is smooth, so $\otimes$ also preserves boundedness. In particular, one can define tensor product on $K^G(X)$ for X smooth:

$$\otimes : K^G(X) \times K^G(X) \xrightarrow{\boxtimes} K^G(X \times X) \xrightarrow{\Delta^*} K^G(X) \qquad \mathcal{F} \otimes \mathcal{F}' = \Delta^*\left(\mathcal{F} \boxtimes \mathcal{F}'\right)$$

Remark 2.2.4. When $f : Y \longrightarrow X$ is an open embedding, the non-derived pullback f^* is exact, so we can define pullback on K-theory automatically.

2.2.3 Restriction with supports

In practice, the varieties we consider are not smooth, but always embed in some ambiance spaces which are smooth.

Definition 2.2.5 (Restriction with supports)**.** *For a triple (X, Y, f) satisfying assumption* (2.2.2)*, and a G-equivariant closed subvariety Z of X, the triple $\left(Z, f^{-1}(Z), f\big|_{f^{-1}(Z)}\right)$ is called a restriction with supports of (X, Y, f).*

We can now define pullback of f in the following assumption:

$$\begin{gathered} f : Y \longrightarrow X \text{ is } G\text{-equivariant, and } f \text{ is a restriction with supports} \\ \text{of some } f' : Y' \longrightarrow X', \text{ where } X',\ Y' \text{ are smooth.} \end{gathered} \tag{2.2.3}$$

Definition 2.2.6 (Pullback with supports)**.** *Let Z, Z' be G-varieties, $h : Z' \longrightarrow Z$ be a G-equivariant closed embedding. Suppose that h is a restriction with support of some (X, Y, f) satisfying the assumption* (2.2.2)*, i.e., we have a G-equivariant closed embedding $\iota_Z : Z \longrightarrow X$ such that $Z' \cong f^{-1}(Z)$ and $h = f\big|_{Z'}$. Let $\iota_{Z'} : Z' \longrightarrow Y$ represent the induced G-equivariant closed embedding, we would like to construct the pullback $h^* : K^G(Z) \longrightarrow K^G(Z')$.*

$$\begin{tikzcd} Z' \arrow[r, hook, "h"] \arrow[d, hook, "\iota_{Z'}"'] & Z \arrow[d, hook, "\iota_Z"] \\ Y \arrow[r, "f"] & X \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} K^G(Z') \arrow[d, "\iota_{Z',*}"'] & K^G(Z) \arrow[l, dashed, "h^*"'] \arrow[d, "\iota_{Z,*}"] \\ K^G(Y) \arrow[u, dotted, bend right, "\mathrm{gr}"'] & K^G(X) \arrow[l, "f^*"'] \end{tikzcd} \tag{2.2.4}$$

Following [3, 5.2.7(ii)], one can construct a morphism

$$\mathrm{gr} : \mathrm{Im}\left(f^* \circ \iota_{Z,*}\right) \longrightarrow K^G(Z'),$$

and the pullback is defined as

$$h^* : K^G(Z) \xrightarrow{\iota_{Z,*}} K^G(X) \xrightarrow{f^*} K^G(Y) \xrightarrow{\mathrm{gr}} K^G(Z').$$

Warning 2.2.7. *The diagram* (2.2.4) *of K-group is usually not commutative. In fact, we will state the excess base change in Section 4.2, in which the Euler class measures the failure of diagram to be commutative. We draw the dashed arrow for h^* to emphasize this noncommutativity.*

Definition 2.2.8 (Tensor product with supports/Intersection product)**.** *Let X be a smooth G-variety, and Z, $Z' \subseteq X$ be two closed G-subvarieties. The tensor product with supports is defined as*

$$\otimes : K^G(Z) \times K^G(Z') \xrightarrow{\boxtimes} K^G(Z \times Z') \xrightarrow{\Delta^*} K^G(Z \cap Z')$$

i.e., $\mathcal{F} \otimes \mathcal{F}' := \Delta^*(\mathcal{F} \boxtimes \mathcal{F}')$.

The following diagram explains the word "restriction with supports":

$$\begin{tikzcd} K^G(Z) \times K^G(Z') \arrow[r, "\boxtimes"] \arrow[d] & K^G(Z \times Z') \arrow[r, dashed, "\Delta^*"] \arrow[d] & K^G(Z \cap Z') \arrow[d] \\ K^G(X) \times K^G(X) \arrow[r, "\boxtimes"] & K^G(X \times X) \arrow[r, "\Delta^*"] & K^G(X) \end{tikzcd}$$

Lemma 2.2.9. *Let X be a smooth variety, $Z \subseteq X$ be a closed G-subvariety, $\pi_Z : Z \longrightarrow \mathrm{pt}$ be the projection map. For any $\alpha \in K^G(Z)$, $\alpha \otimes \pi_Z^* 1_{\mathrm{R}(G)} = \alpha$.*

Proof. This comes from the definition of the tensor product. □

2.2.4 Algebraic structures of K-theory

With enough tools in hand, we can define some extra structures on $K^G(X)$. (A priori $K^G(X)$ is an abelian group)

Proposition 2.2.10 ($\mathrm{R}(G)$-module)**.** *For any G-variety X, $K^G(X)$ is an $\mathrm{R}(G)$-module by*

$$\mathrm{R}(G) \times K^G(X) \cong K^G(\mathrm{pt}) \times K^G(X) \xrightarrow{\boxtimes} K^G(\mathrm{pt} \times X) \cong K^G(X).$$

Under this proposition, these three functors become $\mathrm{R}(G)$-homomorphisms.

Proposition 2.2.11 ($\otimes$ as multiplication)**.** *For any smooth G-variety X, $K^G(X)$ is a unital commutative associative $\mathrm{R}(G)$-algebra, where the multiplication (call the $\otimes$-product on $K^G(X)$) is defined by*

$$K^G(X) \times K^G(X) \xrightarrow{\otimes} K^G(X).$$

Under this proposition, for any morphism $f : Y \longrightarrow X$ of smooth G-varieties, f^* is a ring homomorphism.

Warning 2.2.12. *We will define another product (called the convolution product) on some K-theories in Section 5.1. These two products are essentially different products, and people have to specify which one they are using, when they discuss the "algebra structures on K-theories". The final task is to compute the convolution product of $K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}})$, not the $\otimes$-product.*

After that, whenever we see an isomorphism of K-theories, we need to specify which structures this isomorphism preserve.

2.3 Thom isomorphism

In this section we state Thom isomorphism theorem, which is an analogy of Poincaré lemma in K-theory.

Proposition 2.3.1 (Thom isomorphism, [3, Theorem 5.4.17]). *Let X be a G-variety, $\pi : E \longrightarrow X$ be a G-equivariant affine bundle on X. The pullback*

$$\pi^* : K^G(X) \longrightarrow K^G(E)$$

is an isomorphism of K-theories as $\mathrm{R}(G)$*-modules.*

For a proof, see [3, Theorem 5.4.17].

With Thom isomorphism, we can compute K-theory of affine bundles by the K-theory of the base spaces. Proposition 1.6.7 offers plenty of cases to apply Thom isomorphism. Also, for any $k \in \mathbb{N}_{>0}$,

$$K^G(\mathbb{A}^k) \cong K^G(\mathrm{pt}) \cong \mathrm{R}(G).$$

as an $\mathrm{R}(G)$-module.

2.4 Induction

2.4.1 Balanced product

Before stating the induction isomorphism, let us recall one basic construction of spaces: the balanced product.

Definition 2.4.1 (Balanced product). *Let $H \subseteq G$ be a closed algebraic subgroup and X be an H-variety. The balanced product of G and X over H is defined as*

$$G \times^H X := (G \times X)/\sim$$

where

$$(gh, x) \sim (g, hx) \qquad \text{for any } g \in G,\ h \in H,\ x \in X.$$

$G \times^H X$ has a natural variety structure. G acts on $G \times^H X$ by multiplying from the left side. We have a G-equivariant flat morphism

$$G \times^H X \longrightarrow G/H \qquad (g, x) \longrightarrow gH$$

which realize $G \times^H X$ as an X-bundle over G/H. In particular, for $X = \mathrm{pt}$, we get an isomorphism of G-varieties

$$G \times^H \mathrm{pt} \xrightarrow{\sim} G/H.$$

The balanced product is not only used for the induction isomorphism, but also used in the definition of equivariant cohomology (see Definition 2.6.1) and description of some typical varieties (see the description of $\overline{\boldsymbol{\Omega}}_s$ in 1.1.2).

Example 2.4.2. *In the setting 1.1.1, the* GL_n*-equivariant map*

$$\mathrm{GL}_n \times^B \mathcal{F} \xrightarrow{\sim} \mathrm{GL}_n / B \times \mathcal{F} = \mathcal{F} \times \mathcal{F} \qquad (g, g'B) \longmapsto (gB, gg'B)$$

realizes $\mathcal{F} \times \mathcal{F}$ as a balanced product, and

$$\boldsymbol{\Omega}_{w'} \cong \mathrm{GL}_n \times^B \Omega_{w'}$$

under this isomorphism.

2.4.2 Statement

Proposition 2.4.3 (Induction isomorphism, [3, 5.2.16]). *Let $H \subseteq G$ be a closed algebraic subgroup and X be an H-variety, we have a Cartesian diagram of H-varieties*

$$\begin{tikzcd} X = H \times^H X \arrow[r, "\iota_X"] \arrow[d] & G \times^H X \arrow[d, "\pi"] \\ \mathrm{pt} = H/H \arrow[r, "\iota_{\mathrm{pt}}"] & G/H \end{tikzcd}$$

The functor

$$\mathrm{Res}^G_H : \mathrm{Coh}^G(G \times^H X) \xrightarrow{\textit{forget}} \mathrm{Coh}^H(G \times^H X) \xrightarrow{\iota_X^*} \mathrm{Coh}^H(X)$$

is an equivalence of categories, and descend to an $\mathrm{R}(H)$*-module homomorphism of K-groups:*

$$\mathrm{Res}^G_H : K^G(G \times^H X) \xrightarrow{\textit{forget}} K^H(G \times^H X) \xrightarrow{\iota_X^*} K^H(X)$$

When X is smooth, Res^G_H *is an isomorphism of algebras (for $\otimes$-product).*

We denote the inverse functor of Res^G_H by Ind^G_H, called the induction, which is also explicitly constructed by pulling back and descent argument in [3, 5.2.16].

Remark 2.4.4. The isomorphism Res^G_H also gives $K^G(G \times^H X)$ an $\mathrm{R}(H)$-module structure.

2.4.3 Applications

This induction formula is usually used for computing G-equivariant K-theory of G-orbits. For example, in Setting 1.1.1,

$$K^{\mathrm{GL}_n}(\mathcal{F}) = K^{\mathrm{GL}_n}(\mathrm{GL}_n / B) \cong K^B(\mathrm{pt}) = \mathrm{R}(B)$$

is an isomorphism as $\mathrm{R}(\mathrm{GL}_n)$-modules. Notice that $K^{\mathrm{GL}_n}(\mathcal{F})$ is a free $\mathrm{R}(\mathrm{GL}_n)$-module of rank $\#W = n!$.

Also, the isomorphism

$$K^{\mathrm{GL}_n}(\mathcal{F} \times \mathcal{F}) \cong K^{\mathrm{GL}_n}(\mathrm{GL}_n \times^B \mathcal{F}) \cong K^B(\mathcal{F})$$

gives $K^{\mathrm{GL}_n}(\mathcal{F} \times \mathcal{F})$ an $\mathrm{R}(B)$-module structure.

In the next section we will explore how to reduce B-equivariant K-theory to T-equivariant K-theory.

2.5 Reduction

Let $P = M \ltimes U$ be a linear algebraic group in this section, where M is reductive and $U = R_u(M)$ is the unipotent radical of P.

Proposition 2.5.1 (Reduction isomorphism, [3, 5.2.18]). *For any P-variety X, the forgetful map*

$$K^P(X) \longrightarrow K^M(X)$$

is an isomorphism as $\mathrm{R}(M)$*-modules. (and as algebras for $\otimes$-product, when X is smooth)*

In the proof of the reduction isomorphism, the induction isomorphism and the Thom isomorphism are used in an essential way.

This isomorphism allows us to identify B-equivariant K-theory and T-equivariant K-theory. In particular, $\mathrm{R}(B) \cong \mathrm{R}(T)$ as $\mathbb{Z}$-algebras.

2.6 Equivariant cohomology

The theory of equivariant cohomology is completely parallel with the theory of equivariant K-theory. We shortly sketch the definition and refer readers to see [17, Chapter 2] for details (like the definition of universal principle bundle $\mathrm{E}G \longrightarrow \mathrm{B}G$)

Nearly all the abstract results for K-theory have a corresponding cohomology theory version in [17]. We will mention about the difference of Euler class in Section 4.1, compute some examples in Section 6.2, and compare these two theories in Section 6.3.

2.6.1 G-equivariant cohomology $H_G^*(X;\mathbb{Q})$

Definition 2.6.1 (G-equivariant cohomology, [17, Definition 2.7])**.** *For a G-variety X, the G-equivariant cohomology is defined as the (singular) cohomology ring of the balanced product space $\mathrm{E}G \times^G X$, i.e.,*

$$H_G^*(X;\mathbb{Q}) := H^*\big(\mathrm{E}G \times^G X;\mathbb{Q}\big)\,.$$

Specifically, for a point $\{\mathrm{pt}\} = \operatorname{Spec}\mathbb{C}$ with trivial G-action, denote

$$\mathrm{S}(G) := H_G^*\big(\{\mathrm{pt}\};\mathbb{Q}\big) = H^*(\mathrm{B}G;\mathbb{Q})$$

the cohomology ring of classifying space $\mathrm{B}G$.

We work with coefficient $\mathbb{Q}$ for simplicity, and we may omit $\mathbb{Q}$ for the convenience of writing and typing.

Parallelly, there are two extreme situations worth mentioning about. When $G = \mathrm{Id}$, $\mathrm{E}G = \{\mathrm{pt}\}$. Therefore,

$$H_{\mathrm{Id}}^*(X;\mathbb{Q}) = H^*\Big(\{\mathrm{pt}\} \times^{\mathrm{Id}} X;\mathbb{Q}\Big) \cong H^*(X;\mathbb{Q}).$$

When G acts on X trivially, we get

$$H_G^*(X;\mathbb{Q}) = H^*(\mathrm{B}G \times X;\mathbb{Q}) \cong H^*(\mathrm{B}G;\mathbb{Q}) \otimes_{\mathbb{Q}} H^*(X;\mathbb{Q}).$$

2.6.2 Cohomology ring $\mathrm{S}(G)$

We also list examples in parallel with subsection 2.1.2. Everything is much more sketchy though. We use Setting 1.1.1.

Example 2.6.2. *For trivial group* Id*,* $\mathrm{BId} = \{\mathrm{pt}\}$*, so*

$$\mathrm{S}(\mathrm{Id}) = H^*\big(\{\mathrm{pt}\};\mathbb{Q}\big) \cong \mathbb{Q}.$$

Example 2.6.3 ([17, Example 2.9(i)])**.** *For group T, $\mathrm{B}T = \prod_{j=i}^n \mathbb{CP}^\infty$, so*

$$\mathrm{S}(T) = H^*\left(\prod_{j=1}^n \mathbb{CP}^\infty;\mathbb{Q}\right) \cong \bigotimes_{j=1}^n H^*\left(\mathbb{CP}^\infty\,;\mathbb{Q}\right) \cong \bigotimes_{j=1}^n \mathbb{Q}[\lambda_j] = \mathbb{Q}[\lambda_1,\ldots,\lambda_n]$$

where $\deg t_j = 2$ for any j.

By forgetting T-actions, we get a morphism of $\mathbb{Q}$-algebras

$$\mathrm{S}(T) \longrightarrow \mathrm{S}(\mathrm{Id}) \qquad f(\lambda_1,\ldots,\lambda_n) \longmapsto f(0,\ldots,0).$$

Example 2.6.4. *By using the reduction isomorphism 2.5.1 in the version of cohomology theory, we can show that*

$$\mathrm{S}(N) \cong \mathrm{S}(\mathrm{Id}) \cong \mathbb{Q} \qquad \mathrm{S}(B) \cong \mathrm{S}(T) \cong \mathbb{Q}[\lambda_1, \ldots, \lambda_n]$$

Example 2.6.5 ([17, Example 2.9(ii)])**.** *For group* GL_n*,* $\mathrm{BGL}_n = \mathrm{Gr}(n, \infty)$*, so*

$$\mathrm{S}(\mathrm{GL}_n) = H^*(\mathrm{Gr}(n,\infty); \mathbb{Q}) \cong \mathbb{Q}[\sigma_1, \ldots, \sigma_n] \qquad \deg \sigma_j = 2j$$

We also have the Chevalley restriction theorem in the version of cohomology theory. In this case, it says

$$\mathrm{S}(\mathrm{GL}_n) \cong \mathrm{S}(T)^W \cong \mathbb{Q}[\lambda_1, \ldots, \lambda_n]^{S_n}.$$

See [17, 2.3.2] for the functionalities of equivariant cohomology. Thom isomorphism, induction isomorphism and reduction isomorphism are still true in the equivariant cohomology case. In particular, we have

$$H^*_{\mathrm{GL}_n}(\mathcal{F}) \cong H^*_B(\mathrm{pt}) \cong H^*_T(\mathrm{pt}) \cong \mathbb{Q}[\lambda_1, \ldots, \lambda_n]$$

as an $\mathrm{S}(\mathrm{GL}_n)$-module. $H^*_{\mathrm{GL}_n}(\mathcal{F})$ is a free $\mathrm{S}(\mathrm{GL}_n)$-module with rank $\#W = n!$.

Also, the isomorphism

$$H^*_{\mathrm{GL}_n}(\mathcal{F} \times \mathcal{F}) \cong H^*_{\mathrm{GL}_n}(\mathrm{GL}_n \times^B \mathcal{F}) \cong H^*_B(\mathcal{F})$$

gives $H^*_{\mathrm{GL}_n}(\mathcal{F} \times \mathcal{F})$ an $\mathrm{S}(B)$-module structure.

Chapter 3

Cellular fibration theorem

In this chapter, we state the cellular fibration theorem 3.1.3, and apply it to get the module structure of K-groups, as shown in Table 3.1, 3.2 and 3.3.

3.1 Statement

We first state one general theorem, and then apply it repeatly to get the cellular fibration theorem.

Theorem 3.1.1 (Glueing theorem, [3, Lemma 5.5.1(a)])**.** *Suppose the triple* (X, Y, π) *satisfies assumption* (2.2.3)*. For a* G*-equivariant closed embedding* $i : Z \hookrightarrow Y$*, define* $U := Y \smallsetminus Z$*, and* $j : U \hookrightarrow Y$ *as the open immersion, as follows.*

$$\begin{tikzcd} Z \arrow[r, hook, "i"] & Y \arrow[d, "\pi"] & U \arrow[l, hook', "j"'] \\ & X & \end{tikzcd}$$

Suppose that $\pi|_U = \pi \circ j : U \longrightarrow X$ *realizes* U *as a* G*-equivariant affine bundle on* X*, so*

$$\pi\big|_U^* : K^G(X) \xrightarrow{\cong} K^G(U)$$

as $\mathrm{R}(G)$*-modules.*

(1) We have a canonical short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & K^G(Z) \arrow[r, "i_*"] & K^G(Y) \arrow[r, "j^*"] & K^G(U) \arrow[r] \arrow[l, bend left, dashed, "s"] & 0 \end{tikzcd} \tag{3.1.1}$$

(2) If $K^G(X)$ is a free $\mathrm{R}(G)$-module with basis $\{y_1, \dots y_m\}$, then the short exact sequence (3.1.1) (non-naturally) splits, and

$$K^G(Y) \cong K^G(Z) \oplus K^G(U)$$

as $\mathrm{R}(G)$-modules. The splitting s is defined on basis of $K^G(U)$:

$$s : K^G(U) \longrightarrow K^G(Y) \qquad \pi\big|_U^*(y_l) \longmapsto \iota_{\overline{U},*}\pi\big|_{\overline{U}}^*(y_l)$$

where $\iota_{\overline{U}}$, $\pi\big|_{\overline{U}}$ are defined in the following diagram:

$$\begin{tikzcd} U \arrow[r, hook] \arrow[drr, "\pi|_U"'] & \overline{U} \arrow[r, hook, "\iota_{\overline{U}}"] \arrow[dr, "\pi|_{\overline{U}}"] & Y \arrow[d, "\pi"] \\ & & X \end{tikzcd}$$

In practice, we will use Theorem 3.1.1 by repetition.

Definition 3.1.2 (Cellular fibration)**.** *Let $\pi : E \longrightarrow X$ be a G-equivariant morphism satisfying the assumption* (2.2.3)*. A (G-equivariant)* ***cellular fibration structure*** *of E is a fibration of closed G-equivariant subvarieties*

$$\varnothing = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_k = E$$

such that $\pi_j := \pi\big|_{E_j \setminus E_{j-1}} : E_j \setminus E_{j-1} \longrightarrow X$ is a G-equivariant affine bundle over X, for any $j \in \{1, \ldots, k\}$.

When $X = \mathrm{pt}$, this filtration is called a ***cellular decomposition*** *of E.*

Theorem 3.1.3 (Cellular fibration, [3, Lemma 5.5.1])**.** *Suppose a G-equivariant morphism $\pi : E \longrightarrow X$ has a cellular fibration structure*

$$\varnothing = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_k = E$$

and $K^G(X)$ is a free $\mathrm{R}(G)$-module with basis $\{y_1, \ldots, y_m\}$.

For $j \in \{1, \ldots, k\}$, let $U_j := E_j \setminus E_{j-1}$, $\overline{U}_j$ denotes the closure of U_j in E_j, $\iota_{\overline{U}_j}$ be the embedding of $\overline{U}_j$ in E, $\pi_{\overline{U}_j} := \pi\big|_{\overline{U}_j} = \pi \circ \iota$, as follows.

$$\begin{tikzcd} \overline{U}_j \arrow[r, hook, "\iota_{\overline{U}_j}"] \arrow[d, "\pi_{\overline{U}_j}"'] & E \arrow[dl, dashed, "\pi"] \\ X & \end{tikzcd}$$

- *$K^G(E)$ is a free $\mathrm{R}(G)$-module with basis*

$$\left\{ \iota_{\overline{U}_j,*}\pi^*_{\overline{U}_j}(y_l) \,\middle|\, 1 \leqslant l \leqslant m, 1 \leqslant j \leqslant k \right\}$$

- *In particular, when $X = \mathrm{pt}$ is a point,*

$$K^G(E) \cong \bigoplus_j \mathrm{R}(G)\iota_{\overline{U}_j,*}\pi^*_{\overline{U}_j}(1_{\mathrm{R}(G)}).$$

When $\overline{U}_j$ is smooth, $\pi^*_{\overline{U}_j}(1_{\mathrm{R}(G)}) = 1_{K^G\left(\pi_{\overline{U}_j}\right)}$.

Most stratifications can be (non-canonically) viewed as cellular decompositions, and the theorem gives us the $\mathrm{R}(G)$-module structure of the total space. Readers can compare this theorem with the cellular cohomology of CW-complexs with no cell in odd dimension.

3.2 Application: module structure

Before we really start working, let us make a shorthand for the basis.

Definition 3.2.1. *Let $\iota_Y : Y \longrightarrow X$ be a closed G-equivariant embedding, $\pi_Y : Y \longrightarrow \mathrm{pt}$ be the projection map. Denote*

$$[Y]_X^G := \iota_{Y,*}\pi_Y^* 1_{\mathrm{R}(G)} \in K^G(X).$$

$$\begin{tikzcd} Y \arrow[r, hook, "\iota_Y"] \arrow[d, "\pi_Y"'] & X \\ \mathrm{pt} \end{tikzcd}$$

Warning 3.2.2. *The symbol $[Y]_X^G$ (weakly) depends on X, and we don't want to mention X all the time. In practice, Y will be the closure of some U_i for the stratification $X = \sqcup_i U_i$, so we can read X from the symbol in the bracket.*

Table 3.1 to 3.3 conclude the results in this section.

	pt	$\mathcal{F}$	$\mathcal{F} \times \mathcal{F}$
GL_n	$\mathrm{R}(T)^W$	$\mathrm{R}(T)$	$\bigoplus_{w'} \mathrm{R}(T)\left[\boldsymbol{\Omega}_{w'}\right]^{\mathrm{GL}_n}$
B	$\mathrm{R}(T)$	$\bigoplus_{w} \mathrm{R}(T)\left[\overline{\Omega}_{w}\right]^{B}$	$\bigoplus_{w,w'} \mathrm{R}(T)\left[\overline{\boldsymbol{\Omega}}_{w,w'}\right]^{B}$
Id	$\mathbb{Z}$	$\bigoplus_{w} \mathbb{Z}\left[\overline{\Omega}_{w}\right]$	$\bigoplus_{w,w'} \mathbb{Z}\left[\overline{\boldsymbol{\Omega}}_{w,w'}\right]$

Table 3.1: Initial case

	pt	$\mathcal{F}_{\underline{\mathbf{d}}}$	$\mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}}'}$	$\widetilde{\mathrm{Rep}}_{\underline{\mathbf{d}}}(Q)$	$\mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}'}$
$G_{\mathbf{d}}$	$\mathrm{R}(T_{\mathbf{d}})^{W_{\mathbf{d}}}$	$\mathrm{R}(T_{\mathbf{d}})$	$\bigoplus_{w'} \mathrm{R}(T_{\mathbf{d}})\left[\overline{\boldsymbol{\Omega}}_{w'}^{u,u'}\right]^{G_{\mathbf{d}}}$	$\mathrm{R}(T_{\mathbf{d}})$	$\bigoplus_{w'} \mathrm{R}(T_{\mathbf{d}})\left[\mathcal{Z}_{w'}^{u,u'}\right]^{G_{\mathbf{d}}}$
$B_{\mathbf{d}}$	$\mathrm{R}(T_{\mathbf{d}})$	$\bigoplus_{w} \mathrm{R}(T_{\mathbf{d}})\left[\overline{\Omega}_{w}^{u}\right]^{B_{\mathbf{d}}}$	$\bigoplus_{w,w'} \mathrm{R}(T_{\mathbf{d}})\left[\overline{\boldsymbol{\Omega}}_{w,w'}^{u,u'}\right]^{B_{\mathbf{d}}}$	$\bigoplus_{w} \mathrm{R}(T_{\mathbf{d}})\left[\overline{\widetilde{\Omega}}_{w}^{u}\right]^{B_{\mathbf{d}}}$	$\bigoplus_{w,w'} \mathrm{R}(T_{\mathbf{d}})\left[\overline{\widetilde{\boldsymbol{\Omega}}}_{w,w'}^{u,u'}\right]^{B_{\mathbf{d}}}$
Id	$\mathbb{Z}$	$\bigoplus_{w} \mathbb{Z}\left[\overline{\Omega}_{w}^{u}\right]$	$\bigoplus_{w,w'} \mathbb{Z}\left[\overline{\boldsymbol{\Omega}}_{w,w'}^{u,u'}\right]$	$\bigoplus_{w} \mathbb{Z}\left[\overline{\widetilde{\Omega}}_{w}^{u}\right]$	$\bigoplus_{w,w'} \mathbb{Z}\left[\overline{\widetilde{\boldsymbol{\Omega}}}_{w,w'}^{u,u'}\right]$

Table 3.2: Relative case

	pt	$\mathcal{F}_{\mathbf{d}}$	$\mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}}$	$\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)$	$\mathcal{Z}_{\mathbf{d}}$
$G_{\mathbf{d}}$	$\mathrm{R}(T_{\mathbf{d}})^{W_{\mathbf{d}}}$	$\bigoplus_{\underline{\mathbf{d}}} \mathrm{R}(T_{\mathbf{d}})\left[\mathcal{F}_{\underline{\mathbf{d}}}\right]^{G_{\mathbf{d}}}$	$\bigoplus_{\varpi'} \mathrm{R}(T_{\mathbf{d}})\left[\overline{\boldsymbol{\mathcal{O}}}_{\varpi'}\right]^{G_{\mathbf{d}}}$	$\bigoplus_{\underline{\mathbf{d}}} \mathrm{R}(T_{\mathbf{d}})\left[\widetilde{\mathrm{Rep}}_{\underline{\mathbf{d}}}(Q)\right]^{G_{\mathbf{d}}}$	$\bigoplus_{\varpi'} \mathrm{R}(T_{\mathbf{d}})\left[\mathcal{Z}_{\varpi'}\right]^{G_{\mathbf{d}}}$
$B_{\mathbf{d}}$	$\mathrm{R}(T_{\mathbf{d}})$	$\bigoplus_{\varpi} \mathrm{R}(T_{\mathbf{d}})\left[\overline{\mathcal{O}}_{\varpi}\right]^{B_{\mathbf{d}}}$	$\bigoplus_{\varpi,\varpi'} \mathrm{R}(T_{\mathbf{d}})\left[\overline{\boldsymbol{\mathcal{O}}}_{\varpi,\varpi'}\right]^{B_{\mathbf{d}}}$	$\bigoplus_{\varpi} \mathrm{R}(T_{\mathbf{d}})\left[\overline{\widetilde{\mathcal{O}}}_{\varpi}\right]^{B_{\mathbf{d}}}$	$\bigoplus_{\varpi,\varpi'} \mathrm{R}(T_{\mathbf{d}})\left[\overline{\widetilde{\boldsymbol{\mathcal{O}}}}_{\varpi,\varpi'}\right]^{B_{\mathbf{d}}}$
Id	$\mathbb{Z}$	$\bigoplus_{\varpi} \mathbb{Z}\left[\overline{\mathcal{O}}_{\varpi}\right]$	$\bigoplus_{\varpi,\varpi'} \mathbb{Z}\left[\overline{\boldsymbol{\mathcal{O}}}_{\varpi,\varpi'}\right]$	$\bigoplus_{\varpi} \mathbb{Z}\left[\overline{\widetilde{\mathcal{O}}}_{\varpi}\right]$	$\bigoplus_{\varpi,\varpi'} \mathbb{Z}\left[\overline{\widetilde{\boldsymbol{\mathcal{O}}}}_{\varpi,\varpi'}\right]$

Table 3.3: Absolute case

First, we work over Setting 1.1.1.

Example 3.2.3. *The complete flag variety $\mathcal{F}$ has a stratification $\mathcal{F} = \sqcup_w \Omega_w$. By extending the Bruhat order on W to a total order $\preccurlyeq$, we get a cellular decomposition of $\mathcal{F}$:*

$$0 \subseteq \Omega_{\mathrm{Id}} \subseteq \quad \cdots \quad \subseteq \bigsqcup\nolimits_{x \preccurlyeq w} \Omega_x \subseteq \quad \cdots \quad \subseteq \bigsqcup\nolimits_x \Omega_x = \mathcal{F}$$

By Theorem 3.1.3,

$$K^B(\mathcal{F}) \cong \bigoplus_w \mathrm{R}(B) \left[\overline{\Omega}_w\right]^B \qquad K(\mathcal{F}) \cong \bigoplus_w \mathbb{Z} \left[\overline{\Omega}_w\right].$$

In particular,

$$\begin{aligned} K^{\mathrm{GL}_n}(\mathcal{F} \times \mathcal{F}) \cong K^B(\mathcal{F}) &\cong \bigoplus_w \mathrm{R}(B) \cdot \mathrm{Ind}_B^{\mathrm{GL}_n} \left(\left[\overline{\Omega}_w\right]^B\right) \\ &\cong \bigoplus_{w'} \mathrm{R}(B) \left[\overline{\boldsymbol{\Omega}}_{w'}\right]^{\mathrm{GL}_n} \end{aligned}$$

Example 3.2.4. *$\mathcal{F} \times \mathcal{F}$ has many stratifications. Consider the stratification $\mathcal{F} \times \mathcal{F} = \sqcup_{w,w' \in W} \boldsymbol{\Omega}_{w,w'}$. By extending the Bruhat order on $W \times W$ (i.e., $(x, x') \leqslant (w, w')$ if and only if $x \leqslant w$ and $x' \leqslant w'$) to a total order $\preccurlyeq$, we get a cellular decomposition of $\mathcal{F} \times \mathcal{F}$:*

$$0 \subseteq \boldsymbol{\Omega}_{\mathrm{Id},\mathrm{Id}} \subseteq \quad \cdots \quad \subseteq \bigsqcup\nolimits_{(x,x') \preccurlyeq (w,w')} \boldsymbol{\Omega}_{x,x'} \subseteq \quad \cdots \quad \subseteq \bigsqcup\nolimits_{x,x'} \boldsymbol{\Omega}_{x,x'} = \mathcal{F} \times \mathcal{F}$$

By Theorem 3.1.3,

$$K^B(\mathcal{F} \times \mathcal{F}) \cong \bigoplus_{w,w'} \mathrm{R}(B) \left[\overline{\boldsymbol{\Omega}}_{w,w'}\right]^B \qquad K(\mathcal{F} \times \mathcal{F}) \cong \bigoplus_{w,w'} \mathbb{Z} \left[\overline{\boldsymbol{\Omega}}_{w,w'}\right].$$

Example 3.2.5. *For computing $K^{\mathrm{GL}_n}(\mathcal{F} \times \mathcal{F})$, consider the ($GL_n$-equivariant) stratification $\mathcal{F} \times \mathcal{F} = \sqcup_{w'} \boldsymbol{\Omega}_{w'}$. Again, we get a cellular decomposition of $\pi_2 : \mathcal{F} \times \mathcal{F} \longrightarrow \mathcal{F}$:*

$$0 \subseteq \boldsymbol{\Omega}_{\mathrm{Id}} \subseteq \quad \cdots \quad \subseteq \bigsqcup\nolimits_{x' \preccurlyeq w'} \boldsymbol{\Omega}_{x'} \subseteq \quad \cdots \quad \subseteq \bigsqcup\nolimits_{x'} \boldsymbol{\Omega}_{x'} = \mathcal{F} \times \mathcal{F}$$

By Theorem 3.1.3 and Example 2.4.2, we get

$$\begin{aligned} K^{\mathrm{GL}_n}(\mathcal{F} \times \mathcal{F}) &\cong \bigoplus_{w'} K^{\mathrm{GL}_n}(\boldsymbol{\Omega}_{w'}) \\ &\cong \bigoplus_{w'} K^B(\Omega_{w'}) \\ &\cong \bigoplus_{w'} \mathrm{R}(B) \left[\overline{\boldsymbol{\Omega}}_{w'}\right]^{\mathrm{GL}_n} \end{aligned}$$

The general case can be solved by the same method.

Example 3.2.6. *By repeating Example 2.1.3 to 2.1.6, we get*

$$\mathrm{R}(N_{\mathbf{d}}) \cong \mathrm{R}(\mathrm{Id}) \cong \mathbb{Z} \qquad \mathrm{R}(B_{\mathbf{d}}) \cong \mathrm{R}(T_{\mathbf{d}}) \cong \mathbb{Z}\left[e_1^{\pm 1}, \ldots, e_{|\mathbf{d}|}^{\pm 1}\right]$$

$$\mathrm{R}(G_{\mathbf{d}}) \cong \mathrm{R}(T_{\mathbf{d}})^{W_{\mathbf{d}}} \cong \mathbb{Z}\left[e_1^{\pm 1}, \ldots, e_{|\mathbf{d}|}^{\pm 1}\right]^{W_{\mathbf{d}}}$$

The induction formula tells us

$$K^{G_{\mathbf{d}}}(\mathcal{F}_{\underline{\mathbf{d}}}) \cong K^{B_{\mathbf{d}}}(\mathrm{pt}) = \mathrm{R}(B_{\mathbf{d}}) \cong \mathbb{Z}\Big[e_1^{\pm 1}, \ldots, e_{|\mathbf{d}|}^{\pm 1}\Big].$$

By repeating Example 3.2.3 to 3.2.4, we get

$$K^{B_{\mathbf{d}}}(\mathcal{F}_{\underline{\mathbf{d}}}) \cong \bigoplus_{w} \mathrm{R}(B_{\mathbf{d}}) \left[\overline{\Omega}_w^u\right]^{B_{\mathbf{d}}} \qquad K(\mathcal{F}_{\underline{\mathbf{d}}}) \cong \bigoplus_{w} \mathbb{Z}\left[\overline{\Omega}_w^u\right]$$

$$K^{G_{\mathbf{d}}}(\mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}}'}) \cong \bigoplus_{w'} \mathrm{R}(B_{\mathbf{d}}) \left[\overline{\boldsymbol{\Omega}}_{w'}^{u,u'}\right]^{G_{\mathbf{d}}}$$

$$K^{B_{\mathbf{d}}}(\mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}}'}) \cong \bigoplus_{w,w'} \mathrm{R}(B_{\mathbf{d}}) \left[\overline{\boldsymbol{\Omega}}_{w,w'}^{u,u'}\right]^{B_{\mathbf{d}}} \qquad K(\mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}}'}) \cong \bigoplus_{w,w'} \mathbb{Z}\left[\overline{\boldsymbol{\Omega}}_{w,w'}^{u,u'}\right]$$

Since

$$\mathcal{F}_{\mathbf{d}} = \bigsqcup_{\underline{\mathbf{d}}} \mathcal{F}_{\underline{\mathbf{d}}} \qquad \mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}} = \bigsqcup_{\underline{\mathbf{d}},\underline{\mathbf{d}}'} \mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}}'}$$

as topological spaces, we get K-theory of $\mathcal{F}_{\mathbf{d}}$ and $\mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}}$ for free. (See Table 3.3)

The calculations of incidence spaces use the same method we introduced in Example 3.2.5.

Example 3.2.7. *We compute $G_{\mathbf{d}}$-equivariant K-theory of the Steinberg variety in this example.*

$$\begin{aligned} K^{G_{\mathbf{d}}}(\mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}'}) &\cong \bigoplus_{w'} K^{G_{\mathbf{d}}}(\widetilde{\boldsymbol{\Omega}}_{w'}^{u,u'}) \\ &\cong \bigoplus_{w'} K^{G_{\mathbf{d}}}(\boldsymbol{\Omega}_{w'}^{u,u'}) \\ &\cong \bigoplus_{w'} K^{B_{\mathbf{d}}}(\Omega_{w'}^{u,u'}) \\ &\cong \bigoplus_{w'} \mathrm{R}(B_{\mathbf{d}}) \left[\overline{\widetilde{\boldsymbol{\Omega}}}_{w'}^{u,u'}\right]^{G_{\mathbf{d}}} \end{aligned}$$

The equivariant cohomology theory can be computed in the same way, see [17, Chapter 7].

Chapter 4

Localization theorem

We have already gotten the module structure of K-theories. However, this basis behaves badly with the convolution product (will be introduced in Section 5.1), because "the information is not concentrated enough". In this chapter we will introduce another basis, which "concentrates information in the T-fixed points". The localization formula describes the transition matrix of two basis. Readers with topological background can compare the localization theorem with the Poincaré-Hopf theorem.

4.1 Euler class

In the category of coherent sheaf, the "proper base change" is usually not true. In order to describe the defect of the diagram, we introduce the Euler class.

Definition 4.1.1 (Euler class, for K-group)**.** *Let X be a G-variety, and $\mathcal{T}$ be a G-equivariant vector bundle over X. The Euler class is defined by*

$$\mathrm{eu}(\mathcal{T}) := \sum_{k=0}^{\infty}(-1)^k \left[\Lambda^k \mathcal{T}^*\right] \in K^G(X)$$

In our examples, X are points and G is a torus. In that case, since we know the representation of a torus (see Example 2.1.4), the Euler class can be explicitly written down. For example, ($X = \mathrm{pt}$)

$$\begin{aligned}
&\mathrm{eu}\,(1) = 1 \\
&\mathrm{eu}\left(\frac{e_1}{e_2}\right) = 1 - \frac{e_2}{e_1} \\
&\mathrm{eu}\left(\frac{e_1}{e_2} + \frac{e_2}{e_3} + \frac{e_3}{e_1}\right) = \left(1 - \frac{e_2}{e_1}\right)\left(1 - \frac{e_3}{e_2}\right)\left(1 - \frac{e_1}{e_3}\right)
\end{aligned}$$

Here we abuse the notation for $\mathrm{R}(T)$ and $\mathrm{Rep}(T)$: the elements inside the bracket of Euler class should be viewed as a vector bundle rather than a $\mathbb{Z}$-linear combination of coherent sheaves.

Remark 4.1.2. For Euler classes in K-theory, we have

$$\begin{aligned}
&\mathrm{eu}(\mathcal{T} \oplus \mathcal{T}') \cong \mathrm{eu}(\mathcal{T}) \cdot \mathrm{eu}(\mathcal{T}') \\
&\mathrm{eu}(\mathcal{L}_1 \otimes \mathcal{L}_2) \neq \mathrm{eu}(\mathcal{L}_1) + \mathrm{eu}(\mathcal{L}_2) \qquad \mathrm{eu}(\mathcal{L}^*) \neq -\,\mathrm{eu}(\mathcal{L})
\end{aligned}$$

for line bundles $\mathcal{L}$, $\mathcal{L}_1$, $\mathcal{L}_2$ over X. We also have equivariant Euler class for cohomology, see [17, Chapter 9], [6, Section 22] for more details. In particular, for any T-representation $\mathcal{T}$ with weight space decomposition $\mathcal{T}^* = \oplus \mathcal{T}^*_\lambda$, the Euler class of $\mathcal{T}$ (for cohomology theory) is defined by

$$\mathrm{eu}'(\mathcal{T}) := \prod_{\lambda \in X^*(T)} \lambda^{\dim \mathcal{T}^*_\lambda} \in \mathrm{S}(T)$$

where $X^*(T)$ embeds in $\mathrm{S}(T)$ by

$$X^*(T) \longrightarrow \mathrm{S}(T) \qquad \sum_i k_i \varepsilon_i \longmapsto \sum_i k_i \lambda_i.$$

For example,

$$\begin{aligned}
&\mathrm{eu}'(1) = 1 \\
&\mathrm{eu}'\left(\frac{e_1}{e_2}\right) = \lambda_2 - \lambda_1 \\
&\mathrm{eu}'\left(\frac{e_1}{e_2} + \frac{e_2}{e_3} + \frac{e_3}{e_1}\right) = (\lambda_2 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_1 - \lambda_3)
\end{aligned}$$

4.2 Statement

We first state one general theorem, which will be connected with both localization formula and excess intersection formula.

Theorem 4.2.1 (Excess base change, [23, Théorème 3.1])**.** *Let* (4.2.1) *be a Cartesian square of G-varieties, ϕ, φ are regular embeddings and f, g are of globally finite* Tor-*dimension. Define $\mathcal{N}_\phi$ and $\mathcal{N}_\varphi$ as the normal cone of ϕ and φ, respectively, and $\mathcal{T} := (g^*\mathcal{N}_\varphi)/\mathcal{N}_\phi$ as a vector bundle over W.*

$$\begin{tikzcd}
\mathcal{N}_\phi \arrow[dr, dash] & & g^*\mathcal{N}_\varphi \arrow[dl, dash] & & \mathcal{N}_\varphi \arrow[dl, dash] \\
& W \arrow[rr, "g"] \arrow[d, "\phi"'] & & Z \arrow[d, "\varphi"] & \\
& Y \arrow[rr, "f"] & & X &
\end{tikzcd} \tag{4.2.1}$$

For any $\alpha \in K^G(Z)$, we have the ***excess base change formula****:*

$$f^* \circ \varphi_*(\alpha) = \phi_*\Big(\mathrm{eu}(\mathcal{T}) \cdot g^*(\alpha)\Big) \qquad \text{in } K^G(Y)$$

where the dot product of $\mathrm{eu}(\mathcal{T})$ *is given by the tensor product in $K^G(W)$.*

By applying Theorem 4.2.1 to the Cartesian square (4.2.2), we get the (fake) localization formula:

$$\begin{tikzcd}
X^T \arrow[r, "\mathrm{Id}"] \arrow[d, "\mathrm{Id}"'] & X^T \arrow[d, "i"] \\
X^T \arrow[r, "i"] & X
\end{tikzcd} \tag{4.2.2}$$

Proposition 4.2.2 (Fake localization formula)**.** *For a smooth T-variety X with finite fixed points $\{x_1, \dots, x_m\}$, let $i : X^T \longrightarrow X$ and $i_k : \{x_k\} \longrightarrow X$ be embeddings. For any $\beta \in K^T(X^T)$, $\beta_k \in K^T(\{x_k\})$, we have formulas*

$$i^* i_* \beta = \mathrm{eu}\left(\bigoplus_k T_{x_k} X\right) \cdot \beta \qquad i_k^* i_{k,*}\beta = \mathrm{eu}\left(T_{x_k} X\right) \cdot \beta_k.$$

This proposition explains some technical details in the localization theorem and localization formula. First, we would like to work on a base ring where Euler classes are invertible, so we use the curly font to indicate everything in the fraction field.

$$\begin{aligned} \mathcal{R}(T) &:= \mathrm{Frac}\big(\mathrm{R}(T)\big) & \mathcal{K}^T(X) &:= K^T(X) \otimes_{\mathrm{R}(T)} \mathcal{R}(T) \\ \mathcal{S}(T) &:= \mathrm{Frac}\big(\mathrm{S}(T)\big) & \mathcal{H}_T^*(X) &:= H_T^*(X) \otimes_{\mathrm{S}(T)} \mathcal{S}(T) \end{aligned}$$

Theorem 4.2.3 (Localization theorem, [17, Theorem 10.1] or [3, Corollary 5.11.3])**.** *Let X be a smooth T-variety, $i : X^T \longrightarrow X$ be the embedding. The morphisms i_*, i^* are isomorphisms after tensored over the fraction field, i.e.,*

$$\mathcal{K}^T(X^T) \xrightarrow{i_*} \mathcal{K}^T(X) \xrightarrow{i^*} \mathcal{K}^T(X^T)$$
$$\mathcal{H}_T^*(X^T) \xrightarrow{i_*} \mathcal{H}_T^*(X) \xrightarrow{i^*} \mathcal{H}_T^*(X^T)$$

are isomorphism as $\mathcal{R}(T)$ or $\mathcal{S}(T)$-modules.

The genuine localization formula is stated as follows.

Theorem 4.2.4 (Localization formula, [17, Theorem 10.2] or [10, Proposition 6])**.** *For a smooth T-variety X with finite fixed points $\{x_1, \dots, x_m\}$, let $i_k : \{x_k\} \longrightarrow X$ be embeddings. For any $\alpha \in \mathcal{K}^T(X)$, we have*

$$\alpha = \sum_{k=1}^{m} \eta_k \cdot i_{k,*} i_k^* \alpha$$

where $\eta_k := \big(\mathrm{eu}(T_{x_k} X)\big)^{-1} \in \mathcal{R}(T)$.

More generally, suppose $f : Y \hookrightarrow X$ is a T-equivariant closed subvariety with finite fixed points $\{x_1, \dots, x_{m'}\}$, let $i'_k : \{x_k\} \longrightarrow Y$ be the embeddings. For any $\beta \in \mathcal{K}^T(Y)$, we have

$$\beta = \sum_{k=1}^{m} \eta_k \cdot i'_{k,*} i_k^* f_* \beta.$$

Let us unravel Theorem 4.2.4 a little bit. By the localization theorem, we get

$$\begin{aligned} [Z]_Y^T &= \sum_{k=1}^{m} \eta_k \cdot i'_{k,*} i_k^* f_* [Z]_Y^T \\ &= \sum_{k=1}^{m} \eta_k \cdot i'_{k,*} \left(i_k^* [Z]_X^T \cdot 1_{\mathrm{R}(T)}\right) && \text{definition of } [Z]_X^T \\ &= \sum_{k=1}^{m} \eta_k \cdot \left(i_k^* [Z]_X^T\right) \cdot \left(i'_{k,*} 1_{\mathrm{R}(T)}\right) && i'_{k,*} \text{ is an } \mathrm{R}(T)\text{-module homomorphism} \\ &= \sum_{k=1}^{m} \eta_k \cdot \left(i_k^* [Z]_X^T\right) \cdot [x_k]_Y^T && \text{definition of } [x_k]_Y^T \end{aligned}$$

When Z is smooth at x_k,[1] denote $g : Z \hookrightarrow X$ and $j_k : \{x_k\} \longrightarrow Z$,

$$\begin{aligned} i_k^*[Z]_X^T &= i_k^* g_*\big(\pi_Z^* 1_{\mathrm{R}(T)}\big) \\ &= \mathrm{eu}\,(j_k^* N_Z X) \cdot j_k^*\big(\pi_Z^* 1_{\mathrm{R}(T)}\big) && \text{excess base change} \\ &= \mathrm{eu}\left(\frac{T_{x_k}X}{T_{x_k}Z}\right) \cdot 1_{\mathrm{R}(T)} && \pi_Z \circ j_k = \mathrm{Id}_{\mathrm{pt}} \\ &= \frac{\mathrm{eu}\,(T_{x_k}X)}{\mathrm{eu}\,(T_{x_k}Z)} && \mathrm{Rep}(T) \text{ is semisimple} \end{aligned}$$

Therefore, the coefficient before $[x_k]_Y^T$ is

$$\eta_k \cdot \big(i_k^*[Z]_X^T\big) = \frac{1}{\mathrm{eu}\,(T_{x_k}X)} \cdot \frac{\mathrm{eu}\,(T_{x_k}X)}{\mathrm{eu}\,(T_{x_k}Z)} = \frac{1}{\mathrm{eu}\,(T_{x_k}Z)}.$$

In other word, we computed the transition matrix between two basis, where the matrix coefficient is roughly the inverse of the Euler class. Keep this is mind, and let us see applications now.

4.3 Application: change of basis

Before we really start working, let us make a shorthand for the basis and the Euler class.

Definition 4.3.1 (Another basis)**.** *For ϖ, ϖ', $x \in \mathbb{W}_{|\mathbf{d}|}$, denote*

$$\begin{aligned} \psi_\varpi &:= [\{F_\varpi\}]^{T_{\mathbf{d}}} = (i_\varpi)_* 1_{\mathrm{R}(T_{\mathbf{d}})} && \in K^{T_{\mathbf{d}}}(\mathcal{F}_{\mathbf{d}}) \\ \psi_\varpi^x &:= [\{F_\varpi\}]^{T_{\mathbf{d}}} = (i_\varpi^x)_* 1_{\mathrm{R}(T_{\mathbf{d}})} && \in K^{T_{\mathbf{d}}}(\overline{\mathcal{O}}_x) \\ \psi_{\varpi,\varpi'} &:= \big[\{F_{\varpi,\varpi'}\}\big]^{T_{\mathbf{d}}} = (i_{\varpi,\varpi'})_* 1_{\mathrm{R}(T_{\mathbf{d}})} && \in K^{T_{\mathbf{d}}}(\mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}}) \\ \psi_{\varpi,\varpi'}^x &:= \big[\{F_{\varpi,\varpi'}\}\big]^{T_{\mathbf{d}}} = (i_{\varpi,\varpi'}^x)_* 1_{\mathrm{R}(T_{\mathbf{d}})} && \in K^{T_{\mathbf{d}}}(\overline{\boldsymbol{\mathcal{O}}}_x) \end{aligned}$$

The same symbols are used for

$$\widetilde{\psi}_\varpi \in K^{T_{\mathbf{d}}}\left(\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)\right) \quad \widetilde{\psi}_\varpi^x \in K^{T_{\mathbf{d}}}\left(\widetilde{\overline{\mathcal{O}}}_x\right) \quad \widetilde{\psi}_{\varpi,\varpi'} \in K^{T_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}}) \quad \widetilde{\psi}_{\varpi,\varpi'}^x \in K^{T_{\mathbf{d}}}(\mathcal{Z}_x).$$

Also, we use underline to twist subscripts, like $\underline{\psi}_{\varpi,\varpi'} := \psi_{\varpi,\varpi\varpi'}$.

By Theorem 4.2.3,

$$\mathcal{K}^{T_{\mathbf{d}}}(\mathcal{F}_{\mathbf{d}}) \cong \bigoplus_\varpi \mathcal{R}(T_{\mathbf{d}})\psi_\varpi \qquad \mathcal{K}^{T_{\mathbf{d}}}(\mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}}) \cong \bigoplus_{\varpi,\varpi'} \mathcal{R}(T_{\mathbf{d}})\psi_{\varpi,\varpi'}$$

$$\mathcal{K}^{T_{\mathbf{d}}}\left(\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)\right) \cong \bigoplus_\varpi \mathcal{R}(T_{\mathbf{d}})\widetilde{\psi}_\varpi \qquad \mathcal{K}^{T_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}}) \cong \bigoplus_{\varpi,\varpi'} \mathcal{R}(T_{\mathbf{d}})\widetilde{\psi}_{\varpi,\varpi'}.$$

Definition 4.3.2 (Shorthand for Euler class)**.** *For ϖ, ϖ', $x \in \mathbb{W}_{|\mathbf{d}|}$, we define Euler classes in $\mathrm{R}(T_{\mathbf{d}})$:*

$$\begin{aligned} \Lambda_\varpi &:= \mathrm{eu}\,(\mathcal{T}_\varpi) & \Lambda_\varpi^x &:= \mathrm{eu}\,(\mathcal{T}_\varpi^x) & \Lambda_{\varpi,\varpi'}^x &:= \mathrm{eu}\left(\mathcal{T}_{\varpi,\varpi'}^x\right) \\ \widetilde{\Lambda}_\varpi &:= \mathrm{eu}\left(\widetilde{\mathcal{T}}_\varpi\right) & \widetilde{\Lambda}_\varpi^x &:= \mathrm{eu}\left(\widetilde{\mathcal{T}}_\varpi^x\right) & \widetilde{\Lambda}_{\varpi,\varpi'}^x &:= \mathrm{eu}\left(\widetilde{\mathcal{T}}_{\varpi,\varpi'}^x\right) \end{aligned}$$

[1] The smoothness guarantees the regular embedding condition in Theorem 4.2.1.

For completeness, denote

$$\Lambda_{\varpi,\varpi'} := \mathrm{eu}\left(\mathcal{T}_{\varpi,\varpi'}\right) \qquad \widetilde{\Lambda}_{\varpi,\varpi'} := \mathrm{eu}\left(\widetilde{\mathcal{T}}_{\varpi,\varpi'}\right).$$

Also, we use underline to twist subscripts.

Now we can compute the transition matrix of two basis.

Example 4.3.3. *Let $X = Y = \mathcal{F}_{\mathbf{d}}$, $T = T_{\mathbf{d}}$, $i_\varpi : \{F_\varpi\} \hookrightarrow \mathcal{F}_{\mathbf{d}}$ be the embedding, $y \in W_{\mathbf{d}}$, we get*

$$\left[\overline{\Omega}_y^u\right]^{T_{\mathbf{d}}} = \sum_{w\le y} \Lambda_{wu}^{-1}\left(i_{wu}^*\left[\overline{\Omega}_y^u\right]^{T_{\mathbf{d}}}\right)\cdot \psi_{wu}.$$

When $\overline{\Omega}_y^u$ is smooth at F_{wu}, $\Lambda_{wu}^{-1}\left(i_{wu}^*\left[\overline{\Omega}_y^u\right]^{T_{\mathbf{d}}}\right) = \left(\mathrm{eu}\left(T_{F_{wu}}\overline{\Omega}_y^u\right)\right)^{-1} = \left(\Lambda_{wu}^{yu}\right)^{-1}$. Especially, for $s \in \Pi_{\mathbf{d}}$,

$$\begin{aligned}
\left[\overline{\Omega}_{\mathrm{Id}}^u\right]^{T_{\mathbf{d}}} &= \left(\Lambda_u^u\right)^{-1}\psi_u = \psi_u\\
\left[\overline{\Omega}_s^u\right]^{T_{\mathbf{d}}} &= \left(\Lambda_u^{su}\right)^{-1}\psi_u + \left(\Lambda_{su}^{su}\right)^{-1}\psi_{su}\\
\left[\mathcal{F}_{\underline{u}}\right]^{T_{\mathbf{d}}} &= \sum_w \Lambda_{wu}^{-1}\psi_{wu}\\
\left[\mathcal{F}_{\mathbf{d}}\right]^{T_{\mathbf{d}}} &= \sum_\varpi \Lambda_\varpi^{-1}\psi_\varpi
\end{aligned}$$

Also, for $s \in \Pi$,

$$\left[\overline{\mathcal{O}}_s\right]^{T_{\mathbf{d}}} = \begin{cases} \left(\Lambda_{\mathrm{Id}}^s\right)^{-1}\psi_{\mathrm{Id}} + \left(\Lambda_s^s\right)^{-1}\psi_s, & s \in \Pi_{\mathbf{d}}\\ \psi_s, & s \notin \Pi_{\mathbf{d}}\end{cases}$$

Example 4.3.4. *Let $X = Y = \widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)$, $T = T_{\mathbf{d}}$, $i_\varpi : \{(\rho_0, F_\varpi)\} \hookrightarrow \widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)$ be the embedding, $y \in W_{\mathbf{d}}$, we get*

$$\left[\widetilde{\overline{\Omega}}_y^u\right]^{T_{\mathbf{d}}} = \sum_{w\le y}\widetilde{\Lambda}_{wu}^{-1}\left(i_{wu}^*\left[\widetilde{\overline{\Omega}}_y^u\right]^{T_{\mathbf{d}}}\right)\cdot\widetilde{\psi}_{wu}.$$

When $\widetilde{\overline{\Omega}}_y^u$ is smooth at F_{wu}, $\widetilde{\Lambda}_{wu}^{-1}\left(i_{wu}^*\left[\widetilde{\overline{\Omega}}_y^u\right]^{T_{\mathbf{d}}}\right) = \left(\mathrm{eu}\left(T_{F_{wu}}\widetilde{\overline{\Omega}}_y^u\right)\right)^{-1} = \left(\widetilde{\Lambda}_{wu}^{yu}\right)^{-1}$. Especially, for $s \in \Pi_{\mathbf{d}}$,

$$\begin{aligned}
\left[\widetilde{\overline{\Omega}}_{\mathrm{Id}}^u\right]^{T_{\mathbf{d}}} &= \left(\widetilde{\Lambda}_u^u\right)^{-1}\widetilde{\psi}_u = \widetilde{\psi}_u\\
\left[\widetilde{\overline{\Omega}}_s^u\right]^{T_{\mathbf{d}}} &= \left(\widetilde{\Lambda}_u^{su}\right)^{-1}\widetilde{\psi}_u + \left(\widetilde{\Lambda}_{su}^{su}\right)^{-1}\widetilde{\psi}_{su}\\
\left[\widetilde{\mathrm{Rep}}_{\underline{\mathbf{d}}}(Q)\right]^{T_{\mathbf{d}}} &= \sum_w \widetilde{\Lambda}_{wu}^{-1}\widetilde{\psi}_{wu}\\
\left[\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)\right]^{T_{\mathbf{d}}} &= \sum_\varpi \widetilde{\Lambda}_\varpi^{-1}\widetilde{\psi}_\varpi
\end{aligned}$$

Also, for $s \in \Pi$,

$$\left[\widetilde{\overline{\mathcal{O}}}_s\right]^{T_{\mathbf{d}}} = \begin{cases} \left(\widetilde{\Lambda}_{\mathrm{Id}}^s\right)^{-1}\widetilde{\psi}_{\mathrm{Id}} + \left(\widetilde{\Lambda}_s^s\right)^{-1}\widetilde{\psi}_s, & s \in \Pi_{\mathbf{d}}\\ \widetilde{\psi}_s, & s \notin \Pi_{\mathbf{d}}\end{cases}$$

Example 4.3.5. *Let* $X = Y = \mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}}$, $T = T_{\mathbf{d}}$, $s \in \Pi$. *Since* $\overline{\mathcal{O}}_s$ *is smooth, we get*

$$\left[\overline{\mathcal{O}}_s\right]^{T_{\mathbf{d}}} = \sum_{\varpi \in \mathbb{W}_{|\mathbf{d}|}} \left(\Lambda^s_{\varpi,\varpi s}\right)^{-1} \psi_{\varpi,\varpi s} + \sum_{\substack{\varpi \in \mathbb{W}_{|\mathbf{d}|} \\ \varpi s \varpi^{-1} \in W_{\mathbf{d}}}} \left(\Lambda^s_{\varpi,\varpi}\right)^{-1} \psi_{\varpi,\varpi}.$$

One can also write $\left[\overline{\mathcal{O}}_\varpi\right]$ *in terms of* $\mathcal{R}(T_{\mathbf{d}})$*-linear combination of those* $\psi_{\varpi,\varpi'}$.

Example 4.3.6. *Let* $X = \mathrm{Rep}_{\mathbf{d}}(Q) \times \mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}}$, $Y = \mathcal{Z}_{\mathbf{d}}$, $T = T_{\mathbf{d}}$, $s \in \Pi$. *Since* $\mathcal{Z}_s$ *is smooth, we get*

$$[\mathcal{Z}_s]^{T_{\mathbf{d}}} = \sum_{\varpi \in \mathbb{W}_{|\mathbf{d}|}} \left(\widetilde{\Lambda}^s_{\varpi,\varpi s}\right)^{-1} \widetilde{\psi}_{\varpi,\varpi s} + \sum_{\substack{\varpi \in \mathbb{W}_{|\mathbf{d}|} \\ \varpi s \varpi^{-1} \in W_{\mathbf{d}}}} \left(\widetilde{\Lambda}^s_{\varpi,\varpi}\right)^{-1} \widetilde{\psi}_{\varpi,\varpi}.$$

One can also write $\left[\overline{\mathcal{O}}_\varpi\right]$ *in terms of* $\mathcal{R}(T_{\mathbf{d}})$*-linear combination of those* $\widetilde{\psi}_{\varpi,\varpi'}$.

Chapter 5

Excess intersection formula

Finally, we are able to compute the convolution structure of the Steinberg variety in Theorem 5.3.8. We first introduce the convolution product, then give an explicit intersection formula, and finally apply theorems to our settings.

5.1 Convolution

The construction of the convolution product is similar to a Fourier-Mukai transformation, which is the composition of pullback, tensor product and proper pushforward.

Definition 5.1.1 (Convolution product)**.** *For the convenience of reading, we divide the whole process into three steps.*

Step1. *Setting.*

Let M_1, M_2, M_3 be smooth quasi-projective G-varieties. For convenience, let

$$M_{ij} := M_i \times M_j \qquad M_{123} = M_1 \times M_2 \times M_3$$

and $p_i^{jk}, p_i := p_i^{123}, p_{ij} := p_{ij}^{123}$ be projections onto some factors, as follows.

$$\begin{tikzcd}
 & & M_{123} \arrow[dl, "p_{12}"'] \arrow[dr, "p_{23}"] \arrow[drrr, bend left, "p_{13}"] & & & \\
 & M_{12} \arrow[dl, "p_1^{12}"'] \arrow[dr, "p_2^{12}"] & & M_{23} \arrow[dl, "p_2^{23}"'] \arrow[dr, "p_3^{23}"] & & M_{13} \arrow[dlllll, dashed, "p_1^{13}"'] \arrow[dl, dashed, "p_3^{13}"] \\
M_1 & & M_2 & & M_3 &
\end{tikzcd}$$

Step2. *Convolution product on the level of varieties.*

For closed G-subvarieties $Z_{12} \subseteq M_{12}$, $Z_{23} \subseteq M_{23}$, let

$$Z_{123} := p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23}) \subseteq M_{123}$$

*be the intersection of two preimages. The **convolution product** of Z_{12} and Z_{23} is defined as*

$$Z_{12} \circ Z_{23} := p_{13}(Z_{123}) \subseteq M_{13}$$

which is a closed G-subvariety of M_{13}.

Step3. *Convolution product on the level of K-theories.*

Let

$$\pi_{12} := p_{12}\big|_{p_{12}^{-1}(Z_{12})} \qquad \pi_{23} := p_{23}\big|_{p_{23}^{-1}(Z_{23})} \qquad \pi_{13} := p_{13}\big|_{Z_{123}}$$

be corresponding morphisms restricted to $p_{12}^{-1}(Z_{12})$, $p_{23}^{-1}(Z_{23})$ and Z_{123}, respectively. We assume that π_{13} is proper, so that we can use proper pushforward in K-theory.

We define the convolution product by

$$* : K_0^G(Z_{12}) \times K_0^G(Z_{23}) \longrightarrow K_0^G(Z_{12} \circ Z_{23}) \qquad (\mathcal{F}_{12}, \mathcal{F}_{23}) \longmapsto \mathcal{F}_{12} * \mathcal{F}_{23}$$

$$\mathcal{F}_{12} * \mathcal{F}_{23} = \pi_{13,*}\left(\pi_{12}^* \mathcal{F}_{12} \otimes \pi_{23}^* \mathcal{F}_{23}\right) \in K_0^G(Z_{12} \circ Z_{23})$$

Remark 5.1.2. Those "Z-varieties" (Z_{12}, $p_{12}^{-1}(Z_{12})$, Z_{123}, etc.) are often singular in practice, so π_{12}^*, π_{23}^* and $\otimes$ are defined in the sense of "restriction with supports", under the "M-varieties" which are smooth. The following diagram best illustrates the "actual" definition.

$$\begin{tikzcd}
K_0^G(Z_{12}) \times K_0^G(Z_{23}) \arrow[r, dashed, "\pi_{12}^* \times \pi_{23}^*"] \arrow[d, "\iota_{Z_{12},*}, \iota_{Z_{23},*}"] & K_0^G(p_{12}^{-1}(Z_{12})) \times K_0^G(p_{12}^{-1}(Z_{23})) \arrow[r, dashed, "\otimes"] \arrow[d] & K_0^G(Z_{123}) \arrow[r, "\pi_{13,*}"] \arrow[d] & K_0^G(Z_{12} \circ Z_{23}) \arrow[d, "\iota_{Z_{12} \circ Z_{23},*}"'] \\
K_0^G(M_{12}) \times K_0^G(M_{23}) \arrow[r, "p_{12}^* \times p_{23}^*"] & K_0^G(M_{123}) \times K_0^G(M_{123}) \arrow[r, "\otimes"] & K_0^G(M_{123}) \arrow[r, "p_{13,*}"] & K_0^G(M_{13})
\end{tikzcd} \tag{5.1.1}$$

The diagram in (5.1.1) commutes by the vanishing of the Euler class. Therefore, one can compute

$$\mathcal{F}_{12} * \mathcal{F}_{23} = p_{13,*}\left(p_{12}^* \iota_{Z_{12},*} \mathcal{F}_{12} \otimes p_{23}^* \iota_{Z_{23},*} \mathcal{F}_{23}\right) \in K_0^G(M_{13}),$$

and then find the preimage of it under the map $\iota_{Z_{12} \circ Z_{23},*}$. This technique will be used in Subsection 5.3.2.

The whole process can be concluded in Figure 5.1.

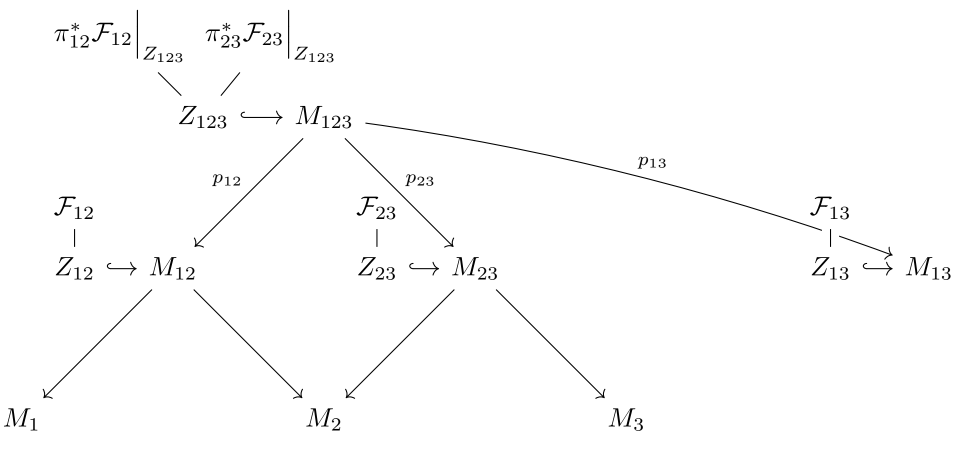


Figure 5.1: Convolution Product

5.2 Statement

To facilitate the computation of intersections (i.e., tensor product in the construction of convolution product), we state the excess intersection formula.

Theorem 5.2.1 (Excess intersection formula, [17, Corollary 9.4])**.** *Let X' be a smooth G-variety, $X \subseteq X'$ be a (maybe singular) closed G-subvariety, and $Y_1, Y_2 \subseteq X$ be closed G-equivariant embeddings (of globally finite* Tor*-dimension). Denote*

$$\begin{aligned} &Y := Y_1 \cap Y_2 \qquad \iota_Y : Y \hookrightarrow X \\ &\mathcal{T} := TX\big|_Y / \left(TY_1\big|_Y + TY_2\big|_Y\right) \end{aligned}$$

$$\begin{tikzcd} N_Y Y_2 \arrow[dr, dash] & \frac{N_Y X}{N_Y Y_1} \arrow[d, dash] & N_{Y_1} X \arrow[d, dash] \\ & Y \arrow[r, "g"] \arrow[d, "\phi"'] & Y_1 \arrow[d, "\varphi"] \\ & Y_2 \arrow[r, "f"] & X \end{tikzcd} \tag{5.2.1}$$

Assume that $TY_1\big|_Y \cap TY_2\big|_Y = TY$. We get the excess intersection formula:

$$[Y_1]^G_X \otimes [Y_2]^G_X = \iota_{Y,*}\left(\mathrm{eu}(\mathcal{T}) \cdot [Y]^G_Y\right).$$

In particular, when $Y = \mathrm{pt}$ is a point, we get simplified formula in $K^G(X)$:

$$[Y_1]^G \otimes [Y_2]^G = \mathrm{eu}(\mathcal{T}) \cdot [Y]^G$$

where $\mathrm{eu}(\mathcal{T}) \in \mathrm{R}(G)$ acts by scalar multiplication.

Readers may find Theorem 5.2.1 as a special case of excess base change theorem. In fact,

$$\begin{aligned} [Y_1]^G_X \otimes [Y_2]^G_X &= [Y_1]^G_X \otimes f_*[Y_2]^G_{Y_2} && \text{definition of } [Y_2]^G_X \\ &= f_*\left(f^*[Y_1]^G_X \otimes [Y_2]^G_{Y_2}\right) && \text{proper projection formula} \\ &= f_*\left(f^*[Y_1]^G_X\right) && \text{Lemma 2.2.9} \\ &= f_*\left(f^*\varphi_*[Y_1]^G_{Y_1}\right) && \text{definition of } [Y_1]^G_X \\ &= f_*\left(\phi_*\left(\mathrm{eu}(\mathcal{T}) \cdot g^*[Y_1]^G_{Y_1}\right)\right) && \text{excess base change to (5.2.1)} \\ &= \iota_{Y,*}\left(\mathrm{eu}(\mathcal{T}) \cdot [Y]^G_Y\right) \end{aligned}$$

The projection formula is stated here.

Proposition 5.2.2 (Projection formula)**.** *For any proper G-equivariant morphism $f : Y \longrightarrow X$ of globally finite* Tor*-dimension, $\alpha \in K^G(Y)$, $\beta \in K^G(X)$, we have proper projection formula:*

$$f_*\alpha \otimes \beta = f_*(\alpha \otimes f^*\beta).$$

5.3 Application: convolution structure

In this section, we will apply Definition 5.1.1 and Theorem 5.2.1 to our typical varieties. In particular, we will get the convolution product formula in terms of basis elements $\widetilde{\phi}_{\varpi}$ and $\widetilde{\phi}_{\varpi,\varpi'}$.

5.3.1 Algebraic structures induced by convolution product

Definition 5.3.1 (Convolution product structure on $K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}})$). *Following notations in 5.1.1, We take $G = G_{\mathbf{d}}$,*

$$\begin{aligned} M_1 &= M_2 = M_3 = \widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q) \\ Z_{12} &= Z_{23} = \mathcal{Z}_{\mathbf{d}} \\ \mathcal{Z}_{\mathbf{d}} &= \widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q) \times_{\mathrm{Rep}_{\mathbf{d}}(Q)} \widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q) \subseteq \widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q) \times \widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q) \end{aligned}$$

By definition, we see that $\mathcal{Z}_{\mathbf{d}} \circ \mathcal{Z}_{\mathbf{d}} = \mathcal{Z}_{\mathbf{d}}$. Therefore, we define a ring structure on $K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}})$:

$$* : K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}}) \times K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}}) \longrightarrow K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}}).$$

Definition 5.3.2 ($K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}})$-module structure on $K^{G_{\mathbf{d}}}(\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q))$). *Following notations in 5.1.1, We take $G = G_{\mathbf{d}}$,*

$$\begin{aligned} M_1 &= M_2 = \widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q) & M_3 &= \{\mathrm{pt}\} \\ Z_{12} &= \mathcal{Z}_{\mathbf{d}} & Z_{23} &= \widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q) \end{aligned}$$

By definition, we see that $\mathcal{Z}_{\mathbf{d}} \circ \widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q) = \widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)$. Therefore, we define a $K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}})$-module structure on $K^{G_{\mathbf{d}}}\left(\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)\right)$:

$$\star : K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}}) \times K^{G_{\mathbf{d}}}\left(\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)\right) \longrightarrow K^{G_{\mathbf{d}}}\left(\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)\right).$$

Remark 5.3.3. Notice that in the construction of the convolution product, pullback, tensor product and proper pushforward are compatible with the forgetful map of groups. Therefore, the following diagrams commute:

$$\begin{tikzcd} K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}}) \times K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}}) \arrow[r, "*"] \arrow[d] & K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}}) \arrow[d] \\ K^{T_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}}) \times K^{T_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}}) \arrow[r, "*"] \arrow[d] & K^{T_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}}) \arrow[d] \\ \mathcal{K}^{T_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}}) \times \mathcal{K}^{T_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}}) \arrow[r, "*"] & \mathcal{K}^{T_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}}) \end{tikzcd}
\qquad
\begin{tikzcd} K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}}) \times K^{G_{\mathbf{d}}}\left(\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)\right) \arrow[r, "\star"] \arrow[d] & K^{G_{\mathbf{d}}}\left(\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)\right) \arrow[d] \\ K^{T_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}}) \times K^{T_{\mathbf{d}}}\left(\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)\right) \arrow[r, "\star"] \arrow[d] & K^{T_{\mathbf{d}}}\left(\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)\right) \arrow[d] \\ \mathcal{K}^{T_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}}) \times \mathcal{K}^{T_{\mathbf{d}}}\left(\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)\right) \arrow[r, "\star"] & \mathcal{K}^{T_{\mathbf{d}}}\left(\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)\right) \end{tikzcd}$$

Definition 5.3.4 ($K^{G_{\mathbf{d}}}(\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q))$-module structure on $K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}})$). *We know that*

$$\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q) \cong \mathcal{Z}_{\mathrm{Id}} \subseteq \mathcal{Z}_{\mathbf{d}}, \qquad \mathcal{Z}_{\mathrm{Id}} \circ \mathcal{Z}_{\mathrm{Id}} = \mathcal{Z}_{\mathrm{Id}},$$

so $K^{G_{\mathbf{d}}}\left(\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)\right)$ can be realized as a $R(G_{\mathbf{d}})$-subalgebra of $K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}})$, and $K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}})$ has the $K^{G_{\mathbf{d}}}\left(\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)\right)$-module structure induced by the convolution product:

$$* : K^{G_{\mathbf{d}}}\left(\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)\right) \times K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}}) \longrightarrow K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}}).$$

5.3.2 Convolution product formula

In this subsection, we compute the convolution product in the bottom line of the diagram in Remark 5.3.3.

Proposition 5.3.5 (Convolution product formula)**.** *For ϖ, ϖ', ϖ'', $\varpi''' \in \mathbb{W}_{|\mathbf{d}|}$, we have*

$$\begin{aligned}\widetilde{\psi}_{\varpi,\varpi'} * \widetilde{\psi}_{\varpi'',\varpi'''} &= \delta_{\varpi',\varpi''}\widetilde{\Lambda}_{\varpi'}\widetilde{\psi}_{\varpi,\varpi'''}\\ \widetilde{\psi}_{\varpi,\varpi'} \star \widetilde{\psi}_{\varpi''} &= \delta_{\varpi',\varpi''}\widetilde{\Lambda}_{\varpi'}\widetilde{\psi}_{\varpi}.\end{aligned}$$

Proof. Follow the Definition 5.1.1 and Theorem 5.2.1 if needed.

For clearance, we divide the proof into four cases.

<u>**Case 1.**</u> Assume $\varpi' \neq \varpi''$, need to show $\widetilde{\psi}_{\varpi,\varpi'} * \widetilde{\psi}_{\varpi'',\varpi'''} = 0$.

Denote

$$Y_{12} := \{(\rho_0, F_\varpi, F_{\varpi'})\} \subseteq \mathcal{Z}_{\mathbf{d}}, \qquad Y_{23} := \{(\rho_0, F_{\varpi''}, F_{\varpi'''})\} \subseteq \mathcal{Z}_{\mathbf{d}}\,.$$

Since $\varpi' \neq \varpi''$, $p_{12}^{-1}(Y_{12}) \cap p_{23}^{-1}(Y_{23}) = \varnothing$, so

$$\begin{aligned}\widetilde{\psi}_{\varpi,\varpi'} * \widetilde{\psi}_{\varpi'',\varpi'''} =&\ [Y_{12}]^{T_{\mathbf{d}}}_{\mathcal{Z}_{\mathbf{d}}} * [Y_{23}]^{T_{\mathbf{d}}}_{\mathcal{Z}_{\mathbf{d}}}\\ &= p_{13,*}\left(p_{12}^*\,[Y_{12}]^{T_{\mathbf{d}}}_{M_{12}} \otimes p_{23}^*\,[Y_{23}]^{T_{\mathbf{d}}}_{M_{23}}\right)\\ &= p_{13,*}\left(\left[p_{12}^{-1}(Y_{12})\right]^{T_{\mathbf{d}}}_{M_{123}} \otimes \left[p_{12}^{-1}(Y_{23})\right]^{T_{\mathbf{d}}}_{M_{123}}\right)\\ &= 0\end{aligned}$$

<u>**Case 2.**</u> Assume $\varpi' \neq \varpi''$, need to show $\widetilde{\psi}_{\varpi,\varpi'} \star \widetilde{\psi}_{\varpi''} = 0$.

Denote

$$Y_{12} := \{(\rho_0, F_\varpi, F_{\varpi'})\} \subseteq \mathcal{Z}_{\mathbf{d}}, \qquad Y_{23} := \{(\rho_0, F_{\varpi''})\} \subseteq \widetilde{\operatorname{Rep}}_{\mathbf{d}}(Q).$$

Since $\varpi' \neq \varpi''$, $p_{12}^{-1}(Y_{12}) \cap p_{23}^{-1}(Y_{23}) = \varnothing$, so

$$\begin{aligned}\widetilde{\psi}_{\varpi,\varpi'} \star \widetilde{\psi}_{\varpi''} =&\ [Y_{12}]^{T_{\mathbf{d}}}_{\mathcal{Z}_{\mathbf{d}}} \star [Y_{23}]^{T_{\mathbf{d}}}_{\widetilde{\operatorname{Rep}}_{\mathbf{d}}(Q)}\\ &= p_{13,*}\left(p_{12}^*\,[Y_{12}]^{T_{\mathbf{d}}}_{M_{12}} \otimes p_{23}^*\,[Y_{23}]^{T_{\mathbf{d}}}_{M_{23}}\right)\\ &= p_{13,*}\left(\left[p_{12}^{-1}(Y_{12})\right]^{T_{\mathbf{d}}}_{M_{123}} \otimes \left[p_{12}^{-1}(Y_{23})\right]^{T_{\mathbf{d}}}_{M_{123}}\right)\\ &= 0\end{aligned}$$

<u>**Case 3.**</u> For ϖ, ϖ', $\varpi'' \in \mathbb{W}_{|\mathbf{d}|}$, need to show that

$$\widetilde{\psi}_{\varpi,\varpi'} * \widetilde{\psi}_{\varpi',\varpi''} = \widetilde{\Lambda}_{\varpi'}\widetilde{\psi}_{\varpi,\varpi''}.$$

Denote

$$Y_{12} := \{(\rho_0, F_\varpi, F_{\varpi'})\} \subseteq \mathcal{Z}_{\mathbf{d}}, \qquad Y_{23} := \{(\rho_0, F_{\varpi'}, F_{\varpi''})\} \subseteq \mathcal{Z}_{\mathbf{d}},$$

then

$$p_{12}^{-1}(Y_{12}) = Y_{12} \times \widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q) \qquad p_{23}^{-1}(Y_{23}) = \widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q) \times Y_{23}$$
$$p_{12}^{-1}(Y_{12}) \cup p_{23}^{-1}(Y_{23}) = Y \qquad Y_{12} \circ Y_{23} = Y_{13},$$

where

$$Y = \{y\} \qquad y = \big((\rho_0, F_\varpi), (\rho_0, F_{\varpi'}), (\rho_0, F_{\varpi''})\big) \quad \in M_{123}$$
$$Y_{13} = \{y_{13}\} \qquad y_{13} = \big((\rho_0, F_\varpi), (\rho_0, F_{\varpi''})\big) \quad \in M_{13}.$$

Therefore,

$$\begin{aligned}
\widetilde{\psi}_{\varpi,\varpi'} * \widetilde{\psi}_{\varpi',\varpi''} &= [Y_{12}]^{T_{\mathbf{d}}}_{\mathcal{Z}_{\mathbf{d}}} * [Y_{23}]^{T_{\mathbf{d}}}_{\mathcal{Z}_{\mathbf{d}}} \\
&= p_{13,*}\Big(p_{12}^*\,[Y_{12}]^{T_{\mathbf{d}}}_{M_{12}} \otimes p_{23}^*\,[Y_{23}]^{T_{\mathbf{d}}}_{M_{23}}\Big) \\
&= p_{13,*}\Big(\big[p_{12}^{-1}(Y_{12})\big]^{T_{\mathbf{d}}}_{M_{123}} \otimes \big[p_{12}^{-1}(Y_{23})\big]^{T_{\mathbf{d}}}_{M_{123}}\Big) \\
&= p_{13,*}\Big(\mathrm{eu}(\mathcal{T}) \cdot [Y]^{T_{\mathbf{d}}}_{M_{123}}\Big) \\
&= \mathrm{eu}(\mathcal{T}) \cdot [Y]^{T_{\mathbf{d}}}_{M_{13}} \\
&= \widetilde{\Lambda}_{\varpi'} \widetilde{\psi}_{\varpi,\varpi''}
\end{aligned}$$

where

$$\mathcal{T} := \frac{T_y M_{123}}{T_y\big(p_{12}^{-1}(Y_{12})\big) \oplus T_y\big(p_{23}^{-1}(Y_{23})\big)} = \frac{\widetilde{\mathcal{T}}_\varpi \oplus \widetilde{\mathcal{T}}_{\varpi'} \oplus \widetilde{\mathcal{T}}_{\varpi''}}{\widetilde{\mathcal{T}}_\varpi \quad \oplus \quad \widetilde{\mathcal{T}}_{\varpi''}} = \widetilde{\mathcal{T}}_{\varpi'}.$$

<u>**Case 4.**</u> For ϖ, $\varpi' \in \mathbb{W}_{|\mathbf{d}|}$, need to show that

$$\widetilde{\psi}_{\varpi,\varpi'} \star \widetilde{\psi}_{\varpi'} = \widetilde{\Lambda}_{\varpi'} \widetilde{\psi}_{\varpi}.$$

Denote

$$Y_{12} := \{(\rho_0, F_\varpi, F_{\varpi'})\} \subseteq \mathcal{Z}_{\mathbf{d}}, \qquad Y_{23} := \{(\rho_0, F_{\varpi'})\} \subseteq \widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q),$$

then

$$p_{12}^{-1}(Y_{12}) = Y_{12} \times \{\mathrm{pt}\} \qquad p_{23}^{-1}(Y_{23}) = \widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q) \times Y_{23}$$
$$p_{12}^{-1}(Y_{12}) \cup p_{23}^{-1}(Y_{23}) = Y \qquad Y_{12} \circ Y_{23} = Y_{13},$$

where

$$Y = \{y\} \qquad y = \big((\rho_0, F_\varpi), (\rho_0, F_{\varpi'})\big) \quad \in M_{123}$$
$$Y_{13} = \{y_{13}\} \qquad y_{13} = (\rho_0, F_\varpi) \quad \in M_{13}$$

Therefore,

$$\begin{aligned}
\widetilde{\psi}_{\varpi,\varpi'} \star \widetilde{\psi}_{\varpi'} &= [Y_{12}]^{T_{\mathbf{d}}}_{\mathcal{Z}_{\mathbf{d}}} \star [Y_{23}]^{T_{\mathbf{d}}}_{\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)} \\
&= p_{13,*}\Big(p_{12}^*\,[Y_{12}]^{T_{\mathbf{d}}}_{M_{12}} \otimes p_{23}^*\,[Y_{23}]^{T_{\mathbf{d}}}_{M_{23}}\Big) \\
&= p_{13,*}\Big(\big[p_{12}^{-1}(Y_{12})\big]^{T_{\mathbf{d}}}_{M_{123}} \otimes \big[p_{12}^{-1}(Y_{23})\big]^{T_{\mathbf{d}}}_{M_{123}}\Big) \\
&= p_{13,*}\Big(\mathrm{eu}(\mathcal{T}) \cdot [Y]^{T_{\mathbf{d}}}_{M_{123}}\Big) \\
&= \mathrm{eu}(\mathcal{T}) \cdot [Y]^{T_{\mathbf{d}}}_{M_{13}} \\
&= \widetilde{\Lambda}_{\varpi'} \widetilde{\psi}_{\varpi}
\end{aligned}$$

where

$$\mathcal{T} := \frac{T_y M_{123}}{T_y\big(p_{12}^{-1}(Y_{12})\big) \oplus T_y\big(p_{23}^{-1}(Y_{23})\big)} = \frac{\widetilde{\mathcal{T}}_\varpi \oplus \widetilde{\mathcal{T}}_{\varpi'} \oplus 0}{\widetilde{\mathcal{T}}_\varpi \quad \oplus \quad 0} = \widetilde{\mathcal{T}}_{\varpi'}.$$

$\square$

5.3.3 Demazure operator

In this subsection, we will compute the action of some elements in $K^{G_{\mathbf{d}}}(\mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}'})$ acting on $K^{G_{\mathbf{d}}}\left(\widetilde{\mathrm{Rep}}_{\underline{\mathbf{d}}'}(Q)\right)$. As a reminder,

$$
\begin{array}{ccc}
K^{G_{\mathbf{d}}}\left(\widetilde{\mathrm{Rep}}_{\underline{\mathbf{d}}'}(Q)\right) & \cong & \mathrm{R}(T_{\mathbf{d}})\left[\widetilde{\mathrm{Rep}}_{\underline{\mathbf{d}}'}(Q)\right]^{G_{\mathbf{d}}} \\
\big\downarrow & & \big\downarrow \\
K^{T_{\mathbf{d}}}\left(\widetilde{\mathrm{Rep}}_{\underline{\mathbf{d}}'}(Q)\right) & \cong & \bigoplus_{w} \mathrm{R}(T_{\mathbf{d}})\left[\widetilde{\Omega}^{u}_{w}\right]^{T_{\mathbf{d}}}
\end{array}
\tag{5.3.1}
$$

where the $R(T_{\mathbf{d}})$-module structure on $K^{G_{\mathbf{d}}}\left(\widetilde{\mathrm{Rep}}_{\underline{\mathbf{d}}'}(Q)\right)$ is induced by the induction formula.

For $f \in \mathrm{R}(T_{\mathbf{d}}) \cong \mathbb{Z}\left[e_1^{\pm 1}, \ldots, e_{|\mathbf{d}|}^{\pm 1}\right]$, denote $f^u := f \cdot \left[\widetilde{\mathrm{Rep}}_{\underline{\mathbf{d}}'}(Q)\right]^{G_{\mathbf{d}}}$. Under the morphism (5.3.1), f is sent to $f \cdot \left[\widetilde{\mathrm{Rep}}_{\underline{\mathbf{d}}'}(Q)\right]^{T_{\mathbf{d}}}$. Viewing f^u as an element in $\mathcal{K}^{G_{\mathbf{d}}}\left(\widetilde{\mathrm{Rep}}_{\underline{\mathbf{d}}'}(Q)\right)$, we get

$$
f^u = \sum_{w} f(e_1, \ldots, e_{|\mathbf{d}|}) \widetilde{\Lambda}_{wu}^{-1} \widetilde{\psi}_{wu}.
$$

Remark 5.3.6. This formula looks not so compatible with the group action. To facilitate our computation, we relable the coefficient ring before $\widetilde{\psi}_{\varpi}$ by $e_i^{\varpi} := e_{\varpi^{-1}(i)}$, which means that

$$
K^{T_{\mathbf{d}}}\left(\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)\right) \cong \bigoplus_{\varpi} \mathbb{Z}\left[e_1^{\varpi,\pm 1}, \ldots, e_{|\mathbf{d}|}^{\varpi,\pm 1}\right] \widetilde{\psi}_{\varpi}
$$

Therefore,

$$
\begin{aligned}
f^u &= \sum_{w} (wuf)(e_1^{wu}, \ldots, e_{|\mathbf{d}|}^{wu}) \widetilde{\Lambda}_{wu}^{-1} \widetilde{\psi}_{wu}. \\
&\hat{=} \sum_{w} (wuf)\, \widetilde{\Lambda}_{wu}^{-1} \widetilde{\psi}_{wu}.
\end{aligned}
$$

Later, every expression before $\widetilde{\psi}_{\varpi}$ should be viewed as an expression in $\mathbb{Z}\left[e_1^{\varpi,\pm 1}, \ldots, e_{|\mathbf{d}|}^{\varpi,\pm 1}\right]$.

Definition 5.3.7 (Demazure operator)**.** *For $i \in \{1, \ldots, |\mathbf{d}|-1\}$, set $s = s_i$, the (absolute) Demazure operator is defined as*

$$
D_i := [\mathcal{Z}_{s_i}]^{G_{\mathbf{d}}} \in K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}}).
$$

View D_i as an element in $\mathcal{K}^{T_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}})$, we get

$$
D_i = \sum_{\varpi \in \mathbb{W}_{|\mathbf{d}|}} \left(\widetilde{\Lambda}^{s}_{\varpi,\varpi s}\right)^{-1} \widetilde{\psi}_{\varpi,\varpi s} + \sum_{\substack{\varpi \in \mathbb{W}_{|\mathbf{d}|} \\ \varpi s \varpi^{-1} \in W_{\mathbf{d}}}} \left(\widetilde{\Lambda}^{s}_{\varpi,\varpi}\right)^{-1} \widetilde{\psi}_{\varpi,\varpi}.
$$

We also have the relative version. Suppose that $W_{\mathbf{d}} u s_i = W_{\mathbf{d}} u'$ (which guarantees the existence of $\mathcal{Z}_{s_i}^{u,u'}$), the (relative) Demazure operator is defined as

$$
D_i^{u,u'} := \left[\mathcal{Z}_{s_i}^{u,u'}\right]^{G_{\mathbf{d}}} \in K^{G_{\mathbf{d}}}(\mathcal{Z}^{u,u'}).
$$

View $D_i^{u,u'}$ as an element in $\mathcal{K}^{T_{\mathbf{d}}}(\mathcal{Z}^{u,u'})$, we get

$$D_i^{u,u'} = \sum_w \left(\widetilde{\Lambda}^s_{wu,wus}\right)^{-1} \widetilde{\psi}_{wu,wus} + \delta_{u,u'} \sum_w \left(\widetilde{\Lambda}^s_{wu,wu}\right)^{-1} \widetilde{\psi}_{wu,wu}.$$

The equivariant cohomology version of Demazure operators are denoted by ∂_i and $\partial_i^{u,u'}$.

Theorem 5.3.8. *We obtain a formula for the Demazure operator:*

$$D_i^{u,u'} \star f^{u'} = \begin{cases} \left[\dfrac{s_i f}{1-\frac{e_{i+1}}{e_i}} + \dfrac{f}{1-\frac{e_i}{e_{i+1}}}\right]^u & u = u', \\ \left[s_i f \left(1 - \dfrac{e_{i+1}}{e_i}\right)^k\right]^u & u \neq u'. \end{cases}$$

and similarly in equivariant cohomology:

$$\partial_i^{u,u'} \star f^{u'} = \begin{cases} \left[\dfrac{s_i f}{\lambda_{i+1}-\lambda_i} + \dfrac{f}{\lambda_i - \lambda_{i+1}}\right]^u & u = u', \\ \left[s_i f \left(\lambda_{i+1} - \lambda_i\right)^k\right]^u & u \neq u'. \end{cases}$$

In the formula, $\lambda_l^u := \lambda_{u^{-1}(l)}$, and k stands for the number of arrows from the vertex associated to $v_{u(i+1)}$ to the vertex associated to $v_{u(i)}$.

In the computation we mainly focus on the K-theory case. Using 5.3.6, one can compute $D_i^{u,u'} \star f^{u'}$ in terms of ϕ's: ($s := s_i$ for simplicity)

$$\begin{aligned} D_i^{u,u'} \star f^{u'} &= \left(\sum_w \left(\widetilde{\Lambda}^s_{wu,wus}\right)^{-1} \widetilde{\psi}_{wu,wus} + \delta_{u,u'} \sum_w \left(\widetilde{\Lambda}^s_{wu,wu}\right)^{-1} \widetilde{\psi}_{wu,wu}\right) \\ &\qquad \star \left(\sum_w (wu'f)\, \widetilde{\Lambda}^{-1}_{wu'} \widetilde{\psi}_{wu'}\right) \\ &= \left(\sum_w \left(\widetilde{\Lambda}^s_{wu,wus}\right)^{-1} \widetilde{\psi}_{wu,wus}\right) \star \left(\sum_w (wusf)\, \widetilde{\Lambda}^{-1}_{wus} \widetilde{\psi}_{wus}\right) \\ &\quad + \delta_{u,u'} \left(\sum_w \left(\widetilde{\Lambda}^s_{wu,wu}\right)^{-1} \widetilde{\psi}_{wu,wu}\right) \star \left(\sum_w (wuf)\, \widetilde{\Lambda}^{-1}_{wu} \widetilde{\psi}_{wu}\right) \\ &= \left(\sum_w (wusf) \left(\widetilde{\Lambda}^s_{wu,wus}\right)^{-1} \widetilde{\psi}_{wu}\right) + \delta_{u,u'} \left(\sum_w (wuf) \left(\widetilde{\Lambda}^s_{wu,wu}\right)^{-1} \widetilde{\psi}_{wu}\right) \\ &= \sum_w \left[\left(\frac{wusf}{\widetilde{\Lambda}^s_{wu,wus}} + \delta_{u,u'} \frac{wuf}{\widetilde{\Lambda}^s_{wu,wu}}\right) \widetilde{\Lambda}_{wu}\right] \widetilde{\Lambda}^{-1}_{wu} \widetilde{\psi}_{wu} \\ &= \sum_w w \left[\left(\frac{usf}{\widetilde{\Lambda}^s_{u,us}} + \delta_{u,u'} \frac{uf}{\widetilde{\Lambda}^s_{u,u}}\right) \widetilde{\Lambda}_u\right] \widetilde{\Lambda}^{-1}_{wu} \widetilde{\psi}_{wu} \\ &= \sum_w wu \left[\left(\frac{sf}{u^{-1}\widetilde{\Lambda}^s_{u,us}} + \delta_{u,u'} \frac{f}{u^{-1}\widetilde{\Lambda}^s_{u,u}}\right) u^{-1}\widetilde{\Lambda}_u\right] \widetilde{\Lambda}^{-1}_{wu} \widetilde{\psi}_{wu} \end{aligned}$$

$$= \left[\left(\frac{sf}{u^{-1}\widetilde{\Lambda}^s_{u,us}} + \delta_{u,u'}\frac{f}{u^{-1}\widetilde{\Lambda}^s_{u,u}}\right)u^{-1}\widetilde{\Lambda}_u\right]^u$$

Recall Subsection 1.6.5 (especially Proposition 1.6.14), we get

$$\widetilde{\mathcal{T}}^s_{u,us} \cong \mathfrak{r}_{u,us} \oplus \mathfrak{n}^-_u \oplus \mathfrak{m}_{u,us} \qquad \widetilde{\mathcal{T}}^s_{u,u} \cong \mathfrak{r}_{u,us} \oplus \mathfrak{n}^-_u \oplus \mathfrak{m}_{us,u} \qquad \widetilde{\mathcal{T}}_u \cong \mathfrak{r}_u \oplus \mathfrak{n}^-_u.$$

Therefore,

$$D_i^{u,u'} \star f^{u'} = \left[\left(\frac{sf}{u^{-1}\operatorname{eu}(\mathfrak{m}_{u,us})} + \delta_{u,u'}\frac{f}{u^{-1}\operatorname{eu}(\mathfrak{m}_{us,u})}\right)u^{-1}\operatorname{eu}(\mathfrak{d}_{u,us})\right]^u. \tag{5.3.2}$$

Recall the computation in 1.4.9 and Section 4.1. We collect needed information in Table 5.1.

Name	$\mathfrak{g}$	$u^{-1}\mathfrak{g}$	$u^{-1}\operatorname{eu}(\mathfrak{g})$	$u^{-1}\operatorname{eu}'(\mathfrak{g})$	
$\mathfrak{m}_{u,us}$	$\frac{e_{u(i)}}{e_{u(i+1)}}$	$\frac{e_i}{e_{i+1}}$	$1-\frac{e_{i+1}}{e_i}$	$\lambda_{i+1}-\lambda_i$	$u=u'$
	0	0	1	1	$u\neq u'$
$\mathfrak{m}_{us,u}$	$\frac{e_{u(i+1)}}{e_{u(i)}}$	$\frac{e_{i+1}}{e_i}$	$1-\frac{e_i}{e_{i+1}}$	$\lambda_i-\lambda_{i+1}$	$u=u'$
	0	0	1	1	$u\neq u'$
$\mathfrak{d}_{u,us}$	$k\frac{e_{u(i)}}{e_{u(i+1)}}$	$k\frac{e_i}{e_{i+1}}$	$\left(1-\frac{e_{i+1}}{e_i}\right)^k$	$(\lambda_{i+1}-\lambda_i)^k$	

Table 5.1

Theorem 5.3.8 is our final destination in this part. We will express its importance in Subsection 5.3.4, see some generalizations in Section 6.1 and compute some examples in Section 6.2.

5.3.4 Miscellaneous

In this subsection, we collect some more results. The arguments in reference work for both K-theory and cohomology theory.

Proposition 5.3.9. *The action of $K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}})$ on $K^{G_{\mathbf{d}}}(\widetilde{\operatorname{Rep}_{\mathbf{d}}}(Q))$ is faithful.*

Sketch of proof. Reduce the problem to the faithfulness for the action of $\mathcal{K}^{T_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}})$ on $\mathcal{K}^{T_{\mathbf{d}}}(\widetilde{\operatorname{Rep}_{\mathbf{d}}}(Q))$. For details, see [17, Theorem 10.10]. □

Proposition 5.3.10. *The elements $\{D_i^{u,u'}\}_{u,u',i}$ generate $K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}})$ as a $K^{G_{\mathbf{d}}}(\widetilde{\operatorname{Rep}_{\mathbf{d}}}(Q))$-algebra.*

Sketch of proof. See [17, Theorem 11.3]. The key observation is [17, Lemma 7.30, 11.4]. □

Combining these propositions with Theorem 5.3.8, we understand the convolution structure of $K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}})$ theoretically.

Chapter 6

Generalizations, examples and connections

This chapter is devoted for further discussions of Theorem 5.3.8. Generalizations of Theorem 5.3.8 are discussed in Section 6.1, while examples are shown by strands in Section 6.2. Finally, we will mention about the connection between equivariant K-theory and equivariant cohomology in Section 6.3.

6.1 Generalizations

In this section we generalize Theorem 5.3.8 in different directions. Quivers with loops are allowed, and the group actions can be replaced by $G \times \mathbb{C}^{\times}$-actions. After the generalization, we are able to cover the result in [3, Theorem 7.2.5].

6.1.1 Quiver with loops

In this section we still assume the quiver has no cycles. For quiver with loops, we need to redefine Definition 1.5.8 in a strict version:

Definition 6.1.1 (Incidence variety for strict flags)**.** *For a quiver Q with flag-type dimension vector $\underline{\mathbf{d}}$, define*

$$\begin{aligned}
\widetilde{\mathrm{Rep}}_{\underline{\mathbf{d}},\mathrm{str}}(Q) &:= \left\{(\rho, F) \in \mathrm{Rep}_{\mathbf{d}}(Q) \times \mathcal{F}_{\underline{\mathbf{d}}} \,\middle|\, \rho(M_j) \subseteq M_{j-1} \textit{ for any } j\right\} \\
\widetilde{\mathrm{Rep}}_{\mathbf{d},\mathrm{str}}(Q) &:= \{(\rho, F) \in \mathrm{Rep}_{\mathbf{d}}(Q) \times \mathcal{F}_{\mathbf{d}} \,|\, \rho(M_j) \subseteq M_{j-1} \textit{ for any } j\} \\
&= \bigsqcup_{\underline{\mathbf{d}}} \widetilde{\mathrm{Rep}}_{\underline{\mathbf{d}},\mathrm{str}}(Q)
\end{aligned}$$

and $\mu_{\underline{\mathbf{d}},\mathrm{str}}$, $\pi_{\underline{\mathbf{d}},\mathrm{str}}$, $\mu_{\mathbf{d},\mathrm{str}}$, $\pi_{\mathbf{d},\mathrm{str}}$ to be the natural morphisms from the incidence varieties to $\mathrm{Rep}_{\mathbf{d}}(Q)$ or flag varieties.

We then replace $\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)$ by $\widetilde{\mathrm{Rep}}_{\mathbf{d},\mathrm{str}}(Q)$. The Lie algebra $\mathfrak{r}_{\varpi}$ (in Definition 1.4.8) is redefined by

$$\begin{aligned}
\mathfrak{r}_{\varpi} &:= \left\{(f_a)_{a\in Q_1} \in \mathrm{Rep}_{\mathbf{d}}(Q) \,\middle|\, f_a(V_{\varpi,j} \cap V_{s(a)}) \subseteq V_{\varpi,j} \text{ for any } j\right\} \\
&\cong \pi_{\mathbf{d},\mathrm{str}}^{-1}(\{F_{\varpi}\})
\end{aligned}$$

then the same formula in (5.3.2) still works.

Theorem 6.1.2. *We obtain a formula for the Demazure operator:*

$$D_i^{u,u'} \star f^{u'} = \begin{cases} \left[\left(\frac{s_i f}{1-\frac{e_{i+1}}{e_i}} + \frac{f}{1-\frac{e_i}{e_{i+1}}}\right)\left(1-\frac{e_{i+1}}{e_i}\right)^k\right]^u & u = u', \\ \left[s_i f\left(1-\frac{e_{i+1}}{e_i}\right)^k\right]^u & u \neq u'. \end{cases}$$

and similarly in equivariant cohomology:

$$\partial_i^{u,u'} \star f^{u'} = \begin{cases} \left[\left(\frac{s_i f}{\lambda_{i+1}-\lambda_i} + \frac{f}{\lambda_i-\lambda_{i+1}}\right)(\lambda_{i+1}-\lambda_i)^k\right]^u & u = u', \\ \left[s_i f\,(\lambda_{i+1}-\lambda_i)^k\right]^u & u \neq u'. \end{cases}$$

6.1.2 $G \times \mathbb{C}^\times$-action

The second generalization is about $G\times\mathbb{C}^\times$-actions. Recall the Remark 1.5.4. Following the same arguments as in Example 2.1.3-2.1.6 and 2.6.2-2.6.5, we get (in the Setting 1.1.1)

$$\mathrm{R}(N\times\mathbb{C}^\times) \cong \mathrm{R}(\mathbb{C}^\times) \cong \mathbb{Z}[q^{\pm 1}] \qquad \mathrm{S}(N\times\mathbb{C}^\times) \cong \mathrm{S}(\mathbb{C}^\times) \cong \mathbb{Q}[t]$$
$$\mathrm{R}(B\times\mathbb{C}^\times) \cong \mathrm{R}(T\times\mathbb{C}^\times) \cong \mathbb{Z}[q^{\pm 1}]\left[e_1^{\pm 1},\dots,e_n^{\pm 1}\right] \qquad \mathrm{S}(B\times\mathbb{C}^\times) \cong \mathrm{S}(T\times\mathbb{C}^\times) \cong \mathbb{Q}[t][\lambda_1,\dots,\lambda_n]$$
$$\mathrm{R}(G\times\mathbb{C}^\times) \cong \mathbb{Z}[q^{\pm 1}]\left[e_1^{\pm 1},\dots,e_n^{\pm 1}\right]^{S_n} \qquad \mathrm{S}(G\times\mathbb{C}^\times) \cong \mathbb{Q}[t][\lambda_1,\dots,\lambda_n]^{S_n}$$

So everything remains the same except for the change of coefficient ring. In particular, for $D_i^{u,u'} := [\mathcal{Z}_{s_i}^{u,u'}]^{G_{\mathbf{d}}\times\mathbb{C}^\times}$, $f^u := f\cdot\left[\widetilde{\mathrm{Rep}}_{\underline{\mathbf{d}},\mathrm{str}}(Q)\right]^{G_{\mathbf{d}}}$, we have formula (5.3.2), with informations in Table 6.1.

Name	$\mathfrak{g}$	$u^{-1}\mathfrak{g}$	$u^{-1}\,\mathrm{eu}(\mathfrak{g})$	$u^{-1}\,\mathrm{eu}'(\mathfrak{g})$	
$\mathfrak{m}_{u,us}$	$\frac{e_{u(i)}}{e_{u(i+1)}}$	$\frac{e_i}{e_{i+1}}$	$1-\frac{e_{i+1}}{e_i}$	$\lambda_{i+1}-\lambda_i$	$u=u'$
	0	0	1	1	$u\neq u'$
$\mathfrak{m}_{us,u}$	$\frac{e_{u(i+1)}}{e_{u(i)}}$	$\frac{e_{i+1}}{e_i}$	$1-\frac{e_i}{e_{i+1}}$	$\lambda_i-\lambda_{i+1}$	$u=u'$
	0	0	1	1	$u\neq u'$
$\mathfrak{d}_{u,us}$	$k\frac{e_{u(i)}}{e_{u(i+1)}}q^{-1}$	$k\frac{e_i}{e_{i+1}}q^{-1}$	$\left(1-\frac{e_{i+1}}{e_i}q\right)^k$	$(\lambda_{i+1}-\lambda_i+t)^k$	

Table 6.1

Theorem 6.1.3. *When the quiver has no cycle, we have a formula of Demazure operator for the $G_{\mathbf{d}}\times\mathbb{C}^\times$-action:*

$$D_i^{u,u'} \star f^{u'} = \begin{cases} \left[\left(\frac{s_i f}{1-\frac{e_{i+1}}{e_i}} + \frac{f}{1-\frac{e_i}{e_{i+1}}}\right)\left(1-\frac{e_{i+1}}{e_i}q\right)^k\right]^u & u = u', \\ \left[s_i f\left(1-\frac{e_{i+1}}{e_i}q\right)^k\right]^u & u \neq u'. \end{cases}$$

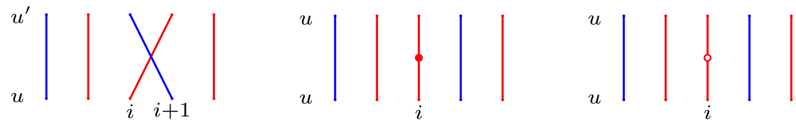
and similar for the equivariant cohomology:

$$\partial_i^{u,u'} \star f^{u'} = \begin{cases} \left[\left(\dfrac{s_i f}{\lambda_{i+1}-\lambda_i} + \dfrac{f}{\lambda_i - \lambda_{i+1}}\right)(\lambda_{i+1}-\lambda_i+t)^k\right]^u & u = u', \\ \left[s_i f\,(\lambda_{i+1}-\lambda_i+t)^k\right]^u & u \neq u'. \end{cases}$$

6.2 From formula to diagram

This section is designed for showing examples. Recall Proposition 1.3.3 that every $\underline{\mathbf{d}}$ or u corresponds to an ordered set of colored points. It can be imagined that the lines connecting two ordered sets represents one element in $K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}})$. Actually, we draw the picture of generators of $K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}})$ in Figure 6.1, where

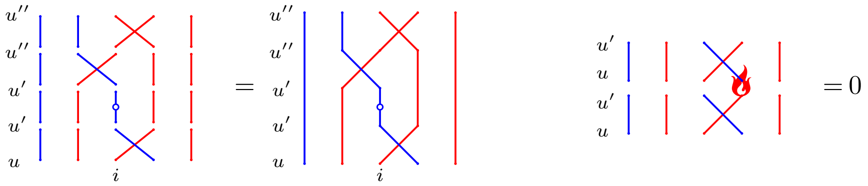
$$e_i^u =: e_{u^{-1}(i)}\left[\widetilde{\mathrm{Rep}}_u(Q)\right]^{G_{\mathbf{d}}} \in K^{G_{\mathbf{d}}}\left(\widetilde{\mathrm{Rep}}_u(Q)\right) \hookrightarrow K^{G_{\mathbf{d}}}\left(\mathcal{Z}^{u,u}\right).$$



$D_i^{u,u'}$ $\quad e_i^u \quad$ $(e_i^u)^{-1}$

Figure 6.1

The convolution product can be then viewed as pictures gluing vertically, where the incompatibility of colors gives 0. For example,



$D_3^{u,u'} * (e_3^{u'})^{-1} * D_2^{u',u''} * D_3^{u'',u''}$ $\qquad$ $D_3^{u,u'} * D_3^{u,u'} = 0$

By Proposition 5.3.10, every element in $K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}}) = \oplus_{u,u'} K^{G_{\mathbf{d}}}(\mathcal{Z}^{u,u'})$ can be expressed as a $\mathbb{Z}$-linear combination of strands. The expressions are not unique, so we need to find out their relations. Some relations are clear from the picture (but still need to check), for example,

$D_3^{u,u'} * D_1^{u',u'''} = D_1^{u,u''} * D_3^{u'',u'''}$ $\qquad$ $D_3^{u,u'} * e_2^{u'} = e_2^u * D_3^{u,u'}$

We won't draw these "obvious" relations later. The first nontrivial relation comes from the following lemma.

Lemma 6.2.1. *For $f \in \mathrm{R}(T_{\mathbf{d}})$, denote $D_i^{u,u'} = \left[\mathcal{Z}_{s_i}^{u,u'}\right]^{G_{\mathbf{d}}}$, $f^u = f \cdot \left[\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)\right]^{G_{\mathbf{d}}} \in K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}})$, we have*

$$D_i^{u,u'} * f^{u'} = (s_i f)^u * D_i^{u,u'} + \delta_{u,u'} \left[(s_i f - f) \frac{e_{i+1}}{e_i} \left(1 - \frac{e_{i+1}}{e_i}\right)^{k-1}\right]^u .$$

Similarly, for the $G_{\mathbf{d}}$-equivariant cohomology, we have

$$\partial_i^{u,u'} * f^{u'} = (s_i f)^u * \partial_i^{u,u'} + \delta_{u,u'} \left[(s_i f - f) (\lambda_{i+1} - \lambda_i)^{k-1}\right]^u .$$

In the formula, k stands for the number of arrows from the vertex associated to $v_{u(i+1)}$ to the vertex associated to $v_{u(i)}$.

Proof. By Proposition 5.3.10, we only need to show, for any $g \in \mathrm{R}(T_{\mathbf{d}})$,

$$D_i^{u,u'} * f^{u'} \star g^{u'} = (s_i f)^u * D_i^{u,u'} \star g^{u'} + \delta_{u,u'} \left[(s_i f - f) \frac{e_{i+1}}{e_i} \left(1 - \frac{e_{i+1}}{e_i}\right)^{k-1}\right]^u \star g^{u'}.$$

Now we apply Theorem 5.3.8. The same argument works for equivariant cohomology. □

Lemma 6.2.1 explains "what happens when a point walk through a crossing". The convolution algebra $H^*_{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}})$ is called the **KLR algebra**. The relations of the KLR algebra can be found in [20, Definition 3.2.2], and we will only show the relations of K-theoretical version.

Warning 6.2.2. *In the following examples, $*$ is often omitted for simplicity.*

6.2.1 One point quiver

We begin with the trivial quiver, which has only one vertex and no arrows. Everything is simplified:

$$\mathbb{W}_{|\mathbf{d}|} = W_{\mathbf{d}}, \quad \mathrm{Min}(\mathbb{W}_{|\mathbf{d}|}, \mathrm{W}_{\mathbf{d}}) = \{\mathrm{Id}\}, \quad \widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q) \cong \mathcal{F}_{\mathbf{d}}, \quad \mathcal{Z}_{\mathbf{d}} \cong \mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}},$$

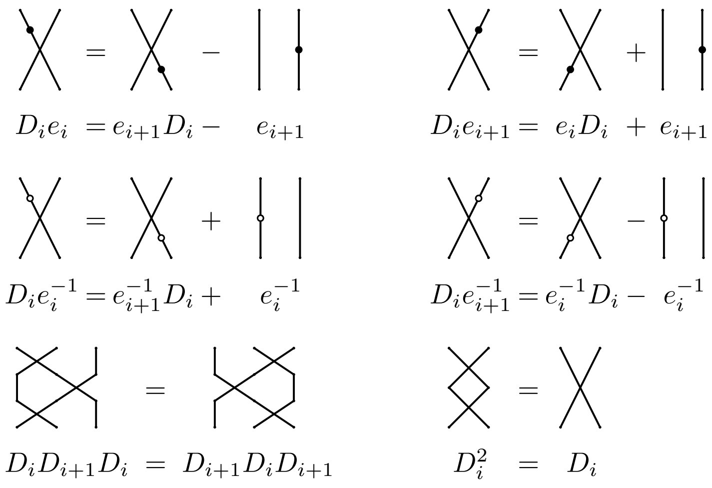
$$K^{G_{\mathbf{d}}}(\mathcal{F}_{\mathbf{d}}) \cong \mathbb{Z}\left[e_1^{\pm 1}, \ldots, e_{|\mathbf{d}|}^{\pm 1}\right], \qquad H^*_{G_{\mathbf{d}}}(\mathcal{F}_{\mathbf{d}}) \cong \mathbb{Q}[\lambda_1, \ldots, \lambda_{|\mathbf{d}|}].$$

In this case, $K^{G_{\mathbf{d}}}(\mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}})$ is called the ***K*-theoretic NilHecke algebra**, and $H^*_{G_{\mathbf{d}}}(\mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}})$ is called the **(cohomological) NilHecke algebra**.

The formulas in Theorem 5.3.8 and Lemma 6.2.1 are simplified: (superscripts are omitted, and functions f in four formulas lie in $K^{G_{\mathbf{d}}}(\mathcal{F}_{\mathbf{d}})$, $K^{G_{\mathbf{d}}}(\mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}})$, $H^*_{G_{\mathbf{d}}}(\mathcal{F}_{\mathbf{d}})$ and $H^*_{G_{\mathbf{d}}}(\mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}})$, respectively)

$$\begin{aligned}
D_i \star f &= \frac{s_i f}{1 - \frac{e_{i+1}}{e_i}} + \frac{f}{1 - \frac{e_i}{e_{i+1}}} \\
\partial_i \star f &= \frac{s_i f}{\lambda_{i+1} - \lambda_i} + \frac{f}{\lambda_i - \lambda_{i+1}} = \frac{f - s_i f}{\lambda_i - \lambda_{i+1}} \\
D_i f &= (s_i f) D_i + \frac{f - s_i f}{1 - \frac{e_i}{e_{i+1}}} \\
\partial_i f &= (s_i f) \partial_i + \frac{f - s_i f}{\lambda_i - \lambda_{i+1}}
\end{aligned}$$

The relations for D_i are shown in Figure 6.2.



$$D_ie_i = e_{i+1}D_i - e_{i+1} \qquad D_ie_{i+1} = e_iD_i + e_{i+1}$$

$$D_ie_i^{-1} = e_{i+1}^{-1}D_i + e_i^{-1} \qquad D_ie_{i+1}^{-1} = e_i^{-1}D_i - e_i^{-1}$$

$$D_iD_{i+1}D_i = D_{i+1}D_iD_{i+1} \qquad D_i^2 = D_i$$

Figure 6.2

6.2.2 A_2-quiver

Now let us consider the A_2-quiver $\bullet \longrightarrow \bullet$. This time we have to color the dots and strands. In this case,

$$K^{G_{\mathbf{d}}}\left(\widetilde{\operatorname{Rep}}_{\mathbf{d}}(Q)\right) \cong \bigoplus_u \mathbb{Z}\left[e_1^{u,\pm 1}, \ldots, e_{|\mathbf{d}|}^{u,\pm 1}\right], \qquad H^*_{G_{\mathbf{d}}}\left(\widetilde{\operatorname{Rep}}_{\mathbf{d}}(Q)\right) \cong \bigoplus_u \mathbb{Q}\left[\lambda_1^u, \ldots, \lambda_{|\mathbf{d}|}^u\right].$$

The formulas in Theorem 5.3.8 and Lemma 6.2.1 are simplified:

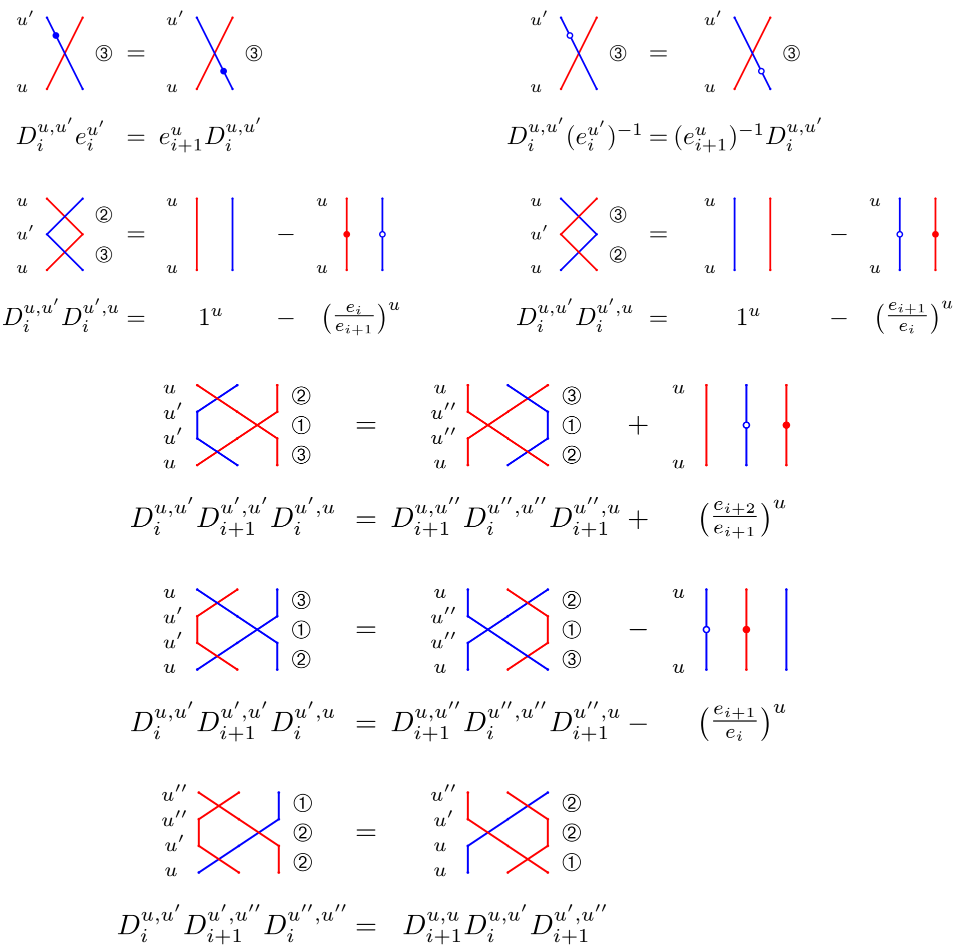
$$D_i^{u,u'} \star f^{u'} = \begin{cases} \left[\dfrac{s_if}{1-\frac{e_{i+1}}{e_i}} + \dfrac{f}{1-\frac{e_i}{e_{i+1}}}\right]^u & ① \ u = u' \ , \\ \left[s_if\left(1-\dfrac{e_{i+1}}{e_i}\right)\right]^u & ② \ u(i+1) \longrightarrow u(i) \ , \\ (s_if)^u & ③ \ u(i) \longrightarrow u(i+1) \ . \end{cases}$$

$$\partial_i^{u,u'} \star f^{u'} = \begin{cases} \left[\dfrac{f-s_if}{\lambda_i-\lambda_{i+1}}\right]^u & ① \ u = u' \ , \\ \left[s_if\left(\lambda_{i+1}-\lambda_i\right)\right]^u & ② \ u(i+1) \longrightarrow u(i) \ , \\ (s_if)^u & ③ \ u(i) \longrightarrow u(i+1) \ . \end{cases}$$

$$D_i^{u,u'}f^{u'} = (s_if)^uD_i^{u,u'} + \left[\frac{f-s_if}{1-\frac{e_i}{e_{i+1}}}\right]^u$$

$$\partial_i^{u,u'}f^{u'} = (s_if)^u\partial_i^{u,u'} + \left[\frac{f-s_if}{\lambda_i-\lambda_{i+1}}\right]^u$$

Part of relations for D_i are shown in Figure 6.3.



$$D_i^{u,u'} e_i^{u'} = e_{i+1}^{u} D_i^{u,u'} \qquad D_i^{u,u'} (e_i^{u'})^{-1} = (e_{i+1}^{u})^{-1} D_i^{u,u'}$$

$$D_i^{u,u'} D_i^{u',u} = 1^u - \left(\tfrac{e_i}{e_{i+1}}\right)^u \qquad D_i^{u,u'} D_i^{u',u} = 1^u - \left(\tfrac{e_{i+1}}{e_i}\right)^u$$

$$D_i^{u,u'} D_{i+1}^{u',u'} D_i^{u',u} = D_{i+1}^{u,u''} D_i^{u'',u''} D_{i+1}^{u'',u} + \left(\tfrac{e_{i+2}}{e_{i+1}}\right)^u$$

$$D_i^{u,u'} D_{i+1}^{u',u'} D_i^{u',u} = D_{i+1}^{u,u''} D_i^{u'',u''} D_{i+1}^{u'',u} - \left(\tfrac{e_{i+1}}{e_i}\right)^u$$

$$D_i^{u,u'} D_{i+1}^{u',u''} D_i^{u'',u''} = D_{i+1}^{u,u} D_i^{u,u'} D_{i+1}^{u',u''}$$

Figure 6.3

6.2.3 1-loop quiver

In this subsection we try to give a simplest example for Section 6.1, which is the 1-loop quiver. In this case,

$$K^{G_{\mathbf{d}}}\left(\widetilde{\mathrm{Rep}}_{\mathbf{d},\mathrm{str}}(Q)\right) \cong \mathbb{Z}\left[e_1^{\pm 1}, \ldots, e_{|\mathbf{d}|}^{\pm 1}\right], \qquad H^*_{G_{\mathbf{d}}}\left(\widetilde{\mathrm{Rep}}_{\mathbf{d},\mathrm{str}}(Q)\right) \cong \mathbb{Q}\left[\lambda_1, \ldots, \lambda_{|\mathbf{d}|}\right].$$

The formulas in Theorem 5.3.8 and Lemma 6.2.1 are simplified:

$$\begin{aligned} D_i \star f &= s_i f + f \cdot \frac{e_{i+1}}{e_i} \\ \partial_i \star f &= f - s_i f \\ D_i f &= (s_i f) D_i + (s_i f - f) \frac{e_{i+1}}{e_i} \\ \partial_i f &= (s_i f) \partial_i + (s_i f - f) \end{aligned}$$

Now for the $G_{\mathbf{d}} \times \mathbb{C}^\times$-action. We have analog of Lemma 6.2.1 for $G_{\mathbf{d}} \times \mathbb{C}^\times$-action:

Lemma 6.2.3. *For $f \in \mathrm{R}(T_{\mathbf{d}} \times \mathbb{C}^\times)$, denote $D_i^{u,u'} = \left[\mathcal{Z}_{s_i}^{u,u'}\right]^{G_{\mathbf{d}} \times \mathbb{C}^\times}$, $f^u = f \cdot \left[\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)\right]^{G_{\mathbf{d}} \times \mathbb{C}^\times} \in$*

$K^{G_{\mathbf{d}}\times\mathbb{C}^\times}(\mathcal{Z}_{\mathbf{d}})$, *we have*

$$D_i^{u,u'} * f^{u'} = (s_i f)^u * D_i^{u,u'} + \delta_{u,u'}\left[(f - s_i f)\frac{\left(1-\frac{e_{i+1}}{e_i}q\right)^k}{1-\frac{e_i}{e_{i+1}}}\right]^u .$$

Similarly, for the $(G_{\mathbf{d}}\times\mathbb{C}^\times)$*-equivariant cohomology, we have*

$$\partial_i^{u,u'} * f^{u'} = (s_i f)^u * \partial_i^{u,u'} + \delta_{u,u'}\left[(f - s_i f)\frac{(\lambda_{i+1}-\lambda_i+t)^k}{\lambda_i-\lambda_{i+1}}\right]^u .$$

In the formula, k *stands for the number of arrows from the vertex associated to* $v_{u(i+1)}$ *to the vertex associated to* $v_{u(i)}$.

In the 1-loop quiver case, notice that

$$K^{G_{\mathbf{d}}\times\mathbb{C}^\times}\left(\widetilde{\mathrm{Rep}}_{\mathbf{d},\mathrm{str}}(Q)\right)\cong\mathbb{Z}\left[q^{\pm1}\right]\left[e_1^{\pm1},\ldots,e_{|\mathbf{d}|}^{\pm1}\right],\quad H^*_{G_{\mathbf{d}}\times\mathbb{C}^\times}\left(\widetilde{\mathrm{Rep}}_{\mathbf{d},\mathrm{str}}(Q)\right)\cong\mathbb{Q}[t]\left[\lambda_1,\ldots,\lambda_{|\mathbf{d}|}\right].$$

The formulas in Theorem 6.1.3 and Lemma 6.2.3 are simplified:

$$\begin{aligned}
D_i\star f &= \left(\frac{s_i f}{1-\frac{e_{i+1}}{e_i}}+\frac{f}{1-\frac{e_i}{e_{i+1}}}\right)\left(1-\frac{e_{i+1}}{e_i}q\right)^k\\
\partial_i\star f &= (f-s_i f)\frac{\lambda_{i+1}-\lambda_i+t}{\lambda_i-\lambda_{i+1}}\\
D_i f &= (s_i f)D_i + (f-s_i f)\frac{1-\frac{e_{i+1}}{e_i}q}{1-\frac{e_i}{e_{i+1}}}\\
\partial_i f &= (s_i f)\partial_i + (f-s_i f)\frac{\lambda_{i+1}-\lambda_i+t}{\lambda_i-\lambda_{i+1}}
\end{aligned}$$

Readers are welcomed to write a complete set of relations.

6.3 Atiyah–Segal completion theorem

Different cohomology theories are connected in a incredible way. In this section, we describe the Atiyah–Segal completion theorem, which connect G-equivariant K-theory with G-equivariant cohomology. Roughly speaking, they are isomorphic after completion (and base change to $\mathbb{Q}$).

For an algebraic group G, we define

$$I := \ker\left(\mathrm{R}(G)\longrightarrow\mathrm{R}(\mathrm{Id})\right)\qquad J := \ker\left(\mathrm{S}(G)\longrightarrow\mathrm{S}(\mathrm{Id})\right)$$

as the augmentation ideals in $\mathrm{R}(G)$ and $\mathrm{S}(G)$, respectively. We define

$$K^G(X)^\wedge_I := \varprojlim_n K^G(X)/\left(I^nK^G(X)\right)\qquad H^*_G(X)^\wedge_J := \varprojlim_n H^*_G(X)/\left(J^nH^*_G(X)\right)$$

as the I-adic (resp. J-adic) completion.

Theorem 6.3.1 (Atiyah–Segal completion theorem)**.** *For a G-variety X, the Atiyah–Segal map from the equivariant K-theory to the ordinary topological K-theory*

$$\mathrm{AS}: K^G(X)^{\wedge}_I \longrightarrow K(\mathrm{E}G \times^G X)$$

is an isomorphism, and the (cohomology) Chern class map (defined in [3, 5.8])

$$\mathrm{ch}^*: K(\mathrm{E}G \times^G X) \longrightarrow H^*_G(X)^{\wedge}_J$$

is an isomorphism after base change to $\mathbb{Q}$.

Instead of explaining terminologies in Theorem 6.3.1, let us see some examples and get a feeling how that works.

Example 6.3.2. *For $G = \mathbb{C}^\times$,*

$$\mathrm{R}(\mathbb{C}^\times) \cong \mathbb{Z}\left[e^{\pm 1}\right], I = (e-1), \quad \mathrm{S}(\mathbb{C}^\times) \cong \mathbb{Q}[\lambda], J = (\lambda),$$

we get the following diagram:

$$\begin{array}{ccccccccc}
K^{\mathbb{C}^\times}(\mathrm{pt}) & \subseteq & K^{\mathbb{C}^\times}(\mathrm{pt})^{\wedge}_I & \xrightarrow{\mathrm{AS}} & K(\mathrm{B}\mathbb{C}^\times) & \xrightarrow{\mathrm{ch}^*} & H^*_{\mathbb{C}^\times}(\mathrm{pt})^{\wedge}_J & \supseteq & H^*_{\mathbb{C}^\times}(\mathrm{pt}) \\
\mathbb{Z}\left[e^{\pm 1}\right] & \subseteq & \mathbb{Z}[[e-1]] & \longrightarrow & \mathbb{Z}[[e-1]] & \longrightarrow & \mathbb{Q}[[\lambda]] & \supseteq & \mathbb{Q}[\lambda] \\
 & & & & e-1 & \longmapsto & e^\lambda - 1 & &
\end{array}$$

This can be generalized to any torus T of rank n.

Example 6.3.3. *For $G = \mathrm{GL}_n$, $X = \mathcal{F}$,*

$$\begin{aligned}
\mathrm{R}(\mathrm{GL}_n) &\cong \mathbb{Z}\left[e_1^{\pm 1}, \ldots, e_n^{\pm 1}\right]^{S_n}, & I' &= \left\{ f \in \mathrm{R}(\mathrm{GL}_n) \;\middle|\; f(1, \ldots, 1) = 0 \right\} \\
\mathrm{S}(\mathrm{GL}_n) &\cong \mathbb{Q}[\lambda_1, \ldots, \lambda_n]^{S_n}, & J' &= \left\{ f \in \mathrm{S}(\mathrm{GL}_n) \;\middle|\; f(0, \ldots, 0) = 0 \right\}.
\end{aligned}$$

We have the commutative diagram

$$\begin{array}{ccccccccc}
K^{\mathrm{GL}_n}(\mathcal{F}) & \subseteq & K^{\mathrm{GL}_n}(\mathcal{F})^{\wedge}_{I'} & \xrightarrow{\mathrm{AS}} & K\left(\mathrm{E}GL_n \times^{\mathrm{GL}_n} \mathcal{F}\right) & \xrightarrow{\mathrm{ch}^*} & H^*_{GL_n}(\mathcal{F})^{\wedge}_{J'} & \supseteq & H^*_{GL_n}(\mathcal{F}) \\
\wr\Vert & & \wr\Vert & & \wr\Vert & & \wr\Vert & & \wr\Vert \\
K^T(\mathrm{pt}) & \subseteq & K^T(\mathrm{pt})^{\wedge}_I & \xrightarrow{\mathrm{AS}} & K(BT) & \xrightarrow{\mathrm{ch}^*} & H^*_T(\mathrm{pt})^{\wedge}_J & \supseteq & H^*_T(\mathrm{pt})
\end{array}$$

which reduce to Example 6.3.3.

Under this isomorphism, the Demazure operator D_i is sent to $\partial_i * \dfrac{\lambda_i - \lambda_{i+1}}{1 - \exp(\lambda_i - \lambda_{i+1})}$, i.e., the diagram (6.3.1) *commutes:*

$$\begin{array}{ccc}
K^{\mathrm{GL}_n}(\mathcal{F})^{\wedge}_{I'} \otimes_{\mathbb{Z}} \mathbb{Q} & \xrightarrow{\mathrm{ch}^* \circ \mathrm{AS}} & H^*_{GL_n}(\mathcal{F})^{\wedge}_{J'} \\
{\scriptstyle D_i}\big\downarrow & & \big\downarrow{\scriptstyle \partial_i * \frac{\lambda_i - \lambda_{i+1}}{1 - e^{\lambda_i - \lambda_{i+1}}}} \\
K^{\mathrm{GL}_n}(\mathcal{F})^{\wedge}_{I'} \otimes_{\mathbb{Z}} \mathbb{Q} & \xrightarrow{\mathrm{ch}^* \circ \mathrm{AS}} & H^*_{GL_n}(\mathcal{F})^{\wedge}_{J'}
\end{array} \tag{6.3.1}$$

As a quotient of two (different types of) Euler class, the **Todd class**

$$\mathrm{Td}_i := \frac{\lambda_i - \lambda_{i+1}}{1 - e^{\lambda_i - \lambda_{i+1}}}$$

measures the noncommutativity of (6.3.1) when $\partial_i * \dfrac{\lambda_i - \lambda_{i+1}}{1 - \exp(\lambda_i - \lambda_{i+1})}$ is replaced by ∂_i.

Part II

Affine pavings of partial flag varieties

Chapter 7

Preliminaries for Affine Pavings

The goal of the two chapters is to fine an affine paving of the partial flag variety $\mathrm{Flag}_{\underline{\mathbf{f}}}(X)$, which generalizes the variety $\mathrm{Flag}_{\underline{\mathbf{d}}}(X)$ defined in 1.5.9. We have constructed some stratifications of most varieties by orbits in Subsection 1.6.1 and 1.6.2, and some are even affine pavings. For $\mathrm{Flag}_{\underline{\mathbf{f}}}(X)$, we have no natural group actions, so we can not find affine pavings by methods in Subsection 1.6.1 and 1.6.2. To achieve our goal, we extend methods from [12] and [16] as well as techniques from Auslander–Reiten theory.

In this chapter, we realize the partial flag variety $\mathrm{Flag}_{\underline{\mathbf{f}}}(X)$ as a quiver Grassmannian of the extended quiver, prove some Ext-vanishing properties, and also give a short introduction of Auslander–Reiten theory.

Setting 7.0.1. *Throughout Chapter 7 and 8, R is a $\mathbb{C}$-algebra with unit, and $\mathrm{Mod}(R)$ denotes the category of R-modules of finite dimension. For a representation $X \in \mathrm{rep}(Q)$, we denote by $X_i := e_i X$ the $\mathbb{C}$-linear space at the vertex $i \in v(Q)$. We denote by $P(i)$, $I(i)$ and $S(i)$ the indecomposable projective, injective, simple modules corresponding to the vertex i, respectively.*

7.1 Extended quiver

In this section, we introduce the notion of extended quiver which allows to view partial flag varieties as quiver Grassmannians. Intuitively, a flag of quiver representations can be encoded as a subspace of a representation of the extended quiver.

Definition 7.1.1 (Extended quiver)**.** *For a quiver Q and an integer $d \geqslant 1$, the extended quiver Q_d is defined as follows:*

- *The vertex set of Q_d is defined as the Cartesian product of the vertex set of Q and $\{1, \ldots, d\}$, i.e.,*
$$v(Q_d) = v(Q) \times \{1, \ldots, d\}.$$
- *There are two types of arrows: for each $(i, r) \in v(Q) \times \{1, \ldots, d-1\}$, there is one arrow from (i, r) to $(i, r+1)$; for each arrow $i \longrightarrow j$ in Q and $r \in \{1, \ldots, d\}$, there is one arrow from (i, r) to (j, r).*

The extended quiver Q_d is exactly the same quiver as $\hat{\Gamma}_d$ in [16, Definition 2.2]. The next definition is a small variation:

Definition 7.1.2 (Strict extended quiver)**.** *For a quiver Q and an integer $d \geqslant 2$, the strict extended quiver $Q_{d,\mathrm{str}}$ is defined as follows:*

- *The vertex set of Q_d is defined as the Cartesian product of the vertex set of Q and $\{1, \ldots, d\}$, i.e.,*
$$v(Q_{d,\mathrm{str}}) = v(Q) \times \{1, \ldots, d\}.$$
- *We have two types of arrows: for each $(i, r) \in v(Q) \times \{1, \ldots, d-1\}$, there is one arrow from (i, r) to $(i, r+1)$; for each arrow $i \longrightarrow j$ in quiver Q and $r \in \{2, \ldots, d\}$, there is one arrow from (i, r) to $(j, r-1)$.*

Example 7.1.3. *The (strict) extended quiver for a Dynkin quiver Q of type A_4 looks as follows.*

$$\begin{tikzcd} \bullet \arrow[r] & \bullet & \bullet \arrow[l] \arrow[r] & \bullet \end{tikzcd}$$

Q

$$\begin{tikzcd} \bullet \arrow[r] & \bullet & \bullet \arrow[l] \arrow[r] & \bullet \\ \bullet \arrow[u] \arrow[r] & \bullet \arrow[u] & \bullet \arrow[u] \arrow[l] \arrow[r] & \bullet \arrow[u] \\ \bullet \arrow[u] \arrow[r] & \bullet \arrow[u] & \bullet \arrow[u] \arrow[l] \arrow[r] & \bullet \arrow[u] \end{tikzcd}$$

Q_3

$$\begin{tikzcd} \bullet \arrow[dr] & \bullet & \bullet \arrow[dl] \arrow[dr] & \bullet \\ \bullet \arrow[u] \arrow[dr] & \bullet \arrow[u] & \bullet \arrow[u] \arrow[dl] \arrow[dr] & \bullet \arrow[u] \\ \bullet \arrow[u] & \bullet \arrow[u] & \bullet \arrow[u] & \bullet \arrow[u] \end{tikzcd}$$

$Q_{3,\mathrm{str}}$

Next, we define the quiver algebras for later use.

Definition 7.1.4 (Algebra of an extended quiver)**.** *For an extended quiver Q_d, let $\mathbb{C}Q_d$ be the corresponding path algebra, and I be the ideal of $\mathbb{C}Q_d$ identifying all the paths with the same sources and targets. The algebra of the extended quiver Q_d is defined as*
$$R_d := \mathbb{C}Q_d/I.$$
Similarly, we define the algebra $R_{d,\mathrm{str}} := \mathbb{C}Q_{d,\mathrm{str}}/I$ for the strict extended quiver.

By abuse of notation, we often abbreviate R_d and $R_{d,\mathrm{str}}$ by R.

7.2 Canonical functor Φ

We follow [16, 2.3] in this section with a few variations.

Definition 7.2.1 (Partial flag variety)**.** *For a quiver representation $X \in \mathrm{rep}(Q)$ and an integer $d \geqslant 1$, we define*
$$\mathrm{Flag_d}(X) := \{0 \subseteq M_1 \subseteq \cdots M_d \subseteq X\}$$
as the collection of all partial flags of length d, and call it the partial flag variety associated to X.

Definition 7.2.2 (Strict partial flag variety)**.** *For a quiver representation $X \in \mathrm{rep}(Q)$ and an integer $d \geqslant 2$, we define*
$$\mathrm{Flag_{d,str}}(X) := \{0 \subseteq M_1 \subseteq \cdots M_d \subseteq X \mid x.M_{k+1} \subseteq M_k\}$$
as the collection of all strict partial flags of length d, and call it the strict partial flag variety associated to X.

Definition 7.2.3 (Grassmannian)**.** *Let R be the bounded quiver algebra defined in Definition 7.2.1 or 7.2.2. For a module $T \in \mathrm{Mod}(R)$, the Grassmannian $\mathrm{Gr}^R(T)$ is defined as the set of all submodules of T, i.e.,*
$$\mathrm{Gr}^R(T) := \{T' \subseteq T \textit{ as the submodule}\}.$$

Definition 7.2.4 (Canonical functor Φ). *The canonical functor $\Phi : \mathrm{rep}(Q) \longrightarrow \mathrm{Mod}(R)$ is defined as follows:*

- $(\Phi(X))_{(i,r)} := X_i$;
- $(\Phi(X))_{(i,r)\to(i,r+1)} := \mathrm{Id}_{X_i}$;
- *Either* $(\Phi(X))_{(i,r)\to(j,r)} := X_{i\to j}$ *for* $R = R_d$,
  or $(\Phi(X))_{(i,r)\to(j,r-1)} := X_{i\to j}$ *for* $R = R_{d,\mathrm{str}}$.

The functor Φ helps to realize a partial flag as a quiver subrepresentation.

Proposition 7.2.5. *For a representation $X \in \mathrm{rep}(Q)$, the canonical functor Φ induces isomorphisms*

$$\mathrm{Flag}_{\mathrm{d}}(X) \cong \mathrm{Gr}^{R_d}(\Phi(X)) \qquad \mathrm{Flag}_{\mathrm{d,str}}(X) \cong \mathrm{Gr}^{R_{d,\mathrm{str}}}(\Phi(X)).$$

Proof. The isomorphism maps a flag $M : M_1 \subseteq \cdots \subseteq M_d$ to a representation $\Phi'(M)$ with $\Phi'(M)_{(i,r)} = M_{i,r}$ and obvious morphisms for arrows. The non-strict case is mentioned in [16, page 4] and the strict case works similarly. □

Example 7.2.6. *Consider the quiver Q: $x \longrightarrow y \longleftarrow z \longrightarrow w$, and let X: $X_x \longrightarrow X_y \longleftarrow X_z \longrightarrow X_w$ be a representation. The varieties $\mathrm{Flag}_3(X), \mathrm{Flag}_{3,\mathrm{str}}(X)$ then arise as quiver Grassmannian as shown in Figure 7.1.*

In many cases, the proof of the strict case and the non-strict case is the same, so we often treat them in the same way. For example, we may abbreviate the formula in Proposition 7.2.5 as

$$\mathrm{Flag}(X) \cong \mathrm{Gr}(\Phi(X)).$$

7.3 Dimension vector

In this section we define some varieties indexed by dimension vectors. Recall Definition 1.2.7 and 1.2.11. Notice that every module $T \in \mathrm{Mod}(\mathbb{C}Q/I)$ can be viewed as a representation of Q, so we automatically have a notion of dimension vector for T.

We can write the (strict) partial flag variety and Grassmannian as disjoint union of several pieces. Since $v(Q_{d,(\mathrm{str})}) = v(Q) \times \{1, \ldots, d\}$, any dimension vector $\underline{\mathbf{f}}$ of R can be viewed as d dimension vectors $(\underline{\mathbf{f}}_1, \ldots, \underline{\mathbf{f}}_d)$. Define

$$\begin{aligned}
\mathrm{Flag}_{\underline{\mathbf{f}}}(X) &:= \{0 \subseteq M_1 \subseteq \cdots M_d \subseteq X \mid \underline{\mathbf{dim}}\, M_k = \underline{\mathbf{f}}_k\} && \subseteq \mathrm{Flag}_{\mathrm{d}}(X),\\
\mathrm{Flag}_{\underline{\mathbf{f}},\mathrm{str}}(X) &:= \{0 \subseteq M_1 \subseteq \cdots M_d \subseteq X \mid x.M_{k+1} \subseteq M_k,\, \underline{\mathbf{dim}}\, M_k = \underline{\mathbf{f}}_k\} && \subseteq \mathrm{Flag}_{\mathrm{d,str}}(X),\\
\mathrm{Gr}^R_{\underline{\mathbf{f}}}(T) &:= \{T' \subseteq T \text{ with } \underline{\mathbf{dim}}\, T' = \underline{\mathbf{f}}\} && \subseteq \mathrm{Gr}^R(T).
\end{aligned}$$

Then from the Proposition 7.2.5 we get

$$\mathrm{Flag}_{\underline{\mathbf{f}}}(X) \cong \mathrm{Gr}^{R_d}_{\underline{\mathbf{f}}}(\Phi(X)) \qquad \mathrm{Flag}_{\underline{\mathbf{f}},\mathrm{str}}(X) \cong \mathrm{Gr}^{R_{d,\mathrm{str}}}_{\underline{\mathbf{f}}}(\Phi(X)).$$

Finally, we need to define the Euler form of two dimension vectors. For this we need to define the set of virtual arrows of the quivers Q_d and $Q_{d,\mathrm{str}}$. Following Example 7.3.3, the virtual arrows of the quivers Q_3 and $Q_{3,\mathrm{str}}$ are depicted in red.

$$
\begin{array}{ccc}
\left\{\begin{array}{c}
X\colon\ X_x \longrightarrow X_y \longleftarrow X_z \longrightarrow X_w \\
\rotatebox[origin=c]{90}{$\subseteq$} \\
X_3\colon X_{3x} \longrightarrow X_{3y} \longleftarrow X_{3z} \longrightarrow X_{3w} \\
\rotatebox[origin=c]{90}{$\subseteq$} \\
X_2\colon X_{2x} \longrightarrow X_{2y} \longleftarrow X_{2z} \longrightarrow X_{2w} \\
\rotatebox[origin=c]{90}{$\subseteq$} \\
X_1\colon X_{1x} \longrightarrow X_{1y} \longleftarrow X_{1z} \longrightarrow X_{1w}
\end{array}\right\}
& \longleftrightarrow &
\left\{\begin{array}{c}
\begin{array}{rccccccc}
 & X_x & \longrightarrow & X_y & \longleftarrow & X_z & \longrightarrow & X_w \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\Phi(X)\colon & X_x & \longrightarrow & X_y & \longleftarrow & X_z & \longrightarrow & X_w \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 & X_x & \longrightarrow & X_y & \longleftarrow & X_z & \longrightarrow & X_w
\end{array} \\
\rotatebox[origin=c]{90}{$\subseteq$} \\
\begin{array}{ccccccc}
X_{3x} & \longrightarrow & X_{3y} & \longleftarrow & X_{3z} & \longrightarrow & X_{3w} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
X_{2x} & \longrightarrow & X_{2y} & \longleftarrow & X_{2z} & \longrightarrow & X_{2w} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
X_{1x} & \longrightarrow & X_{1y} & \longleftarrow & X_{1z} & \longrightarrow & X_{1w}
\end{array}
\end{array}\right\} \\
\mathrm{Flag}_3(X) & \longleftrightarrow & \mathrm{Gr}^{R_3}(\Phi(X))
\end{array}
$$

$$
\begin{array}{ccc}
\left\{\begin{array}{c|c}
\begin{array}{c}
X\colon\ X_x \longrightarrow X_y \longleftarrow X_z \longrightarrow X_w \\
\rotatebox[origin=c]{90}{$\subseteq$} \\
X_3\colon X_{3x} \to X_{3y} \leftarrow X_{3z} \to X_{3w} \\
\rotatebox[origin=c]{90}{$\subseteq$} \\
X_2\colon X_{2x} \to X_{2y} \leftarrow X_{2z} \to X_{2w} \\
\rotatebox[origin=c]{90}{$\subseteq$} \\
X_1\colon X_{1x} \to X_{1y} \leftarrow X_{1z} \to X_{1w}
\end{array} & x.X_{k+1} \subseteq X_k
\end{array}\right\}
& \longleftrightarrow &
\left\{\begin{array}{c}
\begin{array}{rccccccc}
 & X_x & & X_y & & X_z & & X_w \\
 & \uparrow & \searrow & \uparrow & \swarrow & \uparrow & \searrow & \uparrow \\
\Phi(X)\colon & X_x & & X_y & & X_z & & X_w \\
 & \uparrow & \searrow & \uparrow & \swarrow & \uparrow & \searrow & \uparrow \\
 & X_x & & X_y & & X_z & & X_w
\end{array} \\
\rotatebox[origin=c]{90}{$\subseteq$} \\
\begin{array}{ccccccc}
X_{3x} & & X_{3y} & & X_{3z} & & X_{3w} \\
\uparrow & \searrow & \uparrow & \swarrow & \uparrow & \searrow & \uparrow \\
X_{2x} & & X_{2y} & & X_{2z} & & X_{2w} \\
\uparrow & \searrow & \uparrow & \swarrow & \uparrow & \searrow & \uparrow \\
X_{1x} & & X_{1y} & & X_{1z} & & X_{1w}
\end{array}
\end{array}\right\} \\
\mathrm{Flag}_{3,\mathrm{str}}(X) & \longleftrightarrow & \mathrm{Gr}^{R_{3,\mathrm{str}}}(\Phi(X))
\end{array}
$$

Figure 7.1

Definition 7.3.1 (Virtual arrows of the quiver Q_d)**.** *For $d \geqslant 1$, the virtual arrows of the quiver Q_d is defined as a triple $\big(va(Q_d), s, t\big)$, where*

$$va(Q_d) := a(Q) \times \{1, \dots, d-1\}$$

is a finite set, and $s, t : va(Q_d) \longrightarrow v(Q_d)$ are maps defined by

$$s\big((i \to j, r)\big) = (i, r) \qquad t\big((i \to j, r)\big) = (j, r+1).$$

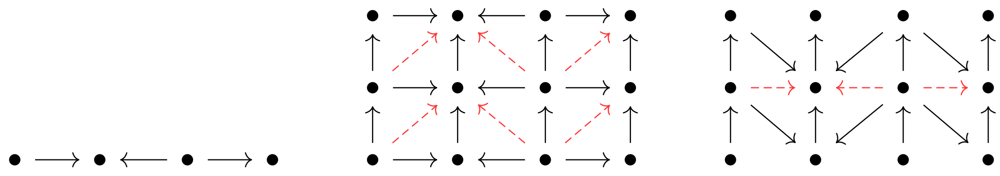
Definition 7.3.2 (Virtual arrows of the quiver $Q_{d,\mathrm{str}}$)**.** *For $d \geqslant 2$, the virtual arrows of the quiver $Q_{d,\mathrm{str}}$ is defined as a triple $\big(va(Q_{d,\mathrm{str}}), s, t\big)$, where*

$$va(Q_{d,\mathrm{str}}) := a(Q) \times \{2, \dots, d-1\}$$

is a finite set, and $s, t : va(Q_{d,\mathrm{str}}) \longrightarrow v(Q_{d,\mathrm{str}})$ are maps defined by

$$s\big((i \to j, r)\big) = (i, r) \qquad t\big((i \to j, r)\big) = (j, r).$$

Example 7.3.3.



Q $\qquad$ Q_3 $\qquad$ $Q_{3,\mathrm{str}}$

Definition 7.3.4 (Euler form of R)**.** *Let R be a bounded quiver algebra defined in Definition 7.1.4. We denote*

$$
\begin{aligned}
v(R) &:= \{\textit{vertices in } Q_d \textit{ or } Q_{d,\mathrm{str}}\},\\
a(R) &:= \{\textit{arrows in } Q_d \textit{ or } Q_{d,\mathrm{str}}\},\\
va(R) &:= \{\textit{virtual arrows in } Q_d \textit{ or } Q_{d,\mathrm{str}}\}.
\end{aligned}
$$

For two dimension vectors $\underline{\mathbf{f}}, \underline{\mathbf{g}}$ of R, the Euler form $\langle \underline{\mathbf{f}}, \underline{\mathbf{g}} \rangle_R$ is defined by

$$\langle \underline{\mathbf{f}}, \underline{\mathbf{g}} \rangle_R := \sum_{i \in v(R)} f_i g_i - \sum_{b \in a(R)} f_{s(b)} g_{t(b)} + \sum_{c \in va(R)} f_{s(c)} g_{t(c)}.$$

7.4 Ext-vanishing properties

We will show that some higher rank extension group are zero, which will be a key ingredient in the proofs of Section 8.1.

For a bounded quiver algebra R defined in Definition 7.1.4, we have a standard resolution for every R-module T:

$$0 \to \bigoplus_{c \in va(Q)} Re_{t(c)} \otimes_{\mathbb{C}} e_{s(c)} T \to \bigoplus_{b \in a(Q)} Re_{t(b)} \otimes_{\mathbb{C}} e_{s(b)} T \to \bigoplus_{i \in v(Q)} Re_i \otimes_{\mathbb{C}} e_i T \to T \to 0$$

$$r \otimes x \longmapsto \begin{smallmatrix} rc_1 \otimes x + r \otimes b_1 x \\ -rc_2 \otimes x - r \otimes b_2 x \end{smallmatrix} \qquad\qquad r \otimes x \longmapsto rx$$

$$r \otimes x \longmapsto rb \otimes x - r \otimes bx$$

There are exactly two paths of length two from $s(c)$ to $t(c)$ for any virtual arrow c, which we denoted by $b_1 c_1$ and $b_2 c_2$ in the above. By definition, these paths are identified in R.

Lemma 7.4.1. *Let $M, N \in \mathrm{rep}(Q)$.*

(1) $\mathrm{gl.dim}\, R \leqslant 2$;
(2) The functor $\Phi : \mathrm{rep}(Q) \longrightarrow \mathrm{Mod}(R)$ is exact and fully faithful;
(3) Φ maps projective module to projective module, and maps injective module to injective module;
(4) $\mathrm{Ext}^i_{\mathbb{C}Q}(M, N) \cong \mathrm{Ext}^i_R(\Phi(M), \Phi(N))$;
(5) $\mathrm{proj.dim}\, \Phi(M) \leqslant 1$. $\mathrm{inj.dim}\, \Phi(M) \leqslant 1$;

Proof.

For (1), this follows from the standard resolution.

For (2), it follows by direct inspection, see [16, Lemma 2.3].

For (3), we reduce to the case of indecomposable projective modules, and observe that

$$\Phi(P(i)) = P\big((i,1)\big), \qquad \Phi(I(i)) = I\big((i,d)\big).$$

For (4), it comes from the fact that Φ is fully faithful and maps projective module to projective module.

For (5), notice that the minimal projective resolution of M is of length 1, and $\Phi(-)$ sends the projective resolution of M to the projective resolution of $\Phi(M)$ by (3), thus we get $\operatorname{proj.dim}\Phi(M) \leqslant 1$. The injective dimension of $\Phi(M)$ is computed in a similar way. $\square$

The following key lemma will be crucial later.

Lemma 7.4.2. *Let $X, S \in \operatorname{rep}(Q)$ and $V \subseteq \Phi(X), W \subseteq \Phi(S)$, $T \in \operatorname{Mod}(R)$. Then $\operatorname{Ext}^2_R(W, T) = 0$ and $\operatorname{Ext}^2_R(T, \Phi(X)/V) = 0$.*

Proof. The short exact sequence

$$0 \longrightarrow W \longrightarrow \Phi(S) \longrightarrow \Phi(S)/W \longrightarrow 0$$

induces the long exact sequence

$$\cdots \longrightarrow \operatorname{Ext}^2_R(\Phi(S), T) \longrightarrow \operatorname{Ext}^2_R(W, T) \longrightarrow \operatorname{Ext}^3_R(\Phi(S)/W, T) \longrightarrow \cdots .$$

By Lemma 7.4.1 (1) and (5), $\operatorname{Ext}^3_R(\Phi(S)/W, T)$ and $\operatorname{Ext}^2_R(\Phi(S), T)$ are both 0, so $\operatorname{Ext}^2_R(W, T) = 0$.

Similarly, from the short exact sequence

$$0 \longrightarrow V \longrightarrow \Phi(X) \longrightarrow \Phi(X)/V \longrightarrow 0$$

we get the induced long exact sequence

$$\cdots \longrightarrow \operatorname{Ext}^2_R(T, \Phi(X)) \longrightarrow \operatorname{Ext}^2_R(T, \Phi(X)/V) \longrightarrow \operatorname{Ext}^3_R(T, V) \longrightarrow \cdots ,$$

so $\operatorname{Ext}^2_R(T, \Phi(X)/V) = 0$. $\square$

We will frequently use extension groups as well as long exact sequences, so we introduce some abbreviations. For Q-representations M, N and R-modules T, T', we denote

$$\begin{aligned} [M,N]^i &:= \dim_{\mathbb{C}} \operatorname{Ext}^i_{\mathbb{C}Q}(M,N) \qquad & [M,N] &:= \dim_{\mathbb{C}} \operatorname{Hom}_{\mathbb{C}Q}(M,N) \\ [T,T']^i &:= \dim_{\mathbb{C}} \operatorname{Ext}^i_R(T,T') \qquad & [T,T'] &:= \dim_{\mathbb{C}} \operatorname{Hom}_R(T,T') \end{aligned}$$

and write the Euler form as

$$\langle T, T' \rangle_R := \sum_{i=0}^{\infty} (-1)^i [T,T']^i \quad = [T,T'] - [T,T']^1 + [T,T']^2.$$

Lemma 7.4.3 (Homological interpretation of the Euler form)**.** *For two R-modules T, T', we have*

$$\langle T, T' \rangle_R = \langle \underline{\mathbf{dim}}\, T, \underline{\mathbf{dim}}\, T' \rangle_R .$$

Proof. Compute $\langle T, T' \rangle_R$ by applying the functor $\operatorname{Hom}_R(-, T')$ to the standard resolution of the R-module T. $\square$

7.5 A crash course on Auslander–Reiten theory

In this section, we will introduce concepts in Auslander–Reiten theory one by one: indecomposable representation, irreducible morphism, Auslander–Reiten translation, Auslander–Reiten sequence, Auslander–Reiten quiver, and minimal sectional mono. The main references for the material covered in this section are [4, 16].

Definition 7.5.1 (Indecomposable module)**.** *Fix an algebra R. A non-zero module $M \in$* mod (R) *is called indecomposable if M can not be written as a direct sum of two non-zero submodules. The set of all indecomposable modules is denoted by* $\mathrm{ind}(R)$*.*

There are several descriptions of the indecomposable representations in special cases. For instance:

- By Gabriel's theorem [11, Theorem 2.1], the functor $\underline{\mathbf{dim}}$ yields a bijection from the indecomposable representations of a Dynkin quiver to the positive roots of the associated Lie algebra.

  There is a unique indecomposable representation of maximal dimension vector which corresponds to the unique maximal positive root. This is shown in Table 7.1.
- By [4, Theorem 2, p34], in the affine case, the functor $\underline{\mathbf{dim}}$ yields a surjective map from the indecomposable representations to the positive roots of the associated affine diagram. The map is ∞-to-1 when the root is imaginary, and is 1-to-1 when the root is real.[1]

  We also have a unique minimal imaginary root δ which controls the whole indecomposable representation theory, as shown in Table 7.1.
- All indecomposable representations of Dynkin quivers and all indecomposable representations of affine quivers corresponding to the positive real roots α with $\alpha < \delta$ or $\langle \alpha, \delta \rangle \neq 0$ are rigid, i.e., $[M, M]^1 = 0$. They are also bricks, i.e., $[M, M]^1 = 0$ and $[M, M] = 1$.[2]

Indecomposable representations form the vertices of Auslander–Reiten quiver, while irreducible morphisms form the arrows.

Definition 7.5.2 (Irreducible morphism)**.** *Given two indecomposable representations $T, T' \in$* $\mathrm{Mod}(R)$*, let*

$$\begin{aligned}\mathrm{rad}(T, T') &:= \left\{ f \in \mathrm{Hom}_R(T, T') \middle| f \text{ is not invertible} \right\} \\ &= \begin{cases} \mathrm{Hom}_R(T, T') & T \ncong T', \\ \mathrm{Jac}(\mathrm{End}_R(T)) & T \cong T'. \end{cases}\end{aligned}$$

be the radical, and let

$$\mathrm{rad}^2(T, T') := \bigcup_{S \in \mathrm{ind}(R)} \mathrm{Im}\left[\mathrm{rad}(T, S) \times \mathrm{rad}(S, T') \longrightarrow \mathrm{rad}(T, T') \right]$$

be the subspace of $\mathrm{rad}(T, T')$*. A morphism $f \in \mathrm{Hom}_R(T, T')$ is called irreducible if $f \in$* $\mathrm{rad}(T, T') \setminus \mathrm{rad}^2(T, T')$*.*

The definition of irreducible morphism applies to any representation, and one can easily show that any irreducible morphism is either injective or surjective.

[1] The root $\alpha \in \underline{\mathbf{dim}}(Q)$ is called real if $\langle \alpha, \alpha \rangle = 1$, and called imaginary if $\langle \alpha, \alpha \rangle = 0$.

[2] Any rigid indecomposable module of a hereditary algebra is a brick.

Type	maximal positive real root(Dynkin)	minimal positive imaginary root δ(affine)
A	1 — 1 — $\cdots$ — 1 — 1	1 1 — 1 — $\cdots$ — 1 — 1
D	1 | 1 — 1 — $\cdots$ — 2 — 1	1 1 | | 1 — 2 — $\cdots$ — 2 — 1
E_6	2 | 1 — 2 — 3 — 2 — 1	1 — 2 | 1 — 2 — 3 — 2 — 1
E_7	2 | 1 — 2 — 3 — 4 — 3 — 2	2 | 1 — 2 — 3 — 4 — 3 — 2 — 1
E_8	3 | 2 — 3 — 4 — 5 — 6 — 4 — 2	3 | 1 — 2 — 3 — 4 — 5 — 6 — 4 — 2

Table 7.1: Roots which control all other roots.

Definition 7.5.3. *Let $R = \mathbb{C}Q/I$ be a bounded quiver algebra. We define the Nakayama functor ν_R, Auslander–Reiten translation τ_R, and inverse Auslander–Reiten translation τ_R^{-1}, as follows:*

$$\begin{aligned}
\nu_R: &\quad \mathrm{Mod}(R) \xrightarrow{\mathrm{Hom}_R(-,{}_RR)} \mathrm{Mod}(R^{op}) \xrightarrow{\mathrm{Hom}_{\mathbb{C}}(-,\mathbb{C})} \mathrm{Mod}(R),\\
\tau_R: &\quad \underline{\mathrm{mod}}(R) \xrightarrow{\mathrm{Ext}^1_R(-,{}_RR)} \underline{\mathrm{mod}}(R^{op}) \xrightarrow{\mathrm{Hom}_{\mathbb{C}}(-,\mathbb{C})} \overline{\mathrm{mod}}(R),\\
\tau_R^{-1}: &\quad \overline{\mathrm{mod}}(R) \xrightarrow{\mathrm{Hom}_{\mathbb{C}}(-,\mathbb{C})} \underline{\mathrm{mod}}(R^{op}) \xrightarrow{\mathrm{Ext}^1_{R^{op}}(-,R_R)} \underline{\mathrm{mod}}(R).
\end{aligned}$$

Here $\underline{\mathrm{mod}}(R)$ and $\overline{\mathrm{mod}}(R)$ denote the stable module categories. The objects are the same as in $\mathrm{Mod}(R)$, and their morphisms are modified by "collapsing" the morphisms passing through projective/injective modules to zero, i.e.,

$$\begin{aligned}
\mathrm{Mor}_{\underline{\mathrm{mod}}(R)}(T,T') &:= \mathrm{Mor}_{\mathrm{Mod}(R)}(T,T')/(f: T \to P \to T', P \text{ is projective}),\\
\mathrm{Mor}_{\overline{\mathrm{mod}}(R)}(T,T') &:= \mathrm{Mor}_{\mathrm{Mod}(R)}(T,T')/(f: T \to I \to T', I \text{ is injective}).
\end{aligned}$$

These modifications guarantee that the Auslander–Reiten translation τ_R is indeed a functor. For convenience, we abbreviate $\mathrm{Mor}_{\underline{\mathrm{mod}}(R)}$, $\mathrm{Mor}_{\overline{\mathrm{mod}}(R)}$, $\mathrm{Mor}_{\mathrm{Mod}(R)}$ as $\underline{\mathrm{Hom}}_R$, $\overline{\mathrm{Hom}}_R$, Hom_R, and ignore the subscription R in the symbol τ_R.

The Auslander–Reiten translation has many magical properties. For example, τ_R induces the one-to-one correspondence between non-projective indecomposable representations and non-injective indecomposable representations. We would also frequently use the Auslander–Reiten formulas: ($(-)^\vee = \mathrm{Hom}_{\mathbb{C}}(-,\mathbb{C})$ is the dual)

$$\big(\overline{\mathrm{Hom}}_R(T,\tau T')\big)^\vee \xrightarrow{\sim} \mathrm{Ext}^1_R(T',T)$$

$$\big(\underline{\mathrm{Hom}}_R(\tau^{-1}T,T')\big)^\vee \xrightarrow{\sim} \mathrm{Ext}^1_R(T',T)$$

which is functorial for any $T, T' \in \mathrm{Mod}(R)$. Especially, when T is not injective, $\overline{\mathrm{Hom}}_R(T,\tau T') = \mathrm{Hom}_R(T,\tau T')$, we get $[T',T]^1 = [T,\tau T']$; when T' is not projective, $\underline{\mathrm{Hom}}_R(\tau^{-1}T,T') = \mathrm{Hom}_R(\tau^{-1}T,T')$, we get $[T',T]^1 = [\tau^{-1}T,T']$.

For the Auslander–Reiten sequence there can be many equivalent definitions, and we only present one due to limitations of space.

Definition 7.5.4 (Auslander–Reiten sequence)**.** *For $X \in \operatorname{ind}(R)$ non-projective, an epimorphism $g : E \longrightarrow X$ is called* ***right almost split*** *if g is not split epi and every homomorphism $h : T \longrightarrow X$ which is not split epi factors through E. The short exact sequence*

$$0 \longrightarrow \tau X \longrightarrow E \xrightarrow{g} X \longrightarrow 0$$

is called an Auslander–Reiten sequence if g is right almost split.

All the concepts introduced in this section can be clearly observed from the Auslander–Reiten quiver. In the Auslander–Reiten quiver the vertices are indecomposable representations, the arrows are irreducible morphisms among indecomposable representations, Auslander–Reiten translation is labeled as the dotted arrow, and the Auslander–Reiten sequence can be read by collecting all paths from τX to X. For instance, in Figure 7.2 we can get an Auslander–Reiten sequence

$$0 \longrightarrow \substack{2\\12321} \longrightarrow \substack{1\\12211} \oplus \substack{1\\11110} \oplus \substack{1\\01221} \longrightarrow \substack{1\\12221} \longrightarrow 0$$

of the corresponding quiver.

Finally we move forward to the definition of minimal sectional mono. The rest can be skipped until Lemma 8.3.1.

Definition 7.5.5 (Sectional morphism)**.** *Let Q be a quiver of Dynkin/affine type, and $M, N \in \operatorname{rep}(Q)$ be two indecomposable representations of Q, which are preprojective*[3] *when Q is affine. A morphism $f \in \operatorname{Hom}_{\mathbb{C}Q}(M, N)$ is called sectional if f can be written as the composition*

$$f : M = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{t-1}} X_{t-1} \xrightarrow{f_t} X_t = N$$

where $f_i \in \operatorname{Hom}_{\mathbb{C}Q}(X_{i-1}, X_i)$ are irreducible morphisms between indecomposable representations, and $\tau X_{i+2} \ncong X_i$ for any suitable i.

Remark 7.5.6. Let f be a sectional morphism. If the underlying quiver Q is a Dynkin/affine quiver without oriented cycles, then $X_0, \ldots, X_t$ are uniquely determined, and $f_1, \ldots, f_t$ are unique up to constant.

Lemma 7.5.7. *Any sectional morphism $f \in \operatorname{Hom}_{\mathbb{C}Q}(M, N)$ is either surjective or injective.*

Proof. When Q is a quiver without oriented cycles, then $[N, M]^1 \leqslant [M, \tau N] = 0$, thus by [16, Lemma 7] we get the result; when Q is of type $\tilde{A}$, the result comes from [16, Lemma 51]. □

Definition 7.5.8 (Sectional mono, minimal sectional mono)**.** *Let Q be a quiver without oriented cycles. A sectional morphism $f \in \operatorname{Hom}_{\mathbb{C}Q}(M, N)$ is called as a sectional mono if f is injective; a sectional mono is called minimal if $f_t \circ \cdots \circ f_{i+1} : X_i \longrightarrow N$ are surjective for any $i \in \{1, 2, \ldots, t\}$.*

[3] A representation $M \in \operatorname{rep}(Q)$ is called preprojective if $\tau^k M$ is projective for some $k \geqslant 0$. Similarly, A representation $M \in \operatorname{rep}(Q)$ is called preinjective if $\tau^{-k} M$ is injective for some $k \geqslant 0$.

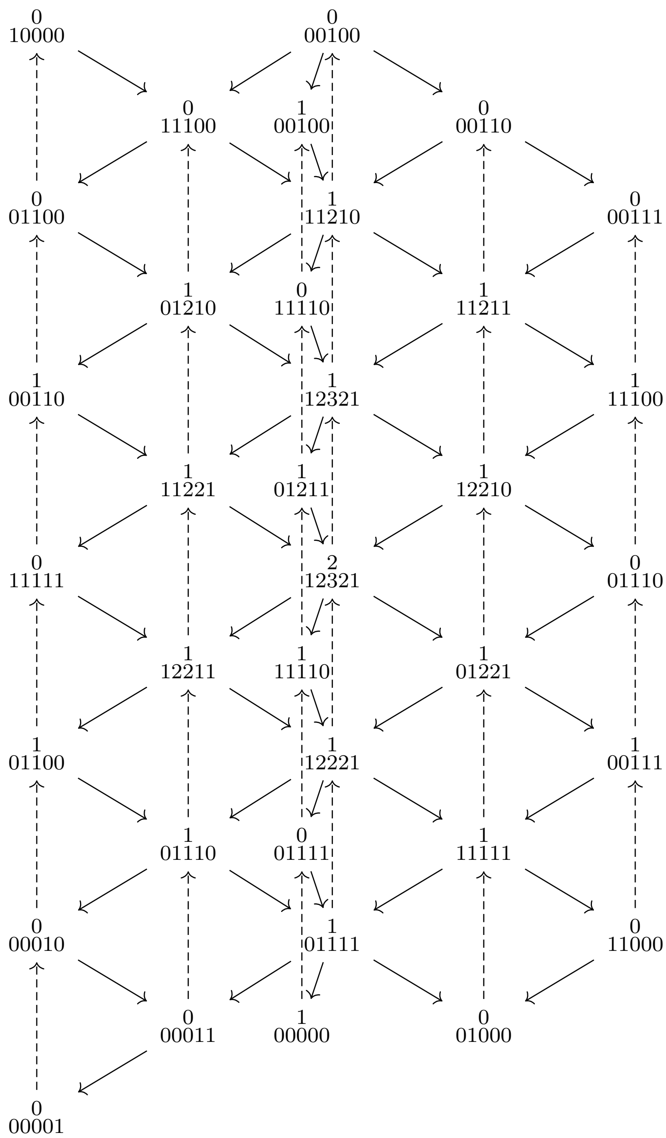


Figure 7.2: The Auslander–Reiten quiver of the quiver

$$\begin{tikzcd} & & \bullet \arrow[d] & & \\ E_6 : \bullet & \bullet \arrow[l] \arrow[r] & \bullet & \bullet \arrow[l] & \bullet \arrow[l] \end{tikzcd}$$

Minimal sectional monos can also be clearly seen from the Auslander–Reiten quiver, and we can check if a sectional morphism is mono by comparing the dimension vectors. In the case of Example E_6 in Figure 7.2, a non-zero morphism from $\substack{1\\00110}$ to $\substack{1\\11110}$ is a minimal sectional mono while a non-zero morphism from $\substack{0\\01100}$ to $\substack{1\\01211}$ is not, since a sectional morphism from $\substack{1\\01210}$ to $\substack{1\\01211}$ is also injective.

Chapter 8

Constructions of Affine Pavings

With tools in hand, we prove the main theorem 8.1.2 and 8.1.3 in Part II, and apply them to obtain the affine paving structure of partial flag varieties, see Theorem B.

8.1 Main theorem in Part II

In this section we state and prove the main theorems, which are essential in Section 8.2 and 8.4.

Let $\eta : 0 \longrightarrow X \xrightarrow{\iota} Y \xrightarrow{\pi} S \longrightarrow 0$ be a short exact sequence in $\operatorname{rep}(Q)$. Consider the canonical **non-continuous** map

$$\Psi : \operatorname{Gr}(\Phi(Y)) \longrightarrow \operatorname{Gr}(\Phi(X)) \times \operatorname{Gr}(\Phi(S)) \qquad U \longmapsto \big([\Phi(\iota)]^{-1}(U), [\Phi(\pi)](U)\big).$$

Denote the set

$$\operatorname{Gr}(\Phi(Y))_{\underline{\mathbf{f}},\underline{\mathbf{g}}} := \Psi^{-1}\Big(\operatorname{Gr}_{\underline{\mathbf{f}}}(\Phi(X)) \times \operatorname{Gr}_{\underline{\mathbf{g}}}(\Phi(S))\Big)$$

and let $\Psi_{\underline{\mathbf{f}},\underline{\mathbf{g}}}$ be the map Ψ restricted to $\operatorname{Gr}(\Phi(Y))_{\underline{\mathbf{f}},\underline{\mathbf{g}}}$, i.e.,

$$\Psi_{\underline{\mathbf{f}},\underline{\mathbf{g}}} : \operatorname{Gr}(\Phi(Y))_{\underline{\mathbf{f}},\underline{\mathbf{g}}} \longrightarrow \operatorname{Gr}_{\underline{\mathbf{f}}}(\Phi(X)) \times \operatorname{Gr}_{\underline{\mathbf{g}}}(\Phi(S)).$$

Remark 8.1.1. Even though Ψ is not continuous, $\Psi_{\underline{\mathbf{f}},\underline{\mathbf{g}}}$ is continuous. Moreover, for any dimension vectors $\underline{\mathbf{f}}, \underline{\mathbf{g}}$, the set

$$\operatorname{Gr}(\Phi(Y))_{\geqslant\underline{\mathbf{f}},\leqslant\underline{\mathbf{g}}} := \left\{ U \in \operatorname{Gr}(\Phi(Y)) \;\middle|\; \begin{array}{l} \underline{\mathbf{dim}}[\Phi(\iota)]^{-1}(U) \geqslant \underline{\mathbf{f}} \\ \underline{\mathbf{dim}}[\Phi(\pi)]\ \ (U) \leqslant \underline{\mathbf{g}} \end{array} \right\}$$

is closed in $\operatorname{Gr}(\Phi(Y))$. This gives us a filtration

$$0 = Z_0 \subset Z_1 \subset \cdots \subset Z_d = \operatorname{Gr}_{\underline{\mathbf{h}}}(\Phi(Y))$$

with Z_i closed and $Z_{i+1} \setminus Z_i$ isomorphic to $\operatorname{Gr}(\Phi(Y))_{\underline{\mathbf{f}},\underline{\mathbf{g}}}$ for some $\underline{\mathbf{f}}, \underline{\mathbf{g}}$. Therefore, from the affine pavings of $\operatorname{Gr}(\Phi(Y))_{\underline{\mathbf{f}},\underline{\mathbf{g}}}$ (for every $\underline{\mathbf{f}}, \underline{\mathbf{g}}$) one can construct one affine paving of $\operatorname{Gr}_{\underline{\mathbf{h}}}(\Phi(Y))$.

Theorem 8.1.2. *If η splits, then Ψ is surjective. Moreover, if $[S, X]^1 = 0$, then $\Psi_{\underline{\mathbf{f}},\underline{\mathbf{g}}}$ is a Zarisky-locally trivial affine bundle of rank $\langle \underline{\mathbf{g}}, \underline{\mathbf{dim}}\,\Phi(X) - \underline{\mathbf{f}} \rangle_R$.*

Theorem 8.1.3 (Generalizes [12, Theorem 32]). *When η does not split and $[S,X]^1=1$,*

$$\operatorname{Im}\Psi_{\underline{\mathbf{f}},\underline{\mathbf{g}}}=\Big(\operatorname{Gr}_{\underline{\mathbf{f}}}(\Phi(X))\times\operatorname{Gr}_{\underline{\mathbf{g}}}(\Phi(S))\Big)\setminus\Big(\operatorname{Gr}_{\underline{\mathbf{f}}}(\Phi(X_S))\times\operatorname{Gr}_{\underline{\mathbf{g}}-\underline{\mathbf{dim}}\,\Phi(S^X)}\big(\Phi(S/S^X)\big)\Big)$$

where

$$X_S:=\max\big\{M\subseteq X \,\big|\, [S,X/M]^1=1\big\}\subseteq X,$$
$$S^X:=\max\big\{M\subseteq S \,\big|\, [M,X]^1=1\big\}\subseteq S.$$

Moreover, $\Psi_{\underline{\mathbf{f}},\underline{\mathbf{g}}}$ is a Zarisky-locally trivial affine bundle of rank $\langle\underline{\mathbf{g}},\underline{\mathbf{dim}}\,\Phi(X)-\underline{\mathbf{f}}\rangle_R$ over $\operatorname{Im}\Psi_{\underline{\mathbf{f}},\underline{\mathbf{g}}}$.

We will spend the rest of the section proving these theorems. We investigate the image as well as the fiber of Ψ respectively.

Lemma 8.1.4 (Follows [12, Lemma 21]). *The element $(V,W)\in\operatorname{Gr}(\Phi(X))\times\operatorname{Gr}(\Phi(S))$ lies in the image of Ψ if and only if the canonical map $\operatorname{Ext}^1(\Phi(S),\Phi(X))\longrightarrow\operatorname{Ext}^1(W,\Phi(X)/V)$ maps η to 0.*

Proof. The canonical map is defined as follows:

$$\begin{array}{ccc}
\eta\in\operatorname{Ext}^1(\Phi(S),\Phi(X)) & & 0\longrightarrow\Phi(X)\longrightarrow\Phi(Y)\xrightarrow{\Phi(\pi)}\Phi(S)\longrightarrow 0\\
\downarrow & & \Vert\qquad\qquad\uparrow\qquad\qquad\uparrow\\
\operatorname{Ext}^1(W,\Phi(X)) & & 0\longrightarrow\Phi(X)\longrightarrow\pi^{-1}(W)\longrightarrow W\longrightarrow 0\\
\downarrow & & \downarrow\qquad\qquad\downarrow\qquad\qquad\Vert\\
\bar\eta\in\operatorname{Ext}^1(W,\Phi(X)/V) & & 0\longrightarrow\Phi(X)/V\longrightarrow\pi^{-1}(W)/V\longrightarrow W\longrightarrow 0
\end{array}$$

so $\bar\eta=0$ if and only if the last short exact sequence splits, that means, there exists a submodule $U\subseteq\Phi(Y)$, such that $\Phi(\pi)(U)=W$ and $U\cap\Phi(X)=V$. ☐

Corollary 8.1.5. *Resume the notations of Lemma 8.1.4 When η splits, then Ψ is surjective.*

Lemma 8.1.6. *The canonical map $\operatorname{Ext}^1(\Phi(S),\Phi(X))\longrightarrow\operatorname{Ext}^1(W,\Phi(X)/V)$ is surjective.*

Proof. By using the long exact sequence of extension groups and the fact that $\operatorname{Ext}^2(\Phi(S)/W,\Phi(X))=0$ and $\operatorname{Ext}^2(W,V)=0$ by Lemma 7.4.2, the maps

$$\operatorname{Ext}^1(\Phi(S),\Phi(X))\longrightarrow\operatorname{Ext}^1(W,\Phi(X))\qquad\operatorname{Ext}^1(W,\Phi(X))\longrightarrow\operatorname{Ext}^1(W,\Phi(X)/V)$$

are both surjective. Thus the composition is also surjective. ☐

Corollary 8.1.7. *Let $W\subseteq\Phi(S),V\subseteq\Phi(X)$ be R-submodules, then*

$$[W,\Phi(X)/V]^1\leqslant[\Phi(S),\Phi(X)]^1=[S,X]^1.$$

In particular, when $[S,X]^1=1$, we get $[W,\Phi(X)/V]^1=0$ or 1; when η generates $\operatorname{Ext}^1(S,X)$, we get

$$(V,W)\in\operatorname{Im}\Psi\iff[W,\Phi(X)/V]^1=0.$$

In the case where η generates $\mathrm{Ext}^1(S,X)$, we want to describe $\mathrm{Im}\,\Psi$ more precisely. For this reason we need to introduce two new R-modules:

$$\begin{aligned}\widetilde{X_S} &:= \max\left\{V \subseteq \Phi(X) \,\middle|\, [\Phi(S), \Phi(X)/V]^1 = 1\right\} \subseteq \Phi(X),\\ \widetilde{S^X} &:= \max\left\{W \subseteq \Phi(S) \,\middle|\, [W, \Phi(X)]^1 = 1\right\} \subseteq \Phi(S).\end{aligned}$$

$\widetilde{X_S}$ and $\widetilde{S^X}$ are well-defined because of the following lemma:

Lemma 8.1.8 (Follows [12, Lemma 27])**.**

(i) Let $V, V' \subset \Phi(X)$ such that $[\Phi(S), \Phi(X)/V]^1 = [\Phi(S), \Phi(X)/V']^1 = 1$. Then $[\Phi(S), \Phi(X)/(V+V')]^1 = 1$.
(ii) Let $W, W' \subset \Phi(S)$ such that $[W, \Phi(X)]^1 = [W', \Phi(X)]^1 = 1$. Then $[W \cap W', \Phi(X)]^1 = 1$.

Proof. We only prove (i). (ii) is similar.

From the short exact sequence

$$0 \longrightarrow \Phi(X)/(V \cap V') \longrightarrow \Phi(X)/V \oplus \Phi(X)/V' \longrightarrow \Phi(X)/(V+V') \longrightarrow 0,$$

we get the long exact sequence

$$\cdots \to \mathrm{Ext}^1\Big(\Phi(S), \tfrac{\Phi(X)}{V\cap V'}\Big) \to \mathrm{Ext}^1\Big(\Phi(S), \tfrac{\Phi(X)}{V}\Big) \oplus \mathrm{Ext}^1\Big(\Phi(S), \tfrac{\Phi(X)}{V'}\Big) \to \mathrm{Ext}^1\Big(\Phi(S), \tfrac{\Phi(X)}{V+V'}\Big) \to \cdots.$$

By Corollary 8.1.7, $[\Phi(S), \Phi(X)/(V \cap V')]^1 \leqslant 1$, $[\Phi(S), \Phi(X)/(V+V')]^1 \leqslant 1$, and this forces $[\Phi(S), \Phi(X)/(V+V')]^1 = 1$. □

Lemma 8.1.9 (Follows [12, Lemma 31(1)(2)], with the same proof)**.** *Let τ be the Auslander–Reiten translation.*
Let $f : X \longrightarrow \tau S$ be a non-zero morphism,[1] *then $X_S = \ker(f)$;*
also, $\Phi(f) : \Phi(X) \longrightarrow \Phi(\tau S)$ is a non-zero morphism, $\widetilde{X_S} = \ker(\Phi(f))$.

Proof. For any $M \subseteq X$, we have

$$\begin{aligned}\mathrm{Ext}^1(S, X/M)^\vee &\cong \overline{\mathrm{Hom}}(X/M, \tau S)\\ &\cong \{g \in \mathrm{Hom}(X, \tau S) |\, g|_M = 0\}\\ &\cong \begin{cases}\mathbb{C}, & M \subseteq \ker f\\ 0, & M \nsubseteq \ker f.\end{cases}\end{aligned}$$

so $[S, X/M]^1 = 1$ exactly when $M \subseteq \ker f$. Thus $X_S = \ker f$.

For $\Phi(f)$ it is similar. For any $V \subseteq \Phi(X)$, we have

$$\begin{aligned}\mathrm{Ext}^1(\Phi(S), \Phi(X)/V)^\vee &\cong \overline{\mathrm{Hom}}(\Phi(X)/V, \tau\Phi(S))\\ &\cong \overline{\mathrm{Hom}}(\Phi(X)/V, \Phi(\tau S))\\ &\cong \{g \in \mathrm{Hom}(\Phi(X), \Phi(\tau S)) |\, g|_V = 0\}\\ &\cong \begin{cases}\mathbb{C}, & V \subseteq \ker \Phi(f)\\ 0, & V \nsubseteq \ker \Phi(f).\end{cases}\end{aligned}$$

so $[\Phi(S), \Phi(X)/V]^1 = 1$ exactly when $V \subseteq \ker \Phi(f)$. Thus $\widetilde{X_S} = \ker(\Phi(f))$. □

[1] Since X is not injective, $[X, \tau S] = [S, X]^1 = 1$, f is uniquely determined up to a constant.

Corollary 8.1.10. $\widetilde{X_S} = \Phi(X_S)$*.(since* $\widetilde{X_S} = \ker(\Phi(f)) = \Phi(\ker(f)) = \Phi(X_S)$*)*

By a dual argument, one can show that $\widetilde{S^X} = \Phi(S^X)$.

Lemma 8.1.11 (Follows [12, Lemma 31(6)])**.** *For $V \subseteq \Phi(X)$ and $W \subseteq \Phi(S)$, we have*

$$[W, \Phi(X)/V]^1 = 0 \iff V \nsubseteq \Phi(X_S) \text{ or } W \nsupseteq \Phi(S^X).$$

Proof. $\Leftarrow$: Without loss of generality suppose $V \nsubseteq \Phi(X_S)$, then

$$V \nsubseteq \Phi(X_S) \iff [\Phi(S), \Phi(X)/V]^1 = 0 \Rightarrow [W, \Phi(X)/V]^1 = 0.$$

$\Rightarrow$: If not, then $V \subseteq \Phi(X_S)$ and $W \supseteq \Phi(S^X)$, and[2]

$$[W, \Phi(X)/V]^1 \geqslant [\Phi(S^X), \Phi(X)/\Phi(X_S)]^1 = [S^X, X/X_S]^1 = 1.$$

□

Corollary 8.1.12. *When η generates* $\mathrm{Ext}^1(S, X)$*, we have*

$$\operatorname{Im}\Psi_{\underline{\mathbf{f}},\underline{\mathbf{g}}} = \Big(\mathrm{Gr}_{\underline{\mathbf{f}}}(\Phi(X)) \times \mathrm{Gr}_{\underline{\mathbf{g}}}(\Phi(S))\Big) \setminus \Big(\mathrm{Gr}_{\underline{\mathbf{f}}}(\Phi(X_S)) \times \mathrm{Gr}_{\underline{\mathbf{g}}-\underline{\mathbf{dim}}\,\Phi(S^X)}\big(\Phi(S/S^X)\big)\Big).$$

Lemma 8.1.13. *For $(V, W) \in \operatorname{Im}\Psi$, the preimage of (V, W) is a torsor of* $\mathrm{Hom}_R(W, \Phi(X)/V)$*. Hence, there is a non-canonical isomorphism*

$$\Psi^{-1}((V, W)) \cong \mathrm{Hom}_R(W, \Phi(X)/V).$$

Proof. Recall the commutative diagram

$$\begin{tikzcd}
\eta \arrow[dd, mapsto] & \in & \mathrm{Ext}^1(\Phi(S),\Phi(X)) \arrow[d] & 0 \arrow[r] & \Phi(X) \arrow[r] \arrow[d, equal] & \Phi(Y) \arrow[r, "\Phi(\pi)"] & \Phi(S) \arrow[r] & 0 \\
 & & \mathrm{Ext}^1(W,\Phi(X)) \arrow[d] & 0 \arrow[r] & \Phi(X) \arrow[r] \arrow[d] & \pi^{-1}(W) \arrow[r] \arrow[u] \arrow[d] & W \arrow[r] \arrow[u] \arrow[d, equal] & 0 \\
\bar\eta & \in & \mathrm{Ext}^1(W,\Phi(X)/V) & 0 \arrow[r] & \Phi(X)/V \arrow[r, "\iota"] & \pi^{-1}(W)/V \arrow[r, "\pi'"] & W \arrow[r] \arrow[l, bend left, "\theta"] & 0
\end{tikzcd}$$

When $(V, W) \in \operatorname{Im}\Psi$, $\bar\eta$ is split, and each split morphism θ give us an element in $\Psi^{-1}((V, W))$. If we fix one split morphism θ_0, then the other split morphisms are all of the form $\theta_0 + \iota \circ f$ where $f \in \mathrm{Hom}_R(W, \Phi(X)/V)$(and this form is unique). So

$$\Psi^{-1}((V, W)) \cong \{\theta : \text{ split morphism}\} \cong \mathrm{Hom}_R(W, \Phi(X)/V).$$

□

Remark 8.1.14. Any point $(V, W) \in \operatorname{Im}\Psi_{\underline{\mathbf{f}},\underline{\mathbf{g}}}$ can be also viewed as a morphism

$$f : \operatorname{Spec}\mathbb{C} \longrightarrow \operatorname{Im}\Psi_{\underline{\mathbf{f}},\underline{\mathbf{g}}} \subseteq \mathrm{Gr}_{\underline{\mathbf{f}}}(\Phi(X)) \times \mathrm{Gr}_{\underline{\mathbf{g}}}(\Phi(S))$$

where Grassmannian are viewed as moduli spaces over $\mathbb{C}$. Essentially by replacing $\operatorname{Spec}\mathbb{C}$ by any locally closed reduced subscheme $\operatorname{Spec} A$ of $\operatorname{Im}\Psi_{\underline{\mathbf{f}},\underline{\mathbf{g}}}$ in Lemma 8.1.13, we can run the machinery of algebraic geometry, and mimic the proof of [12, Theorem 24] to show that $\Psi_{\underline{\mathbf{f}},\underline{\mathbf{g}}}$ is a Zarisky-locally trivial affine bundle over $\operatorname{Im}\Psi_{\underline{\mathbf{f}},\underline{\mathbf{g}}}$ when η generates $\mathrm{Ext}^1(S, X)$.

Proof of Theorem 8.1.2 and 8.1.3. We have already computed $\operatorname{Im}\Psi$ in Corollary 8.1.5 and 8.1.12. In both cases η generates $\mathrm{Ext}^1(S, X)$, so by Corollary 8.1.7 we get

$$\begin{aligned}(V, W) \in \operatorname{Im}\Psi_{\underline{\mathbf{f}},\underline{\mathbf{g}}} &\iff [W, \Phi(X)/V]^1 = 0 \\ &\implies [W, \Phi(X)/V] = \langle W, \Phi(X)/V\rangle_R = \big\langle \underline{\mathbf{f}}, \underline{\mathbf{dim}}\,\Phi(X) - \underline{\mathbf{g}}\big\rangle_R .\end{aligned}$$

From Remark 8.1.14, $\Psi_{\underline{\mathbf{f}},\underline{\mathbf{g}}}$ is a Zarisky-locally trivial affine bundle. □

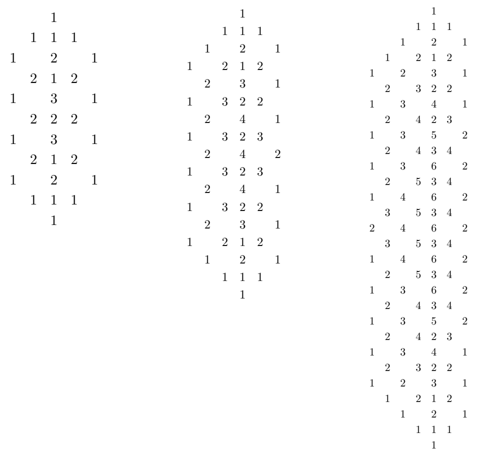
[2] $[S^X, X/X_S]^1 = 1$ follows from [12, Lemma 31(5)].

8.2 Application: Dynkin case

This section (plus Section 8.3) mainly focus on the proof of the following result:

Theorem 8.2.1. *For any Dynkin quiver Q and any representation $M \in \operatorname{rep}(Q)$, the (strict) partial flag variety* $\operatorname{Flag}(M) \cong \operatorname{Gr}(\Phi(M))$ *has an affine paving.*

Before discussing the proof of the affine paving property, we introduce some numerical concepts, which can be seen as a measure of the "complexity" of the representation.



E_6 E_7 E_8

Figure 8.1: The quantity ord_e for indecomposable representations in type E arranged in the Auslander–Reiten quiver.[3]

For an **indecomposable** quiver representation $M \in \operatorname{rep}(Q)$, we define the order of M by

$$\operatorname{ord}(M) := \max_{i \in v(Q)} \dim_{\mathbb{C}} M_i.$$

When the quiver Q is of type E, we denote by $e \in v(Q)$ the unique vertex which is connected to three other vertices, and the number

$$\operatorname{ord}_e(M) := \dim_{\mathbb{C}} M_e = [P(e), M]$$

is equal to $\operatorname{ord}(M)$ unless $\operatorname{ord}_e(M) = 0$.

The next lemma shows the affine paving property for representations of small order.

Lemma 8.2.2 (Follows [16, Lemma 2.22])**.** *Suppose that the underlying graph of Q is a tree. For an indecomposable representation $M \in \operatorname{rep}(Q)$ with $\operatorname{ord}(M) \leqslant 2$, the variety $\operatorname{Gr}_{\underline{\mathbf{f}}}(\Phi(M))$ is either empty or a direct product of some copies of $\mathbb{P}^1$. Especially, the partial flag variety $\operatorname{Gr}_{\underline{\mathbf{f}}}(\Phi(M))$ has an affine paving.*

Proof. For every $i \in v(Q)$, $\dim_{\mathbb{C}} M_i \leqslant 2$. Since Q is a tree and M is indecomposable, for every $b \in a(Q)$ satisfying $\dim_{\mathbb{C}} M_{s(b)} = \dim_{\mathbb{C}} M_{t(b)} = 2$, the map $M_{s(b)} \longrightarrow M_{t(b)}$ is an isomorphism. Therefore, when $\operatorname{Gr}_{\underline{\mathbf{f}}}(\Phi(M)) \neq \varnothing$,[4] we get the natural embedding

$$\operatorname{Gr}_{\underline{\mathbf{f}}}(\Phi(M)) \longrightarrow \prod_{\substack{i \in v(Q) \ s.t. \\ \dim_{\mathbb{C}} M_i = 2 \\ \underline{\mathbf{f}}_{(i,r)} = 1 \text{ for some } r}} \mathbb{P}^1,$$

[3]Some representations M are hidden when $\operatorname{ord}_e(M) = 0$. In [1] the Figure 8.1 is called the starting functions.

[4]This condition imposes very strong restrictions on $\underline{\mathbf{f}}$.

and the information of non-vertical arrows in the extended quiver (see Example 7.1.3) just reduce the number of $\mathbb{P}^1$. $\square$

Now we've nearly prepared every step of the proof of Theorem 8.2.1. By following the process in Figure 8.2, we now prove Theorem 8.2.1 assuming Claim 8.2.3. We will prove Claim 8.2.3 in Section 8.3.

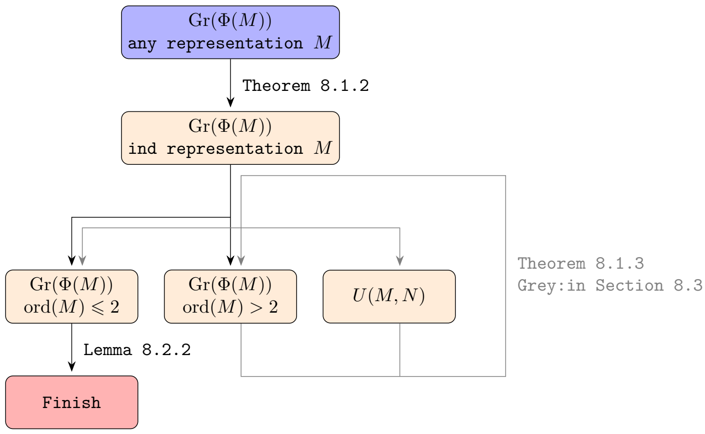


Figure 8.2: the process of induction

Claim 8.2.3. *Suppose Q is of Dynkin type. For any indecomposable representation $M \in \operatorname{rep}(Q)$ with $\operatorname{ord}(M) > 2$, the (strict) partial flag variety $\operatorname{Gr}(\Phi(M))$ has an affine paving.*

Proof of Theorem 8.2.1. First of all, for any indecomposable representation $M \in \operatorname{rep}(Q)$ we obtain an affine paving. This follows from Claim 8.2.3 when $\operatorname{ord}(M) > 2$, and follows from Lemma 8.2.2 when $\operatorname{ord}(M) \leqslant 2$.

The general case follows by induction on the dimension vector. The indecomposable representations $\{N_i\}_{i\in v(Q)}$ of quiver Q can be ordered such that $[N_i, N_j] = 0$ for all $i > j$. Therefore, every non-indecomposable representation M can be decomposed as the direct sum of two nonzero representations M_1, M_2 satisfying $[M_2, M_1]^1 = 0$. By applying Theorem 8.1.2 to the short exact sequence

$$0 \longrightarrow M_1 \longrightarrow M \longrightarrow M_2 \longrightarrow 0,$$

we get an affine paving from the affine pavings of M_1 and M_2, see Remark 8.1.1. $\square$

Remark 8.2.4. By the same technique one can show that, for Dynkin quiver Q and any representation M with $\max\limits_{i\in v(Q)} \dim_{\mathbb{C}} M_i \leqslant 2$, the variety $\operatorname{Gr}_{\underline{\mathbf{f}}}(\Phi(M))$ has an affine paving. This result does not depend on Claim 8.2.3.

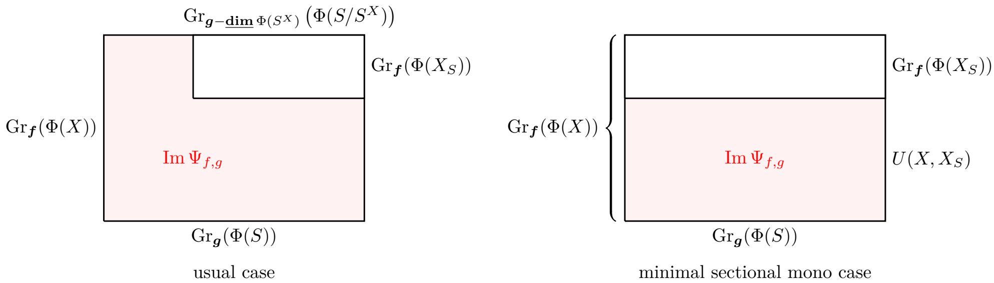
8.3 Proof of Claim 8.2.3

The task of this section is to prove Claim 8.2.3. When the quiver Q is of type A or D, Claim 8.2.3 is trivially true since no indecomposable representation can have order bigger than two. So we only concentrate on type E.

The idea of the proof is as follows. For any indecomposable representation Y with $\operatorname{ord}(Y) > 2$, we put Y into a short exact sequence

$$\eta : 0 \longrightarrow X \longrightarrow Y \longrightarrow S \longrightarrow 0$$

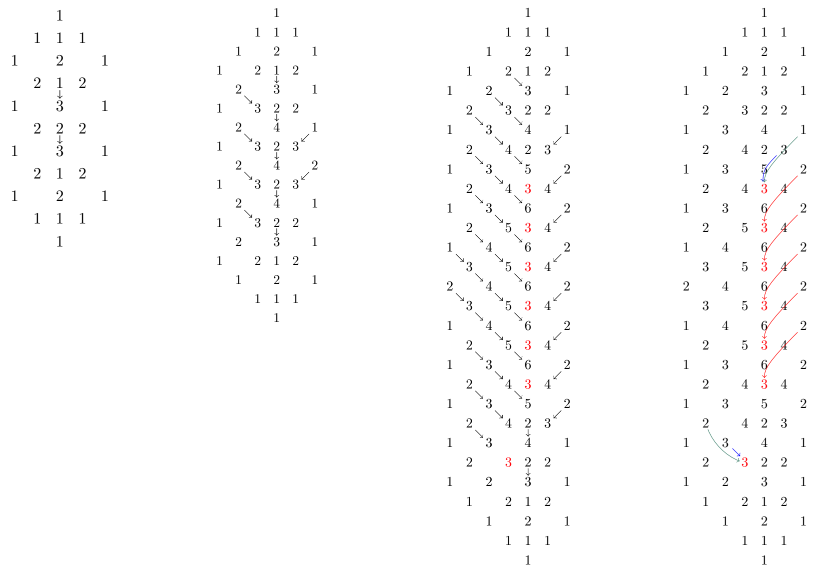
fulfilling the assumptions of Theorem 8.1.3, and then $\mathrm{Gr}(\Phi(Y))$ has an affine paving if $\mathrm{Im}\,\Psi$ has. If additionally the map $X \hookrightarrow Y$ is a minimal sectional mono, then $\mathrm{Im}\,\Psi_{\underline{\mathbf{f}},\underline{\mathbf{g}}}$ can be written as the product space, which makes $\mathrm{Im}\,\Psi$ easier to understand.



The next two lemmas tell us the existence of the desired short exact sequence.

Lemma 8.3.1. *For every indecomposable representation Y of type E with $\mathrm{ord}(Y) > 2$, there is a minimal sectional mono $f : X \longrightarrow Y$.*

Proof. Just observe the Auslander–Reiten quiver. The chosen minimal sectional monos are represented in Figure 8.3. Notice that for the most time $\mathrm{ord}_e(-)$ is enough to guarantee the map to be a mono. □



E_6 E_7 E_8 easy situation E_8 some exceptions

Figure 8.3: minimal sectional monos

Remark 8.3.2. The condition $\mathrm{ord}(Y) > 2$ in the lemma can not be removed.

Lemma 8.3.3. *Let $X \hookrightarrow Y$ be a minimal sectional mono, and $S := Y/X$ be the quotient. Then we have the short exact sequence*

$$\eta : 0 \longrightarrow X \longrightarrow Y \longrightarrow S \longrightarrow 0$$

and the dimensions of extension groups among X, Y, S are as shown in the Table 8.1.

In particular, S is indecomposable and rigid; $[S, X]^1 = 1$, so X_S and S^X are well-defined.

$[M,N]$ $\backslash N$ $[M,N]^1$ M	X	Y	S
X	1 0	1 0	0 0
Y	0 0	1 0	1 0
S	0 1	0 0	1 0

Table 8.1

Proof. Since every indecomposable representation of Dynkin quiver is a brick, we get $[X,X]=[Y,Y]=1$ and $[X,X]^1=[Y,Y]^1=0$. By the definition of minimal sectional mono, we get $[X,Y]=1,[Y,X]=0$ and $[X,Y]^1=[Y,X]^1=0$. By applying the functors $[Y,-],[-,S],[X,-],[-,X],[-Y]$ to the short exact sequence η we get the results. $\square$

In the following two lemmas we will describe the representations S^X and X_S more clearly.

Lemma 8.3.4. *Take the same notations as in Lemma 8.3.3. Then* $S^X = S$.

Proof. Let $\iota : N \longrightarrow S$ be a proper non-zero subrepresentation of S, we need to prove that $\iota^*\eta : 0 \longrightarrow X \longrightarrow Y' \longrightarrow N \longrightarrow 0$ splits.

$$\begin{tikzcd}
\iota^*\eta : & 0 \arrow[r] & X \arrow[r, hook] \arrow[d, equal] & Y' \arrow[r] \arrow[d, hook, "\eta"] & N \arrow[r] \arrow[d, hook, "\iota"] & 0 \\
\eta : & 0 \arrow[r] & X \arrow[r] & Y \arrow[r] & S \arrow[r] & 0
\end{tikzcd}$$

We decompose $Y'=\oplus_i Y_i'$ as the direct sum of indecomposable representations. Since the map $X \longrightarrow Y$ is the minimal sectional mono, we get $Y_i'=X$ or $Y_i'=Y$ or $X \xrightarrow{0} Y_i'$ for all i. If there exists i such that $Y_i'=X$, then ι^* splits; if there exists i such that $Y_i'=Y$, then η is isomorphism, we get ι is isomorphism; if for every i the map $X \longrightarrow Y_i'$ is 0, then the map $X \longrightarrow Y'$ is 0, we also get the contradiction. $\square$

Lemma 8.3.5 (Follows [12, Lemma 36], with the same proof)**.** *Let $E \longrightarrow X$ be the minimal right almost split morphism ending in X, then we can decompose E as $E = E' \oplus \tau X_1$. When Y is not projective, X_S is isomorphic to* $\ker(E \longrightarrow \tau Y) \cong E' \oplus \ker(\tau X_1 \longrightarrow \tau Y)$*; when Y is projective, $X_S \cong E$.*

Corollary 8.3.6. *When $X \longrightarrow Y$ is irreducible monomorphism, the representation X_S is either* 0 *or an indecomposable representation with property that $X_S \longrightarrow X$ is also an irreducible monomorphism.*

Remark 8.3.7. We can not copy everything in [12, Lemma 56], sometimes it would happen that $X_S = F \oplus T$ with F and T indecomposable, $F \hookrightarrow X$ is irreducible but $T \longrightarrow X/F$ is not a good mono.

For example, take the quiver of type E_7:

$$\begin{tikzcd}
 & & & \bullet \arrow[d] & & \\
\bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet & \bullet \arrow[l] & \bullet \arrow[l]
\end{tikzcd}$$

take $Y = {\scriptstyle 122\overset{1}{3}21}$, $X = {\scriptstyle 112\overset{1}{3}21}$, then $X_S = {\scriptstyle 111\overset{1}{2}10} \oplus {\scriptstyle 000\overset{0}{1}11} = F \oplus T$, $X/F = {\scriptstyle 001\overset{0}{1}11}$, the map $T \longrightarrow X/F$ is not a good mono.

Luckily, we can avoid this bad situation by carefully choosing the minimal sectional mono $X \longrightarrow Y$. The minimal sectional monos I chose are presented in Figure 8.3. In section 8.3 we will write down the induction process in detail for some examples.

Now we analyse every case in Figure 8.3, i.e., prove Claim 8.2.3 by cases. For convenience we omit subscripts which indicate the dimension vectors.

Proof of Claim 8.2.3. When the minimal sectional mono $X \longrightarrow Y$ is irreducible, we use Theorem 8.1.3 to get morphism

$$\mathrm{Gr}(\Phi(Y)) \longrightarrow \mathrm{Gr}(\Phi(X)) \times \mathrm{Gr}(\Phi(S)) \quad \text{or} \quad \mathrm{Gr}(\Phi(X)) \setminus \mathrm{Gr}(\Phi(X_S)).$$

By observation of Figure 8.3, $\mathrm{ord}_e(S) = \mathrm{ord}_e(Y) - \mathrm{ord}_e(X)$ is smaller or equal to 2, so by Lemma 8.2.2 $\mathrm{Gr}(\Phi(S))$ has the affine paving property. Let $Y_1 := X$, $X_1 := X_S$, $S_1 := Y_1/X_1$, we again use Theorem 8.1.3 to get Zariski-locally affine maps

$$\begin{aligned} \mathrm{Gr}(\Phi(X)) &\longrightarrow \mathrm{Gr}(\Phi(X_1)) \times \mathrm{Gr}(\Phi(S_1)) \quad \text{or} \quad \mathrm{Gr}(\Phi(X_1)) \setminus \mathrm{Gr}(\Phi(X_{1S_1})) \\ \mathrm{Gr}(\Phi(X)) \setminus \mathrm{Gr}(\Phi(X_S)) &\longrightarrow \mathrm{Gr}(\Phi(X_1)) \times \mathrm{Gr}(\Phi(S_1)). \end{aligned}$$

Luckily $\mathrm{ord}_e(S_1)$ is still smaller or equal to 2. We can continue this process until the order of representations are small enough.

The exceptional cases are similar, but the discussion is a bit more complicated. Let us look at some examples. (We simplify the notations: $\mathrm{Gr}(M)$ as $\mathrm{Gr}_{\underline{\mathbf{f}}}(\Phi(M))$, $U(M,N)$ as $\mathrm{Gr}_{\underline{\mathbf{f}}}(\Phi(M)) \setminus \mathrm{Gr}_{\underline{\mathbf{f}}}(\Phi(N))$, and we also ignore the dimension vectors.)

Example 8.3.8. *In the case of Figure 8.4(a), if $X_1 \longrightarrow Y$ is injective, then we obtain some Zariski-locally affine maps*

$$\begin{aligned} \mathrm{Gr}(Y) &\longrightarrow \mathrm{Gr}(X_1) \times \mathrm{Gr}(Y/X_1) \quad \textit{or} \quad U(X_1, X) \\ \mathrm{Gr}(X_1) &\longrightarrow \mathrm{Gr}(X) \times \mathrm{Gr}(X_1/X) \quad \textit{or} \quad U(X, X_S) \\ U(X_1, X) &\longrightarrow \mathrm{Gr}(X) \times \mathrm{Gr}(X_1/X) \\ U(X, X_S) &\longrightarrow \mathrm{Gr}(X_S) \times \mathrm{Gr}(X/X_S). \end{aligned}$$

When $X_1 \longrightarrow Y$ is not injective, we get

$$\mathrm{Gr}(Y) \longrightarrow \mathrm{Gr}(X) \times \mathrm{Gr}(Y/X) \quad \textit{or} \quad U(X, X_S).$$

Since the map $\tau X_1 \longrightarrow \tau Y$ is injective, from Lemma 8.3.5 we get $X_S \longrightarrow X$ is irreducible monomorphism. Thus

$$U(X, X_S) \longrightarrow \mathrm{Gr}(X_S) \times \mathrm{Gr}(X/X_S).$$

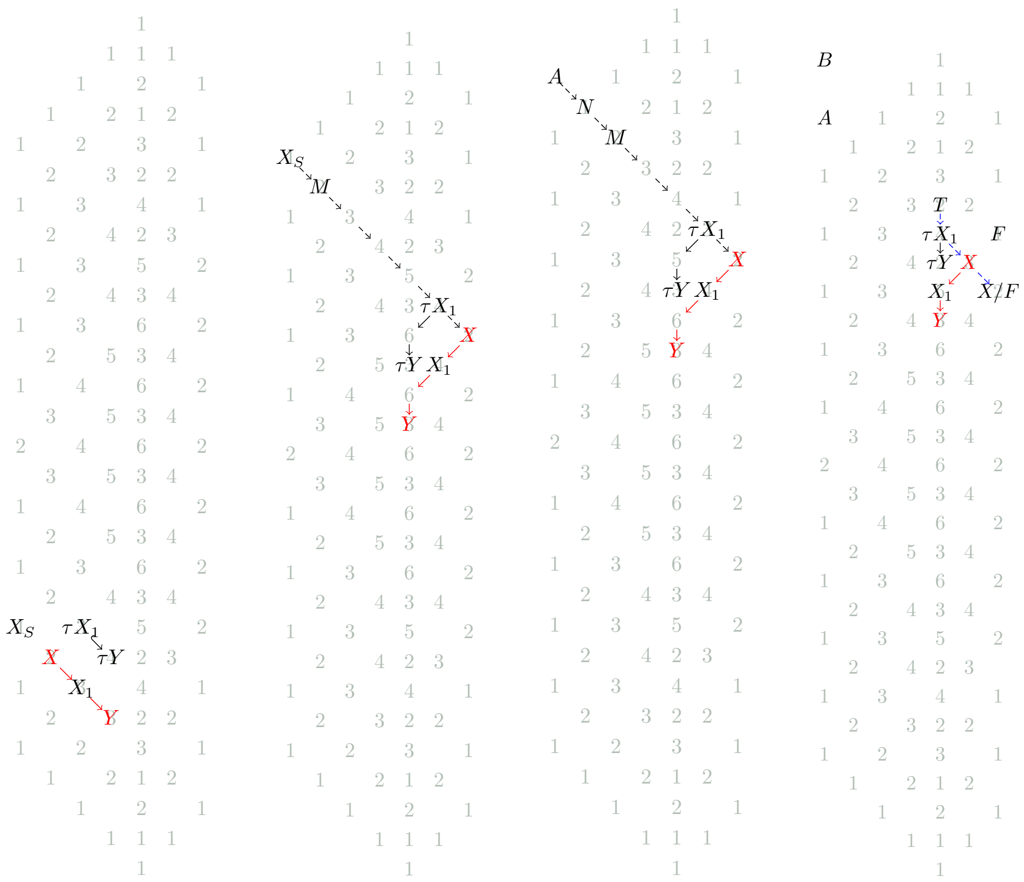
These maps give the variety $\mathrm{Gr}(Y)$ an affine paving from bottom to top.

Example 8.3.9. *In Figure 8.4(b), we would like to prove that $\mathrm{Gr}(Y)$ has the affine paving property. We have*

$$\mathrm{Gr}(Y) \longrightarrow \mathrm{Gr}(X) \times \mathrm{Gr}(Y/X) \quad \textit{or} \quad U(X, X_S).$$

When the map $M \longrightarrow X$ is not monomorphism, we get

$$U(X, X_S) \longrightarrow \mathrm{Gr}(X_S) \times \mathrm{Gr}(X/X_S);$$



(a) (b) (c) (d)

Figure 8.4: special cases

when the map $M \longrightarrow X$ is monomorphism, we get

$$U(X, X_S) = U(X, M) \bigsqcup U(M, X_S)$$
$$U(X, M) \longrightarrow \mathrm{Gr}(M) \times \mathrm{Gr}(X/M)$$
$$U(M, X_S) \longrightarrow \mathrm{Gr}(X_S) \times \mathrm{Gr}(M/X_S).$$

Since the order of X, Y/X, X_S, X/X_S, M, X/M, M/X_S are smaller or equal to 2, the induction process stops, we get $\mathrm{Gr}(Y)$ has the affine paving property.

Example 8.3.10. *In the case of Figure 8.4(c), we have*

$$\mathrm{Gr}(Y) \longrightarrow \mathrm{Gr}(X) \times \mathrm{Gr}(Y/X) \quad or \quad U(X, X_S)$$

where $X_S = \ker(\tau X_1 \longrightarrow \tau Y)$. When $X_S = 0$ we're done; if not, then $A \neq 0$ and $X_S = A$, we decompose $X_S \longrightarrow Y$ as compositions of minimal sectional monos:

Case 1: $M \longrightarrow X$ is not injective, then

$$U(X, X_S) = U(X, N) \bigsqcup U(N, X_S)$$
$$U(X, N) \longrightarrow \mathrm{Gr}(N) \times \mathrm{Gr}(X/N)$$
$$U(N, X_S) \longrightarrow \mathrm{Gr}(X_S) \times \mathrm{Gr}(N/X_S).$$

Case 2: $M \longrightarrow X$ is injective, then

$$\begin{aligned} U(X,X_S) &= U(X,M)\bigsqcup U(M,N)\bigsqcup U(N,X_S)\\ U(X,M) &\longrightarrow \mathrm{Gr}(M)\times\mathrm{Gr}(X/M)\\ U(M,N) &\longrightarrow \mathrm{Gr}(N)\times\mathrm{Gr}(M/N)\\ U(N,X_S) &\longrightarrow \mathrm{Gr}(X_S)\times\mathrm{Gr}(N/X_S). \end{aligned}$$

Since $\mathrm{Gr}(X)$, $\mathrm{Gr}(Y/X)$, $\mathrm{Gr}(N)$, $\dots$ *have affine paving property, we conclude that* $\mathrm{Gr}(Y)$ *has also the affine paving property.*

Example 8.3.11. *Finally we begin to tackle the most difficult case(Figure 8.4(d)). When $X \longrightarrow Y$ is not injective, we get*

$$\mathrm{Gr}(Y) \longrightarrow \mathrm{Gr}(F)\times\mathrm{Gr}(Y/F) \quad \textit{or} \quad U(F,?),$$

and then we get the result.[5]

When $X \longrightarrow Y$ is injective, we have

$$\mathrm{Gr}(Y) \longrightarrow \mathrm{Gr}(X)\times\mathrm{Gr}(Y/X) \quad \textit{or} \quad U(X,X_S)$$

where $X_S = F \oplus \ker(\tau X_1 \longrightarrow \tau Y) = F \oplus T$ by Lemma 8.3.5. Since $X \longrightarrow Y$ is injective, we get $A = 0$, thus $B = 0$ also, and then the sectional map $T \longrightarrow X/F$ in injective. We thus get two short exact sequence satisfying the conditions in 8.1.3:

$$\begin{aligned} \eta: &\qquad 0 \longrightarrow F \longrightarrow X \xrightarrow{\pi} X/F \longrightarrow 0\\ \xi: &\qquad 0 \longrightarrow T \longrightarrow X/F \xrightarrow{\pi'} X/X_S \longrightarrow 0. \end{aligned}$$

Let $N \in \mathrm{Gr}(X)$ be a subrepresentation, it is obvious that $N \in \mathrm{Gr}(X_S) \iff \pi' \circ \pi(N) = 0$, so

$$\begin{aligned} N \in U(X,X_S) &\iff \pi' \circ \pi(N) \neq 0\\ &\iff \pi(N) \notin \mathrm{Gr}(T)\\ &\iff \pi(N) \in U(X/F,T)\\ &\iff \Psi_\eta(N) \in \mathrm{Gr}(F) \times U(X/F,T). \end{aligned}$$

Thus the Zarisky-locally trivial affine bundle map

$$U(X,F) \longrightarrow \mathrm{Gr}(F)\times\mathrm{Gr}(X/F)$$

restricted to the Zarisky-locally trivial affine bundle map

$$U(X,X_S) \longrightarrow \mathrm{Gr}(F) \times U(X/F,T).$$

Finally, by applying the short exact sequence ξ to Theorem 8.1.3, we get the map

$$U(X/F,T) \longrightarrow \mathrm{Gr}(X/F)\times\mathrm{Gr}(T).$$

Since all the Grassmannians $\mathrm{Gr}(X)$, $\mathrm{Gr}(Y/X)$, $\mathrm{Gr}(F)$, $\mathrm{Gr}(X/F)$, $\mathrm{Gr}(T)$ have the affine paving property, we conclude that $\mathrm{Gr}(Y)$ has the affine paving property. □

[5] $\mathrm{Gr}(F)$ is empty or a singleton, so is $U(F,?)$, no matter what representation is in the questionmark.

8.4 Application: affine case

This section tries to explain the difficulty of the Conjecture 8.4.1.

Conjecture 8.4.1. *For any affine quiver Q and any indecomposable representation $M \in \operatorname{rep}(Q)$, the (strict) partial flag variety $\operatorname{Flag}(M) \cong \operatorname{Gr}(\Phi(M))$ has an affine paving.*

Actually, if readers follow the proof in [12, Section 6], and change everything from $\operatorname{Gr}(-)$ to $\operatorname{Gr}(\Phi(-))$, then there is no difference except the Proposition 48, in which the authors proved the affine paving properties of quasi-simple regular representations. So we reduced the question to the case of quasi-simple regular representation. Combined with Lemma 8.4.2, we've proved the affine paving properties for $\tilde{A}, \tilde{D}$ cases.

Lemma 8.4.2. *Assume that Q is an affine quiver of type A or D, $M \in \operatorname{rep}(Q)$ is the **regular quasi-simple** representation, then the Grassmannian $\operatorname{Gr}(\Phi(M))$ has an affine paving.*

Proof. The concept "quasi-simple" is defined in [12, Definition 15]; the concepts "preprojective","preinjective" and "regular" are defined in [12, 2.1.1]. It's shown in [4, Section 9, Lemma 3] that the regular quasi-simple representation M have dimension vector smaller or equal to the minimal positive imaginary root, thus $\operatorname{ord}_e(M) \leqslant 2$ for the quiver of type $\tilde{D}$ and $\operatorname{ord}_e(M) \leqslant 1$ for the quiver of type $\tilde{A}$. □

Theorem 8.4.3.

(1) Assume that Q is an affine quiver of type A or D, then for any indecomposable representation M, the Grassmannian $\operatorname{Gr}(\Phi(M))$ has an affine paving;
(2) Assume that Q is an affine quiver of type E, and $\operatorname{Gr}(\Phi(N))$ has an affine paving for any regular quasi-simple representation $N \in \operatorname{rep}(Q)$. The Grassmannian $\operatorname{Gr}(\Phi(M))$ then has an affine paving for any indecomposable representation M.

For a regular quasi-simple representation Y of type $\tilde{E}$, it's possible that there's no short exact sequence

$$\eta : 0 \longrightarrow X \longrightarrow Y \longrightarrow S \longrightarrow 0$$

such that $[S, X]^1 \leqslant 1$. Then we can no longer use Theorem 8.1.2 or 8.1.3. Hence, the new methods are needed for this case.

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