

Master thesis



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**Warning 0.0.1.** *I made some assumptions during the writing. To avoid confusing readers, these assumptions are listed here:*

- *For quivers, all the quivers we considered (except Auslander–Reiten quivers) are connected and finite (Remark 1.2.2). For simplicity, From ??? to ???, all the quivers have no loops or cycles.*
- *For any  $\varpi \in \mathbb{W}_{|\mathbf{d}|}$ , we always write  $\varpi = wu$ , where  $w \in W_{\mathbf{d}}$  and  $u$  is the shortest element in the coset  $W_{\mathbf{d}}\varpi$ . The flag-type dimension vector  $\underline{\mathbf{d}} \in W_{\mathbf{d}} \setminus \mathbb{W}_{|\mathbf{d}|}$  corresponds to  $u$ , i.e.,  $\underline{\mathbf{d}} = W_{\mathbf{d}}u$ . Whenever  $\tilde{w}$  and  $\tilde{u}$  emerge, they are always defined by  $wu'u' = \tilde{w}\tilde{u}$ .*
- *For the diagram, we always read from top to bottom.*

# Chapter 1

## Variety and stratification

### 1.1 Initial case: $\mathcal{F}$ and $\mathcal{F} \times \mathcal{F}$

We introduce the complete flag variety to give a bird's eye view on the whole section. Actually, the entire difficulty is bundled in this example.

**Setting 1.1.1.** Fix  $n \geq 1$ , we denote  $\mathrm{GL}_n := \mathrm{GL}_n(\mathbb{C})$ ,  $B$ ,  $T$ ,  $N$ ,  $W$  be the standard Borel subgroup, standard torus, unipotent subgroup, Weyl group respectively, i.e.,

$$\mathrm{GL}_n = \begin{pmatrix} * & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & * \end{pmatrix} \quad B = \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix} \quad T = \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \quad N = \begin{pmatrix} 1 & \cdots & * \\ & \ddots & \vdots \\ 0 & & 1 \end{pmatrix}$$
$$W := N_{\mathrm{GL}_n}(T)/T \cong S_n$$

#### 1.1.1 $\mathcal{F}$

**Definition 1.1.2** (Flag). For a finite dimensional  $\mathbb{C}$ -vector space  $V$ , a flag of  $V$  is an increasing sequence of subspaces of  $V$ :

$$F : 0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_k = V.$$

$F$  is called a complete flag if  $\dim V_j = j$  for all  $j$ , otherwise  $F$  is called a partial flag.

**Definition 1.1.3** (Complete flag variety). The complete flag variety  $\mathcal{F}$  is defined as

$$\begin{aligned} \mathcal{F} &= \mathrm{GL}_n / B \\ &\cong \{\text{complete flags of } \mathbb{C}^n\} \\ &= \{0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n = \mathbb{C}^n \mid \dim M_j = j\} \\ &\cong \{\text{Borel subgroups of } \mathrm{GL}_n\} \\ &= \{gBg^{-1} \mid g \in \mathrm{GL}_n\} \end{aligned}$$

*Remark 1.1.4.*

1.  $\mathcal{F}$  is a smooth projective variety of dimension  $\frac{n(n+1)}{2}$ , which can be seen from the embedding

$$\mathcal{F} \hookrightarrow \mathrm{Gr}(1, n) \times \cdots \times \mathrm{Gr}(n-1, n)$$

2. We implicitly give the base point of  $\mathcal{F}$ , which is not considered as the data of  $\mathcal{F}$ . Fix a standard basis of  $\mathbb{C}^n$  by  $\{v_1, \dots, v_n\}$ , we define the standard flag

$$F_{\mathrm{Id}} : 0 \subseteq \langle v_1 \rangle \subseteq \langle v_1, v_2 \rangle \subseteq \cdots \subseteq \langle v_1, \dots, v_n \rangle = \mathbb{C}^n.$$

3. We have the natural  $\mathrm{GL}_n$ -action on  $\mathcal{F}$ , which is considered as the data of  $\mathcal{F}$ .

For  $g \in \mathrm{GL}_n$ , we define the flag attached to  $g$ :

$$F_g \triangleq gF_{\mathrm{Id}} : 0 \subseteq \langle gv_1 \rangle \subseteq \langle gv_1, gv_2 \rangle \subseteq \cdots \subseteq \langle gv_1, \dots, gv_n \rangle = \mathbb{C}^n.$$

Especially, for  $w \in W = N_{\mathrm{GL}_n}(T)/T \cong S_n$ , the flag attached to  $w$

$$\begin{aligned} F_w : 0 \subseteq \langle \tilde{w}v_1 \rangle \subseteq \langle \tilde{w}v_1, \tilde{w}v_2 \rangle \subseteq \cdots \subseteq \langle \tilde{w}v_1, \dots, \tilde{w}v_n \rangle &= \mathbb{C}^n \\ 0 \subseteq \langle v_{w(1)} \rangle \subseteq \langle v_{w(1)}, v_{w(2)} \rangle \subseteq \cdots \subseteq \langle v_{w(1)}, \dots, v_{w(n)} \rangle &= \mathbb{C}^n \end{aligned}$$

does not depend on the choice of the lift  $\tilde{w} \in N_{\mathrm{GL}_n}(T)$  of  $w$ .

Readers can verify that  $\{F_w \mid w \in W\}$  are all  $T$ -fixed points of  $\mathcal{F}$ , while  $\{wBw^{-1} \mid w \in W\}$  are all Borel subgroups of  $G$  containing the standard torus  $T$ .

4. For  $n = 2$ ,  $\mathcal{F} \cong \mathbb{P}^1$ . We encourage readers to use  $\mathbb{P}^1$  as a toy example for the whole theory.

interpretation	$\mathrm{GL}_n/B$	flags	Borel subgroups
base point	$\mathrm{Id}$	$F_{\mathrm{Id}}$	$B$
$\mathrm{GL}_n$ -action	left multiplication	$\{V_i\} \mapsto \{gV_i\}$	conjugation
general point	$g$	$F_g$	$gBg^{-1}$

$\mathcal{F}$  is a well-studied variety, and has many combinatorial properties. For example, from the well-known Bruhat decomposition,<sup>1</sup>

$$\mathrm{GL}_n = \bigsqcup_{w \in W} BwB$$

We get a stratification of  $\mathcal{F}$  by  $B$ -orbits:

$$\mathcal{F} = \mathrm{GL}_n/B \cong \bigsqcup_{w \in W} BwB/B$$

The  $B$ -orbit  $BwB/B$  is called the **Schubert cell**, denoted by  $\Omega_w$ . Since

$$\Omega_w = BwB/B \cong B/(B \cap wBw^{-1}) \cong \mathbb{A}^{l(w)},$$

the Schubert cell is an affine space of dimension  $l(w)$ .

$H^i(\mathcal{F}; \mathbb{C})$	0	2	4	6	8	10	12
1	1						
2	1	1					
3	1	2	2	1			
4	1	3	5	6	5	3	1
5	1	4	9	15	20	22	20

$G$	Orbit	$G$ -fixed points
$GL_n$	$\mathcal{F} \cong GL_n/B$	$\emptyset$
$B$	$\Omega_w \cong B/(B \cap wBw^{-1})$	$\{F_{\text{Id}}\}$
$T$	—	$\{F_w   w \in W\}$

As a result, we know a lot of information of  $\mathcal{F}$ :

$\overline{\Omega}_w \subseteq \mathcal{F}$  is called the **Schubert variety**. It is well-known that

$$\overline{\Omega}_w = \bigsqcup_{w' \leq w} \Omega_{w'}$$

as a set. Especially, for any  $s \in W$  with  $l(s) = 1$ , denote  $P_s = B \sqcup BsB$ ,

$$\overline{\Omega}_s = \Omega_{\text{Id}} \sqcup \Omega_s = B/B \sqcup BsB/B = P_s/B \cong \mathbb{P}^1.$$

For other Schubert variety, the structures are quite dedicate and far away from the scope of this master thesis. For example, most Schubert varieties are not smooth.

### 1.1.2 $\mathcal{F} \times \mathcal{F}$

As a more complicated geometrical object,  $\mathcal{F} \times \mathcal{F}$  works as the base space for the Steinberg variety, which turns out to be the central focus in the thesis.  $\mathcal{F} \times \mathcal{F}$  has naturally a diagonal  $GL_n$ -action:

$$GL_n \times \mathcal{F} \times \mathcal{F} \longrightarrow \mathcal{F} \times \mathcal{F} \quad (g, F_1, F_2) \longmapsto (gF_1, gF_2).$$

Under this action,  $\mathcal{F} \times \mathcal{F}$  has a stratification consisting of  $GL_n$ -orbits, indexed by the Weyl group:

$$GL_n \backslash (\mathcal{F} \times \mathcal{F}) \cong GL_n \backslash (GL_n/B \times GL_n/B) \cong B \backslash GL_n/B \cong W \quad \text{as sets.}$$

Denote  $\Omega_{w'} := GL_n \cdot (F_{\text{Id}}, F_{w'})$ , then  $\mathcal{F} \times \mathcal{F} = \sqcup_{w'} \Omega_{w'}$ . Moreover, by the orbit-stabilizer theorem, we get

$$\Omega_{w'} \cong GL_n / (B \cap w'B(w')^{-1}).$$

Different from  $\mathcal{F}$ , the  $GL_n$ -action on  $\mathcal{F} \times \mathcal{F}$  is not transitive. To facilitate the stratification of  $\mathcal{F} \times \mathcal{F}$ , we introduce the twisted  $GL_n \times GL_n$ -action:

$$GL_n \times GL_n \times \mathcal{F} \times \mathcal{F} \longrightarrow \mathcal{F} \times \mathcal{F} \quad (g_1, g_2, F_g, F_{g'}) \longmapsto (F_{g_1 g}, F_{g_1 (g g_2 g^{-1}) g'}).$$

---

<sup>1</sup>For the most time the formula does not depend on the lift of  $w$ , so we abuse the notation of  $w \in N_{GL_n}(T)/T$  and  $\tilde{w} \in N_{GL_n}(T)$ .

If we write  $\underline{F}_{g,g'} := (F_g, F_{gg'}) \in \mathcal{F} \times \mathcal{F}$ , then

$$(g_1, g_2) \cdot \underline{F}_{g,g'} = \underline{F}_{g_1 g, g_2 g'}.$$

This  $\mathrm{GL}_n \times \mathrm{GL}_n$ -action is now transitive, and decompose  $\mathcal{F} \times \mathcal{F}$  as disjoint union of finite many  $B \times B$ -orbits, which are compatible with  $G$ -orbits:

$$\begin{aligned} \Omega_{w,w'} &:= (B \times B) \cdot \underline{F}_{w,w'} \subseteq \mathcal{F} \times \mathcal{F} \\ \mathcal{F} \times \mathcal{F} &= \bigsqcup_{w,w' \in W} \Omega_{w,w'} \quad \Omega_{w'} = \bigsqcup_{w \in W} \Omega_{w,w'} \\ \Omega_{w,w'} &\cong (B \times B) / \{(g_1, g_2) \in B \times B \mid (g_1, g_2) \cdot (F_w, F_{ww'}) = (F_w, F_{ww'})\} \\ &= (B \times B) / \{(g_1, g_2) \in B \times B \mid g_1 w B = w B, g_1 w g_2 w' B = w w' B\} \\ &= (B \times B) / \{(g_1, g_2) \in B \times B \mid g_1 w B = w B, (w^{-1} g_1 w) g_2 w' B = w' B\} \\ &= (B \times B) / \{(g_1, g_2) \in B \times B \mid g_1 \in w B w^{-1}, g_2 \in (w^{-1} g_1^{-1} w) (w' B w'^{-1})\} \quad (1.1.1) \\ &= (B \times B) / \{(g_1, g_2) \in (B \cap w B w^{-1}) \times (w^{-1} g_1^{-1} w) (B \cap w' B w'^{-1})\} \\ &\cong B / (B \cap w B w^{-1}) \times B / (B \cap w' B w'^{-1}) \cong \mathbb{A}^{l(w)+l(w')} \end{aligned}$$

We conclude the information of orbits and fixed points of  $\mathcal{F} \times \mathcal{F}$  in Table 1.1:

$G$	Orbit	$G$ -fixed points
$\mathrm{GL}_n \times \mathrm{GL}_n$	$\mathcal{F} \times \mathcal{F}$	$\emptyset$
$\mathrm{GL}_n$	$\Omega_{w'}$	$\emptyset$
$B \times B$	$\Omega_{w,w'}$	$\{F_{\mathrm{Id}, \mathrm{Id}}\}$
$T$	$-$	$\{\underline{F}_{w,w'} \mid w, w' \in W\}$

Table 1.1: Orbit and fixed points of  $\mathcal{F} \times \mathcal{F}$

Like  $\mathcal{F}$ , we also study the closure of  $\Omega_{w'}$  and  $\Omega_{w,w'}$  in special case. It can be shown that

$$\overline{\Omega}_{w'} = \bigsqcup_{x' \leq w'} \Omega_{x'} \quad \overline{\Omega}_{w,w'} = \bigsqcup_{x \leq w, x' \leq w'} \Omega_{x,x'}$$

as a set. Especially, for any  $s \in W$  with  $l(s) = 1$ ,<sup>2</sup>

$$\begin{aligned} \overline{\Omega}_s &= \Omega_{\mathrm{Id}} \sqcup \Omega_s \cong \mathrm{GL}_n / (B \cap s B s^{-1}) \sqcup \mathrm{GL}_n / B \\ &\cong \mathrm{GL}_n \times^B (B / (B \cap s B s^{-1})) \sqcup \mathrm{GL}_n \times^B (B / B) \\ &\cong \mathrm{GL}_n \times^B (B s B / B) \sqcup \mathrm{GL}_n \times^B (B / B) \\ &\cong \mathrm{GL}_n \times^B (P_s / B) \end{aligned}$$

<sup>2</sup>Here,  $\mathrm{GL}_n \times^B X$  is called contracted product. Roughly, it is defined by

$$\mathrm{GL}_n \times^B X := \mathrm{GL}_n \times X / ((gb, x) \sim (g, bx))$$

We will discuss contracted product in Subsection 2.4.1 thoroughly.



is an  $\mathcal{F}$ -bundle over  $\mathbb{P}^1$ . Also,

$$\begin{aligned}\overline{\Omega}_{\text{Id},s} &= \Omega_{\text{Id},s} \sqcup \Omega_{\text{Id},\text{Id}} \cong (B/B \times BsB/B) \sqcup (B/B \times B/B) \\ &\cong P_s/B \cong \mathbb{P}^1\end{aligned}$$

Other closure can be highly singular.

**Example 1.1.5.** In the table,  $n = 3$ ,  $t = (12)$ ,  $s = (23)$ . In this case,  $\mathcal{F} \times \mathcal{F}$  has 6  $\text{GL}_3$ -orbits, and each  $\text{GL}_3$ -orbits decompose as 6  $B \times B$ -orbits, with dimensions equal to  $l(w) + l(w')$ .

$\begin{matrix} \text{dim} & B \cdot F_{ww'} \\ (B \times B) \cdot E_{w,w'} \end{matrix}$	$B \cdot F_{ww'}$	0	1	1	2	2	3
$B \cdot F_w$		$\Omega_{\text{Id}}$	$\Omega_t$	$\Omega_s$	$\Omega_{ts}$	$\Omega_{st}$	$\Omega_{sts}$
0	$\Omega_{\text{Id}}$	0 $\Omega_{\text{Id},\text{Id}}$	1 $\Omega_{\text{Id},t}$	1 $\Omega_{\text{Id},s}$	2 $\Omega_{\text{Id},ts}$	2 $\Omega_{\text{Id},st}$	3 $\Omega_{\text{Id},sts}$
1	$\Omega_t$	2 $\Omega_{t,t}$	1 $\Omega_{t,\text{Id}}$	3 $\Omega_{t,ts}$	2 $\Omega_{t,s}$	4 $\Omega_{t,sts}$	3 $\Omega_{t,st}$
1	$\Omega_s$	2 $\Omega_{s,s}$	3 $\Omega_{s,st}$	1 $\Omega_{s,\text{Id}}$	4 $\Omega_{s,sts}$	2 $\Omega_{s,t}$	3 $\Omega_{s,ts}$
2	$\Omega_{ts}$	4 $\Omega_{ts,st}$	3 $\Omega_{ts,s}$	5 $\Omega_{ts,sts}$	2 $\Omega_{ts,\text{Id}}$	4 $\Omega_{ts,ts}$	3 $\Omega_{ts,t}$
2	$\Omega_{st}$	4 $\Omega_{st,ts}$	5 $\Omega_{st,sts}$	3 $\Omega_{st,t}$	4 $\Omega_{st,st}$	2 $\Omega_{st,\text{Id}}$	3 $\Omega_{st,s}$
3	$\Omega_{sts}$	6 $\Omega_{sts,sts}$	5 $\Omega_{sts,ts}$	5 $\Omega_{sts,st}$	4 $\Omega_{sts,t}$	4 $\Omega_{sts,s}$	3 $\Omega_{sts,\text{Id}}$

Table 1.2: stratifications of  $\mathcal{F} \times \mathcal{F}$

We can also see the  $\text{GL}_3$ -orbit (and its closure) from the table, for example,

$$\begin{aligned}\Omega_s &= \Omega_{\text{Id},s} \sqcup \Omega_{t,s} \sqcup \Omega_{s,s} \sqcup \Omega_{ts,s} \sqcup \Omega_{st,s} \sqcup \Omega_{sts,s} \\ \overline{\Omega}_s &= \Omega_s \sqcup \Omega_{\text{Id}} = \bigsqcup_w (\Omega_{w,s} \sqcup \Omega_{w,\text{Id}})\end{aligned}$$

Now we understand a lot about  $\mathcal{F}$  and  $\mathcal{F} \times \mathcal{F}$ , and the whole process of analysis (investigations?) will be applied repeatedly in Section 1.5 and 1.6.

## 1.2 Quiver

To introduce more complicated spaces and discuss their stratifications, we fix notations related to quiver and algebraic group in the following sections.

Roughly speaking, a quiver is a directed multigraph permitting loops.

**Definition 1.2.1** (Quiver). *A quiver is a quadruple*

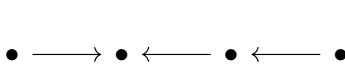
$$Q = (Q_0, Q_1, s, t)$$

where

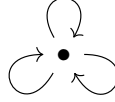
- $Q_0$  is a non-empty set consisting of vertices of  $Q$ ,
- $Q_1$  is a set consisting of arrows of  $Q$ ,
- $s : Q_1 \longrightarrow Q_0$  is a map indicating the start vertex of arrows,
- $t : Q_1 \longrightarrow Q_0$  is a map indicating the terminal vertex of arrows.

*Remark 1.2.2.* In the first part of our master thesis, all the quivers are supposed to be connected and finite (i.e.,  $Q_0, Q_1$  are finite sets). We will only encounter disconnected and infinite quiver as Auslander–Reiten quiver later on.

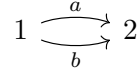
**Example 1.2.3.** *The following graphs are quivers.*



quiver of type  $A_3$



3-loop quiver  $L(3)$



2-Kronecker quiver  $K(2)$

*The reader can easily read the quadruple of quivers from the graphs. Take  $Q = K(2)$  as an example, we have*

$$Q_0 = \{1, 2\}, \quad Q_1 = \{a, b\} \quad s, t : \{a, b\} \longrightarrow \{1, 2\}$$

by  $s(a) = s(b) = 1, t(a) = t(b) = 2$ .

*For convenience, we mainly use simpler quivers as examples:*



trivial quiver



quiver of type  $A_1$



1-loop quiver  $L(1)$

From those quivers we are able to construct relatively complicated algebraic and geometrical objects.

**Definition 1.2.4** (Quiver representation). *Fix a quiver  $Q$ . A representation of  $Q$  consists of the following data:*

- A finite dimensional  $\mathbb{C}$ -vector space  $V_i$  for each vertex  $i \in Q_0$ ;
- A  $\mathbb{C}$ -linear map  $V_a : V_{s(a)} \longrightarrow V_{t(a)}$  for each arrow  $a \in Q_1$ .

**Example 1.2.5.** A representation of 1-loop quiver  $L(1)$  is a 2-tuple

$$(V, \alpha : V \longrightarrow V)$$

which is equivalent to a (finite dimensional)  $\mathbb{C}[t]$ -module.

*Remark 1.2.6.* The equivalence appeared in the example can actually be generalized to arbitrary quivers. For a quiver  $Q$ , we can define the path algebra  $\mathbb{C}Q$ , and view any  $Q$ -representation as  $\mathbb{C}Q$ -module, and vice versa.

For many constructions, we only care about the data of vector space.

**Definition 1.2.7** ( $Q$ -vector space/Vector space with quiver partition). Fix a quiver  $Q$ , a  $Q$ -vector space is a finite dimensional  $\mathbb{C}$ -vector space with the direct sum decomposition

$$V = \bigoplus_{i \in Q_0} V_i.$$

The dimension vector of a  $Q$ -vector space is defined as

$$\underline{\dim} V = (\dim_{\mathbb{C}} V_i)_{i \in Q_0} \subseteq \prod_{i \in Q_0} \mathbb{Z}.$$

On the contrary, given  $\mathbf{d} \in \prod_{i \in Q_0} \mathbb{N}_{\geq 0}$ , we can construct a canonical  $Q$ -vector space of dimension vector  $\mathbf{d}$ , as follows:

$$V = \bigoplus_{i \in Q_0} V_i \quad \text{with } V_i = \mathbb{C}^{\mathbf{d}_i}.$$

**Definition 1.2.8.** The total dimension vector of a  $Q$ -vector space  $V$  is defined as

$$|\underline{\dim} V| := \dim_{\mathbb{C}} V.$$

For  $\mathbf{d} \in \prod_{i \in Q_0} \mathbb{N}_{\geq 0}$ , denote  $|\mathbf{d}| := \sum_{i \in Q_0} \mathbf{d}_i$ .

**Definition 1.2.9** (Space of representations with given dimension vector). For any quiver  $Q$ , dimension vector  $\mathbf{d}$ , fix the canonical  $Q$ -vector space  $V = \bigoplus_{i \in Q_0} V_i$ , the space of representations with dimension vector  $\mathbf{d}$  is defined as

$$\begin{aligned} \text{Rep}_{\mathbf{d}}(Q) &= \{(V_i, V_a : V_{s(a)} \longrightarrow V_{t(a)}) \text{ as a representation of } Q\} \\ &= \bigoplus_{a \in Q_1} \text{Hom}(\mathbb{C}^{\mathbf{d}_{s(a)}}, \mathbb{C}^{\mathbf{d}_{t(a)}}) \end{aligned}$$

Since we encode the information of vector space in  $\mathbf{d}$ ,  $\text{Rep}_{\mathbf{d}}(Q)$  only records the information of linear maps.

For both  $Q$ -vector space and  $Q$ -representations, we can define (complete) flags.

**Definition 1.2.10** (Flag with quiver). *For a quiver representation  $V \in \text{rep}(Q)$ , a flag of  $V$  is defined as an increasing sequence of subrepresentation of  $V$ , i.e.,*

$$F : 0 \subseteq M_1 \subseteq \dots \subseteq M_k = V \quad M_j \in \text{rep}(Q).$$

*For a  $Q$ -vector space  $V = \bigoplus_{i \in Q_0} V_i$ , a (quiver-graded) flag of  $V$  is defined as an increasing sequence of  $Q$ -subspace of  $V$ , i.e.,*

$$F : 0 \subseteq M_1 \subseteq \dots \subseteq M_k = V \quad M_j = \bigoplus_{i \in Q_0} M_{j,i}.$$

*For both  $Q$ -vector space and  $Q$ -representation,  $F$  is called a complete flag if  $k = \dim_{\mathbb{C}} V$  and*

$$\dim_{\mathbb{C}} M_j = j \quad \text{for any } j \in \{1, \dots, |\mathbf{d}|\}$$

For the flag we also have the notation of dimension vector.

**Definition 1.2.11** (flag-type dimension vector). *For any flag  $F : 0 \subseteq M_1 \subseteq \dots \subseteq M_k = V$ , the dimension vector of  $F$  is defined as*

$$\underline{\mathbf{d}} = (\underline{\dim} M_j)_{j \in \{1, \dots, k\}} \subseteq \prod_{\substack{i \in Q_0 \\ j \in \{1, \dots, k\}}} \mathbb{Z}.$$

$\underline{\mathbf{d}}$  is called a flag-type dimension vector if  $\underline{\mathbf{d}}$  is the dimension vector of some complete flag  $F$ , i.e.,<sup>3</sup>

$$|\underline{\dim} M_{j+1}/M_j| = 1 \quad \text{for any } j \in \{0, \dots, |\mathbf{d}| - 1\}.$$

**Example 1.2.12.** *For quiver  $Q : i \longrightarrow i'$ ,  $\mathbf{d} = (3, 2)$ , the canonical  $Q$ -vector space of dimension vector  $\mathbf{d}$  is*

$$\begin{aligned} V &= V_i \oplus V_{i'} \\ &= \langle v_1, v_2, v_3 \rangle_{\mathbb{C}} \oplus \langle v_4, v_5 \rangle_{\mathbb{C}} \end{aligned}$$

*The flag*

$$F : 0 \subseteq \langle v_4 \rangle \subseteq \langle v_4, v_1 \rangle \subseteq \langle v_4, v_1, v_2 \rangle \subseteq \langle v_4, v_1, v_2, v_5 \rangle \subseteq \langle v_4, v_1, v_2, v_5, v_3 \rangle = V$$

*is a complete flag of  $V$ , with dimension vector*

$$\underline{\mathbf{d}} = \begin{pmatrix} 3, 2 \\ 2, 2 \\ 2, 1 \\ 1, 1 \\ 0, 1 \end{pmatrix}.$$

**Remark 1.2.13.** The flag-type dimension vector  $\underline{\mathbf{d}}$  can be viewed as a partition on set  $\{1, \dots, |\mathbf{d}|\}$ , i.e., a map

$$\text{par} : \{1, \dots, |\mathbf{d}|\} \longrightarrow Q_0$$

---

<sup>3</sup>For convenience, we denote  $M_0$  by 0.

such that  $\# \text{par}^{-1}(i) = \mathbf{d}_i$ .<sup>4</sup> As an example,

$$\underline{\mathbf{d}} = \begin{pmatrix} 3, 2 \\ 2, 2 \\ 2, 1 \\ 1, 1 \\ 0, 1 \end{pmatrix} \quad \text{corresponds to} \quad \{1, 2, 3, 4, 5\} = \{2, 3, 5\} \sqcup \{1, 4\}.$$

### 1.3 Symmetric group calculus

As a reminder, we recall some basic diagrams referring to the elements in  $S_n$ , and do some calculations by these diagrams. We will also relate cosets with flag-type dimension vectors.

Fix a quiver  $Q$  and dimension vector  $\mathbf{d}$ . Later (Definition 1.4.2, 1.4.3) we will define

$$\mathbb{W}_{|\mathbf{d}|} = S_{|\mathbf{d}|} \quad W_{\mathbf{d}} = \prod_{i \in Q_0} S_{\mathbf{d}_i} \leq \mathbb{W}_{|\mathbf{d}|}$$

For simplicity, we take  $Q_0 = \{1, \dots, k\}$ , then  $W_{\mathbf{d}} = S_{\mathbf{d}_1} \times \dots \times S_{\mathbf{d}_k}$  embed in  $S_{|\mathbf{d}|}$  in the most natural way.

*Remark 1.3.1.* We have different ways to express  $\varpi \in \mathbb{W}_{|\mathbf{d}|} = S_{|\mathbf{d}|}$ . For example, take  $|\mathbf{d}| = 5$ ,  $\varpi \in S_5$  by

$$\varpi(1) = 4, \quad \varpi(2) = 3, \quad \varpi(3) = 1, \quad \varpi(4) = 5, \quad \varpi(5) = 2,$$

then

$$\begin{aligned} \varpi = (14523) &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 1 & 5 & 2 \end{pmatrix} = \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \end{array} \\ \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \end{array} = \begin{bmatrix} & 1 & & & \\ & & 1 & & \\ 1 & & & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix} \\ &= (23)(34)(45)(12)(23)(12) = \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \end{array} \\ \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \end{array} \end{aligned}$$

Even though all expressions give us the same amount of information, the diagram presents them more vividly. For example, each intersection of strands corresponds to a simple reflection, so we read from the diagram that  $l(\varpi) = 6$ . Readers can also check that

$$\begin{aligned} l(\varpi s_1) &= 5, & l(\varpi s_2) &= 5, & l(\varpi s_3) &= 7, & l(\varpi s_4) &= 5, \\ l(s_1 \varpi) &= 7, & l(s_2 \varpi) &= 5, & l(s_3 \varpi) &= 5, & l(s_4 \varpi) &= 7, \end{aligned}$$

where  $s_i := (i, i+1) \in S_5$  are simple reflections.

---

<sup>4</sup>The partition corresponding to  $\text{par}$  is

$$\{1, \dots, |\mathbf{d}|\} = \bigsqcup_{i \in Q_0} \text{par}^{-1}(i).$$

**Definition 1.3.2** (Special elements in the Weyl group). *For  $i \in \{1, \dots, |\mathbf{d}| - 1\}$ , the simple reflection is defined as*

$$s_i := (i, i + 1) \in S_{|\mathbf{d}|}.$$

We denote

$$\begin{aligned} \Pi &= \left\{ s_i \in S_{|\mathbf{d}|} \mid i \in \{1, \dots, |\mathbf{d}| - 1\} \right\} \\ \Pi_{\mathbf{d}} &= \left\{ s_i \in S_{\mathbf{d}_1} \times \dots \times S_{\mathbf{d}_k} \mid i \in \{1, \dots, |\mathbf{d}| - 1\} \right\} \\ &= \{s_1, \dots, s_{|\mathbf{d}|-1}\} \setminus \{s_{\mathbf{d}_1}, s_{\mathbf{d}_1+\mathbf{d}_2}, \dots, s_{\mathbf{d}_1+\dots+\mathbf{d}_{k-1}}\} \end{aligned}$$

to be the set of simple reflections in  $\mathbb{W}_{|\mathbf{d}|}$  and  $W_{\mathbf{d}}$ , respectively.

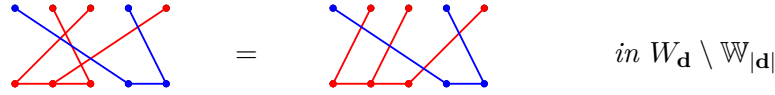
We also denote  $\varpi_{\max} \in \mathbb{W}_{|\mathbf{d}|}$ ,  $w_{\max} \in W_{\mathbf{d}}$  to be the longest elements in  $\mathbb{W}_{|\mathbf{d}|}$ ,  $W_{\mathbf{d}}$ , respectively.

We discuss about right cosets  $W_{\mathbf{d}} \setminus \mathbb{W}_{|\mathbf{d}|}$  and minimal length coset representatives now.

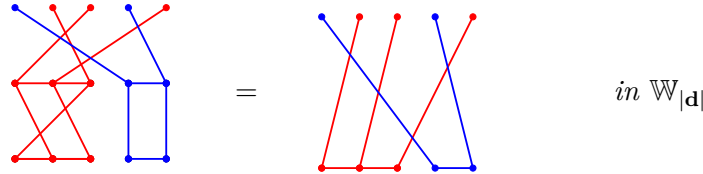
Multiplying on left by  $w \in W_{\mathbf{d}}$  is equivalent to plugging in a diagram representing  $w \in W_{\mathbf{d}}$  underneath the original diagram. Therefore, we connect some bottom points by lines, indicating that switching them will cause no trouble. Furthermore, we color different parts to make the following fact more explicitly.

**Fact 1.3.3.** *Every element  $\varpi_{\max} \in W_{\mathbf{d}} \setminus \mathbb{W}_{|\mathbf{d}|}$  corresponds to a partition on set  $\{1, \dots, |\mathbf{d}|\}$  (of a given number partition  $\mathbf{d}$ ), which corresponds to a flag-type dimension vector  $\underline{\mathbf{d}}$ .*

**Example 1.3.4.**



since



*This coset corresponds to the partition  $\{1, 2, 3, 4, 5\} = \{2, 3, 5\} \sqcup \{1, 4\}$ .*

It is easy to see from the diagram that in every coset, there exists a unique element  $u \in \mathbb{W}_{|\mathbf{d}|}$  of minimal length. We collect these minimal length coset representatives as a set, and denote it by  $\text{Min}(\mathbb{W}_{|\mathbf{d}|}, W_{\mathbf{d}})$ .<sup>5</sup>

**Proposition 1.3.5.** *For any  $\varpi \in \mathbb{W}_{|\mathbf{d}|}$ , exists unique  $w \in W_{\mathbf{d}}$ ,  $u \in \text{Min}(\mathbb{W}_{|\mathbf{d}|}, W_{\mathbf{d}})$  such that  $\varpi = wu$ .*

<sup>5</sup>In some references  $\text{Min}(\mathbb{W}_{|\mathbf{d}|}, W_{\mathbf{d}})$  is also denoted by  $\text{Schuffle}_{\mathbf{d}}$ , since those elements can be thought as ways off riffle shuffling several words together.

**Exercise 1.3.6.** For  $u \in \text{Min}(\mathbb{W}_{|\mathbf{d}|}, W_{\mathbf{d}})$ ,  $s_i \in \Pi$ , show that

$$\begin{aligned} us_i u^{-1} \in W_{\mathbf{d}} &\implies us_i u^{-1} = s_{u(i)} \in \Pi_{\mathbf{d}}, \\ us_i u^{-1} \notin W_{\mathbf{d}} &\implies us_i \in \text{Min}(\mathbb{W}_{|\mathbf{d}|}, W_{\mathbf{d}}). \end{aligned}$$



An example is shown in Table 1.4.

We finish this section with figures and examples.

$$\begin{array}{ccccccc} & & & \text{Min}(\mathbb{W}_{|\mathbf{d}|}, W_{\mathbf{d}}) & & & u \\ & & \swarrow & \downarrow \cong & & & \downarrow \\ 0 \longrightarrow W_{\mathbf{d}} \longrightarrow \mathbb{W}_{|\mathbf{d}|} \longrightarrow W_{\mathbf{d}} \setminus \mathbb{W}_{|\mathbf{d}|} \longrightarrow 0 & & & & & \varpi = wu \longmapsto & \underline{\mathbf{d}} \end{array}$$

**Example 1.3.7.** In this table,  $|\mathbf{d}| = 5$ ,  $\mathbf{d} = (3, 2)$ , typical elements would be

$$\varpi = \begin{array}{c} \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \end{array} \quad w = \begin{array}{c} \bullet \quad \bullet \quad \bullet \quad \bullet \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \\ \bullet \quad \bullet \quad \bullet \quad \bullet \end{array} \quad u = \begin{array}{c} \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \\ \text{red lines} \quad \text{blue lines} \\ \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \end{array}$$

set	element	special element	others
$\mathbb{W}_{ \mathbf{d} } = S_5$	$\varpi, x$	$\varpi_{\max} = \begin{array}{c} \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \end{array}$	$\Pi = \{s_1, s_2, s_3, s_4\}$
$W_{\mathbf{d}} = S_3 \times S_2$	$w$	$w_{\max} = \begin{array}{c} \bullet \quad \bullet \quad \bullet \quad \bullet \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \\ \bullet \quad \bullet \quad \bullet \quad \bullet \end{array}$	$\Pi_{\mathbf{d}} = \{s_1, s_2, s_4\}$
$W_{\mathbf{d}} \setminus \mathbb{W}_{ \mathbf{d} } \cong (S_3 \times S_2) \setminus S_5$	$\varpi, \underline{\mathbf{d}}$		$\text{Comp}_{\mathbf{d}}$
$\text{Min}(\mathbb{W}_{ \mathbf{d} }, W_{\mathbf{d}}) = \left\{ \begin{array}{c} \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \\ \text{red lines} \quad \text{blue lines} \\ \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \end{array}, \dots \right\}$	$u$		$\text{Schuffle}_{\mathbf{d}}$

**Example 1.3.8.** In this table,

$$|\mathbf{d}| = 3, \quad \mathbf{d} = (1, 2), \quad \mathbb{W}_{|\mathbf{d}|} = S_3, \quad W_{\mathbf{d}} = S_1 \times S_2, \quad s = (12), \quad t = (23).$$

The columns "order of basis" and "Borel subgroups" will be introduced in Definition 1.5.5 and Remark 1.4.4.

## 1.4 Algebraic group and Lie algebra

In this section we fix notations of algebraic group and Lie algebras. Later, the algebraic group will act on varieties, and some Lie algebra will serve as tangent spaces.













$\varpi = wu$	$w$	$\underline{\mathbf{d}}, u$	order of basis	$l(\varpi)$	$l(w)$	$\mathbb{B}_\varpi$	$B_\varpi$	$\varpi B_{\underline{\mathbf{d}}} \varpi^{-1}$
Id Id $\begin{pmatrix} 123 \\ 123 \end{pmatrix}$  $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$		abb 	$\{v_1, v_2, v_3\}$	0	0	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$
$t$ (23) $\begin{pmatrix} 123 \\ 132 \end{pmatrix}$  $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$	$\text{I} \times$	abb 	$\{v_1, v_3, v_2\}$	1	1	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$
$s$ (12) $\begin{pmatrix} 123 \\ 213 \end{pmatrix}$  $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$		bab 	$\{v_2, v_1, v_3\}$	1	0	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$
$ts$ (132) $\begin{pmatrix} 123 \\ 312 \end{pmatrix}$  $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$	$\text{I} \times$	bab 	$\{v_3, v_1, v_2\}$	2	1	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$
$st$ (123) $\begin{pmatrix} 123 \\ 231 \end{pmatrix}$  $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$		bba 	$\{v_2, v_3, v_1\}$	2	0	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$
$sts$ (13) $\begin{pmatrix} 123 \\ 321 \end{pmatrix}$  $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$	$\text{I} \times$	bba 	$\{v_3, v_2, v_1\}$	3	1	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$

Table 1.3: basic information of  $(1, 2)$ -case

**Setting 1.4.1.** We fix a quiver  $Q$ , a dimension vector  $\underline{\mathbf{d}}$  and a  $Q$ -vector space

$$V = \bigoplus_{i \in Q_0} V_i \quad \text{with } V_i = \mathbb{C}^{\underline{\mathbf{d}}_i}.$$

When a basis of  $V$  is needed, we fix a total order on  $Q_0$ , and denote

$$V = \langle v_1, \dots, v_{|\underline{\mathbf{d}}|} \rangle$$

where

$$V_i = \langle v_{f_i+1}, \dots, v_{f_i+\underline{\mathbf{d}}_i} \rangle \quad f_i = \sum_{i' < i} \underline{\mathbf{d}}_{i'}.$$

### 1.4.1 Algebraic group

**Definition 1.4.2** (absolute algebraic groups). We set

$$\mathbb{G}_{|\underline{\mathbf{d}}|} := \text{GL}(V) = \text{GL}_{|\underline{\mathbf{d}}|}(\mathbb{C}),$$

and  $\mathbb{B}_{|\underline{\mathbf{d}}|}$ ,  $\mathbb{T}_{|\underline{\mathbf{d}}|}$ ,  $\mathbb{N}_{|\underline{\mathbf{d}}|}$  are corresponding standard Borel, torus and unipotent subgroups.

The Weyl group is

$$\mathbb{W}_{|\underline{\mathbf{d}}|} := N_{\mathbb{G}_{|\underline{\mathbf{d}}|}}(\mathbb{T}_{|\underline{\mathbf{d}}|}) / \mathbb{T}_{|\underline{\mathbf{d}}|} \cong S_{|\underline{\mathbf{d}}|}.$$

For  $\varpi \in \mathbb{W}_{|\underline{\mathbf{d}}|}$ , we define<sup>6</sup>

$$\mathbb{B}_\varpi := \varpi \mathbb{B}_{|\underline{\mathbf{d}}|} \varpi^{-1}.$$

We will view  $\mathbb{B}_\varpi$  as the stabilizer of the flag  $F_\varpi$  with  $\mathbb{G}_{|\underline{\mathbf{d}}|}$ -action.

We also have a series of algebraic groups compatible with the quiver partition of  $V$ , and they're more common in this thesis.

<sup>6</sup>As usual, we abuse the notation of  $\varpi$  and its lift.



**Definition 1.4.3** (relative algebraic groups). *We set*

$$G_{\mathbf{d}} := \bigoplus_{i \in Q_0} \mathrm{GL}(V_i) = \mathrm{GL}_{\mathbf{d}}(\mathbb{C}) \subseteq \mathbb{G}_{|\mathbf{d}|},$$

and  $B_{\mathbf{d}}, T_{\mathbf{d}}, N_{\mathbf{d}}$  are corresponding standard Borel, torus and unipotent subgroups.

The Weyl group is

$$W_{\mathbf{d}} := N_{G_{\mathbf{d}}}(T_{\mathbf{d}})/T_{\mathbf{d}} \cong \prod_{i \in Q_0} S_{\mathbf{d}_i}.$$

For  $\varpi = wu \in W_{\mathbf{d}}$ , we define

$$B_{\varpi} := wB_{\mathbf{d}}w^{-1}.$$

We will view  $B_{\varpi}$  as the stabilizer of the flag  $F_{\varpi}$  with  $G_{\mathbf{d}}$ -action.

*Remark 1.4.4.* Be careful that  $B_{\varpi} \neq \varpi B_{\mathbf{d}}\varpi^{-1}$ . Actually,

$$B_{\varpi} = \varpi \mathbb{B}_{|\mathbf{d}|} \varpi^{-1} \cap B_{\mathbf{d}} = wB_{\mathbf{d}}w^{-1}$$

The difference is clearly shown in Table 1.3.

We also have a series of algebraic groups with subscription as elements in the Weyl group:

**Definition 1.4.5** (more algebraic groups). *For  $\varpi, \varpi'' \in \mathbb{W}_{|\mathbf{d}|}$ , define*

$$\begin{aligned} N_{\varpi} &:= R_u(B_{\varpi}), \\ N_{\varpi, \varpi''} &:= N_{\varpi} \cap N_{\varpi''}, \\ M_{\varpi, \varpi''} &:= N_{\varpi}/N_{\varpi, \varpi''}, \end{aligned}$$

where  $R_u$  denotes for the unipotent radical.

For  $s \in \Pi$  such that  $\varpi s \varpi^{-1} \in W_{\mathbf{d}}$  (i.e.,  $W_{\mathbf{d}}\varpi = W_{\mathbf{d}}\varpi s$ ), define

$$\begin{aligned} P_{\varpi, \varpi s} &:= \overline{\overline{\overline{\varpi = wu}}} w (B_{\mathbf{d}} u s u^{-1} B_{\mathbf{d}} \cup B_{\mathbf{d}}) w^{-1} \\ &= B_{\varpi} \varpi s \varpi^{-1} B_{\varpi} \cup B_{\varpi} \end{aligned}$$

*Remark 1.4.6.* One can easily show that  $N_{\varpi, \varpi s} = R_u(P_{\varpi, \varpi s})$ .

**Example 1.4.7** (Follows Example 1.3.7). For  $|\mathbf{d}| = 5$ ,  $\mathbf{d} = (3, 2)$ ,  $\varpi = \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array}$ ,  $w =$

$\times \times \times$ ,  $s = s_2$ , we compute all the algebraic groups we mentioned:

$$\begin{aligned}
\mathbb{G}_{|\mathbf{d}|} &= \begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{pmatrix} & \mathbb{B}_{|\mathbf{d}|} &= \begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{pmatrix} & \mathbb{T}_{|\mathbf{d}|} &= \begin{pmatrix} * & & & & \\ & * & & & \\ & & * & & \\ & & & * & \\ & & & & * \end{pmatrix} & \mathbb{N}_{|\mathbf{d}|} &= \begin{pmatrix} 1 & * & * & * & * \\ & 1 & * & * & * \\ & & 1 & * & * \\ & & & 1 & * \\ & & & & 1 \end{pmatrix} \\
\mathbb{W}_{|\mathbf{d}|} &\cong S_5 & \mathbb{B}_{\varpi} &= \begin{pmatrix} * & * & & & * \\ * & * & & & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{pmatrix} & \mathbb{B}_{\varpi s} &= \begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{pmatrix} \\
G_{\mathbf{d}} &= \begin{pmatrix} * & * & * & | & * \\ * & * & * & | & * \\ * & * & * & | & * \\ * & * & * & | & * \\ * & * & * & | & * \end{pmatrix} & B_{\mathbf{d}} &= \begin{pmatrix} * & * & * & | & * \\ * & * & * & | & * \\ * & * & * & | & * \\ * & * & * & | & * \\ * & * & * & | & * \end{pmatrix} & T_{\mathbf{d}} &= \begin{pmatrix} * & & & | & * \\ & * & & | & * \\ & & * & | & * \\ & & & | & * \\ & & & | & * \end{pmatrix} & N_{\mathbf{d}} &= \begin{pmatrix} 1 & * & * & | & * \\ & 1 & * & | & * \\ & & 1 & | & * \\ & & & | & 1 \\ & & & | & 1 \end{pmatrix} \\
W_{\mathbf{d}} &\cong S_3 \times S_2 & B_{\varpi} &= \begin{pmatrix} * & * & & | & * \\ * & * & & | & * \\ * & * & * & | & * \\ * & * & * & | & * \\ * & * & * & | & * \end{pmatrix} & B_{\varpi s} &= \begin{pmatrix} * & & & | & * \\ * & * & & | & * \\ * & * & * & | & * \\ * & * & * & | & * \\ * & * & * & | & * \end{pmatrix} \\
N_{\varpi} &= \begin{pmatrix} 1 & * & & | & * \\ & 1 & & | & * \\ * & * & 1 & | & * \\ & & & | & 1 \\ * & 1 & & | & * \end{pmatrix} & N_{\varpi, \varpi s} &= \begin{pmatrix} 1 & & & | & * \\ & 1 & & | & * \\ * & * & 1 & | & * \\ & & & | & 1 \\ * & 1 & & | & * \end{pmatrix} & M_{\varpi, \varpi s} &= \begin{pmatrix} 1 & * & & | & * \\ & 1 & & | & * \\ - & - & 1 & | & * \\ & & & | & 1 \\ & & & | & -1 \end{pmatrix} & P_{\varpi, \varpi s} &= \begin{pmatrix} * & * & & | & * \\ * & * & & | & * \\ * & * & * & | & * \\ * & * & * & | & * \\ * & * & * & | & * \end{pmatrix}
\end{aligned}$$

### 1.4.2 Lie algebra

For the Lie algebra, we use the corresponding Fraktur-font symbols:

$$\begin{aligned}
&\mathfrak{g}_{|\mathbf{d}|}, \quad \mathfrak{b}_{|\mathbf{d}|}, \quad \mathfrak{t}_{|\mathbf{d}|}, \quad \mathfrak{n}_{|\mathbf{d}|}, \quad \mathfrak{b}_{\varpi} \\
&\mathfrak{g}_{\mathbf{d}}, \quad \mathfrak{b}_{\mathbf{d}}, \quad \mathfrak{t}_{\mathbf{d}}, \quad \mathfrak{n}_{\mathbf{d}}, \quad \mathfrak{b}_{\varpi}, \\
&\mathfrak{n}_{\varpi}, \quad \mathfrak{n}_{\varpi, \varpi''}, \quad \mathfrak{m}_{\varpi, \varpi''}, \quad \mathfrak{p}_{\varpi, \varpi s},
\end{aligned}$$

We also have to encode the information of representations as Lie algebra. Notice that

$$\mathrm{Hom}(V_{s(a)}, V_{t(a)}) \hookrightarrow \mathrm{Hom}(V, V) \cong \mathfrak{g}_{|\mathbf{d}|} \quad f \mapsto \iota_{t(a)} \circ f \circ \pi_{s(a)}$$

realizes  $\mathrm{Hom}(V_{s(a)}, V_{t(a)})$  as a Lie subalgebra of  $\mathfrak{g}_{|\mathbf{d}|}$ , so


$$\mathrm{Rep}_{\mathbf{d}}(Q) = \bigoplus_{a \in Q_1} \mathrm{Hom}(\mathbb{C}^{\mathbf{d}_{s(a)}}, \mathbb{C}^{\mathbf{d}_{t(a)}}) \subseteq \bigoplus_{a \in Q_1} \mathfrak{g}_{|\mathbf{d}|}.$$

**Definition 1.4.8** (Lie algebras connected with representations). *For  $\varpi \in \mathbb{W}_{|\mathbf{d}|}$ , denote temperately*

$$V_{\varpi, j} := \langle e_{\varpi(1)}, \dots, e_{\varpi(j)} \rangle \subseteq V.$$

We define Lie subalgebras of  $\mathrm{Rep}_{\mathbf{d}}(Q)$  as follows.

$$\begin{aligned}
\mathfrak{r}_{\varpi} &:= \{ (f_a)_{a \in Q_1} \in \mathrm{Rep}_{\mathbf{d}}(Q) \mid f_a(V_{\varpi, j} \cap V_{s(a)}) \subseteq V_{\varpi, j} \text{ for any } j \}, \\
\mathfrak{r}_{\varpi, \varpi''} &:= \mathfrak{r}_{\varpi} \cap \mathfrak{r}_{\varpi''}, \\
\mathfrak{d}_{\varpi, \varpi''} &:= \mathfrak{r}_{\varpi} / \mathfrak{r}_{\varpi, \varpi''},
\end{aligned}$$

**Example 1.4.9** (Follows Example 1.4.7). Consider the quiver  $\bullet \longrightarrow \bullet$ , and  $u =$  . Table 1.4 gives us an example of the shape of these Lie algebras.






		$\mathfrak{n}_u$	$\mathfrak{m}_{u,u}$	$\mathfrak{r}_u$	$\mathfrak{d}_{u,u}$
	$u =$ 	$\begin{bmatrix} * & * \\ * & \\ \hline & * \end{bmatrix}$	$\begin{bmatrix} & \\ \hline & \end{bmatrix}$	$\begin{bmatrix} & \\ * & * \\ * & * \end{bmatrix}$	$\begin{bmatrix} & \\ \hline & \end{bmatrix}$
$s$	cases	$\mathfrak{n}_{us}$	$\mathfrak{m}_{u,us}$	$\mathfrak{r}_{us}$	$\mathfrak{d}_{u,us}$
$s = s_1$	$us_1 =$  $\notin W_{\mathbf{d}}u$	$\begin{bmatrix} * & * \\ * & \\ \hline & * \end{bmatrix}$	$\begin{bmatrix} & \\ \hline & \end{bmatrix}$	$\begin{bmatrix} & \\ * & * \\ * & * \end{bmatrix}$	$\begin{bmatrix} & \\ * & \\ \hline & \end{bmatrix}$ $\frac{e_4}{e_1}$
$s = s_2$	$us_2 =$  $\in W_{\mathbf{d}}u$	$\begin{bmatrix} * & * \\ * & \\ \hline & * \end{bmatrix}$	$\begin{bmatrix} * & \\ \hline & \end{bmatrix}$ $\frac{e_1}{e_2}$	$\begin{bmatrix} & \\ * & * \\ * & * \end{bmatrix}$	$\begin{bmatrix} & \\ \hline & \end{bmatrix}$
$s = s_3$	$us_3 =$  $\notin W_{\mathbf{d}}u$	$\begin{bmatrix} * & * \\ * & \\ \hline & * \end{bmatrix}$	$\begin{bmatrix} & \\ \hline & \end{bmatrix}$	$\begin{bmatrix} & \\ * & * \\ * & * \end{bmatrix}$	$\begin{bmatrix} & \\ \hline & \end{bmatrix}$
$s = s_4$	$us_4 =$  $\notin W_{\mathbf{d}}u$	$\begin{bmatrix} * & * \\ * & \\ \hline & * \end{bmatrix}$	$\begin{bmatrix} & \\ \hline & \end{bmatrix}$	$\begin{bmatrix} & \\ * & * \\ * & * \end{bmatrix}$	$\begin{bmatrix} & \\ & * \\ \hline & \end{bmatrix}$ $\frac{e_5}{e_3}$

Table 1.4: examples of Lie algebras

*Remark 1.4.10.* We also have twisted notations for Lie algebras. For example,

$$\begin{aligned} \underline{\mathfrak{n}}_{\varpi, \varpi'} &= \mathfrak{n}_{\varpi, \varpi \varpi'}, & \underline{\mathfrak{m}}_{\varpi, \varpi'} &= \mathfrak{m}_{\varpi, \varpi \varpi'}, & \underline{\mathfrak{p}}_{\varpi, s} &= \mathfrak{p}_{\varpi, \varpi s}, \\ \underline{\mathfrak{r}}_{\varpi, \varpi'} &= \mathfrak{r}_{\varpi, \varpi \varpi'}, & \underline{\mathfrak{d}}_{\varpi, \varpi'} &= \mathfrak{d}_{\varpi, \varpi \varpi'}. \end{aligned}$$

Another twist happens when we add minus sign as the superscript:

$$\begin{aligned} \mathfrak{b}_{\varpi}^- &= \mathfrak{b}_{\varpi_{\max} \varpi}, & \mathfrak{n}_{\varpi}^- &= \mathfrak{b}_{w_{\max} \varpi}, \\ \mathfrak{b}_{\varpi}^- &= \mathfrak{b}_{w_{\max} \varpi}, & \mathfrak{m}_{\varpi, \varpi''}^- &= \mathfrak{m}_{w_{\max} \varpi, w_{\max} \varpi''}. \end{aligned}$$

## 1.5 Typical variety

In this section, we define nearly all the varieties we care about in the same spirit as Section 1.1. Their stratifications and related "Schubert" varieties will be defined in Section 1.6.

Recall Setting 1.1 and Definition 1.2.10.

### 1.5.1 Flag variety

**Definition 1.5.1** (Absolute complete flag variety). *The absolute complete flag variety  $\mathcal{F}_{|\mathbf{d}|}$  is defined as*

$$\begin{aligned}\mathcal{F}_{|\mathbf{d}|} &= \mathbb{G}_{|\mathbf{d}|} / \mathbb{B}_{|\mathbf{d}|} \\ &\cong \left\{ \text{complete flags of } \mathbb{C}^{|\mathbf{d}|} \right\} \\ &= \left\{ 0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_{|\mathbf{d}|} = \mathbb{C}^{|\mathbf{d}|} \mid \dim M_j = j \right\} \\ &\cong \left\{ \text{Borel subgroups of } \mathbb{G}_{|\mathbf{d}|} \right\} \\ &= \left\{ g\mathbb{B}_{|\mathbf{d}|}g^{-1} \mid g \in \mathbb{G}_{|\mathbf{d}|} \right\}\end{aligned}$$

Here,  $M_i$  can have no  $Q$ -vector space structure.

**Definition 1.5.2** (complete flag variety with flag-type dimension vector). *For a flag-type dimension vector  $\underline{\mathbf{d}}$ , the flag variety  $\mathcal{F}_{\underline{\mathbf{d}}}$  is defined as*

$$\begin{aligned}\mathcal{F}_{\underline{\mathbf{d}}} &= \left\{ \text{complete flags of } V = \bigoplus_{i \in Q_0} V_i \text{ with dimension vector } \underline{\mathbf{d}} \right\} \\ &= \left\{ F : 0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_{|\mathbf{d}|} = V \mid \underline{\dim} F = \underline{\mathbf{d}} \right\}\end{aligned}$$

**Definition 1.5.3** (Relative complete flag variety). *The relative complete flag variety  $\mathcal{F}_{\mathbf{d}}$  is defined as*

$$\begin{aligned}\mathcal{F}_{\mathbf{d}} &= \left\{ \text{complete flags of } V = \bigoplus_{i \in Q_0} V_i \right\} \\ &= \left\{ 0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_{|\mathbf{d}|} = V \mid |\underline{\dim} M_j| = j \right\} \\ &= \bigsqcup_{\underline{\mathbf{d}}} \mathcal{F}_{\underline{\mathbf{d}}}\end{aligned}$$

Here,  $M_i$  are  $Q$ -vector spaces.

*Remark 1.5.4.*

1.  $\mathcal{F}_{|\mathbf{d}|}$ ,  $\mathcal{F}_{\underline{\mathbf{d}}}$  and  $\mathcal{F}_{\mathbf{d}}$  are smooth varieties, since

$$\mathcal{F}_{|\mathbf{d}|} \cong \mathrm{GL}_{|\mathbf{d}|} / B \quad \mathcal{F}_{\underline{\mathbf{d}}} \cong \prod_{i \in Q_0} \mathrm{GL}_{\mathbf{d}_i} / B$$

are products of usual flag varieties.

2.  $\mathcal{F}_{|\mathbf{d}|}$  is an  $\mathrm{GL}_{|\mathbf{d}|}$ -variety, while  $\mathcal{F}_{\underline{\mathbf{d}}}$ ,  $\mathcal{F}_{\mathbf{d}}$  are  $G_{\mathbf{d}}$ -varieties. The actions are induced by the actions on the vector space  $V$ .

We need to simplify our notations of flags.

**Definition 1.5.5** (Special flags). *For a basis  $\{x_1, \dots, x_{|\mathbf{d}|}\}$ , denote the flag*

$$F_{\{x_1, \dots, x_{|\mathbf{d}|}\}} : 0 \subseteq \langle x_1 \rangle \subseteq \langle x_1, x_2 \rangle \subseteq \dots \subseteq \langle x_1, \dots, x_{|\mathbf{d}|} \rangle = V.$$

For  $g \in \mathbb{G}_{|\mathbf{d}|}$ ,  $\varpi \in \mathbb{W}_{|\mathbf{d}|}$ , define

$$\begin{aligned} F_{\text{Id}} &= F_{\{v_1, \dots, v_{|\mathbf{d}|}\}} && \in \mathcal{F}_{\mathbf{d}} \\ F_g &= gF_{\text{Id}} = F_{\{gv_1, \dots, gv_{|\mathbf{d}|}\}} && \in \mathcal{F}_{|\mathbf{d}|} \\ F_{\varpi} &= \varpi F_{\text{Id}} = F_{\{v_{\varpi(1)}, \dots, v_{\varpi(|\mathbf{d}|)}\}} && \in \mathcal{F}_{\mathbf{d}} \end{aligned}$$

$F_{\text{Id}}$  is called the **standard flag** of  $V$ .

Now we can define flag varieties attached to  $\varpi \in \mathbb{W}_{|\mathbf{d}|}$ .

**Definition 1.5.6.** *For  $\varpi = wu \in \mathbb{W}_{|\mathbf{d}|}$ , define  $\mathcal{F}_{\varpi}$  as the  $G_{\mathbf{d}}$ -orbit of  $F_{\varpi}$ . By the orbit-stabilizer theorem,*

$$\mathcal{F}_{\varpi} \cong G_{\mathbf{d}}/B_{\varpi}.$$

We can generalize it a little bit: for  $g \in G_{\mathbf{d}}$ ,  $F_{g\varpi} \in \mathcal{F}_{\mathbf{d}}$ ,

$$\mathcal{F}_{g\varpi} := G_{\mathbf{d}} \cdot F_{g\varpi} \cong G_{\mathbf{d}}/B_{g\varpi} = G_{\mathbf{d}}/gB_{\varpi}g^{-1}.$$

*Remark 1.5.7.*  $F_{\varpi}$  is the preferred base point of  $\mathcal{F}_{\varpi}$ . Ignoring the base point,

$$\mathcal{F}_{\varpi} = \mathcal{F}_u = \mathcal{F}_{\underline{\mathbf{d}}} \quad \text{for } \varpi = wu \quad \underline{\mathbf{d}} = W_{\mathbf{d}}\varpi.$$

In fact, we are not defining new varieties; we give old varieties new names, so that we can manipulate them more freely.

Like Section 1.1, we also consider the product of two flag varieties. For  $g, g', g'' \in \mathbb{G}_{|\mathbf{d}|}$ ,  $\varpi, \varpi', \varpi'' \in \mathbb{W}_{|\mathbf{d}|}$ , denote

$$\begin{aligned} F_{\text{Id}, \text{Id}} &= (F_{\text{Id}}, F_{\text{Id}}) \\ F_{g, g''} &= (F_g, F_{g''}) & \underline{F}_{g, g'} &= F_{g, gg'} = (F_g, F_{gg'}) \\ F_{\varpi, \varpi''} &= (F_{\varpi}, F_{\varpi''}) & \underline{F}_{\varpi, \varpi'} &= F_{\varpi, \varpi\varpi'} = (F_{\varpi}, F_{\varpi\varpi'}) \end{aligned}$$

Table 1.5 concludes all varieties we get until now.

### 1.5.2 Incidence variety

Now it is time to conclude information about arrows, and construct spaces over varieties in Table 1.5.

	base point		base point
$\mathcal{F}_{ \mathbf{d} } \cong \mathbb{G}_{ \mathbf{d} }/\mathbb{B}_{ \mathbf{d} }$	$F_{\text{Id}}$	$\mathcal{F}_{ \mathbf{d} } \times \mathcal{F}_{ \mathbf{d} }$	$F_{\text{Id}, \text{Id}}$
$\mathcal{F}_{\underline{\mathbf{d}}} \cong G_{\mathbf{d}}/B_{\mathbf{d}}$	$F_u$	$\mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}'}}$	$F_{u, u'}$
$\mathcal{F}_{\varpi} \cong G_{\mathbf{d}}/B_{\varpi}$	$F_{\varpi}$	$\mathcal{F}_{\varpi} \times \mathcal{F}_{\varpi'}$	$F_{\varpi, \varpi'}$
$\mathcal{F}_{\mathbf{d}} = \bigsqcup_{\underline{\mathbf{d}}} \mathcal{F}_{\underline{\mathbf{d}}}$	—	$\mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}} = \bigsqcup_{\underline{\mathbf{d}}, \underline{\mathbf{d}'}} \mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}'}}$	—

Table 1.5: Base varieties and their preferred base point

**Definition 1.5.8** (Incidence variety). *For a quiver  $Q$  with flag-type dimension vector  $\underline{\mathbf{d}}$ , define*

$$\begin{aligned} \widetilde{\text{Rep}}_{\underline{\mathbf{d}}}(Q) &:= \{(\rho, F) \in \text{Rep}_{\mathbf{d}}(Q) \times \mathcal{F}_{\underline{\mathbf{d}}} \mid \rho(M_j) \subseteq M_j \text{ for any } j\} \\ \widetilde{\text{Rep}}_{\mathbf{d}}(Q) &:= \{(\rho, F) \in \text{Rep}_{\mathbf{d}}(Q) \times \mathcal{F}_{\mathbf{d}} \mid \rho(M_j) \subseteq M_j \text{ for any } j\} \\ &= \bigsqcup_{\underline{\mathbf{d}}} \widetilde{\text{Rep}}_{\underline{\mathbf{d}}}(Q) \end{aligned}$$

and  $\mu_{\underline{\mathbf{d}}}$ ,  $\pi_{\underline{\mathbf{d}}}$ ,  $\mu_{\mathbf{d}}$ ,  $\pi_{\mathbf{d}}$  to be the natural morphisms from the incidence varieties to  $\text{Rep}_{\mathbf{d}}(Q)$  or flag varieties, as follows:

$$\begin{array}{ccc} \widetilde{\text{Rep}}_{\underline{\mathbf{d}}}(Q) \subseteq \text{Rep}_{\mathbf{d}}(Q) \times \mathcal{F}_{\underline{\mathbf{d}}} & & \widetilde{\text{Rep}}_{\mathbf{d}}(Q) \subseteq \text{Rep}_{\mathbf{d}}(Q) \times \mathcal{F}_{\mathbf{d}} \\ \mu_{\underline{\mathbf{d}}} \swarrow & \searrow \pi_{\underline{\mathbf{d}}} & \mu_{\mathbf{d}} \swarrow & \searrow \pi_{\mathbf{d}} \\ \text{Rep}_{\mathbf{d}}(Q) & & \text{Rep}_{\mathbf{d}}(Q) & \end{array}$$

*Remark 1.5.9.* For  $M \in \text{Rep}_{\mathbf{d}}(Q)$ , the **Springer fiber**

$$\text{Flag}_{\underline{\mathbf{d}}}(M) := \mu_{\underline{\mathbf{d}}}^{-1}(M) \cong \pi_{\underline{\mathbf{d}}}(\mu_{\underline{\mathbf{d}}}^{-1}(M)) \subseteq \mathcal{F}_{\underline{\mathbf{d}}}$$

records the complete flags of subrepresentations of  $M$ . The partial flag variety version of  $\text{Flag}_{\underline{\mathbf{d}}}(M)$  will become the key object in the second part.

**Definition 1.5.10** (Steinberg variety). *For quiver  $Q$  with flag-type dimension vectors  $\underline{\mathbf{d}}$ ,  $\underline{\mathbf{d}'}$ , define*

$$\begin{aligned} \mathcal{Z}_{\underline{\mathbf{d}}, \underline{\mathbf{d}'}} &:= \widetilde{\text{Rep}}_{\underline{\mathbf{d}}}(Q) \times_{\text{Rep}_{\mathbf{d}}(Q)} \widetilde{\text{Rep}}_{\underline{\mathbf{d}'}}(Q) \\ \mathcal{Z}_{\mathbf{d}} &:= \widetilde{\text{Rep}}_{\mathbf{d}}(Q) \times_{\text{Rep}_{\mathbf{d}}(Q)} \widetilde{\text{Rep}}_{\mathbf{d}}(Q) \\ &= \bigsqcup_{\underline{\mathbf{d}}, \underline{\mathbf{d}'}} \mathcal{Z}_{\underline{\mathbf{d}}, \underline{\mathbf{d}'}} \end{aligned}$$

$\mathcal{Z}_{\mathbf{d}}$  is called the **Steinberg variety**.

$\mathcal{Z}_{\underline{\mathbf{d}}, \underline{\mathbf{d}'}}$  can actually be realized as the incidence variety between  $\text{Rep}_{\underline{\mathbf{d}}}(Q)$  and  $\mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}'}}$ , since

$$\begin{aligned} \mathcal{Z}_{\underline{\mathbf{d}}, \underline{\mathbf{d}'}} &= \widetilde{\text{Rep}_{\underline{\mathbf{d}}}(Q)} \times_{\text{Rep}_{\underline{\mathbf{d}}}(Q)} \widetilde{\text{Rep}_{\underline{\mathbf{d}'}}(Q)} \\ &\subseteq (\text{Rep}_{\underline{\mathbf{d}}}(Q) \times \mathcal{F}_{\underline{\mathbf{d}}}) \times_{\text{Rep}_{\underline{\mathbf{d}}}(Q)} (\text{Rep}_{\underline{\mathbf{d}}}(Q) \times \mathcal{F}_{\underline{\mathbf{d}'}})) \\ &\cong \text{Rep}_{\underline{\mathbf{d}}}(Q) \times \mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}'}} \end{aligned}$$

For that reason, we denote  $\mu_{\underline{\mathbf{d}}, \underline{\mathbf{d}'}}$ ,  $\pi_{\underline{\mathbf{d}}, \underline{\mathbf{d}'}}$ ,  $\mu_{\underline{\mathbf{d}}, \underline{\mathbf{d}}}$ ,  $\pi_{\underline{\mathbf{d}}, \underline{\mathbf{d}}}$  as natural morphisms from  $\mathcal{Z}_{\underline{\mathbf{d}}, \underline{\mathbf{d}'}}$ ,  $\mathcal{Z}_{\underline{\mathbf{d}}}$  to  $\text{Rep}_{\underline{\mathbf{d}}}(Q)$  or product of flag varieties, as follows:

$$\begin{array}{ccc} \mathcal{Z}_{\underline{\mathbf{d}}, \underline{\mathbf{d}'}} \subseteq \text{Rep}_{\underline{\mathbf{d}}}(Q) \times \mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}'}} & & \mathcal{Z}_{\underline{\mathbf{d}}, \underline{\mathbf{d}'}} \subseteq \text{Rep}_{\underline{\mathbf{d}}}(Q) \times \mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}'}} \\ \mu_{\underline{\mathbf{d}}, \underline{\mathbf{d}'}} \swarrow & \searrow \pi_{\underline{\mathbf{d}}, \underline{\mathbf{d}'}} & \mu_{\underline{\mathbf{d}}, \underline{\mathbf{d}'}} \swarrow & \searrow \pi_{\underline{\mathbf{d}}, \underline{\mathbf{d}'}} \\ \text{Rep}_{\underline{\mathbf{d}}}(Q) & \mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}'}} & \text{Rep}_{\underline{\mathbf{d}}}(Q) & \mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}'}} \end{array}$$

*Remark 1.5.11* (Group actions).

1.  $\text{Rep}_{\underline{\mathbf{d}}}(Q) \subseteq \oplus_{a \in Q_1} \mathfrak{g}_{|\underline{\mathbf{d}}|}$  has a natural  $G_{\underline{\mathbf{d}}}$ -action, which is induced by the conjugation action of  $G_{\underline{\mathbf{d}}}$  on  $\mathfrak{g}_{|\underline{\mathbf{d}}|}$ . We have already mentioned the  $G_{\underline{\mathbf{d}}}$ -action on  $\mathcal{F}_{\underline{\mathbf{d}}}$  and  $\mathcal{F}_{\underline{\mathbf{d}'}}$  in Remark 1.5.4. Therefore, by restriction we automatically get  $G_{\underline{\mathbf{d}}}$ -actions on  $\widetilde{\text{Rep}_{\underline{\mathbf{d}}}(Q)}$ ,  $\text{Rep}_{\underline{\mathbf{d}}}(Q)$ ,  $\mathcal{Z}_{\underline{\mathbf{d}}, \underline{\mathbf{d}'}}$  and  $\mathcal{Z}_{\underline{\mathbf{d}}}$ . All the maps we mentioned in Definition 1.5.8 are  $G_{\underline{\mathbf{d}}}$ -equivariant.
2. In Section 6.2 we will also view all the varieties as  $G_{\underline{\mathbf{d}}} \times \mathbb{C}^{\times}$ -varieties, so we also shortly introduce  $\mathbb{C}^{\times}$ -action here. View  $\text{Rep}_{\underline{\mathbf{d}}}(Q)$  as a  $\mathbb{C}$ -vector space,  $\mathbb{C}^{\times}$  acts on  $\text{Rep}_{\underline{\mathbf{d}}}(Q)$  by scalar multiplication. For  $\mathcal{F}_{\underline{\mathbf{d}}}$  and  $\mathcal{F}_{\underline{\mathbf{d}'}}$ ,  $\mathbb{C}^{\times}$  acts trivially, and by restriction we get  $\mathbb{C}^{\times}$ -actions on  $\widetilde{\text{Rep}_{\underline{\mathbf{d}}}(Q)}$ ,  $\text{Rep}_{\underline{\mathbf{d}}}(Q)$ ,  $\mathcal{Z}_{\underline{\mathbf{d}}, \underline{\mathbf{d}'}}$  and  $\mathcal{Z}_{\underline{\mathbf{d}}}$ . Also, all the maps we mentioned above are  $\mathbb{C}^{\times}$ -equivariant.
3. It may worth mentioning that  $\mathcal{F}_{\underline{\mathbf{d}}}$  has an  $\mathbb{W}_{|\underline{\mathbf{d}}|}$ -action which can be extended neither to  $\mathbb{G}_{|\underline{\mathbf{d}}|}$ -action on  $\mathcal{F}_{\underline{\mathbf{d}}}$  nor to  $\mathbb{W}_{|\underline{\mathbf{d}}|}$ -action on  $\text{Rep}_{\underline{\mathbf{d}}}(Q)$ .

## 1.6 Stratification and $T$ -fixed points

Natural defined varieties resemble burr puzzles, they have delicate structures and can be decomposed as relatively easy pieces. In this subsection, we will find stratifications of varieties introduced in Section 1.5, and fix notations of orbits. We will also mention about their  $T$ -fixed points. These stratifications will give us a basis for the  $K$ -theory and cohomology theory in Chapter 2, while those  $T$ -fixed points will give us another "basis" in Chapter 4.

### 1.6.1 Stratification: flag variety

We begin with  $\mathcal{F}_{|\mathbf{d}|}$  and  $\mathcal{F}_{|\mathbf{d}|} \times \mathcal{F}_{|\mathbf{d}|}$ , which is roughly a repetition of Section 1.1.

**Definition 1.6.1** (Twisted action). *We define the twisted  $\mathbb{G}_{|\mathbf{d}|} \times \mathbb{G}_{|\mathbf{d}|}$ -action on  $\mathcal{F}_{|\mathbf{d}|} \times \mathcal{F}_{|\mathbf{d}|}$ :*

$$\mathbb{G}_{|\mathbf{d}|} \times \mathbb{G}_{|\mathbf{d}|} \times \mathcal{F}_{|\mathbf{d}|} \times \mathcal{F}_{|\mathbf{d}|} \longrightarrow \mathcal{F}_{|\mathbf{d}|} \times \mathcal{F}_{|\mathbf{d}|} \quad (g_1, g_2, \underline{F}_{g, g'}) \longmapsto \underline{F}_{g_1 g, g_2 g'}$$

*which is the same as original  $\mathbb{G}_{|\mathbf{d}|}$ -action when we restrict to  $\mathbb{G}_{|\mathbf{d}|} \times \{\text{Id}\}$ -action. Other  $G \times G$ -actions on  $\mathcal{F} \times \mathcal{F}$  are defined in a similar way.*

**Definition 1.6.2** (Stratifications of  $\mathcal{F}_{|\mathbf{d}|}$  and  $\mathcal{F}_{|\mathbf{d}|} \times \mathcal{F}_{|\mathbf{d}|}$ ). *For  $\varpi, \varpi' \in \mathbb{W}_{|\mathbf{d}|}$ , we define*

$$\begin{aligned} \mathcal{V}_{\varpi} &= \mathbb{B}_{|\mathbf{d}|} \cdot F_{\varpi} && \subseteq \mathcal{F}_{|\mathbf{d}|} \\ \mathcal{V}_{\varpi, \varpi'} &= (\mathbb{B}_{|\mathbf{d}|} \times \mathbb{B}_{|\mathbf{d}|}) \cdot \underline{F}_{\varpi, \varpi'} && \subseteq \mathcal{F}_{|\mathbf{d}|} \times \mathcal{F}_{|\mathbf{d}|} \\ \mathcal{V}_{\varpi'} &= \mathbb{G}_{|\mathbf{d}|} \cdot \underline{F}_{\text{Id}, \varpi'} && \subseteq \mathcal{F}_{|\mathbf{d}|} \times \mathcal{F}_{|\mathbf{d}|} \end{aligned}$$

*as  $\mathbb{B}_{|\mathbf{d}|}$ -orbit,  $\mathbb{B}_{|\mathbf{d}|} \times \mathbb{B}_{|\mathbf{d}|}$ -orbit,  $\mathbb{G}_{|\mathbf{d}|}$ -orbit of  $\mathcal{F}_{|\mathbf{d}|}$ ,  $\mathcal{F}_{|\mathbf{d}|} \times \mathcal{F}_{|\mathbf{d}|}$ ,  $\mathcal{F}_{|\mathbf{d}|} \times \mathcal{F}_{|\mathbf{d}|}$ , respectively.*

By Bruhat-decomposition, we are able to show

$$\mathcal{F}_{|\mathbf{d}|} = \bigsqcup_{\varpi} \mathcal{V}_{\varpi} \quad \mathcal{F}_{|\mathbf{d}|} \times \mathcal{F}_{|\mathbf{d}|} = \bigsqcup_{\varpi'} \mathcal{V}_{\varpi'} = \bigsqcup_{\varpi, \varpi'} \mathcal{V}_{\varpi, \varpi'}.$$

We also realize these orbits as quotients of algebraic groups by the orbit-stabilizer theorem, as follows:

$$\begin{aligned} \mathcal{V}_{\varpi} &\cong \mathbb{B}_{|\mathbf{d}|} / (\mathbb{B}_{|\mathbf{d}|} \cap \mathbb{B}_{\varpi}) && \cong \mathbb{A}^{l(\varpi)} \\ \mathcal{V}_{\varpi, \varpi'} &\cong (\mathbb{B}_{|\mathbf{d}|} \times \mathbb{B}_{|\mathbf{d}|}) / (\mathbb{B}_{|\mathbf{d}|} \cap \mathbb{B}_{\varpi} \times \mathbb{B}_{|\mathbf{d}|} \cap \mathbb{B}_{\varpi'}) && \cong \mathbb{A}^{l(\varpi) + l(\varpi')} \\ \mathcal{V}_{\varpi'} &\cong \mathbb{G}_{|\mathbf{d}|} / (\mathbb{B}_{|\mathbf{d}|} \cap \mathbb{B}_{\varpi'}) && \cong \mathbb{A}^{l(\varpi')} \text{-bundle over } \mathcal{F}_{|\mathbf{d}|} \end{aligned}$$

Similar stratifications happen for  $\mathcal{F}_u$  and  $\mathcal{F}_{\mathbf{d}}$ .

**Definition 1.6.3** (Stratifications of  $\mathcal{F}_u$  and  $\mathcal{F}_u \times \mathcal{F}_{u'}$ ). *For  $u, u' \in \text{Min}(\mathbb{W}_{|\mathbf{d}|}, W_{\mathbf{d}})$ ,  $w, w' \in W_{\mathbf{d}}$ , we define*

$$\begin{aligned} \Omega_w^u &= B_{\mathbf{d}} \cdot F_{wu} && \subseteq \mathcal{F}_u \\ \Omega_{w, w'}^{u, u'} &= (B_{\mathbf{d}} \times B_{\mathbf{d}}) \cdot (F_{wu}, F_{ww'u'}) && \subseteq \mathcal{F}_u \times \mathcal{F}_{u'} \\ \Omega_{w'}^{u, u'} &= G_{\mathbf{d}} \cdot (F_u, F_{w'u'}) && \subseteq \mathcal{F}_u \times \mathcal{F}_{u'} \end{aligned}$$

*as  $B_{\mathbf{d}}$ -orbit,  $B_{\mathbf{d}} \times B_{\mathbf{d}}$ -orbit,  $G_{\mathbf{d}}$ -orbit of  $\mathcal{F}_u$ ,  $\mathcal{F}_u \times \mathcal{F}_{u'}$ ,  $\mathcal{F}_u \times \mathcal{F}_{u'}$ , respectively.*

By Bruhat decomposition, we are again able to show

$$\mathcal{F}_u = \bigsqcup_w \Omega_w^u \quad \mathcal{F}_u \times \mathcal{F}_{u'} = \bigsqcup_{w'} \Omega_{w'}^{u, u'} = \bigsqcup_{w, w'} \Omega_{w, w'}^{u, u'}$$

and

$$\begin{aligned} \Omega_w^u &\cong B_{\mathbf{d}} / (B_{\mathbf{d}} \cap B_w) && \cong \mathbb{A}^{l(w)} \\ \Omega_{w, w'}^{u, u'} &\cong (B_{\mathbf{d}} \times B_{\mathbf{d}}) / (B_{\mathbf{d}} \cap B_w \times B_{\mathbf{d}} \cap B_{w'}) && \cong \mathbb{A}^{l(w) + l(w')} \\ \Omega_{w'}^{u, u'} &\cong G_{\mathbf{d}} / (B_{\mathbf{d}} \cap B_{w'}) && \cong \mathbb{A}^{l(w')} \text{-bundle over } \mathcal{F}_u \end{aligned}$$



**Definition 1.6.4** (Stratifications of  $\mathcal{F}_{\mathbf{d}}$  and  $\mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}}$ ). For  $\varpi, \varpi' \in \mathbb{W}_{|\mathbf{d}|}$ , we define

$$\begin{aligned} \mathcal{O}_{\varpi} &= B_{\mathbf{d}} \cdot F_{\varpi} && \subseteq \mathcal{F}_{\varpi} && \subseteq \mathcal{F}_{\mathbf{d}} \\ \mathcal{O}_{\varpi, \varpi'} &= (B_{\mathbf{d}} \times B_{\mathbf{d}}) \cdot \underline{F}_{\varpi, \varpi'} && \subseteq \mathcal{F}_{\varpi} \times \mathcal{F}_{\varpi\varpi'} && \subseteq \mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}} \\ \mathcal{O}_{\varpi'} &= \bigsqcup_u G_{\mathbf{d}} \cdot \underline{F}_{u, \varpi'} && \subseteq \bigsqcup_u \mathcal{F}_u \times \mathcal{F}_{u\varpi'} && \subseteq \mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}} \end{aligned}$$

as  $B_{\mathbf{d}}$ -orbit,  $B_{\mathbf{d}} \times B_{\mathbf{d}}$ -orbit, (union of)  $G_{\mathbf{d}}$ -orbit of  $\mathcal{F}_{\mathbf{d}}$ ,  $\mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}}$ ,  $\mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}}$ , respectively.

Notice that  $\mathcal{O}_{\varpi}$ ,  $\mathcal{O}_{\varpi, \varpi'}$ ,  $\mathcal{O}_{\varpi'}$  are preimages of  $\mathcal{V}_{\varpi}$ ,  $\mathcal{V}_{\varpi, \varpi'}$ ,  $\mathcal{V}_{\varpi'}$  under the maps

$$\mathcal{F}_{\mathbf{d}} \hookrightarrow \mathcal{F}_{|\mathbf{d}|} \quad \mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}} \hookrightarrow \mathcal{F}_{|\mathbf{d}|} \times \mathcal{F}_{|\mathbf{d}|}.$$

Therefore,

$$\mathcal{F}_{\mathbf{d}} = \bigsqcup_{\varpi} \mathcal{O}_{\varpi} \quad \mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}} = \bigsqcup_{\varpi'} \mathcal{O}_{\varpi'} = \bigsqcup_{\varpi, \varpi'} \mathcal{O}_{\varpi, \varpi'}.$$

We still need to care about symbols. For  $\varpi = wu$ ,  $\varpi' = w'u'$ , denote  $uw'u' = \tilde{w}\tilde{u}$  for  $\tilde{w}' \in W_{\mathbf{d}}$ ,  $\tilde{u} \in \text{Min}(\mathbb{W}_{|\mathbf{d}|}, W_{\mathbf{d}})$ , then

$$\underline{F}_{\varpi, \varpi'} = (F_{\varpi}, F_{\varpi\varpi'}) = (F_{wu}, F_{wuw'u'}) = (F_{wu}, F_{w\tilde{w}\tilde{u}}) \in \mathcal{F}_u \times \mathcal{F}_{\tilde{u}}.$$

This incompatibility comes from our twisted  $G_{\mathbf{d}} \times G_{\mathbf{d}}$ -actions. In particular, denote

$$\mathcal{O}_{\varpi'}^u := G_{\mathbf{d}} \cdot \underline{F}_{u, \varpi'} \subseteq \mathcal{F}_u \times \mathcal{F}_{\tilde{u}},$$

we have  $\mathcal{O}_{\varpi'} = \sqcup_u \mathcal{O}_{\varpi'}^u$ , and identifications

$$\mathcal{O}_{\varpi} = \Omega_w^u \quad \mathcal{O}_{\varpi, \varpi'} = \Omega_{w, \tilde{w}}^{u, \tilde{u}} \quad \mathcal{O}_{\varpi'}^u = \Omega_{\tilde{w}}^{u, \tilde{u}}. \quad (\star)$$

We can also describe the closure of orbits, for example,

$$\overline{\Omega}_w^u = \bigsqcup_{x \leq w} \Omega_x^u \quad \overline{\Omega}_{w, w'}^{u, u'} = \bigsqcup_{x \leq w, x' \leq w'} \Omega_{x, x'}^{u, u'} \quad \overline{\Omega}_{w'}^{u, u'} = \bigsqcup_{x' \leq w'} \Omega_{x'}^{u, u'}$$

Especially, for any  $s \in \Pi_{\mathbf{d}}$ ,  $u, u' \in \text{Min}(\mathbb{W}_{|\mathbf{d}|}, W_{\mathbf{d}})$ , we have

$$\overline{\Omega}_s^{u, u'} = \Omega_s^{u, u'} \sqcup \Omega_{\text{Id}}^{u, u'} \cong G_{\mathbf{d}} \times^{B_{\mathbf{d}}} (P_{\text{Id}, s} / B_{\mathbf{d}})$$

when we work over base point  $F_{u, u'}$ . If we work over different base points, we will get different isomorphisms, as follows:

$$\begin{aligned} \overline{\Omega}_s^{u, u'} &= \Omega_{\text{Id}}^{u, u'} \sqcup \Omega_s^{u, u'} \cong G_{\mathbf{d}} / (B_w \cap B_{ws}) && \sqcup G_{\mathbf{d}} / B_w \\ &\cong G_{\mathbf{d}} \times^{B_w} (B_w / (B_w \cap B_{ws})) \sqcup G_{\mathbf{d}} \times^{B_w} (B_w / B_w) \\ &\cong G_{\mathbf{d}} \times^{B_w} (B_w s B_w / B_w) && \sqcup G_{\mathbf{d}} \times^{B_w} (B_w / B_w) \\ &\cong G_{\mathbf{d}} \times^{B_w} (\underline{P}_{w, s} / B_w) && \text{base point } F_{wu, wu'} \\ &\cong G_{\mathbf{d}} \times^{B_w} (\underline{P}_{w, s} / B_{ws}) && \text{base point } F_{wu, wsu'} \end{aligned}$$

Closures of  $\mathcal{O}$ -cells are obtained by identifications  $(\star)$ . To illustrate it, we compute  $\overline{\mathcal{O}}_s$  by hand. Let  $\varpi' = s, us = \tilde{w}\tilde{u}$ ,

$$\begin{aligned} \overline{\mathcal{O}}_s &= \bigsqcup_u \overline{\mathcal{O}}_s^u = \bigsqcup_u \overline{\Omega}_{\tilde{w}}^{u, \tilde{u}} \\ &= \left( \bigsqcup_{u:usu^{-1} \in W_{\mathbf{d}}} \overline{\Omega}_{usu^{-1}}^{u, u} \right) \sqcup \left( \bigsqcup_{u:usu^{-1} \notin W_{\mathbf{d}}} \overline{\Omega}_{\text{Id}}^{u, us} \right) \\ &= \left( \bigsqcup_{u:usu^{-1} \in W_{\mathbf{d}}} \Omega_{usu^{-1}}^{u, u} \right) \sqcup \left( \bigsqcup_{u:usu^{-1} \notin W_{\mathbf{d}}} \Omega_{\text{Id}}^{u, us} \right) \sqcup \left( \bigsqcup_{u:usu^{-1} \in W_{\mathbf{d}}} \Omega_{\text{Id}}^{u, u} \right) \\ &= \mathcal{O}_s \sqcup \left( \bigsqcup_{u:usu^{-1} \in W_{\mathbf{d}}} \mathcal{O}_{\text{Id}}^u \right) \end{aligned}$$

We restrict the result of  $\overline{\Omega}_s^{u, u'}$  to  $\overline{\mathcal{O}}_s^u$  in Lemma 1.6.5.

**Lemma 1.6.5.** *For  $\varpi = wu \in \mathbb{W}_{|\mathbf{d}|}$ ,  $s \in \Pi$  such that  $\varpi s \varpi^{-1} \in W_{\mathbf{d}}$ , we have isomorphisms of  $G_{\mathbf{d}}$ -varieties*

$$\begin{aligned} G_{\mathbf{d}} \times^{B_{\varpi}} (\underline{P}_{\varpi, s} / B_{\varpi}) &\longrightarrow \overline{\mathcal{O}}_s^u & (g, p) &\longmapsto (g \cdot F_{\varpi}, gp \cdot F_{\varpi}) \\ G_{\mathbf{d}} \times^{B_{\varpi}} (\underline{P}_{\varpi, s} / B_{\varpi s}) &\longrightarrow \overline{\mathcal{O}}_s^u & (g, p) &\longmapsto (g \cdot F_{\varpi}, gp \cdot F_{\varpi s}) \end{aligned}$$

*Proof.* Notice that when  $\varpi s \varpi^{-1} \in W_{\mathbf{d}}$ ,  $\mathcal{O}_s^u = \Omega_{usu^{-1}}^{u, u}$ . Therefore,

$$\begin{aligned} \overline{\mathcal{O}}_s^u = \overline{\Omega}_{usu^{-1}}^{u, u} &\cong \begin{cases} G_{\mathbf{d}} \times^{B_w} (\underline{P}_{w, usu^{-1}} / B_w) & \text{base point } F_{wu, wu} \\ G_{\mathbf{d}} \times^{B_w} (\underline{P}_{w, usu^{-1}} / B_{wusu^{-1}}) & \text{base point } F_{wu, wus} \end{cases} \\ &\cong \begin{cases} G_{\mathbf{d}} \times^{B_{\varpi}} (\underline{P}_{\varpi, s} / B_{\varpi}) & \text{base point } F_{\varpi, \varpi} \\ G_{\mathbf{d}} \times^{B_{\varpi}} (\underline{P}_{\varpi, s} / B_{\varpi s}) & \text{base point } F_{\varpi, \varpi s} \end{cases} \end{aligned} \quad \square$$

After so many notations are introduced rapidly, an enlightening example is needed here.

**Example 1.6.6** (Follows Example 1.3.8). *Here,  $\mathbb{W}_{|\mathbf{d}|} = S_3$ ,  $W_{\mathbf{d}} = S_1 \times S_2$ ,*

$$\varpi = ts = t \cdot s, \quad \varpi' = s = \text{Id} \cdot s, \quad \varpi \varpi' = t = t \cdot \text{Id}.$$

$\mathcal{F}_{\mathbf{d}}$  has 3 connected components, each of them has 2  $B_{\mathbf{d}}$ -orbits;

$\mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}}$  has 9 connected components, each of them has 4  $B_{\mathbf{d}} \times B_{\mathbf{d}}$ -orbits. We have

$\begin{array}{c} \text{shape} \\ (B_{\mathbf{d}} \times B_{\mathbf{d}}) \cdot F_{\varpi, \varpi'} \\ B_{\mathbf{d}} \cdot F_{\varpi} \end{array}$		$B_{\mathbf{d}} \cdot F_{\varpi \varpi'}$		$\mathcal{F}_{\text{Id}}$		$\mathcal{F}_s$		$\mathcal{F}_{st}$	
		$\bullet$	$\overline{\phantom{x}}$	$\bullet$	$\overline{\phantom{x}}$	$\bullet$	$\overline{\phantom{x}}$	$\bullet$	$\overline{\phantom{x}}$
$\mathcal{F}_{\text{Id}}$	$\mathcal{O}_{\text{Id}} = \Omega_{\text{Id}}^{\text{Id}}$	$\bullet$	$\overline{\phantom{x}}$	$\Omega_{\text{Id}, \text{Id}}^{\text{Id}, \text{Id}}$	$\Omega_{\text{Id}, t}^{\text{Id}, \text{Id}}$	$\bullet$	$\overline{\phantom{x}}$	$\Omega_{\text{Id}, \text{Id}}^{\text{Id}, st}$	$\Omega_{\text{Id}, t}^{\text{Id}, st}$
	$\mathcal{O}_t = \Omega_t^{\text{Id}}$	$\overline{\phantom{x}}$	$\overline{\phantom{x}}$	$\Omega_{t, t}^{\text{Id}, \text{Id}}$	$\Omega_{t, \text{Id}}^{\text{Id}, \text{Id}}$	$\overline{\phantom{x}}$	$\overline{\phantom{x}}$	$\Omega_{t, t}^{\text{Id}, st}$	$\Omega_{t, \text{Id}}^{\text{Id}, st}$
$\mathcal{F}_s$	$\mathcal{O}_s = \Omega_{\text{Id}}^s$	$\bullet$	$\overline{\phantom{x}}$	$\Omega_{\text{Id}, \text{Id}}^{s, \text{Id}}$	$\Omega_{\text{Id}, t}^{s, \text{Id}}$	$\bullet$	$\overline{\phantom{x}}$	$\Omega_{\text{Id}, \text{Id}}^{s, st}$	$\Omega_{\text{Id}, t}^{s, st}$
	$\mathcal{O}_{ts} = \Omega_t^s$	$\overline{\phantom{x}}$	$\overline{\phantom{x}}$	$\Omega_{t, t}^{s, \text{Id}}$	$\Omega_{t, \text{Id}}^{s, \text{Id}}$	$\overline{\phantom{x}}$	$\overline{\phantom{x}}$	$\Omega_{t, t}^{s, st}$	$\Omega_{t, \text{Id}}^{s, st}$
$\mathcal{F}_{st}$	$\mathcal{O}_{ts} = \Omega_{\text{Id}}^{st}$	$\bullet$	$\overline{\phantom{x}}$	$\Omega_{\text{Id}, \text{Id}}^{st, \text{Id}}$	$\Omega_{\text{Id}, t}^{st, \text{Id}}$	$\bullet$	$\overline{\phantom{x}}$	$\Omega_{\text{Id}, \text{Id}}^{st, st}$	$\Omega_{\text{Id}, t}^{st, st}$
	$\mathcal{O}_{sts} = \Omega_t^{st}$	$\overline{\phantom{x}}$	$\overline{\phantom{x}}$	$\Omega_{t, t}^{st, \text{Id}}$	$\Omega_{t, \text{Id}}^{st, \text{Id}}$	$\overline{\phantom{x}}$	$\overline{\phantom{x}}$	$\Omega_{t, t}^{st, st}$	$\Omega_{t, \text{Id}}^{st, st}$

Table 1.6: stratifications of  $\mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}}$ 

given every orbit a name, and other spaces are finite union of these orbits. For example,

$$\begin{aligned}
\mathcal{O}_{ts, s} &= \Omega_{t, \text{Id}}^{s, \text{Id}} \\
\mathcal{O}_s^s &= \Omega_{\text{Id}}^{s, \text{Id}} = \Omega_{\text{Id}, \text{Id}}^{s, \text{Id}} \sqcup \Omega_{t, \text{Id}}^{s, \text{Id}} \\
\mathcal{O}_s &= \mathcal{O}_s^s \sqcup \mathcal{O}_s^{\text{Id}} \sqcup \mathcal{O}_s^{st} \\
&= \Omega_{\text{Id}}^{s, \text{Id}} \sqcup \Omega_{\text{Id}}^{\text{Id}, s} \sqcup \Omega_{\text{Id}}^{st, st} \\
&= \Omega_{\text{Id}, \text{Id}}^{s, \text{Id}} \sqcup \Omega_{t, \text{Id}}^{s, \text{Id}} \sqcup \Omega_{\text{Id}, \text{Id}}^{\text{Id}, s} \sqcup \Omega_{t, \text{Id}}^{\text{Id}, s} \sqcup \Omega_{\text{Id}, \text{Id}}^{st, st} \sqcup \Omega_{t, \text{Id}}^{st, st}
\end{aligned}$$

Their closures are also clear from the table, for example,

$$\overline{\mathcal{O}}_s = \mathcal{O}_s \sqcup \Omega_{\text{Id}}^{st, st}$$

contains 8  $B_{\mathbf{d}} \times B_{\mathbf{d}}$ -orbits.

### 1.6.2 Stratification: incidence variety

Now comes the stratifications of incidence varieties. Those stratifications are produced by taking the preimage of stratifications on base spaces. They are relatively easy to obtain,

while their closures are quite difficult to analyze.

**Definition 1.6.7** (Stratifications of incidence varieties). *For  $\varpi = wu$ ,  $\varpi' = w'u' \in \mathbb{W}_{|\mathbf{d}|}$ , denote  $uwu' = \tilde{w}\tilde{u}$ ,  $\underline{\mathbf{d}} = W_{\mathbf{d}}u$ ,  $\underline{\mathbf{d}}' = W_{\mathbf{d}}u'$ ,  $\tilde{\underline{\mathbf{d}}} = W_{\mathbf{d}}\tilde{u}$ , we define*

$$\begin{aligned} \tilde{\Omega}_w^u &:= \pi_{\underline{\mathbf{d}}}^{-1}(\Omega_w^u) && \subseteq \widetilde{\text{Rep}_{\underline{\mathbf{d}}}}(Q) \\ \tilde{\Omega}_{w,w'}^{u,u'} &:= \pi_{\underline{\mathbf{d}},\underline{\mathbf{d}}'}^{-1}(\Omega_{w,w'}^{u,u'}) && \subseteq \mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}'} \\ \tilde{\Omega}_{w'}^{u,u'} &:= \pi_{\underline{\mathbf{d}},\underline{\mathbf{d}}'}^{-1}(\Omega_{w'}^{u,u'}) && \subseteq \mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}'} \\ \tilde{\mathcal{O}}_{\varpi'}^u &:= \pi_{\underline{\mathbf{d}},\tilde{\underline{\mathbf{d}}}}^{-1}(\mathcal{O}_{\varpi'}^u) = \tilde{\Omega}_{\tilde{w}}^{u,\tilde{u}} && \subseteq \mathcal{Z}_{\underline{\mathbf{d}},\tilde{\underline{\mathbf{d}}}} \\ \tilde{\mathcal{O}}_{\varpi} &:= \pi_{\underline{\mathbf{d}}}^{-1}(\mathcal{O}_{\varpi}) && \subseteq \widetilde{\text{Rep}_{\underline{\mathbf{d}}}}(Q) \\ \tilde{\mathcal{O}}_{\varpi,\varpi'} &:= \pi_{\underline{\mathbf{d}},\underline{\mathbf{d}}'}^{-1}(\mathcal{O}_{\varpi,\varpi'}) && \subseteq \mathcal{Z}_{\underline{\mathbf{d}}} \\ \tilde{\mathcal{O}}_{\varpi'} &:= \pi_{\underline{\mathbf{d}},\underline{\mathbf{d}}'}^{-1}(\mathcal{O}_{\varpi'}) && \subseteq \mathcal{Z}_{\underline{\mathbf{d}}} \end{aligned}$$

It is not hard to see that they are stratifications:

$$\begin{aligned} \widetilde{\text{Rep}_{\underline{\mathbf{d}}}}(Q) &= \bigsqcup_{\varpi} \tilde{\Omega}_w^u & \mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}'} &= \bigsqcup_w \tilde{\Omega}_{w'}^{u,u'} = \bigsqcup_{w,w'} \tilde{\Omega}_{w,w'}^{u,u'} \\ \widetilde{\text{Rep}_{\mathbf{d}}}(Q) &= \bigsqcup_{\varpi} \tilde{\mathcal{O}}_{\varpi} & \mathcal{Z}_{\mathbf{d}} &= \bigsqcup_{\varpi'} \tilde{\mathcal{O}}_{\varpi'} = \bigsqcup_{\varpi,\varpi'} \tilde{\mathcal{O}}_{\varpi,\varpi'} \end{aligned}$$

**Proposition 1.6.8.** *Those stratifications are affine spaces over corresponding base spaces. To be precise,*

$$\begin{aligned} \tilde{\Omega}_w^u &= \mathbf{r}_{wu}\text{-bundle over } \Omega_w^u \\ \tilde{\Omega}_{w,w'}^{u,u'} &= \mathbf{r}_{wu,ww'u'}\text{-bundle over } \Omega_{w,w'}^{u,u'} \\ \tilde{\Omega}_{w'}^{u,u'} &= \mathbf{r}_{u,w'u'}\text{-bundle over } \Omega_{w'}^{u,u'} \\ \tilde{\mathcal{O}}_{\varpi'}^u &= \mathbf{r}_{u,\varpi'}\text{-bundle over } \mathcal{O}_{\varpi'}^u \\ \tilde{\mathcal{O}}_{\varpi} &= \mathbf{r}_{\varpi}\text{-bundle over } \mathcal{O}_{\varpi} \\ \tilde{\mathcal{O}}_{\varpi,\varpi'} &= \mathbf{r}_{\varpi,\varpi'}\text{-bundle over } \mathcal{O}_{\varpi,\varpi'} \\ \tilde{\mathcal{O}}_{\varpi'} &= \mathbf{r}_{\text{Id},\varpi'}\text{-bundle over } \mathcal{O}_{\varpi'} \end{aligned}$$

*Proof.* The fibers are all computed over the preferred base point. The group action induces the isomorphism between different fibers, and lift affine local charts on base space (viewed as group quotient) to the local charts of fiber bundles.  $\square$

We will frequently use closures of some stratifications, so we give them names.

**Definition 1.6.9.** We define

$$\begin{aligned} \mathcal{Z}_{w'}^{u,u'} &:= \overline{\Omega}_{w'}^{u,u'} \subseteq \mathcal{Z}^{u,u'} := \mathcal{Z}_{\underline{\mathbf{d}}, \underline{\mathbf{d}}'}, \\ \mathcal{Z}_{\varpi'} &:= \overline{\mathcal{O}}_{\varpi'} \subseteq \mathcal{Z}_{\underline{\mathbf{d}}}. \end{aligned}$$

**Proposition 1.6.10** (Properties of the closure).  $\mathcal{Z}_{\varpi'}$  is a Zarisky-locally trivial cone bundle over  $\mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}}}$ . To be precise, under the map

$$\pi_{\underline{\mathbf{d}}, \underline{\mathbf{d}}, \varpi'} : \mathcal{Z}_{\varpi'} \longrightarrow \mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}}},$$

for any  $x, x' \in \mathbb{W}_{|\underline{\mathbf{d}}|}$ ,  $\pi_{\underline{\mathbf{d}}, \underline{\mathbf{d}}, \varpi'}^{-1}(\mathcal{O}_{x, x'})$  is a trivial fiber bundle over  $\mathcal{O}_{x, x'}$ , whose fibers are cones.

I have no idea when  $\mathcal{Z}_{\varpi'}$  is a Zarisky-locally trivial vector bundle. Let us see when we need this fact.

*Remark 1.6.11.* Notice that

$$\begin{aligned} \mathcal{Z}_{w'}^{u,u'} &:= \overline{\Omega}_{w'}^{u,u'} \subseteq \tilde{\Omega}_{w'}^{u,u'} := \pi_{\underline{\mathbf{d}}, \underline{\mathbf{d}}'}^{-1}(\overline{\Omega}_{w'}^{u,u'}), \\ \mathcal{Z}_{\varpi'} &:= \overline{\mathcal{O}}_{\varpi'} \subseteq \tilde{\mathcal{O}}_{\varpi'} := \pi_{\underline{\mathbf{d}}, \underline{\mathbf{d}}}^{-1}(\overline{\mathcal{O}}_{\varpi'}). \end{aligned}$$

Even though these inclusions are usually not equalities, we can still say something when the length of  $w'$  or  $\varpi'$  is small. For example,

$$\begin{aligned} \mathcal{Z}_{\text{Id}}^{u,u'} &= \tilde{\Omega}_{\text{Id}}^{u,u'} \\ \mathcal{Z}_{\text{Id}} &= \tilde{\mathcal{O}}_{\text{Id}} \\ \tilde{\Omega}_s^{u,u'} \sqcup \Omega_{\text{Id}}^{u,u'} &\subseteq \mathcal{Z}_s^{u,u'} \subseteq \tilde{\Omega}_s^{u,u'} \sqcup \tilde{\Omega}_{\text{Id}}^{u,u'} \quad (s \in \Pi_{\underline{\mathbf{d}}}) \\ \tilde{\mathcal{O}}_s \sqcup \left( \bigsqcup_{u:usu^{-1} \in W_{\underline{\mathbf{d}}}} \mathcal{O}_{\text{Id}}^u \right) &\subseteq \mathcal{Z}_s \subseteq \tilde{\mathcal{O}}_s \sqcup \left( \bigsqcup_{u:usu^{-1} \in W_{\underline{\mathbf{d}}}} \tilde{\mathcal{O}}_{\text{Id}}^u \right) \quad (s \in \Pi) \end{aligned}$$

I have no idea if we have any accurate description for  $\mathcal{Z}_s^{u,u'}$  and  $\mathcal{Z}_s$ .

We end this subsection by Table 1.7:

### 1.6.3 $T$ -fixed points

Compare with stratifications,  $T$ -fixed points are easy to compute and have clear structures. Somewhat surprisingly, these  $T$ -fixed points encode most information of varieties.

Recall that the  $T$ -fixed points of a complete flag variety  $\mathcal{F}$  are exactly those coordinate flags  $\{F_w \mid w \in W\}$ . For absolute or relative flag varieties, we have similar results:

$$\mathcal{F}_{|\underline{\mathbf{d}}|}^{\mathbb{T}_{|\underline{\mathbf{d}}|}} = \mathcal{F}_{\underline{\mathbf{d}}}^{T_{\underline{\mathbf{d}}}} = \{F_{\varpi} \mid \varpi \in \mathbb{W}_{|\underline{\mathbf{d}}|}\} \quad \mathcal{F}_u^{T_{\underline{\mathbf{d}}}} = \{F_{wu} \mid w \in W_{\underline{\mathbf{d}}}\}$$

<div style="display: inline-block; transform: rotate(-45deg);"> stratification stabilizer variety base point </div>		type	$B$ -orbit	$B \times B$ -orbit twisted stabilizer	$G$ -orbit	Remark
$\mathcal{F}$	$\mathcal{F} \times \mathcal{F}$		$\Omega_g$	$\Omega_{g,g'}$	$\Omega_{g'}$	
$F_g$	$(F_g, F_{gg'})$		$B \cap gBg^{-1}$	$(B \cap gBg^{-1}) \times (B \cap g'Bg'^{-1})$	$gBg^{-1} \cap gg'B(gg')^{-1}$	
$\mathcal{F}_{ \mathbf{d} }$	$\mathcal{F}_{ \mathbf{d} } \times \mathcal{F}_{ \mathbf{d} }$		$\mathcal{V}_\varpi$	$\mathcal{V}_{\varpi,\varpi'}$	$\mathcal{V}_{\varpi'}$	
$F_\varpi$	$(F_\varpi, F_{\varpi\varpi'})$		$\mathbb{B}_{ \mathbf{d} } \cap \mathbb{B}_\varpi$	$(\mathbb{B}_{ \mathbf{d} } \cap \mathbb{B}_\varpi) \times (\mathbb{B}_{ \mathbf{d} } \cap \mathbb{B}_{\varpi'})$	$\mathbb{B}_\varpi \cap \mathbb{B}_{\varpi\varpi'}$	
$\mathcal{F}_u$	$\mathcal{F}_u \times \mathcal{F}_{u'}$		$\Omega_w^u$	$\Omega_{w,w'}^{u,u'}$	$\Omega_{w'}^{u,u'}$	
$F_{wu}$	$(F_{wu}, F_{ww'u'})$		$B_{\mathbf{d}} \cap B_w$	$(B_{\mathbf{d}} \cap B_w) \times (B_{\mathbf{d}} \cap B_{w'})$	$B_w \cap B_{ww'}$	
$\mathcal{F}_{\mathbf{d}}$	$\mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}}$		$\Omega_w^u$	$\Omega_{w,\tilde{w}}^{u,\tilde{u}}$	$\mathcal{O}_{\varpi'}^u = \Omega_{\tilde{w}}^{u,\tilde{u}}$	
$F_\varpi$	$(F_\varpi, F_{\varpi\varpi'})$		$B_{\mathbf{d}} \cap B_w$	$(B_{\mathbf{d}} \cap B_w) \times (B_{\mathbf{d}} \cap B_{\tilde{w}})$	$B_w \cap B_{w\tilde{w}}$	
$F_{wu}$	$(F_{wu}, F_{w\tilde{w}\tilde{u}})$					
The following may not be single orbit, but derived from the above definition.						
$\mathcal{F}_{\mathbf{d}}$	$\mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}}$		$\mathcal{O}_\varpi$	$\mathcal{O}_{\varpi,\varpi'}$	$\mathcal{O}_{\varpi'}$	preimage of
$F_\varpi$	$(F_\varpi, F_{\varpi\varpi'})$		$\Omega_w^u$	$\Omega_{w,\tilde{w}}^{u,\tilde{u}}$	$\sqcup_u \mathcal{O}_{\varpi'}^u$	$\mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}} \hookrightarrow \mathcal{F}_{ \mathbf{d} } \times \mathcal{F}_{ \mathbf{d} }$
$\widetilde{\text{Rep}}_{\mathbf{d}}(Q)$	$\mathcal{Z}_{\mathbf{d},\mathbf{d}'}$		$\tilde{\Omega}_w^u$	$\tilde{\Omega}_{w,w'}^{u,u'}$	$\tilde{\Omega}_{w'}^{u,u'}$	preimage of
$F_{wu}$	$(F_{wu}, F_{ww'u'})$					$\mathcal{Z}_{\mathbf{d},\mathbf{d}'} \hookrightarrow \mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}'}$
$\widetilde{\text{Rep}}_{\mathbf{d}}(Q)$	$\mathcal{Z}_{\mathbf{d}}$		$\tilde{\Omega}_w^u$	$\tilde{\Omega}_{w,\tilde{w}}^{u,\tilde{u}}$	$\tilde{\mathcal{O}}_{\varpi'}^u = \tilde{\Omega}_{\tilde{w}}^{u,\tilde{u}}$	preimage of
$F_\varpi$	$(F_\varpi, F_{\varpi\varpi'})$					$\mathcal{Z}_{\mathbf{d}} \hookrightarrow \mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}}$
$\widetilde{\text{Rep}}_{\mathbf{d}}(Q)$	$\mathcal{Z}_{\mathbf{d}}$		$\tilde{\mathcal{O}}_\varpi$	$\tilde{\mathcal{O}}_{\varpi,\varpi'}$	$\tilde{\mathcal{O}}_{\varpi'}$	preimage of
$F_\varpi$	$(F_\varpi, F_{\varpi\varpi'})$		$\tilde{\Omega}_w^u$	$\tilde{\Omega}_{w,\tilde{w}}^{u,\tilde{u}}$	$\sqcup_u \tilde{\mathcal{O}}_{\varpi'}^u$	$\mathcal{Z}_{\mathbf{d}} \hookrightarrow \mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}}$

Table 1.7: stratifications of typical varieties

For  $\text{Rep}_{\mathbf{d}}(Q)$ , we get

$$(\text{Rep}_{\mathbf{d}}(Q))^{T_{\mathbf{d}}} = \bigoplus_{a \in Q_1} \left( \text{Hom} \left( \mathbb{C}^{\mathbf{d}_{s(a)}}, \mathbb{C}^{\mathbf{d}_{t(a)}} \right) \right)^{T_{\mathbf{d}}} = \{\rho_0\}$$

where  $\rho_0$  is the zero representation in  $\text{Rep}_{\mathbf{d}}(Q)$ .

Combining these two results, one can easily describe  $T$ -fixed points of varieties constructed over them:

$$\begin{aligned}
(\mathcal{F}_{|\mathbf{d}|} \times \mathcal{F}_{|\mathbf{d}|})^{\mathbb{T}_{|\mathbf{d}|}} &= \{(F_\varpi, F_{\varpi'}) \mid \varpi, \varpi' \in \mathbb{W}_{|\mathbf{d}|}\} \\
(\mathcal{F}_u \times \mathcal{F}_{u'})^{T_{\mathbf{d}}} &= \{(F_{wu}, F_{w'u'}) \mid w, w' \in W_{\mathbf{d}}\} \\
(\mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}})^{T_{\mathbf{d}}} &= \{(F_\varpi, F_{\varpi'}) \mid \varpi, \varpi' \in \mathbb{W}_{|\mathbf{d}|}\} \\
(\widetilde{\text{Rep}}_{\mathbf{d}}(Q))^{T_{\mathbf{d}}} &= \{(\rho_0, F_{wu}) \mid w \in W_{\mathbf{d}}\} \\
(\widetilde{\text{Rep}}_{\mathbf{d}}(Q))^{T_{\mathbf{d}}} &= \{(\rho_0, F_\varpi) \mid \varpi \in \mathbb{W}_{|\mathbf{d}|}\} \\
(\mathcal{Z}_{\mathbf{d},\mathbf{d}'} )^{T_{\mathbf{d}}} &= \{(\rho_0, F_{wu}, F_{w'u'}) \mid w, w' \in W_{\mathbf{d}}\} \\
(\mathcal{Z}_{\mathbf{d}})^{T_{\mathbf{d}}} &= \{(\rho_0, F_\varpi, F_{\varpi'}) \mid \varpi, \varpi' \in \mathbb{W}_{|\mathbf{d}|}\}
\end{aligned}$$

Notice that, each  $B_{\mathbf{d}} \times B_{\mathbf{d}}$ -orbit of  $\mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}}$  contains exactly one  $T_{\mathbf{d}}$ -fixed point. Also, all the  $T$ -fixed points lie in the zero sections. By this reason, we can compute more:

$$\begin{aligned}
(\mathcal{Z}_{\text{Id}}^{u,u'})^{T_{\mathbf{d}}} &= \{(\rho_0, F_{wu}, F_{wu'}) \mid w \in W_{\mathbf{d}}\} \\
(\mathcal{Z}_{\text{Id}})^{T_{\mathbf{d}}} &= \{(\rho_0, F_{\varpi}, F_{\varpi}) \mid \varpi \in \mathbb{W}_{|\mathbf{d}|}\} \\
(\mathcal{Z}_s^{u,u'})^{T_{\mathbf{d}}} &= \{(\rho_0, F_{wu}, F_{wsu'}) \mid w \in W_{\mathbf{d}}\} \sqcup \{(\rho_0, F_{wu}, F_{wu'}) \mid w \in W_{\mathbf{d}}\} \\
(\mathcal{Z}_s)^{T_{\mathbf{d}}} &= \{(\rho_0, F_{\varpi}, F_{\varpi s}) \mid \varpi \in \mathbb{W}_{|\mathbf{d}|}\} \sqcup \{(\rho_0, F_{\varpi}, F_{\varpi}) \mid \varpi \in \mathbb{W}_{|\mathbf{d}|}, \varpi s \varpi^{-1} \in W_{\mathbf{d}}\}
\end{aligned}$$

With huge effort, finally we fixed all the symbols and understand those typical varieties in detail.





## Chapter 2

# $K$ -theory and cohomology theory

From my humble point of view, there is no easy cohomology theory, in a sense that key properties are usually hard to prove. On the other hand, plenty of examples can be quickly computed once we grasp some properties and use them in black boxes. Therefore, we won't prove any properties we stated. We have no choice but to do so, for the restricted space and time.

The main reference for the  $K$ -theory is [1, Chapter 5].

**Setting 2.0.1.** *Throughout abstract results of  $K$ -theory, we use the following notations:*

- $G$  stands for a linear algebraic group, i.e., a closed subgroup of  $\mathrm{GL}_n(\mathbb{C})$ .<sup>1</sup> Denote  $m : G \times G \longrightarrow G$  as the multiplication map of  $G$ .
- $X$  is a variety over  $\mathbb{C}$ , i.e., a reduced, separated scheme of finite type over  $\mathbb{C}$ . We assume  $X$  to be quasi-projective.
- Usually,  $X$  is equipped with an algebraic  $G$ -action (which is compatible with the variety structure of  $G$  and  $X$ ), then we say that  $X$  is a  $G$ -variety. In that case, we will denote  $\alpha : G \times X \longrightarrow X$  as the  $G$ -action map.
- $\mathcal{F}$  is usually a sheaf on  $X$ , which is not flag variety  $\mathrm{GL}_n/B$ .

### 2.1 Definitions and initial examples

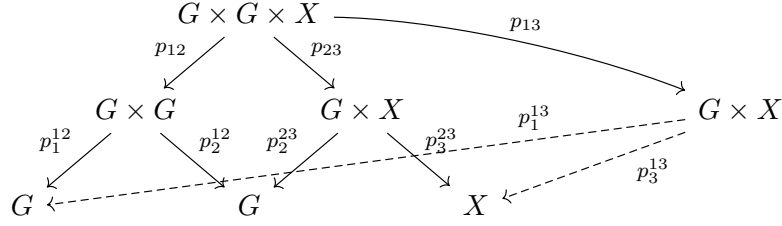
We give definition for  $K$ -theory, which is lengthy already. Roughly speaking, a  $G$ -equivariant coherent sheaf over  $X$  is a sheaf  $\mathcal{F} \in \mathrm{Coh}(X)$  equipped with  $G$ -action which is compatible with the  $G$ -action on  $X$ , and  $K$ -theory is the Grothendieck group of  $G$ -equivariant coherent sheaves over  $X$ .

**Definition 2.1.1** ( $G$ -equivariant sheaf, [1, Definition 5.1.6]). *For a  $G$ -variety  $X$ , denote*

---

<sup>1</sup>The closed embedding  $G \hookrightarrow \mathrm{GL}_n(\mathbb{C})$  is not considered as the data of  $G$ .

$p_i^{jk}, p_i := p_i^{123}, p_{ij} := p_{ij}^{123}$  as projections onto some factors, as follows.<sup>2</sup>



We have morphisms

$$\begin{array}{ccccc} & & m \times \text{Id}_X & \longrightarrow & \\ G \times G \times X & \xrightarrow{p_{23}} & G \times X & \xrightarrow[p_3^{23}=p_3^{13}]{\alpha} & X \\ & \xrightarrow{\text{Id}_G \times \alpha} & & & \end{array}$$

which satisfies the "coequalizer conditions":

$$\begin{aligned} p_3^{23} \circ (m \times \text{Id}_X) &= p_3^{23} \circ p_{23} & (g_1, g_2, x) &\longmapsto x \\ p_3^{23} \circ (\text{Id}_G \times \alpha) &= \alpha \circ p_{23} & (g_1, g_2, x) &\longmapsto g_2 x \\ \alpha \circ (m \times \text{Id}_X) &= \alpha \circ (\text{Id}_G \times \alpha) & (g_1, g_2, x) &\longmapsto g_1 g_2 x \end{aligned}$$

A **G-equivariant (coherent) sheaf**<sup>3</sup> on  $X$  is a sheaf  $\mathcal{F} \in \text{Coh}(X)$  equipped with an isomorphism

$$\phi_{\mathcal{F}} : p_3^{23,*} \mathcal{F} \longrightarrow \alpha^* \mathcal{F}$$

such that the following diagram commutes:

$$\begin{array}{ccc} (m \times \text{Id}_X)^* p_3^{23,*} \mathcal{F} & \xrightarrow{(m \times \text{Id}_X)^* \phi_{\mathcal{F}}} & (m \times \text{Id}_X)^* \alpha^* \mathcal{F} \\ \parallel & & \parallel \\ p_{23}^* p_3^{23,*} \mathcal{F} & & (\text{Id}_G \times \alpha)^* \alpha^* \mathcal{F} \\ \searrow p_{23}^* \phi_{\mathcal{F}} & & \nearrow (\text{Id}_G \times \alpha)^* \phi_{\mathcal{F}} \\ p_{23}^* \alpha^* \mathcal{F} & \xlongequal{\quad} & (\text{Id}_G \times \alpha)^* p_3^{23,*} \mathcal{F} \end{array} \quad (2.1.1)$$

A **(G-equivariant) morphism**  $f : (\mathcal{F}, \phi_{\mathcal{F}}) \longrightarrow (\mathcal{G}, \phi_{\mathcal{G}})$  between two  $G$ -equivariant sheaves is a morphism  $f : \mathcal{F} \longrightarrow \mathcal{G}$  in  $\text{Coh}(X)$  such that the diagram

$$\begin{array}{ccc} p_3^{23,*} \mathcal{F} & \xrightarrow{\phi_{\mathcal{F}}} & \alpha^* \mathcal{F} \\ p_3^{23,*} f \downarrow & & \downarrow \alpha^* f \\ p_3^{23,*} \mathcal{G} & \xrightarrow{\phi_{\mathcal{G}}} & \alpha^* \mathcal{G} \end{array} \quad (2.1.2)$$

<sup>2</sup>Be careful, under this convention, the projection map  $p_3^{23} = p_3^{13} : G \times X \longrightarrow X$  has subscription 3, and  $p_2$  means the projection from  $G \times G \times X$  to the second  $G$ . This convention is different with notations in [1, 5.1].

<sup>3</sup>we will omit the word "coherent" for shorter notation.

commutes.

We denote  $\text{Coh}^G(X)$  as the category of  $G$ -equivariant sheaves.

**Definition 2.1.2** ( $G$ -equivariant  $K$ -theory). *For a  $G$ -variety  $X$ , the  $G$ -equivariant  $K$ -theory is defined as the Grothendieck group of  $G$ -equivariant coherent sheaves over  $X$ , i.e.,*

$$K_0^G(X) := K_0(\text{Coh}^G(X)).$$

Specifically, for a point  $\text{pt} = \text{Spec } \mathbb{C}$  with trivial  $G$ -action, denote

$$R(G) := K_0^G(\text{pt}) = K_0(\text{Rep}(G))$$

as the representation ring of group  $G$ .

Let us unravel this construction a little bit. For (geometrical) points  $g, g_1, g_2 \in G$ , denote that

$$\begin{aligned} \iota_g : X &\longrightarrow G \times X & x &\longmapsto (g, x) \\ \iota_{g_1, g_2} : X &\longrightarrow G \times G \times X & x &\longmapsto (g_1, g_2, x) \\ \alpha_g : X &\xrightarrow{\iota_g} G \times X \xrightarrow{\alpha} X & x &\longmapsto gx \end{aligned}$$

By pulling back along  $\iota_g$  and  $\iota_{g_1, g_2}$ , we can see geometrical meanings in the expressions. Apply  $\iota_g^*$  to  $\phi_{\mathcal{F}}$ , one get

$$\iota_g^* \phi_{\mathcal{F}} : \mathcal{F} \longrightarrow \alpha_g^* \mathcal{F} \quad \rightsquigarrow \quad \phi_{g,x}^{\mathcal{F}} \triangleq (\iota_g^* \phi_{\mathcal{F}})_x : \mathcal{F}_x \longrightarrow \mathcal{F}_{gx}$$

Therefore,  $\phi_{\mathcal{F}}$  encodes information of  $G$ -action on  $\mathcal{F}$ , which is  $G$ -equivariant.

Now we apply  $\iota_{g_1, g_2}^*$  to (2.1.1):

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\iota_{g_1, g_2}^* \phi_{\mathcal{F}}} & \alpha_{g_1, g_2}^* \mathcal{F} = \alpha_{g_1}^* \alpha_{g_2}^* \mathcal{F} \\ \downarrow \iota_{g_2}^* \phi_{\mathcal{F}} & & \uparrow \iota_{g_1}^* \phi_{\alpha_{g_2}^* \mathcal{F}} \\ & \alpha_{g_2}^* \mathcal{F} & \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} \mathcal{F}_x & \xrightarrow{\phi_{g_1 g_2, x}^{\mathcal{F}}} & \mathcal{F}_{g_1 g_2 x} \\ \downarrow \phi_{g_2, x}^{\mathcal{F}} & & \uparrow \phi_{g_1, g_2 x}^{\mathcal{F}} = \phi_{g_1, x}^{\alpha_{g_2}^* \mathcal{F}} \\ & \mathcal{F}_{g_2 x} & \end{array}$$

So (2.1.1) is just the associative constraint of the  $G$ -structure on  $\mathcal{F}$ .

Similarly, apply  $\iota_g^*$  to (2.1.2), we get

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\iota_g^* \phi_{\mathcal{F}}} & \alpha_g^* \mathcal{F} \\ f \downarrow & & \downarrow \alpha_g^* f \\ \mathcal{G} & \xrightarrow{\iota_g^* \phi_{\mathcal{G}}} & \alpha_g^* \mathcal{G} \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} \mathcal{F}_x & \xrightarrow{\phi_{g,x}^{\mathcal{F}}} & \mathcal{F}_{gx} \\ f_x \downarrow & & \downarrow f_{gx} \\ \mathcal{G}_x & \xrightarrow{\phi_{g,x}^{\mathcal{G}}} & \mathcal{G}_{gx} \end{array}$$

So (2.1.2) is just the condition for  $f$  to be  $G$ -equivariant.

There are two extreme situations worth mentioning about. When  $G = \text{Id}$ , there is no  $G$ -action structure constrain on varieties and sheaves. Therefore,

$$\text{Coh}^{\text{Id}}(X) = \text{Coh}(X) \quad K_0^{\text{Id}}(X) = K_0(X) \hat{=} K_0(\text{Coh}(X)).$$

When  $G$  acts on  $X = \text{Spec } A$  trivially, any sheaf  $\mathcal{F} \in \text{Coh}^G(X)$  can be viewed as an (finitely generated)<sup>4</sup>  $A$ -module  $M$  with  $G$ -action, so

$$\text{Coh}^G(X) = \text{rep}_A(G) \xrightarrow{\text{when } G \text{ is finite}} \text{Mod}(A[G]).$$

In particular, any sheaf  $\mathcal{F} \in \text{Coh}^G(\text{pt})$  can be viewed as a finite dimensional complex  $G$ -representation, so

$$\text{Coh}^G(\text{pt}) = \text{rep}_{\mathbb{C}}(G) \xrightarrow{\text{when } G \text{ is finite}} \text{Mod}(\mathbb{C}[G]).$$

???(If I have time I will compute  $K_0(\mathbb{P}^1)$  here.)

### 2.1.1 Representation ring $R(G)$

Now let us try to figure out some examples.

Recall that any coherent sheaf over  $\text{pt}$  is equivalent to a finite dimensional  $\mathbb{C}$ -vector space, and any  $G$ -equivariant coherent sheaf over  $\text{pt}$  is equivalent to a finite dimensional complex  $G$ -representation. Moreover, by Jordan-Hölder theorem, every finite dimensional complex  $G$ -representation can be written as a composition series such that each quotient object is irreducible. Therefore,

$$R(G) = \bigoplus_{\rho \in \text{Irr}(G)} \mathbb{Z}$$

as a free  $\mathbb{Z}$ -module.

For  $R(G)$ , we have the multiplication structure induced by tensor products on complex  $G$ -representations. Let us see some examples now. We use Setting 1.1.1 in these examples.

**Example 2.1.3.** *For trivial group  $\text{Id}$ , every  $\text{Id}$ -representation is just a  $\mathbb{C}$ -vector space, which can be written as the direct sum of 1-dimensional vector spaces. Therefore,*

$$R(\text{Id}) = \mathbb{Z}.$$

**Example 2.1.4.** *For group  $T$ , since  $T$  is abelian, every  $T$ -representation can be written as direct sum of 1-dimensional vector spaces. Furthermore,*

$$\begin{aligned} \text{Irr}(T) &= \{ \rho : T \longrightarrow \mathbb{C}^\times \mid \rho \text{ is an (algebraic) group homomorphism} \} \\ &= \text{Hom}_{\mathbb{C}\text{-Alg } gp}(T, \mathbb{C}^\times) := X^*(T) \end{aligned}$$

---

<sup>4</sup>We already assume  $X$  to be of finite type, so coherent condition is equivalent to finitely generated condition.

We get

$$R(T) = \bigoplus_{\rho \in \text{Irr}(T)} \mathbb{Z} = \mathbb{Z}[X^*(T)].$$

The group structure in  $X^*(T)$  is given by tensor product, so the multiplication structure is induced by the group structure in  $X^*(T)$ . Denote

$$\varepsilon_i : T \longrightarrow \mathbb{C}^\times \quad \left( \begin{matrix} t_1 & \cdots & t_i & \cdots & t_n \end{matrix} \right) \longmapsto t_i$$

as a  $\mathbb{Z}$ -basis of  $X^*(T)$ , then

$$R(T) \cong \mathbb{Z}[\varepsilon_1^{\pm 1}, \dots, \varepsilon_n^{\pm 1}]$$

as a  $\mathbb{Z}$ -algebra.

By forgetting  $T$ -actions, we get a morphism of  $\mathbb{Z}$ -algebra

$$R(T) \longrightarrow R(\text{Id}) \quad f(\varepsilon_1, \dots, \varepsilon_n) \longmapsto f(1, \dots, 1).$$

**Example 2.1.5.** After we prove the reduction theorem ???(ref), we can show that

$$R(N) \cong R(\text{Id}) \cong \mathbb{Z} \quad R(B) \cong R(T) \cong \mathbb{Z}[\varepsilon_1^{\pm 1}, \dots, \varepsilon_n^{\pm 1}]$$

**Example 2.1.6.** By [1, Theorem 6.1.4],

$$R(\text{GL}_n) \cong R(T)^W \cong \mathbb{Z}[\varepsilon_1^{\pm 1}, \dots, \varepsilon_n^{\pm 1}]^{S_n}.$$

This can be viewed as a "group" analogue of Chevalley restriction theorem. Notice that we have clear description of finite dimensional irreducible representations of  $\text{GL}_n$ , and the forget map

$$\text{rep}(\text{GL}_n) \longrightarrow \text{rep}(T) \quad \rightsquigarrow \quad R(\text{GL}_n) \longrightarrow R(T)$$

views  $\text{GL}_n$ -representations as special  $W$ -invariant  $T$ -representations.

From these examples we already see the difficulty of computing  $K$ -theories. Therefore, a series of properties of  $K$ -theories are definitely needed for computations. To state these properties, we need to define some tools (or weapons???) in  $K$ -theory.

## 2.2 Basic constructions: pullback, proper pushforward and tensor product

### 2.2.1 non-derived 3-functors in $\mathrm{Coh}^G(X)$

### 2.2.2 3-functors in $K_0^G(X)$ , smooth case

### 2.2.3 3-functors in $K_0^G(X)$ , restriction with supports

## 2.3 Thom isomorphism

## 2.4 Induction

### 2.4.1 Contracted product

**Definition 2.4.1** (Contracted product). *Let  $H \subseteq G$  be a closed algebraic subgroup and  $X$  be an  $H$ -variety. The contracted product of  $G$  and  $X$  over  $H$  is defined as*

$$G \times^H X := (G \times X) / \sim$$

where

$$(gh, x) \sim (g, hx) \quad \text{for any } g \in G, h \in H, x \in X.$$

$G \times^H X$  has a natural variety structure, which is not easy to construct.  $G$  acts on  $G \times^H X$  by multiplying from the left side. We have a  $G$ -equivariant flat morphism

$$G \times^H X \longrightarrow G/H \quad (g, x) \longrightarrow gH$$

which realize  $G \times^H X$  as an  $X$ -bundle over  $G/H$ . In particular, for  $X = \mathrm{pt}$ , we get an isomorphism of  $G$ -varieties

$$G \times^H \mathrm{pt} \xrightarrow{\sim} G/H.$$

## 2.5 Reduction

## 2.6 Equivariant cohomology theory

## Chapter 3

# Cellular fibration theorem

### 3.1 Statement

### 3.2 Application: module structure





## Chapter 4

# Localization theorem

4.1 Euler class

4.2 Statement

4.3 Application: change of basis



## Chapter 5

# Excess intersection formula

### 5.1 Convolution

The construction of the convolution product has a similar flavor with Fourier-Mukai transformation, which is the composition of pullback, tensor product and proper pushforward.

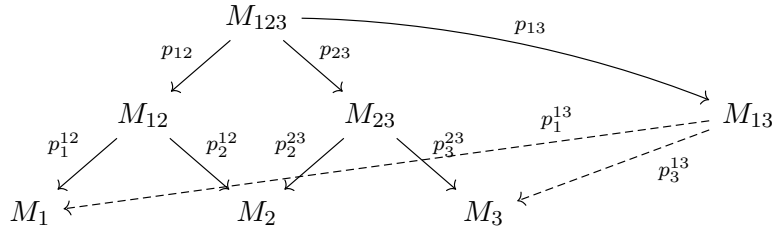
**Definition 5.1.1** (Convolution product). *For the convenience of reading, we divide the whole process into three steps.*

**Step1.** *Setting.*

Let  $M_1, M_2, M_3$  be smooth quasi-projective  $G$ -varieties. For convenience, denote

$$M_{ij} := M_i \times M_j \quad M_{123} = M_1 \times M_2 \times M_3$$

and  $p_i^{jk}, p_i := p_i^{123}, p_{ij} := p_{ij}^{123}$  as projections onto some factors, as follows.



(Check that  $p_i = p_i^{jk} \circ p_{jk}$  for  $1 \leq j < k \leq 3$ ,  $i = j$  or  $i = k$ )

**Step2.** *Convolution product on the level of varieties.*

For closed  $G$ -subvarieties  $Z_{12} \subseteq M_{12}$ ,  $Z_{23} \subseteq M_{23}$ , denote

$$Z_{123} := p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23}) \subseteq M_{123}$$

as the intersection of two preimages. The **convolution product** of  $Z_{12}$  and  $Z_{23}$  is defined as

$$Z_{12} \circ Z_{23} := p_{13}(Z_{123}) \subseteq M_{13}$$

which is a closed  $G$ -subvariety of  $M_{13}$ .

**Step3.** Convolution product on the level of  $K$ -theories.

Denote

$$\pi_{12} := p_{12}|_{Z_{123}} \quad \pi_{23} := p_{23}|_{Z_{123}} \quad \pi_{13} := p_{13}|_{Z_{123}}$$

as corresponding morphisms restricted to  $Z_{123}$ . We assume that  $\pi_{13}$  is proper, so that we can use proper pushforward in  $K$ -theory.

We define the convolution product by

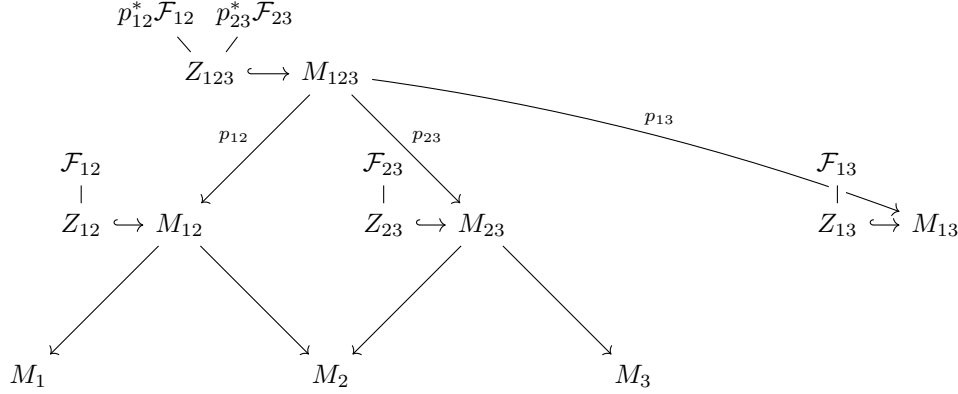
$$* : K_0^G(Z_{12}) \times K_0^G(Z_{23}) \longrightarrow K_0^G(Z_{12} \circ Z_{23}) \quad (\mathcal{F}_{12}, \mathcal{F}_{23}) \longmapsto \mathcal{F}_{12} * \mathcal{F}_{23}$$

$$\mathcal{F}_{12} * \mathcal{F}_{23} = \pi_{13,*}(\pi_{12}^* \mathcal{F}_{12} \otimes \pi_{23}^* \mathcal{F}_{23}) \in K_0^G(Z_{12} \circ Z_{23})$$

*Remark 5.1.2.* Those "Z-varieties" ( $Z_{12}$ ,  $p_{12}^{-1}(Z_{12})$ ,  $Z_{123}$ , etc.) are often singular in practice, so  $\pi_{12}^*$ ,  $\pi_{23}^*$  and  $\otimes$  are defined in the sense of "restriction with supports", under the "M-varieties" which are smooth. The following diagram best illustrate the "actual" definition.

$$\begin{array}{ccccccc} K_0^G(Z_{12}) \times K_0^G(Z_{23}) & \xrightarrow{\pi_{12}^* \times \pi_{23}^*} & K_0^G(p_{12}^{-1}(Z_{12})) \times K_0^G(p_{12}^{-1}(Z_{23})) & \xrightarrow{\otimes} & K_0^G(Z_{123}) & \xrightarrow{\pi_{13,*}} & K_0^G(Z_{12} \circ Z_{23}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ K_0^G(M_{12}) \times K_0^G(M_{23}) & \xrightarrow{p_{12}^* \times p_{23}^*} & K_0^G(M_{123}) \times K_0^G(M_{123}) & \xrightarrow{\otimes} & K_0^G(M_{123}) & \xrightarrow{p_{13,*}} & K_0^G(M_{13}) \end{array}$$

The whole process can be concluded in the following diagram.



## 5.2 Statement

To facilitate the computation of intersection (i.e., tensor product in the construction of convolution product), we state the excess intersection formula.

### **5.3 Application: convolution formula**

#### **5.4 Demazure operator**



## Chapter 6

# Generalization

**6.1** quiver with loops

**6.2**  $G \times \mathbb{C}^\times$ -action





## Chapter 7

# From formula to diagram

7.1 One point quiver

7.2  $A_2$ -quiver

7.3 1-loop quiver



## Chapter 8

# Atiyah-Segal completion theorem



# Bibliography

- [1] Neil Chriss and Victor Ginzburg. *Representation theory and complex geometry*, volume 42. Springer, 1997.