Master thesis

Contents

Warning 0.0.1. *I made some assumptions during the writing. To avoid confusing readers, these assumptions are listed here:*

- *For quivers, all the quivers we considered (except Auslander–Reiten quivers) are connected and finite (Remark 1.2.2). For simplicity, From ??? to ???, all the quivers have no loops or cycles.*

- *For any $\varpi \in \mathbb{W}_{|\mathbf{d}|}$, we always write $\varpi = wu$, where $w \in W_{\mathbf{d}}$ and u is the shortest element in the coset $W_{\mathbf{d}}\varpi$. The flag-type dimension vector $\underline{\mathbf{d}} \in W_{\mathbf{d}} \backslash \mathbb{W}_{|\mathbf{d}|}$ corresponds to u, i.e., $\underline{\mathbf{d}} = W_{\mathbf{d}} u$.*

- *For the diagram, we always read from top to bottom.*

Chapter 1

Variety and stratification

1.1 Initial case: $\mathcal{F}$ and $\mathcal{F} \times \mathcal{F}$

We introduce the complete flag variety to give a bird's eye view on the whole section. Actually, the entire difficulty is bundled in this example.

Fix $n \geqslant 1$, we denote $\mathrm{GL}_n := \mathrm{GL}_n(\mathbb{C})$, B, T, N, W be the standard Borel subgroup, standard torus, unipotent subgroup, Weyl group respectively, i.e.,

$$\mathrm{GL}_n = \begin{pmatrix} * & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & * \end{pmatrix} \quad B = \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix} \quad T = \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \quad N = \begin{pmatrix} 1 & \cdots & * \\ & \ddots & \vdots \\ 0 & & 1 \end{pmatrix}$$

$$W := N_{\mathrm{GL}_n}(T)/T \cong S_n$$

$\mathcal{F}$

Definition 1.1.1 (Flag). *For a finite dimensional $\mathbb{C}$-vector space V, a flag of V is an increasing sequence of subspaces of V:*

$$F : 0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_k = \mathbb{C}^n.$$

F is called a complete flag if $\dim V_i = i$ for all i, otherwise F is called a partial flag.

Definition 1.1.2 (Complete flag variety). *The complete flag variety $\mathcal{F}$ is defined as*

$$\begin{aligned} \mathcal{F} &= \mathrm{GL}_n / B \\ &\cong \{\textit{complete flags of } \mathbb{C}^n\} \\ &= \{0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n = \mathbb{C}^n \mid \dim V_i = i\} \\ &\cong \{\textit{Borel subgroups of } \mathrm{GL}_n\} \\ &= \left\{gBg^{-1} \;\middle|\; g \in \mathrm{GL}_n\right\} \end{aligned}$$

Remark 1.1.3.

1. $\mathcal{F}$ is a smooth projective variety of dimension $\frac{n(n+1)}{2}$, which can be seen from the embedding
$$\mathcal{F} \hookrightarrow \mathrm{Gr}(1,n) \times \cdots \times \mathrm{Gr}(n-1,n)$$

2. We implicitly give the base point of $\mathcal{F}$, which is not considered as the data of $\mathcal{F}$. Fix a standard basis of $\mathbb{C}^n$ by $\{v_1, \dots, v_n\}$, we define the standard flag
$$F_{\mathrm{Id}} : 0 \subseteq \langle v_1 \rangle \subseteq \langle v_1, v_2 \rangle \subseteq \cdots \subseteq \langle v_1, \dots, v_n \rangle = \mathbb{C}^n.$$

3. We have the natural GL_n-action on $\mathcal{F}$, which is considered as the data of $\mathcal{F}$.

   For $g \in \mathrm{GL}_n$, we define the flag attached to g:
$$F_g \triangleq gF_{\mathrm{Id}} : 0 \subseteq \langle gv_1 \rangle \subseteq \langle gv_1, gv_2 \rangle \subseteq \cdots \subseteq \langle gv_1, \dots, gv_n \rangle = \mathbb{C}^n.$$

   Especially, for $w \in W = N_{\mathrm{GL}_n}(T)/T \cong S_n$, the flag attached to w
$$\begin{aligned} F_w : 0 &\subseteq \langle \tilde{w}v_1 \rangle \subseteq \langle \tilde{w}v_1, \tilde{w}v_2 \rangle \subseteq \cdots \subseteq \langle \tilde{w}v_1, \dots, \tilde{w}v_n \rangle = \mathbb{C}^n \\ 0 &\subseteq \left\langle v_{w(1)} \right\rangle \subseteq \left\langle v_{w(1)}, v_{w(2)} \right\rangle \subseteq \cdots \subseteq \left\langle v_{w(1)}, \dots, v_{w(n)} \right\rangle = \mathbb{C}^n \end{aligned}$$
   does not depend on the choice of the lift $\tilde{w} \in N_{\mathrm{GL}_n}(T)$ of w.

   Readers can verify that $\{F_w | w \in W\}$ are all T-fixed points of $\mathcal{F}$, while $\left\{wBw^{-1} \middle| w \in W\right\}$ are all Borel subgroups of G containing the standard torus T.

4. For $n = 2$, $\mathcal{F} \cong \mathbb{P}^1$. We encourage readers to use $\mathbb{P}^1$ as a toy example for the whole theory.

interpretation	GL_n / B	flags	Borel subgroups
base point	Id	F_{Id}	B
GL_n-action	left multiplication	$\{V_i\} \mapsto \{gV_i\}$	conjugation
general point	g	F_g	gBg^{-1}

$\mathcal{F}$ is a well-studied variety, and has many combinatorical properties. For example, from the well-known Bruhat decomposition, [1]
$$\mathrm{GL}_n \cong \bigsqcup_{w \in W} BwB$$

We get a stratification of $\mathcal{F}$ by B-orbits:
$$\mathcal{F} = \mathrm{GL}_n / B \cong \bigsqcup_{w \in W} BwB/B$$

The B-orbit BwB/B is called the **Schubert cell**, denoted by $\mathcal{V}_w$. Since
$$\mathcal{V}_w = BwB/B \cong B/\left(B \cap wBw^{-1}\right) \cong \mathbb{A}^{l(w)},$$
the Schubert cell is an affine space of dimension $l(w)$.

$H^i(\mathcal{F};\mathbb{C})$	0	2	4	6	8	10	12
1	1						
2	1	1					
3	1	2	2	1			
4	1	3	5	6	5	3	1
5	1	4	9	15	20	22	20

G	Orbit	G-fixed points
GL_n	$\mathcal{F} \cong \mathrm{GL}_n /B$	$\varnothing$
B	$\mathcal{V}_w \cong B/(B \cap wBw^{-1})$	$\{F_{\mathrm{Id}}\}$
T	$-$	$\{F_w \mid w \in W\}$

As a result, we know a lot of information of $\mathcal{F}$:
$\overline{\mathcal{V}}_w \subseteq \mathcal{F}$ is called the **Schubert variety**. It is well-known that

$$\overline{\mathcal{V}}_w = \bigsqcup_{w' \leq w} \mathcal{V}_w$$

as a set. Especially, for any $s \in W$ with $l(s) = 1$, denote $P_s = B \sqcup BsB$,

$$\overline{\mathcal{V}}_s = \mathcal{V}_{\mathrm{Id}} \sqcup \mathcal{V}_s = B/B \sqcup BsB/B = P_s/B \cong \mathbb{P}^1.$$

For other Schubert variety, the structures are quite dedicate and far away from the scope of this master thesis. For example, most Schubert variety are not smooth.

$\mathcal{F} \times \mathcal{F}$

As a more complicated geometrical object, $\mathcal{F}\times\mathcal{F}$ works as the base space for the Steinberg variety, which turns out to be the central focus in the thesis. $\mathcal{F}\times\mathcal{F}$ has naturally a diagonal GL_n-action:

$$\mathrm{GL}_n \times \mathcal{F} \times \mathcal{F} \longrightarrow \mathcal{F} \times \mathcal{F} \qquad (g, F_1, F_2) \longmapsto (gF_1, gF_2).$$

Under this action, $\mathcal{F} \times \mathcal{F}$ has a stratification consisting of GL_n-orbits, indexed by the Weyl group:

$$\mathrm{GL}_n \backslash(\mathcal{F} \times \mathcal{F}) \cong \mathrm{GL}_n \backslash(\mathrm{GL}_n /B \times \mathrm{GL}_n /B) \cong B \setminus \mathrm{GL}_n /B \cong W \qquad \text{as sets.}$$

Denote $\boldsymbol{\mathcal{V}}_{w'} := \mathrm{GL}_n \cdot(F_{\mathrm{Id}}, F_{w'})$, then $\mathcal{F}\times\mathcal{F} = \sqcup_{w'}\boldsymbol{\mathcal{V}}_{w'}$. Moreover, by the orbit-stabilizer theorem, we get

$$\boldsymbol{\mathcal{V}}_{w'} \cong \mathrm{GL}_n / \big(B \cap w'B(w')^{-1}\big)$$

Different from $\mathcal{F}$, the GL_n-action on $\mathcal{F} \times \mathcal{F}$ is not transitive. To facilitate the stratification of $\mathcal{F} \times \mathcal{F}$, we introduce the twisted $\mathrm{GL}_n \times \mathrm{GL}_n$-action:

$$\mathrm{GL}_n \times \mathrm{GL}_n \times \mathcal{F} \times \mathcal{F} \longrightarrow \mathcal{F} \times \mathcal{F} \qquad (g_1, g_2, F_g, F_{g'}) \longmapsto (F_{g_1 g}, F_{g_1(gg_2g^{-1})g'}).$$

[1]For the most time the formula does not depend on the lift of w, so we abuse the notation of $w \in N_{\mathrm{GL}_n}(T)/T$ and $\tilde{w} \in N_{\mathrm{GL}_n}(T)$.

If we write $\underline{F}_{g,g'} := (F_g, F_{gg'}) \in \mathcal{F} \times \mathcal{F}$, then

$$(g_1, g_2, \underline{F}_{g,g'}) = \underline{F}_{g_1 g, g_2 g'}.$$

This $\mathrm{GL}_n \times \mathrm{GL}_n$-action is now transitive, and decompose $\mathcal{F} \times \mathcal{F}$ as disjoint union of finite many $B \times B$-orbits, which are compatible with G-orbits:

$$\begin{aligned}
&\boldsymbol{\mathcal{V}}_{w,w'} := (B \times B) \cdot \underline{F}_{w,w'} \subseteq \mathcal{F} \times \mathcal{F} \\
&\mathcal{F} \times \mathcal{F} = \bigsqcup_{w,w' \in W} \boldsymbol{\mathcal{V}}_{w,w'} \qquad \boldsymbol{\mathcal{V}}_{w'} = \bigsqcup_{w \in W} \boldsymbol{\mathcal{V}}_{w,w'} \\
&\boldsymbol{\mathcal{V}}_{w,w'} \cong B/(B \cap wBw^{-1}) \times B/(B \cap w'Bw'^{-1}) \cong \mathbb{A}^{l(w)+l(w')}
\end{aligned}$$

We conclude the information of orbits and fixed points of $\mathcal{F} \times \mathcal{F}$ in Table 1.1:

G	Orbit	G-fixed points
$\mathrm{GL}_n \times \mathrm{GL}_n$	$\mathcal{F} \times \mathcal{F}$	$\varnothing$
GL_n	$\boldsymbol{\mathcal{V}}_{w'}$	$\varnothing$
$B \times B$	$\boldsymbol{\mathcal{V}}_{w,w'}$	$\{F_{\mathrm{Id,Id}}\}$
T	$-$	$\{\underline{F}_{w,w'} \mid w, w' \in W\}$

Table 1.1: Orbit and fixed points of $\mathcal{F} \times \mathcal{F}$

Like $\mathcal{F}$, we also study the closure of $\boldsymbol{\mathcal{V}}_{w'}$ and $\boldsymbol{\mathcal{V}}_{w,w'}$ in special case. It can be shown that

$$\overline{\boldsymbol{\mathcal{V}}}_{w'} = \bigsqcup_{x' \leq w'} \boldsymbol{\mathcal{V}}_{x'} \qquad \overline{\boldsymbol{\mathcal{V}}}_{w,w'} = \bigsqcup_{x \leq w, x' \leq w'} \boldsymbol{\mathcal{V}}_{x,x'}$$

as a set. Especially, for any $s \in W$ with $l(s) = 1$,

$$\begin{aligned}
\overline{\boldsymbol{\mathcal{V}}}_s = \boldsymbol{\mathcal{V}}_{\mathrm{Id}} \sqcup \boldsymbol{\mathcal{V}}_s &\cong \mathrm{GL}_n / B \sqcup \mathrm{GL}_n / (B \cap sBs^{-1}) \\
&\cong \mathrm{GL}_n \times^B (B/B) \sqcup GL_n \times^B (B \cap sBs^{-1}) \\
&\cong \mathrm{GL}_n \times^B (B/B) \sqcup GL_n \times^B (BsB/B) \\
&\cong \mathrm{GL}_n \times^B (P_s/B)
\end{aligned}$$

is an $\mathcal{F}$-bundle over $\mathbb{P}^1$.[2] Also,

$$\begin{aligned}
\overline{\boldsymbol{\mathcal{V}}}_{\mathrm{Id},s} = \boldsymbol{\mathcal{V}}_{\mathrm{Id,Id}} \sqcup \boldsymbol{\mathcal{V}}_{\mathrm{Id},s} &\cong \big(B/B \times B/B\big) \sqcup \big(B/B \times BsB/B\big) \\
&\cong P_s/B \cong \mathbb{P}^1
\end{aligned}$$

Other closure can be highly singular.

Example 1.1.4. *In the table, $n = 3$, $t = (12)$, $s = (23)$. In this case, $\mathcal{F} \times \mathcal{F}$ has* 6 GL_3*-orbits, and each* GL_3*-orbits decompose as* 6 $B \times B$*-orbits, with dimensions equal to* $l(w) + l(w')$.

Now wee understand a lot about $\mathcal{F}$ and $\mathcal{F} \times \mathcal{F}$, and the whole process of analysis(investigations?) will be applied repeatedly in subsection ???.

[2] ??? need to explain $\times^B$

dim $\mathbb{B}_{\|d\|}\cdot F_{\varpi'}$ $\mathbb{B}_{\|d\|}\times\mathbb{B}_{\|d\|}\cdot(F_{\varpi},F_{\varpi'})$ $\varpi\varpi'$ $\mathbb{B}_{\|d\|}\cdot F_{\varpi}$ ϖ	0 ϑ_{Id}	1 ϑ_t	1 ϑ_s	2 ϑ_{ts}	2 ϑ_{st}	3 ϑ_{sts}	$pr_1^{-1}(\vartheta_{ts})$
0 ϑ_{Id}	0 $\vartheta_{Id,Id}$	1 $\vartheta_{Id,t}$	1 $\vartheta_{Id,s}$	2 $\vartheta_{Id,ts}$	2 $\vartheta_{Id,st}$	3 $\vartheta_{Id,sts}$	ϑ_s
1 ϑ_t	2 $\vartheta_{t,t}$	1 $\vartheta_{t,Id}$	3 $\vartheta_{t,ts}$	2 $\vartheta_{t,s}$	4 $\vartheta_{t,sts}$	3 $\vartheta_{t,st}$	
1 ϑ_s	2 $\vartheta_{s,s}$	3 $\vartheta_{s,st}$	1 $\vartheta_{s,Id}$	4 $\vartheta_{s,sts}$	2 $\vartheta_{s,t}$	3 $\vartheta_{s,ts}$	
2 ϑ_{ts}	4 $\vartheta_{ts,st}$	3 $\vartheta_{ts,s}$	5 $\vartheta_{ts,sts}$	2 $\vartheta_{ts,Id}$	4 $\vartheta_{ts,ts}$	3 $\vartheta_{ts,t}$	
2 ϑ_{st}	4 $\vartheta_{st,ts}$	5 $\vartheta_{st,sts}$	3 $\vartheta_{st,t}$	4 $\vartheta_{st,st}$	2 $\vartheta_{st,Id}$	3 $\vartheta_{st,s}$	
3 ϑ_{sts}	6 $\vartheta_{sts,sts}$	5 $\vartheta_{sts,ts}$	5 $\vartheta_{sts,st}$	4 $\vartheta_{sts,t}$	4 $\vartheta_{sts,s}$	3 $\vartheta_{sts,Id}$	

Figure 1.1: stratifications of $\mathcal{F}\times\mathcal{F}$

1.2 Quiver

To introduce more complicated spaces and discuss their stratifications, we fix notations related to quiver and algebraic group in the following subsections.

Roughly speaking, a quiver is a directed multigraph permitting loops.

Definition 1.2.1 (Quiver)**.** *???*

Remark 1.2.2. In the first part of our master thesis, all the quivers are supposed to be connected and finite (i.e., Q_0, Q_1 are finite sets). We will only encounter disconnected and infinite quiver as Auslander–Reiten quiver later on.

Example 1.2.3. *The following graphs are quivers.*

???

The reader can easily writen down the quadruple of these quivers. ???

For convenience, we mainly use simpler quivers as examples:

???

From those quivers we are able to construct relatively complicated algebraic and geometrical objects.

Definition 1.2.4 (Quiver representation)**.** *Fix a quiver Q. A representation of Q consists of the following data:*

- *A finite dimensional $\mathbb{C}$-vector space V_i for each vertex $i \in Q_0$;*

- *A $\mathbb{C}$-linear map $V_a : V_{s(a)} \longrightarrow V_{t(a)}$ for each arrow $a \in Q_1$.*

Example 1.2.5. *A representation of 1-loop quiver $L(1)$ is a 2-tuple*

$$(V,\ \alpha : V \longrightarrow V)$$

which is equivalent to a (finite dimensional) $\mathbb{C}[t]$-module.

Remark 1.2.6. The equivalence appeared in the example can actually be generalized to arbitrary quivers. For a quiver Q, we can define the path algebra $\mathbb{C}Q$, and view any Q-representation as $\mathbb{C}Q$-module, and vice versa.

For many constructions, we only care about the data of vector space.

Definition 1.2.7 (Q-vector space)**.** *Fix a quiver Q, a Q-vector space is a finite dimensional $\mathbb{C}$-vector space with the direct sum decomposition*

$$V = \bigoplus_{i \in Q_0} V_i.$$

The dimension vector of a Q-vector space is defined as

$$\underline{\mathbf{dim}}\, V = (\dim_{\mathbb{C}} V_i)_{i \in Q_0} \subseteq \prod_{i \in Q_0} \mathbb{Z}.$$

On the country, given $\mathbf{d} \in \prod_{i \in Q_0} \mathbb{N}_{\geqslant 0}$, we can construct a canonical Q-vector space of dimension vector $\mathbf{d}$, as follows:

$$V = \bigoplus_{i \in Q_0} V_i \qquad \text{with } V_i = \mathbb{C}^{\mathbf{d}_i}.$$

Definition 1.2.8 (Space of representations with given dimension vector)**.** *For any quiver Q, dimension vector $\mathbf{d}$, fix the canonical Q-vector space $V = \bigoplus_{i \in Q_0} V_i$, the space of representations with dimension vector $\mathbf{d}$ is defined as*

$$\begin{aligned} \mathrm{Rep}_{\mathbf{d}}(Q) &= \left\{ (V_i, V_a : V_{s(a)} \longrightarrow V_{t(a)}) \text{ as a representation of } Q \right\} \\ &= \bigoplus_{a \in Q_1} \mathrm{Hom}\left(\mathbb{C}^{\mathbf{d}_{s(a)}}, \mathbb{C}^{\mathbf{d}_{t(a)}} \right) \end{aligned}$$

Since we encode the information of vector space in $\mathbf{d}$, $\mathrm{Rep}_{\mathbf{d}}(Q)$ only records the information of linear maps.

1.3 Symmetric group calculus

As a reminder, we recall some basic diagrams referring to the elements in S_n, and do some calculations by these diagrams. We will also relate cosets with flag-type dimension vectors.

Fix a quiver Q and dimension vector $\mathbf{d}$. Later (Definition 1.4.1, 1.4.2) we will define

$$\mathbb{W}_{|\mathbf{d}|} = S_{|\mathbf{d}|} \qquad W_{\mathbf{d}} = \prod_{i \in Q_0} S_{\mathbf{d}_i} \leqslant \mathbb{W}_{|\mathbf{d}|}$$

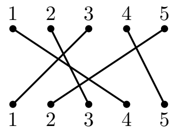
For simplicity, we take $Q_0 = \{1, \ldots, k\}$, then $W_{\mathbf{d}} = S_{\mathbf{d}_1} \times \cdots \times S_{\mathbf{d}_k}$ embed in $S_{|\mathbf{d}|}$ in the most natural way.

Remark 1.3.1. We have different ways to express $\varpi \in \mathbb{W}_{|\mathbf{d}|} = S_{|\mathbf{d}|}$. For example, take $|\mathbf{d}| = 5$, $\varpi \in S_5$ by

$$\varpi(1) = 4, \quad \varpi(2) = 3, \quad \varpi(3) = 1, \quad \varpi(4) = 5, \quad \varpi(5) = 2,$$

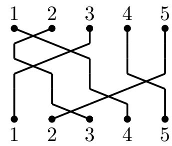
then

$$\varpi = (14523) = \begin{pmatrix} 1\,2\,3\,4\,5 \\ 4\,3\,1\,5\,2 \end{pmatrix} =$$



$$= \begin{bmatrix} & & 1 & & \\ & & & & 1 \\ & 1 & & & \\ 1 & & & & \\ & & & 1 & \end{bmatrix}$$

$$= (23)(34)(45)(12)(23)(12) =$$



Even though all expressions give us the same amount of information, the diagram presents them more vividly. For example, each intersection of strands corresponds to a simple reflection, so we read from the diagram that $l(\varpi) = 6$. Readers can also check that

$$\begin{aligned} &l(\varpi s_1) = 5, \quad l(\varpi s_2) = 5, \quad l(\varpi s_3) = 7, \quad l(\varpi s_4) = 5, \\ &l(s_1 \varpi) = 7, \quad l(s_2 \varpi) = 5, \quad l(s_3 \varpi) = 5, \quad l(s_4 \varpi) = 7, \end{aligned}$$

where $s_i := (i, i+1) \in S_5$ are simple reflections.

Definition 1.3.2 (Special elements in the Weyl group)**.** *For $i \in \{1, \ldots, |\mathbf{d}| - 1\}$, the simple reflection is defined as*

$$s_i := (i, i+1) \in S_{|\mathbf{d}|}.$$

We denote

$$\begin{aligned} \Pi &= \left\{ s_i \in S_{|\mathbf{d}|} \;\middle|\; i \in \{1, \ldots, |\mathbf{d}| - 1\} \right\} \\ \Pi_{\mathbf{d}} &= \left\{ s_i \in S_{\mathbf{d}_1} \times \cdots \times S_{\mathbf{d}_k} \;\middle|\; i \in \{1, \ldots, |\mathbf{d}| - 1\} \right\} \\ &= \left\{ s_1, \ldots, s_{|\mathbf{d}|-1} \right\} \smallsetminus \left\{ s_{\mathbf{d}_1}, s_{\mathbf{d}_1 + \mathbf{d}_2}, \ldots, s_{\mathbf{d}_1 + \cdots + \mathbf{d}_{k-1}} \right\} \end{aligned}$$

to be the set of simple reflections in $\mathbb{W}_{|\mathbf{d}|}$ and $W_{\mathbf{d}}$, respectively.

We also denote $\varpi_{\max} \in \mathbb{W}_{|\mathbf{d}|}$, $w_{\max} \in W_{\mathbf{d}}$ to be the longest elements in $\mathbb{W}_{|\mathbf{d}|}$, $W_{\mathbf{d}}$, respectively.

We discuss about right cosets $W_{\mathbf{d}} \backslash \mathbb{W}_{|\mathbf{d}|}$ and minimal length coset representatives now.

Multiplying on left by $w \in W_{\mathbf{d}}$ is equivalent to plugging in a diagram representing $w \in W_{\mathbf{d}}$ underneath the original diagram. Therefore, we connect some bottom points by lines, indicating that switching them will cause no trouble. Furthermore, we color different parts to make the following fact more explicitly.

Fact 1.3.3. *Every element $\varpi_{\max} \in W_{\mathbf{d}} \backslash \mathbb{W}_{|\mathbf{d}|}$ corresponds to a partition on set $\{1, \dots, |\mathbf{d}|\}$ (of a given number partition $\mathbf{d}$), which corresponds to a flag-type dimension vector $\underline{\mathbf{d}}$.*

Example 1.3.4. *???*

This coset corresponds to the partition $\{1, 2, 3, 4, 5\} = \{2, 3, 5\} \sqcup \{1, 4\}$.

???

It is easy to see from the diagram that in every coset, there exists an unique element $u \in \mathbb{W}_{|\mathbf{d}|}$ of minimal length. We collect these minimal length coset representatives as a set, and denote it by $\mathrm{Min}(\mathbb{W}_{|\mathbf{d}|}, \mathrm{W}_{\mathbf{d}})$.[3]

Proposition 1.3.5. *For any $\varpi \in \mathbb{W}_{|\mathbf{d}|}$, exists unique $w \in W_{\mathbf{d}}$, $u \in \mathrm{Min}(\mathbb{W}_{|\mathbf{d}|}, \mathrm{W}_{\mathbf{d}})$ such that $\varpi = wu$.*

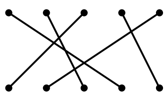
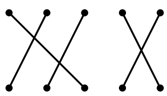
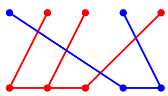
Exercise 1.3.6. *For $u \in \mathrm{Min}(\mathbb{W}_{|\mathbf{d}|}, \mathrm{W}_{\mathbf{d}})$, $s_i \in \Pi$, show that*

$$u s_i u^{-1} \in W_{\mathbf{d}} \quad \Longrightarrow \quad u s_i u^{-1} = s_{u(i)} \in \Pi_{\mathbf{d}}.$$

We finish this subsection with figures and examples.

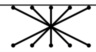
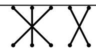
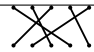
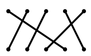
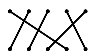
$$\begin{tikzcd}
 & & & \mathrm{Min}(\mathbb{W}_{|\mathbf{d}|}, \mathrm{W}_{\mathbf{d}}) \arrow[dl, hook'] \arrow[d, "\cong"] & & & u \arrow[d, mapsto] \\
0 \arrow[r] & W_{\mathbf{d}} \arrow[r] & \mathbb{W}_{|\mathbf{d}|} \arrow[r] & W_{\mathbf{d}} \backslash \mathbb{W}_{|\mathbf{d}|} \arrow[r] & 0 & \varpi = wu \arrow[r, mapsto] & \underline{\mathbf{d}}
\end{tikzcd}$$

Example 1.3.7. *In this table, $|\mathbf{d}| = 5$, $\mathbf{d} = (3, 2)$, typical elements would be*

$\varpi =$  $w =$  $u =$ 

Example 1.3.8. *In this table, $|\mathbf{d}| = 3$, $\mathbf{d} = (1, 2)$, $s = (12)$, $t = (23)$. The columns "order of basis" and Borelsubgroups have not been introduced yet, and they are here for the future usage.*

[3]In some references $\mathrm{Min}(\mathbb{W}_{|\mathbf{d}|}, \mathrm{W}_{\mathbf{d}})$ is also denoted by $\mathrm{Schuffle}_{\mathbf{d}}$, since those elements can be thought as ways off riffle shuffling several words together.

set	element	special element	others
$\mathbb{W}_{\lvert\mathbf{d}\rvert} = S_5$	ϖ, x	$\varpi_{\max} =$ 	$\Pi = \{s_1, s_2, s_3, s_4\}$
$W_{\mathbf{d}} = S_3 \times S_2$	w	$w_{\max} =$ 	$\Pi_d = \{s_1, s_2, \quad s_4\}$
$W_{\mathbf{d}} \setminus \mathbb{W}_{\lvert\mathbf{d}\rvert} \cong (S_3 \times S_2) \setminus S_5$	$\varpi, \underline{\mathbf{d}}$		$\mathrm{Comp}_{\mathbf{d}}$
$\mathrm{Min}(\mathbb{W}_{\lvert\mathbf{d}\rvert}, \mathrm{W}_{\mathbf{d}}) = \Big\{$  $, \dots \Big\}$	u		$\mathrm{Schuffle}_{\mathbf{d}}$

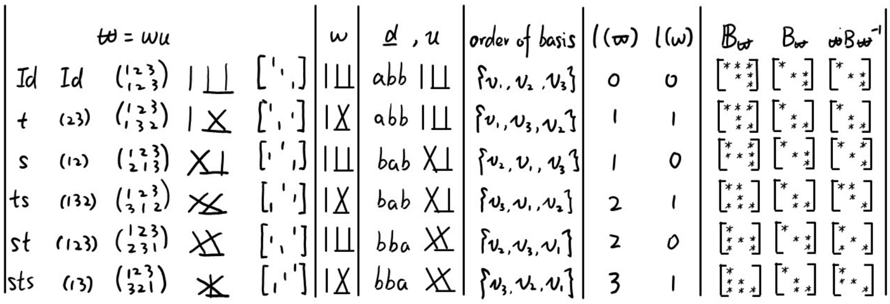


Figure 1.2: desired picture

1.4 Algebraic group and Lie algebra

In this subsection we fix notations of algebraic group and Lie algebras. Later, the algebraic group will act on varieties, and some Lie algebra will serve as tangent spaces.

We fix a quiver Q, a dimension vector $\mathbf{d}$ and a $\mathbb{C}$-vector space with quiver partition

$$V = \bigoplus_{i \in Q_0} V_i \qquad \text{with } V_i = \mathbb{C}^{\mathbf{d}_i}.$$

Definition 1.4.1 (absolute algebraic groups)**.** *We set*

$$\mathbb{G}_{\lvert\mathbf{d}\rvert} := \mathrm{GL}(V) = \mathrm{GL}_{\lvert\mathbf{d}\rvert}(\mathbb{C}),$$

and $\mathbb{B}_{\lvert\mathbf{d}\rvert}$, $\mathbb{T}_{\lvert\mathbf{d}\rvert}$, $\mathbb{N}_{\lvert\mathbf{d}\rvert}$ *are corresponding standard Borel, torus and unipotent subgroups.*

The Weyl group is

$$\mathbb{W}_{\lvert\mathbf{d}\rvert} := N_{\mathbb{G}_{\lvert\mathbf{d}\rvert}}(\mathbb{T}_{\lvert\mathbf{d}\rvert})/\mathbb{T}_{\lvert\mathbf{d}\rvert} \cong S_{\lvert\mathbf{d}\rvert}.$$

For $\varpi \in \mathbb{W}_{\lvert\mathbf{d}\rvert}$, *we define*[4]

$$\mathbb{B}_{\varpi} := \varpi \mathbb{B}_{\lvert\mathbf{d}\rvert} \varpi^{-1}.$$

We will view $\mathbb{B}_{\varpi}$ *as the stabilizer of the flag* F_{ϖ} *with* $\mathbb{G}_{\lvert\mathbf{d}\rvert}$*-action.*

We also have a series of algebraic groups compatible with the quiver partition of V, and they're more common in this thesis.

[4]As usual, we abuse the notation of ϖ and its lift.

Definition 1.4.2 (relative algebraic groups)**.** *We set*

$$G_{\mathbf{d}} := \bigoplus_{i \in Q_0} \mathrm{GL}(V_i) = \mathrm{GL}_{\mathbf{d}_i}(\mathbb{C}) \subseteq \mathbb{G}_{|\mathbf{d}|},$$

and $B_{\mathbf{d}}$, $T_{\mathbf{d}}$, $N_{\mathbf{d}}$ are corresponding standard Borel, torus and unipotent subgroups.

The Weyl group is

$$W_{\mathbf{d}} := N_{G_{\mathbf{d}}}(T_{\mathbf{d}})/T_{\mathbf{d}} \cong \prod_{i \in Q_0} S_{\mathbf{d}_i}.$$

For $\varpi = wu \in W_{\mathbf{d}}$, we define

$$B_{\varpi} := w B_{\mathbf{d}} w^{-1}.$$

We will view B_ϖ as the stabilizer of the flag F_ϖ with $G_{\mathbf{d}}$-action.

We also have a series of algebraic groups with subscription as elements in the Weyl group:

Definition 1.4.3 (more algebraic groups)**.** *For $\varpi, \varpi'' \in \mathbb{W}_{|\mathbf{d}|}$, define*

$$\begin{aligned} N_\varpi &:= R_u(B_\varpi), \\ N_{\varpi,\varpi''} &:= N_\varpi \cap N_{\varpi''}, \\ M_{\varpi,\varpi''} &:= N_\varpi / N_{\varpi,\varpi''}, \end{aligned}$$

where R_u denotes for the unipotent radical.

For $s \in \Pi$ such that $\varpi s \varpi^{-1} \in W_d$ (i.e., $W_{\mathbf{d}}\varpi = W_{\mathbf{d}}\varpi s$), define

$$\begin{aligned} P_{\varpi,\varpi s} &:\overset{\varpi = wu}{=\!=\!=} w\left(B_{\mathbf{d}} u s u^{-1} B_{\mathbf{d}} \cup B_{\mathbf{d}}\right) w^{-1} \\ &=\!=\!= B_\varpi \varpi s \varpi^{-1} B_\varpi \cup B_\varpi \end{aligned}$$

Remark 1.4.4. One can easily show that $N_{\varpi,\varpi s} = R_u(P_{\varpi,\varpi s})$.

Example 1.4.5. *For $|\mathbf{d}| = 5$, $\mathbf{d} = (3,2)$, ???,*

$$\mathbb{G}_{|\mathbf{d}|} = \begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{pmatrix} \quad \mathbb{B}_{|\mathbf{d}|} = \begin{pmatrix} * & * & * & * & * \\ & * & * & * & * \\ & & * & * & * \\ & & & * & * \\ & & & & * \end{pmatrix} \quad \mathbb{T}_{|\mathbf{d}|} = \begin{pmatrix} * & & & & \\ & * & & & \\ & & * & & \\ & & & * & \\ & & & & * \end{pmatrix} \quad \mathbb{N}_{|\mathbf{d}|} = \begin{pmatrix} 1 & * & * & * & * \\ & 1 & * & * & * \\ & & 1 & * & * \\ & & & 1 & * \\ & & & & 1 \end{pmatrix}$$

$$\mathbb{W}_{|\mathbf{d}|} \cong S_5 \quad \mathbb{B}_{\varpi} = \begin{pmatrix} * & * & & & * \\ & * & & & \\ * & * & * & & * \\ * & * & * & * & * \\ & * & & & * \end{pmatrix} \quad \mathbb{B}_{\varpi s} = \begin{pmatrix} * & * & * & & * \\ & * & & & \\ & * & * & & * \\ * & * & * & * & * \\ & * & & & * \end{pmatrix}$$

$$G_{\mathbf{d}} = \left(\begin{array}{ccc|cc} * & * & * & & \\ * & * & * & & \\ * & * & * & & \\ \hline & & & * & * \\ & & & * & * \end{array}\right) \quad B_{\mathbf{d}} = \left(\begin{array}{ccc|cc} * & * & * & & \\ & * & * & & \\ & & * & & \\ \hline & & & * & * \\ & & & & * \end{array}\right) \quad T_{\mathbf{d}} = \left(\begin{array}{ccc|cc} * & & & & \\ & * & & & \\ & & * & & \\ \hline & & & * & \\ & & & & * \end{array}\right) \quad N_{\mathbf{d}} = \left(\begin{array}{ccc|cc} 1 & * & * & & \\ & 1 & * & & \\ & & 1 & & \\ \hline & & & 1 & * \\ & & & & 1 \end{array}\right)$$

$$W_{\mathbf{d}} \cong S_3 \times S_2 \quad B_{\varpi} = \left(\begin{array}{ccc|cc} * & * & & & \\ & * & & & \\ * & * & * & & \\ \hline & & & * & \\ & & & * & * \end{array}\right) \quad B_{\varpi s} = \left(\begin{array}{ccc|cc} * & & & & \\ * & * & & & \\ * & * & * & & \\ \hline & & & * & \\ & & & * & * \end{array}\right)$$

$$N_{\varpi} = \left(\begin{array}{ccc|cc} 1 & * & & & \\ & 1 & & & \\ * & * & 1 & & \\ \hline & & & 1 & \\ & & & * & 1 \end{array}\right) \quad N_{\varpi,\varpi s} = \left(\begin{array}{ccc|cc} 1 & & & & \\ & 1 & & & \\ * & * & 1 & & \\ \hline & & & 1 & \\ & & & * & 1 \end{array}\right) \quad M_{\varpi,\varpi s} = \left(\begin{array}{ccc|cc} 1 & * & & & \\ & 1 & & & \\ - & - & 1 & & \\ \hline & & & 1 & \\ & & & - & 1 \end{array}\right) \quad P_{\varpi,\varpi s} = \left(\begin{array}{ccc|cc} * & * & & & \\ * & * & & & \\ * & * & * & & \\ \hline & & & * & \\ & & & * & * \end{array}\right)$$

For the Lie algebra, we use the corresponding Fraktur-font symbols:

$$\begin{array}{lllll} \mathfrak{g}_{|\mathbf{d}|}, & \mathfrak{b}_{|\mathbf{d}|}, & \mathfrak{t}_{|\mathbf{d}|}, & \mathfrak{n}_{|\mathbf{d}|}, & \mathfrak{b}_{\varpi} \\ \mathfrak{g}_{\mathbf{d}}, & \mathfrak{b}_{\mathbf{d}}, & \mathfrak{t}_{\mathbf{d}}, & \mathfrak{n}_{\mathbf{d}}, & \mathfrak{b}_{\varpi}, \\ \mathfrak{n}_{\varpi}, & \mathfrak{n}_{\varpi,\varpi''}, & \mathfrak{m}_{\varpi,\varpi''}, & \mathfrak{p}_{\varpi,\varpi s}, & \end{array}$$

We also have to encode the information of representations as Lie algebra. Notice that

$$\mathrm{Hom}(V_{s(a)}, V_{t(a)}) \hookrightarrow \mathrm{Hom}(V, V) \cong \mathfrak{g}_{|\mathbf{d}|} \qquad f \longmapsto \iota_{t(a)} \circ f \circ \pi_{s(a)}$$

realizes $\mathrm{Hom}(V_{s(a)}, V_{t(a)})$ as a Lie subalgebra of $\mathfrak{g}_{|\mathbf{d}|}$, so

$$\mathrm{Rep}_{\mathbf{d}}(Q) = \bigoplus_{a \in Q_1} \mathrm{Hom}\left(\mathbb{C}^{\mathbf{d}_{s(a)}}, \mathbb{C}^{\mathbf{d}_{t(a)}}\right) \subseteq \bigoplus_{a \in Q_1} \mathfrak{g}_{|\mathbf{d}|}.$$

Definition 1.4.6 (Lie algebras connected with representations)**.** *For $\varpi \in \mathbb{W}_{|\mathbf{d}|}$, denote temperately*

$$V_{\varpi,i} := \left\langle e_{\varpi(1)}, \dots e_{\varpi(i)} \right\rangle \subseteq V.$$

We define Lie subalgebras of $\mathrm{Rep}_{\mathbf{d}}(Q)$ *as follows.*

$$\mathfrak{r}_{\varpi} := \left\{ (f_a)_{a \in Q_1} \in \mathrm{Rep}_{\mathbf{d}}(Q) \;\middle|\; f_a(V_{\varpi,i} \cap V_{s(a)}) \subseteq V_{\varpi,i} \right\},$$

$$\mathfrak{r}_{\varpi,\varpi''} := \mathfrak{r}_{\varpi} \cap \mathfrak{r}_{\varpi''},$$

$$\mathfrak{d}_{\varpi,\varpi''} := \mathfrak{r}_{\varpi} / \mathfrak{r}_{\varpi,\varpi''},$$

1.5 Typical variety

1.6 Stratification and T-fixed points

Chapter 2

K-theory and cohomology theory

Chapter 3

Localization theorem

Chapter 4

Excess intersection formula

Chapter 5

From formula to diagram

Chapter 6

Generalization

Chapter 7

Atiyah-Segal completion theorem