Chapter 5

Excess intersection formula

Finally, we are able to compute the convolution structure of the Steinberg variety in this Chapter. We first introduce the convolution product, then give an explicit intersection formula, and finally apply theorems to our settings.

5.1 Convolution

The construction of the convolution product has a similar flavor with Fourier-Mukai transformation, which is the composition of pullback, tensor product and proper pushforward.

Definition 5.1.1 (Convolution product)**.** *For the convenience of reading, we divide the whole process into three steps.*

Step1. *Setting.*

Let M_1, M_2, M_3 be smooth quasi-projective G-varieties. For convenience, denote

$$M_{ij} := M_i \times M_j \qquad M_{123} = M_1 \times M_2 \times M_3$$

and $p_i^{jk}, p_i := p_i^{123}, p_{ij} := p_{ij}^{123}$ as projections onto some factors, as follows.

$$\begin{tikzcd}
& & M_{123} \arrow[dl, "p_{12}"'] \arrow[dr, "p_{23}"] \arrow[rrrd, bend left, "p_{13}"] & & & \\
& M_{12} \arrow[dl, "p_1^{12}"'] \arrow[dr, "p_2^{12}"] & & M_{23} \arrow[dl, "p_2^{23}"'] \arrow[dr, "p_3^{23}"] & & M_{13} \arrow[dlllll, dashed, "p_1^{13}"'] \arrow[dl, dashed, "p_3^{13}"] \\
M_1 & & M_2 & & M_3 &
\end{tikzcd}$$

(Check that $p_i = p_i^{jk} \circ p_{jk}$ for $1 \leqslant j < k \leqslant 3$, $i = j$ or $i = k$)

Step2. *Convolution product on the level of varieties.*

For closed G-subvarieties $Z_{12} \subseteq M_{12}$, $Z_{23} \subseteq M_{23}$, denote

$$Z_{123} := p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23}) \subseteq M_{123}$$

as the intersection of two preimages. The ***convolution product*** *of Z_{12} and Z_{23} is defined as*

$$Z_{12} \circ Z_{23} := p_{13}(Z_{123}) \subseteq M_{13}$$

which is a closed G-subvariety of M_{13}.

Step3. *Convolution product on the level of K-theories.*

Denote

$$\pi_{12} := p_{12}\big|_{p_{12}^{-1}(Z_{12})} \qquad \pi_{23} := p_{23}\big|_{p_{23}^{-1}(Z_{23})} \qquad \pi_{13} := p_{13}\big|_{Z_{123}}$$

as corresponding morphisms restricted to $p_{12}^{-1}(Z_{12})$, $p_{23}^{-1}(Z_{23})$ and Z_{123}, respectively. We assume that π_{13} is proper, so that we can use proper pushforward in K-theory.

We define the convolution product by

$$* : K_0^G(Z_{12}) \times K_0^G(Z_{23}) \longrightarrow K_0^G(Z_{12} \circ Z_{23}) \qquad (\mathcal{F}_{12}, \mathcal{F}_{23}) \longmapsto \mathcal{F}_{12} * \mathcal{F}_{23}$$

$$\mathcal{F}_{12} * \mathcal{F}_{23} = \pi_{13,*}\left(\pi_{12}^* \mathcal{F}_{12} \otimes \pi_{23}^* \mathcal{F}_{23}\right) \in K_0^G(Z_{12} \circ Z_{23})$$

Remark 5.1.2. Those "Z-varieties" (Z_{12}, $p_{12}^{-1}(Z_{12})$, Z_{123}, etc.) are often singular in practice, so π_{12}^*, π_{23}^* and $\otimes$ are defined in the sense of "restriction with supports", under the "M-varieties" which are smooth. The following diagram best illustrates the "actual" definition.

$$\begin{tikzcd}
K_0^G(Z_{12}) \times K_0^G(Z_{23}) \arrow[r, dashed, "\pi_{12}^* \times \pi_{23}^*"] \arrow[d, "\iota_{Z_{12},*}, \iota_{Z_{23},*}"] & K_0^G\big(p_{12}^{-1}(Z_{12})\big) \times K_0^G\big(p_{12}^{-1}(Z_{23})\big) \arrow[r, dashed, "\otimes"] \arrow[d] & K_0^G(Z_{123}) \arrow[r, "\pi_{13,*}"] \arrow[d] & K_0^G(Z_{12} \circ Z_{23}) \arrow[d, "\iota_{Z_{12} \circ Z_{23},*}"'] \\
K_0^G(M_{12}) \times K_0^G(M_{23}) \arrow[r, "p_{12}^* \times p_{23}^*"] & K_0^G(M_{123}) \times K_0^G(M_{123}) \arrow[r, "\otimes"] & K_0^G(M_{123}) \arrow[r, "p_{13,*}"] & K_0^G(M_{13})
\end{tikzcd} \tag{5.1.1}$$

Somewhat lucky, the diagram in (5.1.1) commutes by the vanishment of the Euler class. Therefore, one can compute

$$\mathcal{F}_{12} * \mathcal{F}_{23} = p_{13,*}\left(p_{12}^* \iota_{Z_{12},*} \mathcal{F}_{12} \otimes p_{23}^* \iota_{Z_{23},*} \mathcal{F}_{23}\right) \in K_0^G(M_{13}),$$

and then find the preimage of it under the map $\iota_{Z_{12} \circ Z_{23},*}$. This technique will be used in Subsection 5.3.2.

The whole process can be concluded in Figure 5.1.

$$
\begin{tikzcd}
& & \pi^*_{12}\mathcal{F}_{12}\big|_{Z_{123}} \arrow[dr, dash] & & \pi^*_{23}\mathcal{F}_{23}\big|_{Z_{123}} \arrow[dl, dash] & & & & & \\
& & & Z_{123} \arrow[r, hook] & M_{123} \arrow[ddll, "p_{12}"'] \arrow[ddrr, "p_{23}"] \arrow[ddrrrrr, bend left, "p_{13}"] & & & & & \\
& \mathcal{F}_{12} \arrow[d, dash] & & & & \mathcal{F}_{23} \arrow[d, dash] & & & \mathcal{F}_{13} \arrow[d, dash] & \\
& Z_{12} \arrow[r, hook] & M_{12} \arrow[dll] \arrow[drr] & & & Z_{23} \arrow[r, hook] & M_{23} \arrow[dll] \arrow[dr] & & Z_{13} \arrow[r, hook] & M_{13} \\
M_1 & & & & M_2 & & & M_3 & &
\end{tikzcd}
$$

Figure 5.1: Convolution Product

5.2 Statement

To facilitate the computation of intersection (i.e., tensor product in the construction of convolution product), we state the excess intersection formula.

Theorem 5.2.1 (Excess intersection formula, [3, Corollary 9.4])**.** *Let X' be a smooth G-variety, $X \subseteq X'$ be a (maybe singular) closed G-subvariety, and $Y_1, Y_2 \subseteq X$ be closed G-equivariant embeddings (of globally finite* Tor*-dimension). Denote*

$$
\begin{gathered}
Y := Y_1 \cap Y_2 \qquad \iota_Y : Y \hookrightarrow X \\
\mathcal{T} := TX\big|_Y / \left(TY_1\big|_Y + TY_2\big|_Y\right)
\end{gathered}
$$

$$
\begin{tikzcd}
N_Y Y_2 \arrow[dr, dash] & \frac{N_Y X}{N_Y Y_1} \arrow[d, dash] & N_{Y_1} X \arrow[d, dash] \\
& Y \arrow[r, "g"] \arrow[d, "\phi"'] & Y_1 \arrow[d, "\varphi"] \\
& Y_2 \arrow[r, "f"] & X
\end{tikzcd}
\tag{5.2.1}
$$

Assume that $TY_1\big|_Y \cap TY_2\big|_Y = TY$, we get excess intersection formula:

$$
[Y_1]^G_X \otimes [Y_2]^G_X = \iota_{Y,*}\left(\mathrm{eu}(\mathcal{T}) \cdot [Y]^G_Y\right).
$$

In particular, when $Y = \mathrm{pt}$ is a point, we get simplified formula in $K^G(X)$:

$$
[Y_1]^G \otimes [Y_2]^G = \mathrm{eu}(\mathcal{T}) \cdot [Y]^G
$$

where $\mathrm{eu}(\mathcal{T}) \in \mathrm{R}(G)$ acts by scalar multiplication.

Readers may find Theorem 5.2.1 as a special case of excess base change theorem. In fact,

$$
\begin{aligned}
[Y_1]_X^G \otimes [Y_2]_X^G &= [Y_1]_X^G \otimes f_*[Y_2]_{Y_2}^G && \text{definition of } [Y_2]_X^G \\
&= f_*\left(f^*[Y_1]_X^G \otimes [Y_2]_{Y_2}^G\right) && \text{proper projection formula} \\
&= f_*\left(f^*[Y_1]_X^G\right) && \text{Lemma 2.2.9} \\
&= f_*\left(f^*\varphi_*[Y_1]_{Y_1}^G\right) && \text{definition of } [Y_1]_X^G \\
&= f_*\left(\phi_*\left(\mathrm{eu}(\mathcal{T}) \cdot g^*[Y_1]_{Y_1}^G\right)\right) && \text{excess base change to (5.2.1)} \\
&= \iota_{Y,*}\left(\mathrm{eu}(\mathcal{T}) \cdot [Y]_Y^G\right)
\end{aligned}
$$

The projection formula is stated here.

Proposition 5.2.2 (Projection formula)**.** *For any proper G-equivariant morphism $f : Y \longrightarrow X$ of globally finite* Tor*-dimension, $\alpha \in K^G(Y)$, $\beta \in K^G(X)$, we have proper projection formula:*

$$f_*\alpha \otimes \beta = f_*(\alpha \otimes f^*\beta).$$

5.3 Application: convolution structure

In this section, we will apply Definition 5.1.1 and Theorem 5.2.1 to our typical varieties. In particular, we will get the convolution product formula in terms of basis elements $\widetilde{\phi}_\varpi$ and $\widetilde{\phi}_{\varpi,\varpi'}$.

5.3.1 Algebraic structures induced by convolution product

Definition 5.3.1 (Convolution product sturcture on $K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}})$)**.** *Following notations in 5.1.1, We take $G = G_{\mathbf{d}}$,*

$$
\begin{aligned}
M_1 &= M_2 = M_3 = \widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q) \\
Z_{12} &= Z_{23} = \mathcal{Z}_{\mathbf{d}} \\
\mathcal{Z}_{\mathbf{d}} &= \widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q) \times_{\mathrm{Rep}_{\mathbf{d}}(Q)} \widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q) \subseteq \widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q) \times \widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)
\end{aligned}
$$

By definition, we see that $\mathcal{Z}_{\mathbf{d}} \circ \mathcal{Z}_{\mathbf{d}} = \mathcal{Z}_{\mathbf{d}}$. Therefore, we define a ring structure on $K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}})$:

$$* : K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}}) \times K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}}) \longrightarrow K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}}).$$

Definition 5.3.2 ($K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}})$-module sturcture on $K^{G_{\mathbf{d}}}(\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q))$)**.** *Following notations in 5.1.1, We take $G = G_{\mathbf{d}}$,*

$$
\begin{aligned}
M_1 &= M_2 = \widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q) & M_3 &= \{\mathrm{pt}\} \\
Z_{12} &= \mathcal{Z}_{\mathbf{d}} & Z_{23} &= \widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)
\end{aligned}
$$

By definition, we see that $\mathcal{Z}_{\mathbf{d}} \circ \widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q) = \widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)$. Therefore, we define a $K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}})$-module sturcture on $K^{G_{\mathbf{d}}}\left(\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)\right)$:

$$\star : K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}}) \times K^{G_{\mathbf{d}}}\left(\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)\right) \longrightarrow K^{G_{\mathbf{d}}}\left(\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)\right).$$

Remark 5.3.3. Notice that in the construction of the convolution product, pullback, tensor product and proper pushforward are compatible with the forgetful map of groups. Therefore, the following diagrams commute:

$$\begin{tikzcd}
K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}}) \times K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}}) \arrow[r, "*"] \arrow[d, hook] & K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}}) \arrow[d, hook] \\
K^{T_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}}) \times K^{T_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}}) \arrow[r, "*"] \arrow[d, hook] & K^{T_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}}) \arrow[d, hook] \\
\mathcal{K}^{T_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}}) \times \mathcal{K}^{T_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}}) \arrow[r, "*"] & \mathcal{K}^{T_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}})
\end{tikzcd}
\qquad
\begin{tikzcd}
K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}}) \times K^{G_{\mathbf{d}}}\left(\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)\right) \arrow[r, "\star"] \arrow[d, hook] & K^{G_{\mathbf{d}}}\left(\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)\right) \arrow[d, hook] \\
K^{T_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}}) \times K^{T_{\mathbf{d}}}\left(\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)\right) \arrow[r, "\star"] \arrow[d, hook] & K^{T_{\mathbf{d}}}\left(\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)\right) \arrow[d, hook] \\
\mathcal{K}^{T_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}}) \times \mathcal{K}^{T_{\mathbf{d}}}\left(\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)\right) \arrow[r, "\star"] & \mathcal{K}^{T_{\mathbf{d}}}\left(\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)\right)
\end{tikzcd}$$

Definition 5.3.4 ($K^{G_{\mathbf{d}}}(\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q))$-module sturcture on $K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}})$)**.** *We know that*

$$\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q) \cong \mathcal{Z}_{\mathrm{Id}} \subseteq \mathcal{Z}_{\mathbf{d}}, \qquad \mathcal{Z}_{\mathrm{Id}} \circ \mathcal{Z}_{\mathrm{Id}} = \mathcal{Z}_{\mathrm{Id}},$$

so $K^{G_{\mathbf{d}}}\left(\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)\right)$ can be realized as a $R(G_{\mathbf{d}})$-subalgebra of $K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}})$, and $K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}})$ has the $K^{G_{\mathbf{d}}}\left(\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)\right)$-module structure induced by the convolution product:

$$* : K^{G_{\mathbf{d}}}\left(\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)\right) \times K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}}) \longrightarrow K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}}).$$

5.3.2 Convolution product formula

In this subsection, we compute the convolution product in the bottom line of the diagram in Remark 5.3.3.

Proposition 5.3.5 (Convolution product formula)**.** *For ϖ, ϖ', ϖ'', $\varpi''' \in \mathbb{W}_{|\mathbf{d}|}$, we have*

$$\begin{aligned}
\widetilde{\psi}_{\varpi,\varpi'} * \widetilde{\psi}_{\varpi'',\varpi'''} &= \delta_{\varpi',\varpi''}\widetilde{\Lambda}_{\varpi'}\widetilde{\psi}_{\varpi,\varpi'''} \\
\widetilde{\psi}_{\varpi,\varpi'} \star \widetilde{\psi}_{\varpi''} &= \delta_{\varpi',\varpi''}\widetilde{\Lambda}_{\varpi'}\widetilde{\psi}_{\varpi}.
\end{aligned}$$

Proof. Follow the Definition 5.1.1 and Theorem 5.2.1 if needed.

For clearance, we divide the proof into 4 cases.

Case 1. Assume $\varpi' \neq \varpi''$, need to show $\widetilde{\psi}_{\varpi,\varpi'} * \widetilde{\psi}_{\varpi'',\varpi'''} = 0$.

Denote [1]

$$Y_{12} := \{(\rho_0, F_\varpi, F_{\varpi'})\} \subseteq \mathcal{Z}_{\mathbf{d}}, \qquad Y_{23} := \{(\rho_0, F_{\varpi''}, F_{\varpi'''})\} \subseteq \mathcal{Z}_{\mathbf{d}}\,.$$

Since $\varpi' \neq \varpi''$, $p_{12}^{-1}(Y_{12}) \cap p_{23}^{-1}(Y_{23}) = \varnothing$, so

$$\begin{aligned}
\widetilde{\psi}_{\varpi,\varpi'} * \widetilde{\psi}_{\varpi'',\varpi'''} &= [Y_{12}]^{T_{\mathbf{d}}}_{\mathcal{Z}_{\mathbf{d}}} * [Y_{23}]^{T_{\mathbf{d}}}_{\mathcal{Z}_{\mathbf{d}}} \\
&= p_{13,*}\left(p_{12}^*\,[Y_{12}]^{T_{\mathbf{d}}}_{M_{12}} \otimes p_{23}^*\,[Y_{23}]^{T_{\mathbf{d}}}_{M_{23}}\right) \\
&= p_{13,*}\left(\left[p_{12}^{-1}(Y_{12})\right]^{T_{\mathbf{d}}}_{M_{123}} \otimes \left[p_{12}^{-1}(Y_{23})\right]^{T_{\mathbf{d}}}_{M_{123}}\right) \\
&= 0
\end{aligned}$$

Case 2. Assume $\varpi' \neq \varpi''$, need to show $\widetilde{\psi}_{\varpi,\varpi'} \star \widetilde{\psi}_{\varpi''} = 0$.

Denote

$$Y_{12} := \{(\rho_0, F_\varpi, F_{\varpi'})\} \subseteq \mathcal{Z}_{\mathbf{d}}, \qquad Y_{23} := \{(\rho_0, F_{\varpi''})\} \subseteq \widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q).$$

Since $\varpi' \neq \varpi''$, $p_{12}^{-1}(Y_{12}) \cap p_{23}^{-1}(Y_{23}) = \varnothing$, so

$$\begin{aligned}
\widetilde{\psi}_{\varpi,\varpi'} \star \widetilde{\psi}_{\varpi''} &= [Y_{12}]^{T_{\mathbf{d}}}_{\mathcal{Z}_{\mathbf{d}}} \star [Y_{23}]^{T_{\mathbf{d}}}_{\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)} \\
&= p_{13,*}\left(p_{12}^*\,[Y_{12}]^{T_{\mathbf{d}}}_{M_{12}} \otimes p_{23}^*\,[Y_{23}]^{T_{\mathbf{d}}}_{M_{23}}\right) \\
&= p_{13,*}\left(\left[p_{12}^{-1}(Y_{12})\right]^{T_{\mathbf{d}}}_{M_{123}} \otimes \left[p_{12}^{-1}(Y_{23})\right]^{T_{\mathbf{d}}}_{M_{123}}\right) \\
&= 0
\end{aligned}$$

Case 3. For ϖ, ϖ', $\varpi'' \in \mathbb{W}_{|\mathbf{d}|}$, need to show that

$$\widetilde{\psi}_{\varpi,\varpi'} * \widetilde{\psi}_{\varpi',\varpi''} = \widetilde{\Lambda}_{\varpi'}\widetilde{\psi}_{\varpi,\varpi''}.$$

Denote

$$Y_{12} := \{(\rho_0, F_\varpi, F_{\varpi'})\} \subseteq \mathcal{Z}_{\mathbf{d}}, \qquad Y_{23} := \{(\rho_0, F_{\varpi'}, F_{\varpi''})\} \subseteq \mathcal{Z}_{\mathbf{d}},$$

then

$$\begin{aligned}
&p_{12}^{-1}(Y_{12}) = Y_{12} \times \widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q) \qquad && p_{23}^{-1}(Y_{23}) = \widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q) \times Y_{23} \\
&p_{12}^{-1}(Y_{12}) \cup p_{23}^{-1}(Y_{23}) = Y && Y_{12} \circ Y_{23} = Y_{13},
\end{aligned}$$

[1]For some people, the notation

$$Y_{12} := \left\{\left((\rho_0, F_\varpi), (\rho_0, F_{\varpi'})\right)\right\} \subseteq \mathcal{Z}_{\mathbf{d}}$$

is better for understanding. We don't write like that, because too many brackets occupy attentions.

where

$$\begin{array}{lll} Y=\{y\} & y=\big((\rho_0,F_\varpi),(\rho_0,F_{\varpi'}),(\rho_0,F_{\varpi''})\big) & \in M_{123} \\ Y_{13}=\{y_{13}\} & y_{13}=\big((\rho_0,F_\varpi),(\rho_0,F_{\varpi''})\big) & \in M_{13} \end{array}$$

Therefore,

$$\begin{aligned} \widetilde{\psi}_{\varpi,\varpi'} * \widetilde{\psi}_{\varpi',\varpi''} &= [Y_{12}]^{T_{\mathbf{d}}}_{\mathcal{Z}_{\mathbf{d}}} * [Y_{23}]^{T_{\mathbf{d}}}_{\mathcal{Z}_{\mathbf{d}}} \\ &= p_{13,*}\left(p_{12}^*\,[Y_{12}]^{T_{\mathbf{d}}}_{M_{12}} \otimes p_{23}^*\,[Y_{23}]^{T_{\mathbf{d}}}_{M_{23}}\right) \\ &= p_{13,*}\left(\big[p_{12}^{-1}(Y_{12})\big]^{T_{\mathbf{d}}}_{M_{123}} \otimes \big[p_{12}^{-1}(Y_{23})\big]^{T_{\mathbf{d}}}_{M_{123}}\right) \\ &= p_{13,*}\left(\mathrm{eu}(\mathcal{T})\cdot [Y]^{T_{\mathbf{d}}}_{M_{123}}\right) \\ &= \mathrm{eu}(\mathcal{T})\cdot [Y]^{T_{\mathbf{d}}}_{M_{13}} \\ &= \widetilde{\Lambda}_{\varpi'}\widetilde{\psi}_{\varpi,\varpi''} \end{aligned}$$

where

$$\mathcal{T} := \frac{T_y M_{123}}{T_y\big(p_{12}^{-1}(Y_{12})\big)\oplus T_y\big(p_{23}^{-1}(Y_{23})\big)} = \frac{\widetilde{\mathcal{T}}_\varpi \oplus \widetilde{\mathcal{T}}_{\varpi'} \oplus \widetilde{\mathcal{T}}_{\varpi''}}{\widetilde{\mathcal{T}}_\varpi \quad \oplus \quad \widetilde{\mathcal{T}}_{\varpi''}} = \widetilde{\mathcal{T}}_{\varpi'}.$$

Case 4. For ϖ, $\varpi' \in \mathbb{W}_{|\mathbf{d}|}$, need to show that

$$\widetilde{\psi}_{\varpi,\varpi'} \star \widetilde{\psi}_{\varpi'} = \widetilde{\Lambda}_{\varpi'}\widetilde{\psi}_{\varpi}.$$

Denote

$$Y_{12} := \{(\rho_0,F_\varpi,F_{\varpi'})\} \subseteq \mathcal{Z}_{\mathbf{d}}, \qquad Y_{23} := \{(\rho_0,F_{\varpi'})\} \subseteq \widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q),$$

then

$$\begin{array}{ll} p_{12}^{-1}(Y_{12}) = Y_{12}\times\{\mathrm{pt}\} & p_{23}^{-1}(Y_{23}) = \widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)\times Y_{23} \\ p_{12}^{-1}(Y_{12})\cup p_{23}^{-1}(Y_{23}) = Y & Y_{12}\circ Y_{23} = Y_{13}, \end{array}$$

where

$$\begin{array}{lll} Y=\{y\} & y=\big((\rho_0,F_\varpi),(\rho_0,F_{\varpi'})\big) & \in M_{123} \\ Y_{13}=\{y_{13}\} & y_{13}=(\rho_0,F_\varpi) & \in M_{13} \end{array}$$

Therefore,

$$\begin{aligned} \widetilde{\psi}_{\varpi,\varpi'} \star \widetilde{\psi}_{\varpi'} &= [Y_{12}]^{T_{\mathbf{d}}}_{\mathcal{Z}_{\mathbf{d}}} \star [Y_{23}]^{T_{\mathbf{d}}}_{\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)} \\ &= p_{13,*}\left(p_{12}^*\,[Y_{12}]^{T_{\mathbf{d}}}_{M_{12}} \otimes p_{23}^*\,[Y_{23}]^{T_{\mathbf{d}}}_{M_{23}}\right) \\ &= p_{13,*}\left(\big[p_{12}^{-1}(Y_{12})\big]^{T_{\mathbf{d}}}_{M_{123}} \otimes \big[p_{12}^{-1}(Y_{23})\big]^{T_{\mathbf{d}}}_{M_{123}}\right) \\ &= p_{13,*}\left(\mathrm{eu}(\mathcal{T})\cdot [Y]^{T_{\mathbf{d}}}_{M_{123}}\right) \\ &= \mathrm{eu}(\mathcal{T})\cdot [Y]^{T_{\mathbf{d}}}_{M_{13}} \\ &= \widetilde{\Lambda}_{\varpi'}\widetilde{\psi}_{\varpi} \end{aligned}$$

where

$$\mathcal{T} := \frac{T_y M_{123}}{T_y\big(p_{12}^{-1}(Y_{12})\big) \oplus T_y\big(p_{23}^{-1}(Y_{23})\big)} = \frac{\widetilde{\mathcal{T}}_{\varpi} \oplus \widetilde{\mathcal{T}}_{\varpi'} \oplus 0}{\widetilde{\mathcal{T}}_{\varpi} \quad \oplus \quad 0} = \widetilde{\mathcal{T}}_{\varpi'}.$$

□

Readers can think matrix multiplication as an analog of Proposition 5.3.5: denote $E_{ij} \in M^{n\times n}(\mathbb{C})$ as the matrix having 1 in the (i,j)-position and 0 elsewhere, and $e_i \in M^{n\times 1}(\mathbb{C})$ as the standard column vector, then

$$E_{ij}E_{kl} = \delta_{jk}E_{il} \qquad E_{ij}e_k = \delta_{jk}e_i.$$

5.3.3 Demazure operator

In this subsection, we will compute the action of some elements in $K^{G_{\mathbf{d}}}(\mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}'})$ acting on $K^{G_{\mathbf{d}}}\left(\widetilde{\operatorname{Rep}}_{\underline{\mathbf{d}}'}(Q)\right)$. As a reminder,

$$\begin{array}{ccc} K^{G_{\mathbf{d}}}\left(\widetilde{\operatorname{Rep}}_{\underline{\mathbf{d}}'}(Q)\right) & \cong & \mathrm{R}(T_{\mathbf{d}})\left[\widetilde{\operatorname{Rep}}_{\underline{\mathbf{d}}'}(Q)\right]^{G_{\mathbf{d}}} \\ \downarrow & & \downarrow \\ K^{T_{\mathbf{d}}}\left(\widetilde{\operatorname{Rep}}_{\underline{\mathbf{d}}'}(Q)\right) & \cong & \bigoplus_{w}\mathrm{R}(T_{\mathbf{d}})\left[\widetilde{\overline{\Omega}}{}^{u}_{w}\right]^{T_{\mathbf{d}}} \end{array} \tag{5.3.1}$$

where the $R(T_{\mathbf{d}})$-module structure on $K^{G_{\mathbf{d}}}\left(\widetilde{\operatorname{Rep}}_{\underline{\mathbf{d}}'}(Q)\right)$ is induced by the induction formula.

For $f \in \mathrm{R}(T_{\mathbf{d}}) \cong \mathbb{Z}\Big[e_1^{\pm 1}, \dots, e_{|\mathbf{d}|}^{\pm 1}\Big]$, denote $f^u := f \cdot \left[\widetilde{\operatorname{Rep}}_{\underline{\mathbf{d}}'}(Q)\right]^{G_{\mathbf{d}}}$. Under the morphism (5.3.1), f is sent to $f \cdot \left[\widetilde{\operatorname{Rep}}_{\underline{\mathbf{d}}'}(Q)\right]^{T_{\mathbf{d}}}$. Viewing f^u as an element in $\mathcal{K}^{G_{\mathbf{d}}}\left(\widetilde{\operatorname{Rep}}_{\underline{\mathbf{d}}'}(Q)\right)$, we get

$$f^u = \sum_{w} f(e_1, \dots, e_{|\mathbf{d}|})\widetilde{\Lambda}_{wu}^{-1}\widetilde{\psi}_{wu}.$$

Remark 5.3.6. This formula looks not so compatible with the group action. To facilitate our computation, we relable the coefficient ring before $\widetilde{\psi}_{\varpi}$ by $e_i^{\varpi} := e_{\varpi^{-1}(i)}$, which means that

$$K^{T_{\mathbf{d}}}\left(\widetilde{\operatorname{Rep}}_{\mathbf{d}}(Q)\right) \cong \bigoplus_{\varpi}\mathbb{Z}\Big[e_1^{\varpi,\pm 1}, \dots, e_{|\mathbf{d}|}^{\varpi,\pm 1}\Big]\,\widetilde{\psi}_{\varpi}$$

Therefore,

$$\begin{aligned} f^u &= \sum_{w}(wuf)(e_1^{wu}, \dots, e_{|\mathbf{d}|}^{wu})\widetilde{\Lambda}_{wu}^{-1}\widetilde{\psi}_{wu}. \\ &\hat{=} \sum_{w}(wuf)\;\widetilde{\Lambda}_{wu}^{-1}\widetilde{\psi}_{wu}. \end{aligned}$$

Later, every expression before $\widetilde{\psi}_{\varpi}$ should be viewed as an expression in $\mathbb{Z}\Big[e_1^{\varpi,\pm 1}, \dots, e_{|\mathbf{d}|}^{\varpi,\pm 1}\Big]$.

Definition 5.3.7 (Demazure operator)**.** *For $i \in \{1, \dots, |\mathbf{d}|-1\}$, set $s = s_i$, the (absolute) Demazure operator is defined as*

$$D_i := [\mathcal{Z}_{s_i}]^{G_{\mathbf{d}}} \in K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}}).$$

View D_i as an element in $\mathcal{K}^{T_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}})$, we get

$$D_i = \sum_{\varpi \in \mathbb{W}_{|\mathbf{d}|}} \left(\widetilde{\Lambda}^s_{\varpi,\varpi s}\right)^{-1} \widetilde{\psi}_{\varpi,\varpi s} + \sum_{\substack{\varpi \in \mathbb{W}_{|\mathbf{d}|} \\ \varpi s \varpi^{-1} \in W_{\mathbf{d}}}} \left(\widetilde{\Lambda}^s_{\varpi,\varpi}\right)^{-1} \widetilde{\psi}_{\varpi,\varpi}.$$

We also have the relative version. Suppose that $W_{\mathbf{d}} u s_i = W_{\mathbf{d}} u'$ (which guarantees the existence of $\mathcal{Z}^{u,u'}_{s_i}$), the (relative) Demazure operator is defined as

$$D^{u,u'}_i := \left[\mathcal{Z}^{u,u'}_{s_i}\right]^{G_{\mathbf{d}}} \in K^{G_{\mathbf{d}}}(\mathcal{Z}^{u,u'}).$$

View $D^{u,u'}_i$ as an element in $\mathcal{K}^{T_{\mathbf{d}}}(\mathcal{Z}^{u,u'})$, we get

$$D^{u,u'}_i = \sum_{w} \left(\widetilde{\Lambda}^s_{wu,wus}\right)^{-1} \widetilde{\psi}_{wu,wus} + \delta_{u,u'} \sum_{w} \left(\widetilde{\Lambda}^s_{wu,wu}\right)^{-1} \widetilde{\psi}_{wu,wu}.$$

The equivariant cohomology theory version of Demazure operators are denoted by ∂_i and $\partial^{u,u'}_i$.

Theorem 5.3.8. *We have a formula of Demazure operator:*

$$D^{u,u'}_i \star f^{u'} = \begin{cases} \left[\left(\dfrac{s_i f}{1 - \frac{e_{i+1}}{e_i}} + \dfrac{f}{1 - \frac{e_i}{e_{i+1}}}\right)\left(1 - \dfrac{e_{i+1}}{e_i}\right)^k\right]^u & u = u', \\ \left[s_i f \left(1 - \dfrac{e_{i+1}}{e_i}\right)^k\right]^u & u \neq u'. \end{cases}$$

and similar for the equivariant cohomology theory:

$$\partial^{u,u'}_i \star f^{u'} = \begin{cases} \left[\left(\dfrac{s_i f}{\lambda_{i+1} - \lambda_i} + \dfrac{f}{\lambda_i - \lambda_{i+1}}\right)(\lambda_{i+1} - \lambda_i)^k\right]^u & u = u', \\ \left[s_i f \left(\lambda_{i+1} - \lambda_i\right)^k\right]^u & u \neq u'. \end{cases}$$

In the formula, $\lambda^u_l := \lambda_{u^{-1}(l)}$, and k stands for the number of arrows from the vertex associated to $v_{u(i+1)}$ to the vertex associated to $v_{u(i)}$.

In the computation we mainly focus on the K-theory. Using 5.3.6, one can compute $D_i^{u,u'} \star f^{u'}$ in terms of ϕ's: ($s := s_i$ for simplicity)

$$
\begin{aligned}
D_i^{u,u'} \star f^{u'} &= \left(\sum_w \left(\widetilde{\Lambda}^s_{wu,wus}\right)^{-1} \widetilde{\psi}_{wu,wus} + \delta_{u,u'} \sum_w \left(\widetilde{\Lambda}^s_{wu,wu}\right)^{-1} \widetilde{\psi}_{wu,wu} \right) \\
&\qquad\qquad\qquad\qquad \times \left(\sum_w (wu'f)\, \widetilde{\Lambda}^{-1}_{wu'} \widetilde{\psi}_{wu'} \right) \\
&= \left(\sum_w \left(\widetilde{\Lambda}^s_{wu,wus}\right)^{-1} \widetilde{\psi}_{wu,wus} \right) \cdot \left(\sum_w (wusf)\, \widetilde{\Lambda}^{-1}_{wus} \widetilde{\psi}_{wus} \right) \\
&\quad + \delta_{u,u'} \left(\sum_w \left(\widetilde{\Lambda}^s_{wu,wu}\right)^{-1} \widetilde{\psi}_{wu,wu} \right) \cdot \left(\sum_w (wuf)\, \widetilde{\Lambda}^{-1}_{wu} \widetilde{\psi}_{wu} \right) \\
&= \left(\sum_w (wusf) \left(\widetilde{\Lambda}^s_{wu,wus}\right)^{-1} \widetilde{\psi}_{wu} \right) + \delta_{u,u'} \left(\sum_w (wuf) \left(\widetilde{\Lambda}^s_{wu,wu}\right)^{-1} \widetilde{\psi}_{wu} \right) \\
&= \sum_w \left[\left(\frac{wusf}{\widetilde{\Lambda}^s_{wu,wus}} + \delta_{u,u'} \frac{wuf}{\widetilde{\Lambda}^s_{wu,wu}} \right) \widetilde{\Lambda}_{wu} \right] \widetilde{\Lambda}^{-1}_{wu} \widetilde{\psi}_{wu} \\
&= \sum_w w \left[\left(\frac{usf}{\widetilde{\Lambda}^s_{u,us}} + \delta_{u,u'} \frac{uf}{\widetilde{\Lambda}^s_{u,u}} \right) \widetilde{\Lambda}_{u} \right] \widetilde{\Lambda}^{-1}_{wu} \widetilde{\psi}_{wu} \\
&= \sum_w wu \left[\left(\frac{sf}{u^{-1}\widetilde{\Lambda}^s_{u,us}} + \delta_{u,u'} \frac{f}{u^{-1}\widetilde{\Lambda}^s_{u,u}} \right) u^{-1}\widetilde{\Lambda}_{u} \right] \widetilde{\Lambda}^{-1}_{wu} \widetilde{\psi}_{wu} \\
&= \left[\left(\frac{sf}{u^{-1}\widetilde{\Lambda}^s_{u,us}} + \delta_{u,u'} \frac{f}{u^{-1}\widetilde{\Lambda}^s_{u,u}} \right) u^{-1}\widetilde{\Lambda}_{u} \right]^u
\end{aligned}
$$

Recall Subsection 1.6.4 (especially Proposition 1.6.15), we get

$$
\widetilde{\mathcal{T}}^s_{u,us} \cong \mathfrak{r}_{u,us} \oplus \mathfrak{n}_u^- \oplus \mathfrak{m}_{u,us} \qquad \widetilde{\mathcal{T}}^s_{u,u} \cong \mathfrak{r}_{u,us} \oplus \mathfrak{n}_u^- \oplus \mathfrak{m}_{us,u} \qquad \widetilde{\mathcal{T}}_u \cong \mathfrak{r}_u \oplus \mathfrak{n}_u^- .
$$

Therefore,

$$
D_i^{u,u'} \star f^{u'} = \left[\left(\frac{sf}{u^{-1}\,\mathrm{eu}(\mathfrak{m}_{u,us})} + \delta_{u,u'} \frac{f}{u^{-1}\,\mathrm{eu}(\mathfrak{m}_{us,u})} \right) u^{-1}\,\mathrm{eu}(\mathfrak{d}_{u,us}) \right]^u . \tag{5.3.2}
$$

Recall the computation in 1.4.9 and Section 4.1. We collect needed information in Table 5.1:

Theorem 5.3.8 is our final destination in this part. We will express its importance in Subsection 5.3.4, see some generalizations in Section 6.1 and compute some examples in Section 6.2.

5.3.4 Miscellaneous

In this subsection, we collect some results which are of significant importance theoretically. The arguments in reference work for both K-theory and cohomology theory.

Name	$\mathfrak{g}$	$u^{-1}\mathfrak{g}$	$u^{-1}\operatorname{eu}(\mathfrak{g})$	$u^{-1}\operatorname{eu}'(\mathfrak{g})$	
$\mathfrak{m}_{u,us}$	$\frac{e_{u(i)}}{e_{u(i+1)}}$	$\frac{e_i}{e_{i+1}}$	$1-\frac{e_{i+1}}{e_i}$	$\lambda_{i+1}-\lambda_i$	$u=u'$
	0	0	1	1	$u\neq u'$
$\mathfrak{m}_{us,u}$	$\frac{e_{u(i+1)}}{e_{u(i)}}$	$\frac{e_{i+1}}{e_i}$	$1-\frac{e_i}{e_{i+1}}$	$\lambda_i-\lambda_{i+1}$	$u=u'$
	0	0	1	1	$u\neq u'$
$\mathfrak{d}_{u,us}$	$k\frac{e_{u(i)}}{e_{u(i+1)}}$	$k\frac{e_i}{e_{i+1}}$	$\left(1-\frac{e_{i+1}}{e_i}\right)^k$	$(\lambda_{i+1}-\lambda_i)^k$	

Table 5.1

Proposition 5.3.9. *The action of* $K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}})$ *on* $K^{G_{\mathbf{d}}}(\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q))$ *is faithful.*

Sketch of proof. Reduce the problem to the faithfulness for the action of $\mathcal{K}^{T_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}})$ on $\mathcal{K}^{T_{\mathbf{d}}}(\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q))$. For details, see [3, Theorem 10.10]. □

Proposition 5.3.10. *The elements* $\{D_i^{u,u'}\}_{u,u',i}$ *generate* $K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}})$ *as a* $K^{G_{\mathbf{d}}}(\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q))$*-algebra.*

Sketch of proof. See [3, Theorem 11.3]. The key observation is [3, Lemma 7.30, 11.4]. □

Combining these propositions with Theorem 5.3.8, we understand the convolution structure of $K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}})$ theoretically.

Chapter 6

Generalizations, examples and connections

6.1 Generalization

In this section we generalize results in different directions. Generalizing complete flag variety to partial flag variety needs further investigation, so we don't do this. After the generalization, we are able to cover the result in [1, Theorem 7.2.5].

6.1.1 Quiver with loops

In this section we still assume the quiver has no cycles. For quiver with loops, we need to redefine Definition 1.5.8 in a strict version:

Definition 6.1.1 (Incidence variety for strict flags)**.** *For a quiver Q with flag-type dimension vector $\underline{\mathbf{d}}$, define*

$$\begin{aligned}
\widetilde{\mathrm{Rep}}_{\underline{\mathbf{d}},\mathrm{str}}(Q) &:= \left\{(\rho, F) \in \mathrm{Rep}_{\mathbf{d}}(Q) \times \mathcal{F}_{\underline{\mathbf{d}}} \,\middle|\, \rho(M_j) \subseteq M_{j-1} \textit{ for any } j\right\} \\
\widetilde{\mathrm{Rep}}_{\mathbf{d},\mathrm{str}}(Q) &:= \{(\rho, F) \in \mathrm{Rep}_{\mathbf{d}}(Q) \times \mathcal{F}_{\mathbf{d}} \,|\, \rho(M_j) \subseteq M_{j-1} \textit{ for any } j\} \\
&= \bigsqcup_{\underline{\mathbf{d}}} \widetilde{\mathrm{Rep}}_{\underline{\mathbf{d}},\mathrm{str}}(Q)
\end{aligned}$$

and $\mu_{\underline{\mathbf{d}},\mathrm{str}}$, $\pi_{\underline{\mathbf{d}},\mathrm{str}}$, $\mu_{\mathbf{d},\mathrm{str}}$, $\pi_{\mathbf{d},\mathrm{str}}$ to be the natural morphisms from the incidence varieties to $\mathrm{Rep}_{\mathbf{d}}(Q)$ *or flag varieties.*

We then replace $\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)$ by $\widetilde{\mathrm{Rep}}_{\mathbf{d},\mathrm{str}}(Q)$. The Lie algebra $\mathfrak{r}_{\varpi}$ (in Definition 1.4.8) is redefined by

$$\begin{aligned}
\mathfrak{r}_{\varpi} &:= \left\{(f_a)_{a \in Q_1} \in \mathrm{Rep}_{\mathbf{d}}(Q) \,\middle|\, f_a(V_{\varpi,j} \cap V_{s(a)}) \subseteq V_{\varpi,j} \text{ for any } j\right\} \\
&\cong \pi^{-1}_{\mathbf{d},\mathrm{str}}(\{F_{\varpi}\})
\end{aligned}$$

then the same formula in Theorem 5.3.8 still works.

6.1.2 $G \times \mathbb{C}^\times$-action

The second generalization is about $G\times\mathbb{C}^\times$-actions. Recall the Remark 1.5.4. Following the same arguments as in Example 2.1.3-2.1.6 and 2.6.2-2.6.5, we get (in the Setting 1.1.1)

$$\begin{aligned}
&\mathrm{R}(N \times \mathbb{C}^\times) \cong \mathrm{R}(\mathbb{C}^\times) \cong \mathbb{Z}[q^{\pm 1}] && \mathrm{S}(N \times \mathbb{C}^\times) \cong \mathrm{S}(\mathbb{C}^\times) \cong \mathbb{Q}[t]\\
&\mathrm{R}(B \times \mathbb{C}^\times) \cong \mathrm{R}(T \times \mathbb{C}^\times) \cong \mathbb{Z}[q^{\pm 1}][e_1^{\pm 1}, \dots, e_n^{\pm 1}] && \mathrm{S}(B \times \mathbb{C}^\times) \cong \mathrm{S}(T \times \mathbb{C}^\times) \cong \mathbb{Q}[t][\lambda_1, \dots, \lambda_n]\\
&\mathrm{R}(G \times \mathbb{C}^\times) \cong \mathbb{Z}[q^{\pm 1}][e_1^{\pm 1}, \dots, e_n^{\pm 1}]^{S_n} && \mathrm{S}(G \times \mathbb{C}^\times) \cong \mathbb{Q}[t][\lambda_1, \dots, \lambda_n]^{S_n}
\end{aligned}$$

So everything remains the same except for the change of coefficient ring. In particular, for $D_i^{u,u'} := [\mathcal{Z}_{s_i}^{u,u'}]^{G_{\mathbf{d}}\times\mathbb{C}^\times}$, $f^u := f \cdot \left[\widetilde{\mathrm{Rep}}_{\underline{\mathbf{d}},\mathrm{str}}(Q)\right]^{G_{\mathbf{d}}}$, we have formula (5.3.2), with informations in Table 6.1.

Name	$\mathfrak{g}$	$u^{-1}\mathfrak{g}$	$u^{-1}\,\mathrm{eu}(\mathfrak{g})$	$u^{-1}\,\mathrm{eu}'(\mathfrak{g})$	
$\mathfrak{m}_{u,us}$	$\frac{e_{u(i)}}{e_{u(i+1)}}$	$\frac{e_i}{e_{i+1}}$	$1-\frac{e_{i+1}}{e_i}$	$\lambda_{i+1}-\lambda_i$	$u=u'$
	0	0	1	1	$u\neq u'$
$\mathfrak{m}_{us,u}$	$\frac{e_{u(i+1)}}{e_{u(i)}}$	$\frac{e_{i+1}}{e_i}$	$1-\frac{e_i}{e_{i+1}}$	$\lambda_i-\lambda_{i+1}$	$u=u'$
	0	0	1	1	$u\neq u'$
$\mathfrak{d}_{u,us}$	$k\frac{e_{u(i)}}{e_{u(i+1)}}q^{-1}$	$k\frac{e_i}{e_{i+1}}q^{-1}$	$\left(1-\frac{e_{i+1}}{e_i}q\right)^k$	$(\lambda_{i+1}-\lambda_i+t)^k$	

Table 6.1

Theorem 6.1.2. *When the quiver has no cycle, we have a formula of Demazure operator for the $G_{\mathbf{d}} \times \mathbb{C}^\times$-action:*

$$D_i^{u,u'} \star f^{u'} = \begin{cases} \left[\left(\dfrac{s_i f}{1-\frac{e_{i+1}}{e_i}} + \dfrac{f}{1-\frac{e_i}{e_{i+1}}}\right)\left(1-\dfrac{e_{i+1}}{e_i}q\right)^k\right]^u & u=u',\\[2ex] \left[s_i f\left(1-\dfrac{e_{i+1}}{e_i}q\right)^k\right]^u & u\neq u'. \end{cases}$$

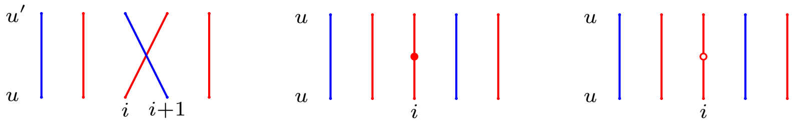
and similar for the equivariant cohomology theory:

$$\partial_i^{u,u'} \star f^{u'} = \begin{cases} \left[\left(\dfrac{s_i f}{\lambda_{i+1}-\lambda_i} + \dfrac{f}{\lambda_i-\lambda_{i+1}}\right)(\lambda_{i+1}-\lambda_i+t)^k\right]^u & u=u',\\[2ex] \left[s_i f\,(\lambda_{i+1}-\lambda_i+t)^k\right]^u & u\neq u'. \end{cases}$$

6.2 From formula to diagram

This section is designed for showing examples. Recall Fact 1.3.3 that every $\underline{\mathbf{d}}$ or u corresponds to an ordered set of colored points. It can be imagined that the lines connecting two ordered sets represents one element in $K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}})$. Actually, we draw the picture of generators of $K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}})$ in Figure 6.1, where

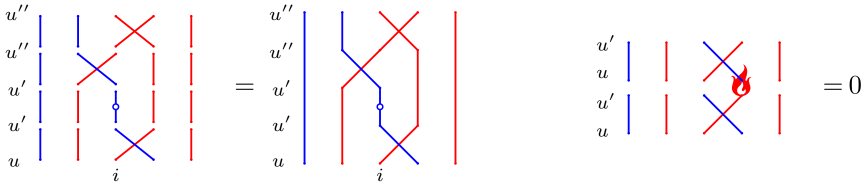
$$e_i^u =: e_{u^{-1}(i)} \left[\widetilde{\mathrm{Rep}}_u(Q)\right]^{G_{\mathbf{d}}} \in K^{G_{\mathbf{d}}}\left(\widetilde{\mathrm{Rep}}_u(Q)\right) \hookrightarrow K^{G_{\mathbf{d}}}\left(\mathcal{Z}^{u,u}\right).$$



$D_i^{u,u'}$ $\quad e_i^u$ $\quad (e_i^u)^{-1}$

Figure 6.1

The convolution product can be then viewed as pictures gluing vertically, where the incompatibility of colors gives 0. For example,



$D_3^{u,u'} * (e_3^{u'})^{-1} * D_2^{u',u''} * D_3^{u'',u''}$ $\qquad D_3^{u,u'} * D_3^{u,u'} = 0$

By Proposition 5.3.10, every element in $K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}}) = \oplus_{u,u'} K^{G_{\mathbf{d}}}(\mathcal{Z}^{u,u'})$ can be expressed as a $\mathbb{Z}$-linear combination of strands. The expressions are not unique, so we need to find out their relations. Some relations are clear from the picture (but still need to check), for example,

$D_3^{u,u'} * D_1^{u',u'''} \quad = \quad D_1^{u,u''} * D_3^{u,u'}$ $\qquad D_3^{u,u'} * e_2^{u'} \quad = \quad e_2^u * D_3^{u,u'}$

We won't draw these "obvious" relations later. The first nontrivial relation comes from the following lemma.

Lemma 6.2.1. *For $f \in \mathrm{R}(T_{\mathbf{d}})$, denote $D_i^{u,u'} = \left[\mathcal{Z}_{s_i}^{u,u'}\right]^{G_{\mathbf{d}}}$, $f^u = f \cdot \left[\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)\right]^{G_{\mathbf{d}}} \in$*

$K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}})$, *we have*

$$D_i^{u,u'} * f^{u'} = (s_i f)^u * D_i^{u,u'} + \delta_{u,u'} \left[(s_i f - f) \frac{e_{i+1}}{e_i} \left(1 - \frac{e_{i+1}}{e_i}\right)^{k-1} \right]^u .$$

Similarly, for the $G_{\mathbf{d}}$-equivariant cohomology theory, we have

$$\partial_i^{u,u'} * f^{u'} = (s_i f)^u * \partial_i^{u,u'} + \delta_{u,u'} \left[(s_i f - f) (\lambda_{i+1} - \lambda_i)^{k-1} \right]^u .$$

In the formula, k stands for the number of arrows from the vertex associated to $v_{u(i+1)}$ to the vertex associated to $v_{u(i)}$.

Proof. By Proposition 5.3.10, we only need to show, for any $g \in \mathrm{R}(T_{\mathbf{d}})$,

$$D_i^{u,u'} * f^{u'} \star g^{u'} = (s_i f)^u * D_i^{u,u'} \star g^{u'} + \delta_{u,u'} \left[(s_i f - f) \frac{e_{i+1}}{e_i} \left(1 - \frac{e_{i+1}}{e_i}\right)^{k-1} \right]^u \star g^{u'} .$$

Now we apply Theorem 5.3.8. The same argument works for equivariant cohomology theory. □

Lemma 6.2.1 explains ”what would happen when a point walk through a crossing”. For other relations, people have to guess by trial-and-error method. The convolution algebra $H^*_{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}})$ is called the **KLR-algebra**. The relations of the KLR-algebra can be found in ???, and we will only show the relations of K-theoretical version.

Warning 6.2.2. *In the following examples, $*$ is often omitted for simplicity.*

6.2.1 One point quiver

We begin with the trivial quiver, which has only one vertex and no arrows. Everything is simplified:

$$\mathbb{W}_{|\mathbf{d}|} = W_{\mathbf{d}}, \quad \mathrm{Min}(\mathbb{W}_{|\mathbf{d}|}, \mathrm{W}_{\mathbf{d}}) = \{\mathrm{Id}\}, \quad \widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q) \cong \mathcal{F}_{\mathbf{d}}, \quad \mathcal{Z}_{\mathbf{d}} \cong \mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}},$$

$$K^{G_{\mathbf{d}}}(\mathcal{F}_{\mathbf{d}}) \cong \mathbb{Z}\left[e_1^{\pm 1}, \ldots, e_{|\mathbf{d}|}^{\pm 1}\right], \qquad H^*_{G_{\mathbf{d}}}(\mathcal{F}_{\mathbf{d}}) \cong \mathbb{Q}[\lambda_1, \ldots, \lambda_{|\mathbf{d}|}].$$

In this case, $K^{G_{\mathbf{d}}}(\mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}})$ is called the ***K*-theoretic NilHecke algebra**, and $H^*_{G_{\mathbf{d}}}(\mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}})$ is called the **(cohomological) NilHecke algebra**.

The formulas in Theorem 5.3.8 and Lemma 6.2.1 are simplified: (superscripts are omitted, and functions f in four formulas lie in $K^{G_{\mathbf{d}}}(\mathcal{F}_{\mathbf{d}})$, $K^{G_{\mathbf{d}}}(\mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}})$, $H^*_{G_{\mathbf{d}}}(\mathcal{F}_{\mathbf{d}})$ and

$H^*_{G_{\mathbf{d}}}(\mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}})$, respectively)

$$
\begin{aligned}
D_i \star f &= \frac{s_i f}{1 - \frac{e_{i+1}}{e_i}} + \frac{f}{1 - \frac{e_i}{e_{i+1}}} \\
\partial_i \star f &= \frac{s_i f}{\lambda_{i+1} - \lambda_i} + \frac{f}{\lambda_i - \lambda_{i+1}} = \frac{f - s_i f}{\lambda_i - \lambda_{i+1}} \\
D_i f &= (s_i f) D_i + \frac{f - s_i f}{1 - \frac{e_i}{e_{i+1}}} \\
\partial_i f &= (s_i f) \partial_i + \frac{f - s_i f}{\lambda_i - \lambda_{i+1}}
\end{aligned}
$$

The relations for D_i are shown in Figure 6.2.

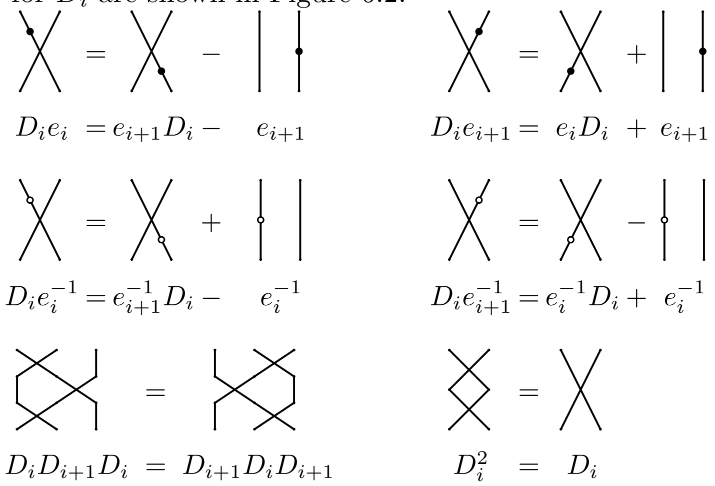


Figure 6.2

6.2.2 A_2-quiver

Now let us consider the A_2-quiver $\color{red}{\bullet} \longrightarrow \color{blue}{\bullet}$. This time we have to color the dots and strands. In this case,

$$
K^{G_{\mathbf{d}}}\left(\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)\right) \cong \bigoplus_u \mathbb{Z}\left[e_1^{u,\pm 1}, \ldots, e_{|\mathbf{d}|}^{u,\pm 1}\right], \qquad H^*_{G_{\mathbf{d}}}\left(\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)\right) \cong \bigoplus_u \mathbb{Q}\left[\lambda_1^u, \ldots, \lambda_{|\mathbf{d}|}^u\right].
$$

The formulas in Theorem 5.3.8 and Lemma 6.2.1 are simplified:

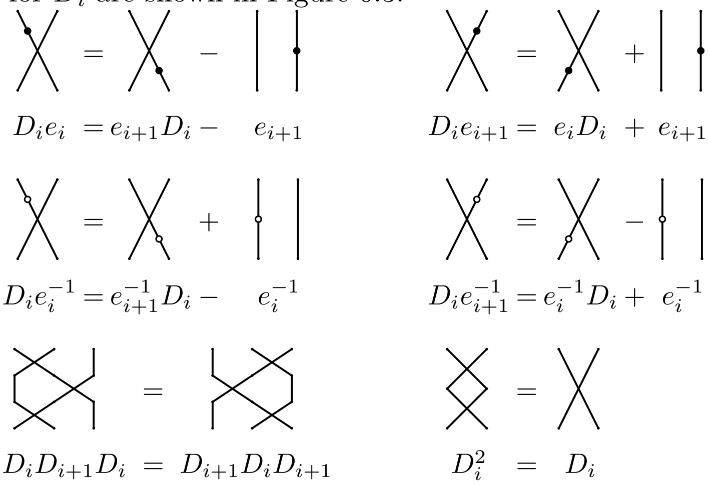
$$
D_i^{u,u'} \star f^{u'} = \begin{cases} \left[\dfrac{s_i f}{1 - \frac{e_{i+1}}{e_i}} + \dfrac{f}{1 - \frac{e_i}{e_{i+1}}}\right]^u & u = u', \\ \left[s_i f \left(1 - \dfrac{e_{i+1}}{e_i}\right)\right]^u & \color{red}{u(i+1)} \longrightarrow \color{blue}{u(i)} \ , \\ (s_i f)^u & \color{red}{u(i)} \longrightarrow \color{blue}{u(i+1)} \ . \end{cases}
$$

$$\partial_i^{u,u'} \star f^{u'} = \begin{cases} \left[\dfrac{f - s_i f}{\lambda_i - \lambda_{i+1}}\right]^u & u = u', \\ [s_i f\,(\lambda_{i+1} - \lambda_i)]^u & u(i+1) \longrightarrow u(i)\ , \\ (s_i f)^u & u(i) \longrightarrow u(i+1)\ . \end{cases}$$

$$D_i^{u,u'} f^{u'} = (s_i f)^u D_i^{u,u'} + \left[\frac{f - s_i f}{1 - \frac{e_i}{e_{i+1}}}\right]^u$$

$$\partial_i^{u,u'} f^{u'} = (s_i f)^u \partial_i^{u,u'} + \left[\frac{f - s_i f}{\lambda_i - \lambda_{i+1}}\right]^u$$

The relations for D_i are shown in Figure 6.3.



$$D_i e_i = e_{i+1} D_i - e_{i+1} \qquad D_i e_{i+1} = e_i D_i + e_{i+1}$$

$$D_i e_i^{-1} = e_{i+1}^{-1} D_i - e_i^{-1} \qquad D_i e_{i+1}^{-1} = e_i^{-1} D_i + e_i^{-1}$$

$$D_i D_{i+1} D_i = D_{i+1} D_i D_{i+1} \qquad D_i^2 = D_i$$

Figure 6.3

6.2.3 1-loop quiver

In this subsection we try to give a simplest example for Section 6.1, which is the 1-loop quiver. In this case,

$$K^{G_{\mathbf{d}}}\left(\widetilde{\operatorname{Rep}}_{\mathbf{d},\mathrm{str}}(Q)\right) \cong \mathbb{Z}\left[e_1^{\pm 1}, \ldots, e_{|\mathbf{d}|}^{\pm 1}\right], \qquad H^*_{G_{\mathbf{d}}}\left(\widetilde{\operatorname{Rep}}_{\mathbf{d},\mathrm{str}}(Q)\right) \cong \mathbb{Q}\left[\lambda_1, \ldots, \lambda_{|\mathbf{d}|}\right].$$

The formulas in Theorem 5.3.8 and Lemma 6.2.1 are simplified:

$$D_i \star f = s_i f + f \cdot \frac{e_{i+1}}{e_i}$$
$$\partial_i \star f = f - s_i f$$
$$D_i f = (s_i f) D_i + (s_i f - f)\frac{e_{i+1}}{e_i}$$
$$\partial_i f = (s_i f)\partial_i + (s_i f - f)$$

The relations for D_i are shown in Figure 6.4.

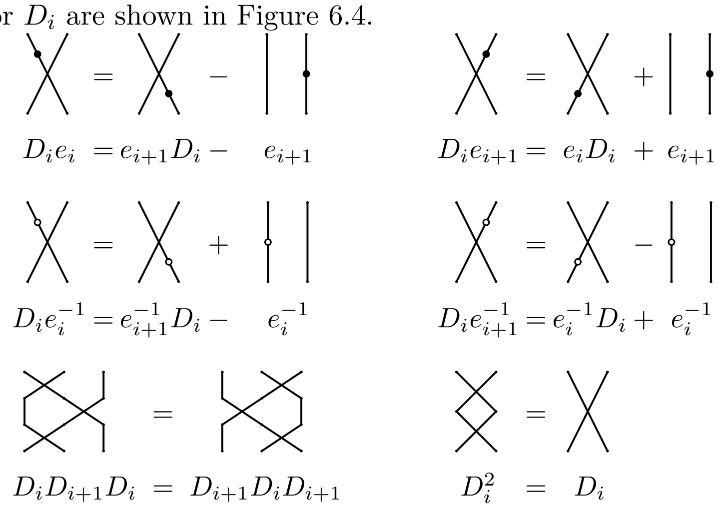


Figure 6.4

Now for the $G_{\mathbf{d}} \times \mathbb{C}^\times$-action. We have analog of Lemma 6.2.1 for $G_{\mathbf{d}} \times \mathbb{C}^\times$-action:

Lemma 6.2.3. *For* $f \in \mathrm{R}(T_{\mathbf{d}} \times \mathbb{C}^\times)$*, denote* $D_i^{u,u'} = \left[\mathcal{Z}_{s_i}^{u,u'}\right]^{G_{\mathbf{d}} \times \mathbb{C}^\times}$, $f^u = f \cdot \left[\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q)\right]^{G_{\mathbf{d}} \times \mathbb{C}^\times} \in K^{G_{\mathbf{d}} \times \mathbb{C}^\times}(\mathcal{Z}_{\mathbf{d}})$*, we have*

$$D_i^{u,u'} * f^{u'} = (s_i f)^u * D_i^{u,u'} + \delta_{u,u'} \left[(f - s_i f) \frac{\left(1 - \frac{e_{i+1}}{e_i} q\right)^k}{1 - \frac{e_i}{e_{i+1}}} \right]^u .$$

Similarly, for the $(G_{\mathbf{d}} \times \mathbb{C}^\times)$*-equivariant cohomology theory, we have*

$$\partial_i^{u,u'} * f^{u'} = (s_i f)^u * \partial_i^{u,u'} + \delta_{u,u'} \left[(f - s_i f) \frac{(\lambda_{i+1} - \lambda_i + t)^k}{\lambda_i - \lambda_{i+1}} \right]^u .$$

In the formula, k *stands for the number of arrows from the vertex associated to* $v_{u(i+1)}$ *to the vertex associated to* $v_{u(i)}$*.*

In the 1-loop quiver case, notice that

$$K^{G_{\mathbf{d}} \times \mathbb{C}^\times}\left(\widetilde{\mathrm{Rep}}_{\mathbf{d},\mathrm{str}}(Q)\right) \cong \mathbb{Z}\left[q^{\pm 1}\right]\left[e_1^{\pm 1}, \ldots, e_{|\mathbf{d}|}^{\pm 1}\right], \quad H^*_{G_{\mathbf{d}} \times \mathbb{C}^\times}\left(\widetilde{\mathrm{Rep}}_{\mathbf{d},\mathrm{str}}(Q)\right) \cong \mathbb{Q}[t]\left[\lambda_1, \ldots, \lambda_{|\mathbf{d}|}\right].$$

The formulas in Theorem 6.1.2 and Lemma 6.2.3 are simplified:

$$D_i \star f = \left(\frac{s_i f}{1 - \frac{e_{i+1}}{e_i}} + \frac{f}{1 - \frac{e_i}{e_{i+1}}} \right) \left(1 - \frac{e_{i+1}}{e_i} q \right)^k$$

$$\partial_i \star f = (f - s_i f) \frac{\lambda_{i+1} - \lambda_i + t}{\lambda_i - \lambda_{i+1}}$$

$$D_i f = (s_i f) D_i + (f - s_i f) \, \frac{1 - \frac{e_{i+1}}{e_i} q}{1 - \frac{e_i}{e_{i+1}}}$$

$$\partial_i f = (s_i f) \partial_i + (f - s_i f) \frac{\lambda_{i+1} - \lambda_i + t}{\lambda_i - \lambda_{i+1}}$$

The relations for D_i are shown in Figure 6.5.

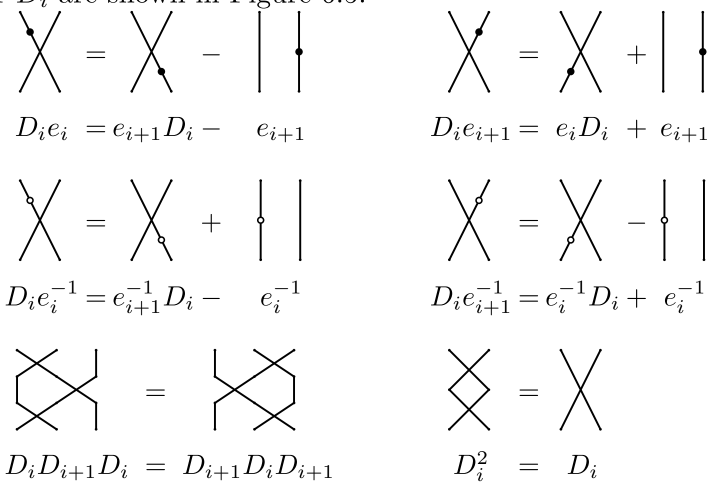


Figure 6.5

6.3 Atiyah-Segal completion theorem

Bibliography

[1] Neil Chriss and Victor Ginzburg. *Representation theory and complex geometry*, volume 42. Springer, 1997.

[2] Dan Edidin and William Graham. Localization in equivariant intersection theory and the bott residue formula. *American Journal of Mathematics*, 120(3):619–636, 1998.

[3] Tomasz Przezdziecki. Geometric approach to klr algebras and their representation theory, 2015.

[4] Robert W Thomason. Les k-groupes d'un schéma éclaté et une formule d'intersection excédentaire. *Inventiones mathematicae*, 112(1):195–215, 1993.

[5] Ravi Vakil. The rising sea: Foundations of algebraic geometry. *preprint*, 2017.

[6] Michela Varagnolo and Eric Vasserot. Canonical bases and klr-algebras. 2011.