

# Geometry of Quiver Flag Varieties

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# Introduction

Year	People	Cohomology theory	Algebra
1980	Kazhdan–Lusztig	$\mathbb{C}[W]$	$H_*^{\text{BM}}(\mathcal{Z})$
1985	Lusztig	$\mathcal{H}_q(W)$	$K^{G \times \mathbb{C}^\times}(\mathcal{Z})$
2011	Varagnolo–Vasserot	KLR algebra	$H_{G_d}^*(\mathcal{Z}_d)$

In the first part, we compute the  $G$ -equivariant  $K$ -theory of the Steinberg variety in the quiver version.

# Variety structure

$$\begin{array}{ccc} \widetilde{\text{Rep}}_{\mathbf{d}}(Q) \subseteq \text{Rep}_{\mathbf{d}}(Q) \times \mathcal{F}_{\mathbf{d}} & & \\ \mu_{\mathbf{d}} \swarrow & \searrow \pi_{\mathbf{d}} & \\ \text{Rep}_{\mathbf{d}}(Q) & & \mathcal{F}_{\mathbf{d}} \end{array}$$

$$\begin{array}{ccc} \mathcal{Z}_{\mathbf{d}} \subseteq \text{Rep}_{\mathbf{d}}(Q) \times \mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}} & & \\ \mu_{\mathbf{d},\mathbf{d}} \swarrow & \searrow \pi_{\mathbf{d},\mathbf{d}} & \\ \text{Rep}_{\mathbf{d}}(Q) & & \mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}} \end{array}$$

# Stratification structure

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# Main theorem

## Theorem

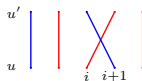
*Under the convolution product,  $K^{G_d}(\mathcal{Z}_d)$  has a  $R(T_d)$ -algebra structure. Moreover,*

- (1) As an  $R(T_d)$ -module,  $K^{G_d}(\mathcal{Z}_d)$  is free of rank  $|d|!$ ;
- (2) As an  $R(T_d)$ -algebra, we can write down generators and relations explicitly.

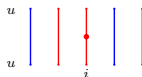
For proving (2), we mainly use the localization formula and the excess intersection formula.

# Generators

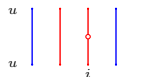
$\bullet \longrightarrow \bullet$ ,  $\mathbf{d} = (3, 2)$ ,  $\text{Min}(\mathbb{W}_{|\mathbf{d}|}, W_{\mathbf{d}}) = \{\dots\}$



$D_i^{u,u'}$

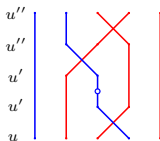


$e_i^u$



$(e_i^u)^{-1}$

A typical element in  $K^{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}})$  is a  $\mathbb{Z}$ -linear combination of diagrams shown below:



# Compositions and trivial relations

$$\begin{array}{c} u'' \\ u'' \\ u' \\ u' \\ u \end{array} \begin{array}{c} | \\ | \\ | \\ | \\ | \end{array} \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \\ \diagup \diagdown \\ \diagdown \diagup \\ \diagup \diagdown \end{array} \begin{array}{c} | \\ | \\ | \\ | \\ | \end{array} \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \\ \diagup \diagdown \\ \diagdown \diagup \\ \diagup \diagdown \end{array} \begin{array}{c} | \\ | \\ | \\ | \\ | \end{array} = \begin{array}{c} u'' \\ u'' \\ u' \\ u' \\ u \end{array} \begin{array}{c} | \\ | \\ | \\ | \\ | \end{array} \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \\ \diagup \diagdown \\ \diagdown \diagup \\ \diagup \diagdown \end{array} \begin{array}{c} | \\ | \\ | \\ | \\ | \end{array} \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \\ \diagup \diagdown \\ \diagdown \diagup \\ \diagup \diagdown \end{array} \begin{array}{c} | \\ | \\ | \\ | \\ | \end{array}$$

$$D_3^{u,u'} * (e_3^{u'})^{-1} * D_2^{u',u''} * D_3^{u'',u''} = 0$$

$$\begin{array}{c} u' \\ u \\ u' \\ u \end{array} \begin{array}{c} | \\ | \\ | \\ | \end{array} \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \\ \diagup \diagdown \\ \diagdown \diagup \end{array} \begin{array}{c} | \\ | \\ | \\ | \end{array} = 0$$

$$D_3^{u,u'} * D_3^{u,u'} = 0$$

$$\begin{array}{c} u''' \\ u' \\ u \end{array} \begin{array}{c} | \\ | \\ | \end{array} \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \\ \diagup \diagdown \end{array} \begin{array}{c} | \\ | \\ | \end{array} = \begin{array}{c} u''' \\ u'' \\ u \end{array} \begin{array}{c} | \\ | \\ | \end{array} \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \\ \diagup \diagdown \end{array} \begin{array}{c} | \\ | \\ | \end{array}$$

$$D_3^{u,u'} * D_1^{u',u''} = D_1^{u,u''} * D_3^{u'',u''}$$

$$\begin{array}{c} u' \\ u' \\ u \end{array} \begin{array}{c} | \\ | \\ | \end{array} \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \\ \diagup \diagdown \end{array} \begin{array}{c} | \\ | \\ | \end{array} = \begin{array}{c} u' \\ u \\ u \end{array} \begin{array}{c} | \\ | \\ | \end{array} \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \\ \diagup \diagdown \end{array} \begin{array}{c} | \\ | \\ | \end{array}$$

$$D_3^{u,u'} * e_2^{u'} = e_2^u * D_3^{u,u'}$$

# Nontrivial relations I

Same color:  $(D_i^{u,u} \hat{=} D_i, e_i^u \hat{=} e_i)$

$$\begin{array}{c} \text{Diagram: } \begin{array}{|c|} \hline \diagup \text{ (black dot)} \diagdown \\ \hline \end{array} = \begin{array}{|c|} \hline \diagdown \text{ (black dot)} \diagup \\ \hline \end{array} - \begin{array}{|c|} \hline | \text{ (black dot)} \\ \hline \end{array} \\ D_i e_i = e_{i+1} D_i - e_{i+1} \end{array}$$

$$\begin{array}{c} \text{Diagram: } \begin{array}{|c|} \hline \diagdown \text{ (black dot)} \diagup \\ \hline \end{array} = \begin{array}{|c|} \hline \diagup \text{ (black dot)} \diagdown \\ \hline \end{array} + \begin{array}{|c|} \hline | \text{ (black dot)} \\ \hline \end{array} \\ D_i e_{i+1} = e_i D_i + e_{i+1} \end{array}$$

$$\begin{array}{c} \text{Diagram: } \begin{array}{|c|} \hline \diagup \text{ (white dot)} \diagdown \\ \hline \end{array} = \begin{array}{|c|} \hline \diagdown \text{ (white dot)} \diagup \\ \hline \end{array} + \begin{array}{|c|} \hline | \text{ (white dot)} \\ \hline \end{array} \\ D_i e_i^{-1} = e_{i+1}^{-1} D_i + e_i^{-1} \end{array}$$

$$\begin{array}{c} \text{Diagram: } \begin{array}{|c|} \hline \diagdown \text{ (white dot)} \diagup \\ \hline \end{array} = \begin{array}{|c|} \hline \diagup \text{ (white dot)} \diagdown \\ \hline \end{array} - \begin{array}{|c|} \hline | \text{ (white dot)} \\ \hline \end{array} \\ D_i e_{i+1}^{-1} = e_i^{-1} D_i - e_i^{-1} \end{array}$$

$$\begin{array}{c} \text{Diagram: } \begin{array}{|c|} \hline \text{Cubic relation diagram} \\ \hline \end{array} = \begin{array}{|c|} \hline \text{Cubic relation diagram} \\ \hline \end{array} \\ D_i D_{i+1} D_i = D_{i+1} D_i D_{i+1} \end{array}$$

$$\begin{array}{c} \text{Diagram: } \begin{array}{|c|} \hline \text{Diamond relation diagram} \\ \hline \end{array} = \begin{array}{|c|} \hline \text{Diamond relation diagram} \\ \hline \end{array} \\ D_i^2 = D_i \end{array}$$



# Nontrivial relations II

Different color:

Diagrammatic equation: A crossing of a blue line (top-left to bottom-right) and a red line (top-right to bottom-left). The blue line has a solid dot at the crossing. This is equal to the same crossing, but the dot is on the red line. Both diagrams are labeled with a circled 3.

$$D_i^{u,u'} e_i^{u'} = e_{i+1}^u D_i^{u,u'}$$

Diagrammatic equation: A crossing of a blue line (top-left to bottom-right) and a red line (top-right to bottom-left). The blue line has a solid dot at the crossing. This is equal to the crossing with the dot on the red line, minus the crossing with the dot on the blue line. Both diagrams are labeled with a circled 2 and a circled 3.

$$D_i^{u,u'} D_i^{u',u} = 1^u - \left(\frac{e_i}{e_{i+1}}\right)^u$$

Diagrammatic equation: A crossing of a blue line (top-left to bottom-right) and a red line (top-right to bottom-left). The red line has a solid dot at the crossing. This is equal to the same crossing, but the dot is on the blue line. Both diagrams are labeled with a circled 3.

$$D_i^{u,u'} (e_i^{u'})^{-1} = (e_{i+1}^u)^{-1} D_i^{u,u'}$$

Diagrammatic equation: A crossing of a blue line (top-left to bottom-right) and a red line (top-right to bottom-left). The blue line has a solid dot at the crossing. This is equal to the crossing with the dot on the red line, minus the crossing with the dot on the blue line. Both diagrams are labeled with a circled 3 and a circled 2.

$$D_i^{u,u'} D_i^{u',u} = 1^u - \left(\frac{e_{i+1}}{e_i}\right)^u$$

# Nontrivial relations III

Different color:

$$\begin{array}{c} u \\ u' \\ u' \\ u \end{array} \begin{array}{c} \text{Diagram 1} \end{array} \begin{array}{c} \textcircled{2} \\ \textcircled{1} \\ \textcircled{3} \end{array} = \begin{array}{c} u \\ u'' \\ u'' \\ u \end{array} \begin{array}{c} \text{Diagram 2} \end{array} \begin{array}{c} \textcircled{3} \\ \textcircled{1} \\ \textcircled{2} \end{array} + \begin{array}{c} u \\ u \\ u \end{array} \begin{array}{c} \text{Diagram 3} \end{array}$$

$$D_i^{u,u'} D_{i+1}^{u',u'} D_i^{u',u} = D_{i+1}^{u,u''} D_i^{u'',u''} D_{i+1}^{u'',u} + \left( \frac{e_{i+2}}{e_{i+1}} \right)^u$$

$$\begin{array}{c} u \\ u' \\ u' \\ u \end{array} \begin{array}{c} \text{Diagram 4} \end{array} \begin{array}{c} \textcircled{3} \\ \textcircled{1} \\ \textcircled{2} \end{array} = \begin{array}{c} u \\ u'' \\ u'' \\ u \end{array} \begin{array}{c} \text{Diagram 5} \end{array} \begin{array}{c} \textcircled{2} \\ \textcircled{1} \\ \textcircled{3} \end{array} - \begin{array}{c} u \\ u \\ u \end{array} \begin{array}{c} \text{Diagram 6} \end{array}$$

$$D_i^{u,u'} D_{i+1}^{u',u'} D_i^{u',u} = D_{i+1}^{u,u''} D_i^{u'',u''} D_{i+1}^{u'',u} - \left( \frac{e_{i+1}}{e_i} \right)^u$$

$$\begin{array}{c} u'' \\ u'' \\ u' \\ u \end{array} \begin{array}{c} \text{Diagram 7} \end{array} \begin{array}{c} \textcircled{1} \\ \textcircled{2} \\ \textcircled{2} \\ \textcircled{2} \end{array} = \begin{array}{c} u'' \\ u' \\ u \\ u \end{array} \begin{array}{c} \text{Diagram 8} \end{array} \begin{array}{c} \textcircled{2} \\ \textcircled{2} \\ \textcircled{2} \\ \textcircled{1} \end{array}$$

$$D_i^{u,u'} D_{i+1}^{u',u''} D_i^{u'',u''} = D_{i+1}^{u,u} D_i^{u,u'} D_{i+1}^{u',u''}$$

# Affine pavings

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# Idea of affine pavings

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# Outlook

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