# Geometry of Quiver Flag Varieties


Xiaoxiang Zhou

Advisor: Prof. Dr. Catharina Stroppel
Second Advisor: Dr. Jens Niklas Eberhardt

Universität Bonn


January 9, 2023

# Introduction

| Year | People | Cohomology theory | Algebra |
|---|---|---|---|
| 1980 | Kazhdan–Lusztig | $\mathbb{C}[W]$ | $H_*^{\mathrm{BM}}(\mathcal{Z})$ |
| 1985 | Lusztig | $\mathcal{H}_q(W)$ | $K^{G\times\mathbb{C}^\times}(\mathcal{Z})$ |
| 2011 | Varagnolo-Vasserot | KLR algebra | $H^*_{G_{\mathbf{d}}}(\mathcal{Z}_{\mathbf{d}})$ |

In the first part, we compute the $G$-equivariant $K$-theory of the Steinberg variety in the quiver version.

# Variety structure

$$\begin{tikzcd}
\widetilde{\mathrm{Rep}}_{\mathbf{d}}(Q) \subseteq \mathrm{Rep}_{\mathbf{d}}(Q) \times \mathcal{F}_{\mathbf{d}} \arrow[d, bend left, "\mu_{\mathbf{d}}"'] \arrow[dr, bend left, "\pi_{\mathbf{d}}"] & \\
\mathrm{Rep}_{\mathbf{d}}(Q) & \mathcal{F}_{\mathbf{d}}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{Z}_{\mathbf{d}} \subseteq \mathrm{Rep}_{\mathbf{d}}(Q) \times \mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}} \arrow[d, bend left, "\mu_{\mathbf{d},\mathbf{d}}"'] \arrow[dr, bend left, "\pi_{\mathbf{d},\mathbf{d}}"] & \\
\mathrm{Rep}_{\mathbf{d}}(Q) & \mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}}
\end{tikzcd}$$

# Stratification structure

$\ldots$

# Main theorem

...

# Generators

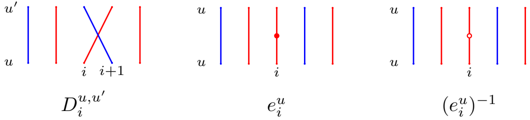
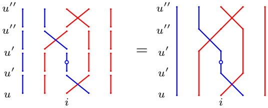
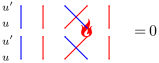



Figure: Generators

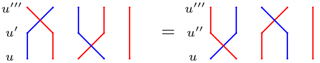
# Compositions and trivial relations



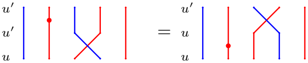

$D_3^{u,u'} * (e_3^{u'})^{-1} * D_2^{u',u''} * D_3^{u'',u''}$




$D_3^{u,u'} * D_3^{u,u'} = 0$




$D_3^{u,u'} * D_1^{u',u'''} \quad = \quad D_1^{u,u''} * D_3^{u'',u'''}$



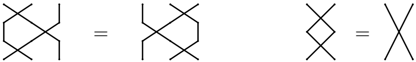

$D_3^{u,u'} * e_2^{u'} \quad = \quad e_2^{u} * D_3^{u,u'}$

# Nontrivial relations I

Same color: $(D_i^{u,u} \hat{=} D_i, e_i^u \hat{=} e_i)$

$$D_i e_i = e_{i+1} D_i - e_{i+1} \qquad D_i e_{i+1} = e_i D_i + e_{i+1}$$

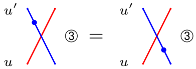
$$D_i e_i^{-1} = e_{i+1}^{-1} D_i + e_i^{-1} \qquad D_i e_{i+1}^{-1} = e_i^{-1} D_i - e_i^{-1}$$



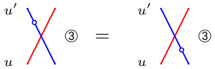
$$D_i D_{i+1} D_i = D_{i+1} D_i D_{i+1} \qquad D_i^2 = D_i$$

# Nontrivial relations II

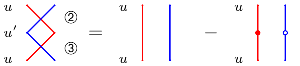
Different color:



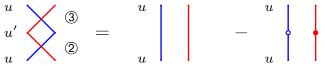

$$D_i^{u,u'} e_i^{u'} = e_{i+1}^{u} D_i^{u,u'}$$




$$D_i^{u,u'} (e_i^{u'})^{-1} = (e_{i+1}^{u})^{-1} D_i^{u,u'}$$



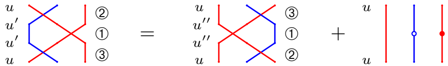

$$D_i^{u,u'} D_i^{u',u} = 1^u - \left(\frac{e_i}{e_{i+1}}\right)^u$$



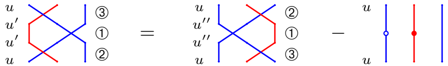

$$D_i^{u,u'} D_i^{u',u} = 1^u - \left(\frac{e_{i+1}}{e_i}\right)^u$$

# Nontrivial relations III

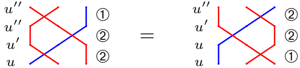
Different color:




$$D_i^{u,u'} D_{i+1}^{u',u'} D_i^{u',u} = D_{i+1}^{u,u''} D_i^{u'',u''} D_{i+1}^{u'',u} + \left(\frac{e_{i+2}}{e_{i+1}}\right)^u$$




$$D_i^{u,u'} D_{i+1}^{u',u'} D_i^{u',u} = D_{i+1}^{u,u''} D_i^{u'',u''} D_{i+1}^{u'',u} - \left(\frac{e_{i+1}}{e_i}\right)^u$$




$$D_i^{u,u'} D_{i+1}^{u',u''} D_i^{u'',u''} = D_{i+1}^{u,u} D_i^{u,u'} D_{i+1}^{u',u''}$$

# Affine pavings

...

# Idea of affine pavings

...

# Outlook

...