MODULI IN ALGEBRAIC GEOMETRY

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Abstract. In this personal survey, we conclude the definitions of moduli functors in the algebraic geometry. Most of the results are in the black box, so it's very possible that they're wrong. And also I'm not responsible for the completeness of the whole theory. I make no claim to originality. However, I'm still happy to improve this document, and make it better over time.

Contents

1. Goal and related concepts

The personal survey is motivated by the three courses in Bonn: "the moduli space of curves", "moduli of elliptic curves" and "moduli of vector bundles". I would also highly recommend the course "Moduli and GIT" in Freie Universität Berlin and the lecture note "Notes on stacks and moduli" in the University of Washington. This is also partially covered by Dinamo Djounvouna's master thesis. I want to construct my personal understanding on the moduli, and find out the details I missed in the courses.

"Some mathematicians are birds, others are frogs." This document is devoted to those "birds" from a "frog" who gets stuck in the mud.

1.1. Conventions and Notations.

In this survey, Sch_k is denoted as the category of **locally Noetherian** schemes over k, where k is a field or $\mathbb{Z}$ or $\mathbb{Z}\left[\frac{1}{6}\right]$.

1.2. Representable functor.

In this subsection we would follow on [12, Definition 2.2.1] in full generality, but you can always think

$$\mathcal{C} = \mathrm{Sch}_k \qquad \mathcal{C}' = \mathrm{Functor}(\mathrm{Sch}_k^{op}, \mathrm{Set})$$

to visualize the statement, and you can refer to [19, 6.6.2] to see basic examples.

Definition 1.1 (Functor category). *Fix a category $\mathcal{C}$, we define the corresponding functor category $\mathcal{C}'$ as follows:*

$$\mathrm{Ob}(\mathcal{C}') := \{\textit{functors } \mathcal{C}^{op} \longrightarrow \mathrm{Set}\} \qquad \mathrm{Mor}(\mathcal{C}') := \left\{ \textit{natural trans } \begin{tikzcd} \mathcal{C}^{op} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \mathrm{Set} \end{tikzcd} \right\}$$

In brief, $\mathcal{C}' = \mathrm{Functor}(\mathcal{C}^{op}, \mathrm{Set})$.[1]

Proposition 1.2. *We have a canonical functor*

$$\iota : \mathcal{C} \longrightarrow \mathcal{C}' \qquad X \longmapsto h_X := \mathrm{Mor}_{\mathcal{C}}(-, X)$$

which embeds $\mathcal{C}$ as a full subcategory of $\mathcal{C}'$.[2]

Proof. Recall that the Yoneda's lemma gives us the isomorphism

$$\mathrm{Mor}_{\mathcal{C}'}(h_X, F) \cong F(X).$$

$\square$

Warning 1. The functor $\iota : \mathcal{C} \longrightarrow \mathcal{C}'$ may not preserve colimits! For example, for $\mathcal{C} = \mathrm{Sch}_{\mathbb{Z}}$, the colimit of schemes

$$\mathrm{Spec}\, \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathrm{Spec}\, \mathbb{Z}/p^2\mathbb{Z} \longrightarrow \mathrm{Spec}\, \mathbb{Z}/p^2\mathbb{Z} \longrightarrow \cdots$$

is $\mathrm{Spec}\, \mathbb{Z}_p$ (in $\mathcal{C}$) or $\mathrm{Spf}\, \mathbb{Z}_p$ (in $\mathcal{C}'$).[3]

Definition 1.3 (Representable functor). *The functor $F \in \mathrm{Ob}(\mathcal{C}')$ is called represented by X if $F \cong h_X$.*

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[1] From here we can also write $\mathcal{C}' = \mathrm{Psh}(C)$, notice that here it is the presheaf on **category**, not on a specific scheme X/k.

[2] h_X is called the functor of points of a scheme X from here. For example, $h_X(\mathrm{Spec}\, \mathbb{Q}) = X(\mathbb{Q})$ gives us the set of rational points in X(under certain conditions). Notice that $h_{-}(-)$ is a bifunctor.

[3] By the universal property we have the canonical map $\mathrm{Spf}\, \mathbb{Z}_p \longrightarrow \mathrm{Spec}\, \mathbb{Z}_p$ in $\mathcal{C}'$.

From this proposition, we can always view the object as some functor satisfying some properties. we get three advantages from this point of view:

- It's easy to see rational points and complex points;
- We can define scheme canonically, without explicit constructions;
- We can enlarge our area of research, and think them as the defective schemes. We will see some reasonable functors which is represented not by schemes, but by stacks. Especially, we can do limits and colimits, which allows us to do quotient as well as to glue objects.

Remark 1.4. By [20, 1.3.6] or [1, B.2], any representable functor (after restricted to specific subcategory of $\mathcal{C} = \mathrm{Sch}_k$) is a sheaf in Zariski/étale/fppf/fpqc topology.

1.3. **Corepresentable functor.** This is the concept "dual to" the representable functor. To motivate, we begin with the equivalent definition of representable functor:

Definition 1.5 (Representatable functor, equivalent definition). *A functor $F \in \mathrm{Ob}(\mathcal{C}')$ is represented by $X \in \mathrm{Ob}(C)$ if F satisfies the following universal properties:*

- *There exists a morphism $\eta^{-1} : h_X \longrightarrow F$ in $\mathrm{Mor}_{\mathcal{C}'}(h_X, F)$;*
- *For any object $X' \in \mathrm{Ob}(\mathcal{C})$ and morphism $\alpha' : h_{X'} \longrightarrow F$ in $\mathrm{Mor}_{\mathcal{C}'}(h_{X'}, F)$, there exists an unique morphism $\beta \in \mathrm{Mor}_{\mathcal{C}}(X', X)$ such that $\alpha' = \eta^{-1} \circ h_\beta$.*

$$\begin{tikzcd} h_{X'} \arrow[d, dashed, "\exists ! h_\beta"'] \arrow[dr, "\alpha'"] & \\ h_X \arrow[r, "\eta^{-1}"'] & F \end{tikzcd}$$

This definition is equivalent to the Definition 1.3 since [4]

$$h_X(X') = \mathrm{Mor}_{\mathcal{C}}(X', X) \cong \mathrm{Mor}_{\mathcal{C}'}(h_{X'}, F) = F(X') \text{ for any } X' \in \mathrm{Ob}(\mathcal{C})$$

thus the functor $\eta^{-1} : h_X \longrightarrow F$ is an isomorphism.

Definition 1.6 (Corepresentable functor). *A functor $F \in \mathrm{Ob}(\mathcal{C}')$ is corepresented by $X \in \mathrm{Ob}(C)$ if F satisfies the following universal properties:*

- *There exists a morphism $\eta : F \longrightarrow h_X$ in $\mathrm{Mor}_{\mathcal{C}'}(F, h_X)$;*
- *For any object $X' \in \mathrm{Ob}(\mathcal{C})$ and morphism $\alpha' : F \longrightarrow h_{X'}$ in $\mathrm{Mor}_{\mathcal{C}'}(F, h_{X'})$, there exists an unique morphism $\beta \in \mathrm{Mor}_{\mathcal{C}}(X, X')$ such that $\alpha' = h_\beta \circ \eta$.*

$$\begin{tikzcd} F \arrow[r, "\eta"] \arrow[dr, "\alpha'"'] & h_X \arrow[d, dashed, "\exists ! h_\beta"] \\ & h_{X'} \end{tikzcd}$$

Definition 1.7 (Universal corepresentable functor). *A functor $F \in \mathrm{Ob}(\mathcal{C}')$ is universal corepresented by $X \in \mathrm{Ob}(C)$ if for any morphism $\gamma : Y \longrightarrow X$, the functor $F \times_{h_X} h_Y$ is*

[4]The middle isomorphism comes from the universal property, while the two equalities come from the Yoneda's lemma.

corepresented by Y.

$$\begin{tikzcd}
F \times_{h_X} h_Y \arrow[r] \arrow[d] & h_Y \arrow[d, "h_\gamma"] \\
F \arrow[r, "\alpha"] & h_X
\end{tikzcd}$$

Proposition 1.8. *Suppose a functor $F \in \mathrm{Ob}(\mathcal{C}')$ is corepresented by $X \in \mathrm{Ob}(C)$, then it is represented by X if and only if $\eta : F \longrightarrow h_X$ is a $\mathcal{C}'$-isomorphism.*

1.4. **Coarse moduli space.** You need to see the definition of the naive moduli problem, extended moduli problem, moduli functor, fine moduli space and corresponding universal family from here. I don't want to copy, and it's well-written.

The following example will check if you really understand these concepts.

Example 1.9 ($\mathcal{M}_{0,3}$ is represented by one point $\mathrm{Spec}\, k$)**.**
Naive moduli problem $(\mathcal{A}, \sim)$*:*

$$\mathcal{A} := \left\{ (p_1, p_2, p_3) \;\middle|\; p_i \in \mathbb{P}^1_k(k) \text{ are distinct } \right\}$$

$$(p_1, p_2, p_3) \sim (q_1, q_2, q_3) \iff \exists \gamma \in \mathrm{PGL}_2(k) \text{ s.t. } q_i = \gamma(p_i)$$

Extended moduli problem $(\mathcal{A}_S, \sim_S)$ *+ pullback:* $(S \in \mathrm{Ob}(\mathrm{Sch}_k))$

$$\mathcal{A}_S := \left\{ (X, \pi, \sigma_1, \sigma_2, \sigma_3) \;\middle|\; \begin{array}{l} X \in \mathrm{Ob}(\mathrm{Sch}_k) \\ \pi : X \longrightarrow S \text{ proper flat and } \pi^{-1}(p) \cong \mathbb{P}^1_{\kappa(p)} \\ \sigma_i : S \longrightarrow X \text{ pairwise disjoint sections of } \pi \end{array} \right\}$$

$(X, \pi, \sigma_1, \sigma_2, \sigma_3) \sim_S (X', \pi', \sigma'_1, \sigma'_2, \sigma'_3)$ *if there exists an isomorphism $f : X \longrightarrow X'$ such that the following diagrams commute:*

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "\pi"'] & & X' \arrow[dl, "\pi'"] \\
& S &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
X \arrow[rr, "f"] & & X' \\
& S \arrow[ul, "\sigma_i"] \arrow[ur, "\sigma'_i"'] &
\end{tikzcd}$$

For a map $f : T \longrightarrow S$, the pullback f^* is defined by

$$f^* : \mathcal{A}_S \longrightarrow \mathcal{A}_T \qquad (X, \pi, \sigma_1, \sigma_2, \sigma_3) \longmapsto (X_T, \pi_T, \sigma_{1,T}, \sigma_{2,T}, \sigma_{3,T})$$

where X_T, π_T are defined as fiber product, and $\sigma_{i,T}$ are defined by the universal property of fiber product, as follows:

$$\begin{tikzcd}
X_T \arrow[r] \arrow[d, "\pi_T"'] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d, "\pi"] \\
T \arrow[r, "f"] & S
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
T \arrow[dr, dashed, "\exists!\,\sigma_{i,T}"] \arrow[drr, bend left, "\sigma_i \circ f"] \arrow[ddr, bend right, "\mathrm{Id}_T"'] & & \\
& X_T \arrow[r] \arrow[d, "\pi_T"'] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d, shift right, "\pi"'] \\
& T \arrow[r, "f"] & S \arrow[u, dashed, shift right, "\sigma_i"']
\end{tikzcd}$$

Remark. We can always view the point $p \in S$ as an affine scheme of its residue field. So the two conditions below are equivalent:

$$\begin{aligned} \pi^{-1}(p) &\cong \mathbb{P}^1_{\kappa(p)} && \text{for any } p \in S \\ \pi^{-1}(\operatorname{Spec} L) &\cong \mathbb{P}^1_L && \text{for any } \operatorname{Spec} L \hookrightarrow S \end{aligned}$$

Remark. In some textbooks, they require $\pi : X \longrightarrow S$ to be a smooth, proper, surjective, locally finitely presented (l.f.p.) morphism of relative dimension $\leqslant 1$ with geometric fibres isomorphic to $\mathbb{P}^1$. These extra conditions are automatically satisfied, since

$$\begin{tikzcd}[row sep=tiny] \pi^{-1}(p) \cong \mathbb{P}^1_{\kappa(p)} \arrow[r] \arrow[dr] & \text{surjective} \\ \text{proper} \arrow[r] \arrow[dr] & \text{relative dimension} \leqslant 1 \\ \text{locally Noetherian} \arrow[r] & \text{locally finitely presented} \end{tikzcd}$$

and by [19, 25.2.2], flat + fiberwise $\mathbb{P}^1$ + l.f.p. $\Rightarrow$ smooth.

Question.
Can we find $(X, \pi, \sigma_1, \sigma_2, \sigma_3)$ satisfying all conditions in $\mathcal{A}_S$ except π is separated?
Can we find $(X, \pi, \sigma_1, \sigma_2, \sigma_3)$ satisfying all conditions in $\mathcal{A}_S$ except π is flat?

Answer. Still unknown. From Dr. Johannes Anschütz, the map

$$\mathbb{P}^1_k \longrightarrow \operatorname{Spec} k \longrightarrow \operatorname{Spec} k[t]/t^2$$

is proper but not flat(sadly we can't find any section). If the base scheme S is reduced, then maybe the flatness can be checked fiberwise?

Another obstruction for the flatness comes from [19, 26.2.11] or stackexchange.

Easy exercise.
Verify that $(\mathcal{A}_S, \sim_S)$ and pullback satisfy (i)-(iv) in [10, Definition 2.10].

Moduli functor $\mathcal{M}_{0,3}$*: The moduli functor* $\mathcal{M}_{0,3} : \mathrm{Sch}_k^{op} \longrightarrow \mathrm{Set}$ *is defined as*

$$\mathcal{M}_{0,3}(S) = \mathcal{A}_S / \sim_S \qquad \mathcal{M}_{0,3}(f : T \longrightarrow S) = f^* : \mathcal{A}_S / \sim_S \longrightarrow \mathcal{A}_T / \sim_T .$$

Question.
Why is the moduli functor $\mathcal{M}_{0,3}$ represented by $\operatorname{Spec} k$?

Answer.
Yes, but it's pretty hard to prove it. The proof of [17, Proposition 4.1] uses [17, Proposition 4.2] whose proof is quite technical.

Actually we construct the natural functor $\eta^{-1} : h_{\operatorname{Spec} k} \longrightarrow \mathcal{M}_{0,3}$ by

$$\eta_S^{-1} : h_{\operatorname{Spec} k}(S) \longrightarrow \mathcal{M}_{0,3}(S) \qquad [S \to \operatorname{Spec} k] \longmapsto (\mathbb{P}^1_S = \mathbb{P}^1_k \times_k S, pr_1, 0, 1, \infty)$$

and then verify that η_S^{-1} is an isomorphism.

Universal family*: By* [10, Definition 2.16]*, the universal family is* $(\mathbb{P}^1_k, \pi, 0, 1, \infty)$.

Remark 1.10. Whenever the fine moduli is mentioned, the following slogan is also mentioned:

The presence of nontrivial automorphisms often prevents the existence of a fine moduli space.

Some detailed explanations can be found in [ncatlab], [17, Remark 2.11] and [7, 2.A]. Usually the slogan is achieved as a proof in this way:

The presence of nontrivial automorphisms may help us to construct a nontrivial locally-trivial family, which prevents the existence of a fine moduli space.

It is, however, generally not easy to construct such a family. According to the comment here, the problem mainly comes from carefully handling the quotient in scheme category, and tricks to show the "non-trivial" keyword. Some discussions about construction have taken place in stackexchange(1257517, rigid) and mathoverflow(15884, 8812).

In concreted cases people can also achieve their goal in a different way. They may produce two non-isomorphic objects which become isomorphic after the field extension, or use descent theory to compute automorphism group(See Lec 13, p5, Application).

Question. The following problems may be not well-stated.

- Construct a scheme X with $\mathbb{Z}$-free action on X, such that the quotient space is still a scheme. (It should be used to construct $E \times X/\mathbb{Z} \longrightarrow X/\mathbb{Z}$)
- Summerize methods to prove if a family is trivial or not.
- Show that any elliptic curve over an intergral scheme S has finite automorphisms. Or give a counterexample. (Maybe solved in [3, Prop 5.3])

coarse moduli space and some related concepts.

1.5. **Bigger category.** I would follow [1]. The guide by Jarod Alper may be also useful.

We need to define Grothendieck Topology and site, discussing how this generalized the language of scheme.

We need to define stack, algebraic stack, Deligne-Munford stack and some related concepts.

Remark 1.11. When is a coarse moduli space also a fine moduli space? Take a look on mathoverflow or stackexchange.

When is an algebraic space a scheme? Take a look on mathoverflow.

1.6. **Goal.** ???

1.7. **Quotient.** We will frequently use the quotient. Here is the list of quotients we've already seen:

- (We could begin at topology: quotient by a subset) linear space-> Abelian category,group,ring(Notice that for this item, we don't take quotient by a group, so sometimes it looks easier)
- topology(This gives us an categorical quotient!)
- manifold
- scheme(Categorical quotient; GIT quotient)

See [10] for the following concepts:

- categorical quotients, good quotients, geometric quotients, and GIT quotients.
- reductive, linearly reductive, and geometrically reductive.(see [10, Theorem 4.16])
- stable, semistable, unstable, and polystable.
- Hilbert-Mumford Criterion and Fundamental Theorem in GIT.

See [4] for the following concepts:

- universal categorical quotients, universal geometric quotients.(4.5(iv), 4.12)
- fppf quotient.(4.28)

$$
\begin{tikzcd}
\text{étale quotient} \arrow[d, Rightarrow] & & \\
\text{fppf quotient} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{fpqc quotient} & \\
\begin{array}{c}\text{universal}\\ \text{geometric quotient}\end{array} \arrow[rr, Rightarrow] \arrow[dd, Rightarrow] & & \begin{array}{c}\text{universal}\\ \text{categorucal quotient}\end{array} \arrow[dd, Rightarrow] \\
 & \text{reductive GIT quotient} \arrow[d, Rightarrow, "\text{Thm 4.30, 5.3}"] \arrow[dl, dashed, "\begin{array}{c}\text{stable}\\ \text{Thm 4.36, 5.6}\end{array}"'] & \\
\text{geometric quotient} \arrow[r, Rightarrow] \arrow[uuu, dashed, bend left, "\begin{array}{c}\text{assumption in}\\ \text{[4, (4.35)(ii)]}\end{array}"] & \text{good quotient} \arrow[r, Rightarrow] & \text{categorical quotient}
\end{tikzcd}
$$

2. Basic object

In this section, we present some algebraic geometric objects which can be viewed as moduli.

Here is a picture showing the relationships of these objects:

$$\begin{tikzcd}
\mathbb{P}V \arrow[d] \arrow[dr] & & \\
\mathbb{P}\mathcal{E} \arrow[dr] & \mathrm{Gr}(r,V) \arrow[d] \arrow[dr] & \\
\mathrm{Hilb}_X \arrow[r, tail] & \mathrm{Quot}_{\mathcal{E}} & \mathrm{Flag}_{\mathrm{d}}(V)
\end{tikzcd}$$

2.1. **Projective space.** We begin with a basic extended moduli problem, and then gradually make some variations.

Example 2.1 (Moduli of line bundle with base-point-free sections is represented by $\mathbb{P}^n_k$)**.** *Fix $n \geqslant 0$, we define a moduli problem:*

$$\mathcal{A}_S := \left\{ (\mathcal{L}, s_0, \dots, s_n) \;\middle|\; \begin{array}{l} \mathcal{L} \in \mathrm{Pic}(S) \\ s_i \in \Gamma(S, \mathcal{L}) \textit{ with no common zero} \end{array} \right\}$$

$(\mathcal{L}, s_0, \dots, s_n) \sim_S (\mathcal{L}', s'_0, \dots, s'_n)$ if there exists an isomorphism of line bundles $\phi : \mathcal{L} \longrightarrow \mathcal{L}'$ such that $\phi(S) : \Gamma(S, \mathcal{L}) \longrightarrow \Gamma(S, \mathcal{L}')$ sends s_i to s'_i.

For a map $f : T \longrightarrow S$, the pullback f^* is defined by

$$f^* : \mathcal{A}_S \longrightarrow \mathcal{A}_T \qquad (\mathcal{L}, s_0, \dots, s_n) \longmapsto (f^*\mathcal{L}, f^*s_0, \dots, f^*s_n)$$

By [19, 15.3.F, 16.4.1]*, the moduli functor defined by this extended moduli problem is represented by $\mathbb{P}^n_k$.*

We also have the coordinate-free version.

Example 2.2 (Coordinate-free projective space $\mathbb{P}V^\vee = \mathrm{Proj}(\mathrm{Sym}^\bullet V)$, see [19, 4.5.12])**.** *Fix a k-vector spcace V of finite dimension. We define a moduli problem:*

$$\mathcal{A}_S := \left\{ (\mathcal{L}, \lambda) \;\middle|\; \begin{array}{l} \mathcal{L} \in \mathrm{Pic}(S) \\ \lambda : V \longrightarrow \Gamma(S, \mathcal{L}) \textit{ is base-point-free} \end{array} \right\}$$

$(\mathcal{L}, \lambda) \sim_S (\mathcal{L}', \lambda')$ if there exists an isomorphism of line bundles $\phi : \mathcal{L} \longrightarrow \mathcal{L}'$ such that $\lambda' = \phi(S) \circ \lambda$.

For a map $f : T \longrightarrow S$, the pullback f^* is defined by

$$f^* : \mathcal{A}_S \longrightarrow \mathcal{A}_T \qquad (\mathcal{L}, \lambda) \longmapsto \big(f^*\mathcal{L}, f^* \circ \lambda : V \to \Gamma(T, f^*\mathcal{L})\big)$$

By [19, 16.4.E]*, the moduli functor defined by this extended moduli problem is represented by $\mathbb{P}V^\vee$.*

Now we generalize it to the projective bundle, for this we should fix a scheme $X \in \mathrm{Ob}(\mathrm{Sch}_k)$ and a locally free coherent sheaf $\mathcal{E} \in \mathrm{Coh}(X)$, and consider the moduli problem in the category Sch_X of locally Noetherian schemes over X, rather than Sch_k.

Example 2.3 (Projective bundle[5] $\mathbb{P}\mathcal{E} = \mathrm{Proj}(\mathrm{Sym}^\bullet \mathcal{E})$, see [19, 17.2.3])**.** *For $(S, \pi_S : S \longrightarrow X) \in \mathrm{Ob}(\mathrm{Sch}_X)$, we define a moduli problem:*

$$\mathcal{A}_S := \left\{ (\mathcal{L}, \lambda) \;\middle|\; \begin{array}{l} \mathcal{L} \in \mathrm{Pic}(S) \\ \lambda : \pi_S^* \mathcal{E} \longrightarrow\!\!\!\!\!\rightarrow \mathcal{L} \end{array} \right\}$$

$(\mathcal{L}, \lambda) \sim_S (\mathcal{L}', \lambda')$ if there exists an isomorphism of line bundles $\phi : \mathcal{L} \longrightarrow \mathcal{L}'$ such that $\lambda' = \phi \circ \lambda$.

For a map $f : T \longrightarrow S$, the pullback f^* is defined by

$$f^* : \mathcal{A}_S \longrightarrow \mathcal{A}_T \qquad (\mathcal{L}, \lambda) \longmapsto \left(f^*\mathcal{L}, f^*\lambda : \pi_T^*\mathcal{E} \twoheadrightarrow f^*\mathcal{L}\right)$$

By [8, Proposition II.7.12]*, the moduli functor defined by this extended moduli problem is represented by $\mathbb{P}\mathcal{E}$ (over X).*

Finally, there is also "another" natural moduli problem of $\mathbb{P}_k^n$, see [17, Example 2.4]. For the convinience of comparison with Grassmannian, we exhibit it and make some small variations here.[6]

Example 2.4 (Moduli of lines through the origin in $\mathbb{A}_k^{n+1}$ is represented by $\mathbb{P}_k^n$)**.** *For $n \geqslant 0$, we define a moduli problem:*

$$\mathcal{A}_S := \left\{ (\mathcal{L}, \pi) \;\middle|\; \begin{array}{l} \mathcal{L} \in \mathrm{Pic}(S) \\ \pi : \mathcal{O}_S^{\oplus n+1} \longrightarrow\!\!\!\!\!\rightarrow \mathcal{L} \end{array} \right\}$$

$(\mathcal{L}, \pi) \sim_S (\mathcal{L}', \pi')$ if there exists an isomorphism of line bundles $\phi : \mathcal{L} \longrightarrow \mathcal{L}'$ such that $\pi' = \phi \circ \pi$.

For a map $f : T \longrightarrow S$, the pullback f^* is defined by

$$f^* : \mathcal{A}_S \longrightarrow \mathcal{A}_T \qquad (\mathcal{L}, \pi) \longmapsto \left(f^*\mathcal{L}, f^*\pi : \mathcal{O}_T^{\oplus n+1} \twoheadrightarrow f^*\mathcal{L}\right)$$

2.2. **Grassmannian.** It's well-written in [19, 16.7]. We just exhibit(copy) the moduli problem and make a short remark about the existence proof (prove the representability without explicite construction of the scheme)

Example 2.5 (Grassmannian $\mathrm{Gr}(k, n)$)**.** *For $n \geqslant k \geqslant 0$, we define a moduli problem:*

$$\mathcal{A}_S := \left\{ (\mathcal{F}, \pi) \;\middle|\; \begin{array}{l} \mathcal{F} \in \mathrm{Coh}(S) \text{ locally free of rank } k \\ \pi : \mathcal{O}_S^{\oplus n} \longrightarrow\!\!\!\!\!\rightarrow \mathcal{F} \end{array} \right\}$$

$(\mathcal{F}, \pi) \sim_S (\mathcal{F}', \pi')$ if there exists an isomorphism of vector bundles $\phi : \mathcal{F} \longrightarrow \mathcal{F}'$ such that $\pi' = \phi \circ \pi$.

For a map $f : T \longrightarrow S$, the pullback f^* is defined by

$$f^* : \mathcal{A}_S \longrightarrow \mathcal{A}_T \qquad (\mathcal{F}, \pi) \longmapsto \left(f^*\mathcal{F}, f^*\pi : \mathcal{O}_T^{\oplus n} \twoheadrightarrow f^*\mathcal{F}\right)$$

By [19, 16.7, page 442-443]*, the moduli functor defined by this extended moduli problem is representable, and we denote it by $\mathrm{Gr}(k, n)$.*

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[5] There is an notational abuse in [19]. From my personal point of view, it's better to replace $\mathbb{P}\mathcal{E}$ with $\mathbb{P}\mathcal{E}^\vee$ or $\mathbb{P}V^\vee$ with $\mathbb{P}V$ to make symbols consistent.

[6] The reason of the variation is already explained in [19, 16.7, page 442].

Remark 2.6. The idea of the proof comes from [19, 9.1.I]. First we prove it to be the Zariski sheaf, then we cover it with open subfunctors that are representable.

Remark 2.7. It may help to extend $(\mathcal{F}, \pi)$ in $\mathcal{A}_S$ to a short exact sequence

$$0 \longrightarrow \ker \pi \longrightarrow \mathcal{O}_S^{\oplus n} \longrightarrow \mathcal{F} \longrightarrow 0$$

and visualize it as the pullback of vector bundles over $\mathrm{Gr}(k, n)$.

From map to sheaf: In each point $[V] \in \mathrm{Gr}(k, n)$, we have a short exact sequence:

$$0 \longrightarrow V \longrightarrow k^n \longrightarrow k^n/V \longrightarrow 0.$$

Gluing fibers, one can get a short exact sequence of vector bundles over $\mathrm{Gr}(k, n)$:

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{O}_{\mathrm{Gr}(k,n)}^{\oplus n} \longrightarrow \mathcal{Q} \longrightarrow 0. \tag{2.1}$$

Here, $\mathcal{S}$ is the tautological vector bundle over $\mathrm{Gr}(k, n)$, and $\mathcal{Q}$ is the quotient vector bundle.

When $k = 1$, $\mathcal{S} = \mathcal{O}(-1)$. In this case, people working in algebraic geometry are not quite satisfied. They want $\mathcal{F}$ to be the pullback of $\mathcal{O}(1)$, and they want nice sections in $\mathcal{O}(1)$ coming from the map $\mathcal{O}^{\oplus n} \longrightarrow \mathcal{O}(1)$.

Luckily, by applying the functor $(-)^\vee = \mathrm{Hom}_{\mathcal{O}}(-, \mathcal{O})$, (2.1) is equivalent to (since all terms are vector bundles)

$$0 \longrightarrow \mathcal{Q}^\vee \longrightarrow \mathcal{O}_{\mathrm{Gr}(k,n)}^{\oplus n} \longrightarrow \mathcal{S}^\vee \longrightarrow 0. \tag{2.2}$$

Pulling back through the map $S \longrightarrow \mathrm{Gr}(k, n)$, one gets

$$0 \longrightarrow \ker \pi \longrightarrow \mathcal{O}_S^{\oplus n} \longrightarrow \mathcal{F} \longrightarrow 0.$$

By taking stalks, this short exact sequence degenerates to

$$0 \longrightarrow (k^n/V)^* \longrightarrow (k^n)^* \longrightarrow V^* \longrightarrow 0.$$

From sheaf to map: A vector bundle $\mathcal{E}$ over S of rank k with n sections provides us with a short exact sequence

$$0 \longrightarrow \ker \pi \longrightarrow \mathcal{O}_S^{\oplus n} \xrightarrow{\pi} \mathcal{E} \longrightarrow 0$$

whose geometry looks like

$$0 \longrightarrow E \longrightarrow \underline{k}^n \xrightarrow{\pi} \underline{k}^n/E \longrightarrow 0.$$

Example 2.8. *For a proper smooth curve $\mathcal{C}$ of genus $g > 1$, the Gaussian map*

$$\phi : \mathcal{C} \longrightarrow \mathbb{P}^{g-1} \qquad p \longmapsto [T_p\mathcal{C}]$$

is not induced by the sheaf $\mathcal{T}_{\mathcal{C}}$, but by $\omega_{\mathcal{C}} = \mathcal{T}_{\mathcal{C}}^\vee$. One can check that $c_1(\omega_{\mathcal{C}}) = \phi^* c_1(\mathcal{O}(1))$.

2.3. **Flag variety, partial flag variety.**

2.4. **Quot scheme.** We refer to wiki and Venkatesh's lecture notes.

The Quot scheme is a relative version of the Grassmannian. Instead of parameterizing subvector spaces, it parameterizes quotient sheaves of a fixed sheaf.

Example 2.9 (Quot scheme $\mathrm{Quot}_{\mathcal{E}}$)**.** *In this example, the field κ can be generalized to a Noetherian base scheme S_0.*

For a scheme X/κ of finite type and $\mathcal{E} \in \mathrm{Coh}(X)$, we define a moduli problem for $(S, \pi_S : S \longrightarrow \operatorname{Spec}\kappa) \in \mathrm{Ob}(\mathrm{Sch}_\kappa)$:

$$\mathcal{A}_S := \left\{ (\mathcal{F}, \pi) \;\middle|\; \begin{array}{l} \mathcal{F} \in \mathrm{Coh}(X_S) \text{ flat over } S \\ \mathrm{supp}(\mathcal{F}) \text{ proper over } S \\ \pi : \mathcal{E}_S \twoheadrightarrow \mathcal{F} \end{array} \right\}$$

$(\mathcal{F}, \pi) \sim_S (\mathcal{F}', \pi')$ if there exists an isomorphism of vector bundles $\phi : \mathcal{F} \longrightarrow \mathcal{F}'$ such that $\pi' = \phi \circ \pi$.

For a map $f : T \longrightarrow S$, the pullback f^* is defined by

$$f^* : \mathcal{A}_S \longrightarrow \mathcal{A}_T \qquad (\mathcal{F}, \pi) \longmapsto \big((\mathrm{Id} \times f)^*\mathcal{F}, f^*\pi : \mathcal{E}_T \twoheadrightarrow (\mathrm{Id} \times f)^*\mathcal{F}\big)$$

The moduli functor defined by this extended moduli problem is representable, and we denote it by $\mathrm{Quot}_{\mathcal{E}}$.

Remark 2.10. The requirements of flatness and properness ensure the well-definedness of the Hilbert polynomial $P_{\mathcal{F}}(t) \in \mathbb{Q}[t]$, defines as follows.

When a line bundle $\mathcal{L}$ over X is fixed, take a closed point $s \in S$, then

$$P_{\mathcal{F}}(m) = \chi(\mathcal{F}_s \otimes \mathcal{L}_s^{\otimes m}) = \sum_{i=0}^{\dim \mathcal{F}} (-1)^i h^i(X_s, \mathcal{F}_s \otimes \mathcal{L}_s^{\otimes m}).$$

By flatness, the Hilbert polynomial $P_{\mathcal{F}}(m)$ does not depend on the choice of $s \in S$. Furthermore, when X is projective, one can take $\mathcal{L} = \mathcal{O}_X(1)$. This provides us a decomposition of $\mathrm{Quot}_{\mathcal{E}}$:

$$\mathrm{Quot}_{\mathcal{E}} = \bigsqcup_{P \in \mathbb{Q}[t]} \mathrm{Quot}_{\mathcal{E}}^P .$$

Example 2.11. *When $X = \operatorname{Spec}\kappa$, $\mathcal{E} = \kappa^n$, one gets*

$$\mathcal{A}_S = \left\{ (\mathcal{F}, \pi) \;\middle|\; \begin{array}{l} \mathcal{F} \in \mathrm{Coh}(S) \text{ flat over } S \\ \pi : \mathcal{O}_S^{\oplus n} \twoheadrightarrow \mathcal{F} \end{array} \right\}$$

and $P_{\mathcal{F}}(t) = \chi(\mathcal{F}_s)$ is constant, indicating the rank of $\mathcal{F}$. Therefore,

$$\mathrm{Gr}(k, n) = \mathrm{Quot}_{\kappa^n}^k, \qquad \mathbb{P}V^\vee = \mathrm{Quot}_V^1 .$$

Example 2.12. *In this example, the base scheme S_0 is X, and $\mathcal{E} \in \mathrm{Coh}(X)$ is locally free. In this setting, the moduli problem for $(S, \pi_S : S \longrightarrow X) \in \mathrm{Ob}(\mathrm{Sch}_X)$ is given by*

$$\mathcal{A}_S = \left\{ (\mathcal{F}, \pi) \;\middle|\; \begin{array}{l} \mathcal{F} \in \mathrm{Coh}(S) \text{ flat over } S \\ \pi : \pi_S^*\mathcal{E} \twoheadrightarrow \mathcal{F} \end{array} \right\}$$

Since any finitely generated flat module over a commutative local, Noetherian ring is free (see SE1812584), $\mathcal{F}$ is locally free of rank r, and $P_{\mathcal{F}}(t) = r$. Therefore,

$$\mathbb{P}\mathcal{E} = \mathrm{Quot}^1_{\mathcal{E}} \ \textit{(over } X/X\textit{)}.$$

Remark 2.13. The tangent space and smoothness of $\mathrm{Quot}_{\mathcal{E}}$ can be described locally, just like $T_V \,\mathrm{Gr}(k,n) = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C}^n/V)$, see [11, II.1.3] for more details.

2.5. **Hilbert scheme.** Since we already defined $\mathrm{Quot}_{\mathcal{E}}$, the Hilbert scheme is just a special case:

$$\mathrm{Hilb}_X := \mathrm{Quot}_{\mathcal{O}_X} \qquad \mathrm{Hilb}^P_X := \mathrm{Quot}^P_{\mathcal{O}_X}$$

where the moduli problem is given by

$$\mathcal{A}_S = \left\{ (\mathcal{F}, \pi) \;\middle|\; \begin{array}{l} \mathcal{F} \in \mathrm{Coh}(X_S) \text{ flat over } S \\ \mathrm{supp}(\mathcal{F}) \text{ proper over } S \\ \pi : \mathcal{O}_{X_S} \twoheadrightarrow \mathcal{F} \end{array} \right\}$$

In this case, $\mathcal{F} \cong i_*\mathcal{O}_Z$ for some subvariety (maybe non-reduced) of X_S (of Hilbert polynomial P).

Example 2.14. *The Hilbert scheme of k points on X is denoted by*

$$X^{[k]} := \mathrm{Hilb}^k_X = \mathrm{Quot}^k_{\mathcal{O}_X}.$$

It is birational equivalent to X^k/S_k.

Example 2.15. *Recall that $P_{\mathbb{P}^m}(t) = \binom{m+t}{m}$. For any closed integral subvariety $X \subset \mathbb{P}^n$, the Fano variety of m-planes $\mathbb{P}^m$ in X is given by*

$$F(X,m) := \mathrm{Hilb}^{P_{\mathbb{P}^m}}_X = \mathrm{Quot}^{P_{\mathbb{P}^m}}_{\mathcal{O}_X}.$$

Example 2.16. *Recall that $P_Z(t) = \binom{n+t}{n} - \binom{n+t-d}{n}$ for a degree d hypersurface Z in $\mathbb{P}^n$. The moduli space of degree d hypersurfaces in $\mathbb{P}^n$ is given by*

$$\mathrm{Hilb}^{P_Z}_{\mathbb{P}^n} = \mathrm{Quot}^{P_Z}_{\mathcal{O}_{\mathbb{P}^n}} \cong \mathbb{P}\left(\Gamma\left(\mathbb{P}^n, \mathcal{O}(d)\right)\right) \cong \mathbb{P}^{\binom{n+d}{d}-1}.$$

2.6. **Misc.** The representable functor is also used to construct the fibered product of schemes, see [19, 9.1.6-7] for more details.

We've already met the idea of moduli in plane geometry. For example, fix two points A, B on the plane, we wanted to find the set of points C such that $\triangle ABC$ is a right triangle (resp. an isosceles triangle). We used the term "locus" in junior high school.

There are still quite a lot of interesting examples of moduli spaces not presented here, see examples in [1, 0.1.1, 0.2].

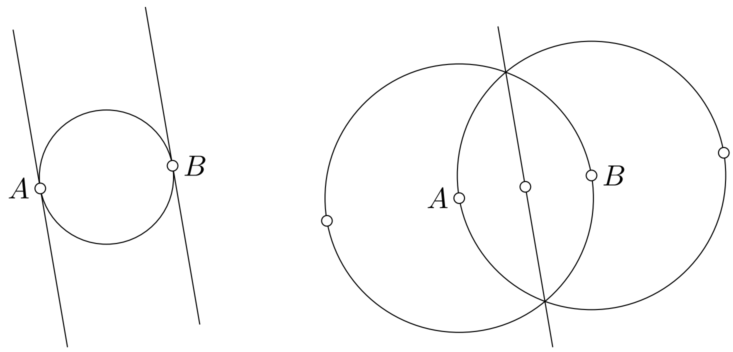

Figure 1. "moduli space" in Euclidean geometry

3. Moduli of curve

The content of this section is already well written in the course "the moduli space of curves". This section is just for the completeness of the survey.

Maybe [7] would be another good reference for this section, especially for the readers concerning about arithmetic. I haven't read about it.

So now comes the conclusion.

3.1. Initial definition.

Definition 3.1 (smooth/stable curve, follow [17, Definition 3.15])**.** *For $g, n \geqslant 0$, a smooth curve of genus g over $S \in \mathrm{Sch}_k$ with n-points is an element in the set*

$$\mathcal{A}_S := \left\{ (X, \pi, \sigma_i) \;\middle|\; \begin{array}{l} X \in \mathrm{Ob}(\mathrm{Sch}_k) \\ \pi : X \longrightarrow S \textit{ proper flat and} \\ \qquad \pi^{-1}(\mathrm{Spec}\, L) \textit{ is a sm proj connected curve of genus } g \\ \sigma_i : S \longrightarrow X \textit{ pairwise disjoint sections of } \pi \end{array} \right\}$$

and a stable curve of genus g over $S \in \mathrm{Sch}_k$ with n-points is an element in the set

$$\overline{\mathcal{A}}_S := \left\{ (X, \pi, \sigma_i) \;\middle|\; \begin{array}{l} X \in \mathrm{Ob}(\mathrm{Sch}_k) \\ \pi : X \longrightarrow S \textit{ proper flat (maybe not smooth)} \\ \qquad \pi^{-1}(\mathrm{Spec}\, L) \textit{ is a stable proj connected curve of genus } g \\ \sigma_i : S \longrightarrow X \textit{ pairwise disjoint sections of } \pi \\ \qquad\qquad \textit{with image in the smooth locus of } \pi \end{array} \right\}$$

Definition 3.2 (moduli of smooth/stable curves)**.** *For $g, n \geqslant 0$, the moduli of smooth curves $\mathcal{M}_{g,n}$ and the moduli of smooth curves $\overline{\mathcal{M}}_{g,n}$ are defined as*

$$\mathcal{M}_{g,n}(S) = \mathcal{A}_S / \sim_S \qquad \mathcal{M}_{g,n}(f : T \longrightarrow S) = f^* : \mathcal{A}_S / \sim_S \longrightarrow \mathcal{A}_T / \sim_T .$$

$$\overline{\mathcal{M}}_{g,n}(S) = \overline{\mathcal{A}}_S / \sim_S \qquad \overline{\mathcal{M}}_{g,n}(f : T \longrightarrow S) = f^* : \overline{\mathcal{A}}_S / \sim_S \longrightarrow \overline{\mathcal{A}}_T / \sim_T .$$

where the equivalent relation $\sim_S$ and the pullback are similar in the Example 1.9, i.e. $(X, \pi, \sigma_i) \sim_S (X', \pi', \sigma_i')$ if there exists an isomorphism $f : X \longrightarrow X'$ such that the following diagrams commute:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\pi"'] & & X' \arrow[dl, "\pi'"] \\ & S & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[rr, "f"] & & X' \\ & S \arrow[ul, "\sigma_i"] \arrow[ur, "\sigma_i'"'] & \end{tikzcd}$$

For a map $f : T \longrightarrow S$, the pullback f^* is defined by

$$f^* : \mathcal{A}_S \longrightarrow \mathcal{A}_T \qquad (X, \pi, \sigma_i) \longmapsto (X_T, \pi_T, \sigma_{i,T})$$

where X_T, π_T are defined as fiber product, and $\sigma_{i,T}$ are defined by the universal property of fiber product, as follows:

$$\begin{tikzcd} X_T \arrow[r] \arrow[d, "\pi_T"'] \arrow[dr, phantom, "\ulcorner"] & X \arrow[d, "\pi"] \\ T \arrow[r, "f"] & S \end{tikzcd} \qquad \begin{tikzcd} T \arrow[dr, dashed, red, "\exists!\,\sigma_{i,T}"] \arrow[drr, bend left, "\sigma_i\circ f"] \arrow[ddr, bend right, "\mathrm{Id}_T"'] & & \\ & X_T \arrow[r] \arrow[d, "\pi_T"'] \arrow[dr, phantom, "\ulcorner"] & X \arrow[d, "\pi"'] \\ & T \arrow[r, "f"] & S \arrow[u, dashed, bend right, "\sigma_i"'] \end{tikzcd}$$

3.2. **Result.** Here is the result coming from [17] which we really care:

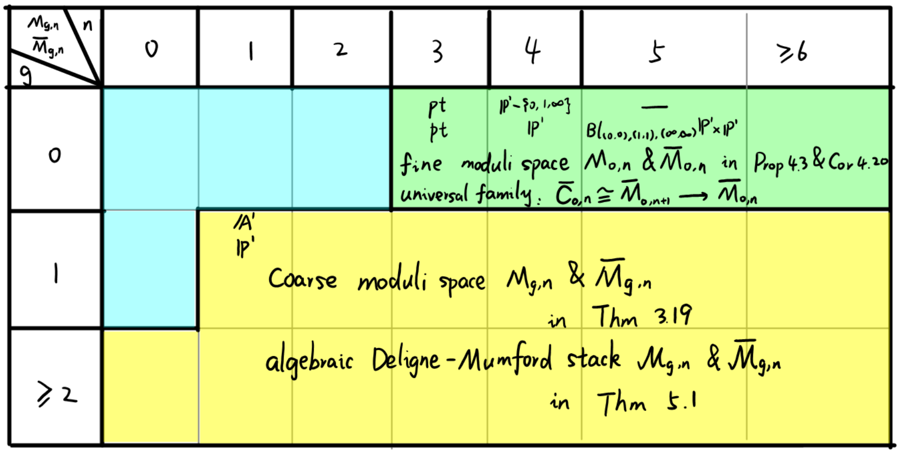

FIGURE 2. The moduli of curves

If you're interested on the automorphism of these moduli spaces, then you can check [13]. If you want to know the Euler characteristic of $\mathcal{M}_{g,n}$, you should check [6] for the Harer-Zagier formula.

4. Moduli of elliptic curve

The elliptic curve theory is especially rich compared to the other curves. That's why we'd like to put it a special section.

4.1. **Definition and basic properties of elliptic curve.**

Definition 4.1 (Relative elliptic curve)**.** *Let $S \in \mathrm{Ob}(\mathrm{Sch}_k)$. An S-group scheme (E, m) is called an elliptic curve over S if the structure map $\pi : E \longrightarrow S$ is proper, smooth of relative dimension 1, and has connected fibers. Two elliptic curves are considered as the same if they are isomorphic as the S-group scheme.*

This data is equivalent to what we described for $\mathcal{M}_{1,1}(S)$ in Definition 3.2, and also equivalent to the definition in [18, Tag 072J].

Recall the Cohomology and Base Change Theorem in [19, 28.1.6]. Here we just give a simplified version:

Theorem 4.2 (Cohomology and Base Change Theorem, simplified version)**.**

Suppose

- *S is locally Noetherian;*
- *$\pi : X \longrightarrow S$ is proper;*
- *The coherent sheaf $\mathcal{F}$ over X is flat over S.*

$$\begin{tikzcd} & & \mathcal{F} \arrow[dl, dash] \\ X_q \arrow[r] \arrow[d, "\pi_q"'] & X \arrow[d, "\pi"] & \\ \operatorname{Spec} \kappa(q) \arrow[r, "\iota"] & S & \end{tikzcd}$$

then for any $q \in S$ and $p \in \mathbb{Z}$, we have the natural map

$$\phi_q^p : (\mathrm{R}^p\pi_*\mathcal{F})|_q \longrightarrow H^p(X_q, \mathcal{F}|_{X_q})$$

which has the following properties:

$$\phi_q^p \text{ is surj} \implies \begin{cases} \phi_q^p \text{ is iso} \\ \left(\phi_q^{p-1} \text{ is surj } \Leftrightarrow \mathrm{R}^p\pi_*\mathcal{F} \text{ is l.f.}\right) \end{cases}$$

We can use Theorem 4.2 to compute some pushforward of sheaves, and the results are shown here:

n	-3	-2	-1	0	1	2	3
$\mathrm{R}^1\pi_*\mathcal{O}_E(ne)$	rank 3	rank 2	l.b.	l.b.	0	0	0
$\pi_*\mathcal{O}_E(ne)$	0	0	0	$\mathcal{O}_S$	l.b.	rank 2	rank 3
$\mathrm{R}^1\pi_*\Omega_{E/S}(ne)$	rank 3	rank 2	l.b.	$\mathcal{O}_S$	0	0	0
$\pi_*\Omega_{E/S}(ne)$	0	0	0	l.b.	l.b.	rank 2	rank 3

Table 1. results of higher pushforward

Example 4.3 (case of $\mathrm{R}^p\pi_*\mathcal{O}_E$)**.** *Let $X = E$ be an elliptic curve over S, $\mathcal{F} = \mathcal{O}_X$. Obviously ϕ_q^2, ϕ_q^{-1} are surjective; the map ϕ_q^0 is also surjective[7]. By using Theorem 4.2 (see the Figure 3) we obtain that*

$$\mathrm{R}^p\pi_*\mathcal{O}_X = \begin{cases} 0 & p \geqslant 2 \\ \textit{line bundle} & p = 0, 1 \end{cases}$$

[7] see the hint in [19, 28.1.H].

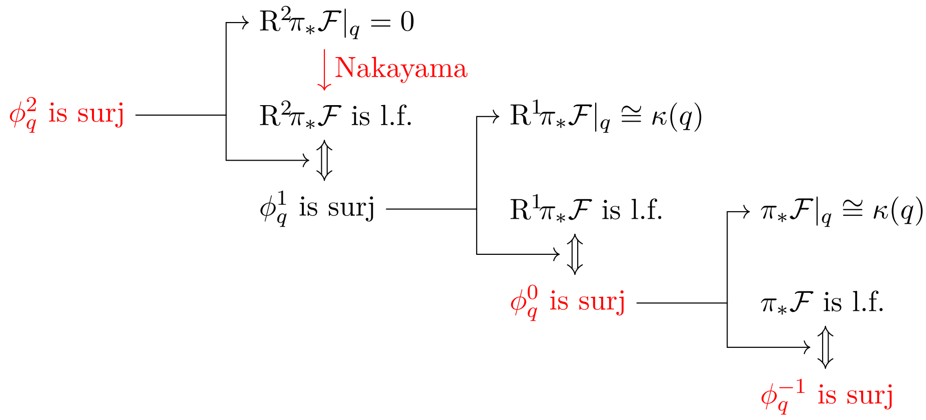

FIGURE 3. the process; red color is the initial condition

Lemma 4.4. $\pi_*\mathcal{O}_X \cong \mathcal{O}_S$.

Proof. The morphism $\pi : X \longrightarrow S$ induces the map of sheaves

$$\pi^\# : \mathcal{O}_S \longrightarrow \pi_*\mathcal{O}_X$$

which corresponds a section of $\pi_*\mathcal{O}_X$. Since $\pi_*\mathcal{O}_X$ is a line bundle, this section defines a Cartier divisor D of S, i.e. $\pi_*\mathcal{O}_X \cong \mathcal{O}_S(D)$. From the isomorphism

$$\pi^\#|_q : \kappa(q) \longrightarrow \pi_*\mathcal{O}_X \times_{\mathcal{O}_S} \kappa(q) \xrightarrow{\phi_q^0} H^0(X_q, \mathcal{O}_{X_q}) \cong \kappa(q)$$

we get $D = 0$, thus $\pi_*\mathcal{O}_X \cong \mathcal{O}_S$. $\square$

Remark 4.5. The other cases are similarly solved except $\mathcal{F} = \Omega_{E/S}$. Actually, if we admit the Grothendieck-Serre duality

$$R^p\pi_*\Omega_{E/S} \cong (R^{1-p}\pi_*\mathcal{O}_E)^\vee$$

this would be easily solved. You can see the discussion here for Exercise [19, 28.1.N], and [9, 2.1.2] for the "proof" of Grothendieck-Serre duality. For me the question in stackexchange is still unsolved, and for solving Exercise [19, 28.1.N] one need to assume the base scheme is reduced to use the Grauert's Theorem in [19, 28.1.5].(The Figure 4 shows where we use the Grauert's Theorem.)

Method	Result	Requirement
Grothendieck-Serre duality	$R^1\pi_*\mathcal{F} \cong \mathcal{O}_E$	haven't checked
Grauert's theorem	$R^1\pi_*\mathcal{F}$ is l.f.	S is reduced
$\Omega_{E/S} \cong \mathcal{O}_E$ locally	$R^1\pi_*\mathcal{F}$ is l.f.	E/S is group scheme

TABLE 2.

Remark 4.6. In the case that $X = E$ is an elliptic curve over S, we know more about the Hodge bundle $\pi_*\Omega_{E/S}$ than just a line bundle:

- $(\pi_*\Omega_{E/S})^E = \pi_*\Omega_{E/S}$ since any global differential on an Abelian variety is invariant;

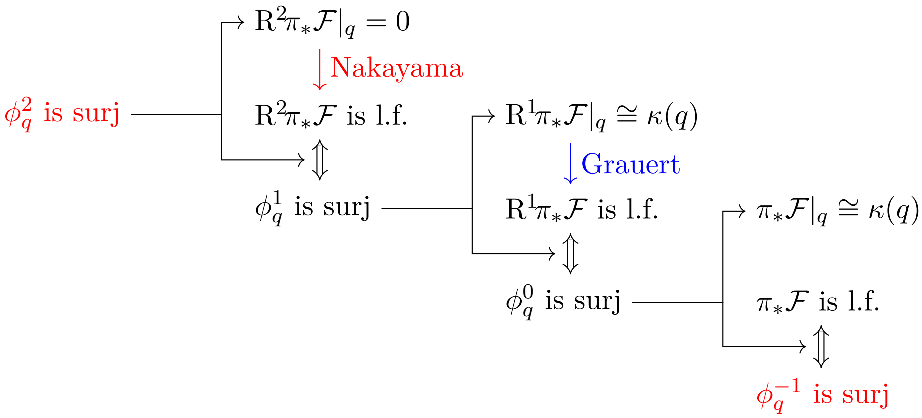

FIGURE 4. $\mathcal{F} = \Omega_{E/S}$; red color is the initial condition

- By [16, Proposition 3.15](wrong reference), the line bundle $\omega_{E/S} := e^*\Omega_{E/S}$ is isomorphic to the Hodge bundle $\pi_*\Omega_{E/S}$. By the conormal exact sequence, $\omega_{E/S}$ is also isomorphic to $\mathcal{I}/\mathcal{I}^2 := e^*\mathcal{I}$ where $\mathcal{I} := \mathcal{O}_E(-e)$ corresponds to the Cartier divisor cut by e. So

$$\pi_*\Omega_{E/S} \quad\cong\quad \omega_{E/S} := e^*\Omega_{E/S} \quad\cong\quad \mathcal{I}/\mathcal{I}^2 := e^*\mathcal{I}.$$

As a corollary, we get

$$\Omega_{E/S} \overset{\text{gp sch}}{\cong} \pi^*e^*\Omega_{E/S} = \pi^*\omega_{E/S} \cong \pi^*\pi_*\Omega_{E/S}.$$

and

$$\begin{aligned} e^*\big(\mathcal{O}_E(ne)\big) &\cong e^*\big(\mathcal{I}^{\otimes(-n)}\big) \cong (e^*\mathcal{I})^{\otimes(-n)} \cong \omega_{E/S}^{\otimes(-n)} \\ e^*\big(\Omega_{E/S}(ne)\big) &\cong e^*\Omega_{E/S} \otimes e^*\big(\mathcal{O}_E(ne)\big) \cong \omega_{E/S}^{\otimes(-n+1)} \end{aligned} \tag{4.1}$$

4.2. **Differential.** We know that $\mathcal{M}_{1,1}$ is not representable, so we have to introduce extra structures to rigidify elliptic curves. The first possible extra structure is the differential.

Definition 4.7 (Weierstrass moduli $\widetilde{\mathcal{M}}[\frac{1}{6}]$). *For a base scheme $S \in \mathrm{Ob}(\mathrm{Sch}_{\mathbb{Z}[\frac{1}{6}]})$, we define a moduli problem*

$$\mathcal{A}_S := \left\{ (E,\omega) \;\middle|\; \begin{array}{l} E:\ \textit{elliptic curve over } S \\ \omega \in \Gamma(E,\Omega_{E/S}) \textit{ global generator} \end{array} \right\}$$

$(E,\omega) \sim_S (E',\omega')$ if there exists an isomorphism of elliptic curves $\phi : E \longrightarrow E'$ over S such that $\omega = \phi^*(\omega')$, where [8]

$$\phi^* : \Gamma(E',\Omega_{E'/S}) \longrightarrow \Gamma(E',\phi_*\Omega_{E/S}) \cong \Gamma(E,\Omega_{E/S})$$

is got by pulling back 1-forms from E' to E.

For a map $f : T \longrightarrow S$, the pullback f^* is defined by

$$f^* : \mathcal{A}_S \longrightarrow \mathcal{A}_T \qquad (E,\omega) \longmapsto \big(f^*E, f^*\omega\big)$$

[8] See [19, 21.2.27] and [19, 18.2.E] for the construction of ϕ^*.

By doing so we define the moduli functor

$$\widetilde{\mathcal{M}}\left[\tfrac{1}{6}\right] : \mathrm{Sch}_{\mathbb{Z}[\frac{1}{6}]} \longrightarrow \mathrm{Set} \qquad S \longmapsto \mathcal{A}_S/\sim_S$$

Theorem 4.8 (Originally in [3, Prop 2.5])**.** *The moduli functor $\widetilde{\mathcal{M}}\left[\frac{1}{6}\right]$ is represented by* $\operatorname{Spec}\mathbb{Z}\left[\frac{1}{6}\right]\left[a,b,\Delta^{-1}\right]$ *where* $\Delta = -16(4a^3+27b^2)$*. Denote* $R = \mathbb{Z}\left[\frac{1}{6}\right]\left[a,b,\Delta^{-1}\right]$*, the universal family is* $(E_R, \omega_R) \in \widetilde{\mathcal{M}}\left[\frac{1}{6}\right](R)$*, where*

$$\begin{aligned} E_R &= \operatorname{Proj} R[x,y,z]/\left(y^2z-(x^3+axz^2+bz^3)\right) \\ \omega_R &= \frac{x\mathrm{d}z - z\mathrm{d}x}{2yz} = \frac{y\mathrm{d}z - z\mathrm{d}y}{3x^2+az^2} = \frac{x\mathrm{d}y - y\mathrm{d}x}{y^2-2axz-3bz^2} \qquad \textit{whenever it's defined.} \end{aligned}$$

Proof. Fix a scheme $S \in \mathrm{Ob}(\mathrm{Sch}_{\mathbb{Z}[\frac{1}{6}]})$, we need to construct an isomorphism

$$\Psi : \widetilde{\mathcal{M}}\left[\tfrac{1}{6}\right](S) \xrightarrow{\sim} \{a, b \in \mathcal{O}_S(S) | \Delta \text{ is invertible}\}$$

Let $(E,\omega) \in \widetilde{\mathcal{M}}\left[\frac{1}{6}\right](S)$ in the following steps.

Step1. We give an description of $\pi_*\mathcal{O}_E(ne)$ by the following lemma:

Lemma 4.9. *We have a canonical filstration for $\pi_*\mathcal{O}_E(ne)$:*

$$0 \overset{\mathcal{O}_S}{\hookrightarrow} \pi_*\mathcal{O}_E = \mathcal{O}_E(e) \overset{\omega_{E/S}^{\otimes(-2)}}{\hookrightarrow} \pi_*\mathcal{O}_E(2e) \overset{\omega_{E/S}^{\otimes(-3)}}{\hookrightarrow} \pi_*\mathcal{O}_E(3e) \hookrightarrow \cdots$$

Proof of Lemma 4.9. We begin with the short exact sequence

$$0 \longrightarrow \mathcal{O}_E(-e) \longrightarrow \mathcal{O}_E \longrightarrow e_*\mathcal{O}_S \longrightarrow 0$$

apply the functor $-\otimes_{\mathcal{O}_E} \mathcal{O}_E(ne)$, since

$$e_*\mathcal{O}_S \otimes_{\mathcal{O}_E} \mathcal{O}_E(ne) \cong e_*\Big(\mathcal{O}_S \otimes_{\mathcal{O}_S} e^*\big(\mathcal{O}_E(ne)\big)\Big) \cong e_*e^*\big(\mathcal{O}_E(ne)\big) \overset{(4.1)}{\cong} e_*\omega_{E/S}^{\otimes(-n)}$$

we get short exact sequence

$$0 \longrightarrow \mathcal{O}_E\big((n-1)e\big) \longrightarrow \mathcal{O}_E(ne) \longrightarrow e_*\omega_{E/S}^{\otimes(-n)} \longrightarrow 0$$

apply π_*, we get long exact sequence

$$0 \longrightarrow \pi_*\mathcal{O}_E\big((n-1)e\big) \longrightarrow \pi_*\mathcal{O}_E(ne) \longrightarrow \omega_{E/S}^{\otimes(-n)} \longrightarrow \mathrm{R}^1\pi_*\mathcal{O}_E\big((n-1)e\big) \longrightarrow \mathrm{R}^1\pi_*\mathcal{O}_E(ne)$$

thus we get the result. □

Step2. From the filstration we construct the Weierstrass equation locally.

For any point $p \in S$, choose small enough open affine neighbourhood $U \subseteq S$ such that all the sheaf considered here are all free when restricted to U. Then from Lemma 4.9 we get split exact sequences of $\mathcal{O}_S(U)$-modules $(k \geqslant 3)$

$$\begin{tikzcd}
0 \arrow[r] & \Gamma(U,\mathcal{O}_S) \arrow[r] & \Gamma\big(U,\pi_*\mathcal{O}_E(2e)\big) \arrow[r, "p_2"] & \Gamma(U,\omega_{E/S}^{\otimes(-2)}) \arrow[r, "0"] \arrow[l, dashed, bend left, "l_2"] & H^1(U,\mathcal{O}_S)
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & \Gamma\Big(U,\pi_*\mathcal{O}_E\big((k-1)e\big)\Big) \arrow[r] & \Gamma\big(U,\pi_*\mathcal{O}_E(ke)\big) \arrow[r, "p_k"] & \Gamma(U,\omega_{E/S}^{\otimes(-k)}) \arrow[r, "0"] \arrow[l, dashed, bend left, "l_k"] & H^1\Big(U,\pi_*\mathcal{O}_E\big((k-1)e\big)\Big)
\end{tikzcd}$$

Since

$$\Gamma(U,\omega_{E/S}^{\otimes}) \cong \Gamma(U,\pi_*\Omega_{E/S}) \overset{\text{def}}{=} \Gamma(\pi^{-1}(U),\Omega_{E/S})$$

is generated by $\omega|_{\pi^{-1}(U)}$ as a free $\mathcal{O}_E(\pi^{-1}(U)) = \mathcal{O}_S(U)$-module, we can choose

$$x_0 = l_2(\omega^{-2}|_{\pi^{-1}(U)}) \in \Gamma(U,\pi_*\mathcal{O}_E(2e)) = \Gamma(\pi^{-1}(U),\mathcal{O}_E(2e))$$
$$y_0 = l_3(\omega^{-3}|_{\pi^{-1}(U)}) \in \Gamma(U,\pi_*\mathcal{O}_E(3e)) = \Gamma(\pi^{-1}(U),\mathcal{O}_E(3e))$$

By a usual dimension argument and Lemma 4.9, (x_0,y_0) satisfy a Weierstrass equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

Since 6 is invertible, we can find x_0', y_0' such that

$$p_2(x_0') = \omega^{-2}|_{\pi^{-1}(U)} \qquad p_3(y_0') = \omega^{-3}|_{\pi^{-1}(U)}$$

and (x_0', y_0') satisfies the reduced Weierstrass equation

$$y^2 = x^3 + ax + b \qquad a,b \in \mathcal{O}_S(U).$$

All the sections x_0', y_0', a, b are uniquely determined by these conditions.

Step3. We prove that locally E is defined by this Weierstrass equation.

Proposition 4.10. *The sections $x_0', y_0', 1$ in $\mathcal{O}_E(3e)|_{\pi^{-1}(U)}$ give a closed embedding*

$$\pi^{-1}(U) \longrightarrow \operatorname{Proj}\mathcal{O}_S(U)[x,y,z]$$

which induces an isomorphism

$$\psi_U : \pi^{-1}(U) \longrightarrow E_U := \operatorname{Proj}\mathcal{O}_S(U)[x,y,z]/\left(y^2z - (x^3 + axz^2 + bz^3)\right).$$

Furthermore, $\omega|_{\pi^{-1}(U)} = \psi_U^*(\omega_U)$, where

$$\omega_U := \frac{x\mathrm{d}z - z\mathrm{d}x}{2yz} = \frac{y\mathrm{d}z - z\mathrm{d}y}{3x^2 + az^2} = \frac{x\mathrm{d}y - y\mathrm{d}x}{y^2 - 2axz - 3bz^2} \qquad \text{whenever it's defined}$$

is a global generator of $\Omega_{E_U/U}$.

Proof of Proposition 4.10. The closed embedding can be checked fiberwise(2627476), and the surjectivity of ψ_U can be also checked fiberwise, so ψ_U is an isomorphism.

Notice that E_U is an elliptic curve over U, by Lemma 4.9 $\pi_*\mathcal{O}_{E_U}(ne)$ has an canonical filstration, and the symbol p_2, p_3 are also used for E_U. From the Weierstrass equation we obtain

$$\tilde{x_0} := \frac{x}{z} \in \Gamma(E_U,\mathcal{O}_{E_U}(2e)) = \Gamma(U,\pi_*\mathcal{O}_{E_U}(2e))$$
$$\tilde{y_0} := \frac{y}{z} \in \Gamma(E_U,\mathcal{O}_{E_U}(3e)) = \Gamma(U,\pi_*\mathcal{O}_{E_U}(3e))$$

where $e : U \longrightarrow E_U$ is given by $[x : y : z] = [0 : 1 : 0]$.

Let $\tilde{\omega} := \frac{p_2(\tilde{x_0})}{p_3(\tilde{y_0})} \in \Gamma(U, \omega_{E_U/U})$, then $\psi_U^*(\tilde{\omega}) = \omega|_{\pi^{-1}(U)}$ by definition, and $\tilde{\omega} = \omega_U$ follows from carefully checking up the definition of $\tilde{\omega}$:

(1) Initially, we have
$$\tilde{\omega} = \frac{p_2(\tilde{x_0})}{p_3(\tilde{y_0})} = \frac{e^*\left(\dfrac{x}{z}\right)}{e^*\left(\dfrac{y}{z}\right)} = e^*\left(\frac{x}{y}\right) \in \Gamma(U, e^*\mathcal{O}_{E_U}(-e))$$

(2) The isomorphism [9]
$$e^*\mathcal{O}_{E_U}(-e) = \mathcal{I}/\mathcal{I}^2 \xrightarrow{\sim} e^*\Omega_{E_U/U} \qquad \tilde{\omega} \longmapsto e^*\left(d\left(\frac{x}{y}\right)\right) = \frac{xdy - ydx}{y^2}\Big|_e = -dx|_e$$
realizes $\tilde{\omega}$ as a section on $e^*\Omega_{E_U/U}$ over U.

(3) The isomorphism
$$e^*\Omega_{E_U/U} \cong \omega_{E_U/U} \cong \pi_*\omega_{E_U/U}$$
shows that $\tilde{\omega}|_e = -dx|_e = \omega_U|_e$ when $\tilde{\omega}$ is viewed as a section of $\omega_{E_U/U}$, so $\tilde{\omega} = \omega_U$.

□

Step4. Now we glue local informations to get global informations.

Suppose $S = \cup_{i\in I} U_i$, and the Weierstrass equation on U_i is
$$y^2 = x^3 + a_i x + b_i \qquad a_i, b_i \in \mathcal{O}_S(U_i).$$
By the uniqueness of Weierstrass equation,
$$a_i|_{U_i\cap U_j} = a_j|_{U_i\cap U_j} \qquad b_i|_{U_i\cap U_j} = b_j|_{U_i\cap U_j}$$
so a_i, b_i glue to two global sections $a, b \in \mathcal{O}_S(S)$, and the global Weierstrass equation is
$$y^2 = x^3 + ax + b \qquad a, b \in \mathcal{O}_S(S).$$
Δ is invertible since $\Delta|_{U_i}$ is invertible for any $i \in I$.

Moreover, the local isomorphism
$$\psi_U : \pi^{-1}(U) \longrightarrow \operatorname{Proj} \mathcal{O}_S(U)[x, y, z]/\left(y^2 z - (x^3 + axz^2 + bz^3)\right)$$
glues to the global isomorphism
$$\psi : E \longrightarrow \operatorname{Proj} \mathcal{O}_S[x, y, z]/\left(y^2 z - (x^3 + axz^2 + bz^3)\right)$$
with $\psi^*(\omega_R) = \omega$. Now it's easy to construct an inverse map Ψ^{-1} and prove that Ψ is an isomorphism. □

To be compatible with the notations in modular form, we rewrite
$$R = \mathbb{Z}\left[\tfrac{1}{6}\right]\left[a, b, \Delta^{-1}\right] \cong \mathbb{Z}\left[\tfrac{1}{6}\right]\left[g_2, g_3, \Delta^{-1}\right] \qquad a = -\frac{1}{4}g_2, b = -\frac{1}{4}g_3, \Delta = g_3^2 - 27g_3^2$$
$$E_R = \operatorname{Proj} R[x, y, z]/\left(y^2 z - (4x^3 - g_2 xz^2 - g_3 z^3)\right) \qquad \text{here } x \text{ is different}$$
$$\omega_R = \frac{x\mathrm{d}z - z\mathrm{d}x}{yz} = \frac{2(y\mathrm{d}z - z\mathrm{d}y)}{12x^2 - g_2 z^2} = \frac{2(x\mathrm{d}y - y\mathrm{d}x)}{y^2 + 2g_2 xz + 3g_3 z^2} \qquad \text{whenever it's defined.}$$

[9] Notice that $d(x/y)$ is not a global differential on $\Omega_{E_U/U}$.

As an application, we prove that the coarse moduli of functor $\mathcal{M}\left[\frac{1}{6}\right] := \mathcal{M}_{1,1}\left[\frac{1}{6}\right]$ is $\mathbb{A}^1_{\mathbb{Z}\left[\frac{1}{6}\right]}$. Recall that $\mathbb{G}_m$ acts on $\widetilde{\mathcal{M}}\left[\frac{1}{6}\right]$ by

$$R \longrightarrow R \times_{\mathbb{Z}} \mathbb{Z}[t, t^{-1}] \cong R[t, t^{-1}] \qquad g_2 \longmapsto t^{-4} g_2 \quad g_3 \longrightarrow t^{-6} g_3,$$

thus $\mathbb{G}_m(S)$ acts on $\widetilde{\mathcal{M}}\left[\frac{1}{6}\right](S)$ by

$$\mathbb{G}_m(S) \times \widetilde{\mathcal{M}}\left[\tfrac{1}{6}\right](S) \longrightarrow \widetilde{\mathcal{M}}\left[\tfrac{1}{6}\right](S) \qquad \big(u, (E, \omega)\big) \longmapsto \big(E, u^{\#}(t) \cdot \omega\big).$$

Define

$$j := 1728 \frac{g_2^3}{\Delta} = 1728 \frac{g_2^3}{g_3^2 - 27 g_3^2} \in R^{\mathbb{G}_m},$$

by tedious check we get

$$R^{\mathbb{G}_m} = \mathbb{Z}\left[\tfrac{1}{6}\right][j]$$

which induce the isomorphism

$$j : \mathbb{G}_m \backslash \widetilde{\mathcal{M}}\left[\tfrac{1}{6}\right] \longrightarrow \mathbb{A}^1_{\mathbb{Z}\left[\frac{1}{6}\right]}.$$

Claim 4.11. *The scheme $\mathbb{G}_m \backslash \widetilde{\mathcal{M}}\left[\frac{1}{6}\right] \cong \mathbb{A}^1_{\mathbb{Z}\left[\frac{1}{6}\right]}$ is the course moduli of $\mathcal{M}\left[\frac{1}{6}\right]$.*

$$\begin{tikzcd}
 & \widetilde{\mathcal{M}}\left[\frac{1}{6}\right] \cong h_{\operatorname{Spec} R} \arrow[dl, "forget"'] \arrow[dr, "quotient"] & \\
\mathcal{M}\left[\frac{1}{6}\right] \arrow[rr, "\eta"] \arrow[drr, "\alpha'"'] & & \mathbb{G}_m \backslash \widetilde{\mathcal{M}}\left[\frac{1}{6}\right] \cong h_{\mathbb{A}^1_{\mathbb{Z}[1/6]}} \arrow[d, dashed, "\exists !\, h_\beta"] \\
 & & h_{X'}
\end{tikzcd}$$

FIGURE 5. verification of coarse moduli

Proof. We check it by the definition of coarse moduli, see Figure 5.

Step1. Construct η. To define $(S \in \mathrm{Sch}_{\mathbb{Z}\left[\frac{1}{6}\right]})$

$$\eta(S) : \mathcal{M}\left[\tfrac{1}{6}\right](S) \longrightarrow \mathbb{G}_m \backslash \widetilde{\mathcal{M}}\left[\tfrac{1}{6}\right](S) \cong \Gamma(S, \mathcal{O}_S),$$

we first find differential locally (locally lift to $\widetilde{\mathcal{M}}\left[\frac{1}{6}\right]$)[10], and then take quotient. You need to check:

- j-function doesn't depend on the choice of differential, so $\eta(S)$ is well-defined;
- η is really a functor.

Step2. We know that GIT quotient is a categorical quotient, so there exists unique h_β such that

$$h_\beta \circ \text{quotient} = \alpha' \circ \text{forget}.$$

You need to check $\alpha' = h_\beta \circ \eta$.

[10]We know that $\Omega_{E/S} \cong \pi^* e^* \Omega_{E/S} \overset{loc}{\cong} \pi^* \mathcal{O}_S \cong \mathcal{O}_X$ locally.

Step3. For any closed field $k = \bar{k}$, $\operatorname{char} k \neq 2, 3$, the map

$$\eta(k) : \mathcal{M}\left[\tfrac{1}{6}\right](k) \longrightarrow \mathbb{G}_m \backslash \widetilde{\mathcal{M}}\left[\tfrac{1}{6}\right](k) \cong k$$

is an isomorphism. □

4.3. **Crash course on Abelian variety.** When I was reading some materials about the level structure of elliptic curve, I realised that I'm still not so familiar with some fine structure of elliptic curve, such as $E[n]$. It's usually done as the special case of the Abelian variety[11]. The standard reference is [15, 14], but we would follow instead [4][12] since it's much less disgusting to read.

As a lazy guy, I would instead list main tasks in [4]:

- State the theorem of cubic, and show that
$$[n]^*\mathcal{L} \cong \mathcal{L}^{\otimes \frac{n(n+1)}{2}} \otimes ([-1]^*\mathcal{L})^{\otimes \frac{n(n-1)}{2}}$$
- Understand torsion points $X[n]$.
  - Define isogeny(5.3) and isogenous(5.13);
  - Give the canonical factorization of an isogeny(5.8)[13];
  - Examples of isogeny: $[n], F, V$[14]. Find their relationships(5.19, 5.20);
  - Describe the kernel of isogeny: $X[n], X[F], X[V]$(5.11, 5.22, not completed).
    In char p case, define the relevant notions: p-rank, ordinary, supersingular(5.23, 5.25).
    When $p = 2, 3$, describe the criterian for an elliptic curve to be supersingular(5.26, 5.27).
    When $p = 2$, describe the action α_2 on $X \subseteq \mathbb{P}^2_k : y^2z + yz^2 = x^3$, and show that $X[F] \cong \alpha_2$(5.28).
    For more informations about supersingular curve, see wiki and [16, Proposition 8.2].
- Understand Picard group, dual, Weil pairing and Tate modules. I haven't read about it.

4.4. **Level structure.** The second possible extra structure is the level structure.

4.5. **Complex case.** In this subsection, we will show that how the moduli is connected to the modular curve $\mathcal{H}/\operatorname{SL}_2(\mathbb{Z})$.

[11] There are thousands of books talking about elliptic curves, but most of them are unrelavent to us: some restrict themselves to the complex field $\mathbb{C}$, some focus on the applications of cryptography, and some prove every result by the Weierestrass equation, in a very down to earth but ugly way.

[12] Be careful that there are a lot of versions in the internet, and the latest verion (as far as I know) is here. Unless otherwise specified we cite for the main part of the text rather than the exercises.

[13] Analog: any field extension can be uniquely written as an inseparable extension with a separable extension.

[14] $[n]$ is an isogeny when $n \neq 0$; F and V are defined when $\operatorname{char} k = p > 0$.

5. Moduli of higher dimensional variety, MMP

I would add something here if I know.

5.1. **Moduli of algebraic surfaces.** We know the Enriques–Kodaira classification, which contributes to a better understanding of algebraic surfaces. But that's not enough. Do we know the classification of K3-surfaces?

5.2. **MMP.** Here we refer to the survey [21], or the updated version. You can get a glimpse of the current progress in MMP (especially in the table of page 63).

6. Moduli of vector bundle

The course lecture note "Moduli and GIT" would be a perfect survey to begin with. We also refer to [12]. It's not easy to read, but It's in some sense completed, and everybody refers it.

For a variant, you may get some informations of moduli of vector bundles over elliptic curve in [2] and G-bundles over elliptic curve in [5].

7. After FOAG: the future plan

When I finished reading the book [19], I felt the confidence of understanding everything in algebraic geometry field. However, I felt soon so confused and helpless, because of the superabundance of topics and articles which are not linearly developed, and they intersected with each other. I had totally no idea what goal to set and what to read. Is it still possible to organize all the (relatively advanced) basics in algebraic geometry, just like Prof. Vakil did in [19]?

This survey is one important part of the whole plan, which aimed to fill in everything well-known to experts but unknown for me. Chinese discussion can be found here.

- A series of classes, such as complex algebraic surfaces, toric varieties, Abelian varieties, finite group schemes, ...
- Moduli theory. It is this survey, even though we missed still a lot:
  - Modular curve and Schimura variety
  - Fermat's last theorem

  I wonder If I would say anything about these in this survey.
- Cohomology theory. In my mind derived category as well as six-functors formalism are the basic tool boxes, and Prof. Scholze's picture concludes the cohomology theories in a magical way. In particular, we need to show:
  - Étale cohomology
  - The proof of Weil's conjectures
- Use scheme-analogic models to solve problems.
  - Berkovich spaces, p-adic spaces, and formal schemes.
  - Theory of height. The typical example is Faltings's theorem.
- Anything with analysis. Index theory and symplectic geometry can be related.
- Anything with representation theory and number theory. Langlands program is for me the central goal. Class field theory and Iwasawa theory can be also relative topics.
- Anything with combination. For example, the dessin d'enfant, knot, number field, ...

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