MODULI IN ALGEBRAIC GEOMETRY

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ABSTRACT. In this personal survey, we conclude the definitions of moduli functors in the algebraic geometry. Most of the results are in the black box, so it's very possible that they're wrong. And also I'm not responsible for the completeness of the whole theory. However, I'm still happy to improve this document, and make it better over time.

CONTENTS

1. GOAL AND RELATED CONCEPTS

The personal survey is motivated by the three courses in Bonn: "the moduli space of curves", "moduli of elliptic curves" and "moduli of vector bundles". I would also highly recommend the course "Moduli and GIT" in Freie Universität Berlin. I want to construct my personal understanding on the moduli, and find out the details I missed in the courses.

"Some mathematicians are birds, others are frogs." This document is devoted to those "birds" from a "frog" who gets stuck in the mud.

1.1. **Conventions and Notations.** In this survey, Sch_k is denoted as the category of **locally Noetherian** schemes over k, where k is a field or $\mathbb{Z}$ or $\mathbb{Z}\left[\frac{1}{6}\right]$.

1.2. **Representable functor.** In this subsection we would follow on [**?**, Definition 2.2.1] in full generality, but you can always think

$$\mathcal{C} = \mathrm{Sch}_k \qquad \mathcal{C}' = \mathrm{Functor}(\mathrm{Sch}_k^{op}, \mathrm{Set})$$

to visualize the statement, and you can refer to [**?**, 6.6.2] to see basic examples.

Definition 1.1 (Functor category)**.** *Fix a category $\mathcal{C}$, we define the corresponding functor category $\mathcal{C}'$ as follows:*

$$\mathrm{Ob}(\mathcal{C}') := \{functors\ \mathcal{C}^{op} \longrightarrow \mathrm{Set}\} \qquad \mathrm{Mor}(\mathcal{C}') := \left\{ natural\ trans\ \ \mathcal{C}^{op} \underset{\longrightarrow}{\overset{\longrightarrow}{\Downarrow}} \mathrm{Set} \right\}$$

In brief, $\mathcal{C}' = \mathrm{Functor}(\mathcal{C}^{op}, \mathrm{Set})$.[1]

Proposition 1.2. *We have a canonical functor*

$$\iota : \mathcal{C} \longrightarrow \mathcal{C}' \qquad X \longmapsto h_X := \mathrm{Mor}_{\mathcal{C}}(-, X)$$

which embeds $\mathcal{C}$ as a full subcategory of $\mathcal{C}'$.[2]

Proof. Recall that the Yoneda's lemma gives us the isomorphism

$$\mathrm{Mor}_{\mathcal{C}'}(h_X, F) \cong F(X).$$

$\square$

Definition 1.3 (Representable functor)**.** *The functor $F \in \mathrm{Ob}(\mathcal{C}')$ is called represented by X if $F \cong h_X$.*

From this proposition, we can always view the object as some functor satisfying some properties. we get three advantages from this point of view:

- It's easy to see rational points and complex points;
- We can define scheme canonically, without explicit constructions;
- We can enlarge our area of research, and think them as the defective schemes. We will see some reasonable functors which is represented not by schemes, but by stacks.

1.3. **Corepresentable functor.** This is the concept "dual to" the representable functor. To motivate, we begin with the equivalent definition of representable functor:

Definition 1.4 (Representatable functor, equivalent definition)**.** *A functor $F \in \mathrm{Ob}(\mathcal{C}')$ is represented by $X \in \mathrm{Ob}(C)$ if F satisfies the following universal properties:*

- *There exists a morphism $\eta^{-1} : h_X \longrightarrow F$ in $\mathrm{Mor}_{\mathcal{C}'}(h_X, F)$;*
- *For any object $X' \in \mathrm{Ob}(\mathcal{C})$ and morphism $\alpha' : h_{X'} \longrightarrow F$ in $\mathrm{Mor}_{\mathcal{C}'}(h_{X'}, F)$, there exists an unique morphism $\beta \in \mathrm{Mor}_{\mathcal{C}}(X', X)$ such that $\alpha' = \eta^{-1} \circ h_\beta$.*

$$
\begin{array}{ccc}
 & h_{X'} & \\
{\scriptstyle \exists! h_\beta}\big\downarrow & & \searrow {\scriptstyle \alpha'} \\
 & h_X \xrightarrow{\ \eta^{-1}\ } & F
\end{array}
$$

This definition is equivalent to the Definition 1.3 since [3]

$$h_X(X') = \mathrm{Mor}_{\mathcal{C}}(X', X) \cong \mathrm{Mor}_{\mathcal{C}'}(h_{X'}, F) = F(X') \text{ for any } X' \in \mathrm{Ob}(\mathcal{C})$$

thus the functor $\eta^{-1} : h_X \longrightarrow F$ is an isomorphism.

Definition 1.5 (Corepresentable functor)**.** *A functor $F \in \mathrm{Ob}(\mathcal{C}')$ is corepresented by $X \in \mathrm{Ob}(C)$ if F satisfies the following universal properties:*

---

[1]From here we can also write $\mathcal{C}' = \mathrm{Psh}(C)$, notice that here it is the presheaf on **category**, not on a specific scheme X/k.

[2]h_X is called the functor of points of a scheme X from here. For example, $h_X(\mathrm{Spec}\,\mathbb{Q}) = X(\mathbb{Q})$ gives us the set of rational points in X(under certain conditions). Notice that $h_-(-)$ is a bifunctor.

[3]The middle isomorphism comes from the universal property, while the two equalities come from the Yoneda's lemma.

- *There exists a morphism $\eta : F \longrightarrow h_X$ in $\mathrm{Mor}_{\mathcal{C}'}(F, h_X)$;*
- *For any object $X' \in \mathrm{Ob}(\mathcal{C})$ and morphism $\alpha' : F \longrightarrow h_{X'}$ in $\mathrm{Mor}_{\mathcal{C}'}(F, h_{X'})$, there exists an unique morphism $\beta \in \mathrm{Mor}_{\mathcal{C}}(X, X')$ such that $\alpha' = h_\beta \circ \eta$.*

$$
\begin{array}{ccc}
F & \xrightarrow{\ \eta\ } & h_X \\
& \searrow{\alpha'} & \downarrow{\color{red}\exists! h_\beta} \\
& & h_{X'}
\end{array}
$$

Definition 1.6 (Universal corepresentable functor)**.**

Proposition 1.7. *Suppose a functor $F \in \mathrm{Ob}(\mathcal{C}')$ is corepresented by $X \in \mathrm{Ob}(C)$, then it is represented by X if and only if $\eta : F \longrightarrow h_X$ is a $\mathcal{C}'$-isomorphism.*

1.4. **Coarse moduli space.** You need to see the definition of the naive moduli problem, extended moduli problem, moduli functor, fine moduli space and corresponding universal family from here. I don't want to copy, and it's well-written.

The following example will check if you really understand these concepts.

Example 1.8 ($\mathcal{M}_{0,3}$ is represented by one point $\mathrm{Spec}\, k$)**.**
  Naive moduli problem $(\mathcal{A}, \sim)$***:***

$$\mathcal{A} := \left\{ (p_1, p_2, p_3) \mid p_i \in \mathbb{P}^1_k(k) \ are\ distinct \right\}$$

$$(p_1, p_2, p_3) \sim (q_1, q_2, q_3) \Longleftrightarrow \exists\, \gamma \in \mathrm{PGL}_2(k) \ s.t. \ q_i = \gamma(p_i)$$

  Extended moduli problem $(\mathcal{A}_S, \sim_S)$ *+ pullback:* $(S \in \mathrm{Ob}(\mathrm{Sch}_k))$

$$
\mathcal{A}_S := \left\{ (X, \pi, \sigma_1, \sigma_2, \sigma_3) \ \middle| \ \begin{array}{l} X \in \mathrm{Ob}(\mathrm{Sch}_k) \\ \pi : X \longrightarrow S \ proper\ flat\ and\ \pi^{-1}(p) \cong \mathbb{P}^1_{\kappa(p)} \\ \sigma_i : S \longrightarrow X \ pairwise\ disjoint\ sections\ of\ \pi \end{array} \right\}
$$

$(X, \pi, \sigma_1, \sigma_2, \sigma_3) \sim_S (X', \pi', \sigma'_1, \sigma'_2, \sigma'_3)$ *if there exists an isomorphism $f : X \longrightarrow X'$ such that the following diagrams commute:*

$$
\begin{array}{ccc}
X & \xrightarrow{\ f\ } & X' \\
{\scriptstyle\pi}\searrow & & \swarrow{\scriptstyle\pi'} \\
& S &
\end{array}
\qquad\qquad
\begin{array}{ccc}
X & \xrightarrow{\ f\ } & X' \\
{\scriptstyle\sigma_i}\nwarrow & & \nearrow{\scriptstyle\sigma'_i} \\
& S &
\end{array}
$$

For a map $f : T \longrightarrow S$, the pullback f^* is defined by

$$f^* : \mathcal{A}_S \longrightarrow \mathcal{A}_T \qquad (X, \pi, \sigma_1, \sigma_2, \sigma_3) \longmapsto (X_T, \pi_T, \sigma_{1,T}, \sigma_{2,T}, \sigma_{3,T})$$

where X_T, π_T are defined as fiber product, and $\sigma_{i,T}$ are defined by the universal property of fiber product, as follows:

$$
\begin{array}{ccc}
X_T & \longrightarrow & X \\
{\scriptstyle\pi_T}\downarrow & \llcorner & \downarrow{\scriptstyle\pi} \\
T & \xrightarrow{\ f\ } & S
\end{array}
\qquad\qquad
\begin{array}{ccc}
T & \xrightarrow{\ \sigma_i \circ f\ } & \\
& {\color{red}\exists! \sigma_{i,T}} & \\
{\scriptstyle\mathrm{Id}_T}\searrow & X_T \longrightarrow & X \\
& {\scriptstyle\pi_T}\downarrow \ \llcorner & \uparrow\downarrow{\scriptstyle\pi}\ {\scriptstyle\sigma_i} \\
& T \xrightarrow{\ f\ } & S
\end{array}
$$

Remark. We can always view the point $p \in S$ as an affine scheme of its residue field. So the two conditions below are equivalent:

$$\pi^{-1}(p) \cong \mathbb{P}^1_{\kappa(p)} \qquad \text{for any } p \in S$$
$$\pi^{-1}(\operatorname{Spec} L) \cong \mathbb{P}^1_L \quad \text{for any } \operatorname{Spec} L \hookrightarrow S$$

Remark. In some textbooks, they require $\pi : X \longrightarrow S$ to be a smooth, proper, surjective, locally finitely presented (l.f.p.) morphism of relative dimension $\leqslant 1$ with geometric fibres isomorphic to $\mathbb{P}^1$. These extra conditions are automatically satisfied, since

$$\pi^{-1}(p) \cong \mathbb{P}^1_{\kappa(p)} \longrightarrow \text{surjective}$$
$$\text{proper} \longrightarrow \text{relative dimension} \leqslant 1$$
$$\text{locally Noetherian} \longrightarrow \text{locally finitely presented}$$

and by [**?**, 25.2.2], flat + fiberwise $\mathbb{P}^1$ + l.f.p. $\Rightarrow$ smooth.

Question.
Can we find $(X, \pi, \sigma_1, \sigma_2, \sigma_3)$ satisfying all conditions in $\mathcal{A}_S$ except π is seperated?
Can we find $(X, \pi, \sigma_1, \sigma_2, \sigma_3)$ satisfying all conditions in $\mathcal{A}_S$ except π is flat?

Answer. Still unknown. From Dr. Johannes Anschütz, the map

$$\mathbb{P}^1_k \longrightarrow \operatorname{Spec} k \longrightarrow \operatorname{Spec} k[t]/t^2$$

is proper but not flat(sadly we can't find any section). If the base scheme S is reduced, then maybe the flatness can be checked fiberwise?

Easy exercise.
Verify that $(\mathcal{A}_S, \sim_S)$ and pullback satisfy (i)-(iv) in [**?**, Definition 2.10].

Moduli functor $\mathcal{M}_{0,3}$: *The moduli functor* $\mathcal{M}_{0,3} : \operatorname{Sch}_k^{op} \longrightarrow \operatorname{Set}$ *is defined as*

$$\mathcal{M}_{0,3}(S) = \mathcal{A}_S / \sim_S \qquad \mathcal{M}_{0,3}(f : T \longrightarrow S) = f^* : \mathcal{A}_S / \sim_S \longrightarrow \mathcal{A}_T / \sim_T .$$

Question.
Why is the moduli functor $\mathcal{M}_{0,3}$ represented by $\operatorname{Spec} k$?

Answer.
Yes, but it's pretty hard to prove it. The proof of [**?**, Proposition 4.1] uses [**?**, Proposition 4.2] whose proof is quite technical.
Actually we construct the natural functor $\eta^{-1} : h_{\operatorname{Spec} k} \longrightarrow \mathcal{M}_{0,3}$ by

$$\eta_S^{-1} : h_{\operatorname{Spec} k}(S) \longrightarrow \mathcal{M}_{0,3}(S) \qquad [S \to \operatorname{Spec} k] \longmapsto (\mathbb{P}^1_S = \mathbb{P}^1_k \times_k S, pr_1, 0, 1, \infty)$$

and then verify that η_S^{-1} is an isomorphism.

Universal family: *By* [**?**, *Definition 2.16*]*, the universal family is* $(\mathbb{P}^1_k, \pi, 0, 1, \infty)$.

coarse moduli space and some related concepts.

1.5. **Stack.** We need to define stack, algebraic stack, Deligne-Munford stack and some related concepts.
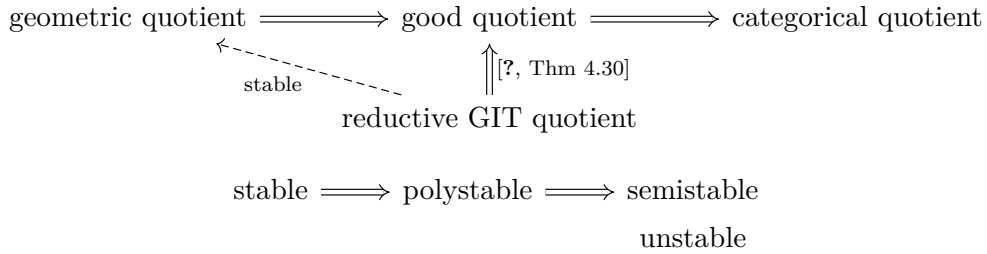
1.6. **Goal.** ???

1.7. **Quotient.** We will frequently use the quotient. Here is the list of quotients we've already seen:

- (We could begin at topology:quotient by a subset) linear space-> Abelian category,group,ring(Notice that for this item, we don't take quotient by a group, so sometimes it looks easier)
- topology(This gives us an categorical quotient!)
- manifold
- scheme(Categorical quotient; GIT quotient)
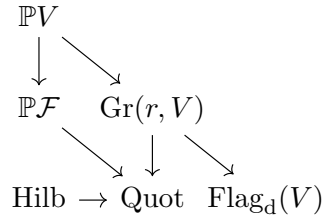
See [**?**] for the following concepts:

- categorical quotients, good quotients, geometric quotients, and GIT quotients.
- reductive, linearly reductive, and geometrically reductive.(see [**?**, Theorem 4.16])
- stable, semistable, unstable, and polystable.
- Hilbert-Mumford Criterion and Fundamental Theorem in GIT.

$$\text{geometric quotient} \Longrightarrow \text{good quotient} \Longrightarrow \text{categorical quotient}$$

$$\text{stable} \qquad \text{[?, Thm 4.30]}$$

$$\text{reductive GIT quotient}$$

$$\text{stable} \Longrightarrow \text{polystable} \Longrightarrow \text{semistable}$$

$$\text{unstable}$$

2. Basic object

In this section, we present some algebraic geometric objects which can be viewed as moduli.

Here is a picture showing the relationships of these objects:

$$
\begin{array}{c}
\mathbb{P}V \\
\downarrow \searrow \\
\mathbb{P}\mathcal{F} \qquad \mathrm{Gr}(r, V) \\
\searrow \downarrow \searrow \\
\mathrm{Hilb} \to \mathrm{Quot} \quad \mathrm{Flag}_{\mathrm{d}}(V)
\end{array}
$$

2.1. **Projective space.** We begin with a basic extended moduli problem, and then gradually make some variations.

Example 2.1 (Moduli of line bundle with base-point-free sections is represented by $\mathbb{P}_k^n$)**.** *Fix $n \geqslant 0$, we define a moduli problem:*

$$\mathcal{A}_S := \left\{ (\mathcal{L}, s_0, \ldots, s_n) \;\middle|\; \begin{array}{l} \mathcal{L} \in \mathrm{Pic}(S) \\ s_i \in \Gamma(S, \mathcal{L}) \text{ with no common zero} \end{array} \right\}$$

$(\mathcal{L}, s_0, \ldots, s_n) \sim_S (\mathcal{L}', s_0', \ldots, s_n')$ if there exists an isomorphism of line bundles $\phi : \mathcal{L} \longrightarrow \mathcal{L}'$ such that $\phi(S) : \Gamma(S, \mathcal{L}) \longrightarrow \Gamma(S, \mathcal{L}')$ sends s_i to s_i'.

For a map $f : T \longrightarrow S$, the pullback f^* is defined by

$$f^* : \mathcal{A}_S \longrightarrow \mathcal{A}_T \qquad (\mathcal{L}, s_0, \ldots, s_n) \longmapsto (f^*\mathcal{L}, f^*s_0, \ldots, f^*s_n)$$

By [**?**, 15.3.F, 16.4.1], *the moduli functor defined by this extended moduli problem is represented by* $\mathbb{P}_k^n$.

We also have the coordinate-free version.

Example 2.2 (Coordinate-free projective space $\mathbb{P}V^\vee = \mathrm{Proj}(\mathrm{Sym}^\bullet V)$, see [**?**, 4.5.12]). *Fix a k-vector spcace V of finite dimension. We define a moduli problem:*

$$\mathcal{A}_S := \left\{ (\mathcal{L}, \lambda) \;\middle|\; \begin{array}{l} \mathcal{L} \in \mathrm{Pic}(S) \\ \lambda : V \longrightarrow \Gamma(S, \mathcal{L}) \text{ is base-point-free} \end{array} \right\}$$

$(\mathcal{L}, \lambda) \sim_S (\mathcal{L}', \lambda')$ *if there exists an isomorphism of line bundles* $\phi : \mathcal{L} \longrightarrow \mathcal{L}'$ *such that* $\lambda' = \phi(S) \circ \lambda$.
For a map $f : T \longrightarrow S$, *the pullback* f^* *is defined by*

$$f^* : \mathcal{A}_S \longrightarrow \mathcal{A}_T \qquad (\mathcal{L}, \lambda) \longmapsto \left(f^*\mathcal{L}, f^* \circ \lambda : V \to \Gamma(T, f^*\mathcal{L}) \right)$$

By [**?**, 16.4.E], *the moduli functor defined by this extended moduli problem is represented by* $\mathbb{P}V^\vee$.

Now we generalize it to the projective bundle, for this we should fix a scheme $X \in \mathrm{Ob}(\mathrm{Sch}_k)$ and a locally free coherent sheaf $\mathcal{F} \in \mathrm{Coh}(X)$, and consider the moduli problem in the category Sch_X of locally Noetherian schemes over X, rather than Sch_k.

Example 2.3 (Projective bundle[4] $\mathbb{P}\mathcal{F} = \mathrm{Proj}(\mathrm{Sym}^\bullet \mathcal{F})$, see [**?**, 17.2.3]).
For $(S, \pi_S : S \longrightarrow X) \in \mathrm{Ob}(\mathrm{Sch}_X)$, *we define a moduli problem:*

$$\mathcal{A}_S := \left\{ (\mathcal{L}, \lambda) \;\middle|\; \begin{array}{l} \mathcal{L} \in \mathrm{Pic}(S) \\ \lambda : \pi_S^*\mathcal{F} \longrightarrow\!\!\!\rightarrow \mathcal{L} \end{array} \right\}$$

$(\mathcal{L}, \lambda) \sim_S (\mathcal{L}', \lambda')$ *if there exists an isomorphism of line bundles* $\phi : \mathcal{L} \longrightarrow \mathcal{L}'$ *such that* $\lambda' = \phi \circ \lambda$.
For a map $f : T \longrightarrow S$, *the pullback* f^* *is defined by*

$$f^* : \mathcal{A}_S \longrightarrow \mathcal{A}_T \qquad (\mathcal{L}, \lambda) \longmapsto \left(f^*\mathcal{L}, f^*\lambda : \pi_T^*\mathcal{F} \twoheadrightarrow f^*\mathcal{L} \right)$$

By [**?**, Proposition 7.12], *the moduli functor defined by this extended moduli problem is represented by* $\mathbb{P}\mathcal{F}$.

Finally, there is also "another" natural moduli problem of $\mathbb{P}_k^n$, see [**?**, Example 2.4]. For the convinience of comparison with Grassmannian, we exhibit it and make some small variations here.[5]

Example 2.4 (Moduli of lines through the origin in $\mathbb{A}_k^{n+1}$ is represented by $\mathbb{P}_k^n$). *For* $n \geqslant 0$, *we define a moduli problem:*

$$\mathcal{A}_S := \left\{ (\mathcal{L}, \pi) \;\middle|\; \begin{array}{l} \mathcal{L} \in \mathrm{Pic}(S) \\ \pi : \mathcal{O}_S^{\oplus n+1} \longrightarrow\!\!\!\rightarrow \mathcal{L} \end{array} \right\}$$

---

[4]There is an notational abuse in [**?**]. From my personal point of view, it's better to replace $\mathbb{P}\mathcal{F}$ with $\mathbb{P}\mathcal{F}^\vee$ or $\mathbb{P}V^\vee$ with $\mathbb{P}V$ to make symbols consistent.

[5]The reason of the variation is already explained in [**?**, 16.7, page 442].

$(\mathcal{L}, \pi) \sim_S (\mathcal{L}', \pi')$ *if there exists an isomorphism of line bundles* $\phi : \mathcal{L} \longrightarrow \mathcal{L}'$ *such that* $\pi' = \phi \circ \pi$.

For a map $f : T \longrightarrow S$, *the pullback* f^* *is defined by*

$$f^* : \mathcal{A}_S \longrightarrow \mathcal{A}_T \qquad (\mathcal{L}, \pi) \longmapsto \left(f^*\mathcal{L}, f^*\pi : \mathcal{O}_T^{\oplus n+1} \twoheadrightarrow f^*\mathcal{L} \right)$$

By [**?**, Proposition 7.12], *the moduli functor defined by this extended moduli problem is represented by* $\mathbb{P}\mathcal{F}$.

2.2. **Grassmannian.**

2.2. **Grassmannian.** It's well-written in [**?**, 16.7]. We just exhibit(copy) the moduli problem and make a short remark about the existence proof (prove the representability without explicite construction of the scheme)

Example 2.5 (Grassmannian $\mathrm{Gr}(k, n)$)**.** *For* $n \geqslant k \geqslant 0$, *we define a moduli problem:*

$$\mathcal{A}_S := \left\{ (\mathcal{F}, \pi) \;\middle|\; \begin{array}{l} \mathcal{F} \in \mathrm{Coh}(S) \text{ locally free of rank } k \\ \pi : \mathcal{O}_S^{\oplus n} \longrightarrow\!\!\!\!\!\rightarrow \mathcal{F} \end{array} \right\}$$

$(\mathcal{F}, \pi) \sim_S (\mathcal{F}', \pi')$ *if there exists an isomorphism of line bundles* $\phi : \mathcal{F} \longrightarrow \mathcal{F}'$ *such that* $\pi' = \phi \circ \pi$.

For a map $f : T \longrightarrow S$, *the pullback* f^* *is defined by*

$$f^* : \mathcal{A}_S \longrightarrow \mathcal{A}_T \qquad (\mathcal{F}, \pi) \longmapsto \left(f^*\mathcal{F}, f^*\pi : \mathcal{O}_T^{\oplus n} \twoheadrightarrow f^*\mathcal{F} \right)$$

By [**?**, 16.7, page 442-443], *the moduli functor defined by this extended moduli problem is representable, and we denote it by* $\mathrm{Gr}(k, n)$.

Remark 2.6. The idea of the proof comes from [**?**, 9.1.I]. First we prove it to be the Zariski sheaf, then we cover it with open subfunctors that are representable.

2.3. **Flag variety, partial flag variety.**

2.4. **Hilbert scheme.**

2.5. **Quot scheme.**

2.5. **Quot scheme.** The moduli space of projective hypersurfaces is a special case for this.

2.6. **Misc.**

2.6. **Misc.** The representable functor is also used to construct the fibered product of schemes, see [**?**, 9.1.6-7] for more details.

We've already met the idea of moduli in plane geometry. For example, fix two points A, B on the plane, we wanted to find the set of points C such that $\triangle ABC$ is a right triangle (resp. an isosceles triangle). We used the term "locus" in junior high school.

3. MODULI OF CURVE

The content of this section is already well written in the course "the moduli space of curves". This section is just for the completeness of the survey.
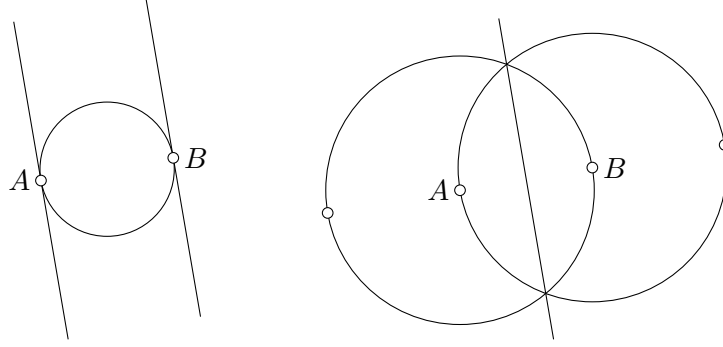
So now comes the conclusion.

FIGURE 1. "moduli space" in Euclidean geometry

3.1. **Initial definition.**

Definition 3.1 (smooth/stable curve, follow [**?**, Definition 3.15]). *For $g, n \geqslant 0$, a smooth curve of genus g over $S \in \mathrm{Sch}_k$ with n-points is an element in the set*

$$\mathcal{A}_S := \left\{ (X, \pi, \sigma_i) \,\middle|\, \begin{array}{l} X \in \mathrm{Ob}(\mathrm{Sch}_k) \\ \pi : X \longrightarrow S \text{ proper flat and} \\ \qquad \pi^{-1}(\mathrm{Spec}\, L) \text{ is a sm proj connected curve of genus } g \\ \sigma_i : S \longrightarrow X \text{ pairwise disjoint sections of } \pi \end{array} \right\}$$

and a stable curve of genus g over $S \in \mathrm{Sch}_k$ with n-points is an element in the set

$$\overline{\mathcal{A}}_S := \left\{ (X, \pi, \sigma_i) \,\middle|\, \begin{array}{l} X \in \mathrm{Ob}(\mathrm{Sch}_k) \\ \pi : X \longrightarrow S \text{ proper flat } \textit{(maybe not smooth)} \\ \qquad \pi^{-1}(\mathrm{Spec}\, L) \text{ is a } \textit{stable} \text{ proj connected curve of genus } g \\ \sigma_i : S \longrightarrow X \text{ pairwise disjoint sections of } \pi \\ \qquad \text{with image in the smooth locus of } \pi \end{array} \right\}$$

Definition 3.2 (moduli of smooth/stable curves). *For $g, n \geqslant 0$, the moduli of smooth curves $\mathcal{M}_{g,n}$ and the moduli of smooth curves $\overline{\mathcal{M}}_{g,n}$ are defined as*

$$\mathcal{M}_{g,n}(S) = \mathcal{A}_S / \sim_S \qquad \mathcal{M}_{g,n}(f : T \longrightarrow S) = f^* : \mathcal{A}_S / \sim_S \longrightarrow \mathcal{A}_T / \sim_T \, .$$

$$\overline{\mathcal{M}}_{g,n}(S) = \overline{\mathcal{A}}_S / \sim_S \qquad \overline{\mathcal{M}}_{g,n}(f : T \longrightarrow S) = f^* : \overline{\mathcal{A}}_S / \sim_S \longrightarrow \overline{\mathcal{A}}_T / \sim_T \, .$$

where the equivalent relation $\sim_S$ and the pullback are similar in the Example 1.8, i.e. $(X, \pi, \sigma_i) \sim_S (X', \pi', \sigma_i')$ if there exists an isomorphism $f : X \longrightarrow X'$ such that the following diagrams commute:

$$\begin{array}{ccc} X & \xrightarrow{\;\;f\;\;} & X' \\ {\scriptstyle \pi} \searrow & & \swarrow {\scriptstyle \pi'} \\ & S & \end{array} \qquad\qquad \begin{array}{ccc} X & \xrightarrow{\;\;f\;\;} & X' \\ {\scriptstyle \sigma_i} \nwarrow & & \nearrow {\scriptstyle \sigma_i'} \\ & S & \end{array}$$

For a map $f : T \longrightarrow S$, the pullback f^* is defined by

$$f^* : \mathcal{A}_S \longrightarrow \mathcal{A}_T \qquad (X, \pi, \sigma_i) \longmapsto (X_T, \pi_T, \sigma_{i,T})$$

where X_T, π_T are defined as fiber product, and $\sigma_{i,T}$ are defined by the universal property of fiber product, as follows:

3.2. **Result.** Here is the result coming from [**?**] which we really care:

FIGURE 2. The moduli of curves

4. MODULI OF ELLIPTIC CURVE

The elliptic curve theory is especially rich compared to the other curves. That's why we'd like to put it a special section.

4.1. **Definition and basic properties of elliptic curve.**

Definition 4.1 (Relative elliptic curve)**.** *Let $S \in \mathrm{Ob}(\mathrm{Sch}_k)$. An S-group scheme (E, m) is called an elliptic curve over S if the structure map $\pi : E \longrightarrow S$ is proper, smooth of relative dimension 1, and has connected fibers. Two elliptic curves are considered as the same if they are isomorphic as the S-group scheme.*

This data is equivalent to what we described for $\mathcal{M}_{1,1}(S)$ in Definition 3.2, and also equivalent to the definition in [**?**, Tag 072J].

Recall the Cohomology and Base Change Theorem in [**?**, 28.1.6]. Here we just give a simplified version:

Theorem 4.2 (Cohomology and Base Change Theorem, simplified version)**.**

Suppose

- *S is locally Noetherian;*
- *$\pi : X \longrightarrow S$ is proper;*
- *The coherent sheaf $\mathcal{F}$ over X is flat over S.*

$$
\begin{array}{ccc}
X_q & \longrightarrow & X \\
\pi_q \downarrow & & \downarrow \pi \\
\mathrm{Spec}\ \kappa(q) & \overset{\iota}{\longrightarrow} & S
\end{array}
\qquad \mathcal{F}
$$

then for any $q \in S$ and $p \in \mathbb{Z}$, we have the natural map

$$\phi_q^p : (\mathrm{R}^p\pi_*\mathcal{F})|_q \longrightarrow H^p(X_q, \mathcal{F}|_{X_q})$$

which has the following properties:

$$
\phi_q^p \text{ is surj} \implies
\begin{cases}
\phi_q^p \text{ is iso} \\
\left(\phi_q^{p-1} \text{ is surj } \Leftrightarrow \mathrm{R}^p\pi_*\mathcal{F} \text{ is l.f.} \right)
\end{cases}
$$

We can use Theorem 4.2 to compute some pushforward of sheaves, and the results are shown here:

n	-3	-2	-1	0	1	2	3
$\mathrm{R}^1\pi_*\mathcal{O}_E(ne)$	rank 3	rank 2	l.b.	l.b.	0	0	0
$\pi_*\mathcal{O}_E(ne)$	0	0	0	$\mathcal{O}_S$	l.b.	rank 2	rank 3
$\mathrm{R}^1\pi_*\Omega_{E/S}(ne)$	rank 3	rank 2	l.b.	$\mathcal{O}_S$	0	0	0
$\pi_*\Omega_{E/S}(ne)$	0	0	0	l.b.	l.b.	rank 2	rank 3

Example 4.3 (case of $\mathrm{R}^p\pi_*\mathcal{O}_E$). *Let $X = E$ be an elliptic curve over S, $\mathcal{F} = \mathcal{O}_X$. Obviously ϕ_q^2, ϕ_q^{-1} are surjective; the map ϕ_q^0 is also surjective[6]. By using Theorem 4.2 (see the Figure 3) we obtain that*

$$
\mathrm{R}^p\pi_*\mathcal{O}_X =
\begin{cases}
0 & p \geqslant 2 \\
\text{line bundle} & p = 0, 1
\end{cases}
$$

FIGURE 3. the process; red color is the initial condition

Lemma 4.4. $\pi_*\mathcal{O}_X \cong \mathcal{O}_S$.

---

[6]see the hint in [**?**, 28.1.H].

Proof. The morphism $\pi : X \longrightarrow S$ induces the map of sheaves

$$\pi^{\#} : \mathcal{O}_S \longrightarrow \pi_*\mathcal{O}_X$$

which corresponds a section of $\pi_*\mathcal{O}_X$. Since $\pi_*\mathcal{O}_X$ is a line bundle, this section defines a Cartier divisor D of S, i.e. $\pi_*\mathcal{O}_X \cong \mathcal{O}_S(D)$. From the isomorphism

$$\pi^{\#}|_q : \kappa(q) \longrightarrow \pi_*\mathcal{O}_X \times_{\mathcal{O}_S} \kappa(q) \xrightarrow{\phi_q^0} H^0(X_q, \mathcal{O}_{X_q}) \cong \kappa(q)$$

we get $D = 0$, thus $\pi_*\mathcal{O}_X \cong \mathcal{O}_S$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 4.5. The other cases are similarly solved except $\mathcal{F} = \Omega_{E/S}$. Actually, if we admit the Grothendieck-Serre duality

$$R^p\pi_*\Omega_{E/S} \cong (R^{1-p}\pi_*\mathcal{O}_E)^{\vee}$$

this would be easily solved. You can see the discussion here for Exercise [**?**, 28.1.N], and [**?**, 2.1.2] for the "proof" of Grothendieck-Serre duality. For me the question in stackexchange is still unsolved, and for solving Exercise [**?**, 28.1.N] one need to assume the base scheme is reduced to use the Grauert's Theorem in [**?**, 28.1.5].(The Figure 4 shows where we use the Grauert's Theorem.)

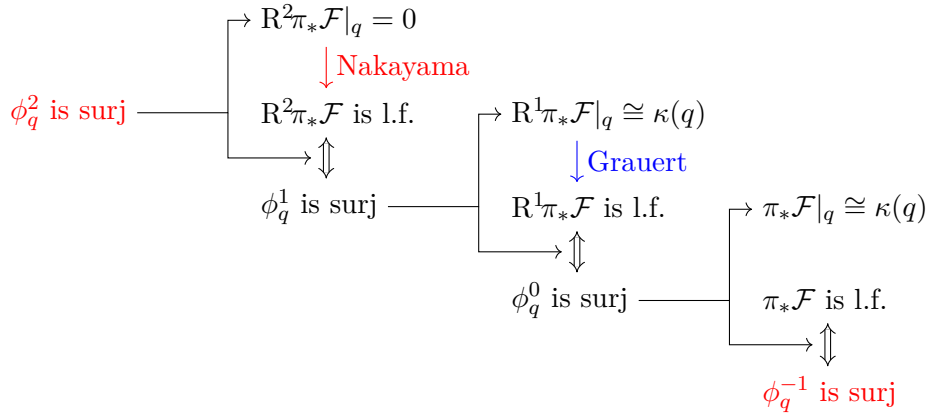| Method	Result	Requirement
Grothendieck-Serre duality	$R^1\pi_*\mathcal{F} \cong \mathcal{O}_E$	haven't checked
Grauert's theorem	$R^1\pi_*\mathcal{F}$ is l.f.	S is reduced
$\Omega_{E/S} \cong \mathcal{O}_E$ locally	$R^1\pi_*\mathcal{F}$ is l.f.	E/S is group scheme

<div align="center">TABLE 1.</div>

FIGURE 4. $\mathcal{F} = \Omega_{E/S}$; red color is the initial condition

4.2. **Differential.** We know that $\mathcal{M}_{1,1}$ is not representable, so we have to introduce extra structures to rigidify elliptic curves. The first possible extra structure is the differential.

Definition 4.6 (Weierstrass moduli $\widetilde{\mathcal{M}}\big[\frac{1}{6}\big]$)**.** *For a base scheme $S \in \mathrm{Ob}(\mathrm{Sch}_{\mathbb{Z}[\frac{1}{6}]})$, we define a moduli problem*

$$\mathcal{A}_S := \left\{ (E, \omega) \, \middle| \, \begin{array}{l} E : \ \text{elliptic curve over } S \\ \omega \in \Gamma(E, \Omega_{E/S}) \ \text{global generator} \end{array} \right\}$$

$(E, \omega) \sim_S (E', \omega')$ if there exists an isomorphism of elliptic curves $\phi : E \longrightarrow E'$ such that $\omega' = \phi^* \circ \omega'$.

For a map $f : T \longrightarrow S$, the pullback f^* is defined by

$$f^* : \mathcal{A}_S \longrightarrow \mathcal{A}_T \qquad (E, \omega) \longmapsto \big(f^* E, f^* \omega\big)$$

By doing so we define the moduli functor

$$\widetilde{\mathcal{M}}\big[\tfrac{1}{6}\big] : \mathrm{Sch}_{\mathbb{Z}[\frac{1}{6}]} \longrightarrow \mathrm{Set} \qquad S \longmapsto \mathcal{A}_S / \sim_S$$

Theorem 4.7. *The moduli functor $\widetilde{\mathcal{M}}\big[\frac{1}{6}\big]$ is represented by $\mathrm{Spec}\, \mathbb{Z}\big[\frac{1}{6}\big]\big[a, b, \Delta^{-1}\big]$ where $\Delta = -\frac{1}{256}(4a^3 + 27b^2)$. Denote $R = \mathbb{Z}\big[\frac{1}{6}\big]\big[a, b, \Delta^{-1}\big]$, the universal family is $(E_R, \omega_R) \in \widetilde{\mathcal{M}}\big[\frac{1}{6}\big](R)$, where*

$$E_R = \mathrm{Proj}\, R[x, y, z]/\big(y^2 z - (x^3 + axz^2 + bz^3)\big)$$

$$\omega_R = \frac{x\mathrm{d}z - z\mathrm{d}x}{2yz} = \frac{y\mathrm{d}z - z\mathrm{d}y}{3x^2 + az^2} = \frac{x\mathrm{d}y - y\mathrm{d}x}{y^2 - 2axz - 3bz^2} \quad \text{whenever it's defined.}$$

Proof. need complement. The idea is, consider it locally, find appropriate $1, x, y$ with respect to the differential ω, and finally glue them. $\qquad\square$

To be compatible with the notations in modular form, we rewrite

$$R = \mathbb{Z}\big[\tfrac{1}{6}\big]\big[a, b, \Delta^{-1}\big] \cong \mathbb{Z}\big[\tfrac{1}{6}\big]\big[g_2, g_3, \Delta^{-1}\big] \qquad\qquad a = -4g_2, b = -16g_3, \Delta = g_3^2 - 27g_3^2$$

$$E_R = \mathrm{Proj}\, R[x, y, z]/\big(y^2 z - (4x^3 - g_2 x z^2 - g_3 z^3)\big) \qquad \text{here } x \text{ is different}$$

$$\omega_R = \frac{x\mathrm{d}z - z\mathrm{d}x}{2yz} = \frac{y\mathrm{d}z - z\mathrm{d}y}{12x^2 - g_2 z^2} = \frac{x\mathrm{d}y - y\mathrm{d}x}{y^2 + 2g_2 xz + 3g_3 z^2} \quad \text{whenever it's defined.}$$

As an application, we prove that the coarse moduli of functor $\mathcal{M}\big[\frac{1}{6}\big] := \mathcal{M}_{1,1}\big[\frac{1}{6}\big]$ is $\mathbb{A}^1_{\mathbb{Z}[\frac{1}{6}]}$. Recall that $\mathbb{G}_m$ acts on $\widetilde{\mathcal{M}}\big[\frac{1}{6}\big]$ by

$$R \longrightarrow R \times_{\mathbb{Z}} \mathbb{Z}[t, t^{-1}] \cong R[t, t^{-1}] \qquad g_2 \longmapsto t^{-4} g_2 \quad g_3 \longrightarrow t^{-6} g_3,$$

thus $\mathbb{G}_m(S)$ acts on $\widetilde{\mathcal{M}}\big[\frac{1}{6}\big](S)$ by

$$\mathbb{G}_m(S) \times \widetilde{\mathcal{M}}\big[\tfrac{1}{6}\big](S) \longrightarrow \widetilde{\mathcal{M}}\big[\tfrac{1}{6}\big](S) \qquad \big(u, (E, \omega)\big) \longmapsto \big(E, u^{\#}(t) \cdot \omega\big).$$

Define

$$j := 1728 \frac{g_2^3}{\Delta} = 1728 \frac{g_2^3}{g_3^2 - 27g_3^2} \in R^{\mathbb{G}_m},$$

by tedious check we get

$$R^{\mathbb{G}_m} = \mathbb{Z}\big[\tfrac{1}{6}\big][j]$$

which induce the isomorphism

$$j : \mathbb{G}_m \backslash \widetilde{\mathcal{M}}\big[\tfrac{1}{6}\big] \longrightarrow \mathbb{A}^1_{\mathbb{Z}[\frac{1}{6}]}.$$

Claim 4.8. *The scheme $\mathbb{G}_m \backslash \widetilde{\mathcal{M}}\left[\frac{1}{6}\right] \cong \mathbb{A}^1_{\mathbb{Z}\left[\frac{1}{6}\right]}$ is the course moduli of $\mathcal{M}\left[\frac{1}{6}\right]$.*
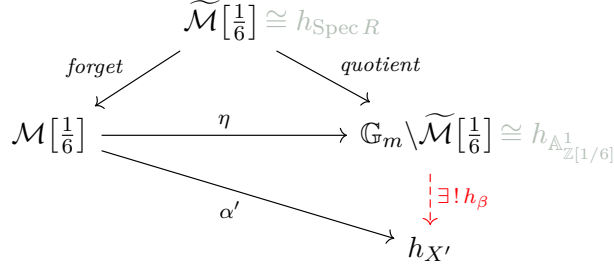

FIGURE 5. verification of coarse moduli

Proof. We check it by the definition of coarse moduli, see Figure 5.

Step1. Construct η. To define $(S \in \mathrm{Sch}_{\mathbb{Z}\left[\frac{1}{6}\right]})$

$$\eta(S) : \mathcal{M}\left[\tfrac{1}{6}\right](S) \longrightarrow \mathbb{G}_m \backslash \widetilde{\mathcal{M}}\left[\tfrac{1}{6}\right](S) \cong \Gamma(S, \mathcal{O}_S),$$

we first find differential locally (locally lift to $\widetilde{\mathcal{M}}\left[\frac{1}{6}\right]$)[7], and then take quotient. You need to check:

- j-function doesn't depend on the choice of differential, so $\eta(S)$ is well-defined;
- η is really a functor.

Step2. We know that GIT quotient is a categorical quotient, so there exists unique h_β such that

$$h_\beta \circ \text{quotient} = \alpha' \circ \text{forget}.$$

You need to check $\alpha' = h_\beta \circ \eta$. $\qquad \square$

Step3. For any closed field $k = \bar{k}$, char $k \neq 2, 3$, the map

$$\eta(k) : \mathcal{M}\left[\tfrac{1}{6}\right](k) \longrightarrow \mathbb{G}_m \backslash \widetilde{\mathcal{M}}\left[\tfrac{1}{6}\right](k) \cong k$$

is an isomorphism.

4.3. **Level structure.** The second possible extra structure is the level structure.

4.4. **Complex case.** In this subsection, we will show that how the moduli is connected to the modular curve $\mathcal{H}/\operatorname{SL}_2(\mathbb{Z})$.

5. MODULI OF HIGHER DIMENSIONAL VARIETY, MMP

I would add something here if I know.

5.1. **Moduli of algebraic surfaces.** We know the Enriques–Kodaira classification, which contributes to a better understanding of algebraic surfaces. But that's not enough. Do we know the classification of K3-surfaces?

5.2. **MMP.** Here we refer to the survey [**?**], or the updated version. You can get a glimpse of the current progress in MMP.

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[7]We know that $\Omega_{E/S} \cong \pi^* e^* \Omega_{E/S} \overset{loc}{\cong} \pi^* \mathcal{O}_S \cong \mathcal{O}_X$ locally.

6. MODULI OF VECTOR BUNDLE

The course lecture note "Moduli and GIT" would be a perfect survey to begin with. We also refer to [?]. It's not easy to read, but It's in some sense completed, and everybody refers it.

For a variant, you may get some informations of moduli of G-bundles over elliptic curve in [?].

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