MODULI IN ALGEBRAIC GEOMETRY

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Abstract. In this personal survey, we conclude the definitions of moduli functors in the algebraic geometry. Most of the results are in the black box, so it's very possible that they're wrong. And also I'm not responsible for the completeness of the whole theory. However, I'm still happy to improve this document, and make it better over time.

Contents

1. Goal and related concepts

The personal survey is motivated by the three courses in Bonn: "the moduli space of curves", "moduli of elliptic curves" and "moduli of vector bundles". I want to construct my personal understanding on the moduli, and find out the details I missed in the courses.

"Some mathematicians are birds, others are frogs." This document is devoted to those "birds" from a "frog" who gets stuck in the mud.

1.1. **representable functor.** In this subsection we would follow on [1, Definition 2.2.1] in full generality, but you can always think

$$\mathcal{C} = \mathrm{Sch}_k \qquad \mathcal{C}' = \mathrm{Functor}(\mathrm{Sch}_k^{op}, \mathrm{Set})$$

to visualize the statement, and you can refer to [2, 6.6.2] to see basic examples.

Definition 1.1 (functor category)**.** *Fix a category $\mathcal{C}$, we define the corresponding functor category $\mathcal{C}'$ as follows:*

$$\mathrm{Ob}(\mathcal{C}') := \{\textit{functors } \mathcal{C}^{op} \longrightarrow \mathrm{Set}\} \qquad \mathrm{Mor}(\mathcal{C}') := \left\{\textit{natural trans } \begin{tikzcd} \mathcal{C}^{op} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \mathrm{Set} \end{tikzcd} \right\}$$

In brief, $\mathcal{C}' = \mathrm{Functor}(\mathcal{C}^{op}, \mathrm{Set})$.

Proposition 1.2. *We have a canonical functor*

$$\iota : \mathcal{C} \longrightarrow \mathcal{C}' \qquad X \longmapsto h_X := \mathrm{Mor}_{\mathcal{C}}(-, X)$$

which embeds $\mathcal{C}$ as a full subcategory of $\mathcal{C}'$.

Definition 1.3 (representable functor)**.** *The functor $F \in \mathrm{Ob}(\mathcal{C}')$ is called represented by X if $F \cong h_X$.*

From this proposition, we can always view the object as some functor satisfying some properties. we get three advantages from this point of view:

- It's easy to see rational points and complex points;
- We can define scheme canonically, without explicit constructions;
- We can enlarge our area of research, and think them as the defective schemes. We will see some reasonable functors which is represented not by schemes, but by stacks.

1.2. **coarse moduli space.** We need to define the fine moduli space, coarse moduli space and some related concepts.

1.3. **corepresentable functor.** Now this is the second concept "dual to" the representable functor. To motivate, we begin with the equivalent definition of representable functor:

Definition 1.4 (representatable functor, equivalent definition)**.** ...

Definition 1.5 (corepresentable functor)**.** *A functor $F \in \mathrm{Ob}(\mathcal{C}')$ is corepresented by $X \in \mathrm{Ob}(C)$ if F satisfies the following universal properties:*

- *There exists a morphism $\alpha : F \longrightarrow h_X$ in $\mathrm{Mor}_{\mathcal{C}'}(F, h_X)$;*
- *For any object $X' \in \mathrm{Ob}(\mathcal{C})$ and morphism $\alpha' : F \longrightarrow h_{X'}$ in $\mathrm{Mor}_{\mathcal{C}'}(F, h_{X'})$, there exists an unique morphism $\beta \in \mathrm{Mor}_{\mathcal{C}}(X, X')$ such that $\alpha' = h_\beta \circ \alpha$.*

$$\begin{tikzcd} F \arrow[r, "\alpha"] \arrow[dr, "\alpha'"'] & h_X \arrow[d, dashed, "\exists!\, h_\beta"] \\ & h_{X'} \end{tikzcd}$$

Definition 1.6 (universal corepresentable functor)**.**

Proposition 1.7. *Suppose a functor $F \in \mathrm{Ob}(\mathcal{C}')$ is corepresented by $X \in \mathrm{Ob}(C)$, then it is represented by X if and only if $\alpha : F \longrightarrow h_X$ is a $\mathcal{C}'$-isomorphism.*

1.4. **stack.** We need to define stack, algebraic stack, Deligne-Munford stack and some related concepts.

1.5. **goal.**

2. Basic object

In this section, we present some algebraic geometric objects which can be viewed as moduli.

Here is a picture showing the relationships of these objects:

???

2.1. **projective space.**

2.2. **Grassmannian.** It's well-written in [2, 16.7].

2.3. flag variety, partial flag variety.

2.4. Hilbert scheme.

2.5. Quot scheme.

2.6. Misc. The representable functor is also used to construct the fibered product of schemes, see [2, 9.1.6-7] for more details.

3. MODULI OF CURVE

The content of this section is already well written in the course "the moduli space of curves". This section is just for the completeness of the survey.

4. MODULI OF ELLIPTIC CURVE

The elliptic curve theory is especially rich compared to the other curves. That's why we'd like to put it a special section.

4.1. differential.

4.2. level structure.

4.3. complex case. In this subsection, we will show that how the moduli is connected to the modular curve $\mathcal{H}/\operatorname{SL}_2(\mathbb{Z})$.

5. MODULI OF VECTOR BUNDLE

Here we refer to [1]. It's not easy to read, but I don't know the other better reference.

References

[1] Daniel Huybrechts and Manfred Lehn. *The geometry of moduli spaces of sheaves.* Cambridge University Press, 2010.

[2] Ravi Vakil. MATH 216: Foundations of Algebraic Geometry.

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