

MODULI IN ALGEBRAIC GEOMETRY

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ABSTRACT. In this personal survey, we conclude the definitions of moduli functors in the algebraic geometry. Most of the results are in the black box, so it's very possible that they're wrong. And also I'm not responsible for the completeness of the whole theory. I make no claim to originality. However, I'm still happy to improve this document, and make it better over time.

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7. AFTER FOAG: THE FUTURE PLAN

When I finished reading the book [16], I felt the confidence of understanding everything in algebraic geometry field. However, I felt soon so confused and helpless, because of the superabundance of topics and articles which are not linearly developed, and they intersected with each other. I had totally no idea what goal to set and what to read. Is it still possible to organize all the (relatively advanced) basics in algebraic geometry, just like Prof. Vakil did in [16]?

This survey is one important part of the whole plan, which aimed to fill in everything well-known to experts but unknown for me. Chinese discussion can be found [here](#).

- A series of classes, such as complex algebraic surfaces, toric varieties, Abelian varieties, finite group schemes, ...
- Moduli theory. It is this survey, even though we missed still a lot:
 - Modular curve and Shimura variety
 - Fermat's last theorem

I wonder If I would say anything about these in this survey.

- Cohomology theory. In my mind Prof. Scholze's picture illustrates it in a magical way. In particular, we need to show:
 - Étale cohomology
 - Derived category?
 - The proof of Weil's conjectures
- Use scheme-analogic models to solve problems.
 - Berkovich spaces, p-adic spaces, and formal schemes.
 - Theory of height. The typical example is Faltings's theorem.
- Anything with analysis. Index theory and symplectic geometry can be related.
- Anything with representation theory and number theory. Langlands program is for me the central goal. Class field theory and Iwasawa theory can be also relative topics.
- Anything with combination. For example, the dessin d'enfant, knot, number field, ...

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