LaTeX TEMPLATE

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Contents

0.1. **Quot scheme.** We refer to wiki and Venkatesh's lecture notes.

The Quot scheme is a relative version of the Grassmannian. Instead of parameterizing subvector spaces, it parameterizes quotient sheaves of a fixed sheaf.

Example 0.1 (Quot scheme $\mathrm{Quot}_{\mathcal{E}}$)**.** *In this example, the field κ can be generalized to a Noetherian base scheme S_0.*

For a scheme X/κ of finite type and $\mathcal{E} \in \mathrm{Coh}(X)$, we define a moduli problem for $(S, \pi_S : S \longrightarrow \operatorname{Spec} \kappa) \in \mathrm{Ob}(\mathrm{Sch}_\kappa)$:

$$\mathcal{A}_S := \left\{ (\mathcal{F}, \pi) \;\middle|\; \begin{array}{l} \mathcal{F} \in \mathrm{Coh}(X_S) \text{ flat over } S \\ \mathrm{supp}(\mathcal{F}) \text{ proper over } S \\ \pi : \mathcal{E}_S \longrightarrow\!\!\!\!\!\rightarrow \mathcal{F} \end{array} \right\}$$

$(\mathcal{F}, \pi) \sim_S (\mathcal{F}', \pi')$ if there exists an isomorphism of vector bundles $\phi : \mathcal{F} \longrightarrow \mathcal{F}'$ such that $\pi' = \phi \circ \pi$.

For a map $f : T \longrightarrow S$, the pullback f^* is defined by

$$f^* : \mathcal{A}_S \longrightarrow \mathcal{A}_T \qquad (\mathcal{F}, \pi) \longmapsto \big((\mathrm{Id} \times f)^*\mathcal{F}, f^*\pi : \mathcal{E}_T \twoheadrightarrow (\mathrm{Id} \times f)^*\mathcal{F}\big)$$

The moduli functor defined by this extended moduli problem is representable, and we denote it by $\mathrm{Quot}_{\mathcal{E}}$.

Remark 0.2. The requirements of flatness and properness ensure the well-definedness of the Hilbert polynomial $P_{\mathcal{F}}(t) \in \mathbb{Q}[t]$, defines as follows.

When a line bundle $\mathcal{L}$ over X is fixed, take a closed point $s \in S$, then

$$P_{\mathcal{F}}(m) = \chi(\mathcal{F}_s \otimes \mathcal{L}_s^{\otimes m}) = \sum_{i=0}^{\dim \mathcal{F}} (-1)^i h^i(X_s, \mathcal{F}_s \otimes \mathcal{L}_s^{\otimes m}).$$

By flatness, the Hilbert polynomial $P_{\mathcal{F}}(m)$ does not depend on the choice of $s \in S$. Furthermore, when X is projective, one can take $\mathcal{L} = \mathcal{O}_X(1)$. This provides us a decomposition of $\mathrm{Quot}_{\mathcal{E}}$:

$$\mathrm{Quot}_{\mathcal{E}} = \bigsqcup_{P \in \mathbb{Q}[t]} \mathrm{Quot}_{\mathcal{E}}^P .$$

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Example 0.3. *When $X = \operatorname{Spec}\kappa$, $\mathcal{E} = \kappa^n$, one gets*

$$\mathcal{A}_S = \left\{ (\mathcal{F}, \pi) \,\middle|\, \begin{array}{l} \mathcal{F} \in \operatorname{Coh}(S) \text{ flat over } S \\ \pi : \mathcal{O}_S^{\oplus n} \twoheadrightarrow \mathcal{F} \end{array} \right\}$$

and $P_{\mathcal{F}}(t) = \chi(\mathcal{F}_s)$ is constant, indicating the rank of $\mathcal{F}$. Therefore,

$$\operatorname{Gr}(k, n) = \operatorname{Quot}^k_{\kappa^n}, \qquad \mathbb{P}V^\vee = \operatorname{Quot}^1_V.$$

Example 0.4. *In this example, the base scheme S_0 is X, and $\mathcal{E} \in \operatorname{Coh}(X)$ is locally free. In this setting, the moduli problem for ??? is given by*

$$\mathcal{A}_S = \left\{ (\mathcal{F}, \pi) \,\middle|\, \begin{array}{l} \mathcal{F} \in \operatorname{Coh}(S) \text{ flat over } S \\ \pi : \pi_S^* \mathcal{E} \twoheadrightarrow \mathcal{F} \end{array} \right\}$$

Since any finitely generated flat module over a commutative local, Noetherian ring is free (see SE1812584), $\mathcal{F}$ is locally free of rank r, and $P_{\mathcal{F}}(t) = r$. Therefore,

$$\mathbb{P}\mathcal{E} = \operatorname{Quot}^1_{\mathcal{E}} \ \text{(over } X/X\text{)}.$$

Remark 0.5. The tangent space and smoothness of $\operatorname{Quot}_{\mathcal{E}}$ can be described locally, just like $T_V \operatorname{Gr}(k, n) = \operatorname{Hom}_{\mathbb{C}}(V, \mathbb{C}^n/V)$, see [?] for more details.

0.2. **Hilbert scheme.** Since we already defined $\operatorname{Quot}_{\mathcal{E}}$, the Hilbert scheme is just a special case:

$$\operatorname{Hilb}_X := \operatorname{Quot}_{\mathcal{O}_X} \qquad \operatorname{Hilb}^P_X := \operatorname{Quot}^P_{\mathcal{O}_X}$$

where the moduli problem is given by

$$\mathcal{A}_S = \left\{ (\mathcal{F}, \pi) \,\middle|\, \begin{array}{l} \mathcal{F} \in \operatorname{Coh}(X_S) \text{ flat over } S \\ \operatorname{supp}(\mathcal{F}) \text{ proper over } S \\ \pi : \mathcal{O}_{X_S} \twoheadrightarrow \mathcal{F} \end{array} \right\}$$

In this case, $\mathcal{F} \cong i_* \mathcal{O}_Z$ for some subvariety (maybe non-reduced) of X_S (of Hilbert polynomial P).

Example 0.6. *The Hilbert scheme of k points on X is denoted by*

$$X^{[k]} := \operatorname{Hilb}^k_X = \operatorname{Quot}^k_{\mathcal{O}_X}.$$

It is birational equivalent to X^k/S_k.

Example 0.7. *Recall that $P_{\mathbb{P}^m}(t) = \binom{m+t}{m}$. For any closed integral subvariety $X \subset \mathbb{P}^n$, the Fano variety of m-planes $\mathbb{P}^m$ in X is given by*

$$F(X, m) := \operatorname{Hilb}^{P_{\mathbb{P}^m}}_X = \operatorname{Quot}^{P_{\mathbb{P}^m}}_{\mathcal{O}_X}.$$

Example 0.8. *Recall that $P_Z(t) = \binom{n+t}{n} - \binom{n+t-d}{n}$ for a degree d hypersurface Z in $\mathbb{P}^n$. The moduli space of degree d hypersurfaces in $\mathbb{P}^n$ is given by*

$$\operatorname{Hilb}^{P_Z}_{\mathbb{P}^n} = \operatorname{Quot}^{P_Z}_{\mathcal{O}_{\mathbb{P}^n}} \cong \mathbb{P}\left(\Gamma\left(\mathbb{P}^n, \mathcal{O}(d)\right)\right) \cong \mathbb{P}^{\binom{n+d}{d}-1}.$$

References

[1] Jens Niklas Eberhardt. K-motives and Koszul duality. *Bulletin of the London Mathematical Society*, 54(6):2232–2253, 2022.

[2] Ravi Vakil. The rising sea: Foundations of algebraic geometry. *preprint*, 2017.

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