Analysis and Differential Equations (1)

Problem1. Show that there is a function $f : \mathbb{R}^2 \to \mathbb{R}$ that is not continuous, but is continuous when restricted to any straight line.

Problem2. Prove that there is no one-to-one conformal map from the punctured unit disk $\{z : 0 < |z| < 1\}$ onto the annulus $\{z : 1 < |z| < 2\}$

Problem3. Let $1 \le p < \infty$. Suppose $U(z)$ is a harmonic function on the complex plane such that

$$\int\int_{\mathbb{R}\times\mathbb{R}} |U(x+iy)|^p dxdy < \infty.$$

Prove that $U(z) = 0$ for all $z \in \mathbb{C}$.

Problem4. Consider the following parabolic equation

$$\theta_t = \Delta((|x|^2+1)\theta) + |D\theta| - 4n\theta \quad for\ (x,t) \in \mathbb{R}^n \times (0,+\infty)$$

(a) Let $\theta_1(x,t)$ and $\theta_2(x,t)$ be two smooth, nonnegative solutions of the above equation which vanishes at infinity, with ordered initial data $\theta_1(x,0) \le \theta_2(x,0)$. Show that then $\theta_1(x,t) \le \theta_2(x,t)$ for all $t > 0$.
(b) Let θ be a smooth, nonnegative, integrable solution of above equation, where all its derivatives and its products with $|x|^2$ vanish at $|x| \to +\infty$. Show that $\int \theta(\cdot,t)dx$ exponentially decays to zero as $t \to +\infty$.