Analysis and Differential Equations (4)

Problem1. Let $\hat{f}(\xi) = \int_{\mathbb{R}} e^{-i\xi x} f(x)dx$ be the Fourier transform on Schwartz functions $f \in \mathcal{S}(R)$. Suppose $f \in \mathcal{S}(R)$ satisfies $f(2\pi n) = 0$ and $\hat{f}(n) = 0$ for all integers n. Prove or disprove that f must be the zero function.

Problem2.Let S be a closed subspace of $C[0,1]$ (with the sup norm). Suppose that $f \in S$ implies that f is continuously differentiable. Prove that S is finite dimensional.

Problem3. Prove that there exists a constant C such that for every polynomial P of degree 2014,

$$P(0) \leq C \int_0^1 |P(x)|dx$$

Problem4. Let f be a meromorphic function on $\mathbb{C}$ that is analytic in a neighborhood of 0. Let its Maclaurin series be

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

Assume that all k, a_k are real and $a_k \geq 0$, that there are no poles at z with $|z| < r$, and that there is at least one pole at z with $|z| = r$. Show that there is a pole at $z = r$.

Problem5. Let Q be the unit cube in $\mathbb{R}^n$, and let $H_0^2(Q)$, be the closure of C^∞ functions supported in Q with respect to the norm whose square is $||f||^2_{H_0^2(Q)} = \int_Q |f|^2 + |\nabla f|^2 + |\nabla^2 f|^2$.

(a) Prove that for every $\epsilon > 0$, there exists a constant C_ϵ so that for all $f \in H_0^2(Q)$, $\int_Q |\nabla f|^2 \leq \epsilon \int_Q |\nabla^2 f|^2 + C_\epsilon \int_Q |f|^2$.

(b) Suppose that u is in the Hölder space $C^{1,\alpha}(I)$, $\alpha \in (0,1), I = [a,b]$. Prove that for every $\epsilon > 0$, there exists C_ϵ such that

$$\sup_I |u'(x)| \leq \epsilon [u]_{I;1,\alpha} + C_\epsilon \sup_I |u|, \ \ \textit{where}\ [u]_{I;1,\alpha} = \sup_{x \neq y, x,y \in I} \frac{|u'(x) - u'(y)|}{|x-y|^\alpha}.$$

(c) Let $L = L_0 + \sum_{i=1}^n b_i(x) \frac{\partial}{\partial x_i} + c(x)$, $L_0 = \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j}$,with C^2 coefficients. Suppose that for every $u \in H_0^2(Q)$, $||u||^2_{H_0^2(Q)} \leq C(||L_0 u||^2_{L^2(Q)} + ||u||^2_{L^2(Q)})$. Prove, using (a) that a similar estimate holds if L_0 is replaced by L.