Analysis and Differential Equations (3)

Problem1. Show that

$$\int_4^9 \sqrt{-6+5\sqrt{-6+5\sqrt{-6+5\sqrt{-6+5\sqrt{x}}}}}dx$$

is a rational number.

Problem2. Suppose g and f_n are non-negative integrable functions such that $\int f_n \to 0$ as $n \to \infty$ and $f_n^2 \leq g$ for all n. Prove or find a counterexample to the statement that $\int f_n^4 \to 0$ as $n \to \infty$.

Problem3. Show that if $f : \mathbb{C} \to \mathbb{C} \cup \infty$ is a meromorphic function, such that there exist $R, C > 0$ so that for $|z| > R, |f(z)| \leq C|z|^n$, then f is a rational function.

Problem4. Prove or disprove that there is a sequence $\{f_n\}$ of continuous functions on $\mathbb{R}$ such that for any rational x the sequence $f_n(x)$ is bounded but the sequence $f_n(x+\sqrt{2})$ is unbounded.

Problem5

(1) Suppose $f \in L^1(\mathbb{R}), f \geq 0$ and $f \neq 0$ as an element in $L^1(\mathbb{R})$. Let $\hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-i\xi x}dx$ be the Fourier transformation of f. Prove that $\sup|\hat{f}|$ is attained exactly at 0.

(2) Suppose $f \in \mathcal{S}(\mathbb{R}), f \geq 0, \int_{\mathbb{R}} f(x)dx = 1, \int_{\mathbb{R}} xf(x)dx = 0$. Let $f_1 = f, f_k = f_{k-1} * f, k \geq 2$. Show that with $g_k(x) = kf_k(kx), g_k \to \delta_0$ in the distributions as $k \to \infty$. Here $\mathcal{S}(\mathbb{R})$ denotes the Schwartz space on $\mathbb{R}$, i.e.

$$S(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) \big| \sup_{x\in\mathbb{R}} |x^k f^{(j)}(x)| < \infty, \forall\, k, j \in \mathbb{N}\}$$