EXTRA KNOWLEDGE

XIAOXIANG ZHOU

ABSTRACT. We will talk about some actions induced by the Lie group structure.

1. Lie Group Action

$G \circlearrowright M$ is a group + smooth action. Now suppose $g \in G; X, Y \in \mathfrak{g}; \beta \in \mathfrak{g}^*$.

$$\begin{aligned} c_g : G &\longrightarrow G \\ a &\mapsto gag^{-1} \end{aligned}$$

$$\rightsquigarrow \begin{bmatrix} Ad_g \hat{=} (dc_g)_e : T_eG \longrightarrow T_eG \\ (\ \mathfrak{g} \longrightarrow \mathfrak{g}\ ) \\ Ad : G \longrightarrow Aut(\mathfrak{g}) \quad (G \circlearrowright \mathfrak{g}) \\ g \ \mapsto \ Ad_g \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} Ad_g^* : \mathfrak{g}^* \longrightarrow \quad \mathfrak{g}^* \\ \beta \ \mapsto \begin{bmatrix} Ad_g^*\beta : \mathfrak{g} \longrightarrow \quad \mathbb{R} \\ X \mapsto \beta(Ad_{g^{-1}}X) \end{bmatrix} \\ Ad : G \longrightarrow Aut(G) \quad (G \circlearrowright \mathfrak{g}) \\ g \ \mapsto \ Ad_g \end{bmatrix}$$

$$☆ : \left\langle Ad_g^*\beta, X \right\rangle = \left\langle \beta, Ad_{g^{-1}}X \right\rangle$$

$$Ad^* : G \circlearrowright \mathfrak{g}^*$$

- Coadjoint orbit of $\beta \in \mathfrak{g}^*$:
  $\mathcal{O}_\beta = \{Ad_g^*\beta \mid g \in G\} \subseteq \mathfrak{g}^*$
- Stabilizer of $\beta \in \mathfrak{g}^*$:
  $G_\beta = \{g \in G \mid Ad_g^*\beta = \beta\} \subseteq G$
- $\mathcal{O}_\beta = G/G_\beta$

$$\begin{aligned} [\ ,\ ] : \mathfrak{g} \times \mathfrak{g} &\longrightarrow \mathfrak{g} \\ (X, Y) &\mapsto [X, Y] \end{aligned}$$

$$\rightsquigarrow \begin{bmatrix} ad_X : \mathfrak{g} \longrightarrow \mathfrak{g} \\ Y \longrightarrow [X, Y] \\ ad : \mathfrak{g} \longrightarrow Aut(\mathfrak{g}) \quad (\mathfrak{g} \circlearrowright \mathfrak{g}) \\ X \ \mapsto \ ad_X \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} ad_X^* : \mathfrak{g}^* \longrightarrow \quad \mathfrak{g}^* \\ \beta \ \mapsto \begin{bmatrix} ad_X^*\beta : \mathfrak{g} \longrightarrow \quad \mathbb{R} \\ Y \mapsto \beta(ad_{-X}Y) \\ = -\beta([X, Y]) \end{bmatrix} \\ ad^* : \mathfrak{g} \longrightarrow Aut(\mathfrak{g}) \quad (\mathfrak{g} \circlearrowright \mathfrak{g}) \\ X \ \mapsto \ ad_X^* \end{bmatrix}$$

$$☆ : \langle ad_X^*\beta, Y \rangle = \langle \beta, ad_{-X}(Y) \rangle = -\beta([X, Y])$$

$$ad^* : \mathfrak{g} \circlearrowright \mathfrak{g}^*$$

- Orbit of $\beta \in \mathfrak{g}^*$:
  $\mathfrak{g} \cdot \beta = \{ad_X^*\beta \mid X \in \mathfrak{g}\} \xlongequal{Verify} T_\beta\mathcal{O}_\beta$
- Stabilizer of $\beta \in \mathfrak{g}^*$:
  $\mathfrak{g}_\beta = \{X \in \mathfrak{g} \mid ad_X^*\beta = 0\}$
- $T_\beta\mathcal{O}_\beta = \mathfrak{g}/\mathfrak{g}_\beta$

References

SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA, HEFEI, 230026, P.R. CHINA,

Email address: email:xx352229@mail.ustc.edu.cn