Exercise of Yau contest

1 Let G be a group with a normal subgroup N such that $N \cong \mathbb{Z}$ and $G/N \cong \mathbb{Z}/n\mathbb{Z}$. Show that G is abelian if n is odd.

2 Let R be an integral domain. Prove that R is a UFD if and only if each nonzero prime ideal of R contains a prime element.

3 Let p be an odd prime number. Define complex number $\zeta = e^{\frac{2i\pi}{p^2}}$ and $\alpha = \sqrt[p]{p}\zeta$, where $\sqrt[p]{p}$ denotes the p-th root of p in the field of real numbers.

(1)Determine that $[\mathbb{Q}(\alpha, \zeta) : \mathbb{Q}]$ and $[\mathbb{Q}(\alpha) : \mathbb{Q}]$.

(2)Compute $Gal(\mathbb{Q}(\alpha)/\mathbb{Q})$.

(3)Find the number of intermediate fields F of the extension $\mathbb{Q}(\alpha, \zeta)/\mathbb{Q}$ such that $[F : \mathbb{Q}] = p^2$.

4 Find all the orders of finite subgroups of $SL_2(\mathbb{Z})$.

5 Let ζ be a root of unity satisfying an equation $\zeta = 1 + N\eta$ for an integer $N \geqslant 3$ and an algebraic integer η. Show that $\zeta = 1$.

6 Let $\mathbb{C}[x, y, z]$ and $\mathbb{C}[s, t]$ be the rings of polynomials, and φ is a $\mathbb{C}$-algebra homomorphism:

$$\mathbb{C}[x, y, z] \to \mathbb{C}[s, t], x \mapsto s, y \mapsto st, z \mapsto t^2$$

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Let R be the image of $\mathbb{C}[x, y, z]$ by φ.

(1)Find the system of generators of the ideal $ker(\varphi)$ whose cardinality is minimal.

(2)For all $(a, b, c) \in \mathbb{C}^3$ such that the ideal $\mathfrak{m} = (s - a, st - b, t^2 - c)$ of R is maximal and the maximal ideal $\mathfrak{m}R_{\mathfrak{m}}$ of $R_{\mathfrak{m}}$ is not generated by two elements.