Analysis and Differential Equations(1)

Problem1. Show that there is a function $f : \mathbb{R}^2 \to \mathbb{R}$ that is not continuous, but is continuous when restricted to any straight line.

Solution1： Take $f(x, y) = \dfrac{xy^2}{x^2 + y^4}$ when $(x, y) \neq (0, 0)$ and 0 when $(x, y) = (0, 0)$.
Solution2: Take f to be zero when $y = 2x^2$ or $y \leq 0$ and one when $y = x^2 (x \neq 0)$, then extend f continuously to $\mathbb{R}^2$

Problem2. Prove that there is no one-to-one conformal map from the punctured unit disk $\{z : 0 < |z| < 1\}$ onto the annulus $\{z : 1 < |z| < 2\}$

Solution： Suppose f is such a map, extend f to the unit disk then use Rouche theorem at 0.

Problem3. Let $1 \leq p < \infty$. Suppose $U(z)$ is a harmonic function on the complex plane such that

$$\int\int_{\mathbb{R}\times\mathbb{R}} |U(x + iy)|^p dxdy < \infty.$$

Prove that $U(z) = 0$ for all $z \in \mathbb{C}$.

Solution: Use mean-value equality and Hölder inequality.

Problem4. Consider the following parabolic equation

$$\theta_t = \Delta((|x|^2 + 1)\theta) + |D\theta| - 4n\theta \quad for\ (x, t) \in \mathbb{R}^n \times (0, +\infty)$$

(a) Let $\theta_1(x, t)$ and $\theta_2(x, t)$ be two smooth, nonnegative solutions of the above equation which vanishes at infinity, with ordered initial data $\theta_1(x, 0) \leq \theta_2(x, 0)$. Show that then $\theta_1(x, t) \leq \theta_2(x, t)$ for all $t > 0$.
(b) Let θ be a smooth, nonnegative, integrable solution of above equation, where all its derivatives and its products with $|x|^2$ vanish at $|x| \to +\infty$. Show that $\int \theta(\cdot, t)dx$ exponentially decays to zero as $t \to +\infty$.

Solution: (1)Let $\theta = \theta_1 - \theta_2$ and $L\theta = \theta_t - \Delta((|x|^2 + 1)\theta) + 4n\theta$, then $L\theta \leq |D\theta|$. Prove θ attains its maximum on the parabolic boundary.

(2)