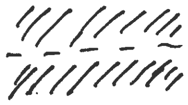
Last time. Affine scheme $(\operatorname{Spec} A, \mathcal{T}_{Zar}, \mathcal{O}_{\operatorname{Spec} A})$

	Scheme		A	$\mathbb{C}[x,y]$	$\mathbb{C}^2$
7.1	open subscheme (not nessesary affine)		A_f	$\mathbb{C}[x,y]_{xy}$	 $y=0$
8	close subscheme	reduced	$A/\sqrt{I}$	$\mathbb{C}[x,y]/(x)$	 $y=0$
		non-reduced	A/I	$\mathbb{C}[x,y]/(x^2)$	 $y=0$
	disjoint union		$A \times B$	$\mathbb{Z}/12\mathbb{Z} = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$	$\bullet$ (2) $\bullet$ (3)
9	fiber product		$A \otimes_{\mathbb{C}} B$	$\mathbb{C}[x,y] = \mathbb{C}[x] \otimes_{\mathbb{C}} \mathbb{C}[y]$	$\mathbb{C}^2 = \mathbb{C} \times \mathbb{C}$

6.3

Sheaf

Notation : $\mathcal{F}(U) = \Gamma(U, \mathcal{F}) = H^0(U, \mathcal{F})$

$\Gamma(U, \mathcal{F})$: stress element as a FUNCTION (Section)

E.g. $X = [0,1]$

$\Gamma([0,1], C^1) = C^1([0,1])$

$\Gamma([0,1], C^\infty) = C^\infty([0,1])$

$X \subseteq \mathbb{C}$ open, $U \subseteq X$ open $(X \rightsquigarrow RS)$.

$\Gamma(U, \mathcal{O}_X) = \mathcal{O}_X(U) = \{$holo ~~funct~~ fcts on $U\}$

$\Gamma(U, M_X) = M_X(U) = \{$mero fcts on $U\}$

$X = \operatorname{Spec} A$

$\Gamma(U, \mathcal{O}_X) = \mathcal{O}_X(U)$

e.p. $\Gamma(X, \mathcal{O}_X) = \mathcal{O}_X(X) = A$ "global fcts on X"

The value of $a \in A$ at $\mathfrak{p} \in \operatorname{Spec} A$ is $\pi(a)$, where

$$\begin{array}{ccccc} \pi : A & \longrightarrow & A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} & = (A/\mathfrak{p})_\mathfrak{p} = & K(A/\mathfrak{p}) \\ \| & & \| & & \\ \mathcal{O}_X(X) & & \mathcal{O}_{X,\mathfrak{p}}/m_{X,\mathfrak{p}} & & \end{array}$$

e.g. $X = \mathbb{A}^1_{\mathbb{C}}$, the function x^3+2x-1.

take value $\underline{i-1 \quad \text{in} \quad \mathbb{C}[x]/(x-i)}$ at $(x-i)$

$\underline{x^3+2x-1 \quad \text{in} \quad \mathbb{C}(x)}$ at (0)

Isomorphism of Sheaves (c.f. 2.3.1)

$\phi: \mathcal{F} \to \mathcal{G}$ is isomorphism, if $\exists \psi: \mathcal{G} \to \mathcal{F}$, s.t

$\forall U \overset{open}{\subset} X$, $\phi(U) \circ \psi(U) = Id_{\mathcal{G}(U)}$

$\psi(U) \circ \phi(U) = Id_{\mathcal{F}(U)}$

This time Scheme $(X, \mathcal{T}_{Zar}, \mathcal{O}_X)$

After considering the affine scheme (domain in ~~$\mathbb{R}$~~ $\mathbb{R}$ or $\mathbb{R}^n$) we naturally consider the scheme (~~≈~~ smooth mfld).

Smooth Structure	~~She~~ Structure Sheaf
|	|
Topo (Euclid)	Topo (Zariski)
|	|
Set	Set
Manifold	Scheme

Remark: $(\mathbb{C}, \mathcal{T}_{Euclid}, \mathcal{O}_{\mathbb{C}})$ is not an (affine) scheme!

holo fcts

1. Def of scheme

Def A Scheme is a Topo space X with a sheaf $\mathcal{O}_X$ on X, satisfying:

$\forall p \in X$, $\exists_{p \in} U \subseteq X$ open s.t

$\Phi: (U, \mathcal{O}_X|_U) \longrightarrow (\mathrm{Spec}A, \mathcal{O}_{\mathrm{Spec}A})$ is an isomorphism, i.e.

① $\Phi: U \xrightarrow{\sim} \mathrm{Spec}A$ iso of topo spaces

② $\Phi^\#: \pi^{-1}(\mathcal{O}_{\mathrm{Spec}A}) \xrightarrow{\sim} \mathcal{O}_X|_U$ iso of ~~scheme~~ sheaves

(or $\mathcal{O}_{\mathrm{Spec}A} \xrightarrow{\sim} \pi_*(\mathcal{O}_X|_U)$)

We'll call $\mathcal{O}_X$ as the structure sheaf of scheme X.

Remarks 1. By defining iso of schemes, we can prove

$\pi: \mathrm{Spec}\, A \longrightarrow \mathrm{Spec}\, B$ iso as scheme

$\Leftrightarrow \pi^\#: B \longrightarrow A$ iso as ring.

$\mathrm{Spec}\,\mathbb{Q} \neq \mathrm{Spec}\,\mathbb{C}$ because of the difference of sheaves!

2. Sheaf is a globally geometry object, so we do not need to make compatibility conditions like smooth mfld.
(However, we still need them when we glue schemes, because we need to construct NEW ~~sheaf~~ sheaf.)

3. Calculation of $\mathcal{O}_X(U)$

Suppose $U = \bigcup_{i\in\Lambda} \operatorname{Spec} A_i$, then we have the LES.

$$0 \longrightarrow \mathcal{O}_X(U) \longrightarrow \prod_{i\in\Lambda} \underbrace{\mathcal{O}_X(\operatorname{Spec} A_i)}_{= A_i} \xrightarrow{\Phi} \prod_{(i,j)\in\Lambda\times\Lambda} \mathcal{O}_X(\operatorname{Spec} A_i \cap \operatorname{Spec} A_j)$$

Then $\mathcal{O}_X(U) \cong \operatorname{Ker} \Phi$.

We especially concern about $\mathcal{O}_X(X) = \Gamma(X, \mathcal{O}_X)$ (global fcts)

2. Constructing New Schemes

2.1. By Universal property (morphisms? $\Rightarrow$ postponed)

① disjoint union of (affine) schemes $(X_1, \mathcal{O}_{X_1}), (X_2, \mathcal{O}_{X_2})$

Scheme structure of $X_1 \amalg X_2$:

$$\begin{tikzcd} & X_1 \arrow[dl] \arrow[dr, "?"] & \\ A & & X_1 \amalg X_2 \arrow[ll, dashed, "\exists!"'] \\ & X_2 \arrow[ul] \arrow[ur, "?"'] & \end{tikzcd}$$

- Set: $X_1 \amalg X_2 = X_1 \sqcup X_2$
- Topo: $U \subset X_1 \amalg X_2$ is open $\Leftrightarrow U \cap X_1 \overset{\text{open}}{\subset} X_1$ & $U \cap X_2 \overset{\text{open}}{\subset} X_2$
- Sheaf.

$$\begin{array}{ccc} \mathcal{O}(U_1 \sqcup U_2) & := & \mathcal{O}_{X_1}(U_1) \times \mathcal{O}_{X_2}(U_2) \\ \downarrow & := & \downarrow \rho_1 \times \rho_2 \\ \mathcal{O}(V_1 \sqcup V_2) & := & \mathcal{O}_{X_1}(V_1) \times \mathcal{O}_{X_2}(V_2) \end{array}$$

- locally $\cong \operatorname{Spec} A$

Eg 1. $\operatorname{Spec} A \amalg \operatorname{Spec} B \cong \operatorname{Spec}(A \times B)$

Similarly, for ~~$\{X_i\}_{i\in\Lambda}$~~ a family of schemes $\{X_i\}_{i\in\Lambda}$, we define schemes $\coprod_{i\in\Lambda} X_i$

Eg 2. $\coprod_{i\in\mathbb{Z}} \operatorname{Spec} \mathbb{C}$ is not an affine scheme because it is not quasicompact.

② Open subscheme (Later: for glue)

$$\left.\begin{array}{l}(X,\mathcal{O}_X) \text{ a scheme} \\ \mathcal{U} \overset{\text{open}}{\subset} X\end{array}\right\} \rightsquigarrow (\mathcal{U}, \mathcal{O}_X|_{\mathcal{U}}) \text{ a scheme}$$

Locally: Suppose $p \in \mathcal{U}$, choose $p \in \operatorname{Spec} A \subset X$

$$\mathcal{U} \cap \operatorname{Spec} A \overset{\text{open}}{\subset} \operatorname{Spec} A$$

$$\Rightarrow_{p\in} \mathcal{U} \cap \operatorname{Spec} A = \bigcup_{i\in\Lambda} D(f_i) = \bigcup_{i\in\Lambda} \operatorname{Spec} A_{f_i}$$

$$\Rightarrow \exists i,\ p \in \operatorname{Spec} A_{f_i}$$

E.g. 3. $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) \rightsquigarrow (\operatorname{Spec} A_f, \mathcal{O}_{\operatorname{Spec} A_f} = \mathcal{O}_{\operatorname{Spec} A}|_{\operatorname{Spec} A_f})$

E.g. 4. $A = k[x,y]$, $\operatorname{Spec} A = \mathbb{A}_k^2$

$\mathcal{U} := \mathbb{A}_k^2 - \{(0,0)\} = \underbrace{\operatorname{Spec} k[x,y]_x}_{\mathcal{U}_1} \cup \underbrace{\operatorname{Spec} k[x,y]_y}_{\mathcal{U}_2}$ is open

$\rightsquigarrow (\mathcal{U}, \mathcal{O}_{\operatorname{Spec} A}|_{\mathcal{U}})$ is a scheme.

Remark $\mathcal{U}$ is not an affine scheme because

$$\mathcal{U} \neq \operatorname{Spec}(\mathcal{O}_{\mathcal{U}}(\mathcal{U}))$$

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathcal{O}_{\mathbb{A}_k^2}(\mathcal{U}) & \longrightarrow & \mathcal{O}_{\mathbb{A}_k^2}(\mathcal{U}_1) \oplus \mathcal{O}_{\mathbb{A}_k^2}(\mathcal{U}_2) & \longrightarrow & \mathcal{O}_{\mathbb{A}_k^2}(\mathcal{U}_1 \cap \mathcal{U}_2) \\
& \| & & \| \qquad \| & & \| \\
0 \longrightarrow & \operatorname{Ker}\varphi & \longrightarrow & k[x,y]_x \oplus k[x,y]_y & \xrightarrow{\varphi} & k[x,y]_{xy} \\
& \| & & \left(\frac{f(x,y)}{x^m}, \frac{g(x,y)}{y^n}\right) & \longmapsto & \frac{f(x,y)}{x^m} - \frac{g(x,y)}{y^n} \\
& k[x,y] & & & &
\end{array}$$

$$\mathcal{O}_{\mathcal{U}}(\mathcal{U}) = \mathcal{O}_{\mathbb{A}_k^2}(\mathcal{U}) = k[x,y]$$

If $(\mathcal{U}, \mathcal{O}_{\mathcal{U}}) = (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$, then

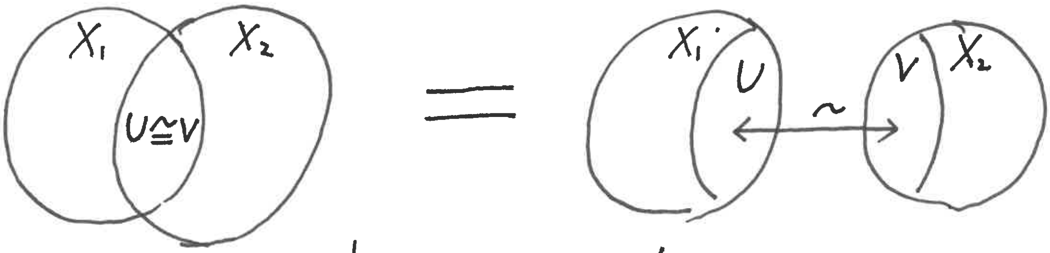
$$\begin{array}{l} A = \mathcal{O}_{\mathcal{U}}(\mathcal{U}) = k[x,y] \\ \operatorname{Spec} A = \operatorname{Spec} k[x,y] \end{array} \Bigg\| \ \emptyset \overset{\mathcal{U}}{=} V(x) \cap V(y) \overset{\operatorname{Spec} A}{=} \{(0,0)\}, \text{ Contradiction!}$$

Sub Remark. 1. We ~~常~~ often prove the failure of a scheme to be an affine scheme by computing the global sections.

2. Hartogs's Lemma: extend over points.

2.2. By gluing (affine) schemes

Recall the glue of two topo spaces: $X_1 \underset{glue}{\cup} X_2 = X_1 \amalg X_2 / \sim$



$$X_1 \xhookrightarrow{l_1} X_1 \underset{glue}{\cup} X_2 \xhookleftarrow{l_2} X_2$$

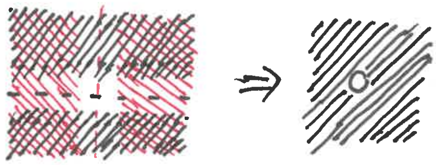
Similarly, if we have

- Schemes $(X_1, \mathcal{O}_{X_1})$, $(X_2, \mathcal{O}_{X_2})$
- $U \subset X_1$ $V \subset X_2$ open
- $(U, \mathcal{O}_{X_1}|_U) \cong (V, \mathcal{O}_{X_2}|_V)$

$\rightsquigarrow$ A new scheme: $Z := X_1 \underset{glue}{\cup} X_2$

- Set & Topo: ✓
- Sheaf: 2.5.D
- Locally affine: ✓

$$\begin{tikzcd} U \arrow[r, phantom, "="] \arrow[d, hook] & V \arrow[r, hook] & X_2 \arrow[d, hook] \\ X_1 \arrow[rr, hook] & & X_1 \underset{glue}{\cup} X_2 \end{tikzcd}$$



E.g. 4' By gluing $\operatorname{Spec} k[x,y]_x$ and $\operatorname{Spec} k[x,y]_y$ with

$$\Phi: \operatorname{Spec}(k[x,y]_x)_y \xrightarrow{\sim} \operatorname{Spec}(k[x,y]_y)_x$$

We get $\mathbb{A}_k^2 - \{(0,0)\}$.

E.g. 5 By gluing $\operatorname{Spec} k[u]$ with $\operatorname{Spec} k[v]$ by

$$\Phi: \operatorname{Spec} k[u]_u \xrightarrow{\sim} \operatorname{Spec} k[v]_v$$

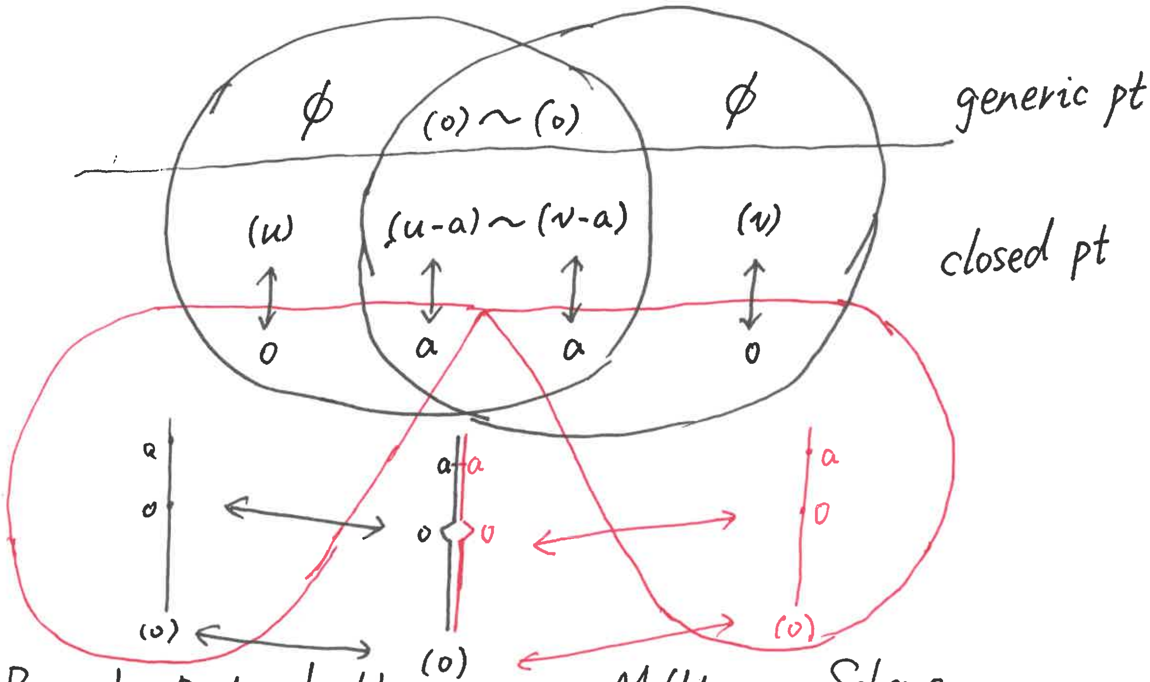
induced by

$$\begin{array}{ccc} k[u]_u & \longleftarrow & k[v]_v \quad : \Phi^\# \\ u & \longleftarrow\!\shortmid & v \end{array}$$

We get the <u>affine line with doubled origin</u>.

When $K = \mathbb{C}$,

$$\Phi : (\mathbb{C} - \{0\}) \sqcup \{(0)\} \xrightarrow{\sim} (\mathbb{C} - \{0\}) \sqcup \{(0)\}$$



<u>Remark</u>. Bad subset!

Mfld $\qquad$ Scheme

Hausdorff $\rightsquigarrow$ separatedness.

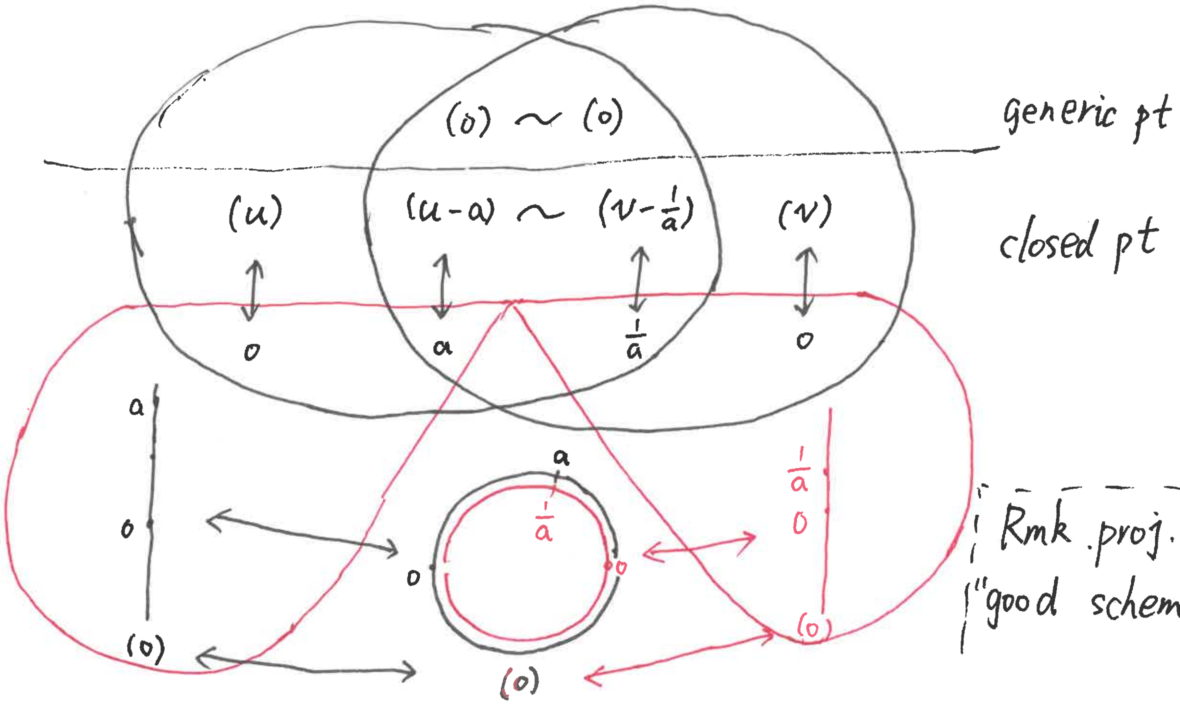
Eg 6. ...... We get the projective line.

When $K = \mathbb{C}$,

$$\Phi : (\mathbb{C} - \{0\}) \sqcup \{(0)\} \xrightarrow{\sim} (\mathbb{C} - \{0\}) \sqcup \{(0)\}$$

$$(0) \longmapsto (0)$$

$$a \longmapsto \frac{1}{a}$$

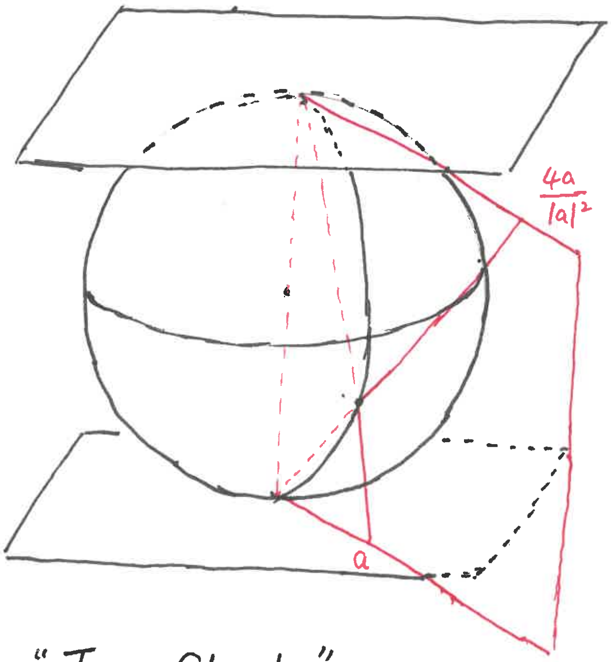


Rmk. proj. line!
"good scheme".

E.x. Compute $\mathcal{O}_X(X)$ in Eg 5. Eg 6, and prove it's not an affine scheme.

Compared with $\mathbb{C}\mathbb{P}^1$:

$$\begin{array}{ccc} & \mathbb{C}\mathbb{P}^1 & \\ & [z_0, z_1] & \\ {\scriptstyle z_1 \neq 0} \swarrow & & \searrow {\scriptstyle z_0 \neq 0} \\ [\frac{z_0}{z_1}, 1] & & [1, \frac{z_1}{z_0}] \\ \mathbb{C} & & \mathbb{C} \end{array}$$



"Two Charts"

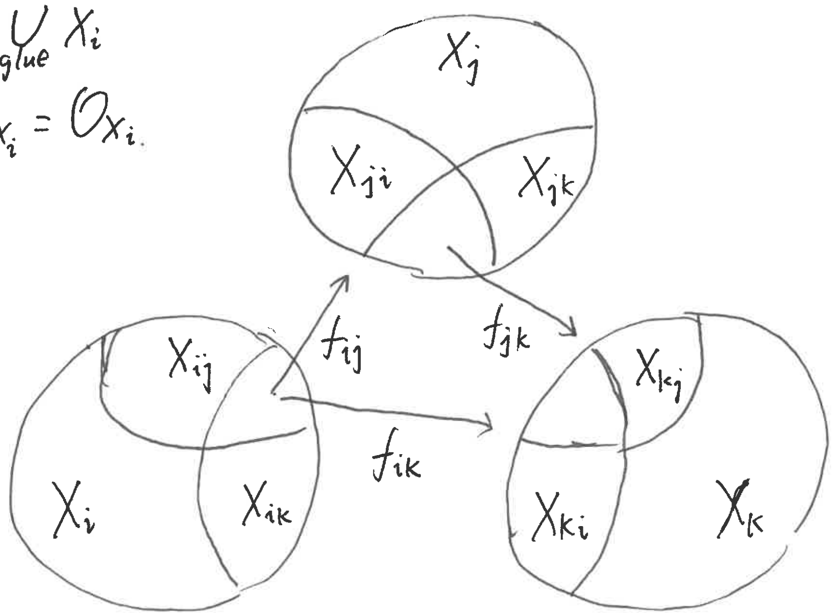
Generalization: gluing schemes. (not just two!)

Thm (4.4.A) Suppose we're given:

- schemes X_i $(i \in I)$
- open subschemes $X_{ij} \subset X_i$ with $X_{ii} = X_i$
- iso $f_{ij} : X_{ij} \longrightarrow X_{ji}$ with $f_{ii} = Id_{X_i}$
- They satisfy the cocycle condition:
  - $f_{ij}(X_{ik} \cap X_{ij}) \subset X_{jk}$
  - $f_{ik}|_{X_{ij} \cap X_{ik}} = f_{jk}|_{X_{ji} \cap X_{jk}} \circ f_{ij}|_{X_{ij} \cap X_{ik}}$.

Then we get a unique scheme $(X, \mathcal{O}_X)$, satisfying

- $X = \bigcup_{glue} X_i$
- $\mathcal{O}_X|_{X_i} = \mathcal{O}_{X_i}$.



Remark. $\operatorname{Spec} A = \bigcup_{f\in A} \operatorname{Spec} A_f = \bigcup_{i\in\Lambda} \operatorname{Spec} A_{f_i}$ if $\langle f_i\rangle_{i\in\Lambda} = A$

can be seen as the glued scheme.

Eg 7. projective space $\mathbb{P}^n_A$

Recall the property of $\mathbb{CP}^n$:

$\binom{4.4.F}{8.2.0}$ • $\mathbb{CP}^n = \mathbb{C}^{n+1}\setminus\{0\}/\sim$

(4.4.9) • $\mathbb{CP}^n$ has local chart $(\mathcal{U}_i, \varphi_i)$

$$\varphi_i : \mathcal{U}_i = \{[t_0,\cdots,t_n] \mid t_i \neq 0\} \longrightarrow \mathbb{C}^n$$
$$[t_0,\cdots,t_n] \longmapsto \left(\frac{t_0}{t_i},\cdots,\widehat{\frac{t_i}{t_i}},\cdots,\frac{t_n}{t_i}\right)$$

and

$$\begin{tikzcd}
 & \mathcal{U}_i \cap \mathcal{U}_j \arrow[dl] \arrow[dr] & \\
\mathbb{C}^n & {[s_{0/i},\cdots,\underset{\uparrow i}{1},\cdots,s_{n/i}]} \arrow[dl, mapsto] \arrow[dr, mapsto] & \mathbb{C}^n \\
(s_{0/i},\cdots,\widehat{s_{i/i}},\cdots,s_{n/i}) \arrow[rr, mapsto, "\psi_{ij}"] & & \left(\frac{s_{0/i}}{s_{j/i}},\cdots,\widehat{\frac{s_{i/i}}{s_{j/i}}},\cdots,\frac{s_{n/i}}{s_{j/i}}\right)
\end{tikzcd}$$

where ψ_{ij} is the transition map.

• $\mathbb{CP}^n = \underset{\substack{[1,\cdots] \\ \mathbb{C}^n}}{\mathbb{C}^n} \sqcup \underset{\substack{[0,\cdots] \\ \mathbb{CP}^{n-1}}}{\mathbb{CP}^{n-1}} = \mathbb{C}^n \sqcup \mathbb{C}^{n-1} \sqcup \cdots \sqcup \mathbb{C}^1 \sqcup \{0\}$

Construction of $\mathbb{P}^n_k$

• Open cover (Thinking $x_{i/j} = \frac{x_i}{x_j}$)

$$X_i = \operatorname{Spec} k[x_{0/i}, x_{1/i}, \cdots, x_{n/i}]/(x_{i/i}-1) \cong \operatorname{Spec} k[x_{0/i}, x_{1/i}, \cdots, \widehat{x_{i/i}}, \cdots, x_{n/i}]$$

• open subschemes

$$X_{ij} = \operatorname{Spec}(A_i)_{x_{j/i}} = \operatorname{Spec} k[x_{0/i}, \cdots, x_{n/i}, \tfrac{1}{x_{j/i}}]/(x_{i/i}-1)$$

$$X_{ii} = \operatorname{Spec}(A_i)_{x_{i/i}} = \operatorname{Spec} A_i = X_i$$

. isomorphisms.

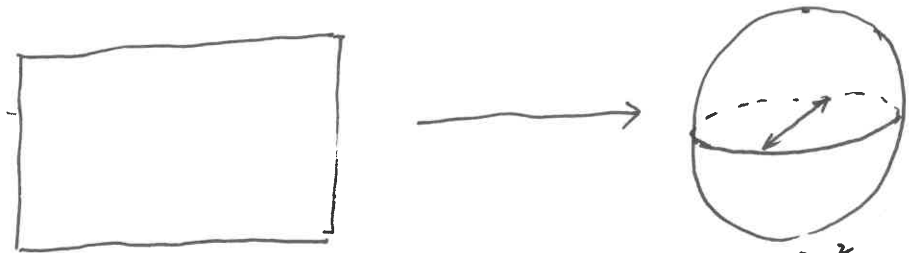
$$f_{ij} : X_{ij} \longrightarrow X_{ji} \text{ induced by}$$

$$k[x_{0/j}, \ldots, x_{n/j}, \tfrac{1}{x_{i/j}}]/(x_{j/j}-1) \longrightarrow k[x_{0/i}, \ldots, x_{n/i}, \tfrac{1}{x_{j/i}}]/(x_{i/i}-1)$$

$$x_{t/j} \longmapsto \frac{x_{t/i}}{x_{j/i}}$$

You can check f_{ij} satisfy the cocycle condition.

Remark We construct proj. space as the compactation of $\mathbb{R}^n$ or $\mathbb{C}^n$, as it is shown below.



$\mathbb{C}^2 \longrightarrow \mathbb{C}\mathbb{P}^2$

$(x, y) \longmapsto [x : y : 1]$

$\operatorname{Spec} k[x, y]$ $\qquad$ $\mathbb{P}_k^2$.

$\{(x,y) \in \mathbb{C}^2 \mid y^2 = 4x^3 + ax + b\} \longrightarrow X := \{[x:y:z] \mid y^2 z = 4x^3 + axz^2 + bz^3\}$

$\operatorname{Spec} k[x,y]/(y^2 - 4x^3 - ax - b)$ $\qquad$?

We want to construct the alg. geo. obj. as the analog of X. But how?

Method 1. Let

$$X_0 = \operatorname{Spec}\left(K[1,y,z]/(y^2z-4-az^2-bz^3)\right)$$
$$X_1 = \operatorname{Spec}\left(K[x,1,z]/(z-4x^3-axz^2-bz^3)\right)$$
$$X_2 = \operatorname{Spec}\left(K[x,y,1]/(y^2-4x^3-ax-b)\right)$$

Then glue them.

Ex. To convince yourself, write down X_{ij} and f_{ij}.

Method 2. Proj construction.

3. Proj Construction.

Advantages: • Glue :(

• See the underlying set & Topo clearly.

• A similar analogue compared with

$A \longleftrightarrow \operatorname{Spec} A$.

Def ($\mathbb{N}$-graded rings)

A $\mathbb{N}$-graded ring is a ring $S_\cdot = \bigoplus_{n\in\mathbb{N}} S_n$, where

- $S_0 \leq S_\cdot$
- S_n is an S_0-module $(S_n + S_n \subseteq S_n)$
- $S_m \times S_n \subseteq S_{m+n}$

E.g. $\mathbb{C}[x,y,z]$ & $\mathbb{C}[x,y,z]/(y^2z-4x^3-axz^2-bz^3)$ are $\mathbb{N}$-graded ring.

Def (homogeneous element/ideal)

Let $S_\cdot = \bigoplus_{n\in\mathbb{N}} S_n$ is a graded ring.

- $a \in S_\cdot - \{0\}$ is called the homo element if $a \in S_n$. define $\deg a := n$
- $I \triangleleft S_\cdot$ is called homo ideal if I can be written as $\langle a_i \rangle_{i\in\Lambda}$ where $a_i \in \bigcup_{n\in\mathbb{N}} S_n$ $\cancel{a_i \in \bigsqcup_{n\in\mathbb{N}} S_n}$.

Prop. (4.5.C) Suppose $I \lhd S$. Define $S_+ = \bigoplus_{i>0} S_i$

- I is homo $\Leftrightarrow I = \bigoplus_{n\in\mathbb{N}} I \cap S_n$
- homo ideals are closed under

  $I+J$, IJ, $I\cap J$, $\sqrt{I}$

- I is prime $\Leftrightarrow \forall a,b \in \bigcup_{n\in\mathbb{N}} S_n$ $\left.\begin{matrix} a\notin I \\ b\notin I\end{matrix}\right\} \Rightarrow ab \notin I$
- I is homo $\Rightarrow S/I$ is graded ring
- $f \in \bigcup_{n\in\mathbb{N}^+} S_n$, then $(S_\cdot)_f$ is graded ring with

  $\{\text{homo prime ideal of } (S_\cdot)_f\} \xleftrightarrow{1:1} \cancel{\mathbb{f}}\ \mathrm{Spec}((S_\cdot)_f)_0$.

Def (f.g. graded ring & generated in deg 1)

Constructing Proj S.

- Set: $\mathrm{Proj}\, S := \{ I \underset{\text{prime}}{\overset{\text{homo}}{\lhd}} S \mid I \not\supseteq S_+ \}$
- Topo: $V(T) := \{ I \in \mathrm{Proj}\, S \mid I \supseteq T \}$ $(T = \{y_i\}_{i\in\Lambda}, \deg y_i > 0)$

  defined as a close set

- Local chart: If $f \in \bigcup_{n\in\mathbb{N}^+} S_n$, then

  $X_f := \mathrm{Spec}((S_\cdot)_f)_0 \leftrightarrow$ homo prime ideals of $(S_\cdot)_f \subseteq \mathrm{Proj}\, S$.

Please verify the compatibility of local chart. (2.5.D)

Left: Topo properties & Hilbert correspondence about Proj S.

E.g. 7. $\mathbb{P}^n_A := \mathrm{Proj}\ A[x_0, \dots, x_n]$

E.g. 8 $\mathbb{P}^2_{\mathbb{C}} := \mathrm{Proj}\ \mathbb{C}[x,y,z]$

$f(x,y,z) = y^2 z - 4x^3 - axz^2 - b^3$

$\rightsquigarrow V(f) = \mathrm{Proj}(\mathbb{C}[x,y,z]/(f))$.

~~Rmemark~~ Remark When $4x^3+ax+b$ has no multiple root,

$\left(\mathrm{Proj}(\mathbb{C}[x,y,z]/(f)),\ (x,\underset{\substack{\uparrow \\ \text{base pt}}}{y-1},z)\right)$ is called the

Complex elliptic curve. Denote the closed points

of $\mathrm{Proj}(\mathbb{C}[x,y,z]/(f))$ as C, then

- C has a group structure
  (compatible with scheme $\rightsquigarrow$ C is an group scheme)
- C is iso to $\mathbb{C}/\Lambda$ as a topo group.

Q. Can we def an analogue of $\mathbb{C}/\Lambda$ in alg geo?

Finally remark. Why do we need to study scheme?

1. to get the geometric understanding of some algebraic concepts.
   e.g. A M graded rings $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ (9.2.E)
   Chinese Remainder Theorem (4.4.1).
2. algebraic function are easier to handle then smooth/analytic function
   e.g. $\mathcal{O}_{\mathbb{C}}(\mathbb{C})$ compared with $\mathcal{O}_{\mathbb{A}^1_{\mathbb{C}}}(\mathbb{A}^1_{\mathbb{C}})$
3. We can generalize it a lot.
   e.g. • treat curves/surfaces with singularity
   • $\mathbb{C} \rightsquigarrow \mathbb{Q}$. $\mathbb{Q}_p$ etc
   $\rightsquigarrow$ arithmetic geometry
   (the application of techniques from AG to problems in number theory)