Solution to Commutative Ring Theory

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This is the solution of the homework in the course 《Commutative Ring Theory》. Problems can be founded on `http://www.wwli.url.tw/downloads/CommRing-2019-HW.pdf`, and I'm glad to any errata to this document.

We suppose A is a commutative ring, $p > 0$ is a prime number.

1 Ring Theory Revisited

1. Suppose $x = 0$, then

$$\begin{aligned} &(1+x)(1-x+x^2-\cdots+(-1)^{n-1}x^{n-1}) = 1 \\ \Rightarrow &1+x \text{ is a unit.} \end{aligned}$$

   If $a \in A^\times, b \in Nil(A)$, then $a^{-1}b \in Nil(A)$,
   $\Rightarrow 1 + a^{-1}b \in A^\times \quad \Rightarrow \qquad a + b = a(1 + a^{-1}b) \in A^\times$

2. (1) ($\Longleftarrow$):
$$\begin{aligned} &a_1, \ldots, a_n \in Nil(A) \\ \Rightarrow &a_1, \ldots, a_n \in Nil(A[x]) \\ \Rightarrow &a_1x, \ldots, a_nx^n \in Nil(A[x]) \qquad a_0 \in A^\times \subset (A[x])^\times \\ \Rightarrow &f \in (A[x])^\times \end{aligned}$$

   ($\Longrightarrow$): If there exists $g = b_0 + b_1x + \cdots + b_mx^m$ such that $fg = 1$, then (denote $a_k = 0$ when $k > n$, $b_k = 0$ when $k > m$)

$$\begin{aligned}
&a_0b_0 = 1 && \Rightarrow a_0, b_0 \in A^\times \\
&a_0b_1 + a_1b_0 = 0 \\
&\quad\vdots \quad \vdots \\
&\sum_{k=0}^{t} a_k b_{t-k} = 0 \\
&\quad\vdots \quad \vdots && \Rightarrow \begin{cases} a_n^{m+1}b_0 = 0 & \Rightarrow a_n^{m+1} = 0 \quad \Rightarrow a_n \in Nil(A) \\ \quad\vdots \quad \vdots \\ a_n^2 b_{m-1} = 0 \\ a_n b_m = 0 \end{cases} \\
&a_{n-1}b_m + a_nb_{m-1} = 0 \\
&a_nb_m = 0
\end{aligned}$$

then $a_0 + a_1x + \cdots + a_{n-1}x^{n-1} = f - a_nx^n$ is nilpotent. Continue the previous process, we obtain that $a_1, \dots, a_n$ are nilpotent.

(2) ($\Longleftarrow$):
$$\begin{aligned}
&a_1, \dots, a_n \in Nil(A) \\
\Rightarrow &a_0, \dots, a_n \in Nil(A[x]) \\
\Rightarrow &a_0, a_1x, \dots, a_nx^n \in Nil(A[x]) \\
\Rightarrow &f = a_0 + a_1x + \cdots + a_nx^n \in Nil(A[x])
\end{aligned}$$

($\Longrightarrow$):
$$\begin{aligned}
& f \in Nil(A) && \Rightarrow \quad f^n = 0 \\
\Rightarrow \quad & a_0^n = 0 && \Rightarrow \quad a_0 \in Nil(A) \\
\Rightarrow \quad & f - a_0 \in Nil(A) \\
\Rightarrow \quad & a_1 + a_2x + \cdots + a_nx^{n-1} \in Nil(A)
\end{aligned}$$

Continue the previous process, we obtain that $a_1, \dots, a_n$ are nilpotent.

(3) ($\Longleftarrow$):Obvious.

($\Longrightarrow$):If $g = b_0 + b_1x + \cdots + b_mx^m$ such that $b_0 \neq 0$ and $fg = 0$, then

$$\begin{aligned}
&a_0b_0 = 0 \\
&a_0b_1 + a_1b_0 = 0 \\
&\quad\vdots \quad\ \vdots \\
&\sum_{k=0}^{t} a_k b_{t-k} = 0 \\
&\quad\vdots \quad\ \vdots \\
&a_{n-1}b_m + a_nb_{m-1} = 0 \\
&a_nb_m = 0
\end{aligned}
\qquad \Rightarrow \begin{cases} a_1b_0^2 = 0 \\ a_2b_0^3 = 0 \\ \ \vdots \quad\ \vdots \\ a_nb_0^{n+1} = 0 \end{cases}$$

$\Rightarrow fb_0^{n+1} = a_0b_0^{n+1} + a_1b_0^{n+1}x + \cdots + a_nb_0^{n+1}x^n = 0$

(4) Denote $fg = \sum_{i=0}^{m+n} c_ix^i$

($\Longleftarrow$):If $I = (a_0, a_1, \ldots, a_n) \neq (1)$, then

$$\begin{aligned}
&(c_0, \ldots, c_{m+n}) \subseteq (a_0, \ldots, a_n) \subsetneqq (1) \\
\Rightarrow &fg \text{ is not primitive, Absurd.}
\end{aligned}$$

($\Longrightarrow$):If $I = (c_0, \ldots, c_{m+n}) \neq (1)$, then we consider the maximal ideal $\mathfrak{M} \supseteq I$, and $A/\mathfrak{M}[x]$, we have:

$$\begin{aligned}
&\bar{f}\bar{g} = 0 \qquad \text{in } (A/\mathfrak{M})[x] \\
\Rightarrow &\exists\ \bar{a} \neq 0 \text{in } A/\mathfrak{M} \text{ such that } \bar{a}\bar{f} = 0 \\
\Rightarrow &(A/\mathfrak{M} \text{ is a field})\bar{f} = 0 \\
\Rightarrow &(a_0, \ldots, a_n) \subseteq \mathfrak{M} \neq (1). \qquad \text{Absurd.}
\end{aligned}$$

3. We prove it by contradiction. Suppose there exists $e \neq 0, 1$ such that $e^2 = e$.

- Denote the unique maximal ideal by $\mathfrak{M}$, then

$$\begin{aligned}
&\forall\ a \notin A^\times, (a) \subsetneqq A \Rightarrow (a) \subseteq \mathfrak{M} \\
\Rightarrow &A \setminus A^\times \subseteq \mathfrak{M} \\
\Rightarrow &A \setminus A^\times = \mathfrak{M} \text{ is an ideal}
\end{aligned}$$

- Let $e' := 1 - e$, then $e' \neq 0, 1$ and $(e')^2 = e'$.
- $\quad e, e'$ are nonunits and $e + e' = 1$

  $\Rightarrow e, e' \in \mathfrak{M}$ while $e + e' \notin \mathfrak{M}$, contradiction!

2 Zariski Topology

4. Suppose $U \subseteq X$ is closed, then there exists $I \lhd A$,

$$\begin{aligned} A \smallsetminus U = V(I) = \bigcap_{f\in I} V(f) \\ \Rightarrow U = A \smallsetminus \bigcap_{f\in I} V(f) = \bigcup_{f\in I} A \smallsetminus V(f) = \bigcup_{f\in I} X_f \end{aligned}$$

Thus $\{X_f\}$ form a basis of open sets s for the Zariski topology.

(1)

$$\begin{aligned} X_f \cap X_g =& (A \smallsetminus V(f)) \cap (A \smallsetminus V(g)) \\ =& A \smallsetminus (V(f) \cup V(g)) \\ =& A \smallsetminus V(fg) \\ =& X_{fg} \end{aligned}$$

(2)

$$\begin{aligned} X_f = \varnothing \Leftrightarrow & V(f) = X \\ \Leftrightarrow & f \in \bigcap_{\mathfrak{p}\in Spec(A)} \mathfrak{p} = Nil(A) \\ \Leftrightarrow & f \text{ is nilpotent.} \end{aligned}$$

(3)

$$\begin{aligned} X_f = X \Leftrightarrow & V(f) = \varnothing \\ \Leftrightarrow & f \notin \bigcup_{\mathfrak{p}\in Spec(A)} \mathfrak{p} = \bigcup_{\mathfrak{M}\in Max(A)} \mathfrak{M} = A \smallsetminus A^{\times} \\ \Leftrightarrow & f \text{ is a unit.} \end{aligned}$$

3 Prime Avoidance

5. (1) Denote

$$\begin{aligned} A :=& k[x,y]/(x,y)^2 \\ =& \{0, 1, x, y, 1+x, 1+y, x+y, 1+x+y\} \end{aligned}$$

Then

$$\begin{aligned} (x,y) =& \{0, x, y, x+y\} \\ =& \{0, x\} \cup \{0, y\} \cup \{0, x+y\} \end{aligned}$$

to be the union of 3 properly small ideals.

(2) Suppose $a \in J$ is a homogeneous element of degree k.

- k is odd: each monomial must have y, so $a \in I_2$.
- If a monomial have y^2, then it equals to 0 in $k[x,y]/(xy,y^2)$, so $a \in I_1$.

Obvious $y \in J \setminus I_1, x^2 \in J \setminus I_2$ and I_2 is prime, because $(k[x,y]/(xy,y^2))/I_2 \cong k[x]$ is a domain.

4 Localization of rings and modules

6. Suppose $0 \notin T$, then T is the multiplication subset, and

$$\begin{aligned} B[S^{-1}] &= \left\{ \frac{b}{f(s)} \mid b \in B, s \in S \right\} \\ &= \left\{ \frac{b}{t} \mid b \in B, t \in T \right\} \end{aligned}$$

as sets. Moreover, $A[S^{-1}]$ acts on $B[S^{-1}]$ and $B[T^{-1}]$ by the exactly same way:

$$A[S^{-1}] \times B[S^{-1}] \longrightarrow B[S^{-1}] \qquad \left(\frac{a}{s_1}, \frac{b}{f(s_2)}\right) \longmapsto \frac{f(a)b}{f(s_1 s_2)}$$

$$A[S^{-1}] \times B[T^{-1}] \longrightarrow B[T^{-1}] \qquad \left(\frac{a}{s_1}, \frac{b}{t}\right) \longmapsto \frac{f(a)b}{f(s_1)t}$$

thus $B[S^{-1}]$ and $B[T^{-1}]$ are isomorphic as $A[S^{-1}]$-modules.

7. (1)⇒(2)⇒(3):Obviously.
(3)⇒(1):Fix $m \in M$, then

$$\begin{aligned} &\frac{m}{1} = \frac{0}{1} \text{ in } M_{\mathfrak{M}} \Leftrightarrow \exists\, s \in A \setminus \mathfrak{M}, sm = 0 \\ \Rightarrow\, & s \in Ann(m) \Rightarrow Ann(m) \nsubseteq \mathfrak{M} \text{ for any } \mathfrak{M} \in Max(A) \\ \Rightarrow\, & Ann(m) = R \Rightarrow 0 = 1 \cdot m = m. \end{aligned}$$

5 Nakayama’s lemma

8. N is a finite generated A-module. So suppose N is generated by $n_1, \ldots, n_k$, then $N/\mathfrak{a}N$ is generated by $\bar{n}_1, \ldots, \bar{n}_k$, suppose $\bar{n}_i = \bar{u}(\bar{m}_i)$, then $\bar{n}_i = \pi \circ u(m_i)$.

So by Nakayama's lemma,

$$\begin{aligned}&\bar{n}_1, \ldots, \bar{n}_k \text{ generate } N/\mathfrak{a}N\\ \Rightarrow& u(m_1), \ldots, u(m_k) \text{ generate} N\\ \Rightarrow& u \text{ is surjective.}\end{aligned}$$

6 Radicals

1. Obviously $Nil(A) \subseteq Rad(A)$.
   If $Nil(A) \subsetneqq Rad(A)$, Let $x \in Rad(A) \setminus Nil(A)$, We obtain

   $$Ax \nsubseteq \sqrt{0} \text{ is an ideal of } A$$

   So there exists $a \in A$ such that $(ax)^2 = ax \neq 0 \Rightarrow ax(1-ax) = 0$
   $\Rightarrow 1 - ax$ is not invertible, thus lies is one maximal ideal $\mathfrak{M}$ of A
   $\Rightarrow 1 = 1 - ax + ax \in \mathfrak{M}$, Contradiction!

7 Noetherian and Artinian rings

2. For any $I \lhd A$, $\sqrt{I}$ is finitely generated, suppose $\sqrt{I} = \langle a_1, \ldots, a_n \rangle$, and $a_i^{r_i} \in I$, then $(\sqrt{I})^{\sum r_i} \subseteq I$.

3. First, we know

   $$\varphi : M/(N_1 \cap N_2) \longrightarrow M/N_1 \times M/N_2 \qquad \bar{m} \longmapsto (\bar{m}, \bar{m})$$

   is a monomorphism,so we can view $M/(N_1 \cap N_2)$ as a submodule of $M/N_1 \times M/N_2$. then

   $$\begin{aligned}&M/N_1 \& M/N_2 \text{ are Noetherian(Artinian)}\\ \Rightarrow& M/N_1 \times M/N_2 \text{ are Noetherian(Artinian)}\\ \Rightarrow& M/(N_1 \cap N_2) \text{ are Noetherian(Artinian)}\end{aligned}$$

8 Support of a Module

4. (1) We have a short exact sequence

   $$0 \longrightarrow \mathfrak{M} \longrightarrow A \longrightarrow k \longrightarrow 0$$

thus induce a exact sequence

$$\mathfrak{M}M \longrightarrow M \longrightarrow k \otimes_A M \longrightarrow 0$$

thus $M_k = k \otimes_A M \cong M/\mathfrak{M}M$.

$$\text{Wrong}[0 = k \otimes_A M \otimes_A N \otimes_A k = M_k \otimes_A N_k \cong M_k \otimes_k N_k]$$

$$0 = M \otimes_A (k \otimes_k k) \otimes_A N \cong M_k \otimes_k N_k$$

$$\Rightarrow M_k = 0 \text{ or } N_k = 0. \text{ Suppose } M_k = 0$$

$$\Rightarrow M = \mathfrak{M}M$$

Then we use the Nakayama's lemma, and we obtain

$$\exists\, a \in \mathfrak{M}, \text{ such that } (1+a)M = 0 \;\Rightarrow\; M = 0$$

(2) We know that

$$\begin{aligned} M_\mathfrak{p} \otimes_{A_\mathfrak{p}} N_\mathfrak{p} &= (A_\mathfrak{p} \otimes_A M) \otimes_{A_\mathfrak{p}} (A_\mathfrak{p} \otimes_A N) \\ &= M \otimes_A A_\mathfrak{p} \otimes_A N \\ &= A_\mathfrak{p} \otimes_A M \otimes_A N \\ &= (M \otimes_A N)_\mathfrak{p} \end{aligned}$$

$$\begin{aligned} &\mathfrak{p} \in Supp(M \otimes_A N) \\ \Leftrightarrow\; & 0 \neq (M \otimes_A N)_\mathfrak{p} = M_\mathfrak{p} \otimes_{A_\mathfrak{p}} N_\mathfrak{p} \\ \Leftrightarrow\; & M_\mathfrak{p} \neq 0 \;\&\; N_\mathfrak{p} \neq 0 \\ \Leftrightarrow\; & \mathfrak{p} \in Supp(M) \cap Supp(N) \end{aligned}$$

9 Support, Associated primes and Primary decompositions

Facts.

- *$Spec(\mathbb{Z}) = \{(p) \mid p \text{ is prime } \} \cup \{(0)\}$*
- *Let $\mathfrak{p} = (p)$, then*

$$\mathbb{Z}_\mathfrak{p} = \left\{ \frac{a}{b} \;\middle|\; p \nmid b \right\}$$

and

$$(\mathbb{Z}/n\mathbb{Z})_\mathfrak{p} \cong \mathbb{Z}_\mathfrak{p} \otimes_\mathbb{Z} \mathbb{Z}/n\mathbb{Z} = \begin{cases} 0, & p \nmid n \\ \mathbb{Z}/p^{\alpha_0}\mathbb{Z}, & n = p^{\alpha_0} p_1^{\alpha_1} \cdots p_n^{\alpha_n} \end{cases}$$

- *Let* $\mathfrak{p} = (0)$,*then*
$$\mathbb{Z}_{\mathfrak{p}} = \left\{ \frac{a}{b} \;\middle|\; b \neq 0 \right\} = \mathbb{Q}$$
and
$$(\mathbb{Z}/n\mathbb{Z})_{\mathfrak{p}} \cong \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} = 0$$
- *localization preserves the exactness.*
- *localization commutes with (infinitely) direct sum.*
- $Spec(\mathbb{C}[T]) = \{(T-a) \mid a \in \mathbb{C}\} \cup \{(0)\}$

1. (Another way: Use $Supp(M) = V(ann_R(M))$ when M is finitely generated)
We know that
$$M_{\mathfrak{p}} \cong \bigoplus_{n>0} (\mathbb{Z}/n\mathbb{Z})_{\mathfrak{p}} \begin{cases} = 0, & \mathfrak{p} = (0) \\ \neq 0, & \mathfrak{p} = (p) \end{cases}$$
So
$$Supp(M) = Spec(\mathbb{Z}) \setminus \{(0)\}$$
is not a closed subset of $Spec(\mathbb{Z})$.
[If $V(I) = Spec(\mathbb{Z}) \setminus \{(0)\} \;\Rightarrow\; (p) \supseteq I$ for any $p \;\Rightarrow\; (0) \supseteq I$]
We also have
$$V(ann_{\mathbb{Z}}(M)) = V(\{0\}) = Spec(\mathbb{Z}) = \overline{Supp(M)}$$

2. We know
$$\varphi : \mathbb{Z} \longrightarrow \prod_{a=1}^{\infty} \mathbb{Z}/p^{\alpha}\mathbb{Z} \qquad n \longmapsto (n, \ldots, n, \ldots)$$
is injective, so we have an exact sequence
$$0 \longrightarrow \mathbb{Z} \longrightarrow \prod_{a=1}^{\infty} \mathbb{Z}/p^{\alpha}\mathbb{Z}$$
and then an exact sequence
$$0 \longrightarrow \mathbb{Z}_{\mathfrak{p}} \longrightarrow \left(\prod_{a=1}^{\infty} \mathbb{Z}/p^{\alpha}\mathbb{Z} \right)_{\mathfrak{p}}$$
So $Supp\left(\prod_{a=1}^{\infty} \mathbb{Z}/p^{\alpha}\mathbb{Z} \right) = Spec(\mathbb{Z})$. While
$$\bigcup_{a=1}^{\infty} Supp(\mathbb{Z}/p^{\alpha}\mathbb{Z}) = \bigcup_{a=1}^{\infty} \{(p)\} = \{(p)\}$$

So

$$\overline{\bigcup_{a=1}^{\infty} Supp(\mathbb{Z}/p^{\alpha}\mathbb{Z})} = \{(p)\} \neq Spec(\mathbb{Z}) = Supp\left(\prod_{a=1}^{\infty} \mathbb{Z}/p^{\alpha}\mathbb{Z}\right)$$

3. (1) Naturally $\bigoplus_{a=0}^{\infty} \mathfrak{p} \cap R_a \subseteq \mathfrak{p}$. The other direction is followed by this method:

$$\forall p = \sum_{i=0}^{k} p_i \in \mathfrak{p}, pm = \sum_{i=0}^{k} p_i m = 0$$

$\Rightarrow$ Use the double induction: first, shows $p_k m = 0$,

to prove it needs the induction of degree of m replace m by $p_k m$.

$\Rightarrow p_i m = 0$ for any $i \in \{0, \ldots, k\}$

$\Rightarrow p_i \in ann_R(m)$

$\Rightarrow \mathfrak{p} \subseteq \bigoplus_{a=0}^{\infty} \mathfrak{p} \cap R_a$

(2)Suppose $\mathfrak{p} = ann_R(m), m = \sum_{i=0}^{k} m_i$,we have

$$\begin{aligned}
&(\mathfrak{p} \cap R_a)m = 0 \\
\Rightarrow\ &(\mathfrak{p} \cap R_a)m_i = 0 \\
\Rightarrow\ &\mathfrak{p} m_i = 0 \qquad \text{for any } i \in \{0, \ldots, k\} \\
\Rightarrow\ &\mathfrak{p} \subseteq ann_R(m_i) \\
\Rightarrow\ &\mathfrak{p} \subseteq \cap_{i=1}^{k} ann_R(m_i)
\end{aligned}$$

If for any i, $\mathfrak{p} \subsetneqq ann_R(m_i)$, then there exists $p_i \in ann_R(m_i) \setminus \mathfrak{p}$, so

$$p := \prod_{i=0}^{n} p_i \notin \mathfrak{p}$$

but $pm = 0$,contradiction!

4. We know $(T-1)e_1 = (T-2)e_3 = 0$, and $dim(Ker(T-a)) = 0$ when $a \notin \{1, 2\}$. so

$$Ass(\mathbb{C}^3) = \{(T-1), (T-2)\}$$

5. We have

$$(x^3y, xy^4) = (x) \cap (y) \cap (x^3, y^4)$$

They are all primary ideals.($(x, y)^4 \subseteq (x^3, y^4) \subseteq (x, y)$)
Method:$(x^3y, xy^4) = (x^3, xy^4) \cap (y, xy^4) = (x^3, x) \cap (x^3, y^4) \cap (y, x) \cap (y, y^4)$...

10 Integral dependence, Nullstellensatz

1. Prove that the integral closure of $R := \mathbb{C}[X,Y]/(Y^2 - X^2 - X^3)$ in $\mathrm{Frac}(R)$ equals $\mathbb{C}[t]$ with $t := \bar{Y}/\bar{X}$, where $\bar{X}, \bar{Y}$ denote the images of X, Y in R.

证明. Let

$$\varphi' : \mathbb{C}[X,Y] \longrightarrow \mathbb{C}[T] \qquad X \longmapsto T^2 - 1 \quad Y \longmapsto T^3 - T$$

We have

$$\varphi'(Y^2 - X^2 - X^3) = (T^3 - T)^2 - (T^2 - 1)^2 T^2 = 0$$

induces the map

$$\varphi' : \mathbb{C}[X,Y]/(Y^2 - X^2 - X^3) \longrightarrow \mathbb{C}[T]$$

Suppose $f(X,Y) = a_0(X) + a_1(X)Y \in Ker\ \varphi$ where

- $a_0(X) = 0$ or $a_1(X) = 0$, easy to know $f(X,Y) = 0$.
- $a_0, a_1 \in \mathbb{C}[X]$ has no common nontrivial factors. Then

$$\varphi(f) = \big(a_0(T^2 - 1)\big) + \big(a_1(T^2 - 1)\big)(T^3 - T) \Rightarrow \quad X|a_0, X|a_1$$

  Contradiction!

So φ is injective, thus we can view $R := \mathbb{C}[X,Y]/(Y^2 - X^2 - X^3)$ as a subring of $\mathbb{C}[T]$,and

$$\begin{aligned} XT - Y &= 0 \\ \Rightarrow &\ T \text{ is integral over } R \\ \Rightarrow &\ R[T] \text{ is integral over } R \end{aligned}$$

$R[T] = \mathbb{C}[T^2 - 1, T^3 - T, T] = \mathbb{C}[T]$ is a UFD, so normal. □

2. Consider a Noetherian ring R with $K := \mathrm{Frac}(R)$. Show that $y \in K$ is integral over R if and only if there exists $u \in R$ such that $u \neq 0$ and $uy^n \in R$ for all n.

证明. Suppose R is a domain.

($\Longrightarrow$): $\qquad y \in K$ is integral over R

$$\Rightarrow \exists a_{m-1}, \ldots, a_0 \in R, y^m + a_{m-1}x^{m-1} + \cdots + a_0 = 0$$

If $x = \frac{v}{w}$, then there exists $u := w^m \neq 0$, such that for any $n \in \mathbb{N}^+$,

$$ux^n \in u(R + Rx + \cdots + Rx^{m-1}) \subseteq R + R \cdots + R = R$$

($\Longleftarrow$):

$$
\begin{aligned}
& R \text{ is a Noetherian ring} \\
\Rightarrow\ & u^{-1}R \text{ is a Noetherian } R\text{-module} \\
\Rightarrow\ & R[x] \text{ is a finitely generated } R\text{-module} \\
\Rightarrow\ & x \text{ is integral over } R
\end{aligned}
$$

□

3. Let $R = \mathbb{Q}[X_1, X_2, \ldots]$ (finite or infinitely many variables). Show that $\mathrm{nil}(R) = \mathrm{rad}(R) = \{0\}$.

证明. Claim: For any $f \in \mathbb{Q}[X_1, \ldots, X_n]$, there exists $(a_1, \ldots, a_n) \in \mathbb{Q}$, such that $f(a_1, \ldots, a_n) \neq 0$.
We can prove it by induction on n. Suppose it holds for $k < n$.
WLOG, suppose $f = \sum_{i=0}^{m} g_i x_n^{\beta_i}$ where $g_i \in \mathbb{Q}[X_1, \ldots, X_n]$ and $g_m \neq 0$.
Then by induction, there exists $(a_1, \ldots, a_{k-1})$ such that $g_k(a_1, \ldots, a_{k-1}) \neq 0$,now

$$\bar{f}(x) := f(a_1, \ldots, a_{k-1}, x)$$

has at most m roots. We can choose $a_k \in \mathbb{Q}$ such that $\bar{f}(a_k) \neq 0$ For any $f \in R$ nonzero, there exists $n \in \mathbb{N}$, such that $f \in \mathbb{Q}[X_1, \ldots, X_n]$.Let

$$\varphi_f : R \longrightarrow \mathbb{Q} \qquad g \longrightarrow g(a_1, \ldots, a_k, 0, \ldots)$$

It is a surjective homomorphism.

$$
\begin{aligned}
\Rightarrow\ & R/Ker\varphi_f \cong \mathbb{Q} \\
\Rightarrow\ & Ker\varphi_f \text{ is a maximal ideal not containing} f \\
\Rightarrow\ & \mathrm{rad}(R) \subseteq \{0\} \\
\Rightarrow\ & \{0\} \subseteq \mathrm{nil}(R) \subseteq \mathrm{rad}(R) \subseteq \{0\} \\
\Rightarrow\ & \mathrm{nil}(R) = \mathrm{rad}(R) = \{0\}
\end{aligned}
$$

□

4. Let $R = \mathbb{Q}[X_1, X_2, \ldots]$ (infinitely many variables). Show that R is not a Jacobson ring.

证明. Denote $\mathfrak{p} = (X)$ in $\mathbb{Q}[X]$, $R' = (\mathbb{Q}[X])_{\mathfrak{p}}$ and an bijection map $\Psi : \mathbb{N}^+ \longrightarrow \mathbb{Q} \setminus \{0\}$
We constrct

$$\psi : R \longrightarrow R' \qquad X_i \longmapsto \frac{1}{X - \Psi(i)}$$

which is surjective.

Consider all the irreducible polynomials.

If R is a Jacobson ring, then so is R', but

$$\operatorname{Max} R' = \{\mathfrak{p}\} \neq \{\mathfrak{p}, (0)\} = \operatorname{Spec} R'$$

Contradiction! □

5. (1) Let A be a subring of an integral domain B, and let C be the integral closure of A in B. Let f, g be monic polynomials in $B[x]$ such that $fg \in C[x]$. Then f, g are in $C[x]$.
(2) Prove the same result without assuming that B (or A) is an integral domain.

证明. 偷懒抄书，此题不算！
[2]Take a field $(\overline{Frac(B)})$ containing B in which the polynomials f, g split into linear factors; say $f = \Pi\left(x - \xi_1\right), g = \Pi\left(x - \eta_j\right)$. Each ξ_i andeach η, is a root of fg, hence is integral over C. Hence the coefficients of f and g are integral over C.
[1]LEMMA (14.7).–Let $R \subset R'$ be a ring extension, X a variable, $f \in R[X]$ a monic polynomial. Suppose $f = gh$ with $g, h \in R'[X]$ monic. Then the coefficients of g and h are integral over R.
Proof: Set $R_1 := R'[X]/\langle g\rangle$. Let x_1 be the residue of X. Then $1, x_1, x_1^2, \ldots$ form a free basis of R_1 over R' by (10.25) as g is monic; hence, $R' \subset R_1$. Now,$g\left(x_1\right) = 0$; so g factors as $\left(X - x_1\right) g_1$ with $g_1 \in R_1[X]$ monic of degree 1 less than g. Repeat this process, extending R_1. Continuing, obtain $g(X) = \prod\left(X - x_i\right)$and $h(X) = \prod\left(X - y_j\right)$ with all x_i and y_j in an extension of R'. The x_i and y_i are integral over R as they are roots of f. But the coefficients of q and h are polynomials in the x_i and y_j; so they too are integral over R. □

6. Let $f : A \to B$ be an injective map, with A Noetherian and B integral over A. Assume that neither A nor B have zero divisors.
(1) Show that if A is a field, then so is B.
(2) Deduce that a field k is algebraically closed (i.e., every polynomial has a root) if and only if for every finite field extension $k \subset k'$ i.e., k' is f.d. as a k -vector space, we have $k = k'$.
(3) Show that if B is a field, then so is A.

证明. (1) For any $x \in B$, there exists $a_0, \ldots, a_{n-1} \in A$, $a_0 \neq 0$, such that

$$\begin{aligned}
& x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0 \\
\Rightarrow\ & x \cdot (-\frac{1}{a_0})(x^{n-1} + a_{n-1}x^{n-1} + \cdots + a_1) = 1 \\
\Rightarrow\ & B \text{ is a field.}
\end{aligned}$$

(2) ($\Longrightarrow$): For any $x_0 \in k'$, there exists $x_1, \ldots, x_n \in k$, such that

$$f(x) = (x - x_1)\cdots(x - x_n)\ \&\ f(x_0) = 0$$

So $x_0 \in k$. ($\Longleftarrow$):If not, there exists an irreducible polynomial f which has no root. Then $k[t]/(f(t))$ is a finite field extension of degree $degf$.

(3) For any $x \in A$, there exists $y \in B$ such that $xy = 1$.

$$\begin{aligned}
\Rightarrow\ & y^n + b_{n-1}y^{n-1} + \cdots + b_0 = 0 \\
\Rightarrow\ & y + b_{n-1} + \cdots + b_0 x^{n-1} = 0 \\
\Rightarrow\ & y \in A
\end{aligned}$$

So k is a field. □

11 Flatness

Facts.

- *$k[[t]]$ is a PID. Its ideal can be written as (t^m), while for any $f \in k[[t]]$, there exists $m \in \mathbb{Z}_{\geqslant 0}, g \in (k[[t]])^*$, such that $f = gt^m$.*
- *A beautiful diagram:*

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{Tor}_3^A(A,M) \arrow[r] & \mathrm{Tor}_3^A(K,M) \arrow[r] & \mathrm{Tor}_3^A(K/A,M) \arrow[dll] \\
 & \mathrm{Tor}_2^A(A,M) \arrow[r] & \mathrm{Tor}_2^A(K,M) \arrow[r] & \mathrm{Tor}_2^A(K/A,M) \arrow[dll] \\
 & \mathrm{Tor}_1^A(A,M) \arrow[r] & \mathrm{Tor}_1^A(K,M) \arrow[r] & \mathrm{Tor}_1^A(K/A,M) \arrow[dll] \\
 & A \otimes_A M \arrow[r] & K \otimes_A M \arrow[r] & K/A \otimes_A M \arrow[r] & 0
\end{tikzcd}$$

1. For a field k, show that $k[[t]][Y,Z]/(YZ-t)$ is flat over $k[[t]]$.

证明. We only need to prove $k[[t]][Y,Z]/(YZ-t)$ has no zero divisors except 0. This is easy: We claim that every item in $k[[t]][Y,Z]/(YZ-t)$ can be uniquely written as the form

$$\sum_{n=0}^{+\infty}(f_n(Y)+g_n(Z)+a_n)t^n$$

where

$$f_n \in k[Y], g_n \in k[Z], a_n \in k, \qquad f_n(0)=g_n(0)=0$$

If

$$\sum_{n=0}^{+\infty}(f_n(Y)+g_n(Z)+a_n)t^n \neq 0$$

then

$$t^m\sum_{n=0}^{+\infty}(f_n(Y)+g_n(Z)+a_n)t^n \neq 0$$

$$gt^m\sum_{n=0}^{+\infty}(f_n(Y)+g_n(Z)+a_n)t^n \neq 0 \qquad g \in (k[[t]])^*$$

□

2. Let N', N, N'' be A -modules, and $0 \to N' \to N \to N'' \to 0$ be an exact sequence, with N'' flat. Prove that N' is flat $\Leftrightarrow$ N is flat.

证明. If $0 \to M \xrightarrow{\varphi} M'$ is a A-modular exact sequence, then We have the exact sequences:

$$\begin{tikzcd}
 & 0 \arrow[r] & Ker\varphi \otimes Id_{N'} \arrow[r] \arrow[d] & Ker\varphi \otimes Id_N \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
 & Tor_1(M,N'') \arrow[r] & M \otimes N' \arrow[r] \arrow[d] & M \otimes N \arrow[r] \arrow[d] & M \otimes N'' \arrow[r] \arrow[d] & 0 \\
 & Tor_1(M',N'') \arrow[r] & M' \otimes N' \arrow[r] & M' \otimes N \arrow[r] & M' \otimes N'' \arrow[r] & 0
\end{tikzcd}$$

Because $Tor_1(M,N'') = Tor_1(M',N'')$, the upper line is exact. So N' is flat $\Leftrightarrow$ N is flat.

Better solution:

$$0 = Tor_{i+1}(M,N'') \longrightarrow Tor_i(M,N) \longrightarrow Tor_i(M,N') \longrightarrow Tor_i(M,N'') = 0$$

□

3. A ring A is absolutely flat if every A -module is flat. Prove that the following are equivalent:

   (1) A is absolutely flat.

   (2) Every principal ideal is idempotent.

   (3) Every finitely generated ideal is a direct summand of A.

   证明.

(1)⇒(2): Consider the exact sequence

$$0 \longrightarrow I \longrightarrow A \longrightarrow A/I \longrightarrow 0$$

Tensoring with I, we obtain the exact sequence ($I \otimes_A I = I^2$ because I is principal ideal)

$$0 \longrightarrow I^2 \longrightarrow I \longrightarrow I \otimes_A A/I \longrightarrow 0$$

So $I/I^2 = I \otimes_A A/I = 0$.($I \otimes_A A/I = 0$ because $I \otimes_A A/I \hookrightarrow A \otimes_A A/I \cong A/I$ is a zero map(using the fact that A/I is flat))
Especially, when $I = (x)$, then $(x)^2 = (x)$, there exists $a \in A$ such that $x = ax^2$, now $(x) = (ax)$ is idempotent.

(2)⇒(3): We just need to prove that every finitely generated ideal is principal ideal.(Then, use the decomposition $A = (a) \oplus (1 - a)$)
We only need to prove $\langle a, b\rangle = \langle a + b - ab\rangle$ when $a^2 = a, b^2 = b$.

(3)⇒(1): Suppose any $I \lhd A$ is the direct summand, then A/I is flat, we obtain

$$Tor_1(N, R/I) = 0 \qquad \text{for any } N, I$$

. So A is absolutely flat.

□

4. Prove the following properties of absolutely flat:

   (1) Every homomorphic image of an absolutely flat ring is absolutely flat.

   (2) If a local ring is absolutely flat, then it is a field.

   (3) If a ring A is absolutely flat, then every non-unit in A is a zero-divisor.

证明. (1)

$$\begin{aligned} A \text{ is absolutely flat } &\Rightarrow \langle x\rangle^2 = \langle x\rangle \text{ in } A \\ &\Rightarrow \langle x\rangle^2 = \langle x\rangle \text{ in } A/I \\ &\Rightarrow A/I \text{ is absolutely flat} \end{aligned}$$

(2) Suppose $\mathfrak{m}$ is the unique maximal ideal of A, then $A/\mathfrak{m}$ is a field. If there exists $\langle x\rangle \subsetneq A, x \neq 0$, then $\langle x\rangle \subseteq \mathfrak{m} \subsetneq A \Rightarrow x = ax^2 \Rightarrow x(1-ax) = 0$.
However, $1 - ax \notin \mathfrak{m} \Rightarrow (ax-1) \in A^\times$, so we obtain $x \in A^\times$.

(3) If there exists $\langle x\rangle \subsetneq A, x \neq 0$, then there exists $e \in A$ such that $e^2 = e, (x) = (e)$. we get $a \in A$ such that $x = ae(x \neq a) \Rightarrow x(x-a) = 0$.
So every non-unit in A is a zero-divisor.

□

12 Going-up and Going-down

5. Let $f : A \to B$ be an integral homomorphism of rings, i.e. B is integra over its subring $f(A)$. Show that $f^\# : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a closed mapping, i.e. that it maps closed sets to closed sets.

证明. When $f(B) \neq B$, It may fail. Consider

$$f : \mathbb{Z} \Longrightarrow \mathbb{Z}/3\mathbb{Z} \qquad f^\#((0)) = (3)$$

is the counterexample.
(Notice that $\mathrm{Spec}\, f(A) \cong V(\ker f)$ is closed set of $\mathrm{Spec}\, B$)
When $A \subseteq B$, Suppose $V(I) \subseteq \mathrm{Spec}(B)$ is the closed set $(I \triangleleft B)$, then we claim: $f^\#(V(I)) = V(f^{-1}I)$.

(a) For all $\mathfrak{p} \in V(I) \Rightarrow \mathfrak{p} \supseteq I \Rightarrow f^\#(\mathfrak{p}) = f^{-1}(\mathfrak{p}) \supseteq f^{-1}(I) \Rightarrow f^\#(\mathfrak{p}) \in V(f^{-1}(I))$

(b) For all $\mathfrak{q} \in V(f^{-1}(I)) \subseteq \mathrm{Spec}(A)$, there exists $\mathfrak{p} \in \mathrm{Spec}(B)$ such that $\mathfrak{p} \cap \mathrm{Spec}\, A = \mathfrak{q}$. Easy to find that $\mathfrak{p} \supseteq I$.

□

6. Let $A \subset B$ be an extension of rings, making B integral over A, and let $\mathfrak{p}$ be a prime ideal of A. Suppose there is a unique prime ideal $\mathfrak{q}$ of B with $\mathfrak{q} \cap A = \mathfrak{p}$. Show that

(a) $\mathfrak{q}B_\mathfrak{p}$ is the unique maximal ideal of $B_\mathrm{p} := B\left[(A \setminus \mathfrak{p})^{-1}\right]$

(b) $B_\mathfrak{q} = B_\mathfrak{p}$

(c) $B_\mathfrak{q}$ is integral over $A_\mathfrak{p}$

证明. (a) We have the following commutative diagram:

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A_\mathfrak{p} & \longrightarrow & B_\mathfrak{p} \end{array} \qquad \begin{array}{ccc} \mathfrak{p} & \longleftarrow & \mathfrak{q} \\ \uparrow & & \uparrow \\ \mathfrak{p}A_\mathfrak{p} & \longleftarrow & \mathfrak{q}B_\mathfrak{p} \end{array}$$

$B_\mathfrak{p}$ is integral over $A_\mathfrak{p}$, and $\mathfrak{p}A_\mathfrak{p}$ is maximal in $A_\mathfrak{p}$
$\Rightarrow \mathfrak{q}B_\mathfrak{p}$ is maximal in $B_\mathfrak{p}$
If there exists $\mathfrak{m} \lhd B_\mathfrak{p}$ is maximal, then $f_\mathfrak{p}^{\#}(m) = \mathfrak{p}A_\mathfrak{p}$, thus $\mathfrak{m} = \mathfrak{q}B_\mathfrak{p}$.

(b) $\mathfrak{p} \subseteq \mathfrak{q} \Rightarrow B_\mathfrak{q} \supseteq B_\mathfrak{p}$.
By the universal property, we only need to show

$$\forall\, x \in B \setminus \mathfrak{q}, x \text{ is invertible.}$$

If $\dfrac{x}{1} = q\dfrac{b}{a_1}$ where $a_1 \in A \setminus \mathfrak{p}$, there exists $a_2 \in A \setminus \mathfrak{p}$ such that $a_1a_2x = a_2bq \Rightarrow x \in \mathfrak{q}$.
So $\dfrac{x}{1} \notin \mathfrak{q}B_\mathfrak{p}$ is invertible.

(c) $B_\mathfrak{q} = B_\mathfrak{p}$ is integral over $A_\mathfrak{p}$.

□

7. Let the integral extension $A \subset B$ and the prime ideal $\mathfrak{p}$ be as above. Suppose that A is a domain and $\mathfrak{q}, \mathfrak{q}'$ are distinct prime ideals of B, both mapping to $\mathfrak{p}$ under $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$. Show that $B_\mathfrak{q}$ is not integral over $A_\mathfrak{p}$.

证明. Take $y \in \mathfrak{q}' \setminus \mathfrak{q}$. If $y^{-1} \in B_\mathfrak{q}$ is integral over $A_\mathfrak{p}$, then there exists $a_0, \dots a_{n-1} \in A_\mathrm{p}$ with $a_0 \neq 0$ such that $a_0y^n + a_1y^{n-1} + \cdots + a_{n-1}y = -1$. Now there exists $s \in A \setminus \mathfrak{p}, a_0', \dots a_{n-1}' \in A$ with

$$a_0'y^n + a_1'y^{n-1} + \cdots + a_{n-1}'y = -s$$

Now the left is in $\mathfrak{q}'$ while $-s \notin \mathfrak{q}'$, contradiction! □

Graded rings and modules, Filtrations

1. Many basic operations on ideals, when applied to homogeneous ideals in $\mathbb{Z}$ -graded rings, lead to homogeneous ideals. Let I be a homogeneous ideal in a $\mathbb{Z}$-graded ring R. Show that:

   (1) The radical of I is homogeneous, that is, the radical of I is generated by all the homogeneous elements f such that $f^n \in I$ for some n.

   (2) If I and J are homogeneous ideals of R, then $(I : J) := \{f \in R | fJ \subset I\}$ is a homogeneous ideal.

   (3) Suppose that for all f, g homogeneous elements of R such that $fg \in I$ one of f and g is in I. Show that I is prime.

证明. We need to show that:

Claim 12.1. *For any $r = \sum_{i\in\mathbb{Z}} r_i \in \sqrt{I}, r_i \in R_i$, we have $r_i \in \sqrt{I}$*

Consider $i_0 = \min\{i \mid r_i \neq 0\}$, we know that

$$r_{i_0}^n = (r^n)_{ni_0} \in I \quad \Rightarrow \; r_{i_0} \in R_{i_0} \cap \sqrt{I}$$

then consider $r - r_{i_0}$. By induction, we can show that $r \in \langle f \in R_i \mid f^n \in I\rangle$.

We need to show that (the difficult part):

Claim 12.2. *For any $f = \sum_{i\in\mathbb{Z}} f_i \in (I : J), f_i \in R_i$, we have $f_i \in (I : J)$*

Now

$$\begin{aligned} fJ \subseteq I \Rightarrow fg \subseteq I \quad &\text{for any } g \in J_j \\ \Rightarrow f_i g \subseteq I_{i+j} \subseteq I \quad &\text{for any } g \in J_j \\ \Rightarrow f_i J \subseteq I \end{aligned}$$

When $f = \sum_{i\in\mathbb{Z}} f_i \notin I, g = \sum_{j\in\mathbb{Z}} g_j \notin I$, we need to show that

$$fg = \sum_{i,j\in\mathbb{Z}} f_i g_j \notin I.$$

Choose

$$i_0 = \min\{i \mid f_i \notin I\} \qquad j_0 = \min\{j \mid f_j \notin I\}$$

then

$$(fg)_{i_0+j_0} = \sum_i f_i g_{i_0+j_0-i} \in f_{i_0} g_{j_0+I} \subseteq A \setminus I$$

so $fg \notin I$. □

$$\Rightarrow$$

2. Suppose R is a $\mathbb{Z}$ -graded ring and $0 \neq f \in R_1$

   (1) Show that $R\left[f^{-1}\right]$ is again a Z-graded ring.

   (2) Let $S = R\left[f^{-1}\right]_0$, show that $S \cong R/(f-1)$, and $R\left[f^{-1}\right] \cong S\left[x, x^{-1}\right]$ where x is a new variable.

证明. (1) $R[f^{-1}]$ is a ring which is graded by

$$(R[f^{-1}])_i = \left\langle \frac{a_{i+j}}{f^j} \right\rangle_{a_{i+j} \in R_{i+j}}.$$

(2) We know

$$(R[f^{-1}])_0 = \left\langle \frac{a_j}{f^j} \right\rangle_{a_j \in R_j}.$$

and we have the surjective ring homomorphism

$$R \longrightarrow (R[f^{-1}])_0 \qquad r_i \in R_i \longmapsto \frac{r_i}{f^i}$$

and the kernel of which is $(f-1)$.
Now the homomorphism

$$R\left[f^{-1}\right] \longrightarrow S\left[x, x^{-1}\right] \qquad \frac{r_{i+j}}{f^j} \in (R[f^{-1}])_i \longmapsto \frac{r_{i+j}}{f^{i+j}} x^i$$

is an isomorphism.

$\square$

3. Show that if R is a graded ring with no nonzero homogeneous prime ideals, then R_0 is a field and either $R = R_0$ or $R = R_0\left[x, x^{-1}\right]$.

证明. We first state a lemma which can be proved using Zorn's lemma:*

Lemma 12.3. *Let I be a homogeneous ideal of a graded ring $R, I \neq R$, then there exists a homogeneous prime ideal which contains I.*

Using the lemma to Ra, we get:

*for details, see: https://math.stackexchange.com/questions/385292/homogeneous-ideals-are-contained-in-homogeneous-prime-ideals

Corollary 12.4. *If R is a graded ring with no nonzero homogeneous prime ideals, then any homogeneous item $a \in R \setminus \{0\}$ is invertible.*

Now R_0 is a field. if $R = R_0$, then everything was done; Otherwise, Suppose $i_0 = \{i \in \mathbb{Z}_{>0} \mid R_i \neq 0\}$ there exists $x \in R_{i_0} \setminus \{0\}$, which is invertible.Now:

- If $i_0 \nmid r$, then $R_r = 0$ by the euclidean division;
- If $i_0 \mid r, a \in R_r$, then $a = ax^{-r/i_0} \cdot x^{-r/i_0} \in R_0 [x, x^{-1}]$.

So we're done.

Another point: you can take the homogeneous items from prime ideal $\mathfrak{p}$ to construct a homogeneous prime ideal. □

4. Taking the associated graded ring can also simplify some features of the structure of R. For example, let k be a field, and let $R = k [x_1, \ldots, x_r] \subset R_1 = k [[x_1, \ldots, x_r]]$ be the rings of polynomials in r variables and formal power series in r variables over k, and write $I = (x_1, \ldots, x_r), I'$ for the ideal generated by the variables in either ring. Show that $\operatorname{gr}_I R = \operatorname{gr}_{I'} R_1$

证明. We know that

$$I^k/I^{k+1} \cong I'^k/I'^{k+1}$$

so

$$\operatorname{gr}_I R = \bigoplus_{k \in \mathbb{Z}} I^k/I^{k+1} \cong \bigoplus_{k \in \mathbb{Z}} I'^k/I'^{k+1} = \operatorname{gr}_{I'} R_1$$

□

13 Completions

1. Let A be a local ring, m its maximal ideal. Assume that A is m -adically complete. For any polynomial $f(x) \in A[x]$, let $f(x) \in (A/m)[x]$ denote its reduction mod. m. Prove Hensel's lemma: if $f(x)$ is monic of degree n and if there exist coprime monic polynomials $\overline{g}(x), \overline{h}(x) \in (A/\mathrm{m})[x]$ of degrees $r, n-r$ with $\overline{f}(x) = \overline{g}(x)\overline{h}(x)$, then we can lift $\overline{g}(x), \overline{h}(x)$ back to monic polynomials $g(x), h(x) \in A[x]$ such that $f(x) = g(x)h(x)$

证明. See [Hensel's Lemma, Theorem 1][†] Using induction makes sense. □

[†] html:http://therisingsea.org/notes/HenselsLemma.pdf

2. (a) With the notation of Exercise 1, deduce from Hensel's lemma that if $\bar{f}(x)$ has a simple root $\alpha \in A/\mathfrak{M}$, then $f(x)$ has a simple root $a \in A$ such that $\alpha = a \bmod \mathfrak{M}$
   (b) Show that 2 is a square in the ring of 7 -adic integers.
   (c) Let $f(x, y) \in k[x, y]$, where k is a field, and assume that $f(0, y)$ has $y = a_0$ as a simple root. Prove that there exists a formal power series $y(x) = \sum_{n=0}^{\infty} a_n x^n$ such that $f(x, y(x)) = 0$. This gives the "analytic branch " of the curve $f = 0$ through the point $(0, a_0)$.

证明. (a) Suppose $\bar{f} = (x - \alpha)\bar{h}(x)$, then by the Hensel's lemma, there exists $g(x) \in A[x], h(x) \in A[x]$ such that

$$f(x) = g(x)h(x)$$
$$A[x] \longrightarrow A/\mathfrak{M}[x] \qquad g \longmapsto (x - \alpha) \quad h \longmapsto \bar{h}$$

As a consequence, there exists $a \in A, g(x) = x - a, a \equiv \alpha \mod \mathfrak{M}$ (the root is evidently simple.)

(b) $x^2 - 2$ has a simple root $5 \in \mathbb{Z}/7\mathbb{Z}$. By (a), there exists $a \in A$ to be a simple root of $x^2 - 2$ (in $\mathbb{Z}_{(7)}$).

(c) Using (a), let $A = k[[x]], \mathfrak{M} = (x)$, we get

Corollary 13.1. *If $f(0, y) \in k[y]$ has a simple root a_0, then $f(x, y) \in k[[x]][y]$ has a simple root $y(x) = \sum\limits_{n=0}^{\infty} b_n x^n \in k[[x]]$ such that*

$$f(x, y(x)) = 0 \qquad b_0 = a_0$$

□

3. Let A be a Noetherian ring, a an ideal in A, and $\hat{A}$ the a-adic completion. For any $x \in A$, let $\hat{x}$ be the image of x in $\hat{A}$. Show that x not a zero-divisor in A implies $\hat{x}$ not a zero-divisor in $\hat{A}$. Does this imply that if A is an integral domain then $\hat{A}$ is an integral domain?

证明. suppose x is not a zero-divisor in A, then

$$0 \longrightarrow A \xrightarrow{\times x} A$$
$$\Rightarrow 0 \longrightarrow \hat{A} \xrightarrow{\times \hat{x}} \hat{A}$$
$$\Rightarrow 0 \text{ is not a zero-divisor in } \hat{A}$$

To take the example where A is an integral domain but $\hat{A}$ is not an integral domain, we take

$$A = \mathbb{Q}[x,y]/(y^2 - x^2 - x^3)$$

which is a domain because $(y^2 - x^2 - x^3)$ is irreducible in $\mathbb{Q}[x,y]$,but

$$\hat{A} = \mathbb{Q}[[x,y]]/(y^2 - x^2 - x^3)$$

is not a domain because

$$[y - x(1 + \frac{1}{2}x - \frac{1}{8}x^2 \cdots)][y + x(1 + \frac{1}{2}x - \frac{1}{8}x^2 \cdots)] = 0 \qquad \text{in } \hat{A}$$

□

4. Let k be a field and consider the quotient of infinite polynomial ring

$$R := \frac{\mathrm{k}\,[X, Z, Y_1, Y_2, Y_3, \ldots]}{(X - ZY_1, X - Z^2Y_2, X - Z^3Y_3, \ldots)}$$

Denote by $\overline{Z}$ the image of Z in R. Show that the ideal $I := (\overline{Z})$ of R satisfies $\bigcap_{n\geq 1} I^n \neq \{0\}$. Why is this consistent with Krull's intersection theorem?

证明. For convinience, we omit the bar.

$$X = Z^n Y_n \in I^n \Rightarrow X \in \bigcap_{n\geq 1} I^n$$

R is not Noetherian because

$$(Y_1) \subsetneq (Y_1, Y_2) \subsetneq (Y_1, Y_2, Y_3) \subsetneq \cdots$$

is a infinite ascending chain in R. □

14 *Mittag-Leffler* systems and Completions for non-Noetherian rings

1. Consider an inverse system of sets $\cdots \longleftarrow A_n \xleftarrow{\varphi_{n+1}} A_{n+1} \longleftarrow \cdots$ (where $n = 1, 2, \ldots$. For each $j > i$, let $\varphi_{j,i} : A_j \longrightarrow A_i$ be the composition of $\varphi_j, \ldots, \varphi_i$. We say that *Mittag-Leffler* conditions holds for $(A_n, \varphi_n)_{n\geq 1}$ if for each i, we have

$$\varphi_{k,i}(A_k) = \varphi_{j,i}(A_j) \qquad \text{whenever } j, k \gg i$$

Show that if $(A_n, \varphi_n)_n$ is *Mittag-Leffler* and $A_n \neq \emptyset$ for each n, then the limit

$$\varprojlim_n A_n := \left\{ (a_n)_n \in \prod_n A_n : \forall n, \varphi_{n+1}(a_{n+1}) = a_n \right\}$$

is nonempty as well.

证明. Let $D_n = \cap_{k=n+1}^{\infty} \varphi_{k,n}(A_k)$ be a set which is nonempty (because

$$\varphi_{n+1,n}(A_{n+1}) \supseteq \varphi_{n+1,n}(A_{n+1}) \supseteq \cdots \supseteq \cdots$$

is stable and nonempty). We have the inverse system

$$\cdots \longleftarrow D_n \overset{\varphi_{n+1}}{\longleftarrow} D_{n+1} \longleftarrow \cdots$$

and each φ_n is surjective. So $\varprojlim_n A_n$ is nonempty.(Find the PREIMAGE). $\square$

Remark 14.1. Moreover,

$$\tilde{\varphi}_D : \prod D_n \longrightarrow \prod D_n \qquad (a_n)_{n\in\mathbb{Z}} \longmapsto (a_n - \varphi_{n+1}(a_{n+1}))_{n\in\mathbb{Z}}$$

is surjective. Then

$$\varprojlim{}^1 D_n := \operatorname{coker} \tilde{\varphi}_D = 0$$

2. Suppose we are given an inverse system of short exact sequences of abelian groups, i.e. a commutative diagram with exact rows, where $n = 1, 2, \ldots$ Show that if $(A_n, \varphi_n)_n$ is Mittag- Leffler, then

$$0 \longrightarrow \varprojlim A_n \xrightarrow{\lim f_n} \varprojlim B_n \xrightarrow{\lim g_n} \varprojlim C_n \longrightarrow 0$$

is exact. You only have to show the surjectivity of $\lim g_n$.

证明. We have the exact sequence:

$$\begin{tikzcd} 0 \arrow[r] & \prod_n A_n \arrow[r] \arrow[d, "\theta'"] & \prod_n B_n \arrow[r] \arrow[d, "\theta"] & \prod_n C_n \arrow[r] \arrow[d, "\theta''"] & 0 \\ 0 \arrow[r] & \prod_n A_n \arrow[r] & \prod_n B_n \arrow[r] & \prod_n C_n \arrow[r] & 0 \end{tikzcd}$$

where

$$\theta' : \prod_n A_n \longrightarrow \prod_n A_n \qquad (a_n)_{n\in\mathbb{Z}} \longmapsto (a_n - \varphi_{n+1}(a_{n+1}))_{n\in\mathbb{Z}}$$

So by the Snake lemma, we have the exact sequence:

$$0 \longrightarrow \varprojlim A_n \overset{\lim f_n}{\longrightarrow} \varprojlim B_n \overset{\lim g_n}{\longrightarrow} \varprojlim C_n \longrightarrow \varprojlim{}^1 A_n$$

We only need to prove that θ' is surjective. Define D_n as in Problem 1, we get the exact sequence

$$\varprojlim D_n \longrightarrow \varprojlim A_n \longrightarrow \varprojlim A_n/D_n$$

from

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \prod_n D_n & \longrightarrow & \prod_n A_n & \longrightarrow & \prod_n A_n/D_n & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \tau'} & & \downarrow{\scriptstyle \theta'} & & \downarrow{\scriptstyle \tau''} & & \\ 0 & \longrightarrow & \prod_n D_n & \longrightarrow & \prod_n A_n & \longrightarrow & \prod_n A_n/D_n & \longrightarrow & 0 \end{array}$$

By 14.1, $\varprojlim D_n = 0$; we will show that $\varprojlim A_n/D_n = 0$, thus $\varprojlim A_n = 0$, thus the proof ends. □

Remark 14.2. The *Mittag-Leffler* conditions tell us that

$$\forall\, n \in \mathbb{Z}, \exists\, m > n \quad s.t. \qquad \varphi_{m,n}(A_m) = D_n$$
$$\Rightarrow \bar{\varphi}_{m,n} : A_m/D_m \longrightarrow A_n/D_n \text{ is the zero map.}$$

So $\varprojlim A_n/D_n = 0$ (ONLY ZERO SOLUTION)

3. Let R be a ring (not necessarily Noetherian), I be a proper ideal, and $\varphi : M \longrightarrow N$ be a homomorphism of R -modules. Prove the following statements.
   (a) If $M/IM \to N/IN$ is surjective, then so is $\hat{\varphi} : \hat{M} \to \hat{N}$. Here $\hat{M} = \lim_{n\geq 1} M/I^n M$ stands for the I-adic completion.
   (**Hint**: First, show $M/I^n M \to N/I^n N$ is surjective by Nakayama's lemma. Next, set $K_n = \operatorname{Ker}[M \to N/I^n N]$ to get exact sequences 0

   $$\to K_n/I^n M \to M/I^n M \to N/I^n N \to 0$$

   then try to establish the surjectivity of $K_{n+1}/I^{n+1}M \to K_n/I^n M$ in order to apply Mittag-Leffler.)
   (b) If $0 \to K \to M \to N \to 0$ is an exact sequence of R -modules and N is flat, then $0 \to \hat{K} \to \hat{M} \to \hat{N} \to 0$ is exact.
   (c) If M is finitely generated, then the natural homomorphism $M \otimes_R \hat{R} \to \hat{M}$ given by $m \otimes (r_n)_{n=1}^{\infty} \mapsto (r_n m)_{n=1}^{\infty}$ is surjective.

证明. (a) We'd like to point out the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & * \arrow[r] \arrow[d] & Kerf_{n+1} \arrow[r] \arrow[d] & Kerf_n \arrow[d] \arrow[dddll, rounded corners] & \\
0 \arrow[r] & I^nM/I^{n+1}M \arrow[r] \arrow[d] & M/I^{n+1}M \arrow[r] \arrow[d, "f_{n+1}"] & M/I^nM \arrow[r] \arrow[d, "f_n"] & 0 \\
0 \arrow[r] & I^nN/I^{n+1}N \arrow[r] \arrow[d] & N/I^{n+1}N \arrow[r] \arrow[d] & N/I^nN \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & \operatorname{coker} f_{n+1} \arrow[r] & \operatorname{coker} f_n \arrow[r] & 0
\end{tikzcd}$$

Then by induction we know the two surjectivity in the Hint.

(b) We have

$$\begin{tikzcd}
0 \arrow[r] & K/I^{n+1}K \arrow[r] \arrow[d, "\varphi"] & M/I^{n+1}M \arrow[r] \arrow[d] & N/I^{n+1}N \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K/I^nK \arrow[r] & M/I^nM \arrow[r] & N/I^nN \arrow[r] & 0
\end{tikzcd}$$

and φ is surjective, so $0 \to \hat{K} \to \hat{M} \to \hat{N} \to 0$ is exact.

(c) Just see the commutative diagram

$$\begin{tikzcd}
\hat{R}^{\oplus N} \arrow[r] \arrow[dr, two heads] & M \otimes_R \hat{R} \arrow[d, "\exists\ surj"] \\
 & \hat{M}
\end{tikzcd}$$

□

4. Suppose I is finitely generated. Let M be an R -module. Prove that

$$I^n\hat{M} = \operatorname{Ker}\left[\hat{M} \to M/I^nM\right] = \widehat{I^nM}$$

for all $n \in \mathbb{Z}_{\geq 1}$, and $\hat{M}$ is I -adically complete as an R -module.

证明. (**Hint**: Fix n and take generators $f_1, \ldots, f_r$ of I^n. This yields a surjective homomorphism of R -modules $(f_1, \ldots, f_r) : M^{\oplus r} \to I^nM \subset M$ Pass to completions and show that

$$(f_1, \ldots, f_r)^\wedge : \hat{M}^{\oplus r} \to \widehat{I^nM} = \lim_{m \geq n} \frac{I^nM}{I^mM} \simeq \operatorname{Ker}\left[\hat{M} \to M/I^nM\right] \subset \hat{M}$$

which is surjective by the previous exercise. The image of $(f_1, \ldots, f_r) : \hat{M}^{\oplus r} \to \hat{M}$ is both $I^n\hat{M}$ and $\widehat{I^n M}$ to infer that

$$\hat{M}/I^n\hat{M} \cong \widehat{M/I^n M} \cong M/I^n M$$

then we have $M^{\wedge\wedge} \cong M^{\wedge}$, which shows that $M^{\wedge}$ is complete. □

15 dimension theory

1. Let $\Bbbk$ be a field and $R = \Bbbk[X_0, \ldots, X_n]$, graded by total degree. Consider the graded R -module $S = R/(f)$ where f is a homogeneous poly-nomial of total degree $d \geq 1$. Show that when $m \geq d$,

$$\chi(S, m) := \dim_{\Bbbk} S_m = \left(\begin{array}{c} m+n \\ n \end{array} \right) - \left(\begin{array}{c} m+n-d \\ n \end{array} \right)$$

证明. We have the SES of the graded $\Bbbk$-linear spaces:

$$0 \longrightarrow (f) \longrightarrow R \longrightarrow S \longrightarrow 0$$

Thus

$$\dim_{\Bbbk} S_m = \dim_{\Bbbk} R_m - \dim_{\Bbbk}(f)_m = \left(\begin{array}{c} m+n \\ n \end{array} \right) - \left(\begin{array}{c} m+n-d \\ n \end{array} \right)$$

□

2. Let $R = \bigoplus_n R_n$ be a $\mathbb{Z}_{\geq 0}$-graded ring, finitely generated over R_0. Assume R_0 is Artinian (for example, a field) and let $M = \oplus_n M_n$ be a finitely generated $\mathbb{Z}_{\geq 0^-\text{graded}} R$ -module. Define the Hilbert series in the variable T as

$$H_M(T) := \sum_{m \geq 0} \chi(M, m) T^m \in \mathbb{Z}[[T]]$$

where $\chi(M, m)$ denotes the length of the R_0 -module M_m, as usual. In what follows, graded means graded by $\mathbb{Z}_{\geq 0}$.

   (a) Show that if $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of graded R -modules, then $H_M(T) = H_{M'}(T) + H_{M''}(T)$.

   (b) Relate $H_M(T)$ and $H_{M(k)}(T)$ for arbitrary $k \in \mathbb{Z}$, where $M(k)_d := M_{d+k}$.

(c) Suppose that R is generated as an R_0 -algebra by homogeneous elements $x_1, \ldots, x_n$ with $d_i := \deg x_i \geq 1$. Show that there exists $Q \in \mathbb{Q}[T]$ such that

$$H_M(T) = \frac{Q(T)}{(1-T^{d_1})\cdots(1-T^{d_n})}$$

as elements of $\mathbb{Q}[T]$.

(**Hint**: you may imitate the arguments for the quasi-polynomiality of $\chi(M,n)$.)

(d) What can be said about the $\mathbb{Z}^m_{\geq 0}$ -graded case, for general m?

证明. (a) We just need to prove

$$l(M_n) = l(M_n') + l(M_n'')$$

this is true even if R_0 is not Artinian. [‡]

(b)

$$\begin{aligned} T^k H_{M(k)}(T) &= \sum_{m\geq 0} \chi(M(k),m)T^{m+k} \\ &= \sum_{m\geq 0} \chi(M,m+k)T^{m+k} \\ &= H_M(T) - \sum_{0\leqslant n<k} \chi(M,n)T^n \end{aligned}$$

(c) This is followed by [2]. We prove by induction on n. When $n=0$, $R=R_0$ and M is f.g., so $M_n = 0$ for $n >> 0$; now suppose $n>0$ and the theorem was true for $n-1$, we consider the ES induced by the map $\times x_n$:

$$0 \longrightarrow K_m \longrightarrow M_m \longrightarrow M_{m+d_n} \longrightarrow L_{m+d_n}$$

We have

$$H_K(T) - H_M(T) = H_{M(d_n)}(T) - H_{L(d_n)}(T)$$

then easy to find the required form.

(d) Suppose that R is generated as an R_0 -algebra by homogeneous elements $x_1, \ldots, x_n$ with $d_i := \deg x_i \neq (0,0,\ldots,0)$. Then there exists $Q \in \mathbb{Q}[T_1,\ldots,T_m]$ such that $(T := (T_1,\ldots,T_m))$

$$H_M(T) = \frac{Q(T)}{(1-T^{d_1})\cdots(1-T^{d_n})}$$

as elements of $\mathbb{Q}[T_1,\ldots,T_m]$.

□

[‡] see `https://math.stackexchange.com/questions/145564`.

1. Let $\mathbb{Z}_3$ be the 3 -adic completion of the ring $\mathbb{Z}$, so that $\mathbb{Z} \hookrightarrow \mathbb{Z}_3$ naturally. Evaluate $1+3+3^2+3^3+\cdots$ in $\mathbb{Z}_3$.

证明.
$$1+3+3^2+3^3+\cdots = \frac{1}{1-3} = -\frac{1}{2}$$
□

2. Let R be a Noetherian local ring. Suppose that there exists a principal prime ideal p in R such that ht $(\mathfrak{p}) \geq 1$. Prove that R is an integral domain.
(**Hint**: Below is one possible approach. Suppose p $= (x)$ for some $x \in R$. Let $\mathfrak{q} \subset \mathfrak{p}$ be a minimal prime in R. Argue that (i)$x \notin \mathfrak{q}$, (ii) $\mathfrak{q} = x\mathfrak{q}$, and finally (iii) $\mathfrak{q} = \{0\}$.

证明. (i) Otherwise,
$$x \in \mathfrak{q} \;\Rightarrow\; \mathfrak{q} = \mathfrak{p} \;\Rightarrow\; ht(\mathfrak{p}) = 0$$
(ii) $x\mathfrak{q} \subseteq \mathfrak{q}$ is a prime ideal because
$$xf \notin x\mathfrak{q},\; xg \notin x\mathfrak{q} \implies xfxg \notin x\mathfrak{q}$$
Just verify $\mathfrak{q} \subseteq x\mathfrak{q}$ is easier and quicker.

(iii) Nakayama lemma applied to (ii).

□

3. Let $\Bbbk$ be a field and $R = \Bbbk[[X]] \times \Bbbk[[X]]$. Prove that R is a Noetherian semi-local ring, R contains a principal prime ideal of height 1, but R is not an integral domain.
(**Hint**: It is known that $\Bbbk[[X]]$ is Noetherian local with maximal ideal (X). Argue that the ideals in $\Bbbk[[X]] \times \Bbbk[[X]]$ take the form $I \times J$ where I, J are ideals in $\Bbbk[[X]]$. Show that $(X) \times \Bbbk[[X]]$ and $\Bbbk[[X]] \times (X)$ are the only maximal ideals, and both are of height 1.)

证明. (a) R is not an integral domain.

(b) $\Bbbk[[X]]$ is a Dedekind domain, thus Noetherian local with maximal ideal (X).

(c) We know that $I = \pi_1(I) \times \pi_2(I)$, where $\pi_1(I), \pi_2(I) \lhd \Bbbk[[X]]$ and $(\pi_1(a), \pi_2(b)) = (1,0)a + (0,1)b$.
Obviously(by contradiction) the maximal ideals are $(X) \times \Bbbk[[X]]$ and $\Bbbk[[X]] \times (X)$;

thus R is semilocal; moreover,$(X) \times \Bbbk[[X]] = ((X, 1))$ is principal.
the prime ideals are

$$(0) \times \Bbbk[[X]],\ \Bbbk[[X]] \times (0),\ (X) \times \Bbbk[[X]],\ \Bbbk[[X]] \times (X)$$

So the height of $(X) \times \Bbbk[[X]]$ is 1.

$\square$

参考文献

[1] Allen Altman and Steven Kleiman. *A term of commutative algebra*. Worldwide Center of Mathematics, 2013.

[2] Michael Atiyah. *Introduction to commutative algebra*. CRC Press, 2018.