Exercise of algebra 3————Galois Theory

Exercise 1: Let k be a field and let $K = k(t)$ be the field of rational functions in t over k.

(a) Let $u \in K$ with $u \notin k$. Calculate $[K : k(u)]$.

(b) Show that $K = k(u)$ if and only if $u = (ax + b)/(cx + d)$ for some $a,\ b,\ c,\ d \in k$ with det $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0$.

(c) Conclude that $Gal(K/k) \cong PGL_2(k)$.

Exercise 2: Let k be a field and let $K = k(x_1, ..., x_n)$ be the field of rational functions in n variables over k. Let $s_1, ..., s_n$ be the elementary symmetric functions in the x_i, that is, $s_k = \sum_{1 \leqslant i_1 < i_2 < ... < i_k \leqslant n} x_{i_1} ... x_{i_k}$.

(a) We view the symmetric group S_n as a subgroup of $Aut(K)$ by defining

$$\sigma(\frac{f(x_1, ..., x_n)}{g(x_1, ..., x_n)}) = \frac{f(x_{\sigma(1)}, ..., x_{\sigma(n)})}{g(x_{\sigma(1)}, ..., x_{\sigma(n)})}$$

for $\sigma \in S_n$. Let $F = Inv(S_n)$, prove that $[K : F] = n!$.

(b) Use (a) to prove that $k(s_1, ..., s_n) = F$.

(c) Use (b) to prove that $k[s_1, ..., s_n] = F \bigcap k[x_1, ..., x_n]$.

Exercise 3:

(a) Let F be a field, and let $f(x) \in F[x]$ be a polynomial of prime degree. Suppose for every field extension K of F that if $f(x)$ has a root in K, then $f(x)$ splits over K. Prove that either $f(x)$ is irreducible over F or $f(x)$ has a root in F.

(b) Let K be a normal extension of F, and let $f(x)$ be an irreducible polynomial in $F[x]$. Show that $f(x)$ is irreducible over K or can factor into a product of irreducible polynomials of the same degree over K.

Exercise 4: Let k be a field of characterisitic $p > 0$. Let $K = k(x,\ y)$

be the rational function field in two variables over k, and let $F = (x^p,\ y^p)$.

(a) Prove that $[K : F] = p^2$ and $K^p \subseteq F$.

(b) Prove that there is no $\alpha \in K$ with $K = F(\alpha)$.

(c) Exhibit an infinite number of intermediates fields of K/F.

Exercise 5: Suppose that K/F is a finite extension with K algebraically closed.

(a) If $char(F) = p > 0$ and $\beta \in F - F^p$, then $x^{p^r} - \beta$ is irreducible over F for all $r > 0$.

(b) If $char(F) = p > 0$ and there is a cyclic extension of degree p, then there are cyclic extensions of F of degree p^r for all $r > 0$.

(c) Let p be a prime, suppose that F contains a primitive p-th root of unity for p odd or 4-th root of unity for $p = 2$. If there is an $a \in F$ with $x^p - a$ irreducible over F, then $(x^p)^2 - a$ is irreducible over F.

(d) Show that $char(F) = 0$ and $K = F(\sqrt{-1})$.

Exercise 6: Let p be an odd prime number. Define complex number $\zeta = e^{\frac{2i\pi}{p^2}}$ and $\alpha = \sqrt[p]{p}\zeta$, where $\sqrt[p]{p}$ denotes the p-th root of p in the field of real numbers.

(1) Determine that $[\mathbb{Q}(\alpha, \zeta) : \mathbb{Q}]$ and $[\mathbb{Q}(\alpha) : \mathbb{Q}]$.

(2) Find the number of intermediate fields F of the extension $\mathbb{Q}(\alpha, \zeta)/\mathbb{Q}$ such that $[F : \mathbb{Q}] = p^2$.

Yau 2017: Let L/F be a Galois extension, and $x \in L$.

(a) Show that the set $\mathcal{P}$ of subextensions of L/F not containing x has a maximal element E. Let K/E be a non-trivial finite extensions contained in L. Show that $x \in K$.

(b) Let K' be the Galois closure of K/E in L. Show that there exists $g \in Gal(K'/E)$ such that $gx \neq x$.

(c) Show that K/E is a cyclic Galois extension.