$\forall$模M $\exists$ $F \to M \to 0$ F自由(投射) $0 \to M \to E$ E内射

$$\cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$$

投射分解

$$0 \to M \xrightarrow{f_0} E_0 \xrightarrow{f_1} E_1 \xrightarrow{f_2} \cdots$$

内射分解

做张量积,

$$\cdots \xrightarrow{d_2\otimes Id_N} P_1\otimes N \xrightarrow{d_1\otimes Id_N} P_0\otimes N \xrightarrow{d_0\otimes Id_N} M\otimes_R N \to 0$$

应用$Hom_R(N,-)$函子,

$$0 \to Hom_R(N,M) \xrightarrow{f_{0*}} Hom_R(N,E_0) \xrightarrow{f_{1*}} Hom_R(N,E_1) \xrightarrow{f_{2*}} \cdots$$

均为链复形

考虑同调群

$$Tor_n^R(M,N) = H_n = \ker d_{n-1}\otimes Id_N / Im\, d_n\otimes Id_N \quad (n\geq 1)$$

特别地, $Tor_0^R(M,N) \cong M\otimes_R N$

$$= \ker d_0$$

$$Tor_1^R(M,N) = H_1 = \ker d_0\otimes Id_N / Im\, d_n\otimes Id_N \quad (n\geq 1)$$

$$Ext_R^n(N,M) = \ker f_{n-1*} / Im f_{n*} \quad (n\geq 1)$$

$$Ext_R^0(N,M) = \ker f_{0*} \cong Hom_R(N,M)$$

$$Ext_R^1(N,M) = \ker f_{1*} / Im f_{0*}$$

对于正合列 $0 \to M' \to M \to M'' \to 0$

有正合列

$$\cdots \to Tor_2^R(M'',N) \to Tor_1^R(M',N) \to Tor_1^R(M,N) \to Tor_1^R(M'',N) \to M'\otimes N \to M\otimes N \to M''\otimes N \to 0$$

$$0 \to Hom_R(N,M') \to Hom_R(N,M) \to Hom_R(N,M'') \to Ext_R^1(N,M') \to Ext_R^1(N,M) \to Ext_R^1(N,M'') \to \cdots$$

若R为PID $\Rightarrow Tor_n^R(M,N) = 0$ for $\forall n\geq 2$

M,N投射 $\Rightarrow Tor_n^R(M,N) = 0$ for $\forall n\geq 1$

M平坦 $\Leftrightarrow Tor_1^R(M,N) = 0$ $\forall N$

$K = Frac(R)/R$, 则 $T(M) = Tor_1^R(K,M)$

$Ext_R^1(R/I,E) = 0$

不记得E满足什么条件了…… (E内射)

应用

1. 引理1.123 ($Tor_1^R(F,E) = 0$)
2. 命题1.124 ($Tor_1^R(F',N) = 0 = Tor_1^R(F'',N) \Rightarrow Tor_1^R(F,N) = 0$)
3. 习题1.34 (M_{tor}正合 $\Leftrightarrow K\otimes_R M' = 0 \Leftarrow M'$为torsion/divisible)
4. 习题1.27 ($0 \to Hom_R(R/I,E) \to Hom_R(R/I,B) \to Hom_R(R/I,R/I) \to Ext^1(R/I,E)$)

R/I 同构于 0 (E内射)