Geometry and Topology(1)

Problem1. Prove that $\mathbb{CP}^2$ does not cover any manifold other than itself.

Solution: If $\pi : \mathbb{CP}^2 \to M$ is a n-sheet cover and $n > 1$, then $3 = \chi(\mathbb{CP}^2) = n\chi(M)$. Hence $\chi(M) = 1$ and $n = 3$. $\pi_1(M)$ is a group of order 3, hence $\mathbb{Z}/3\mathbb{Z}$. It follows that $H_1(M, \mathbb{R}) = 0$. Moreover, $H_4(M, \mathbb{R}) = \mathbb{R}$ since M is orientable(Otherwise let $\tilde{M}$ be its double over, we then get a compostion of cover $\mathbb{CP}^2 \to \tilde{M} \xrightarrow{2} M$, contradicting that π is a 3-sheet cover). Using Poincáre duality, one has $1 = \chi(M) = 1 - 0 + \dim_{\mathbb{R}} H_2(M, \mathbb{R}) - 0 + 1 > 1$, which is a contradiction.

Problem2. Prove that T^2 admits a Lorentz metric.

Solution: View T^2 as $\mathbb{R}^2/\mathbb{Z}^2$, then $dx \otimes dx - dy \otimes dy$ is a Lorentz metric on T^2.
Remark: This problem is trivial when the space in T^2. Generally, when M is a smooth manifold admiting a nowhere-vanishing vector field X(In particular when M is compact and orientable with $\chi(M) = 0$), one can construct a Lorentz metric on M as follows: Fix a Riemann metric g on M,suppose $g(X, X) = 1$, we seek a symmetric tensor $A \in \Gamma(T^*M \otimes TM)$ w.r.t. g s.t A is of sign(1,...,1,-1) everywhere. Locally on U_α, choose $\{E_1, ..., E_{n-1}, X\}$ as an orthonormal frame and define $A_\alpha E_i = E_i, 1 \leq i \leq n-1$ and $A_\alpha X = -X$. Let $A = \sum\limits_{\alpha} \varphi_\alpha A_\alpha$,where $\{\varphi_\alpha\}$ is a partition of unity subordinate to the open cover $\{U_\alpha\}$. Define $\tilde{g}(Y, Z) = g(Y, AZ)$, then $\tilde{g}$ is a Lorentz metric on M.

Problem3. Let $r(s)$ be a smooth closed curve in E^3 with positive curvature, where s denotes the arc parameter of r. Prove that if the principle normal vector $n(s)$ is a simple closed curve, then it divides the unit sphere to two portions of the same area.

Solution: See 彭家贵，陈卿，《微分几何》 P175 定理2.6(Jacobi).

Problem4. Prove the Cartan-Hadamard theorem: If (M, g) is a complete, simply connected Riemannian manifold with non-positive sectional curvature, then $\exp_p : T_pM \to M$ is a diffeomorphism for any $p \in M$.

Solution: See Do Carmo, *Riemannian Geometry* P149-151

$\boldsymbol{e}_2(s)$ 的终点在单位球面上画出一条曲线，称为曲线 $\boldsymbol{r}(s)$ 的主法标线. 由 Frenet 公式

$$\frac{d\boldsymbol{e}_2}{ds} = -\kappa \boldsymbol{e}_1 + \tau \boldsymbol{e}_3. \tag{2.11}$$

所以 $\left|\dfrac{d\boldsymbol{e}_2}{ds}\right|^2 = \kappa^2 + \tau^2 > 0$ ，这说明 $\boldsymbol{e}_2(s)$ 是一条正则曲线.

定理 2.6(Jacobi) 设 C 为 E^3 的光滑闭曲线，曲率 $\kappa(s) > 0$. 如果 C 的主法标线 $\boldsymbol{e}_2(s)$ 是一条简单闭曲线，则它将单位球面分成面积相等的两部分.

证明 设 ρ 是主法标线的弧长参数，则它在单位球面上的测地曲率为

$$k_g = \left(\boldsymbol{e}_2, \frac{d\boldsymbol{e}_2}{d\rho}, \frac{d^2\boldsymbol{e}_2}{d\rho^2}\right). \tag{2.12}$$

从 (2.11) 式可知

$$d\rho = \sqrt{k^2 + \tau^2} ds,$$

因此由 Frenet 公式有

$$\begin{aligned} k_g &= \left(\kappa \frac{d\tau}{ds} - \tau \frac{d\kappa}{ds}\right)\left(\frac{ds}{d\rho}\right)^3 \\ &= \frac{1}{\kappa^2 + \tau^2}\left(\kappa \frac{d\tau}{ds} - \tau \frac{d\kappa}{ds}\right)\frac{ds}{d\rho} \\ &= \frac{d}{ds}\left(\arctan \frac{\tau}{\kappa}\right)\frac{ds}{d\rho}. \end{aligned} \tag{2.13}$$

设 D 为主法标线所界定的区域之一，在 D 上应用 Gauss-Bonnet 公式，注意到单位球面的 Gauss 曲率为 1 ，有

$$\int_D dA + \int_{\partial D} k_g d\rho = 2\pi. \tag{2.14}$$

但由 (2.13) 式知

$$\int_{\partial D} k_g d\rho = \int_{\partial D} d\left(\arctan \frac{\tau}{\kappa}\right) = 0,$$

所以 D 的面积为 2π ，这恰好是球面面积的一半.

for s_o. Let $\{s_n\}$ be a convergent sequence, converging to s_o with $s_n < s_o$. Given $\varepsilon > 0$, there exists an index n_o such that if $n, m > n_o$ then $|s_n - s_m| < \varepsilon$. It follows that

$$d(\gamma(s_n), \gamma(s_m)) \leq |s_n - s_m| < \varepsilon,$$

and hence the sequence $\{\gamma(s_n)\}$ is a Cauchy sequence in M. Since M is complete in the metric d, $\{\gamma(s_n)\} \to p_o \in M$.

Let (W, δ) be a totally normal neighborhood of p_o. Choose n_1 such that if $n, m > n_1$, then $|s_m - s_n| < \delta$ and $\gamma(s_n)$, $\gamma(s_m)$ belong to W. Then, there exists a unique geodesic g whose length is less than δ joining $\gamma(s_n)$ to $\gamma(s_m)$. It is clear that g coincides with γ, wherever γ is defined. Since $\exp_{\gamma(s_n)}$ is a diffeomorphism on $B_\delta(0)$ and $\exp_{\gamma(s_n)}(B_\delta(0)) \supset W$, g extends γ beyond s_0.

d) $\Rightarrow$ a). Obvious.

b) $\Leftrightarrow$ e). General topology. □

2.9 COROLLARY. *If M is compact then M is complete.*

2.10 COROLLARY. *A closed submanifold of a complete Riemannian manifold is complete in the induced metric; in particular, the closed submanifolds of Euclidean space are complete.*

3. The Theorem of Hadamard

As an application of the theorem of Hopf-Rinow, we are going to prove the following global fact.

3.1 THEOREM. *(Hadamard). Let M be a complete Riemannian manifold, simply connected, with sectional curvature $K(p, \sigma) \leq 0$, for all $p \in M$ and for all $\sigma \subset T_p(M)$. Then M is diffeomorphic to $\mathbf{R}^n$, $n = \dim M$; more precisely $\exp_p: T_pM \to M$ is a diffeomorphism.*

Before starting the proof, we need a few lemmas. The following lemma shows that the exponential map of a manifold with non-positive curvature is a local diffeomorphism.

3.2 LEMMA. *Let M be a complete Riemannian manifold with $K(p, \sigma) \leq 0$, for all $p \in M$ and for all $\sigma \subset T_pM$. Then for all $p \in M$, the conjugate locus $C(p) = \phi$; in particular the exponential map $\exp_p: T_pM \to M$ is a local diffeomorphism.*

Proof. Let J be a non-trivial (that is, not identically zero) Jacobi field along a geodesic $\gamma\colon [0,\infty) \to M$, where $\gamma(0) = p$ and $J(0) = 0$. Then from the hypothesis on the curvature and from the Jacobi equation

$$\begin{aligned}\langle J, J\rangle'' &= 2\langle J', J'\rangle + 2\langle J'', J\rangle \\ &= 2\langle J', J'\rangle - 2\langle R(\gamma', J)\gamma', J\rangle \\ &= 2|J'|^2 - 2K(\gamma', J)\,|\gamma' \wedge J|^2 \geq 0.\end{aligned}$$

Therefore $\langle J, J\rangle'(t_2) \geq \langle J, J\rangle'(t_1)$ whenever $t_2 > t_1$. Since $J'(0) \neq 0$ and $\langle J, J\rangle'(0) = 0$, it follows, in addition, that for t a sufficiently small positive number

$$\langle J, J\rangle(t) > \langle J, J\rangle(0).$$

It follows that for all $t > 0$, $\langle J, J\rangle(t) > 0$, and $\gamma(t)$ is not conjugate to $\gamma(0)$ along γ. $\square$

The crucial point in the proof of the Hadamard theorem is given in the lemma below which is of independent interest.

3.3 Lemma. *Let M be a complete Riemannian manifold and let $f\colon M \to N$ be a local diffeomorphism onto a Riemannian manifold N which has the following property: for all $p \in M$ and for all $v \in T_pM$, we have $|df_p(v)| \geq |v|$. Then f is a covering map.*

Proof. By a general property of covering spaces (Cf. M. do Carmo, [dC 2], p. 383), it suffices to show that f has the path lifting property for curves in N, that is, given a differentiable curve $c\colon [0,1] \to N$ and a point $q \in M$ with $f(q) = c(0)$, there exists a curve $\bar{c}\colon [0,1] \to M$ with $\bar{c}(0) = q$ and $f \circ \bar{c} = c$.

To prove what is required, observe that, since f is a local diffeomorphism at q, there exists an $\varepsilon > 0$ such that it is possible to define $\bar{c}\colon [0,\varepsilon] \to M$ with $\bar{c}(0) = q$ and $f \circ \bar{c} = c$; that is, c can be lifted to a small interval starting from q. Because f is a local diffeomorphism over all of M, the set of values $A \subset [0,1]$, such that c can be lifted on A starting from q, is an open interval on the right; that is, $A = [0, t_o)$. If we can show that $t_o \in A$, we shall have A open and closed in $[0,1]$, therefore $A = [0,1]$ and c can be lifted on the entire interval.

To show that $t_o \in A$, let $\{t_n\}$, $n = 1, \ldots$, be an increasing sequence in A with $\lim t_n = t_o$. Then the sequence $\{\bar{c}(t_n)\}$ is contained in a compact set $K \subset M$. Indeed, if this were not the case, since M is complete, the distance from $\bar{c}(t_n)$ to $\bar{c}(0)$ would be arbitrarily large. However, by hypothesis,

$$\begin{aligned}\ell_{0,t_n}(c) &= \int_o^{t_n} \left|\frac{dc}{dt}\right| dt = \int_0^{t_n} \left|df_{\bar{c}(t)}\left(\frac{d\bar{c}}{dt}\right)\right| dt \\ &\geq \int_0^{t_n} \left|\frac{d\bar{c}}{dt}\right| dt \geq d(\bar{c}(t_n), \bar{c}(0)),\end{aligned}$$

implying that the length of c between 0 and t_o is arbitrarily large, which is absurd, and proves the assertion.

Since $\{\bar{c}(t_n)\} \subset K$, $n = 1, \ldots$, there exists an accumulation point $r \in M$ of $\{\bar{c}(t_n)\}$. Let V be a neighborhood of r such that $f \mid V$ is a diffeomorphism. Then $c(t_o) \in f(V)$ and, by continuity, there exists an interval $I \subset [0,1]$, $t_o \in I$, such that $c(I) \subset f(V)$. Choose an index n such that $\bar{c}(t_n) \in V$ and consider the lifting g of c on I passing through r. The liftings g and $\bar{c}$ coincide on $[0, t_n) \cap I$, because $f \mid V$ is bijective. Therefore, g is an extension of $\bar{c}$ to I, hence $\bar{c}$ is defined at t_o and $t_o \in A$.

Proof of the Hadamard theorem. Since M is complete, $\exp_p: T_pM \to M$ is defined for all $p \in M$ and is surjective. By Lemma 3.2, $\exp_p$ is a local diffeomorphism. This allows us to introduce a Riemannian metric on T_pM in such a way that $\exp_p$ is a local isometry. Such a metric is complete, because the geodesics of T_pM passing through the origin are straight lines (Cf. Theorem 2.8, (a) $\Rightarrow$ (d)). From Lemma 3.3, $\exp_p$ is a covering map. Since M is simply connected, $\exp_p$ is a diffeomorphism. $\square$

3.4 REMARK. The above proof gives a little more than what is stated. Call a point p of a complete Riemannian manifold M a *pole* if it has the property that it has no conjugate points. Any point of a complete manifold M with non-positive sectional curvature is a pole of M. However, poles can exist in non-compact manifolds which have positive sectional curvature (See Exercise 13). What we have just proved is the following general fact. *If a complete simply connected Riemannian manifold M has a pole, then M is diffeomorphic to* $\mathbf{R}^n$, $n = \dim M$.