Analysis and Differential Equations (6)

Problem1. Let u be a positive harmonic function on the punctured complex plane $\mathbb{C}\backslash\{0\}$, prove that u is a constant.

Problem2. Prove that any measurable function on $[0,1]$ can be approximated by simple functions almost everywhere.

Problem3. Show that $L^p(\mathbb{R}^n) * L^q(\mathbb{R}^n) \subset L^r(\mathbb{R}^n)$ for $1 \leq p, q, r \leq \infty$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$. What about the reverse inclusion?

Problem4. Let U be the puncture disk $\{z \in \mathbb{C} | 0 < |z| < 1\}$. If f is holomorphic on U and $||f||_{L^2(U)} < \infty$. Show that 0 is a removable singularity of f.

Problem5. Let f be an entire function. Suppose that for each $z_0 \in \mathbb{C}$, the power series expansion $f(z) = \sum_{i=0}^{\infty} c_n (z - z_0)^n$ has at least one coefficient $c_n = 0$. Show that f is a polynomial.

Problem6. Suppose f is holomorphic in an annulus $\{r < |z| < R\}$, and there exists a sequence of holomorphic polynomials p_n converging to f on every compact subset of the annulus. Show that f can be extended to the disc$\{|z| < R\}$ as a holomorphic function.

Problem 7 Let U be an open subset of $\mathbb{C}$. Let $z_0 \in U$ and f is a meromorphic function on U with a pole at z_0. Prove that there does not exist a holomorphic function $g : U\backslash\{z_0\}$ such that $e^{g(z)} = f(z)$.

Problem 8. State and prove the Possion summation formula for Schwarz functions on $\mathbb{R}$.