Geometry and Topology (3)

Problem1. Suppose N is a smoothly embedded submanifold of a smooth manifold M. A vector field X on M is called tangent to N if $X_p \in T_pN \subset T_pM$ for all $p \in M$.
(a) Show that if X and Y are vector fields on M both tangent to N, then $[X, Y]$ is also tangent to N.
(b) Illustrate this principle by choosing two vector fields X, Y tangent to $S^2 \subset \mathbb{R}^3$ (such that $[X, Y]$ is not identically zero), computing $[X, Y]$ and checking that it is tangent to S^2.

Problem2. Let M and N be smooth, connected, orientable n-manifolds for $n \geq 3$, and let $M \sharp N$ denote their connect sum.
(a) Compute the fundamental group of $M \sharp N$ in terms of that of M and of N (you may assume that the basepoint is on the boundary sphere along which we glue M and N).
(b) Compute the homology groups of $M \sharp N$.
(c) For part (a), what changes if $n = 2$? Use this to describe the fundamental groups of orientable surfaces.

Problem3 Calculate the homology with coefficients in $\mathbb{Z}$ of the Lens space $L(a, b)$

Problem4 What is π_2 of $S^2 \vee S^1$?

Problem5 Let (M, g) be a Riemannian manifold of dimension n with $n \geq 3$. If there exists a smooth function on M such that $\mathrm{Ric}(X, Y) = fg(X, Y)$ for all smooth vector fields X, Y on M, prove that f is a constanst.