Geometry and Topology (2)

Problem1. A symplectic form on an eight-dimensional manifold is defined to be a closed two-form ω such that $\omega \wedge \omega \wedge \omega \wedge \omega$ is a volume form (that is, everywhere nonvanishing). Determine which of the following manifolds admit symplectic forms: (a) S^8; (b) $S^2 \times S^6$; (c) $S^2 \times S^2 \times S^2 \times S^2$

Problem2. Suppose that M and N are connected smooth manifolds of the same dimension and $f : M \to N$ is a smooth submersion.
(a) Prove that if M is compact, then f is onto and f is a covering map.
(b) Give an example of a smooth submersion $f : M \to N$ such that M and N have the same dimension, N is compact, and f is onto, but f is not a covering map.

Problem3. For $n \geq 1$, consider the subset $X \subset \mathbb{CP}^{2n}$ given by

$$X = \{[z_0 : z_1 : \cdots : z_{2n}] \in \mathbb{CP}^{2n} | z_{n+1} = z_{n+2} = \cdots = z_{2n} = 0\}$$

(a) Show that X is a smooth submanifold.
(b) Calculate the mod 2 intersection number of X with itself.

Problem4. Let $\Phi_N, \Phi_S : \mathbb{R} \times S^2 \to S^2$ be two global flows on the sphere S^2. Show that there exist $\epsilon > 0$, a neighborhood U of the North pole, a neighborhood V of the South pole, and a global flow $\Phi : \mathbb{R} \times S^2 \to S^2$ such that $\Phi(t, q) = \Phi_N(t, q)$ for all $t \in (-\epsilon, \epsilon), q \in U$, and $\Phi(t, q) = \Phi_S(t, q)$ for all $t \in (-\epsilon; \epsilon), q \in V$.

Problem5. Point S^2 via the south pole, and consider the Cartesian product $S^2 \times S^2$.
(a) Describe a cell structure on $S^2 \times S^2$ that is compatible with the inclusion of $S^2 \vee S^2 \hookrightarrow S^2 \times S^2$ as those pairs where one coordinate is the south pole.
(b) Let X be $(S^2 \times S^2) \bigcup_{S^2} D^3$, where we attach the 3-disk via the map $S^2 \to S^2 \vee S^2$ which crushes a great circle connecting the north and south poles. Compute the homology groups of X.