一、例子：如何用组合性质算 De Rham 上同调（注：之后的谱序列是更强的工具）

e.g 圆 S^1　找一簇好覆盖

	$0 \longrightarrow$	$C^0(\mathcal{U},\mathbb{R}) \longrightarrow$	$C^1(\mathcal{U},\mathbb{R}) \longrightarrow$	$C^2(\mathcal{U},\mathbb{R}) \to \cdots$
	0	$\mathbb{R}_0\oplus\mathbb{R}_1\oplus\mathbb{R}_2$	$\mathbb{R}_{01}\oplus\mathbb{R}_{02}\oplus\mathbb{R}_{12}$	0
Ker	0	$\mathbb{R}$	$\mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R}$	0
Im	0	0	$\mathbb{R}\oplus\mathbb{R}$	0
$H^*(S^1)$	0	$\mathbb{R}$	$\mathbb{R}$	0
generator		$\omega=(1,1,1)$	$\eta=(1,0,0)$	

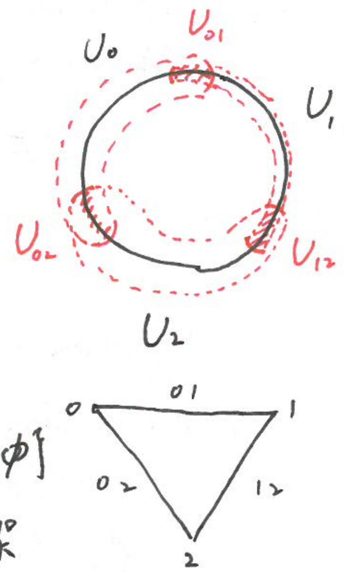


FIG 1　圆的好覆盖

Concept：$\mathcal{U}$ 的神经 $N(\mathcal{U})=\{$无序字符串 $\alpha_1\cdots\alpha_r \mid \bigcap_{i=1}^{r} U_{\alpha_i}\neq\phi\}$

对在三维欧氏空间中的曲面 $\mathcal{U}$ 的神经也可视作一个骨架

Question：如何构造出好覆盖？

Answer：当给定一个较好的骨架时(三角剖分)，我们可以构造出好覆盖

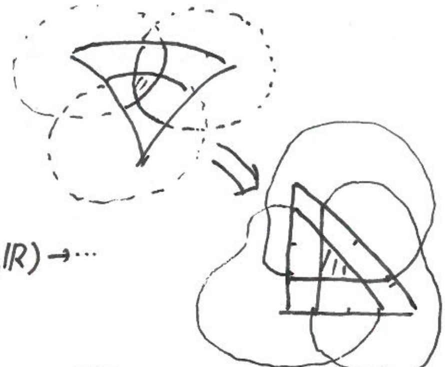


FIG 2　三角剖分时构造好覆盖的方法

e.g. 球面 S^2

	$0 \longrightarrow$	$C^0(\mathcal{U},\mathbb{R}) \to$	$C^1(\mathcal{U},\mathbb{R}) \longrightarrow$	$C^2(\mathcal{U},\mathbb{R}) \to$	$C^3(\mathcal{U},\mathbb{R}) \to \cdots$
	0	$\mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R}$	$\underbrace{\mathbb{R}\oplus\mathbb{R}\oplus\cdots\mathbb{R}}_{6个}$	$\mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R}$	0
Ker	0	$\mathbb{R}$	$\mathbb{R}^3$	$\mathbb{R}^4$	0
Im	0	0	$\mathbb{R}^3$	$\mathbb{R}^3$	0
$H^*(S^2)$	0	$\mathbb{R}$	0	$\mathbb{R}$	0
generator		$\omega=(1,1,1,1)$		$\eta=(0,0,0,1)$	

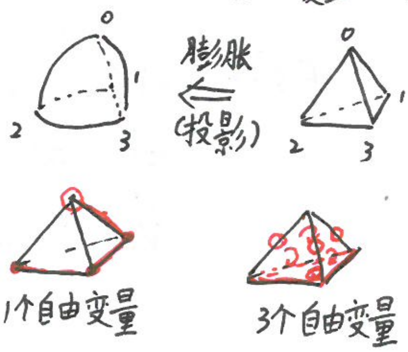


FIG 3　球面的骨架

e.g. 三角剖分的例子（c.f《基础拓扑学讲义》尤承业 P177-178）

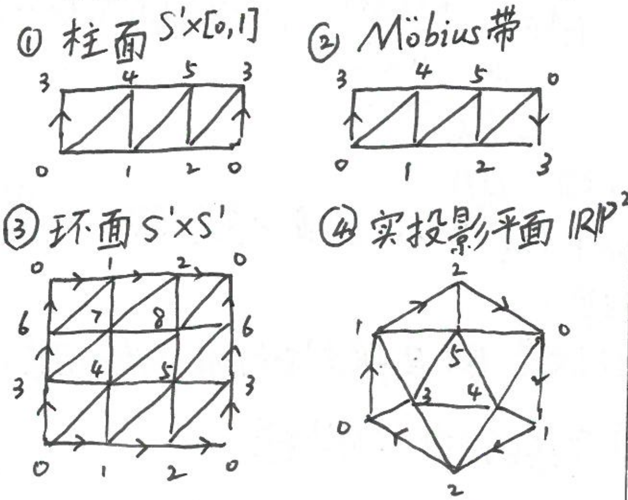


FIG 4　三角剖分平面图

Remark：利用第2小节的校对公式(Prop 9.5) 我们可以用 $f(\eta)=(-1)^n(D''K)^n\eta$ 算出上同调群的一组生成元

e.g　$H^1(S^1)$的生成元

$$f(\eta)=-D''K\eta=-D''(-\rho_1,\rho_0,0)=-d(-\rho_1)=d\rho_1$$

二. Künneth公式

Künneth公式描述的是~~向量~~纤维丛与底空间之间的关系，当然这个公式只描述了平凡纤维丛的情形。在描述Künneth公式之前，我们先来看一个特例。

设(E,M,π)为向量丛，π满 $\Rightarrow$ $(U_\alpha\cap U_\beta\neq\phi \Leftrightarrow \pi^{-1}U_\alpha\cap\pi^{-1}U_\beta\neq\phi)$

E与M有相同的组合性质 $\Rightarrow$ 相同的Čech上同调

$\Rightarrow H^*_{DR}(E)\simeq H^*_{DR}(M)$ (*)

Remark. 结论亦可从"E同伦等价于M"中推出

$$\begin{tikzcd} E \arrow[d, "\pi"'] & \coprod\pi^{-1}U_\alpha \arrow[l] & \coprod\pi^{-1}U_{\alpha\beta} \arrow[l, shift left] \arrow[l, shift right] & \cdots \arrow[l, shift left] \arrow[l] \arrow[l, shift right] \\ M & \coprod U_\alpha \arrow[l] & \coprod U_{\alpha\beta} \arrow[l, shift left] \arrow[l, shift right] & \cdots \arrow[l, shift left] \arrow[l] \arrow[l, shift right] \end{tikzcd}$$

FIG 5

Prop 9.12 (Künneth公式) M.F为流形，F为有限维上同调群，则有

$$H^*(M\times F)=H^*(M)\otimes H^*(F)$$

具体地，$H^n(M\times F)=\bigoplus_{k=0}^{n} H^k(M)\otimes H^{n-k}(F)$

Remark. 1. 当$F=\mathbb{R}^m$时化为(*)的特例：$H^n(M\times\mathbb{R}^m)=H^n(M)$

2. 当F上同调群无限维时，结论可能不成立。

e.g. $H^*(\mathbb{N}^+\times\mathbb{N}^+)\neq H^*(\mathbb{N}^+)\otimes H^*(\mathbb{N}^+)$

注意.张量积是有限和运算，当将生成元视作$\infty\times\infty$的矩阵时，右边矩阵的秩为有限维的（可以定义行秩：行作为元素生成的线性空间的维数）

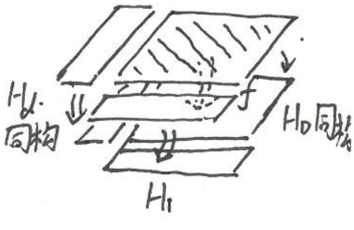
Proof of Prop 9.12

$$\begin{tikzcd} F & M\times F \arrow[l, dashed, "\rho"'] \arrow[d, "\pi"] \\ & M \end{tikzcd} \Rightarrow \begin{tikzcd} C^*(\pi^{-1}\mathcal{U},\Omega^*) \\ C^*(\mathcal{U},\Omega^*) \arrow[u, "\pi^*"'] \end{tikzcd} \Rightarrow \begin{tikzcd} C^*(\pi^{-1}\mathcal{U},\Omega^*) & \rho^*\omega_\alpha\wedge\pi^*\phi \\ H^*(F)\otimes C^*(\mathcal{U},\Omega^*) \arrow[u, "\pi^*_{\mathcal{U}}"'] & {[\omega_\alpha]\otimes\phi} \arrow[u] \end{tikzcd} \Rightarrow \begin{tikzcd} H_D\{C^*(\mathcal{U},\Omega^*)\} \\ H^*(F)\otimes H_D\{C^*(\mathcal{U},\Omega^*)\} \arrow[u] \end{tikzcd} \text{ i.e } \begin{tikzcd} H^*(E) \\ H^*(F)\otimes H^*(M) \arrow[u] \end{tikzcd}$$

$$H^*(F)\otimes C^*(\mathcal{U},\Omega^*)\cong(C^*(\mathcal{U},\Omega^*))^n$$

① $\pi^*\mathcal{U}$诱导复形的同构 (H_d)

② 若双复形的~~映~~映射诱导H_d-同构，则亦~~诱~~诱导H_D-同构



Remark (Leray-Hirsch定理)

设$e_1,\cdots e_r$为全局上同调类，~~且~~，且限制在纤维上为一组基，则$H^*(E)$为以$\{e_1,\cdots e_r\}$为基的自由$H^*(M)$-模

Def $K: C^p(\mathcal{U}, \Omega^q) \longrightarrow C^{p-1}(\mathcal{U}, \Omega^q)$

$$(K\omega)_{\alpha_0\cdots\alpha_{p-1}} = \sum_\alpha \rho_\alpha \omega_{\alpha\alpha_0\cdots\alpha_{p-1}}$$

It satisfies $\delta K + K\delta = 1$

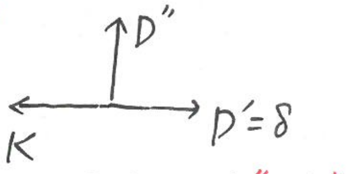


FIG 1 $\delta D'' = -D''\delta$

Def $f: C^*(\mathcal{U}, \Omega^*) \longrightarrow \cancel{\Omega^*(M)}\, C^0(\mathcal{U}, \Omega^*)$

$$f(\alpha) = \sum_{i=0}^{n} (-D''K)^i \alpha_i - \sum_{i=1}^{n+1} K(-D''K)^{i-1}\beta_i$$

where $\beta = D\alpha = (\delta + D'')\alpha$

It satisfies: $\beta_{i+1} = \delta\alpha_i + D''\alpha_{i+1}$, $0 = \delta\beta_i + D''\beta_{i+1}$

① global form, i.e $\delta f(\alpha) = 0$

② f is a chain map, i.e $f(D\alpha) = df(\alpha)$

③ $f \circ r = 1$

④ $1 - r\circ f = DL + LD$ $\cancel{D = D' + D}$ $D = D'' + \delta$

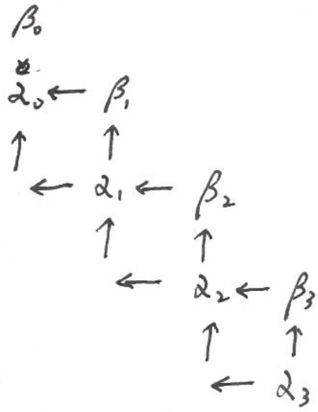


FIG 2 f

where $L: C^*(\mathcal{U}, \Omega^*) \longrightarrow C^*(\mathcal{U}, \Omega^*)$

$$L(\alpha) = \sum_{p=0}^{n-1} (L(\alpha))_p = \sum_{p=0}^{n-1} \sum_{i=p+1}^{n} K(-D''K)^{i-(p+1)}\alpha_i$$

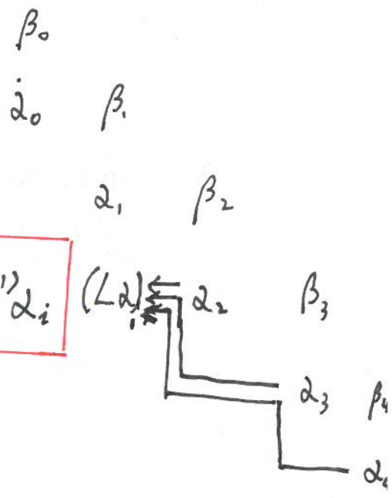


FIG 3 L

Lemma ⓪ for $i \geq 1$

$$\delta(D''K)^i = (D''K)^i\delta - (D''K)^{i-1}D''$$

i.e $\delta(-D''K)^i = (-D''K)^i\delta + (-D''K)^{i-1}D''$

Denote $\mathcal{A} = -D''K$, then

$$f(\alpha) = \sum_{i=0}^{n} \mathcal{A}^i\alpha_i - \sum_{i=1}^{n+1} K\mathcal{A}^{i-1}\beta_i$$

$$L(\alpha) = \sum_{p=0}^{n-1}\sum_{i=p+1}^{n} K\mathcal{A}^{i-(p+1)}\alpha_i$$

$$\delta\mathcal{A}^i = \mathcal{A}^i\delta + \mathcal{A}^{i-1}D'' \quad (i \geq 1)$$

We are going to proof from ⓪ to ④

⓪: By induction

$i = 1$, then $\cancel{\delta(D''}\ \delta\mathcal{A} = -\delta D''K = \cancel{D''(-\delta K) = D''(K\delta - 1)} =$

$= D''\delta K = D''(1 - K\delta) = \mathcal{A}\delta + D''$

$i = k+1$, then $\delta\mathcal{A}^{k+1} = \delta\mathcal{A}\mathcal{A}^k = \mathcal{A}\delta\mathcal{A}^k + D''\mathcal{A}^k = \mathcal{A}\delta\mathcal{A}^k$

$\overset{\text{induction}}{=} \mathcal{A}(\mathcal{A}^k\delta + \mathcal{A}^{k-1}D'') = \mathcal{A}^{k+1}\delta + \mathcal{A}^kD''$

① : $\delta f(\alpha) = \delta\left(\sum_{i=0}^{n} \mathcal{A}^i \alpha_i - \sum_{i=1}^{n+1} K \mathcal{A}^{i-1} \beta_i\right)$

$$= \sum_{i=0}^{n} \delta \mathcal{A}^i \alpha_i - \sum_{i=1}^{n+1} \delta K \mathcal{A}^{i-1} \beta_i$$

$$= \sum_{i=1}^{n} (\mathcal{A}^i \delta \alpha_i + \mathcal{A}^{i-1} D'' \alpha_i) + \delta \alpha_0 - \sum_{i=1}^{n+1} \delta K \mathcal{A}^{i-1} \beta_i$$

$$= \left(\sum_{i=1}^{n} \mathcal{A}^i \delta \alpha_i + \sum_{i=0}^{n-1} \mathcal{A}^i D'' \alpha_{i+1} + \delta \alpha_0\right) - \sum_{i=1}^{n+1} \delta K \mathcal{A}^{i-1} \beta_i$$

FIG 4

$$= \sum_{i=1}^{n+1} \mathcal{A}^{i-1} \beta_i - \sum_{i=1}^{n+1} \delta K \mathcal{A}^{i-1} \beta_i$$

$$= \sum_{i=1}^{n+1} K \delta \mathcal{A}^{i-1} \beta_i$$

$$= \sum_{i=2}^{n+1} (-K \mathcal{A}^{i-1} D'' \beta_{i+1} + K \mathcal{A}^{i-2} D'' \beta_i) + K \delta \beta_1$$

$$= K D'' \beta_2 + K \delta \beta_1$$

$$= 0$$

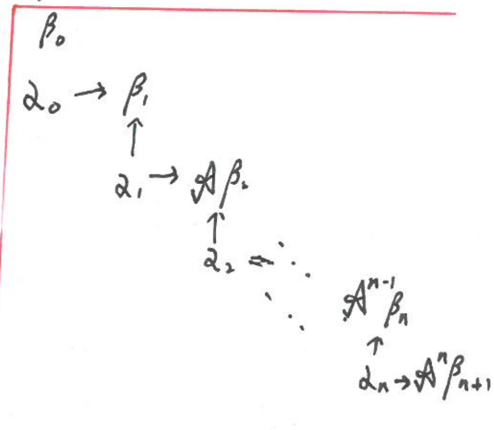


FIG 4

② $f(D\alpha) = f(\beta) = \sum_{i=0}^{n+1} \mathcal{A}^i \beta_i$

$$d f(\alpha) = D'' f(\alpha) = D''\left(\sum_{i=0}^{n} \mathcal{A}^i \alpha_i - \sum_{i=1}^{n+1} K \mathcal{A}^{i-1} \beta_i\right)$$

$$= \beta_0 + \sum_{i=1}^{n+1} (-D''K) \mathcal{A}^{i-1} \beta_i$$

$$= \sum_{i=0}^{n-1} \mathcal{A}^i \beta_i$$

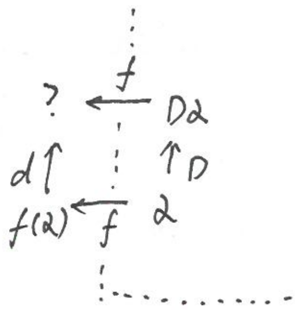


FIG 5

③ $f o r(\alpha) = f(\alpha_0) = \alpha_0 - \beta_1 = \alpha_0 = \alpha$

④ By the linear behaviour of r.f.D.L

we only consider $\alpha = \alpha_t$

$$(1 - r \circ f)(\alpha_t) = \alpha_t - r(\mathcal{A}^t \alpha_t - K \mathcal{A}^{t-1} \beta_t - K \mathcal{A}^t \beta_{t+1})$$

$$= (1 - \mathcal{A}^t - K \mathcal{A}^{t-1} D'' - K \mathcal{A}^t \delta)(\alpha_t)$$

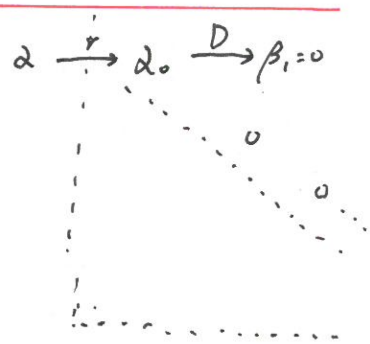


FIG 6

$$DL(\alpha_t) = D \sum_{p=0}^{t-1} K \mathcal{A}^{t-(p+1)} \alpha_t$$

$$= \sum_{p=0}^{t-1} \delta K \mathcal{A}^{t-(p+1)} \alpha_t - \sum_{p=0}^{t+1} \mathcal{A}^{t-p} \alpha_t$$

$$= \left(\sum_{p=0}^{t-1} (1 - K\delta) \mathcal{A}^{t-(p+1)} - \sum_{p=0}^{t-1} \mathcal{A}^{t-p}\right) \alpha_t$$

$$= \left(1 - \mathcal{A}^t - \sum_{p=0}^{t-1} K \delta \mathcal{A}^{t-(p+1)}\right) \alpha_t$$

$$= \left[1 - \mathcal{A}^t - \sum_{p=0}^{t-1} K \mathcal{A}^{t-(p+1)} \delta + K \mathcal{A}^{t-(p+2)} D''\right] \alpha_t$$

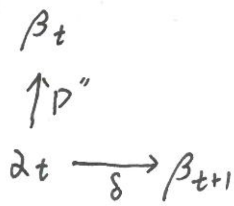


FIG 7

$$LD(\alpha_t) = L(\beta_t + \beta_{t+1})$$

$$= \sum_{p=0}^{t-1} K A^{t-(p+1)} \beta_t + \sum_{p=0}^{t} K A^{t-\cancel{(p+1)}} \beta_{t+1}$$

$$= \left(\sum_{p=0}^{t-1} K A^{t-(p+1)} D'' + \sum_{p=0}^{t} K A^{\overset{t-p}{\cancel{t-(p+1)}}} \delta \right) \alpha_t$$

$$= \left(\sum_{p=-1}^{t-2} K A^{t-(p+2)} D'' + \sum_{p=-1}^{t-1} K A^{t-(p+1)} \delta \right) \alpha_t$$

~~LD7~~

$$(DL + LD)(\alpha_t) = \left(1 - A^t + K A^{\overset{t-1}{\cancel{t+1}}} D'' - K A^t \delta \right) \alpha_t$$

$$= (1 - r \circ f) \alpha_t$$

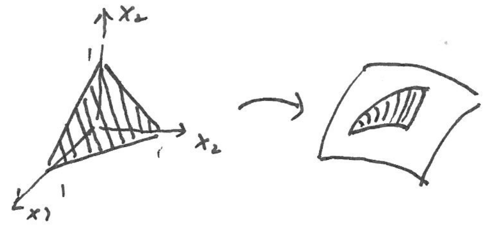
GTM 082 §13 单值性 讲稿

一. 一些定义

n-单形　单纯复形　紧支集$|K|$

重心　第一次重心划分

k-骨架　星形　$st(v)$



二. 好覆盖上的预层

Review 预层 Top $\to$ Grp 反变函子

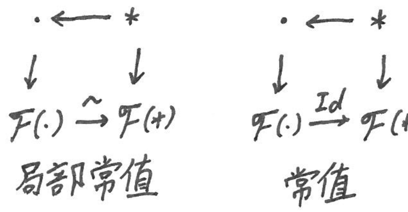
弱化: 只需于 $U_{\alpha_0\cdots\alpha_p}$ 上定义即可

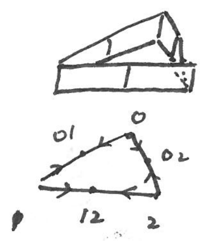
强化: 局部常值: $\mathcal{F}(\cdot)\xrightarrow{\sim}\mathcal{F}(*)$ (逆!)

常值: $\mathcal{F}(\cdot)\xrightarrow{Id}\mathcal{F}(*)$ (or $\mathcal{F}\cong G$)

同构: $h_W:\mathcal{F}(W)\to\mathcal{G}(W)$

(Denote $\rho^{\alpha}_{\alpha\beta}:\mathcal{F}(U_\alpha)\to\mathcal{F}(U_{\alpha\beta})$)





E.g. 局部常值非常值

三. 预层 F 上的单值表示

我们的目的:

$$\begin{tikzcd} \{\text{loops}\} \arrow[r, "\tilde{\rho}"] \arrow[d] & \mathrm{Aut}\, G \\ \pi_1(N(\mathcal{U})) \arrow[ur, dashed, "\exists!\rho"'] & \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{F}(U_\alpha)=G \arrow[d, "\rho^{\alpha}_{\alpha\beta}"] \\ \mathcal{F}(U_{\alpha\beta}) \arrow[d, dashed, shift right, "\rho^{\alpha\beta}_{\beta}"'] \\ \mathcal{F}(U_\beta)=G \arrow[u, shift right, "\rho^{\beta}_{\alpha\beta}"'] \end{tikzcd}$$

$\downarrow\rho^{\alpha}_{\beta}:\ \rho^{\alpha\beta}_{\beta}\circ\rho^{\alpha}_{\alpha\beta}$

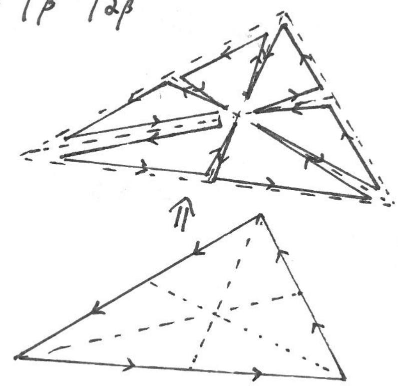
为此,需要构造:

(1) $\{(\alpha,\beta)\mid U_{\alpha\beta}\neq\emptyset\}\to\mathrm{Aut}\, G$

$(\alpha,\beta)\mapsto\rho^{\alpha}_{\beta}$

(2) $\tilde{\rho}:\{\text{loops}\}\to\mathrm{Aut}\, G$

(3) $\tilde{\rho}(\text{bounding loops})=0$ (边界回路)

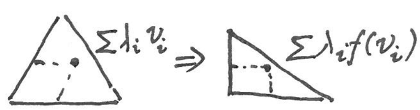


Thm. $\pi_1(N(\mathcal{U}))=0\Rightarrow$ 局部常值预层为常值预层

Remark. 单纯映射: 单纯复形 K 至 L 的单纯映射

$f:\mathrm{vert}(K)\to\mathrm{vert}(L)\xRightarrow{\text{线性延拓}} f:|K|\to|L|$

(将单形映为单形并保持单纯形与边界的关系)



四. 引理

a) $\pi_1(K)=\pi_1(\mathrm{border}(K))$

b) $\pi_1(K)=\pi_1(|K|)$ ← 拓扑的基本群定义

c) 单纯逼近定理

$f:|K|\to|L|\overset{\text{同伦}}{\sim}$ 单纯映射 $g:|K^{(k)}|\to|L|$

d) 延拓原理: $\partial I_n\to X$ 可缩 $\xrightarrow{\text{延拓}} I_n\to X$

(or $\pi_q(X)=0\ \forall q\leq n-1$)

Def. 拓扑空间上的好覆盖: 有限交可缩

Remark. 反例: 需额外的条件

$\Rightarrow$ ☆ Thm 13.4　$\pi_1(X)\cong\pi_1(N(\mathcal{U}))$

五. 单值表示的例子

1. $\pi: S^1\to S^1$, $z\mapsto z^2$

$\rightsquigarrow\ \mathcal{F}:\mathcal{U}(S^1)\to\mathrm{Grp}$, $U_i\mapsto(\mathbb{R}^2,+)$

2. $\pi:\mathbb{R}\to S^1$, $x\mapsto e^{2\pi i x}$

$\rightsquigarrow\ \mathcal{F}:\mathcal{U}(\mathbb{R})\to\mathrm{Grp}$, $U_i\mapsto(\mathbb{R}^2,+)$

3. $S^m\vee S^n$　e.g. $S^3\vee S^2$