SCHUR-WEYL DUALITY

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Abstract. This article mainly concerns about the Schur-Weyl Duality. We will prove it by using the Double Centralizer Theorem. After that, we will give some commentary about the relations to other field including the Invariant Theory.

Suppose V is a $\mathbb{C}$-linear space, we consider two group actions on $V^{\otimes n}$:

$$GL(V) \circlearrowright V^{\otimes n} \qquad \mathcal{A}(v_1, \dots, v_n) = (\mathcal{A}v_1, \dots, \mathcal{A}v_n)$$

$$V^{\otimes n} \circlearrowleft S_n \qquad g(v_1, \dots, v_n) = (v_{g^{-1}(1)}, \dots, v_{g^{-1}(n)})$$

Notice that these two actions commutes each other, we can abbriviate it as

$$GL(V) \xrightarrow{\rho_1} \mathrm{End}(V^{\otimes n}) \xleftarrow{\rho_2} S_n$$

We state the central theorem, which connects the irreducible representations of these two groups:

Theorem 0.1. *For $V^{\otimes n}$, we have the decomposition*

$$V^{\otimes n} \cong \bigoplus_{\lambda \in \Lambda} V_\lambda \otimes \mathbb{S}_\lambda V$$

where V_λ runs over all irreducible representations of S_n,

$$\mathbb{S}_\lambda V := \mathrm{Hom}_{\mathbb{C}[S_n]}(V_\lambda, V^{\otimes n})$$

is 0 or the irreducible representation of $GL(V)$.

We will need some knowledges from the Representation Theory, which can be founded in [1]

And recall the following theorems:

Lemma 0.2. *Suppose G is a finite group(or compact Lie group), V is a representation of G, while W is its subrepresentation. Then there exists a subrepresentation $W^\perp$ of V, such that*

$$V \cong W \oplus W^\perp$$

is the isomorphism of representations.

Theorem 0.3 (Maschke's Theorem)**.** *Suppose A is finite dimensional $\mathbb{C}$-algebra, then A has finitely many irreducible finite dimensional representations V_i up to isomorphism, then*

$$A \cong \bigoplus_{i=1}^{n} \mathrm{End}_{\mathbb{C}}(V_i)$$

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1. The Double Centralizer Theorem

Denote

- A is a semisimple $\Bbbk$-algebra with $dim_{\Bbbk} A < +\infty$.
- E is an A-module of $dim_{\Bbbk} E < +\infty$.
- $B := \mathrm{End}_A(E) = \{\mathcal{B} : E \longrightarrow E \mid \mathcal{B} \text{ is an } A\text{-invariant map}\}$
- $V_1, \ldots, V_k$ are all the irreducible representations of A.

then we have

- B is semisimple.
- the space
$$W_i := \mathrm{Hom}_A(V_i, E) \qquad (1 \leqslant i \leqslant k)$$
are irreducible representations of B or 0.
Moreover, we have the decompositions of E:
$$E = \bigoplus_{i=1}^{k} (V_i \otimes W_i)$$
and a description of B:
$$B = \bigoplus_{i=1}^{k} \mathrm{End}(W_i).$$

Remark 1.1. When the representation
$$\rho : A \longrightarrow \mathrm{End}_{\Bbbk}(E)$$
is faithful (i.e. ρ is injective), then we can view A as a subspace of $\mathrm{End}_{\Bbbk}(E)$, and B as the centralizer of A. In this condition we can show even more: W_i is never 0, and $A = \mathrm{End}_B(E)$ where
$$\mathrm{End}_B(E) := \{\mathcal{A} : E \longrightarrow E \mid \mathcal{A} \text{ is a } B\text{-invariant map}\}$$
this is why we call it the "Double Centralizer Theorem".

This special case (ρ is faithful) of the Double Centralizer Theorem is well illustrated in [5, 6]. In this situation, the decomposition
$$E = \bigoplus_{i=1}^{k} V_i \otimes W_i$$
gives a bijection between the irreducible representations of A and the irreducible representations of B.

However, this version of theorem is not strong enough to prove the Schur-Weyl theorem. In the proof of Schur-Weyl theorem, we study the representation of $\mathbb{C}[S_n]$, and this representation is not faithful.

For example, the representation $\mathbb{C}[S_n] \longrightarrow \mathrm{End}_{\mathbb{C}}(\mathbb{C}^n)$ is not faithful when $n > 3$, for
$$\dim_{\mathbb{C}} \mathbb{C}[S_n] = n! \quad \text{while} \quad \dim_{\mathbb{C}} \mathrm{End}_{\mathbb{C}}(\mathbb{C}^n) = n^2$$

Proof. We divide it into two steps.

Step1. We will show that, for a fixed i, if $W_i \neq 0$,then W_i is an irreducible representation of B.

We define the (left)B-action:

$$B \circlearrowright W_i = \mathrm{Hom}_A(V_i, E) \qquad \mathcal{B}(f) = \mathcal{B} \circ f.$$

This is well-defined because of the following diagram:

$$\begin{tikzcd}
V_i \arrow[r, "f"] \arrow[d, "\mathcal{A}"'] & E \arrow[r, "\mathcal{B}"] \arrow[d, "\mathcal{A}"'] & E \arrow[d, "\mathcal{A}"'] \\
V_i \arrow[r, "f"] & E \arrow[r, "\mathcal{B}"] & E
\end{tikzcd}$$

To show that W_i is irreducible, we only need:

Fact. *Suppose* $f_1, f_2 \in W_i, f_1 \neq 0$. *Then there exists* $\mathcal{B} \in B$ *such that*

$$f_2 = \mathcal{B}(f_1) = \mathcal{B} \circ f_1$$

Idea. *When* $w = f_1(v), \mathcal{B}(w)$ *can be only defined by*

$$\mathcal{B}(w) = \mathcal{B}(f_1(v)) = \mathcal{B} \circ f_1(v) = f_2(v)$$

So we only need to worry about elements not in $\mathrm{Im}\ f_1$.

Proof of the fact. Choose $v \neq 0 \in V_i$, then $Av = V_i$ because V_i is irreducible representation of A. From this,

- f_1, f_2 is uniquely defined by $f_1(v), f_2(v)$.
- $f_1 \neq 0 \Longrightarrow f_1(v) \neq 0$.
- $\mathrm{Im}\ f_1 = f_1(V_i) = f_1(Av) = A(f_1(v))$ is A-invariant.

By the lemma 0.2, we can decompose E into

$$E = \mathrm{Im}\ f_1 \oplus (\mathrm{Im}\ f_1)^{\perp}$$

□

then we can easily define

$$\mathcal{B} : E \longrightarrow E \qquad f_1(v) \longmapsto f_2(v) \quad v' \in (\mathrm{Im}\ f_1)^{\perp} \longmapsto v'$$

Now $\mathcal{B} \in B, f_2 = \mathcal{B} \circ f_1$.

Step2. What remains are the simple but interesting algebraic calculations. We will show them step by step:

- $E \cong \bigoplus_{i=1}^{n} (V_i \otimes_{\Bbbk} W_i)$.
- $B \cong \bigoplus_{i=1}^{n} \mathrm{End}(W_i)$, thus semisimple.
- When ρ is faithful, W_i is nonzero and $A \cong \mathrm{End}_B(E)$

$$\begin{aligned} E &\cong A \otimes_A E \\ &\cong \bigoplus_{i=1}^n (\mathrm{End}_{\Bbbk}(V_i) \otimes_A E) \\ &\cong \bigoplus_{i=1}^n (V_i \otimes_{\Bbbk} V_i^* \otimes_A E) \\ &\cong \bigoplus_{i=1}^n (V_i \otimes_{\Bbbk} \mathrm{Hom}_A(V_i, E)) \\ &\cong \bigoplus_{i=1}^n (V_i \otimes_{\Bbbk} W_i) \end{aligned}$$

$$\begin{aligned} B &\cong \mathrm{End}_A(E) \\ &\cong \mathrm{Hom}_A(E, E) \\ &\cong \mathrm{Hom}_A(\textstyle\bigoplus_{i=1}^n V_i \otimes_{\Bbbk} W_i, E) \\ &\cong \bigoplus_{i=1}^n \mathrm{Hom}_A(V_i \otimes_{\Bbbk} W_i, E) \\ &\overset{(*)}{\cong} \bigoplus_{i=1}^n \mathrm{Hom}_A(W_i \otimes_{\Bbbk} V_i, E) \\ &\cong \bigoplus_{i=1}^n \mathrm{Hom}_{\Bbbk}(W_i, \mathrm{Hom}_A(V_i, E)) \\ &\cong \bigoplus_{i=1}^n \mathrm{Hom}_{\Bbbk}(W_i, W_i) \cong \bigoplus_{i=1}^n \mathrm{End}_{\Bbbk}(W_i) \end{aligned}$$

Remark 1.2. Though $W_i = \mathrm{Hom}_A(V_i, E)$, $V_i \neq \mathrm{Hom}_A(W_i, E)$ in general. So we can't skip (*) to get "$B \cong \bigoplus_{i=1}^n \mathrm{Hom}_A(V_i \otimes_{\Bbbk} W_i, E) \cong A$". But we do have $V_i \cong \mathrm{Hom}_B(W_i, E)$ when $W_i \neq 0$ because of the isomorphism

$$E \cong \bigoplus_{i=1}^n (W_i \otimes \mathrm{Hom}_B(W_i, E)) \cong \bigoplus_{i=1}^n (W_i \otimes V_i)$$

Now let us suppose ρ is faithful. Because V is a faithful representation of A, it must contain all irreducible representations of A. So there is always a nonzero map $V_i \longrightarrow V$, i.e. $W_i \neq 0$.

In a similar way, we calculate

$$\begin{aligned} A &\cong \bigoplus_{i=1}^n \mathrm{End}_{\Bbbk}(V_i) \\ &\cong \bigoplus_{i=1}^n \mathrm{Hom}_{\Bbbk}(V_i, V_i) \\ &\cong \bigoplus_{i=1}^n \mathrm{Hom}_{\Bbbk}(V_i, \mathrm{Hom}_B(W_i, E)) \\ &\cong \bigoplus_{i=1}^n \mathrm{Hom}_B(V_i \otimes_{\Bbbk} W_i, E) \\ &\cong \mathrm{End}_B(E) \end{aligned}$$

□

2. proof of the Schur-Weyl Duality

Recall the statement of the theorem:

Theorem 2.1. *For $V^{\otimes n}$, we have the decomposition*

$$V^{\otimes n} \cong \bigoplus_{\lambda \in \Lambda} V_\lambda \otimes \mathbb{S}_\lambda V$$

where V_λ runs over all irreducible representations of S_n,

$$\mathbb{S}_\lambda V := \mathrm{Hom}_{\mathbb{C}[S_n]}(V_\lambda, V^{\otimes n})$$

is 0 or an irreducible representation of $GL(V)$.

Proof. We will use the Double Centralizer Theorem by applying

- $E = V^{\otimes n}$
- $A = \mathbb{C}[S_n]$ (semisimple by the Maschke's Theorem)
- $B = \tilde{\rho}\big(\mathcal{U}(\mathfrak{gl}(V))\big)$ where $\tilde{\rho}$ is induced by the group action

  $$\rho : GL(V) \circlearrowleft V^{\otimes n}$$

  through the following isomorphism:

  $$\begin{aligned}\mathrm{Hom}_{Grp}(GL(V), (\mathrm{End}(V^{\otimes n})^{\times})) &\cong \mathrm{Hom}_{LieAlg}(\mathfrak{gl}(V), \mathrm{End}(V^{\otimes n})) \\ &\cong \mathrm{Hom}_{Alg}(\mathcal{U}(\mathfrak{gl}(V)), \mathrm{End}(V^{\otimes n}))\end{aligned}$$

What remains to prove is listed as follows.

- $$\mathrm{End}_{\mathbb{C}[S_n]}(V^{\otimes n}) = \mathrm{Im}\ \tilde{\rho} = \langle \mathrm{Im}\ \rho \rangle_{Alg}$$
- Any irreducible representation of B is a irreducible representation of $GL(V)$.

$\underline{\mathrm{End}_{\mathbb{C}[S_n]}(V^{\otimes n}) = \mathrm{Im}\ \tilde{\rho} = \tilde{\rho}\big(\mathcal{U}(\mathfrak{gl}(V))\big)}$

" $\supseteq$ ": For any $X \in \mathfrak{gl}(V)$, the action of X on $V^{\otimes n}$ is

$$X(v_1 \otimes \cdots \otimes v_n) = \sum_{i=1}^{n} v_1 \otimes \cdots \otimes Xv_i \otimes \cdots \otimes v_n$$

thus

$$\tilde{\rho}(X) = \sum_{i=1}^{n} Id \otimes \cdots \otimes X \otimes \cdots \otimes Id \in \mathrm{End}_{\mathbb{C}[S_n]}(V^{\otimes n})$$

We get $\mathrm{Im}\ \tilde{\rho} \subseteq \mathrm{End}_{\mathbb{C}[S_n]}(V^{\otimes n})$.

" $\subseteq$ ": Abbrieviate

$$X^{\otimes n} = X \otimes X \otimes \cdots \otimes X \in \mathrm{End}_{\mathbb{C}[S_n]}(V^{\otimes n})$$

We know that any elementary symmetric polynomial can be expressed as the polynomial of

$$p_j = x_1^j + x_2^j + \cdots + x_n^j.$$

Especially, there exists a polynomial $\mathcal{P}$ such that

$$\prod_{i=1}^{n} x_i = \mathcal{P}(p_1, \ldots, p_n).$$

Then we have

$$X^{\otimes n} = \mathcal{P}\big(\,\tilde{\rho}(X), \tilde{\rho}(X^2), \ldots, \tilde{\rho}(X^n)\,\big) \in \mathrm{Im}\ \tilde{\rho}$$

Moreover, the set $\{X^{\otimes n} \mid X \in \mathrm{End}(V)\}$ span

$$\mathrm{Sym}^n \mathrm{End}(V) \simeq \big(\mathrm{End}(V)^{\otimes n}\big)^{S_n} \simeq \big(\mathrm{End}\left(V^{\otimes n}\right)\big)^{S_n} = \mathrm{End}_{\mathbb{C}[S_n]}\left(V^{\otimes n}\right)$$

So we get $\mathrm{End}_{\mathbb{C}[S_n]}(V^{\otimes n}) \subseteq \mathrm{Im}\ \tilde{\rho}$.

(The fact that $\mathrm{Sym}^n \mathrm{End}(V)$ was spanned by $\{X^{\otimes n}\}$ is due to the polarization theorem, the technique similar to the construction of the inner product from a normed vector space with the parallelogram law.)

$\underline{\mathrm{Im}\ \tilde{\rho} = \langle \mathrm{Im}\ \rho \rangle_{Alg}}$

" $\supseteq$ ":

$$\begin{aligned} &\text{for any } g \in GL(V),\ g \text{ commutes with } S_n \\ \Rightarrow\ &\rho(g) \in \mathrm{End}_{\mathbb{C}[S_n]}(V^{\otimes n}) = \mathrm{Im}\ \tilde{\rho} \\ \Rightarrow\ &\langle \mathrm{Im}\ \rho \rangle_{Alg} \subseteq \mathrm{Im}\ \tilde{\rho} \end{aligned}$$

" $\subseteq$ ": For any $X \in \mathrm{End}(V)$, we want to show $X^{\otimes n} \in \langle \mathrm{Im}\ \rho \rangle_{Alg}$.

$$\begin{aligned} &\det(X + tI) \neq 0 \text{ for all but finite } t \in \mathbb{C} \\ \Rightarrow &(X + tI)^{\otimes n} \in \mathrm{Im}\ \rho \text{ for all but finite } t \in \mathbb{C} \\ \Rightarrow &(X + tI)^{\otimes n} \in \langle \mathrm{Im}\ \rho \rangle_{Alg} \text{ for all } t \in \mathbb{C} \\ \Rightarrow &X^{\otimes n} \in \langle \mathrm{Im}\ \rho \rangle_{Alg} \end{aligned}$$

<u>Any irreducible representation of B is a irreducible representation of $GL(V)$.</u>

$B = \langle \mathrm{Im}\ \rho \rangle_{Alg} = \rho(\mathbb{C}[GL(V)])$, so

- Any representation of B is a representation of $GL(V)$.
- Any irreducible representation of B is a irreducible representation of $GL(V)$.

□

As a Corrollary,

Corollary 2.2. [3, Thm 2.4.2] *The algebras spanned by the images of $GL(V)$ and of S_k, each acting on $V^{\otimes n}$ as described in the beginning of this essay, are mutual centralizer in* $\mathrm{End}(V^{\otimes n})$.

3. Commentary

Remark 3.1. This theorem is of much interest because it connects the representation of symmetic group and the representation of general linear group.

For the symmetric group part, we have an algorithm (though finicky) To obtain all its irreducible representations Using the Young tableau.(This gives us many examples of the Schur-Weyl Duality, for reference,[5, Example 7.0.3]. The algorithm can be founded in [5, 5]) For a vivid introduction about the Young tableau, you can see [6].

For the general linear group part, we can generalize it to other classical groups including O_n, Sp_{2n}.(for reference,[7, 3.4])

The Schur-Weyl duality is closely connected to the invariant theory. In [2], the author gives an equivalent propositions of the Schur-Weyl duality Theorem:

Theorem 3.2 ((GL_n, GL_m)-duality)**.** *Let U, V be two linear spaces of dimension n, m. we consider two group actions on $U \times V$:*

$$GL(U) \circlearrowright U \times V \qquad g(u \otimes v) = g(u) \otimes v$$

$$GL(V) \circlearrowright U \times V \qquad g(u \otimes v) = u \otimes g(v)$$

Notice that these two actions commutes each other.

Denote $\mathcal{S}(U \otimes V)$ to be the symmetric algebra of $U \otimes V$, then we have a decomposition

$$\mathcal{S}(U \otimes V) = \sum_D \rho_U^D \otimes \rho_V^D$$

of $GL(U) \otimes GL(V)$-modules where ρ_U^D is some representaion of $GL(U)$, ρ_V^D is some representaion of $GL(V)$.

You can see the equivalence from [3, 2.4.5].

These theories are related to the invariant theory because the invariant theory studies the invariants of a group action, and the decomposition offers a method to find these invariants.

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