Analysis and Differential Equations (5)

Problem1. Suppose f is a function from the reals to the reals satisfying $2f(x) = f(2x)$ for all x.
(a) Prove that if f is differentiable at 0 then f is linear.
(b) Give an example of such a function f that is continuous but not linear.

Problem2.(2012 Individual) Let $\mathbb{R}_n^+ = \{x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n | x_n > 0\}$. Show that the formula

$$u(x) = \frac{2x_n}{n\alpha_n} \int_{\partial \mathbb{R}_+^n} \frac{g(y)}{|x-y|^n} dy, x \in \mathbb{R}^n$$

is a solution of the problem

$$\Delta u = 0 \ \ in \ \ \mathbb{R}_n^+, \ \ u = g \ \ on \ \ \partial \mathbb{R}_+^n;$$

where α_n is the volume of the unit n dimensional sphere.

Problem3.(2016 Individual) Suppose that F is continuous on $[a, b]$, $F'(x)$ exists for every $x \in (a, b)$, $F'(x)$ is integrable. Prove that F is absolutely continuous and

$$F(b) - F(a) = \int_a^b F'(x)dx.$$

Problem4. Let $\{f_n\}$ be a sequence of injective holomorphic functions in a domain $\Omega \subset \mathbb{C}$. If f_n converges uniformly on every compact subset of Ω, to a non-constant function f as $n \to \infty$, prove that f is an injective holomorphic function on Ω.

Problem5.(2011 Individual) For $s > 0$, let $H^s(T)$ be the space of L^2 functions f on the circle $T = R/2\pi\mathbb{Z}$ whose Fourier coefficients $\hat{f}_n = \int_0^{2\pi} e^{-inx} f(x)dx$ satisfy $\sum(1+n^2)^s ||\hat{f}_n||^2 < \infty$; with norm $||f||_s^2 = \frac{1}{2\pi} \sum(1+n^2)^s ||\hat{f}_n||^2$.
(a) Show that for $r > s \geq 0$, the inclusion map $i : H^r(T) \to H^s(T)$ is compact.
(b) Show that if $s > \frac{1}{2}$, then $H^s(T)$ includes continuously into $C(T)$, the space of continuous functions on T, and the inclusion map is compact.