Analysis and Differential Equations (2)

Problem1. Compute $\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots$

Problem2. A number $\xi \in \mathbb{R}$ is *diophantine* (with exponent k, $k > 0$) if for some constant $C > 0$, there are no rational numbers $\frac{p}{q}$ ($p, q \in \mathbb{Z}$, $q > 0$) such that $|\xi - \frac{p}{q}| < Cq^{-k}$. ξ is a *Liouville number* if it is not diophantine, i.e. if for every $k > 0$ there exist integers p, q such that $q > k$ and $|\xi - \frac{p}{q}| \leq q^{-k}$.

(1) Prove that the set of diophantine numbers is of the first category in $\mathbb{R}$.
(2) Prove that the set L of Liouville numbers has zero Hausdorff dimension.

Problem3. Prove or disprove: If $f(z)$ is an entire function on $\mathbb{C}$ such that the image of every unbounded set under f is unbounded, then f is a polynomial.

Problem4. Let $B_1(0)$ be the unit ball in $\mathbb{R}^n$, centered at the origin, and let u be a smooth solution of

$$\begin{cases} u_{tt} + a^2(x)u_t - \Delta u = 0 & \textit{in } B_1(0) \times (0, +\infty) \\ u(x,t) = f(x) & \textit{on } \partial B_1(0) \times (0, +\infty) \\ u(x,0) = g(x), u_t(x,0) = h(x) & \textit{on } B_1(0) \times \{0\} \end{cases}$$

Here g, h and a are smooth functions and g, h vanish on $\partial B_1(0)$. Prove that

$$\int_{B_1(0)} u^2(x,t)dx \leq C \exp(-At),$$

where $A = \min\{a^2(x)\}$ and C only depends on g, h and n.

Problem5. Let $\phi : \mathbb{R}^n \to \mathbb{C}$ be a measurable function such that $\phi(x+y) = \phi(x)\phi(y)$ and $|\phi| = 1$. Prove that there exists $\xi \in \mathbb{R}^n$ s.t. $\phi(x) = e^{i\xi \cdot x}$.