Tate Cohomology & Brauer Groups of Local Fields

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Abstract

A short essay regarding Brauer groups of local fields and the resulting theory obtained for division algebras.

Contents

1 Introduction

Brauer groups are fascinating objects; they classify central simple algebras over a field but also tell us a huge amount about the field itself and are in some sense foundational to class field theory. In developing local class field theory it is useful to compute the Brauer groups of local fields, and we do this here. However, we go further and relate these statements back to statements about central division algebras (CDAs) over local fields to see what this tells us. The result of this is that the structure of division algebras over these local fields is surprisingly generic, and that in fact the canonical valuation on a local field is somehow what forces this to be the case.

This essay is largely an expansion of Appendix 1 of Serre [2, VI], filling in the details ignored in the discussion. Throughout, all local fields will be non-archimedean and so we may take this to mean that they are complete with respect to a discrete valuation and have finite residue field. This is simply because the archimedean local fields $\mathbb{R}$ and $\mathbb{C}$ have easily understood Brauer groups, the former is cyclic of order 2 with generator given by Hamiltons quaternions and the latter is trivial as $\mathbb{C}$ is algebraically closed.

We begin with §2, which covers the basic material of Tate cohomology. The prepared reader who has studied [2, IV,V] and/or [4, I] can safely ignore this section. §3 then goes on to develop the theory of central division algebras over local fields, in particular the analogue of ramification theory here, and concludes that the Brauer group consists of unramified classes. Finally, in §4 we compute the Brauer group of a local field, and then unify the cohomological and division algebraic perspectives of the Brauer group in the context of this isomorphism. We conclude with a pleasant application of this theory; there are division algebras which contain solutions to all irreducible polynomials of a fixed degree. The final section §5 very briefly mentions some work of Yaun and Witt and the relations to our discussion.

A notation guide is provided in the appendix for the convenience of the reader, but in particular we wish to be clear that the cohomology H^q will always be taken to be Tate cohomology groups.

2 Tate Cohomology

Galois cohomology is frequently developed as group cohomology for finite groups and then for infinite groups is taken to be the direct limit of the system given by finite Galois subextensions [2, V]. Of course there is a corresponding homology theory, and these fuse in the situation of Galois cohomology to give the far more elegant **Tate cohomology**. This section will describe briely the construction of Tate cohomology, taking the viewpoint of Neukirch [4]. Throughout this section, G is a finite group.

2.1 Basic Definitions and Properties

Definition 2.1. *Let G be a finite group. A complete projective resolution of $\mathbb{Z}$ by G-modules is an exact, commuting diagram of G-modules of shape*

$$\begin{tikzcd}
\cdots & X_{-2} \arrow[l, "d_{-2}"'] & X_{-1} \arrow[l, "d_{-1}"'] & & X_0 \arrow[ll, "d_0"'] \arrow[dl, "\varepsilon"] & X_1 \arrow[l, "d_1"'] & X_2 \arrow[l, "d_2"'] & \cdots \arrow[l, "d_3"'] \\
& & & \mathbb{Z} \arrow[ul, "\mu"] \arrow[dl] & & & & \\
& & 0 & & 0 \arrow[ul] & & &
\end{tikzcd} \tag{1}$$

where the X_i are projective G-modules.

Note that the full subdiagram given by X_q for $q \geq 0$ and $\mathbb{Z}$ is a sequence of the correct shape (a projective resolution) to give group cohomology.

Definition 2.2. *Let G be a finite group, then the* ***Tate cohomology*** *groups for a G-module A are given by applying the functor $\mathrm{Hom}_G(-, A)$ to the top row of (1) and taking cohomology. We label these as H^q and the group (co)homology as $\hat{H}^q$ and $\hat{H}_q$ in this section to distinguish between the two.*

These Tate cohomology groups are independent of choice of resolution for exactly the same reasons as the standard group (co)homology groups are. Note further that in almost all degrees these are exactly the same as (co)homology, so that this really is fusing the two.

$$H^q(G, A) = \hat{H}^q(G, A) \quad (q > 0),$$
$$H^q(G, A) = \hat{H}_{-q-1}(G, A) \quad (q < -1).$$

As in the case of group (co)homology, we will of course appeal to a standard complete resolution to make this more explicit.

Definition 2.3 (The standard complete resolution)**.** *We give the objects for the standard resolution by*

- *For $q \geq 1$, consider all $\sigma = (\sigma_1, \dots, \sigma_q) \in G^q$. Then we set*

$$X_q = X_{-q-1} = \bigoplus_{\sigma \in G^q} \mathbb{Z}[G](\sigma_1, \dots, \sigma_q)$$

with the G-action on these being the extension of $\tau \cdot (\alpha(\sigma_1, \dots, \sigma_1)) = \tau\alpha(\sigma_1, \dots, \sigma_q)$. In other words, G acts trivially on our basic elements $(\sigma_1, \dots, \sigma_q)$.
- *For $q = 0$, we put $X_0 = X_{-1} = \mathbb{Z}[G]$ with the canonical G action.*

Further we define the maps for the standard resolution to be

- *$\varepsilon : X_0 \to \mathbb{Z}$ is the augmentation map,*

$$\varepsilon(\sum_{\sigma \in G} n_\sigma \sigma) = \sum_{\sigma \in G} n_\sigma.$$

- *$\mu : \mathbb{Z} \to X_{-1}$ is the coaugmentation map*

$$\mu(n) = nN_G$$

where $N_G = \sum_{\sigma \in G} \sigma$ is the norm element.
- *$d_q : X_q \to X_{q-1}$ are given by their action on the basis elements, so*

$$\begin{array}{lrl}
\mathbf{q = 0}: & d_0(1) = & N_G \\
\mathbf{q = 1}: & d_1(\sigma) = & \sigma - 1 \\
\mathbf{q > 1}: & d_q(\sigma_1, \dots, \sigma_q) = & \sigma_1(\sigma_2, \dots, \sigma_q) \\
& & + \sum_{i=1}^{q-1} (-1)^i (\sigma_1, \dots, \sigma_{i-1}, \sigma_i\sigma_{i+1}, \sigma_{i+2}, \dots, \sigma_q) \\
& & + (-1)^q (\sigma_1, \dots, \sigma_{q-1}) \\
\mathbf{q = -1}: & d_{-1}1 = & \sum_{\sigma \in G} (\sigma^{-1} - 1)(\sigma) \\
\mathbf{q < -1}: & d_q(\sigma_1, \dots, \sigma_{-q-1}) = & \sum_{\sigma \in G} \sigma^{-1}(\sigma, \sigma_1, \dots, \sigma_{-q-1}) \\
& & + \sum_{\sigma \in G} \sum_{i=1}^{-q-1} (-1)^i (\sigma_1, \dots, \sigma_{i-1}, \sigma_i\sigma, \sigma^{-1}, \sigma_{i+1}, \dots, \sigma_{-q-1}) \\
& & + \sum_{\sigma \in G} (-1)^{q+1} (\sigma_1, \dots, \sigma_{-q-1}, \sigma).
\end{array}$$

Note that X_q are all free as $\mathbb{Z}[G]$-modules, so certainly projective. It remains to check that the maps for the complex satify the exactness and commutativity required for (1).

Proposition 2.4. *The standard resolution is indeed a complete resolution as in* (1).

Proof. It is immediate from the way we defined the maps on basic elements that they are G-equivariant, i.e. G-homomorphisms. Further, $\mu \circ \varepsilon(1) = N_G = d_0(1)$ so the commutativity is immediate. For exactness we split into the two long exact sequences:

$$0 \longleftarrow \mathbb{Z} \xleftarrow{\varepsilon} X_0 \xleftarrow{d_1} X_1 \xleftarrow{d_2} X_2 \xleftarrow{d_3} \dots \tag{2}$$

$$\dots \xleftarrow{d_{-2}} X_{-2} \xleftarrow{d_{-1}} X_{-1} \xleftarrow{\mu} \mathbb{Z} \longleftarrow 0 \tag{3}$$

It is clear that if each of these is exact then the whole diagram (1) is exact by commutativity.

Let us begin with (2). It is immediate from the definitions of the maps that this is at least a complex, i.e. any composition of maps is 0. Note that we only need exactness as a $\mathbb{Z}$-module diagram so define $\mathbb{Z}$-homomorphisms (group homomorphisms but not necessarily G-equivariant) on the $\mathbb{Z}$-bases:

$$\begin{aligned}
E : \mathbb{Z} \to X_0; &\quad E(1) = 1, \\
D_0 : X_0 \to X_1; &\quad D_0(\sigma_0) = (\sigma_0), \\
D_q : X_q \to X_{q+1}; &\quad D_q(\sigma_0(\sigma_1, \dots, \sigma_q)) = (\sigma_0, \dots, \sigma_q)
\end{aligned}$$

Now it is immediate from the definitions that $E \circ \varepsilon + d_1 \circ D_0 = \mathrm{Id}$. Thus we know that $x \in \ker \varepsilon$ must satisfy $x = d_1 \circ D_0(x) \in \mathrm{im}(d_1)$, so (2) is exact at X_0. Further for $q > 0$ we have that $d_{q+1} \circ D_q + D_{q-1} \circ d_q = \mathrm{Id}$, since

$$\begin{aligned}
D_{q-1} \circ d_q(\sigma(\sigma_1, \ldots, \sigma_q)) &= (\sigma\sigma_1, \sigma_2, \ldots, \sigma_q) \\
&\quad + \sum_{i=1}^{q-1} (-1)^i (\sigma, \sigma_1, \ldots, \sigma_{i-1}, \sigma_i\sigma_{i+1}, \sigma_{i+1}, \ldots \sigma_q) \\
&\quad + (-1)^q (\sigma, \sigma_1, \ldots, \sigma_{q-1}) \\
d_{q+1} \circ D_q(\sigma(\sigma_1, \ldots, \sigma_q)) &= d_{q+1}((\sigma, \sigma_1, \ldots, \sigma_q)) \\
&= \sigma(\sigma_1, \ldots, \sigma_q) - (\sigma\sigma_1, \ldots, \sigma_q) \\
&\quad + \sum_{i=1}^{q-1} (-1)^{i+1} (\sigma, \sigma_1, \ldots, \sigma_{i-1}, \sigma_i\sigma_{i+1}, \sigma_{i+1}, \ldots \sigma_q) \\
&\quad + (-1)^{q+1} (\sigma, \sigma_1, \ldots, \sigma_{q-1})
\end{aligned}$$

and again this gives us exactness at X_q for $q > 0$. The surjectivity of ε is obvious from its definition and so (2) is exact.

Moving to the sequence (3), we see that if we take $\mathrm{Hom}(-, \mathbb{Z})$ of (2) then firstly we get an exact sequence (since the X_q of (2) are free) and secondly this is canonically isomorphic to the sequence (3) using the $\mathbb{Z}[G]$-dual basis. Thus we have exactness. □

Now that we have an explicit model of Tate cohomology, we should prove that it has at least the usual properties shown by group cohomology; in particular, the extension of short exact sequences. Firstly, recall that a map $f : A \to B$ induces a map $f^* : \mathrm{Hom}_G(X_q, A) \to \mathrm{Hom}_G(X_q, B)$ and note that by the way we defined the d_q maps we have that these commute. In other words $f^* \circ d_q = d_q \circ f^*$.

Theorem 2.5. *Given a short exact sequence of G-modules*

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$$

there is a canonical long exact sequence of cohomology

$$\ldots \longrightarrow H^q(G, A) \xrightarrow{\bar{i}_q} H^q(G, B) \xrightarrow{\bar{j}_q} H^q(G, C) \xrightarrow{\delta} H^{q+1}(G, A) \longrightarrow \ldots$$

Proof. Firstly we construct this δ in the usual way. Take the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_G(X_{q-1}, A) \arrow[r, "i^*"] \arrow[d, "d^*"] & \mathrm{Hom}_G(X_{q-1}, B) \arrow[r, "j^*"] \arrow[d, "d^*"] & \mathrm{Hom}_G(X_{q-1}, C) \arrow[r] \arrow[d, "d^*"] & 0 \\
0 \arrow[r] & \mathrm{Hom}_G(X_q, A) \arrow[r, "i^*"] \arrow[d, "d^*"] & \mathrm{Hom}_G(X_{q-1}, B) \arrow[r, "j^*"] \arrow[d, "d^*"] & \mathrm{Hom}_G(X_{q-1}, C) \arrow[r] \arrow[d, "d^*"] & 0 \\
0 \arrow[r] & \mathrm{Hom}_G(X_{q-1}, A) \arrow[r, "i^*"] & \mathrm{Hom}_G(X_{q-1}, B) \arrow[r, "j^*"] & \mathrm{Hom}_G(X_{q-1}, C) \arrow[r] & 0
\end{tikzcd}$$

where we neglect the indices on d from the standard complex. Note that this diagram commutes, and further the rows are exact since X_i are free and so $\mathrm{Hom}_G(X_i, -)$ is an exact functor. An application of the snake lemma gives the connecting map δ and the maps $\bar{i}_q, \bar{j}_q$. It is immediately clear that the sequence of cohomology is at least a complex, and we need only prove that at each point there is actually exactness. For this the argument is wholly explicit, and the reader is referred to [4, §3 p23]. □

2.2 Dimension Shifting

Dimension shifting is one of the main elegances of Tate cohomology, a way to fluidly define maps and prove results for all cohomology degrees by simply considering one degree explicitly. The essential idea is to use short exact sequences like in Theorem 2.5 where the middle term has trivial cohomology in all degrees to establish isomorphisms between different degrees of modules. By far the best reference for this is Neukirch [4], which we follow closely.

Recall the definition of an induced G-module,

Definition 2.6. *Let $H \leq G$ be a subgroup. We say a G-module A is induced from an H-module Λ if*

$$A \cong \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} \Lambda.$$

Further we say this is G-induced if $H = \{1\}$ is trivial.

G-induced modules prove to be a vital concept in both Tate cohomology and group cohomology.

Lemma 2.7. *Let A be a G-induced module, and $H \leq G$ a subgroup. Then A is a H-induced H-module.*

Proof. We can just rewrite this with isomorphisms of $\mathbb{Z}[H]$-modules. Fix R a set of right coset representatives of $H \backslash G$.

$$\begin{aligned}\mathbb{Z}[G] \otimes_{\mathbb{Z}} \Lambda &= \bigoplus_{\sigma \in G} \sigma\Lambda \\ &= \bigoplus_{\sigma \in H} \sigma \bigoplus_{\tau \in R} \tau\Lambda \\ &\cong \mathbb{Z}[H] \otimes_{\mathbb{Z}} \bigoplus_{\tau \in R} \tau\Lambda\end{aligned}$$

So this is H-induced as an H module.

$$\oplus_{\sigma \in G}\Lambda = \bigoplus_{h \in H} \bigoplus_{\tau \in H \backslash G} \Lambda = \mathbb{Z}$$

$\square$

Definition 2.8. *We say a G-module A has **trivial cohomology** if for every subgroup $H \leq G$ and $q \in \mathbb{Z}$ we have*

$$H^q(H, A) = 0.$$

Proposition 2.9. *Every G-induced module A has trivial cohomology.*

Proof. By Lemma 2.7 we have that it is sufficient to prove that $H^q(G, A) = 0$, since A is also induced as an H module for $H \leq G$. Thus we need exactness of the sequence

$$\ldots \longrightarrow \mathrm{Hom}_G(X_{q-1}, A) \longrightarrow \mathrm{Hom}_G(X_q, A) \longrightarrow \mathrm{Hom}_G(X_{q+1}, A) \longrightarrow \ldots.$$

Note that since $A = \mathbb{Z}[G] \otimes_{\mathbb{Z}} \Lambda$, we have a natural projection $\pi : A \to \Lambda$ given by

$$\pi(\sigma \otimes x) = \begin{cases} x & \sigma = \mathrm{Id} \\ 0 & \text{otherwise} \end{cases}$$

which induces an obvious isomorphism of groups

$$\mathrm{Hom}_G(X_q, A) \cong \mathrm{Hom}(X_q, \Lambda); \quad f \mapsto \pi \circ f$$

Thus, identifying our sequence through these isomorphisms we have the natural sequence

$$\ldots \longrightarrow \mathrm{Hom}(X_{q-1}, \Lambda) \longrightarrow \mathrm{Hom}(X_q, \Lambda) \longrightarrow \mathrm{Hom}(X_{q+1}, \Lambda) \longrightarrow \ldots.$$

which is exact as it is the application of $\mathrm{Hom}(-, \Lambda)$ to a sequence of free $\mathbb{Z}$-modules. $\square$

Using this we can now describe dimension shifting completely explicitly. Define G-modules

$$\begin{aligned} I_G &= \langle \sigma - 1 \rangle_{\sigma \in G} \lhd \mathbb{Z}[G], \\ J_G &= \mathbb{Z}[G]/\mathbb{Z}N_G. \end{aligned}$$

and for any $m \in \mathbb{Z}$ define

$$A^m = \begin{cases} I_G \otimes \cdots \otimes I_G \otimes A & m \leq 0 \\ J_G \otimes \cdots \otimes J_G \otimes A & m \geq 0 \end{cases}$$

where $\otimes$ is $\otimes_{\mathbb{Z}}$ for brevity.

Theorem 2.10 (Dimension Shifting)**.** *For all G-modules A and $q \in \mathbb{Z}$ there is a natural isomorphism*

$$H^{q-m}(G, A^m) \xrightarrow{\delta^m} H^q(G, A)$$

given by the concatenation of connecting maps of cohomology induced from natural short exact sequences.

Proof. We have canonical exact sequences of G-modules induced by our augmentation and coaugmentation maps,

$$0 \longrightarrow I_G \longrightarrow \mathbb{Z}[G] \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\mu} \mathbb{Z}[G] \longrightarrow J_G \longrightarrow 0$$

which are all exact sequences of free $\mathbb{Z}$-modules so $- \otimes_{\mathbb{Z}} A$ acts exactly to give short exact sequences

$$0 \longrightarrow A^{-1} \longrightarrow \mathbb{Z}[G] \otimes_{\mathbb{Z}} A \xrightarrow{\varepsilon \otimes \mathrm{Id}} A \longrightarrow 0$$

$$0 \longrightarrow A \xrightarrow{\mu \otimes \mathrm{Id}} \mathbb{Z}[G] \otimes_{\mathbb{Z}} A \longrightarrow A^1 \longrightarrow 0 \ .$$

Now, note that the middle term of each of these is $\mathbb{Z}[G] \otimes A$ which is clearly G-induced, and so by Lemma 2.9 has trivial cohomology. Thus the connecting maps from Theorem 2.5 are isomorphisms for all $q \in \mathbb{Z}$

$$H^{q+1}(G, A^{-1}) \xrightarrow{\delta^{-1}} H^q(G, A)$$

$$H^{q-1}(G, A^1) \xrightarrow{\delta} H^q(G, A)$$

Concatenating these isomorphisms we have the result as required. $\square$

This is a very useful result. There are many uses of this dimension shifting in literature [4, 2], and so we shall be sure to be explicit when using it to illustrate its power. Below is an example.

Proposition 2.11. *Let A be a G-module. Then $H^q(G, A)$ are torsion groups with exponent dividing $n = |G|$. In other words,*

$$n \cdot H^q(G, A) = 0$$

for all $q \in \mathbb{Z}$.

Proof. Using dimension shifting, we have that if this result is true in degree 0 then $H^0(G, A^q)$ is n-torsion and the isomorphisms $H^0(G, A^q) \xrightarrow{\delta^q} H^q(G, A)$ ensure that it is true for all $q \in \mathbb{Z}$. The degree 0 case is obvious since

$$H^0(G, A) = A^G / N_G A$$

and for $a \in A^G$ we have $n \cdot a = N_G a$. $\square$

This allows us to make a useful statement about uniquely divisible modules, which are those modules Q such that $n \cdot x = a$ always has a unique solution $x \in Q$.

Corollary 2.12. *If Q is a uniquely divisible G-module, then it has trivial cohomology.*

Proof. Let $H \leq G$ be a subgroup and $n = |H|$, then multiplication by n is an isomorphism on Q and so is an isomorphism on cohomologies $H^q(G, Q)$. However by Proposition 2.11 multiplication by n is also the zero map on cohomology, and so the cohomology must be trivial. $\square$

2.3 Restriction and Corestriction

As in the case of standard group cohomology, there is the notion of restriction and corestriction maps here. In fact, dimension shifting allows us to define these naturally so that they commute with connecting maps. In this subsection we give no proofs, as the material required would take us too far from the topic at hand, and instead summarise the main definitions and results of use to us. The reader who craves detail is referred to [4, I§4].

Definition 2.13. *For a finite group G, subgroup $H \leq G$ and G-module A, the restriction maps*

$$\mathrm{res}_q : H^q(G, A) \to H^q(H, A)$$

are given by

1. *If $q = 0$ we have,*
$$\mathrm{res}_0 : H^0(G, A) \to H^0(H, A); \quad a + N_G A \mapsto a + N_H A$$
2. *For every short exact sequence $0 \to A \to B \to C \to 0$ of G-modules, the diagram below commutes*

$$\begin{tikzcd} H^q(G,C) \arrow[r, "\delta"] \arrow[d, "\mathrm{res}_q"] & H^{q+1}(G,A) \arrow[d, "\mathrm{res}_{q+1}"] \\ H^q(H,C) \arrow[r, "\delta"] & H^{q+1}(H,A) \end{tikzcd} .$$

In a similar manner, one defines corestriction.

Definition 2.14. *For a finite group G, subgroup $H \leq G$ and G-module A, the corestriction maps*

$$\mathrm{cores}_q : H^q(H, A) \to H^q(G, A)$$

are given by

1. *If $q = 0$ we have,*
$$\mathrm{cores}_0 : H^0(G, A) \to H^0(H, A); \quad a + N_H A \mapsto N_{G/H} a + N_G A$$
where $N_{G/H} a = \sum_{\sigma \in R} \sigma a$ for R a system of left coset representatives of G/H is well defined since $a \in A^H$.
2. *For every short exact sequence $0 \to A \to B \to C \to 0$ of G-modules, the diagram below commutes*

$$\begin{tikzcd} H^q(H,C) \arrow[r, "\delta"] \arrow[d, "\mathrm{cores}_q"] & H^{q+1}(H,A) \arrow[d, "\mathrm{cores}_{q+1}"] \\ H^q(G,C) \arrow[r, "\delta"] & H^{q+1}(G,A) \end{tikzcd} .$$

Note that these definitions allow us to compute (co)restriction maps explicitly with the dimension shifting result of Theorem 2.10. Further algebraic manipulations would show that for $q \geq 1$ we get that these notions of (co)restriction precisely agree with that of (co)restriction in group cohomology.

3 Central Division Algebras Over Local Fields

We proceed towards the main goal of this essay – computing the Brauer groups of local fields. Throughout, we will take K a local field with uniformiser $\pi = \pi_K$, maximal ideal $\mathfrak{p} = \langle \pi_K \rangle \lhd \mathcal{O}_K$ and residue field k_K. Let v be the normalised valuation on K, and note that we can extend this to any CDA D/K by noting that v extends uniquely to any finite field extension, and for any $\alpha \in D$ we can extend to $K(\alpha)/K$ and take the valuation here. Uniqueness of extension assures us that these valuations 'fit together' to define a function on D which satisfies the usual properties of a valuation.

3.1 Structure of CDA's Over Local Fields

The structure theory of CDA's over a local field K is very similar to that of field extensions, it turns out that the lack of commutativity can be circumvented to obtain the same ramification theory we know and love from the commutative case. This material largely follows some notes of Lipnowski [3], and concludes with some of Serres appendix [2].

Definition 3.1. *For D/K a CDA we define the **integers of** D*

$$\mathcal{O}_D = \{x \in D \mid v(x) \geq 0\}$$

*and the **valuation ideal***

$$\mathfrak{P}_D = \{x \in D \mid v(x) > 0\}.$$

We first note that the usual theory holds true for these rings as in the case of finite commutative extensions.

Proposition 3.2. *For D/K a CDA, $\mathcal{O}_D$ is a local principal ideal ring, with $\mathfrak{P}_D = \langle \pi_D \rangle$ the unique maximal 2-sided ideal. Further, every 2-sided ideal of $\mathcal{O}_D$ is of the form $\mathfrak{P}_D^m$ for some $m \geq 0$*

Proof. That $\mathcal{O}_D$ is a ring and $\mathfrak{P}_D$ is an ideal are immediate from the properties of the valuation v on D. Now since any subfield L/K of D has degree $[L : K] \leq n$ we know that $v(x) \in \frac{1}{n}\mathbb{Z}$ for any $x \in D$ and so in $\mathfrak{P}_D$ there exists $\pi_D$with minimal valuation. Now, exactly as for discrete valuation rings, for any $x \in \mathcal{O}_D$ we can write $x = u \cdot \pi_D^m$ for some $m \geq 0$ and $v(u) = 0$. Further since $v(u) = 0$ is clearly equivalent to $u \in \mathcal{O}_D^\times$, we see that this uniquely identifies every $x \in \mathcal{O}_D$ and so we have the result. □

We can use the result of Proposition 3.2 immediately to define ramification as in the case of extensions of local fields.

Definition 3.3. *Let D/K be a CDA, then say that the two-sided ideal $\mathcal{O}_D \mathfrak{p} \mathcal{O}_D = \mathfrak{P}_D^e$. We define the **ramification index** of D/K to be*

$$e_{D/K} := e.$$

Now we need some analogue of inertia degree to continue in the same way as the case of field extensions. We begin with the obvious definitions.

Definition 3.4. *The **residue field** of a CDA D/K is*

$$k_D = \frac{\mathcal{O}_D}{\mathfrak{P}_D}.$$

Lemma 3.5. *For D/K a CDA, the residue field k_D is indeed a field, and is a finite extension of k_K. The degree $f_{D/K} := [k_D : k_K]$ will be called the **inertia degree** as in the commutative case.*

Proof. Firstly, note that k_D is certainly an extension of k_K in the obvious way. Further k_D is a finite dimensional k_K division algebra, so by Wedderburns little theorem is a field. □

Proposition 3.6. *Let D/K be a CDA with $[D : K] = n^2$, then $\mathcal{O}_D$ is a free $\mathcal{O}_K$-module of rank n^2. Moreover*

$$n^2 = e_{D/K} f_{D/K}.$$

Proof. Note that $\mathcal{O}_D$ is a finitely generated, torsionfree (since it is a division algebra), $\mathcal{O}_K$-module. As $\mathcal{O}_K$ is a PID, the structure theory of such modules immediately gives us that $\mathcal{O}_D$ is a free $\mathcal{O}_K$ module of some rank. Now extending scalars we have $\mathcal{O}_D \otimes_{\mathcal{O}_K} K = D$ and we see that $\mathrm{rank}_{\mathcal{O}_K}(\mathcal{O}_D) = \dim_K(D) = n^2$ as required. To relate this rank to ramification and inertia, we extend scalars to the residue field and since $\mathcal{O}_D \otimes_{\mathcal{O}_K} k_K = \mathcal{O}_D/\mathfrak{p}\mathcal{O}_D$ then

$$n^2 = \dim_{k_K}(\mathcal{O}_D/\mathfrak{p}\mathcal{O}_D).$$

However, just as for the commutative case we have a filtration for this module given by

$$\mathcal{O}_D\mathfrak{p}\mathcal{O}_D = \mathfrak{P}_D^{e_{D/K}} \subset \cdots \subset \mathfrak{P}_D^2 \subset \mathfrak{P}_D \subset \mathcal{O}_D$$

with the successive quotients all having size $f_{D/K}$ and this the result $n^2 = e_{D/K}f_{D/K}$. □

In fact we can use this to prove a seemingly stronger relation, and give a pleasant result about central division algebras over local fields!

3.2 Brauer Group As Unramified Classes

We would like to use our newly aquired structure theory for central division algebras over a local field K to make some statements about the Brauer group. In fact we can sharpen the result of Proposition 3.6 to show that the splitting behaviour of CDAs is determined by unramified extensions.

Proposition 3.7. *Let D/K be a CDA with $[D : K] = n^2$, then in fact $n = e_{D/K} = f_{D/K}$. Further D contains a maximal subfield L_D which is unramified over K.*

Proof. By Proposition 3.6 it is sufficient to show that $e_{D/K}, f_{D/K} \leq n$ as then $n^2 = e_{D/K}f_{D/K}$ gives that this must actually be an equality.

$\mathbf{e_{D/K} \leq n}$: The uniformizer π_D of D must be in some subfield $F = K(\pi_D)$ which has ramification index $e_{D/K}$. However maximal subfields are of degree n, so using $e_{D/K}f_{F/K} = [F : K] \leq n$ in the commutative case then we have this inequality.

$\mathbf{f_{D/K} \leq n}$: Since k_D/k_K is a finite extension of finite fields we have that $k_D = k_K(\bar{\alpha})$ with a lift $\alpha \in D$. Set $L = K(\alpha)$, and as $k_L = k_D = k_K(\bar{\alpha})$, we see $f_{L/K} = f_{D/K}$. Maximal subfields have degree n so $e_{L/K}f_{D/K} = [L : K] \leq n$, and the result follows.

Now since $f_{D/K} = n = f_{L/K} = [L : K]$ from the previous result and Proposition 3.6 we obtain that L is a maximal subfield of D which is unramified over K. □

Remark 3.8. *Despite our choice of notation, the unramified subfield L_D is not unique. However unramified extensions constructed as L_D are unique up to unique K-isomorphism, so let L'_D be a second such and $f : L_D \to L'_D$. The Noether-Skolem theorem states that f extends to an inner automorphism of D. In other words, L_D is unique up to conjugation in D.*

Proposition 3.7 provides an immediate result for the Brauer group.

Theorem 3.9. *For a local field K, the Brauer group satisfies*

$$\mathrm{Br}(K) = \mathrm{Br}(K_{nr}/K).$$

In more direct language, every central simple algebra over a local field splits over the maximal unramified extension K_{nr}.

Proof. Take a class $\delta \in \mathrm{Br}(K)$, and let D/K be the CDA representing this class. For any choice of $L_D \subseteq K_{nr}$, D splits over the maximal subfield L_D and so over K_{nr}. □

4 Brauer Groups of Local Fields

We shall now switch tracks to take the cohomological viewpoint on Brauer groups. By Theorem 3.9 we know that there is an isomorphism

$$\mathrm{Br}(K) \cong H^2(\mathrm{Gal}(K_{nr}/K), K_{nr}^\times) = H^2(\Gamma_K, K_{nr}^\times),$$

where K_{nr}/K is the maximal unramified extension of K in some seperable closure. For convenience we will write $H^2(L/K)$ to mean $H^2(\mathrm{Gal}(L/K), L^\times)$. Our goal shall be to compute the Brauer group of the local field K using the cohomological machinery available to us. In order to motivate what lies ahead, consider L/K a finite unramified extension with normalised valuation v. The exact sequence below is of fundamental importance to our computation:

$$0 \longrightarrow \mathcal{O}_L^\times \longrightarrow L^\times \xrightarrow{v} \mathbb{Z} \longrightarrow 0. \tag{4}$$

The $\mathbb{Z}$ term is generally easy to work with, so we study the $\mathcal{O}_L^\times$ term of this to get a better grip on $L^\times$.

4.1 Unit Groups

Recall that for a finite Galois extension of local fields L/K with Galois group G there is a family of groups $U_L^{(i)}$ given by

$$U_L^{(i)} := \begin{cases} \mathcal{O}_L^\times & i = 0 \\ 1 + \pi^i \mathcal{O}_L & i \geq 1 \end{cases}.$$

These give a useful filtration of the unit group, as the successive quotients are easily understood as Galois modules.

Lemma 4.1. *For L/K a finite unramified extension of local fields, there are Galois isomorphisms*

$$U_L^{(i)}/U_L^{(i+1)} \cong \begin{cases} k_L^\times & i = 0 \\ k_L^+ & i \geq 1 \end{cases}.$$

Proof. For $i = 0$ this is obviously just the group isomorphism given by reduction modulo π, and the fact that it is a Galois isomorphism is immediate. For $i > 0$ take the map given by $(1 + \pi^i x) \mapsto x$ and the result is immediate since

$$(1 + \pi^i x)(1 + \pi^i y) = 1 + \pi^i (x + y) + \pi^{2i} xy \mapsto x + y \mod \pi^{i+1}$$

and Galois action commutes with this map since for $\sigma \in G$

$$\sigma(1 + \pi^i x) = 1 + \pi^i \sigma(x).$$

□

Remark 4.2. *Note that being unramified was necessary to show that these isomorphisms are Galois equivariant. π can be chosen to already be in K as L/K is unramified, but were the extension to ramify then the uniformizer of L would be acted on nontrivially.*

In fact these $U_L^{(i)}$ are fundamental to the unit group $U_L^{(0)} = \mathcal{O}_L^\times$, as the quotients by these determine the unit group.

Lemma 4.3. *For L/K a finite Galois extension of local fields,*

$$\mathcal{O}_L^\times \cong \varprojlim_i \frac{\mathcal{O}_L}{U_L^{(i)}}$$

is given by the natural map.

Proof. The kernel of the natural map is $\left\{x \in \mathcal{O}_L^\times \mid x \in U_L^{(i)} \quad \forall i \in \mathbb{N}\right\} = \{1\}$, so it is sufficient to show the surjection. But note that if $(x_i)_i \in \varprojlim_i \frac{\mathcal{O}_L}{U_L^{(i)}}$ then since the maps that the limit is over are given by reduction, $x_i \equiv x_{i-1} \mod U_L^{(i-1)}$ and so these converge to some $x \in \mathcal{O}_L^\times$. □

Now since we will be taking cohomology on the sequence (4), it will be useful to actually compute the cohomology of the unit group. For this we can reduce to the cohomology of successive quotients of Lemma 4.1 using the following technical lemma.

Lemma 4.4. *Let G be a finite group and M a G-module such that there is a decreasing sequence of G-submodules M^i for $i \geq 0$ with $M^0 = M$ and*

$$M = \varprojlim_i \frac{M}{M^i}.$$

Then if for some $q \in \mathbb{Z}$ we have that $H^q(G, M^i/M^{i+1}) = 0$ for all $i \geq 0$ then $H^q(G, M) = 0$.

This lemma allows us to check if the cohomology of a module with a good filtration is trivial just by checking the successive quotients, which is often (and indeed is in our case) easier to do.

Proof. We will denote by d the map on cochains inducing cohomology. Let $f \in Z^q(G, M)$ be a q-cocycle. Since $H^q(G, M/M^1) = 0$ we know that the reduction map $M \to M/M^1$ will render f a coboundary and so we can write

$$f = d\psi_1 + f_1$$

for $f_1 \in Z^q(G, M^1)$ and $\psi_1 \in C^{q-1}(G, M)$. Now since $H^q(G, M^1/M^2) = 0$ we do the same again to f_1, so that

$$f_1 = d\psi_2 + f_2$$

for $f_2 \in Z^q(G, M^2)$ and $\psi_2 \in C^{q-1}(M^1)$. Continuing in this way we obtain a sequence (ψ_n, f_n) such that $\psi_n \in C^{q-1}(G, M^{n-1})$ and $f_n \in Z^q(G, M^n)$, and with the relation

$$f_n = d\psi_{n+1} + f_{n+1}.$$

Now set $\psi = \sum_{n \geq 1} \psi_n$, which is a well defined convergent sum as only finitely many terms of the sum are nonzero in any term of the limit $\varprojlim_i M/M^i$. Further expanding $f = f_1 + d\psi_1 = f_2 + d(\psi_1 + \psi_2) = \ldots$ we see that $f = d\psi$ is a coboundary and so is trivial in the cohomology. $\square$

We will use this technical result to compute the cohomology of the unit group.

Proposition 4.5. *For a finite unramified extension L/K of local fields with Galois group G,*

$$H^q(G, \mathcal{O}_L^\times) = 0$$

for all $q \in \mathbb{Z}$.

Proof. Using Lemmas 4.3 and 4.4, it is sufficient to show that

$$H^q(G, U_L^{(i)}/U_L^{(i+1)}) = 0 \qquad \forall q \in \mathbb{Z}, \forall i \geq 0 \tag{5}$$

Further L/K is unramified, so G is cyclic and the cohomology satisfies

$$H^q(G, U_L^{(i)}/U_L^{(i+1)}) = H^{q+2}(U_L^{(i)}/U_L^{(i+1)}).$$

so it is sufficient to prove the statement (5) for $q \in \{1, 2\}$. We can restrict the region of proof even further, by Lemma 4.1 we know $U_L^{(i)}/U_L^{(i+1)}$ is finite. Thus the herbrand quotient $h_{2/1}(G, U_L^{(i)}/U_L^{(i+1)}) = 1$, so it is sufficient to prove the statement (5) for $q = 1$. Using Lemma 4.1, the case $i = 0$ reduces to

$$H^1(G, k_L^\times) = H^1(\mathrm{Gal}(k_L/k_K), k_L^\times)$$

which is trivial by Hilbert 90. For $i \geq 1$ we have

$$H^1(G, k_L) = H^1(\mathrm{Gal}(k_L/k_K), k_L).$$

As a $k_K[G]$-module, the normal basis theorem states that k_L is free on one generator. As $k_K[G] = \mathbb{Z}[G] \otimes k_K$, this means that as a $\mathbb{Z}[G]$-module, k_L is induced from k_K and

$$H^1(G, k_L) = H^1(\{1\}, k_K) = 0$$

as required. $\square$

4.2 The Invariant Map

Recall the short exact sequence induced by the extension of the valuation v of K to an unramified extension L/K (4),

$$0 \longrightarrow \mathcal{O}_L^\times \longrightarrow L^\times \xrightarrow{v} \mathbb{Z} \longrightarrow 0$$

Using the result of Proposition 4.5 and the induced long exact sequence on cohomology we immediately obtain

Theorem 4.6. *The valuation map $v : K_{nr}^\times \to \mathbb{Z}$ induces an isomorphism for all $q \in \mathbb{Z}$*

$$H^q(K_{nr}/K) \xrightarrow{v} H^q(\hat{\mathbb{Z}}, \mathbb{Z})$$

where $\mathbb{Z}$ is a trivial $\hat{\mathbb{Z}}$-module, and $\hat{\mathbb{Z}}$ is the profinite completion of $\mathbb{Z}$.

Proof. Note that by Proposition 4.5 we get an isomorphism for all $q \in \mathbb{Z}$ and L/K unramified,

$$H^q(L/K) \xrightarrow{v} H^q(\mathrm{Gal}(L/K), \mathbb{Z})$$

so taking the direct limit we get that v induces an isomorphism

$$H^q(K_{nr}/K) \xrightarrow{v} H^q(\mathrm{Gal}(K_{nr}, K, \mathbb{Z})\ .$$

Recall that $\mathrm{Gal}(L/K) \cong \mathrm{Gal}(k_L/k_K)$ canonically and the unramified extensions are in natural bijection with the residue extensions, so

$$\mathrm{Gal}(K_{nr}/K) = \mathrm{Gal}(k_K^s/k_K) = \hat{\mathbb{Z}},$$

where k_K^s is the seperable closure. □

Now we have reduced the problem of computing Brauer groups of local fields to that of computing $H^2(\hat{\mathbb{Z}}, \mathbb{Z})$. In particular, we already have shown that the Brauer groups of local fields are all the same. Now recall the canonical short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

where each has the trivial G-action.

Proposition 4.7. *For a profinite group G, the G-module $\mathbb{Q}$ has trivial cohomology. In particular the connecting map induces an isomorphism*

$$H^2(\hat{\mathbb{Z}}, \mathbb{Z}) \xrightarrow{\delta^{-1}} H^1(\hat{\mathbb{Z}}, \mathbb{Q}/\mathbb{Z})\ .$$

Proof. This is an immediate application of Corollary 2.12 since $\mathbb{Q}$ is uniquely divisible. □

Finally, we note that $H^1(\hat{\mathbb{Z}}, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}_{cts}(\hat{\mathbb{Z}}, \mathbb{Q}/\mathbb{Z})$ and since $1 \in \hat{\mathbb{Z}}$ is a topological generator we have a canonical isomorphism

$$\begin{aligned} \eta : H^1(\hat{\mathbb{Z}}, \mathbb{Q}/\mathbb{Z}) &\to \mathbb{Q}/\mathbb{Z} \\ f &\mapsto f(1) \end{aligned}$$

This finally allows us to define the invariant isomorphism.

Definition 4.8. *The* ***invariant map*** *for a local field K is given by*

$$\mathrm{inv}_K :\ \mathrm{Br}(K) = H^2(K_{nr}/K) \xrightarrow{v} H^2(\hat{\mathbb{Z}}, \mathbb{Z}) \xrightarrow{\delta^{-1}} H^1(\hat{\mathbb{Z}}, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\eta} \mathbb{Q}/\mathbb{Z}$$

where the first equality is Theorem 3.9 and the remaining maps are as in Theorem 4.6 Proposition 4.7 and above.

Remark 4.9. *Notice that each of the composite maps in* inv_K *is an isomorphism, so we have an isomorphism* $\mathrm{Br}(K) \cong \mathbb{Q}/\mathbb{Z}$ *for any local field.*

4.3 The Invariant Map on CSAs

The definition of inv_K is very much an example of some cohomological yoga, and it isn't strictly clear what this does to a Brauer class of central simple K-algebras. In fact, we can show that the invariant map is very naturally defined for central division algebras.

Let D/K be a central division algebra. Then by Noether-Skolem, for every choice of maximal subfield $L \subset D$, unramified over K, there exists $\gamma \in D$ such that the Frobenius of L/K is given by conjugation by γ.

Remark 4.10. *Our choice of notation is deceptive, but for our purposes will be optimal. In fact γ is only defined by to multiplication by L^*. The reason for this is that $\gamma x \gamma^{-1} = \varepsilon x \varepsilon^{-1}$ tells us that $\gamma\varepsilon^{-1}$ commutes with $x \in L$. As L is a maximal subfield, this can only mean that $\gamma\varepsilon^{-1} \in L$.*

Definition 4.11. *Let D/K be a central division algebra with $[D : K] = n^2$, and maximal subfield L which is unramified over K then*

$$i(D) := v_D(\gamma) \in \frac{1}{n}\mathbb{Z}/\mathbb{Z}$$

where v_D is the extension of the normalised valuation on K to D, and γ is a choice of Noether-Skolem element corresponding to the Frobenius $\mathrm{Fr}_{L/K}$ as above.

Given our attention to detail so far regarding well definedness of L and γ in Remarks 3.8 and 4.10, it should be clear that there is a real question regarding well-definedness here.

Lemma 4.12. *The quantity $i(D)$ is independent of choice of L or γ, and so i can be interpreted as a well defined function on the collection of D/K division algebras of degree n.*

Proof. Fix $L_D = L$, then $\gamma_{L_D} = \gamma$ is defined up to multiplication by some $w \in L_D$. However $v(L_D) = \mathbb{Z}$ since this is an unramified extension, and so

$$v(\gamma_{L_D} w) = v(\gamma_{L_D}) + v(w) \equiv v(\gamma_{L_D}) \mod \mathbb{Z}.$$

Now, if L'_D is another choice for L_D then as in Remark 3.8 there is $\alpha \in D$ such that $L'_D = \alpha L_D \alpha^{-1}$. In particular, we can see that one choice for $\gamma_{L'_D}$ is $\alpha\gamma_{L_D}\alpha^{-1}$ and $v(\alpha\gamma_{L_D}\alpha^{-1}) = v(\gamma_{L_D})$ so that this quantity is independent of choice of L_D also. □

In order to compare this i map to the invariant map, we will need to make explicit the isomorphism between $\mathrm{Br}(L/K)$ and $H^2(L/K)$. This map is detailed below:

Definition 4.13. *Let L/K be a Galois extension such that $L \subset D$ is a maximal subfield for D/K a CDA. Then by Noether-Skolem we know that for all $\sigma \in \mathrm{Gal}(L/K)$ there is $e_\sigma \in D$ such that for all $x \in L$*

$$e_\sigma x e_\sigma^{-1} = \sigma(x).$$

Further, as in Remark 4.10, we have that e_σ is well defined up to multiplication by $L^\times$. In particular we have that for any pair $\sigma, \tau \in G$

$$e_\sigma e_\tau = \varphi(\sigma,\tau) e_{\sigma\tau}$$

for some $\varphi(\sigma,\tau) \in L^\times$. Then the natural isomorphism is

$$\begin{gathered}\mathrm{Br}(L/K) \to H^2(L/K)\\ D \mapsto \varphi.\end{gathered}$$

With this isomorphism we note that finite unramified extensions of local fields are not only Galois but cyclic, generated by the Frobenius element. Thus, if $\mathrm{Fr}_{L/K}$ is the Frobenius with order n then the 2-cocycle associated to a division algebra with this extension as a maximal subfield is given by

$$\varphi(\mathrm{Fr}_{L/K}^r, \mathrm{Fr}_{L/K}^s) = \gamma^{(r \mod n)}\gamma^{(s \mod n)}(\gamma^{r+s \mod n})^{-1}$$

where $\mathrm{Fr}_{L/K}(x) = \gamma x \gamma^{-1}$ via Noether-Skolem, and by $a \mod n$ we mean to take a minimal representative $0 \le a' < n$ of $a \mod n$.

Theorem 4.14. *The i map on central division algebras extends to the invariant map on the Brauer group. i.e.*

$$i(D) = \mathrm{inv}_K([D])$$

where $[D]$ is the Brauer class of the central division algebra D/K.

Proof. Let L/K be a maximal subfield, unramified over K, of a degree n central division algebra D . Let $\mathrm{Fr}_K \in \Gamma_K$ be the Frobenius element, where $L \subset K_{nr}$. Further let $\gamma \in D$ be such that $\mathrm{Fr}_K(x) = \gamma x \gamma^{-1}$ for all $x \in L$. Then we want to prove that, in the language of Definitions 4.11, 4.8 and 4.13,

$$v_D(\gamma) = \eta \circ \delta^{-1} \circ v(\varphi).$$

To do so is equivalent (since the composite maps on the right hand side are all isomorphisms) to proving

$$\delta \circ \eta^{-1}(v_D(\gamma)) = v(\varphi) \in H^2(\Gamma_K, \mathbb{Z}).$$

Firstly we compute the natural 2-cocycle on $\mathrm{Gal}(L/K)$ with values in $L^\times$ given by $v(\varphi)$, let $0 \leq r, s < n$

$$\begin{aligned} v(\varphi)(\mathrm{Fr}^r_{L/K}, \mathrm{Fr}^s_{L/K}) &= v(\gamma^r \gamma^s (\gamma^{r+s \mod n})^{-1}) \\ &= \begin{cases} v(\gamma^n) & r+s \geq n \\ v(1) & r+s < n \end{cases} \\ &= \begin{cases} n v_D(\gamma) & r+s \geq n \\ 0 & r+s < n \end{cases}, \end{aligned}$$

Secondly, let $\varepsilon : \Gamma_K \to \mathbb{Q}/\mathbb{Z}$ be the cocycle in $Z^1(G, \mathbb{Q}/\mathbb{Z})$ such that

$$\varepsilon(\mathrm{Fr}_K) = v_D(\gamma)$$

and so $\eta(\varepsilon) = v_D(\gamma)$. Then lifting ε to $\varepsilon' : G \to \mathbb{Q}$ we fix $\varepsilon'(\mathrm{Fr}_K) = v_D(\gamma)$, and extend $\mathbb{Z}$-linearly and continuously, so for all $g_1, g_2 \in G$ we have

$$\begin{aligned} \delta(\varepsilon)(g_1, g_2) &= g_1 \varepsilon'(g_2) - \varepsilon'(g_1 g_2) + \varepsilon'(g_1) \\ &= \varepsilon'(g_2) - \varepsilon'(g_1 g_2) + \varepsilon'(g_1) \end{aligned}$$

Now, since $H^2(G, \mathbb{Z}) = \varinjlim_{\substack{M/K finite \\ unramified}} H^2(\mathrm{Gal}(M/K), \mathbb{Z})$, we need only check that $v(\varphi) = \delta(\varepsilon)$ on $\mathbb{Z}/n\mathbb{Z}$ (equivalently on $\mathrm{Gal}(L/K)$), and this is immediate since for $0 \leq r, s < n$

$$\begin{aligned} \delta(\varepsilon)(\mathrm{Fr}^r_{L/K}, \mathrm{Fr}^s_{L/K}) &= r v_D(\gamma) - (r+s \mod n) v_D(\gamma) + s v_D(\gamma) \\ &= \begin{cases} n v_D(\gamma) & r+s \geq n \\ 0 & r+s < n \end{cases} \\ &= v(\varphi)(\mathrm{Fr}^r_{L/K}, \mathrm{Fr}^s_{L/K}) \end{aligned}$$

and we have the required result. □

4.4 Change of Fields

Now that we have the isomorphism between Brauer groups of local fields and $\mathbb{Q}/\mathbb{Z}$ defined in terms of our two viewpoints, it remains to see how the isomorphism relates to our natural maps. If L/K is a finite extension of local fields then $L_{nr} \supseteq K_{nr}$, so if we view $\Gamma_L = \mathrm{Gal}(L_{nr}/L)$ as the absolute Galois group of the residue field then there is a natural inclusion

$$\Gamma_L \to \Gamma_K,$$

thus we have a natural restriction map

$$\mathrm{res} : H^2(\Gamma_K, A) \to H^2(\Gamma_L, A),$$

for any continuous Γ_K-module A. We would like to see how this map is acting on $\mathbb{Q}/\mathbb{Z}$ via the invariant maps.

Theorem 4.15. *Let L/K be a finite extension of local fields of degree n, then the diagram below commutes*

$$\begin{tikzcd} \mathrm{Br}(K) \arrow[r, "\mathrm{inv}_K"] \arrow[d, "\mathrm{res}"] & \mathbb{Q}/\mathbb{Z} \arrow[d, "n"] \\ \mathrm{Br}(L) \arrow[r, "\mathrm{inv}_L"] & \mathbb{Q}/\mathbb{Z} \end{tikzcd}$$

Proof. Let $\mathrm{Fr}_K, \mathrm{Fr}_L$ be the frobenii of Γ_K and Γ_L and note that since

$$\Gamma_K/\Gamma_L = \mathrm{Gal}(k_L/k_K) \cong \mathbb{Z}/f_{L/K}\mathbb{Z}$$

we have $\mathrm{Fr}_L = (\mathrm{Fr}_K)^{f_{L/K}}$. Recall from Theorem 3.9 that $\mathrm{Br}(K) = H^2(K_{nr}/K)$, and so if we extend the diagram above in the language of Definition 4.8, we need to prove that the squares of the diagram below commute

$$\begin{tikzcd} H^2(K_{nr}/K) \arrow[r, "v_K"] \arrow[d, "\mathrm{res}"] & H^2(\Gamma_K, \mathbb{Z}) \arrow[r, "\delta^{-1}"] \arrow[d, "e_{L/K}\,\mathrm{res}"] & H^1(\Gamma_K, \mathbb{Q}/\mathbb{Z}) \arrow[r, "\eta"] \arrow[d, "e_{L/K}\,\mathrm{res}"] & \mathbb{Q}/\mathbb{Z} \arrow[d, "n"] \\ H^2(L_{nr}/L) \arrow[r, "v_L"] & H^2(\Gamma_L, \mathbb{Z}) \arrow[r, "\delta^{-1}"] & H^1(\Gamma_L, \mathbb{Q}/\mathbb{Z}) \arrow[r, "\eta"] & \mathbb{Q}/\mathbb{Z}. \end{tikzcd}$$

For the first square, note that $v_L = ev_K$ on K_{nr}, and since res corresponds directly to restriction of the cocycles viewed as continuous maps then this square clearly commutes. The second square commutes obviously since res commutes with δ, either by definition or by the explicit construction of δ depending on the way in which one defines δ. For the third and final square, we simply note that if $\phi : \Gamma_K \to \mathbb{Q}/\mathbb{Z}$ is a 1-cocycle then it is just a continuous homomorphism and

$$n\eta(f) = nf(\mathrm{Fr}_K) = e_{L/K} f_{L/K} f(\mathrm{Fr}_K) = e_{L/K} f(\mathrm{Fr}_L) = \eta(e_{L/K}\,\mathrm{res}(f)).$$

Thus the diagram commutes. ☐

This result allows us to state some surprising corollaries.

Corollary 4.16. *For L/K an extension of degree n of local fields,*

$$\mathrm{Br}(L/K) \cong \frac{\mathbb{Z}}{n\mathbb{Z}}$$

and is generated by $u_{L/K}$, the unique element of $\mathrm{Br}(L/K)$ *such that* $\mathrm{inv}_K(u_{L/K}) = \frac{1}{n}$.

Proof. Note that $\mathrm{Br}(L/K)$ is precisely the kernel of $\mathrm{res} : H^2(K_{nr}/K) \to H^2(L_{nr}/L)$ and the result is immediate from Theorem 4.15. ☐

Corollary 4.17. *For any central division algebra D/K over a local field, and $n \in \mathbb{Z}$, the following are equivalent:*

1. *$n[D] = 0$ in* $\mathrm{Br}(K)$,
2. *$D \in \mathrm{Br}(L/K)$ for L/K of degree n.*

Proof. This is immediate from Theorem 4.15. ☐

Corollary 4.18. *For all $n \in \mathbb{N}$, there exists a central division algebra D/K such that every extension L/K of degree n has a K-embedding into D. In particular, all degree n irreducible polynomials over K have a solution in such D.*

Proof. Let $\mathrm{inv}_K(D) = \frac{1}{n}$. Then note that $n[D] = 0$ and for $0 < r < n$, $r[D] \neq 0$. In particular by Corollary 4.17 we see that D is not split by any extensions of degree $r < n$ and is split by one of degree n. But then $\deg(D)$ must thus divide n and be at least n and so $\deg(D) = n$. In particular, since again by Corollary 4.17, $[D] \in \mathrm{Br}(L/K)$ for any L/K of degree n, we must have that L has a K embedding into D. ☐

5 Final Comments

The computation of the Brauer groups shown in this essay is, in a sense, a specialisation of a result of Witt which says the following

Theorem 5.1 (Witt [5], translated in [6])**.** *Let K be a field, complete with respect to a discrete valuation, with residue field k_K. If k_K is perfect then there is a short exact sequence*

$$0 \longrightarrow \mathrm{Br}(k_K) \longrightarrow \mathrm{Br}(K) \longrightarrow \chi(\Gamma_K) \longrightarrow 0,$$

where Γ_K in this setting is the absolute Galois group of k_K, and $\chi(\Gamma_L)$ is the character group of Γ_K. Moreover this splits, decomposing $\mathrm{Br}(K)$ into the two direct summands.

Note that in this theorem K is *almost* a local field, and the thing it is missing is finiteness of k_K. In fact, k_K being finite just tells us that $\mathrm{Br}(k_K) = 0$ and so for local fields this is an isomorphism

$$\mathrm{Br}(K) \cong \chi(\Gamma_K) = H^1(\Gamma_K, \mathbb{Q}/\mathbb{Z}).$$

We established such an isomorphism in defining the invariant map in Definition 4.8. This is a more general statement however, as we can consider an example.

Example 1. *Let $K = \mathbb{R}((t))$ be the field of Laurent series over $\mathbb{R}$. This is certainly complete with respect to the obvious valuation (uniformiser t) and the residue field is $k_K = \mathbb{R}$ which is perfect, but not finite. This tells us that*

$$\mathrm{Br}(\mathbb{R}((t))) \cong \mathrm{Br}(\mathbb{R}) \oplus \chi(\mathrm{Gal}(\mathbb{C}/\mathbb{R})) \cong \frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}}.$$

Whereas if we had taken $K = \mathbb{C}((t))$ we would have reached the statement that $\mathrm{Br}(K) = 0$ purely on the basis that $\mathbb{C}$ is algebraically closed (and perfect). Note we can use these statements inductively to obtain

$$\mathrm{Br}(\mathbb{R}((t_1, \dots, t_n))) = (\mathbb{Z}/2\mathbb{Z})^{n+1} \qquad\qquad \mathrm{Br}(\mathbb{C}((t_1, \dots, t_n))) = 0$$

Further, some work of Yuan involving the generalisation of central simple algebras over commutative rings (and not just fields) (see [1]) admits the following interesting Corollary in this vein. [6, Cor 5.3]

Theorem 5.2 ([6], Cor 5.3)**.** *Let K be a field. Then K is a perfect field if and only if every element of $\mathrm{Br}(K((t)))$ has an unramified splitting field.*

Explicitly, for $K = \mathbb{F}_q$ a finite field then this is really just repeating the statement of Theorem 3.9. However, this result generalises it to the characteristic 0 situation and tells us that $\mathrm{Br}(K((t)))$ is always split by unramified extensions for $\mathrm{char}(K) = 0$.

A Notation

Below is a table of notation that persists in this essay, but may not be included in any definition.

$H^q(G, A)$	Tate cohomology of the G-module A.
N_G	$\sum_{\sigma \in G} \sigma \in \mathbb{Z}[G]$
$\mathrm{Hom}_G(X, Y)$	G-module homomorphisms $X \to Y$
I_G	$\langle \sigma - 1 \rangle_{\sigma \in G} \lhd \mathbb{Z}[G]$
J_G	$\mathbb{Z}[G]/\mathbb{Z}N_G$
$h_{2/1}(G, A)$	Herbrand quotient of the G-module A for G cyclic
$K, \mathcal{O}_K, v, \pi, \mathfrak{p}, k_K$	Local field, valuation ring, valuation, uniformiser, maximal ideal, residue field
$e_{L/K}, f_{L/K}$	Ramification degree, inertia degree of extension L/K of local fields
K_{nr}	Maximal unramified extension of K
Γ_K	$\mathrm{Gal}(K_{nr}/K)$
Fr_K	Frobenius of Γ_K
$\mathrm{Fr}_{L/K}$	Frobenius of $\mathrm{Gal}(L/K)$ for L/K unramified

References

[1] M. Auslander and O. Goldman. The brauer group of a commutative ring. *Transactions of the American Mathematical Society*, 97(3):367–409, 1960.

[2] J. Cassels and A. Fröhlich. *Algebraic Number Theory: Proceedings of an Instructional Conference Organized by the London Mathematical Society (a NATO Advanced Study Institute) with the Support of the International Mathematical Union*. London Mathematical Society, 2010.

[3] M. Lipnowski. Central simple algebras. virtualmath1.stanford.edu/ conrad/modseminar/pdf/L13.pdf.

[4] J. Neukirch and A. Schmidt. *Class Field Theory: -The Bonn Lectures- Edited by Alexander Schmidt*. Class Field Theory: The Bonn Lectures. Springer Berlin Heidelberg, 2013.

[5] E. Witt. Schiefkörper über diskret bewerteten körpern. *Journal für die reine und angewandte Mathematik*, 176:153–156, 1937.

[6] S. Yuan. On the brauer groups of local fields. *Annals of Mathematics*, 82(3):434–444, 1965.