Workshop on Arithmetic and Algebra of Rational Points
Project Proposals

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Projects

Galois module structure of Néron-Severi groups

Alex Bartel

Description: This is a project that can take many different directions. For concreteness, here is a special case that, I believe, is already not well understood: take a curve over a number field (you know what, let's say over $\mathbb{Q}$) that you care about. Its Jacobian is an abelian variety whose arithmetic is important to understand if you truly care about your curve. One of the invariants attached to this Jacobian is its Néron-Severi group. This is a free abelian group of finite rank that comes with a natural action of the absolute Galois group of $\mathbb{Q}$. What can we say about the Galois module structure? This is a vague question that can be made more precise in many different ways: can we compute the Galois module structure reasonably efficiently? Are there some restrictions on which lattices with Galois action can occur as the Néron-Severi groups of Jacobians of curves over $\mathbb{Q}$, say? Is this Galois module structure linked to some other interesting invariants of the curve?

I know very little about this problem. All I have done is compute the Galois module structure of the Néron-Severi group of the modular curve $X_1(13)$, and this was enough to convince me that there is some fun to be had here.

Variants: Some variants that could be explored: instead of Jacobians of curves, one could look at other interesting classes of varieties, e.g. other abelian varieties, or K3 surfaces. The Néron-Severi group of any non-singular projective variety is finitely generated, but in general it need not be free. In those cases, the question of Galois module structure is likely even more subtle.

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Brauer-Manin obstructions on log K3 surfaces

Martin Bright

Summary: Log K3 surfaces include affine cubic surfaces and more generally affine del Pezzo surfaces and are a natural setting for studying local-global principles for integral points. An important tool in this study is the Brauer-Manin obstruction.

Goal: The broad aim of this project is to use geometric methods to study the Brauer-Manin obstruction on log K3 surfaces. For example, which primes can be involved in the obstruction? One interesting starting point would be to look at the example of an affine degree 5 del Pezzo surface given in Julian Lyczak's thesis, from the point of view of recent work of Bright and Newton.

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Symmetric Squares of Quadric Hypersurfaces

Tim Browning

Goal: Prove a lower bound for the number of rational points of bounded height on the symmetric square of X, when X is a quadric hypersurface of dimension at least two.

Context: Manin's conjecture is concerned with counting rational points of bounded height on appropriate Fano varieties defined over the rational numbers. In 2014, le Rudulier studied the conjecture for $\mathrm{Sym}^2\mathbb{P}^2$, finding that the asymptotic formula only conforms to the Manin conjecture once one is allowed to remove thin sets from the variety. The resolution of Manin's conjecture symmetric powers of varieties essentially boils down to counting rational points of bounded height and bounded degree on the variety. It remains a challenging and significant problem to find new cases.

Approach: Let X be a smooth quadric hypersurface defined over the rational numbers. The idea will be to use a number field version of the circle method to count K-points of bounded height on X, for a fixed quadratic number field K, taking care to make sure all estimates are uniform in K. We can then get try to get a lower bound for the counting function associated to $\mathrm{Sym}^2 X$ by summing this over many number fields of relatively small discriminant (compared to the height of the points).

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Reduction types of curves of genus 2

Vladimir Dokchitser

Description: Silverman's "Advanced topics..." contains an extremely useful table of possible reduction types of elliptic curves, which, in particular, lists their special fibres, Neron component groups and minimal discriminants. The aim of the project is to produce a similarly handy reference table for curves of genus 2. The precise list of invariants we try to tabulate will depend on what the project participants are interested in! Like in the elliptic curve table, we will concentrate on curves with tame reduction.

References: "A user's guide to the arithmetic of hyperelliptic curves", by authors too numerous to list, is probably a good place to start, although its main focus is curves with semistable reduction. The old classification of Namikawa and Ueno, which lists the possible special fibres of genus 2 curves but doesn't explain how to obtain them, is also worth a serious look, although we should really aim to come up with something that's more intuitive and less cumbersome to use. A somewhat more practical classification for curves of genus 2 with tame reduction can be found at the end of Nowell's PhD thesis.

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Normic equations over function fields and Laurent series fields

Diego Izquierdo

Let K be a field. A normic equation over K is an equation of the form:

$$N_{L_1/K}(\mathbf{x}_1)N_{L_2/K}(\mathbf{x}_2)...N_{L_r/K}(\mathbf{x}_r) = a$$

for some finite extensions $L_1, ..., L_r$ of K and some $a \in K^\times$. It defines a principal homogeneous space under the normic torus given by the same equation with $a = 1$. Normic equations have been vastly studied over number fields. They do satisfy the local-global principle when $r = 1$ and the extension L_1/K is cyclic but not in general.

In this project, we aim at studying normic equations over various fields other than number fields. We are more precisely interested in function fields of varieties or Laurent series fields in several variables defined over $\mathbb{C}$, $\mathbb{C}((t))$, finite fields, p-adic fields or number fields.

There are usually several ways to make sense of the local-global principle over such fields, depending on whether one takes into account all of their places or only those that have a natural geometric interpretation. For instance, if K is the function field of a smooth projective curve over some base field k, one may take into account all places of K or only those that are induced by a closed point of the curve C. Similarly, if K is the Laurent series field in 2 variables $k((x, y))$ over some base field k, one can take into account all places of K or only those that are induced by a codimension one point of $\mathrm{Spec}(k[[x, y]])$.

Studying normic equations in general is hard. Here are some questions that we could investigate:

1) For which function fields and Laurent series fields can one ensure that the local-global principle holds for normic equations with respect to a single cyclic extension (ie a normic equation with $r = 1$ and L_1/K cyclic)? When the answer is negative, can one explicitly compute the corresponding obstructions to the local-global principle (either theoretically or computationally)?

2) Same as question 1) for normic equations with respect to a single biquadratic extension or more generally to a single bicyclic extension.

3) Same as question 1) for normic equations with respect to several cyclic extensions (ie a normic equation with $r > 1$ and L_1/K, ..., L_r/K all cyclic)?

4) Depending on the kind on field on which one works, under which assumptions on a finite group G can one ensure that the local-global principle holds for normic equations with respect to a single Galois extension with group G?

5) Given a normic torus, are there finitely many principal homogeneous spaces under that torus that contradict the local-global principle?

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Character sum methods in arithmetic statistics

Peter Koymans

Summary: In a seminal paper, Heath-Brown [8] obtained the distribution of 2-Selmer groups of the congruent number curve. The proof can roughly be divided in three steps

- write down the algebraic problem in terms of a character sum. A typical example of a character sum is

$$\sum_{d_0d_1d_2d_3=n} \mu^2(n) \left(\frac{d_2d_3}{d_0}\right) \left(\frac{d_0d_1}{d_3}\right).$$

- show oscillation of the character sum once there is at least one non-principal character. The principal tools in this step are the large sieve and the Siegel–Walfisz theorem.
- find the main term of the character sum. This typically involves some combinatorial problem.

These techniques are applicable to a wide number of arithmetic statistics problems, and this has been done successfully by a number of authors, most notably by Fouvry–Klüners [4, 5]. These problems range from number field counting, class group statistics and rational points. The main prerequisite for the method is the first step. There are many interesting open problems that can be expressed in terms of character sums, and the goal of this project would be to investigate some of these problems.

Depending on the interest of the participants, there are various projects that could be pursued with these methods.

Number field counting: For a fixed finite group G, Malle [13] has predicted an asymptotic for the number of G-extensions with discriminant bounded by X. Malle's conjecture is considered to be of exceptional difficulty as it implies the inverse Galois problem. Even if one restricts to nilpotent groups, which is a relatively well-understood class of groups, Malle's conjecture is wide open with the only known cases being abelian groups by Wright [15], nonic Heisenberg groups by Fouvry–Koymans [6], quartic D_4-extensions by Cohen–Diaz y Diaz–Olivier [3] (see also [1] for the related problem of ordering by Artin conductor) and nilpotent groups where all elements of minimal order are central by Koymans–Pagano [10]. If G is a nilpotent group of nilpotency class 2, then one is able to express this count in terms of a character sum.

Subgoal: prove Malle's conjecture for Galois D_4-extensions when ordered by product of ramified primes over any number field with an explicit leading constant.
Main goal: prove Malle's conjecture for nilpotent groups of nilpotency class 2.

Rational points: Local solubility of families of varieties has been an important topic in rational points initiated by Loughran–Smeets [12]. In favorable situations this can also be detected with character sums.

Subgoal: obtain the count of integers $a, b, c \in$ with $\max(|a|, |b|, |c|) \leq B$ for which the equation

$$aX^3 + bY^3 = cZ^3$$

is everywhere locally soluble. This would prove a new instance of Loughran–Smeets [12], would improve previous work of Browning–Dietmann [2] and would be a natural followup to recent work of Loughran–Rome–Sofos [11], who considered $aX^2 + bY^2 = cZ^2$ with $\gcd(a, b, c) = 1$.
Main goal: obtain the count of integers $a, b, c \in \mathbb{Z}$ with $\max(|a|, |b|, |c|) \leq B$ for which the equation

$$aX^d + bY^d = cZ^d$$

is everywhere locally soluble.

Class groups: Fixing a number field K, we consider the 4-rank of the field $K(\sqrt{n})$ as n varies over *rational integers*. This problem has been considered in [7], [9] for quadratic number fields K and an elliptic curve analogue was considered in [14]. Both these problems reduce to character sum questions.

Subgoal: find the distribution of $\mathbb{Q}(\zeta_{12}, \sqrt{n})$.
Main goal: find the distribution of $K(\sqrt{n})$ for any Galois extension K of $\mathbb{Q}$.

Another topic of interest is the class group structure of abelian extensions, in particular that of $\mathbb{Z}/p^n\mathbb{Z}$-extensions ordered by product of ramifiied primes.

Main goal: find the distribution of the first few layers of $\mathrm{Cl}(K)[p^\infty]$ as K varies over all $\mathbb{Z}/p^n\mathbb{Z}$-extensions ordered by product of ramified primes.

Recommended reading: Some familiarity with [4] or [8] is highly recommended. A good background in algebraic number theory will also be valuable.

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Thin sets of rational points on weighted projective stacks

Dan Loughran

Description: A result of Serre and Cohen [3, §13] states that any thin subset of $\mathbb{P}^n(\mathbb{Q})$ contains $O(B^{n-1/2}(\log B))$ rational points of height at most B. This saves a factor of $B^{1/2}$ over the total number of points in $\mathbb{P}^n(\mathbb{Q})$.

The aim of the project is to prove a version of this for weighted projective stacks and closely related stacks (e.g. M-curve structures on $\mathbb{P}^!$).

The word "stack" may seem scary, but actually in this case everything can be made very explicit and one is just counting lattice points with a slightly different height function. This viewpoint is expounded in the papers [1, 2]. The method will be based on the same as Serre using the large sieve [3, §12, §13].

Subgoals: The simplest non-trivial case is the weighted project stack $\mathbb{P}(1,2)$. This is a copy of $\mathbb{P}^1$ with one stacky point of order 2 at infinity. So we'll study this first in detail and make sure we understand it. Once this case is done we'll look at more general cases and see how far the method can be pushed.

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Rational points on quartic del Pezzo surfaces through 2 descent on genus 2 curves

Julian Lyczak

There is connection between between del Pezzo surface of degree 4 and curves of genus 2, described in [4]. In [2] arithmetic properties of quartic del Pezzo surfaces were used to produce families of genus 2 curves with explicit non-trivial classes in the Tate–Shafarevich group. In this project we will go in the opposite direction: using the understanding of the Selmer and Tate-Shafarevich group of Jacobians of curves, we will derive arithmetic properties of families of del Pezzo surfaces.

From a curve to a surface:

Let C be the genus 2 given by

$$y^2 = f(x)$$

for a separable quintic polynomial $f \in \mathbb{Q}[x]$. Let us consider the quintic $\mathbb{Q}$-algebra $\mathcal{A} = \mathbb{Q}[x]/(f)$. We consider the Jacobian $J = \mathrm{Jac}(C)$ and its 2-Selmer group

$$S^{(2)}(J) := \mathrm{Ker}\left(\mathrm{H}^1(\mathbb{Q}, J[2]) \to \prod_p \mathrm{H}^1(\mathbb{Q}_p, J[2])\right)$$

which is effectively computable as subset $\mathrm{H}^1(\mathbb{Q}, J[2]) \subseteq \mathcal{A}^*/(\mathcal{A}^*)^2$. Here any $\delta \in \mathrm{H}^1(\mathbb{Q}, J[2])$ corresponds to a homogeneous space $H_\delta \to J$, and those in the 2-Selmer group are the everywhere locally soluble homogeneous spaces under $J[2]$. For each δ one can construct a del Pezzo surface $X_\delta \subseteq \mathbb{P}^4$, explicitly given by two quadratic equations in five variables, such that $X_\delta(K) \neq \emptyset \Rightarrow H_\delta(K) \neq \emptyset$ for any $\mathbb{Q}$-algebra K.

From a surface to a curve:

Now suppose we have a quartic del Pezzo surface given by two quadratic equations Q_1 and Q_2 in two variables. We can represent those two quadratic equations by symmetric matrices $M_1, M_2 \in M_2(\mathbb{Q})$. It is known that the quintic polynomial $f(x) = \det(M_1 + xM_2)$ is separable. Flynn [4] proved that for the curve C given by $y^2 = f(x)$ there is always a $\delta \in \mathrm{H}^1(\mathbb{Q}, J[2])$ for which X_δ is isomorphic to X, and such pair (C, δ) is essentially unique.

Main idea of the project:

Using the exact sequence

$$0 \to J(\mathbb{Q})/2J(\mathbb{Q}) \to S^{(2)}(J) \to Ш(J)[2] \to 0$$

and conjectures on either the parity of rk $J(\mathbb{Q})$ or rk $Ш(J)[2]$ (for example the Tate–Shafarevich conjecture) will allow us to conclude that in specific situations there is a $\delta \in S^{(2)}(J)$ which comes from $J(\mathbb{Q})/2J(\mathbb{Q})$ of which we didn't know this explicitely. Hence δ lies in the kernel of $S^{(2)}(J) \to Ш(J)[2]$. Hence $H_\delta(\mathbb{Q}) \neq \emptyset$ and we conclude $X_\delta(\mathbb{Q}) \neq \emptyset$.

This would conditionally prove that some quartic del Pezzo surface has a rational point. Note that existing results on the existence of rational points on quartic del Pezzo surfaces are also conditional on Shinzel's hypothesis and the Tate–Shafarevich conjecture on elliptic curves, for example [3] and [6].

Main goal of the project:

- Produce a family of quartic del Pezzo surfaces for which the above method established the existence of a rational point.
- Determine how this construction relates to other invariants of quartic del Pezzo surfaces.

A possible subgoal:

- An interesting invariant of a quartic del Pezzo surface is the quintic polynomial f. It also comes with $\epsilon_i \in \mathbb{Q}[x]/(f_i)$ for any factor f_i of f. In [5] there is an algorithm to compute BrX as an abstract group from these invariants. How are these invariants determined by $\delta \in \mathrm{H}^1(\mathbb{Q}, J[2])$?

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A step toward the modular method for hyperelliptic curves: conductor computations in families.

Céline Maistret

Diophantine equations are a major topic in modern number theory and the most known example of these is Fermat's last theorem. It states that for any $n \geq 3$ the equation

$$x^n + y^n = z^n$$

has no integer solutions. It is known that it suffices to prove the statement for $n = p$ a prime number. Many mathematicians like Frey, Ribet, Serre and Wiles worked on proving it and their approach is nowadays known as the modular method. The proof goes by contradiction: assume one has a solution to the equation, the first step is to associate to this putative solution an elliptic curve defined over $\mathbb{Q}$, often known as a Frey curve. By the modularity theorem, the Frey curve is modular and by Ribet's level lowering theorem, the residual galois representation arises from a newform of weight 2 and level 2. But the space of such newforms has dimension 0, thus the contradiction!

In 2000, Darmon proposed a program generalising the modular method to other diophantine equations. In this generalisation, one considers hyperelliptic curves with their associated Hilbert modular forms. Ribet's level lowering theorem remains a key step in the method and one needs to compute the conductor of the Jacobian of the hyperelliptic curve to achieve it. With this project, we propose to do so for the Fermat-type equation:

$$x^r + y^r = Cz^p$$

where $r > p$ are prime numbers and $C \in \mathbb{Z}$. Following Kraus' ideas, we can construct a Frey curve defined over a totally real field, associated to a solution of this equation as follows.

Consider ζ_r a primitive r-th root of unity and define $\omega_j := \zeta_r^j + \zeta_r^{-j}$ for $1 \leq j \leq \frac{r-1}{2}$, so that $K := \mathbb{Q}(\zeta_r)^+ = \mathbb{Q}(\omega_1)$ is the maximal totally real subfield of $\mathbb{Q}(\zeta_r)$. Define

$$h(x) := \prod_{j=1}^{\frac{r-1}{2}} (x - \omega_j).$$

For a putative solution (a, b, c) of the equation we consider the hyperelliptic curve

$$C_r(a,b) : y^2 = (ab)^{\frac{r-1}{2}} xh\left(\frac{x^2}{ab} + 2\right) + b^r - a^r.$$

This curve is actually defined over $\mathbb{Q}$ but one can also see it as defined over K through base change (which has advantages when proving modularity). Its discriminant is given by

$$\Delta(C_r(a,b)) = (-1)^{\frac{r-1}{2}} 2^{2(r-1)} r^r (a^r + b^r)^{r-1}$$

The goal of this project is to compute the conductor of the jacobian of $C_r(a, b)$ over K, i.e. the level of the Hilbert newforms before level lowering.

For $r = 5$ we computed explicitly the roots of the RHS polynomial and extracted the conductor exponents from the cluster pictures at every place of odd residual characteristic. We propose to apply a similar method, or any other that could be more fruitful, to compute the conductor for a wider range of values of the parameters.

References: I'd recommend reading the users guide to hyperelliptic curves, especially sections $1, 2, 3, 12$ to prepare.

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4-Echardt Points on Low Degree del Pezzo Surfaces

Rosa Winter & Sam Streeter

We will study the occurrence of 4 lines intersecting in one point on del Pezzo surfaces of degrees 1 and 2 with conic bundles. By work of Demeio and Streeter, this has implications for such surfaces to satisfy weak weak approximation. Depending on where participant's interests and expertise lie, this project could involve arithmetic statistics through counting how often this occurs in a suitable moduli space, or constructing explicit (families of) examples of such surfaces that don't contain any points in 4 lines, and therefore satisfy weak weak approximation, or both.

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