COMPLEX MULTIPLICATION

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Disclaimer. These notes were taken live during lectures. Any errors are the fault of the transcriber and not of the lecturer.

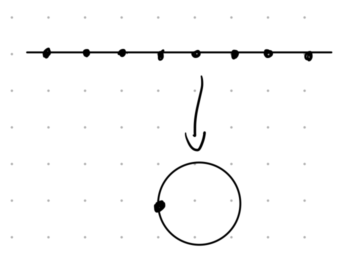
Notation 1.
- $\mathbb{Z}' = \mathbb{Z}/\setminus\{0\}$

Lecture 1 (Jan Vonk)

We begin, very classically, with a viewpoint due to Eisenstein. Forget everything you know about trigonometric functions!

1. Cyclotomy

Consider $\mathbb{Z} \subseteq \mathbb{R}$, and think about the quotient $\mathbb{R}/\mathbb{Z}$ which we usually think of as the circle group. We'd like to think of this quotient algebraically.



To do this we shall look at the invariant functions for $k \geq 2$

$$\alpha_k(z) = \sum_{\lambda \in \mathbb{Z}} \frac{1}{(z-\lambda)^k}.$$

Many polynomial relations exist between these (for example $\alpha_2^2 = \alpha_4 + \Omega_2\alpha_2$) with coeficients equal to combinations of

$$\Omega_k := \sum_{\lambda \in \mathbb{Z}'} \frac{1}{\lambda^k}.$$

There are extra terms to add:

- Consider the case $k = 1$, and define in pretty much the same way

$$\alpha_1(z) := \frac{1}{z} + \sum_{\lambda \in \mathbb{Z}'} \frac{1}{z-\lambda} + \frac{1}{\lambda}.$$

This is absolutely convergent (unlike what we would have had if we hadn't modified for $k = 1$) and is translation invariant. It satisfies the relation

$$\alpha_1^2 = \alpha_2 - 3\Omega_2. \tag{1}$$

- We want a multiplicative lift for
$$\mathrm{d}\log/\mathrm{d}z : f \mapsto f'/f$$
for our function α_1. We take
$$\sigma(z) := \pi z \prod_{\lambda \in \mathbb{Z}'} \left(1 - \frac{z}{\lambda}\right) \exp\left(\frac{z}{\lambda}\right),$$
and note that we can prove formally the following two identities:
$$(\mathrm{d}\log/\mathrm{d}z)(\sigma) = \sigma'(z)/\sigma(z) = \alpha_1(z)$$
$$\sigma(z+1) = -\sigma(z)$$

1.1. **Periods.** Euler realised that

$$\sigma(z) = \sin(\pi z),$$

so that

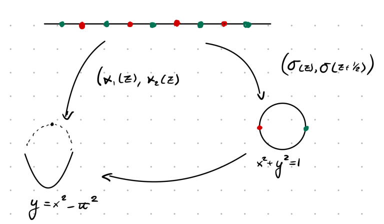
$$\begin{aligned}\alpha_1(z) &= \frac{1}{z} - \sum_{k \geq 2} \Omega_k z^{k-1} \\ &= \pi \cot(\pi z) \\ &= -\pi i (e^{2\pi i z} + 1)/(e^{2\pi i z} - 1).\end{aligned}$$

From this we deduce that for $k \geq 2$

$$\Omega_k = \frac{(2\pi)^k}{k!} |B_k|$$

where B_k are Bernoulli numbers. This leads us nicely on to special values.

1.2. **Special Values.** Consider the set of vaues at division points of $\mathbb{R}/\mathbb{Z}$, i.e. $z \in \mathbb{Q}/\mathbb{Z}$.

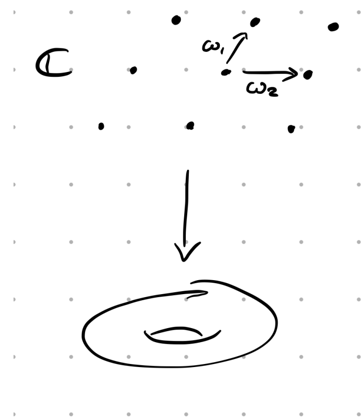


We have the Chebyshev polynomials

$$T_n(\cos(\theta)) = \cos(n\theta),$$

so find that the values of $\sigma(z)$ at division points are algebraic.

Example 2. *Consider $z = 2/17$, then we get $\frac{1}{2n}(\zeta_{17} - \zeta_{17}^{-1}) \in \mathbb{Q}(\zeta_{68}) =: K$. It is half of a 17-unit, i.e. it is half of an element in $\mathcal{O}_K[1/17]^\times$.*



2. Elliptic Functions

Consider a rank 2 lattice $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \subseteq \mathbb{C}$
Again, we want to find invariant functions. For $k \geq 3$ we define

$$\alpha_k(\Lambda, z) = \sum_{\lambda \in \Lambda} \frac{1}{(z-\lambda)^k}.$$

Outside the range of convergence we define as follows.

- for $k = 2$ we write
$$\alpha_2(\Lambda, z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda'} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right),$$
which is usually known as the Weierstrass $\mathfrak{p}$-function. This is an invariant function.
- For $k = 1$ we define
$$\alpha_2(\Lambda, z) = \frac{1}{z} + \sum_{\lambda \in \Lambda'} \left(\frac{1}{(z-\lambda)} + \frac{1}{\lambda} + \frac{z}{\lambda^2} \right).$$

This is often called the Weierstrass ζ-function, but it is **NOT** invariant!

We have a transformation law:

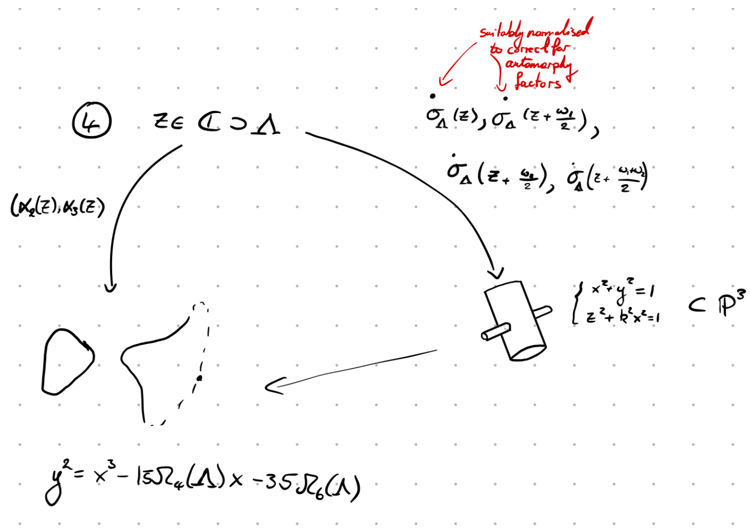
$$\alpha_1(\Lambda, z + \omega_i) = \alpha_1(\Lambda, z) + \eta_i.$$

We have multiplicative lifts given by

$$\sigma(\Lambda, z) := z \prod_{\lambda \in \Lambda'} \left(1 - \frac{z}{\lambda}\right) \exp\left(\frac{z}{\lambda} + \frac{z^2}{2\lambda^2}\right),$$

and it satisfies

$$(\mathrm{d}\log/\mathrm{d}z)(\sigma) = \sigma'(z)/\sigma(z) = \alpha_1(\Lambda, z)$$
$$\sigma(\Lambda, z + \omega_i) = -\exp\left(\eta_i\left(z + \frac{\omega_i}{2}\right)\right)\sigma(\Lambda, z)$$



2.1. **Special Values.** The Values at division points of $\mathbb{C}/\Lambda$

We will study values at division points when Λ has complex multiplication, i.e.

$$\{\alpha \in \mathbb{C} \ : \ \alpha\Lambda \subseteq \Lambda\} \supsetneq \mathbb{Z}.$$

We will look at:

(1) singular moduli, e.g. the j-invariant $j(\Lambda) = \frac{(60\Omega_4(\Lambda))^3}{(60\Omega_4(\Lambda))^3 - (140\Omega_6(\Lambda))62}$;

(2) elliptic units, i.e. quotients of σ-functions (Klein forms), for example

$$(\Delta|\gamma)/\Delta$$

for $\gamma \in M_2(\mathbb{Z})$ and Δ the usual Ramanujan modular form.

Some remarks on CM theory:

- Heegner (1952) used CM theory to construct integral points on modular curves $X_{\mathrm{ns}}(p)$, solving the class number 1 problem for imaginary quadratic fields.
- Coates–Wiles (1976) used elliptic units to prove the Birch–Swinnerton-Dyer conjecture in the analytic rank 0 case.
- Gross–Zagier (1985) determine factorisation of (differences of) singular moduli to obtain the Birch–Swinnerton-Dyer conjecture in the analytic rank 1 case.