LOCAL-GLOBAL PRINCIPLES PART II

COURSE: RACHEL NEWTON
NOTES: ROSS PATERSON

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Note: all rings and algebras are assumed to be associative and unital.

Lecture 1

1. The Brauer Groups of a Local Ring

Throughout this section, let R denote a commutative local ring, with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$.

Definition 1. An Azumaya algebra A over R is a finitely generated free R-algebra such that $A \otimes_R k$ is a central simple algebra (CSA) over k.

Definition 2. Two Azumaya algebras $A, B/R$ are equivalent if there exist $n, m \in \mathbb{Z}_{>0}$ such that

$$M_n(A) \cong M_m(B)$$

as R-algebras. Write $[A]$ for the equivalence class of A with respect to this relation.

Definition 3. The Brauer group $\mathrm{Br}(R)$ is the set of equivalence classes of Azumaya algebras over R with group operation induced by tensor product over R. The identity element is clearly $[R]$, and it is easy to show that the inverse of $[A]$ is, in general, $[A^{\mathrm{opp}}]$.

Theorem 4. *Let R be a regular integral domain. Then*

$$\begin{aligned} \mathrm{Br}(R) &\to \mathrm{Br}(\mathrm{Frac}(R)) \\ A &\mapsto A \otimes_R \mathrm{Frac}(R) \end{aligned}$$

is an injection.

Proof. Auslander & Goldman, Theorem 7.2. □

2. The Brauer Group of a Variety

Let X/K be a nice (i.e. smooth, quasi-projective, and irreducible) variety over a number field.

Definition 5. $\mathrm{Br}_{\mathrm{Az}}(X) := \bigcap_{P \in X(\overline{K})} \mathrm{Br}(\mathcal{O}_{X,P})$, where the intersection is taken in $\mathrm{Br}(k(X))$ via Theorem 4.

Theorem 6 (Grothendieck)**.** *In fact,*

$$\mathrm{Br}_{\mathrm{Az}}(X) = \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m) =: \mathrm{Br}(X),$$

and $\mathrm{Br}(X)$ *is a torsion abelian group.*

Remark 7. We are using the niceness assumptions here: in general there is a difference between the Brauer group (in terms of Azumaya algebras) and the cohomological Brauer group.

Definition 8. For $A \in \mathrm{Br}(\mathcal{O}_{X,P})$, write $\mathcal{A}(P)$ for the image of A under the map

$$\begin{aligned} \mathrm{Br}(\mathcal{O}_{X,P}) &\to \mathrm{Br}(k(P)) \\ A &\mapsto A \otimes_{\mathcal{O}_{X,P}} k(P) \end{aligned}$$

where $k(P)$ is the residue field at P: $\mathcal{O}_{X,P}/\mathfrak{m}_{X,P}$.

Thus for all $P \in X(\overline{K})$ we have evaluation maps

$$\mathrm{Br}(X) \to \mathrm{Br}(k(P))$$

coinciding with the maps

$$\mathrm{Br}(X) = \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m) \to H^2(k(P), \mathbb{G}_m) = \mathrm{Br}(k(P)).$$

Notation 9. For a number field K, we write

$$\mathbb{A}_K = \prod_{v \in \Omega_K}{}' K_v := \left\{ (x_v)_v \in \prod_{v \in \Omega_K} K_v \ : \ x_v \in \mathcal{O}_v \text{ for all but finitely many } v \right\}.$$

Remark 10. If X is projective, then $X(\mathbb{A}_K) = \prod_{v \in \Omega_K} X(K_v)$

Theorem 11 (Manin)**.** *There is a pairing (called the Brauer-Manin pairing)*

$$\begin{aligned} \langle , \rangle_{\mathrm{BM}} : X(\mathbb{A}_K) \times \mathrm{Br}(X) &\to \mathbb{Q}/\mathbb{Z} \\ \langle (P_v)_v, \mathcal{A} \rangle_{\mathrm{BM}} &= \sum_{v \in \Omega_K} \mathrm{inv}_v(\mathcal{A}(P_v)), \end{aligned}$$

such that $\overline{X(K)} \subseteq X(\mathbb{A}_K)^{\mathrm{Br}} := \{(P_v)_v \in X(\mathbb{A}_K) \ : \ \forall A \in \mathrm{Br}(X), \langle (P_v)_v, A \rangle_{\mathrm{BM}} = 0\}$.

Remark 12. Here $\overline{X(K)}$ means the closure in the adelic points, and the inclusion follows from Albert–Brauer–Hasse–Noether together with the continuity of evaluation maps.

Definition 13. If $X(\mathbb{A}_K) \neq \emptyset$, but $X(\mathbb{A}_K)^{\mathrm{Br}} = \emptyset$ then immediately from Theorem 11 we see $X(K) = \emptyset$, and in this case we say that there is a Brauer–Manin obstruction to the Hasse principle.

Definition 14. Weak approximation holds for X if $\overline{X(K)} = X(\mathbb{A}_K)$. Note that if $X(\mathbb{A}_K)^{\mathrm{Br}} \neq X(\mathbb{A}_K)$ then by Theorem 11 weak approximation cannot hold, and we say there is a Brauer–Manin obstruction to weak approximation.

Definition 15. The constant elements of the Brauer group are

$$\mathrm{Br}_0(X) := \mathrm{im}(\mathrm{Br}(K) \to \mathrm{Br}(X)).$$

The algebraic elements of the Brauer group are

$$\mathrm{Br}_1(X) := \ker(\mathrm{Br}(X) \to \mathrm{Br}(\overline{X})),$$

where $\overline{X} := X \times_K \overline{K}$ for $\overline{K}$ a seperable closure. One can show (indeed, it is in the exercises!) that

$$\mathrm{Br}_0(X) \subseteq \mathrm{Br}_1(X).$$

The transcendental elements of the Brauer group are

$$\mathrm{Br}(X) \backslash \mathrm{Br}_1(X).$$

Exercise 16. Show that the Brauer–Manin pairing factors through $\mathrm{Br}(X)/\mathrm{Br}_0(X)$.

3. $\mathrm{Br}_1(X)$

Let X/K be a smooth, projective, geometrically irreducible variety over a field of characteristic 0. Let $\overline{K}/K$ be a choice of algebraic closure, and let $\Gamma := \mathrm{Gal}(\overline{K}/K)$. The Hochschield–Serre spectral sequence yields an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Pic}(X) \arrow[r] & \mathrm{Pic}(\overline{X})^{\Gamma} \arrow[r, "(*)"] & \mathrm{Br}(K) \arrow[dlll] \\ \mathrm{Br}_1(X) \arrow[r] & H^1(K, \mathrm{Pic}(\overline{X})) \arrow[r, "(*)"] & H^3(K, \overline{K}^{\times}). \end{tikzcd}$$

The rightmost group is 0 if K is a number field. Moreover, the $(*)$ maps are 0 if $X(K) \neq \emptyset$. So $\mathrm{Br}_1(X)/\mathrm{Br}_0(X) \hookrightarrow H^1(K, \mathrm{Pic}(\overline{X}))$, and is an isomorphism if K is a number field.

4. $\mathrm{Br}(X)/\mathrm{Br}_1(X)$

Let X/K be a smooth projective geometrically irreducible variety over a field $K \subseteq \mathbb{C}$. Recall that, by definition,

$$\mathrm{Br}(X)/\mathrm{Br}_1(X) \hookrightarrow \mathrm{Br}(\overline{X}).$$

Thus, to bound and control the non-algebraic part of the Brauer group it makes sense to study the geometric Brauer group. For this there is an exact sequence of Grothendieck.

Theorem 17 (Grothendieck)**.** *There is an exact sequence*

$$0 \longrightarrow \mathbb{Q}/\mathbb{Z}^{b_2-\rho} \longrightarrow \mathrm{Br}(\overline{X}) \longrightarrow H^3(\overline{X}, \mathbb{Z})_{\mathrm{tors}} \longrightarrow 0,$$

where b_2 *is the second Betti number, and* $\rho = \mathrm{rk}(\mathrm{NS}(\overline{X}))$

Of course, even now, just knowing the structure of the geometric Brauer group as a group isn't going to give us explicit Azumaya algebras and so we're going to want to work at this.

Remark 18. If $b_2 - \rho \neq 0$ then $\mathrm{Br}(\overline{X})$ is infinite.

Note that since our variety is defined over K we actually have

$$\mathrm{Br}(X)/\mathrm{Br}_1(X) \hookrightarrow \mathrm{Br}(\overline{X})^{G_K}.$$

In particular, we don't need the whole geometric Brauer group to be finite, just the set of fixed points. This was observed and used by Skorobogatov–Zahrin.

Theorem 19 (Skorobogatov–Zahrin)**.** *Let* $K/\mathbb{Q}$ *be finitely generated. If* X/K *is a principal homogeneous space of an abelian variety* A/K*, so that* $\overline{X} \cong \overline{A}$ *over* $\overline{K}$*, or a* $K3$*-surface, then in fact*

$$\#\mathrm{Br}(\overline{X})^{G_K} < \infty$$

In particular, we obtain the following.

Corollary 20. *If* X/K *is a* $K3$*-surface over a finitely generated extension of* $\mathbb{Q}$ *then* $\mathrm{Br}(X)/\mathrm{Br}_0(X)$ *is finite (see exercises).*

Structure of the proof of Theorem 19.

Step 1. Show that for every prime ℓ,

$$\mathrm{Br}(\overline{X})^{G_K}[\ell^\infty] := \bigcup_{n\geq 1} \mathrm{Br}(\overline{X})^{G_K}[\ell^n]$$

is finite. See the exercises.

Step 2. Show that for all but finitely many primes ℓ,

$$\mathrm{Br}(\overline{X})^{G_K}[\ell] = 0$$

See Skorobogatov–Zahrin 2008. □

5. Uniform Bounds

We will proceed by analogy to elliptic curves, where a famous uniformity result is known.

Theorem 21 (Merel: strong uniform boundedness for elliptic curves)**.** *Given $d \in \mathbb{Z}_{>0}$, there is a constant $c(d)$ such that*

$$\#E(K)_{\mathrm{tors}} < c(d)$$

for every elliptic curve E/K and number field $K/\mathbb{Q}$ with $[K:\mathbb{Q}] = d$.

For a curve of genus g, the canonical class has degree $2g-2$, so curves of genus 1 are precisely those with trivial canonical sheaf. Surfaces with trivial canonical sheaf come in two flavours:

- Abelian surfaces, where $\dim H^1(X, \mathcal{O}_X) = 2$;
- $K3$-surfaces where $\dim H^1(X, \mathcal{O}_X) = 0$.

Maybe these should be analogous to elliptic curves, and there should be some analogue to Merel's theorem here. What is the correct analogue of torsion in the elliptic curve here though? Well note that for an elliptic curve over a number field, denoted E/K, we have

$$\begin{aligned} E(K)_{\mathrm{tors}} &\cong \mathrm{Pic}^0(E)^{G_K}_{\mathrm{tors}} \\ P &\mapsto [P] - [O_E]. \end{aligned}$$

Moreover, noting the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Pic}^0 E \arrow[r] & \mathrm{Pic} E \arrow[r, "\deg"] & \mathrm{NS}(E) \arrow[r] \arrow[d, "\sim"] & 0 \\ & & & \mathbb{Z} & \end{tikzcd}$$

we have

$$\begin{aligned} \mathrm{Pic}^0(E)^{G_K}_{\mathrm{tors}} &= \mathrm{Pic}(E)^{G_K}_{\mathrm{tors}} \\ &= \mathrm{H}^1_{\text{ét}}(E, \mathbb{G}_m) \\ &= \mathrm{im}\left(\mathrm{H}^1_{\text{ét}}(E, \mathbb{G}_m)_{\mathrm{tors}} \to \mathrm{H}^1_{\text{ét}}(\overline{E}, \mathbb{G}_m)_{\mathrm{tors}}\right) \\ &= \mathrm{im}\left(\mathrm{H}^{\dim E}_{\text{ét}}(E, \mathbb{G}_m)_{\mathrm{tors}} \to \mathrm{H}^{\dim E}_{\text{ét}}(\overline{E}, \mathbb{G}_m)_{\mathrm{tors}}\right). \end{aligned}$$

So for X an abelian or $K3$ surface, perhaps

$$\mathrm{im}\left(\mathrm{H}^{\dim X}_{\text{ét}}(X, \mathbb{G}_m)_{\mathrm{tors}} \to \mathrm{H}^{\dim X}_{\text{ét}}(\overline{X}, \mathbb{G}_m)_{\mathrm{tors}}\right) = \mathrm{im}\left(\mathrm{Br}(X) \to \mathrm{Br}(\overline{X})\right)$$

might be the right analogue of $E(K)_{\text{tors}}$. Indeed, for each X, $\mathrm{Br}(X)/\mathrm{Br}_1(X)$ is finite by Theorem 19, so we can actually ask for uniformity.

Question 22 (Varilly-Alvarado & Viray)**.** *Let X be a smooth projective surface over a number field K with trivial canonical sheaf. Is there a bound for $\mathrm{Br}(X)/\mathrm{Br}_1(X)$ that is independent of X, depending only on $\dim H^1(X, \mathcal{O}_X)$ (i.e. whether X is $K3$ or abelian), $\mathrm{NS}(\overline{X})$ (as an absolute lattice, no Galois action!) and $[K : \mathbb{Q}]$?*

In fact, X/K with K characteristic 0 it turns out that $\mathrm{Br}_1(X)/\mathrm{Br}_0(X)$ is uniformly bounded over all $K3$-surfaces, so really we could ask about $\mathrm{Br}(X)/\mathrm{Br}_0(X)$.

Conjecture 23 (Varilly-Alvarado–Viray)**.** *The answer to Question 22 is yes for $K3$-surfaces, and in fact this bounds $\mathrm{Br}(X)/\mathrm{Br}_0(X)$ uniformly.*

The first main result on this conjecture was then the following.

Theorem 24 (Orr–Skorobogatov)**.** *Conjecture 23 holds for all CM $K3$-surfaces over number fields with a constant $C = C([K : \mathbb{Q}])$ which is even independent of the lattice $\mathrm{NS}(\overline{X})$.*

Why should we want to bound this? One reason is the following.

Theorem 25 (Kresch–Tschinkel, Poonen–Testa–van Luijk)**.** *Let X/K be a $K3$-surface over a number field, given by a system of homogeneous polynomial equations. Then the Brauer–Manin set $X(\mathbb{A}_k)^{\mathrm{Br}(X)}$ is effectively computable, provided that*

$$|\mathrm{Br}(X)/\mathrm{Br}_0(X)|$$

can be bounded effectively.

There is then a conjecture of Skorobogatov.

Conjecture 26 (Skorobogatov)**.** *For X/K a $K3$-surface over a number field, the Brauer–Manin obstruction is the only one. That is:*

$$\overline{X(K)} = X(\mathbb{A}_K)^{\mathrm{Br}(X)}.$$

If Conjecture 26 holds then together with Theorem 25 and effective bounds we would have an effective method to determine whether $X(K) \neq \emptyset$ for X/K a $K3$-surface over a number field – c.f. Hilbert's 10th problem!

For recent progress on effective bounds, see e.g.

- Hanset–Kresch–Tschinkel 2012;
- Cantoral-Farfan–Tang–Tanimoto–Visse 2016;
- Varilly-Alvarado–Viray 2020;
- Balestrieri–Johnson–Newton 2022.

Lecture 2

6. The Brauer–Manin Pairing

Question 27 (Swinnerton-Dyer)**.** *Let X/K be a nice (smooth proj. geom. int.) variety over a number field. Assume that $\mathrm{Pic}(\overline{X})$ is finitely generated and torsion-free. Let $S \subset \Omega_K$ be a finite set of places of K containing all archimedean places and all places of bad reduction for X.*

Is there an open and closed set $Z \subset \prod_{v \in S} X(K_v)$ such that the Brauer-Manin set is given by

$$X(\mathbb{A}_K)^{\mathrm{Br}(X)} = Z \times \prod_{v \notin S} X(K_v)?$$

That is, are the only places that play a role in the Brauer–Manin obstruction the archimedean ones and those of bad reduction?

Theorem 28 (Colliot-Thélène–Skorobogatov)**.** *Let X/K be as in Question 27, and assume that the transcendental part of the Brauer group, $\mathrm{Br}(X)/\mathrm{Br}_1(X)$, is finite. If v is a finite places of good reduction for X and its residue characteristic is coprime to $\#\mathrm{Br}(X)/\mathrm{Br}_1(X)$ then for all $\mathcal{A} \in \mathrm{Br}(X)$*

$$\begin{aligned} |\mathcal{A}| : X(K_v) &\to \mathbb{Q}/\mathbb{Z} \\ P &\mapsto \mathrm{inv}_v(\mathcal{A}(P)) \end{aligned}$$

is constant. In particular the answer to Question 27 is yes if S contains all primes dividing $\mathrm{Br}(X)/\mathrm{Br}_1(X)$.

However it turns out that the answer to the question is no in general.

Theorem 29 (M. Pagano)**.** *Let $X \subseteq \mathbb{P}^3_{\mathbb{Q}}$ be the $K3$-surface with equation*

$$x^3y + y^3z + z^3w + w^3x + xyzw = 0.$$

Then 2 is a prime of good reduction for X,

$$\mathcal{A} = \left(\frac{z^3 + w^2x + xyz}{x^3}, \frac{-z}{x} \right) \in \mathrm{Br}(X)[2]$$

and $|\mathcal{A}| : X(\mathbb{Q}_2) \to \frac{1}{2}\mathbb{Z}/\mathbb{Z}$ is non-constant. So in particular the answer to Question 27 is no, even for $K3$-surfaces over $\mathbb{Q}$.

We can ask *is there any* finite set S which does all the work?

Theorem 30 (Bright–Newton)**.** *Let X/K be as in Question 27. Let $S \subseteq \Omega_K$ be a finite set of places comprising:*

(i) archimedean v;
(ii) places of bad reduction for X;
(iii) v such that $e_{v/p} \geq p - 1$, where p is the residue characteristic of v and e_v the ramification index at v of $K/\mathbb{Q}$;
(iv) v such that $H^0(\mathscr{X}(v), \Omega^1) \neq 0$ where $\mathscr{X}(v)$ is the special fibre at v of a smooth model for X.

Then $\forall \mathcal{A} \in \mathrm{Br}(X)$ and all primes $v \notin S$,

$$|\mathcal{A}| : X(K_v) \to \mathbb{Q}/\mathbb{Z}$$

is constant.

Remark 31. (1) If X/K is a $K3$-surface and v is a place of good reduction then $\mathscr{X}(v)$ is $K3$ and hence $H^0(\mathscr{X}(v), \Omega^1) = 0$.
(2) If $X/\mathbb{Q}$ is $K3$ then the set of finite good places satisfying (iii) is simply 2. In particular Theorem 29 is optimal in its case.

Ingredients of proof of Theorem 30. Let $v \notin S$, with residue characteristic p, and let $\mathcal{A} \in \mathrm{Br}(X)$. We want to show that

$$|\mathcal{A}| : X(K_v) \to \mathbb{Q}/\mathbb{Z}$$

is constant. This is a purely local question.

Notation 32. We adopt the following notation:

- k a p-adic field;
- π a uniformizer;
- $\mathbb{F}$ the residue field;
- X/k smooth geometrically irreducible variety;
- $\mathscr{X}/\mathcal{O}_k$ a smooth model;
- $Y = \mathscr{X} \times_{\mathcal{O}_k} \mathbb{F}$ special fibre, assumed to be geometrically irreducible.

Suppose $Q \in X(k)$ extends to $\mathscr{X}(\mathcal{O}_k)$ and has reduction $Q_0 \in Y(\mathbb{F})$. Then the following diagram commutes, where $\mathrm{Br}(X)(p')$ is the prime-to-p torsion in $\mathrm{Br}(X)$:

$$\begin{array}{ccc} \mathrm{Br}(X)(p') & \xrightarrow{\delta} & H^1(Y, \mathbb{Q}/\mathbb{Z}) \\ \big\downarrow Q & & \big\downarrow Q_0 \\ \mathrm{Br}(k) & \xrightarrow{\delta \cong} & H^1(\mathbb{F}, \mathbb{Q}/\mathbb{Z}) \end{array}$$

where the left vertical map is $\mathcal{A} \mapsto \mathcal{A}(Q)$, δ is the residue map, and the right vertical is evalutation at Q_0.

Corollary 33. *If $\mathcal{A}$ has order coprime to p then $\mathcal{A}(Q)$ only depends on Q_0.*

This is not true for $\mathcal{A} \in \mathrm{Br}(X)[p^r]$. Classifying elements of the Brauer group $\mathrm{Br}(X)$ according to the π-adic accuracy needed to evaluate them yields a filtration on $\mathrm{Br}(X)$. We then relate this to Kato's filtration by Swan conductor (see Kato 1989):

$$\mathrm{fil}_0\mathrm{Br}(X) \subseteq \mathrm{fil}_1\mathrm{Br}(X) \subseteq \mathrm{fil}_2\mathrm{Br}(X) \subseteq \dots$$

$\mathrm{sw}(\mathcal{A})$ is then the least n such that $\mathcal{A} \in \mathrm{fil}_n\mathrm{Br}(X)$. Kato extends the residue map to $\mathrm{fil}_0\mathrm{Br}(X) \supseteq \mathrm{Br}(X)(p')$.

Theorem 34 (Kato)**.** *For $n \geq 1$, there exists an injection*

$$\mathrm{rsw}_n : \frac{\mathrm{fil}_n\mathrm{Br}(X)}{\mathrm{fil}_{n-1}\mathrm{Br}(X)} \hookrightarrow H^0(Y, \Omega^2) \oplus H^0(Y, \Omega^1)$$
$$\mathcal{A} \mapsto (\alpha, \beta) = \mathrm{rsw}(\mathcal{A}),$$

which we call the 'refined swan conductor'.

To prove Theorem 30 we use properties of rsw (for example that $n\alpha = d\beta$). Let $\mathcal{A} \in \mathrm{fil}_n\mathrm{Br}(X)$, for $v \notin S$ we have $H^0(Y, \Omega^1) = 0$ so $\beta = 0$. When $e_{v/p} < p - 1$, we show that $\beta = 0 \implies \alpha = 0$ and hence $\mathrm{fil}_n\mathrm{Br}(X) = \mathrm{fil}_{n-1}\mathrm{Br}(X)$. □