JACOBIANS & MODELS PART I

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Disclaimer. These notes were taken live during lectures at the Introduction to SAGA winter school held at the CIRM from 30th January to 3rd February 2023. Any errors are the fault of the transcriber and not of the lecturer.

Lecture 1: Curves

Good notes on algebraic curves: Cristophe Ritzenthaler, Algebraic Curves (Link).

1. Description of Low Genus Curves

Let k be an algebraically closed field. Recall that for a curve C/k, writing $K = \operatorname{div}(\omega)$ for the canonical divisor class, Riemann-Roch tells us that

$$\ell(D) - \ell(K - D) = \deg(D) - g + 1, \tag{1}$$

where $\ell(D)$ is the dimension of the Riemann–Roch space

$$\mathcal{L}(D) = \{f \in K(C) \ : \ \operatorname{div}(f) + D \geq 0\} \cup \{0\}.$$

We now use this to construct equations for abstract curves of various genera.

1.1. **Genus 0.** If C has genus 0, then consider $-K$. Applying (1) with $D = 0$ we obtain $\ell(K) = 0$. Now apply (1) with $D = K$ to arrive at $\deg(K) = -2$. In particular, since $-K \geq 0$, we write $-K = P_1 + P_2$. Using that $\ell(D) = 0$ when $\deg(D) < 0$, we have $\ell(2K) = \ell(3K) = 0$ and so by (1)

$$\ell(-K) = \ell(2K) + \deg(-K) + 1 = 3,$$
$$\ell(-2K) = \ell(3K) + \deg(-2K) + 1 = 5.$$

Thus write

$$\mathcal{L}(-K) = \langle x, y, z\rangle$$
$$\mathcal{L}(-2K) = \left\langle x^2, y^2, z^2, xy, xz, yz\right\rangle.$$

The latter of these has 6 generators but is 5 dimensional, so there is a relation given by a quadratic form

$$Q(x, y, z) = 0.$$

Q is irreducible, as there is no relation on x, y, z, and so is smooth. Take the map

$$\phi : C \to V(Q) \subseteq \mathbb{P}^2$$
$$P \mapsto (x(P) : y(P) : z(P)).$$

We claim that this provides an isomorphism. Indeed, let $(x_0 : y_0 : z_0) \in V(Q)$. Then consider the equation $f(P) = x_0y(P) - y_0x(P)$ on C. Note that this can have at most two zeroes as it has at most 2 poles by virtue of being in the Riemann-Roch space $\mathcal{L}(-K) = \mathcal{L}(P_1 + P_2)$. The same is true for $g(P) = y_0 z(P) - z_0 x(P)$.

If both zeroes were equal for f and g then they would have the same divisor, and in this case $\exists \lambda \in K$ such that

$$f = \lambda g \implies x_0 y + (\lambda z_0 - y_0)x = \lambda y_0 z$$

which cannot happen as the three functions x, y, z are lineary independent by assumtion, and so $\deg \phi = 1$.

1.2. **Genus 1.** Let $O \in C$. Similar to the genus 0 case, we compute that $\ell(K) = 1$, $\deg(K) = 0$, so $K =$

$$\begin{aligned} \ell(O) = 1 - 1 + 1 = 1 &\quad \mathcal{L}(O) = \langle 1 \rangle \\ \ell(2O) = 2 &\quad \mathcal{L}(O) = \langle 1, x \rangle \\ \ell(3O) = 3 &\quad \mathcal{L}(O) = \langle 1, x, y \rangle \\ \ell(6O) = 6 &\quad \mathcal{L}(O) = \left\langle 1, x, x^2, x^3, y, xy, y^2 \right\rangle \end{aligned}$$

Note that there are 7 elements in this final space, and that moreover the coefficients of x^3 and y^2 cannot be zero in the relation since then we would arrive at a relation seen lower down, so get equation

$$E : y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6.$$

Once more, we'd like this to be our curve. Take again the map

$$\begin{aligned} \phi : C &\to E \\ P &\mapsto (x(P) : y(P) : 1) \end{aligned}$$

By definition, O is a pole for x, y, so we'll complete the map by $O \mapsto [0 : 1 : 0]$. Now consider the diagram of function fields

$$\begin{tikzcd} & K(C) \arrow[d, dash] \arrow[ddl, dash] \arrow[ddr, dash] & \\ & K(x,y) \arrow[dl, dash] \arrow[dr, dash] & \\ K(x) & & K(y). \end{tikzcd}$$

We claim that $K(C)/K(y)$ is degree 3 and $K(C)/K(x)$ is degree 2: note that $\phi_x : C \to \mathbb{P}^1; P \mapsto (x(P) : 1)$ is degree 2 as $x \in \mathcal{L}(2O)$. Similarly for y the conclusion holds. Thus $K(C)/K(x,y)$ is degree 1 and so E and C are birationally equivalent.

Moreover, we are smooth at $(0 : 1 : 0)$ because the maximal ideal satisfies

$$\mathfrak{m}_{[0:1:0]} = (x, z), \mathfrak{m}^2_{[0:1:0]} = (x^2, xz, z^2)$$

so $\dim \mathfrak{m}_{[0:1:0]}/\mathfrak{m}^2_{[0:1:0]} = 1$. For points $[x_0 : y_0 : 1]$ we can apply a projective linear transformation to assume $x_0 = y_0 = 0$ (so $a_6 = 0$) and then to be singular would mean

$$a_3 = a_4 = 0,$$

and so

$$y^2 + a_1 xy = x^3 + a_2 x^2.$$

Then $\operatorname{div}(x) = P_1+P_2-2O$ and $\operatorname{div}(y) = P_1+P_2+Q_3-3O$. Thus $\operatorname{div}(y/x) = Q_3-O$ and so $y/x \in \mathcal{L}(O)$ but this space is the space of constants and so x and y would be linearly dependent, a contradiction. Thus our curve is smooth and we have isomorphism.

1.3. **Genus 2.** In this case $\deg(K) = 2$, $\ell(K) = 2$, $K = P_1 + P_2$.

$$\mathcal{L}(K) = \langle 1, x\rangle$$

Take the map

$$\begin{aligned} \phi : C &\to \mathbb{P}^1 \\ P &\mapsto (x(P) : 1) \end{aligned}$$

note $\deg(\phi) \leq 2$. Apply Riemann–Hurwitz to get

$$2g - 2 = -2\deg(\phi) + \sum_{P\in C} (e_\phi(P) - 1)$$

so $2 = 2(-2) + 6$. If $\operatorname{char}(K) \neq 2$ then we obtain

$$y^2 = f(x)$$

where $\deg(f) = 5$ or 6.

Definition 1. C is hyperelliptic if ϕ_K, the canonical map, is not an embedding. in this case it is a degree 2 map to $\mathbb{P}^2$.

$$\begin{aligned} \mathcal{L}(K) &= \langle \phi_1, \dots, \phi_g\rangle \\ \phi_K : C &\to \mathbb{P} \qquad\qquad\qquad\qquad g-1 \\ P &\mapsto [\phi_1(P) : \cdots : \phi_g(P)] \end{aligned}$$

1.4. **Genus 3.** We get genus 3 non-hyperelliptic curves given by plane quartics, essentially because $\deg(K) = 4$ and we do the calculations.

1.5. **Genus 4.** Non-hyperelliptic:

$$C \to \mathbb{P}^3$$

do Riemann-Roch again

$$\ell(2K) = 9$$

Denote by $\mathcal{P}(n, m)$ degree m polynomials in n variables, then the space $\mathcal{P}(3, 2)$ is 10-dimensional. So we obtain a quadratic Q that vanishes on this curve. Again look at

$$\ell(3K) = 15$$

Then $\mathcal{P}(3, 3)$ has dimension 20 and so we obtain a cubic R which vanishes (we already have xQ, yQ, zQ, wQ so obtain one more, which is our cubic R). Then we can show that

$$C : \{Q = R = 0\} \subseteq \mathbb{P}^3$$

is a degree 6 complete intersection.

Lecture 2: Jacobians

2. The Jacobian

Definition 2. An abelian variety over a field k is a smooth projective algebraic variety A/K of dimension g such that there exists:

- a morphism defined over k: $+ : A \times A \to A$ (addition)
- a morphism defined over k: $- : A \to A$ (inversion)
- a k-rational point $O \in A(k)$ for which $A(\overline{k})$ is a commutative goup (under the operations above) with identity O

Remark 3. Mumford proved that any abelian variety of dimension g over a field $\overline{k}$ of characteristic not 2 can be given by quadrics in $\mathbb{P}^{4^g-1}$.

Definition 4. A complex torus is a group of the form

$$\mathbb{C}^g/\Lambda$$

where $\Lambda \subseteq \mathbb{C}^g \cong \mathbb{R}^{2g}$ is the $\mathbb{Z}$-span of an $\mathbb{R}$-basis.

Example 5. *If $g = 1$ then have $\Lambda = \langle 1, \tau\rangle$ (up to isometry), and have the Weierstrass p-function*

$$\wp(z,\Lambda) = \frac{1}{z^2} + \sum_{\lambda\in\Lambda\setminus\{0\}} \frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}$$

which induces

$$\begin{aligned} \mathbb{C}/\Lambda &\to E : y^2 = 4x^3 - g_2(z)x + g_3(z) \\ z &\mapsto (\wp(z,\Lambda), \wp'(z,\Lambda)). \end{aligned}$$

Definition 6. A Riemann fomr H on a complex torus $\mathbb{C}^g/\Lambda$ is a hermitian form

$$H : \mathbb{C}^g \times \mathbb{C}^g \to \mathbb{C}$$

such that its imaginary part is integer valued on Λ.

Remark 7. Recall that a hermitian form is a pairing such that

$$H(au + bv, w) = aH(u,w) + bH(v,w)$$

and

$$H(v,u) = \overline{H(u,v)},$$

for every $u, v, w \in \mathbb{C}^g$ and $a, b \in \mathbb{C}$.

Theorem 8. *A complex torus is (isomorphic to) the complex points of hte abelian variety if and only if it admits a positive definition Riemann form.*

Remark 9. A polarisation is a morphism $\lambda : A \to A^\vee$. Have the Poincaré bundle $P \to A \times A^\vee$.

Proposition 10. *If $A/\mathbb{C}$ is an abelian variety of dimension g, then*

$$A[n] \cong (\mathbb{Z}/n\mathbb{Z})^{2g}.$$

This is still true over any $k = \overline{k}$, not just $\mathbb{C}$, so long as $\mathrm{char} k \nmid n$. *If, however,* $\mathrm{char} k = p > 0$ *then*

$$A[p^i] \cong (\mathbb{Z}/p^i\mathbb{Z})^\gamma$$

for some $0 \le \gamma \le g$ called the p-rank of A.

Example 11. *Recall that a CM field is a totally imaginary extension of a totally real number field. Assume that k is a CM field, with totally real subfield K^+ which is degree g over $\mathbb{Q}$. Let $\phi_1, \dots, \phi_g$ be a choice of embeddings $K \to \mathbb{C}$, one for each place of K. Let*

$$\begin{aligned} A &= \mathbb{C}^g/\Lambda, \\ \Lambda &= \{(\phi_1(a), \dots, \phi_g(a)) \ : \ a \in \mathcal{O}_K\}. \end{aligned}$$

For any $t \in K^\times$ such that $\bar{t} = -t$ and $\Im(\phi_i(t)) > 0$, we then have a Riemann form

$$H(x, y) = \mathrm{tr}_{K/\mathbb{Q}}(tx\overline{y})$$

Definition 12. Let C/k be a nice curve of genus $g > 0$. Then $\mathrm{Jac}(C)$ is a principally polarised abelian variety called the Jacobian of C which satisfies

$$\mathrm{Jac}(C)(k) = \mathrm{Pic}^0(C)(\overline{k})^{\mathrm{Gal}(\overline{k}/k)}.$$

Remark 13. Actually, $\mathrm{Jac}(C)(k') = \mathrm{Pic}^0(C)(k')$ as soon as $C(k') \neq \emptyset$.

Example 14. *$E : y^2 = -x^4 - 1$ a genus one curve, then let ζ_8 be an 8th root of unity. We have points (over $\overline{Q}$)*

$$P_1 = (\zeta_8, 0),\ P_2 = (-\zeta_8, 0),\ P_3 = (\zeta_8^3, 0),\ P_4 = (-\zeta_8^3, 0).$$

Consider the divisor $D = P_1 + P_2 - P_3 - P_4$. Then for $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_8)/\mathbb{Q})$ such that $\sigma(\zeta_8) = \zeta_8^3$ we have $\sigma(D) = -D$, and

$$D - \sigma(D) = 2D = \mathrm{div}\left(\frac{x^2 - \zeta_8^2}{x^2 + \zeta_8^2}\right).$$

Proposition 15. *Assume that $C(k) \neq \emptyset$, and let $P_0 \in C(k)$. Then we have a map*

$$\begin{aligned} \phi : \mathrm{Sym}^g C &\to \mathrm{Pic}^0(C) \\ D &\mapsto D - g \cdot P_0. \end{aligned}$$

is surjective and there exists an open subset of $\mathrm{Sym}^g(C)$ on which it is bijective.

Example 16. *For curve C of genus 2 people tend to use Mumford coordinates. Indeed, since the curve is necessarily hyperelliptic, write*

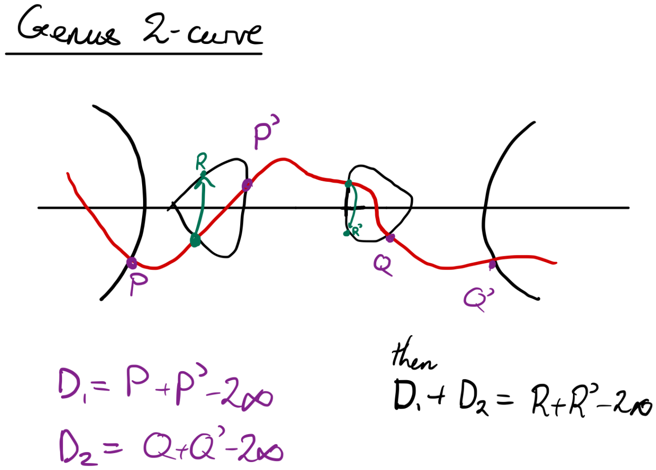
$$C/k : y^2 = f(x) = x^5 + a_3x^3 + a_2x^2 + a_1x + a_0.$$

Say $P \in \mathrm{Jac}(C)(k)$, with $P = (x_1, y_1) + (x_2, y_2) - 2\infty$, then we write two polynomials

$$(u(x), v(x)) = (x^2 + qx + r, sx + t) \leftrightarrow (q, r, s, t) \in \mathbb{A}_k^4$$

such that $u(x_i) = 0$, $v(x_i) = y_i$.

Finally, the geometric diagram for adding points on Jacobians of genus 2-curves:



Lecture 3: Models

References: Romagny: *models of curves*; Silverman: *Advanced Topics...*.

3. Models

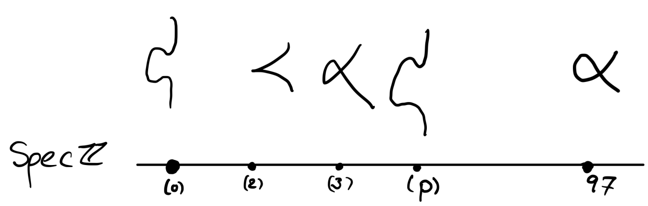
We'd like to find nice models of curves, so that we can say useful things about some kind of well defined 'reduction modulo p'.

Example 17. *Consider $C : 9x^2 + y^2 = z^2$, then maybe the reduction mod 3 should be two lines (just by reducing this equation). But then also we could factor the 9 into the x^2 in which case we should get something irreducible.*

Let R be a Dedekind domain (e.g. $\mathbb{Z}, \mathbb{Z}_p, k[[t]]$ etc), let K be the fraction field of R, and k be the residue field at a chosen prime $\mathfrak{p}$.

Remark 18. An arithmetic surface $\mathscr{C}$ should be a one-dimensional family of varieties, so a scheme of dimension 2. Alternatively, we might call this a 'curve over R'. We'll want properties like propernes, finite type, and other nice things to ensure that this plays nicely.

Example 19. *Consider $\mathscr{C} : y^2 = x^3 + 2x^2 + 6$ over $R = \mathbb{Z}$, then we have, as well as the generic fibre (i.e. this equation considered as a curve over $\mathbb{Q}$), curves over $\mathbb{F}_p$ for every prime p given by reducing this equation mod p. The behaviours at various primes are different, we illustrate some below.*



Definition 20. Say that a curve C/K has good reduction at $\mathfrak{p}$ if there exists a model $\mathscr{C}/R_{(\mathfrak{p})}$ such that the generic fibre is C and the special fibre is smooth.

Definition 21. A model $\mathscr{C}$ is regular if it is regular at every point, i.e.

$$\dim \mathcal{O}_{\mathscr{C},x} = \dim \mathfrak{m}_x/\mathfrak{m}_x^2$$

From now on let us assume that R is a DVR with valuation v, uniformizer π etc.

Example 22. *$\mathscr{C} : y^2 = x^3 + a$ for some $a \in R$. Then if $v(a) = 0$ we have that this model is smooth over R. If $v(a) = 1$ we are still regular: the point which might be an issue is $\mathfrak{m} = \langle \pi, x, y\rangle$, we verify the dimension is fine. If $v(a) > 1$ then we are no longer regular.*

Definition 23. $\mathscr{C}$ is stable (rep. semistable) if it is reduced, connected, has only nodal singularities, and all irreducible components which are isomorphic to $\mathbb{P}^1$ meed the other components in at least 3 points (resp. 2 points).

Remark 24. We have a nodal singularity at x if $\tilde{\mathcal{O}}_{\mathscr{C},x} \cong k[[u, v]]/\langle uv\rangle$, where the $\sim$ denotes completion.

Theorem 25. *Let C be a smooth, geometrically connected curve over K of genus at least 1. Then C has a minimal regular model (with normal crossings) over R and this is unique up to isomorphism.*

Remark 26. $\mathscr{C}$ is a minimal regular model if every dominant map $\mathscr{C} \to \mathscr{C}'$ to another refular model is in fact an isomorphism.

We will not discuss the proof, but instead look at an example shortly. Firstly: there is a similar result for stable models, after a possible finite base extension.

Theorem 27. *Let C/K be a smooth, geometrically connected curve of genus at least 2. Then there exists a finite field extension L/K such that C/L has stable reduction. The stable model is unique up to isomorphism.*

Definition 28. Let $\mathscr{C} \subseteq \mathbb{A}^2_R$ be an arithmetic surface defined by $f(x, y) = 0$, $\mathscr{C} = \mathrm{Spec}\,(R[x, y]/\langle f(x, y)\rangle)$. Assume that we have a singularity at $\pi = x = y = 0$. The blow-up at $\mathfrak{m} = \langle x, y, \pi\rangle$ is formed by taking the following three schemes and gluing them:

- **Chart 1:** $\pi = \pi_1$ $x = \pi_1 x_1$, $y = \pi_1 y_1$, $f(\pi x_1, \pi y_1) = \pi_1^v f_1(\pi_1 x_1, \pi_1 y_1)$,
$$\mathscr{C}_1 : \mathrm{Spec} R[\pi_1, x_1, y_1]/\langle f_1,\ \pi_1 - \pi\rangle .$$
- **Chart 2:** $\pi = \pi_2 y_2$, $x = x_2 y_2$, $y = y_2$, so $f(x_2 y_2, y_2) = y_2^{J_2} f_2(x_2, y_2)$,
$$\mathscr{C}_2 : \mathrm{Spec} R[\pi_2, x_2, y_2]/\langle f_2,\ \pi - \pi_2 y_2\rangle .$$
- **Chart 3:** $\pi = \pi_3 x_3$, $x = x_3$, $y = x_3 y_3$, so $f(x_3, x_3 y_3) = x_3^{J_3} f_3(x_3, y_3)$
$$\mathscr{C}_3 : \mathrm{Spec} R[\pi_3, x_3, y_3]/\langle f_3,\ \pi - \pi_3 x_3\rangle .$$

Glue these in the natural way using the identifications induced by our substitutions above.

Example 29. *Consider $\mathscr{C} : x^2 + y^5 = \pi^4$. (The generic fibre of) $\mathscr{C}$ is a genus 2 curve. We now construct the blowup at the point $\mathfrak{m} = \langle x, y, \pi\rangle$. We have the schemes as in the definition*

- ***Chart 1:*** *We make the substitutions $x = \pi x_1$, $y = \pi y_1$, and then following our recipe we obtain that $\mathscr{C}_1$ is given by*
$$x_1^2 + \pi^3 y_1^5 = \pi^2.$$
That is to say: the special fibre of $\mathscr{C}_1$ is a double line at $x_1 = 0$.

- ***Chart 2:*** *We make substitutions* $\pi = \pi_2 y_2$, $x = x_2 y_2$, $y = y_2$, *so* $\mathscr{C}_2$ *is given by*
$$x_2^2 + y_2^3 = \pi_2^4 y_2^2, \qquad \pi = \pi_2 y_2.$$
That is to say: the special fibre of $\mathscr{C}_2$ *is a cuspidal cubic given by* $x_2^2 = -y_2^3$.
- ***Chart 3:*** *We make substitutions* $\pi = \pi_3 x_3$, $y = y_3 x_3$, $x = x_3$, *so* $\mathscr{C}_3$ *is given by*
$$1 + x_3^3 y_3^5 = \pi_3^4 x_3^2, \qquad \pi = \pi_3 x_3.$$
That is to say: the special fibre of $\mathscr{C}_3$ *is* $1 + x_3^3 y_3^5 = 0$.

We then glue these schemes, noting that this identifies the special fibre of $\mathscr{C}_3$ *with the nodal cubic in that of* $\mathscr{C}_2$ *and so the special fibre of the gluing is a double line together with a cuspidal cubic in which the double line meets the cuspidal cubic at the singular point with full multiplicity.*

For the stable model we can extend our base field so just extend by $\pi^{1/5}$ *to do the change of variables*
$$x = \pi^2 x_1,\ y = \pi^{4/5} y_1$$
giving the stable model $x^2 + y^5 = 1$.