Week 9

Let $\mathbb{F}$ be a field and let $m, n \in \mathbb{Z}_{>0}$. Consider a map

$$T : \mathbb{F}^n \longrightarrow \mathbb{F}^m$$

with the property $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{F}^n$. Show that $T(\mathbf{0}) = \mathbf{0}$.

Answers

1. Since this is a linear map between fields, it is also a group homomorphism, and thus has the property $T(\mathbf{0}) = \mathbf{0}$.
2. Set $\mathbf{u} = \mathbf{v} = \mathbf{0}$. Then we have $T(\mathbf{0}) = T(\mathbf{0}+\mathbf{0}) = T(\mathbf{0}) + T(\mathbf{0}) = 2T(\mathbf{0})$. By subtracting $T(\mathbf{0})$ from both sides it follows that $T(\mathbf{0}) = \mathbf{0}$.
3. $T(\mathbf{0}) = T(\mathbf{0}+\mathbf{0}) = T(\mathbf{0}) + T(\mathbf{0}) = 2T(\mathbf{0})$
4. The claim is not true in general: If $T(\mathbf{0}) = \mathbf{0}$ then T would be a linear map, but we do not know whether $T(\lambda\mathbf{v}) = \lambda T(\mathbf{v})$ for all $\lambda \in \mathbb{F}$ and all $\mathbf{v} \in \mathbb{F}^n$.