Workshop on Arithmetic and Algebra of Rational Points
Workshop Booklet

WARP
2023

University of Bristol

Date: 11th–15th September 2023

Organisers:
Julian Lyczak and Ross Paterson

Supported by:

Contents

Research Groups

Version: September 11, 2023

Workshop Schedule

Registration for the workshop is 09:00 – 09:20 on Monday 11th September. There will be a welcome talk at 09:20.

	Monday	Tuesday	Wednesday	Thursday	Friday
09:30 – 10:30	Jennifer Park	Lilybelle Cowland Kellock	H. Uppal	Margaret Bilu	Elvira Lupoian
10:30 – 11:00	Coffee Break	Coffee Break	Coffee Break	Coffee Break	Coffee Break
11:00 – 12:00	Jef Laga	Celine Maistret	Judith Ortmann	Stephanie Chan	Mar Curcó-Iranzo
12:00 – 13:00	Lunch	Lunch	Kevin Destagnol	Lunch	Alex Bartel
13:00 – 15:00	Research Groups	Research Groups	FREE	Research Groups	
15:00 – 15:30	Coffee Break	Coffee Break		Coffee Break	
15:30 – 18:00	Research Groups	Research Groups		Research Groups	

Locations

- **Registration & Catering:** this will be provided in the foyer of the Fry building, which is directly ahead as you come in the front door.
- **Talks:** these will be held in LG.02, which is located at the bottom of the stairs behind the foyer.
- **Research Groups:** Each research group has allocated space in the building for the research group sessions. We list the allocations below, note that the first character denotes the floor (G=ground floor, 1=first floor)
  - LG.02 (lecture room)
    - *Normic equations over function fields and Laurent series fields*
      Diego Izquierdo, Damián Gvirtz-Chen, and Margherita Pagano
    - *Reduction types of curves of genus 2*
      Vladimir Dokchitser, Lillybelle Cowland Kellock, Pip Goodman, and Elvira Lupoian
  - G.07
    - *Galois Module Structure of Néron-Severi Groups*
      Alex Bartel, Matt Bisatt, Francesco Campagna, Mar Curcó Iranzo
    - *Thin sets of rational points on weighted projective stacks*
      Dan Loughran, Stephanie Chan, Sebastian Monnet, and Nick Rome
    - *A step toward the modular method for hyperelliptic curves: conductor computations in families*
      Céline Maistret, Martin Azon, Maleeha Khawaja, and Diana Mocanu
  - G.11
    - *Symmetric Squares of Quadric Hypersurfaces*
      Tim Browning, Jef Laga, Judith Ortmann, and Jennifer Park
  - G.16
    - *Brauer-Manin obstructions on log K3 surfaces*
      Martin Bright, Abdulmuhsin Alfaraj, Margaret Bilu, and H. Uppal
    - *Rational points on quartic del Pezzo surfaces through 2 descent on genus 2 curves*
      Julian Lyczak, Adam Morgan, Katerina Santicola, and Haowen Zhang
  - 1.11
    - *Character sum methods in arithmetic statistics*
      Peter Koymans, Ross Paterson, Tim Santens, Alec Shute
    - 4-*Eckardt points on low-degree del Pezzo surfaces*
      Sam Streeter, Rosa Winter, Kevin Destagnol, and Jakob Glas

Please note: there are many more (unbookable) spaces in the building, such as the common room (floor 2), the research labs (floors 1 and 2), and several open spaces scattered throughout the building. You are very welcome to use these also, please speak to the organisers or a local if you would like directions.

In order to make moving around the building easier, each research group leader has been provided a Bristol university card to open the card-access areas.

Talks

Jennifer Park

Counting points on weighted projective spaces

Abstract: Counting points on stacks by height is often translated into a statement in arithmetic statistics, yet the study of the general theory is only just beginning. I will report on the progress made on counting points of fixed degree on weighted projective spaces, one of the simplest kinds of stacks, with respect to various heights. This is joint work with Soumya Sankar.

Jef Laga

Rational torsion on abelian surfaces with quaternionic multiplication

Abstract: Mazur classified all possible rational torsion subgroups of elliptic curves over $\mathbb{Q}$. In joint work with Ciaran Schembri, Ari Shnidman and John Voight, we put strong constraints on torsion subgroups of abelian surfaces which acquire quaternionic multiplication over $\bar{\mathbb{Q}}$. Along the way, we will meet a fun family of such surfaces: the Prym surfaces associated to curves of the form $y^3 = x^4 + ax^2 + b$, called bielliptic Picard curves.

Lilybelle Cowland Kellock

A generalisation of Tate's algorithm for hyperelliptic curves

Abstract: Tate's algorithm tells us that, for an elliptic curve E over a discretely valued field K with residue characteristic ≥ 5, the dual graph of the special fibre of the minimal regular model of E over K^{unr} can be read off from the valuation of $j(E)$ and Δ_E. For a hyperelliptic curve C/K, we can ask if we can give a similar algorithm that gives important data related to the curve and its Jacobian from polynomials in the coefficients of a Weierstrass equation for C/K. This talk will be split between being an introduction to cluster pictures of hyperelliptic curves, from which the important data can be gathered, and a presentation of how the cluster picture can be recovered from polynomials in the coefficients of a Weierstrass equation.

Celine Maistret

Euler factors of curves

Abstract: L-functions of abelian varieties are objects of great interest. In particular, they are believed (and known in some cases) to carry key arithmetic information of the variety via the Birch and Swinnerton-Dyer conjecture. As such, it is useful to be able to compute them in practice. In this talk, we will address the case of a genus 2 curve C/Q with bad reduction at an odd prime p where Jac(C) has good reduction. Our approach relies on counting points on the special fibre of the minimal regular model of the curve, which we extract using the theory of cluster pictures of hyperelliptic curves. Our method yields a fast algorithm in the sense that all computations occur in at most quadratic extensions of Q or finite fields. This is joint work with Andrew Sutherland.

H. Uppal

Integral points on diagonal affine cubic surfaces

Abstract: The sum of three cubes conjecture state if n is an integer not congruent to 4 or 5 mod 9 then the affine surface

$$\mathcal{U} : x^3 + y^3 + z^3 = n \subset \mathbb{A}^3_{\mathbb{Z}}$$

always has an integral point. Colliot-Thélène and Wittenberg showed that $\mathcal{U}$ has no "cohomological obstruction" to the existence of integral points (more formally there is no integral Brauer-Manin obstruction to the integral Hasse principle). In this talk we will study all diagonal affine cubic surfaces and show that there are instances of the integral Hasse principle failing in this more general setting. If time permits we will also see some counting results on how often the integral Hasse principle fails these surfaces. This is joint work with Vladimir Mitankin and Julian Lyczak.

Judith Ortmann

Conics over $\mathbb{F}_2((t))$ that have a rational point

Abstract: We consider a family C of conics in a variable $y \in \mathbb{A}^1_K$ over the global function field $K = \mathbb{F}_2((t))$. We are interested in an asymptotic formula for the number of conics of bounded height in C that have a rational point. To obtain such an asymptotic formula, the main idea is to use harmonic analysis to compute the height zeta function and then use a Tauberian theorem.

In this talk, we will answer the question for which y the corresponding conic C_y does have a rational point. If time permits, we will use this to compute the height zeta function.

Kevin Destagnol

Sums of arithmetic functions over values of polynomials and applications to the Loughran-Smeets conjecture

Abstract: I will explain how one can estimate sums of arithmetic functions over values of polynomials provided that the arithmetic functions is well-behaved in arithmetic progressions and that the number of variables of the polynomials is big enough. I will then give a few applications of this result to the Loughran-Smeets problem regarding the probability for a random algebraic variety among a family to admit a rational point. This is joint work with Efthymios Sofos and Leonhard Hochfilzer.

Margaret Bilu

A motivic circle method

Abstract: The Hardy–Littlewood circle method is a well-known technique of analytic number theory that has successfully solved several major number theory problems. More recently, a version of the method over function fields, combined with spreading out techniques, has led to new results about the geometry of moduli spaces of rational curves on hypersurfaces of low degree. I will explain how one can implement a circle method with an even more geometric flavour, where the computations take place in a suitable Grothendieck ring of varieties, leading thus to a more precise description of the geometry of the above moduli spaces. This is joint work with Tim Browning.

Stephanie Chan

The 6-torsion of class groups of quadratic fields

Abstract: For quadratic number fields, the 2-torsion of the narrow class group is well understood through Gauss' genus theory. The asymptotic for the average 3-torsion was proved by Davenport—Heilbronn using the geometry of numbers. By generalising the method of Nair—Tenenbaum, we obtain the correct order of magnitude of the average 6-torsion. This is joint work with Peter Koymans, Carlo Pagano, and Efthymios Sofos.

Elvira Lupoian

Cuspidal Points on Modular Jacobians

Abstract: The rational points of an arbitrary modular curve are studied by looking at their image in the Jacobian, under the Abel-Jacobi embedding. Although it's often the non-cuspidal points that are of interest, studying the cuspidal points is often accessible and more effective than one might first think.
The theorem of Manin and Drinfeld tell us that the difference of two cusps is a torsion point on the Jacobian of a modular curve, and hence we can consider the finite subgroup generated by the cusps in the Jacobian. On the classical modular curve $X_0(p)$ of prime level $p \geq 5$, there are two cusps, both defined over $\mathbb{Q}$, and Ogg proved that their difference generates a cyclic subgroup of order the numerator of $\frac{p-1}{12}$. Mazur later proved that this subgroup is in fact the entire rational torsion subgroup of the Jacobian $J_0(p)$. This phenomenon is expected to be more general. In this talk, we will consider the intermediate modular curve $X_H(p)$, where H is a proper subgroup of $(\mathbb{Z}/p\mathbb{Z})^*$ and describe abstractly some cuspidal subgroups corresponding to these curves.

Mar Curcó-Iranzo

Computational efficiency towards power sums of arithmetic progressions

Abstract: In this talk, we will discuss tools for solving (families of) Diophantine equations constructed from power sums. In particular, we will focus on sums of like powers of an n-term arithmetic progression (AP). We will see how computational efficiency to solve certain equations can be achieved by combining refinements of existing methodologies. As an application, we will present the recent resolution of a set of Diophantine equations related to 9-term APs. This is joint work with Nirvana Coppola, Maleeha Khawaja, Vandita Patel and Özge Ülkem.

Alex Bartel

Hasse principle for genus 1 hyperelliptic curves

Abstract: Fix a quartic polynomial $f(x)$ with rational coefficients and non-zero discriminant. As d runs over square-free integers, how often does the hyperelliptic curve with equation $C_d : dy^2 = f(x)$ have a rational point? It is not too hard to determine the asymptotic count of those d for which C_d has a point everywhere locally, and the question becomes how often C_d satisfies the Hasse principle. Under an assumption on the resolvent cubic of f, I will explain a heuristic that predicts a precise proportion, strictly between 0 and 1, among all d for which C_d has a point everywhere locally of those for which the Hasse principle holds; and will present a result towards that conjecture. The main technical ingredient is a determination of the distribution of ranks of 2-Selmer groups of elliptic curves in certain thin quadratic twist families. This is joint work with Adam Morgan.

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Galois module structure of Néron-Severi groups

Alex Bartel, Matt Bisatt, Francesco Campagna, and Mar Curcó Iranzo

Description: This is a project that can take many different directions. For concreteness, here is a special case that, I believe, is already not well understood: take a curve over a number field (you know what, let's say over $\mathbb{Q}$) that you care about. Its Jacobian is an abelian variety whose arithmetic is important to understand if you truly care about your curve. One of the invariants attached to this Jacobian is its Néron-Severi group. This is a free abelian group of finite rank that comes with a natural action of the absolute Galois group of $\mathbb{Q}$. What can we say about the Galois module structure? This is a vague question that can be made more precise in many different ways: can we compute the Galois module structure reasonably efficiently? Are there some restrictions on which lattices with Galois action can occur as the Néron-Severi groups of Jacobians of curves over $\mathbb{Q}$, say? Is this Galois module structure linked to some other interesting invariants of the curve?

I know very little about this problem. All I have done is compute the Galois module structure of the Néron-Severi group of the modular curve $X_1(13)$, and this was enough to convince me that there is some fun to be had here.

Variants: Some variants that could be explored: instead of Jacobians of curves, one could look at other interesting classes of varieties, e.g. other abelian varieties, or K3 surfaces. The Néron-Severi group of any non-singular projective variety is finitely generated, but in general it need not be free. In those cases, the question of Galois module structure is likely even more subtle.

References

[1] C. W. Curtis and I. Reiner, Methods of Representation Theory *Wiley Classics Library*, John Wiley & Sons, New York, 1990.
[2] J. Milne. Abelian Varieties. `https://www.jmilne.org/math/CourseNotes/av.html`.

Brauer-Manin obstructions on log K3 surfaces

Martin Bright, Abdulmuhsin Alfaraj, Margaret Bilu, and H. Uppal

Summary: Log K3 surfaces include affine cubic surfaces and more generally affine del Pezzo surfaces and are a natural setting for studying local-global principles for integral points. An important tool in this study is the Brauer-Manin obstruction.

Goal: The broad aim of this project is to use geometric methods to study the Brauer-Manin obstruction on log K3 surfaces. For example, which primes can be involved in the obstruction? One interesting starting point would be to look at the example of an affine degree 5 del Pezzo surface given in Julian Lyczak's thesis, from the point of view of recent work of Bright and Newton.

References

[1] Colliot-Thélène, Skorobogatov. Good reduction of the Brauer–Manin obstruction. *Trans. Amer. Math. Soc.* **365** (2013), 579-590.

[2] Lyczak. Order 5 Brauer-Manin obstructions to the integral Hasse principle on log K3 surfaces. *Ann. Inst. Fourier, Grenoble*, **73**, 2 (2023) 447-478. (arXiv:2005.14013)

[3] Pagano. An example of a Brauer-Manin obstruction to weak approximation at a prime with good reduction. *Research in Number Theory*, **8**, 63 (2022)

Symmetric Squares of Quadric Hypersurfaces

Tim Browning, Jef Laga, Judith Ortmann, and Jennifer Park

Goal: Prove a lower bound for the number of rational points of bounded height on the symmetric square of X, when X is a quadric hypersurface of dimension at least two.

Context: Manin's conjecture is concerned with counting rational points of bounded height on appropriate Fano varieties defined over the rational numbers. In 2014, le Rudulier studied the conjecture for $\mathrm{Sym}^2\mathbb{P}^2$, finding that the asymptotic formula only conforms to the Manin conjecture once one is allowed to remove thin sets from the variety. The resolution of Manin's conjecture symmetric powers of varieties essentially boils down to counting rational points of bounded height and bounded degree on the variety. It remains a challenging and significant problem to find new cases.

Approach: Let X be a smooth quadric hypersurface defined over the rational numbers. The idea will be to use a number field version of the circle method to count K-points of bounded height on X, for a fixed quadratic number field K, taking care to make sure all estimates are uniform in K. We can then get try to get a lower bound for the counting function associated to Sym^2X by summing this over many number fields of relatively small discriminant (compared to the height of the points).

References

[1] T. Browning and P. Vishe. Cubic hypersurfaces and a version of the circle method for number fields. *Duke Math. J.* **163** (2014), 1825-1883.

[2] L.C. Helfrich. Quadratische Diophantische Gleichungen über algebraischen Zahlkörpern. Ph.D. Thesis, Göttingen University, 2015.

[3] C. Le Rudulier. Points algébriques de hauteur bornée. Ph.D. Thesis, Université Rennes, 2014

Reduction types of curves of genus 2

Vladimir Dokchitser, Lillybelle Cowland Kellock, Pip Goodman, and Elvira Lupoian

Description: Silverman's "Advanced topics..." contains an extremely useful table of possible reduction types of elliptic curves, which, in particular, lists their special fibres, Neron component groups and minimal discriminants. The aim of the project is to produce a similarly handy reference table for curves of genus 2. The precise list of invariants we try to tabulate will depend on what the project participants are interested in! Like in the elliptic curve table, we will concentrate on curves with tame reduction.

References: "A user's guide to the arithmetic of hyperelliptic curves", by authors too numerous to list, is probably a good place to start, although its main focus is curves with semistable reduction. The old classification of Namikawa and Ueno, which lists the possible special fibres of genus 2 curves but doesn't explain how to obtain them, is also worth a serious look, although we should really aim to come up with something that's more intuitive and less cumbersome to use. A somewhat more practical classification for curves of genus 2 with tame reduction can be found at the end of Nowell's PhD thesis.

References

[1] A. J. Best, L. A. Betts, M. Bisatt, R. van Bommel, V. Dokchitser, O. Faraggi, S. Kunzweiler, C. Maistret, A. Morgan, S. Muselli, S. Nowell A user's guide to the local arithmetic of hyperelliptic curves. *Bull. Lond. Math. Soc.* 54(3):825-867, 2022. (arXiv:200701749)

[2] S. Nowell. Models of Hyperelliptic Curves Over p-adic Fields Ph.D. Thesis, University College London, `https://discovery.ucl.ac.uk/id/eprint/10152113/2/SCNowell%20-%20PhD%20Thesis.pdf`

[3] K. Ueno and Y. Namikawa. The Complete Classification of Fibres in Pencils of Curves of Genus Two. *Manuscripta mathematica* 9 (1973): 143-186.

Normic equations over function fields and Laurent series fields

Diego Izquierdo, Damián Gvirtz-Chen, and Margherita Pagano

Let K be a field. A normic equation over K is an equation of the form:

$$N_{L_1/K}(\mathbf{x}_1)N_{L_2/K}(\mathbf{x}_2)...N_{L_r/K}(\mathbf{x}_r) = a$$

for some finite extensions $L_1, ..., L_r$ of K and some $a \in K^\times$. It defines a principal homogeneous space under the normic torus given by the same equation with $a = 1$. Normic equations have been vastly studied over number fields. They do satisfy the local-global principle when $r = 1$ and the extension L_1/K is cyclic but not in general.

In this project, we aim at studying normic equations over various fields other than number fields. We are more precisely interested in function fields of varieties or Laurent series fields in several variables defined over $\mathbb{C}$, $\mathbb{C}((t))$, finite fields, p-adic fields or number fields.

There are usually several ways to make sense of the local-global principle over such fields, depending on whether one takes into account all of their places or only those that have a natural geometric interpretation. For instance, if K is the function field of a smooth projective curve over some base field k, one may take into account all places of K or only those that are induced by a closed point of the curve C. Similarly, if K is the Laurent series field in 2 variables $k((x, y))$ over some base field k, one can take into account all places of K or only those that are induced by a codimension one point of $\mathrm{Spec}(k[[x, y]])$.

Studying normic equations in general is hard. Here are some questions that we could investigate:

1) For which function fields and Laurent series fields can one ensure that the local-global principle holds for normic equations with respect to a single cyclic extension (ie a normic equation with $r = 1$ and L_1/K cyclic)? When the answer is negative, can one explicitly compute the corresponding obstructions to the local-global principle (either theoretically or computationally)?

2) Same as question 1) for normic equations with respect to a single biquadratic extension or more generally to a single bicyclic extension.

3) Same as question 1) for normic equations with respect to several cyclic extensions (ie a normic equation with $r > 1$ and L_1/K, ..., L_r/K all cyclic)?

4) Depending on the kind on field on which one works, under which assumptions on a finite group G can one ensure that the local-global principle holds for normic equations with respect to a single Galois extension with group G?

5) Given a normic torus, are there finitely many principal homogeneous spaces under that torus that contradict the local-global principle?

References

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[HS16] **D. Harari, T. Szamuely.** Local-global questions for tori over p-adic function fields. *J. Algebraic Geom.* 25(3) (2016), 571–605.

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[Izq19] **D. Izquierdo.** Dualité et principe local-global pour les anneaux locaux henséliens de dimension 2. With an appendix by Joël Riou. *Algebr. Geom.* 6(2), 148–176, 2019.

Character sum methods in arithmetic statistics

Peter Koymans, Ross Paterson, Tim Santens, Alec Shute

Summary: In a seminal paper, Heath-Brown [8] obtained the distribution of 2-Selmer groups of the congruent number curve. The proof can roughly be divided in three steps

- write down the algebraic problem in terms of a character sum. A typical example of a character sum is
$$\sum_{d_0 d_1 d_2 d_3 = n} \mu^2(n) \left(\frac{d_2 d_3}{d_0}\right) \left(\frac{d_0 d_1}{d_3}\right).$$
- show oscillation of the character sum once there is at least one non-principal character. The principal tools in this step are the large sieve and the Siegel–Walfisz theorem.
- find the main term of the character sum. This typically involves some combinatorial problem.

These techniques are applicable to a wide number of arithmetic statistics problems, and this has been done successfully by a number of authors, most notably by Fouvry–Klüners [4, 5]. These problems range from number field counting, class group statistics and rational points. The main prerequisite for the method is the first step. There are many interesting open problems that can be expressed in terms of character sums, and the goal of this project would be to investigate some of these problems.

Depending on the interest of the participants, there are various projects that could be pursued with these methods.

Number field counting: For a fixed finite group G, Malle [13] has predicted an asymptotic for the number of G-extensions with discriminant bounded by X. Malle's conjecture is considered to be of exceptional difficulty as it implies the inverse Galois problem. Even if one restricts to nilpotent groups, which is a relatively well-understood class of groups, Malle's conjecture is wide open with the only known cases being abelian groups by Wright [15], nonic Heisenberg groups by Fouvry–Koymans [6], quartic D_4-extensions by Cohen–Diaz y Diaz–Olivier [3] (see also [1] for the related problem of ordering by Artin conductor) and nilpotent groups where all elements of minimal order are central by Koymans–Pagano [10]. If G is a nilpotent group of nilpotency class 2, then one is able to express this count in terms of a character sum.

Subgoal: prove Malle's conjecture for Galois D_4-extensions when ordered by product of ramified primes over any number field with an explicit leading constant.
Main goal: prove Malle's conjecture for nilpotent groups of nilpotency class 2.

Rational points: Local solubility of families of varieties has been an important topic in rational points initiated by Loughran–Smeets [12]. In favorable situations this can also be detected with character sums.

Subgoal: obtain the count of integers $a, b, c \in$ with $\max(|a|, |b|, |c|) \leq B$ for which the equation
$$aX^3 + bY^3 = cZ^3$$
is everywhere locally soluble. This would prove a new instance of Loughran–Smeets [12], would improve previous work of Browning–Dietmann [2] and would be a natural followup to recent work of Loughran–Rome–Sofos [11], who considered $aX^2 + bY^2 = cZ^2$ with $\gcd(a, b, c) = 1$.
Main goal: obtain the count of integers $a, b, c \in \mathbb{Z}$ with $\max(|a|, |b|, |c|) \leq B$ for which the equation
$$aX^d + bY^d = cZ^d$$
is everywhere locally soluble.

Class groups: Fixing a number field K, we consider the 4-rank of the field $K(\sqrt{n})$ as n varies over *rational integers.* This problem has been considered in [7], [9] for quadratic number fields K and an elliptic curve analogue was considered in [14]. Both these problems reduce to character sum questions.

Subgoal: find the distribution of $\mathbb{Q}(\zeta_{12}, \sqrt{n})$.
Main goal: find the distribution of $K(\sqrt{n})$ for any Galois extension K of $\mathbb{Q}$.

Another topic of interest is the class group structure of abelian extensions, in particular that of $\mathbb{Z}/p^n\mathbb{Z}$-extensions ordered by product of ramified primes.

Main goal: find the distribution of the first few layers of $\mathrm{Cl}(K)[p^\infty]$ as K varies over all $\mathbb{Z}/p^n\mathbb{Z}$-extensions ordered by product of ramified primes.

Recommended reading: Some familiarity with [4] or [8] is highly recommended. A good background in algebraic number theory will also be valuable.

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Thin sets of rational points on weighted projective stacks

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Description: A result of Serre and Cohen [3, §13] states that any thin subset of $\mathbb{P}^n(\mathbb{Q})$ contains $O(B^{n-1/2}(\log B))$ rational points of height at most B. This saves a factor of $B^{1/2}$ over the total number of points in $\mathbb{P}^n(\mathbb{Q})$.

Goal: The aim of the project is to prove a version of this for weighted projective stacks and closely related stacks (e.g. M-curve structures on $\mathbb{P}^!$).
The word "stack" may seem scary, but actually in this case everything can be made very explicit and one is just counting lattice points with a slightly different height function. This viewpoint is expounded in the papers [1, 2]. The method will be based on the same as Serre using the large sieve [3, §12, §13].

Subgoals: The simplest non-trivial case is the weighted project stack $\mathbb{P}(1,2)$. This is a copy of $\mathbb{P}^1$ with one stacky point of order 2 at infinity. So we'll study this first in detail and make sure we understand it. Once this case is done we'll look at more general cases and see how far the method can be pushed.

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Rational points on quartic del Pezzo surfaces through 2 descent on genus 2 curves

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There is connection between between del Pezzo surface of degree 4 and curves of genus 2, described in [4]. In [2] arithmetic properties of quartic del Pezzo surfaces were used to produce families of genus 2 curves with explicit non-trivial classes in the Tate–Shafarevich group. In this project we will go in the opposite direction: using the understanding of the Selmer and Tate-Shafarevich group of Jacobians of curves, we will derive arithmetic properties of families of del Pezzo surfaces.

From a curve to a surface:

Let C be the genus 2 given by

$$y^2 = f(x)$$

for a separable quintic polynomial $f \in \mathbb{Q}[x]$. Let us consider the quintic $\mathbb{Q}$-algebra $\mathcal{A} = \mathbb{Q}[x]/(f)$. We consider the Jacobian $J = \mathrm{Jac}(C)$ and its 2-Selmer group

$$S^{(2)}(J) := \mathrm{Ker}\left(\mathrm{H}^1(\mathbb{Q}, J[2]) \to \prod_p \mathrm{H}^1(\mathbb{Q}_p, J[2])\right)$$

which is effectively computable as subset $\mathrm{H}^1(\mathbb{Q}, J[2]) \subseteq \mathcal{A}^*/(\mathcal{A}^*)^2$. Here any $\delta \in \mathrm{H}^1(\mathbb{Q}, J[2])$ corresponds to a homogeneous space $H_\delta \to J$, and those in the 2-Selmer group are the everywhere locally soluble homogeneous spaces under $J[2]$. For each δ one can construct a del Pezzo surface $X_\delta \subseteq \mathbb{P}^4$, explicitly given by two quadratic equations in five variables, such that $X_\delta(K) \neq \emptyset \Rightarrow H_\delta(K) \neq \emptyset$ for any $\mathbb{Q}$-algebra K.

From a surface to a curve:

Now suppose we have a quartic del Pezzo surface given by two quadratic equations Q_1 and Q_2 in two variables. We can represent those two quadratic equations by symmetric matrices $M_1, M_2 \in M_2(\mathbb{Q})$. It is known that the quintic polynomial $f(x) = \det(M_1 + xM_2)$ is separable. Flynn [4] proved that for the curve C given by $y^2 = f(x)$ there is always a $\delta \in \mathrm{H}^1(\mathbb{Q}, J[2])$ for which X_δ is isomorphic to X, and such pair (C, δ) is essentially unique.

Main idea of the project:

Using the exact sequence

$$0 \to J(\mathbb{Q})/2J(\mathbb{Q}) \to S^{(2)}(J) \to \text{Ш}(J)[2] \to 0$$

and conjectures on either the parity of rk $J(\mathbb{Q})$ or rk Ш$(J)[2]$ (for example the Tate–Shafarevich conjecture) will allow us to conclude that in specific situations there is a $\delta \in S^{(2)}(J)$ which comes from $J(\mathbb{Q})/2J(\mathbb{Q})$ of which we didn't know this explicitely. Hence δ lies in the kernel of $S^{(2)}(J) \to \text{Ш}(J)[2]$. Hence $H_\delta(\mathbb{Q}) \neq \emptyset$ and we conclude $X_\delta(\mathbb{Q}) \neq \emptyset$.

This would conditionally prove that some quartic del Pezzo surface has a rational point. Note that existing results on the existence of rational points on quartic del Pezzo surfaces are also conditional on Shinzel's hypothesis and the Tate–Shafarevich conjecture on elliptic curves, for example [3] and [6].

Main goal of the project:

- Produce a family of quartic del Pezzo surfaces for which the above method establishes the existence of a rational point.
- Determine how this construction relates to other invariants of quartic del Pezzo surfaces.

A possible subgoal:

- An interesting invariant of a quartic del Pezzo surface is the quintic polynomial f. It also comes with $\epsilon_i \in \mathbb{Q}[x]/(f_i)$ for any factor f_i of f. In [5] there is an algorithm to compute BrX as an abstract group from these invariants. How are these invariants determined by $\delta \in \mathrm{H}^1(\mathbb{Q}, J[2])$?

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A step toward the modular method for hyperelliptic curves: conductor computations in families.

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Diophantine equations are a major topic in modern number theory and the most known example of these is Fermat's last theorem. It states that for any $n \geq 3$ the equation

$$x^n + y^n = z^n$$

has no integer solutions. It is known that it suffices to prove the statement for $n = p$ a prime number. Many mathematicians like Frey, Ribet, Serre and Wiles worked on proving it and their approach is nowadays known as the modular method. The proof goes by contradiction: assume one has a solution to the equation, the first step is to associate to this putative solution an elliptic curve defined over $\mathbb{Q}$, often known as a Frey curve. By the modularity theorem, the Frey curve is modular and by Ribet's level lowering theorem, the residual galois representation arises from a newform of weight 2 and level 2. But the space of such newforms has dimension 0, thus the contradiction!

In 2000, Darmon proposed a program generalising the modular method to other diophantine equations. In this generalisation, one considers hyperelliptic curves with their associated Hilbert modular forms. Ribet's level lowering theorem remains a key step in the method and one needs to compute the conductor of the Jacobian of the hyperelliptic curve to achieve it. With this project, we propose to do so for the Fermat-type equation:

$$x^r + y^r = Cz^p$$

where $r > p$ are prime numbers and $C \in \mathbb{Z}$. Following Kraus' ideas, we can construct a Frey curve defined over a totally real field, associated to a solution of this equation as follows.

Consider ζ_r a primitive r-th root of unity and define $\omega_j := \zeta_r^j + \zeta_r^{-j}$ for $1 \leq j \leq \frac{r-1}{2}$, so that $K := \mathbb{Q}(\zeta_r)^+ = \mathbb{Q}(\omega_1)$ is the maximal totally real subfield of $\mathbb{Q}(\zeta_r)$. Define

$$h(x) := \prod_{j=1}^{\frac{r-1}{2}} (x - \omega_j).$$

For a putative solution (a, b, c) of the equation we consider the hyperelliptic curve

$$C_r(a,b) : y^2 = (ab)^{\frac{r-1}{2}} x h\left(\frac{x^2}{ab} + 2\right) + b^r - a^r.$$

This curve is actually defined over $\mathbb{Q}$ but one can also see it as defined over K through base change (which has advantages when proving modularity). Its discriminant is given by

$$\Delta(C_r(a,b)) = (-1)^{\frac{r-1}{2}} 2^{2(r-1)} r^r (a^r + b^r)^{r-1}$$

The goal of this project is to compute the conductor of the jacobian of $C_r(a, b)$ over K, i.e. the level of the Hilbert newforms before level lowering.

For $r = 5$ we computed explicitly the roots of the RHS polynomial and extracted the conductor exponents from the cluster pictures at every place of odd residual characteristic. We propose to apply a similar method, or any other that could be more fruitful, to compute the conductor for a wider range of values of the parameters.

References: I'd recommend reading the users guide to hyperelliptic curves, especially sections $1, 2, 3, 12$ to prepare.

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4-Eckardt points on low-degree del Pezzo surfaces

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We will study the occurrence of 4 lines intersecting in one point on del Pezzo surfaces of degrees 1 and 2 with conic bundles. By work of Demeio and Streeter, this has implications for such surfaces to satisfy weak weak approximation. Depending on where participant's interests and expertise lie, this project could involve arithmetic statistics through counting how often this occurs in a suitable moduli space, or constructing explicit (families of) examples of such surfaces that don't contain any points in 4 lines, and therefore satisfy weak weak approximation, or both.

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