Date: 07.01.25

** Analytical Approach:

$$2x+3=7$$
$$\Rightarrow 2x = 7-3$$
$$\Rightarrow 2x = 4$$
$$\therefore x = 2$$

** Numerical Approach:

$$2x+3=7$$
$$\downarrow$$

→ We will keep guessing values of x untill both sides are approximately equal. This method is called approximation.

** System of Linear Equations:

$$3x+4y+5z=8$$
$$6x-5y+3z=9$$
$$2x+y-z=7$$

* calculation uses numerical not analytical

Numerical analysis done when the calculation becomes complex.

Ques: What is Numerical Analysis?

Ans: Numerical Analysis is a branch of mathematics that solves continuous problems (Problems involving continuous variables) using numeric approximation.

Difference between Numerical Analysis and Numerical Methods.

* Numerical Analysis – Focuses on the study and development of methods for solving mathematical problems numerically.

* Numerical Methods – Consists of the actual techniques or algorithms used in numerical analysis.

Discrete Variable, $x = 1, 2, 3, 4$

$x \in R \rightarrow$ Continuous

Error Analysis:

True value = approximation + error

$\Rightarrow$ error = True Value − approximation.

$\Delta x = x - x_1$

Types of error −

1. Relative Error.
2. Percentage Error.
3. Absolute Error.

** $x = 3.5211$
$x_1 = 3.5231$

$E_A = X - X_1 = \Delta X$

$\therefore E_A = |3.5211 - 3.5231|$ ⟶ Absolute error

$E_R = \frac{E_A}{x} = \frac{\Delta x}{x}$ ⟶ Relative Error

$E_P = E_R \times 100$ ⟶ Percentage Error

Significant Figure : $\boxed{3.5213}000$

$\downarrow$

Significant figure

Ques:-01: Find the percentage error, if 625.483 is approximated to 3 significant figure

Soln:- $X = 625.483$

$X_1 = 625$

$\therefore \Delta x = x - x_1$

$\Rightarrow \Delta x = 625.483 - 625$

$\therefore \Delta x = 0.483$

Now,

$E_p = E_R \times 100$

$\Rightarrow E_p = \frac{\Delta x}{x} \times 100$ $\left[\because E_R = \frac{\Delta x}{x}\right]$

$\Rightarrow E_p = \frac{0.483}{625.483} \times 100$

$\Rightarrow E_p = 0.077\%$

Floating point representation:

625.483

$m \cdot b^{e}$

Exponent

mantissa

base

$= 625483 \times 10^{-3}$

$= 62548.3 \times 10^{-2}$

$= 6254.83 \times 10^{-1}$

$= 62.5483 \times 10^{1}$

* Modified Normalized form $\longrightarrow \frac{1}{m} < m < 1$

$$\frac{1}{10} < m < 1$$

$$0.1 < m < 1$$

$$\therefore (0.625483 \times 10^{3})$$

* True Normalized form $\longrightarrow 1 < m < 10$

$$\therefore 6.25483 \times 10^{2}$$

Date: 14.01.25

Error Analysis

Round-off the number: 965.25278 to six significant figures and compute E_p.

Absolute error $= |\text{True value} - \text{Approximation}|$

$\therefore \Delta x = |x - x_1|$

Relative Error $= \dfrac{\text{Absolute Error}}{\text{True value}} = \dfrac{E_A}{x}$

Percentage Error $= E_R \times 100$

** Error calculation:

Let,

True Value be x

Approximation value be x_1

Then, The absolute Error, $\Delta x = |x - x_1|$

Realative Error, $E_R = \dfrac{\Delta x}{x}$

Percentage Error, $E_p = E_R \times 100\ \%$

$= \dfrac{\Delta x}{x} \times 100\ \%$

True value, $x = 965.25278$

App. value, $x_1 = 965.253$ (Rounding off to six significant figures)

$\therefore$ Error, $E_P = E_R \times 100\%$

$\Rightarrow$ Error, $E_P = \frac{\Delta x}{x} \times 100\%$

$\Rightarrow$ Error, $E_P = \frac{|x - x_1|}{x} \times 100\%$

$\Rightarrow$ Error, $E_P = \frac{|965.25278 - 965.253|}{965.25278} \times 100\%$

$\therefore$ Error, $E_P = 2.279 \times 10^{-5}\ \%$

** Round-off the number: 625.483 to two decimal digits and compute E_R and E_P.

Solⁿ: Given,

True value, $x = 625.483$

App. value, $x_1 = 625.48$

$\therefore \Delta x = |x - x_1|$

$\Rightarrow \Delta x = |625.483 - 625.48|$

$\Rightarrow \Delta x = |0.003|$

$\therefore \Delta x = 0.003$

Now,

Relative Error, $E_R = \dfrac{\Delta x}{x}$

$= \dfrac{0.003}{625.483}$

$= 0.0000047963$

Percentage Error, $E_P = E_R \times 100\ \%$

$= 0.0000047963 \times 100\ \%$

$= 0.00047963\ \%$

Ans:

Types of Error:

- Inherent Error
- Round-off Error
- Truncation Error

Machine Epsilon:

$$\frac{|\Delta x|}{|x|} \leq \epsilon$$

ϵ → Epsilon

$$\epsilon = b^{1-t}$$

t → No. of significant digits in the Mantissa

b → Number base (10)

$$25.253 = 25253 \times 10^{-3}$$

$$\boxed{mb^e}$$

↓

$$10^e$$

* দশমিক পরে গেলে 10-এর power "–" হবে।

* " আগে " 10 " power "+" হবে।

$$1 < m < 10$$

↑

→ True Normalized : 2.5253×10^1

→ Modified Normalized : 0.25253×10^2

↓

$$0.1 < m < 1$$

Date: 16.01.25

Floating point Representation:

$m.b^e$

$e \rightarrow$ Exponent

$m \rightarrow$ mantissa

$b \rightarrow$ Number base

* Normalized $\rightarrow$ modified: $\frac{1}{10} \leq m < 1$

Normalized $\rightarrow$ True: $1 \leq m \leq 10$

* $105.532 = 105532 \times 10^{-3}$

$= 0.105532 \times 10^{3}$

$= 1.05532 \times 10^{2}$

Mathematical formula of error propagation for function of single variable:

Let,

$$\Delta f(x) = |f(x) - f(\bar{x})|$$

using Taylor's Series,

$$f(x) = f(\bar{x}) + (x-\bar{x}) f'(x) + \underbrace{\frac{(x-\bar{x})^2}{2!} f'(\bar{x}) + \cdots\cdots}_{\text{very much trivial}}$$

Since, $|\bar{x}-x| \sim 0$, So higher order terms in the series because trivial and that's why we can omit these part for our convenience of calculation —

$$f(x) = f(\bar{x}) + |(\bar{x}-x)| f'(\bar{x})$$

$$\Rightarrow f(x) = f(\bar{x}) \mp \Delta x f'(\bar{x})$$

$$\Rightarrow f(x) - f(\bar{x}) = |\Delta x| f'(\bar{x})$$

$$\therefore \Delta f(\bar{x}) = \Delta x . f'(\bar{x})$$

* Taylor Series:

$$f(b) = f(a) + (b-a) f'(a) + \frac{(b-a)^2}{2!} . f'(a) + \frac{(b-a)^3}{3!} . f'(a) + \cdots\cdots$$

Problem-01: Given a value of $\bar{x} = 2.5$ with an error of $\Delta \bar{x} = 0.01$ estimate the resulting error in the function $f(x) = x^3$ and determine the possible values.

Sol^n:— $\Delta f(\bar{x}) = \Delta \bar{x} . f'(\bar{x})$

$$\Rightarrow \Delta f(\bar{x}) = 0.01 \cdot 3(\bar{x})^2$$

$$\Rightarrow \Delta f(\bar{x}) = 0.01 \times 3(2.5)^2$$

$$\Rightarrow \Delta f(\bar{x}) = 0.01 \times 18.75$$

$$\therefore \Delta f(\bar{x}) = 0.1875$$

$$\therefore f(\bar{x}) = (2.5)^3 \quad [\because f(\bar{x}) = x^3]$$

$$= 15.625$$

$$f(x) = 15.625 \pm 0.1875$$

$$f(x) \in [15.4375, 15.8125]$$

Multiple Variable error propagation formula:

$$\Delta f(\bar{x}) = (\Delta \bar{x}) |f(\bar{x})|$$

$$\Delta f(\bar{x}_1, \bar{x}_2, \bar{x}_3) = \Delta \bar{x}_1 \left| \frac{\partial f}{\partial x_1} \right| + \Delta \bar{x}_2 \left| \frac{\partial f}{\partial x_2} \right| + \Delta \bar{x}_3 \left| \frac{\partial f}{\partial x_3} \right| + \cdots + \Delta \bar{x}_n \left| \frac{\partial f}{\partial x_n} \right|$$

$x_1 \quad x_2 \quad x_3$

Problem-02: The deflection y of the top of a sailboat mast is

$$y = \frac{FL^4}{8EI} \quad ; \text{ Where,}$$

F = a uniform side loading (Nm^{-1})

L = height (m)

E = the modulus of elasticity (Nm^{-2})

I = The moment of inertia (m^4)

Esitmate the error in y given the following data:-

$\bar{F} = 750\ Nm^{-1}$	$\Delta\bar{F} = 30\ Nm^{-1}$
$\bar{L} = 9\ m$	$\Delta\bar{L} = 0.03\ m$
$\bar{E} = 7.5\times10^{9}\ Nm^{-2}$	$\Delta\bar{E} = 5\times10^{7}\ Nm^{-2}$
$\bar{I} = 0.0005\ m^4$	$\Delta\bar{I} = 0.000005\ m^4$

Soln:- Given,

$$\Delta y(\bar{F},\bar{L},\bar{E},\bar{I}) = \Delta\bar{F}\left|\frac{\partial y}{\partial F}\right| + \Delta\bar{L}\left|\frac{\partial y}{\partial L}\right| + \Delta\bar{E}\left|\frac{\partial y}{\partial E}\right| + \Delta\bar{I}\left|\frac{\partial y}{\partial I}\right|$$

$$\therefore y = \frac{FL^4}{8EI}$$

$$\frac{\partial y}{\partial F} = \frac{\bar{L}^4}{8\bar{E}\bar{I}} = \frac{(9)^4}{8\times(7.5\times10^9)\times0.0005} = \frac{6561}{3\times10^7}$$

$$= 2.187\times10^{-4}$$

$$\frac{\partial y}{\partial \bar{L}} = \frac{4\bar{F}\bar{L}^3}{8\bar{E}\bar{I}} = \frac{4\times750\times(9)^3}{8\times(7.5\times10^9)\times0.0005} = \frac{2187000}{3\times10^7}$$

$$= 0.0729$$

$$\frac{\partial y}{\partial \bar{E}} = \frac{\bar{F}\bar{L}^4}{8\bar{E}^2\bar{I}} = \frac{750\times(9)^4}{8\times(7.5\times10^9)^2\times0.0005} = \frac{4920750}{2.25\times10^{17}}$$

$$= 2.187\times10^{-11}$$

$$\frac{\partial y}{\partial \bar{I}} = \frac{\bar{F}\bar{L}^4}{8\bar{E}\bar{I}^2} = \frac{750\times(9)^4}{8\times(7.5\times10^9)\times(0.0005)^2} = \frac{4920750}{15000}$$

$$= 328.05$$

Now,

$$\Delta y(\bar{F},\bar{L},\bar{E},\bar{I}) = \Delta\bar{F}\left|\frac{\partial y}{\partial \bar{F}}\right| + \Delta\bar{L}\left|\frac{\partial y}{\partial \bar{L}}\right| + \Delta\bar{E}\left|\frac{\partial y}{\partial \bar{E}}\right| + \Delta\bar{I}\left|\frac{\partial y}{\partial \bar{I}}\right|$$

$$\Rightarrow \Delta y = (30\times2.187\times10^{-4}) + (0.03\times0.0729) + (5\times10^7\times2.187\times10^{-11})$$

$$+ (0.000005\times328.05)$$

$$\Rightarrow \Delta y = 0.006561 + 0.002187 + 0.0010935 + 0.00164$$

$$\therefore \Delta y = 0.011482$$

$$y = \frac{\bar{F}\bar{L}^4}{8\bar{E}\bar{I}} = \frac{750 \times (9)^4}{8 \times (7.5 \times 10^9) \times 0.0005} = 0.164025$$

$$y = 0.164025 \pm 0.011482$$

$$y \in [0.152543, 0.175507]$$

$\therefore$ Deflection y is between,

$$0.152543 \leq y \leq 0.175507$$

And,

$$y_{min} = \frac{\bar{F}_{min} \bar{L}^4_{min}}{8 \bar{E}_{max} \bar{I}_{max}} = \frac{720 \times (8.97)^4}{8 \times (7.55 \times 10^9) \times 0.000505}$$

$$= \frac{4661248.629}{30502000}$$

$$= 0.152818$$

$$y_{max} = \frac{\bar{F}_{max} \bar{L}^4_{max}}{8 \bar{E}_{min} \bar{I}_{min}} = \frac{780 \times (9.03)^4}{8 \times (7.45 \times 10^9) \times 0.000495}$$

$$= 0.175789 \simeq 0.175790$$

Thus, the first-order estimates are reasonably close to the exact values.

Date: 21.01.25

Problem-02: Use Taylor series expansions with $n=0$ to 6 to approximate $f(x)=\cos x$ at $x_{i+1}=\frac{\pi}{3}$ on the basis of the value of $f(x)$ and its derivatives at $x_i=\frac{\pi}{4}$

Solⁿ:- Given,

$$x_{i+1}=\frac{\pi}{3}$$

$$x_i=\frac{\pi}{4}$$

Here,

$$h=x_{i+1}-x_i$$

$$\Rightarrow h=\frac{\pi}{3}-\frac{\pi}{4}$$

$$\Rightarrow h=\frac{4\pi-3\pi}{12}$$

$$\therefore h=\frac{\pi}{12}$$

The Taylor Series is,

$$f(b)=f(a)+(b-a)f'(a)+\frac{(b-a)^2}{2!}f''(a)+\frac{(b-a)^3}{3!}f'''(a)+\frac{(b-a)^4}{4!}f''''(a)+\ldots\ldots$$

$$\Rightarrow f(x_{i+1})=f(x_i)+\frac{\pi}{12}f'(x_i)+\frac{(\pi/12)^2}{2!}f''(x_i)+\frac{(\pi/12)^3}{3!}f'''(x_i)+$$

$$+\frac{(\pi/12)^4}{4!} f''''(x_i) + \cdots\cdots\cdots$$

When, $n=0$,

$$f(x_{i+1}) = f(x_i)$$

$$\Rightarrow f\left(\frac{\pi}{3}\right) = f\left(\frac{\pi}{4}\right)$$

$$\Rightarrow f\left(\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{4}\right)$$

$$\therefore f(x_{i+1}) = 0.707107$$

True value,

$$f(x_{i+1}) = \cos\left(\frac{\pi}{3}\right) = 0.5$$

$$\text{Error, } E_1 = \left(\frac{|\text{True value} - \text{Approximation value}|}{\text{True value}}\right) \times 100\%$$

$$\Rightarrow \text{Error, } E_1 = \left(\frac{|0.5 - 0.7071071|}{0.5}\right) \times 100\%$$

$$\Rightarrow \text{Error, } E_1 = \left(\frac{|-0.2071071|}{0.5}\right) \times 100\%$$

$$\Rightarrow \text{Error, } E_1 = \left(\frac{0.207107}{0.5}\right) \times 100\%$$

$$\Rightarrow \text{Error, } E_1 = 0.4142 \times 100\%$$

$\therefore$ Error, $E_1 = 41.4\%$

When, $n=1$; $f(x_i) = \cos x$

$$f(x_{i+1}) = f(x_i) + \frac{\pi}{12} f'(x_i)$$

$$\Rightarrow f(x_i+1) = \cancel{f\left(\frac{\pi}{4}\right)} \cos x_i + \frac{\pi}{12}(-\sin x_i)$$

$$\Rightarrow f(x_i+1) = \cos\left(\frac{\pi}{4}\right) - \frac{\pi}{12} \sin\left(\frac{\pi}{4}\right)$$

$$\Rightarrow f(x_i+1) = 0.707107 - \frac{\pi}{12}(0.707107)$$

$$\Rightarrow f(x_i+1) = 0.707107 - 0.185120$$

$$\therefore f(x_i+1) = 0.521987$$

True value,

$$f(x_i+1) = \cos\left(\frac{\pi}{3}\right) = 0.5$$

$$\text{Error, } E_2 = \left(\frac{|\text{True value} - \text{Approximation value}|}{\text{True value}}\right) \times 100\%$$

$$\Rightarrow \text{Error, } E_2 = \left(\frac{|0.5 - 0.521987|}{0.5}\right) \times 100\%$$

$$\Rightarrow \text{Error, } E_2 = 0.044 \times 100\%$$

$$\therefore \text{Error, } E_2 = 4.4\%$$

order, n	Derivatives of f(x)	$f(x_{i+1}) = f(\cancel{\frac{\pi}{3}})(x_i)$	value of $f(x_{i+1})$	Error
0	$\cos x$	$f(x_{i+1}) = \cos(\frac{\pi}{4})$	0.707107	$\frac{\lvert 0.5-0.707107 \rvert}{0.5} \times 100\%$ $= 41.4\%$
1	$-\sin x$	$f(x_i) + \frac{\pi}{12} f'(x_i)$ $0.707107 - \frac{\pi}{12} \cdot \sin(\frac{\pi}{4})$	0.521987	$\frac{\lvert 0.5-0.521987 \rvert}{0.5} \times 100\%$ $= 4.4\%$
2	$-\cos x$	$0.521987 - \frac{(\pi/12)^2}{2!} \cos(\frac{\pi}{4})$	0.497754	$\frac{\lvert 0.5-0.497754 \rvert}{0.5} \times 100\%$ $= 0.449\%$
3	$\sin x$	$0.497754 + \frac{(\pi/12)^3}{3!} \sin(\frac{\pi}{4})$	0.499869	$\frac{\lvert 0.5-0.499869 \rvert}{0.5} \times 100\%$ $= 0.0262\%$
4	$\cos x$	$0.499869 + \frac{(\pi/12)^4}{4!} \cos(\frac{\pi}{4})$	0.50000740	$\frac{\lvert 0.5-0.50000740 \rvert}{0.5} \times 100\%$ $= 0.00148\%$
5	$-\sin x$	$0.500007 - \frac{(\pi/12)^5}{5!} \sin(\frac{\pi}{4})$	~~0.499999~~ 0.50000015	$\cancel{\frac{\lvert 0.5-0.499999 \rvert}{0.5} \times 100\% = 2\times10^{-4}}$ $\frac{\lvert 0.5-0.50000015 \rvert}{0.5} \times 100\%$ $= 3\times10^{-5}$
6	$-\cos x$	$0.50000015 - \frac{(\pi/12)^6}{6!} \cos(\frac{\pi}{4})$	0.49999983	$\frac{\lvert 0.5-0.49999983 \rvert}{0.5} \times 100\%$ $= 3.4\times10^{-5}$