Probability and Random Processes

Keivan Mallahi-Karai

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Jacobs University

Markov chains: motivation

sequence of independent identically distributed random variables $X_1, X_2, X_3, \ldots X_n,$

full history

$$\mathbb{P}[X_{n+1} = s_{n+1} \mid X_1 = s_1, X_2 = s_2, \ldots X_n = s_n]$$

$$= \mathbb{P}[X_{n+1} = s_{n+1}]$$

Sequence of independent random variable $\longrightarrow$ Markov Property.

Markov property

Definition

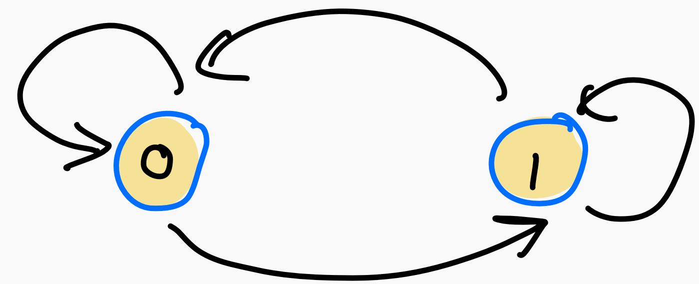
Let S be a set/ A sequence $X_1, X_2, \ldots$ random variables taking values in S is called a Markov chain if for all $n \geq 1$ and all $s_1, \ldots, s_n \in S$ we have

$$\mathbb{P}\left[X_n = s_n | X_{n-1} = s_{n-1}, \ldots, X_1 = s_1\right] = \mathbb{P}\left[X_n = s_n | X_{n-1} = s_{n-1}\right].$$

$$\mathbb{P}\left[X_n = s_n \,|\, X_1 = s_1, X_2 = s_2, \ldots\ldots X_{n-1} = s_{n-1}\right]$$

$$= \mathbb{P}\left[X_n = s_n\right]$$

A Markov chain with two states:



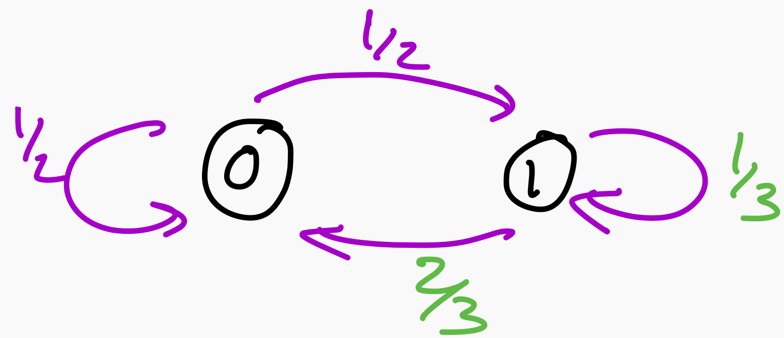
$X_1, X_2, \ldots$ -- Take values 1, 2

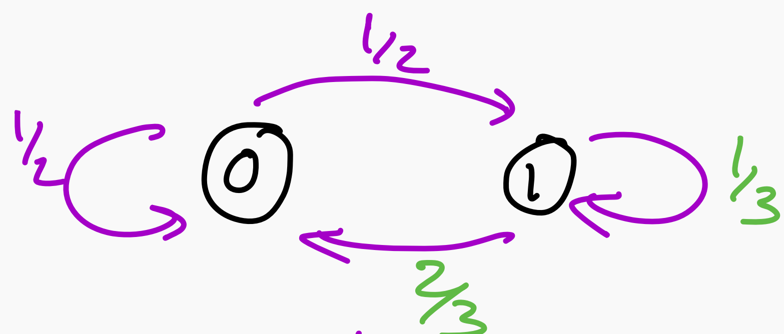
$$p_{00}^{[n]} = \mathbb{P}[X_{n+1}=0 \mid X_n=0]$$

$$p_{01}^{[n]} = \mathbb{P}[X_{n+1}=1 \mid X_n=0]$$

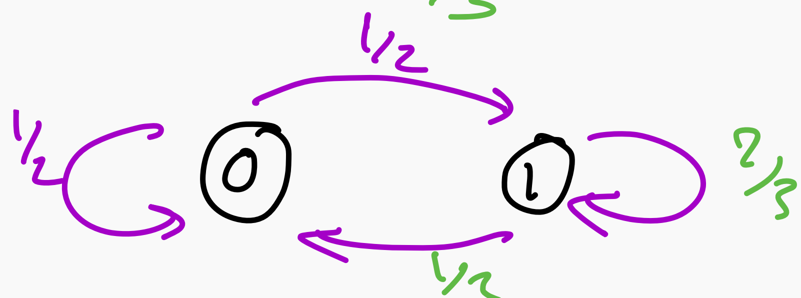
$$p_{11}^{[n]} = \mathbb{P}[X_{n+1}=1 \mid X_n=1]$$

$$p_{10}^{[n]} = \mathbb{P}[X_{n+1}=0 \mid X_n=1]$$





n even



n odd

Time homogenous Markov chains

$$y' = \lambda y$$

$$y' = \lambda y + \underline{f(t)}$$

$y(t)$

The transition matrix of a Markov chain

Definition

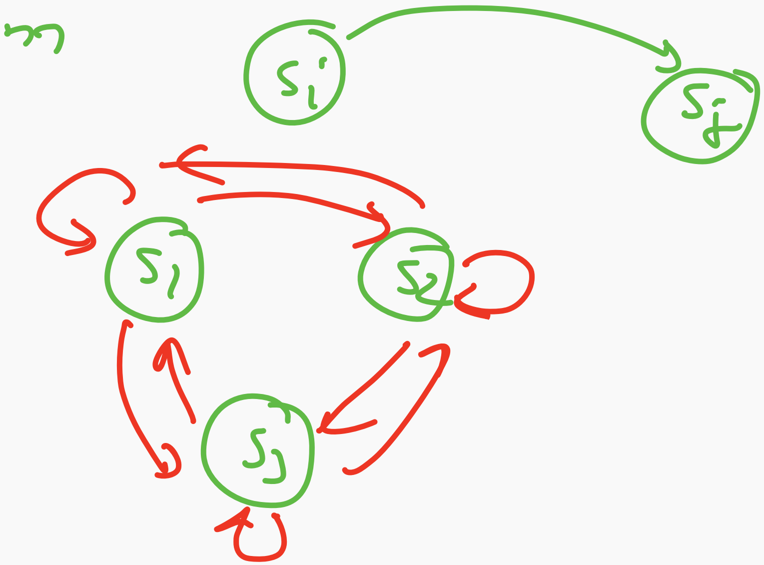
Let X_n be a time-homogenous Markov chain. For states s_i and s_j of the chain we write

$$p_{ij} = \mathbb{P}\left[X_n = s_j | X_{n-1} = s_i\right].$$

The matrix $P = [p_{ij}]$ is called the transition matrix of the Markov chain.

States : $s_1, s_2, \ldots, s_m$

$$\begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} p_{11} & p_{12} & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \end{array}$$

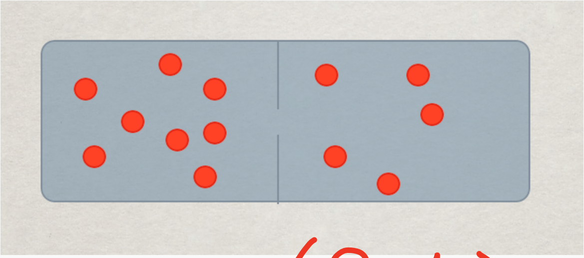


Transition matrix of a Markov chain with two states

Example: Ehrenfest urns

Consider N identical balls places into two urns.

$(8\ ,\ 5)$

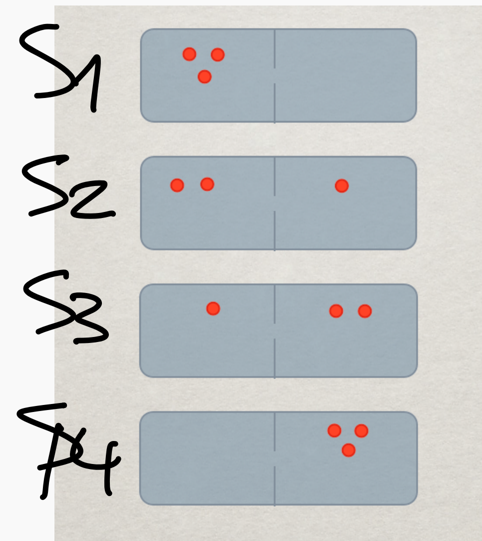


$(8,5) \to (9,4) \quad \frac{5}{13}$

$\quad\ \ \searrow (7,6) \quad \frac{8}{13}$

Moves: In transition from time $n-1$ to time n, one ball is selected randomly and is moved to the other urn.

$$P = \begin{array}{c|cccc} & s_1 & s_2 & s_3 & s_4 \\ \hline s_1 & 0 & 1 & 0 & 0 \\ s_2 & \frac{1}{3} & 0 & \frac{2}{3} & 0 \\ s_3 & 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ s_4 & 0 & 0 & 1 & 0 \end{array}$$



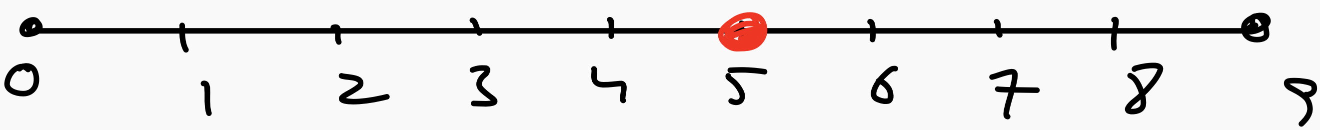
Example: modeling of weather

Suppose that weather in Bremen can be cloudy, sunny, or rainy. We will denote these possible states of weather by C, S, R, respectively. Assume that the state of the weather tomorrow can be reasonably predictable by the weather today (in fact, it has been shown that a weather forecast that announces tomorrow's weather to be the same is today's weather is about sixty percent of the days accurate!). Let us assume that the transition matrix of the chain is given by

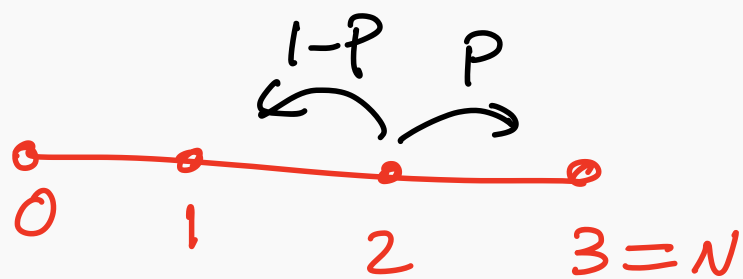
$$P = \begin{pmatrix} 1/6 & 1/3 & 1/2 \\ 1/8 & 3/4 & 1/8 \\ 1/4 & 1/4 & 1/2 \end{pmatrix}$$

A gambler has a wealth of n Euros and plays a game in which they win with probability p and lose with probability $1-p$. Quitting condition: reach N euros or go bankrupt.

5 euros



$$\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & & & & \\ 1 & & & & \\ 2 & & & & \\ 3 & & & & \end{array} \quad \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1-p & 0 & p & 0 \\ 0 & 1-p & 0 & p \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Transition matrix for gambler's ruin: $N = 4$

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1-p & 0 & p & 0 & 0 \\ 0 & 1-p & 0 & p & 0 \\ 0 & 0 & 1-p & 0 & p \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

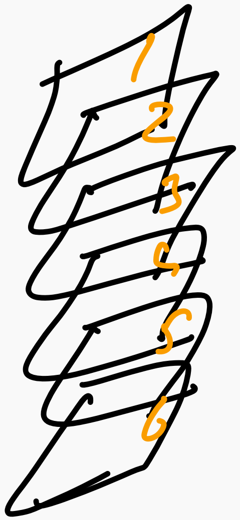
Coupon collector problem

\# coupon collected $= \{0, 1, 2, 3\}$

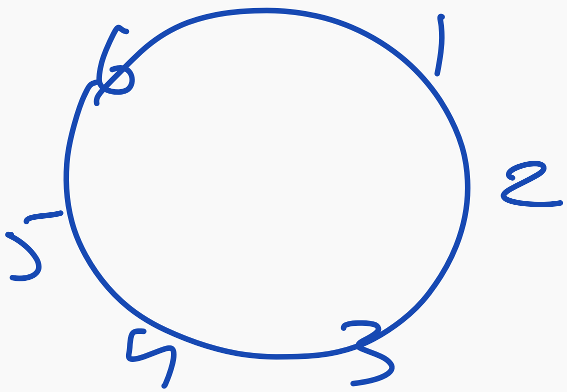
$$
\begin{array}{c c}
 & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} &
\begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & \frac{1}{3} & \frac{2}{3} & 0 \\
0 & 0 & \frac{2}{3} & \frac{1}{3} \\
0 & 0 & 0 & 1
\end{bmatrix}
\end{array}
$$

Shuffling cards

1 | 2 | 3 4 5 6



2 3 4 5 6 1
3 4 5 6 1 2
4 5 6 1 2 3
5 6 1 2 3 4
6 1 2 3 4 5



Properties of the transition matrix

Theorem

The transition matrix P of a Markov chain has the following two properties:

1. $p_{ij} \geq 0$, for all i, j.
2. $\sum_{j=1}^{n} p_{ij} = 1$ for all $i \in S$. In other words, each row of the matrix adds up to one.

$$P_{ij} = \mathbb{P}(X_n = s_j \mid X_{n-1} = s_i)$$

$$\sum_{j=1}^{N} P_{ij} = \sum_{j=1}^{N} \mathbb{P}(X_n = s_j \mid X_{n-1} = s_i) = 1.$$

Initial distribution of the Markov chain

Definition

Consider a Markov chain with n states $s_1, \ldots, s_n$. The initial distribution of the Markov chain is given by a vector $\pi = (\pi_1, \ldots, \pi_n)$ with $\pi_i \geq 0$ and $\sum_{i=1}^{n} \pi_i = 1$.

initial distribution

$$\pi_i = \mathbb{P}(X_0 = s_i)$$

$$\pi = (\pi_1, \pi_2, \ldots, \pi_N).$$

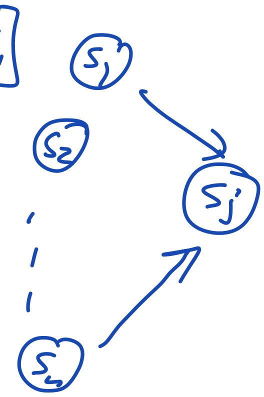
<u>of course</u>

deterministic initial state $\quad \pi = (0 \cdots 0, \overset{\downarrow}{1}, 0 \cdots 0)$

<u>Question</u> Suppose that the initial state X_0 of a Markov chain is given by the vector π.

what is the distribut of X_1 ??

$$P[X_1 = s_j] = \sum_{i=1}^{N} P[X_1 = s_j , X_0 = s_i]$$



$$= \sum_{i=1}^{N} P[X_1 = s_j \mid X_0 = s_i] \cdot \underbrace{P[X_0 = s_i]}$$

$$= \sum_{i=1}^{N} P_{ij} \cdot \pi_i = \sum_{i=1}^{N} \pi_i \cdot P_{ij}$$

$$P(A \cap B) = P(A|B) . P(B)$$

$$[\pi_1 \; \pi_2 \; \cdot \quad \pi_N] \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1N} \\ P_{21} & P_{22} & & \vdots \\ \vdots & \vdots & & \\ P_{N1} & P_{N2} & & P_{NN} \end{bmatrix} = [\, \_ \;\; \_ \qquad\qquad]$$

$P(X_1 = s_1)$ $\qquad\qquad$ $P(X_1 = s_N)$

distribut of X_0 $\quad \pi$

" $\quad X_1$ $\quad \pi . P$

Distribution after a move

Theorem

Suppose that the distribution of X_k is given by the row vector π. Then the distribution of X_{k+1} is given by the row vector πP.

$$\boxed{\pi = \left(\frac{1}{2}, \frac{1}{3}, \frac{1}{6}\right)}$$

↑ distribute of X_0

$$P = \begin{pmatrix} 1/6 & 1/3 & 1/2 \\ 1/8 & 3/4 & 1/8 \\ 1/4 & 1/4 & 1/2 \end{pmatrix}$$

distribute of X_1

$$\pi P = \left(\frac{1}{2} \quad \frac{1}{3} \quad \frac{1}{6}\right) \begin{pmatrix} 1/6 & 1/3 & 1/2 \\ 1/8 & 3/4 & 1/8 \\ 1/4 & 1/4 & 1/2 \end{pmatrix}$$

$$= \left(\frac{1}{12} + \frac{1}{24} + \frac{1}{24}, \frac{1}{6} + \frac{1}{4} + \frac{1}{24}, \frac{1}{4} + \frac{1}{24} + \frac{1}{12}\right)$$

$$\underline{\left(\frac{4}{24}, \frac{11}{24}, \frac{9}{24}\right)}$$

$$(\pi P) . P = \pi P^2.$$

$$\pi P^7$$

Transition in more than one steps

n large $\quad \pi P^n \longrightarrow$

$\pi, \ \pi P, \ \pi P^2, \ \pi P^3, \cdots \longrightarrow \pi^*$

$\pi' = (1 \ 0 \ 0)$

$\pi', \ \pi' P, \ \pi' P^2, \cdots \longrightarrow \pi^*$

π^* stationary distribution

Kolmogorov-Chapmen theorem

Theorem

Let P denote the transition matrix of a finite state Markov chain. Then

$$p_{ij}^{(n)} = (P^n)_{ij},$$

i.e. the multiple state transition probabilities are given by the entries of the powers of the transition matrix.