Probability and Random Processes

(3.1) A pair of fair dice are tossed. Let X denote the sum and Y be the maximum of the outcomes. For instance, if the outcomes are 1 and 4 then $X = 5$ and $Y = 4$.
(a) Find the probability mass functions of X and Y.
(b) Determin $\mathbb{E}[X]$ and $\mathbb{E}[Y]$.

Solution. The probability mass function for Y has been computed in problem (3.4) below. The probability mass function for X can be computed as follows. It is clear that X can take values from 2 to 12. Moreover,

$$\mathbb{P}[X = 2] = \mathbb{P}[X_1 = X_2 = 1] = \frac{1}{36}.$$

$$\mathbb{P}[X = 3] = \mathbb{P}[X_1 = 1, X_2 = 2] + \mathbb{P}[X_1 = 2, X_2 = 1] = \frac{2}{36}.$$

Similarly, we can compute $\mathbb{P}[X = j]$ for other values of j and record the results as follows

X	2	3	4	5	6	7	8	9	10	11	12
$P(X = x)$	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

Using the tables we can compute $\mathbb{E}[X]$ and $\mathbb{E}[Y]$:

$$\mathbb{E}[X] = 2 \cdot \frac{1}{36} + 3 \cdot \frac{2}{36} + \cdots + 12 \cdot \frac{1}{36} = 7$$
$$\mathbb{E}[Y] = 1 \cdot \frac{1}{36} + 2 \cdot \frac{2}{36} + \cdots + 6 \cdot \frac{11}{36} = \frac{161}{36}$$

(3.2) Suppose X is a discrete random variable taking values $x \in \{1, 2, 3, 4, 5\}$ such that

$$\mathbb{P}[X = x] = kx^2$$

for some k.
(a) Find the value of k
(b) Determine $\mathbb{P}[X > 2]$.

Solution.

$$1 = \sum_{x=1}^{5} \mathbb{P}[X = x] = \sum_{x=1}^{5} kx^2 = 55k.$$

Hence $k = 1/55$. This implies

$$\mathbb{P}[X > 2] = \frac{1}{55}(9 + 16 + 25) = \frac{50}{55} = \frac{10}{11}.$$

(3.3) Let X be a continuous random variable with the density function

$$f_X(x) = \begin{cases} 2x & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Determine the distribution function $F_X(t)$.
(b) Let $Y = X^2$. Find the distribution and density function of Y.

Solution. It is clear that for $t \le 0$ we have $F_X(t) = 0$ and for $t \ge 1$ we have $F_X(t) = 1$. For $0 \le t \le 1$ we have

$$F_X(t) = \int_0^t 2x \, dx = x^2 \Big|_0^t = t^2.$$

This gives

$$F_X(t) = \begin{cases} 0 & \text{if } t \le 0 \\ t^2 & \text{if } 0 \le t \le 1 \\ 1 & \text{if } t \ge 1 \end{cases}$$

First note that Y can only take values in $[0, 1]$. Moreover, for $0 \le t \le 1$ we have

$$F_Y(t) = \mathbb{P}[Y \le t] = \mathbb{P}\left[X^2 \le t\right] = \mathbb{P}\left[X \le \sqrt{t}\right] = (\sqrt{t})^2 = t.$$

For $t \le 0$ we have $F_Y(t) = 0$ and for $t \ge 1$ we have $F_Y(t) = 1$. Hence we have

$$F_X(t) = \int_0^t 2x \, dx = x^2 \Big|_0^t = t^2.$$

This gives

$$F_Y(t) = \begin{cases} 0 & \text{if } t \le 0 \\ t & \text{if } 0 \le t \le 1 \\ 1 & \text{if } t \ge 1 \end{cases}$$

Differentiating this gives

$$f_Y(t) = \begin{cases} 1 & \text{if } 0 \le t \le 1 \\ 0 & otherwise \end{cases}$$

(3.4) A fair die is thrown twice with outcomes X_1, X_2. Let $Y = \max(X_1, X_2)$. Find the probability mass function of X.

Solution. Note that since X_1 and X_2 can take integer values $1, 2, \ldots, 6$, it follows that Y can also takes values $1, 2, \ldots, 6$. If $Y = 1$ then $X_1 = X_2 = 1$, hence $\mathbb{P}[Y = 1] = \frac{1}{36}$. In a similar fashion, if $Y = i$ then either $X_1 = i$ and X_2 takes a values from $1, 2, \ldots, i$ or, conversely, $X_2 = i$ and X_1 takes a values from $1, \ldots, i$. Hence, for $1 \le i \le 6$, we have

$$\mathbb{P}[Y = i] = \frac{i}{36} + \frac{i}{36} - \frac{1}{36} = \frac{2i - 1}{36}.$$

In particular, the value are given by

$$\mathbb{P}[Y = 2] = \frac{3}{36}, \mathbb{P}[Y = 3] = \frac{5}{36}, \mathbb{P}[Y = 4] = 7/36, \mathbb{P}[Y = 5] = 9/36, \mathbb{P}[Y = 6] = 11/36.$$

(3.5) Let X be a continuous random variable with the density function

$$f_X(x) = \begin{cases} \lambda x^2 & \text{if } -2 < x < 2 \\ & \text{otherwise} \end{cases}$$

(a) Determine the value of λ.
(b) Find $\mathbb{P}[X > 1]$.

Solution. We have

$$1 = \int_{-\infty}^{+\infty} f(x)dx = \int_{-2}^{2} \lambda x^2 \, dx = 16\lambda/3$$

which implies $\lambda = 3/16$.

$$\mathbb{P}[X > 1] = \int_{1}^{\infty} f(x) \, dx = \int_{1}^{2} \frac{3}{16} x^2 \, dx = \frac{1}{16} x^3 \Big|_1^2 = \frac{7}{16}.$$

(3.6) Let X and Y be Bernoulli random variables with parameters p and q, respectively. Show that $Z := XY$ is a Bernoulli random variable with a parameter r with $p + q - 1 \leq r \leq \min(p, q)$.

Solution. Since X and Y can only take values 0 and 1 and $Z = XY$, it is clear that the only possible values of Z are $0 \cdot 0, 0 \cdot 1, 1 \cdot 0, 1 \cdot 1$, hence Z is itself a Bernoulli random variable with a parameter r. Also notes that $Z = 1$ iff $X = Y = 1$. Hence

$$r = \mathbb{P}[Z = 1] = \mathbb{P}[(X = 1) \cap (Y = 1)] \leq \mathbb{P}[X = 1] = p.$$

Similarly, $\mathbb{P}[Z = 1] \leq q$, hence $r \leq p$ and $r \leq q$, implying $r \leq \min(p, q)$. For the reverse inequality, note that

$$1 - r = \mathbb{P}[Z = 0] = \mathbb{P}[(X = 0) \cup (Y = 0)] \leq \mathbb{P}[X = 0] + \mathbb{P}[Y = 0] = (1 - p) + (1 - q)$$