Probability and Random Processes

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Announcements

1. Problem Set 6 is due today at 23:59.
2. Assessment phase will start tomorrow noon.
3. Problem set 7 will be posted today.

The Central limit theorem

Let $X_1, X_2, \ldots$ be i.i.d. with $\mathbb{E}[X_i] = \mu$ and $\mathrm{Var}[X_i] = \sigma^2$. Then the distribution of

$$Z_n = \frac{X_1 + \cdots + X_n - n\mu}{\sqrt{n}\sigma}$$

converges to the distribution of a standard normal distribution. In other words:

$$\lim_{n\to\infty} \mathbb{P}(a \leq Z_n \leq b) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-y^2/2} dy. \tag{1}$$

$$S_n = X_1 + X_2 + \cdots + X_n \qquad \frac{S_n}{n} \to \mu$$

$$\frac{S_n}{n} - \mu = \frac{S_n - n\mu}{n} \to 0 \qquad \frac{S_n - n\mu}{\sqrt{n}}$$

Application

24 numbers are randomly and independently chosen from the interval $[0, 1]$ according to the uniform distribution. Find the approximate value of the probability that the sum of the numbers is at least 8.

Moment generating functions of a random variable

Definition

Consider the following expected value:

$$M_X(t) = \mathbb{E}\left[e^{tX}\right].$$

It is called the moment generating functionfor X.

Example

Find the moment generating function for a Bernoulli random variable with parameter p.

X Bernoulli

$$X \begin{cases} 1 & p \\ 0 & 1-p \end{cases}$$

$$M_X(t) = \mathbb{E}[e^{tX}]$$

$$X=1 \Rightarrow e^{tX} = e^{t}$$

$$X=0 \Rightarrow e^{tX} = e^{t\cdot 0} = 1$$

$$M_X(t) = p \cdot e^{t} + (1-p) \cdot 1$$

$$= pe^{t} + 1 - p$$

Example

Find the moment generating function for a Poisson random variable with parameter λ.

Let X be a continuous random variable with the density function $f(x)$.

$$\boxed{M_X(t) = \mathbb{E}[e^{tX}] = \int_{-\infty}^{+\infty} e^{tx} \cdot f(x)\, dx}$$

X continuous RV with density $f(x)$

$$\mathbb{E}[h(X)] = \int_{-\infty}^{+\infty} h(x) f(x)\, dx$$

$i = \sqrt{-1}$ Fourier transform

$$\mathbb{E}[e^{itX}] = \int_{-\infty}^{+\infty} e^{itx} f(x)\, dx.$$

Show that the moment generating function for a standard normal random variable is given by

$$M_X(t) = e^{t^2/2}.$$

X standard normal $\quad f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$

$$M_X(t) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} e^{tx}\, dx = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x^2-2tx)} dx$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x^2-2tx+t^2)} e^{t^2/2}\, dx$$

$dx = dy$

$x - t = y$

$$= e^{t^2/2} \underbrace{\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-t)^2} dx} \to e^{t^2/2} \underbrace{\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy}_{1} = e^{t^2/2}.$$

Theorem

For independent random variables $X_1, X_2, \ldots, X_n$ we have

$$M_{X_1+\cdots+X_n}(t) = M_{X_1}(t)\cdots M_{X_n}(t).$$

$$M_{X_1+\cdots+X_n}(t) = \mathbb{E}\left[e^{t(X_1+\cdots+X_n)}\right]$$

$$= \mathbb{E}\left[e^{tX_1} \cdot e^{tX_2} \cdots e^{tX_n}\right]$$

$$= \mathbb{E}\left[e^{tX_1}\right] \cdots \mathbb{E}\left[e^{tX_n}\right]$$

$$= M_{X_1}(t) \cdots M_{X_n}(t).$$

$X_1, \ldots, X_n$ indep.

$$\mathbb{E}[f_1(X_1) \cdots f_n(X_n)]$$

$$= \mathbb{E}[f_1(X_1)] \cdots \mathbb{E}[f_n(X_n)]$$

Example

Find the moment generating function of a binomial random variable with parameters (n, p)

X binomial
count the number of success in n indep. Bernoulli trials
so prob. success p.

$$X = X_1 + X_2 + \cdots + X_n$$

$$\uparrow \quad \uparrow \qquad\qquad \uparrow$$

Bernoulli with parameter p / indep.

$$M_X(t) = M_{X_1}(t) \cdots M_{X_n}(t) = (pe^t + 1 - p)^n$$

Scaling

Theorem

If $Y = aX + b$, then

$$M_Y(t) = e^{bt} M_X(at).$$

$$M_Y(t) = \mathbb{E}[e^{tY}] = \mathbb{E}[e^{t(aX+b)}]$$

$$= \mathbb{E}[e^{atX+bt}] = \mathbb{E}[e^{bt} \cdot e^{atX}]$$

$$= e^{bt}\,\mathbb{E}[e^{atX}] = e^{bt} M_X(at).$$

Uniquness

Theorem

If X and Y are random variables with $M_X(t) = M_Y(t)$ then X and Y have the same distribution.

Let X and Y be independent random variables having Poisson distribution with parameters λ and μ, respectively. Then $X+Y$ has Poisson distribution with parameter $\lambda+\mu$.

$$M_X(t) = e^{\lambda(e^t-1)}$$

X Poisson with paramtr $\lambda \Rightarrow M_X(t) = e^{\lambda(e^t-1)}$

Y Poisson with paramtr $\mu \Rightarrow M_Y(t) = e^{\mu(e^t-1)}$

X, Y independent $\Rightarrow M_{X+Y}(t) = M_X(t)\, M_Y(t)$

$$= e^{\lambda(e^t-1)} \cdot e^{\mu(e^t-1)} = e^{(\lambda+\mu)(e^t-1)}$$

MGF for a Poisson RV with paramt $\lambda+\mu$

$\xRightarrow{\text{uniqueness Thm}}$ $X+Y$ has Poiss. dist with parameter $\lambda+\mu$.

The moment generating function of a sum of independent random variables

Toy version of the CLT

Assume $X_1, X_2, \dots$ are Bernoulli with parameter $1/2$

$$S_n = X_1 + \dots + X_n$$

$$\mu = E[X_i] = 1/2$$

$$\sigma^2 = Var[X_i] = E[X_i^2] - E[X_i]^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$$\boxed{\mu = \frac{1}{2}} \qquad \boxed{\sigma = \frac{1}{2}}$$

$$Z_n = \frac{X_1 + \dots + X_n - \frac{n}{2}}{\frac{1}{2}\sqrt{n}}$$

$$Z_n = \frac{X_1 + \cdots + X_n - n\mu}{\sqrt{n}\sigma}$$

n large distribution of $Z_n \longrightarrow$ distribution of a standard normal RV

$$Z_n = \underbrace{(X_1 + \dots + X_n)}_{S_n}\frac{2}{\sqrt{n}} - \sqrt{n} = \overset{a}{\frac{2}{\sqrt{n}}} S_n \overset{b}{\boxed{-\sqrt{n}}}$$

$Y = aX + b$

$$M_{Z_n}(t) = e^{-\sqrt{n}t} M_{S_n}\left(\frac{2}{\sqrt{n}}t\right)$$

$$M_Y(t) = e^{bt} M_X(at).$$

$$\left(e^{-\frac{t}{\sqrt{n}}}\right)^n = e^{-\sqrt{n}\,t} \cdot \left(\frac{1}{2}e^{\frac{2}{\sqrt{n}}t} + \frac{1}{2}\right)^n$$

$$M_{S_n}(t) = M_{X_1+\cdots+X_n}(t) = \left(\frac{1}{2}e^{t} + \frac{1}{2}\right)^n$$

$$= \left(\frac{1}{2}e^{+\frac{t}{\sqrt{n}}} + \frac{1}{2}e^{-\frac{t}{\sqrt{n}}}\right)^n \qquad \underline{n \to \infty}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$= \left(\frac{1}{2}\left(1 + \frac{t}{\sqrt{n}} + \frac{t^2}{2n} + \frac{t^3}{6n^{3/2}} + \cdots\right) + \frac{1}{2}\left(1 - \frac{t}{\sqrt{n}} + \frac{t^2}{2n} - \frac{t^3}{6n^{3/2}} + \cdots\right)\right)^n$$

$$= \left(1 + \frac{t^2}{2n} + \frac{t^4}{24n^2} + \cdots\right)^n \approx \left(1 + \frac{t^2}{2n}\right)^n$$

$$\left(1 + \frac{x}{n}\right)^n \to e^x$$

$$\xrightarrow{n \to \infty} \quad e^{t^2/2} = M_N(t)$$

$$\left(1 + \frac{1}{n}\right)^n \longrightarrow e$$

$x > 0$

$$\left(1 + \frac{1}{n}\right)^{n^2} \longrightarrow \infty$$

$$\left(1 + \frac{1}{n}\right)^{\sqrt{n}} \longrightarrow 1$$

<u>Berry – Esseen Theorem</u> (error term in the CLT)

$$\left| P(a \le Z_n \le b) - \frac{1}{2\pi}\int_a^b e^{-\frac{1}{2}t^2}\,dt \right| \le \frac{\mathbb{E}[|X_i|^3]}{\sqrt{n}\,\sigma^3}$$

Proof of the Central limit theorem

$X_1, X_2, \ldots$ independent RV

$$\begin{cases} \mathbb{E}[X_i] = 0. \\ \mathrm{Var}[X_i] = 1 \end{cases}$$

Compute the MGF of $Z_n = \dfrac{X_1 + \cdots + X_n - n\mu}{\sqrt{n}\,\sigma}$

$$= \frac{X_1 + \cdots + X_n}{\sqrt{n}}$$

$$M_{Z_n} = \mathbb{E}\left[e^{t\left(\frac{X_1 + \cdots + X_n}{\sqrt{n}}\right)}\right] = \mathbb{E}\left[e^{\frac{tX_1}{\sqrt{n}}} \cdots e^{\frac{tX_n}{\sqrt{n}}}\right] = \underbrace{M_{X_1}}\left(\frac{t}{\sqrt{n}}\right)^n$$

$$\underline{\underline{h}} = M_{X_1} \qquad = h\left(\frac{t}{\sqrt{n}}\right)^n$$

$$h(t) = \mathbb{E}[\underbrace{e^{tX}}] = \mathbb{E}\left[1 + tX + \frac{t^2X^2}{2!} + \cdots\right]$$

$$= 1 + t\,\underbrace{\mathbb{E}[X]}_{0} + \frac{t^2}{2}\underbrace{\mathbb{E}[X^2]}_{1} + \frac{t^3}{3!}\underline{\underline{\mathbb{E}[X^3]}} + \cdots$$

$$= \left(1 + \frac{1}{2}\left(\frac{t}{\sqrt{n}}\right)^2 + \frac{1}{6}\left(\frac{t}{\sqrt{n}}\right)^3 \mathbb{E}[X^3]\right)^n$$

ignore!

$$= \left(1 + \frac{t^2}{2n} + \frac{t^3}{6!\, n^{3/2}} E[x^3] + \cdots \right)^n$$

$$\xrightarrow[n \to \infty]{} e^{t^2/2}$$

$$\left(1 + \frac{t^2}{2n} + \frac{t^3}{6 n^{3/2}} \right)^n$$

$$\left(1 + \frac{a}{n} \right)^n \longrightarrow e^a$$

$$\left(1 + \frac{a}{n^{3/2}} \right)^{n^{3/2}} \longrightarrow e^a$$