Probability and Random Processes

(5.1) For $\alpha > 1$, suppose that X has the density function given by

$$f_X(t) = \begin{cases} \alpha e^{-\alpha t} & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Compute $\mathbb{E}\left[e^X\right]$.

Solution. Setting $h(x) = e^x$, we have

$$\mathbb{E}\left[e^X\right] = \int_0^\infty \alpha e^t e^{-\alpha t} dt = \int_0^\infty \alpha e^{-(\alpha-1)t} dt = \frac{\alpha}{\alpha - 1}.$$

(5.2) A die has been rolled twice. Let X denote the outcome of the first throw and Y denote the smaller of the two outcomes. For instance, if the outcomes are $2, 3$ then $X = 2$ and $Y = 2$ and if the outcomes are $4, 3$ then $X = 4$ and $Y = 3$.

(a) Describe the joint probability mass function of X and Y by drawing a table.
(b) Compute the marginal probability mass functions of X and Y.
(c) What are the possible values of $Z = X - Y$? Compute the probability mass function of Z and use it to find $\mathbb{E}[Z]$.

Solution. Note that X can take values $1, 2, 3, 4, 5, 6$ and Y can also take the same values. It is clear that no matter the outcome we have $Y \leq X$. Let us compute $\mathbb{P}[X = i, Y = j]$. For this to be possible, we must have $j \leq i$. Suppose this condition is satisfied. If $j < i$, then this is only possible if the first die is i and the second one is j. This has probability $1/36$. If $i = j$. Then there is one option for the outcome of the first die (namely i) and exactly $7 - i$ options for the outcome of the second die. Hence

$$\mathbb{P}[X = i, Y = i] = \frac{7 - i}{36}.$$

We can now form the table

	$Y = 1$	$Y = 2$	$Y = 3$	Y=4$	$Y = 5$	$Y = 6$
$X = 1$	6/36	0	0	0	0	0
$X = 2$	1/36	5/36	0	0	0	0
$X = 3$	1/36	1/36	4/36	0	0	0
$X = 4$	1/36	1/36	1/36	3/36	0	0
$X = 5$	1/36	1/36	1/36	1/36	2/36	0
$X = 6$	1/36	1/36	1/36	1/36	1/36	1/36

It is clear that X takes each values $1, 2, \ldots, 6$ with probability $1/6$, so its marginals are simply $p_X(i) = 1/6$ for all $1 \leq i \leq 6$. For Y we have

$$p_Y(1) = \frac{11}{36}, \quad p_Y(2) = \frac{9}{36}, \quad p_Y(3) = \frac{7}{36}, \quad p_Y(4) = \frac{5}{36}, \quad p_Y(5) = 3/36, \quad p_Y(6) = 1/36.$$

(c) It is clear from the table that the possible values of $Z = X - Y$ are $0, 1, 2, 3, 4, 5$. We can seee

$$\mathbb{P}[Z = 0] = \mathbb{P}[X - Y = 0] = \sum_{i=1}^{6} \mathbb{P}[X = i, Y = i] = \sum_{i=1}^{6} p_{X,Y}(i, i) = \frac{21}{36}.$$

For $k = 1, 2, 3, 4, 5$ we have

$$\mathbb{P}[Z = k] = \mathbb{P}[X - Y = k] = \sum_{i=1}^{6-k} \mathbb{P}[X = i + k, Y = i] = \sum_{i=1}^{6-k} p_{X,Y}(i + k, i) = \sum_{i=1}^{6-k} \frac{1}{36} = \frac{6 - k}{36}.$$

These values are given in the following table

	$Z = 0$	$Z = 1$	$Z = 2$	$Z = 3$	$Z = 4$	$Z = 5$
$p_Z(z)$	21/36	5/36	4/36	3/36	2/36	1/36

(5.3) A coin is flipped three times. Let X denote the number of heads and Y denote the number of streaks of heads of length 2. For instance, if the outcome is HTH, then $X = 2$ and $Y = 0$, while if the outcome is HHT, then $X = 2$ and $Y = 1$.

(a) Find the joint probability mass function of X and Y.
(b) Determine $\mathrm{Cov}(X, Y)$.
(c) Are X and Y independent?

Solution. It is clear that $0 \leq X \leq 3$ and $0 \leq Y \leq 2$. If $Y = 2$, then clearly $X = 3$, and this only happens when the outcome is HHH, hence

$$\mathbb{P}[X = 3, Y = 2] = \frac{1}{8}, \quad \mathbb{P}[X = j, Y = 2] = 0 j = 0, 1, 2.$$

Consider $Y = 1$, this corresponding to two outcomes HHT and THH. Hence

$$\mathbb{P}[X = 2, Y = 1] = \frac{2}{8}, \quad \mathbb{P}[X = j, Y = 1] = 0, \quad j = 0, 1, 3.$$

Finally assume that $Y = 0$. Then X can take any of values 0,1, 2. We have $X = 0$ for TTT, we have $X = 1$ for HTT, THT, TTH, and we have $X = 2$ for THT. Hence

$$\mathbb{P}[X = 0, Y = 0] = \frac{1}{8}, \quad \mathbb{P}[X = 1, Y = 0] = \frac{3}{8}, \quad \mathbb{P}[X = 2, Y = 0] = \frac{1}{8}.$$

These numbers can be summarized in the following table:

	$Y = 0$	$Y = 1$	$Y = 2$
$X = 0$	1/8	0	0
$X = 1$	3/8	0	0
$X = 2$	1/8	1/4	0
$X = 3$	0	0	1/8

Since X has a binomial distribution with $n = 3$ and $p = 1/2$ we have $\mathbb{E}[X] = 3/2$. A simple computation shows that

$$\mathbb{E}[Y] = 1 \cdot \frac{1}{4} + 2 \cdot \frac{1}{8} = \frac{1}{2}.$$

Also

$$\mathbb{E}[XY] = 2 \cdot \frac{1}{4} + 6 \cdot \frac{1}{8} = \frac{5}{4}.$$

Hence

$$\mathrm{Cov}(X, Y) = \frac{5}{4} - \frac{1}{2}\frac{3}{2} = \frac{1}{2} > 0$$

from which it follows that X and Y are not independent.

(5.4) If X and Y are two random variables prove that

$$\mathrm{Var}[X+Y] = \mathrm{Var}[X] + \mathrm{Var}[Y] + 2\,\mathrm{Cov}(X,Y).$$

Solution.

$$\mathrm{Var}[X+Y] = \mathbb{E}\left[(X+Y)^2\right] - (\mathbb{E}[X]+\mathbb{E}[Y])^2 = \mathbb{E}\left[X^2+Y^2+2XY\right] - (\mathbb{E}[X]^2+\mathbb{E}[Y]^2+2\mathbb{E}[X]\,\mathbb{E}[Y]).$$

Hence

$$\mathrm{Var}[X+Y] = \mathbb{E}\left[X^2\right] - \mathbb{E}[X]^2 + \mathbb{E}\left[Y^2\right] - \mathbb{E}[Y]^2 + 2(\mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y]) = \mathrm{Var}[X] + \mathrm{Var}[Y] + 2\,\mathrm{Cov}(X,Y).$$

(5.5) Let

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

be a 2×2 matrix where A_{ij} are independent and each is uniformly chosen from the set $\{1,2,3,4,5\}$. Set $D = \det A$. Find $\mathrm{Var}[D]$.

Solution. Note that

$$D = A_{11}A_{22} - A_{12}A_{21}.$$

A simple computation shows that $\mathbb{E}[A_{ij}] = 3$ and

$$\mathbb{E}\left[A_{ij}^2\right] = \frac{1}{5}\left(1^2+2^2+\cdots+5^2\right) = 11.$$

Using the independence of A_{ij} we have

$$\mathbb{E}[D] = \mathbb{E}[A_{11}A_{22} - A_{12}A_{21}] = \mathbb{E}[A_{11}]\,\mathbb{E}[A_{22}] - \mathbb{E}[A_{12}]\,\mathbb{E}[A_{21}] = 0.$$

We also hae

$$\mathbb{E}\left[D^2\right] = \mathbb{E}\left[(A_{11}A_{22} - A_{12}A_{21})^2\right] = \mathbb{E}\left[A_{11}^2A_{22}^2 + A_{12}^2A_{21}^2 - 2A_{11}A_{22}A_{12}A_{21}\right].$$

Using independence of A_{ij} we have

$$\mathbb{E}\left[A_{11}^2A_{22}^2\right] = \mathbb{E}\left[A_{11}^2\right]\mathbb{E}\left[A_{22}^2\right] = 11\times 11 = 121.$$

Similarly we have $\mathbb{E}\left[A_{12}^2A_{21}^2\right] = 121$. Finally, we using independence again we have

$$\mathbb{E}[A_{11}A_{22}A_{12}A_{21}] = \mathbb{E}[A_{11}]\,\mathbb{E}[A_{22}]\,\mathbb{E}[A_{12}]\,\mathbb{E}[A_{21}] = 3^4 = 81.$$

It follows that

$$\mathbb{E}\left[D^2\right] = 121 + 121 - 162 = 80.$$

Hence $\mathrm{Var}[D] = \mathbb{E}\left[D^2\right] - \mathbb{E}[D]^2 = 80$.