Probability and Random Processes

(4.1) Consider a discrete random variable X with the probability mass function given by

$$p_X(x) = \begin{cases} k \cdot 2^x & \text{if } x = -1, 0, 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Compute the value of k.
(b) Find $\mathbb{P}[X \text{ is even}]$.
(c) Determine $\mathbb{E}[X]$.

Solution. (a) We have

$$1 = \sum_{x \in \{-1,0,1,2\}} k \cdot 2^x = \frac{15}{2}k.$$

From here it follows that $k = \frac{2}{15}$.

(b) We have

$$\mathbb{P}[X \text{ is even}] = \frac{2}{15}(1+4) = \frac{2}{3}.$$

(c) Note that $Y = X^2$ takes values $0, 1, 4$, and

$$\mathbb{P}[Y = 1] = \mathbb{P}[X = 1] + \mathbb{P}[X = -1] = \frac{1}{2}.$$

Similarly,

$$\mathbb{P}[Y = 0] = \mathbb{P}[X = 0] = \frac{2}{15}.$$

Finally,

$$\mathbb{P}[Y = 4] = 1 - \frac{2}{15} - \frac{1}{2} = \frac{3}{5}.$$

In a similar fashion, one can see that the probability mass function of Z is given by

$$p_Z(0) = \frac{1}{15}, \quad p_Z(1) = \frac{2}{15}, \quad p_Z(2) = \frac{4}{15}, \quad p_Z(3) = \frac{8}{15}.$$

(4.2) Suppose X is a discrete random variable with $\mathbb{E}[X] = 5$ and $\mathrm{Var}[X] = 15$. Find the values of $\mathbb{E}[X^2]$, $\mathbb{E}[2 - X]$, $\mathrm{Var}[3X + 1]$.

Solution. Note that

$$\mathbb{E}\left[X^2\right] = \mathrm{Var}[X] + \mathbb{E}[X]^2 = 15 + 25 = 40.$$

Also

$$\mathbb{E}[2 - X] = 2 - \mathbb{E}[X] = -3.$$

and

$$\mathrm{Var}[3X + 1] = \mathrm{Var}[3X] = 9\mathrm{Var}[X] = 135.$$

(4.3) Let X be a Poisson random variable with parameter λ. Show that $\mathbb{E}[X] = \lambda$.

Solution.

$$\mathbb{E}[X] = \sum_{i=0}^{\infty} i \cdot e^{-\lambda}\frac{\lambda^i}{i!} = e^{-\lambda}\sum_{1=1}^{\infty}\frac{\lambda^i}{(i-1)!} = \lambda e^{-\lambda}\sum_{j=0}^{\infty}\frac{\lambda^j}{j!} = \lambda.$$

Here the last equality uses $\sum_{j=0}^{\infty}\frac{\lambda^j}{j!} = e^{\lambda}$ and we have used the substitution $j = i - 1$.

(4.4) The number of items arriving daily in the lost-and-found section of Bremen train station has a Poisson distribution with parameter $\lambda = 4$.

(a) Find the probability of the event that on a given day no items arrive.

(b) Knowing that at least one item has arrived today, find the probability that there is at least one more item arriving today?

(c) Let W denote the number of items arriving from January 1st to January 15th. What is $\mathbb{E}[W]$?

Solution. Let us denote the number of items arriving on a given day by X. Since X has Poisson distribution with $\lambda = 4$ we have

$$\mathbb{P}[X = 0] = e^{-4}\frac{\lambda^0}{0!} = e^{-4}.$$

For part (b) we are interested in

$$\mathbb{P}[X \geq 2 | X \geq 1] = \frac{\mathbb{P}[X \geq 2]}{\mathbb{P}[X \geq 1]}.$$

Note that

$$\mathbb{P}[X \geq 1] = 1 - \mathbb{P}[X = 0] = 1 - e^{-4}.$$

Similarly,

$$\mathbb{P}[X \geq 2] = 1 - \mathbb{P}[X = 0] - \mathbb{P}[X = 1] = 1 - e^{-4} - 4e^{-4} = 1 - 5e^{-4}.$$

Hence

$$\mathbb{P}[X \geq 2 | X \geq 1] = \frac{1 - 5e^{-4}}{1 - e^{-4}}.$$

If X_i denotes the number of items arriving on ith day of January, then we know that $\mathbb{E}[X_i] = \lambda = 4$. Hence

$$\mathbb{E}[X_1 + \cdots + X_{15}] = \sum_{i=1}^{15}\mathbb{E}[X_i] = 15 \times 4 = 60.$$

(4.5) There are 5 people in an elevators. An elevator goes up a building with 10 floors and stops at each floor where at least one person wants to get off. If X denote the number of stops, find $\mathbb{E}[X]$.

Hint: For each $1 \leq i \leq n$, set X_i to denote the Bernoulli random variable defined by $X_i = 1$ when the elevator stops at the i-th floor. Use $X = X_1 + \cdots + X_{10}$.

Solution. Set X_i to denote the Bernoulli random variable defined by $X_i = 1$ when the elevator stops at the i-th floor. Set $X = X_1 + \cdots + X_{10}$. Note that $X_i = 0$ if none of the 5 people in the elevator wants to get off at the i-th floor. This mens that

$$\mathbb{P}[X_i = 1] = 1 - \mathbb{P}[X_i = 0] = 1 - (9/10)^5.$$

Since X_i is a Bernoulli random variables with parameter $p = 1 - (9/10)^5$, it follows that

$$\mathbb{E}[X] = \sum_{i=1}^{10} \mathbb{E}[X_i] = 10 \left(1 - (9/10)^5\right) \approx 4.09$$