Probability and Random Processes

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Conditional probability mass function

Definition)

Let X and Y be discrete random variables. The conditional probability mass function of X given $Y = y$ is defined by

$$p_{X|Y}(x|y) = \frac{\mathbb{P}[X = x, Y = y]}{\mathbb{P}[Y = y]} = \frac{p_{X,Y}(x,y)}{p_Y(y)}.$$

$$\overset{\shortparallel}{\mathbb{P}[X=x \mid Y=y]}$$

Remark: This is only defined when $p_Y(y) = \mathbb{P}[Y = y] > 0$.

$$p_{X,Y}(x,y) = \mathbb{P}[X=x, Y=y]$$

$$p_{X|Y}(x|y) = \frac{p_{X,Y}(x,y)}{p_Y(y)} \Rightarrow \sum_x p_{X|Y}(x|y) = \frac{\sum_x p_{X,Y}(x,y)}{p_Y(y)} = 1$$

$$p_Y(y) = \sum_x p_{X,Y}(x,y)$$

Examples

Let X and Y be chosen randomly from the set $\{-1, 0, 1\}$ such that the joint probability mass function of X and Y is given by

	$Y = -1$	$Y = 0$	$Y = 1$	
$X = -1$	$1/10$	0	$1/10$	$2/10$
$X = 0$	$1/10$	$2/10$	$2/10$	$5/10$
$X = 1$	$3/10$	0	0	$3/10$
	$5/10$	$2/10$	$3/10$	

Find the conditional probability mass functions of X given Y and Y given X.

$X \mid Y$	$Y=-1$	$Y=0$	$Y=1$
$X=-1$	$1/5$	0	$1/3$
$X=0$	$1/5$	1	$2/3$
$X=1$	$3/5$	0	0

	$Y=-1$	$Y=0$	$Y=1$
$X=-1$	$1/2$	0	$1/2$
$X=0$	$1/5$	$2/5$	$2/5$
$X=1$	1	0	0

Example

A fair coin is flipped three times. Let N denote the number of Heads and S denote the length of the longest streak of Heads. Determine the joint probability mass function of N given S and S given N.

Connection to independence

Theorem

Discrete random variable X and Y are independent if for all values y of Y we have

$$p_{X|Y}(x, y) = p_X(x).$$

Conditional expectation

Definition)

Let X and Y be a random variable. The conditional expectation of X given $Y = y$ is defined by

$$\mathbb{E}\left[X|Y = y\right] = \sum_{x_i} x_i p_{X|Y}(x, y).$$

$X \quad x_1 \quad \cdots \quad x_n$

$$\mathbb{E}[X] = \sum x_i \cdot p_X(x_i)$$

$$\mathbb{E}[X|Y=y] = \sum x_i \cdot \mathbb{P}[X=x_i|Y=y]$$

Let X and Y be random variables with the joint probability mass function given by

	$Y=-1$	$Y=0$	$Y=1$	
$X=-1$	$1/10$	0	$1/10$	$2/10$
$X=0$	$1/10$	$2/10$	$2/10$	$5/10$
$X=1$	$3/10$	0	0	$3/10$

Find $\mathbb{E}[X|Y=y]$.

X|Y	Y=-1	Y=0	Y=1
X=-1	$1/5$	0	$1/3$
X=0	$1/5$	1	$2/3$
X=1	$3/5$	0	0

	Y=-1	Y=0	Y=1
X=-1	$1/2$	0	$1/2$
X=0	$1/5$	$2/5$	$2/5$
X=1	1	0	0

$$\mathbb{E}[X] = \frac{-2}{10} + 0 + \frac{3}{10} = \frac{1}{10}.$$

$$\mathbb{E}[X \mid Y=-1] = \frac{1}{5}\cdot(-1) + \frac{1}{5}\cdot 0 + \frac{3}{5}\cdot(1) = \frac{2}{5}$$

$$\mathbb{E}[X \mid Y=0] = 0\cdot(-1) + 1\cdot 0 + 0\cdot 1 = 0$$

$$\mathbb{E}[X \mid Y=1] = 1/3\cdot(-1) + 2/3\cdot 0 + 0\cdot 1 = -1/3$$

Conditional expectation as a random variable

Definition

Consider the function

$$y \mapsto \mathbb{E}[X|Y=y].$$

This map can be viewed as a random variable. This random variable is denoted by $\mathbb{E}[X|Y]$.

Y	-1	0	1
$\mathbb{E}[X\|Y]$	$\frac{2}{5}$	0	$\frac{-1}{3}$

what are the values of $\mathbb{E}[X|Y]$: $\frac{2}{5}, 0, \frac{-1}{3}$

$\mathbb{P}[\mathbb{E}[X|Y]=\frac{2}{5}] = \mathbb{P}[Y=-1] = 5/10$

$\mathbb{P}[\mathbb{E}[X|Y]=0] = 2/10$

$\mathbb{P}[\mathbb{E}[X|Y]=\frac{-1}{3}] = 3/10$

$$\mathbb{E}[\mathbb{E}[X|Y]] = \frac{5}{10} \cdot \frac{2}{5} + \frac{2}{10} \cdot 0 + \frac{3}{10} \cdot \frac{-1}{3}$$

$$= \frac{2}{10} - \frac{1}{10} = \frac{1}{10}$$

$$= \mathbb{E}[X]$$

$$\mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}[X]$$

Conditional expectation and expectation

Theorem (Law of iterated expectations)

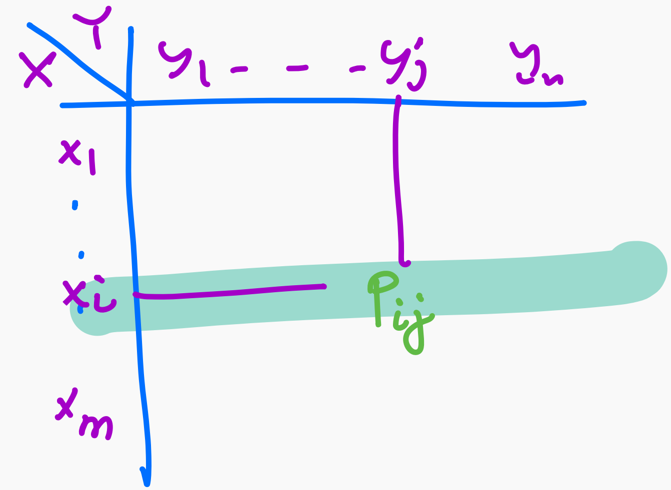
For discrete random variables X and Y we have

$$\mathbb{E}\left[\mathbb{E}\left[X|Y\right]\right] = \mathbb{E}\left[X\right].$$

Proofs

$$p_{ij} = \mathbb{P}[X = x_i, Y = y_j]$$

$$= p_{X,Y}(x_i, y_j)$$



$$\mathbb{E}[X|Y=y_j] = \sum x_i \cdot p_{X|Y}(x|y_j)$$

$$= \sum_{i=1}^{m} x_i \cdot \frac{p_{X,Y}(x, y_j)}{p_Y(y_j)} = \frac{1}{p_Y(y_j)} \sum_{i=1}^{m} x_i \cdot p_{X,Y}(x_i, y_j)$$

$$\mathbb{E}[\mathbb{E}[X|Y]] =$$

$$\sum_{j=1}^{n} \cancel{P_Y(y_j)} \cdot \frac{1}{\cancel{P_Y(y_j)}} \sum_{i=1}^{m} x_i \, P_{X,Y}(x_i, y_j)$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{m} x_i \, P_{X,Y}(x_i, y_j)$$

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_i \, P_{X,Y}(x_i, y_j) =$$

$$= \sum_{i=1}^{m} x_i \sum_{j=1}^{n} P_{X,Y}(x_i, y_j)$$

$$= \sum_{i=1}^{m} x_i \cdot P_X(x_i) = \mathbb{E}[X].$$

Example

Michael chooses an integer Y randomly from the set $\{1, 2, \ldots, 100\}$. He then chooses an integer X from the set $\{1, 2, \ldots, Y\}$. Find $\mathbb{E}[X]$.

direct $E[X]$ computationally
computi heavy

X	1	2	3	. . .	100

$$E[X] = E\left[E[X|Y]\right]$$

$E[X|Y=j]$ possible values of X are $1, 2, \ldots, j$
Probably of each $= 1/j$

$$E[X|Y=j] = 1 \cdot \frac{1}{j} + 2 \cdot \frac{1}{j} + \cdots + j \cdot \frac{1}{j}$$

$$= \frac{1}{j}\left(\underbrace{1+2+3+\cdots+j}\right)$$

$$\boxed{1+2+3+\cdots+N = \frac{N(N+1)}{2}}$$

$$\rightarrow \frac{1}{j}\left(\frac{j(j+1)}{2}\right) = \frac{j+1}{2}.$$

$$\mathbb{E}[X \mid Y=j] = \frac{j+1}{\underset{\sim}{2}}.$$

$$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X \mid Y]]$$

$\mathbb{E}[X \mid Y]$	$\frac{1+1}{2}$	$\frac{2+1}{2}$	$\cdots$	$\frac{100+1}{2}$
Prob.	$\frac{1}{100}$	$\frac{1}{100}$	$\cdots$	$\frac{1}{100}$

$$= \frac{1}{100}\cdot\left(\frac{1+1}{2} + \frac{1+2}{2} + \cdots + \frac{1+100}{2}\right)$$

$$= \frac{1}{200}\left[2+3+\cdots+101\right]$$

$$= \frac{1}{200}\cdot\left(\frac{101\times 102}{2} - 1\right)$$

$$= \frac{1}{200}\left[5150\right] = \frac{515}{20} \approx 25.75$$

Example

Tamara chooses a random number N from the set $\{1, 2, \ldots, 5\}$ and then throws a fair die N times and records the numbers shown. Let S denote the sum of these numbers. Determine $\mathbb{E}[S]$?

$$\mathbb{E}[S] = \mathbb{E}[\mathbb{E}[S|N]]$$

$$\mathbb{E}[S|N=1] = \frac{1+2+3+4+5+6}{6} = \underline{3.5}$$

$$\mathbb{E}[S|N=2] = \mathbb{E}[Y_1 + Y_2 | N=2] = 7$$

$$\mathbb{E}[S|N=j] = 3.5 \times j$$

$$\mathbb{E}[S] = \mathbb{E}[\mathbb{E}[S|N]] = \sum_{j=1}^{5} (3.5 \times j) \times \frac{1}{5}$$

$$= \frac{3.5}{5} \cdot \sum_{1}^{5} j = \underline{10.5}$$

Group testing

- Identify defect objects
- Identify positive cases of Covid

Population

$\underline{128}$

Probability of a random person testing positive = 0.01

Expected number of tests for a group of size 2^n

Patient are divided into two groups of equal size and samples from all patients in each group is tested. If the outcome is positive then the process continues in that group. Find the expected number of tests needed to determine whether each patient is positive or negative.

Negative ① $(0.99)^2 \approx 0.98$

Positive ③ $1-(0.99)^2 \approx 0.02$

$$.98 + 0.06 = 1.04$$

T outcome of the first test

$$E[N] = E[E[N \mid T]]$$

N: # tests to be carried out

T negative $\quad \mathbb{E}[N|T] = 1$

T positive $\quad \mathbb{E}[N|T] = 2 \times 1.04$

$$\mathbb{E}[N] = 1 \times (0.99)^4 + 2 \times 1.04 \times (1 - (0.99)^4)$$

$$1 \times (0.96) + 2.08 \times 0.04$$

$$= 0.96 + 0.0832$$

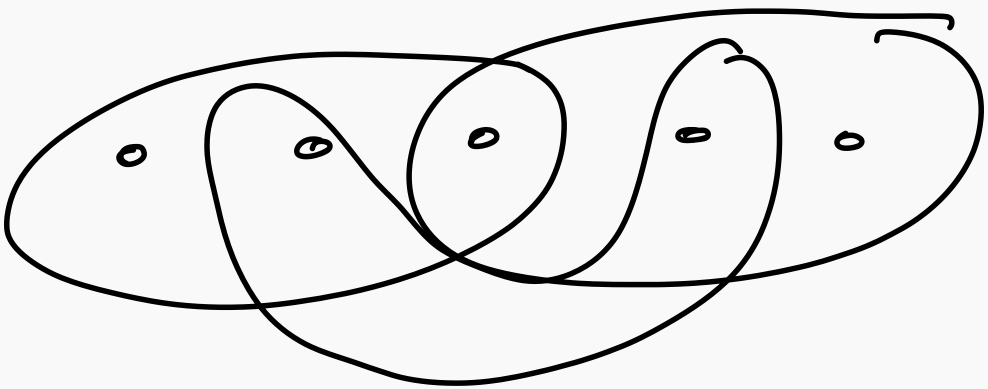
8 patient $\quad \approx \cancel{0} \; 1.0432.$

$$\mathbb{E}[N] = (0.99)^8 \times 1 + (1 - (0.99)^8) \times 2 \times 1.0432$$

$$= 0.92 + 0.08 \times 2.08$$

$$= 0.92 + 0.16 = 1.08$$

$$(0.99)^{128} = \underline{\underline{0.27}}$$



Combinatorial design

Expected value and variance of a sum

Example

Let $X_1, X_2, \ldots$ be a collection of independent random variables with identical distribution. Let N be another random variable which is independent of $X_1, \ldots$. Consider the random sum:

$$S = \sum_{i=1}^{N} X_i.$$

Show that

$$\mathbb{E}[S] = \mathbb{E}[N]\,\mathbb{E}[X_1].$$

$$\mathrm{Var}[S] = \mathbb{E}[X_1]^2\,\mathrm{Var}[N] + \mathbb{E}[N]\,\mathrm{Var}[X_1].$$