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Due: November 25, 2022
Assignment 6

Probability and Random Processes

(6.1) A fair coin is flipped twice. Let X be the number of Heads in the two tosses, and Y denote the random variable whose value is 1 if the outcomes are the same and zero otherwise.
(a) Find the joint probability mass function of X and Y.
(b) Find the marginal probability mass functions of X and Y.
(c) Are X and Y independent?
(d) Find the conditional probability mass functions of Y given $X = x$ and X given $Y = y$.

(6.2) For two random variables X and Y define the covariance by

$$\mathrm{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y].$$

Random variables X and Y are said to be uncorrelated if $\mathrm{Cov}(X, Y) = 0$.
(a) Show that

$$\mathrm{Var}\,[X + Y] = \mathrm{Var}\,[X] + \mathrm{Var}\,[Y] + 2\,\mathrm{Cov}(X, Y).$$

(b) If X and Y are independent, show that they are uncorrelated.

(6.3) Random variables X, Y have the joint density function

$$f(x, y) = \begin{cases} e^{-x-y} & x, y > 0 \\ 0 & \text{otherwise} \end{cases}$$

Compute $\mathbb{P}(X + Y \leq 1)$ and $\mathbb{P}(X < Y)$.

(6.4) For $0 < p < 1$, suppose X and Y are independent discrete random variables with Poisson distributions with parameters $p\lambda$ and $(1 - p)\lambda$, respectively. Let $N = X + Y$.
(a) Show that N has a Poisson distribution with parameter λ
(b) Show that the conditional distribution of X given $N = n$ is binomial with parameters (n, p):

$$p_{X|N}(x|n) = \binom{n}{x}\frac{n!}{x!(n-x)!}p^x(1-p)^{n-x}.$$

(6.5) An integer N randomly from the set $\{1, 2, \dots, 4\}$. Once N is chosen, we throw N fair dice and denote by X the product of scores obtained. For instance, if $N = 3$, three dice will be thrown and if the outcomes are $2, 3, 3$ then we set $X = 18$. Compute $\mathbb{E}\,[X]$ by using the law of iterated expectations.