Probability and Random Processes

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Announcements

1.

Joint probability mass function of discrete random variables

Definition
For n discrete random variables $X_1, X_2, \ldots, X_n$, the joint probability mass function of $X_1, \ldots, X_n$ the function defined by

$$p(x_1, x_2, \ldots, x_n) = \mathbb{P}[X_1 = x_1, \ldots, X_n = x_n].$$

X one discrete RV $\rightsquigarrow$ pmf prob. mass function

$$p_X(x) = \mathbb{P}[X = x]$$

$$p_{X_1, \ldots X_n}(x_1, \ldots, x_n) = \mathbb{P}\left[X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n\right]$$

Suppose a chicken lays N eggs, where N has a Poisson distribution with parameter λ. Each egg independently hatches with probability p and do not hatch with probability $1 - p$. Denote the number of eggs that hatch by X and those that do not by Y. Find the joint probability mass function of X and Y.

$$\boxed{\mathbb{P}(N=n) = e^{-\lambda} \cdot \frac{\lambda^n}{n!}} \qquad n = 0, 1, 2, \ldots$$

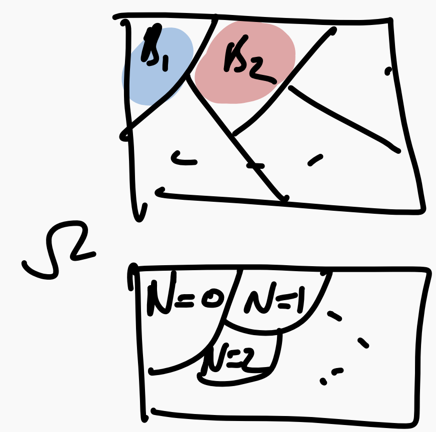
$$p_{X,Y}(x,y) = \mathbb{P}[X=x, Y=y].$$

$$\mathbb{P}[\underbrace{X=x, Y=y}_{A}] = \sum_n \underline{\mathbb{P}[X=x, Y=y \mid N=n]}\, \mathbb{P}[N=n]$$

condition on N

Recall

$$\mathbb{P}(A) = \sum_i \mathbb{P}(A \mid B_i)\, \mathbb{P}(B_i)$$



$$\mathbb{P}[X=x, Y=y \mid N=n] = \begin{cases} 0 & x+y \neq n \\ \binom{n}{x} p^x (1-p)^{n-x} & x+y=n \end{cases}$$

$$\mathbb{P}[X=x, Y=y] = \mathbb{P}[X=x, Y=y \mid N=x+y]\ \mathbb{P}[N=x+y]$$

$$= \binom{x+y}{x} p^x (1-p)^y \cdot e^{-\lambda} \cdot \frac{\lambda^{x+y}}{(x+y)!}$$

$$\boxed{n-x = x+y-x = y}$$

$$= \frac{\cancel{(x+y)!}}{x!\ y!} p^x (1-p)^y e^{-\lambda} \cdot \frac{\lambda^{x+y}}{\cancel{(x+y)!}}$$

$$= e^{-\lambda} \cdot \frac{(\lambda p)^x}{x!} \frac{(\lambda(1-p))^y}{y!}$$

$$= e^{-\lambda p} \frac{(\lambda p)^x}{x!} \cdot e^{-\lambda(1-p)} \frac{(\lambda(1-p))^y}{y!}$$

$$P_{X,Y}(x,y) = \underbrace{e^{-\lambda p} \cdot \frac{(\lambda p)^x}{x!}}_{\substack{\text{PMF of a} \\ \text{Poisson with parameter} \\ \lambda p}} \cdot \underbrace{e^{-\lambda(1-p)} \frac{(\lambda(1-p))^y}{y!}}_{\substack{\text{PMF of} \\ \text{a Poisson with} \\ \text{parameter } \lambda(1-p).}}$$

$$P_{X,Y}(x,y) = P_X(x) \; P_Y(y)$$

<u>X, Y independent</u>

Independence of discrete random variables

Definition

Discrete random variables $X_1, \ldots, X_n$ are *independent* if for all values of $x_1, \ldots, x_n$ we have

$$\mathbb{P}(X_1 = x_1, \ldots X_n = x_n) = \mathbb{P}(X_1 = x_1) \cdots \mathbb{P}(X_n = n)$$

$$p_{X_1,\ldots,X_n}(x_1 \ldots, x_n) = p_{X_1}(x_1) \cdots p_{X_n}(x_n).$$

Here $p_{X_1,\ldots,X_n}$ is the joint probability mass function of $X_1, \ldots, X_n$ and p_{X_i} is the marginal density function of X_i, for $1 \leq i \leq n$.

Theorem

Suppose $X_1, \ldots, X_n$ are independent random variables. Then

$$\mathbb{E}[X_1 \ldots X_n] = \mathbb{E}[X_1] \ldots \mathbb{E}[X_n].$$

More generally, for any choice of functions $h_1, \ldots, h_n$ we have

$$\mathbb{E}[h_1(X_1) \ldots h_n(X_n)] = \mathbb{E}[h_1(X_1)] \ldots \mathbb{E}[h_n(X_n)].$$

- $\mathbb{E}$ is linear $\mathbb{E}[X_1 + X_2] = \mathbb{E}[X_1] + \mathbb{E}[X_2]$

not a surprise.

show $\mathbb{E}[XY] = \mathbb{E}[X]\,\mathbb{E}[Y]$

if X, Y independent

XY	x_1y_1	x_1y_2	x_2y_1	x_2y_2
	p_1q_1	p_1q_2	p_2q_1	p_2q_2

$Y \backslash X$	x_1	x_2	
y_1	p_1q_1	p_2q_1	q_1
y_2	p_1q_2	p_2q_2	q_2
	p_1	p_2	

$$\mathbb{E}[XY] = x_1y_1p_1q_1 + x_1y_2p_1q_2 + x_2y_1p_2q_1 + x_2y_2p_2q_2$$

$$= (x_1p_1 + x_2p_2)(y_1q_1 + y_2q_2)$$

$$= \mathbb{E}[X]\,\mathbb{E}[Y].$$

Example

Suppose A is a 2×2 matrix whose entries are independent random variables with uniform distribution over $\cancel{[1, 2]}$ and $D = \det A$. Find $\mathbb{E}[D]$.

$\{1, 2, 3\}$

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \xrightarrow{\quad a_{ij} \text{ random} \quad} D$$

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \qquad D = \det A = 1 - 2 = -1$$

$$\mathbb{E}[\det A] = \mathbb{E}[D] = \mathbb{E}[A_{11} A_{22} - A_{12} A_{21}]$$

$$\underset{\text{linearity}}{=} \mathbb{E}[A_{11} A_{22}] - \mathbb{E}[A_{12} A_{21}] = \mathbb{E}[A_{11}]\,\mathbb{E}[A_{22}] - \mathbb{E}[A_{12}]\,\mathbb{E}[A_{21}]$$

$$= 2 \times 2 - 2 \times 2 = 0$$

$$\mathbb{E}[A_{11}] = \frac{1+2+3}{3} = 2$$

Definition

Let X and Y be random variables. The covariance of X and Y is defined by

$$\mathrm{Cov}(X, Y) = \mathbb{E}\left[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])\right].$$

X : $\mathbb{E}[X]$ average

$\mathrm{Var}[X]$ how scattered values of X are?

X, Y two random variable $\mathbb{E}[X]$ $\mathbb{E}[Y]$

$\mathrm{Var}(X)$ $\mathrm{Var}[Y]$

Assume: $\mathbb{E}[X]=0$, $\mathbb{E}[Y]=0$.

$$\mathrm{Cov}(X, Y) = \mathbb{E}\left[(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])\right] = \mathbb{E}[XY].$$

Special case X ± 1 with prob $\frac{1}{2}$ $\mathbb{E}[X]=0$

Y ± 1 with prob $\frac{1}{2}$ $\mathbb{E}[Y]=0$

$\mathbb{E}[XY]$

X \ Y	1	−1	
1	1/2	0	1/2
−1	0	1/2	1/2
	1/2	1/2	

X \ Y	1	−1
1	0	1/2
−1	1/2	0

X \ Y	1	−1
1	1/4	1/4
−1	1/4	1/4

$\mathbb{P}[XY = 1] = 1$

$\mathbb{P}[XY = -1] = 0$

$Cov(X,Y) = \mathbb{E}[XY] = 1$

X, Y are positively correlated

$Cov(X,Y) > 0$

$\mathbb{P}[XY = -1] = 1$

$\mathbb{P}[XY = 1] = 0$

$Cov(X,Y) = \mathbb{E}[XY] = -1$

X, Y are negatively correlated!

$Cov(X,Y) < 0$

$\mathbb{E}[XY] = \mathbb{E}[X]\,\mathbb{E}[Y] = 0$

X, Y are uncorrelated

$Cov(X,Y) = 0$

Example

The joint probability mass function of X and Y is given by

	$Y=-1$	$Y=0$	$Y=1$	
$X=-1$	$1/10$	$1/10$	$1/10$	$3/10$
$X=0$	$1/10$	$2/10$	$1/10$	$4/10$
$X=1$	$1/10$	$1/10$	$1/10$	$3/10$
	$3/10$	$4/10$	$3/10$	

Find $\mathrm{Cov}(X, Y)$.

$$\mu_X = \mathbb{E}[X]$$
$$\mu_Y = \mathbb{E}[Y]$$

$$\mathrm{Cov}(X,Y) =$$

$$\mathbb{E}[(X-\mu_X)(Y-\mu_Y)] = \mathbb{E}[XY - \mu_Y \cdot X - \mu_X Y + \mu_X \mu_Y]$$

$$= \mathbb{E}[XY] - \mu_Y \underbrace{\mathbb{E}[X]}_{\mu_X} - \mu_X \underbrace{\mathbb{E}[Y]}_{\mu_Y} + \mu_X \mu_Y$$

$$= \mathbb{E}[XY] - \mu_X \mu_Y = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y]$$

$$\mathrm{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y].$$

X	-1	0	1
	$3/10$	$4/10$	$3/10$

$$\mathbb{E}[X] = -1\cdot\frac{3}{10} + 0\cdot\frac{4}{10} + 1\cdot\frac{3}{10}$$
$$= 0$$

Y	-1	0	1
	$3/10$	$4/10$	$3/10$

$$\mathbb{E}[Y] = 0.$$

XY	-1	0	1
	$2/10$	$6/10$	$2/10$

$$\mathbb{E}[XY] = 0.$$

$$\mathrm{Cov}(X,Y) = 0\cdot 0 - 0 = 0.$$

<u>X, Y are uncorrelated.</u>

Theorem

For discrete random variables X and Y we have

1. $\mathrm{Cov}(X, X) = \mathrm{Var}[X]$.
2. $\mathrm{Cov}(X, Y) = \cancel{\mathrm{Var}}\,\mathbb{E}[XY] - \cancel{\mathrm{Var}}\,\mathbb{E}[X]\,\cancel{\mathrm{Var}}\,\mathbb{E}[Y]$.
3. If X and Y are independent, then $\mathrm{Cov}(X, Y) = 0$.
4. If X and Y are independent then $\mathrm{Var}[X + Y] = \mathrm{Var}[X] + \mathrm{Var}[Y]$.

$$\mathrm{Cov}(X, X) = \mathbb{E}[X.X] - \mathbb{E}[X]\mathbb{E}[X]$$
$$= \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \mathrm{Var}[X]$$

X, Y independent

$$\mathrm{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$$
$$= \mathbb{E}[XY] - \mathbb{E}[XY] = 0.$$

$$\mathrm{Var}(X+Y) = \mathbb{E}[(X+Y)^2] - \mathbb{E}[X+Y]^2$$

$$= \mathbb{E}[X^2 + Y^2 + 2XY] - (\mathbb{E}[X] + \mathbb{E}[Y])^2$$

$$= \mathbb{E}[X^2] + \mathbb{E}[Y^2] + 2\mathbb{E}[XY]$$
$$- \mathbb{E}[X]^2 - \mathbb{E}[Y]^2 - 2\,\mathbb{E}[X]\mathbb{E}[Y]$$

$$= \mathrm{Var}[X] + \mathrm{Var}[Y] + 2\,\mathrm{Cov}(X,Y)$$

$$\underset{\uparrow}{=} \mathrm{Var}[X] + \mathrm{Var}[Y]$$

X, Y are uncorrelated

Correlated

$\Uparrow \quad \Downarrow$

independent

$Y \backslash X$	1	2	3	
1	p_{11}	p_{12}	p_{13}	r_1
2	p_{21}	p_{22}	p_{23}	r_2
3	p_{31}	p_{32}	p_{33}	r_3
	q_1	q_2	q_3	

$\mathrm{Cov}(X,Y) = 0 \rightsquigarrow$ Some linear equation involving $p_{11}, \ldots, p_{33}$

Uncorrelated random variables

Definition

Two random variables X and Y with $\mathrm{Cov}(X, Y) = 0$ are called uncorrelated.

$$X, Y \text{ independent} \underset{\overset{?}{\Longleftarrow}}{\Longrightarrow} X, Y \text{ uncorrelated}$$

Suppose $X_1, \ldots, X_n$ are independent random variables, each with Bernoulli distribution with parameter p. Let

$$S_n = X_1 + \cdots + X_n.$$

What is the probability mass function of S_n?

$\mathbb{P}(X_i = 1) = p \qquad \mathbb{P}(X_i = 0) = 1 - p$

Possible values of S_n are $0, 1, 2, \ldots, n$

$$\mathbb{P}(S_n = k) = \mathbb{P}(X_1 = 1, X_2 = 1, \ldots, X_k = 1, X_{k+1} = \cdots = X_n = 0)$$
$$+ \mathbb{P}(X_1 = 1, X_2 = 1, \ldots X_{k-1} = 1, X_k = 0, X_{k+1} = 1, \underset{\text{Rest}=0}{})$$

$$\mathbb{P}(X_1 = 1, X_2 = 1, \ldots, X_k = 1, X_{k+1} = 0, \ldots, X_n = 0)$$

independence $\mathbb{P}(X_1=1)\,\mathbb{P}(X_2=1)\cdots\mathbb{P}(X_k=1)\,\mathbb{P}(X_{k+1}=0)\cdots\mathbb{P}(X_n=0)$

$$= p^k(1-p)^{n-k}$$

So $\mathbb{P}(S_n=k) = \binom{n}{k} p^k (1-p)^{n-k}$.

distribution of S_n is binomial with parameters (n, p).

Application: Compute the expected value of a binomial dist. with parameter n, p.

binomial distribution $S_n = X_1 + \cdots + X_n$

Bernoulli RV with parameter p

$$\mathbb{E}[S_n] = \mathbb{E}[X_1+\cdots+X_n] = \mathbb{E}[X_1]+\cdots+\mathbb{E}[X_n]$$
$$= np$$

Variance of a binomial distribution

$$Var[S_n] = Var[X_1 + X_2 + \cdots + X_n]$$

Bernoulli RVs with parameter p

independence of X_i

$$= Var[X_1] + Var[X_2] + \cdots + Var[X_n]$$

$$p - p^2 \qquad p(1-p) \qquad p(1-p)$$

$$Var[X_1] = \mathbb{E}[X_1^2] - \mathbb{E}[X_1]^2 \qquad = np(1-p)$$

$$= \mathbb{E}[X_1] - \mathbb{E}[X_1]^2$$

$$= p - p^2$$

$$= p(1-p)$$

$$Var\left[\begin{matrix}\text{Bernoulli RVL} \\ \text{with para } (n,p)\end{matrix}\right] = np(1-p).$$