Jacobs University Bremen
Keivan Mallahi-Karai

Due: September 25, 2022
Assignment 1

Probability and Random Processes

In each one of the problems below, if you are asked to compute a probability, first identify the sample space and the event in question explicitly.

(1.1) A fairs die is thrown twice.

(a) What is the sample space?

(b) Determine the probability p_k that the sum of the numbers shown is k. Plot p_k as a function of k.

(c) What is the probability that the absolute value of the difference between outcomes of the first and the second throw is less than 3?

Solution. Part (8) has 4 points and parts (b) and (c) each 8 points (a) The sample space consists of all pairs (i, j) with $1 \le i, j \le 6$. We can also write:

$$\Omega = \{(i, j) : 1 \le i, j \le 6\}.$$

(b) The possible values of $i + j$ are $2, 3, \dots, 12$. A simple counting shows that p_k is given by the following table

k	2	3	4	5	6	7	8	9	10	11	12
p_k	1/36	2/36	3/36	4/36	5/35	6/36	5/36	4/36	3/36	2/36	1/36

The outcomes corresponding to the event that the absolute value of the difference between outcomes of the first and the second throw is less than 3 are given by

$$A = \{(6, 6)(6, 5), (5, 6), (6, 4), (4, 6), (5, 5), (5, 4), (4, 5), (5, 3), (3, 5), (4, 4), (4, 3),$$
$$(3, 4), (4, 2), (2, 4), (3, 3), (3, 2), (2, 3), (3, 1), (1, 3), (2, 2), (2, 1), (1, 2), (1, 1)\}$$

From here, we see that $\mathbb{P}[A] = \frac{24}{36} = \frac{2}{3}$.

(1.2) A fair die is rolled three times. We say that a match has occurred if the outcome of the first throw is 1, or the outcome of the second throw is 2, or the outcome of the third throw is 3. Find the probability of the event that a match occurs.

Solution. For $i = 1, 2, 3$, let us denote the event that the outcome of throw i is i by A_i. We are interested in computing $\mathbb{P}[A_1 \cup A_2 \cup A_3]$. It is clear that

$$\mathbb{P}[A_1] = \mathbb{P}[A_2] = \mathbb{P}[A_3] = \frac{1}{6}.$$

Similarly,

$$\mathbb{P}[A_1 \cap A_2] = \mathbb{P}[A_1 \cap A_3] = \mathbb{P}[A_2 \cap A_3] = \frac{1}{36}.$$

Finally, since $\mathbb{P}[A_1 \cap A_2 \cap A_3] = \frac{1}{6^3}$, using inclusion-exclusion principle, we can write

$$\mathbb{P}[A_1 \cup A_2 \cup A_3] = \frac{3}{6} - \frac{3}{6^2} + \frac{1}{6^3} = \frac{91}{6^3}.$$

Alternatively, if no matches occur, then 5 options are left for each round. Hence

$$\mathbb{P}[A] = 1 - \frac{5^3}{6^3} = \frac{91}{6^3}.$$

(1.3) An ordinary deck of playing cards (containing 52 standard cards, 13 of each suit) is randomly divided into two parts, each containing at least one card.

(a) What is the probability that each part contains at least one ace.

(b) Find the probability that each part contains exactly two aces.

Solution. Each part 10 points

Denote the cards in the left and the right parts by D_1 and D_2. It is clear that D_1 must be a subset of the set of all cards, which is non-empty and also is not equal to the set of all cards, for otherwise, D_2 would be empty. This implies that that the sample space contains $2^{52} - 2$ elements.

Let A be the event that each part contains an ace. Note that A^c will be the event that all the aces are in D_1 or in D_2. If all aces are in D_1, then there are 48 cards left to be distributed between D_1 and D_2 with the constraint that not all of them go to D_1. This implies that the number of possibilities is $2^{48} - 1$. There is the same number of distributions in which all aces go to D_2. Hence

$$\mathbb{P}[A] = 1 - \frac{2 \times (2^{48} - 1)}{2^{52} - 2} = 1 - \frac{2^{48} - 1}{2^{51} - 1}.$$

For part (b), note that there are $\binom{4}{2} = 6$ aces to choose two out of four aces for D_1. The remaining 48 cards can be distributed arbitrarily between D_1 and D_2. Hence the probability of the event B that each part contains exactly two aces equals:

$$\mathbb{P}[B] = \frac{6 \times 2^{48}}{2^{52} - 2}.$$

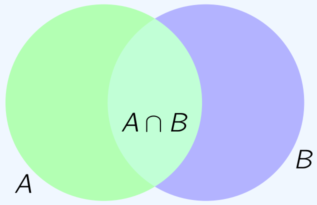
(1.4) Suppose A and B are two events. Let S be the event that A or B occur, but not both. Show that

$$\mathbb{P}[S] = \mathbb{P}[A] + \mathbb{P}[B] - 2\,\mathbb{P}[A \cap B].$$

Hint: Draw a Venn diagram and use it to describe S as Boolean combination of the given events.

Solution. It is clear that S consists of those elements of $A \cup B$ which are *not* in $A \cap B$. Hence

$$\begin{aligned} \mathbb{P}[S] &= \mathbb{P}[A \cup B] - \mathbb{P}[A \cap B] = (\mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[A \cap B]) - \mathbb{P}[A \cap B] \\ &= \mathbb{P}[A] + \mathbb{P}[B] - 2\mathbb{P}[A \cap B]. \end{aligned} \tag{1}$$



(1.5) Alice and Bob are supposed to meet in the cafeteria. Alice arrives at a random time between noon and 1pm, and wait for 15 minutes upon her arrival and then leaves. Bob also also arrives at a random time between noon and 1 pm, but waits up to 20 minutes and then leaves.

(a) What is the probability that Bob arrives before 12:20?

(b) What is the probability that Alice and Bob meet?

(c) If Bob arrives later than Alice, what is the probability that they meet?

(d) Suppose that Alice and Bob have managed to meet. What is the probability that Bob has arrived before 12:20?

Solution. Each part 5 points

Let us represent the time between the noon and 1 pm. by the interval $[0, 1]$. Then the event A that Bob arrives before 12:20 corresponds to the interval $[0, 1/3]$ and hence has probability $1/3$.

For part (b), the sample space can be described by

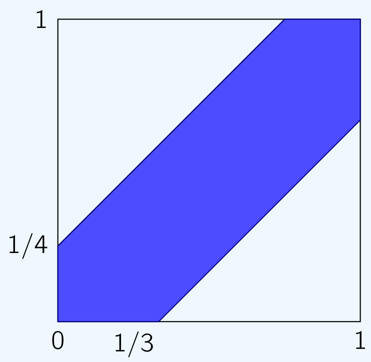
$$\Omega = \{(t_1, t_2) \big| \; 0 \le t_1 \le 1, \quad 0 \le t_2 \le 1\}$$

where t_1, t_2 are the times that Alice and Bob show up. If M is the event that they meet, then

$$M = \{(t_1, t_2) \big| \; t_1 \le t_2 \le t_1 + \frac{1}{4} \text{ or } t_2 \le t_1 \le t_2 + \frac{1}{3}\}.$$

This can be seen as the shaded area in the square:

$$\mathbb{P}[M] = \text{Area}(M) = 1 - \frac{1}{2}(\frac{2}{3})^2 - \frac{1}{2}(\frac{3}{4})^2 = \frac{143}{288}.$$



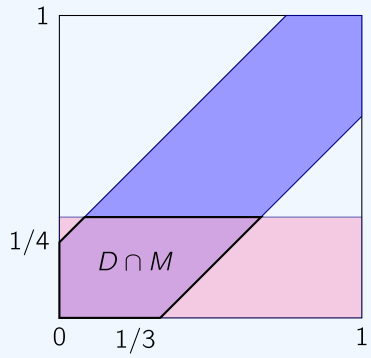
Note that the probability of the event B that Bob arrives after Alice is $1/2$. Hence

$$B \cap M = \{(t_1, t_2) \big| \; t_1 \le t_2 \le t_1 + \frac{1}{4}\}.$$

Hence

$$\mathbb{P}[M|B] = \frac{\mathbb{P}[B \cap M]}{\mathbb{P}[B]} = \frac{\frac{1}{2} - \frac{1}{2}(\frac{3}{4})^2}{1/2} = \frac{7}{16}$$

(d) The area corresponding to Alice and Bob meeting is in blue. The event D that Bob arrives before 12:20 is colored in magenta:



We would like to compute $\mathbb{P}[D|M]$. In order to compute $\mathbb{P}[D \cap M]$, we need to compute the area of the region

$$\mathbb{P}[D \cap M] = \frac{1}{3} - \frac{1}{2}(\frac{1}{12})^2 - \frac{1}{9} - \frac{1}{18} = \frac{47}{288} \approx 0.16.$$

This gives

$$\mathbb{P}[D|M] = \frac{\mathbb{P}[D \cap M]}{\mathbb{P}[M]} = \frac{47/288}{143/288} = \frac{47}{143} \approx 0.33.$$