Probability and Random Processes

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Agenda

1.

Announcements

1. Problem Set ~~6~~ 5 is due tonight 23:59.
2. Problem Set ~~7~~ 6 will be posted today and is due on 25.11.22.
3. Practice exam II in one week.

$x \geq t$ or $x \leq -t$

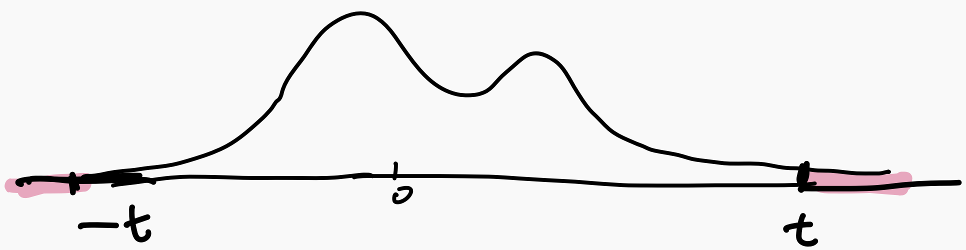
Let X be a random variable. Define

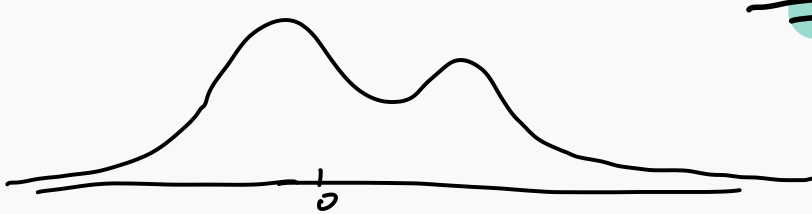
$$N(t) = \mathbb{P}\left[|X| \geq t\right]$$

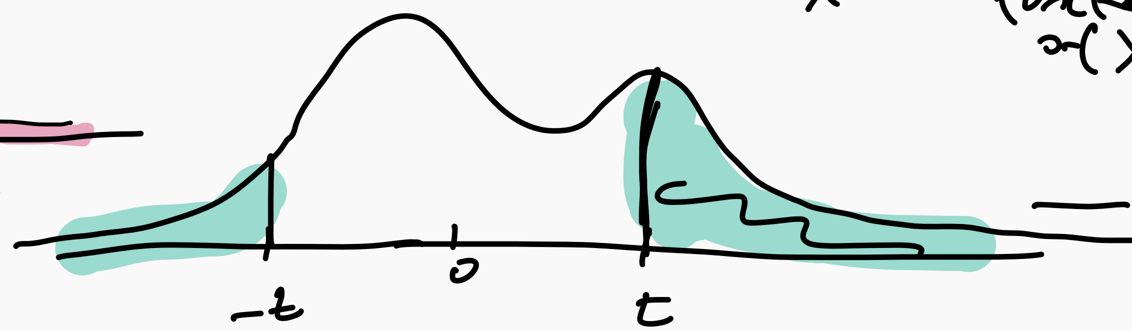
$t \geq 0$

This function describes the probability of X taking large values.

$f_X(x)$ density function of X







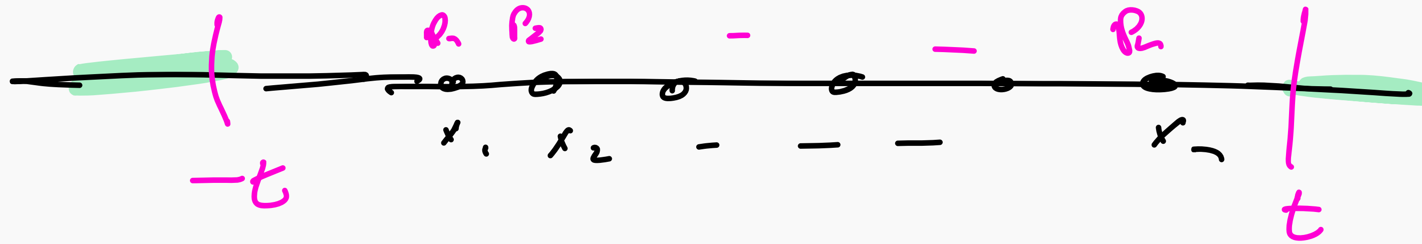
$t \to \infty$

$N(t) \to 0$

Examples

Example 1: X takes only finitely many values. Determine $N(t)$.

t large



$N(t) = 0$

for t sufficiently large.

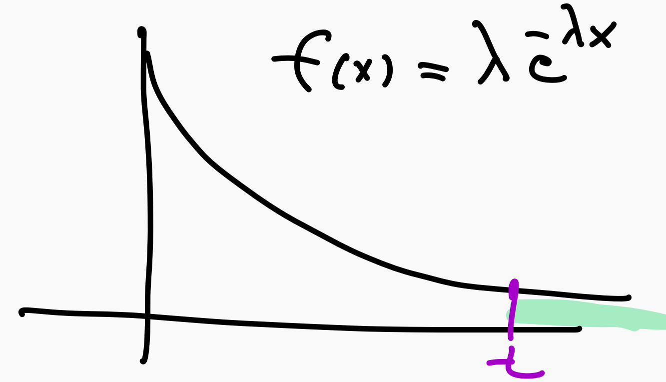
values of X are bounded

X uniform distribution on $[0,1]$

$N(t) = 0$ if $t > 1$

Examples 2

Let X be an exponential random variable with parameter λ. Compute $N(t)$.



$$N(t) = \mathbb{P}(|X| \geq t)$$

$$= \mathbb{P}(X \geq t)$$

$$= \int_t^\infty \lambda e^{-\lambda x}\, dx$$

$$= -e^{-\lambda x}\Big|_t^\infty = e^{-\lambda t}$$

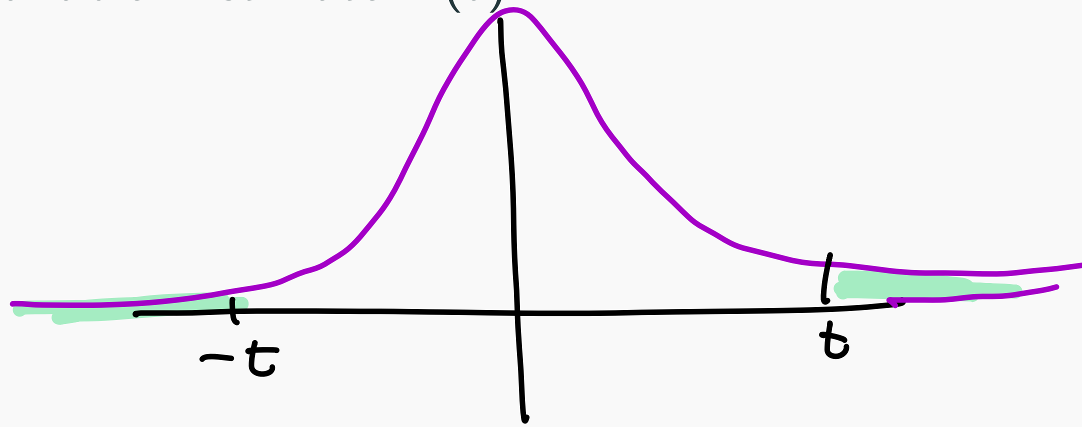
exponential decay

thin tails

Normal (Gaussian) random variables

$\mu = 0, \sigma = 1$

Let X be a standard Normal random variable. Estimate $N(t)$



$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$N(t) = \mathbb{P}(|X| \geq t)$$

Symmetry

$$= 2\,\mathbb{P}(X \geq t) = 2 \int_t^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_t^\infty e^{-x^2/2}\,dx \leq \frac{2}{\sqrt{2\pi}} \int_t^\infty \frac{x}{t} e^{-x^2/2}\,dx$$

$$= \frac{2}{\sqrt{2\pi}\,t} \int_t^\infty x\,\underline{e^{-x^2/2}}\,dx$$

Substitution $\frac{+x^2}{2} = u \qquad du = \frac{2x}{2}\,dx = x\,dx$

$$= \frac{2}{\sqrt{2\pi}\,t} \int_{t^2/2}^{\infty} e^{-u}\,du$$

$x = t \longrightarrow u = t^2/2$

$x = \infty \longrightarrow u = \infty$

$$= \frac{2}{\sqrt{2\pi}\,t} \left. -e^{-u} \right|_{t^2/2}^{\infty} = \frac{2}{\sqrt{2\pi}\,t}\, e^{-t^2/2}.$$

Compare

$e^{-\lambda t}$ exponential decay

$t \to \infty$

$$\lambda t < \frac{t^2}{2}$$

Markov's inequality

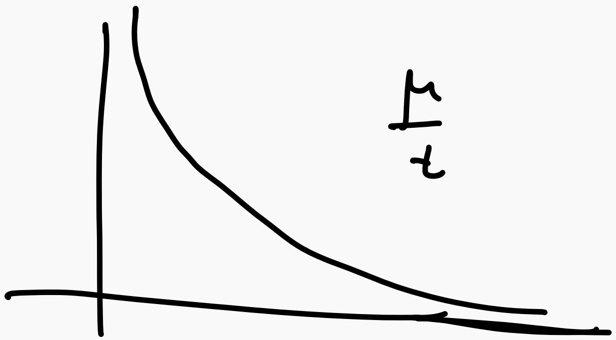
Theorem

Suppose that a positive random variable X has a finite expectation μ. Then

$$N(t) = \mathbb{P}[X \geqslant t] \leq \frac{\mu}{t}.$$

for $t > 0$.

why?



Compare random variable X with another random variable Y

$$Y = \begin{cases} 0 & X \leq t \\ t & X \geqslant t \end{cases}$$

relation between X, Y

notice $Y \leq X$ true when $X \leq t$

when $X \geqslant t$

$Y = t$, $X \geqslant t$

summary

$$X \rightsquigarrow Y = \begin{cases} 0 & X < t \\ t & X \geq t \end{cases}$$

and

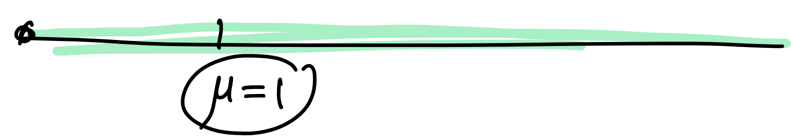
observation: $Y \leq X$

$$\Rightarrow \mathbb{E}[Y] \leq \underbrace{\mathbb{E}[X]}_{\mu}$$

$$\mathbb{E}[Y] = 0 \cdot \mathbb{P}(X < t) + t \cdot \mathbb{P}(X \geq t)$$

$$= t\, \mathbb{P}(X \geq t)$$

$$t\, \mathbb{P}(X \geq t) \leq \mu \Longrightarrow \boxed{\mathbb{P}(X \geq t) \leq \frac{\mu}{t}}$$



$$\boxed{\mathbb{P}(X \geq 10) \leq \frac{1}{10}}$$

Example

A coin is weighted so that its probability of landing on heads is 1/6. Suppose the coin is flipped 24 times. Find a bound for the probability it lands on heads at least 20 times.

$$\mathbb{P}(\#\,\text{head} \geq 20) = \binom{24}{20}\left(\frac{1}{6}\right)^{20}\left(\frac{5}{6}\right)^{4} + \binom{24}{21}\left(\frac{1}{6}\right)^{21}\left(\frac{5}{6}\right)^{3} + \cdots + \binom{24}{24}\left(\frac{1}{6}\right)^{24}.$$

$$\mathbb{P}(\underset{\overset{\shortparallel}{X}}{\#\,\text{head}} \geq 20) \leq \frac{\mathbb{E}[X]}{\underset{20}{t}} = \frac{4}{20} = \frac{1}{5} = 0.2$$

$$\mathbb{E}[X] = 24 \cdot \frac{1}{6} = 4$$

$$\mathbb{P}(\text{heads} \leq 20) = \mathbb{P}(\text{tails} \geq 4) \leq \frac{20}{4} = 5$$

Proof of Markov's inequality

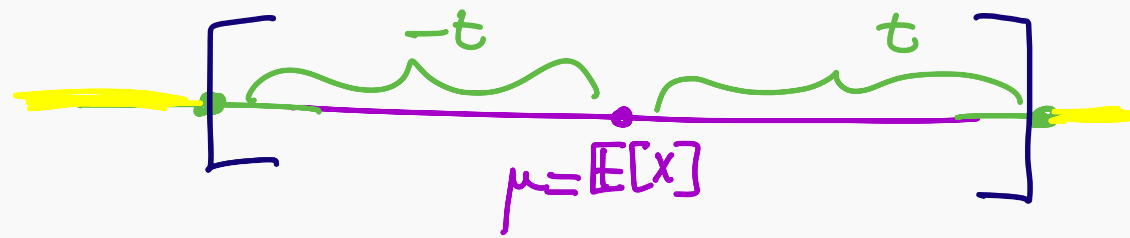
Chebyshev's inequality

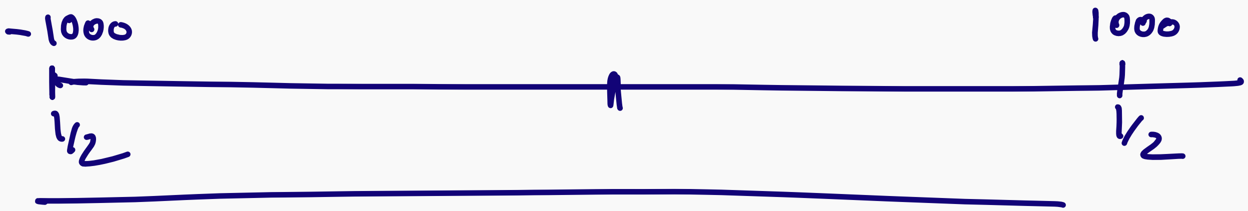
Theorem

μ

Suppose that X has finite expectation and finite variance. Then

$$\mathbb{P}\left[|X - \mathbb{E}(X)| > t\right] \leq \frac{\mathrm{Var}\,[X]}{t^2}.$$





$$\mathbb{E}[X] = \frac{1}{2} \cdot 1000 + \frac{1}{2}(-1000) = 0.$$

standard deviation $\sigma = \sqrt{\text{Varian}(x)} \implies \mathrm{Var}[X] = \sigma^2$

$$\mathbb{P}\left[\,|X-\mathbb{E}[X]|\geq t\,\right]\leq\frac{\mathrm{Var}[X]}{t^2}=\frac{\sigma^2}{t^2}$$

$t=10\sigma$

$$\mathbb{P}\left[\,|X-\mathbb{E}(X)|\geq 10\sigma\,\right]\leq\frac{\sigma^2}{(10\sigma)^2}=\frac{1}{100}.$$

Proof of Chebyshev's inequality

Sum of Independent Random Variables

Definition

A sequence of random variables $X_1, X_2, \dots$ is called *independent and identically distributed* (*in short, i.i.d.*) if X_i are independent and have a common distribution function.

Suppose $X_1, X_2, \dots$ be an i.i.d sequence or random variables. Then define

$$S_n = X_1 + \cdots + X_n$$

X_n Bernoulli $p = 1/2$

$$X_n = \begin{cases} 0 & 1/2 \\ 1 & 1/2 \end{cases}$$

and consider

$$\frac{S_n}{n} = \frac{X_1 + \cdots + X_n}{n}.$$

$X_1 + \cdots + X_n$

expect $\frac{S_n}{n}$ must be around $1/2$.

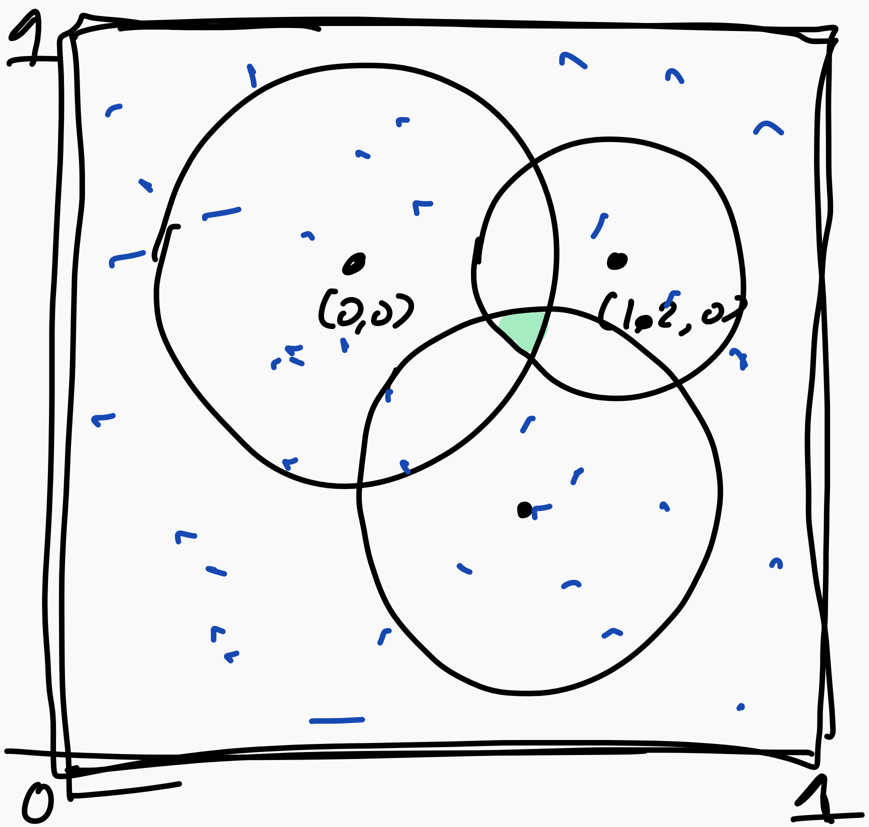
The Weak Law of Large Numbers

Consider a sequence X_n of identically distributed independent random variables. Suppose that they have finite expectation μ and finite variance . Then, for every $\epsilon > 0$, we have

$$\lim_{n \to \infty} \mathbb{P}\left[\left|\frac{X_1 + \cdots + X_n}{n} - \mu\right| > \epsilon\right] = 0.$$

Proof

Region in $\mathbb{R}^3$ defined by some algebraic inequalities



$x^2 + y^2 \leq 1$

$$X_n = \begin{cases} 1 & n\text{th sample is in the region } R \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{X_1 + \cdots + X_n}{n} = \frac{\#\text{region } s}{n}$$

n very large

$$\downarrow$$

$\mathbb{E}[X_n] =$ <u>area of region</u>

Concentration of measure phenomenon