Probability and Random Processes

(6.1) A fair coin is flipped twice. Let X be the number of Heads in the two tosses, and Y denote the random variable whose value is 1 if the outcomes are the same and zero otherwise.

(a) Find the joint probability mass function of X and Y.
(b) Find the marginal probability mass functions of X and Y.
(c) Are X and Y independent?
(d) Find the conditional probability mass functions of Y given $X = x$ and X given $Y = y$.

Solution. It is clear that X can take values $0, 1, 2$ and Y can take values $1, 2$. If the outcomes is HH then $X = 2$ and $Y = 1$, if it is HT or TH then $X = 1$ and $Y = 0$ and if it is TT then $X = 0$ and $Y = 1$. Hence we have the following table

	$Y=0$	$Y=1$
$X=0$	0	1/4
$X=1$	1/2	0
$X=2$	0	1/4

The marginals are given by

	$Y=0$	$Y=1$	
$X=0$	0	1/4	1/4
$X=1$	1/2	0	1/2
$X=2$	0	1/4	1/4
	1/2	1/2	

(c) X and Y are not independent. For instance

$$p_{X,Y}(0,0) = 0 \neq \frac{1}{4} \cdot \frac{1}{2} = p_X(0)p_Y(0).$$

(d) The conditionals of X given Y are given by

$$p_{X|Y}(0,0) = 0 \quad p_{X|Y}(1,0) = 1, \quad p_{X|Y}(2,0) = 0.$$
$$p_{X|Y}(0,1) = 1/2, \quad p_{X|Y}(1,1) = 0, \quad p_{X|Y}(2,1) = 1/2.$$

The conditionals of Y given X are given by

$$p_{Y|X}(0,0) = 0, \quad p_{Y|X}(1,0) = 1$$
$$p_{Y|X}(0,1) = 1, \quad p_{Y|X}(1,1) = 0$$
$$p_{Y|X}(0,2) = 0, \quad p_{Y|X}(1,2) = 1$$

Note that the value of Y is determined by X, that is, Y is a function of X.

(6.2) For two random variables X and Y define the covariance by

$$\mathrm{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y].$$

Random variables X and Y are said to be uncorrelated if $\mathrm{Cov}(X,Y) = 0$.

(a) Show that

$$\mathrm{Var}\,[X+Y] = \mathrm{Var}\,[X] + \mathrm{Var}\,[Y] + 2\,\mathrm{Cov}(X,Y).$$

(b) If X and Y are independent, show that they are uncorrelated.

Solution.
(a) By the definition of variance and the linearity of the expected-value operator,

$$\mathrm{Var}\,[X+Y] = \mathbb{E}\Big[\big(X+Y-\mathbb{E}[X+Y]\big)^2\Big] = \mathbb{E}\left[\Big(\big(X-\mathbb{E}[X]\big)+\big(Y-\mathbb{E}[Y]\big)\Big)^2\right].$$

Expand the square and use once again the linearity of $\mathbb{E}$ to obtain

$$\mathrm{Var}\,[X+Y] = \mathbb{E}\left[\big(X-\mathbb{E}[X]\big)^2\right] + \mathbb{E}\left[\big(Y-\mathbb{E}[Y]\big)^2\right] + 2\cdot\mathbb{E}\Big[\big(X-\mathbb{E}[X]\big)\big(Y-\mathbb{E}[Y]\big)\Big].$$

By the definition of variance and the linearity of $\mathbb{E}$, we get

$$\begin{aligned}\mathrm{Var}\,[X+Y] &= \mathrm{Var}\,[X] + \mathrm{Var}\,[Y] + 2\big(\mathbb{E}[XY]-\mathbb{E}[X]\mathbb{E}[Y]\big)\\ &= \mathrm{Var}\,[X] + \mathrm{Var}\,[Y] + 2\cdot\mathrm{Cov}(X,Y)\,.\end{aligned}$$

(b) If X and Y are statistically independent, then $\mathbb{E}[XY] = \mathbb{E}[X]\cdot\mathbb{E}[Y]$. The rest follows immediately.

(**6.3**) Random variables X, Y have the joint density function

$$f(x,y) = \begin{cases} e^{-x-y} & x,y>0\\ 0 & \text{otherwise}\end{cases}$$

Compute $\mathbb{P}(X+Y\le 1)$ and $\mathbb{P}(X<Y)$.

Solution. It is clear that since X and Y are both non-negative, if $X+Y\le 1$ then $0\le X\le 1$ and $0\le Y\le 1-X$. Hence we have

$$\mathbb{P}\,[X+Y\le 1] = \int_0^1\int_0^{1-x} e^{-x-y}dy\ dx = \int_0^1 e^{-x}(1-e^{x-1})\ dx.$$

Hence,

$$\mathbb{P}\,[X+Y\le 1] = \int_0^1 (e^{-x}-e^{-1})\ dx = (1-e^{-1})-e^{-1} = 1-2e^{-1}.$$

Note that X and Y are independent exponential random variables with parameter 1. A similar computation shows that if X and Y are independent exponential random variables with parameter λ, then for any $t>0$ we have

$$\mathbb{P}\,[X+Y\le t] = 1-e^{-\lambda t}-\lambda t e^{-\lambda t}.$$

To evaluate the second probability note that

$$\mathbb{P}\,[X<Y] = \int_0^\infty\int_0^y e^{-x-y}dxdy = \int_0^\infty e^{-y}(1-e^{-y})dy = 1-\frac{1}{2} = \frac{1}{2}.$$

Alternatively, one can compute marginals

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,\mathrm{d}y$$

to see that X and Y are both exponentially distributed random variable with parameter (mean) 1. Moreover, X and Y are independent since $f_{X,Y}(x,y) = f(x,y) = f_X(x)f_Y(y)$. For two independent, identically distributed random variables, we have $\mathbb{P}[X<Y]=\mathbb{P}[X>Y]$. As X and Y are continuous, $\mathbb{P}[X>Y]=\mathbb{P}[X\geq Y]$. Hence,

$$\mathbb{P}[X<Y]=\frac{\mathbb{P}[X<Y]+\mathbb{P}[X>Y]}{2}=\frac{\mathbb{P}[X<Y]+\mathbb{P}[X\geq Y]}{2}=\frac{1}{2}.$$

(6.4) For $0<p<1$, suppose X and Y are independent discrete random variables with Poisson distributions with parameters $p\lambda$ and $(1-p)\lambda$, respectively. Let $N=X+Y$.

(a) Show that N has a Poisson distribution with parameter λ

(b) Show that the conditional distribution of X given $N=n$ is binomial with parameters (n,p):

$$p_{X|N}(x|n)=\binom{n}{x}\frac{n!}{x!(n-x)!}p^x(1-p)^{n-x}.$$

Solution. We compute $\mathbb{P}[N=n]$. Note that $N=n$ if $X+Y=n$. This means that X can take a value $0\leq x\leq n$ and then $Y=n-x$. Hence using the independence of X and Y we have

$$\begin{aligned}p_N(n)=\mathbb{P}[N=n]&=\sum_{x=0}^{n}p_{X,Y}(x,n-x)=\sum_{x=0}^{n}\mathbb{P}[X=x,Y=n-x]\\&=\sum_{x=0}^{n}e^{-\lambda p}\frac{(\lambda p)^x}{x!}e^{-\lambda(1-p)}\frac{(\lambda(1-p))^{(n-x)}}{(n-x)!}\end{aligned}\tag{2}$$

Simplification followed by inserting $n!$ in the numerator and denominator yields

$$\mathbb{P}[N=n]=\sum_{x=0}^{n}e^{-\lambda}\lambda^n\frac{1}{x!(n-x)!}p^x(1-p)^{1-x}=\frac{1}{n!}e^{-\lambda}\lambda^n\sum_{x=0}^{n}\binom{n}{x}p^x(1-p)^{n-x}=\frac{1}{n!}e^{-\lambda}\lambda^n,$$

where the last equality follows from the binomial expansion

$$\sum_{x=0}^{n}\binom{n}{x}p^x(1-p)^{n-x}=(p+(1-p))^n=1.$$

This shows that N has a Poisson distribution with parameter λ.

(b) By definition and using again the fact that $X=n$ and $N=n$ is equivalent to $X=x$ and $Y=N-x$ we have

$$\begin{aligned}p_{X|N}(x|n)&=\frac{\mathbb{P}[X=x,N=n]}{\mathbb{P}[N=n]}=\frac{\mathbb{P}[X=x,Y=n-x]}{\mathbb{P}[N=n]}\\&=\frac{p_{X,Y}(x,n-x)}{p_N(n)}=\frac{e^{-\lambda p}\frac{(\lambda p)^x}{x!}e^{-\lambda(1-p)}\frac{(\lambda(1-p))^{(n-x)}}{(n-x)!}}{\frac{1}{n!}e^{-\lambda}\lambda^n}\\&=\frac{e^{-\lambda}\lambda^n p^x(1-p)^{n-x}}{\frac{x!(n-x)!}{n!}e^{-\lambda}\lambda^n}=\frac{n!}{x!(n-x)!}p^x(1-p)^{n-x}.\end{aligned}\tag{3}$$

This proves the claim.

(6.5) An integer N randomly from the set $\{1,2,\ldots,4\}$. Once N is chosen, we throw N fair dice and denote by X the product of scores obtained. For instance, if $N=3$, three dice will be thrown and if the outcomes are $2,3,3$ then we set $X=18$. Compute $\mathbb{E}[X]$ by using the law of iterated expectations.

Solution. We compute $\mathbb{E}[X|N=k]$. Note that if $N=k$ then $X = X_1 \cdots X_k$ where each X_i takes values $1, 2, \ldots, 6$ with probability $1/6$. Hence $\mathbb{E}[X_i] = 7/2$ and since $X_1, \ldots, X_k$ are independent, it follows that $\mathbb{E}[X|N=k] = (7/2)^k$. It follows that

$$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|N]] = \frac{1}{4}\left((7/2)^1 + (7/2)^2 + (7/2)^3 + (7/2)^4\right).$$