# *Probability and Random Processes*

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## Recap

States $1, 2, \dots, N$

$\pi = (\pi_1, \pi_2, \dots \pi_N)$

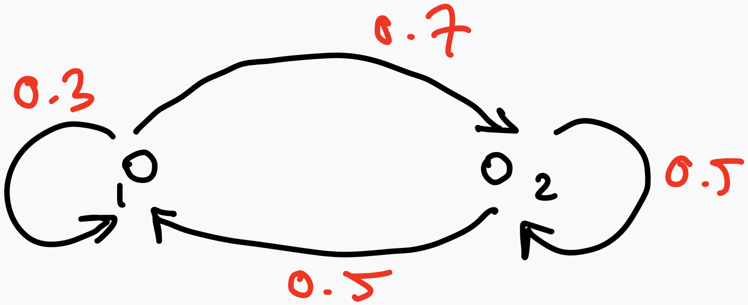
- Transition probabilities of a Markov chain

$$p_{ij} = \mathbb{P}\left[X_n = s_j | X_{n-1} = s_i\right]. \qquad P = (p_{ij})$$

- If the distribution of $X_k$ is given by the row vector $\pi$, then the distribution of $X_{k+1}$ is given by the row vector $\pi P$. More generally, the distribution of $X_{k+n}$ is given by $\pi P^n$.
- The transition probabilities after $n$ steps:

$$p_{ij}^{(n)} = (P^n)_{ij}.$$

# Long term behavior of Markov chains: Example 1




$$P = \begin{pmatrix} 0.3 & 0.7 \\ 0.5 & 0.5 \end{pmatrix}$$

# Long term behavior of Markov chains: Example 2

$$P = \begin{pmatrix} 1/6 & 1/3 & 1/2 \\ 1/8 & 3/4 & 1/8 \\ 1/4 & 1/4 & 1/2 \end{pmatrix}$$

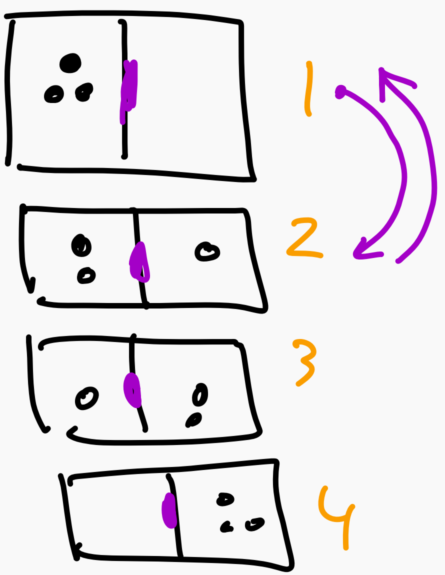
# Long term behavior of Markov chains: Example 3

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/\cancel{2}3 & 0 & 2/3 & 0 & 0 \\ 0 & 1/3 & 0 & 2/3 & 0 \\ 0 & 0 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/3 & 0 & 2/3 & 0 \\ 0 & 2/3 & 0 & 1/3 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Ehrenfest model

$$\begin{array}{c|cccc} & 1 & 3 & 2 & 4 \\ \hline 1 & 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 2/3 & 1/3 \\ 2 & 1/3 & 2/3 & 0 & 0 \\ 4 & 0 & 1 & 0 & 0 \end{array}$$



# Stationary distributions

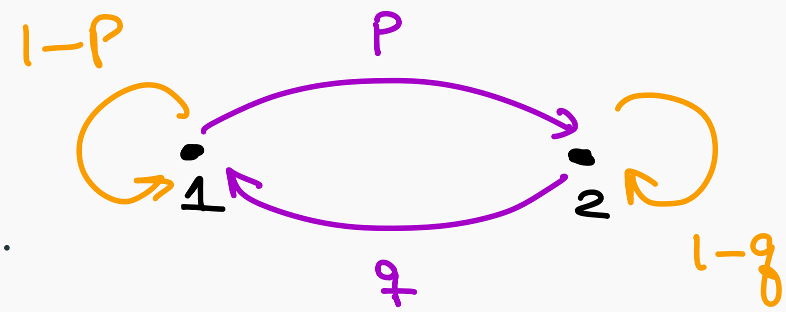
**Definition**

A distribution $\pi$ is called stationary if $\pi = (\pi_1 \; . \; . \; . \; \pi_N)$

$$\pi P = \pi.$$

# Example

Consider the Markov chain with 2 states:




$$P = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}.$$

Determine the stationary distribution.

$\pi = (x\ ,\ y)$ $\qquad x + y = 1$

$\pi P = \pi \qquad (x \quad y)\begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix} = (x \quad y)$

$$\begin{cases} x(1-p) + yq = x \Longrightarrow & yq = x - x + px \\ xp + y(1-q) = y & yq = px \Rightarrow \begin{cases} \frac{x}{y} = \frac{q}{p} \\ x + y = 1 \end{cases} \end{cases}$$

$$x = \frac{q}{p+q} \qquad y = \frac{p}{p+q}$$

$$\pi = \left(\frac{q}{p+q} \quad \frac{p}{p+q}\right)$$

$0 < p, q < 1$

Consider a Markov chain with 2 states:

$$P = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}.$$

Study $P^n$.

$$\pi = \left( \frac{q}{p+q}, \frac{p}{p+q} \right)$$

$$\theta = (1, -1)$$

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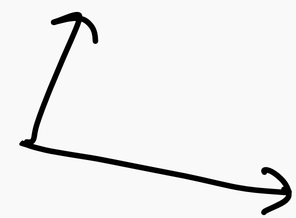


$\pi = \left( \frac{q}{p+q}, \frac{p}{p+q} \right)$

$$\pi P = \pi$$

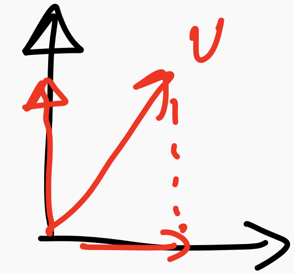
$$\theta P = (1 \quad -1) \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix} = (1-p-q \quad p-1+q)$$

$$= (1-p-q)(1 \quad -1)$$

$$\lambda = 1-p-q \qquad = \lambda (1, -1).$$

$$-1 < \boxed{\overset{\lambda}{1-p-q}} < 1 \qquad -1<\lambda<1$$

$$\theta = (1 \;\; -1)$$



$$\pi = \left(\frac{q}{p+q}, \frac{p}{p+q}\right)$$

Suppose $v$ is an arbitrary vector

$$v = c_1\pi + c_2\theta$$

$$vP^n = (c_1\pi + c_2\theta)P^n$$

$$= c_1 \pi P^n + c_2 \underbrace{\theta P^n}$$

$$= c_1 \pi + \underbrace{c_2 \lambda^n \theta}_{\to 0}$$

$n \to \infty$

$$vP^n \longrightarrow c_1\pi$$

$[1 \;\; 0] P^n = [\text{first row of } P^n] = \pi$

$[0 \;\; 1] P^n = [\text{~~first~~ second row of } P^n] = \pi$

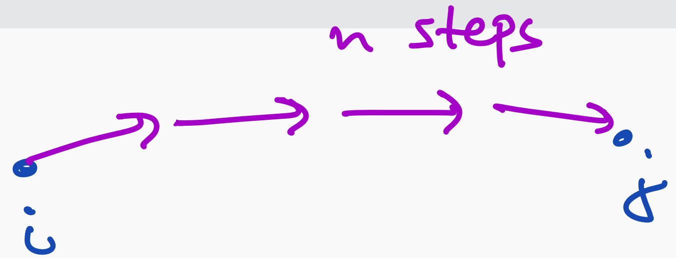
$$\underset{n \text{ large}}{P^n \approx} \begin{bmatrix} \pi \\ \hline \pi \end{bmatrix}$$

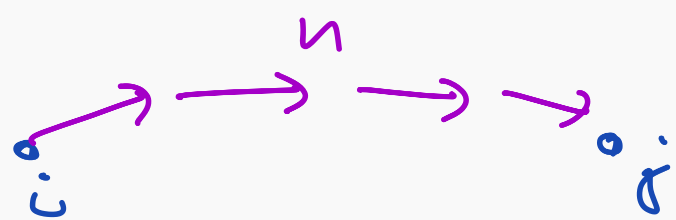
**Definition**

A Markov chain with transition matrix $P$ is called

- irreducible if for every $i$ and $j$ there exists $n$ such that $P^n_{ij} > 0$.
- ergodic ~~if for every $i$ and $j$ there exists $n$ such that~~ $P^n_{ij} > 0$.

if there is an $n > 0$ s.t. for every states $i, j$ $P^n_{ij} > 0$



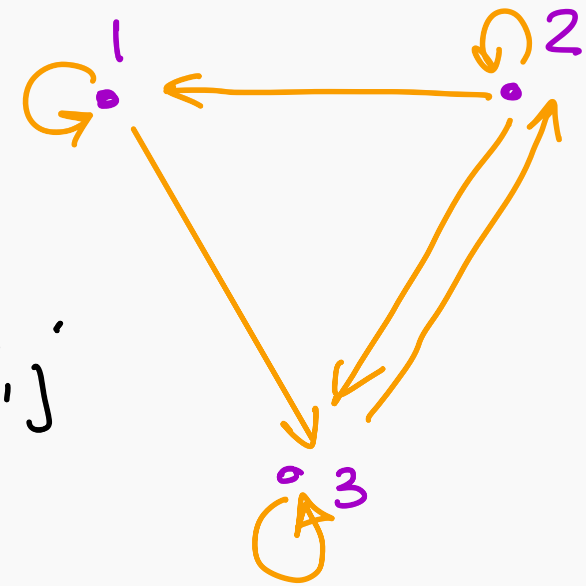



# Example

Determine whether the Markov chain with the matrix $P$ below is irreducible/ergodic.

$$P = \begin{pmatrix} 1/3 & 0 & 2/3 \\ 1/3 & 1/3 & 1/3 \\ 0 & 1/2 & 1/2 \end{pmatrix}$$

$$P^2 = \begin{pmatrix} + & 0 & + \\ + & + & + \\ 0 & + & + \end{pmatrix} \begin{pmatrix} + & 0 & + \\ + & + & + \\ 0 & + & + \end{pmatrix} = \begin{pmatrix} + & + & + \\ + & + & + \\ + & + & + \end{pmatrix}$$




$\underline{n=2}$ $\quad P^2_{ij} > 0$ for all $i,j$

$\Longrightarrow$ Markov chain is $\underline{\text{ergodic}}$.

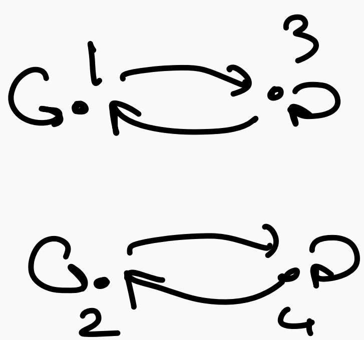
# Example

Determine whether the Markov chain with the matrix $P$ below is irreducible/ergodic.

$$P = \begin{pmatrix} 1/3 & 0 & 2/3 & 0 \\ 0 & 1/3 & 0 & 2/3 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/3 & 0 & 2/3 \end{pmatrix}$$

Markov chain is not irreducible $\Longrightarrow$ not ergodic.

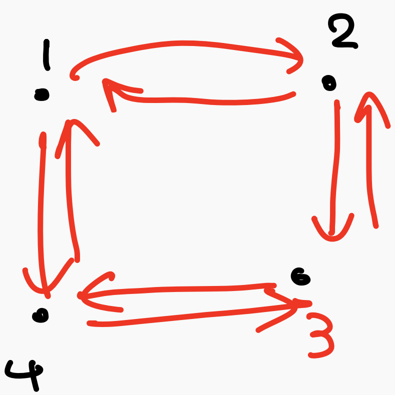
$$\begin{pmatrix} + & 0 & + & 0 \\ 0 & + & 0 & + \\ + & 0 & + & 0 \\ 0 & + & 0 & + \end{pmatrix}$$

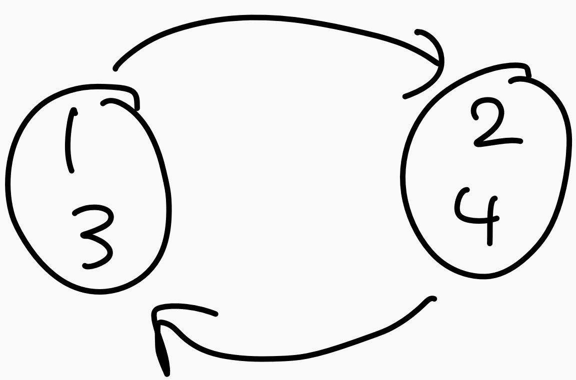


Determine whether the Markov chain with the matrix $P$ below is irreducible/ergodic.

$$P = \begin{pmatrix} 0 & 2/3 & 0 & 1/3 \\ 1/3 & 0 & 2/3 & 0 \\ 0 & 1/3 & 0 & 2/3 \\ 1/2 & 0 & 1/2 & 0 \end{pmatrix}$$

$$\begin{array}{c|c} & \begin{array}{cccc} 1 & 3 & 2 & 4 \end{array} \\ \hline \begin{array}{c} 1 \\ 3 \\ 2 \\ 4 \end{array} & \left(\begin{array}{cc|cc} 0 & 0 & \ast & \\ 0 & 0 & & \ast \\ \hline \ast & & 0 & 0 \\ & \ast & 0 & 0 \end{array}\right) \end{array}$$



## Convergence to the stationary distribution

**Theorem**

Let $P$ be the transition matrix of an ergodic Markov chain. Then

- There exists a unique stationary distribution $\pi$.
- When $n \to \infty$,

$$p_{ij}^{(n)} \to \pi_j.$$

In other words, the matrix $P^n$ converges to the matrix whose all rows are equal to $\pi$.

$$P = \begin{bmatrix} p_{ij} \end{bmatrix} \qquad P^n \to \begin{bmatrix} \pi \\ \pi \\ \pi \\ \vdots \\ \pi \end{bmatrix}$$

# Example

Compute the stationary distribution for the Markov chain with the transition matrix

$$P = \begin{bmatrix} 1/6 & 1/3 & 1/2 \\ 1/8 & 3/4 & 1/8 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}.$$

# An important special cases

**Definition**

Let $P$ be the transition matrix of a Markov chain. Then $P$ is called doubly stochastic if every column of $P$ adds up to 1. In particular, if $P_{ij} = P_{ji}$ then $P$ is doubly stochastic.

$$P = \begin{bmatrix} 0.3 & 0.7 \\ 0.7 & 0.3 \end{bmatrix}$$

# An important special cases

**Theorem**

Suppose that the transition matrix of an ergodic Markov chain is doubly stochastic. Then the stationary ~~measure~~ distributi $\pi$ is the ~~stationary~~ uniform ~~measure~~ distributi.

in other words

$$\pi = \left[\frac{1}{N} \; \frac{1}{N} \; \cdots \; \frac{1}{N}\right]$$

$$\left[\frac{1}{N} \; \frac{1}{N} \; \cdots \; \frac{1}{N}\right] \begin{bmatrix} P_{11} & \cdots & P_{1N} \\ P_{N1} & \cdots & P_{N1} \end{bmatrix} = \left[\frac{1}{N}\underbrace{(P_{11} + \cdots + P_{N1})}_{1}, \; \cdots \; , \right]$$

$$\left[\frac{1}{N}, \; \frac{1}{N} \; \cdots \; \frac{1}{N}\right].$$

# Sampling from a large unstructured set

0 1 Sequences of length K

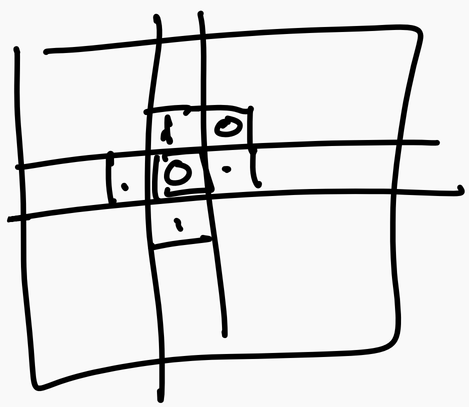
$(1, 0, 1 1 1 1, 0)$ $\qquad \frac{1}{2^K}$.

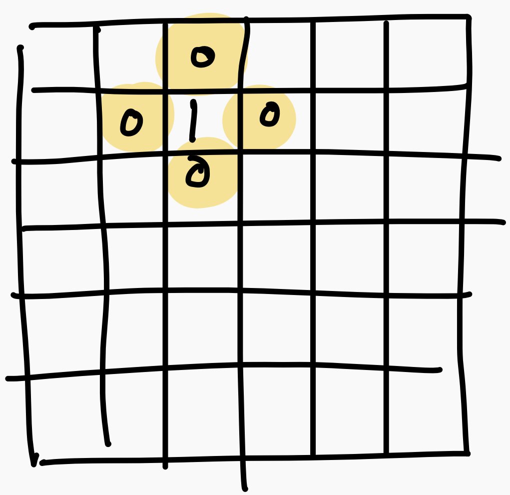
## Example

Let $K$ be a large positive integer and $M_K$ denote the set of $K \times K$ matrices $M$ of 0 and 1 with the constraint that no two 1s are in next to each other in the same row or the same column. In other words, if $M$ is such a matrix then $M(i,j) = 1$ implies

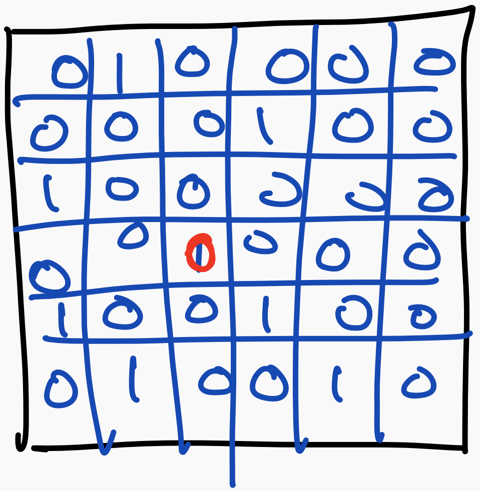
$$M(i,j+1) = M(i,j-1) = M(i-1,j) = M(i+1,j) = 0.$$

Find an algorithm to sample randomly from $M_K$.





# MCMC: Markov chain Monte Carlo

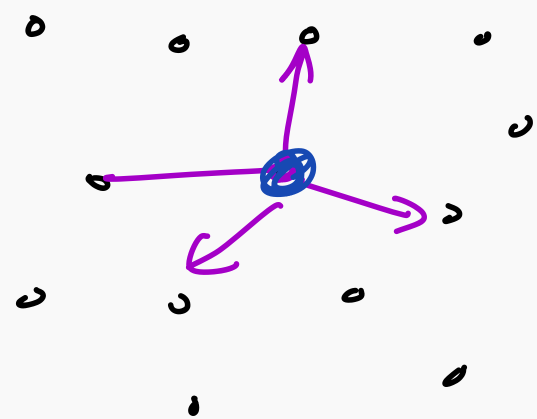


$P_{ij} = P_{ji}$

general scheme

$X$ set from which we want to sample

Define a ergodic Markov chain with space $X$ such that the stationary distribution is the uniform distribution



# Theorem of Perron-Frobenius

**Theorem**

Suppose $P$ is the transition matrix of an ergodic Markov chain. Then there exists a vector $\pi$ with positive entries such that

1. $\pi P = \pi$.
2. If $v$ is a vector with $vP = v$ then $v$ is a multiple of $\pi$.
3. For any other eigenvalue $\lambda \neq 1$ of $P$ we have $|\lambda| < 1$.