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Due: October 7, 2022
Assignment 2

Probability and Random Processes

(2.1) A bias coin has the probability 2/3 of turning up heads. The coin is thrown 4 times.

(a) What is the probability that the total number of heads shown is 3?

(b) Suppose that we know that outcome of the first throw is a head. Find the probability that the total number of heads shown is 3.

(c) If we know that the total number of heads shown is 3, find the probability that the outcome of the first throw was a head.

Solution. (a) 4, (b) and (c) each 8 points. Denote the event that the total number of heads shown is 3 by A. Then

$$\mathbb{P}[A] = \binom{4}{3}(2/3)^3(1/3) = \frac{32}{81}.$$

Suppose B denotes the event that outcome of the first throw is a head, so $\mathbb{P}[B] = 2/3$. Then $A \cap B$ is the event that the outcome of the first throw is a head and there was a total of 3 heads implying 2 heads in the remaining 3 throws. Hence

$$\mathbb{P}[A \cap B] = (2/3)\binom{3}{2}(2/3)^2(1/3) = \frac{8}{27}.$$

This gives

$$\mathbb{P}[A|B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]} = \frac{8/27}{2/3} = \frac{4}{9}.$$

For (c), note that

$$\mathbb{P}[B|A] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[A]} = \frac{8/27}{32/81} = \frac{3}{4}.$$

(2.2) Suppose that 15 percent of the messages arriving at a mailxbox are spam and that 20 percent of spam messages arriving there contain the word “winner”. Suppose also that the probability that the word “winner” appears in a non-spam message is 5 percent.

(a) What percentage of the received emails contain the word “winner”?

(b) Suppose that a message is tagged as spam based on containing the word “winner”. Find the probability that the message is indeed a spam.

Solution. each part 10 points. Denote the event that a message is spam by S and the event that it contains the word “winner” by W. The problem gives

$$\mathbb{P}[S] = 0.15, \quad \mathbb{P}[W|S] = 0.2, \quad \mathbb{P}[W|S^c] = 0.05.$$

Then

$$\mathbb{P}[W] = \mathbb{P}[W|S]\,\mathbb{P}[S] + \mathbb{P}[W|S^c]\,\mathbb{P}[S^c] = 0.2 \times 0.15 + 0.05 \times 0.85 = 0.03 + 0.0425 = 0.0725.$$

Using Bayes' formula, we can write

$$\mathbb{P}[S|W] = \frac{\mathbb{P}[W|S]\,\mathbb{P}[S]}{\mathbb{P}[W]} = \frac{0.2 \times 0.15}{0.0725} \approx 0.41.$$

(2.3) Suppose M is an integer randomly chosen from the set $\{1, 2 \ldots, 10\}$. Once M is chosen, the integer N is chosen from the set $\{1, 2, \ldots, M\}$. For instance if it turns out that $M = 7$, then N can take one of the values $1, \ldots, 7$, each with probability $1/7$.
(a) Find the probability that $N = 7$.
(b) Find the probability of the event $M = N$.

Solution. Each part 10 points. (a) It is clear that always $N \leq M$. Hence if $N = 7$, then M can take one of the values $7, 8, 9, 10$. This gives

$$\mathbb{P}[N = 7] = \sum_{i=7}^{10} \mathbb{P}[M = i]\, \mathbb{P}[N = 7 | M = i] = \frac{1}{10}\left(\frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10}\right).$$

(b) Again, we have

$$\mathbb{P}[M = N] = \sum_{i=1}^{10} \mathbb{P}[N = M | M = i]\, \mathbb{P}[M = i].$$

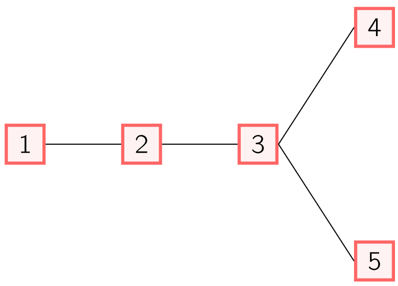
If $M = i$, then there are i options for N, one of which is i. Hence

$$\mathbb{P}[N = M | M = i] = \frac{1}{i}.$$

From here we have

$$\mathbb{P}[M = N] = \sum_{i=1}^{10} \frac{1}{10} \cdot \frac{1}{i}.$$

(2.4) Suppose $1 \leq i \leq 5$. A mouse starts at station i of the network below. At each junction, she moves to one of the adjacent stations with equal probability. For instance, once at 1, she can move to 0 or 2 with probability $1/2$, or from 0 she can move to 1 with probability 1. She will stop when she arrived at one of the stations 4 or 5. Let p_i denote the probability that the mouse eventually ends up at the station 4. Find a formula for p_i for $0 \leq i \leq 5$.



Solution. Computing each p_i has 4 points. We will use conditioning. Let A denote the event that the mouse ends up at 4 and B_i denote she is at station i. If the mouse starts at $i = 2$ then it can move to 1 and 3 with probability $1/2$. Hence, we have

$$p_2 = \mathbb{P}[A|B_2] = \frac{1}{2}\mathbb{P}[A|B_1] + \frac{1}{2}\mathbb{P}[A|B_3] = \frac{1}{2}p_1 + \frac{1}{2}p_3.$$

Similarly, we have

$$p_3 = \mathbb{P}[A|B_3] = \frac{1}{3}\mathbb{P}[A|B_2] + \frac{1}{3}\mathbb{P}[A|B_4] + \frac{1}{3}\mathbb{P}[A|B_5] = \frac{1}{3}p_2 + \frac{1}{3}.$$

We also have $p_1 = p_2$, $p_4 = 1$ and $p_5 = 0$. Solving this system of equation provides $p_1 = p_2 = p_3 = \frac{1}{2}$, $p_4 = 1$ and $p_5 = 0$.

(2.5) We say that an event A attracts B if $\mathbb{P}[B|A] > \mathbb{P}[B]$.

(a) Show that if A attracts B then B also attracts A.

(b) Suppose A attracts B and B attracts C. Does this imply that A attracts C?

Solution. 8 point for (a) and 12 points for (b). Note that $\mathbb{P}[B|A] > \mathbb{P}[B]$ holds when $\frac{\mathbb{P}[A \cap B]}{\mathbb{P}[A]} > \mathbb{P}[B]$, which is equivalent to $\mathbb{P}[A \cap B] > \mathbb{P}[A]\,\mathbb{P}[B]$. This form is clearly symmetric in A and B proving part (a).

(b) The answer in general is no. For instance, consider the event of throwing a die. Let $B = \{1, 2, 3\}$, $A = \{1\}$ and $C = \{2\}$. Then $\mathbb{P}[B|A] = 1 > \mathbb{P}[B]$ and $\mathbb{P}[C|B] = 1/3 > 1/6 = \mathbb{P}[C]$. Hence A attracts B and B attracts C. However, $\mathbb{P}[C|A] = 0$, while $\mathbb{P}[C] = 1/6$, hence A does not attract C.