

### Probability and Random Processes

(5.1) For  $\alpha > 1$ , suppose that  $X$  has the density function given by

$$f_X(t) = \begin{cases} \alpha e^{-\alpha t} & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Compute  $\mathbb{E}[e^X]$ .

(5.2) A die has been rolled twice. Let  $X$  denote the outcome of the first throw and  $Y$  denote the smaller of the two outcomes. For instance, if the outcomes are 2, 3 then  $X = 2$  and  $Y = 2$  and if the outcomes are 4, 3 then  $X = 4$  and  $Y = 3$ .

- (a) Describe the joint probability mass function of  $X$  and  $Y$  by drawing a table.
- (b) Compute the marginal probability mass functions of  $X$  and  $Y$ .
- (c) What are the possible values of  $Z = X - Y$ ? Compute the probability mass function of  $Z$  and use it to find  $\mathbb{E}[Z]$ .

(5.3) A coin is flipped three times. Let  $X$  denote the number of heads and  $Y$  denote the number of streaks of heads of length 2. For instance, if the outcome is  $HTH$ , then  $X = 2$  and  $Y = 0$ , while if the outcome is  $HHT$ , then  $X = 2$  and  $Y = 1$ .

- (a) Find the joint probability mass function of  $X$  and  $Y$ .
- (b) Determine  $\text{Cov}(X, Y)$ .
- (c) Are  $X$  and  $Y$  independent?

(5.4) If  $X$  and  $Y$  are two random variables prove that

$$\text{Var}[X + Y] = \text{Var}[X] + \text{Var}[Y] + 2\text{Cov}(X, Y).$$

(5.5) Let

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

be a  $2 \times 2$  matrix where  $A_{ij}$  are independent and each is uniformly chosen from the set  $\{1, 2, 3, 4, 5\}$ . Set  $D = \det A$ . Find  $\text{Var}[D]$ .