Probability and Random Processes

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The weak law of large numbers

The Weak Law of Large Numbers

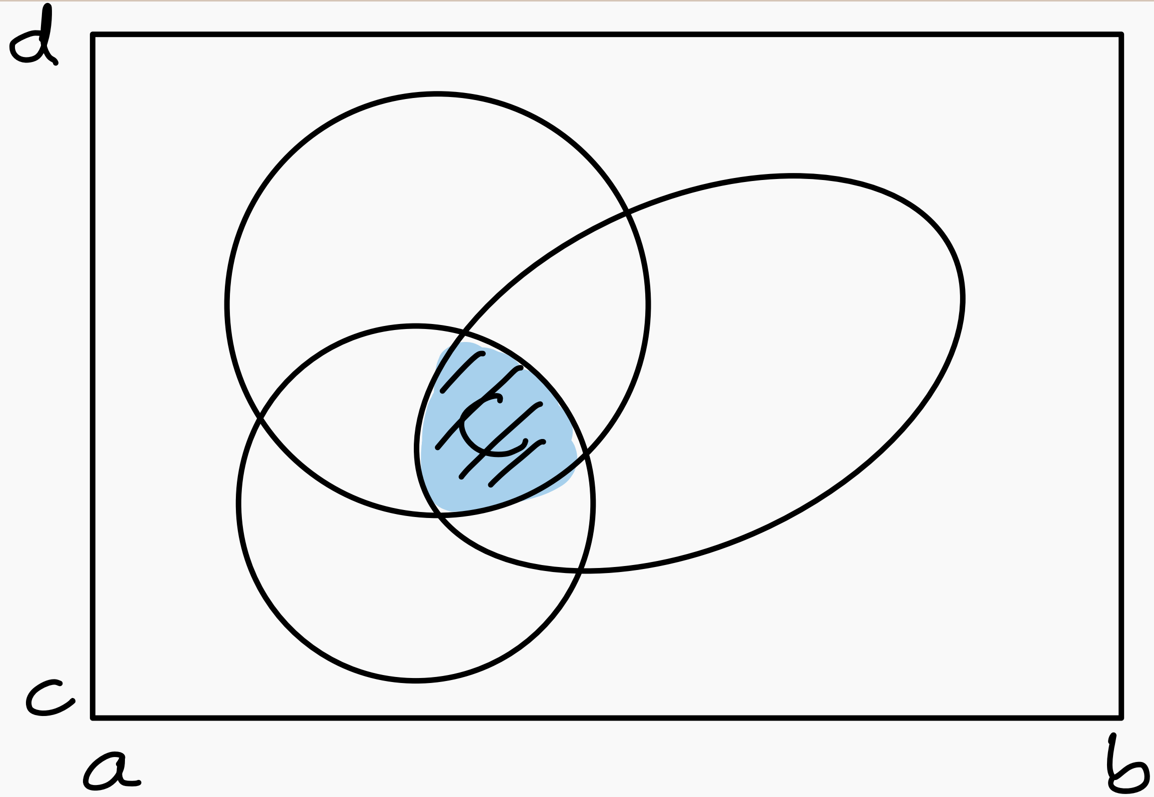
Consider a sequence X_n of identically distributed independent random variables. Suppose that they have finite expectation μ and finite variance . Then, for every $\epsilon > 0$, we have

$$\lim_{n\to\infty} \mathbb{P}\left[\left|\frac{X_1 + \cdots + X_n}{n} - \mu\right| > \epsilon\right] = 0.$$

empirical average

Proof

Application: Monte Carlo algorithm (toy version)



Area of box

$= (d-c)(b-a)$

$x \in [a,b]$ randomly uniform

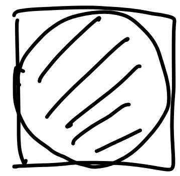
y $[c,d]$ random uniform.

$$\mathbb{P}[(X,Y) \text{ is in } C] = \frac{\text{area of } C}{(d-c)(b-a)}$$

$$\text{area of } C = (d-c)(b-a)\, \underbrace{\mathbb{P}[(X,Y) \text{ is in } C]}_{p}.$$

Suppose $(X_1, Y_1), (X_2, Y_2), \ldots$ are independently chosen from the box according to the uniform distribution

$Z_1, Z_2, \ldots$ all Bernoulli



$$Z_i = \begin{cases} 1 & (X_n, Y_n) \text{ is in } C \\ 0 & (X_n, Y_n) \text{ is not in } C. \end{cases}$$

$$\mathbb{P}[Z_i = 1] = \mathbb{P}[(X_n, Y_n) \text{ is in } C] = p$$

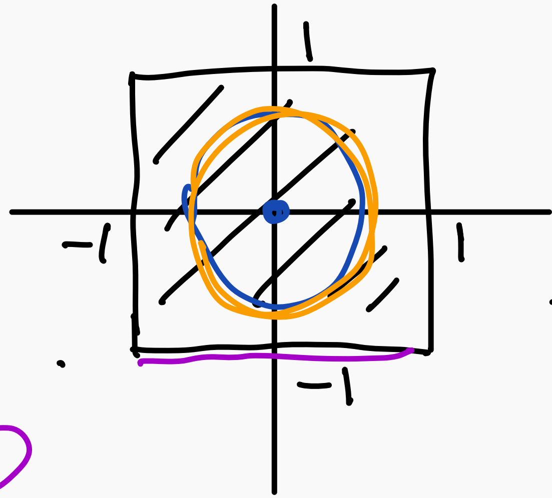
Z_i is a Bernoulli RV with parameter p.

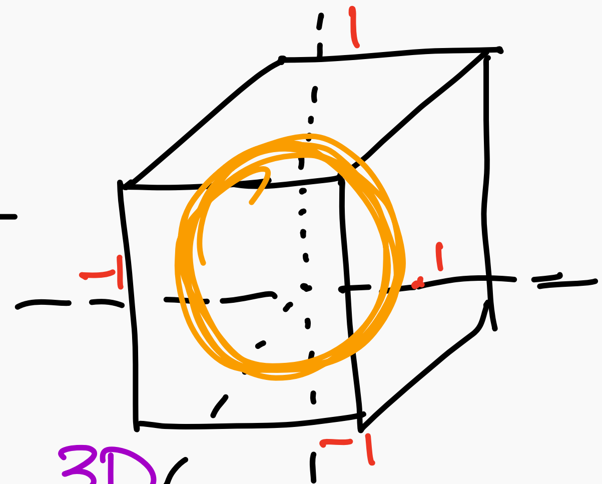
$$\mu = \mathbb{E}[Z_i] = p$$

<u>LLN</u>

$$\mathbb{P}\left[\left|\frac{Z_1 + Z_2 + \cdots + Z_n}{n} - p\right| > \varepsilon\right] \to 0$$

when $n \to \infty$





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n-dim hypercube/cube

2D

3D

nD

2D

(x_1, x_2)

$-1 \le x_1 \le 1$

$-1 \le x_2 \le 1$

2-dim volume area $= 2^n$

$0 \le \text{dist} \le \sqrt{2}$

3D

$-1 \le x_1 \le 1$

$-1 \le x_2 \le 1$

$-1 \le x_3 \le 1$

volume $= 2^3$

$0 \le \text{distance} \le \sqrt{3}$

nD

$-1 \le x_1 \le 1$

$-1 \le x_2 \le 1$

$\vdots$

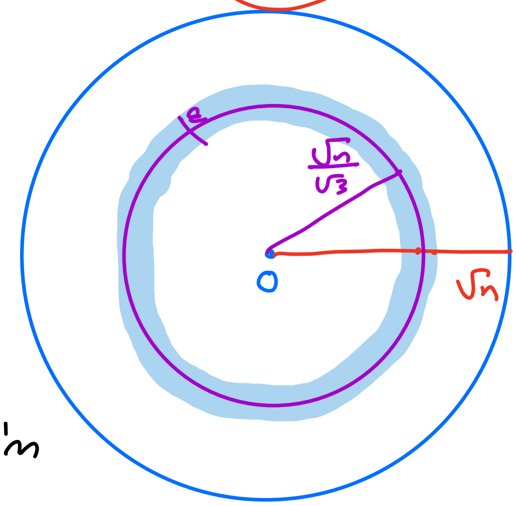
$-1 \le x_n \le 1$

volume $= 2^n$.

$0 \le \text{distance} \le \sqrt{n}$

- Most of the volume of the n dim cube is near the sphere of radius $\sqrt{\frac{n}{3}}$ centred at the origin.

$\frac{\sqrt{n}}{\sqrt{3}}$



$X=(X_1, X_2, \dots, X_n)$ n-dim cube

$X_1, X_2, \dots X_n$ are picked from $[-1, 1]$ independently.

distance from the point X to the origin is

$$D = \sqrt{X_1^2+X_2^2+\dots+X_n^2} \quad \text{random variable}$$

Consider the random variable

$$Y_1 = X_1^2, \quad Y_2 = X_2^2, \dots \quad Y_n = X_n^2.$$

X_i^2 comes from $[0, 1]$.

$$E[x_i^2] = \int_{-1}^{1} x^2 \cdot f(x)\,dx = \int_{-1}^{1} \frac{1}{2} x^2\,dx$$

$$= \frac{1}{2} \cdot \frac{x^3}{3} \Big|_{-1}^{1} = \frac{1}{3}$$

$$\boxed{\mathbb{E}[h(X)] = \int_{-\infty}^{+\infty} h(x)\, f(x)\, dx}$$ $h(x) = x^2$

$n \to \infty$.

LLN

$$\mathbb{P}\left[\left| \frac{X_1^2 + X_2^2 + \cdots + X_n^2}{n} - \frac{1}{3} \right| > \varepsilon \right] \to 0$$

n large, with a prob. close to 1

$$\frac{1}{3} - \varepsilon < \frac{X_1^2 + \cdots + X_n^2}{n} < \frac{1}{3} + \varepsilon$$

$$\sqrt{n\left(\frac{1}{3} - \varepsilon\right)} < \underbrace{\sqrt{X_1^2 + \cdots + X_n^2}}_{D} < \sqrt{n\left(\frac{1}{3} + \varepsilon\right)}$$

Concentration of measure phenomenon II

A point is randomly chosen from

$$Q_n = \{x = (x_1, \dots, x_n) : -1 \leq x_i \leq 1.\}$$

Set

$$A_n = \left\{ x \mid (1-\epsilon)\sqrt{\frac{n}{3}} < \|x\| < (1+\epsilon)\sqrt{\frac{n}{3}} \right\}.$$

Then for any given $\epsilon > 0$ we have $\mathbb{P}[A_n] \to 1$ as $n \to \infty$.

The idea of the central limit theorem

The Central limit theorem

Let $X_1, X_2, \ldots$ be i.i.d. with $\mathbb{E}[X_i] = \mu$ and $\mathrm{Var}[X_i] = \sigma^2$. Then the distribution of

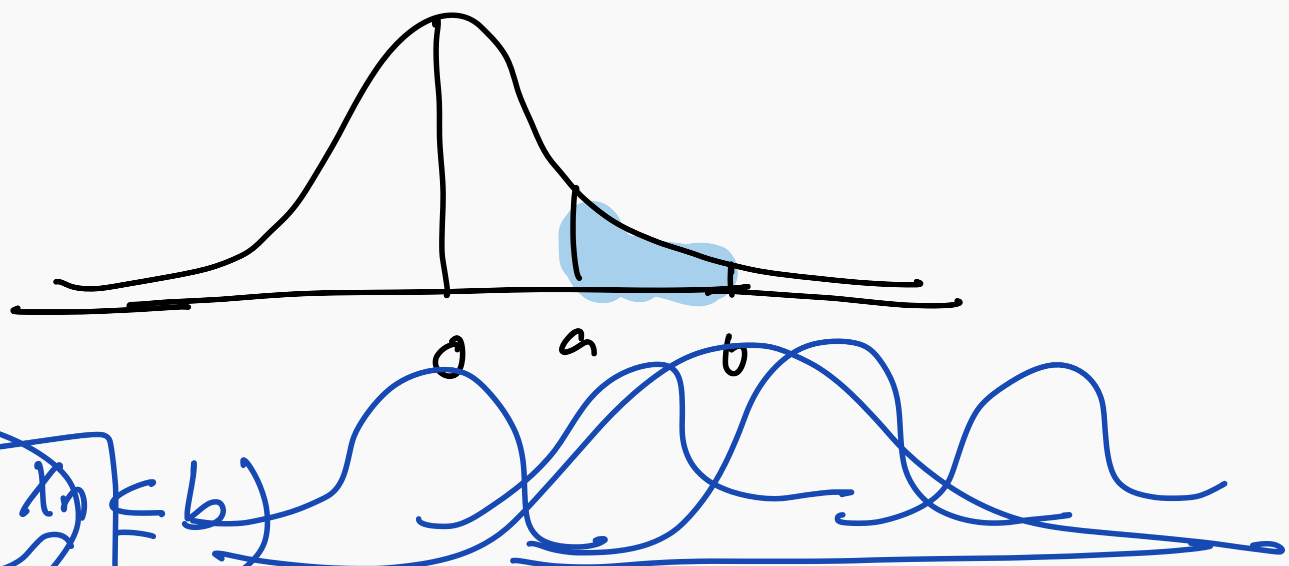
$$Z_n = \frac{\overbrace{\boxed{X_1 + \cdots + X_n}}^{\text{sum}} - n\mu}{\sqrt{n}\sigma}$$

centered sum

converges to the distribution of a standard normal distribution. In other words:

$$\lim_{n\to\infty} \mathbb{P}(a \le Z_n \le b) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-y^2/2} dy. \tag{1}$$

$\mu = 1$



$n\mu$

$P(a \le \boxed{X_1 + \cdots + X_n} \le b)$

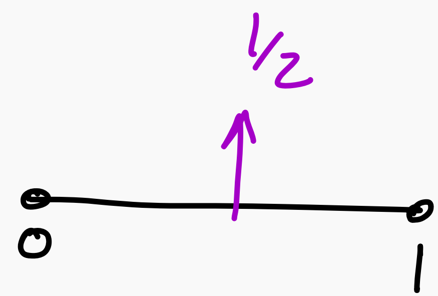
$\mathrm{Var}(X_1 + \cdots + X_n) = n \mathrm{Var} X_i$
$= n\sigma^2$

24 numbers are randomly and independently chosen from the interval $[0, 1]$ according to the uniform distribution. Find the approximate value of the probability that the sum of the numbers is at least 8.

$$S_{24} = X_1 + X_2 + \cdots + X_{24}$$

X_i has uniform distribution in $[0,1]$

$$E[X_i] = \frac{1}{2} = \mu$$



$$Var[X_i] = E[X^2] - E[X]^2$$

$$E[X^2] = \int_0^1 x^2 \, dx = \frac{1}{3}$$

$$Var[X_i] = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} = \sigma^2 \Rightarrow \sigma = \sqrt{\frac{1}{12}} = \frac{1}{2}\sqrt{\frac{1}{3}}$$

$$S_{24} = \frac{X_1 + \cdots + X_{24}}{24} \approx \frac{1}{2} \qquad n = 24, \quad \mu = \frac{1}{2}$$

$$\frac{X_1 + \cdots + X_{24} - 12}{\sqrt{24} \cdot \sqrt{\frac{1}{12}}} \sim \text{standard normal distribution}$$

standard normal

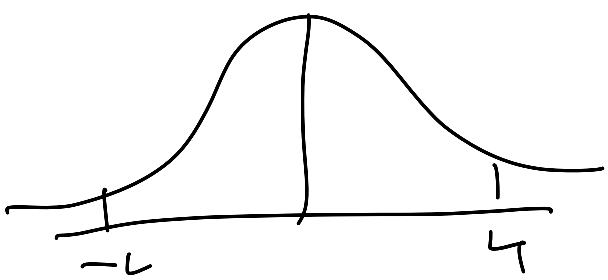
$$P\left(a < \frac{\overbrace{\boxed{X_1 + \cdots + X_{24}}}^{S} - 12}{\sqrt{2}} < b\right) \approx P(a < N < b)$$

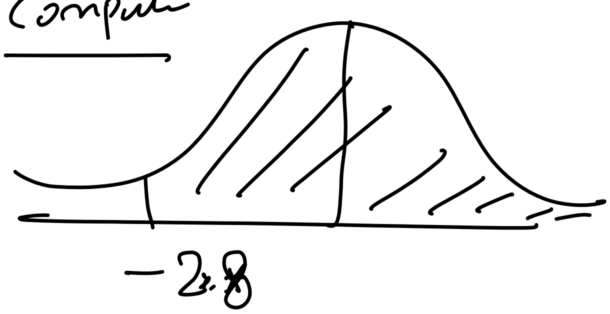
$$\frac{S_n - n\mu}{\sqrt{n}\,\sigma}$$

$$P(S \geq 8) = P\left(\boxed{\frac{S-12}{\sqrt{2}}} > \frac{8-12}{\sqrt{2}}\right)$$

$$= P\left(\frac{S-12}{\sqrt{2}} > \frac{-4}{\sqrt{2}}\right) \approx P\left(N > \frac{-4}{\sqrt{2}}\right)$$

$$\approx P(N > -2.8) \to \underline{\text{compute}}$$





Moment generating functions of a random variable

Definition

Consider the following expected value:

$$M_X(t) = \mathbb{E}\left[e^{tX}\right].$$

It is called the moment generating functionfor X.

X any variable.

Example

Find the moment generating function for a Bernoulli random variable with parameter p.

$$M_X(t) = \mathbb{E}[e^{tX}] = p \cdot e^t + (1-p) \cdot 1$$

$$\boxed{M_X(t) = pe^t + 1 - p}$$

X Bernoulli with parameter p

X:

- 0 $\quad 1-p \quad \longrightarrow e^{tX} = 1$
- 1 $\quad p \quad \longrightarrow e^{tX} = e^t$

Example

Find the moment generating function for a Poisson random variable with parameter λ.

Poisson : $k = 0, 1, 2, \ldots$

$$\mathbb{P}(X=k) = \frac{e^{-\lambda} \cdot \lambda^k}{k!} \qquad X=k \Longrightarrow e^{tX} = e^{tk}$$

$$\mathbb{E}[e^{tX}] = \sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!} e^{tk} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^t)^k}{k!}$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$= e^{-\lambda} \cdot e^{\lambda e^t} \to x$$

$$\boxed{M_X(t) = e^{\lambda(e^t - 1)}}$$