Probability and Random Processes

Keivan Mallahi-Karai

4.11.2022

Jacobs University

Agenda

1. joint probability density function of continuous random variables
2. Independence
3. Computing the pdf of functions of random variables
4. Conditional probability mass functions
5. Law of iterated expectations and its applications

Recall: definition of continuous random variables

Definition

X is a continuous random variable if there exists a non-negative $f : \mathbb{R} \to \mathbb{R}$ such that

1. (version 1) For all real values of a we have

   t

   $$\mathbb{P}[X \leq t] = \int_{-\infty}^{t} f(x)\ dx.$$

2. (version 2) For all intervals $J = [t_1, t_2]$ we have

   $X \in [t_1, t_2]$

   t_1 t_2

   $$\mathbb{P}[t_1 \leq X \leq t_2] = \int_J f(x)\ dx = \int_{t_1}^{t_2} f(x)\ dx.$$

3. (version 3) For all subsets $B \subseteq \mathbb{R}$ we have

   B

   $$\mathbb{P}[X \in B] = \int_B f(x)\ dx.$$

Two continuous random variable X, Y

$$X \longrightarrow f_X(x)$$

$$Y \longrightarrow f_Y(y)$$

discrete version

$Y \backslash X$	x_1	x_2	$\cdots$	x_n
y_1	$\cdot$	$\cdot$		$\cdot$
$\vdots$		$p_{X,Y}(x,y)$		
y_n				

joint probability density function

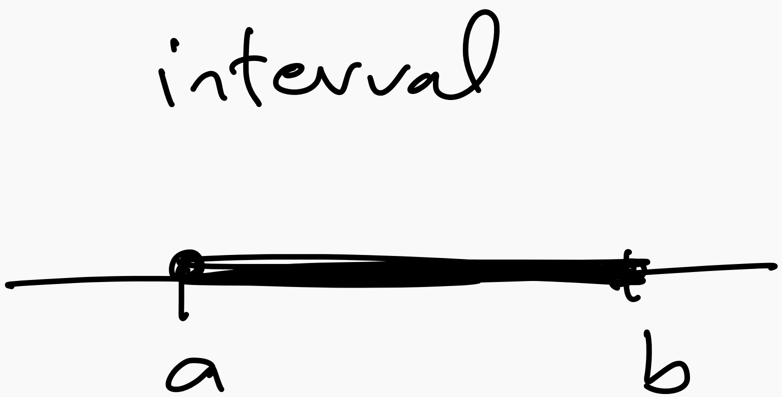
$$f_{X,Y}(x,y)$$

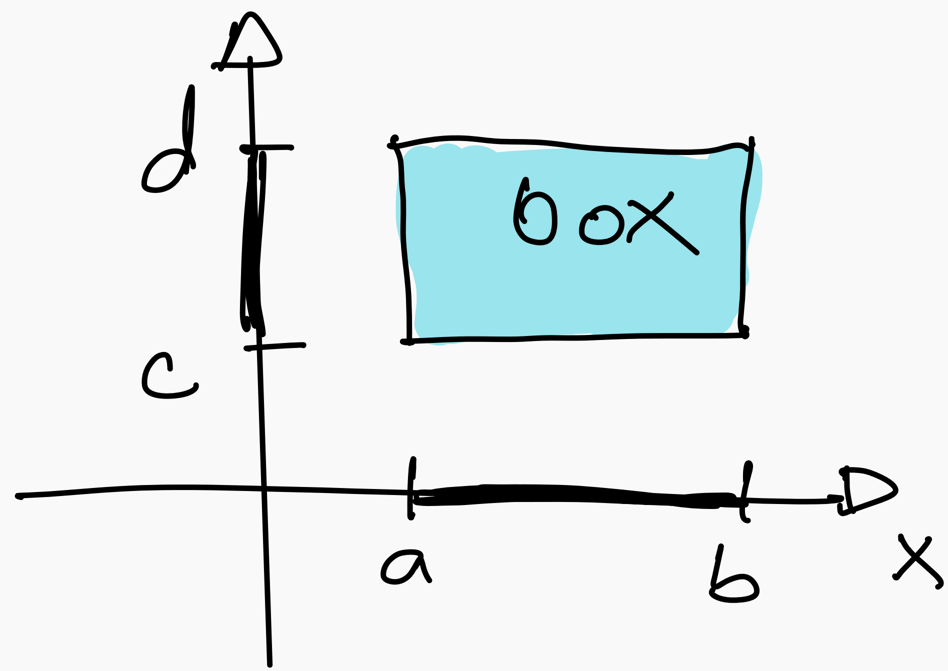
Definition of two-dimensional boxes

Definition

A two-dimensional box is a subset of $\mathbb{R}^2$ defined by

$$J = [a, b] \times [c, d]$$





Joint probability density function of two random variables

Definition

The joint probability density function of two random continuous variables X and Y a non-negative $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that $\to$ two variable function

1. (version 1) For real values of s, t we have $\int_{y=-\infty}^{t} \int_{x=-\infty}^{s} f(x,y)\,dx\,dy$

   $(X,Y) \in C_{s,t}$

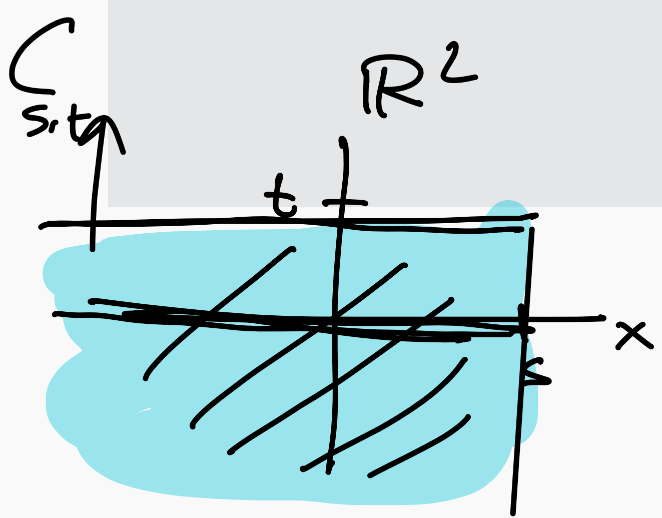
$$\mathbb{P}[X \le s, Y \le t] = \int_{x=-\infty}^{s} \int_{y=-\infty}^{t} f(x,y)\, \cancel{dxdy.}\ dy\, dx$$

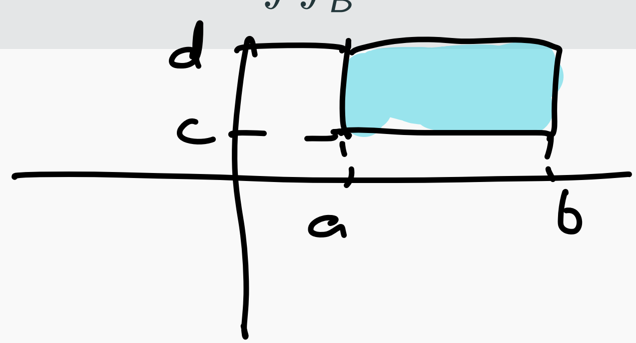
2. (version 1) For a two-dimensional box $J = [s_1, s_2] \times [t_1, t_2]$ we have

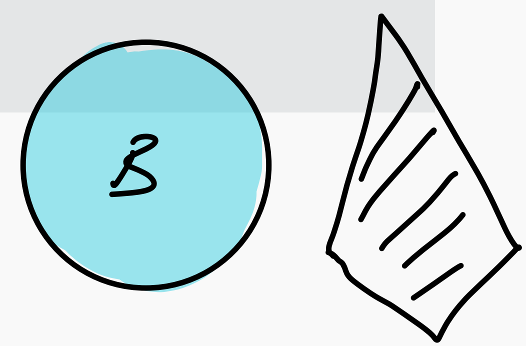
$$\mathbb{P}[a \le X \le b, c \le Y \le d] = \int_{y=c}^{y=d} \left(\int_{x=a}^{x=b} f(x,y)\, dx \right) dy.$$

3. (version 3) For all subsets $B \subseteq \mathbb{R}$ we have

$$\mathbb{P}[\overset{(X,Y)}{\cancel{X}} \in B] = \iint_B f(x,y)\, dx\, dy.$$

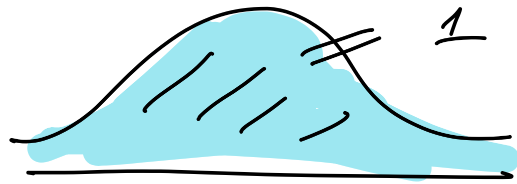






1-dim case

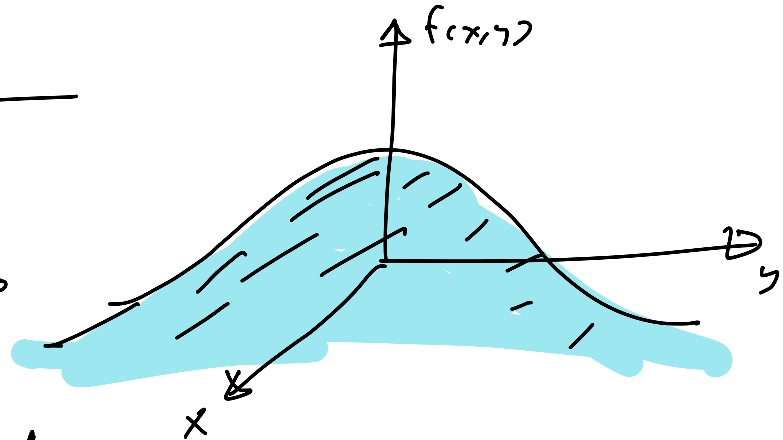
- $f(x) \geq 0$
- $\int_{-\infty}^{+\infty} f(x)\, dx = 1$



2-dim

$f(x, y) \geq 0$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\, dx\, dy = 1$$

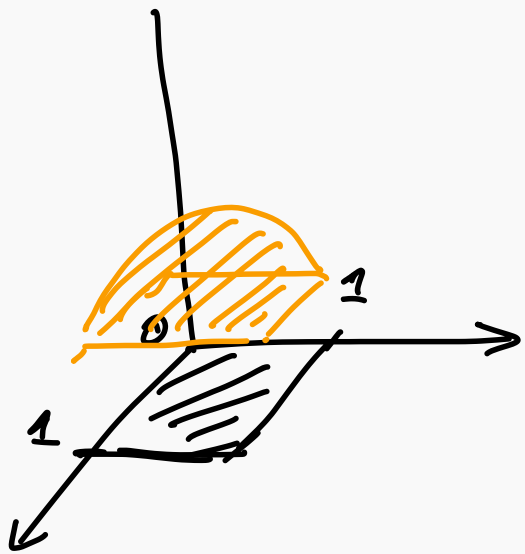


Example

Suppose X and Y have the joint probability density function given by

$$f(x,y) = \begin{cases} c(x^2+y^2) & \text{if } 0 \le x, y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

1. Determine the value of c.
2. Find the marginal pdfs of X and Y.



$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\,dx\,dy = 1$$

$$c\left(\underbrace{\int_0^1\int_0^1 (x^2+y^2)\,dx\,dy}_{2/3}\right) = 1 \Rightarrow c = \frac{3}{2}$$

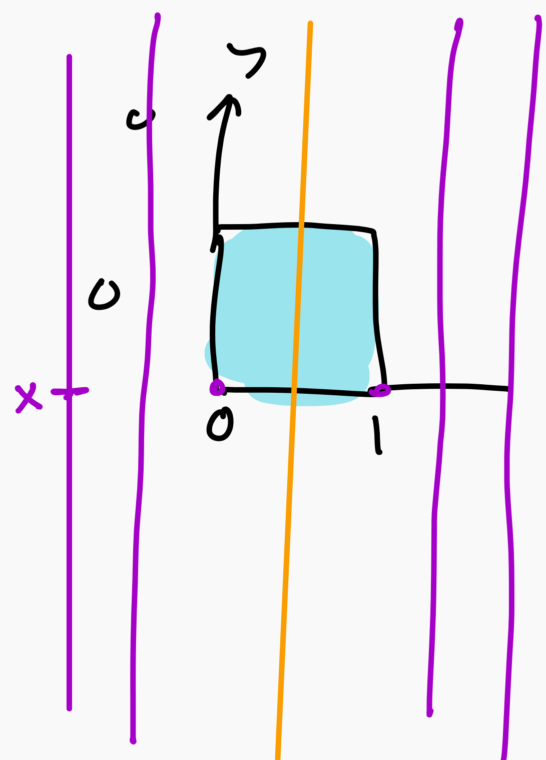
$$\int_0^1 \left(\int_0^1 (x^2+y^2)\, dx \right) dy =$$

$$\int_0^1 \left(\frac{x^3}{3} + y^2 x \Big]_{x=0}^{1} \right) dy$$

$$= \int_0^1 \left(\frac{1}{3} + y^2\right) dy = \frac{1}{3} y + \frac{y^3}{3} \Big|_0^1$$

$$= \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

$$f_{X,Y}(x,y) = \begin{cases} \frac{3}{2}(x^2+y^2) & 0 \le x, y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$



Discrete world

$$P_X(x) = \sum_y P_{X,Y}(x,y)$$

$$P_Y(y) = \sum_x P_{X,Y}(x,y)$$

Continuous world

$$f_X(x) = \int_{y=-\infty}^{+\infty} f_{X,Y}(x,y)\, dy$$

$$f_Y(y) = \int_{x=-\infty}^{+\infty} f_{X,Y}(x,y)\, dx$$

$$f_X(x) = \int_{y=-\infty}^{+\infty} f_{X,Y}(x,y)\, dy = \begin{cases} & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\underset{0 \le x \le 1}{\overset{\uparrow}{=}} \int_{y=0}^{1} \frac{3}{2}(x^2+y^2)\, dy$$

$$= \frac{3}{2}\left(x^2 y + \frac{y^3}{3}\right)\Bigg|_{y=0}^{1}$$

$$= \frac{3}{2}\left(x^2 + \frac{1}{3}\right)$$

$$f_X(x) = \begin{cases} \frac{3}{2}\left(x^2+\frac{1}{3}\right) & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{3}{2}\left(y^2+\frac{1}{3}\right) & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Let B be a subset of $\mathbb{R}^2$. Suppose that the joint probability density function of X and Y is given by

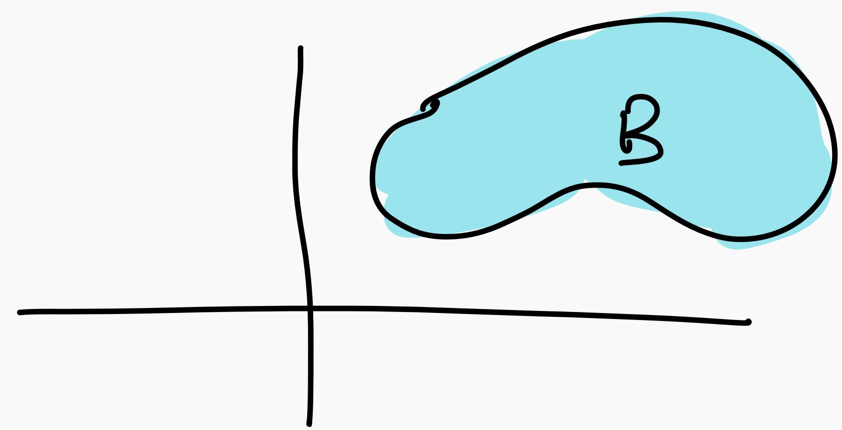
$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{\text{area}(B)} & \text{if } (x,y) \in B \\ 0 & \text{otherwise} \end{cases}$$

1-dim uniform distribution over $[a,b]$

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

$b-a =$ length of $[a,b]$

$$f_{X,Y}(x,y) = \begin{cases} c & (x,y) \in B \\ 0 & (x,y) \notin B \end{cases}$$



$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\,dx\,dy = 1$$

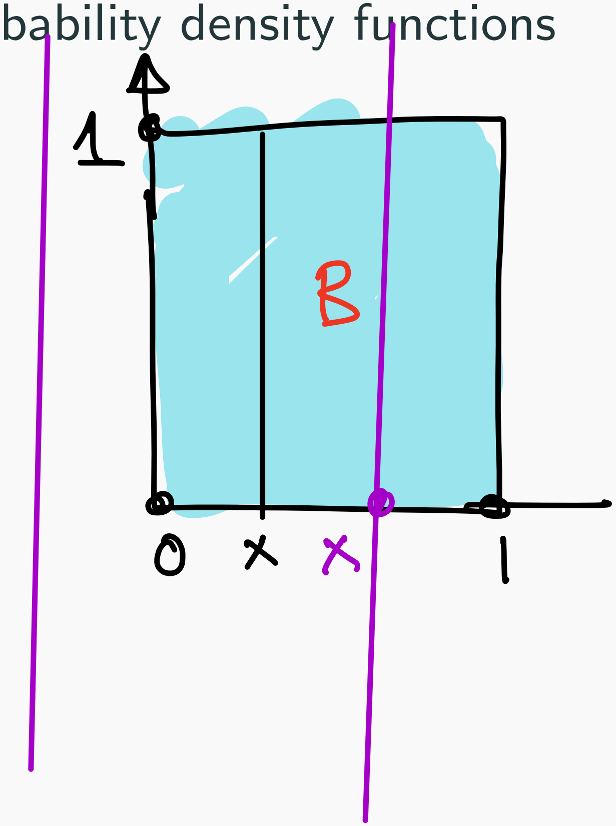
$$\iint_B c\,dx\,dy = 1 \Rightarrow c \underbrace{\iint_B dx\,dy}_{\text{area of } B} = 1$$

$$c = \frac{1}{\text{area of } B}$$

$P = (X, Y)$

A point P is chosen uniformly and randomly from the unit square $[0,1]^2$. Let X and Y denote the coordinates of P. Find the joint probability density functions of X and Y.

$$f_{X,Y}(x,y) = \begin{cases} 1 & \underbrace{0 \le x \le 1,\ 0 \le y \le 1}_{(x,y) \in B} \\ 0 & \text{otherwise} \end{cases}$$



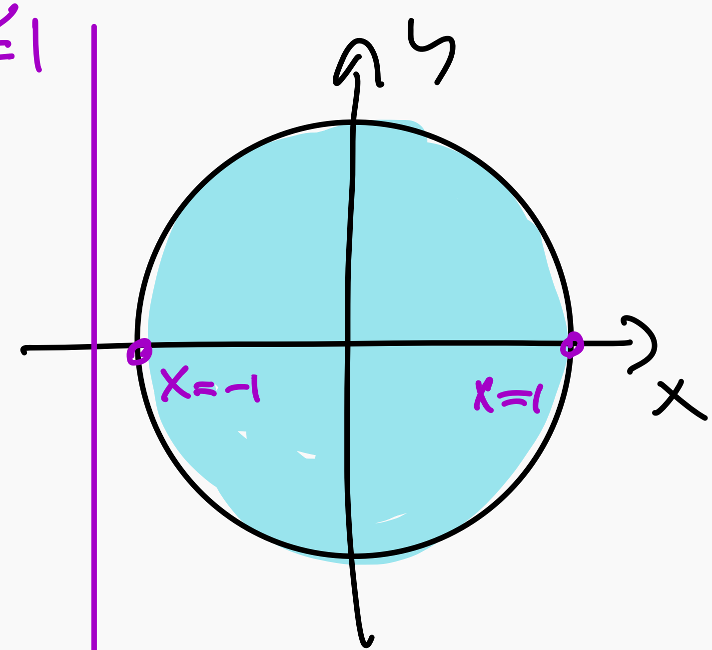
$$f_X(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

density function for a uniform random variable over $[0,1]$

Example

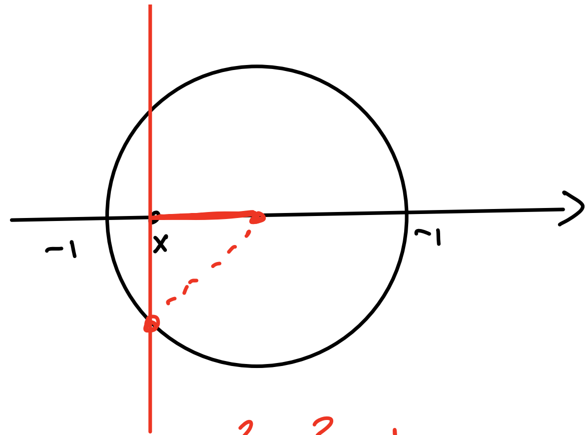
A random point (X, Y) is chosen in the unit circle $B = \{(x, y) : x^2 + y^2 \leq 1\}$. Find the joint probability density function of X and Y and individual probability density functions of X and Y. $x^2+y^2 \leq 1$

$$f_{X,Y} = \begin{cases} \frac{1}{\pi} & (X,Y) \text{ inside the circle} \\ 0 & (X,Y) \text{ outside the circle } x^2+y^2>1 \end{cases}$$



$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dy = \begin{cases} \cdots & -1 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$= \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi}\, dy$$



$$x^2 + y^2 = 1$$
$$y^2 = 1 - x^2$$
$$y = \pm\sqrt{1-x^2}$$

$$\frac{1}{\pi} y \Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}$$

$$= \frac{2}{\pi}\sqrt{1-x^2}$$

$$f_{X}(x) = \begin{cases} \frac{2}{\pi}\sqrt{1-x^2} & -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \qquad f_Y(y) = \begin{cases} \frac{2}{\pi}\sqrt{1-y^2} & -1 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_X(0) = \frac{2}{\pi}$$
$$f_X\left(\frac{4}{5}\right) = \frac{2}{\pi}\sqrt{1-\left(\frac{4}{5}\right)^2} = \frac{2}{\pi}\sqrt{\frac{9}{25}} = \frac{6}{5\pi} = \frac{2}{\pi}\cdot\frac{3}{5}$$

$$f_X(x)\, f_Y(y) = \begin{cases} \frac{4}{\pi^2}\sqrt{1-x^2}\sqrt{1-y^2} & -1 \le x \le 1 \quad -1 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$
$$\neq f_{X,Y}(x,y).$$

Independence

Definition

Continuous random variables X and Y are independent if

$$f_{X,Y}(x,y) = f_X(x) f_Y(y).$$

Ex ① (X,Y) Pick randomly from the unit square

$$f_X(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases} \qquad f_Y(y) = \begin{cases} 1 & 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_X(x) f_Y(y) = \begin{cases} 1 & \begin{matrix} 0 \leq x \leq 1 \\ 0 \leq y \leq 1 \end{matrix} \\ 0 & \text{otherwise} \end{cases} = f_{X,Y}(x,y)$$

② (X,Y) picked randomly from the unit circle.

Example

Suppose X and Y are independent exponentially distributed random variables with ~~mean~~ Paramter 1. Determine the joint probability density function of X and Y and use it to compute the probability distribution and probability density function of $Z = \max(X, Y)$.

$$\exp(\lambda) \rightsquigarrow \lambda e^{-\lambda x}$$

$$f_X(x) = \begin{cases} e^{-x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$f_Y(y) = \begin{cases} e^{-y} & y \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$f_{X,Y}(x,y) = f_X(x) f_Y(y) = \begin{cases} e^{-x} \cdot e^{-y} & x \geq 0,\ y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$