Probability and Random Processes

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Joint probability density function of two random variables

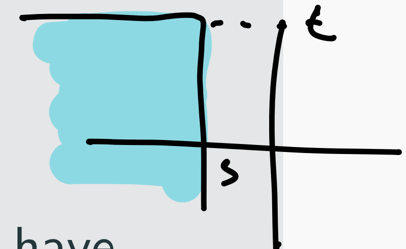
Definition

The joint probability density function of two random continuous variables X and Y a non-negative $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that

1. (version 1) For real values of s, t we have

   $\mathbb{P}(X \leq t)$

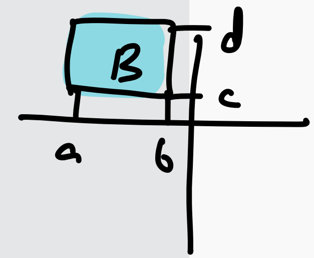
   $$\mathbb{P}[X \leq s, Y \leq t] = \int_{x=-\infty}^{s} \int_{y=-\infty}^{t} f(x, y)\ dxdy.$$

2. (version 2) For a two-dimensional box $J = [s_1, s_2] \times [t_1, t_2]$ we have

   $\mathbb{P}(a \leq X \leq b)$

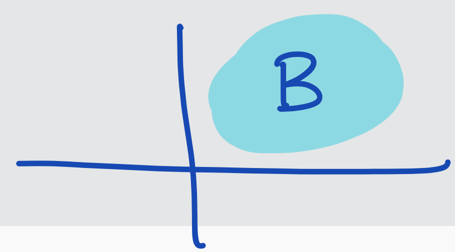
   $$\mathbb{P}[a \leq X \leq b, c \leq Y \leq d] = \int_{c}^{d} \int_{a}^{b} f(x, y)\ dx\ dy.$$

3. (version 3) For all subsets $B \subseteq \mathbb{R}$ we have

   $\mathbb{P}($

   (X, Y) point in $\mathbb{R}^2$ $\quad$ $$\mathbb{P}[(X, Y) \in B] = \iint_B f(x, y)\ dx\ dy.$$

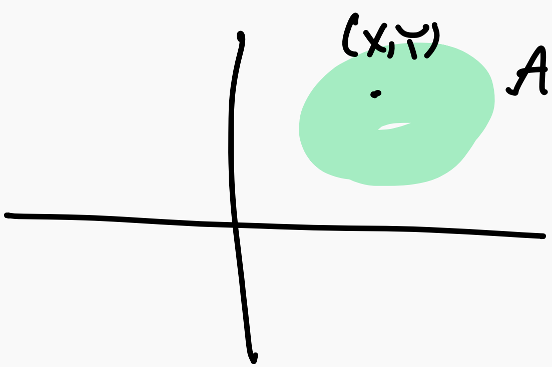
Key properties

- $$f(x, y) \geq 0.$$
- $$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx \; dy = 1.$$

Example: uniform distribution on a region in the plane:

Let A be a subset of $\mathbb{R}^2$. Suppose that the joint probability density function of X and Y is given by

$$f_{X,Y}(x, y) = \begin{cases} \frac{1}{\text{area}(A)} & \text{if } (x, y) \in A \\ 0 & \text{otherwise} \end{cases}$$

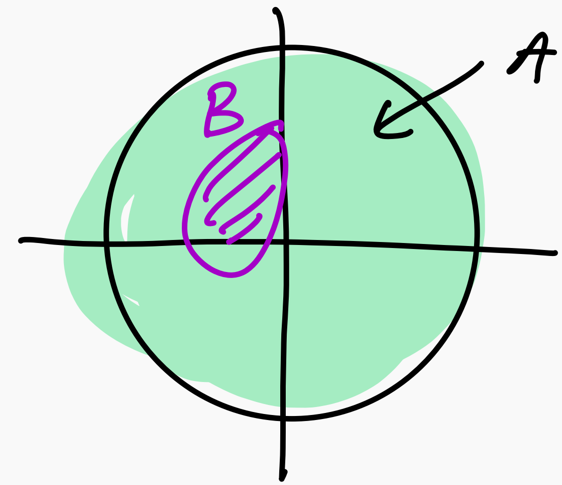


Example

A random point (X, Y) is chosen in the unit circle $A = \{(x, y) : x^2 + y^2 \leq 1\}$. Find the joint probability density function of X and Y and individual probability density functions of X and Y.

$$X^2 + Y^2 \leq 1$$

$$\mathbb{P}\left((X, Y) \in B \right) = \frac{\text{area } B}{\text{area } A}$$



Independence

Definition

Continuous random variables X and Y are independent if

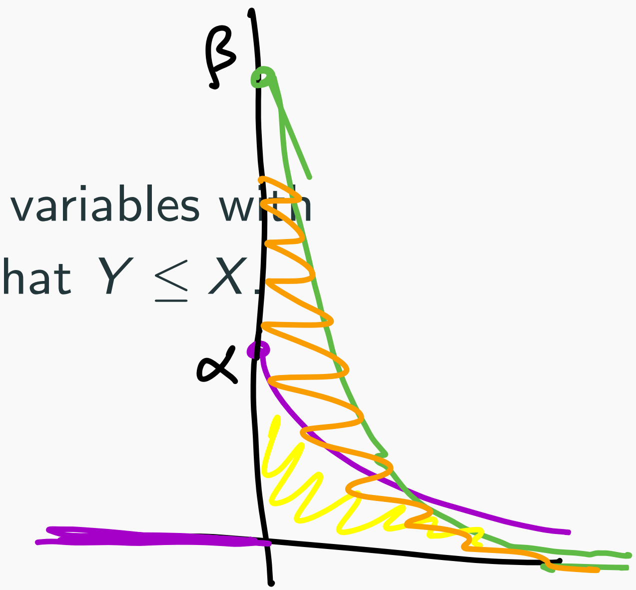
$$f_{X,Y}(x, y) = f_X(x)f_Y(y).$$

Suppose X and Y are independent exponential random variables with parameters α and β. Find the probability of the event that $Y \leq X$.

$$f_X(x) = \begin{cases} \alpha e^{-\alpha x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$\beta > \alpha$

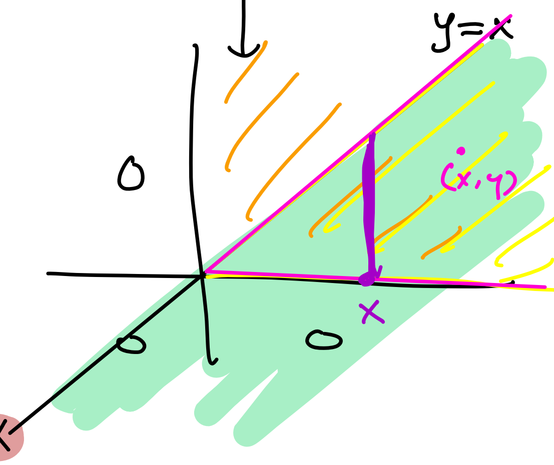
$$f_Y(y) = \begin{cases} \beta e^{-\beta y} & y \geq 0 \\ 0 & x < 0 \end{cases}$$



$$f_{X,Y}(x,y) = f_X(x)\, f_Y(y) = \begin{cases} \alpha\beta\, e^{-(\alpha x+\beta y)} & x>0,\ y\geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{P}(Y \leq X)$$

$$= \mathbb{P}\big((X,Y) \in \_\_\_\big)$$



$$e^{-\alpha x}\cdot e^{-\beta y}$$

$$= \int_{x=0}^{\infty} \int_{y=0}^{x} \alpha\beta\, e^{-(\alpha x+\beta y)}\, dy\, dx$$

$$= \int_{x=0}^{\infty} \left[\alpha\beta \cdot e^{-\alpha x} \cdot \frac{e^{-\beta y}}{-\beta} \Bigg|_{y=0}^{x} \right] dx$$

$$= \int_{x=0}^{\infty} \left[\frac{\alpha\cancel{\beta}\, e^{-\alpha x}\cdot e^{-\beta x}}{-\cancel{\beta}} \cancel{+} \alpha\cancel{\beta}\, e^{-\alpha x}\cdot \frac{1}{\cancel{+\beta}} \right] dx$$

$$= \int_{x=0}^{\infty} \left(\alpha e^{-\alpha x} - \alpha e^{-(\alpha+\beta)x}\right) dx = \frac{\alpha e^{-\alpha x}}{-\alpha}\Bigg|_{x=0}^{\infty} + \frac{\alpha e^{-(\alpha+\beta)x}}{\alpha+\beta}\Bigg|_{x=0}^{\infty}$$

$$= (0+1) + \left(0 - \frac{\alpha}{\alpha+\beta}\right) = 1 - \frac{\alpha}{\alpha+\beta} = \frac{\alpha+\beta-\alpha}{\alpha+\beta} = \frac{\beta}{\alpha+\beta}$$

Example

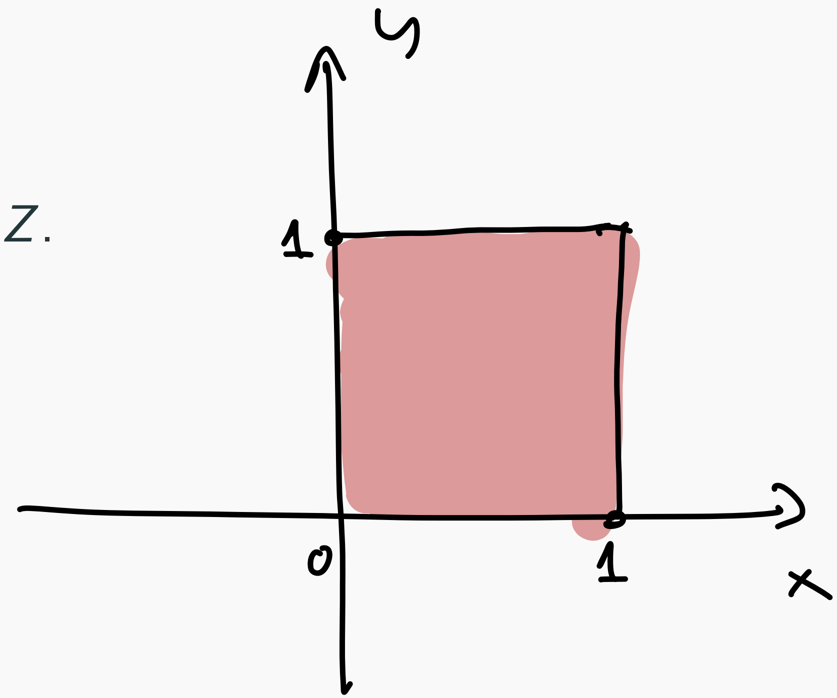
Suppose X and Y are independent exponentially distributed random variables with mean 1. Determine the joint probability density function of X and Y and use it to compute the probability distribution and probability density function of $Z = \max(X, Y)$.

Example

Suppose that X and Y are chosen randomly and independently according to the uniform distribution from the interval $(0, 1)$. Define

$$Z = \frac{Y}{X}.$$

Compute the probability distribution function of Z.



$$f_X(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & x \notin (0,1) \end{cases}$$

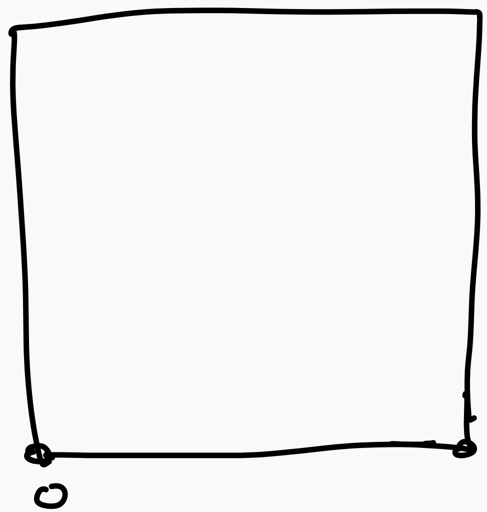
$$f_Y(y) = \begin{cases} 1 & 0 < y < 1 \\ 0 & y \notin (0,1) \end{cases}$$

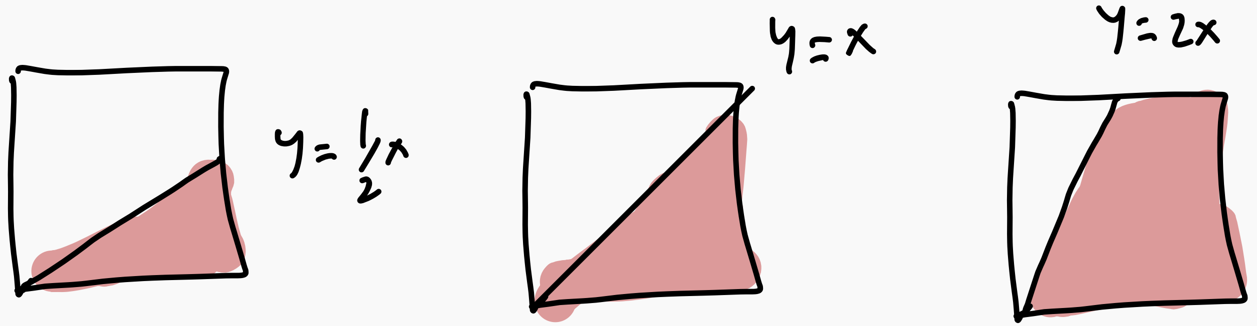
$$f_{X,Y}(x,y) = f_X(x)\, f_Y(y) = \begin{cases} 1 & 0 < x < 1,\quad 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

$Z = Y/X$ can take values $0 < Z < \infty$

compute the pdf of Z

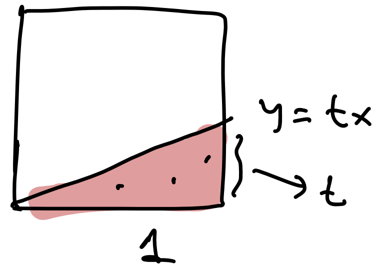
$$F_Z(t) = \mathbb{P}(Z \leq t)$$
$$\mathbb{P}(Y/X \leq t)$$
$$= \mathbb{P}(Y \leq tX)$$





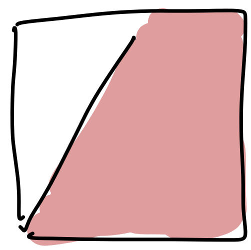
$t = 1/2$ $t = 1$ $t = 2$

$$F(Z \leq t) = \begin{cases} \frac{t}{2} & t<1 \\ 1-\frac{1}{2t} & t>1 \end{cases}$$



Case 1 $t<1$

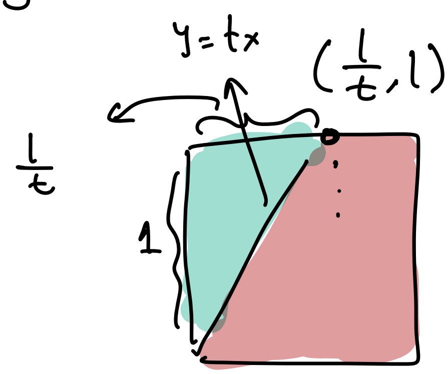
$$F(Z \leq t) = \iint 1 \, dx\,dy$$



$$= \text{area of} = \frac{t \cdot 1}{2} = \frac{t}{2}$$

Case 2 $t>1$

$$F(Z \leq t) = \iint 1 \, dx\,dy$$



$= \text{Total area}$

$- \text{ area}$

$$= 1 - \frac{1}{2t}$$

$$f_Z(t) = \begin{cases} \frac{1}{2} & t<1 \\ \frac{1}{2t^2} & t>1 \end{cases}$$

joint probability density function for n random variables

Definition

Continuous random variables $X_1, \ldots, X_n$ have the joint probability density function $f(x_1, \ldots, x_n)$ if for every subset $B \subseteq \mathbb{R}^n$ we have

$$\mathbb{P}[\underbrace{(X_1, \ldots, X_n)}_{\text{in } \mathbb{R}^n} \in B] = \int \cdots \int_B f(x_1, \ldots, x_n)\, dx_1 \ldots dx_n.$$

Joint probability density function

Definition
Continuous random variables $X_1, \ldots, X_n$ with the joint density function $f_{X_1,\ldots,X_n}$ are *independent* if

$$f_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = f_{X_1}(x_1) \cdots f_{X_n}(x_n)$$

for all $t_1, \ldots, t_n \in \mathbb{R}$. Here $f_{X_i}(t_i)$ is the marginal density function of X_i.

Conditional probability mass function

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

Definition)

Let X and Y be discrete random variables. The conditional probability mass function of X given $Y = y$ is defined by

$$p_{X|Y}(x|y) = \frac{\mathbb{P}[X = x, Y = y]}{\mathbb{P}[Y = y]} = \frac{p_{X,Y}(x, y)}{p_Y(y)}.$$

$$\mathbb{P}(X = x \mid Y = y) = \frac{\mathbb{P}(X = x \cap Y = y)}{\mathbb{P}(Y = y)}$$

Remark: This is only defined when $p_Y(y) = \mathbb{P}[Y = y] > 0$.

Let X and Y be chosen randomly from the set $\{-1, 0, 1\}$ such that the joint probability mass function of X and Y is given by

	$Y = -1$	$Y = 0$	$Y = 1$	
$X = -1$	$1/10$	0	$1/10$	$2/10$
$X = 0$	$1/10$	$2/10$	$2/10$	$5/10$
$X = 1$	$3/10$	0	0	$3/10$
	$5/10$	$2/10$	$3/10$	

Find the conditional probability mass functions of X given Y and Y given X.

$p_{X,Y}(x,y)$

$X|Y$

	$Y=-1$	$Y=0$	$Y=1$
$X=-1$	$1/5$	0	$1/3$
$X=0$	$1/5$	1	$2/3$
$X=1$	$3/5$	0	0

$Y|X$

	$Y=-1$	$Y=0$	$Y=1$
$X=-1$	$1/2$	0	$1/2$
$X=0$	$1/5$	$2/5$	$2/5$
$X=1$	1	0	0

Example

A fair coin is flipped three times. Let N denote the number of Heads and S denote the length of the longest streak of Heads. Determine the joint probability mass function of N given S and S given N.

N & S and conditional pmf

N \ S	0	1	2	3	
0	1/8	0	0	0	1/8
1	0	3/8	0	0	3/8
2	0	1/8	2/8	0	3/8
3	0	0	0	1/8	1/8
	1/8	4/8	2/8	1/8	

HHH	HHT	HTH	HTT
(3,3)	(2,2)	(2,1)	(1,1)
THH	**THT**	**TTH**	**TTT**
(2,2)	(1,1)	(1,1)	(0,0)

N|S

N \ S	0	1	2	3
0	1	0	0	0
1	0	3/4	0	0
2	0	1/4	1	0
3	0	0	0	1

S|N

	0	1	2	3
0	1	0	0	0
1	0	1	0	0
2	0	1/3	2/3	0
3	0	0	0	1

Questes :

if X are independent

$$P_{X|Y}(x|y) = P_X(x)$$

$$P_{X|Y}(x,y) = \frac{P_{X,Y}(x,y)}{P_Y(y)} = \frac{P_X(x) \cdot \cancel{P_Y(y)}}{\cancel{P_Y(y)}}$$

$$= P_X(x)$$

Connection to independence

Theorem

Discrete random variable X and Y are independent if for all values y of Y we have

$$p_{X|Y}(x, y) = p_X(x).$$

$$P_{Y|X}(y, x) = P_Y(y).$$