Design of IIR Filters Using Bilinear Transformation Method

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Abstract

The primary object of this report is to describe how a digital, causal, IIR, low-pass, high-pass, band-pass and band-stop filter may be designed from a set of specifications. As filter types, Butterworth and Chebyshev-1 are chosen. The design process is accomplished in three steps. First the parameters for an analog profile are deduced, then an analog profile with unity as the cut-off frequency calculated. Finally the decimal coefficients of a digital filter are obtained by mapping back this profile. In steps 1 and 3, the famous Bilinear Transformation is applied, and the method called Direct Calculations.

Often, the efficiency of a microprocessor may be increased by restricting calculations on integer numbers. To obtain integer coefficients the above results are scaled properly and both the decimal and integer coefficients saved into text files. The integer coefficients are then used to construct a truncated impulse-response and the metod called Backward Calculations.

The magnitude response of both methods may then be obtained and compared. We test the Direct against the Backward method for all filter functions and both filter types, i.e., low-pass, high-pass, band-pass and bans-stop as well as for the Butterworth and Chebyshev-1 types.

The report then concludes with some comments on the results.

Chapter 1

Introduction

In this report we describe in detail the mathematical background of digital Infinite Impulse Response (IIR) filter design. As filter types Butterworth and Chebyshev-1 analog filters are chosen. To make the process more specific we design first a digital low-pass filter and show that a high-pass, band-pass and band-stop filter may be deduced by a frequency transformation. Then we normalize the filter coefficients such that the magnitude response is in the interval $[0,\ 1]$.

To increase the efficiency of some microprocessors the decimal integers are scaled properly to integers. By using these coefficients in a method called Backward Calculations, we may then compare the magnitude responses.

Definition 1.1 *The output of a time-invariant system (LTI), is defined by convolving the system impulse response and its input, i.e.,*

$$y = h * x = x * h, \tag{1.1a}$$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k], \tag{1.1b}$$

where $x[n]$, $y[n]$ and $h[n]$ denote the system input, output and its impulse response, respectively.

Definition 1.2 *Denote by $x_a(t)$ a continuous function and construct a discrete function $x[n]$ by sampling $x_a(t)$ with sampling period T. Thus*

$$x[n] = x_a(nT). \tag{1.2}$$

Definition 1.3 *Suppose that $x_a(x)$ is a continuous function, and denote by $\mathcal{L}$ the Laplace Transformation. Then $X_a(s)$ is defined by*

$$X_a(s) = \mathcal{L}\Big\{x_a(t)\Big\}, \tag{1.3}$$

$$X_a(s) = \int_{-\infty}^{\infty} x_a(t)\, e^{-st}\ \mathrm{d}t. \tag{1.4}$$

Definition 1.4 *Suppose that $x_a(x)$ is a discrete function, and denote by $\mathcal{F}$ the Fourier Transformation. Then $X\left(e^{j\omega}\right)$ is defined by*

$$X\left(e^{j\omega}\right) = \mathcal{F}\Big\{x[n]\Big\} \tag{1.5a}$$

$$X\left(e^{j\omega}\right) = \sum_{k=-\infty}^{\infty} x[k]\, e^{-j\omega k}. \tag{1.5b}$$

Definition 1.5 *Suppose that $x_a(x)$ is a discrete function, and denote by $\mathcal{Z}$ the Z-Transformation. Then $X(z)$ is defined by*

$$X(z) = \mathcal{Z}\Big\{x[n]\Big\}, \tag{1.6a}$$

$$X(z) = \mathcal{Z}\Big\{x[n]\Big\} = \sum_{k=-\infty}^{\infty} x[k]\, z^{-k}. \tag{1.6b}$$

Remark 1.1 *Comparing Definitions 1.4 and 1.5 we see that the $\mathcal{F}$-transform of a sequence is obtained from its $\mathcal{Z}$-transform by replacing the complex variable z with $e^{j\omega}$. Both transformations map convolution in one domain into multiplication in the other domain. Thus, the input and output in a linear, time-invariant system are related by*

$$y = h * x \Longleftrightarrow Y(z) = H(z)\, X(z), \tag{1.7a}$$

$$y = h * x \Longleftrightarrow Y\left(e^{j\omega}\right) = H\left(e^{j\omega}\right) X\left(e^{j\omega}\right), \tag{1.7b}$$

where $H(z)$ and $Y(z)$ are $\mathcal{Z}$-transforms of $h[n]$, while $H\left(e^{j\omega}\right)$ and $Y\left(e^{j\omega}\right)$ are $\mathcal{F}$-transforms of $h[n]$ and $y[n]$, respectively.

Remark 1.2 *From Definitions 1.3 – 1.5 we see that for a continuous or discrete, real function in the time domain, the complex conjugate transform is obtained by replacing the independent variable with its complex conjugate counterpart,*

$$x_a(t) \text{ is real} \Longleftrightarrow \Big|X_a(\sigma + j\omega)\Big|^2 = X_a(\sigma + j\omega)\, X_a(\sigma - j\omega), \tag{1.8a}$$

$$x[n] \text{ is real} \Longleftrightarrow \Big|X\left(re^{j\omega}\right)\Big|^2 = X\left(re^{j\omega}\right) X\left(re^{-j\omega}\right). \tag{1.8b}$$

1.1 An Infinite Impulse-Response Filter

Definition 1.6 *An Infinite Impulse-Response (IIR), causal filter of order N may be realized by the Nth order difference equation of the form*

$$y[n] = \sum_{k=0}^{N} a_k x[n-k] + \sum_{k=1}^{N} b_k y[n-k]. \tag{1.9a}$$

Thus, to completely determine an Nth order IIR filter, we have to calculate coefficients of filter input ans output, i.e., $\mathbf{a}$ *and* $\mathbf{b}$

$$\mathbf{a} = \big[a_0, \quad a_1, \quad \ldots, \quad a_{N-1}, \quad a_N,\big] \tag{1.9b}$$

$$\mathbf{b} = \big[0, \quad b_1, \quad \ldots, \quad b_{N-1}, \quad b_N.\big] \tag{1.9c}$$

Applying the $\mathcal{Z}$ and $\mathcal{F}$-transformations to Eq. (1.9a), its system function and frequency-response can be obtained as

$$H\left(z\right) = \frac{Y\left(z\right)}{X\left(z\right)} = \frac{\sum_{k=0}^{N} a_k\, z^{-k}}{1 - \sum_{k=1}^{N} b_k\, z^{-k}}, \tag{1.10}$$

$$H\left(e^{j\,\omega}\right) = \frac{Y\left(e^{j\,\omega}\right)}{X\left(e^{j\,\omega}\right)} = \frac{\sum_{k=0}^{N} a_k\, e^{-j\,\omega\,k}}{1 - \sum_{k=1}^{N} b_k\, e^{-j\,\omega\,k}}. \tag{1.11}$$

It is customary to use dB-scale when attenuation of a frequency-response variable is concerned, i.e.

$$\begin{aligned} A_{dB}\left(\omega\right) &= -10\log_{10}\left|H\left(e^{j\,\omega}\right)\right|^2 \\ &= -20\log_{10}\left|H\left(e^{j\,\omega}\right)\right|. \end{aligned} \tag{1.12}$$

A more detailed discussion of the these and related subjects may be found in Chapters 2, 3, 4, 5, 7 and 8 of [2].

Chapter 2

Design Requirements for Digital Filters

2.1 Design Specifications

2.1.1 A General Low-Pass Filter

Specification 2.1 *Consider following frequencies and attenuations*

$$0 < \omega_1 < \omega_2 < \pi, \tag{2.1a}$$
$$0 < A_1 \ll A_2. \tag{2.1b}$$

Design specifications of a normalized, digital, low-pass filter may be formulated as

$$0 \leq A_{dB}(\omega) \leq A_1, \qquad 0 \leq \omega \leq \omega_1, \tag{2.1c}$$
$$A_2 \leq A_{dB}(\omega), \qquad \omega_2 \leq \omega \leq \pi, \tag{2.1d}$$

Figure 2.1 illustrates design specifications for a general low-pass filter, where

$$A_1 = -20\log_{10}(\varepsilon_2), \tag{2.2a}$$
$$A_2 = -20\log_{10}(1-\varepsilon_1). \tag{2.2b}$$

Note that we have normalized the frequency variable with respect to the sampling frequency, i.e.,

$$f = \frac{\omega}{2\,\pi}. \tag{2.3}$$

Remark 2.1 *The low-pass filter in Specification 2.1 gives:*

- *a negligible attenuation i.e.,* $0 \leq A_{dB} \leq A_1$, *for all frequencies* $\omega \in [0,\, \omega_1]$,
- *a strong attenuation i.e.,* $A_2 \leq A_{db}$, *for all frequencies* $\omega \in [\omega_2,\, \pi\rangle$.

2.1.2 A General High-Pass Filter

Specification 2.2 *Consider the following frequencies and attenuations*

$$0 < \omega_1 < \omega_2 < \pi, \tag{2.4a}$$
$$0 < A_2 \ll A_1. \tag{2.4b}$$

Design specifications of a normalized, digital, high-pass filter are of the form

$$A_2 \leq A_{dB}(\omega), \qquad 0 \leq \omega \leq \omega_1, \tag{2.4c}$$
$$0 \leq A_{dB}(\omega) \leq A_1, \qquad \omega_2 \leq \omega \leq \pi, \tag{2.4d}$$

Figure 2.2 illustrates design specifications for a general high-pass filter where

$$A_1 = -20\log_{10}(\varepsilon_2), \tag{2.5a}$$
$$A_2 = -20\log_{10}(1-\varepsilon_1), \tag{2.5b}$$

and a normalized frequency variable given in Eq. (2.3).

Remark 2.2 *The high-pass filter in Specification 2.2 gives:*

- *a strong attenuation i.e.,* $A_1 \leq A_{dB}$, *for all frequencies* $\omega \in [0,\, \omega_1]$,
- *a negligible attenuation i.e.,* $0 \leq A_{dB} \leq A_2$, *for all frequencies* $\omega \in\, [\omega_2,\, \pi\rangle$.

2.1.3 A General Band-Pass Filter

Specification 2.3 *Consider the following frequencies and attenuations*

$$0 < \omega_1 < \omega_2 < \omega_3 < \omega_4 < \pi, \tag{2.6a}$$
$$\omega_4 - \omega_3 = \omega_2 - \omega_1, \tag{2.6b}$$
$$0 < A_2 \ll A_1, \tag{2.6c}$$

A normalized, digital, band-pass filter is specified by

$$A_1 \leq A_{dB}(\omega), \qquad \omega \in [0,\, \omega_1], \tag{2.6d}$$
$$0 \leq A_{dB}(\omega) \leq A_2, \qquad \omega \in [\omega_2,\, \omega_3], \tag{2.6e}$$
$$A_1 \leq A_{dB}(\omega), \qquad \omega \in\, [\omega_4,\, \pi\rangle. \tag{2.6f}$$

Figure 2.3 illustrates design specifications for a general band-pass filter where

$$A_1 = -20\log_{10}(\varepsilon_2), \tag{2.7a}$$
$$A_2 \leq -20\log_{10}(1-\varepsilon_1), \tag{2.7b}$$

and a normalized frequency variable given in Eq. (2.3).

Remark 2.3 *The band-pass filter in Specification 2.3 gives:*

- *a strong attenuation i.e.,* $A_1 \leq A_{dB}$, *for all frequencies* $\omega \in [0,\, \omega_1]$,
- *a negligible attenuation, i.e.,* $0 \leq A_{dB} \leq A_2$, *for all frequencies* $\omega \in [\omega_2,\, \omega_3]$,
- *a strong attenuation i.e.,* $A_1 \leq A_{dB}$, *all frequencies* $\omega \in\, [\omega_4,\, \pi\rangle$.

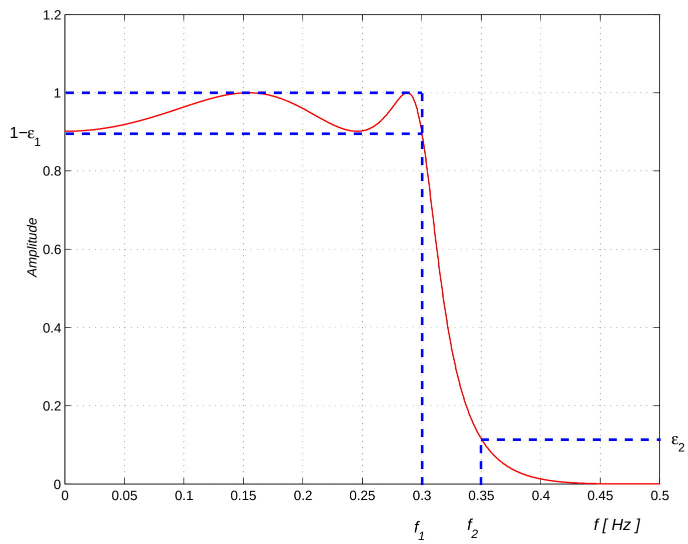


Figure 2.1: Specification of a Digital Low-Pass Filter

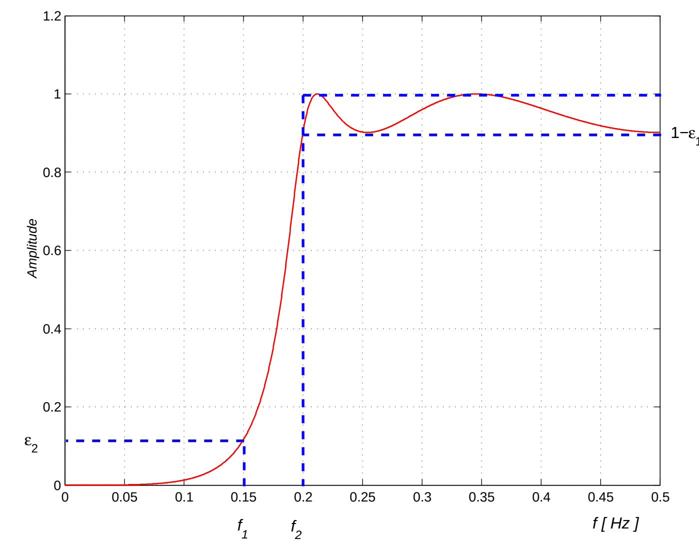


Figure 2.2: Specification of a Digital High-Pass Filter

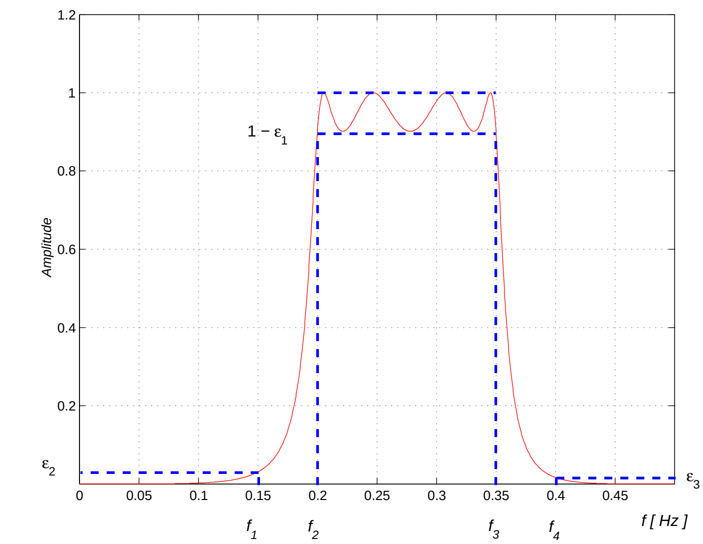


Figure 2.3: Specification of a Digital Band-Pass Filter

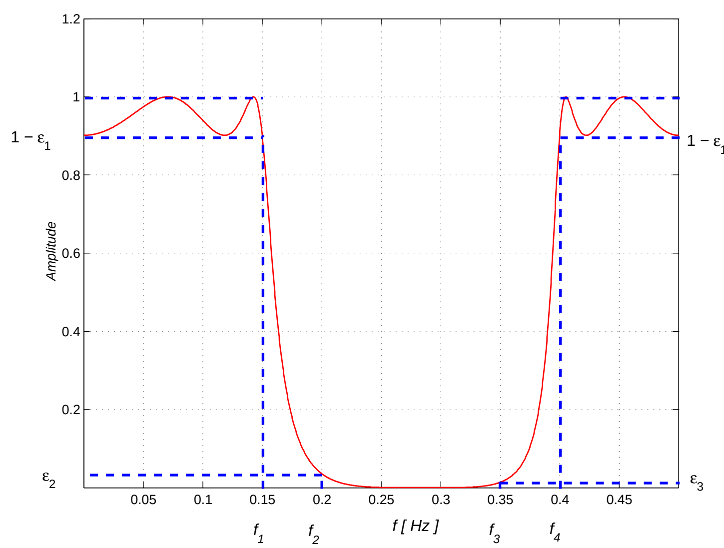


Figure 2.4: Specification of a Digital Band-Stop Filter

2.1.4 A General Band-Stop Filter

Specification 2.4 *Consider the following frequencies and attenuations*

$$0 < \omega_1 < \omega_2 < \omega_3 < \omega_4 < \pi, \tag{2.8a}$$
$$\omega_4 - \omega_3 = \omega_2 - \omega_1, \tag{2.8b}$$
$$0 < A_1 \ll A_2, . \tag{2.8c}$$

A normalized, digital, band-stop filter is specified by

$$0 \leq A_{dB}(\omega) \leq A_1, \qquad \omega \in [0,\, \omega_1], \tag{2.8d}$$
$$A_2 \leq A_{dB}(\omega), \qquad \omega \in [\omega_2,\, \omega_3], \tag{2.8e}$$
$$0 \leq A_{dB}(\omega) \leq A_1, \qquad \omega \in [\omega_4,\, \pi\rangle. \tag{2.8f}$$

Figure 2.4 illustrates design specifications for a general band-stop filter where

$$A_1 = -20\log_{10}(1-\varepsilon_1), \tag{2.9a}$$
$$A_2 = -20\log_{10}(\varepsilon_2), \tag{2.9b}$$

and a normalized frequency variable given in Eq. (2.3).

Remark 2.4 *The band-stop filter in Specification 2.4 gives:*

- *a negligible attenuation i.e.,* $0 \leq A_{dB} \leq A_1$ *all frequencies* $\omega \in [0,\, \omega_1]$,
- *a strong attenuation i.e.,* $A_2 \leq A_{dB}$, *for all frequencies* $\omega \in [\omega_2,\, \omega_3]$,
- *a negligible attenuation i.e.,* $0 \leq A_{dB} \leq A_1$ *all frequencies* $\omega \in [\omega_4,\, \pi\rangle$.

Remark 2.5 *The band-pass and band-stop in Specifications 2.3 and 2.4 are symmetrical. This is clearly seen from*

$$\omega_2 - \omega_1 = \omega_4 - \omega_3, \tag{2.10a}$$
$$A_4 = A_{dB}(\omega)\Big|_{\omega=\omega_4} = A_1, \qquad \textit{in a band-pass filter,} \tag{2.10b}$$
$$A_3 = A_{dB}(\omega)\Big|_{\omega=\omega_3} = A_2 \qquad \textit{in a band-stop filter,} \tag{2.10c}$$

or from Figs. 2.3 and 2.4.

2.2 Analogy between Filter Types

Filter types in Section 2.1 have clear analogies. In fact given a low-pass filter, we may derive the other three types, i.e., high-pass, band-pass and band-stop filters. Below we discuss these relations.

Analogy 2.1 *Compare now a low-pass and a high-pass filter, i.e., Specifications 2.1 and 2.2. Observe that a low-pass filter may be derived from a high-pass filter by changing the frequency variable from* ω_{hp} *to* $\omega_{lp} = \pi - \omega_{hp}$. *Specifically,*

$$\omega_{1,\,lp} = \pi - \omega_{2,\,hp}, \tag{2.11a}$$
$$\omega_{2,\,lp} = \pi - \omega_{1,\,hp}, \tag{2.11b}$$
$$A_{2,\,lp} = A_{1,\,hp}, \tag{2.11c}$$
$$A_{1,\,lp} = A_{2,\,hp}. \tag{2.11d}$$

Analogy 2.2 *By cascading a high-pass and a low-pass filter, we may derive a band-pass filter, see Specifications 2.1 – 2.3. Specifically,*

$$\omega_{i,\,bp} = \omega_{i,\,hp}, \qquad i = 1,2, \tag{2.12a}$$
$$\omega_{i,\,bp} = \omega_{i-2,\,lp}, \qquad i = 3,4, \tag{2.12b}$$
$$A_{i,\,bp} = A_{i,\,hp}, \qquad i = 1,2, \tag{2.12c}$$
$$A_{i,\,bp} = A_{i-2,\,lp}, \qquad i = 3,4. \tag{2.12d}$$

This is most clearly seen by comparing Figs. 2.1 and 2.2 with Fig. 2.3.

Analogy 2.3 *In the same manner, a band-stop filter is derived by cascading a low-pass with a high-pass filter, see Specifications 2.1, 2.2 and 2.4. Specifically,*

$$\omega_{i,\,bp} = \omega_{i,\,lp}, \qquad i = 1,2, \tag{2.13a}$$
$$\omega_{i,\,bp} = \omega_{i-2,\,hp}, \qquad i = 3,4, \tag{2.13b}$$
$$A_{i,\,bp} = A_{i,\,lp}, \qquad i = 1,2, \tag{2.13c}$$
$$A_{i,\,bp} = A_{i-2,\,hp}, \qquad i = 3,4. \tag{2.13d}$$

This is most clearly seen by comparing Figs. 2.1 and 2.2 with Fig. 2.4.

As we see later on, we take advantage of Analogy 2.1 to transform a low-pass filter to a high-pass one.

Chapter 3

Analog Filter Types

3.1 Butterworth

The squared response of an analog, low-pass, Nth order Butterworth filter is

$$\left| H_a\left(j\,\Omega\right) \right|^2 = H_a\left(s\right) H_a\left(-s\right) = \frac{1}{1+\left(-s^2\right)^N}. \tag{3.1}$$

By analytic continuation Eq. (3.1) can be extended to the complex s-domain, i.e.,

$$\left| H_a\left(j\,\Omega\right) \right|^2 = H_a\left(j\,\Omega\right) H_a\left(-j\,\Omega\right) = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}}. \tag{3.2}$$

Poles of the squared magnitude response may thus be found by equating the denominator of Eq. (3.2) to zero, i.e.,

$$s_k = \Omega_c \exp\left[j\pi\left(\frac{1}{2}+\frac{2k+1}{2N}\right)\right], \qquad k = 0, 1, \ldots, 2N-1. \tag{3.3a}$$

The poles lie equally spaced on a circle of radius Ω_c centered about $s = 0$, i.e.,

$$s_k = \sigma_k + j\,\Omega_k, \qquad k = 0, 1, \ldots, 2N-1, \tag{3.4a}$$

$$\sigma_k = -\Omega_c \sin\left[\frac{(2k+1)\,\pi}{2N}\right], \qquad k = 0, 1, \ldots, 2N-1, \tag{3.4b}$$

$$\Omega_k = -\Omega_c \cos\left[\frac{(2k+1)\,\pi}{2N}\right], \qquad k = 0, 1, \ldots, 2N-1. \tag{3.4c}$$

Observe from Eq. (3.2) that:

1. Regardless of order N, the squared magnitude value at $\Omega = \Omega_c$ is given by

$$\left| H_a(j\,\Omega) \right|^2_{\Omega=\Omega_c} = \frac{1}{2}. \tag{3.5a}$$

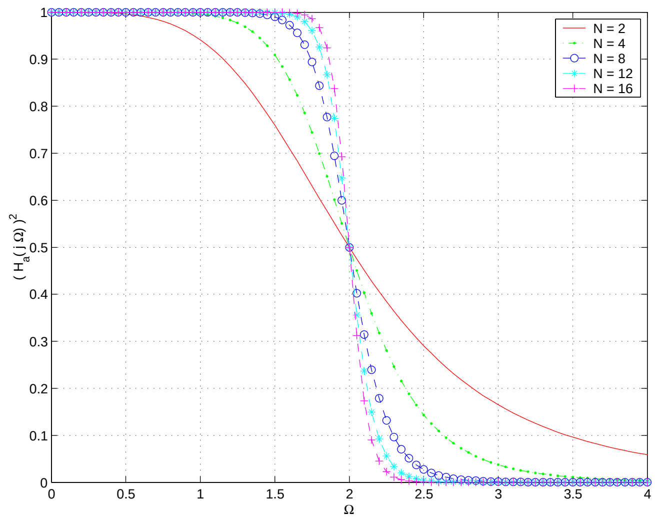


Figure 3.1: Squared Response of Analog, Low-Pass, Butterworth Filter with $\Omega_1 = 2$ and $\Omega_2 = 3$

2. A Butterworth low-pass filter is maximally flat at $\Omega = 0$; that is the first $2N - 1$ derivatives of the squared magnitude at $\Omega = 0$ are equal to zero, i.e.,

$$\frac{d^{(k)}}{d\,\Omega^{(k)}} \left| H_a\left(j\,\Omega\right) \right|^2_{\Omega=0} = 0, \qquad k = 1, 2, \ldots, 2N - 1. \tag{3.5b}$$

In Fig. 3.1 we have depicted a general low-pass, Butterworth filter for some different values of N. From discussion above and this figure it is clear that

1. Thus the magnitude value at $\Omega = \Omega_c$, is down by $3\,\mathrm{dB}$, compared to the corresponding value at $\Omega = 0$. Also the magnitude is a monotonically decreasing function of $|\Omega|$.

2. The following filter characteristics is observed with increasing filter order N:

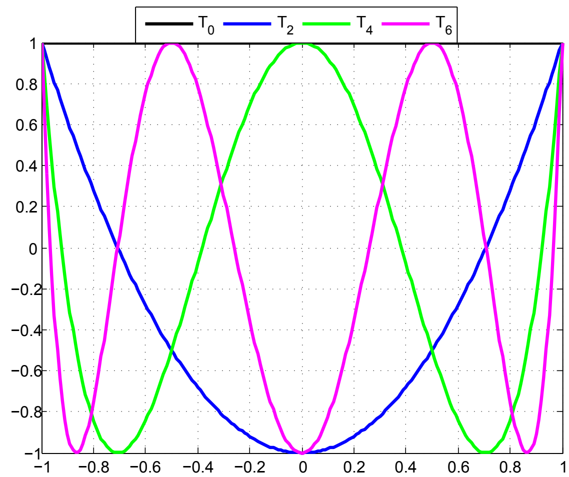
   (a) More rapid transition from the pass-band to the stop-band

   (b) A magnitude value of approximately 1 for more of the pass-band

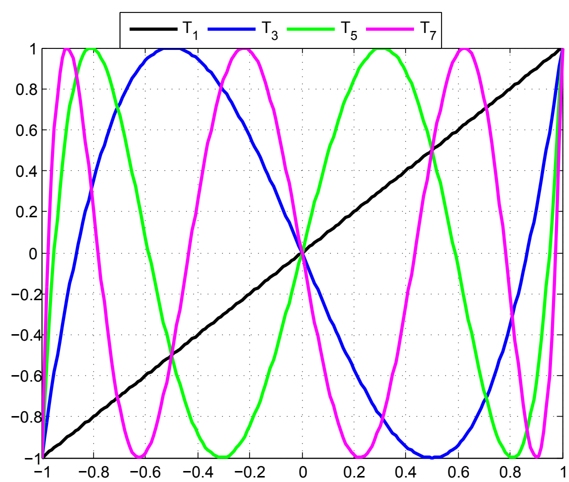
   (c) A magnitude value of approximately 0 for more of the stop-band.

3.2 Chebyshev-1

Chebyshev filters relax the constraint of monotonicity over either the pass-band or the stop-band. For a given filter order N, the Chebyshev filter gives



(a) The First Four Even Chebyshev-1 Polynomials



(b) The First Four Odd Chebyshev-1 Polynomials

Figure 3.2: The First Eight Chebyshev-1 Polynomials in $x \in [-1,\, 1]$

- Minimum equiripple deviation of the magnitude characteristic over one prescribed band of frequencies and
- Monotonic behavior over the remaining band of frequencies.

Depending on the design choice, the minimum equiripple error property can be associated with either the pass-band (Chebyshev-1 filter) or the stop-band (Chebyshev-2 filter). For a given set of filter specifications, this concession of allowing ripple in a band of frequencies leads to lower-order than the Butterworth design choice. Chebyshev-2 filters are rarely used and therefore not included here.

To specify the behavior of a Chebyshev-1 filter we need some background in Chebyshev Polynomials.

Definition 3.1 *The nth order Chebyshev Polynomial is given by*

$$T_n\left(x\right) = \begin{cases} \left(-1\right)^n \cosh\left[n \cosh^{-1}\left(-x\right)\right], & x \leq -1, \\ \cos\left(n \cos^{-1} x\right), & x \in \left[-1,\, 1\right], \\ \cosh\left(n \cosh^{-1} x\right), & x \geq 1. \end{cases} \qquad n \geq 0. \tag{3.6}$$

Figures 3.2a and 3.2b illustrate even and odd Chebyshev-1 polynomials in $x \in [-1,\, 1]$.

Remark 3.1 *The Chebyshev polynomial in Eq.* (3.6) *can be derived from the recurrence relation*

$$T_0\left(x\right) = 1, \tag{3.7a}$$

$$T_1\left(x\right) = x, \tag{3.7b}$$

$$T_{n+1}\left(x\right) = 2x\, T_n\left(x\right) - T_{n-1}\left(x\right), \quad n \geq 1. \tag{3.7c}$$

Remark 3.2 *Observe, from Definition 3.1, following characteristics of the Chebyshev Polynomials:*

$$\left|T_n\left(x\right)\right| < 1, \qquad \left|x\right| \leq 1, \tag{3.8}$$

$$\left|T_n\left(x\right)\right| \geq 1, \qquad \left|x\right| \geq 1. \tag{3.9}$$

Lemma 3.1 *Define the following function*

$$f\left(x\right) = \frac{1}{2^{n-1}} T_n\left(x\right), \quad n \geq 1. \tag{3.10}$$

Note that function $f\left(x\right)$ in Eq. (3.10) *always has a leading coefficient 1, i.e., the coefficient of the highest x-power is always 1. Then function $f\left(x\right)$ in Eq.* (3.10) *has the least maximal absolute value of all polynomial functions with leading coefficient 1.*

Having studied Chebyshev-1 Polynomials shortly, consider the following definition:

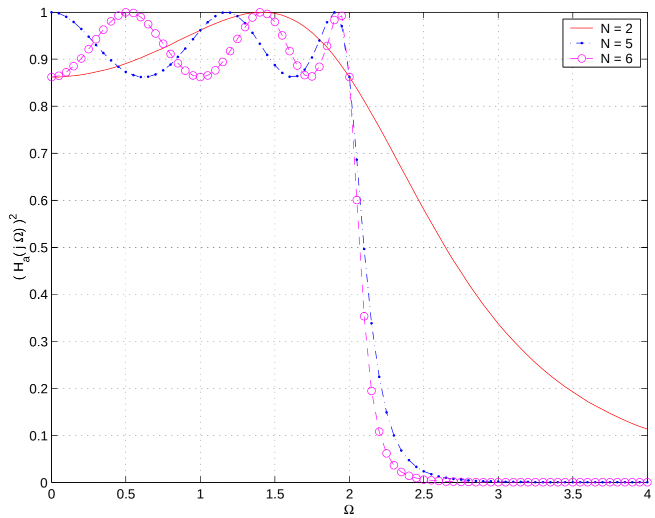


Figure 3.3: Squared Response of Analog, Low-Pass, Chebyshev-1 Filter with $\Omega_c = 2$

Definition 3.2 *The squared response of an analog, low-pass, Chebyshev-1, filter is defined by*

$$\left| H_a\left(j\,\Omega\right) \right|^2 = \frac{1}{1 + \varepsilon^2 \left[T_n \left(\frac{\Omega}{\Omega_1} \right) \right]^2}, \tag{3.11}$$

where T_n is the Chebyshev Polynomial given in Definition 3.1.

Remark 3.3 *Observe from Definition 3.2, Lemma 3.1, and Fig. 3.3, that the analog, Nth order, low-pass, Chebyshev-1 filter exhibits*

1. *A squared response value at $\Omega = 0$ given by*

$$\left| H\left(j\,\Omega\right) \right|^2_{\Omega=0} = \begin{cases} 1, & \textit{for an odd } N, \\ \frac{1}{1+\varepsilon^2}, & \textit{for an even } N. \end{cases} \tag{3.12}$$

2. *A squared response value at $\Omega = \Omega_p$ given by*

$$\left| H\left(j\,\Omega\right) \right|^2_{\Omega=\Omega_l} = \frac{1}{1+\varepsilon^2}. \tag{3.13}$$

3. *An equiripple error and a total of N local extreme values in the pass-band, i.e.,*

$$\left| H_a\left(j\,\Omega\right) \right|^2 \in \left[\frac{1}{1+\varepsilon^2}, 1 \right], \quad \Omega \in \left[0,\, \Omega_l\right], \tag{3.14}$$

4. *A monotonically decreasing response in the stop-band.*

The location of poles in an analog, Nth order, low-pass, Chebyshev-1 filter are obtained by equating the denominator of Eq. (3.11) to zero, i.e.,

$$1 + \varepsilon^2 \left[T_n \left(\frac{\Omega}{\Omega_1} \right) \right]^2 = 0. \tag{3.15a}$$

The pole locations are then given by

$$s_k = \sigma_k + j\,\Omega_k, \qquad k = 0, 1, 2, \ldots, 2N-1. \tag{3.15b}$$

$$\sigma_k = -\mu\,\Omega_1 \sin\left[\frac{(2k+1)\,\pi}{2N} \right], \qquad k = 0, 1, 2, \ldots, 2N-1, \tag{3.15c}$$

$$\Omega_k = \nu\,\Omega_1 \cos\left[\frac{(2k+1)\,\pi}{2N} \right], \qquad k = 0, 1, 2, \ldots, 2N-1, \tag{3.15d}$$

where

$$\mu = \frac{1}{2} \left(\gamma - \gamma^{-1} \right), \tag{3.15e}$$

$$\nu = \frac{1}{2} \left(\gamma + \gamma^{-1} \right), \tag{3.15f}$$

$$\gamma = \left(\frac{1 + \sqrt{1+\varepsilon^2}}{\varepsilon} \right)^{1/N}. \tag{3.15g}$$

As we see from Eq. (3.15) these poles are equally spaced about the origin on an ellipse with minor-axis length $\mu\,\Omega_1$ and major-axis length $\nu\,\Omega_1$ in the s-plane. Thus

$$\frac{\sigma_k^2}{(\mu\,\Omega_1)^2} + \frac{\Omega_k^2}{(\nu\,\Omega_1)^2} = 1. \tag{3.16}$$

For further details on these filter types, consult [1], sections 5-3-1 and 5-3-2.

Chapter 4

Method

To design an IIR digital filter we begin by an analog profile and then map it to a digital system function. This is because, as we have already seen, the theory of analog filter design is well established and many closed-form design formulas exist. For any given design specifications of a digital filter, derivation of the digital filter system function requires the folloowing three steps:

1. Map the desired digital filter specifications into a corresponding analog filter.
2. Derive a corresponding transfer function for the analog prototype.
3. Convert the analog transfer function into an equivalent digital system function.

Remark 4.1 *Note that the above steps are not independent. Specifically, Item 1 transform the digital specifications to an analog counterpart, while Item 3 converts the resulting analog prototype to a digital system function.*

4.1 Bilinear Transformations

To transform an analog to a digital profile or vice versa, we use the most popular method, namely the *Bilinear Transformation*. It relates the two prototypes through the following relations

$$H(z) = H_a(s)\Big|_{s=\alpha\frac{1-z^{-1}}{1+z^{-1}}}, \tag{4.1a}$$

$$H_a(s) = H(z)\Big|_{z=\frac{1+s/\alpha}{1-s/\alpha}}. \tag{4.1b}$$

Remark 4.2 *The resulting digital system function is independent of the value of α, so it may be chosen arbitrary. This is because Item 1 and Item 3 are inverse operations, see Remark 4.1.*

The transformation in Eq. (4.1a) maps the s-plane into the $z-$plane as follows:

1. Left-half s-plane $\longrightarrow$ interior of unit circle in $z-$plane
2. Right-half s-plane $\longrightarrow$ exterior of unit circle in $z-$plane
3. Imaginary axis, $s = j\,\Omega \longrightarrow$ unit circle in $z-$plane , $z = e^{j\,\omega}$. .

Furthermore, the mapping in Item 3 above is non-linear. Substituting $z = e^{j\,\omega}$ into Eq. (4.1a) results in

$$H\left(e^{j\,\omega}\right) = H_a\left(j\alpha\tan\frac{\omega}{2}\right), \tag{4.2}$$

which gives the following relation between the frequency variables

$$\Omega = \alpha\;\tan\left(\frac{\omega}{2}\right), \tag{4.3a}$$

$$\omega = 2\tan^{-1}\left(\frac{\Omega}{\alpha}\right). \tag{4.3b}$$

To normalize the transformation we have chosen α as

$$\alpha = \frac{1}{\tan\left(\frac{1}{2}\right)}. \tag{4.3c}$$

This maps

$$\Omega = 1 \longleftrightarrow \omega = 1. \tag{4.3d}$$

Equation (4.3a) is commonly referred to as *Frequency Prewarping*. Thus the entire Bilinear Transformation Method is as follows:

1. Apply Eq. (4.3a) to prewarp all critical digital frequencies given in the filter specifications, i.e. ω_1 and ω_2.
2. Given the prewarped analog critical frequencies, derive appropriate analog transfer function $H_a\,(s)$.
3. Apply the bilinear transformation, Eq. (4.1a), to obtain the desired digital system function $H\,(z)$.

For further details on the Bilinear Transformation, refer to [3], pages 626-628 and [1], section 5-4-1.

4.2 Mapping Design Specifications

Remember that a low-pass and a high-pass filter had 2 critical frequencies and 2 critical attenuations, while in the case of band-pass and band-stop filters we need 4 critical frequencies and 4 critical attenuations. The analog critical frequencies may be obtained from Eq. (4.3a).

1. An analog low-pass filter is derived by mapping Specification 2.1, i.e.,

$$\Omega_i = \alpha\ \tan\left(\frac{\omega_i}{2}\right), \qquad i = 1, 2, \tag{4.4a}$$
$$0 < \Omega_1 < \Omega_2 < \infty, \tag{4.4b}$$
$$0 < A_1 < A_2 < \infty, \tag{4.4c}$$
$$-A_1 \le 10 \log_{10} |H(j\,\Omega)|^2 \le 0, \qquad \Omega \le \Omega_1, \tag{4.4d}$$
$$-\infty < 10 \log_{10} |H(j\,\Omega)|^2 \le -A_2, \qquad \Omega_2 \le \Omega. \tag{4.4e}$$

2. An analog high-pass filter is specified by mapping Specification 2.2. Thus

$$\Omega_i = \alpha\ \tan\left(\frac{\omega_i}{2}\right), \qquad i = 1, 2, \tag{4.5a}$$
$$0 < \Omega_1 < \Omega_2 < \infty, \tag{4.5b}$$
$$0 < A_2 < A_1 < \infty, \tag{4.5c}$$
$$-\infty < 10 \log_{10} |H(j\,\Omega)|^2 \le -A_1, \qquad 0 \le \Omega \le \Omega_1, \tag{4.5d}$$
$$-A_2 \le 10 \log_{10} |H(j\,\Omega)|^2, \qquad \Omega_2 \le \Omega. \tag{4.5e}$$

3. An analog band-pass filter is specified by mapping Specification 2.3. Thus

$$\Omega_i = \alpha\ \tan\left(\frac{\omega_i}{2}\right), \qquad i = 1, \ldots, 4, \tag{4.6a}$$
$$0 < \Omega_i < \Omega_{i+1}, \qquad i = 1, 2, 3, \tag{4.6b}$$
$$0 < A_2 < A_1, \tag{4.6c}$$
$$0 < A_3 < A_4, \tag{4.6d}$$
$$10 \log_{10} \left|H(j\,\Omega)\right|^2 \le -A_1 \qquad 0 \le \Omega \le \Omega_1, \tag{4.6e}$$
$$-\min(A_2,\ A_3) \le 10 \log_{10} |H(j\,\Omega)|^2, \qquad \Omega_2 \le \Omega \le \Omega_3, \tag{4.6f}$$
$$10 \log_{10} |H(j\,\Omega)|^2 \le -A_4, \qquad \Omega_4 \le \Omega. \tag{4.6g}$$

4. An analog band-stop filter is specified by mapping Specification 2.4. Thus

$$\Omega_i = \alpha\ \tan(\omega_i/2), \qquad i = 1, \ldots, 4, \tag{4.7a}$$
$$0 < \Omega_1 < \Omega_2 < \Omega_3 < \Omega_4 < \infty, \tag{4.7b}$$
$$0 < A_1 < A_2 < \infty, \tag{4.7c}$$
$$0 < A_4 < A_3 < \infty, \tag{4.7d}$$
$$-A_1 < 10 \log_{10} |H(j\,\Omega)|^2 \le 0, \qquad 0 \le \Omega \le \Omega_1, \tag{4.7e}$$
$$-\infty < 10 \log_{10} |H(j\,\Omega)|^2 \le -\max(A_2,\ A_3), \qquad \Omega_2 \le \Omega \le \Omega_3, \tag{4.7f}$$
$$-A_4 \le 10\ \log_{10} |H(j\,\Omega)|^2 \le 0, \qquad \Omega_4 \le \Omega. \tag{4.7g}$$

4.2.1 Butterworth

From Eq. (3.2), we observe that to fully determine an analog, low-pass, Butterworth filter we need to specify Ω_c and N. Now, insert Eq. (3.2) into Eqs. (4.4d) and (4.4e),

$$-A_1 \leq -10\log_{10}\left[1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}\right] \leq 0, \qquad 0 \leq \Omega \leq \Omega_1, \tag{4.8a}$$

$$-\infty < -10\log_{10}\left[1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}\right] \leq -A_2, \qquad \Omega_2 \leq \Omega \leq \infty, \tag{4.8b}$$

Observe that the right side of Eq. (4.8a) and left side of Eq. (4.8b) are automatically satisfied. Solving the left side of Eq. (4.8a) and the right side of Eq. (4.8b) for design parameters N and Ω_c we obtain

$$N = \frac{\log_{10}\left(\frac{C_2}{C_1}\right)}{2\log_{10}\left(\frac{\Omega_2}{\Omega_1}\right)}, \tag{4.9a}$$

$$\Omega_c = \Omega_1\, C_1^{-1/(2N)}, \tag{4.9b}$$

where

$$C_1 = 10^{A_1/10} - 1, \tag{4.9c}$$

$$C_2 = 10^{A_2/10} - 1. \tag{4.9d}$$

Remark 4.3 *Filter order N must be chosen as an even integer greater than or equal to the value given by Eq. (4.9a). With the choice of Ω_c given by Eq. (4.9b), we satisfy the pass-band requirement of Eq. (4.4d), while the stop-band requirement of Eq. (4.4e) is exceeded. This comes from the choice of N which is greater than the value specified by Eq. (4.9a).*

Remark 4.4 *Because of the analogy mentioned in Section 2.2, the parameters of a high-pass Butterworth filter may be derived by first transforming it to a corresponding low-pass filter and then calculate N and Ω_c. The order is the same for the high-pass filter, while Ω_c found in this way must be mapped back. Specifically, the Butterworth parameters of a high-pass filter may be derived in the following order: Transform the limiting frequencies such that*

$$\Omega_{1,\,lp} = \pi - \Omega_{1,\,hp}, \tag{4.10a}$$

$$\Omega_{2,\,lp} = \pi - \Omega_{1,\,hp}, \tag{4.10b}$$

$$A_{1,\,lp} = A_{2,\,hp}, \tag{4.10c}$$

$$A_{2,\,lp} = A_{1,\,hp}. \tag{4.10d}$$

Then calculate N and $\Omega_{c,\,lp}$ as explained above. Finally determine

$$\Omega_{c,\,hp} = \pi - \Omega_{c,\,lp}. \tag{4.10e}$$

Remark 4.5 *Our knowledge of determining the low-pass and high-pass filter parameters may now be applied to the band-pass and band-stop parameters determination. The process may be summarized as follows: Divide the specification to corresponding high-pass and low-pass requirements, as explained in Section 2.2. Then obtain the corresponding filter parameters as explained above. This gives two even filter orders N_1 and N_2 as well as two cut-off frequencies Ω_{c_1} and Ω_{c_2}. Band-pass or band-stop filter parameters are given by*

$$N = 2\max\left(N_1,\, N_2\right), \tag{4.11a}$$

$$\Omega_c = \left[\Omega_{c_1},\, \Omega_{c_2}\right]. \tag{4.11b}$$

Note that we calculate Ω_{c_1} and Ω_{c_2} based on an even integer found for the low-pass or high-pass requirements. Since the actual order is twice the maximum of N_1 and N_2, both the pass-band and the stop-band requirements will be exceeded.

4.2.2 Chebyshev-1

An analog, low-pass, Chebyshev-1 filter has parameters N and ε. Inserting Eq. (3.11) into Eqs. (4.4d) and (4.4e) we get

$$-A_1 \leq -10\log_{10}\left[1+\varepsilon^2 T_N^2\left(\frac{\Omega}{\Omega_1}\right)\right] \leq 0, \qquad 0 \leq \Omega \leq \Omega_1, \tag{4.12a}$$

$$-\infty < -10\log_{10}\left[1+\varepsilon^2 T_N^2\left(\frac{\Omega}{\Omega_1}\right)\right] \leq -A_2, \qquad \Omega_2 \leq \Omega < \infty. \tag{4.12b}$$

Observe from the properties of Chebyshev polynomials, Eqs. (3.8) and (3.9), that the inequalities in right side of Eq. (4.12a) and left side of Eq. (4.12b) are automatically satisfied. Solving inequalities in left side of Eq. (4.12a) and right side of Eq. (4.12b) for N and ε, we obtain

$$\varepsilon = C_1^{1/2}, \tag{4.13a}$$

$$N = \frac{\cosh^{-1}\left(\frac{C_2^{1/2}}{\varepsilon}\right)}{\cosh^{-1}\left(\frac{\Omega_2}{\Omega_1}\right)}, \tag{4.13b}$$

where

$$C_1 = 10^{A_1/10} - 1, \tag{4.13c}$$

$$\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right). \tag{4.13d}$$

Remark 4.6 *Again, N must be chosen as the nearest even integer greater than or equal to the value given by Eq.* (4.13b). *With the choice of ε given by Eq.* (4.13a) *the pass-band requirement of Eq.* (4.4d) *is exactly satisfied, while the stop-band requirement of Eq.* (4.4e) *is exceeded.*

Remark 4.7 *There is a subtle difference between Butterworth and Chebyshev-1 analog low-pass filters. While cut-off frequency* Ω_c*, in a Butterworth filter must be calculated from the specifications in Eq.* (4.9b)*, its counterpart* Ω_1*, in a Chebyshev-1 filter is directly found by frequency prewarping of Eq.* (4.3a).

Remark 4.8 *Referring to Section 2.2 again, observe that the mapping of a low-pass to a high-pass Chebyshev-1 does not change* N *and* ε*. Thus the parameters of a digital high-pass filter may be derived by the following steps*

- *Set*

$$\Omega_{1,\,lp} = \pi - \Omega_{1,\,hp}, \tag{4.14a}$$

$$\Omega_{2,\,lp} = \pi - \Omega_{2,\,hp}, \tag{4.14b}$$

$$A_{1,\,lp} = A_{2,\,hp}, \tag{4.14c}$$

$$A_{2,\,lp} = A_{1,\,hp}. \tag{4.14d}$$

- *Determine* N *and* ε *as explained above.*

Remark 4.9 *Again, we use our knowledge of determining the low-pass and high-pass filter parameters for the band-pass and band-stop cases. First, transform the given specifications to the corresponding high-pass and low-pass requirements, as explained in Section 2.2. Then obtain the corresponding filter parameters for each case as explained above. This gives two even filter orders* N_1 *and* N_2 *as well as* ε_1 *and* ε_2*. Band-pass and band-stop filter parameters are then obtained by*

$$N = 2\max\left(N_1,\, N_2\right), \tag{4.15a}$$

$$\varepsilon = \min\left(\varepsilon_1,\, \varepsilon_2\right). \tag{4.15b}$$

Since the cut-off frequencies for a band-pass and band-stop Chebyshev filter are found directly from the filter specifications, Eq. (4.15a) *has no effect in the pass-band and stop-band values. Thus, as it was the case for the low-pass and high-pass Chebyshev-1 filters, the pass-band requirements are exactly satisfied, while the stop-band requirements will be exceeded.*

4.3 Analog Transfer Function

To determine the analog, low-pass transfer function, we note from Eqs. (3.4b) and (3.4c) that the $2N$ poles of $|H_a(s)|^2$ are complex conjugate pairs and may be separated to N poles for $H_a(\sigma + j\,\omega)$ and N poles for $H_a(\sigma - j\,\omega)$. Since we are interested in a stable transfer function, we may choose those poles with negative real parts, i.e. the poles of $H_a(s)$ are found from

$$s_k = \sigma_k + j\,\Omega_k, \qquad k = 0,\, 1, \ldots, N-1, \tag{4.16}$$

where σ_k and Ω_k are calculated from Eqs. (3.4b) and (3.4c) for a Butterworth low-pass filter and from Eqs. (3.15c) – (3.15g) for a Chebyshev-1 low-pass filter. Suppose that we have chosen poles s_k for $k = 0, 1, \ldots, N/2 - 1$ such that the complex parts Ω_k have the same sign. Then $H_a(s)$ may be thought of as a cascade of $N/2$ transfer functions each of order 2, so that

$$H_a(s) = \prod_{k=0}^{\frac{N}{2}-1} \frac{d_k}{s^2 + c_k s + d_k}, \tag{4.17}$$

where c_k and d_k are real. This is also a numerically advantageous approach because it reduces the error associated with the value of coefficients.

4.4 Digital System Function

Each second order factor in Eq. (4.17) is obtained by multiplying two complex conjugate poles together. Suppose that the complex conjugate pair

$$s_k = \sigma_k + j\,\Omega, \tag{4.18}$$
$$s_k^* = \sigma_k - j\,\Omega, \tag{4.19}$$

are poles of a transfer function. Then the normalized transfer function is given by

$$\begin{aligned} H_k(s) &= \frac{r_k^2}{\Big[s - (\sigma_k + j\Omega_k)\Big]\Big[s - (\sigma_k - j\Omega_k)\Big]} \\ &= \frac{r_k^2}{s^2 - 2\sigma_k s + r_k^2}, && (4.20a) \\ r_k^2 &= \sigma_k^2 + \Omega_k^2. && (4.20b) \end{aligned}$$

To determine the corresponding system function by the Bilinear Transformation we substitute $s = \alpha \frac{1-z^{-1}}{1+z^{-1}}$ into Eq. (4.20a) and simplify to get

$$\begin{aligned} H(z) &= \frac{x_0 + x_1 z^{-1} + x_2 z^{-2}}{1 - (y_1 z^{-1} + y_2 z^{-2})}, && (4.21a) \\ x_0 &= \frac{1}{2} x_1 = x_2 = \frac{1}{E}, && (4.21b) \\ y_1 &= \frac{2r_k^2 - 2\alpha^2}{E}, && (4.21c) \\ y_2 &= -\frac{\alpha^2 + 2\alpha\sigma_k + r_k^2}{E}, && (4.21d) \\ E &= \alpha^2 - 2\sigma_k + r_k^2. && (4.21e) \end{aligned}$$

The digital filter with system function given in Eq. (4.21a) is a low-pass filter with cut-off frequency at $\omega_c' = 1$. We need in addition frequency transformations to design other types of digital filters and/or with other cut-off frequencies.

4.4.1 Low-pass to Low-Pass Frequency Transformation

To design a low-pass digital filter with another cut-off frequency, we need a low-pass to low-pass frequency transformation given by

$$z^{-1} \longrightarrow \frac{z^{-1} - \beta}{1 - \beta z^{-1}}, \tag{4.22a}$$

where

$$\beta = \frac{\sin\left(\frac{1}{2} - \frac{\omega_c}{2}\right)}{\sin\left(\frac{1}{2} + \frac{\omega_c}{2}\right)}. \tag{4.22b}$$

The resulting system function of the digital low-pass filter is

$$H_{lp}\left(z^{-1}\right) = \frac{\Sigma_{k=0}^{2} a_k z^{-k}}{1 - \Sigma_{k=1}^{2} b_k z^{-k}}, \tag{4.23a}$$

with numerator coefficients,

$$a_0 = \frac{1}{D}\left(x_0 - x_1\beta + x_2\beta^2\right), \tag{4.23b}$$

$$a_1 = \frac{1}{D}\left(-2x_0\beta + x_1 + x_1\beta^2 - 2x_2\beta\right), \tag{4.23c}$$

$$a_2 = \frac{1}{D}\left(x_0\beta^2 - x_1\beta + x_2\right), \tag{4.23d}$$

and denominator coefficients,

$$b_1 = \frac{1}{D}\left(2\beta + y_1 + y_1\beta^2 - 2y_2\beta\right), \tag{4.23e}$$

$$b_2 = \frac{1}{D}\left(-\beta^2 - y_1\beta + y_2\right), \tag{4.23f}$$

where

$$D = 1 + y_1\beta + y_2\beta^2. \tag{4.23g}$$

4.4.2 Low-pass to to High-Pass Frequency Transformation

To design a high-pass digital filter, we need a low-pass to high-pass frequency transformation given by

$$z^{-1} \longrightarrow \frac{-z^{-1} - \beta}{1 + \beta z^{-1}}, \tag{4.24a}$$

where

$$\beta = -\frac{\cos\left(\frac{1}{2} + \frac{\omega_c}{2}\right)}{\cos\left(\frac{1}{2} - \frac{\omega_c}{2}\right)}. \tag{4.24b}$$

The resulting system function of the digital high-pass filter is

$$H_{hp}\left(z^{-1}\right) = \frac{\Sigma_{k=0}^{2} a_k z^{-k}}{1 - \Sigma_{k=1}^{2} b_k z^{-k}}, \tag{4.25a}$$

with numerator coefficients,

$$a_0 = \frac{1}{D}\left(x_0 - x_1\beta + x_2\beta^2\right), \tag{4.25b}$$

$$a_1 = -\frac{1}{D}\left(-2x_0\beta + x_1 + x_1\beta^2 - 2x_2\beta\right), \tag{4.25c}$$

$$a_2 = \frac{x_0\beta^2 - x_1\beta + x_2}{D}, \tag{4.25d}$$

and denominator coefficients,

$$b_1 = -\frac{1}{D}\left(2\beta + y_1 + y_1\beta^2 - 2y_2\beta\right), \tag{4.25e}$$

$$b_2 = \frac{1}{D}\left(-\beta^2 - y_1\beta + y_2\right), \tag{4.25f}$$

where

$$D = 1 + y_1\beta + y_2\beta^2. \tag{4.25g}$$

4.4.3 Low-pass to to Band-Pass Frequency Transformation

To design a band-pass digital filter, we need a low-pass to band-pass frequency transformation given by

$$z^{-1} \longrightarrow -\frac{z^{-2} + \delta z^{-1} + \beta}{\beta z^{-2} + \delta z^{-1} + 1}, \tag{4.26a}$$

where

$$\beta = \frac{\zeta - 1}{\zeta + 1}, \tag{4.26b}$$

$$\delta = -\frac{2\eta\zeta}{\zeta + 1}, \tag{4.26c}$$

$$\eta = \frac{\cos\left(\frac{\omega_u + \omega_l}{2}\right)}{\cos\left(\frac{\omega_u - \omega_l}{2}\right)}, \tag{4.26d}$$

$$\zeta = \cot\left(\frac{\omega_u - \omega_l}{2}\right)\tan\left(\frac{1}{2}\right). \tag{4.26e}$$

Note that ω_u and ω_l denote the upper and lower cut-off frequencies. The resulting system function of the digital band-pass filter is

$$H_{bp}\left(z^{-1}\right) = \frac{\Sigma_{k=0}^{4} a_k z^{-k}}{1 - \Sigma_{k=1}^{4} b_k z^{-k}}, \tag{4.27a}$$

with numerator coefficients,

$$a_0 = \frac{1}{D}\left[x_0 - \beta x_1 + \beta^2 x_2\right], \tag{4.27b}$$

$$a_1 = \frac{1}{D}\left[2\delta x_0 - (1+\beta)\,\delta x_1 + 2\beta\delta x_2\right], \tag{4.27c}$$

$$a_2 = \frac{1}{D}\left[\left(2\beta + \delta^2\right)(x_0 + x_2) - \left(1 + \beta^2 + \delta^2\right) x_1\right], \tag{4.27d}$$

$$a_3 = \frac{1}{D}\left[2\beta\delta x_0 - (1+\beta)\,\delta x_1 + 2\delta x_2\right], \tag{4.27e}$$

$$a_4 = \frac{1}{D}\left[\beta^2 x_0 - \beta x_1 + x_2\right], \tag{4.27f}$$

and denominator coefficients,

$$b_1 = -\frac{1}{D}\left[2\delta + (1+\beta)\,\delta y_1 - 2\beta\delta y_2\right], \tag{4.27g}$$

$$b_2 = -\frac{1}{D}\left[\left(2\beta + \delta^2\right)(1 - y_2) + \left(1 + \beta^2 + \delta^2\right) y_1\right], \tag{4.27h}$$

$$b_3 = -\frac{1}{D}\left[2\beta\delta + (1+\beta)\,\delta y_1 - 2\delta y_2\right], \tag{4.27i}$$

$$b_4 = -\frac{1}{D}\left[\beta^2 + \beta y_1 y_2\right], \tag{4.27j}$$

where

$$D = 1 + \beta y_1 - \beta^2 y_2. \tag{4.27k}$$

4.4.4 Low-pass to Band-Stop Frequency Transformation

To design a band-stop digital filter, we need a low-pass to band-stop frequency transformation. This is given by

$$z^{-1} \longrightarrow \frac{z^{-2} + \delta z^{-1} + \beta}{\beta z^{-2} + \delta z^{-1} + 1}, \tag{4.28a}$$

where

$$\beta = \frac{1-\zeta}{1+\zeta}, \tag{4.28b}$$

$$\delta = -\frac{2\eta}{1+\zeta}, \tag{4.28c}$$

$$\eta = \frac{\cos\left(\frac{\omega_u+\omega_l}{2}\right)}{\cos\left(\frac{\omega_u-\omega_l}{2}\right)}, \tag{4.28d}$$

$$\zeta = \tan\left(\frac{\omega_u-\omega_l}{2}\right)\tan\left(\frac{1}{2}\right). \tag{4.28e}$$

Again ω_u and ω_l are upper and lower cut-off frequencies. The resulting system function of the digital low-pass filter is

$$H_{bp}\left(z^{-1}\right) = \frac{\Sigma_{k=0}^{4} a_k z^{-k}}{1-\Sigma_{k=1}^{4} b_k z^{-k}}, \tag{4.29a}$$

with numerator coefficients,

$$a_0 = \frac{1}{D}\left[x_0 + \beta x_1 + \beta^2 x_2\right], \tag{4.29b}$$

$$a_1 = \frac{1}{D}\left[2\delta x_0 + (1+\beta)\,\delta x_1 + 2\beta\delta x_2\right], \tag{4.29c}$$

$$a_2 = \frac{1}{D}\left[\left(2\beta+\delta^2\right)(x_0+x_2) + \left(1+\beta^2+\delta^2\right)x_1\right], \tag{4.29d}$$

$$a_3 = \frac{1}{D}\left[2\beta\delta x_0 + (1+\beta)\,\delta x_1 + 2\delta x_2\right], \tag{4.29e}$$

$$a_4 = \frac{1}{D}\left[\beta^2 x_0 + \beta x_1 + x_2\right], \tag{4.29f}$$

and denominator coefficients,

$$b_1 = -\frac{1}{D}\left[2\delta - (1+\beta)\,\delta y_1 - 2\beta\delta y_2\right], \tag{4.29g}$$

$$b_2 = -\frac{1}{D}\left[\left(2\beta+\delta^2\right)(1-y_2) - \left(1+\beta^2+\delta^2\right)y_1\right], \tag{4.29h}$$

$$b_3 = -\frac{1}{D}\left[2\beta\delta - (1+\beta)\,\delta y_1 - 2\delta y_2\right], \tag{4.29i}$$

$$b_4 = -\frac{1}{D}\left[\beta^2 - \beta y_1 - y_2\right], \tag{4.29j}$$

where

$$D = 1 - \beta y_1 - \beta^2 y_2. \tag{4.29k}$$

Remark 4.10 *The transformation given by Eq.* (4.22a) *just changes the frequency from* 1 *to* ω_c*, while the frequency mapping of Eqs.* (4.24b)*,* (4.26b) *and* (4.28b)*, not only changes the filter property from low-pass to high-pass band-pass or band-stop, but also the cut-off frequency from* 1 *to* ω_c *for a high-pass and from 1 to* ω_u *and* ω_l *for a band-pass or band-stop filter.*

Remark 4.11 *By comparing Eqs.* (4.23b) – (4.23g) *with Eqs.* (4.25b) – (4.25g)*, we see that by applying the correct value of* β*, high-pass coefficients may be obtained just by changing the signs of low-pass coefficients* a_1 *and* b_1*.*

Remark 4.12 *By comparing Eqs.* (4.22a) *and* (4.24a) *with Eqs.* (4.26a) *and* (4.28a)*, it is clear that the first two transformations keep the filter order, while the last two doubles it. This is the reason for doubling maximum* N *to determine the band-pass and band-stop filter order in Eqs.* (4.11a) *and* (4.15a)*.*

Frequency transformations are also discussed in [3] p. 628-630 and in [1], section 5-6.

4.4.5 Cascading System Functions

Remember that we calculate coefficients of a second order system function in each step and then put it in series with the result of previous system by integrating the new coefficients into global vectors $\mathbf{a}$ and $\mathbf{b}$. Suppose now that two second order system functions are defined by

$$H_1(z) = \frac{\sum_{k=0}^{4} \alpha_k z^{-k}}{1 - \sum_{k=1}^{4} \beta_k z^{-k}}, \tag{4.30a}$$

$$H_2(z) = \frac{\sum_{k=0}^{n} \delta_k z^{-k}}{1 - \sum_{k=1}^{n} \eta_k z^{-k}}, \tag{4.30b}$$

and define the third system function by cascading $H_1(z)$ and $H_2(z)$, i.e.,

$$\begin{aligned} H(z) &= H_1(z) H_2(z) \\ &= \frac{\sum_{k=0}^{n+4} c_k z^{-k}}{1 - \sum_{k=1}^{n+2} d_k z^{-k}}. \end{aligned} \tag{4.30c}$$

Fortunately, polynomial multiplication is accomplished efficiently by convolution.

Lemma 4.1 *Define now two polynomials of degree* m *and* n*. Then set up a third polynomial of degree* $m+n$ *by multiplying the first two polynomials, i.e.,*

$$P_1(z) = \sum_{k=0}^{m} \theta_k z^{-k}, \tag{4.31a}$$

$$P_2(z) = \sum_{k=0}^{n} \lambda_k z^{-k}, \tag{4.31b}$$

$$P_3(z) = P_1(z) P_2(z) = \sum_{k=0}^{m+n} \varphi_k z^{-k}. \tag{4.31c}$$

It is straightforward to verify that coefficients of P_3 may be obtained by convolving coefficients of P_1 and P_2, i.e.,

$$\boldsymbol{\varphi} = \boldsymbol{\theta} * \boldsymbol{\lambda}, \tag{4.32a}$$

$$\varphi_k = \sum_{i=0}^{m+n} \theta_i \lambda_{k-i}, \qquad k = 0,\, 1,\, \ldots,\, m+n, \tag{4.32b}$$

where

$$\boldsymbol{\varphi} = \left(\varphi_0,\, \varphi_1,\, \ldots,\, \varphi_m\right), \tag{4.32c}$$

$$\boldsymbol{\theta} = \left(\theta_0,\, \theta_1,\, \ldots,\, \theta_n\right), \tag{4.32d}$$

$$\boldsymbol{\lambda} = \left(\lambda_0,\, \lambda_1,\, \ldots,\, \lambda_{m+n}\right). \tag{4.32e}$$

Applying Lemma 4.1 to coefficients of $H\left(z\right)$ in Eq. (4.30c) we get

$$c_k = \sum_{i=0}^{n+4} \alpha_i \delta_{k-i}, \quad k = 0,\, 1,\, \ldots,\, n+4, \tag{4.33a}$$

$$d_k = \sum_{i=0}^{n+4} \beta_i \eta_{k-i}, \quad k = 1,\, \ldots,\, n+2, \tag{4.33b}$$

where

$$\boldsymbol{\alpha} = \left(a_0,\, a_1,\, a_2,\, a_3,\, a_4\right), \tag{4.33c}$$

$$\boldsymbol{\beta} = \left(1,\, -b_1,\, -b_2,\, -b_3,\, -b_4\right), \tag{4.33d}$$

$$\boldsymbol{\delta} = \left(A_0,\, A_1,\, \ldots,\, A_n\right), \tag{4.33e}$$

$$\boldsymbol{\eta} = \left(1,\, -B_1,\, \ldots,\, -B_n\right). \tag{4.33f}$$

Remark 4.13 *For a low-pass and high-pass filter, both numerator and denominator of $H_1\left(z\right)$ in Eq.* (4.30a) *are of order 2, i.e.,*

$$a_3 = a_4 = b_3 = b_4 = 0. \tag{4.34}$$

Remark 4.14 *Since we chose a stable analog prototype and the Bilinear Transformation preserves the stability property, we are guaranteed the unconditional-stability of the resultant digital filter.*

4.5 Amplitude Normalization

By now we have found the frequency-response of the form

$$H\left(z = e^{j\,\omega}\right) = \frac{\sum_{k=0}^{N} a_k e^{-j\,k\,\omega}}{1 - \sum_{k=1}^{N} b_k^{-j\,k\,\omega}}. \tag{4.35}$$

Initially, the absolute value of H at $\omega = \omega_c$ may be obtained by

$$\begin{aligned} A_i &= \left|H(\omega)\right|_{\omega=\omega_c} \\ &= \left| \frac{\sum_{k=0}^{N} a_k e^{-j\,k\,\omega_c}}{1 - \sum_{k=1}^{N} b_k e^{-j\,k\,\omega_c}} \right|_{\omega=\omega_c}. \end{aligned} \tag{4.36}$$

It is sensible to normalize Eq. (4.35) such that

$$\left|H(\omega)\right|_{\omega=\omega_c} = \left|H_a(\Omega)\right|_{\Omega=\Omega_c}. \tag{4.37}$$

Denote by A_d the desired amplitude for a digital filter at $\omega = \omega_c$ and remember from Eqs. (3.2) and (3.13) that for an even N

$$\begin{aligned} A_d &= \left|H_a(\Omega = \Omega_c)\right| \\ &= \begin{cases} 1, & \text{for a Butterworth filter} \\ \frac{1}{\sqrt{1+\varepsilon^2}}, & \text{for a Chebyshev-1 filter.} \end{cases} \end{aligned} \tag{4.38}$$

The amplitude may now be normalized by

$$H_{nornal} = \frac{A_d}{A_i} H\left(e^{j\,\omega}\right). \tag{4.39}$$

Inserting Eq. (4.35) into Eq. (4.39), this is equivalent to normalizing the numerator coefficients in Eq. (4.35). Thus, the normalized coefficients of the digital filter are calculated from

$$\boldsymbol{a}_n = \frac{A_d}{A_i} \boldsymbol{a}, \tag{4.40}$$

$$\boldsymbol{b}_n = \boldsymbol{b}, \tag{4.41}$$

where A_i and A_d are obtained from Eq. (4.36) and Eq. (4.38), while $\boldsymbol{a}$ and $\boldsymbol{b}$ are the previous coefficient vectors.

4.6 Impluse-Response Construction

Having found the digital filter coefficients, we may obtain the impulse-response in one of the two ways:

- *Direct or exact method*: Calculate $H\left(e^{j\,\omega}\right)$ by inserting $\boldsymbol{a}$ and $\boldsymbol{b}$ into Eq. (1.11), then apply $\mathcal{F}^{-1}$-transform to $H\left(e^{j\,\omega}\right)$,

$$h[n] = \mathcal{F}^{-1}\left\{H\left(e^{j\,\omega}\right)\right\}. \tag{4.42a}$$

- *Backward method*: Send an impulse into the system defined by Difference Eq. (1.9a) and calculate the output, i.e,

$$x[n] = \delta[n] \Longleftrightarrow y[n] = h[n]. \tag{4.42b}$$

Here, we apply the second method to construct the $h[n]$. Then we compare its magnitude response with the results of Direct Calculations.

Remark 4.15 *Clearly, to calculate $h[n]$ in the second method, we have to send a finite length impulse. As we see later this truncation is an error source which can be seen when we compare the magnitude responses of the two methods. Intuitively, we expect smaller deviation when impulse-response length increases.*

4.7 Scaling Filter Coefficients

In some cases we are just able to handle integer coefficients, so we must find a way to convert all coefficients to integers. Suppose now that L is the largest representable integer in the actual processor and C is some unknown constant. Denote by a_k and b_k the normalized coefficients obtained above and define a new filter by

$$\acute{a}_k = Ca_k, \quad k = 0, 1, 2, \ldots, N, \tag{4.43}$$

$$\acute{b}_k = Cb_k, \quad k = 0, 1, 2, \ldots, N, \tag{4.44}$$

$$\tilde{y}[n] = \sum_{k=0}^{N} \acute{a}_k x[n-k] + \sum_{k=1}^{N} \acute{b}_k \tilde{y}[n-k], \quad \forall n, \tag{4.45}$$

$$\acute{y}[n] = \frac{1}{C}\tilde{y}[n], \quad \forall n. \tag{4.46}$$

It may be verified that $\acute{y}[n]$ found in Eq. (4.46) is the same as the output of Eq. (1.9a). Our goal is now to determine scaling factor C. As we see from Eq. (4.46), we must divide the output found from Eq. (4.45) by C in each step. Since the original decimal coefficients a_k and b_k may be quite small, we first divide them by the minimum positive magnitude coefficient larger than a predefined value δ. Then multiply all new coefficients by a positive integer such that the largest coefficient obtained is still inside the integer range. Thus

$$m_1 = \max\left[\min_{k=0,\,1,\,\dots,\,N}\Big(|a_k|,\,\delta\Big)\right], \tag{4.47}$$

$$m_2 = \max\left[\min_{k=1,\,2,\,\dots,\,N}\Big(|b_k|,\,\delta\Big)\right], \tag{4.48}$$

$$m_3 = \min\left[m_1,\,m_2\right], \tag{4.49}$$

$$M_1 = \max_{k=0,\,1,\,\dots,\,N}\left|\frac{a_k}{m_3}\right|, \tag{4.50}$$

$$M_2 = \max_{k=1,\,2,\,\dots,\,N}\left|\frac{b_k}{m_3}\right|, \tag{4.51}$$

$$M_3 = \max\left(M_1,\,M_2\right), \tag{4.52}$$

$$C = \left\lfloor\frac{L}{m_3 M_3}\right\rfloor, \tag{4.53}$$

$$\acute{a}_k = \lfloor C a_k \rfloor, \quad k = 0,\,1,\,\dots,\,N, \tag{4.54}$$

$$\acute{b}_k = \lfloor C b_k \rfloor, \quad k = 1,\,\dots,\,N. \tag{4.55}$$

Remark 4.16 *Note that function $\lfloor x \rfloor$ return an integer by rounding x downward to the nearest integer. The new coefficients obtained from Eqs.* (4.54) *and* (4.55) *are used in the Backward Method to calculate the impulse-response. Thus in the Backward Calculations we have two sources of numeric error, i.e. the truncation of the impulse-response and the rounding effect introduced by Eq.* (4.53)*.*

Remark 4.17 *We have done more than just finding the magnitude of the smallest coefficient in Eqs.* (4.47) *and* (4.48)*. The reason is the case where dividing by the smallest coefficient is enough to go outside the integer range of the actual processor. Furthermore we may do all these calculations in a while-loop, where we increase δ to obtain a value for the minimum coefficient, which after division gives a maximum coefficient inside the range. If it still is impossible keep the integers inside the range we have no choice than stop calculations and give a warning message about the problem. In this case we are left with the decimal coefficients which will be used as described above.*

Chapter 5

Results

In this chapter we present results of the digital filter implementation. Because the input data are obtained by sampling a continuous function, it is more natural to use the normalized frequency with respect to the sampling frequency, i.e. $0 \leq f = \frac{\omega}{2\pi} \leq 0.5$. To be able to compare Butterworth with Chebyshev-1 we applied the same input data for low-pass, high-pass, band-pass and bans-stop filters. Furthermore we will compare the results of the Direct with the Backward calculations. In the first method, we find filter coefficients a_k and b_k, then obtain the frequency-response from

$$H\left(e^{j\,\omega}\right) = \frac{\sum_{k=0}^{N} a_k\, e^{-j\,\omega\,k}}{1 - \sum_{k=1}^{N} b_k\, e^{-j\,\omega\,k}}.$$

In the second method, we begin by reading the data files containing integer vectors $\acute{a}_k$, $\acute{b}_k$ and integer scale factor C, apply Eq. (1.9a) with $\delta\,[n]$ as the input to construct $h\,[n]$ and finally apply $\mathcal{F}$-operator to obtain its frequency-response. Then we plot both magnitude responses and compare them with each other.

Problem 5.1 *Design a low-pass filter satisfying the following conditions:*

- *At most* $1\,dB$ *response attenuation for frequencies less than* $0.20\,Hz$.
- *At least* $20\,dB$ *response attenuation for frequencies greater than* $0.25\,Hz$.

Solution 5.1 *Choosing a butterworth filter for solution of Problem 5.1, the frequency response of the lowest filter order is plotted in Fig. 5.1. In Figs. 5.1a and 5.1b we see amplitude response in magnitude and dB-scale, respectively. Note that amplitudes are plotted with applying both direct and backward methods. Furthermore, a 16-bits processor is used in backward calculations. Then in Fig. 5.1c we compare results by plotting the amplitude deviation. The filter order in this case is* $N = 10$.

Remark 5.1 *This is the only filter result with application of a 16-bits processor. In all other results in this chapter we use a 32-bits processor.*

Solution 5.2 *Problem 5.1 may also be solved with a Chebyshev-1 filter choice. Figure 5.2 shows the result. Again the magnitude and dB-magnitude of the filter response are plotted in Figs. 5.2a and 5.2b, while Fig. 5.2c compares the amplitude deviation. In this case we satisfy the design requirements with filter order $N = 6$.*

Problem 5.2 *Design a high-pass filter with the following design requirements:*

- *At least $40\,dB$ response attenuation for frequencies less than $0.30\,Hz$.*
- *At most $1\,dB$ response attenuation for frequencies greater than $0.35\,Hz$.*

Solution 5.3 *The first filter choice for solving Problem 5.2 is again Butterworth. The result is plotted in Figure 5.3. Magnitude and dB-magnitude of the butterworth filter are plotted in Figs. 5.2a and 5.2b, while Fig. 5.3c shows the response magnitude deviation. In this case we need a filter order $N = 16$.*

Solution 5.4 *If we choose a Chebyshev-1 filter type for solution of Problem 5.2, we are able to reduce the filter order to $N = 8$. The result is plotted in Fig. 5.3.*

Problem 5.3 *Design a band-pass filter satisfying the following conditions:*

- *At least $20\,dB$ response attenuation for frequencies less than $0.20\,Hz$.*
- *At most $1\,dB$ response attenuation for frequencies in the interval $[0.25,\ 0.35]\ Hz$.*
- *At least $30\,dB$ response attenuation for frequencies greater than $0.40\,Hz$.*

Solution 5.5 *Problem 5.3 may be solved by a Butterworth choice of filter. The least filter order is then $N = 20$, see Fig. 5.5.*

Solution 5.6 *Choosing a Chebyshev-1 for solution of Problem 5.3 we may reduce the filter order to $N = 12$, see Fig. 5.6.*

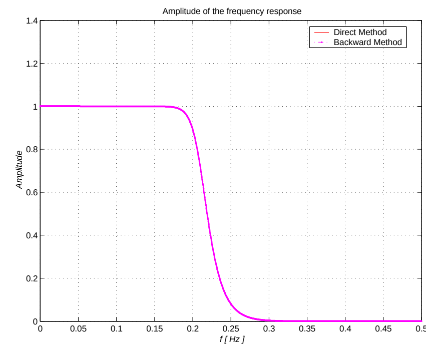
Problem 5.4 *Design a band-stop filter satisfying the following conditions:*

- *At most $1\,dB$ response attenuation for frequencies less than $0.20\,Hz$.*
- *At least $20\,dB$ response attenuation for $f_n = 0.30\,Hz$.*
- *At least $35\,dB$ response attenuation for $f_n = 0.40\,Hz$.*
- *At most $1\,dB$ response attenuation for frequencies greater than $0.45\,Hz$.*

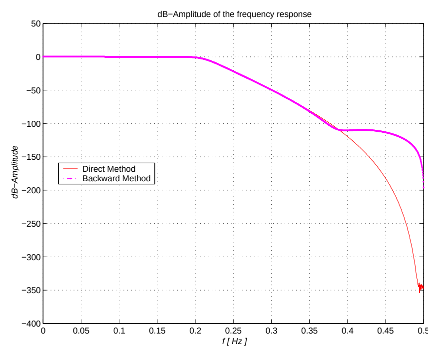
Solution 5.7 *Choosing a Butterworth filter type for solution of Problem 5.4 we need at least a filter order of $N = 20$, see Fig. 5.7.*

Solution 5.8 *By choosing a Chebyshev-1 filter type for solution of Problem 5.4 we may reduce the filter order to $N = 12$, see Fig. 5.8.*

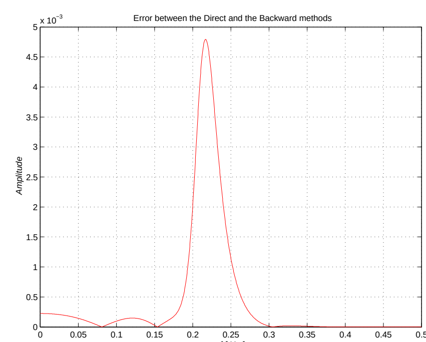
The input and design parameters of Butterworth and Chebyshev-1 filters are gathered in Tabs. 5.1 and 5.2.



(a) Low-pass, Butterworth, Magnitude Response

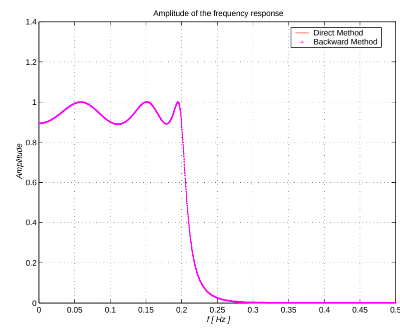


(b) Low-pass, Butterworth, dB-Magnitude Response

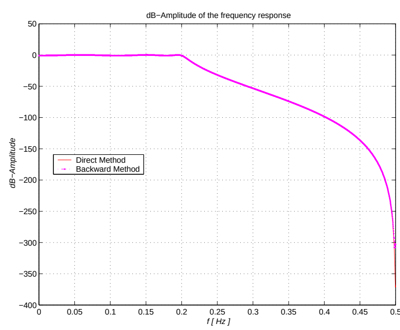


(c) Deviation in Magnitude Response

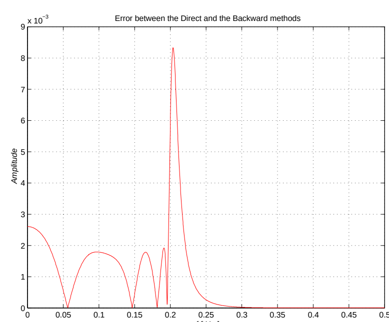
Figure 5.1: Direct and Backward Calculation of a Digital Low-Pass, Butterworth Filter, as well as Deviation Between the Methods. Limiting frequencies and attenuations are $f_{\lim} = [0.20,\ 0.25]\ Hz$ and $A_{\lim} = [1,\ 20]\ dB$, respectively. Backward calculations are performed on a 16-bits processor.



(a) Low-Pass, Chebyshev-1, Magnitude Response

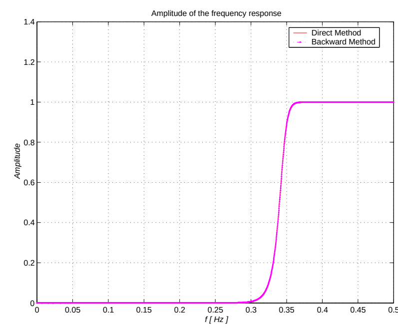


(b) Low-Pass, Chebyshev-1, dB-Magnitude Response

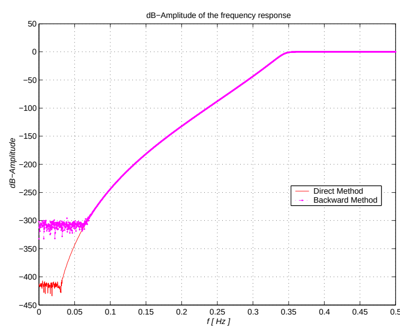


(c) Deviation in Magnitude Response

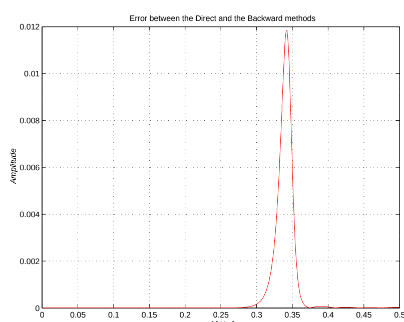
Figure 5.2: Direct and Backward Calculation of a Digital Low-Pass, Chebyshev-1 Filter, as well as Deviation between the Methods. Limiting frequencies and attenuations are $f_{\text{lim}} = [0.20,\ 0.25]\ Hz$ and $A_{\text{lim}} = [1,\ 20]\ dB$, respectively. Backward calculations are performed on a 16-bits processor.



(a) High-Pass, Butterworth, Magnitude Response

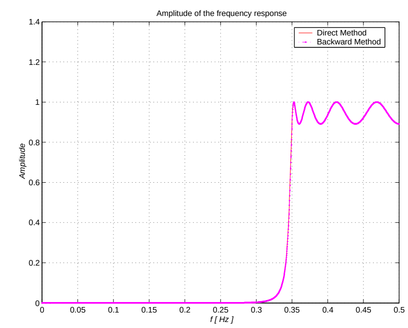


(b) High-Pass, Butterworth, dB-Magnitude Response

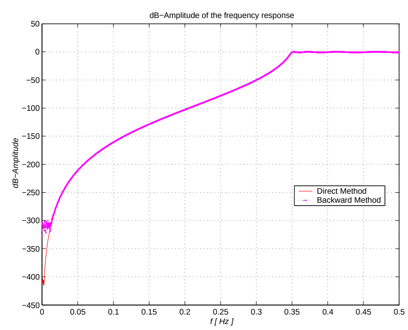


(c) Deviation in Magnitude Response

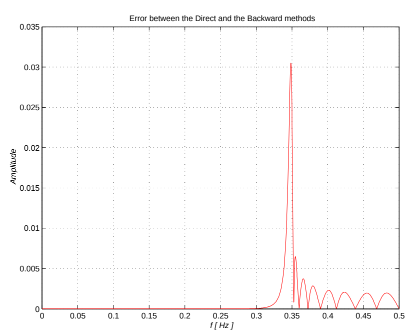
Figure 5.3: Direct and Backward Calculation of a Digital High-Pass, Butterworth Filter, as well as Deviation between the Methods. Limiting frequencies and attenuations are $f_{\lim} = [0.30,\ 0.35]\ Hz$ and $A_{\lim} = [40,\ 1]\ dB$, respectively. Backward calculations are performed on a 32-bits processor.



(a) High-Pass, Chebyshev-1, Magnitude Response

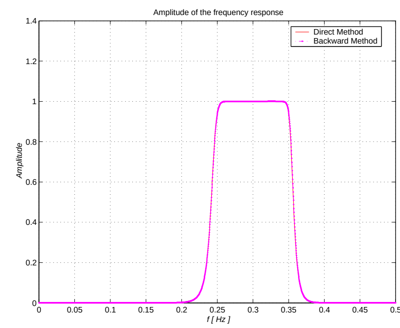


(b) High-Pass, Chebyshev-1 dB-Magnitude Response

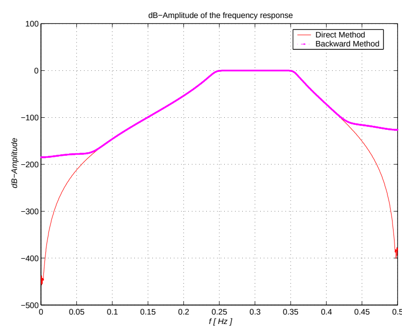


(c) Deviation in Magnitude Response

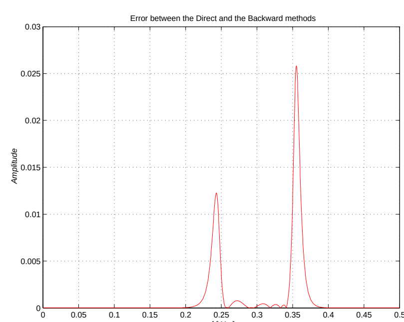
Figure 5.4: Direct and Backward cClculation of a Digital High-Pass, Chebyshev-1 Filter, as well as Deviation between the Methods. Limiting frequencies and attenuations are $f_{\text{lim}} = [0.30,\ 0.35]\ Hz$ and $A_{\text{lim}} = [40,\ 1]\ dB$, respectively. Backward calculations are performed on a 32-bits processor.



(a) Band-Pass, Butterworth, Magnitude Response

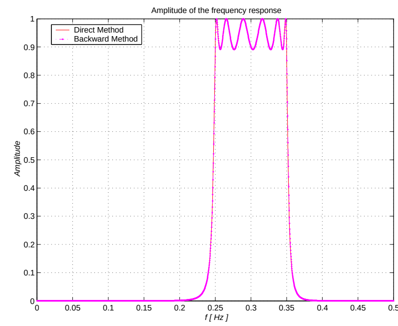


(b) Band-Pass, Butterworth, dB-Magnitude Response

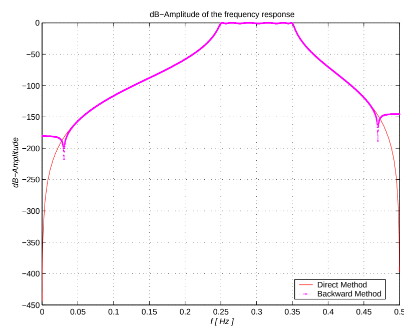


(c) Deviation in Magnitude Response

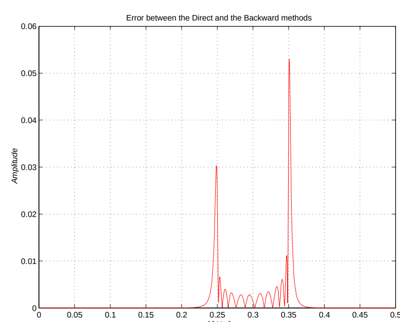
Figure 5.5: Direct and Backward Calculation of a Digital Band-Pass, Butterworth Filter, as well as Deviation between the Methods. Limiting frequencies and attenuations are $f_{\lim} = [0.20,\ 0.25,\ 0.35,\ 0.40]\ Hz$ and $A_{\lim} = [20,\ 1,\ 1,\ 30]\ dB$, respectively. Backward calculations are performed on a 32-bits processor.



(a) Band-Pass, Chebyshev-1, Magnitude Response

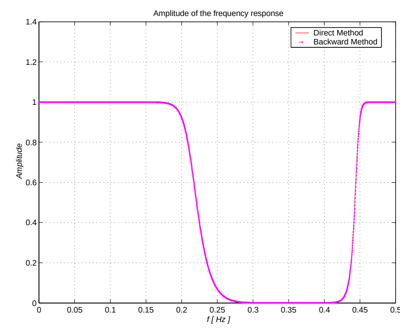


(b) Band-Pass, Chebyshev-1 dB-Magnitude Response

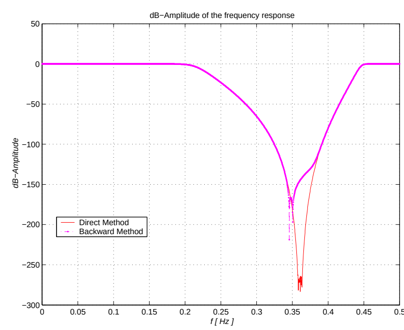


(c) Deviation in Magnitude Response

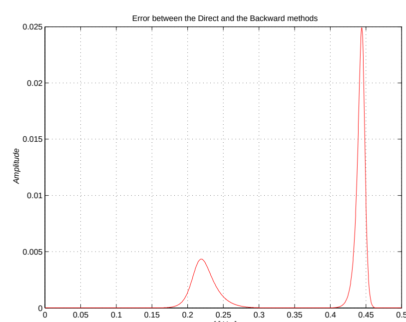
Figure 5.6: Direct and Backward Calculation of a Digital Band-Pass, Chebyshev-1 Filter, as well as Deviation between the Methods. Limiting frequencies and attenuations are $f_{\lim} = [0.20,\ 0.25,\ 0.35,\ 0.40]\ Hz$ and $A_{\lim} = [20,\ 1,\ 1,\ 30]\ dB$, respectively. Backward calculations are performed on a 32-bits processor.



(a) Band-Stop, Butterworth, Magnitude Response

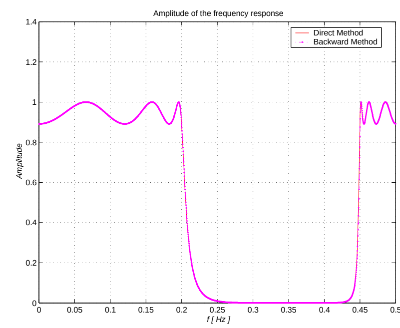


(b) Band-Stop, Butterworth dB-Magnitude Response

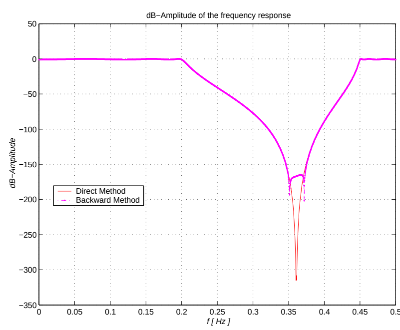


(c) Deviation in Magnitude Response

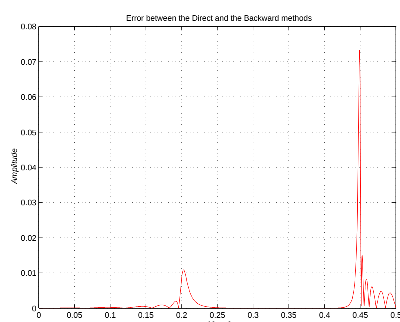
Figure 5.7: Direct and Backward Calculation of a Digital Band-Stop, Butterworth Filter, as well as Deviation between the Methods. Limiting frequencies and attenuations are $f_{\text{lim}} = [0.20,\ 0.30,\ 0.40,\ 0.45]\ Hz$ and $A_{\text{lim}} = [1,\ 20,\ 35,\ 1]\ dB$, respectively. Backward calculations are performed on a 32-bits processor.



(a) Band-Stop, Chebyshev-1, Magnitude Response



(b) Band-Stop, Chebyshev-1 dB-Magnitude Response



(c) Deviation in Magnitude Response

Figure 5.8: Direct and Backward Calculation of a Digital Band-Stop, Chebyshev-1 Filter, as well as Deviation between the Methods. Limiting frequencies and attenuations are $f_{\lim} = [0.20,\ 0.30,\ 0.40,\ 0.45]\ Hz$ and $A_{\lim} = [1,\ 20,\ 35,\ 1]\ dB$, respectively. Backward calculations are performed on a 32-bits processor.

	$f_{\lim}$	$A_{\lim}\,(dB)$	Word Length	N	f_c
Low-pass	$[0.2,\ 0.25]$	$[1,\ 20]$	16	10	0.2103
High-pass	$[0.3,\ 0.35]$	$[40,\ 1]$	32	16	0.3063
Band-pass	$[0.2,\ 0.25,\ 0.35,\ 0.4]$	$[20,\ 1,\ 1,\ 30]$	32	20	$[0.2393,\ 0.3585]$
Band-stop	$[0.2,\ 0.3,\ 0.4,\ 0.45]$	$[1,\ 20,\ 35,\ 1]$	32	16	$[0.2129,\ 0.4457]$

Table 5.1: Input and Design Parameters for Butterworth Filters

	$f_{\lim}$	$A_{\lim}\,(dB)$	Word Length	N	ε
Low-pass	$[0.2,\ 0.25]$	$[1,\ 20]$	16	6	0.5088
High-pass	$[0.3,\ 0.35]$	$[40,\ 1]$	32	8	0.5088
Band-pass	$[0.2,\ 0.25,\ 0.35,\ 0.4]$	$[20,\ 1,\ 1,\ 30]$	32	12	0.5088
Band-stop	$[0.2,\ 0.3,\ 0.4,\ 0.45]$	$[1,\ 20,\ 35,\ 1]$	32	12	0.5088

Table 5.2: Input and design Parameters for Chebyshev-1 Filters

Chapter 6

Summary and Conclusions

We began this report by describing how a digital low-pass, high-pass, band-pass and band-stop filters might be specified. Then we introduced Butterworth and Chebyshev-1 analog filters and explained how their parameters could be deduced from the design specifications. By applying the Bilinear Method we transformed this prototype to a digital filter.

The results were of course the decimal filter coefficients. We called this approach the Direct calculations. Then scaled the decimal coefficients to get integer values for the filter coefficients. We saved also these coefficients and called the method Backward Calculations. Then applied the integer coefficients to construct the impulse-response and Fourier-transform it to obtain the frequency-response.

The magnitude-responses of both methods were then calculated. Finally, solved different design problems by applying the Bilinear transformation method described in this report. We solved each problem with two type of filters and for both we compared the results and plotted the deviation between the two methods for some different set of input parameters. We may conclude the following:

- All digital filters implemented here are unconditionally stable and causal.
- All filters are implemented by transforming a corresponding analog low-pass filter to a digital low-pass using the Bilinear transformation, then applying frequency mapping to convert this prototype to a low-pass, high-pass, band-pass and band-stop filters with different cut-off frequencies.
- The coefficients of a band-pass filter may also be calculated by cascading a high-pass and a low-pass filter. We have not used this approach because of two reasons. First, it is not consistent with the high-pass and band-stop implementation. Second, it gives less satisfactory results.
- While a low-pass Butterworth filter has a flat amplitude in both ends, the frequency of oscillation in the pass-band side of a low-pass Chebyshev-1 grows as we approach the ω_1, just as their analog counter-parts.

- Given the same set of specifications, Chebyshev-1 filters are advantageous because of lower filter order. The price is, some oscillation in the pass-band, as well as a little more complexity of implementing them.

- For a low-pass Butterworth filter, the deviation between the two methods is flat in both ends and reaches its maximum value at Ω_c. For a low-pass Chebyshev-1 filter, the deviation is flat in the pass-band, oscillatory in the stop-band and reaches its maximum value at Ω_1.

- For the low-pass and high-pass filters, the results are quite satisfactory, and the deviation could actually be seen just around the critical frequencies and in the dB-plot for extremely low values of the magnitude responses.

- For the band-pass and band-stop cases we get a little more deviation between the two methods, although still quite satisfactory. This is specially the case if we require steep transitions near the edges of the frequency interval. It comes from the fact that these cases have a filter order twice the corresponding low-pass filter, resulting very small coefficients and more numeric problems.

- Not surprisingly, deviation between direct and bacward method decreases as we increase the impulse-response length and the processor capabilities.

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