

Hashing with Binary Autoencoders



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Large Scale Image Retrieval

Searching a large database for images that match a query.
Query is an image that you already have.

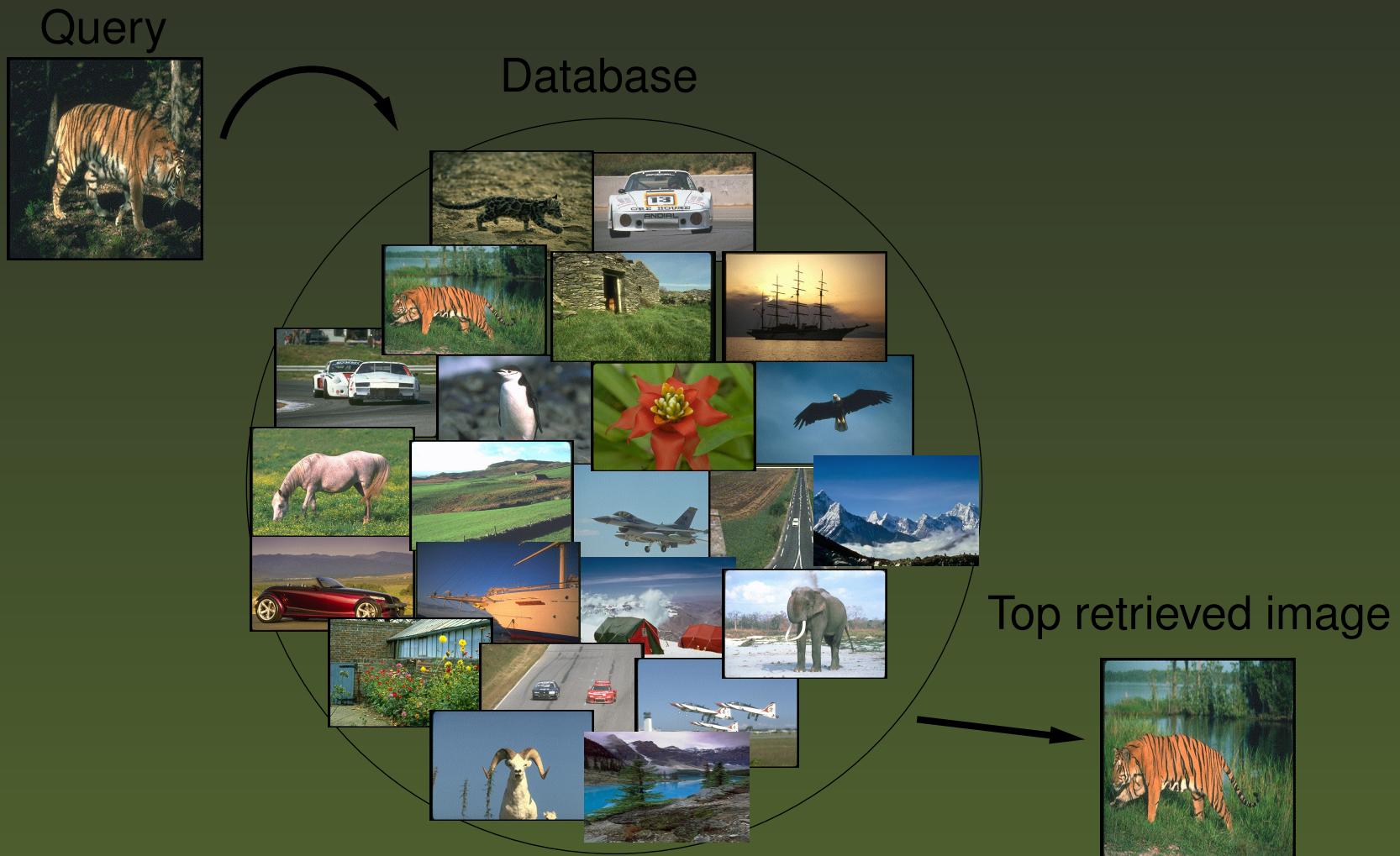
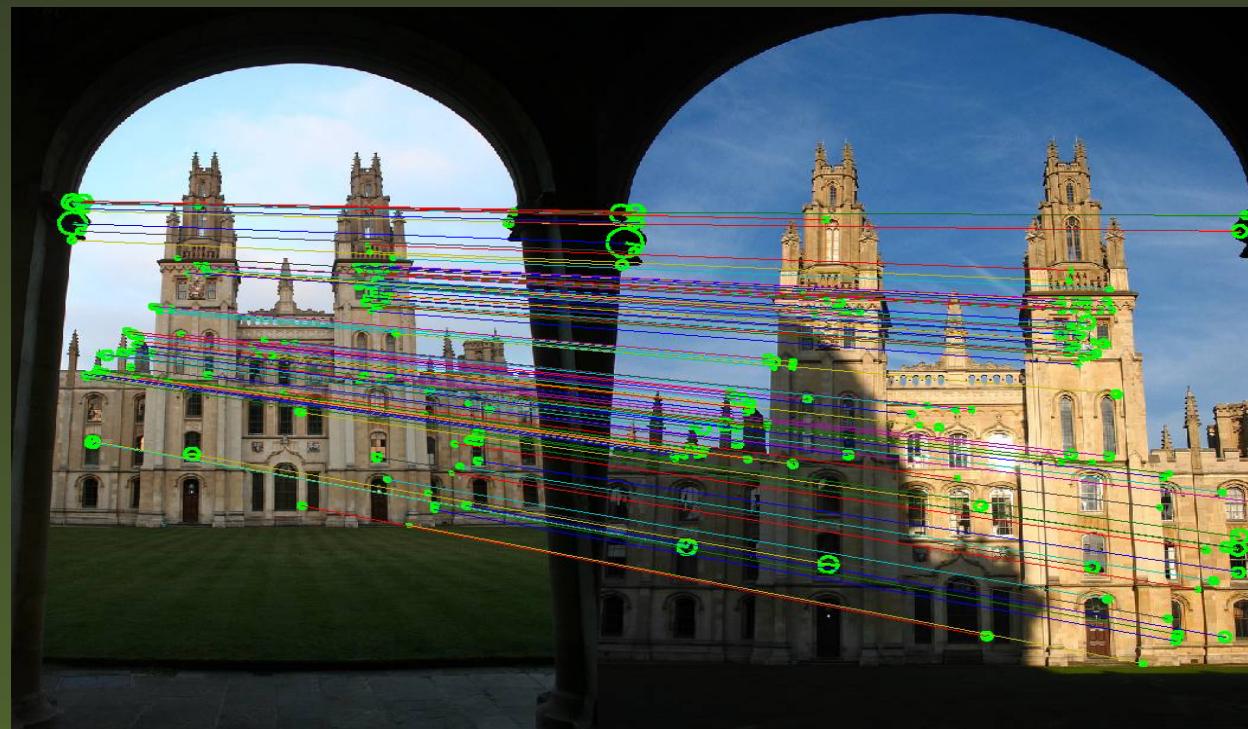


Image Representations

We Compare images by comparing their **feature vectors**.

- ❖ Extract features from images and represent each image by the feature vector.

Common features in image retrieval problem are SIFT, GIST, wavelet.

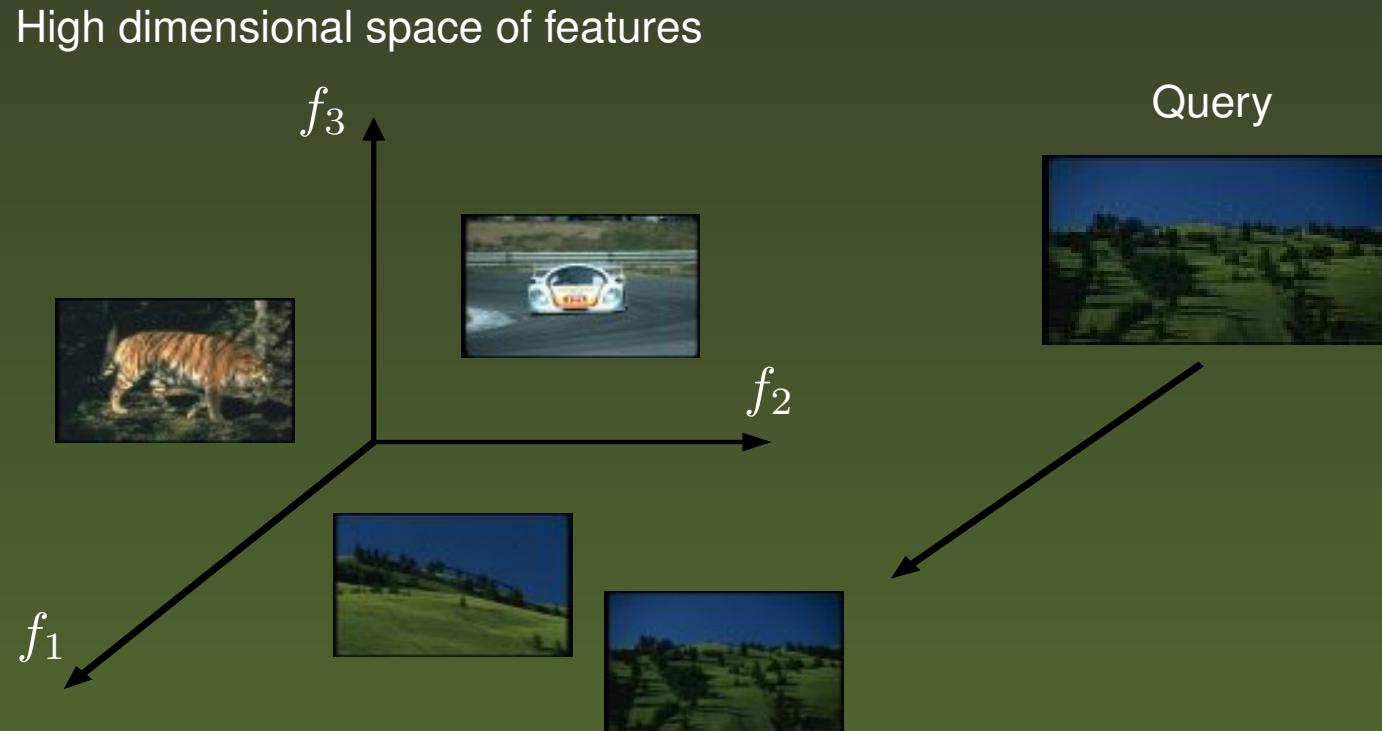


K Nearest Neighbors Problem

We have N training points in D dimensional space (usually $D > 100$)
 $\mathbf{x}_i \in \mathbb{R}^D, i = 1, \dots, N$.

Find the K nearest neighbors of a query point $x_q \in \mathbb{R}^D$.

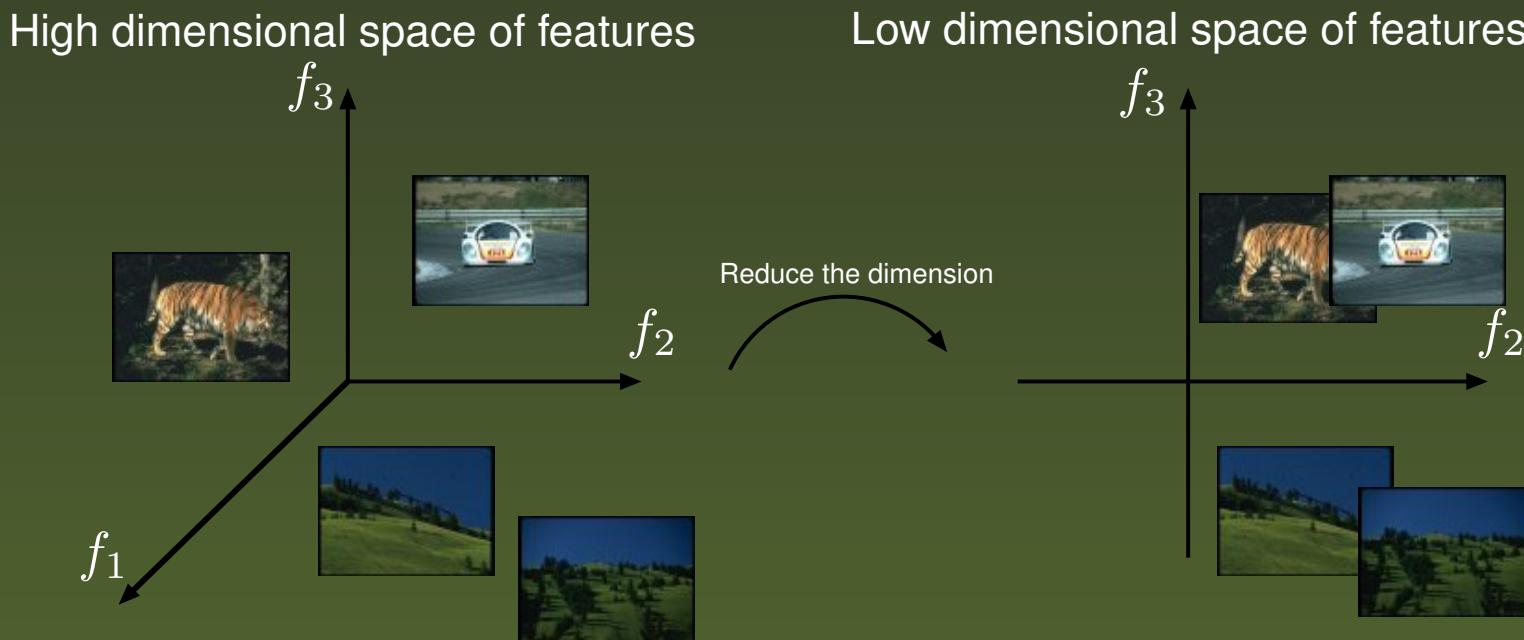
- ❖ Two applications are image retrieval and classification.
- ❖ Neighbors of a point are determined by the Euclidean distance.



Exact vs Approximate Nearest Neighbors

Exact search in the original space is $\mathcal{O}(ND)$ in both time and space.
This does not scale to large, high-dimensional datasets. Algorithms for approximate nearest neighbors:

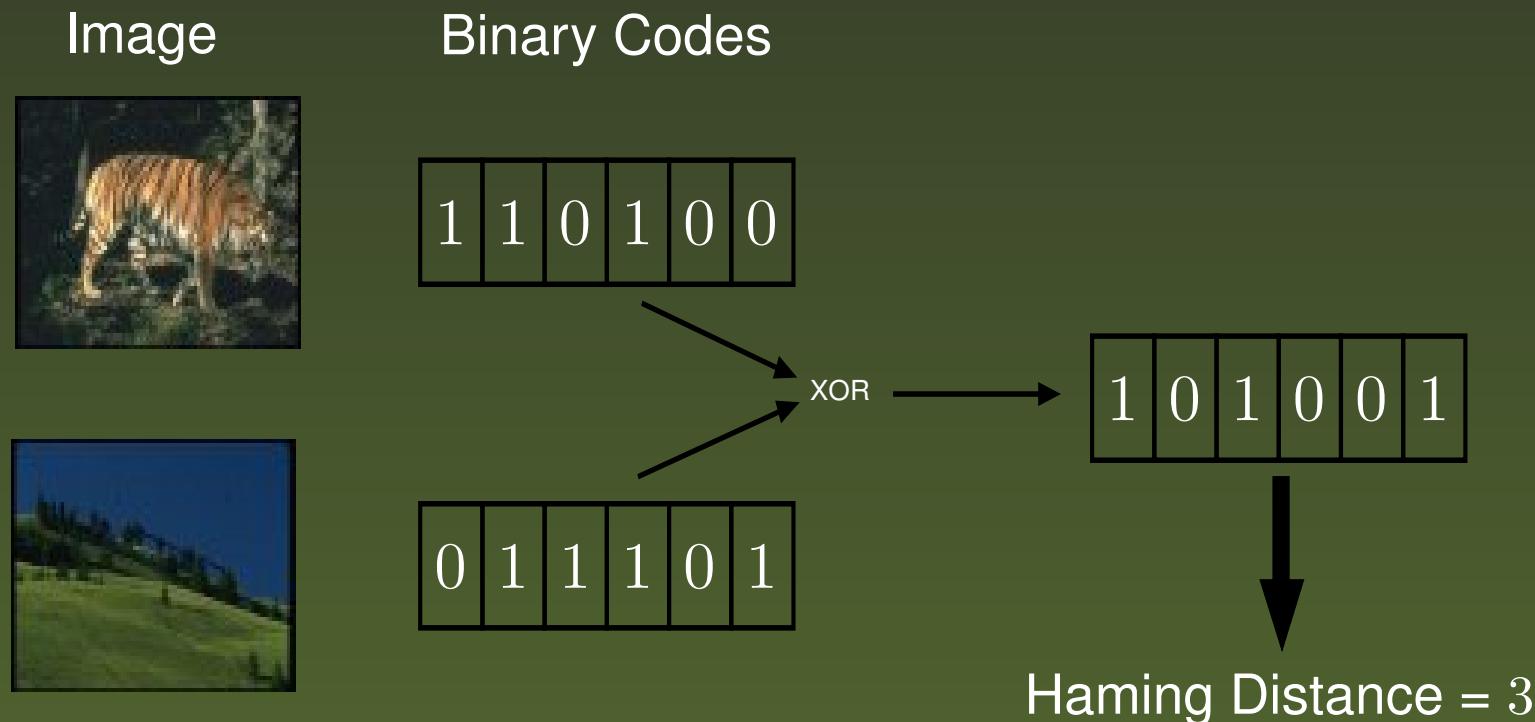
- ❖ Tree based methods
- ❖ Dimensionality reduction
- ❖ Binary hash functions



Binary Hash Functions

A binary hash function h takes as input a high-dimensional vector $\mathbf{x} \in \mathbb{R}^D$ and maps it to an **L -bit** vector $\mathbf{z} = h(\mathbf{x}) \in \{0, 1\}^L$.

- ❖ Main goal: preserve neighbors, i.e., assign (dis)similar codes to (dis)similar patterns.
- ❖ Hamming distance computed using **XOR** and then **counting**.



Binary Hash Function in Large Scale Image Retrieval

Scalability: we have millions or billions of high-dimensional images.

- ❖ Time complexity: $\mathcal{O}(NL)$ instead of $\mathcal{O}(ND)$ with small constants.
 - ◆ Bit operations to compute Hamming distance instead of floating point operations to compute Euclidean distance.
- ❖ Space complexity: $\mathcal{O}(NL)$ instead of $\mathcal{O}(ND)$ with small constants.
We can fit the binary codes of the entire dataset in memory, further speeding up the search.

Ex: $N = 1\,000\,000$ points, $D = 300$ and $L = 32$:

	Space	Time
Original space	1.2 GB	20 ms
Hamming space	4 MB	30 μ s

Previous Works on Binary Hashing

Binary hash functions have attained a lot of attention in recent years:

- ❖ Locality-Sensitive Hashing (Indyk and Motwani 2008)
- ❖ Spectral Hashing (Weiss et al. 2008)
- ❖ Kernelized Locality-Sensitive Hashing (Kulis and Grauman 2009)
- ❖ Semantic Hashing (Salakhutdinov and Hinton 2009)
- ❖ Iterative Quantization (Gong and Lazebnik 2011)
- ❖ Semi-supervised hashing for scalable image retrieval (Wang et al. 2012)
- ❖ Hashing With Graphs (Liu et al. 2011)
- ❖ Spherical Hashing (Heo et al. 2012)

Categories of hash functions:

- ❖ Data-independent methods (e.g. LSH: threshold a random projection).
- ❖ Data-dependent methods: learn hash function from a training set.
 - ◆ Unsupervised: no labels
 - ◆ Semi-supervised: some labels
 - ◆ Supervised: all labels

Objective Functions in Dimensionality Reduction

Learning hash functions is often done with dimensionality reduction:

- ❖ We can optimize an objective over the hash h function directly, e.g.:
 - ◆ **Autoencoder**: encoder (h) and decoder (f) can be linear, neural nets, etc.

$$\min_{\mathbf{h}, \mathbf{f}} \sum_{n=1}^N \|\mathbf{x}_n - \mathbf{f}(\mathbf{h}(\mathbf{x}_n))\|^2$$

- ❖ Or, we can optimize an objective over the projections \mathbf{Z} and then use these to learn the hash function h , e.g.:
 - ◆ **Laplacian Eigenmaps** (spectral problem):

$$\min_{\mathbf{Z}} \sum_{i,j=1}^N \mathbf{W}_{ij} \|\mathbf{z}_i - \mathbf{z}_j\|^2 \quad \text{s.t.} \quad \sum_{i=1}^N \mathbf{z}_i = 0, \quad \mathbf{Z}^T \mathbf{Z} = \mathbf{I}$$

- ◆ **Elastic Embedding** (nonlinear optimization):

$$\min_{\mathbf{Z}, \lambda} \sum_{i,j=1}^N \mathbf{W}_{ij}^+ \|\mathbf{z}_i - \mathbf{z}_j\|^2 + \lambda \sum_{i,j=1}^N \mathbf{W}_{ij}^- \exp(-\|\mathbf{z}_i - \mathbf{z}_j\|^2)$$

Learning Binary Codes

These objective functions are difficult to optimize because the codes are binary. Most existing algorithms approximate this as follows:

1. **Relax** the binary constraints and solve a continuous problem to obtain continuous codes.
2. **Binarize** these codes. Several approaches:
 - ❖ Truncate the real values using threshold zero
 - ❖ Find the best threshold for truncation
 - ❖ Rotate the real vectors to minimize the quantization loss:

$$E(\mathbf{B}, \mathbf{R}) = \|\mathbf{B} - \mathbf{VR}\|_F^2 \quad \text{s.t.} \quad \mathbf{R}^T \mathbf{R} = \mathbf{I}, \quad \mathbf{B} \in \{0, 1\}^{NL}$$

3. **Fit a mapping** to (patterns,codes) to obtain the hash function h .
Usually a classifier.

This is a **suboptimal, “filter” approach**: find approximate binary codes first, then find the hash function. We seek an **optimal, “wrapper” approach**: optimize over the binary codes and hash function **jointly**.

Our Hashing Models: Continuous Autoencoder

Consider first a well-known model for continuous dimensionality reduction, the **continuous autoencoder**:

- ❖ The **encoder** $h: \mathbf{x} \rightarrow \mathbf{z}$ maps a **real vector** $\mathbf{x} \in \mathbb{R}^D$ onto a low-dimensional **real vector** $\mathbf{z} \in \mathbb{R}^L$ (with $L < D$).
- ❖ The **decoder** $f: \mathbf{z} \rightarrow \mathbf{x}$ maps \mathbf{z} back to \mathbb{R}^D in an effort to reconstruct \mathbf{x} .

The objective function of an autoencoder is the **reconstruction error**:

$$E(h, f) = \sum_{n=1}^N \|\mathbf{x}_n - f(h(\mathbf{x}_n))\|^2$$

We can also define the following two-step objective function:

$$\text{first } \min E(f, Z) = \sum_{n=1}^N \|\mathbf{x}_n - f(\mathbf{z}_n)\|^2 \quad \text{then } \min E(h) = \sum_{n=1}^N \|\mathbf{z}_n - h(\mathbf{x}_n)\|^2$$

In both cases, if f and h are linear then the optimal solution is PCA.

Our Hashing Models: Binary Autoencoder

We consider **binary autoencoders** as our hashing model:

- ❖ The **encoder** $h: \mathbf{x} \rightarrow \mathbf{z}$ maps a **real vector** $\mathbf{x} \in \mathbb{R}^D$ onto a low-dimensional **binary vector** $\mathbf{z} \in \{0, 1\}^L$ (with $L < D$). **This will be our hash function.** We consider a thresholded linear encoder (hash function) $\mathbf{h}(\mathbf{x}) = \sigma(\mathbf{W}\mathbf{x})$ where $\sigma(t)$ is a step function elementwise.
- ❖ The **decoder** $f: \mathbf{z} \rightarrow \mathbf{x}$ maps \mathbf{z} back to \mathbb{R}^D in an effort to reconstruct \mathbf{x} . We consider a linear decoder in our method.

Binary autoencoder: optimize jointly over h and f the reconstruction error:

$$E_{BA}(h, f) = \sum_{n=1}^N \|\mathbf{x}_n - f(h(\mathbf{x}_n))\|^2 \quad \text{s.t.} \quad h(\mathbf{x}_n) \in \{0, 1\}^L$$

Binary factor analysis: first optimize over f and Z :

$$E_{BFA}(Z, f) = \sum_{n=1}^N \|\mathbf{x}_n - f(z_n)\|^2 \quad \text{s.t.} \quad z_n \in \{0, 1\}^L, \quad n = 1, \dots, N$$

then fit the hash function h to (\mathbf{X}, \mathbf{Z}) .

Optimization of Binary Autoencoders: “filter” approach

A simple but **suboptimal** approach:

1. Minimize the following objective function over linear functions f, g :

$$E(\mathbf{g}, \mathbf{f}) = \sum_{n=1}^N \|\mathbf{x}_n - \mathbf{f}(\mathbf{g}(\mathbf{x}_n))\|^2$$

which is equivalent to doing PCA on the input data.

2. Binarize the codes $\mathbf{Z} = \mathbf{g}(\mathbf{X})$ by an optimal rotation:

$$E(\mathbf{B}, \mathbf{R}) = \|\mathbf{B} - \mathbf{R}\mathbf{Z}\|_F^2 \quad \text{s.t.} \quad \mathbf{R}^T \mathbf{R} = \mathbf{I}, \quad \mathbf{B} \in \{0, 1\}^{LN}$$

The resulting hash function is $h(\mathbf{x}) = \sigma(\mathbf{R}\mathbf{g}(\mathbf{x}))$.

This is what the Iterative Quantization algorithm (ITQ, Gong et al. 2011), a leading binary hashing method, does.

Can we obtain better hash functions by doing a better optimization, i.e., respecting the binary constraints on the codes?

Optimization of Binary Autoencoders using MAC

Minimize the autoencoder objective function to find the hash function:

$$E_{\text{BA}}(\mathbf{h}, \mathbf{f}) = \sum_{n=1}^N \|\mathbf{x}_n - \mathbf{f}(\mathbf{h}(\mathbf{x}_n))\|^2 \quad \text{s.t.} \quad \mathbf{h}(\mathbf{x}_n) \in \{0, 1\}^L$$

We use the **method of auxiliary coordinates (MAC)** (Carreira-Perpiñán & Wang 2012, 2014). The idea is to break nested functional relationships judiciously by introducing variables as equality constraints, apply a penalty method and use alternating optimization.

We introduce as **auxiliary coordinates** the outputs of \mathbf{h} , i.e., the codes for each of the N input patterns and obtain a constrained problem:

$$\min_{\mathbf{h}, \mathbf{f}, \mathbf{z}} \sum_{n=1}^N \|\mathbf{x}_n - \mathbf{f}(\mathbf{z}_n)\|^2 \quad \text{s.t.} \quad \mathbf{z}_n = \mathbf{h}(\mathbf{x}_n), \quad \mathbf{z}_n \in \{0, 1\}^L, \quad n = 1, \dots, N.$$

Optimization of Binary Autoencoders (cont.)

We now apply the quadratic-penalty method (we could also apply the augmented Lagrangian):

$$E_Q(\mathbf{h}, \mathbf{f}, \mathbf{Z}; \mu) = \sum_{n=1}^N \left(\|\mathbf{x}_n - \mathbf{f}(\mathbf{z}_n)\|^2 + \mu \|\mathbf{z}_n - \mathbf{h}(\mathbf{x}_n)\|^2 \right) \text{ s.t. } \begin{cases} \mathbf{z}_n \in \{0, 1\}^L \\ n = 1, \dots, N. \end{cases}$$

Effects of the new parameter μ on the objective function:

- ❖ During the iterations, we allow the encoder and decoder to be mismatched.
- ❖ When μ is small, there will be a lot of mismatch. As μ increases, the mismatch is reduced.
- ❖ As $\mu \rightarrow \infty$ there will be no mismatch and E_Q becomes like E_{BA} .
- ❖ In fact, this occurs for a finite value of μ .

Optimization of Binary Autoencoders using MAC (cont.)

In order to minimize:

$$E_Q(\mathbf{h}, \mathbf{f}, \mathbf{Z}; \mu) = \sum_{n=1}^N \left(\|\mathbf{x}_n - \mathbf{f}(\mathbf{z}_n)\|^2 + \mu \|\mathbf{z}_n - \mathbf{h}(\mathbf{x}_n)\|^2 \right)$$

s.t. $\mathbf{z}_n \in \{0, 1\}^L, n = 1, \dots, N.$

we apply **alternating optimization**. The algorithm learns the hash function \mathbf{h} and the decoder \mathbf{f} given the current codes, and learns the patterns' codes given \mathbf{h} and \mathbf{f} :

- ❖ Over (\mathbf{h}, \mathbf{f}) for fixed \mathbf{Z} , we obtain $L + 1$ independent problems for each of the L single-bit hash functions, and for \mathbf{f} .
- ❖ Over \mathbf{Z} for fixed (\mathbf{h}, \mathbf{f}) , the problem separates for each of the N codes. The optimal code vector for pattern \mathbf{x}_n tries to be close to the prediction $\mathbf{h}(\mathbf{x}_n)$ while reconstructing \mathbf{x}_n well.

We have to solve each of these steps.

Optimization over f for fixed Z (decoder given codes)

We have to minimize the following over the linear decoder \mathbf{f} (where $\mathbf{f}(\mathbf{x}) = \mathbf{Ax} + \mathbf{b}$):

$$E_Q(\mathbf{h}, \mathbf{f}, \mathbf{Z}; \mu) = \sum_{n=1}^N \left(\|\mathbf{x}_n - \mathbf{f}(\mathbf{z}_n)\|^2 + \mu \|\mathbf{z}_n - \mathbf{h}(\mathbf{x}_n)\|^2 \right) \text{ s.t. } \begin{cases} \mathbf{z}_n \in \{0, 1\}^L \\ n = 1, \dots, N. \end{cases}$$

A simple linear regression with data (\mathbf{Z}, \mathbf{X}) :

$$\min_{\mathbf{f}} \sum_{n=1}^N \|\mathbf{x}_n - \mathbf{f}(\mathbf{z}_n)\|^2 = \min_{\mathbf{A}, \mathbf{b}} \sum_{n=1}^N \|\mathbf{x}_n - \mathbf{Az}_n - \mathbf{b}\|^2$$

The solution is (ignoring the bias for simplicity) $\mathbf{A} = \mathbf{XZ}^T(\mathbf{ZZ}^T)^{-1}$ and can be computed in $\mathcal{O}(NDL)$.

The constant factor in the \mathcal{O} -notation is small because Z is binary, e.g. \mathbf{XZ}^T involves only sums, not multiplications.

Optimization over \mathbf{h} for fixed \mathbf{Z} (encoder given codes)

We have to minimize the following over the linear hash function \mathbf{h} (where $\mathbf{h}(\mathbf{x}) = \sigma(\mathbf{W}\mathbf{x})$):

$$E_Q(\mathbf{h}, \mathbf{f}, \mathbf{Z}; \mu) = \sum_{n=1}^N \left(\|\mathbf{x}_n - \mathbf{f}(\mathbf{z}_n)\|^2 + \mu \|\mathbf{z}_n - \mathbf{h}(\mathbf{x}_n)\|^2 \right) \text{ s.t. } \begin{cases} \mathbf{z}_n \in \{0, 1\}^L \\ n = 1, \dots, N. \end{cases}$$

The hash function has the following form:

$$\begin{aligned} \min_{\mathbf{h}} \sum_{n=1}^N \|\mathbf{z}_n - \mathbf{h}(\mathbf{x}_n)\|^2 &= \min_{\mathbf{W}} \sum_{n=1}^N \|\mathbf{z}_n - \sigma(\mathbf{W}\mathbf{x}_n)\|^2 \\ &= \sum_{l=1}^L \min_{\mathbf{w}_l} \sum_{n=1}^N (\mathbf{z}_{nl} - \sigma(\mathbf{w}_l^T \mathbf{x}_n))^2 \end{aligned}$$

so it separates for each bit $l = 1 \dots L$.

The subproblem for each bit is a **binary classification** problem with data $(\mathbf{X}, \mathbf{Z}_{\cdot l})$ using the number of misclassified patterns as loss function. We approximately solve it with a linear SVM.

Optimization over \mathbf{Z} for fixed (\mathbf{h}, \mathbf{f}) (adjust codes given encoder/decoder)

This is a **binary optimization** on NL variables, but it separates into N independent optimizations each on only L variables:

$$\min_{\mathbf{z}} e(\mathbf{z}) = \|\mathbf{x} - \mathbf{f}(\mathbf{z})\|^2 + \mu \|\mathbf{z} - \mathbf{h}(\mathbf{x})\|^2 \quad \text{s.t.} \quad \mathbf{z} \in \{0, 1\}^L$$

This is a quadratic objective function on binary variables, which is NP-complete in general, but L is small.

With $L \lesssim 16$ we can afford an **exhaustive search** over the 2^L codes. Besides, we don't need to evaluate every code vector, or every bit of every code vectors:

- ❖ Intuitively, the optimum will not be far from $\mathbf{h}(\mathbf{x})$, at least if μ is large.
- ❖ We don't need to test vectors beyond a Hamming distance $\|\mathbf{x} - \mathbf{f}(\mathbf{h}(\mathbf{x}))\|^2 / \mu$ (they cannot be optima).

Z Step for Large L : Approximate Solution

For larger L , we use **alternating optimization** over groups of g bits.

- ❖ The optimization over a g -bit group is done by enumeration using the accelerations described earlier.
- ❖ Consider an example where $L = 8$ and $g = 4$:

initialization	1	1	0	0	0	0	1	0
step over z_1 to z_4	?	?	?	?	0	0	1	0
step over z_5 to z_8	1	0	1	0	?	?	?	?

How to initialize z ? We have used the following two approaches:

- ❖ **Warm start**: Initialize z to the code found in the previous iteration's **Z step**. Convenient in later iterations, when the codes change slowly.
- ❖ Solve the **relaxed problem** on $z \in [0, 1]^L$ and then truncate it. We use an ADMM algorithm, caching one matrix factorization for all $n = 1, \dots, N$. Convenient in early iterations, when the codes change fast.

Summary of the Binary Autoencoder MAC Algorithm

input $\mathbf{X}_{D \times N} = (\mathbf{x}_1, \dots, \mathbf{x}_N)$, $L \in \mathbb{N}$

Initialize $\mathbf{Z}_{L \times N} = (\mathbf{z}_1, \dots, \mathbf{z}_N) \in \{0, 1\}^{LN}$

for $\mu = 0 < \mu_1 < \dots < \mu_\infty$

for $l = 1, \dots, L$ h step

$h_l \leftarrow$ fit SVM to $(\mathbf{X}, \mathbf{Z}_{\cdot l})$

$\mathbf{f} \leftarrow$ least-squares fit to (\mathbf{Z}, \mathbf{X}) f step

for $n = 1, \dots, N$ Z step

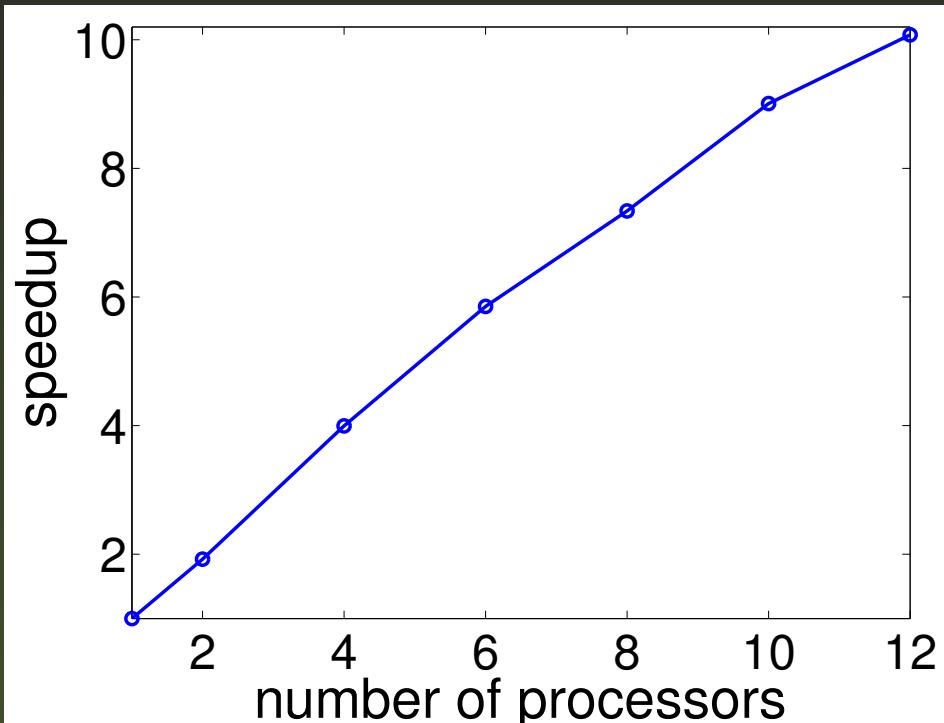
$\mathbf{z}_n \leftarrow \arg \min_{\mathbf{z}_n \in \{0,1\}^L} \|\mathbf{y}_n - \mathbf{f}(\mathbf{z}_n)\|^2 + \mu \|\mathbf{z}_n - \mathbf{h}(\mathbf{x}_n)\|^2$

if $\mathbf{Z} = \mathbf{h}(\mathbf{X})$ then stop

return $\mathbf{h}, \mathbf{Z} = \mathbf{h}(\mathbf{X})$

Repeatedly solve: classification (h), regression (f), binarization (Z).

Optimization of Binary Autoencoders using MAC (cont.)



The steps can be parallelized:

- ❖ Z step: N independent problems, one per binary code vector \mathbf{z}_n .
- ❖ f and h steps are independent.
h step: L independent problems, one per binary SVM.

Schedule for the penalty parameter μ :

- ❖ With exact steps, the algorithm terminates at a finite μ .
This occurs when the solution of the Z step equals the output of the hash function, and gives a practical termination criterion.
- ❖ We start with a small μ and increase it slowly until termination.

Experimental Setup: Precision and Recall

The performance of binary hash functions is usually reported using precision and recall.

Retrieved set for a query point can be defined in two ways:

- ❖ The K nearest neighbors in the Hamming space.
- ❖ The points in the Hamming radius of r .

Ground-truth for a query point contains the first K nearest neighbors of the point in the original(D -dimensional) space.

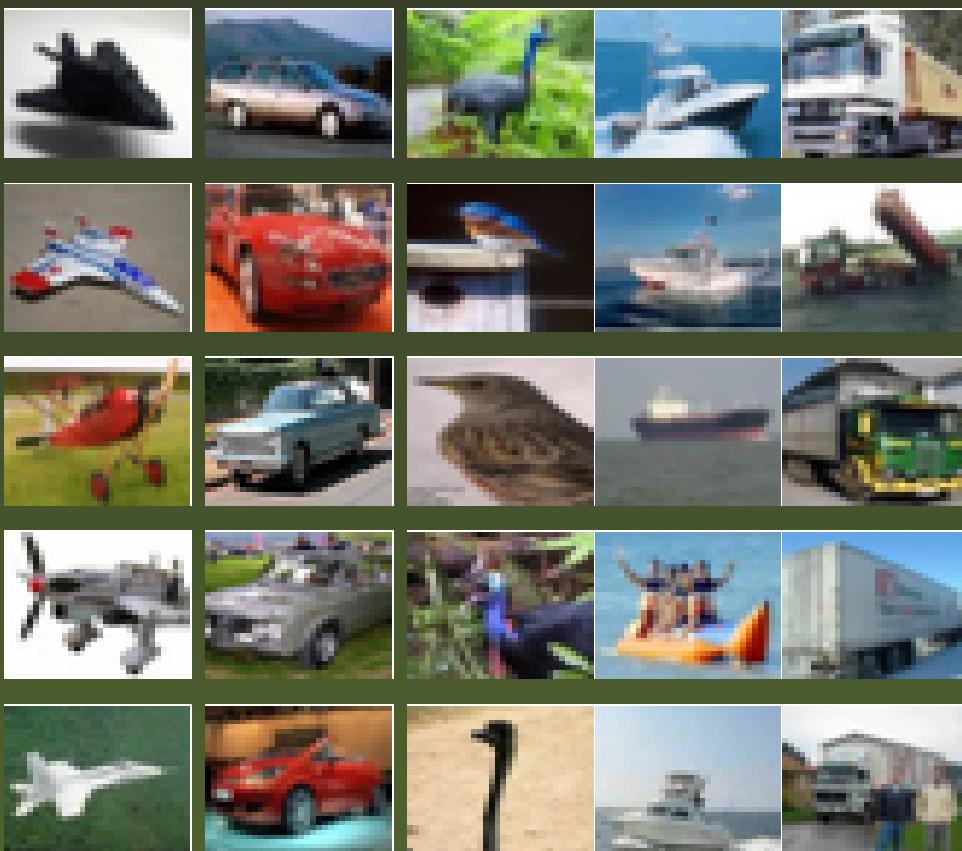
$$\text{precision} = \frac{|\{\text{retrieved points}\} \cap \{\text{groundtruth}\}|}{|\{\text{groundtruth}\}|}$$

$$\text{recall} = \frac{|\{\text{retrieved points}\} \cap \{\text{groundtruth}\}|}{|\{\text{retrieved points}\}|}$$

Experiment: Datasets

CIFAR-10 dataset: 60 000 32×32 color images in 10 classes; training/test 50 000/10 000, 320 GIST features.

airplane automobile bird ship truck



NUS-WIDE dataset: 269 648 high resolution color images in 81 concepts; training/test 161 789/107 859, 128 Wavelet features.

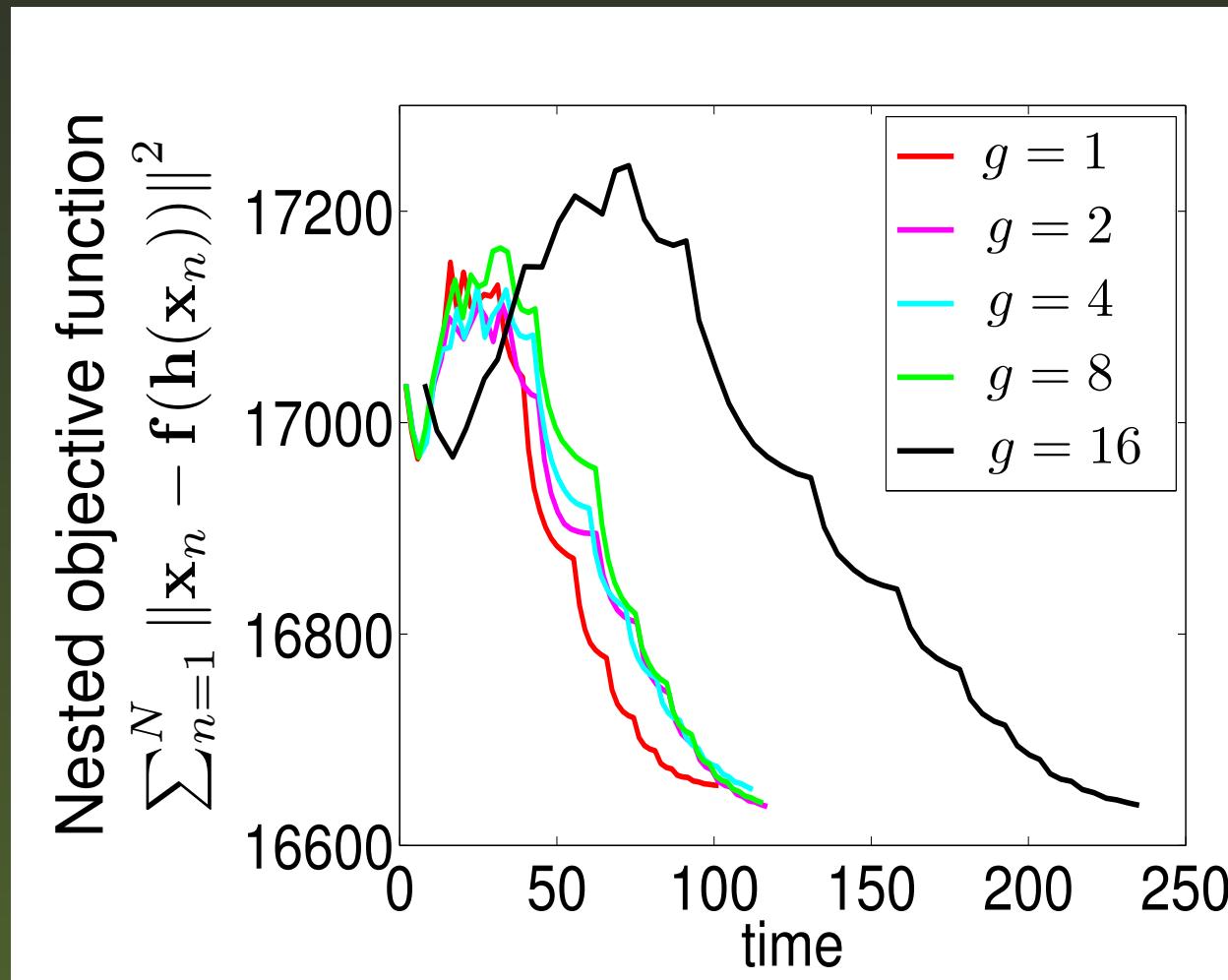
SIFT-1M dataset: 1 010 000 high resolution color images; training/test 1 000 000/10 000, 128 SIFT features.

actor bicycle eagle ship airplane



Experiment: Exact vs. Inexact Optimization

Inexact Z steps achieve solutions of similar quality than exact steps but much faster. Best results occur for $g \approx 1$ in alternating optimization.

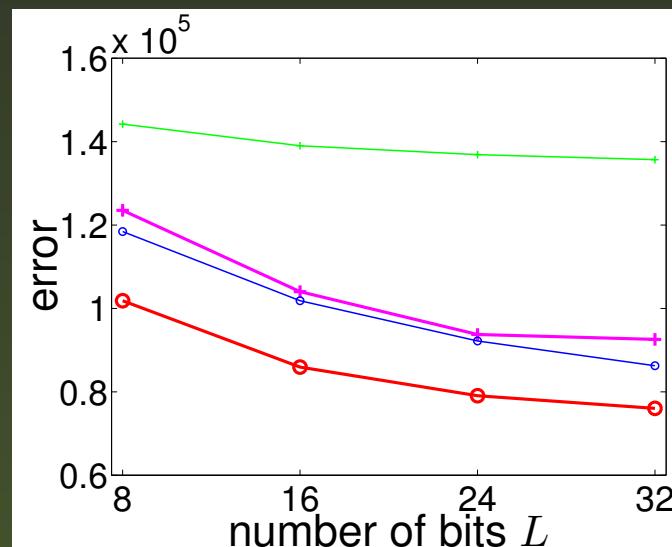


$N = 50\,000$ images of CIFAR dataset, $L = 16$ bits, relaxed initial \mathbf{Z} .

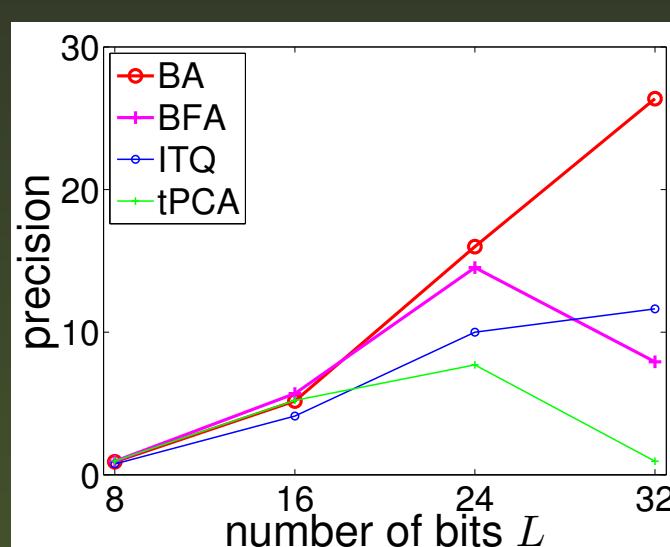
Optimizing Binary Autoencoders Improves Precision

NUS-WIDE-LITE dataset, $N = 27\,807$ training/ 27 808 test images,
 $D = 128$ wavelet features.

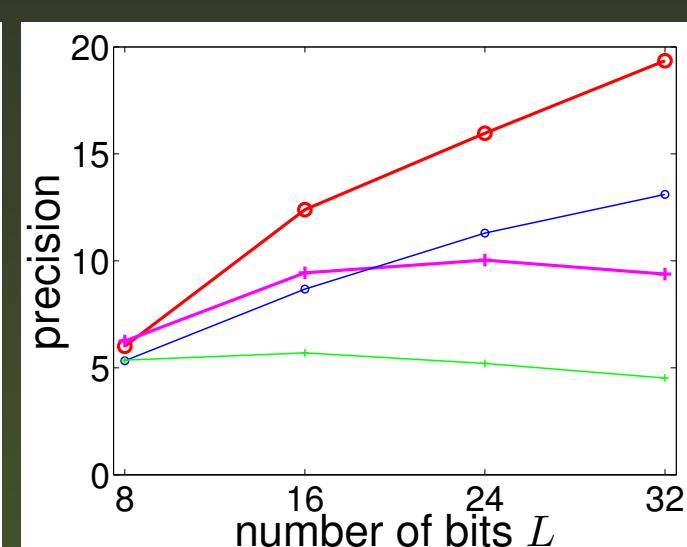
autoencoder error



precision within $r \leq 2$



$k = 50$ nearest neighbors



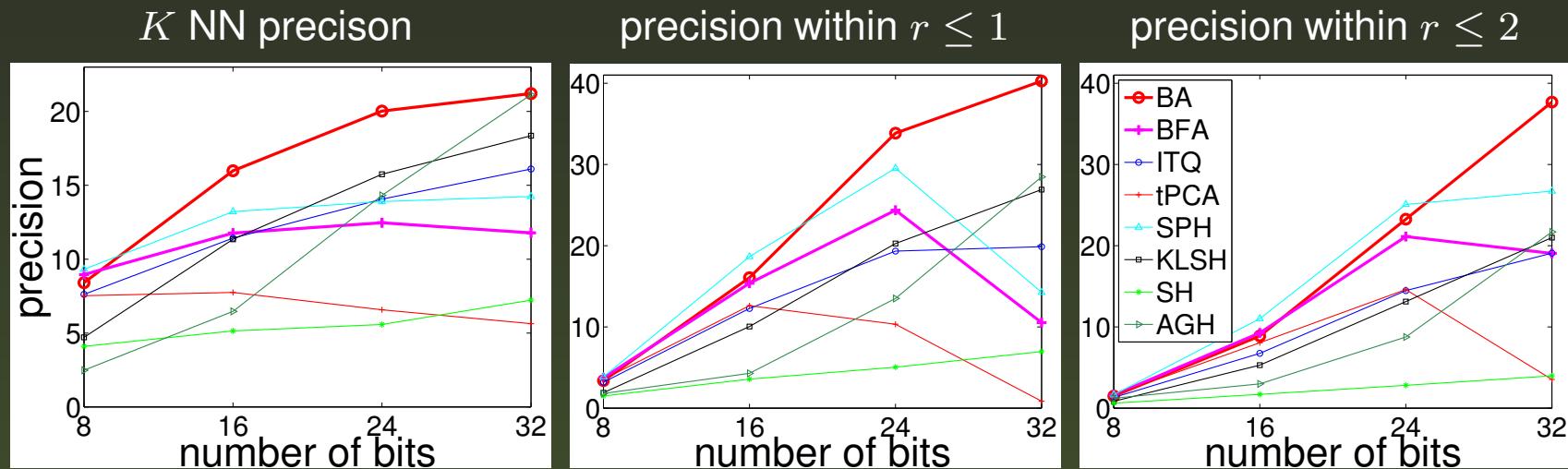
ITQ and tPCA use a filter approach (suboptimal): They solve the continuous problem and truncate the solution.

BA uses a wrapper approach (optimal): It optimizes the objective function respecting the binary nature of the codes.

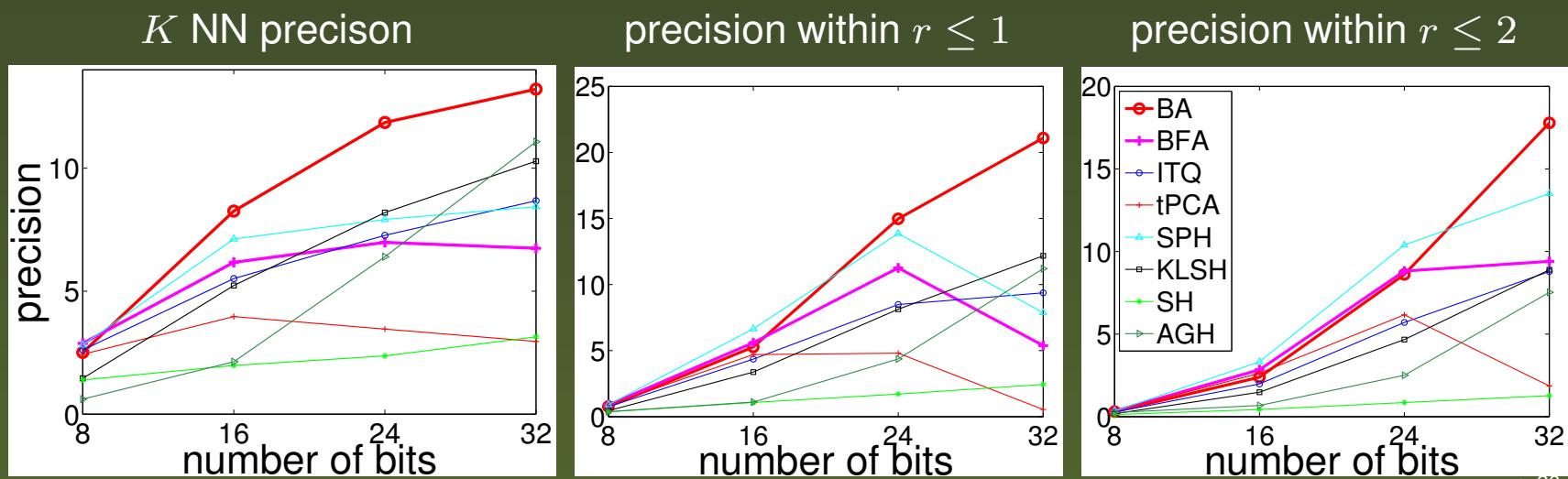
BA achieves lower reconstruction error and also better precision/recall.

Experimental Results on NUS-WIDE Dataset (cont.)

Ground truth: $K = 500$ nearest neighbors of each query point:

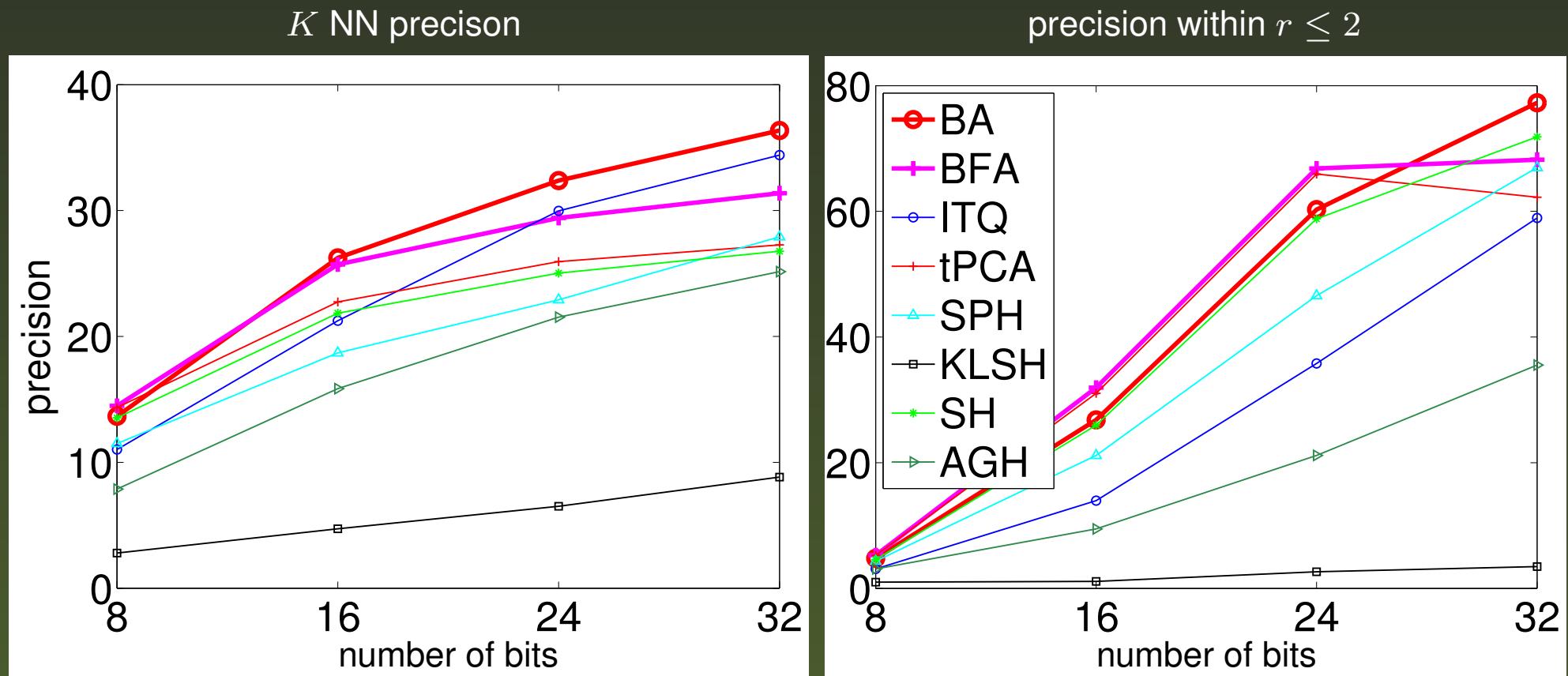


Ground truth: $K = 100$ nearest neighbors of each query point:



Experimental Results On ANNSIFT-1m

Ground truth: $K = 10000$ nearest neighbors of each query point:



A well-optimized binary autoencoder with a linear hash function consistently beats state-of-the-art methods.

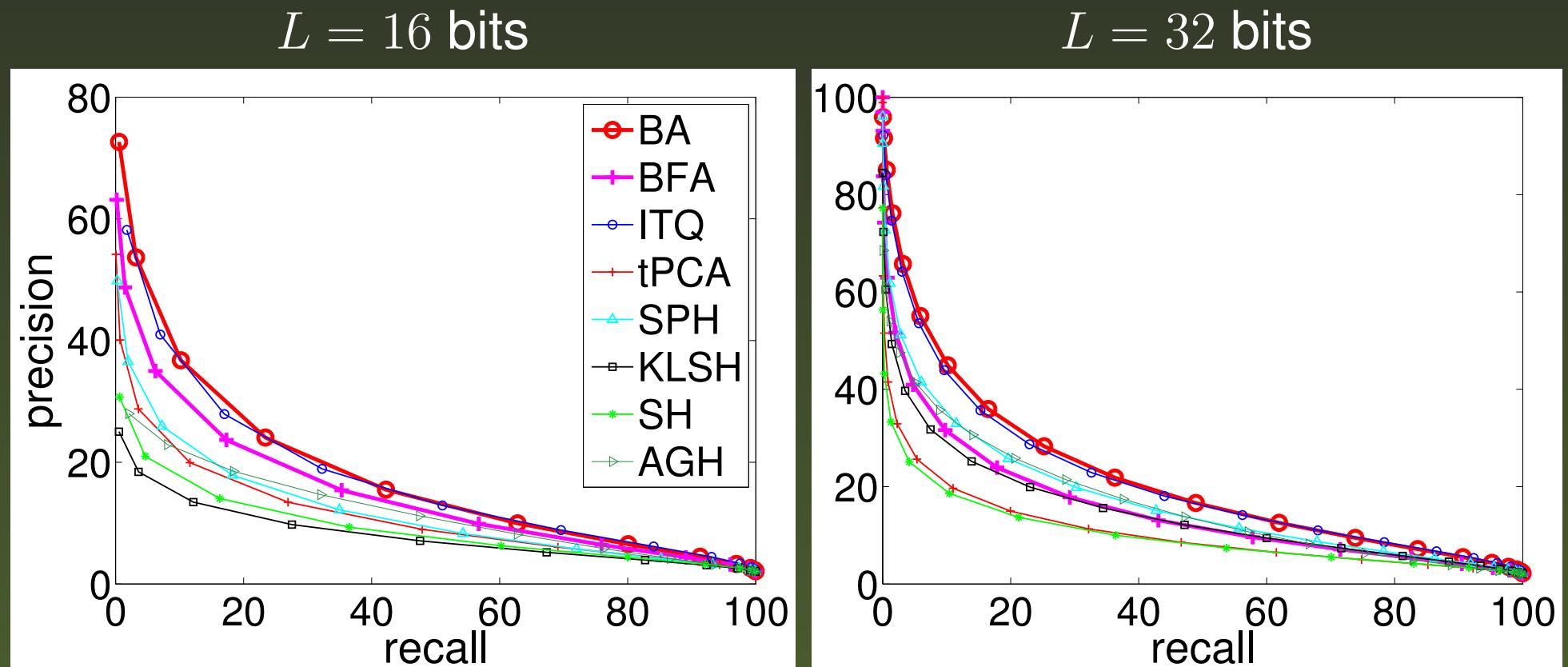
Conclusion

- ❖ A fundamental difficulty in learning hash functions is binary optimization.
 - ◆ Most existing methods relax the problem and find its continuous solution. Then, they threshold the result to obtain binary codes, which is sub-optimal.
 - ◆ Using the method of auxiliary coordinates, we can do the optimization correctly and efficiently for binary autoencoders.
 - ★ Encoder (hash function): train one SVM per bit.
 - ★ Decoder: solve a linear regression problem.
 - ★ Highly parallel.
- ❖ Remarkably, with proper optimization, a simple model (autoencoder with linear encoder and decoder) beats state-of-the-art methods using nonlinear hash functions and/or better objective functions.

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Experimental Results on CIFAR Dataset

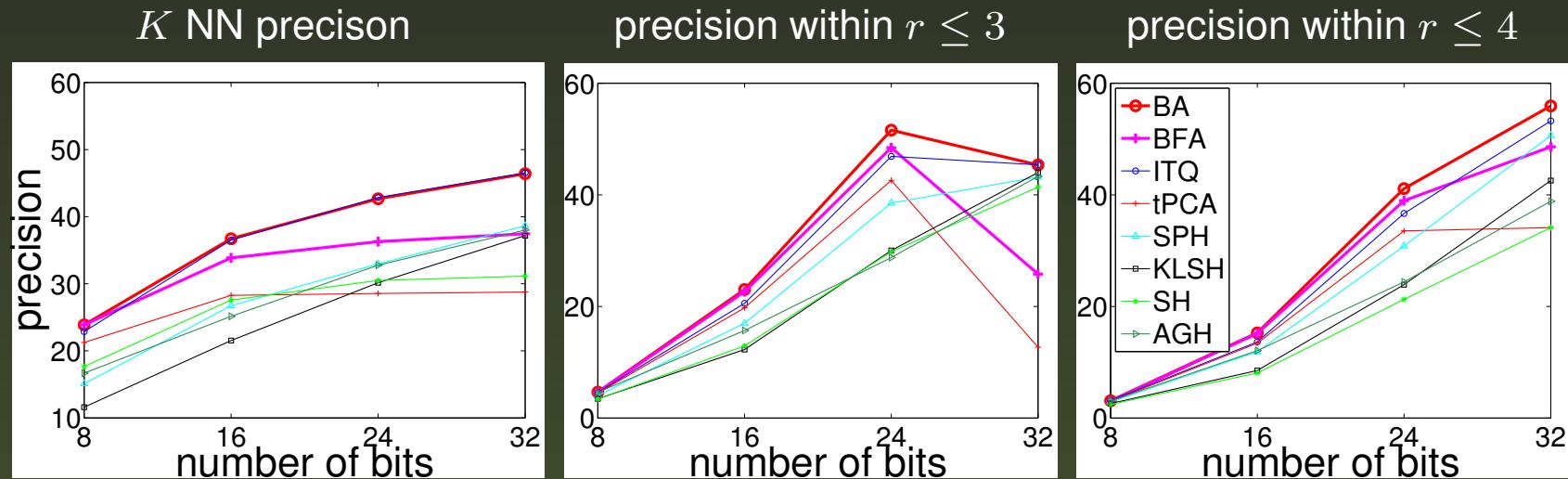
Ground truth: $K = 1000$ nearest neighbors of each query point.



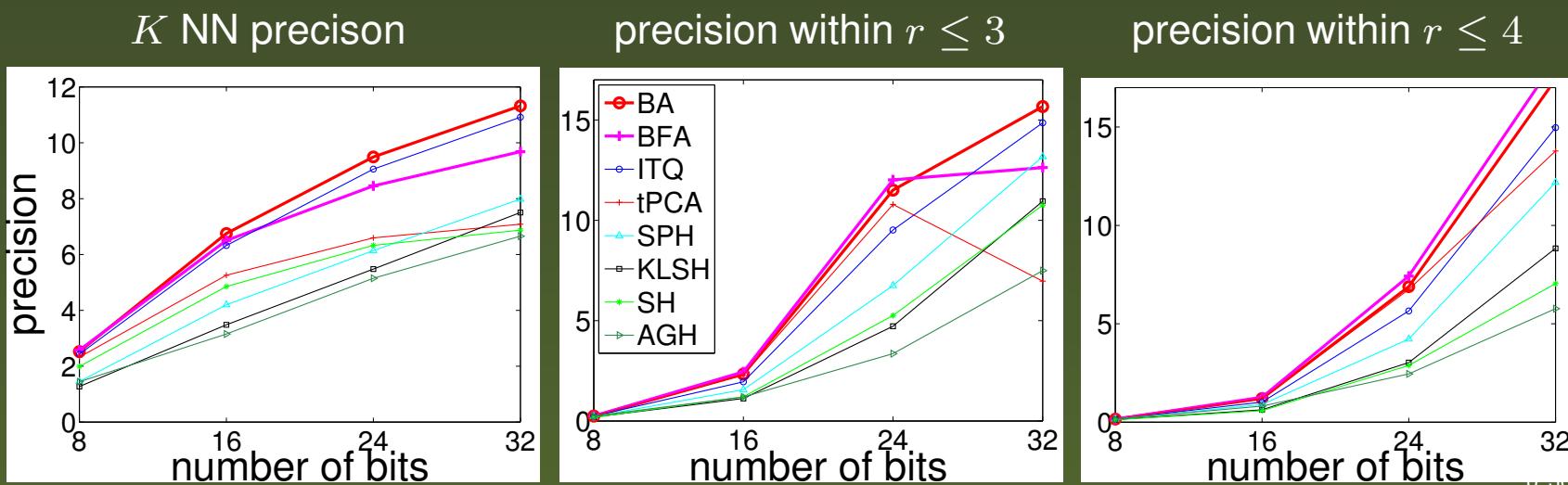
A well-optimized binary autoencoder with a linear hash function consistently beats state-of-the-art methods.

Experimental Results on CIFAR Dataset (cont.)

Ground truth: $K = 1000$ nearest neighbors of each query point:

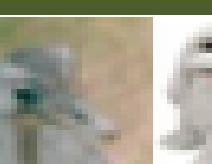
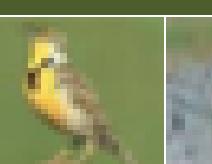
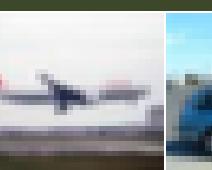
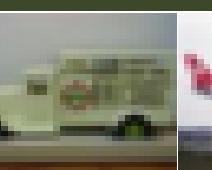
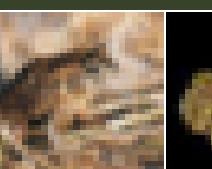
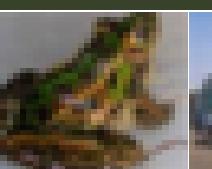
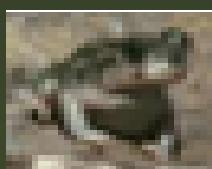
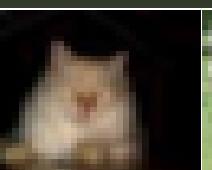
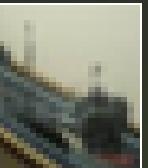
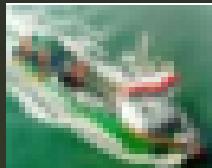


Ground truth: $K = 50$ nearest neighbors of each query point:



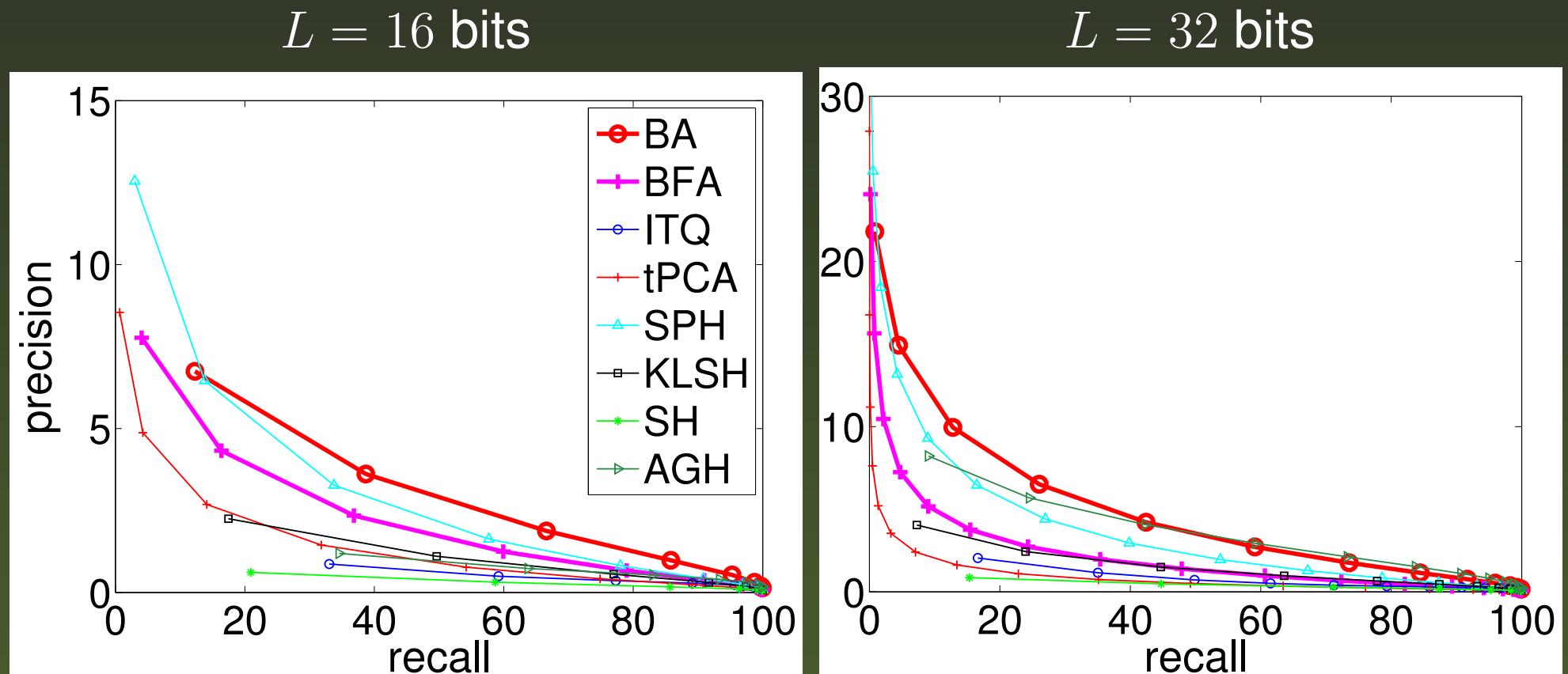
Top retrieved images from CIFAR Dataset

input



Experimental Results on NUS-WIDE Dataset

Ground truth: $K = 100$ nearest neighbors of each query point:



A well-optimized binary autoencoder with a linear hash function consistently beats state-of-the-art methods using more sophisticated objectives and (nonlinear) hash functions.

Comparison Algorithms

Algorithm with Kernel hash functions:

- ❖ KLSH(Kulis et al. 2009): Generalizes locality-sensitive hashing to accommodate arbitrary kernel functions.

Algorithms with embedding objective function(laplacian eigenmap):

- ❖ SH(Weiss et al. 2008): Finds the relaxed solution of laplacian eigenmap and truncates it.
- ❖ AGH(Liu et al. 2011): Approximates eigenfunctions using K points and finds thresholds to make the codes binary.

Algorithms that maximize the variance:

- ❖ ITQ(Gong et al.) and tPCA: First compute PCA on the input patterns and then truncate the continuous solution.
- ❖ SPH(Heo et al. 2012): Iteratively refines the thresholds and pivots to maximize the variance of binary codes.