Chapter 3
Quadcopter Modelling

3.1 Reference frames definition

Most of the paper dealing with flying robots ([3], [4]), start to define the reference frames used for the robot state description. In the context of this project, the experiments are realized in a room defined by a fixed inertial frame named the World frame (W) whose origin is the middle of the room. A rotating frame attached to the centre of the quadcopter is moreover used to follow its attitude. It is commonly called the Body frame (B). Both frames are represented in figure 3.1:

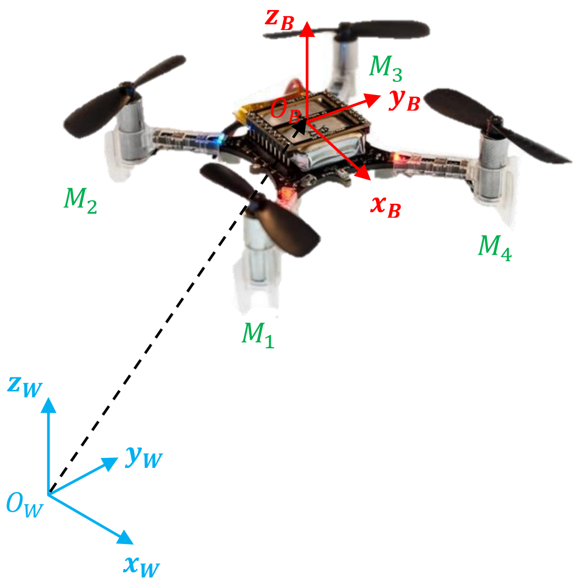

Figure 3.1: Representation of the reference frames: World frame & Body frame

As shown in the figure 3.1, the coordinates' system used for the World frame and the Body frame are respectively denoted by $[\boldsymbol{x_W}\ \boldsymbol{y_W}\ \boldsymbol{z_W}]$ and $[\boldsymbol{x_B}\ \boldsymbol{y_B}\ \boldsymbol{z_B}]$. The alignment of the $\boldsymbol{x_B}$ and $\boldsymbol{y_B}$ axis is chosen according to the quadcopter front definition and to the embedded IMU axis-system in order to simplify the sensor data preprocessing.

3.2 Attitude representation

3.2.1 Euler angles-based representation

The orientation of the quadrotor can be first defined using the Euler angles representation $\boldsymbol{\Theta} = [\phi \; \theta \; \psi]^T$ which allows to map the World frame to the Body frame orientation through a sequence of three rotations:

- A right-hand rotation around $\boldsymbol{z_W}$ giving positive yaw angle ψ.
- A right-hand rotation around $\boldsymbol{y_W}$ giving positive pitch angle θ.
- A right-hand rotation around $\boldsymbol{x_W}$ gives positive roll angle ϕ.

Matrices describing these rotations are defined by:

$$\boldsymbol{R_\phi} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix}$$

$$\boldsymbol{R_\theta} = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \tag{3.1}$$

$$\boldsymbol{R_\psi} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Applying successively these rotations gives the rotation matrix from the World to the Body frame ${}^{\boldsymbol{B}}\boldsymbol{R_W}$:

$$\begin{gathered} {}^{\boldsymbol{B}}\boldsymbol{R_W} = \boldsymbol{R_\phi} \cdot \boldsymbol{R_\theta} \cdot \boldsymbol{R_\psi} \\ \Leftrightarrow \\ {}^{\boldsymbol{B}}\boldsymbol{R_W} = \begin{pmatrix} c\psi c\theta & s\psi c\theta & -s\theta \\ c\psi s\theta s\phi - s\psi c\phi & s\psi s\theta s\phi + c\psi c\phi & c\theta s\phi \\ c\psi s\theta c\phi + s\psi s\phi & s\psi s\theta c\phi - c\psi s\phi & c\theta c\phi \end{pmatrix} \end{gathered} \tag{3.2}$$

With $c\phi = \cos\phi$, $s\phi = \sin\phi$ and similarly for θ and ψ.

Others rotation sequences can be chosen as well but the one used above is the most common when working with Unmanned Aerial Vehicles (UAVs).

As ${}^{B}\boldsymbol{R_W}$ is orthonormal, the corresponding inverse transformation ${}^{W}\boldsymbol{R_B}$ (the rotation matrix from the Body to the World frame) is simply obtained through the transpose operation:

$$ {}^{W}\boldsymbol{R_B} = {}^{B}\boldsymbol{R_W}^{\boldsymbol{T}} = \begin{pmatrix} c\psi c\theta & c\psi s\theta s\phi - s\psi c\phi & c\psi s\theta c\phi + s\psi s\phi \\ s\psi c\theta & s\psi s\theta s\phi + c\psi c\phi & s\psi s\theta c\phi - c\psi s\phi \\ -s\theta & c\theta s\phi & c\theta c\phi \end{pmatrix} \tag{3.3} $$

The column vectors of ${}^{W}\boldsymbol{R_B}$ represent the vectors' coordinates of the Body frame described in the World frame:

$$ {}^{W}\boldsymbol{R_B} = [\boldsymbol{x_B}\ \boldsymbol{y_B}\ \boldsymbol{z_B}] \tag{3.4} $$

According to the Euler angles order convention chosen before, the Euler rates vector $\dot{\boldsymbol{\Theta}} = [\dot{\phi}\ \dot{\theta}\ \dot{\psi}]^T$ is related to the body-fixed angular velocity vector $\boldsymbol{\omega_B} = [p\ q\ r]^T$ through a matrix operation represented in equation 3.5:

$$ \begin{gathered} \boldsymbol{\omega_B} = \begin{pmatrix} \dot{\phi} \\ 0 \\ 0 \end{pmatrix} + \boldsymbol{R_\phi}\left(\begin{pmatrix} 0 \\ \dot{\theta} \\ 0 \end{pmatrix} + \boldsymbol{R_\theta}\begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix}\right) \\ \Leftrightarrow \\ \boldsymbol{\omega_B} = \begin{pmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\phi & \sin\phi\cos\theta \\ 0 & -\sin\phi & \cos\phi\cos\theta \end{pmatrix}\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \boldsymbol{T^{-1}}\dot{\boldsymbol{\Theta}} \end{gathered} \tag{3.5} $$

Where $\boldsymbol{T^{-1}}$ is the rotation matrix mapping the Euler rates to the Body rates. The inverse transformation is simply derived by inverting $\boldsymbol{T^{-1}}$:

$$ \dot{\boldsymbol{\Theta}} = \boldsymbol{T}\boldsymbol{\omega_B} = \begin{pmatrix} 1 & \tan\theta\sin\phi & \tan\theta\cos\phi \\ 0 & \cos\phi & -\sin\phi \\ 0 & \dfrac{\sin\phi}{\cos\theta} & \dfrac{\cos\phi}{\cos\theta} \end{pmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix} \tag{3.6} $$

3.2.2 Quaternion-based representation

The Euler angles-based representation presented before suffers from two main disadvantages:

- The use of trigonometric functions which implies heavy computations.
- The gimbal lock problematic resulting in the loss of a degree of freedom for some angles values.

The quaternion-based representation [5], [6] is an elegant alternative way to get rid of these two problems. It is close to the Euler axis/angle parametrization where a rotation is described as a unit vector $\boldsymbol{e}$ perpendicular to the rotation plane with an angle θ representing the magnitude of the rotation. In the case of quaternions, rotation axis and angle are combined in a single four-dimensional vector $\boldsymbol{q}$ to which is attached different representations:

$$\boldsymbol{q} = \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} q_0 \\ \underline{\boldsymbol{q}} \end{pmatrix} = q_0 + q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k} \tag{3.7}$$

A quaternion is commonly called a hyper complex number in the sense that it has one real part q_0 and a three-dimensional imaginary vector $\overline{\boldsymbol{q}}$ describing the rotation direction. The conversion between the Euler axis/angle and quaternion representations are shown in equations 3.8.

$$\begin{gathered} \boldsymbol{q} = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right)\boldsymbol{e} \end{pmatrix} \\ \boldsymbol{e} = \frac{\underline{\boldsymbol{q}}}{\left\|\underline{\boldsymbol{q}}\right\|} \quad \theta = 2\cos^{-1} q_0 \end{gathered} \tag{3.8}$$

A quaternion used to represent a rotation has a unit norm $\|\boldsymbol{q}\|$, its conjugate and inverse defined in equations 3.9:

$$\begin{gathered} \overline{\boldsymbol{q}} = \begin{pmatrix} q_0 \\ -\underline{\boldsymbol{q}} \end{pmatrix} \\ \boldsymbol{q}^{-\boldsymbol{1}} = \frac{\overline{\boldsymbol{q}}}{\|\boldsymbol{q}\|} = \overline{\boldsymbol{q}} \end{gathered} \tag{3.9}$$

In order to combine rotations, quaternions can be multiplied together through the quaternion product operation denoted by the symbol $*$:

$$\begin{gathered} \boldsymbol{p} * \boldsymbol{q} = \begin{pmatrix} p_0 q_0 - \underline{\boldsymbol{q}}^{\boldsymbol{T}}\underline{\boldsymbol{p}} \\ p_0\underline{\boldsymbol{q}} + q_0\underline{\boldsymbol{p}} + \underline{\boldsymbol{p}} \times \underline{\boldsymbol{q}} \end{pmatrix} \\ \Leftrightarrow \\ \boldsymbol{p} * \boldsymbol{q} = \begin{pmatrix} p_0 & -\underline{\boldsymbol{p}}^{\boldsymbol{T}} \\ \underline{\boldsymbol{p}} & p_0\boldsymbol{I}_{\boldsymbol{3}\times\boldsymbol{3}} + \left[\underline{\boldsymbol{p}}\,\times\right] \end{pmatrix}\boldsymbol{q} = \begin{pmatrix} p_0 & -\underline{\boldsymbol{q}}^{\boldsymbol{T}} \\ \underline{\boldsymbol{q}} & q_0\boldsymbol{I}_{\boldsymbol{3}\times\boldsymbol{3}} + \left[\underline{\boldsymbol{q}}\,\times\right]^T \end{pmatrix}\boldsymbol{p} \end{gathered} \tag{3.10}$$

Where $\boldsymbol{I_{3\times3}}$ is the 3×3 identity matrix and $[\underline{\boldsymbol{p}}\times]$ is the skew-symmetric cross product matrix applied to vector $\underline{\boldsymbol{p}}$:

$$\boldsymbol{p} * \boldsymbol{q} = \begin{pmatrix} p_0 q_0 - \underline{\boldsymbol{q}}^{\boldsymbol{T}} \underline{\boldsymbol{p}} \\ p_0 \underline{\boldsymbol{q}} + q_0 \underline{\boldsymbol{p}} + \underline{\boldsymbol{p}} \times \underline{\boldsymbol{q}} \end{pmatrix} \tag{3.11}$$

This operation is however not commutative: $\boldsymbol{p} * \boldsymbol{q} \neq \boldsymbol{q} * \boldsymbol{p}$.

An important transformation that needs to be described is the vector rotation using quaternions. For this, the desired Euclidean vector $\boldsymbol{u}$ to rotate is written as a normal quaternion with zero real part:

$$\boldsymbol{u_q} = \begin{pmatrix} 0 \\ \boldsymbol{u} \end{pmatrix} \tag{3.12}$$

The transformed vector $\boldsymbol{v_q}$ is also a zero real part quaternion and is commonly obtained using equations 3.13 and 3.14:

$$\boldsymbol{v_q} = \begin{pmatrix} 0 \\ \boldsymbol{v} \end{pmatrix} = \boldsymbol{q^{-1}} * \boldsymbol{u_q} * \boldsymbol{q} = \begin{pmatrix} 0 \\ A(\boldsymbol{q})\boldsymbol{u} \end{pmatrix} \tag{3.13}$$

Where:

$$A(\boldsymbol{q}) = 2\underline{\boldsymbol{q}}\,\underline{\boldsymbol{q}}^{\boldsymbol{T}} + \left(q_0{}^2 - \underline{\boldsymbol{q}}^{\boldsymbol{T}} \underline{\boldsymbol{q}}\right) \boldsymbol{I_{3x3}} - 2q_0 \left[\underline{\boldsymbol{q}}\times\right] \tag{3.14}$$

Thus, if $\boldsymbol{q}$ is the quaternion representing the attitude of the Body frame with respect to the World frame, the corresponding rotation matrix ${}^{\boldsymbol{B}}\boldsymbol{R_W}$ in terms of quaternion components is obtained by writing $A(\boldsymbol{q})$ in compact matrix form:

$${}^{\boldsymbol{B}}\boldsymbol{R_W} = \begin{pmatrix} q_0{}^2 + q_1{}^2 - q_2{}^2 - q_3{}^2 & 2(q_1q_2 + q_0q_3) & 2(q_1q_3 - q_0q_2) \\ 2(q_1q_2 - q_0q_3) & q_0{}^2 - q_1{}^2 + q_2{}^2 - q_3{}^2 & 2(q_2q_3 + q_0q_1) \\ 2(q_1q_3 + q_0q_2) & 2(q_2q_3 - q_0q_1) & q_0{}^2 - q_1{}^2 - q_2{}^2 + q_3{}^2 \end{pmatrix} \tag{3.15}$$

The inverse transformation from the Body to the World frame is simply the transposed matrix:

$${}^{\boldsymbol{W}}\boldsymbol{R_B} = \begin{pmatrix} q_0{}^2 + q_1{}^2 - q_2{}^2 - q_3{}^2 & 2(q_1q_2 - q_0q_3) & 2(q_1q_3 + q_0q_2) \\ 2(q_1q_2 + q_0q_3) & q_0{}^2 - q_1{}^2 + q_2{}^2 - q_3{}^2 & 2(q_2q_3 - q_0q_1) \\ 2(q_1q_3 - q_0q_2) & 2(q_2q_3 + q_0q_1) & q_0{}^2 - q_1{}^2 - q_2{}^2 + q_3{}^2 \end{pmatrix} \tag{3.16}$$

From these results, it is possible to link the Euler angles to the quaternion representation:

$$\begin{pmatrix} \phi \\ \theta \\ \psi \end{pmatrix} = \begin{pmatrix} \tan^{-1} \dfrac{2(q_2 q_3 + q_0 q_1)}{{q_0}^2 - {q_1}^2 - {q_2}^2 + {q_3}^2} \\ -\sin^{-1} 2(q_1 q_3 - q_0 q_2) \\ \tan^{-1} \dfrac{2(q_1 q_2 + q_0 q_3)}{{q_0}^2 + {q_1}^2 - {q_2}^2 - {q_3}^2} \end{pmatrix} \tag{3.17}$$

Or inversely:

$$\begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} \cos\frac{\phi}{2}\cos\frac{\theta}{2}\cos\frac{\psi}{2} + \sin\frac{\phi}{2}\sin\frac{\theta}{2}\sin\frac{\psi}{2} \\ \sin\frac{\phi}{2}\cos\frac{\theta}{2}\cos\frac{\psi}{2} - \cos\frac{\phi}{2}\sin\frac{\theta}{2}\sin\frac{\psi}{2} \\ \cos\frac{\phi}{2}\sin\frac{\theta}{2}\cos\frac{\psi}{2} + \sin\frac{\phi}{2}\cos\frac{\theta}{2}\sin\frac{\psi}{2} \\ \cos\frac{\phi}{2}\cos\frac{\theta}{2}\sin\frac{\psi}{2} - \sin\frac{\phi}{2}\sin\frac{\theta}{2}\cos\frac{\psi}{2} \end{pmatrix} \tag{3.18}$$

Finally, it can be shown that the time derivative of the quaternion with respect to the angular velocity in the Body frame is obtained as follows:

$$\dot{\boldsymbol{q}} = \frac{1}{2}\boldsymbol{q} * \begin{pmatrix} 0 \\ \boldsymbol{\omega_B} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & -p & -q & -r \\ p & 0 & r & -q \\ q & -r & 0 & p \\ r & q & -p & 0 \end{pmatrix}\boldsymbol{q} = \boldsymbol{Qq} \tag{3.19}$$

3.3 Input and state vector

Before developing the quadcopter motion equations, it is important to define the inputs/outputs of the system. The control is performed by varying the angular speed ω_i of the motors. V. Streit has shown in its semester project [7] that the angular speed of a motor can be considered proportional to its PWM value. The propeller rotation induced produces a force F_i and a moment M_i represented in figure 3.2.

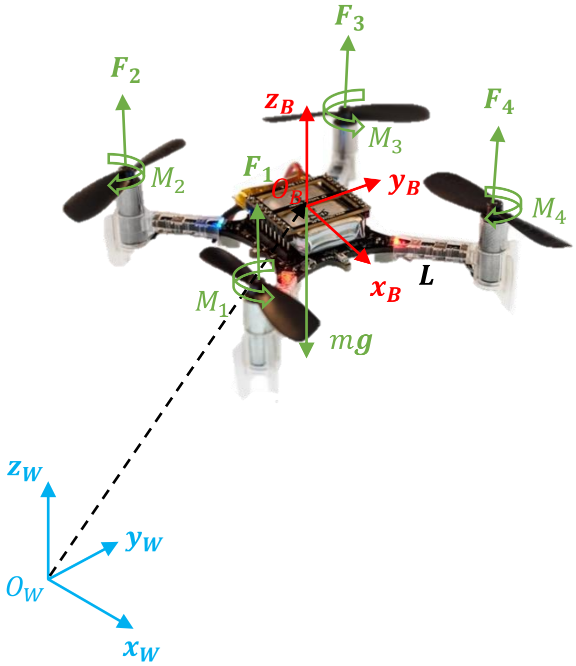

Figure 3.2: Forces and moments applied to the Crazyflie

Thanks to [7], these forces and moments can be simply modeled as follows:

$$F_i = C_F PWM_i{}^2 \qquad M_i = C_M PWM_i{}^2 \tag{3.20}$$

Where C_F and C_M are commonly named the propeller lift and drag coefficient respectively and supposed to be constant, while L is the length of the Crazyflie arm.

From this, an input vector $\boldsymbol{u}$ of four entries is typically defined in order to represent the following statements:

- The total thrust: $u_1 = \sum_{i=1}^{4} F_i$
- The rolling moment: $u_2 = L\cos\frac{\pi}{4}(F_2 + F_4 - F_1 - F_3)$
- The pitching moment: $u_3 = L\cos\frac{\pi}{4}(F_2 + F_3 - F_1 - F_4)$
- The yawing moment: $u_4 = M_1 - M_2 + M_3 - M_4$

In a more compact matrix form, these relations become:

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} C_F & C_F & C_F & C_F \\ -C_F L\cos\frac{\pi}{4} & -C_F L\cos\frac{\pi}{4} & C_F L\cos\frac{\pi}{4} & C_F L\cos\frac{\pi}{4} \\ -C_F L\cos\frac{\pi}{4} & C_F L\cos\frac{\pi}{4} & C_F L\cos\frac{\pi}{4} & -C_F L\cos\frac{\pi}{4} \\ -C_M & C_M & -C_M & C_M \end{pmatrix} \begin{pmatrix} PWM_1{}^2 \\ PWM_2{}^2 \\ PWM_3{}^2 \\ PWM_4{}^2 \end{pmatrix} \tag{3.21}$$

The cosines factor appearing in this input transformation matrix is due to the X formation of the quadcopter.

For the model output, in the case where an Euler-based representation is chosen for the attitude, a 12-dimensional state vector $\boldsymbol{X}$ is usually built to gather the main information about the quadcopter state:

$$\boldsymbol{X} = [x\ y\ z\ \dot{x}\ \dot{y}\ \dot{z}\ \phi\ \theta\ \psi\ p\ q\ r]^T \tag{3.22}$$

Where we can find:

- The position vector in the World frame $\boldsymbol{r} = [x\ y\ z]^T$.
- The speed vector in the World frame $\boldsymbol{v} = [\dot{x}\ \dot{y}\ \dot{z}]^T$.
- The Euler angles vector $\boldsymbol{\Theta} = [\phi\ \theta\ \psi]^T$.
- The angular velocity vector in the Body frame $\boldsymbol{\omega_B} = [p\ q\ r]^T$.

3.4 Dynamic state equations

In classical mechanics, the translational and rotational dynamics of any rigid body can be described using the well-known Newton-Euler equations. From the force description performed previously, the acceleration of the center of mass in the World Frame can be expressed such that:

$$\dot{\boldsymbol{v}} = \boldsymbol{g} + \frac{u_1}{m}\boldsymbol{z_B} \tag{3.23}$$

Which after development becomes:

$$\begin{cases} \ddot{x} = \dfrac{u_1}{m}(\cos\psi \sin\theta \cos\phi + \sin\psi \sin\phi) \\ \ddot{y} = \dfrac{u_1}{m}(\sin\psi \sin\theta \cos\phi - \cos\psi \sin\phi) \\ \ddot{z} = -g + \dfrac{u_1}{m}\cos\theta \cos\phi \end{cases} \tag{3.24}$$

Similarly, the angular acceleration of the drone in the Body reference frame can be written such that:

$$\dot{\boldsymbol{\omega}}_{\boldsymbol{B}} = \boldsymbol{I}^{-1}\left(\begin{pmatrix} u_2 \\ u_3 \\ u_4 \end{pmatrix} - \boldsymbol{\omega_B} \times \boldsymbol{I}\boldsymbol{\omega_B}\right) \tag{3.25}$$

Where $\boldsymbol{I}$ is the inertia matrix along the Body frame's axis: $\boldsymbol{I} = \begin{pmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{pmatrix}$.

Here, the use of a diagonal inertia matrix is a valid approximation because the Crazyflie platform is symmetric.

Developing equation 3.25 leads to:

$$\begin{cases} \dot{p} = \dfrac{u_2}{I_x} - qr\dfrac{(I_z - I_y)}{I_x} \\ \dot{q} = \dfrac{u_3}{I_y} - pr\dfrac{(I_x - I_z)}{I_y} \\ \dot{r} = \dfrac{u_4}{I_z} - pq\dfrac{(I_y - I_x)}{I_z} \end{cases} \tag{3.26}$$

Now from equations 3.23, 3.25 and 3.6 the state dynamic equations using the state vector expression (3.22) can be written as a continuous-time nonlinear model of the form: $\dot{\boldsymbol{X}} = \boldsymbol{f}(\boldsymbol{X}, \boldsymbol{u})$:

$$\begin{cases} \dot{\boldsymbol{r}} = \boldsymbol{v} \\ \dot{\boldsymbol{v}} = \boldsymbol{g} + \dfrac{u_1}{m}\boldsymbol{z_B} \\ \dot{\boldsymbol{\Theta}} = \boldsymbol{T}\boldsymbol{\omega_B} \\ \dot{\boldsymbol{\omega_B}} = \boldsymbol{I}^{-1}\left(\begin{pmatrix} u_2 \\ u_3 \\ u_4 \end{pmatrix} - \boldsymbol{\omega_B} \times \boldsymbol{I}\boldsymbol{\omega_B}\right) \end{cases} \tag{3.27}$$

Where $\boldsymbol{u}$ is obtained from equation 3.21.

In this model there are some parameters that can be measured accurately like the quadcopter mass m and arm Length L appearing in equations 3.23 and 3.6. Conversely, some parameters are quite difficult to estimate such as the diagonal inertia matrix I or the lift and drag coefficients C_F and C_M appearing in the input transformation matrix. These last parameters form the set of unknown parameters of the model that has to be identified. This subject is covered in the next chapter.