

Cauchy's Theorem

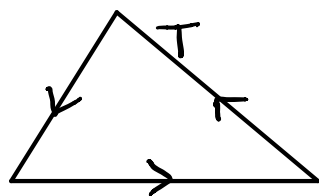
Suppose $f(z)$ is analytic in a simply connected region R ,
 γ is any closed curve in R .

Then

$$\int_{\gamma} f(z) dz = 0$$

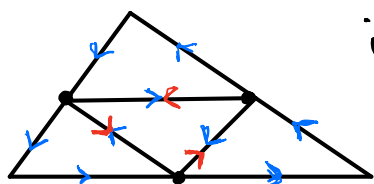
Proof: (Goursat)

Do it for a triangle first. Then generalize to any polygonal curve and then to a smooth curve.



Suppose T has perimeter length L ,
diameter D . D is the farthest distance
between two points. (i.e. $|z-w| \leq D$ $\forall z, w$ on T).

1) Bisect sides of triangle, making 4 new triangles.

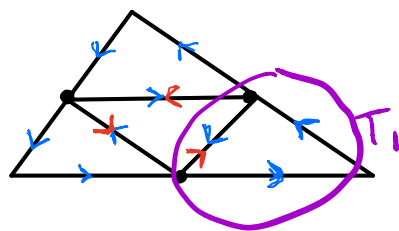


inner edges are traversed twice

contribution from inner edges
cancel out.

$$\int_T f(z) dz = \int_{\text{4 little triangles}} f(z) dz$$

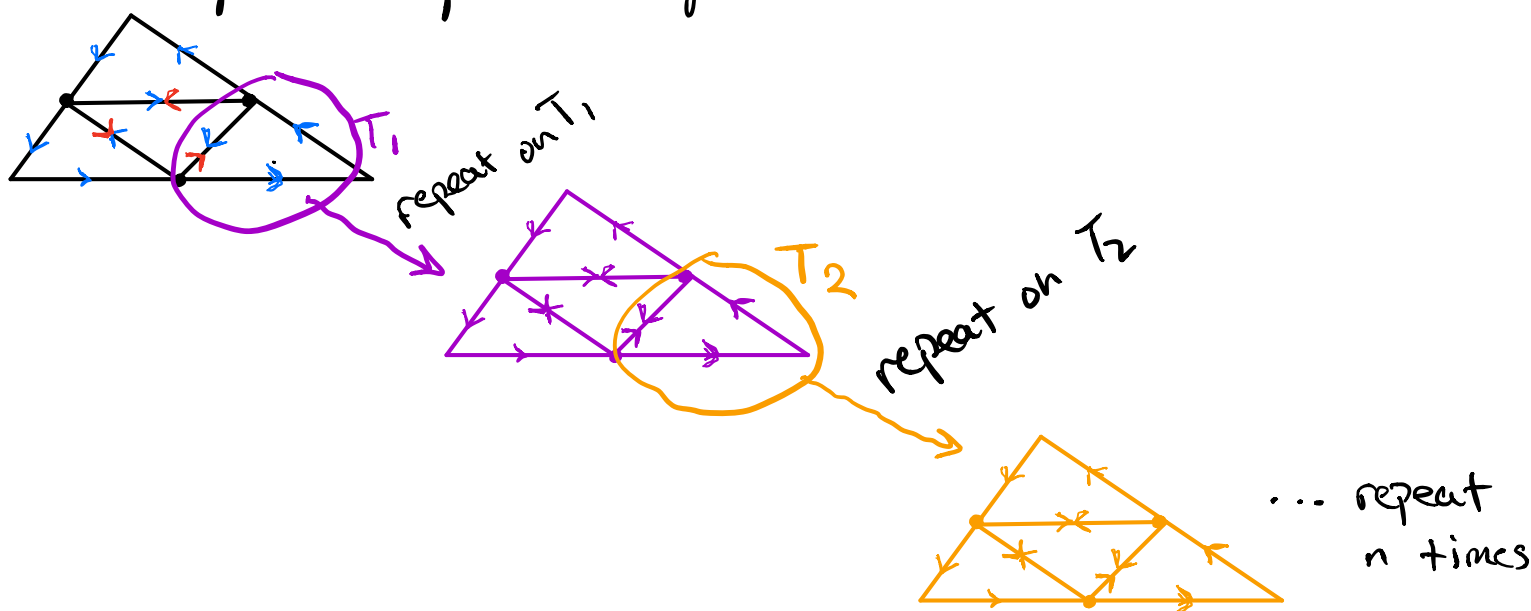
2) Choose little triangle with largest magnitude integral. Call it T_1 .



By triangle inequality,

$$\left| \int_T f(z) dz \right| \leq 4 \left| \int_{T_1} f(z) dz \right|$$

3) Play the same game again and again and again. At the n^{th} stage of doing this, let T_n be the triangle with the largest magnitude integral.



$$\left| \int_T f(z) dz \right| \leq 4^n \left| \int_{T_n} f(z) dz \right|$$

4) There exists a point, a , trapped in the infinite nested intersection of all the interiors of the T_n

$a \subset \dots \subset T_2 \subset T_1 \subset T$ \uparrow not obvious, requires proof
uses "compactness" of closed triangle T

5) Estimate $\int_{T_n} f(z) dz$. On triangle interior T_n , we

can write $f(z)$ as

$$f(z) = f(a) + \underbrace{f'(a)(z-a)}_{\text{linear}} + \underbrace{\varepsilon(z)(z-a)}_{\text{small correction}}$$

Here $\varepsilon(z)$ is a function defined by

$$\varepsilon(z) = \frac{f(z) - f(a)}{z - a} - f'(a) \quad \left\{ \begin{array}{l} \text{equivalent to} \\ \text{saying } f \text{ is} \\ \text{analytic at } a! \end{array} \right.$$

ie

$\lim_{z \rightarrow a} \varepsilon(z) = 0$ because f is analytic throughout R and

here at a .

So,

$$\int_{T_n} f(z) dz = \int_{T_n} f(a) dz + \int_{T_n} f'(z)(z-a) dz + \int_{T_n} \varepsilon(z)(z-a) dz$$

antiderivatives exist
integrated on closed loop

So,

$$\left| \int_{T_n} f(z) dz \right| = \left| \int_{T_n} \varepsilon(z)(z-a) dz \right|$$

$$\text{Recall } \left| \int_T f(z) dz \right| \leq 4^n \left| \int_{T_n} f(z) dz \right| = 4^n \left| \int_{T_n} \varepsilon(z)(z-a) dz \right|$$

6) Use ML estimate.

$$\left| \int_{T_n} \varepsilon(z)(z-a) dz \right| \leq \max_{z \in T_n} |\varepsilon(z)| \max_{z \in T_n} |z-a| \text{ length}(T_n)$$

length $T_n = \frac{L}{2^n}$ } each bisect yields a similar triangle w/ perimeter $\frac{L}{2}$. n reductions

$\max |z-a| = \frac{D}{2^n}$ } hypotenuse halves in length w/ each reduction, $(\frac{D}{2})$, n times.

↑ realized by T_n being constructed by repeated bisection of T .

7) Summarize.

$$\left| \int_T f(z) dz \right| \leq 4^n \left| \int_{T_n} f(z) dz \right| = 4^n \left| \int_{T_n} \varepsilon(z)(z-a) dz \right|$$

$$= 4^n \frac{DL}{2^n 2^n} \max_{z \in T_n} |\varepsilon(z)| = DL \max_{z \in T_n} |\varepsilon(z)|$$

Know that

$$\varepsilon(z) \rightarrow 0 \text{ as } z \rightarrow a \quad \left. \vphantom{\varepsilon(z)} \right\} \text{ as } n \rightarrow \infty, T_n \rightarrow a$$

As $n \rightarrow \infty$, $\max_{z \in T_n} |\varepsilon(z)|$ goes to zero by analyticity of

f .

So,

$$\left| \int_T f(z) dz \right| \leq 4^n \left| \int_{T_n} \varepsilon(z)(z-a) dz \right| \leq 0$$

↑ LHS is independent of n , so LHS estimate is exactly 0!