

Useful definitions and facts:

- The cumulative distribution function (cdf) of a continuous random variable X with probability density function (pdf) $f_X(x)$ is

$$F_X(x) = \int_{-\infty}^x f_X(t) dt .$$

- In the same context, the expected value of X is

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x f_X(x) dx .$$

- In the same context, the variance of X is

$$\text{Var}(X) = \mathbb{E}((X - \mathbb{E}(X))^2) .$$

1. This one is a variation on Problem 1 of Homework VII. Frodo tosses a coin repeatedly and independently. The probability that the coin comes up heads on any toss is p . Every time Frodo tosses a tail followed by a head, Sam pays him \$1. Let X be the total amount of money Frodo has received after 8 tosses. Find $\mathbb{E}(X)$ and $\text{Var}(X)$.

2. A continuous random variable X has pdf

$$f_X(x) = \begin{cases} \frac{1}{\alpha} e^{-(x-\beta)/\alpha} & \text{when } x \geq \beta \\ 0 & \text{otherwise,} \end{cases}$$

where $\alpha > 0$ and β are given real numbers.

- Find the cdf $F_X(x)$ of X .
- Find $\mathbb{E}(X)$ and $\text{Var}(X)$.
- Find the real number m that satisfies $F_X(m) = 1/2$. This number m is known as the median of the random variable X .

3. A random variable X has cdf

$$F_X(x) = \begin{cases} 1 - e^{-(x/\lambda)^\beta} & \text{when } x \geq 0 \\ 0 & \text{otherwise,} \end{cases}$$

where β and λ are given positive real numbers.

- Find the pdf $f_X(x)$ of X .
- Plot the cdf and pdf of X for $\lambda = 1$ and $\beta = 0.5, 1.5$, and 5 .

4. Aragorn and Boromir are bidding on a construction project. The low bidder will win the contract and then pay Gimli's stonemason crew \$100,000 to do the work. Aragorn believes that Boromir's bid is a random variable X distributed uniformly on the interval $[70,000, 140,000]$ dollars. What should Aragorn bid to maximize his expected profit?

5. Let X have exponential pdf

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{when } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

with $\lambda = 3$. Let $Y = \lfloor X \rfloor$. Find $\mathbb{E}(Y)$ and $\text{Var}(Y)$.

6. This is a continuous version of Problem 6 from Homework V. Let X be a nonnegative continuous random variable with pdf $f_X(x)$ and cdf $F_X(x)$. Assuming $\mathbb{E}(X)$ exists, show that

$$\mathbb{E}(X) = \int_0^\infty (1 - F_X(x)) dx .$$

Note that $1 - F_X(x) = \mathbb{P}(\{X > x\})$.

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EEG 3100

HW8 3/29/19

(late submission granted)

① $P(\text{win is heads}) = p$

8 tosses.

If a head is tossed immediately following a tail get \$1.

So Max amount paid is \$4.

$X = \text{amount received after 8 tosses} \in \{0, 1, 2, 3, 4\}$

$$E(X) = \cancel{0[(1-p)^8 + p^8]}^0 + 1 \left[\binom{8}{1} (1-p)^7 p \right] + 2 \left[\binom{8}{2} (1-p)^6 p^2 \right] + 3 \left[\binom{8}{3} (1-p)^5 p^3 \right] + 4 \left[\binom{8}{4} (1-p)^4 p^4 \right]$$

$$E(X^2) = \cancel{0^2[(1-p)^8 + p^8]}^0 + 1^2 \left[\binom{8}{1} (1-p)^7 p \right] + 2^2 \left[\binom{8}{2} (1-p)^6 p^2 \right] + 3^2 \left[\binom{8}{3} (1-p)^5 p^3 \right] + 4^2 \left[\binom{8}{4} (1-p)^4 p^4 \right]$$

$$\text{Var}(X) = 2 \left[\binom{8}{2} (1-p)^6 p^2 \right] + 6 \left[\binom{8}{3} (1-p)^5 p^3 \right] + 12 \left[\binom{8}{4} (1-p)^4 p^4 \right]$$

... It makes sense in my head

Looking back don't need $\binom{n}{k}$ I was trying to make order matter

②

$$f_X(x) = \begin{cases} \frac{1}{\alpha} e^{-(x-\beta)/\alpha} & , x \geq \beta \\ 0 & , \text{otherwise} \end{cases}$$

$$\alpha > 0, \beta \in \mathbb{R}$$

$$(a) F_X(x) = \begin{cases} \int_{\beta}^x \frac{1}{\alpha} e^{-(t-\beta)/\alpha} dt = e^{-(t-\beta)/\alpha} \Big|_{\beta}^x = 1 - e^{-(x-\beta)/\alpha} & , x \geq \beta \\ 0 & , x < \beta \end{cases}$$

$$(b) E(X) = \int_{\beta}^{\infty} x \cdot \frac{1}{\alpha} e^{-(x-\beta)/\alpha} dx$$

$$= \frac{e^{\beta/\alpha}}{\alpha} \int_{\beta}^{\infty} x e^{-x/\alpha} dx$$

$$= \frac{e^{\beta/\alpha}}{\alpha} \left[-\alpha e^{-x/\alpha} - \alpha^2 e^{-x/\alpha} \Big|_{\beta}^{\infty} \right] = \beta + \alpha$$

$$\begin{array}{rcl} & & du \\ & & e^{-x/\alpha} \\ x & \searrow & \\ -1 & \searrow & -\alpha e^{-x/\alpha} \\ 0 & \searrow & \alpha^2 e^{-x/\alpha} \end{array}$$

$$\text{Var}(X) = E(X^2) - (E(X))^2$$

$$E(X^2) = \int_{\beta}^{\infty} \frac{x^2}{\alpha} e^{-(x-\beta)/\alpha} dx$$

$$= \frac{e^{\beta/\alpha}}{\alpha} \left[-x^2 e^{-x/\alpha} - 2x \alpha e^{-x/\alpha} - 2\alpha^2 e^{-x/\alpha} \right] \Big|_{\beta}^{\infty}$$

$$\begin{array}{lcl} u & dv & \\ x^2 & \searrow & e^{-x/\alpha} \\ -2x & \searrow & -\alpha e^{-x/\alpha} \\ 2 & \searrow & \alpha^2 e^{-x/\alpha} \\ 0 & \searrow & -\alpha^3 e^{-x/\alpha} \end{array}$$

Thus

$$\text{Var}(X) = \beta^2 + 2\alpha\beta + 2\alpha^2 - (\alpha + \beta)^2 = \alpha^2$$

(c) Need

$$F_X(m) = 1/2$$

$$1 - e^{-\frac{(m-\beta)}{\alpha}} = \frac{1}{2}$$

$$e^{-\frac{(m-\beta)}{\alpha}} = \frac{1}{2}$$

$$-\frac{(m-\beta)}{\alpha} = \ln\left(\frac{1}{2}\right)$$

$$\frac{m-\beta}{\alpha} = \ln(2)$$

$$m = \alpha \ln(2) + \beta$$

③

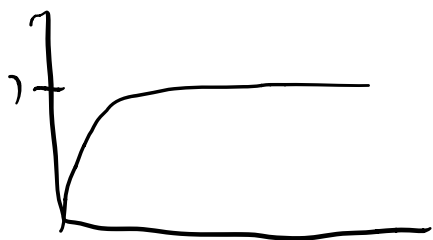
$$F_X(x) = \begin{cases} 1 - e^{-(\frac{x}{\lambda})^\beta} & , x \geq 0 \\ 0 & , \text{otherwise} \end{cases}$$

$$\beta, \lambda \in \mathbb{R}$$

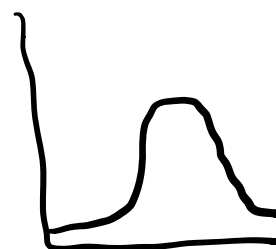
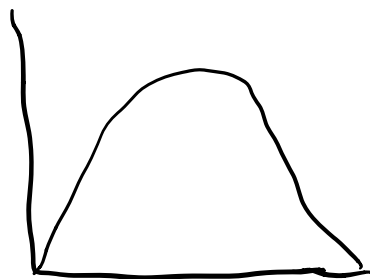
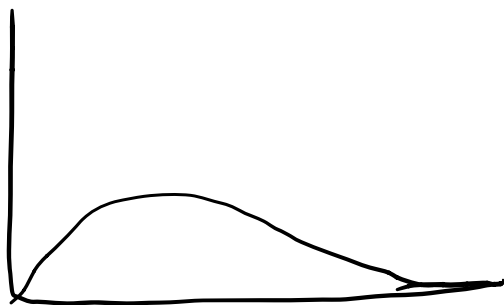
$$(a) \quad f_X(x) = \frac{d}{dx} F_X(x) = \begin{cases} \frac{\beta x^{\beta-1}}{\lambda^\beta} e^{-(\frac{x}{\lambda})^\beta} & , x \geq 0 \\ 0 & , \text{otherwise} \end{cases}$$

$$(b) \quad \lambda = 1$$

$$\text{CDF} \quad \beta = \frac{1}{2} \quad \beta = \frac{3}{2} \quad \beta = 5$$



PDF



- ④ If Boromir bid 70,000, then Aragorn cannot win.
Similarly, if Boromir bid over 140,000 then Aragorn always wins.

$x \in [70,000, 140,000]$ ← Boromir's bid

$$R(\text{Aragorn wins}) = \begin{cases} \frac{x - 70,000}{70,000} & 70,000 \leq x \leq 140,000 \\ 0, & \text{otherwise} \end{cases}$$

Maximize profit bid?

middle of interval
↓

$$E[\text{expected profit}] = \frac{x - 70,000}{70,000} (105,000 - x)$$

Max @ $x = 87,500$ (Wolfram)

$$\text{profit} = 105,000 - 87,500 = \boxed{17,500}$$

⑤

$$p_x(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\lambda = 3.$$

$$Y = \lfloor X \rfloor. \quad E(Y)? \quad \text{Var}(Y)?$$

$= g(x)$

$$E(Y) = \int_{-\infty}^{+\infty} g(x) p_x(x) dx = \int_0^{\infty} \lfloor x \rfloor \lambda e^{-\lambda x}$$

Online



.052

Calculator

$$(Y) = .044$$

⑥ X a nonnegative continuous rv w/ pdf $f_X(x)$;
cdf $F_X(x)$.

Assuming $IE(X)$ exists, show

$$IE(X) = \int_0^{\infty} (1 - F_X(x)) dx$$

$$1 - F_X(x) = P(\{X > x\}) = \int_x^{\infty} f_X(t) dt$$

Taking integral from $0 \rightarrow \infty$ with respect to x we get

$$\int_0^{\infty} \left(\int_x^{\infty} f_X(t) dt \right) dx$$

Using the indicator function,

$$= \int_{x=0}^{x=\infty} \left[\int_{t=0}^{t=\infty} f_X(t) 1_{\{t \geq x\}} dt \right] dx$$

$$= \int_{t=0}^{t=\infty} f_X(t) \left[\int_{x=0}^{\infty} 1_{\{t \geq x\}} dx \right] dt$$

$$= \int_{t=0}^{t=\infty} f_X(t) \left[\int_{x=0}^t dx \right] dt$$

$$= \int_{t=0}^{\infty} t f_X(t) dt = \int_{t=-\infty}^{+\infty} t f_X(t) dt$$