

Recall

Independence - Σ , IP

A, B - events - independent when $P(A \cap B) = P(A)P(B)$

Same as

$$P(A|B) = P(A) \text{ when } P(B) \neq 0$$

$$P(B|A) = P(B) \text{ when } P(A) \neq 0$$

Looked at examples - one was 2 coin flips, all 4 outcomes equally probable

$A \rightarrow$ first head

$B \rightarrow$ second head

$C \rightarrow$ two flips different

Any pair of these is independent

Generalization: say events A_1, A_2, \dots, A_n are independent when

$P(\text{conjunction of any nonempty subsets of these events})$
||

product of $P(\text{events in the conjunction})$

i.e. A_1, A_2, A_3 are indep. when

$$P(A_1 \cap A_2) = P(A_1)P(A_2) \text{ and } P(A_1 \cap A_3) = P(A_1)P(A_3) \text{ and}$$

$$P(A_2 \cap A_3) = P(A_2)P(A_3) \text{ and } P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$$

CAUTION: Independence doesn't follow from either

- para
- $P(A_1 \cap A_2 \dots \cap A_n) = P(A_1) \dots P(A_n)$

Conditional Independence: Ω, \mathcal{P} ; say events A and B are conditionally dependent given (event) C when

$$P(A \cap B | C) \neq P(A | C) P(B | C)$$

$$P(A | B \cap C) = P(A | C) ; \quad P(B | A \cap C) = P(B | C)$$

\uparrow

knowledge of B gives no function info about probability of A on top of knowledge of C

To see this just play w/ formulas

$$P(A | B \cap C) = \frac{P(A \cap B \cap C)}{P(B \cap C)} = \frac{P(A \cap B | C) P(B | C)}{P(B | C)} = P(A | C)$$

Suppose A and B are independent; are they conditionally dependent given some random variable C ?

NO! - eg - flip one fair coin twice - all outcomes equally likely.

A - first flip head
B - second flip head
C - two flips different } independent

A, B \rightarrow independent

A, C \rightarrow independent

B, C \rightarrow independent

Can't have both
 $P(A \cap B | C) = 0$ \leftarrow be heads if they're different!

If they were

$$P(A \cap B | C) = P(A | C)P(B | C) = \frac{1}{4}$$

So we conclude, A + B are independent, but NOT Conditionally dependent given C.

Similarly, even when A and B are conditionally independent given C, A and B need not be independent.

Example - Coins

Have two unfair coins

$$\text{Coin 1: } P(H) = \frac{2}{3}$$

$$\text{Coin 2: } P(H) = \frac{1}{3}$$

Pick one coin and flip it twice.

Let

$C = \text{event "pick coin 1"}$

Assume

$$P(C) = 1/2$$

Let

$A \rightarrow \text{head on first flip}$

$B \rightarrow \text{head on second flip}$

Claim: A, B are conditionally dependent given C .

$$P(A \cap B | C) = P(A | C) P(B | C)$$

If you know what coin you're flipping

$$\Rightarrow P(2 \text{ heads} | C), \text{ then } \frac{2}{3} \cdot \frac{2}{3} \text{ or } \frac{1}{3} \cdot \frac{1}{3}, \text{ etc}$$

Aside: why is $P(A \cap B | C) = \frac{2}{3} \cdot \frac{2}{3}$? Because of independence of flips knowing you're using coin 1

$$\Rightarrow P(\text{Head second} \cap \text{Head first})$$

$$= \underbrace{P(H \text{ on 2nd} | H \text{ on 1st})}_{2/3} \underbrace{P(H \text{ on first})}_{2/3}$$

$$2/3$$

$$2/3$$

However, can easily show that A and B not independent.
Do this by computing $P(A)$, $P(B)$ and $P(A \cap B)$ using
Total Probability Theorem; find $P(A \cap B) \neq P(A)P(B)$.

Eg

$$\begin{aligned}P(A) &= P(A|C)P(C) + P(A|C^c)P(C^c) \\&= \frac{2}{3} \cdot \frac{1}{2} + \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{2}, \text{ etc}\end{aligned}$$

Last "foundational" topic: Counting / Combinatorics

Useful especially in contexts where "all outcomes in Ω are equally probable"

Example - Counting (Elementary)

Given $\Omega = \{s_1, s_2, \dots, s_n\}$, how many events are there? I.e how many subsets of Ω are there?

Claim: Set of subsets of Ω is in 1-to-1 correspondence / set of all binary n -strings

Idea: Given $A \subset \Omega$, each s_i claim is or isn't in A . Define in binary string.

$\text{Str}(A) = \text{String } w \left\{ \begin{array}{ll} \text{Zero in position } k \text{ when } s_k \notin A \\ 1 \text{ in position } k \text{ when } s_k \in A \end{array} \right.$

Similarly, every binary n string "defines" a subset of Σ ,
and "everybody knows" there are 2^n binary n -strings.
Thus,

$$\#\text{(Subsets of } \Sigma) = 2^n$$

Generally, interested in counting up objects that are new
as arising from a sequential buildup _____ of the
following type:

Stage 1: Make a choice from among n_1 options

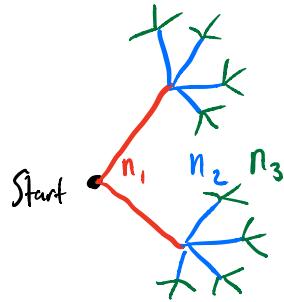
Stage 2: Make a choice from among n_2 options
no matter what choice you made at stage 1

:

Stage r : Make a choice from among n_r options
no matter what choice you made at stage $(r-1)$

Example - Tree Building

Say you have $r=3$. $n_1=2$, $n_2=4$, $n_3=3$



each object you build by this process corresponds w/ a terminal node on the tree. # terminal nodes = $n_1 n_2 \cdots n_r$

Counting Principle:

Re-count # of subsets of $\Omega = \{s_1, \dots, s_n\}$.

We can view each subset as arising from a multi stage building process.

Stage 1: Choose whether to put s_1 in the subset
- either yes or no ($n_1=2$)

Stage 2: Choose whether to put s_2 in the subset
- either yes or no ($n_2=2$)

Stage n : Choose whether to put s_n in the subset
- either yes or no ($n_n=2$)

$$\#(\text{Subsets}) = n_1 n_2 \cdots n_n = 2^n$$