Recall

- Bayes Rule/Independence/Conditional Independence
- Started discussing counting

Counting Principle: In a process with a finite sequence of stages $1, 2, 3, \dots, r$ with n_1 choices at stage 1; an for every such choice at stage 1, n_2 choices at stage 2; $\cdots$; and for every sequence of choices on stages 1 through $r-1$, n_r choices at stage r, the number of "outcomes" of this process is $n_1 n_2 \cdots n_{r-1} n_r$

Derivable by induction

Can use counting principle to rederive

$$\#(\text{subsets of } \Omega) = 2^{\#(\text{elements in } \Omega)} \quad \text{for finite } \Omega$$

Derivation not shown in class

Next Counting Item: k-permutations of n objects

Here, $n > 0$, $n \in \mathbb{Z}$, $k \in \mathbb{Z}$, $0 \leq k \leq n$

A k-permutation of the n-objects is an ordered string of distinct objects (from among the n total) of length k.

Given k, can view building a k-permutation as a multi-stage process:

Stage 1: pick first object in string (n choices)

Stage 2: pick second object in string ($n-1$ choices)

no matter outcome of stage 1

⋮ Keep this up

k-stage process is

$$n(n-1)(n-2)\cdots(n-k+1)$$

Special Case: #(n-permutations of n objects) $= n!$

k-combinations of n-objects: "combinations of n objects taken k at a time"

Question: given a set of k objects from the original set, how many k-permutations of the original n objects features the k objects in this subset?

Answer: The number of k-permutations of the k objects in the set
$= k!$

Conclude: The number of k-combinations of n objects is

$$\frac{n(n-1)\cdots(n-k+1)}{k!}$$

We have just defined "n choose k"!

$$\binom{n}{k} := \frac{n(n-1)\cdots(n-k+1)}{k!}$$

Comment: Amazing that this is an integer!
Can also write

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

This comes up in situations involving <u>independent trials</u>

Idea: Perform a random experiment <u>repeatedly</u> + <u>independently</u>

If experiment has 2 outcomes, call them **Bernoulli Trials**. Generally visualize coin flips in this case - H, T possible outcomes at each stage - if $\mathbb{P}(\{H\}) = p$, $\mathbb{P}(\{T\}) = 1-p$ and you perform experiment n times, for each k, $0 \le k \le n$,

$$\mathbb{P}(\text{sequence you get}) = p^k (1-p)^{n-k}, \text{ where } k = \#(H) \text{ you flip}$$

This is true **REGARDLESS** of the order each H, T appears in the sequence of flips.

$$\mathbb{P}(\text{you get } k \text{ H's}) = \#(\text{ways of picking } k \text{ spots in } n \text{ available spots})$$

$$= \binom{n}{k} p^k (1-p)^{n-k} \quad 0 \le k \le n$$

Note: The events

$$A_k = \text{get } k \text{ H's} \quad 0 \le k \le n$$

are disjoint and partition the whole set of possible outcome sequences.

Thus,

$$\sum_{k=0}^{n} \mathbb{P}(A_k) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k}$$

Example – Team Formation, Again

12 agents: Frodo, Sam, Gandalf, Legolas and 8 miscellaneous hobbits to be assigned randomly to four teams of three via:

- hat containing twelve slips – three 1's, three 2's, etc.
- draw a slip "at random", give to Frodo, then S, G, L
- then do it for rest of the hobbits.

Find: $\mathbb{P}$(each team has one of the "leaders")

= $\mathbb{P}$(all four "leaders" are assigned to different teams)

Can do this using combinatorics because all allocations of characters to teams ("team formations) are equally probable

Think of each distribution of slips of paper as an outcome.

How many ways to hand out slips? $12!$ ← By counting principle

How many slip distributions put main characters on different teams?

$$12 \cdot 9 \cdot 6 \cdot 3 \cdot 8!$$

remaining hobbit draws

draw Frodo's slip

remaining ways to draw Sam's slip

remaining ways to draw Gandalf's slip

remaining ways to draw Legalos' slip

P(all four "leaders" are assigned to different teams)

$$= \frac{12 \cdot 9 \cdot 6 \cdot 3 \cdot 8!}{12!}$$

$$= \left. \frac{9}{11} \cdot \frac{6}{10} \cdot \frac{3}{9} \right\}$$ SAME AS LAST TIME!

Example - Counting Up

Given n objects, have r labeled bins; given $n_1, n_2, \ldots, n_r$ whose sum is $n_1 + n_2 + \cdots + n_r = n$, how many ways to distribute objects among bins so n_1 in Bin 1, n_2 in Bin 2, ..., n_r in Bin r?

Calculate this via counting principle

First fill bin 1: $\frac{n(n-1)(n-2)\cdots(n-r+1)}{r_1!} = \binom{n}{r_1}$

Having filled Bin 1, $n - r_1$ objects remain; $\binom{n-r_1}{r_2}$ ways of picking r_2 objects in Bin 2.

Keep this up, you find

Total # of all allocations $= \binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-\cdots-n_{r-1}}{n_r}$

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-\cdots-n_{r-1}}{n_r} = \frac{n!}{n_1!\,n_2!\cdots n_r!}$$

$$\frac{n!}{n_1!\,n_2!\cdots n_r!} = \binom{n}{n_1\; n_2 \cdots n_r}$$

Example - Re-Revisit Team Formation Problem

12 agents: Frodo, Sam, Gandalf, Legolas and 8 miscellaneous hobbits to be assigned randomly to four teams of three via:

- hat containing twelve slips - three 1's, three 2's, etc.
- draw a slip "at random", give to Frodo, then S, G, L
- then do it for rest of the hobbits.

Total # team setups $= \dfrac{12!}{3!\,3!\,3!\,3!}$

Total # of admissable team setups?

$$4 \cdot 3 \cdot 2 \cdot 1 \cdot \binom{8}{2}$$

remaining ways to split hobbits among 4 teams remaining slots

assign Frodo's slip

remaining ways to assign Sam's slip

remaining ways to assign Gandalfs slip

remaining ways to assign Legalos' slip

$$\frac{4\cdot 3\cdot 2\cdot 1\cdot \dfrac{8!}{2!\,2!\,2!\,2!}}{\dfrac{12!}{3!\,3!\,3!\,3!}} = \frac{9}{11}\cdot\frac{6}{10}\cdot\frac{3}{9}$$