Math 4310

Homework 2

Due 9/18/19

Name: ____________________

Collaborators: ____________________

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Please print out these pages. I encourage you to work with your classmates on this homework. Please list your collaborators on this cover sheet. (Your grade will not be affected.) Even if you work in a group, you should write up your solutions yourself! You should include all computational details, and proofs should be carefully written with full details. As always, please write neatly and legibly.

Please follow the instructions for the "extended glossary" on separate paper (LaTeX it if you can!) Hand in your final draft, including full explanations and write your glossary in complete, mathematically and grammatically correct sentences. Your answers will be assessed for style and accuracy.

Please **staple** *this cover sheet, your exercise solutions, and your glossary together, in that order, and hand in your homework in class.*

GRADES

Exercises ____________________ / 50

Extended Glossary

Component	Correct?	Well-written?
Definition	/6	/6
Example	/4	/4
Non-example	/4	/4
Theorem	/5	/5
Proof	/6	/6
Total	/25	/25

Some of the exercises below use the following (important) notion:

Definition. (Internal) direct sum. Suppose that U_1 and U_2 are subspaces of a vector space V, such that
(1) $U_1 + U_2 = V$, and
(2) $U_1 \cap U_2 = \{0\}$.
In this case, we say that V is the **direct sum** of U_1 and U_2, and we write: $V = U_1 \oplus U_2$.
Given a vector subspace $U \subset V$, a **complementary subspace** of U is a vector subspace $W \subset V$, satisfying

$$V = U \oplus W.$$

Exercises.

1. Let V be a vector space over $\mathbb{F}$. Prove from the axioms/properties:
   (a) (Cancellation) For all $u, v, w \in V$, if $u + v = u + w$, then $v = w$.
   (b) For all $a \in \mathbb{F}$, and $u, v \in V$, if $au = av$, then either $a = 0_{\mathbb{F}}$ or $u = v$.
   (c) For all $a, b \in \mathbb{F}$, and $u \in V$, if $au = bu$, then either $a = b$ or $u = 0_V$.
   (d) $0_{\mathbb{F}} \cdot v = 0_V$.
   (e) If $c \in \mathbb{F}$, then $c \cdot 0_V = 0_V$.
   (f) $(-1)v = -v$, for all $v \in V$.

2. Let
$$U = \left\{ \begin{pmatrix} x \\ y \\ x \\ y \end{pmatrix} \in \mathbb{F}^4 \mid x, y \in F \right\}.$$
   (a) Show that $U \subset \mathbb{F}^4$ is a subspace.
   (b) Find a list of vectors of U which spans U, and which is linearly independent (i.e. a basis of U).
   (c) Find another subspace $W \subset \mathbb{F}^4$ such that $\mathbb{F}^4 = U \oplus W$.

3. Let $V = \mathbb{F}^{2\times 2}$ be the vector space of 2 by 2 matrices, with entries in $\mathbb{F}$.

   Determine if the following subsets are subspaces (justify your answer either way). For those that are subspaces, find a complement W: i.e. a subspace $W \subset V$, such that $V = U \oplus W$.
   (a) $U = \{A \in V \mid A^2 = A\}$.
   (b) $U = \{A \in V \mid AB = BA\}$, where $B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.
   (c) $U = \{A \in V \mid A \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\}$.

4. Find values $a, b \in \mathbb{Q}$ so that $\begin{pmatrix} 2 \\ a-b \\ 1 \end{pmatrix}$ and $\begin{pmatrix} a \\ b \\ 3 \end{pmatrix}$ are linearly dependent in $\mathbb{Q}^3$.

5. Determine which of the following lists of vectors in $\mathrm{Fun}(\mathbb{R}, \mathbb{R})$ are linearly independent, and which are linearly dependent.
   (a) $(\sin^2 x, \cos^2 x)$.
   (b) $(1, \sin^2 x, \cos^2 x)$.
   (c) (e^x, e^{2x}).

6. In this problem, assume that U_1, U_2 (and U_3 in the last two parts) are subspaces of V.
   (a) Is $U_1 \cap U_2$ a subspace? Either prove it, or give a counter-example.
   (b) Is $U_1 \cup U_2$ a subspace if neither contains the other? Either prove it, or give a counter-example.
   (c) (Optional) If $\mathbb{F} = \mathbb{R}$, show that $U_1 \cup U_2 \cup U_3$ is not a subspace of V, unless one of the subspaces contains the other two.
   (d) Suppose that $\mathbb{F} = \mathbb{F}_2$ is the field with two elements. Find an example of a vector space V, and subspaces U_1, U_2, U_3 with no one containing any other, such that $U_1 \cup U_2 \cup U_3$ is a subspace. (i.e. surprising things can happen sometimes with finite fields!)

Extended Glossary. There is no extended glossary this week.

You may work in groups, but please write up your solutions **yourself**. If you do work together, your group should come up with at least two examples, two non-examples, and two theorems. Each one (example/non-example/theorem) should be included in some group member's extended glossary. Your solutions should be written formally, so that we could cut and paste them into a textbook.