Last Time

LI Lemma: Suppose $(v_1, \dots, v_m)$ is LD in V.

Then $\exists\ j \in 1, \dots, m$ s.t

ⓐ $v_j \in \operatorname{span}(v_1, \dots, v_{j-1})$

ⓑ if j^{th} element is removed from $v_1, \dots, v_m$

then the span of the remaining $= \operatorname{span}(v_1, \dots, v_m)$

Main Goal: If V has two bases, $(v_1, \dots, v_m)$, $(w_1, \dots, w_m)$ then $m = n$.

Want to figure this out.

Key Theorem: Suppose V is finite dimensional and

ⓐ $(u_1, \dots, u_m)$ is LI $(u_i \in V)$

ⓑ $(v_1, \dots, v_n)$ spans V $(v_i \in V)$

then $m \leq n$

IDEA

Let $B = (v_1, \dots, v_n)$ (spans V)

① add in $u_1 \notin B$, remove one element of B

to get $B = (u_1, \underbrace{v_2', \dots, v_n'}_{\text{elements of } B})$ still spans V

② add in $u_2 \notin B$, remove one element of B

to get $B = (u_1, u_2, \underbrace{v_3'', \dots, v_n'}_{\text{elements of } B})$ still spans V

⋮ keep going

$B = (u_1, u_2, \dots, u_m, v_{m+1}, \dots, v_n)$ still spans V

PROOF

Step 1: At beginning of step 1, $B = (v_1, \dots, v_n)$ spans V.

$\therefore$ $(u_1, v_1, v_2, \dots, v_n)$ is LD $\rightarrow$ $u_1 = a_1 v_1 + \dots + a_n v_n$

$-u_1 + a_1 v_1 + \dots + a_n v_n = 0$

Thus, LI Lemma $\Rightarrow \exists v_j \in (u_1, v_1, \dots, v_n)$ such that $v_j \in \text{span}(u_1, \dots, v_{j-1})$ and removing it keeps span$(u_1, v_1, \dots, \hat{v_j}, \dots, v_n)$.

So: span(u_1, all v_i's except v_j) = span $B = V$

Step i: At beginning $B = (\underbrace{u_1, \dots, u_{i-1}}_{\text{Note: these are LI}}, \underbrace{v_i', \dots, v_n'}_{\text{Some subset of } (v_1, \dots, v_n)})$

Consider u_i: if $i > m$, then we are done. $m \leq n$

Otherwise $\exists u_i \in (u_1, \dots, u_m)$.

Consider $(u_1, \dots, u_{i-1}, u_i, v_i', \dots, v_n')$ spans V and is LD.

$\therefore \exists j$ s.t span[($u_1, \dots, u_{i-1}, u_i$, all v_i except v_j)] $= V$

At end of step i

$B = (u_1, \dots, u_i, \underbrace{v_{i+1}', \dots, v_n'}_{\text{Some subset of } (v_1, \dots, v_n)})$ Spans V

What happens if at the beginning of step i: $B = (u_1, \dots, u_{i-1})$ spans V $i \leq m$

- This CANNOT happen since then $u_i \in \text{span}(u_1, \dots, u_{i-1})$, since $(u_1, \dots, u_n)$ is LI. $\Rightarrow m \leq n$

Example

Is $\begin{pmatrix}1\\1\\-1\end{pmatrix}, \begin{pmatrix}2\\1\\3\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}42\\e^{\pi^2}+3\\17.1-e\end{pmatrix}$ LI $\in \mathbb{R}^3$?

No!

$\mathbb{R}^3$ is spanned by $\begin{pmatrix}1\\0\\0\end{pmatrix} \begin{pmatrix}0\\1\\0\end{pmatrix} \begin{pmatrix}0\\0\\1\end{pmatrix}$ $\therefore 4 \not\leq 3$ c!

Corollary: If V has two bases, $(u_1, \ldots, u_m)$, $(v_1, \ldots, v_n)$ then $m = n$

Proof

$(u_1, \ldots, u_m)$ is LI ; $(v_1, \ldots, v_n)$ spans $V \Rightarrow m \leq n$ $\Rightarrow m = n$

$(v_1, \ldots, v_n)$ is LI ; $(u_1, \ldots, u_m)$ spans $V \Rightarrow n \leq m$

Q: If V is $\overbrace{\text{finite dimensional}}^{\exists \text{ a spanning set}}$, does V have a basis.

A: Yup.

Theorem: If $(v_1, \ldots, v_n)$ spans V, then some subset of $(v_1, \ldots, v_n)$ is a basis of V.

Proof: Think about it