Math 4310 Name: ______________________________

Homework 10 Collaborators: ______________________________

Due 11/25/19 ______________________________

Please print out these pages. I encourage you to work with your classmates on this homework. Please list your collaborators on this cover sheet. (Your grade will not be affected.) Even if you work in a group, you should write up your solutions yourself! You should include all computational details, and proofs should be carefully written with full details. As always, please write neatly and legibly.

Please follow the instructions for the "extended glossary" on separate paper (LaTeX it if you can!) Hand in your final draft, including full explanations and write your glossary in complete, mathematically and grammatically correct sentences. Your answers will be assessed for style and accuracy.

Please **staple** *this cover sheet, your exercise solutions, and your glossary together, in that order, and hand in your homework in class.*

GRADES

Exercises ______________________________ / 50

Extended Glossary

Component	Correct?	Well-written?
Definition	/6	/6
Example	/4	/4
Non-example	/4	/4
Theorem	/5	/5
Proof	/6	/6
Total	/25	/25

Exercises.

1. Suppose that V is a finite dimensional vector space over $\mathbb{R}$, equipped with an inner product. Suppose that $\phi \in V^*$ (dual vector space), (i.e. $\phi : V \longrightarrow \mathbb{R}$ is a linear transformation). Show that there is a unique vector $\mathbf{w} \in V$ such that for every $\mathbf{v} \in V$,
$$\phi(\mathbf{v}) = \langle \mathbf{v}, \mathbf{w} \rangle.$$

2. Suppose that $T : V \longrightarrow W$ is a linear transformation where V and W are finite dimensional vector spaces over $\mathbb{R}$, each equipped with an inner product.
   (a) Prove that there is a function $T^* : W \longrightarrow V$ which satisfies for all $\mathbf{v} \in V$ and $\mathbf{w} \in W$, that
   $$\langle T(v), w \rangle \ = \ \langle v, T^*(w) \rangle.$$
   Note that the first inner product in this formula refers to the inner product in W, and the second inner product refers to the inner product in V. T^* is called the **adjoint** of T. (You may find the HW problem right before this one to be useful!)
   (b) Define $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ by $T(x, y, z) = (y + 2z, 3x)$. Find a formula for $T^* : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$.

3. In the same setting as the previous problem, suppose that $T : V \longrightarrow W$ is a linear transformation where V and W are finite dimensional vector spaces over $\mathbb{R}$, each equipped with an inner product.
   (a) Show that T^* is a linear transformation.

(b) Show that $(T^*)^* = T$.

(c) Show that $\ker(T^*) = \mathrm{im}(T)^\perp$.

(d) Show that $\mathrm{im}(T^*) = \ker(T)^\perp$.

(e) Suppose that $\mathcal{A}$ is an orthonormal basis for V and $\mathcal{B}$ is an orthonormal basis for W, and that $A = [T]_{\mathcal{B}\leftarrow\mathcal{A}}$. Then show that

$$A^T = [T^*]_{\mathcal{A}\leftarrow\mathcal{B}}.$$

4. **Rotation matrices**

(a) Show that in the plane $\mathbb{R}^2$, rotation by the angle θ (counterclockwise), is a linear transformation with matrix

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

(One way: argue that it is a linear transformation, what does it do on the standard basis vectors?)

(b) Show that $R_\alpha R_\beta = R_{\alpha+\beta}$. Argue geometrically. What does this say about trig angle addition formulas?

(c) Given $1 \le i < j \le n$, and θ an angle. A **Givens rotation** $G(i, j, \theta)$ on $\mathbb{R}^n$ is the linear transformation which fixes e_ℓ for $\ell \ne i, j$, and in the (x_i, x_j)-plane, is given by rotation by angle θ (i.e. in x_i, x_j part, the matrix is R_θ). Write down the matrices of all three Givens rotations in $\mathbb{R}^3$.

(d) Show that $G(i, j, \theta)$ is orthogonal.

5. **Reflection matrices**. Let $V = \mathbb{R}^n$, equipped with the standard inner product. Given a nonzero vector v and unit vector $u = \frac{v}{\|v\|}$, let $H \in \mathbb{R}^{n\times n}$ be the square matrix defined by

$$H = I - 2\frac{vv^T}{v^Tv} = I - 2uu^T.$$

Notice that the displayed fraction is a matrix divided by a scalar, so H is an $n \times n$ matrix.

(a) Show that H is symmetric, and H is an orthogonal $n \times n$ matrix.

(b) Show that H is a reflection: $H(v) = -v$, and if $w \in v^\perp$, then $H(w) = w$.

Extended Glossary. Define the notion of a **permutation matrix**. Give an example, a non-example, and state and prove a theorem about permutation matrices.

You may work in groups, but please write up your solutions **yourself**. If you do work together, your group should come up with at least two examples, two non-examples, and two theorems. Each one (example/non-example/theorem) should be included in some group member's extended glossary. Your solutions should be written formally, so that we could cut and paste them into a textbook.