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Homework 2

Due 9/18/19

Please print out these pages. I encourage you to work with your classmates on this homework. Please list your collaborators on this cover sheet. (Your grade will not be affected.) Even if you work in a group, you should write up your solutions yourself! You should include all computational details, and proofs should be carefully written with full details.
As always, please write neatly and legibly.

Please follow the instructions for the "extended glossary" on separate paper (LaTeX it if you can!) Hand in your final draft, including full explanations and write your glossary in complete, mathematically and grammatically correct sentences. Your answers will be assessed for style and accuracy.

Please **staple** *this cover sheet, your exercise solutions, and your glossary together, in that order, and hand in your homework in class.*

GRADES

Exercises ______________________ / 50

Extended Glossary

Component	Correct?	Well-written?
Definition	/6	/6
Example	/4	/4
Non-example	/4	/4
Theorem	/5	/5
Proof	/6	/6
Total	/25	/25

Some of the exercises below use the following (important) notion:

Definition. (Internal) direct sum. Suppose that U_1 and U_2 are subspaces of a vector space V, such that
(1) $U_1 + U_2 = V$, and
(2) $U_1 \cap U_2 = \{\mathbf{0}\}$.
In this case, we say that V is the **direct sum** of U_1 and U_2, and we write: $V = U_1 \oplus U_2$.
Given a vector subspace $U \subset V$, a **complementary subspace** of U is a vector subspace $W \subset V$, satisfying

$$V = U \oplus W.$$

Exercises.

1. Let V be a vector space over $\mathbb{F}$. Prove from the axioms/properties:

   (a) (Cancellation) For all $u, v, w \in V$, if $u + v = u + w$, then $v = w$.

   $$v = 0_V + v = (u + -u) + v = -u + (u + v) = -u + (u + w) = (-u + u) + w = 0_V + w = w$$

   (b) For all $a \in \mathbb{F}$, and $u, v \in V$, if $au = av$, then either $a = \mathbf{0}_{\mathbb{F}}$ or $u = v$.
   If $a \neq \mathbf{0}_{\mathbb{F}}$, let $\frac{1}{a}$ denote the inverse of $a \in \mathbb{F}$. Then,

   $$u = u \cdot 1 = u \cdot (\frac{a}{a}) = \frac{u \cdot a}{a} = \frac{v \cdot a}{a} = v \cdot \frac{a}{a} = v \cdot 1 = v$$

(c) For all $a, b \in \mathbb{F}$, and $u \in V$, if $au = bu$, then either $a = b$ or $u = \mathbf{0}_V$.
If $a = b$, we are done since $au = bu = au$.
If $a \neq b$,
$$au + -bu = \mathbf{0}_V \implies u = \mathbf{0}_V$$

(d) $\mathbf{0}_\mathbb{F} \cdot v = \mathbf{0}_V \; \forall v \in V$.
$$\mathbf{0}_\mathbb{F} v = \mathbf{0}_\mathbb{F} v + \mathbf{0}_V = \mathbf{0}_\mathbb{F} + (v + -v) = (\mathbf{0}_\mathbb{F} v + v) + -v$$
$$((\mathbf{0}_\mathbb{F} + 1_\mathbb{F})v) + -v = v + -v = \mathbf{0}_V$$

(e) If $c \in \mathbb{F}$, then $c \cdot \mathbf{0}_V = \mathbf{0}_V$.
$$c \cdot \mathbf{0}_V = c \cdot (\mathbf{0}_V + \mathbf{0}_V) = c \cdot \mathbf{0}_V + c \cdot \mathbf{0}_V = \mathbf{0}_V$$
Add $c \cdot \mathbf{0}_V$ to both sides to get
$$\mathbf{0}_V = c \cdot \mathbf{0}_V$$

(f) $(-1)v = -v$, for all $v \in V$.
$$\mathbf{0}_V = \mathbf{0}_\mathbb{F} \cdot v = (1 - 1) \cdot v = 1v + (-1)v = v + (-1)v$$
We add -v to both sides to obtain
$$-v = (v + -v) + (-1)v = \mathbf{0}_V + (-1)v = (-1)v$$

2. Let
$$U = \left\{ \begin{pmatrix} x \\ y \\ x \\ y \end{pmatrix} \in \mathbb{F}^4 \mid x, y \in \mathbb{F} \right\}.$$

(a) Show that $U \subset \mathbb{F}^4$ is a subspace.
We must show that (i). the additive identity exists in U, (ii). that U is closed under addition, and (iii). that U is closed under scalar multiplication.

i. $\mathbf{0}_\mathbb{F} \in U$ since $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \in \mathbb{F}^4$ is also in U when $x = y = 0$

ii. For any $w, v \in U$ with elements $\begin{pmatrix} x_1 \\ y_1 \\ x_1 \\ y_1 \end{pmatrix}$ and $\begin{pmatrix} x_2 \\ y_2 \\ x_2 \\ y_2 \end{pmatrix}$ such that $w_1, w_2, v_1, v_2 \in \mathbb{F}$, we have
$$w + v = \begin{pmatrix} x_1 \\ y_1 \\ x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ x_1 + x_2 \\ y_1 + y_2 \end{pmatrix} \in U$$
since $\mathbb{F}$ is closed under addition ($x_1 + x_2, y_1 + y_2 \in \mathbb{F}$).

iii. For any $c \in \mathbb{F}$ and $v \in U$, we have

$$c \cdot v = c \cdot \begin{pmatrix} x \\ y \\ x \\ y \end{pmatrix} = \begin{pmatrix} c \cdot x \\ c \cdot y \\ c \cdot x \\ c \cdot y \end{pmatrix} \in U$$

since $\mathbb{F}$ is closed under multiplication ($c \cdot x, c \cdot y \in \mathbb{F}$)

(b) Find a list of vectors of U which spans U, and which is linearly independent (i.e. a basis of U).

The set of all linear combinations of vectors $u_1, ..., u_m \in U$ is the span of U. In other words,

$$\text{span}(u_1, ..., u_m) = \{a_1 v_1 + ... + a_m v_m | a_1, ..., a_m \in F\}$$

For the set of vectors $u_1, ..., u_m$ to be called ***linearly independent***, then

$$a_1 v_1 + ... + a_m v_m = 0 \text{ only if } \ a_i = 0 \in \mathbb{F} \ \ \forall i$$

Building off what I know from a previous linear algebra course, there will only be two vectors which span U. This is because each element of U is a column vector with 4 field values where each row has a repeat. Thus if we treat one of these vectors as the vector which varies the x values and the other which varies the y values we have

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

(c) Find another subspace $W \subset \mathbb{F}^4$ such that $\mathbb{F}^4 = U \oplus W$.

The subspace W which satisfies this criteria is the vectors with 4 distinct elements of $\mathbb{F}$.

3. Let $V = \mathbb{F}^{2\times 2}$ be the vector space of 2 by 2 matrices, with entries in $\mathbb{F}$.

Determine if the following subsets are subspaces (justify your answer either way). For those that are subspaces, find a complement W: i.e. a subspace $W \subset V$, such that $V = U \oplus W$.

(a) $U = \{A \in V \mid A^2 = A\}$.

The described set consists only of the elements $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. If we add the second element to itself the result $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \notin U$. Thus U is not a subspace of V.

(b) $U = \{A \in V \mid AB = BA\}$, where $B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

For $B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$, $a_1, a_2, a_3, a_4 \in \mathbb{F}$

$AB = BA$ occurs only when

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$$

$$\begin{pmatrix} a_1 + a_2 & a_2 \\ a_3 + a_4 & a_4 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ a_1 + a_3 & a_2 + a_4 \end{pmatrix}$$

Thus the $A \in V$ which satisfy this condition have the property

$$a_2 = 0$$

$$a_1 = a_4,$$

meaning they have the form

$$A = \begin{pmatrix} a_1 & 0 \\ a_3 & a_1 \end{pmatrix}$$

For this subspace, we

i. have the zero element $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ from V in U
ii. have that the sum $w + v$ of two elements $w, v \in U$ is in U
Proof: the coefficients of $w, v \in U$ are in $\mathbb{F}$ thus their sum is in F and thus the $w + v \in U$
iii. have that the scalar multiplication $a \cdot v$, $a \in \mathbb{F}$, of $v \in U$ is in U Proof: similar to b.

Thus U is a subspace of V.
The complement W of U such that $V = U \oplus W$ needs to include the zero element and every element not in U such that when these subsets are added they comprise of all the elements in V.
We note that the elements of U not included are those where the diagonal entries are different, as well as those with a nonzero a_2 value.
So

$$W = \begin{pmatrix} w_1 & w_2 \\ 0 & w_4 \end{pmatrix}$$

where $w_1 \neq w_2 \neq w_4 \in \mathbb{F}$.

(c) $U = \{A \in V \mid A \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\}$.
In a similar fashion to part (b) above, we derive that the elements of V are of the form

$$A = \begin{pmatrix} x & x \\ y & y \end{pmatrix}$$

where $x, y \in \mathbb{F}$. For this subspace, we

i. have the zero element $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ from V in U
ii. have that the sum $w + v$ of two elements $w, v \in U$ is in U
iii. have that the scalar multiplication $a \cdot v$, $a \in \mathbb{F}$, of $v \in U$ is in U

The proof is analogous to that in (b) and is thus not shown. Thus U is a subspace of V. We note that the elements of U not included are those where all entries are different. So

$$W = \begin{pmatrix} w_1 & w_2 \\ w_3 & w_4 \end{pmatrix}$$

where $w_1, w_2, w_3, w_4 \in \mathbb{F}$.

4. Find values $a, b \in \mathbb{Q}$ so that $\begin{pmatrix} 2 \\ a-b \\ 1 \end{pmatrix}$ and $\begin{pmatrix} a \\ b \\ 3 \end{pmatrix}$ are linearly dependent in $\mathbb{Q}^3$.

   Need $c_1, c_2 \in \mathbb{Q}$ such that

$$c_1 \begin{pmatrix} 2 \\ a-b \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} a \\ b \\ 3 \end{pmatrix} = 0$$

   Divide both sides by c_2, we define $d = \frac{c_1}{c_2}$,

$$d \begin{pmatrix} 2 \\ a-b \\ 1 \end{pmatrix} + \begin{pmatrix} a \\ b \\ 3 \end{pmatrix} = 0$$

   We rearrange the equation to solve for d,a, and b. (note: we absorb the negative into our constant d to avoid working with it).

$$d \begin{pmatrix} 2 \\ a-b \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ b \\ 3 \end{pmatrix}$$

   Thus,

$$d = 3$$

$$a = 2d = 2 \cdot 3 = 6$$

$$b = da - db = 3a - 3b \implies b = \frac{9}{2}$$

5. Determine which of the following lists of vectors in $\mathrm{Fun}(\mathbb{R}, \mathbb{R})$ are linearly independent, and which are linearly dependent.

   (a) $(\sin^2 x, \cos^2 x)$. Linearly **Independent**.
   Proof:

$$a_1 \sin^2 x + a_2 \cos^2 x = 0 \text{ only if } a_1 = a_2 = 0$$

   (b) $(1, \sin^2 x, \cos^2 x)$. Linearly **Dependent**.
   Proof:

$$a_1 \sin^2 x + a_2 \cos^2 x + a_3 \cdot 1 = 0 \text{ when } a_1 = a_2 = -1$$

(c) (e^x, e^{2x}). Linearly **Independent**

$$a_1e^x + a_2e^{2x} = 0$$

For x = 0,

$$a_1 + a_2 = 0 \implies a_1 = -a_2$$

Whereas for x = 1,

$$a_1e + a_2e^2 = 0 \implies a_1 = -a_2e$$

Implying that for each x we have a different set of constants. Thus

$$a_1e^x + a_2e^{2x} = 0 \text{ only if } a_1 = a_2 = 0$$

6. In this problem, assume that U_1, U_2 (and U_3 in the last two parts) are subspaces of V.

   (a) Is $U_1 \cap U_2$ a subspace? Either prove it, or give a counter-example.
   $U_1 \cap U_2$ is a subspace since

   i. $0_V \in U_1$ and $0_V \in U_2 \implies 0_V \in U_1 \cap U_2$

   ii. $u + v \in U_1$ for $u, v \in U_1$ and $w + z \in U_2$ for $w, z \in U_2$
   So for $x, y \in U_1 \cap U_2$ x and y are both in U_1 and U_2. Thus $x+y \in U_1$ and $x+y \in U_2$ since both are subspaces of V. Thus $x + y \in U_1 \cap U_2$

   iii. $a \cdot v \in U_1 \cap U_2$ for $v \in U_1 \cap U_2$
   The proof is so similar to that of part (ii) that we omit it.

   (b) Is $U_1 \cup U_2$ a subspace if neither contains the other? Either prove it, or give a counter-example.
   For 'disjoint' subspaces, U_1, U_2, $U_1 \cup U_2$ is NOT a subspace.
   To see this we show that $U_1 \cup U_2$ is NOT closed under addition. For $x \in U_1$ and $y \in U_2$, $x + y \notin U_1$ (since U_2 has no overlap with U_1. Similarly $x + y \notin U_2$.
   Thus, $x + y \notin U_1 \cup U_2$

   (c) Suppose that $\mathbb{F} = \mathbb{F}_2$ is the field with two elements. Find an example of a vector space V, and subspaces U_1, U_2, U_3 with no one containing any other, such that $U_1 \cup U_2 \cup U_3$ is a subspace. (i.e. surprising things can happen sometimes with finite fields!)
   First we note that the finite field with two elements only contains the elements 0 and 1. If we take

$$V = \mathbb{F}^3 = \left\{ \begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix} \right\}$$

   with

$$U_1 = \left\{ \begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\}$$

$$U_2 = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

$$U_3 = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

Then their union

$$U_1 \cup U_2 \cup U_3 = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\} = V$$

The zero element is clearly in the union. Need to show that the union is closed under addition and scalar multiplication.

For addition, take any two elements in the union, and the result is also in the union, since it covers the whole vector space V.

Similarly, for multiplication, any scalar times a vector in $\mathbb{F}^3$ will also be in $\mathbb{F}^3$ and thus in our union.