If the function

$$f(z) = u(x,y) + iv(x,y)$$

is differentiable at $z_0 = x_0 + iy_0$, then the limit

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

can be computed by allowing $\Delta z = \Delta x + i\Delta y$ to approach zero from any convenient direction in the complex plane.

If it approaches horizontally, then $\Delta z = \Delta x$, and we obtain

$$f'(z_0) = \lim_{\Delta x \to 0} \frac{u(x_0+\Delta x, y_0) + iv(x_0+\Delta x, y_0) - u(x_0, y_0) - iv(x_0, y_0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left[\frac{u(x_0+\Delta x, y_0) - u(x_0, y_0)}{\Delta x}\right] + i \lim_{\Delta x \to 0} \left[\frac{v(x_0+\Delta x, y_0) - v(x_0, y_0)}{\Delta x}\right]$$

We deduce that

$$f'(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0) \qquad (1)$$

On the other hand, if $\Delta z = i\Delta y$ and approaches zero vertically

$$f'(z_0) = \lim_{\Delta y \to 0} \left[\frac{u(x_0, y_0+\Delta y) - u(x_0, y_0)}{i\Delta y}\right] + i \lim_{\Delta y \to 0} \left[\frac{v(x_0, y_0+\Delta y) - v(x_0, y_0)}{i\Delta y}\right]$$

Hence,

$$f'(z_0) = -i\frac{\partial u}{\partial y}(x_0, y_0) + \frac{\partial v}{\partial y}(x_0, y_0) \qquad (2)$$

But (1) = (2) so,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad (3)$$

must hold at $z_0 = x_0 + iy_0$. Eqn (3) are called the Cauchy-Riemann equations.

We've thus established

Theorem 4: A necessary condition for a function $f(z) = u(x,y) + iv(x,y)$ to be differentiable at a point z_0 is that the Cauchy-Riemann equations hold at z_0.

Consequently, if f is analytic in an open set G, then the Cauchy-Riemann equations must hold at every point G.

A tip to remember Cauchy-Riemann equations.

Horizontal = Vertical

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial (iy)} \longrightarrow \frac{\partial (u+iv)}{\partial x} = \frac{1}{i}\frac{\partial (u+iv)}{\partial y}$$

and equate real and imaginary parts

Example 1: Show that the function

$$f(z) = (x^2+y^2) + i(y^2 - x^2)$$

is NOT analytic at any point.

$u(x,y) = x^2+y$ $\qquad\qquad$ $v(x,y) = y^2 - x$

$$\frac{\partial u}{\partial x} = 2x \qquad\qquad \frac{\partial v}{\partial y} = 2y$$

$$\frac{\partial u}{\partial y} = 1 \qquad\qquad \frac{\partial v}{\partial x} = -1$$

Hence the Cauchy-Riemann equations are simultaneously satisfied only on the line $x=y$ and therefore in no open disk. Thus by Theorem 4 the function $f(z)$ is nowhere analytic.

Being mathematically precise, we point out that the Cauchy-Riemann equations alone are NOT sufficient to ensure differentiability.

You also need the additional hypothesis of continuity of the first partial derivatives of u and v.

The complete story is the following theorem

Theorem 5. Let $f(z) = u(x,y) + iv(x,y)$ be defined in some open set G containing the point z_0. If the first partial derivatives of u and v exist in G, are continuous at z_0, and satisfy the Cauchy-Riemann equations at z_0, then f is differentiable at z_0.

Consequently, if the first partial derivatives are continuous and satisfy the Cauchy-Riemann equations at all points of G, then f is analytic in G.

PROOF

The difference quotient for f at z_0 can be written as

$$\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

$$= \frac{[u(x_0+\Delta x, y_0+\Delta y) - u(x_0,y_0)] + i[v(x_0+\Delta x, y_0+\Delta y) - v(x_0,y_0)]}{\Delta x + i\Delta y} \qquad (4)$$

Where $z_0 = x_0 + iy_0$ and $\Delta z = \Delta x + i\Delta y$.

* The above expressions are well defined if $|\Delta z|$ is so small that the closed disk with center z_0 and radius $|\Delta z|$ lies entirely in G.

Let us rewrite the difference

$$u(x_0+\Delta x, y_0+\Delta y) - u(x_0, y_0)$$

as

$\left[u(x_0+\Delta x, y_0+\Delta y) - u(x_0, y_0+\Delta y)\right] + \left[u(x_0, y_0+\Delta y) - u(x_0, y_0)\right]$ (5)

Because the partial derivatives exist in G, the mean-value theorem says that there is a number x^* between x_0 and $x_0+\Delta x$ such that

???

$$u(x_0+\Delta x, y_0+\Delta y) - u(x_0, y_0+\Delta y) = \Delta x \frac{\partial u}{\partial x}(x^*, y_0+\Delta y)$$

Furthermore, since the partial derivatives are continuous at (x_0, y_0), we can write

$$\frac{\partial u}{\partial x}(x^*, y_0+\Delta y) = \frac{\partial u}{\partial x}(x_0, y_0) + \varepsilon_1$$

where

$$\varepsilon_1 \to 0 \quad \text{as} \quad \left.\begin{aligned} x^* &\to x_0 \\ \Delta y &\to 0 \end{aligned}\right\} \text{in particular as } \Delta z \to 0$$

Thus,

$$u(x_0+\Delta x, y_0+\Delta y) - u(x_0, y_0+\Delta y) = \Delta x \left[\frac{\partial u}{\partial x}(x_0, y_0) + \varepsilon_1\right]$$

(Next three proofs were done solo)

Similarly, because the partial derivatives exist in G, the mean-value theorem says that there is a number y^* between y_0 and $y_0+\Delta y$ such that

$$u(x_0, y_0+\Delta y) - u(x_0, y_0) = \Delta y \left[\frac{\partial u}{\partial y}(x_0, y^*)\right]$$

Furthermore, since the partial derivatives are continuous at (x_0, y_0), we can write

$$\frac{\partial u}{\partial x}(x_0, y^*) = \left[\frac{\partial u}{\partial y}(x_0, y_0) + \varepsilon_2\right]$$

where

$$\varepsilon_2 \to 0 \qquad \text{as} \qquad \Delta z \to 0$$

Thus,

$$u(x_0, y_0 + \Delta y) - u(x_0, y_0) = \Delta y\left[\frac{\partial u}{\partial y}(x_0, y_0) + \varepsilon_2\right]$$

In a similar fashion, Let us rewrite the difference

$$i\left[v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)\right]$$

as

$$i\left[\left[v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0 + \Delta y)\right] + \left[v(x_0, y_0 + \Delta y) - v(x_0, y_0)\right]\right] \quad (6)$$

Because the partial derivatives exist in G, the mean-value theorem says that there is a number x^* between x_0 and $x_0 + \Delta x$ such that

$$i\left[v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0 + \Delta y)\right] = i\,\Delta x\,\frac{\partial v}{\partial x}(x^*, y_0 + \Delta y)$$

Furthermore, since the partial derivatives are continuous at (x_0, y_0), we can write

$$\frac{\partial v}{\partial x}(x^*, y_0 + \Delta y) = \frac{\partial v}{\partial x}(x_0, y_0) + \varepsilon_3$$

where

$$\varepsilon_3 \to 0 \qquad \text{as} \qquad \left.\begin{matrix} x^* \to x_0 \\ \Delta y \to 0 \end{matrix}\right\} \begin{matrix} \text{in particular} \\ \text{as } \Delta z \to 0 \end{matrix}$$

Thus,

$$i\left[v(x_0+\Delta x, y_0+\Delta y) - v(x_0, y_0+\Delta y)\right] = i\Delta x\left[\frac{\partial v}{\partial x}(x_0,y_0) + \varepsilon_3\right]$$

Similarly, because the partial derivatives exist in G, the mean-value theorem says that there is a number y^* between y_0 and $y_0+\Delta y$ such that

$$i\left[v(x_0, y_0+\Delta y) - v(x_0,y_0)\right] = i\Delta y\left[\frac{\partial u}{\partial y}(x_0, y^*)\right]$$

Furthermore, since the partial derivatives are continuous at (x_0, y_0), we can write

$$\frac{\partial v}{\partial x}(x_0, y^*) = \left[\frac{\partial v}{\partial y}(x_0,y_0) + \varepsilon_4\right]$$

where

$$\varepsilon_4 \to 0 \qquad \text{as} \qquad \Delta z \to 0$$

Thus,

$$i\left[v(x_0, y_0+\Delta y) - v(x_0,y_0)\right] = i\Delta y\left[\frac{\partial v}{\partial y}(x_0,y_0) + \varepsilon_4\right]$$

Solo work ends here

We ultimately have

$$\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} = \frac{\Delta x\left[\frac{\partial u}{\partial x} + \varepsilon_1 + i\frac{\partial v}{\partial x} + i\varepsilon_3\right] + \Delta y\left[\frac{\partial u}{\partial y} + \varepsilon_2 + i\frac{\partial v}{\partial y} + i\varepsilon_4\right]}{\Delta x + i\Delta y}$$

where each partial derivative is evaluated at (x_0, y_0) and where each $\varepsilon_i \to 0$ as $\Delta z \to 0$.

Now we use the Cauchy-Riemann equations to express the difference quotient as

$$\frac{\Delta x\left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right] + i\Delta y\left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right]}{\Delta x + i\Delta y} + \frac{\lambda}{\Delta x + i\Delta y} \qquad (7)$$

where

$$\lambda := \Delta x(\varepsilon_1 + i\varepsilon_3) + \Delta y(\varepsilon_2 + i\varepsilon_4)$$

Since

$$\left|\frac{\lambda}{\Delta x + i\Delta y}\right| \le \left|\frac{\Delta x}{\Delta x + i\Delta y}\right| |\varepsilon_1 + i\varepsilon_3| + \left|\frac{\Delta y}{\Delta x + i\Delta y}\right| |\varepsilon_2 + i\varepsilon_4|$$

$$\le |\varepsilon_1 + i\varepsilon_3| + |\varepsilon_2 + i\varepsilon_4|$$

We see that the last term in (7) approaches zero as $\Delta z \to 0$, and so

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0)$$

i.e.

$f'(z_0)$ exists!

Example 2: Prove that the function $f(z) = e^z$ is entire and find its derivative.

$$f(z) = e^z = e^x e^{iy} = e^x [\cos y + i \sin y]$$

$$u(x,y) = e^x \cos y \qquad v(x,y) = e^x \sin y$$

$$\frac{\partial u}{\partial x} = e^x \cos y \qquad \frac{\partial v}{\partial y} = e^x \cos y$$

$$\frac{\partial u}{\partial y} = -e^x \sin y \qquad \frac{\partial v}{\partial x} = e^x \sin y$$

Since the first partial derivatives are continuous and satisfy the Cauchy-Riemann equations at every point in the plane.

Hence $f(z)$ is entire. From (1) we see that

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = e^x [\cos y + i \sin y]$$

Not surprisingly,

$$f'(z) = f(z)$$

Theorem 6. If $f(z)$ is analytic in a domain D and if $f'(z)=0$ everywhere in D, then $f(z)$ is constant in D.

Before the proof, observe that the connectedness property of the domain is essential. Indeed, if $f(z)$ is defined by

$$f(z) = \begin{cases} 0 & \text{if } |z|<1 \\ 1 & \text{if } |z|>2 \end{cases}$$

then f is analytic and $f'(z)=0$ on its domain of definition WHICH IS NOT ITS DOMAIN.

Proof of Theorem 6

Since $f'(z)=0$ in D, we see from Eqn (1) & Eqn (2) that all the first partial derivatives of u and v vanish in D; that is

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$$

Thus by Theorem 1 (in 1.6) we have $u=v=\text{constant}$ in D. Consequently, $f=u+iv$ is also constant in D.

A consequence of theorem 6 is the fact that if f and g are two functions analytic in a domain D, then $f = g +$ constant in D. (Prob 7)

Using theorem 6 and the Cauchy-Riemann equations, it can be shown that an analytic function $f(z)$ must be constant when any of the following conditions hold in a domain D:

$$\begin{aligned} &\mathrm{Re}\, f(z) \text{ is constant;} \\ &\mathrm{Im}\, f(z) \text{ is constant;} \\ &|f(z)| \text{ is constant} \end{aligned} \qquad (8)$$

Prob 8, 13