Informally, when we have an infinite sequence $z_1, z_2, z_3, \ldots$ of complex numbers, we say that the number z_0 is the limit of the sequence if the z_n eventually (i.e for large n) stay arbitrarily close to z_0. More precisely,

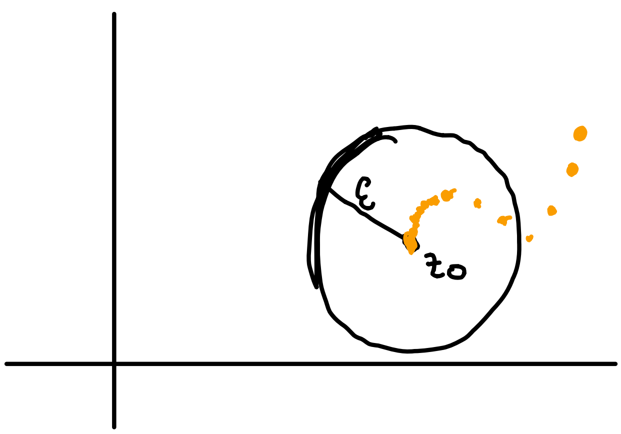
Definition 1: A sequence of complex numbers $\{z_n\}_1^\infty$ is said to have the **limit** z_0 or to **converge** to z_0, and we write

$$\lim_{n \to \infty} z_n = z_0$$

or, equivalently,

$$z_n \to z_0 \quad \text{as} \quad n \to \infty$$

if for any $\varepsilon > 0$ there exists an integer N such that $|z_n - z_0| < \varepsilon$ for all $n > N$



A convergent sequence

Geometrically, this means that each term z_n, for $n > N$, lies in the open disk of radius ε about z_0.

Example 1: Find the limit of the sequence

(a) $\left(\frac{i}{3}\right)^n$ (b) $\frac{2 + in}{1 + 3n}$ (c) $z_n = i^n$

(a) Since $|(i/3)^n| = 1/3^n \to 0$, it follows that

$$\lim_{n\to\infty} \left(\frac{i}{3}\right)^n = 0$$

(b) Dividing numerator and denominator by n we get

$$\frac{2+in}{1+3n} = \frac{2/n+i}{1/n+3} \to \frac{0+i}{0+3} = \frac{i}{3} \text{ as } n\to\infty$$

(c) The sequence i^n consists of infinitely many repetitions of $i, -1, -i$, and 1. Thus

$$\lim_{n\to\infty} i^n \quad \underline{\text{DNE}}$$

- A related concept is the limit of a complex-valued function $f(z)$.
- Roughly speaking, we say that the number w_0 is the limit of the function $f(z)$ as z approaches z_0, if
- $f(z)$ stays arbitrarily close to w_0 whenever z is sufficiently near z_0. In precise terms we give,

Definition 2: Let f be a function defined in some neighborhood z_0, with the possible exception of the point z_0 itself. We say the **limit of $f(z)$ as z approaches z_0 is the number** w_0 and write

$$\lim_{z \to z_0} f(z) = w_0$$

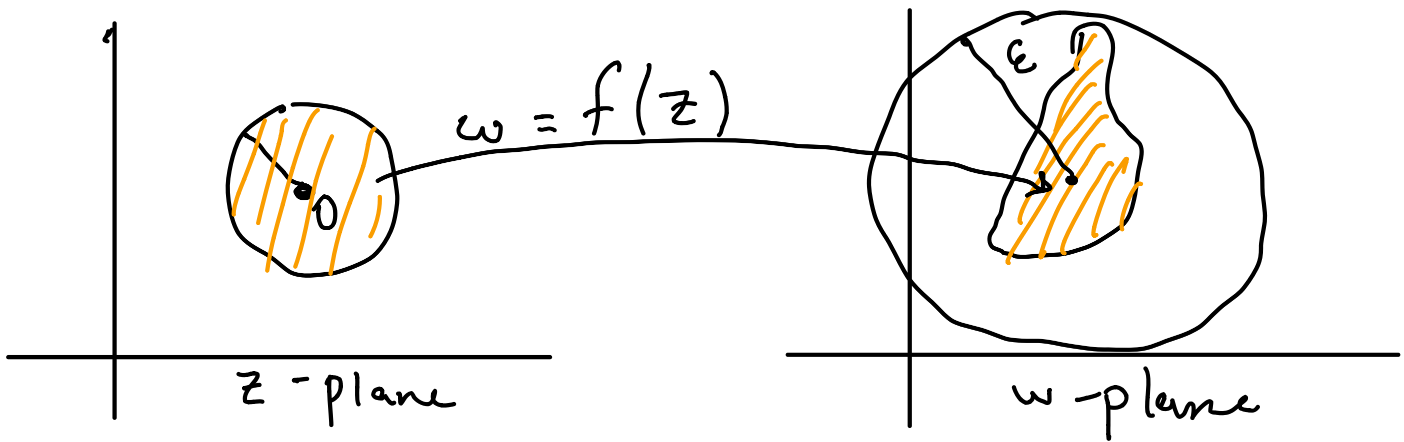
or, equivalently,

$$f(z) \to w_0 \quad \text{as} \quad z \to z_0$$

if for any $\varepsilon > 0$ there exists a positive number δ such that

$$|f(z) - w_0| < \varepsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta$$

Geometrically, this says any neighborhood of w_0 contains all the values assumed by f in some full neighborhood of z_0, except possibly the value $f(z_0)$



Example 2: Use Definition 2 to prove that

$$\lim_{z \to i} z^2 = -1$$

We must show that for any $\varepsilon > 0$ there is a positive number δ such that

$$|z^2 - (-1)| < \varepsilon \quad \text{whenever} \quad 0 < |z-i| < \delta$$

So we express $|z^2 - (-1)|$ in terms of $|z-i|$:

$$z^2 - (-1) = z^2 + 1 = (z-i)(z+i) = (z-i)(z-i+2i)$$

It follows from the properties of absolute value (triangle inequality) that

$|z_1 + z_2| \le |z_1| + |z_2|$

$$|z^2 - (-1)| = |z-i|\,|z-i+2i| \le |z-i|\left(|z-i| + |2i|\right)$$

$\Rightarrow$

$$|z^2 - (-1)| = |z-i|\,|z-i+2i| \le |z-i|\left(|z-i| + 2\right) \qquad (1)$$

Now if $|z-i| < \delta$ the right-hand side of (1) is less than $\delta(\delta+2)$; so to ensure that it is less than ε, we choose δ to be smaller than each either of the numbers $\varepsilon/3$ and 1:

$$|z-1|\left(|z-1|+2\right) < \frac{\varepsilon}{3}(1+2) = \varepsilon \qquad ???$$

There is an obvious relationship b/t the limit of a function and the limit of a sequence; namely, if

$$\lim_{z \to z_0} f(z) = \omega_0$$

then for every sequence $\{z_n\}_1^\infty$ converging to z_0 ($z_n \neq z_0$) the sequence $\{f(z_n)\}_1^\infty$ converges to ω_0.

Converse is also valid.

The condition of continuity is expressed in

Definition 3: Let f be a function defined in the neighborhood of z_0. Then f is **continuous at** z_0 if

$$\lim_{z \to z_0} f(z) = f(z_0)$$

In other words, for f to be continuous at z_0, it must have a limiting value at z_0, and this limiting value must be $f(z_0)$.

A function f is said to be continuous on a set S if it is continuous at each point of S.

See (Prob 18)

Theorem 1: If $\lim_{z \to z_0} f(z) = A$ and $\lim_{z \to z_0} g(z) = B$

(i) $\lim_{z \to z_0} (f(z) \pm g(z)) = A \pm B$

(ii) $\lim_{z \to z_0} f(z)g(z) = AB$

(iii) $\lim_{z \to z_0} \dfrac{f(z)}{g(z)} = \dfrac{A}{B}$ if $B \neq 0$

$\downarrow$ Immediate Consequence

Theorem 2: If $f(z)$ and $g(z)$ are continuous at z_0, then so are $f(z) \pm g(z)$ and $f(z)g(z)$. The quotient $f(z)/g(z)$ is also continuous at z_0 provided $g(z_0) \neq 0$

Note: constant functions as well as $f(z) = z$ are continuous on the whole plane $\mathbb{C}$.

From Theorem 2 we can deduce that the polynomial functions in z

$$a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

where the a_i are constants continuous on the whole plane. Rational functions in z, which are defined as quotients of polynomials, i.e.

$$\frac{a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n}{b_0 + b_1 z + b_2 z^2 + \cdots + b_n z^n}$$

are therefore continuous at each point the denominator vanishes.

Example 3: Find the limits as $z \to 2i$ of the functions

$f_1(z) = z^2 - 2z + 1$

$f_2(z) = (z+2i)/z$

$f_3(z) = (z^2+4) / z(z-2i)$

Since $f_1(z)$ and $f_2(z)$ are continuous at $z = 2i$,

$$\lim_{z \to 2i} f_1(z) = f_1(2i) = (2i)^2 - 2(2i) + 1 = 3 - 4i$$

$$\lim_{z \to 2i} f_2(z) = f_2(2i) = \frac{2i + 2i}{2i} = 2$$

$f_3(z)$ is NOT continuous at $z = 2i$ because its not defined there.

However, for $z \neq 2i$, $z \neq 0$

$$f_3(z) = \frac{(z+2i)(z-2i)}{z(z-2i)} = \frac{z+2i}{z} = f_2(z)$$

and so

$$\lim_{z \to 2i} f_3(z) = \lim_{z \to 2i} f_2(z) = 2$$

In general, if a function can be redefined/defined at a single point z_0 so as to be continuous there, we say this function has a removable discontinuity at z_0.

Limits involving infinity are very useful at describing the behavior of certain sequences and functions. We say "$z_n \to \infty$" if, for each positive number M (no matter how large), there is an integer N such that $|z_n| > M$ whenever $n > N$;

Similarly, "$\lim_{z \to z_0} f(z) = \infty$" means that for each positive number M (no matter how large), there is a $\delta > 0$ such that $|f(z)| > M$ whenever $0 < |z - z_0| < \delta$.

Essentially we are saying that complex numbers approach infinity when their magnitudes approach infinity.

Therefore,

$$\lim_{z \to 3i} \frac{z}{z^2+9} = \lim_{z \to 3i} \frac{z}{(z+3i)(z-3i)} = \infty$$

$$\lim_{z \to \infty} \frac{iz-2}{4z+i} = \frac{i}{4} \quad (\text{L'hospitals})$$

$$\lim_{z \to \infty} \frac{z^3+3i}{z^2+5z} = \infty$$