

Recall

Started discrete random variables (r.v.'s).

Given Ω, P ; a r.v. X is a function $X: \Omega \rightarrow \mathbb{R}$ that takes on a finite or countably infinite number of values.

Associated with a discrete r.v. is its pmf P_X defined for each possible value of x for X

$$p_X(x) = P(\{s \in \Omega : X(s) = x\}) = P(\{X=x\}) = P(X=x)$$

On a given Ω w/ a given P , many discrete r.v.'s are defined

For a given pmf there are many Ω 's, P 's, $X: \Omega \rightarrow \mathbb{R}$ that give rise to p_X = that pmf.

Examples - constant, Bernoulli p , Binomial(n, p) (from last time)
discrete uniform on $0 \leq k \leq b$

Geometric Random Variable

Geometric(p):

$$P_X(k) = p(1-p)^{k-1}, \quad 1 \leq k < \infty$$

A possible Ω, P ? You have a coin where $P(H) = p$;
 $\Omega = \text{set of all semi-infinite } (1 \leq k < \infty) \text{ of H, T}$; $X(s) = \text{index of first head}$
for all $s \in \Omega$

Reality check: recall $\sum_{x \in X} P_X(x) = 1$ for any discrete r.v. X .

Verify this for geometric r.v.'s.

$$\sum_{x \in X} P_X(x) = \sum_{k=1}^{\infty} p(1-p)^{k-1} = p \sum_{m=0}^{\infty} (1-p)^m = p \frac{1}{1-(1-p)} = 1 \quad \checkmark$$

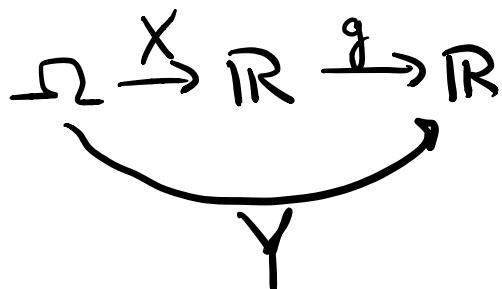
Poisson Random Variable

Poisson (λ):

$$P_X(k) = e^{-\lambda} \frac{\lambda^k}{k!} \quad 0 \leq k < \infty \quad (k \in \mathbb{N})$$

$$\sum_{x \in X} P_X(k) = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} = e^{\lambda} e^{-\lambda} = 1 \quad \checkmark \quad (e^{\lambda} = \sum_{n=0}^{\infty} \frac{1}{n!})$$

Given a random variable X and any function $g: \mathbb{R} \rightarrow \mathbb{R}$, can define another r.v $Y = g(X)$



$\forall s \in \Omega, Y(s) = g(X(s))$
possible X values being finite
 \Rightarrow possible Y values are finite.

"Function of a discrete r.v. is another discrete r.v."

Generally nontrivial to get pmf of $Y = g(X)$ from pmf of X .

BUT, sometimes easy.

Example - p_X discrete uniform on say $-3 \leq k \leq 3$

$$p_X(k) = \begin{cases} \frac{1}{7}, & -3 \leq k \leq 3 \\ 0, & \text{else} \end{cases}$$

Say

$g(x) = x^2$ for all $x \in \mathbb{R}$ - set $Y = g(X) = X^2$
what is p_Y ?

Possible values of Y are $0, 1, 4, 9$.

Easy to figure out probability!
i.e

$$p_Y(0) = P(\{Y=0\}) = P(\{X=0\}) = \frac{1}{7}$$

$$P_Y(4) = P(\{Y=4\}) = P(\{X=-2\}) \cup P(\{X=2\}) = \frac{2}{7}$$

↗
disjoint! X can't
bc -2 and 2 at
once.

Aside

A frequently encountered function: affine:

given a r.v X , define

$$Y = \alpha X + \beta, \quad \alpha, \beta \in \mathbb{R} \text{ given.}$$

Expected Value

Given a discrete r.v X w/ $P_X(x)$ pmf, define the expected value (or expectation)

$$\mathbb{E}(X) = \sum_{x \in X} x P_X(x)$$

Motivation? Say Ω is set of outcomes of a spinner - say outcomes are $\textcircled{1}, \textcircled{2}, \dots, \textcircled{n}$ - say $P(\textcircled{k}) = p_k, k \in \mathbb{N} \setminus \{0\}$

Define $X: \Omega \rightarrow \mathbb{R}$ by

$$X(k) = x_k, \quad k \in \mathbb{N} \setminus \{0\}$$

↗ think of this as payment you get when spinner lands on k .

pmf of X :

$$p_X(x_k) = p_k, \quad k \in \mathbb{N} \setminus \{0\}$$

Suppose you spin again and again repeatedly - collect payments each time - over long haul, what is avg win per spin.

If $L > 0$ is large integer, winnings/spin over L spins is

$$\frac{x_1 \#(1) + x_2 \#(2) + \cdots + x_n \#(n)}{L}$$

Intuition = for L large, $\frac{\#(k)}{L} \sim p_k$ (as $L \rightarrow \infty$)

$$\text{this} \rightarrow \sum_k k \underbrace{p_X(x_k)}_{p_k} = \mathbb{E}(X)$$

i.e $\mathbb{E}(X)$ = "average winnings per spin over long haul"

Terms (synonymous: expected value, expectation, mean, average, ...)

Example - X is Bernoulli(p)

$$P_X(x) = \begin{cases} p & \text{if } x=1 \\ 1-p & \text{if } x=0 \end{cases}$$

$$\mathbb{E}(X) = \sum_{x \in X} x P_X(x) = 1 \cdot p + 0 \cdot (1-p) = p$$

when $p \neq 0, 1$, $\mathbb{E}(X)$ is "not a possible value of X "

□

Given X , $Y = g(X)$, what is $\mathbb{E}(Y)$? One way to find $\mathbb{E}(Y)$: figure out $P_Y(y)$ for all possible values $y \in Y$ - then

$$\mathbb{E}(Y) = \sum_{y \in Y} y P_Y(y)$$

and can get P_Y from P_X - not trivial in general....

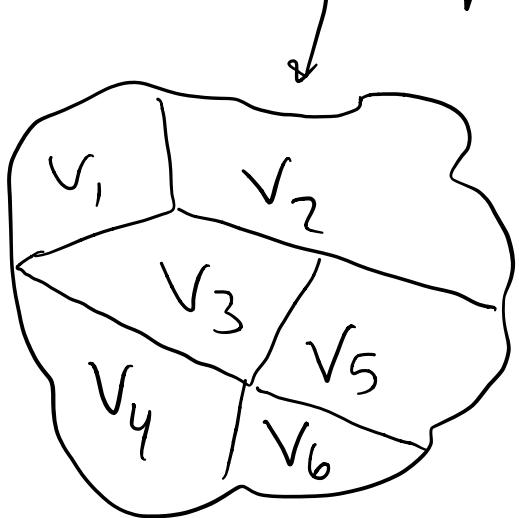
Another easier way: **Expected Value Rule**

Given X , P_X , and $Y = g(X)$,

$$\mathbb{E}(Y) = \sum_{x \in X} g(x) P_X(x)$$

Why?

, set of all possible X -values - say



y_1, \dots, y_6 are possible Y values

$V_k = \text{set of } x\text{-values for which } g(x) = y_k$

$$P_Y(y_k) = \sum_{x \in V_k} P_X(x)$$

Thus,

$$\begin{aligned} E(Y) &= \sum_{y_k} y_k P_Y(y_k) \\ &= \sum_{y_k} y_k \sum_{x \in V_k} P_X(x) \quad \begin{matrix} \text{for all } x \in V_k, g(x) = y_k \\ (\text{for } 1 \leq k \leq 6) \end{matrix} \\ &= \sum_{y_k} \sum_{x \in V_k} g(x) P_X(x) \\ &= \sum_{x \in X} g(x) P_X(x) \end{aligned}$$

Special Case: $Y = \alpha X + \beta$

Use expected value rule to get $E(Y)$.

$$E(Y) = \sum_{x \in X} g(x) P_X(x)$$

$$= \sum_{x \in X} (\alpha x + \beta) P_X(x) = \underbrace{\alpha \sum_{x \in X} x P_X(x)}_{E(X)} + \underbrace{\beta \sum_{x \in X} P_X(x)}_1$$

So,

$$\mathbb{E}(Y) = \alpha \mathbb{E}(X) + \beta$$

✓

This is the one case Professor Delchamps knows of where

$$\mathbb{E}(g(X)) = g(\mathbb{E}(X))$$

Generally, knowing $\mathbb{E}(X)$ doesn't suffice for finding $\mathbb{E}(g(X))$.

VARIANCE

Variance of a r.v X w/ pmf p_X

measures spread of p_X about $\mathbb{E}(X)$ → $\text{Var}(X) = \mathbb{E}((X - \mathbb{E}(X))^2)$

Think of this as $E(g(X))$
where $g(X) = (X - \mathbb{E}(X))^2$

we define

$$\sigma_X = \sqrt{\text{Var } X}$$

to be the standard deviation of X !

Example - Calculations

$$P_X(k) = \begin{cases} \frac{1}{7}, & -3 \leq k \leq 3 \\ 0, & \text{otherwise} \end{cases}$$

$$\mathbb{E}(X) = 0$$

$$\text{Var}(X) = \mathbb{E}((X - \mathbb{E}(X))^2) = \mathbb{E}(X^2)$$

$$\mathbb{E}(X^2) = \sum_{x \in X} x^2 P_X(x)$$

$$\begin{aligned} &= 9 \cdot \frac{1}{7} + 4 \cdot \frac{1}{7} + 1 \cdot \frac{1}{7} + 0 \cdot \frac{1}{7} + 1 \cdot \frac{1}{7} + 4 \cdot \frac{1}{7} + 9 \cdot \frac{1}{7} \\ &= \frac{28}{7} = 4 \end{aligned}$$

$$\text{So, } \text{Var}(X) = 4$$

$$\sigma_X = 2$$

Fact: If $Y = \alpha X + \beta$, then $\text{Var}(Y) = \alpha^2 \text{Var}(X)$

so,

$$\sigma_Y = \sqrt{\text{Var} Y} = |\alpha| \sigma_X$$

\uparrow
easy to show