

Cayley Hamilton Theorem

Minimal-Characteristic Polynomials

Applications to Jordan Canonical Form

Situation: $T \in \mathcal{L}(V)$

- V a v.s. over \mathbb{C}
- $\dim V = n < \infty$

Definition: The characteristic polynomial $q_T(x) \in \mathbb{C}[x]$

is

$$q_T(x) = (x - \lambda_1)^{a_1} (x - \lambda_2)^{a_2} \cdots (x - \lambda_m)^{a_m}$$

where $\lambda_1, \dots, \lambda_m$ are the distinct eigenvalues
of $a_i = \dim G_{\lambda_i}(T) = \dim(\ker(T - \lambda_i)^n)$

Thus, $\deg q_T(x) = a_1 + \dots + a_m = n$

$$(\text{note: } q_A(x) = \det(xI_n - A))$$

Definition: The minimal polynomial of T is the unique monic polynomial $m_T(x) \in \mathbb{C}[x]$ of smallest degree s.t. $m_T(T) = 0$.

"recall"

A polynomial is said to be monic if its lead coefficient is 1.

Recall: The annihilator of T

$$\text{ann}(T) = \{f(x) \in \mathbb{Q}[x] \mid f(T) = 0\}$$

← zero transformation

"this is an ideal in $\mathbb{Q}[x]$!"

Proposition: Let I be a non-zero ideal.

Then

① $I = \langle f(x) \rangle$ for $f(x)$ the unique monic polynomial of lowest degree in I

② If $g(x) \in I$, then $f(x) \mid g(x)$

So, m_T is the generator of ①.

One way to compute the minimal polynomial

Let $A \in \mathbb{Q}^{n \times n}$

Consider $I, A, A^2, \dots, A^{n^2}, \dots$ all in $\mathbb{Q}^{n \times n}$

Choose smallest m s.t. I, A, A^2, \dots, A^m of $\dim n^2$ are L.D. so

$$a_0 I + a_1 A + \dots + a_m A^m = 0.$$

Take $a_m = 1$ and

$$I, A, \dots, A^{m-1}$$

are L.I.

Then

$$m_A(x) = a_m x^m + a_{m-1} x^{m-1} + \dots + a_0$$

So

$\deg m_A \leq n^2$ and m_A exists.

Examples

$$A = J(\lambda, 3) = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$$

$$q_A(x) = (x - \lambda)^3 \leftarrow \dim G_\lambda(T)$$

$$m_A(x) =$$

Cayley - Hamilton Theorem

If $T \in \mathcal{L}(V)$, $\dim V = n$ (over \mathbb{C}), then $q_T(T) = 0$.

Notes: this means $m_T(T) \mid q_T$

Proof

Let $\lambda_1, \dots, \lambda_m$ be the distinct eigenvalues of T . Know

$$V = G_{\lambda_1}(T) \oplus G_{\lambda_2}(T) \oplus \dots \oplus G_{\lambda_m}(T)$$

and

$$a_i = \dim G_{\lambda_i}(T).$$

Also,

$$q_T = (x - \lambda_1)^{a_1} \dots (x - \lambda_m)^{a_m}$$

Need to show

$$q_T(T) = (T - \lambda_1 I)^{a_1} \dots (T - \lambda_m I)^{a_m}$$

it suffices to show, if $v_i \in G_{\lambda_i}(T)$, then $q_T(T)v_i = 0$.

$\vec{v} \in V$ can be written as $v_1 + \dots + v_m$ } from we want
 $\rightarrow q_T(T)v = 0 + \dots + 0 = 0$

Let $v_i \in G_{\lambda_i}(t) \Rightarrow v_i \in \ker(T - \lambda_i I)^{\text{big power}}$

$$\therefore v_i \in \ker(T - \lambda_i I)^{a_i}$$

take, and rearrange g_T , (proved $p(T)g(T) = g(T)p(T)$)

$$g_T(T) v_i = (T - \lambda_1 I)^{a_1} \cdots (T - \lambda_m I)^{a_m} (T - \lambda_i I) v_i = 0$$

Q.E.D

Know $m_T(x) \mid g_T(x) = (x - \lambda_1)^{a_1} \cdots (x - \lambda_m)^{a_m}$

so

$$m_T(x) = (x - \lambda_1)^{b_1} \cdots (x - \lambda_m)^{b_m} \quad 0 \leq b_i \leq a_i$$

Proposition: If $\lambda = \text{eigenvalue of } T$, then $m_T(\lambda) = 0$.

$$\Rightarrow 1 \leq b_i \leq a_i \quad \forall i$$

Proof

Let \vec{v} be an eigenvector for T , $\vec{v} \neq 0$,

$$T(\vec{v}) = \lambda \vec{v}$$

$$Iv = v \quad 0 = m_T(T)(v)$$

$$Tv = \lambda v \quad = c_0 + c_1 T + c_2 T^2 + \cdots + c_r T^r$$

$$T^2 v = \lambda^2 v \quad = c_0 I + c_1 T + c_2 T^2 + \cdots + c_r T^r$$

\vdots

Apply \vec{v}

$$c_0 Iv + c_1 Tv + c_2 T^2 v + \cdots + c_r T^r v \quad (v \neq 0)$$

$$= c_0 v + c_1 \lambda v + \cdots + c_r \lambda^r v = m_T(\lambda) v \Rightarrow \underline{m_T(\lambda) = 0}$$

Examples

$$\textcircled{1} \quad A = J(\lambda, n) = \begin{pmatrix} \lambda & & \\ & \ddots & \\ & & \lambda \end{pmatrix}$$

$$f_A(x) = (x - \lambda)^n$$

$$m_A(x) = (x - \lambda)^a, \quad 1 \leq a \leq n$$

$a??$

e.g.

$$n = 3$$

$$(x - \lambda)^a$$

$$(A - \lambda I)^a = 0$$

$$A - \lambda I = N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$N^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$N^3 = \mathbf{0}$$

$$a = 3 \quad (\text{in general } a = n \text{ if } A = J(\lambda, n))$$

$\textcircled{2}$ Suppose

$$A = A_1 \oplus A_2 = \begin{bmatrix} A_1 & \\ & A_2 \end{bmatrix}$$

then

$$f_A(x) = f_{A_1}(x) f_{A_2}(x)$$

$$m_A(x) = \text{LCM}(m_{A_1}(x), m_{A_2}(x))$$

(e.g. $\text{LCM} \begin{matrix} (x-1)^2(x-2)(x-3) \\ \downarrow \\ (x-1)^2(x-2)^4(x-3)(x-7)^2 \end{matrix}, (x+1)(x-2)^4(x-7)^2 \leftarrow \text{take highest degree in both} \right)$

then

$$f(A) = \begin{bmatrix} f(A_1) & 0 \\ 0 & f(A_2) \end{bmatrix}$$

example $J(\lambda, \underline{k})$, $\underline{k} = (k_1, \dots, k_r)$
 $k_1 \geq k_2 \geq \dots \geq k_r$

then

$$\begin{aligned} q_A(x) &= q_{J(\lambda, k_1)} \cdots q_{J(\lambda, k_r)} \\ &= (x - \lambda)^n \end{aligned}$$

$$\begin{aligned} m_A(x) &= \text{LCM}((x - \lambda)^{k_1}, (x - \lambda)^{k_2}, \dots, (x - \lambda)^{k_m}) \\ &= (x - \lambda)^{k_1} \end{aligned}$$

Example

A is 6×6

$$q_A = (x - 3)^3 (x - 7)^3$$

$$m_A = (x - 3)^2 (x - 7)$$

Possible JCF's of A ?