Why study complex analysis?

Algebra: need complex numbers to solve

$$P_2(x) = ax^2 + bx + c = 0 \rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Similarly, there are formulas for roots of cubic and quartic equations.

$$P_3(x) = 0 \quad , \quad P_4(x) = 0$$

No formula for the roots of the fifth-degree polynomial.

↳ Solutions do exist as complex numbers

$P_n(x) = 0$ has n solutions in complex numbers $\mathbb{C}$

"Fundamental Theorem of Algebra" (we'll prove it)

Exponential & Trig Functions

$e^{i\theta} = \cos\theta + j\sin\theta$ "Euler's formula"

$e^{i\pi} = -1$

$e^{i\pi} + 1 = 0$

Eulers formula is of great use for solving linear differential equations

Power Series

Complex numbers give a much deeper understanding - in Calculus,

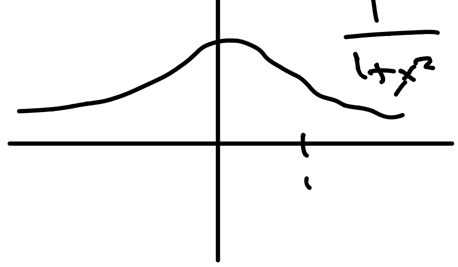
$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 \ldots$$

converges if $|x| < 1$

diverges if $|x| > 1$ ← gives silly results!

What about $x = 1$?

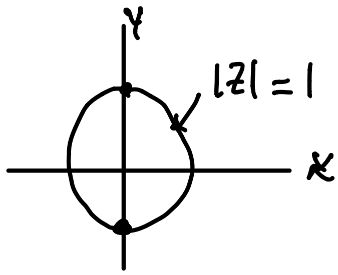
$$\frac{1}{2} = 1 - 1 + 1 - 1 + 1 \cdots$$



Something weird happens at $x = \pm 1$.

However, in the complex plane bad things happen at $\pm i$!

$$\frac{1}{1+z^2} = 1 - z^2 + z - z^6 \cdots \text{ valid in circle } |z| < 1$$



Behavior at $\pm i$ controls behavior on the real axis.

Definite Integrals

$$\int_0^{\infty} \frac{\sin x}{x}\,dx, \quad \int_{-\infty}^{+\infty} \frac{1}{1+x^4}\,dx, \quad \int_0^{\infty} \frac{\cos x}{1+x^2}\,dx$$

Will be able to do the above via residue theory / contour integrals.

Fourier & Laplace Transforms

- best to understand w/ complex numbers.

2-D potential Theory

- Solving Laplaces equation: $\nabla^2 \phi = 0$

where

$$\nabla^2 \phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \phi$$

this is what governs airflow over a wing

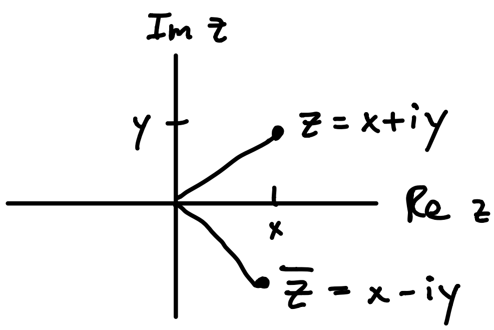
Now the course begins

Basics of i

class exercise on division

$$\frac{2+i}{3+4i} \cdot \frac{(3-4i)}{(3-4i)} = \frac{6 - 8i + 3i + 4}{9 + 16} = \frac{10 - 5i}{25}$$

$$= \frac{10}{25} - \frac{5}{25} i = \frac{2}{5} - \frac{i}{5}$$



$y = \text{Im } z$

$x = \text{Re } z$

$|z| = \text{Magnitude} = \sqrt{x^2+y^2}$

Note: Multiplying by i is a $90°$ rotation CCW

sanity check

$$z = x + iy$$

$$iz = ix + i^2 y = -y + ix$$

Class exercise

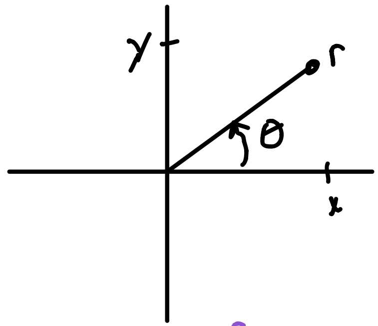
Which points in $\mathbb{C}$ satisfy

$$|z - 1| = \text{Re } z + 1$$

$$\sqrt{(x-1)^2 + y^2} = x + 1$$

$$y^2 = 4x$$

Polar Form



$r \geq 0$ by convention

$x = r\cos\theta$ $\qquad z = r(\cos\theta + i\sin\theta)$

$y = r\sin\theta$ $\qquad$ "=" $r \operatorname{cis} \theta$

θ: argument of z

$\theta = \arg z$ $\qquad$ Ex: $\arg(i) = \pi/2$

But there's ambiguity, since θ, $\theta \pm 2\pi n$, $n \in \mathbb{Z}$ refer to the same z.

The function $\arg(z)$ is "multiple-valued" in this sense.

(not really a function)

To make it single valued we would restrict θ to lie in $-\pi < \theta \leq \pi$

After restriction this is now $\operatorname{Arg}(z)$.

$\operatorname{Arg}(z)$ is <u>discontinuous</u> w/ this definition.

YUCK!

What is i^i though?

$e^{i\theta} = \cos\theta + i\sin\theta$

$e^{i\pi/2} = 0 + i$

$e^{i^2\pi/2} = i^i$

$e^{-\pi/2} = i^i$

For complete correctness however $e^{i(\pi/2 + 2\pi k)}$, $k \in \mathbb{Z}$ is the more correct answer.