Recall

Continuous rv's.

rv X; pdf $f_X(x)$; $\mathbb{P}(\{X \in V\}) = \int_V f_X(x)\,dx \quad \forall\ V \subset \mathbb{R}$

X can be uniform(a,b); exponential(λ), etc.

$$\mathbb{E}(X) = \int_{-\infty}^{+\infty} x f_X(x)\,dx$$

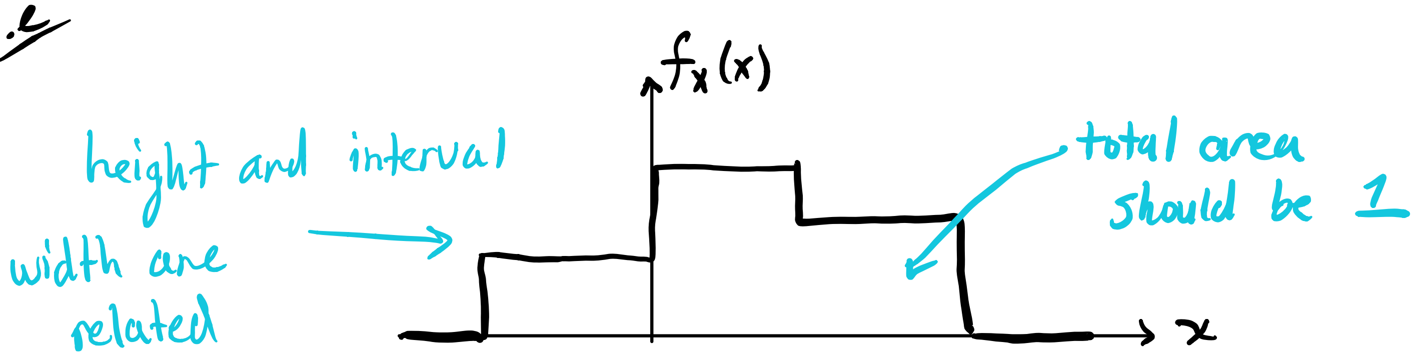
Expected Value Rule:

$$\mathbb{E}(g(X)) = \int_{-\infty}^{+\infty} g(x) f_X(x)\,dx$$

$$\mathrm{Var}(X) = \mathbb{E}\left((X - \mathbb{E}(X))^2\right)$$

Another type of pdf we encounter frequently (especially when doing conditional stuff): **piecewise uniform**

i.e



Next, define – for **ANY** rv X (discrete OR continuous) – the cumulative distribution function (cdf) by

$$F_X(x) = \mathbb{P}(\{X \leq x\}) \quad \forall\, x \in \mathbb{R}$$

Observation: If X is a continuous rv w/ pdf $f_X(x)$, then since

$$\mathbb{P}(\{X \leq x\}) = \int_{-\infty}^{x} f_X(t)\,dt$$

we have

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt \quad \text{and} \quad f_X(x) = \frac{d}{dx} F_X(x)$$

Discrete version: If X is a discrete rv w/ pmf $p_X(x)$ we have

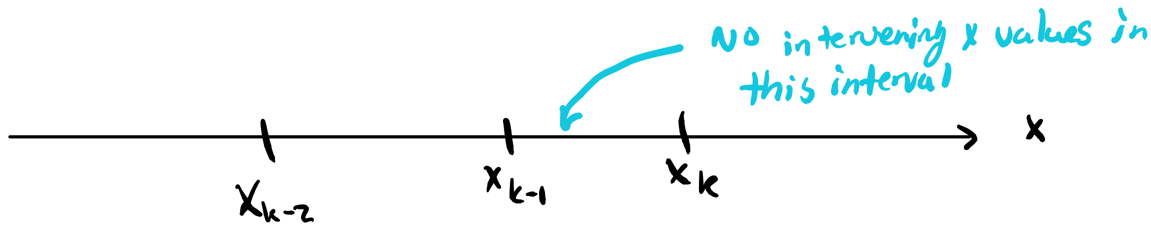
$$F_X(x) = \sum_{\{x_k : x_k \leq x\}} p_X(x_k)$$

set of all possible X-values that don't exceed x

Can invert this formula to get $p_X(x)$ in terms of $F_X(x)$:

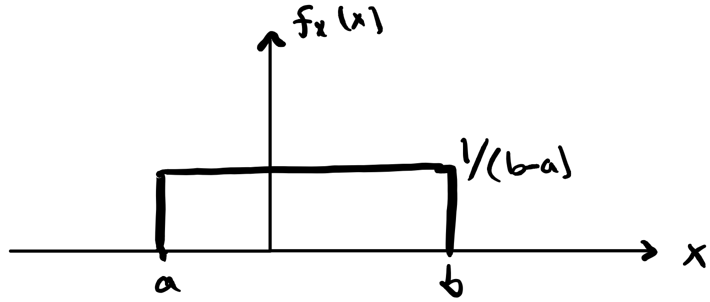
$$p_X(x_k) = F_X(x_k) - F_X(x_{k-1})$$

where x_{k-1} is the "**NEXT** largest value" of X below x_k.



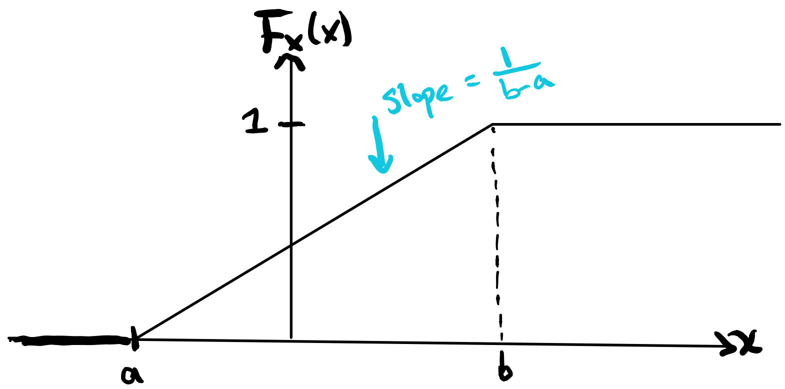
Example – $X \sim \text{Uniform}(a,b)$

$X \sim \text{Uniform}(a,b) \Rightarrow$



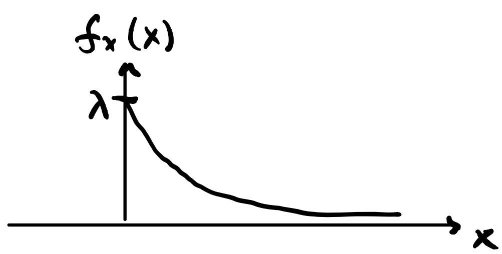
$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt = \begin{cases} 0, & x < a \\ \dfrac{x-a}{b-a}, & a \le x < b \\ 1, & b \le x \end{cases}$$

inequalities NOT too strict

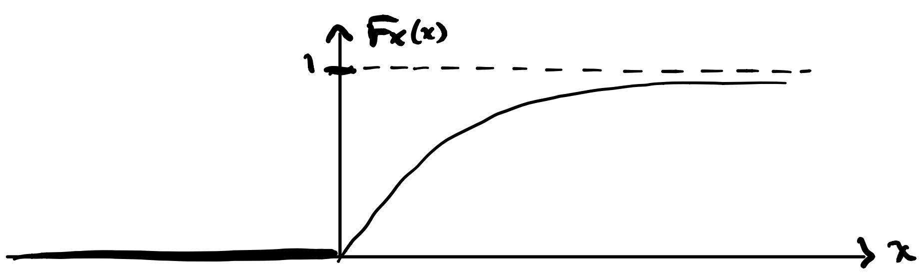


Example – $X \sim \text{exponential}(\lambda)$

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{else} \end{cases} \Rightarrow$$



$$F_X(x) = \int_{-\infty}^{x} \lambda e^{-\lambda t}\,dt = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0 \\ 0, & \text{else} \end{cases}$$



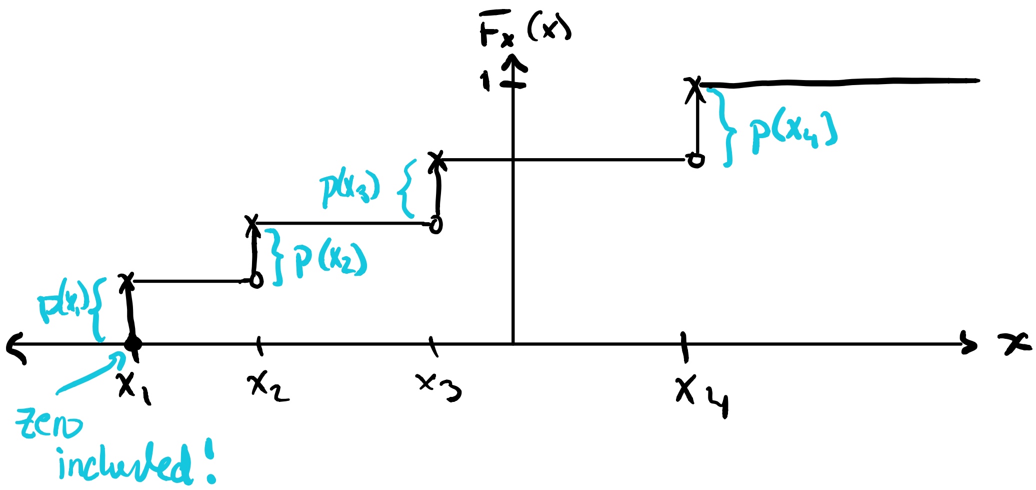
General Properties of CDFs

① $\lim_{x \to 0} F_X(x) = 0$ and $\lim_{x \to \infty} F_X(x) = 1$

② When X is a continuous rv, $F_X(x)$ is continuous in x and differentiable "almost everywhere" (corners in $F_X(x)$ correspond to jumps in $f_X(x)$)

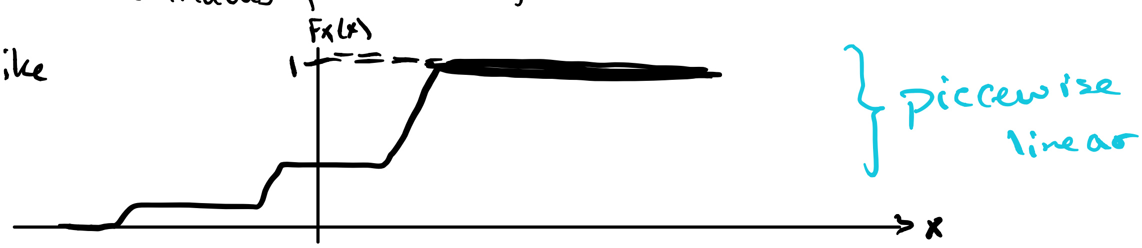
③ X is a discrete iff $f_X(x)$ is a piecewise constant.

Therefore $F_X(x)$ looks like



Caution: don't conflate piecewise constant cdf for any discrete rv w/ piecewise constant for a continuous piecewise uniform rv.

The latter looks like



④ $F_X(x)$ is **monotonically** increasing in x.

$$x_1 \leq x_2 \Rightarrow F_X(x_1) \leq F_X(x_2)$$

CDFs are useful for a variety of reasons, not least of which is getting a pdf of X by first computing $F_X(x)$, then taking d/dx.

Example - Discrete rv Version of this Phenomenon

Have 3 prelims - scores X_1, X_2, X_3 - on each you get a score of 0 to 10 (integer) distributed uniformally.

i.e $\quad p_{X_1}(k) = 1/11 \quad , \quad 0 \le k \le 10$

The events

$$\{X_1 = k\}, \{X_2 = l\}, \{X_3 = m\}$$

are independent of eachother

"Prelim score" at the end $= \max(X_1, X_2, X_3) = X$

Find pmf of X.

Given k, $\mathbb{P}(\{X \le k\}) = \mathbb{P}(\{X_1 \le k\} \cap \{X_2 \le k\} \cap \{X_3 \le k\})$

$= F_{X_1}(k) F_{X_2}(k) F_{X_3}(k)$ ← by independence and definition of cdfs

Note: $F_{X_1}(k) = \mathbb{P}(\{X_1 \le k\}) = \dfrac{(k+1)}{11} = F_{X_2}(k) = F_{X_3}(k)$

Conclusion: $\mathbb{P}(\{X \le k\}) = F_X(k) = \left(\dfrac{k+1}{11}\right)^3, \quad 0 \le k \le 10$

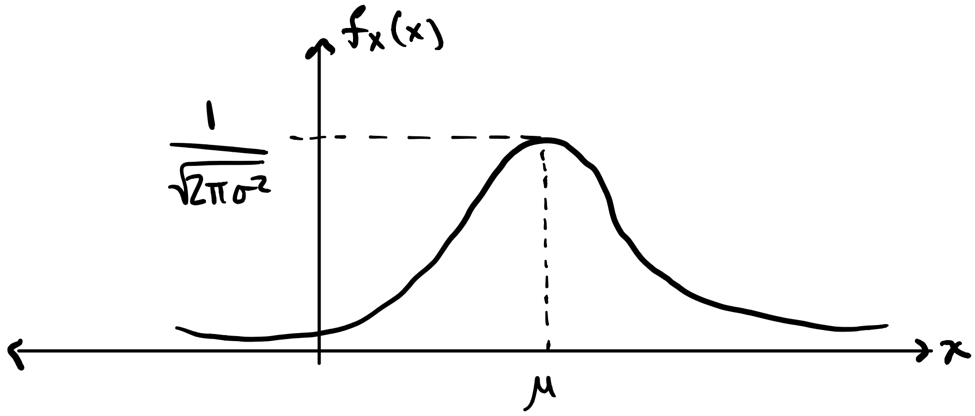
PMF is thus

$$p_X(k) = F_X(k) - F_X(k-1) = \left(\frac{k+1}{11}\right)^3 - \left(\frac{k}{11}\right)^3, \quad 1 \le k \le 10$$

Another important continuous rv: **Gaussian**

The pdf of such a rv is:

$$f_X = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$



Check that:

$$\int_{-\infty}^{+\infty} f_X(x) = 1 \qquad \text{(by "standard" math trick)}$$

"Standard" Math Trick

$$I = \int_{-\infty}^{+\infty} e^{-x^2}\,dx$$

$$I^2 = \left(\int_{-\infty}^{+\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{+\infty} e^{-y^2}\,dy\right)$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-(x^2+y^2)}\,dx\,dy \underset{\text{polarize}}{=} \int_0^{2\pi} d\theta \int_0^{\infty} r e^{-r^2}\,dr = \pi$$

$$\mathbb{E}(X) = \int_{-\infty}^{+\infty} x\, f_X(x)\, dx = \mu$$

$$\mathrm{Var}(X) = \mathbb{E}((X-\mu)^2)$$

$$= \int_{-\infty}^{+\infty} (x-\mu)^2 f_X(x)\, dx = \sigma^2 \qquad \text{(integration by parts)}$$

Why are Gaussian rvs so persuasive?

Turns out that the sum of a bunch of independent random variables, all of which have same pdf (not necessarily Gaussian) "converges" to a Gaussian rv.

CDF of a Gaussian X;

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{x} e^{\frac{-(t-\mu)^2}{2\sigma^2}}\, dt$$

Can't calculate in closed form

Standard Normal (synonym for Gaussian) pdf is

$$\frac{1}{\sqrt{2\pi}} e^{-x^2/2} \qquad \therefore\ \mu = 0,\ \sigma = 1$$

CDF of this

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt$$

book has a table of values for this

Given a Gaussian w/ mean μ and var σ^2,

$$Y = \frac{X-\mu}{\sigma}$$

has the standard normal pdf

For example, say you want $\mathbb{P}(\{X > 17\})$

$$\mathbb{P}(\{X>17\}) = \mathbb{P}(\{\sigma Y + \mu > 17\})$$
$$= \mathbb{P}\left(Y > \frac{17-\mu}{\sigma}\right) \leftarrow \text{Get from table!}$$
$$= 1 - \Phi\left(\frac{17-\mu}{\sigma}\right) = \mathbb{P}\left(\left\{Y \leq \frac{17-\mu}{\sigma}\right\}\right)$$

$$\begin{tikzcd}
f_{X,Y}(x,y) \arrow[d, "\frac{\partial}{\partial x}\frac{\partial}{\partial y}"'] \\
F_{X,Y}(x,y)
\end{tikzcd}$$