Quotients of Vector Spaces

$v+U$

Suppose $v \in V$ and $U \subseteq V$. Then $v+U$ is the subset of V defined by
$$v+U = \{v+u \mid u \in U\}$$

Quotient Space, V/U

Suppose U is a subspace of V. Then the **quotient space** V/U is the set of all affine subsets of V parallel to U.
i.e.
$$V/U = \{v+U \mid v \in V\}$$

Two Affine Subsets Parallel to U are Equal or Disjoint

Suppose U is a subspace of V and $v, w \in V$. Then the following are equivalent:

(a) $v-w \in U$
(b) $v+U = w+U$
(c) $(v+U) \cap (w+U) \neq \emptyset$

Proof → (a) ⇒ (b)
Suppose $v-w \in U$.
If $u \in U$, then
$$v+u = w+((v-w)+u) \in w+U$$
Thus $v+U \subset w+U$
If $u \in U$, then
$$w+u = v+((w-v)+u) \in v+U$$
Thus $w+U \subset v+U$ and $v+U = w+U$

(b) ⇒ (c)
$v+U = w+U \Rightarrow$ NOT disjoint. QED

(c) ⇒ (a)
Suppose $(v+U) \cap (w+U) \neq \emptyset$
Then $\exists\, u_1, u_2 \in U$ s.t. $v+u_1 = w+u_2$
and thus $v-w = u_2-u_1 \Rightarrow v-w \in U$.

Addition and Scalar Multiplication on V/U

$$(v+U)+(w+U) = (v+w)+U$$
$$\lambda(v+U) = \lambda v + U$$

for $v, w \in V$, $\lambda \in \mathbb{F}$

Quotient Space is a Vector Space

Quotient Map, π

Suppose U a subspace of V. The **quotient map** π is the **linear** map
$$\pi: V \to V/U$$
defined by
$$v \to v+U$$
for $v \in V$

Dimension of a Quotient Space

Suppose V is finite dimensional and U is a subspace of V. Then
$$\dim V/U = \dim V - \dim U$$

Proof → let $\pi: V \to V/U$
$\ker(\pi) = U$, $\operatorname{im}(\pi) = V/U$
∴ $\dim V = \dim \ker(\pi) + \dim \operatorname{im}(\pi)$
$\dim V = \dim U + \dim V/U$

$\tilde{T}$

Suppose $T \in \mathcal{L}(V,W)$. Define
$$\tilde{T}: V/\operatorname{Null}(T) \to W$$
by
$$v+\operatorname{Null}(T) \mapsto Tv$$

Null Space and Range of $\tilde{T}$

Suppose $T \in \mathcal{L}(V,W)$. Then

(a) $\tilde{T}$ is a linear map from $V/\operatorname{Null}(T) \to W$
(b) $\tilde{T}$ is injective (i.e. $\ker \tilde{T} = \{0\}$)
(c) range $\tilde{T}$ = range T
(d) $V/\operatorname{Null}(T)$ is isomorphic to range T

Proof → (a)

$\tilde{T}(v_1+\operatorname{Null} T + v_2+\operatorname{Null} T)$
$= \tilde{T}((v_1+v_2)+\operatorname{Null}(T))$
$= T(v_1+v_2)$
$= T(v_1)+T(v_2)$
$= \tilde{T}(v_1+\operatorname{Null}(T)) + \tilde{T}(v_2+\operatorname{Null}(T))$

$\tilde{T}(\lambda(v_1+\operatorname{Null}(T)))$
$= \tilde{T}(\lambda v_1+\operatorname{Null}(T))$
$= T(\lambda v_1)$
$= \lambda T(v_1)$
$= \lambda \tilde{T}(v_1+\operatorname{Null}(T))$

(b) Suppose $v \in V$ and $\tilde{T}(v+\operatorname{Null} T) = 0$
Then $Tv = 0 \Rightarrow v = 0 \Rightarrow \tilde{T}$ injective

(c)
range T = range $\tilde{T}$ by definition

(d) View $\tilde{T}: V/\ker(T) \to \operatorname{range}(T)$
→ $\tilde{T}$ an isomorphism from $V/\ker(T)$ onto range T

Linear Functional

A linear functional on V is a linear map from V to $\mathbb{F}$.
i.e. an element of $\mathcal{L}(V,\mathbb{F})$.

Dual Space, V'

The dual space of V, denoted V', is the vector space of all linear functionals on V. In other words, $V' = \mathcal{L}(V,\mathbb{F})$

$\dim V' = \dim V$

$$\dim \mathcal{L}(V,\mathbb{F}) = (\dim V)(\dim \mathbb{F})$$
$$\dim V' = (\dim V)(1)$$

Dual Basis

If $v_1,\dots,v_n$ is a basis of V, then the **dual basis** of $v_1,\dots,v_n$ is the list $\varphi_1,\dots,\varphi_n$ of elements of V', where each φ_j is the linear functional on V such that
$$\varphi_j(v_k) = \begin{cases} 1, & k=j \\ 0, & k \neq j \end{cases}$$

Note: φ_j well defined since a unique linear map exists for a basis of domain

Dual Basis is a Basis of the Dual Space

Suppose V is finite dimensional. Then the dual basis of a basis of V is a basis of V'

Proof → Suppose $v_1,\dots,v_n$ is a basis of V.
Let $\varphi_1,\dots,\varphi_n$ denote dual basis.
Suppose $a_1,\dots,a_n \in \mathbb{F}$ such that
$a_1\varphi_1+\dots+a_n\varphi_n = 0$
Now $(a_1\varphi_1+\dots+a_n\varphi_n)(v_j) = a_j$ for $j=1,\dots,n$.
Thus $a_1=\dots=a_n=0$ and $\varphi_1,\dots,\varphi_n$ is LI.
A LI list of of V' with length $\dim V'$ is a basis of V'.

Transpose, A^t

The transpose of a matrix A, denoted A^t, is the matrix obtained from A by interchanging rows and columns.
$$(A^t)_{k,j} = A_{j,k}$$

Transpose of Products of Matrices

$$(AC)^t = C^t A^t$$

Row Rank, Column Rank

Suppose A is an $\boxed{m\text{-by-}n}$ matrix with entries in $\mathbb{F}$.
- The row rank of A is the dimension of the span of the rows of A in $\mathbb{F}^{1,n}$
- The column rank of A is the dimension of the span of the columns of A is $\mathbb{F}^{m,1}$

Dimension of range T equals column rank of $\mathcal{M}(T)$.

Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V,W)$. Then $\dim \operatorname{range} T$ = column rank of $\mathcal{M}(T)$.

Row rank equals column rank

Suppose $A \in \mathbb{F}^{m,n}$. Then the row rank of A equals the column rank of A.

Proof → Define $T: \mathbb{F}^{n,1} \to \mathbb{F}^{m,1}$ by $Tx = Ax$.
Thus $\mathcal{M}(T) = A$, where $\mathcal{M}(T)$ is computed w/ respect to the standard bases of $\mathbb{F}^{n,1}$, $\mathbb{F}^{m,1}$.
Then
column rank A = column rank $\mathcal{M}(T)$
= dim range T
= dim range T'
= column rank of $\mathcal{M}(T')$
= column rank of A^t
= row rank of A

The rank of a matrix $A \in \mathbb{F}^{m,n}$ is the column rank of A.

Polynomials

$p: \mathbb{F} \to \mathbb{F}$ is called a polynomial with coefficients in $\mathbb{F}$ if there exist $a_0,\dots,a_m \in \mathbb{F}$ such that
$$p(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_m z^m \quad \forall z \in \mathbb{F}$$

If a polynomial is the zero function, then all coefficients are 0

Suppose $a_0,\dots,a_m \in \mathbb{F}$. If
$$a_0 + a_1 z + \dots + a_m z^m = 0$$
for every $z \in \mathbb{F}$, then $a_0 = \dots = a_m = 0$

Uniqueness of Coefficients for Polynomials

This is implied from result above.

Division Algorithm for Polynomials

Suppose $p, s \in \mathcal{P}(\mathbb{F})$, with $s \neq 0$. Then there exist unique polynomials $q, r \in \mathcal{P}(\mathbb{F})$ such that
$$p = sq + r$$
and $\deg r < \deg s$.

$n<m$
$\frac{p}{s} = q + \frac{r}{s}$
$\frac{x}{x^2} = 0 + \frac{x}{x^2}$

Proof → Let $n = \deg p$ and $m = \deg s$.
If $n<m$, then take $q=0$ and $r=p$ to get desired result.
Thus we can assume $n \geq m$.
Define (linear map)
$$T: \mathcal{P}_{n-m}(\mathbb{F}) \times \mathcal{P}_{m-1}(\mathbb{F}) \to \mathcal{P}_n(\mathbb{F})$$
by
$$(q,r) \mapsto sq+r$$
If $(q,r) \in \ker(T)$, then $sq + r = 0$.
Thus $(q=0, r=0)$ $\dim \ker(T) = 0$ and we have proved uniqueness.
Have
$$\dim(\mathcal{P}_{n-m}(\mathbb{F}) \times \mathcal{P}_{m-1}(\mathbb{F})) = (n-m+1)+(m-1+1) = n+1$$
Thus
$$\dim \mathcal{P}_n(\mathbb{F}) = \dim \operatorname{im} T = n+1$$
→ $\operatorname{im} T = \mathcal{P}_n(\mathbb{F})$, and hence $\exists\, q \in \mathcal{P}_{n-m}(\mathbb{F})$ and $r \in \mathcal{P}_{m-1}(\mathbb{F})$ s.t. $p = sq+r = T(q,r)$.

Zero of a Polynomial

$\lambda \in \mathbb{F}$ is called a root of $p \in \mathcal{P}(\mathbb{F})$ if $p(\lambda) = 0$

Factor

A polynomial $s \in \mathcal{P}(\mathbb{F})$ is called a factor of $p \in \mathcal{P}(\mathbb{F})$ if there exists a polynomial $q \in \mathcal{P}(\mathbb{F})$ such that $p = sq$.

Each Zero of a Polynomial Corresponds to a Degree-1 Factor

Suppose $p \in \mathcal{P}(\mathbb{F})$ and $\lambda \in \mathbb{F}$. Then $p(\lambda) = 0$ iff $\exists\, q \in \mathcal{P}(\mathbb{F})$ s.t.
$$p(z) = (z-\lambda)q(z) \quad \forall z \in \mathbb{F}$$

Proof → (⇒) $p(\lambda) = 0$
$\deg(z-\lambda) = 1$.
division algorithm → $\exists\, q \in \mathcal{P}(\mathbb{F})$ s.t.
$p(z) = (z-\lambda)q(z) + r \quad \forall z \in \mathbb{F}$.
$p(\lambda) = 0 \Rightarrow r = 0$.
$p(z) = (z-\lambda)q(z) \quad \forall z \in \mathbb{F}$.

(⇐) $p(z) = (z-\lambda)q(z)$
$p(\lambda) = (\lambda-\lambda)q(\lambda) = 0$

A polynomial has at most as many zeros as its degree.

Suppose $p \in \mathcal{P}(\mathbb{F})$ is a polynomial with degree $m \geq 0$. Then p has at most m distinct zeros in $\mathbb{F}$.

Proof → Induction on m using last result.

Fundamental Theorem of Algebra

Every nonconstant polynomial w/ complex coefficients has a zero.

Factorization of a polynomial over $\mathbb{C}$

If $p \in \mathcal{P}(\mathbb{C})$ is a nonconstant polynomial, then p has a unique factorization of the form
$$p(z) = c(z-\lambda_1)\cdots(z-\lambda_m)$$
where $c, \lambda_1, \dots, \lambda_m \in \mathbb{C}$

Polynomials with real coefficients have zeros in pairs

Suppose $p \in \mathcal{P}(\mathbb{C})$ is a polynomial with real coefficients. If $\lambda \in \mathbb{C}$ is a zero of p, then so is $\bar{\lambda}$.

Factorization of a polynomial over $\mathbb{R}$

If $p \in \mathcal{P}(\mathbb{R})$ is a nonconstant polynomial, then p has a unique factorization of the form
$$p(x) = c(x-\lambda_1)\cdots(x-\lambda_m)(x^2+b_1x+c_1)\cdots(x^2+b_Mx+c_M)$$
where $c, \lambda_1, \dots, \lambda_m, b_1, \dots, b_M, c_1, \dots, c_M \in \mathbb{R}$ with $b_j^2 < 4c_j$ for each j.

Invariant Subspaces

Suppose $T \in \mathcal{L}(V)$. $U \subseteq V$ is called invariant under T if $u \in U \Rightarrow Tu \in U$.

Take any $v \in V$ with $v \neq 0$ and let U = scalar multiples of v:
$$U = \{\lambda v : \lambda \in \mathbb{F}\} = \operatorname{span}(v)$$
Then $\dim U = 1$.
If U is invariant under $T \in \mathcal{L}(V)$, then $Tv \in U$ and hence $\exists \lambda \in \mathbb{F}$ such that
$$Tv = \lambda v.$$
Conversely, if $Tv = \lambda v$ for some $\lambda \in \mathbb{F}$, then $\operatorname{span}(v)$ is a 1-dimensional subspace of V invariant under T.

Eigenvalue

Suppose $T \in \mathcal{L}(V)$. A number $\lambda \in \mathbb{F}$ is called an eigenvalue of T if $\exists v \in V$ s.t. $v \neq 0$ and $Tv = \lambda v$.

Equivalent Conditions to Be An Eigenvalue

Suppose V is finite dimensional, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbb{F}$. Then the following are equivalent:

(a) λ is an eigenvalue of T
(b) $T-\lambda I$ is NOT injective
(c) $T-\lambda I$ is NOT surjective
(d) $T-\lambda I$ is NOT invertible

Eigenvector

Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$ is an eigenvalue of T. A vector $v \in V$ is called an **eigenvector** of T corresponding to λ if $v \neq 0$ and $Tv = \lambda v$.
Note: $v \in \ker(T-\lambda I)$

Linearly Independent Eigenvectors

Let $T \in \mathcal{L}(V)$. Suppose $\lambda_1,\dots,\lambda_m$ are distinct eigenvalues of T and $v_1,\dots,v_m$ are corresponding eigenvectors. Then $v_1,\dots,v_m$ are linearly independent.

Proof → Assume $v_1,\dots,v_m$ linearly dependent.
Then
$v_k \in \operatorname{span}(v_1,\dots,v_{k-1})$ for **smallest** k
thus
$$v_k = a_1v_1 + \dots + a_{k-1}v_{k-1} \quad (1)$$
Apply T
$$Tv_k = T(a_1v_1 + \dots + a_{k-1}v_{k-1})$$
$$\lambda_k v_k = a_1\lambda_1 v_1 + \dots + a_{k-1}\lambda_{k-1}v_{k-1} \quad (2)$$
Multiply (1) by λ_k
$$\lambda_k v_k = a_1v_1\lambda_k + \dots + a_{k-1}v_{k-1}\lambda_k$$
Subtract (2)
$$-\lambda_k v_k = a_1\lambda_1 v_1 + \dots + a_{k-1}v_{k-1}\lambda_{k-1}$$
$$\Rightarrow 0 = a_1v_1(\lambda_k-\lambda_1) + \dots + a_{k-1}v_{k-1}(\lambda_k-\lambda_{k-1})$$
But $v_1,\dots,v_{k-1}$ LI. $\Rightarrow \lambda_k = \lambda_1, \dots, \lambda_k = \lambda_{k-1}$. λ_j distinct. ↯

Number of Eigenvalues

Suppose V is finite-dimensional. Then each operator on V has at most $\dim V$ distinct eigenvalues.

Restriction and Quotient Operators

Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V invariant under T.

- The restriction operator $T|_U \in \mathcal{L}(U)$ is defined by
$$T|_U(u) = Tu \quad \text{for } u \in U$$
- The quotient operator $T/U \in \mathcal{L}(V/U)$ is defined by
$$(T/U)(v+U) = Tv+U \quad \text{for } v \in V$$

✓ Makes sense
$v+U = w+U \Rightarrow v-w \in U$
$T(v-w) \in U$
$Tv - Tw \in U$
$\Rightarrow Tv+U = Tw+U$

$p(T)$

Suppose $T \in \mathcal{L}(V)$ and $p \in \mathcal{P}(\mathbb{F})$ is a polynomial given by
$$p(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_m z^m \quad \text{for } z \in \mathbb{F}.$$
Then $p(T)$ is the operator defined by
$$p(T) = a_0 I + a_1 T + a_2 T^2 + \dots + a_m T^m$$

Product of Polynomials

If $p, q \in \mathcal{P}(\mathbb{F})$, then $pq \in \mathcal{P}(\mathbb{F})$ is the polynomial defined by
$$(pq)(z) = p(z)q(z) \quad \text{for } z \in \mathbb{F}$$

Multiplicative Properties

Suppose $p, q \in \mathcal{P}(\mathbb{F})$, $T \in \mathcal{L}(V)$.
Then
(a) $(pq)(T) = p(T)q(T)$
(b) $p(T)q(T) = q(T)p(T)$

Operators on Complex Vector Spaces Have an Eigenvalue

Every operator on a finite-dimensional, non-zero, complex vector space has an eigenvalue.

Diagonal Matrix

A diagonal matrix is a square matrix that is 0 everywhere except possibly along the diagonal

Orthonormal Basis

An orthonormal basis of V is an orthonormal list of vectors that is also a basis of V.

Eigenspace - $E(\lambda, T)$

Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$. The eigenspace of T corresponding to λ, denoted $E(\lambda,T)$, is defined by
$$E(\lambda,T) = \ker(T-\lambda I)$$
In other words, $E(\lambda,T)$ is the set of all eigenvectors of T corresponding to λ, along w/ zero vector.

Sum of Eigenspaces is a Direct Sum

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Suppose also that $\lambda_1,\dots,\lambda_m$ are distinct eigenvalues of T. Then
$$E(\lambda_1,T) + \dots + E(\lambda_m,T)$$
is a direct sum. Furthermore,
$$\dim E(\lambda_1,T) + \dots + \dim E(\lambda_m,T) \leq V$$

Diagonalizable

An operator $T \in \mathcal{L}(V)$ is called **diagonalizable** if the operator has a diagonal matrix with respect to some basis of V.

Conditions Equivalent to Diagonalizability

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Let $\lambda_1,\dots,\lambda_m$ denote the distinct eigenvalues of T. Then the following are equivalent.

(a) T is diagonalizable
(b) V has a basis consisting of eigenvectors of T
(c) $\exists$ 1-dimensional subspaces $U_1,\dots,U_n$ of V, each invariant under T, such that
$$V = U_1 \oplus \dots \oplus U_n$$
(d) $V = E(\lambda_1,T) \oplus \dots \oplus E(\lambda_m,T)$
(e) $\dim V = \dim E(\lambda_1,T) + \dots + \dim E(\lambda_m,T)$

Enough Eigenvalues Implies Diagonalizability

If $T \in \mathcal{L}(V)$ has $\dim V$ distinct eigenvalues, then T is diagonalizable.

Inner Product

An inner product on V is a function that takes each ordered pair (u,v) of elements of V to a number $\langle u,v \rangle \in \mathbb{F}$ and has the following properties:

positivity
$$\langle v,v \rangle \geq 0 \quad \forall v \in V$$
definiteness
$$\langle v,v \rangle = 0 \text{ iff } v = 0$$
additivity in the second slot
$$\langle u+v, w \rangle = \langle u+w \rangle + \langle v,w \rangle \quad \forall u,v,w \in V$$
homogeneity in the first slot
$$\langle \lambda u, v \rangle = \lambda \langle u,v \rangle \quad \forall \lambda \in \mathbb{F},\ u,v \in V$$
conjugate symmetry
$$\langle u,v \rangle = \overline{\langle v,u \rangle} \quad \forall u,v \in V$$

Inner Product Space

An inner product space is a vector space V along with an inner product on V.

Basic Properties of Inner Product

(a) For each fixed $u \in V$, the function that takes v to $\langle v,u \rangle$ is a linear map from V to $\mathbb{F}$
(b) $\langle 0,u \rangle = 0 \quad \forall u \in V$
(c) $\langle u,0 \rangle = 0 \quad \forall u \in V$
(d) $\langle u, v+w \rangle = \langle u,v \rangle + \langle u,w \rangle \quad \forall u,v,w \in V$
(e) $\langle u, \lambda v \rangle = \bar{\lambda}\langle u,v \rangle \quad \forall \lambda \in \mathbb{F}$ and $u,v \in \mathbb{F}$

Norm, $\|v\|$

For every $v \in V$, the **norm** of v, denoted $\|v\|$, is defined by
$$\|v\| = \sqrt{\langle v,v \rangle}$$

Basic Properties of Norm

Suppose $v \in V$.
(a) $\|v\| = 0$ iff $v = 0$
(b) $\|\lambda v\| = |\lambda|\,\|v\| \quad \forall \lambda \in \mathbb{F}$

Orthogonal

Two vectors u, v are called orthogonal if $\langle u,v \rangle = 0$.

Orthogonality and 0

- 0 is orthogonal to every vector in V
- 0 is the only vector in V orthogonal to itself

Pythagorean Theorem

u, v orthogonal vectors in V. Then
$$\|u+v\|^2 = \|u\|^2 + \|v\|^2$$

An Orthogonal Decomposition

$u, v \in V$, $v \neq 0$. Set
$$c = \frac{\langle u,v \rangle}{\|v\|^2}$$
and
$$w = u - \frac{\langle u,v \rangle}{\|v\|^2} v.$$
Then $\langle w,v \rangle = 0$ and $u = cv + w$

Cauchy-Schwarz Inequality

Suppose $u, v \in V$. Then
$$|\langle u,v \rangle| \leq \|u\|\,\|v\|$$
equality iff $u = cv$, $c \in \mathbb{F}$

Triangle Inequality

$u, v \in V$
$$\|u+v\| \leq \|u\| + \|v\|$$
equality iff $u = cv$, $c>0$

Parallelogram Equality

Suppose $u, v \in V$. Then
$$\|u+v\|^2 + \|u-v\|^2 = 2(\|u\|^2 + \|v\|^2)$$

Orthonormal

A list of vectors is called orthonormal if each vector in the list has norm 1 and is orthogonal to all other vectors in the list.

The norm of an orthonormal linear combination

If $e_1,\dots,e_m$ is an orthonormal list of vectors in V, then
$$\|a_1e_1 + \dots + a_me_m\|^2 = |a_1|^2 + \dots + |a_m|^2$$
$\forall a_1,\dots,a_m \in \mathbb{F}$

An orthonormal list is linearly independent

Proof → $e_1,\dots,e_m$ orthonormal list of vectors in V
$a_1,\dots,a_m \in \mathbb{F}$ s.t.
$$a_1e_1 + \dots + a_ne_m = 0$$
then $|a_1|^2 + \dots + |a_m|^2 = 0 \Rightarrow$ all $a_j = 0$
∴ $e_1,\dots,e_m$ LI

$U \subset V$ a subspace if
(i) $0_V \in U$
(ii) $u_1+u_2 \in U \quad \forall u_1,u_2 \in U$
(iii) $\lambda u_1 \in U$, $u_1 \in U$, $\lambda \in \mathbb{F}$

$T: V \to W$
$\dim V > \dim W \to$ NOT Inj.
$\dim W > \dim V \to$ NOT surj.

injective iff $\ker = \{0\}$
surjective iff $W = \operatorname{range} T$

$\dim(\mathcal{L}(V,W)) = (\dim V)(\dim W)$

$U_1, U_2 \subset V \to \dim(U_1+U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2)$

A real positive matrix is positive definite if for all $x \neq 0$ $x^TAx > 0$

Fundamental Theorem of Linear Maps
$T: V \to W$ $\dim V = \dim \operatorname{im}(T) + \dim \ker(T)$

An orthonormal list of the right length is an orthonormal basis

Writing a vector as a linear combination of orthonormal basis

Suppose $e_1,\dots,e_n$ is an orthonormal basis of V and $v \in V$. Then

$$v = \langle v, e_1\rangle e_1 + \cdots + \langle v, e_n\rangle e_n$$

and

$$\|v\|^2 = |\langle v, e_1\rangle|^2 + \cdots + |\langle v, e_n\rangle|^2$$

Gram-Schmidt Procedure

Suppose $v_1,\dots,v_m$ is LI list in V. Let $e_1 = \frac{v_1}{\|v_1\|}$. For $j = 2,\dots,m$, define e_j inductively by

$$e_j = \frac{v_j - \langle v_{j-1}, e_1\rangle e_1 - \cdots - \langle v_j, e_{j-1}\rangle e_{j-1}}{\|v_j - \langle v_{j-1}, e_1\rangle e_1 - \cdots - \langle v_j, e_{j-1}\rangle e_{j-1}\|}$$

Then $e_1,\dots,e_m$ is an orthonormal list of vectors in V s.t.

$$\operatorname{span}(v_1,\dots,v_j) = \operatorname{span}(e_1,\dots,e_j)$$

for $j = 1,\dots,m$.

Every finite-dimensional vector space has an orthonormal basis

Orthonormal list extends to orthonormal basis

Riesz Representation Theorem

Suppose V is finite dimensional and $\varphi \in \mathcal{L}(V)$. Then $\exists$ a unique vector $u \in V$ such that

$$\varphi(v) = \langle v, u\rangle \qquad \forall v \in V$$

Proof

Let $e_1,\dots,e_n$ be an orthonormal basis for V. Then

$\exists$

$$\begin{aligned}\varphi(v) &= \varphi(\langle v, e_1\rangle e_1 + \cdots + \langle v, e_n\rangle e_n) \\ &= \varphi(e_1)\langle v, e_1\rangle + \cdots + \varphi(e_n)\langle v, e_n\rangle \\ &= \langle v, \overline{\varphi(e_1)}e_1\rangle + \cdots + \langle v, \overline{\varphi(e_n)}e_n\rangle \\ &= \langle v, \underbrace{\overline{\varphi(e_1)}e_1 + \cdots + \overline{\varphi(e_n)}e_n}_{u}\rangle\end{aligned}$$

!

Suppose $u_1, u_2 \in V$ such that

$$\varphi(v) = \langle v, u_1\rangle = \langle v, u_2\rangle \qquad \forall v \in V.$$

Then

$$0 = \langle v, u_1 - u_2\rangle \to u_1 - u_2 = 0 \to u_1 = u_2$$

Orthogonal Complement, $U^\perp$

If $U \subseteq V$, then the **orthogonal** complement of U, denoted $U^\perp$, is set of all vectors in V that are orthogonal to every vector in U

$$U^\perp = \{v \in V \mid \langle v, u\rangle = 0 \ \forall u \in U\}$$

Properties of Orthogonal Complement

(a) If $U \subseteq V$, then $U^\perp \subseteq V$

(b) $\{0\}^\perp = V$

(c) $V^\perp = \{0\}$

(d) If $U \subseteq V$, then $U \cap U^\perp \subset \{0\}$

(e) If U, W subsets of V and $U \subset W$, then $W^\perp \subset U^\perp$

Direct Sum of Subspace and Orthogonal Complement

Suppose U is a finite-dimensional subspace of V. Then

$$V = U \oplus U^\perp \quad \Rightarrow \dim U^\perp = \dim V - \dim U$$

Proof $v = u + v - u \qquad u = \langle v, e_1\rangle e_1 + \cdots + \langle v, e_m\rangle e_m$

Orthogonal Complement of Orthogonal Complement

$U \subseteq V$, $\dim U = m < \infty$

$U \quad (U^\perp)^\perp$

Orthogonal Projection, P_U

Suppose U is a finite-dimensional subspace of V. The **orthogonal projection** of V onto U is the operator $P_U \in \mathcal{L}(V)$ defined as follows: For $v \in V$, write $v = u + w$, where $u \in U$ and $w \in U^\perp$. Then $P_U v = u$

Properties of the Orthogonal Projection, P_U

Suppose U is a finite-dimensional subspace of V and $v \in V$. Then

(a) $P_U \in \mathcal{L}(V)$ (c) $P_U w = 0 \quad \forall w \in U^\perp$

(b) $P_U u = u \quad \forall u \in U$ (d) range $P_U = U$

(e) null $P_U = U^\perp$ (g) $P_U^2 = P_U$

(f) $v - P_U v \in U^\perp$ (h) $\|P_U v\| \le \|v\|$

(i) For every orthonormal basis $e_1,\dots,e_m$ of U,

$$P_U v = \langle v, e_1\rangle e_1 + \cdots + \langle v, e_m\rangle e_m$$

Minimizing the Distance to a Subspace

Suppose U is a finite-dimensional subspace of V, $v \in V$, $u \in U$. Then

$$\|v - P_U v\| \le \|v - u\|$$

Furthermore, the inequality above is an equality iff $u = P_U v$.

Proof

$$\begin{aligned}\|v - P_U v\|^2 &\le \|v - P_U v\|^2 + \|P_U v - u\|^2 \\ &= \|v - P_U v + P_U v - u\|^2 \\ &= \|v - u\|^2\end{aligned}$$

Adjoint

Suppose $T \in \mathcal{L}(V,W)$. The adjoint of T is the function $T^*: W \to V$ such that

$$\langle Tv, w\rangle = \langle v, T^* w\rangle$$

$\forall v \in V, w \in W$.

Example

Define $T: \mathbb{R}^3 \to \mathbb{R}^2$ by $(x_1, x_2, x_3) \mapsto (x_2 + 3x_3, 2x_1)$.

Find a formula for $T^*: \mathbb{R}^2 \to \mathbb{R}^3$.

Fix $(y_1, y_2) \in \mathbb{R}^2$. Then for every $(x_1, x_2, x_3) \in \mathbb{R}^3$ we have

$$\begin{aligned}\langle (x_1,x_2,x_3), T^*(y_1,y_2)\rangle &= \langle T(x_1,x_2,x_3), (y_1,y_2)\rangle \\ &= \langle (x_1,x_2,x_3), (2y_2, y_1, 3y_1)\rangle\end{aligned}$$

Thus $T^*(y_1,y_2) = (2y_2, y_1, 3y_1)$

The Adjoint is a Linear Map

If $T \in \mathcal{L}(V,W)$, then $T^* \in \mathcal{L}(W,V)$.

Proof

Suppose $T \in \mathcal{L}(V,W)$, $w_1, w_2 \in W$, $\lambda_1, \lambda_2 \in \mathbb{R}$.

Then $\forall v \in V$,

$$\begin{aligned}\langle v, T^*(\lambda_1 w_1 + \lambda_2 w_2)\rangle &= \langle Tv, \lambda_1 w_1 + \lambda_2 w_2\rangle \\ &= \langle Tv, \lambda_1 w_1\rangle + \langle Tv, \lambda_2 w_2\rangle = \bar\lambda_1\langle Tv, w_1\rangle + \bar\lambda_2\langle Tv, w_2\rangle \\ &= \bar\lambda_1\langle v, T^* w_1\rangle + \bar\lambda_2\langle v, T^* w_2\rangle = \langle v, \lambda_1 T^* w_1 + \lambda_2 T^* w_2\rangle\end{aligned}$$

Properties of the Adjoint

(a) $(S+T)^* = S^* + T^* \quad \forall S, T \in \mathcal{L}(V,W)$

(b) $(\lambda T)^* = \bar\lambda T^* \quad \forall \lambda \in \mathbb{F}, T \in \mathcal{L}(V,W)$

(c) $(T^*)^* = T \quad \forall T \in \mathcal{L}(V,W)$

(d) $I^* = I$, I is the identity operator on V

(e) $(ST)^* = T^* S^* \quad \forall T \in \mathcal{L}(V,W)$ and $S \in \mathcal{L}(W,U)$

(here, U an inner product space over $\mathbb{F}$)

Proof (all similar)

(e) For $T \in \mathcal{L}(V,W)$, $S \in \mathcal{L}(W,U)$; if $v \in V$, $u \in U$

$$\langle v, (ST)^* u\rangle = \langle (ST)v, u\rangle = \langle S[Tv], u\rangle = \langle Tv, S^* u\rangle = \langle v, T^* S^* u\rangle$$

Null Space and Range of T^*

Suppose $T \in \mathcal{L}(V,W)$. Then

(a) $\ker T^* = (\operatorname{range} T)^\perp$

(b) range $T^* = (\ker T)^\perp$

(c) $\ker T = (\operatorname{range} T^*)^\perp$

(d) range $T = (\ker T^*)^\perp$

Matrix of T^*

Suppose $\mathcal{A}$ is an orthonormal basis for V and β is an orthonormal basis for W. Then if $A = [T]_{\beta \leftarrow \mathcal{A}} \rightarrow [T^*]_{\mathcal{A} \leftarrow \beta} = A^T$

[illegible]

An operator $T \in \mathcal{L}(V)$ is called self-adjoint if $T = T^*$. In other words, $T \in \mathcal{L}(V)$ is self-adjoint if and only if

$$\langle Tv, w\rangle = \langle v, Tw\rangle \qquad \forall v, w \in \mathbb{F}$$

Eigenvalues of Self-Adjoint Operators are Real

Every eigenvalue of a self-adjoint operator is real.

Proof

$$\lambda\|v\|^2 = \langle \lambda v, v\rangle = \langle Tv, v\rangle = \langle v, Tv\rangle = \langle v, \lambda v\rangle = \bar\lambda\|v\|^2$$

$$\Rightarrow \lambda = \bar\lambda \Rightarrow \lambda \in \mathbb{R}$$

Self-adjoint Operators Have Eigenvalues

Suppose $V \ne \{0\}$ and $T \in \mathcal{L}(V)$ is a self-adjoint operator. Then T has an eigenvalue.

Self-adjoint Operators and Invariant Subspaces

Suppose $T \in \mathcal{L}(V)$ is self-adjoint and U is a subspace of V that is invariant under T. Then

(a) $U^\perp$ is invariant under T

(b) $T|_U$ is self-adjoint

(c) $T|_{U^\perp}$ is self-adjoint

Real Spectral Theorem

Suppose $\mathbb{F} = \mathbb{R}$ and $T \in \mathcal{L}(V)$. Then the following are equivalent:

(a) T is self-adjoint

(b) V has an orthonormal basis consisting of eigenvectors of T

(c) T has a diagonal matrix with respect to some orthonormal basis of V

($\exists$ orthogonal matrix Q, diagonal matrix Σ s.t. $[T] = Q\Sigma Q^T$)

Sequence of Increasing Null Spaces

Suppose $T \in \mathcal{L}(V)$. Then

$$\{0\} = \ker T^0 \subset \ker T^1 \subset \cdots \subset \ker T^k \subset \ker T^{k+1} \subset \cdots$$

Proof

Suppose $k \in \mathbb{Z}_{\ge 1}$ and $v \in \ker T^k$. Then $T(T^k v) = T^{k+1} v = 0$

$\to v \in \ker T^{k+1} \to \ker T^k \subset \ker T^{k+1}$

Equality in the Sequence of Null Spaces

Suppose $T \in \mathcal{L}(V)$. Suppose m is a nonnegative integer such that $\ker T^m = \ker T^{m+1}$. Then

$$\ker T^m = \ker T^{m+1} = \ker T^{m+2} = \cdots$$

Proof

Let $k \in \mathbb{Z}_{\ge 1}$. Want

$$\ker T^{m+k} = \ker T^{m+k+1}$$

Know $\ker T^{m+k} \subset \ker T^{m+k+1}$

Need $\ker T^{m+k+1} \subset \ker T^m$

Suppose $v \in \ker T^{m+k+1}$.

$$T^{m+1}(T^k v) = T^{m+1+k} v = 0$$

$$\to T^k v \in \ker T^{m+1} = \ker T^m$$

$$\Rightarrow v \in \ker T^{m+k}$$

So $\ker T^{m+k+1} \subset \ker T^m$ as desired.

Null Spaces Stop Growing

Suppose $T \in \mathcal{L}(V)$. Let $n = \dim V$. Then

$$\ker T^n = \ker T^{n+1} = \ker T^{n+2} = \ker T^{n+3} = \cdots$$

Proof

Only need $\ker T^n = \ker T^{n+1}$.

Suppose $\ker T^n \ne \ker T^{n+1}$.

Then $\{0\} = \ker T^0 \subsetneq \ker T \subsetneq \subsetneq \cdots \subsetneq \ker T^n \subsetneq \ker T^{n+1}$

At each strict inclusion, dim ker increases by at least one. This give $\dim\ker T^{n+1} \ge n+1$. !

V is the Direct Sum of $\ker T^{\dim V}$ and $\operatorname{im} T^{\dim V}$

Suppose $T \in \mathcal{L}(V)$. Let $n = \dim V$. Then

$$V = \ker T^n \oplus \operatorname{im} T^n.$$

Proof

First show $\ker T^n \cap \operatorname{im} T^n = \{0\}$

Suppose $v \in \ker T^n \cap \operatorname{im} T^n$

then $T^n v = 0$ and $\exists u \in V$ s.t. $T^n v = u$

Applying T^n, $T^{2n} v = T^n u$

$$T^{2n} = T^n \to T^{2n} v = 0 \to T^n u = 0 \to v = 0$$

Generalized Eigenvector

Suppose $T \in \mathcal{L}(V)$ and λ an eigenvalue of T. A vector $v \in V$ is called a **generalized** eigenvector of T corresponding to λ if $v \ne 0$ and

$$(T - \lambda I)^j v = 0$$

for some positive integer j.

Generalized Eigenspace

Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$. The **generalized** eigenspace of T corresponding to λ, denoted $G(\lambda, T)$, is defined to be the set of all generalized eigenvectors of T corresponding to λ, along with the zero vector.

Note: $E(\lambda, T) \subset G(\lambda, T)$

Description of Generalized Eigenspaces

Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$. Then $G(\lambda, T) = \ker(T - \lambda I)^{\dim V}$

Proof

Suppose $v \in \ker(T - \lambda I)^{\dim V}$. Then $v \in G(\lambda, T) \Rightarrow \ker(T - \lambda I)^{\dim V} \subset G(\lambda, T)$

Conversely, suppose $v \in G(\lambda, T)$. Then $\exists j \in \mathbb{Z}_+$ s.t.

$$v \in \ker(T - \lambda I)^j \subset \ker(T - \lambda I)^{\dim V}$$

$$\Rightarrow G(\lambda, T) \subset \ker(T - \lambda I)^{\dim V}$$

Nilpotent

An operator is called **nilpotent** if some power of it equals 0.

Nilpotent Operator Raised to Dimension of Domain is 0

Suppose $N \in \mathcal{L}(V)$ is nilpotent. Then $N^{\dim V} = 0$.

Proof

Because N is nilpotent, $G(0, N) = V$.

Then $\ker N^{\dim V} = V$

Matrix of a Nilpotent Operator

Suppose N is a nilpotent operator on V. Then there is a basis of V with respect to which the matrix N has the form

$$\begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix}$$

The Null Space and Range of $p(T)$ are Invariant Under T

Suppose $T \in \mathcal{L}(V)$ and $p \in \mathcal{P}(\mathbb{F})$. Then $\ker p(T)$ and range $p(T)$ are invariant under T.

Description of Operators on Complex Vector Spaces

Suppose V is a complex vector space and $T \in \mathcal{L}(V)$. Let $\lambda_1,\dots,\lambda_m$ be the distinct eigenvalues of T. Then

(a) $V = G(\lambda_1, T) \oplus \cdots \oplus G(\lambda_m, T)$

(b) each $G(\lambda_j, T)$ in invariant under T

(c) each $(T - \lambda_j I)|_{G(\lambda_j, T)}$ is nilpotent

A Basis of Generalized Eigenvectors

Suppose V is a complex vector space and $T \in \mathcal{L}(V)$. Then there is a basis of V consisting of generalized eigenvectors of T.

Multiplicity

Suppose $T \in \mathcal{L}(V)$. The multiplicity of an eigenvalue λ of T is defined to be the dimension of the corresponding generalized eigenspace $G(\lambda, T)$.

Sum of Multiplicities Equals $\dim V$

Suppose V is a complex vector space and $T \in \mathcal{L}(V)$. Then the sum of all the eigenvalues of T equals $\dim V$.

Note: Geometric Multiplicity $= \dim E(\lambda, T)$

Algebraic Multiplicity $= \dim G(\lambda, T)$

Characteristic Polynomial

Suppose V is a complex vector space and $T \in \mathcal{L}(V)$. Let $\lambda_1,\dots,\lambda_m$ denote the distinct eigenvalues of T, with multiplicities $d_1,\dots,d_m$. The polynomial

$$(z - \lambda_1)^{d_1} \cdots (z - \lambda_m)^{d_m}$$

is called the characteristic polynomial of T.

Degrees and Zeros of Characteristic Polynomial

Suppose V is a complex vector space and $T \in \mathcal{L}(V)$. Then

(a) the characteristic polynomial of T has degree $\dim V$

(b) the zeros of the characteristic polynomial of T are the eigenvalues of T

Cayley Hamilton Theorem

Suppose V is a complex vector space and $T \in \mathcal{L}(V)$. Let q denote the characteristic polynomial of T. Then

$$q(T) = 0$$

Proof

Let $\lambda_1,\dots,\lambda_m$ be the distinct eigenvalues of T with $d_1,\dots,d_m$ algebraic multiplicity each

For each $j \in \{1,\dots,m\}$ we have $(T - \lambda_j I)|_{G(\lambda_j, T)}$ is nilpotent.

Thus

$$(T - \lambda_j I)^{d_j}|_{G(\lambda_j, T)} = 0$$

Every $v \in V$ can be written as

$$v = g_1 + \cdots + g_m \ ; \ g_j \in G(\lambda_j, T)$$

Suffices to show

$$q(T)|_{G(\lambda_j, T)} = 0 \quad \forall j$$

Fix $j \in \{1,\dots,m\}$. Then

$$q(T) = (T - \lambda_1 I)^{d_1} \cdots (T - \lambda_m I)^{d_m}$$

All operators on left-hand side commute.

Thus can rearrange to

$$q(T) = (T - \lambda_1 I) \cdots (T - \lambda_m I)^{d_m} (T - \lambda_j I)^{d_j}$$

Since

$$(T - \lambda_j I)|_{G(\lambda_j, T)} = 0 \quad \forall j$$

then

$$q(T)|_{G(\lambda_j, T)} = 0 \quad \forall j$$

Monic Polynomial

A monic polynomial is a polynomial whose highest-degree coefficient equals 1.

Minimal Polynomial

Suppose $T \in \mathcal{L}(V)$. Then the minimal polynomial of T is the unique monic polynomial p of smallest degree such that $p(T) = 0$.

Note: has at MOST degree $\dim V$

Characteristic Polynomial is a Multiple of Minimal Polynomial

Eigenvalues are the Zeros of the Minimal Polynomial

Let $T \in \mathcal{L}(V)$. Then the zeros of the minimal polynomial of T are precisely the eigenvalues of T.

Proof

Let

$$p(z) = a_0 + a_1 z + \cdots + a_{m-1} z^{m-1} + z^m$$

be the minimal polynomial of T.

Suppose λ is a zero of p. Then

$$p(z) = (z - \lambda) q(z)$$

where q is a monic polynomial with coefficients in $\mathbb{F}$ and $\deg q = \deg p - 1$. Since $p(T) = 0$, we have

$$0 = (T - \lambda I)(q(T)v)$$

Then $\exists v \in V$ such that $q(T)v \ne 0$, and

$$T(q(T)v) = \lambda(q(T)v)$$

This gives λ is an eigenvalue of T.

Suppose now $\lambda \in \mathbb{F}$ is an eigenvalue of T.

Then $\exists v \ne 0 \in V$ such that

$$Tv = \lambda v.$$

Applying T to both sides repeatedly gives

$$T^j v = \lambda^j v \qquad j = 1, 2, \dots$$

Thus

$$\begin{aligned}0 = p(T)v &= (a_0 I + a_1 T + \cdots + a_{m-1} T^{m-1} + T^m)v \\ &= (a_0 + a_1\lambda + \cdots + a_{m-1}\lambda^{m-1} + \lambda^m)v \\ &= p(\lambda)v \Rightarrow p(\lambda) = 0\end{aligned}$$

Q.E.D

SVD

A $m \times n$ matrix over $\mathbb{R}$, rank r

$A^T A$ $n \times n$ symmetric, rank r.

Eigenvalues $\lambda_1 \ge \lambda_2 \ge \cdots \lambda_r \ge \lambda_{r+1} = 0 = \cdots = \lambda_n$

Singular values of A:

① $\sigma_1 = \sqrt{\lambda_1} \ge \sigma_2 = \sqrt{\lambda_2} \ge \cdots \ge \sigma_r = \sqrt{\lambda_r}$

② Let $\vec v_1,\dots,\vec v_n$ be an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of $A^T A$.

So

$$A^T A \vec v_i = \lambda_i \vec v_i, \quad i = 1,\dots,n$$

③ Let

$$\vec u_i = \frac{1}{\sigma_i} A \vec v_i, \quad i = 1,\dots,r$$

NOTE

(note $\vec u_i \in \mathbb{R}^m$)

These are orthonormal, and extend to orthonormal basis $\vec u_1,\dots,\vec u_m$ of $\mathbb{R}^m$.

An SVD of A is a factorization

$$A = U\Sigma V^T$$

such that

① U, V are orthogonal matrices

② $\Sigma = \begin{pmatrix} \sigma_1 & & & & \\ & \sigma_2 & & & \\ & & \ddots & & \\ & & & \sigma_r & \\ & & & & 0 \ddots 0 \end{pmatrix}$,

$$\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_r > 0$$

Jordan Stuff

N: $J(0; k_1,\dots,k_r) = J(0, k_1) \oplus \cdots \oplus J(0, k_r)$

$k_1 \ge k_2 \ge \cdots \ge k_r > 0$

$\dim\ker N = r$, $\dim\operatorname{im} N = \sum k_i - r$

index of Nilpotence, k_1

For each Jordan block $J(0, k_i)$,

$$\operatorname{rank}(J(0, k_i)^m) = \max\{0, k_i - m\}$$

Thus $\operatorname{rank} N^m = \sum_{i=1}^{r} \max\{0, k_i - m\}$

and

$$\dim\ker N = n - \operatorname{rank} N^m$$

Knowing $\dim\ker N^m$ $\forall m$ determines $(k_1,\dots,k_r)$

Assume $d_m = \dim\ker N^m$.

Then $\Delta_m = d_m - d_{m-1}$ is $\#J(0,k)$ w/ $k \ge m$

So

$$\Delta_m - \Delta_{m-1} = 2d_m - d_{m+1} - d_{m-1}$$

is number of $J(0,k)$ in N with $k = N$.

definition: Let $\lambda \in \mathbb{C}$, $n \in \mathbb{Z}_{\ge 1}$.

Define

$$J(\lambda, n) = \begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix} \quad (n \times n)$$

$J(\lambda, n)$ is called a Jordan Block.

$$J(\lambda, 4) = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$$

Note: if $A = J(\lambda, n)$, then λ is the only eigenvalue of A, and

$$V = \mathbb{C}^n = \ker(A - \lambda I)^n$$

so $N = A - \lambda I$ is nilpotent

Theorem: If $\dim V = n < \infty$, V a v.s over $\mathbb{C}$, and $T \in \mathcal{L}(V)$.

Then $\exists$ Jordan basis of T.

Corollary: Every $A \in \mathbb{C}^{n \times n}$ is similar to a direct sum of Jordan blocks.

$A = \begin{bmatrix} A_1 & & \\ & A_2 & \\ & & \ddots A_r \end{bmatrix}$ A_i $n_i \times n_i$, A $n \times n$, $n = n_1 + \cdots + n_r$

Remark: If $A = A_1 \oplus \cdots \oplus A_r$ and π is a permutation of $1,\dots,r$ then $B = A_{\pi(1)} \oplus A_{\pi(2)} \oplus \cdots \oplus A_{\pi(r)}$ is similar to A.

Definition: $(k_1,\dots,k_r)$ is called a partition of $n \in \mathbb{Z}_{>0}$ if

ⓐ $k_1 + \cdots + k_r = n$

ⓑ $k_1 \ge k_2 \ge \cdots \ge k_r > 0$

Definition: If $\underline{k} = (k_1,\dots,k_r)$ is a partition of n, let $J(\lambda, \underline{k}) = J(\lambda, k_1) \oplus \cdots \oplus J(\lambda, k_r)$

Example: Suppose A is 8×8 with eigenvalues

Always $\leq$

$\lambda_1 = 7$, $\dim E_{\lambda_1} = 1$, $\dim G_{\lambda_1} = 3$

$\lambda_2 = 15$, $\dim E_{\lambda_2} = 1$, $\dim G_{\lambda_2} = 1$

$\lambda_3 = -1$, $\dim E_{\lambda_3} = 3$, $\dim G_{\lambda_3} = 4$

Find all possible JCF's of A

<u>Sol'n</u>

3x3 only has eigenvalue $\lambda = \lambda_1 = 7$

$$A \sim A_1 \oplus A_2^{1\times 1} \oplus A_3^{4 \times 4}$$

where

$$A \sim \underset{A_1}{J(\lambda, \underline{k}^{(1)})} \oplus \underset{A_2}{J(\lambda, \underline{k}^{(2)})} \oplus \underset{A_3}{J(\lambda, k^{(3)})}$$

$\underline{k}^{(1)}$ is a partition of 3 $\rightarrow$ 111, 21, 3

$\underline{k}^{(2)}$ is a partition of 1 $\rightarrow$ 1

$\underline{k}^{(3)}$ is a partition of 4 $\rightarrow$ 1111, 211, 22, 31, 4

NOTE: 4, 2, 2, 3, 1 $\}$ $\dim E_{\lambda_3}$

minimal polynomials is

$$m_T(x) = (x-7)(x-15)(x+1)^3$$