

Recall: Discrete r.v's

Given X w/ pmf $P_X(x)$, expected value/mean of X is

$$E(X) = \sum_{x \in X} x P_X(x)$$

Sum over all x in X

Expected Value Rule

If $Y = g(X)$, then

$$E(Y) = \sum_x g(x) P_X(x)$$

Variance of X

$$\text{Var}(X) = E((X - E(X))^2)$$

$$\sigma_X = \sqrt{\text{Var}(X)} \quad \leftarrow \text{Standard deviation}$$

- Some:
- X is Bernoulli(p) $\Rightarrow E(X) = p$; $\text{Var}(X) = p(1-p)$
 - X is discrete uniform on $a \leq k \leq b \Rightarrow E(X) = \frac{a+b}{2}$
 - X is Poisson(λ) $\Rightarrow E(X) = \lambda$; $\text{Var}(X) = \lambda$

↑ Proof

Proof NOT shown

$$P(X=k) = \frac{e^{-\lambda} \lambda^k}{k!}, \quad k \in \mathbb{N}$$

$$E(X) = \sum_k k P_X(k) = \sum_{k=1}^{\infty} e^{-\lambda} \frac{\lambda^k}{(k-1)!} = \lambda \sum_{m=0}^{\infty} e^{-\lambda} \frac{\lambda^m}{m!} = \lambda \cdot 1 = \lambda$$

- X is Geometric(p) : $P_X(k) = p(1-p)^{k-1}$ $1 \leq k < \infty$

Compute useful math using useful math trick

$$\begin{aligned} \text{IE}(X) &= \sum_k k p_X(k) = \sum_{k=1}^{\infty} k p(1-p)^{k-1} \\ &= p \sum_{k=1}^{\infty} k(1-p)^{k-1} = p \left(-\frac{d}{dp} \left[\sum_{k=0}^{\infty} (1-p)^k \right] \right) \\ &= p \left(-\frac{d}{dp} \left[\frac{1}{1-(1-p)} \right] \right) = p \left(-\frac{d}{dp} \left[\frac{1}{p} \right] \right) = \frac{1}{p} \end{aligned}$$

"Forget about Variance for now"

Super Useful Formula:

$$\text{Var}(X) = \text{IE}(X^2) - (\text{IE}(X))^2$$

Proof:

$$\text{Var}(X) = \text{IE} \left(\underbrace{(X - \text{IE}(X))^2}_{g(x)} \right)$$

$$= \sum_x (x - \text{IE}(X))^2 p_X(x) \text{ by expected value rule.}$$

$$= \underbrace{\sum_x x^2 p_X(x)}_{\text{IE}(X^2)} + \underbrace{-2 \text{IE}(X) \sum_x x p_X(x)}_x + \underbrace{(\text{IE}(X))^2 \sum_x p_X(x)}_x$$

\uparrow \uparrow \uparrow

$\text{IE}(X^2)$ $-2(\text{IE}(X))^2$ $(\text{IE}(X))^2$

So,

$$\text{Var}(X) = \text{IE}(X^2) - (\text{IE}(X))^2$$

This is known as the "variance-in-terms-of-moments formula" because, if $k > 0$, define k^{th} moment of X as $\mathbb{E}(X^k)$

Expected Value comes up in Optimization problems -

Making choices to maximize $\mathbb{E}(\frac{\text{gain}}{\text{payoff}})$ or minimize $\mathbb{E}(\text{loss})$

Example - Game Show

There are two questions Q_1, Q_2 available.

$$\Pr(\text{you answer } Q_1) = p_1 \quad \begin{matrix} \checkmark \\ \text{answer correctly!} \end{matrix} \quad \begin{matrix} Q_1 \text{ easy} \Rightarrow p_1 \text{ high} \\ Q_2 \text{ hard} \Rightarrow p_2 \text{ low} \end{matrix}$$
$$\Pr(\text{you answer } Q_2) = p_2$$

Need to choose which question to address first. Get nothing if you answer first question wrong. If right you get a payoff - V_1 for Q_1 , V_2 for Q_2 ; then you address the other question.

Answering the two Q's independent.

Problem: Which Q to address first?

Let

$$X = \text{payoff to given strategy}$$

If Q_1 is chosen first,

$$X = 0 \quad V_1 \quad V_1 + V_2 \quad \leftarrow \text{possible values}$$

$$(1-p_1) \quad p_1(1-p_2) \quad p_1 p_2 \quad \leftarrow \text{probabilities of possible values}$$

Note probabilities add to 1

If Q_2 is chosen first,

$$\begin{array}{lll} X = 0 & v_2 & v_1 + v_2 \\ (1-p_2) & p_2(1-p_1) & p_1 p_2 \end{array} \quad \begin{array}{l} \leftarrow \text{possible values} \\ \leftarrow \text{probabilities of possible values} \end{array}$$

$E(X)$ when Q_1 is first is

$$E(X) = v_1 p_1 (1-p_2) + (v_1 + v_2) p_1 p_2$$

$E(X)$ when Q_2 is first is

$$E(X) = v_2 p_2 (1-p_1) + (v_1 + v_2) p_2 p_1$$

Choosing Q_1 first is better when

$$\frac{p_1 v_1}{1-p_1} > \frac{p_2 v_2}{1-p_2}$$

Else Q_2 first is at least as good

Can also use expectation to answer "how much would you pay to play a game of chance"?

Example- Pay to Play

If you play, you collect \$ k when face k comes up in fair die roll.
Let

X = "outcome of die roll" dollars

So

$$P_X(k) = \frac{1}{6}, \quad 1 \leq k \leq 6$$

$$\mathbb{E}(X) = \frac{1}{6}(1+2+3+4+5+6) = 3.5$$

Bottom Line? A rational agent would play the game for any price less than \$3.50.

Example- Pay to Play Part 2 (St. Petersburg Paradox)

The game: flip a fair coin repeatedly. You get \$ 2^k when k is the index of the first head.

Let

X = payoff if you choose to play

Then

$$P_X(2^k) = \left(\frac{1}{2}\right)^{k-1} \left(\frac{1}{2}\right)^k = 2^{-k}$$

$\uparrow \quad \uparrow$
 $(k-1)T \quad (k)H$

$$\mathbb{E}(X) = \sum_{x \in X} x P_X(x) = \sum_{x \in X} x P_X(x) = \sum_k (2^{-k})(2^k) = \sum_k 1 = \infty$$

Thus a "rational agent" would pay an arbitrary large sum to play?!?
 Daniel Bernoulli's proposed "resolution": instead of judging the worth of playing by $E(X)$, use instead $E(\log_2(X))$.
 can use any base really

In the above case it turns out

$$E(\log_2(X)) = \sum_{k=1}^{\infty} k 2^{-k} = 2$$

So Daniel Bernoulli reasons as follows: I will play if

$$\log_2(\text{price to play}) < 2 \quad \text{--- same as price to play } < 2^2$$

Makes sense that the minimum you will ever make is $\$2$ if you play - so acceptable price to play $> \$2$ is expected

Next BIG Topic:

Multiple Discrete Random Variables

Starting point: Ω, P ; two discrete rvs X, Y defined on Ω .

For any pair (x, y) of possible (X, Y) -values, we have the event

$$\{X=x\} \cap \{Y=y\}$$

define for each (x, y) pair the joint pmf of X and Y at (x, y) as

$$P_{XY}(x, y) = P(\{X=x\} \cap \{Y=y\})$$

$P(X=x \text{ and } Y=y)$ or $P(X=x, Y=y)$ when we're lazy