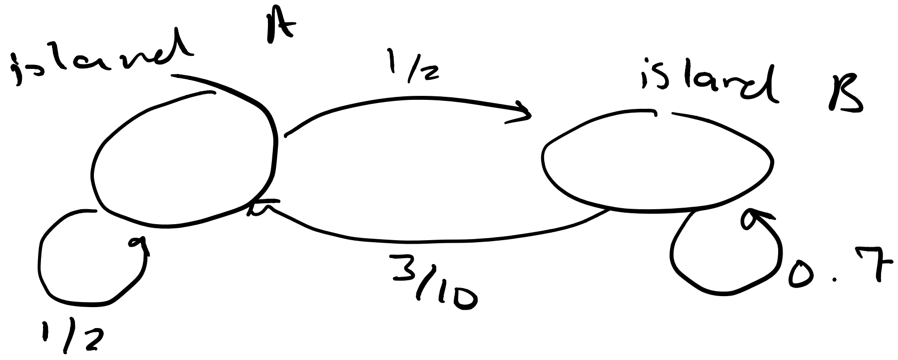
Markov Chains

Zoombinis



at time step 0: fraction of Zoombinis

at island A is a

at island B is b

note: $a, b \geq 0$

$a + b = 1$

at time step 1:
$$\begin{pmatrix} \frac{1}{2}a + \frac{3}{10}b \\ \frac{1}{2}a + \frac{7}{10}b \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{3}{10} \\ \frac{1}{2} & \frac{7}{10} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

at time step 2:
$$A\left(A\begin{pmatrix} a \\ b \end{pmatrix}\right) = A^2\begin{pmatrix} a \\ b \end{pmatrix}$$

$\vdots$

at time step n:
$$A^m\begin{pmatrix} a \\ b \end{pmatrix}$$

Question: what happens long term?

i.e. what can we say about $\lim_{m\to\infty} A^m\begin{pmatrix} a \\ b \end{pmatrix}$?

Definition: A stochastic matrix is an $n \times n$ matrix over $\mathbb{R}$ such that

ⓐ every entry of $A \geq 0$

ⓑ the sum of all entries in each column is one.

$$\sum_{i=1}^{n} A_{ij} = 1 \quad \forall\, j = 1, \dots, n$$

Definition: A vector $\vec{p} = \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix}$ is called a probability vector if

ⓐ $p_i \geq 0 \quad \forall i$

ⓑ $p_1 + \cdots + p_n = 1$

Remark: if $A_{ij} \geq 0 \ \forall\, i,j$ and $\vec{u} = (1 \ 1 \ \cdots \ 1)$ $(1 \times n)$

then A is a stochastic matrix $\Leftrightarrow \vec{u} A = \vec{u}$

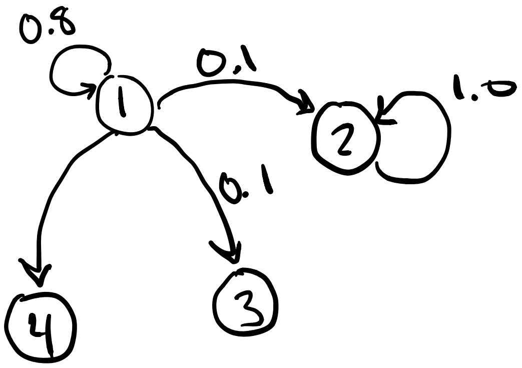
Corollary: if A, B are both stochastic matrices, then so is AB

proof. $AB_{ij} \geq 0 \ \forall\, i,j$ easy to see

$\vec{u}(AB) = (\vec{u}A)B = \vec{u}B = \vec{u} \Rightarrow AB$ a stochastic matrix

Markov Chain

Have n states



get matrix A $n \times n$, stochastic.
Interested in long term behavior.

Definition: Suppose $A_{n \times n}$ is a stochastic matrix.
A is called <u>regular</u> if there is some power of A with all entries <u>positive</u>.

We will assume A is diagonalizable

Goals: ① Show $\lim_{m \to \infty} A^m = L$ exists

② Each column of L is identical

③ If $\vec{p}$ = ANY probability vector, then $\lim_{m \to \infty} A^m \vec{p}$ exists and equals $\vec{r}$.

Aside on limits: given a matrix A, consider $(A^m)_{ij}$.
$\lim_{m \to \infty} (A^m)_{ij} = L_{ij}$ **MIGHT** exist or **MIGHT** not.

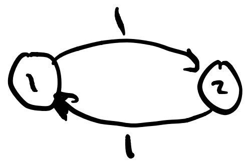
If all exist, write $\lim_{m \to \infty} A^m = L$

Some basic facts (prove on your own):

① If B is a matrix, then $\lim_{m\to\infty} BA^m = BL$

② $\lim_{m\to\infty} A^m C = LC$, C a matrix OR vector

example: $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$



When does $\lim_{m\to\infty} A^m$ exist?

Idea: if

$$A = Q \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & \lambda_m \end{pmatrix} Q^{-1}$$

then

$$A^m = Q \begin{pmatrix} \lambda_1^m & & \\ & \ddots & \\ & & \lambda_m^m \end{pmatrix} Q^{-1}$$

this limit exists $\Leftrightarrow \lim_{m\to\infty} \lambda_i^m$ exists

In our 2×2 matrix, have $\lambda_1 = 0.2$, $\lambda_2 = 1$. So limit exists.

Thus

$$\lim_{m\to\infty} A^m = \begin{bmatrix} 0.375 & 0.375 \\ 0.625 & 0.625 \end{bmatrix}$$

Assume: A is an $n \times n$ matrix, over $\mathbb{C}$. Want to understand where eigenvalues lie.

Definition: Let $\rho_i(A) = \sum_{j=1}^{n} |A_{ij}|$ Sum of absolute values in row i

Let $\nu_j(A) = \sum_{i=1}^{n} |A_{ij}|$ Sum of absolute values in col j

$$\rho(A) = \max(\rho_1(A), \dots, \rho_n(A))$$
$$\nu(A) = \max(\nu_1(A), \dots, \nu_n(A))$$

example

$$A = \begin{pmatrix} 2 & -i & 1 \\ 3+4i & 0 & -3 \\ 1 & 2 & i \end{pmatrix}$$

$\nu = 8$

$\rho = 8$

Definition: Given A, the i^{th} Gershgorin disk is

$$C_i = \{ z \in \mathbb{C} : |z - A_{ii}| \leq r_i \}$$

where $r_i = \rho_i(A) - |A_{ii}|$

ROC

example (back to zoombini)

$$A = \begin{pmatrix} 0.5 & 0.3 \\ 0.5 & 0.7 \end{pmatrix}$$

$$C_1 = \{ z \in \mathbb{C} \mid |z - 0.5| \leq 0.3 \}$$

$$C_2 = \{ z \in \mathbb{C} \mid |z - 0.7| \leq 0.5 \}$$

Theorem: Let $A \in \mathbb{C}^{n \times n}$, and $C_1, \ldots, C_n$ be Gerschgorin disks.

Then if λ is an eigenvalue of A, then $\lambda \in C_i$ for some i.