ECE 3100 **HOMEWORK ASSIGNMENT XI** **Spring 2019**

1. Let X and Y be integer-valued random variables with joint pmf $p_{X,Y}(k,m)$, and let $Z = X + Y$.

(a) Show that for every integer n we have

$$\mathbb{P}(\{Z \leq n\}) = \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{n-k} p_{X,Y}(k,m)$$

and therefore that

$$p_Z(n) = \sum_{k=-\infty}^{\infty} p_{X,Y}(k, n-k) \ .$$

(b) Conclude that when X and Y are independent we have

$$p_Z(n) = \sum_{k=-\infty}^{\infty} p_X(k) p_Y(n-k) \ .$$

2. Recall that the correlation coefficient of two random variables X and Y defined on the same probability space is

$$\rho = \frac{\mathrm{Cov}(X,Y)}{\sigma_X \sigma_Y} \ ,$$

where σ_X and σ_Y are the respective standard deviations of X and Y. In class I asserted that $|\rho| \leq 1$, with equality if and only if $X = cY$ for some nonzero constant c. Let's prove it.

(a) For any real number c, we know that

$$\mathrm{Var}(X - cY) \geq 0$$

with equality if and only if $X = cY$. Show that this is the same as

$$\mathrm{Var}(X) + c^2 \mathrm{Var}(Y) - 2c\mathrm{Cov}(X,Y) \geq 0$$

with equality if and only if $X = cY$.

(b) The result of part (a) holds in particular for

$$c^* = \frac{\mathrm{Cov}(X,Y)}{\mathrm{Var}(Y)} \ .$$

From this fact conclude that $|\rho| \leq 1$ with equality if and only if $X = c^*Y$.

(c) Show from scratch that if $X = cY$, then necessarily $c = c^*$, with c^* defined as in (b).

3. P is a random variable distributed uniformly on $[0,1]$. You have a coin whose probability of coming up heads is P. You toss the coin n times and set $X_k = 1$ if the kth toss comes up heads and $X_k = 0$ if the kth toss comes up tails, where we assume that the tosses — and hence the X_k — are conditionally independent given the value of P. Let $Y = \sum_{k=1}^{n} X_k$.

(a) Find $\mathbb{E}(Y)$ and $\mathrm{Var}(Y)$.

(b) Find $\mathrm{Cov}(P,Y)$.

(c) Find the linear minimum mean-square estimator (LMMSE) of P given Y. That is, find constants a and and b that minimize

$$\mathbb{E}\left((P - b - aY)^2\right) \ .$$

4. Let X be uniform on $[0,1]$ and N be geometric with parameter $p \in (0,1)$. Assume that X and N are independent.

(a) Find $\mathbb{E}\left(X^N \mid N = n\right)$.

(b) Find $\mathbb{E}\left(X^N\right)$. You might want to use the identity

$$\sum_{k=1}^{\infty} \frac{a^k}{k} = -\ln(1-a) \quad \text{for all} \quad a \in (0,1) \ .$$

5. Consider a random sinusoidal signal of the form

$$X(t) = A\cos(\omega_o t + \Theta) \ ,$$

where ω_o is fixed and known, while A and Θ are independent random variables with Θ a continuous random variable uniformly distributed on $[0, 2\pi]$. Find the correlation coefficient ρ between $X(t_0)$ and $X(t_1)$, where t_0 and t_1 are specified sampling times. Note that the random variable A can be continuous or discrete and that the final answer turns out not to depend on how A is distributed.

6. Cornell has N ;living alumni, where N is a discrete random variable with pmf

$$p_N(n) = p^{n-1}(1-p) \quad \text{for all} \quad n > 0 \ ,$$

where $p \in (0,1)$ is given. Every year Cornell holds a fundraiser that each living alum attends with probability $q \in (0,1)$ independent of all other alumni. An alum attending the fundraiser donates an amount of money X that has exponential pdf

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{when } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

The different alums' donations are independent of each other, of the random variable N, and of whether each alum attends the fundraiser or not.

(a) Find the expected value and variance of the number of alumni attending the fundraiser.

(b) Find the expected value and variance of the total amount of money raised at the fundraiser.

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ECE3100 HW11

NOT TURNED IN

1. X, Y integer valued random variables with joint pmf $p_{X,Y}(k,m)$ and let

$$Z = X + Y$$

(a) Show $\forall n \in \mathbb{Z}$ we have

$$\mathbb{P}(\{Z \le n\}) = \sum_{k=-\infty}^{+\infty} \sum_{m=-\infty}^{n-k} p_{X,Y}(k,m)$$

and therefore that

$$p_Z(n) = \sum_{k=-\infty}^{+\infty} p_{X,Y}(k, n-k)$$

$\mathbb{P}(Z \le n) =$ sum of $p_{X,Y}(k,m)$ over all pairs (k,m) such that $k+m \le n$. This is shown in double sum. For k anything, $m \le n-k$.

$$p_Z(n) = \mathbb{P}(Z \le n) - \mathbb{P}(Z \le n-1)$$

$$= \sum_{k=-\infty}^{+\infty} \sum_{m=-\infty}^{n-k} p_{X,Y}(k,m) - \sum_{k=-\infty}^{+\infty} \sum_{m=-\infty}^{n-1-k} p_{X,Y}(k,n)$$

$$= \sum_{k=-\infty}^{+\infty} \left[\sum_{m=-\infty}^{n-k} p_{X,Y}(k,m) - \sum_{m=-\infty}^{n-1-k} p_{X,Y}(k,n) \right]$$

$$= \sum_{k=-\infty}^{+\infty} \left[p_{X,Y}(k,n-k) + \sum_{m=-\infty}^{n-k-1} \left(p_{X,Y}(k,m) - p_{X,Y}(h,m) \right) \right]$$

$$= \sum_{k=-\infty}^{+\infty} p_{X,Y}(k,n-k)$$

(b) X, Y independent $\iff p_{X,Y}(k, n-k) = p_X(k)\, p_Y(n-k)$

Thus

$$\sum_{k=-\infty}^{+\infty} p_{X,Y}(k, n-k) = \sum_{k=-\infty}^{+\infty} p_X(k)\, p_Y(n-k)$$

(2)

$$\rho = \frac{\mathrm{Cov}(X,Y)}{\sigma_X \sigma_Y} \quad \Big\} \text{ Correlation coefficient}$$

(a) $\forall c \in \mathbb{R}$, know

$$\mathrm{Var}(X - cY) \geq 0$$

with equality iff $X = cY$.

Show this is the same as

$$\mathrm{Var}(X) + c^2 \mathrm{Var}(Y) - 2c\, \mathrm{Cov}(X,Y) \geq 0$$

with equality iff $X = cY$

$$\mathrm{Var}(X - cY) = \mathbb{E}[(X-cY)^2] - (\mathbb{E}[X - cY])^2 \geq 0$$

$$= \mathbb{E}[X^2] - \mathbb{E}[2XcY] + \mathbb{E}[c^2Y^2] - (\mathbb{E}[X] - \mathbb{E}[cY])^2$$

$$= \mathbb{E}(X^2) - (\mathbb{E}(X))^2 + \mathbb{E}[c^2Y^2] - (\mathbb{E}(cY))^2 - [2c\,\mathbb{E}(XY) - 2c\,\mathbb{E}(X)\mathbb{E}(Y)]$$

$$= \mathrm{Var}(X) + c^2 \mathrm{Var}(Y) - 2c\, \mathrm{Cov}(X,Y) \geq 0$$

if $X = cY$

$$c^2 \mathrm{Var}(Y) + c^2 \mathrm{Var}(Y) - 2c^2 \mathrm{Var}(Y) = 0$$

(b)

$$c^* = \frac{\mathrm{Cov}(X,Y)}{\mathrm{Var}(Y)}$$

Conclude $|\rho| \le 1$ with equality iff $X = c^* Y$

$$\mathrm{Var}(X) + (c^*)^2\,\mathrm{Var}(Y) - 2c^*\,\mathrm{Cov}(X,Y) \geq 0$$

$$\mathrm{Var}(X) - \frac{(\mathrm{Cov}(X,Y))^2}{\mathrm{Var}(Y)} \geq 0$$

$$1 \geq (\rho(X,Y))^2$$

$$|\rho(X,Y)| \le 1$$

(c) Let X, Y be two random variables such that $X = cY$; $c \in \mathbb{R}$.

Thus

$$X - \mathbb{E}(X) = c(Y - \mathbb{E}(Y))$$

and

$$(X - \mathbb{E}(X))(Y - \mathbb{E}(Y)) = c(Y - \mathbb{E}(Y))^2$$

$$\mathbb{E}[(X - \mathbb{E}(X))(Y - \mathbb{E}(Y))] = \mathbb{E}[c(Y - \mathbb{E}(Y))^2]$$

$$\mathrm{Cov}(X,Y) = c\,\mathrm{Var}(Y)$$

$$c = \frac{\mathrm{Cov}(X,Y)}{\mathrm{Var}(Y)}$$

③ $P \sim$ Uniform $[0,1]$.

Coin whose probability of being heads is P.

Toss coin n times; set $X_k = 1$ if k^{th} toss comes up heads, else 0 - $1 \le k \le n$.

-Thus X_k conditionally independent given value P.

Let

$$Y = \sum_{k=1}^{n} X_k$$

(a) $\mathbb{E}(X)$, $Var(Y)$?

Given $P=p$, Y is the sum of n Bernoulli p random variables.

So

$$\mathbb{E}(Y \mid P=p) = np \rightarrow \mathbb{E}(Y \mid P) = nP$$

Thus

$$\mathbb{E}(Y) = \int_{-\infty}^{+\infty} \mathbb{E}(Y \mid P=p) f_P(p)\, dp$$

$$= \int_0^1 np\, dp = \frac{1}{2} n p^2 \Big|_{p=0}^{p=1} = \frac{n}{2}$$

Similarly,

$$Var(Y \mid P=p) = np(1-p) \rightarrow Var(Y \mid P) = nP(1-P)$$

$$Var(Y) = \mathbb{E}(Var(Y \mid P)) + Var(\mathbb{E}(Y \mid P))$$

$$= \mathbb{E}(nP(1-P)) + Var(nP)$$

$$= \frac{n}{6} + \frac{n^2}{12} = \frac{n(n+2)}{12}$$

b)

$$
\begin{aligned}
\mathrm{Cov}(P,Y) &= \mathbb{E}\big[(P-\overset{A}{\mathbb{E}(P)})(Y-\overset{B}{\mathbb{E}(Y)})\big] \\
&= \mathbb{E}\left[\left(P-\tfrac{1}{2}\right)\left(Y-\tfrac{n}{2}\right)\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\left(P-\tfrac{1}{2}\right)\left(Y-\tfrac{n}{2}\right)\right] \,\middle|\, P=p\right] \\
&= \mathbb{E}\left[\left(p-\tfrac{1}{2}\right)\mathbb{E}\left[Y-\tfrac{n}{2}\right] \,\middle|\, P=p\right] \\
&= \mathbb{E}\left[\left(p-\tfrac{1}{2}\right)\left(\mathbb{E}\left[Y \mid P=p\right]-\tfrac{n}{2}\right)\right] \\
&= \mathbb{E}\left[\left(p-\tfrac{1}{2}\right)\left(np-\tfrac{n}{2}\right)\right] \qquad p \to P \\
&\to \mathbb{E}\left[\left(P-\tfrac{1}{2}\right)n\left(P-\tfrac{1}{2}\right)\right] \\
&= n\,\mathbb{E}\left[\left(P-\tfrac{1}{2}\right)^2\right] \\
&= n\,\mathrm{Var}(P) \\
&= n/12
\end{aligned}
$$

(c) Linear Minimum Mean Square Estimator of P given Y.?

i.e ; a,b that minimize

$$\mathbb{E}\left[(P-b-aY)^2\right]$$

$$\hat{a} = \frac{\mathrm{Cov}(P,Y)}{\mathrm{Var}(Y)} = \frac{n/12}{n(n+2)/12} = \boxed{\frac{1}{n+2}}$$

$$\hat{b} = \mathbb{E}(P) - \hat{a}\,\mathbb{E}(Y) = \frac{1}{2} - \frac{n}{2(n+2)} = \boxed{\frac{1}{n+2}}$$

④ $X \sim \text{Uniform}[0,1]$

$N \sim \text{Geometric}(p)$; $p \in (0,1)$

X, N independent.

$f_{X|N}(x|n) = f_X(x) \ \forall x, n$

by independence

(a) $\mathbb{E}[X^N | N=n] = \displaystyle\int_{-\infty}^{+\infty} x^n f_{X|N}(x|n)\,dx$

$$= \int_{-\infty}^{+\infty} x^n f_X(x)\,dx$$

$$= \int_0^1 x^n\,dx = \frac{1}{n+1}$$

(b) $\mathbb{E}(X^N) = \displaystyle\sum_{n=-\infty}^{+\infty} \mathbb{E}(X^N | N=n)\, p_N(n)$

$$\sum_{n=1}^{\infty} \frac{1}{n+1} p(1-p)^{n-1}$$

$m = n+1$

$$= \sum_{m=2}^{\infty} \frac{p(1-p)^{m-2}}{m}$$

$$= \frac{p}{(1-p)^2} \sum_{m=2}^{\infty} \frac{(1-p)^m}{m}$$

$$= \frac{-p}{(1-p)^2}\left(\ln(p) + (1-p)\right)$$

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$$X(t) = A\cos(\omega_0 t + \Theta)$$

ω_0 fixed; $\Theta \sim \text{Uniform}[0, 2\pi]$

A a random variable

Given t_0, t_1 find ρ between $X(t_0)$ and $X(t_1)$.

$$\rho = \frac{\text{Cov}(X(t_0), X(t_1))}{\sqrt{\text{Var}(X(t_0))}\sqrt{\text{Var}(X(t_1))}}$$

$$\text{Var}(X(t_0)) = \mathbb{E}\left[(X(t_0))^2\right] + \underbrace{\cancel{\left(\mathbb{E}[X(t_0)]\right)^2}}_{\to 0}$$

$0 \leftarrow \frac{1}{2\pi}\int_0^{2\pi} A\cos(\omega_0 t + \theta)\, d\theta = 0$

$$= \mathbb{E}\left[A^2\cos^2(\omega_0 t + \Theta)\right]$$

$\leftarrow A, \Theta$ independent

$$= \mathbb{E}(A^2)\,\mathbb{E}\left[\cos^2(\omega_0 t + \Theta)\right]$$

$$= \mathbb{E}(A^2)\left(\frac{1}{2\pi}\int_0^{2\pi} \cos^2(\omega_0 t + \Theta)\, d\theta\right)$$

$$= \mathbb{E}(A^2)\left(\tfrac{1}{2}\right) = \boxed{\tfrac{1}{2}\mathbb{E}(A^2)}$$

Thus,

$\text{Var}(X(t_1)) = \frac{1}{2}\mathbb{E}(A^2)$ as well.

$$\text{Cov}(X(t_0), X(t_1)) = \mathbb{E}[X(t_0)X(t_1)] - \underbrace{\cancel{\mathbb{E}[X(t_0)]\,\mathbb{E}[X(t_1)]}}_{\to 0}$$

$$= \mathbb{E}\left[A^2\cos(\omega_0(t_0 - t_1)\Theta)\right]$$

$$= \frac{1}{2}\mathbb{E}(A^2)\cos(\omega_0(t_0 - t_1))$$

Thus

$$\rho = \frac{\frac{1}{2}\mathbb{E}(A^2)\cos(\omega_0(t_0 - t_1))}{\sqrt{\frac{1}{2}\mathbb{E}(A^2)}\sqrt{\frac{1}{2}\mathbb{E}(A^2)}} = \boxed{\cos(\omega_0(t_0 - t_1))}$$

⑥ Cornell has N living alumni. N a discrete random variable with pmf

$$P_N(n) = p^{n-1}(1-p) \quad \forall\, n>0 \ ; \ p \in (0,1) \text{ given}$$

Every year Cornell holds a fundraiser that each living alum attends w/ probability $q \in (0,1)$ independent of other alumni.
An alum attending the fundraiser donates an amount of money X that has exponential pdf

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & , x \geq 0 \\ 0 & , \text{else} \end{cases}$$

Different alum donations independent of eachother, of N, and of whether each alum attends the fundraiser or not.

(a) Let Q = number of alumni attending fundraiser.
$\mathbb{E}(Q)$? $Var(Q)$?

Note some integer amount

$$\mathbb{P}(\{Q=k\} \mid \{N=n\}) = \binom{n}{k} q^k (1-p)^{n-k} \leftarrow \text{Binomial pmf!}$$

$$\mathbb{E}(Q) = \sum_{n=1}^{\infty} \underbrace{\mathbb{E}(\{Q=k\} \mid \{N=n\})}_{nq} \underbrace{\mathbb{P}(\{N=n\})}_{P_N(n)} \quad \text{Law of Total Expectation}$$

$$= q \sum_{n=1}^{\infty} n P_N(n) = q\,\mathbb{E}[N] = \frac{q}{(1-p)}$$

OR

$$\mathbb{E}(Q) = \mathbb{E}(\mathbb{E}(Q|N)) = \mathbb{E}(qN) = q\,\mathbb{E}[N] = \frac{q}{1-p} \quad \leftarrow \text{Law of Iterated Expectation}$$

$$Var(Q \mid N=n) = nq(1-q)$$
$$Var(Q \mid N) = Nq(1-q)$$

$$\begin{aligned} Var(Q) &= \mathbb{E}(Var(Q \mid N)) + Var(\mathbb{E}(Q \mid N)) \\ &= \mathbb{E}(Nq(1-q)) + Var(Nq) \\ &= q(1-q)\mathbb{E}[N] + q^2 Var(N) \\ &= \boxed{\frac{q(1-q)}{1-p} + q^2 \frac{p}{(1-p)^2}} \end{aligned}$$

(b) Given $Q = k$

Let

Z = Total amount fundraised $= X_1 + X_2 + \cdots + X_k$

where X_m is the donation of attendee m for $1 \le m \le k$.

X_m are independent exponential random variable w/ rate parameter λ.

So,

$$\mathbb{E}(Z \mid \{Q=k\}) = \sum_{m=1}^{k} \mathbb{E}(X_m \mid Q=k) \overset{X_m \text{ independent of } Q}{=} \sum_{m=1}^{k} \mathbb{E}(X_m) = \frac{k}{\lambda}$$

Thus $\mathbb{E}(Z \mid Q) = Q/\lambda$

$$\mathbb{E}(Z) \overset{\text{Law of iterated expectations}}{=} \mathbb{E}(\mathbb{E}(Z \mid Q)) = \mathbb{E}\left(\frac{Q}{\lambda}\right) = \frac{1}{\lambda}\mathbb{E}(Q) = \frac{q}{(1-p)\lambda}$$

$$Var(Z \mid Q=k) = \sum_{m=1}^{k} Var(X_m \mid Q=k) = \sum_{m=1}^{k} Var(X_m) = k/\lambda^2$$

Thus

$$\mathrm{Var}(Z|Q) = Q/\lambda^2$$

$$\begin{aligned}
\mathrm{Var}(Z) &= \mathbb{E}(\mathrm{Var}(Z|Q)) + \mathrm{Var}(\mathbb{E}(Z|Q)) \\
&= \mathbb{E}(Q/\lambda^2) + \mathrm{Var}(Q/\lambda) \\
&= \frac{1}{\lambda^2}\mathbb{E}(Q) + \frac{1}{\lambda^2}\mathrm{Var}(Q) \\
&= \frac{q}{\lambda^2(1-p)} + \frac{1}{\lambda^2}\left[\frac{q(1-q)}{1-p} + q^2\frac{p}{(1-p)^2}\right]
\end{aligned}$$