Last Time

$$\beta_1 = \left\{ \underset{\vec{u}_1}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}, \underset{\vec{u}_2}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} \right\}$$

$$\beta_2 = \left\{ \underset{\vec{w}_1}{\begin{pmatrix} -1 \\ 1 \end{pmatrix}}, \underset{\vec{w}_2}{\begin{pmatrix} 1 \\ 2 \end{pmatrix}} \right\}$$

Wrote β_2 in terms of β_1

$$\vec{w}_1 = [\vec{u}_1 \;\; \vec{u}_2] \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\vec{w}_2 = [\vec{u}_1 \;\; \vec{u}_2] \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\Rightarrow B = \begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix}$$

Then

$$\vec{x} = \begin{pmatrix} 10 \\ 20 \end{pmatrix} = [\vec{w}_1 \;\; \vec{w}_2] \begin{bmatrix} 0 \\ 10 \end{bmatrix}$$

$$= [\vec{u}_1 \;\; \vec{u}_2] \begin{bmatrix} 20 \\ -10 \end{bmatrix}$$

Point: $B \underbrace{\begin{bmatrix} 0 \\ 10 \end{bmatrix}}_{\text{coords for } \vec{x} \text{ w.r.t. } \beta_2} = \underbrace{\begin{bmatrix} 20 \\ -10 \end{bmatrix}}_{\text{coords for } \vec{x} \text{ w.r.t. } \beta_1}$ } This works $\forall \vec{x}$

Note this is NOT specific to 2 dim's.

Give V & bases $\beta_1 = \{\vec{u}_1, \dots, \vec{u}_d\}$ $\xrightarrow{\text{expand}}$ $w_i = [u_1, \dots, u_d] \begin{bmatrix} \beta_{i0} \\ \vdots \\ \beta_{id} \end{bmatrix}$

$\beta_2 = \{\vec{w}_1, \dots, \vec{w}_d\}$

Forms $B = (B)_{ij}$ which is $d \times d$

Confusion from Last Time

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ 3y \end{pmatrix}$$

$$[T]_{\beta_1} = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}, \quad [T]_{\beta_2} = \begin{bmatrix} 7 & 2 \\ 1 & 8 \end{bmatrix}$$

Question: Does

$$[T]_{\beta_1} = B\,[T]_{\beta_2}\,B^{-1}$$

$$\text{OR } B^{-1}[T]_{\beta_2} B$$

Answer: $B\,[T]_{\beta_2}\,B^{-1}$

LHS inputs $\vec{x}$ in β_1 coords. Outputs β_1 coord of $T\vec{x}$.
RHS input $\vec{x}$ in β_1 coords, B^{-1} outputs β_2 coord of $\vec{x}$.
$[T]_{\beta_2}$ then outputs β_2 coord of $T\vec{x}$. B outputs β_1 coords of $T\vec{x}$.

Theorem (In General)

$$[T]_{\beta_1} = B\,[T]_{\beta_2}\,B^{-1}$$

(1) What do you need to prove this?
Check B does what we claim.

(2) What have we suppressed?
The fact B must be invertible!
You can !ly expand vecs of β_1 in terms of β_2 **AND** vice-versa

Definition

V, W are isomorphic vector spaces is there exists a bijective linear transform $T: V \to W$

Fact

V, W are isomorphic $\iff \dim V = \dim W$ ← this is an isomorphism invariant

↳ finite dimensional

If $T: V \to W$ an isomorphism then $T^{-1}: W \to V$ is a LT

Example

Let $S, T: V \to V$ be LTs, $\dim V < \infty$

Then $ST = I \iff TS = I$

Let's Break This - Work w/ $\dim V = \infty$

$$V = \{\text{All infinitely differentiable function } f: \mathbb{R} \to \mathbb{R}\}$$

Define $T: V \to V$ by $T(f) = \int f(x)\,dx = F(x)$ ← insist $F(0) = 0$

$$S(f) = \frac{d}{dx} f$$

So

$$ST(f) = f$$

$$TS(\text{constant}) = 0_{\text{function}}$$

Laplace Transforms

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt$$ ← This is a Linear Transformation

Solve

$$f'' - 10f' + 9f = 5t \quad \text{subject to} \quad f(0) = -1, \; f'(0) = 2$$

Laplace turns differentiation into polynomials.

Get

$$(s^2 - 10s + 9)F(s) + s - 12 = \frac{5}{s^2}$$

Solve for $F(s)$, use inverstion to get $f(t)$