

Recall

- Conditional pmfs, conditional expectation.
- Total expectation = avg of conditional expectations
 - ("Law of Total Expectation")

Example - Envelopes (from last time)

Have 2 envelopes. One has twice the amount of money as the other.

i.e
one has m dollars
one has $2m$ dollars

Call the envelopes X and Y .

let Z = total amount of money between the two

Say you find

$$X = m$$

Get to keep m if you want. Should you?

Figure out $E(Y|X=m)$ using following **NAIVE** approach:

$$P(\{Y=2m\} | \{X=m\}) = P\left\{Y=\frac{m}{2}\right\} | \{X=m\} = \frac{1}{2}$$

This line of reasoning is incorrect

This leads you to believe you should switch.

$$E(Y|X=m) = 2m\left(\frac{1}{2}\right) + \frac{m}{2}\left(\frac{1}{2}\right) = \frac{5}{4}m > m$$

The reasoning is specious because it is based on a woefully incomplete probability model.

How's it incomplete?

- Are X, Y integers? Continuous valued?

• Eg if integers, if X is 19 you should definitely switch

- What's $P_Z(z)$? Any complete model would enable you to find this...

Let us make it more complete.

- Dealer picks z "at random" - a multiple of 3 dollars
- Divides into two piles. $\frac{z}{3}$; $\frac{2z}{3}$.
- Flip a fair coin to decide how to allot piles to 2 envelopes, X , Y .

Say you open X and find m dollars.

$$Z = \begin{cases} 3m & , \text{ if } X \text{ is smaller envelope} \\ \frac{3}{2}m & , \text{ if } X \text{ is larger envelope} \end{cases}$$

If m is odd, can't have $Z = \frac{3}{2}m$ - but say m is even.

Fact: It's impossible to have

$$P_{Z|X}(3m|m) = P_{Z|X}\left(\frac{3}{2}m|m\right) = \frac{1}{2}$$

so you ALWAYS switch

If m is even, Y contains either $2m$ or $\frac{m}{2}$.

In terms of Z , $Z = 3m$ or $\frac{3}{2}m$.

$$P_{Z|X}(3m|m) = \frac{P(X=m|Z=3m) / P(Z=3m)}{P(X=m)}$$

$$P_{Z|X}\left(\frac{3}{2}m|m\right) = \frac{P(X=m|Z=\frac{3}{2}m) / P(Z=\frac{3}{2}m)}{P(X=m)}$$

If

$$P_{Z|X}(3m|m) = P_{Z|X}\left(\frac{3}{2}m|m\right) = \frac{1}{2}$$

then must have

$$P(Z=3m) = P\left(Z=\frac{3}{2}m\right); m=2n, n \in \mathbb{N} \setminus \{0\}$$

Thus need

$$m=2 \quad P(Z=6) = P(Z=2)$$

$$m=4 \quad P(Z=12) = P(Z=6)$$

$$m=8 \quad P(Z=24) = P(Z=12)$$

i.e

$$P(Z=2) = P(Z=6) = P(Z=12) = P(Z=24) = \dots$$

infinite
chain of
equalities

So if Z is chosen w/ positive probability or all positive integer multiples of 3, can't have this chain of equalities

Moral: If you flesh out the probability model - by putting a pmf on Z - find that specious reasoning fails

i.e $P(Z=3m|x=m) \neq P(Z=\frac{3}{2}m|x=m) \neq \frac{1}{2} \neq \text{even } m$

Example - Conditional Expectation

A tricky way to get $\text{IE}(X)$ and $\text{Var}(X)$ when X is geometric.

i.e.

$$P_x(k) = p(1-p)^{k-1}, k \geq 0$$

Figured out $\text{IE}(X) = 1/p$ already "by math"

Here's another way.

Let

$$A = \{X=1\}$$

$$\Rightarrow A^c = \{X > 1\}$$

Observe that

$$X = 1 + (X-1)$$

Use linearity of IE to get

$$\begin{aligned} \text{IE}(X) &= 1 + \text{IE}(X-1) \\ &= 1 + \underbrace{\text{IE}(X-1|A)}_0 \underbrace{P(A)}_p + \underbrace{\text{IE}(X-1|A^c)}_{\text{IE}(X)} \underbrace{P(A^c)}_{1-p} \end{aligned}$$

"Law of Total Expectation"

Observe that $X-1|A^c$ is distributed the same as $X|0$ i.e it's geometric! Thus $\text{IE}(X-1|A^c) = \text{IE}(X)$

So,

$$\text{IE}(X) - \text{IE}(X)(1-p) = 1$$

$$\text{IE}(X)[1 - (1-p)] = 1$$

$$\text{IE}(X) = \frac{1}{p}$$

How about $\text{Var}(X)$? "By Math" is hard
Let's use the conditional set up.
Know,

$$\text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$$

Need $\mathbb{E}(X^2)$.

Let

$$X = 1 + (X-1)$$

Then

$$X^2 = 1 + 2(X-1) + (X-1)^2$$

$$\begin{aligned}\mathbb{E}(X^2) &= \mathbb{E}(1) + \mathbb{E}(2(X-1)) + \mathbb{E}((X-1)^2) \\ &= 1 + 2\mathbb{E}(X-1) + \mathbb{E}((X-1)^2) \\ &= 1 + 2\mathbb{E}((X-1)|A)\mathbb{P}(A) + 2\mathbb{E}(X-1|A^c)\mathbb{P}(A^c) \\ &\quad + \mathbb{E}((X-1)^2|A)\mathbb{P}(A) + \mathbb{E}((X-1)^2|A^c)\mathbb{P}(A^c)\end{aligned}$$

Neater,

$$\mathbb{E}(X^2) = 1 + \underbrace{2\mathbb{E}((X-1)|A)\mathbb{P}(A)}_0 + \underbrace{2\mathbb{E}(X-1|A^c)\mathbb{P}(A^c)}_{\mathbb{E}(X^2)} + \underbrace{\mathbb{E}((X-1)^2|A)\mathbb{P}(A)}_0 + \underbrace{\mathbb{E}((X-1)^2|A^c)\mathbb{P}(A^c)}_{(1-p)}$$

Again since $p_{X-1|A}(k)$ same as $p_X(k)$, we have ↑

So,

$$\mathbb{E}(X^2) = 1 + 2 \cdot \frac{1-p}{p} + \mathbb{E}(X^2)(1-p)$$

$$\mathbb{E}(X^2)[1 - (1-p)] = 1 + 2 \cdot \frac{1-p}{p}$$

$$\text{Var}(X) = \frac{1}{p} + 2 \frac{1-p}{p^2} - \left(\frac{1}{p}\right)^2 = \frac{1-p}{p^2}$$

Independence

Given Ω, \mathbb{P} and a discrete $\sim X$ defined on Ω .

Given an event A , say X is independent of A when every event $\{X=x\}$ is independent of A (event-wise).

i.e

$$\mathbb{P}(\{X=x\} \cap A) = p_x(x) \mathbb{P}(A)$$

Note: When $\mathbb{P}(A) > 0$, same as $p_{X|A}(x) = p_x(x) \neq x$

Say two rvs X and Y are independent when

- equivalent statements
- X is independent of every event $\{Y=y\}$
 - Y is independent of every event $\{X=x\}$
 - $p_{X,Y}(x,y) = p_X(x)p_Y(y) \quad \forall x \in X, y \in Y$

Extends to multiple rvs X_1, \dots, X_n - Say they're independent when

$$p_{X_1, X_2, \dots, X_n}(x_1, \dots, x_n) = p_{X_1}(x_1)p_{X_2}(x_2) \cdots p_{X_n}(x_n) \quad \forall x_1, \dots, x_n$$

Joint = Product of Marginals

Important Fact: X, Y independent $\Rightarrow \mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$

To see this,

$$\begin{aligned} \mathbb{E}(XY) &= \sum_{x \in X, y \in Y} xy p_{X,Y}(x,y) = \sum_{x \in X, y \in Y} xy (p_X(x)p_Y(y)) = \sum_{x \in X} x \left(\sum_{y \in Y} y p_Y(y) \right) p_X(x) \\ &= \mathbb{E}(X)\mathbb{E}(Y) \end{aligned}$$

Extension of this: X, Y independent $\Rightarrow \mathbb{E}(g(x)h(y)) = \mathbb{E}(g(x))\mathbb{E}(h(y))$

Another Important Fact: X, Y independent. Then

$$\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y)$$