Recap

Want to estimate rv X using observation of rv Y

$$\hat{X}_{MMSE} = \mathbb{E}[X|Y]$$

$$MMSE = \mathbb{E}_Y\left[Var(X|Y)\right]$$

what's added is a zero-mean term

$$\hat{X}_{LMMSE} = \mathbb{E}[X] + \frac{Cov(X,Y)}{Var(Y)}\left(Y - \mathbb{E}[Y]\right)$$

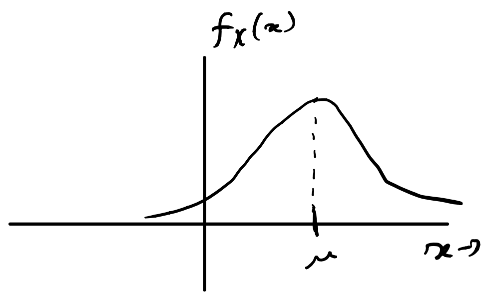
so centered about mean of X

$$LMMSE \triangleq \mathbb{E}\left[(X - \hat{X}_{LMMSE})^2\right] = Var(X) - \frac{(Cov(X,Y))^2}{Var(Y)}$$

$$= Var(X)(1 - \rho_{X,Y}^2)$$

Jointly Gaussian

$X \sim \mathcal{N}(\mu, \sigma^2)$ if $\quad f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$



$\mathbb{E}[X] = \mu$

$Var(X) = \sigma^2$

X and Y are jointly Gaussian if

$Z = aX + bY$ is Gaussian $\forall\, a, b \neq 0 \in \mathbb{R}$

For jointly Gaussian X and Y w/ $|\rho_{XY}| < 1$

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{|K|}} e^{-\frac{1}{2}\left(\begin{pmatrix} x-\mu_X \\ y-\mu_Y \end{pmatrix}^T K^{-1} \begin{pmatrix} x-\mu_X \\ y-\mu_Y \end{pmatrix}\right)}$$

where

$$K = \begin{bmatrix} Cov(X,X) & Cov(X,Y) \\ Cov(X,Y) & Cov(Y,Y) \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_X^2 & \rho_{XY}\sigma_X\sigma_Y \\ \rho_{XY}\sigma_X\sigma_Y & \sigma_Y^2 \end{bmatrix}$$

Note: (not done in class)

$$K^{-1} = \frac{1}{|K|}\begin{bmatrix} \sigma_Y^2 & -\rho_{XY}\sigma_X\sigma_Y \\ -\rho_{XY}\sigma_X\sigma_Y & \sigma_X^2 \end{bmatrix}$$

Inverse of 2x2 matrix

Properties of Jointly Gaussian Random Variables

① J.G. $\Rightarrow$ M.G. ($\not\Leftarrow$)

② Uncorrelated JG rv's are independent ← special!

independence $\Rightarrow$ uncorrelated

$\not\Leftarrow$ in general

③ If X, Y jointly Gaussian, then

$a_1X + b_1Y + c_1$, $a_2X + b_2Y + c_2$ are jointly Gaussian

④ For JG X, Y

$$\hat{X}_{MMSE} = \mathbb{E}[X \mid Y] = \hat{X}_{LMMSE}$$

$$= \mathbb{E}[X] + \frac{Cov(X,Y)}{Var(Y)}(Y - \mathbb{E}[Y])$$

$$MMSE = Var(X)(1-\rho_{XY}^2)$$

$$= Var(X) - \frac{Cov^2(X,Y)}{Var(Y)}$$

<u>Proof!</u>

$$W = X - \hat{X}_{LMMSE}$$

$$= X - \left(\mathbb{E}[X] + \frac{Cov(X,Y)}{Var(Y)}(Y - \mathbb{E}[Y]\right)$$

Gaussian!

$$= X - \frac{Cov(X,Y)}{Var(Y)}Y - \left(\mathbb{E}[X] + \frac{Cov(X,Y)}{Var(Y)}\mathbb{E}[Y]\right)$$

Know

W & Y are Jointly Gaussian

$\mathbb{E}[W] = 0$

$\mathbb{E}[WY] = 0$ (orthogonality)

So $Cov(W,Y) = 0 \Rightarrow$ independence by property 2

Need

$\mathbb{E}[Wg(Y)] = 0 \quad \forall g(Y) \Rightarrow W$ is error given by best MMSE

Trivially true since W independent of Y.

$$\mathbb{E}[Wg(Y)] = \mathbb{E}[W]\,\mathbb{E}[g(Y)] = 0$$

⑤ For J.G X, Y

$$\mathbb{E}[X|Y=y] = \mathbb{E}[X] + \frac{Cov(X,y)}{Var(y)}(y - \mathbb{E}[y])$$

$$\mathbb{E}[X|Y] = \mathbb{E}[X] + \frac{Cov(X,Y)}{Var(Y)}(Y - \mathbb{E}[Y])$$

$$Var(X|Y=y) = Var(X) - \frac{Cov^2(X,y)}{Var(y)}$$

$$Var(X|Y) = Var(X) - \frac{Cov^2(X,Y)}{Var(Y)}$$

Follows from 4

⑥ The conditional distribution of X given $Y=y$ is Gaussian

$$f_{X|Y}(x|y) = \frac{1}{\sqrt{2\pi Var(X|Y)}} e^{-(\dots)}$$

Requires Proof

Proof| $X = \hat{X}_{MMSE} + W$

where W is independent of Y

Given $Y=y$ (an observation),

- $\hat{X}_{MMSE}$ is a constant
- W maintains its distribution

Thus $X \sim \mathcal{N}(\mu_w + \hat{x}_{MMSE}, \sigma_w^2)$

Example: X, Y jointly Gaussian w/ zero mean and

$$K = \begin{pmatrix} 4 & 3 \\ 3 & 9 \end{pmatrix}$$

Find $\mathbb{E}[X|Y]$ and $\mathbb{E}[X^2|Y]$

$$\mathbb{E}[X|Y] = 0 + \frac{3}{9}(Y - 0) = \frac{1}{3}Y$$

$$\begin{aligned} \mathbb{E}[X^2|Y] &= \mathrm{Var}(X|Y) + (\mathbb{E}[X|Y])^2 \\ &= \left(4 - \frac{3^2}{9}\right) + \left(\frac{1}{3}Y\right)^2 \\ &= 3 + \frac{1}{9}Y^2 \end{aligned}$$

Random Vectors

$$\underline{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_m \end{bmatrix} \qquad \mathbb{E}[\underline{X}] = \begin{bmatrix} \mathbb{E}[X_1] \\ \vdots \\ \mathbb{E}[X_m] \end{bmatrix}$$

Correlation MATRIX

$$\mathbb{E}[\underline{X}\underline{X}^T] = \begin{bmatrix} \mathbb{E}[X_1^2] & \mathbb{E}[X_1X_2] & \cdots & \mathbb{E}[X_1X_m] \\ \mathbb{E}[X_2X_1] & \mathbb{E}[X_2^2] & \cdots & \mathbb{E}[X_2X_m] \\ \vdots & & \ddots & \vdots \\ \mathbb{E}[X_mX_1] & \cdots & \cdots & \mathbb{E}[X_m^2] \end{bmatrix}_{m \times m}$$

$$\mathrm{Cov}(\mathbf{X}) = \mathbb{E}\left[(\mathbf{X}-\mathbb{E}[\mathbf{X}])(\mathbf{X}-\mathbb{E}[\mathbf{X}])^T\right]$$

$$= \begin{bmatrix} \mathrm{Var}(X_1) & \mathrm{Cov}(X_1,X_2) & \cdots & \mathrm{Cov}(X_1,X_m) \\ \mathrm{Cov}(X_2,X_1) & \mathrm{Var}(X_2) & & \mathrm{Cov}(X_2,X_m) \\ \vdots & & \ddots & \vdots \\ \mathrm{Cov}(X_m,X_1) & \cdots & \cdots & \mathrm{Var}(X_m) \end{bmatrix}_{m\times m}$$

$$\mathbb{E}[\mathbf{X}\mathbf{Y}^T] = \{\mathbb{E}[X_i Y_j]\}_{\substack{i=1,\dots,m \\ j=1,\dots,n}} \quad m\times n$$

$$\mathrm{Cov}(\mathbf{X},\mathbf{Y}) = \{\mathrm{Cov}(X_i, Y_j)\}_{\substack{i=1,\dots,m \\ j=1,\dots,n}} \quad m\times n$$