

## Recall

$V$  a vector space over  $\mathbb{C}$ ,  $\dim V = n < \infty$

$$E_\lambda(T) = \{v \in V \mid (T - \lambda I)v = 0\} \\ = \ker(T - \lambda I)$$

← same when  
T is diagonalizable

$$G_\lambda(T) = \ker(T - \lambda I)^n$$

$\dim E_\lambda(T)$  is called the "geometric multiplicity" of  $\lambda$  in  $T$ .

$\dim G_\lambda(T)$  is called the "algebraic multiplicity" of  $\lambda$  in  $T$ .

Definition: The characteristic polynomial of  $T$  is

$$q(x) = (x - \lambda_1)^{a_1} (x - \lambda_2)^{a_2} \cdots (x - \lambda_m)^{a_m}$$

where  $\lambda_1, \dots, \lambda_m$  are distinct eigenvalues of  $T$  and

$$a_i = \dim G_{\lambda_i}(T)$$

Last time:

$$V = G_{\lambda_1}(T) \oplus \cdots \oplus G_{\lambda_m}(T)$$

So

$$\deg q(x) = a_1 + a_2 + \cdots + a_m = \dim V = n$$

Want to prove this now:

$$p(x) = \det(xI_n - A)$$

where

$$A = [T]_{\beta} \quad \text{for some basis } \beta$$

Recall

$$\dim V = n$$

$N$   $n \times n$  matrix

$N$  is nilpotent if  $N^r = 0$  for some  $r$   
 $r$  s.t.  $N^r = 0, N^{r-1} \neq 0$  is called the index of  $n$

Proved:  $N$  nilpotent  $\Rightarrow N^n = 0$

Example

Suppose  $N$  is nilpotent and  $\exists v \in V = \mathbb{C}^n$  s.t.

$$\alpha = (\vec{v}, N\vec{v}, N^2\vec{v}, \dots, N^{n-1}\vec{v})$$

is a basis of  $V$ , and

$$\beta = (N^{n-1}\vec{v}, N^{n-2}\vec{v}, \dots, N\vec{v}, \vec{v})$$

be another basis.

Find  $[N]_q, [N]_\beta$

Soln (a)

$$\begin{matrix} & v & Nv & N^2v & \dots & N^{n-1}v \\ \begin{matrix} v \\ Nv \\ N^2v \\ \vdots \\ N^{n-1}v \end{matrix} & \begin{bmatrix} 0 & 0 & & & 0 & 0 \\ 1 & 0 & & & 0 & 0 \\ 0 & 1 & & & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & & & 1 & 0 \\ 0 & 0 & & & 0 & 0 \end{bmatrix} \end{matrix}$$

(b)

$$\begin{matrix} & N^{n-1}v & N^{n-2}v & \dots & v \\ \begin{matrix} N^{n-1}v \\ \vdots \\ v \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & & 0 \end{bmatrix} \end{matrix}$$

(transpose of a!)

e.g.  $n=4$

a

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$\beta$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

definition: Let  $\lambda \in \mathbb{C}$ ,  $n \in \mathbb{Z}_{>0}$ .

Define

$$J(\lambda, n) = \begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix} \quad (n \times n)$$

$J(\lambda, n)$  is called a Jordan Block.

$$J(\lambda, 4) = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$$

Note: if  $A = J(\lambda, n)$ , then  $\lambda$  is the only eigenvalue of  $A$ , and

$$V = \mathbb{C}^n = \ker(A - \lambda I)^n$$

so  $N = A - \lambda I$  is nilpotent

(4x4)  
example

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Definition: Let  $T \in \mathcal{L}(V)$ . A basis  $\beta$  of  $V$  is called a Jordan basis of  $T$  if

$$[T]_{\beta} = \begin{bmatrix} J(\lambda_1, n_1) & & & \\ & J(\lambda_2, n_2) & & \\ & & \ddots & \\ & & & J(\lambda_n, n_n) \end{bmatrix}$$

(n x n)

called Jordan canonical form

of  $T$ . It is unique.

$$= J(\lambda_1, n_1) \oplus \dots \oplus J(\lambda_n, n_n)$$

Theorem: If  $\dim V = n < \infty$ ,  $V$  a v.s over  $\mathbb{C}$ , and  $T \in \mathcal{L}(V)$ .

Then  $\exists$  Jordan basis of  $T$ .

Corollary: Every  $A \in \mathbb{C}^{n \times n}$  is similar to a direct sum of Jordan blocks.

$$A = \begin{bmatrix} A_1 & A_2 & \dots & A_r \end{bmatrix} \begin{matrix} A_i \text{ } n_i \times n_i \\ A \text{ } n \times n \\ n = n_1 + \dots + n_r \end{matrix}$$

Remark: If  $A = A_1 \oplus \dots \oplus A_r$  and

$\pi$  is a permutation of  $1, \dots, r$

then  $B = A_{\pi(1)} \oplus A_{\pi(2)} \oplus \dots \oplus A_{\pi(r)}$  is similar to  $A$ .

Example: Suppose  $A$  is  $n \times n$ , has only one eigenvalue  $\lambda$ .

For small  $n$ , find the possible Jordan Canonical form's of  $A$ .

Sol'n

①  $n=2$ ,  $A_{2 \times 2}$

$$A \underset{\substack{\uparrow \\ \text{similar}}}{\sim} \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \text{ or } \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

	$J(\lambda, 1) \oplus J(\lambda, 1)$	$J(\lambda, 2)$
$\dim E_\lambda(A)$	2	1
$\dim G_\lambda(A)$	2	2

②  $n=3$ ,  $A_{3 \times 3}$

$$A \sim \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \text{ or } \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} \text{ or } \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

$J(\lambda, 1) \oplus J(\lambda, 1) \oplus J(\lambda, 1)$ 
 $J(\lambda, 3)$ 
 $J(\lambda, 2) \oplus J(\lambda, 1)$

$\dim E_\lambda(A)$	3	1	2
$\dim G_\lambda(A)$	3	3	3

Definition:  $(k_1, \dots, k_r)$  is called a partition of  $n \in \mathbb{Z}_{>0}$  if

(a)  $k_1 + \dots + k_r = n$

(b)  $k_1 \geq k_2 \geq \dots \geq k_r > 0$

Definition: If  $\underline{k} = (k_1, \dots, k_r)$  is a partition of  $n$ ,

$$\text{let } J(\lambda, \underline{k}) = J(\lambda, k_1) \oplus \dots \oplus J(\lambda, k_r)$$

$$n=4$$

$$A \sim \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \quad \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$$

$J(\lambda, 1111) \qquad J(\lambda, 21)$

$$\begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ \hline 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \quad \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 0 \\ \hline 0 & 0 & 0 & \lambda \end{pmatrix}$$

$J(\lambda, 22) \qquad J(\lambda, 31)$

$$\begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix} J(\lambda, 4)$$

$$\dim E_\lambda(A) = 4, 3, 2, 2, 1 \quad (\text{in order listed})$$

$$\dim G_\lambda(A) = \underline{4} \quad (\text{always } n)$$