

Solve for π ?

$$\bar{\pi} = [\pi_1, \pi_2, \dots, \pi_m]$$

Use condition

$$\bar{\pi} P = \bar{\pi}$$

$$\sum_i \pi_i = 1$$

Example: Two Coins A and B.

let

$$\begin{array}{ll} A: \text{fair}; \Pr[H] = \frac{1}{2} & H=1 \\ B: \Pr[H] = \frac{1}{4} & T=2 \end{array}$$

$$\begin{array}{l} X_0 \leftarrow A \\ X_{n+1} \leftarrow \begin{cases} A & \text{if } X_n = 2 \\ B & \text{if } X_n = 1 \end{cases} \end{array}$$

$$P(0) = [\Pr[X_0=1] \quad \Pr[X_0=2]] = \left[\frac{1}{2} \quad \frac{1}{2} \right]$$

$$P(1) = [\Pr[X_1=1] \quad \Pr[X_1=2]] = P(0)P$$

$$P = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$[\pi_1 \quad \pi_2] \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = [\pi_1 \quad \pi_2]$$

$$\begin{cases} \frac{1}{4}\pi_1 + \frac{1}{2}\pi_2 = \pi_1 \\ \frac{3}{4}\pi_1 + \frac{1}{2}\pi_2 = \pi_2 \end{cases}$$

and

$$\pi_1 + \pi_2 = 1$$

$$\bar{\pi} = \left[\frac{2}{5}, \frac{3}{5} \right]$$

Stationary Markov Chain

If X_0 is distributed at a stationary distribution π , then $\{X_n\}_{n \geq 1}$ is stationary.

$$\Pr[X_n, X_{n-1}, \dots, X_0 = x_0] = \Pr[X_{n+k} = x_n, X_{n-1+k} = x_{n-1}, \dots, X_k = x_0]$$

Why?



$$\Pr[X_0 = x_0] \Pr[X_1 = x_1 | X_0 = x_0] \cdots \Pr[X_n = x_n | X_{n-1} = x_{n-1}]$$

Stationary Distribution as an Equilibrium

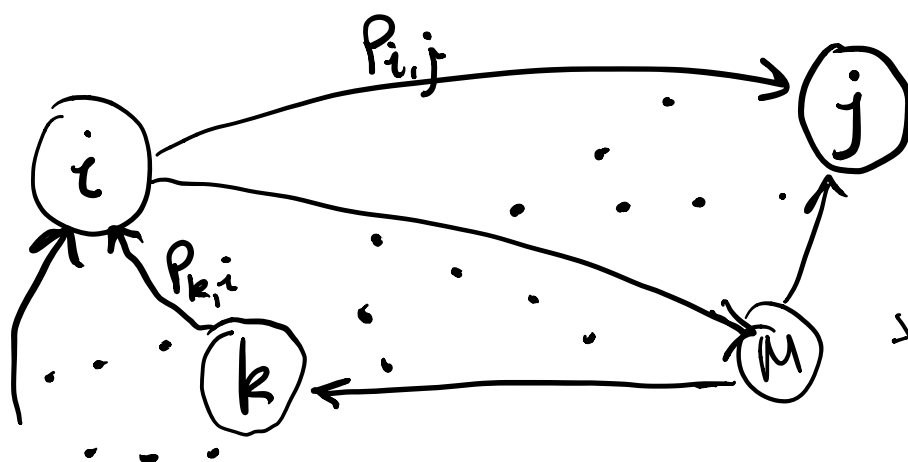


Diagram of possibilities, don't take too seriously

1 gallon of water

X_0 — total probability of 1

Total amount of water given out by i at n ?

$$\sum_{j \neq i} p_i(n) P_{i,j}$$

Water in:

$$\sum_{k \neq i} P_k(n) P_{k,i}$$

If $p_i(0) = \pi_i$

At time $n=1$

Water out of i :

$$\begin{aligned} \sum_{j \neq i} \pi_i P_{i,j} &= \pi_i \sum_{j \neq i} P_{i,j} \\ &= \pi_i (1 - P_{i,i}) \end{aligned}$$

Water in:

$$\sum_{k \neq i} \pi_k P_{k,i} = \pi_k (1 - P_{i,i})$$

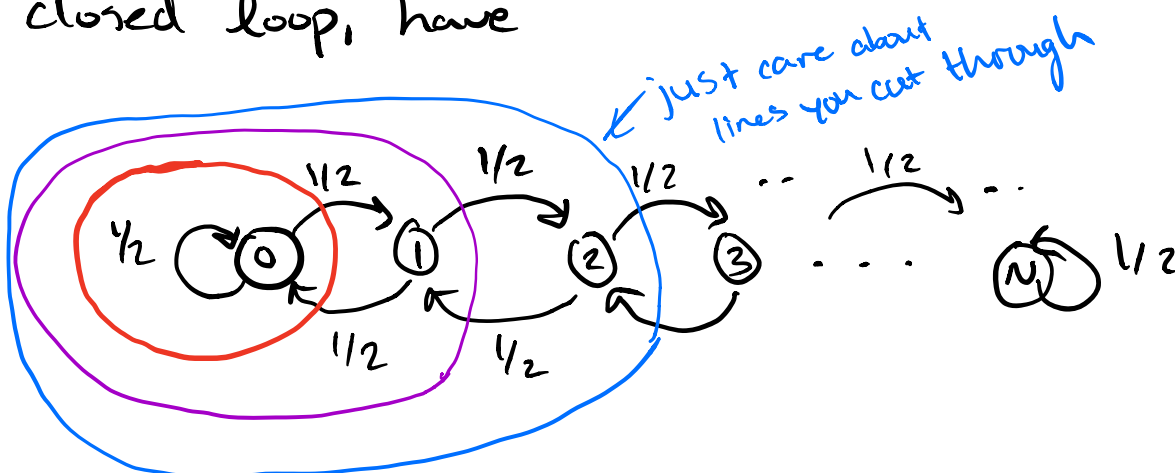
We know

$$\bar{\pi} = \pi P$$

$$\pi_i = \sum_{k \in X} \pi_k P_{k,i} = \sum_{k \neq i} \pi_k P_{k,i} + \pi_i P_{i,i}$$

$$\pi_i (1 - P_{i,i}) = \sum_{k \neq i} \pi_k P_{k,i} \quad !!$$

Generalize, states $0, \dots, N$, ANY arbitrary closed loop, have



$$\frac{1}{2} \pi_0 = \frac{1}{2} \pi_1 \Rightarrow \pi_0 = \pi_1$$

$$\frac{1}{2} \pi_1 = \frac{1}{2} \pi_2 \Rightarrow \pi_1 = \pi_2$$

$$\frac{1}{2} \pi_2 = \frac{1}{2} \pi_3 \Rightarrow \pi_2 = \pi_3$$

⋮

$$\pi_{N-1} = \pi_N$$

$$\Rightarrow \pi_i = \frac{1}{N+1} \quad \forall i$$

Accessible: A state j is accessible from state i if $P_{i,j}^{(n)} > 0$ for some $n \geq 0$

probability of going from $i \rightarrow j$ in n steps

By definition: $i \rightarrow i$ (i is always accessible from itself)

We say i communicates with j if $i \rightarrow j$ and $j \rightarrow i$ (denote $i \leftrightarrow j$).

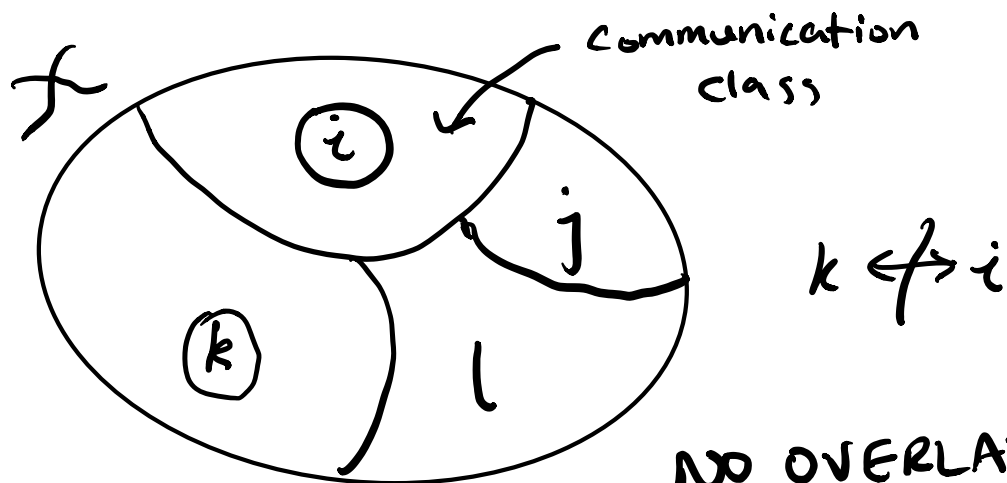
"Communication" is an equivalence relation

① Reflexive: $i \leftrightarrow i$

② Symmetric: if $i \leftrightarrow j$ then $j \leftrightarrow i$

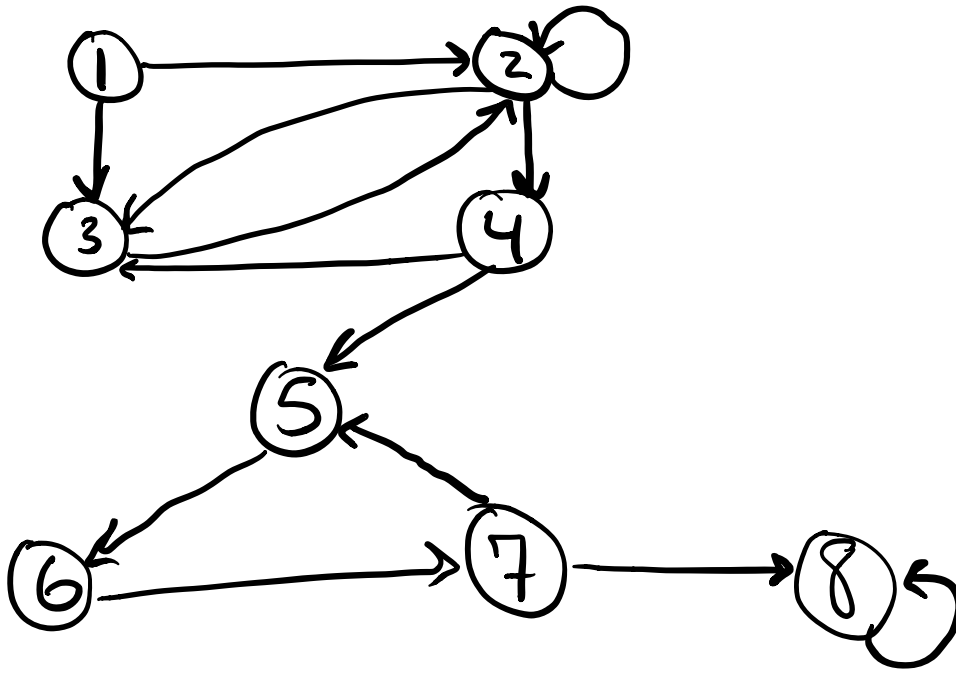
③ Transitive: $i \leftrightarrow j$ and $j \leftrightarrow k$ then $i \leftrightarrow k$

Note that communication partitions our state space.



NO OVERLAP b/t
two different classes by definition.

Example



Step 1: Pick ①

$C_1 = \{①\}$ ← no return state

Pick ②

$C_2 = \{②, ③, ④\}$

Pick ⑤

$C_3 = \{⑤, ⑥, ⑦\}$

$C_4 = \{⑧\}$ ← called an absorbing state