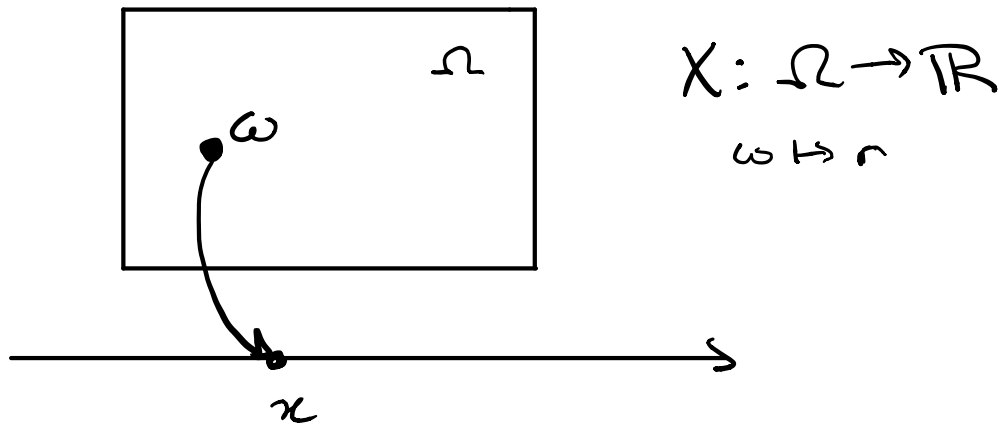
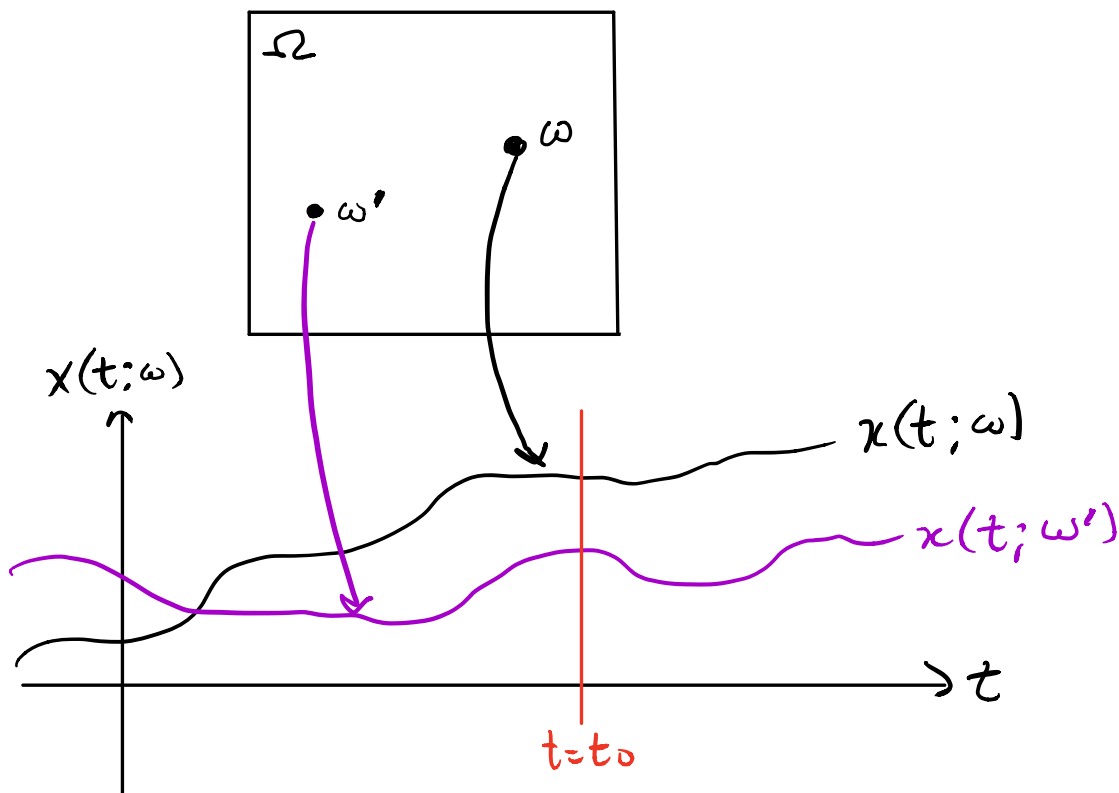


Have a r.v.  $X$



## Random Process

Each sample  $\omega \in \Omega$  in the sample space is mapped to a time function  $X(t, \omega)$



Sampling a random process at a particular time  $t_0$ , you get a random variable  $X(t_0, \omega')$ ,  $X(t_0, \omega)$  random variables

For a fixed  $\omega$ ,

$X(t; \omega)$  is a deterministic function of  $t$ , called a realization or a sample path of this random process

$X(t) \rightarrow$  r.p.       $x(t) \rightarrow$  sample path

## Examples of Random Processes

### Discrete-Time

① An i.i.d sequence of discrete random variables

$$X_1, X_2, X_3, \dots$$

where  $X_i \sim \text{Bernoulli}(p)$

② An i.i.d sequence of continuous random variables

$$X_1, X_2, X_3$$

where  $X_i \sim \mathcal{N}(\mu, \sigma^2)$

③ Binomial Counting Process

(counts # "heads" in a sequence of coin flips)

$$S_n = \sum_{i=1}^n X_i$$

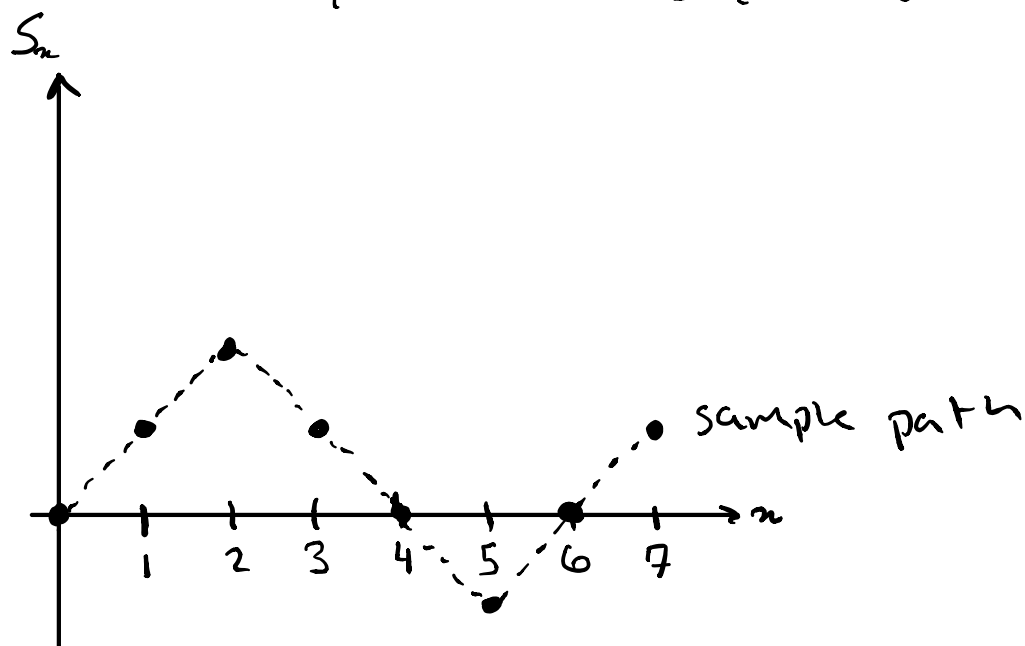
$X_i$  i.i.d Bernoulli( $p$ )

④ Random Walk:

$$S_n = \sum_{i=1}^n Y_i$$

where  $\{Y_i\}_{i=1}^{\infty}$  i.i.d w/ pmf

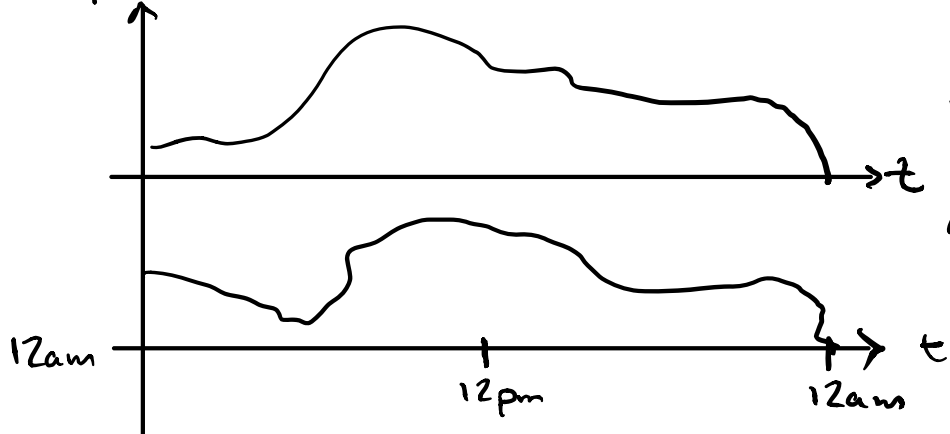
$$\Pr[Y_i=1] = 1 - \Pr[Y_i=-1] = 1/2$$



Continuous - Time

① Temperature in Ithaca on Jan. 1

Temperature



## ② Sinusoid w/ a Random Phase

$$X(t) = \cos(2\pi t + \Theta)$$

where

$$\Theta \sim \text{Unif}[0, 2\pi)$$

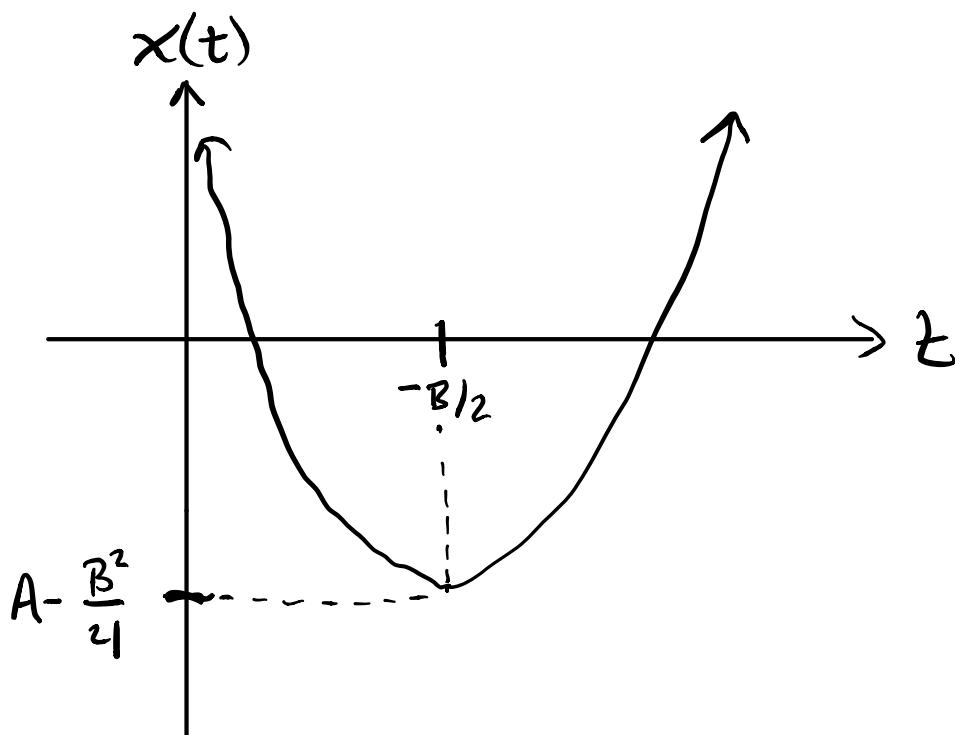
different sample paths  
for each realization of  
 $\Theta$

## ③ Random Parabolas

$$X(t) = A + Bt + t^2$$

where

$A, B \sim \mathcal{N}(0, 1)$  and  $A$  independent  $B$



# Description of Random Processes

Complete Description: A random process is fully specified by the joint CDF

$$F_{x(t_1), x(t_2), \dots, x(t_n)}(x_1, x_2, \dots, x_n) \quad \forall n=1, 2, 3, \dots$$

and all sets of  $t_1, t_2, \dots, t_n$

Example: Bernoulli iid

$$\textcircled{1} X_i \text{ iid}, X_i \sim \text{Bernoulli}(p)$$

joint pmf  $\forall n, k_1, k_2, \dots, k_n$

$$\begin{aligned} P_{X_{k_1}, X_{k_2}, \dots, X_{k_n}}(x_1, x_2, \dots, x_n) &\triangleq \Pr[X_{k_1} = x_1 \cap \dots \cap X_{k_n} = x_n] \\ &= p^{(x_1 + x_2 + \dots + x_n)} (1-p)^{n - (x_1 + x_2 + \dots + x_n)} \end{aligned}$$

## Example: Random Parabolas

$$n=1$$

$$X(t_1) = A + Bt_1 + t_1^2 \sim \mathcal{N}(t_1^2, 1+t_1^2)$$

Why?  $A, B$  independent Gaussian  $\Rightarrow$  Jointly Gaussian

Any linear combination of J.G. r.v.'s is Gaussian.

$$n=2$$

$$X(t_1) = A + Bt_1 + t_1^2$$

$$X(t_2) = A + Bt_2 + t_2^2$$

$\{X(t_1), X(t_2)\}$  Jointly Gaussian!

$$\{X(t_1), X(t_2)\} \sim \mathcal{N}\left(\begin{bmatrix} t_1^2 \\ t_2^2 \end{bmatrix}, \begin{bmatrix} 1+t_1^2 & 1+t_1t_2 \\ 1+t_1t_2 & 1+t_2^2 \end{bmatrix}\right)$$

$$K = \begin{bmatrix} \text{Var}(X_1) & \text{Cov}(X_1, X_2) \\ \text{Cov}(X_2, X_1) & \text{Var}(X_2) \end{bmatrix}$$

$$\text{Cov}(X_1, X_2) = \mathbb{E}[(X(t_1) - \mathbb{E}[X(t_1)])(X(t_2) - \mathbb{E}[X(t_2)])]$$

$$= \mathbb{E}[(A + Bt_1)(A + Bt_2)]$$

$$\begin{aligned} \mathbb{E}[A^2] &= \text{Var}(A) + (\mathbb{E}[A])^2 &= \mathbb{E}[A^2] + \mathbb{E}[ABt_2] + \mathbb{E}[ABt_1] + \mathbb{E}[t_1t_2] \\ &= 1 + t_1t_2 \end{aligned}$$

$n \geq 2$

$$f_{X_1(t_1), \dots, X_n(t_n)}(x_1, x_2, \dots, x_n) = \begin{cases} f_{AB}(a, b) & \text{if } a = x_1 - t_1^2 - \frac{x_2 - t_2^2 - t_1^2 - x_1}{t_2 - t_1} t_1 \\ & b = \frac{(x_2 - t_2^2 - t_1^2 - x_1)}{t_2 - t_1} \\ & x_i = a + b t_i + t_i^2 \quad i \geq 2 \\ 0 & \text{o/w} \end{cases}$$

## Moments of a Random Process

The mean function

$$\mu(t) = \mathbb{E}[X(t)]$$

The auto-covariance function

$$\begin{aligned} C_X(t_1, t_2) &\triangleq \mathbb{E}[(X(t_1) - \mu(t_1))(X(t_2) - \mu(t_2))] \\ &= R_X(t_1, t_2) - \mu(t_1)\mu(t_2) \end{aligned}$$

Example: iid  $X_i \sim \text{Bernoulli}(p)$

$$\begin{aligned} \mu_X(i) &= p \\ R_X(i_1, i_2) &= \begin{cases} p, & i_1 = i_2 \\ p^2, & i_1 \neq i_2 \end{cases} \end{aligned}$$

Example:  $X(t) = \cos(2\pi t + \Theta)$ ,  $\Theta \sim U[0, 2\pi)$

$$\mu_X(t) = \mathbb{E}[\cos(2\pi t + \Theta)]$$

$$= \int_0^{2\pi} \frac{1}{2\pi} \cos(2\pi t + \theta) d\theta = 0$$

$$R_X(t_1, t_2) = \mathbb{E}[X(t_1)X(t_2)]$$

$$= \int_0^{2\pi} \frac{1}{2\pi} \cos(2\pi t_1 + \theta) \cos(2\pi t_2 + \theta) d\theta$$