Recall

VS1 $x+y=y+x$

VS2 $x+(y+z)=(x+y)+z$

VS3 $x+0_V=x$

VS4 $\forall x\in V\ \exists y\in V$ s.t. $x+y=0$

VS5 $1\cdot x=x$

VS6 $a(bx)=(ab)x$

VS7 $a(x+y)=ax+ay$

VS8 $(a+b)x=ax+bx$

OH

Prof: Tuesday 3-5PM MT 503

basic facts (will be on HW)

V is a vector space over $\mathbb{F}$

basic $\begin{cases} \text{(a) the additive identity, } 0_V \text{ is unique} \\ \text{(b) the additive inverse } y \text{ of } x \text{ is unique - denote it by } -x \\ \text{(c) Cancellation: if } x+z=y+z\text{, then } x=y \end{cases}$

(d) $0_{\mathbb{F}}\cdot x=0_V\quad \forall x\in V$

(e) $(-1)\cdot x=-x$ $\quad$ (-1) always exists b/c 1 always exists

(f) $a\cdot 0_V=0_V\quad \forall a\in\mathbb{F}$

Examples of Vector Spaces

① $\mathbb{F}^n$

② $m \times n$ matrices on $\mathbb{F}$, $\mathbb{F}^{m \times n}$

③ $\mathrm{Fun}(X, \mathbb{F}) = \{ f : X \to \mathbb{F} \}$ ← **IMPORTANT**

given a set X function

everything else can be written in terms of this

④ Polynomials : $P(\mathbb{F})$, $\mathbb{F}[x]$

⑤ $\mathbb{F}^\infty = \{ (a_0, a_1, \ldots) \mid a_i \in \mathbb{F} \ \forall i \}$ infinite sequences

$\mathrm{Fun}(X, \mathbb{F})$ Examples

$X = [n] = \{1, 2, \ldots, n\}$

$X = \mathbb{N} = \{0, 1, 2, \ldots\}$

$X = [m] \times [n] = \{ (i, j) \mid i \in [m], \ j \in [n] \}$

We make $\mathrm{Fun}(X, \mathbb{F})$ into a vector space over $\mathbb{F}$

ⓐ $0 : X \to \mathbb{F}$

$x \mapsto 0$

ⓑ If $f, g : X \to \mathbb{F}$, define $f + g : X \to \mathbb{F}$

$(f+g)(x) = f(x) + g(x)$

elements of field; can be added

ⓒ If $a \in \mathbb{F}$, $f : X \to \mathbb{F}$, define

$(af)(x) = a f(x)$

Proposition: Fun$(X, \mathbb{F})$ with these operations (and 0) is a vector space over $\mathbb{F}$

PROOF

VS1 If $f, g : X \to \mathbb{F}$, show $f+g = g+f$

b/c its a field

for $x \in X$, $(f+g)(x) = f(x)+g(x) = g(x)+f(x) = (g+f)(x)$

Since x is arbitrary, applies to all x, thus

$f+g = g+f$ in Fun$(X, \mathbb{F})$

Should always know which space 0 is located

VS4 Given $f : X \to \mathbb{F}$, find $g : X \to \mathbb{F}$ such that $f+g = 0_{\text{Fun}(X,\mathbb{F})}$

Let $x \in X$ Define $g(x) = -f(x)$

Check $(f+g)(x)$

$= f(x)+g(x) = 0$

$= f(x) + (-f(x)) = 0_{\mathbb{F}}$

Because x is arbitrary, $f+g$ is thus $0_{\text{Fun}(X,\mathbb{F})}$ $\forall x$

VS5 If $f : X \to \mathbb{F}$ is in Fun$(X, \mathbb{F})$ $1 \cdot f$ is the function

$1 \cdot f(x) \Rightarrow$ $x \mapsto (1 \cdot f)(x) = 1 \cdot f(x) = f(x)$

Rest of proof is similar

Examples

(1) If $X = [n] = \{1, 2, \ldots, n\}$, what is Fun$(X, \mathbb{F})$?

To give $f : [n] \to \mathbb{F}$ is the same as giving

$f(1), f(2), \ldots, f(n) \in \mathbb{F}$ which is the same as an element of $\mathbb{F}^n$!

Subspaces

Is

$$\left\{ \begin{pmatrix} a \\ b \\ a \end{pmatrix} : a, b \in \mathbb{F} \right\} \subseteq \mathbb{F}^3$$

a vector space (over $\mathbb{F}$)?

$(+)$, scalar multiplication come from $\mathbb{F}^3$.

Definition: Let V be a vector space over $\mathbb{F}$.

A subset $U \subseteq V$ is called a subspace of V if it is a vector space under the operations $+$, scalar mult., and $0_V \in U$ is the additive identity

Theorem: A subset U is a subspace of V if and only if

1. $0_V \in U$
2. U is closed under addition.

   $\forall x, y \in U$; $x + y \in U$ (note $x+y \in V$ as well)
3. U is closed under scalar multiplication

   $\forall a \in \mathbb{F}$, $x \in U$, $ax \in U$

Proof

VS1, VS2 follow immediately

VS3: Must show $(U, +, \cdot, 0_V)$ is a vector space over $\mathbb{F}$

0_V is the additive inverse! ✓

VS4: ?

The rest are automatic b/c they hold in V