Thm: Let $T: V \to W$ be a linear map.

Then $\dim(V) = \dim \underset{\substack{\uparrow \\ \text{Null space }(T)}}{\mathrm{Ker}(T)} + \dim \underset{\substack{\uparrow \\ \text{Range }(T)}}{\mathrm{Im}(T)}$

Rks.

Prose Summary of Ideas

① Take bases B of kernel T and B' of $\mathrm{Im}(T)$. But $B' \in W$, NOT V, so take inverse images under T to get $B'' \subseteq V$. Hope $B \cup B''$ is a basis of V.

② If $\dim V = \infty$ or $\dim W = \infty$ or both?

Q: If $\dim V = \infty$, then is one of $\dim \mathrm{Ker}(T)$, $\dim \mathrm{Im}(T)$ ∞ and other ϕ?

A: NO rsn. One must be ∞, other could be anything.

Example

$P\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$ $\quad P: \mathbb{R}^3 \to \mathbb{R}^3$ (onto xy-plane)

dim=1 Ker P: z-axis $\quad$ $\mathrm{Ker}(I-P) = xy$-plane dim = 2

dim=2 Image P: xy-plane $\quad$ $\mathrm{Im}(I-P) = z$-plane dim = 1

Conjecture

$$\mathrm{Ker}(P) = \mathrm{Im}(I-P)$$

$$\mathrm{Ker}(I-P) = \mathrm{Im}(P)$$

when P is a projection i.e. $P^2 = P$

Proof: Let $\vec{v} \in \mathrm{Ker}(P)$. $P\vec{v} = \vec{0}$

Then $(I-P)\vec{v} = \vec{v} - P\vec{v} = \vec{v} \Rightarrow \vec{v} \in \mathrm{Image}(I-P)$

So $\mathrm{Ker}(P) \subseteq \mathrm{Image}(I-P)$

Let $\vec{v} \in \mathrm{Im}(I-P)$, thus $\vec{v} = (I-P)\vec{w}$

and $P\vec{v} = P(I-P)\vec{w}$

$\vec{0} = P\vec{w} - P^2\vec{w}$

Use hypothesis that $P = P^2$ $\longrightarrow$ $= P\vec{w} - P\vec{w} = \vec{0}$

So $\mathrm{Image}(I-P) \subseteq \mathrm{Ker}(P)$

Thus

$$\mathrm{Ker}(P) = \mathrm{Image}(I-P)$$

Definitions: $T: V \to W$ is injective if $\vec{v} \neq \vec{w} \Rightarrow T\vec{v} \neq T\vec{w}$

surjective if $\vec{w} \in W \Rightarrow \exists \vec{v} \in V$ s.t. $T\vec{v} = \vec{w}$

Q: How does injectivity relate to Kernel(T)?

A) T is injective exactly when dim Ker(T) is 0.

Statement) T is surjective $\Leftrightarrow$ $\dim \mathrm{Im}(T) = \dim W$

Fact: $T: \mathbb{R}^{17} \to \mathbb{R}^{432}$ can NOT be surjective

$17 \neq \dim \operatorname{Ker} T + 432$

can NOT be negative

Similarly, $T: \mathbb{R}^{432} \to \mathbb{R}^{17}$ can NOT be injective.

$$432 = \dim \operatorname{Ker} T + \dim \operatorname{Im} T$$

$$\neq 0 + 17$$

Remark: $\exists$ a continuous function

$f: [0,1] \to [0,1]^2$ that is onto

We will use Matrices to describe Linear Transformations.
We can compute

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix} \to T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$$

2×2 $\quad 2 \times 1 \to 2 \times 1$

must match

result

Recall) Characteristic Polynomial of $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is $\det\begin{pmatrix} \lambda & -1 \\ -1 & \lambda \end{pmatrix}$

$= \lambda^2 - 1 = (\lambda - 1)(\lambda + 1) \Rightarrow$ eigenvalues $1, -1$

$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

We have a basis of $\mathbb{R}^2$ consisting of eigenvalues of T

Example

$$\begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 8 & -1 \\ 1 & 1 \end{pmatrix}$$

ij entry is row i of matrix 1, column j of matrix 2

This bizarre rule of matrix multiplication __is__ composition of LTs

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x+2y \\ y \end{pmatrix}$$

$$S\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2u-v \\ u+v \end{pmatrix}$$

$$T \circ S\begin{pmatrix} u \\ v \end{pmatrix} = T\begin{pmatrix} 2u-v \\ u+v \end{pmatrix} = \begin{pmatrix} 3(2u-v)+2(u+v) \\ u+v \end{pmatrix} = \begin{pmatrix} 8u-v \\ u+v \end{pmatrix}$$

CARE about __this__

Writing down a matrix $T: V \to W$ w.r.t bases B_V & B_W will give varying matrices as B_V, B_W vary but NO NEW info!