Recall

$$1+2+3+4+\cdots \text{ "="} -\frac{1}{12}$$

$$\sum \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$ Basel Problem

Euler realized

Zeta function (2)

$$\sum \frac{1}{n^2} = \frac{\pi^2}{6} = \zeta(2)$$

Where

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$$

is the zeta function

How did Euler realize

$$\sum \frac{1}{n^2} = \frac{\pi^2}{6} ?$$

Well, realize that a function with simple zeros at $0, \pi$ has to be

$$f(x) = C(x-0)(x-\pi)$$

So, $\sin x$ can be written as

$$\sin x = C(x-0)(x-\pi)(x+\pi)(x-2\pi)(x+2\pi)(x-3\pi)(x+3\pi)\cdots$$

{factor out x} $$= Cx\left(1-\frac{x}{\pi}\right)\left(1+\frac{x}{\pi}\right)\left(1-\frac{x}{2\pi}\right)\left(1+\frac{x}{2\pi}\right)\left(1-\frac{x}{3\pi}\right)\left(1+\frac{x}{3\pi}\right)\cdots$$

So, $C=1$ since $\sin x \simeq x$ for small x

Simplifying,

$$\sin x = x\left(1-\frac{x^2}{\pi^2}\right)\left(1-\left[\frac{x}{2\pi}\right]^2\right)\left(1-\left[\frac{x}{3\pi}\right]^2\right)$$

So,

$$\sin x = x\prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2\pi^2}\right)$$ } Euler's infinite product for $\sin(x)$

Next, match powers of x^3 on both sides!

We know,

$$\sin x = x - \boxed{\frac{x^3}{3!}} + \frac{x^5}{5!} - \cdots = x\prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2\pi^2}\right)$$

$$x - \boxed{x^3\left[\frac{1}{\pi^2}+\frac{1}{2^2\pi^2}+\frac{1}{3^2\pi^2}+\cdots\right]} + (\;)x^5 - \cdots$$

stuff

COMPARE $$\frac{x^3}{6} = \frac{x^3}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

So, $\longrightarrow$ $$\boxed{\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2}}$$

Similar could be done for all even $\mathbb{Z}$!

Back to,

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$$

Euler found a product form for this!

Recall how to find prime numbers!

Civ of Aristophanes

1 2 3 ~~4~~ 5 ~~6~~ 7 ~~8~~ ~~9~~ ~~10~~

$$\frac{1}{2^z}\zeta(z) = \frac{1}{2^z}\left(1 + \frac{1}{2^z} + \frac{1}{3^z} + \frac{1}{4^z} + \cdots\right)$$

$$= \frac{1}{2^z} + \frac{1}{4^z} + \frac{1}{6^z} + \frac{1}{8^z}$$

So,

$$\zeta(z) - \frac{1}{2^z}\zeta(z)$$ kills off all even denominators raised to the z-power.

$$1 + \frac{1}{3^z} + \frac{1}{5^z} + \frac{1}{7^z} + \frac{1}{9^z} + \cdots$$

$$= \zeta(z)\left[1 - \frac{1}{2^z}\right]$$

Now take

$$\frac{1}{3^z}\left[\zeta(z)\left[1-\frac{1}{2^z}\right]\right]$$

$$= \frac{1}{3^z} + \frac{1}{9^z} + \frac{1}{15^z} + \frac{1}{21^z} + \cdots$$

So, $\zeta(z)\left[1-\frac{1}{2^z}\right] - \frac{1}{3^z}\left[\zeta(z)\left[1-\frac{1}{2^z}\right]\right]$

$$= \zeta(z)\left[1-\frac{1}{2^z}\right]\left[1-\frac{1}{3^z}\right]$$

$$= 1 + \frac{1}{5^z} + \frac{1}{7^z} + \frac{1}{11^z} + \frac{1}{13^z} + \cdots$$

Got rid of denominators which are multiples of three to the z

$\vdots$ Doing this to infinity

$\vdots$

$$1 = \zeta(z)\prod_{\substack{\text{all primes} \\ p}}\left(1-\frac{1}{p^z}\right)$$

So, $$\zeta(z) = \frac{1}{\prod_{\text{all primes } p}\left(1 - \frac{1}{p^z}\right)}$$

Prime Obsession
John Derbyshire

i.e $$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} = \frac{1}{\prod_{\text{all primes } p}\left(1 - \frac{1}{p^z}\right)}$$