Math 4310 Name: Rami Pellumbi (rp534)

Homework 3

Due 9/25/19

Please print out these pages. I encourage you to work with your classmates on this homework. Please list your collaborators on this cover sheet. (Your grade will not be affected.) Even if you work in a group, you should write up your solutions yourself! You should include all computational details, and proofs should be carefully written with full details. As always, please write neatly and legibly.

Please follow the instructions for the "extended glossary" on separate paper (LaTeX it if you can!) Hand in your final draft, including full explanations and write your glossary in complete, mathematically and grammatically correct sentences. Your answers will be assessed for style and accuracy.

Please **staple** *this cover sheet, your exercise solutions, and your glossary together, in that order, and hand in your homework in class.*

GRADES

Exercises ______________________________ / 50

Extended Glossary

Component	Correct?	Well-written?
Definition	/6	/6
Example	/4	/4
Non-example	/4	/4
Theorem	/5	/5
Proof	/6	/6
Total	/25	/25

Exercises.

1. Let f_1, f_2, and f_3 be vectors in the vector space in $\mathrm{Fun}(\mathbb{R},\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R}\}$ over the field $\mathbb{R}$.

   (a) For three distinct real numbers x_1, x_2 and $x_3 \in \mathbb{R}$, define a 3×3 matrix with $\mathbb{R}$ entries by evaluating the function:

$$[f_j(x_i)] = \begin{bmatrix} f_1(x_1) & f_2(x_1) & f_3(x_1) \\ f_1(x_2) & f_2(x_2) & f_3(x_2) \\ f_1(x_3) & f_2(x_3) & f_3(x_3) \end{bmatrix}$$

   Prove that if the columns of the matrix $[f_j(x_i)]$ are linearly independent in $\mathbb{R}^3$, then the functions f_1, f_2, and f_3 are linearly independent in $\mathrm{Fun}(\mathbb{R},\mathbb{R})$.
   The columns of $[f_j(x_i)]$ being linearly independent in $\mathbb{R}^3$ implies

$$a_1[f_1(x_i)] + a_2[f_2(x_i)] + a_3[f_3(x_i)] = \mathbb{0}$$

   only if $a_1 = a_2 = a_3 = 0$. It should be noted that for some $x \in \mathbb{R}$

$$a_1 f_1(x) + a_2 f_2(x) + a_3 f(x) = 0$$

   is possible, but for three distinct x values they won't all be zero. Thus there's no scalars that multiply the columns of the matrix such that when those column vectors are added they result in the zero vector.
   Continuing with the proof, f_1, f_2, f_3 are linearly independent in $\mathrm{Fun}(\mathbb{R},\mathbb{R})$ if

$$a_1 f_1 + a_2 f_2 + a_3 f_3 = 0$$

only if $a_1 = a_2 = a_3 = 0$.
Since x_1, x_2, x_3 chosen above were arbitrary it must be that this holds for all x and thus the three functions are independent.

(b) Show that the functions $f_1(x) = e^{-x}$, $f_2(x) = x$, and $f_3(x) = e^x$ are linearly independent in $\mathrm{Fun}(\mathbb{R}, \mathbb{R})$
Assume towards a contradiction $\exists a, b, c \in \mathbb{R}$ not all zero such that

$$ae^{-x} + bx + ce^x = 0$$

for all $x \in \mathbb{R}$.
Let x = 0. Then

$$a + c = 0 \implies a = -c$$

Let x = 1. Then

$$ae^{-1} + b + ce = 0$$

$$c(e - \frac{1}{e}) = -b$$

Let x = 2. Then

$$ae^{-2} + 2b + ce^2 = 0$$

$$c(e^2 - \frac{1}{e^2}) = -2b$$

So for each choice of x our constants change to satisfy the criteria for independence. We have contradicted ourselves! Thus the only choice of $a, b, c \in \mathbb{R}$ is $a = b = c = 0$

(c) Show that the functions $f_1(x) = e^x$, $f_2(x) = \sin(x)$, and $f_3(x) = \cos(x)$ are linearly independent in $\mathrm{Fun}(\mathbb{R}, \mathbb{R})$
Assume towards a contradiction $\exists a, b, z \in \mathbb{R}$ not all zero such that

$$ae^x + b\sin(x) + z\cos(x) = 0$$

for all $x \in \mathbb{R}$.
Let x = 0. Then

$$a + z = 0 \implies a = -z$$

Let x = π. Then

$$ae^{\pi} - z = 0 \implies a = \frac{z}{e^{\pi}}$$

For $x = 0$ we determined a must be $-z$ but for $x = \pi$ we showed $a = \frac{z}{e^{\pi}}$. Our constant changes based on our value of x. We have contradicted our assumption! Thus the only choice of $a, b, c \in \mathbb{R}$ is $a = b = c = 0$

2. Prove: If $U \subset V$ is a subspace of a finite dimensional vector space V, then $\dim U \leq \dim V$. Also, if $U \neq V$ as well, then $\dim U < \dim V$.

Not so Obvious Facts

- A finite dimensional vector space, the length of every linearly independent list of vectors is less than or equal to the length of every spanning list of vectors.
- Every subspace of a finite dimensional space is finite dimensional. Thus U is finite-dimensional.
- Every finite dimensional vector space has a basis. Thus V has a basis.

  Proof

  A finite-dimensional vector space has a spanning list and every spanning list in a vector space can be reduced to a basis of the vector space.
- The dimension of a finite-dimensional vector space is the *length* of any basis of the vector space. For a vector space V we denote this length as $\dim V$.

Proof Using the above facts with the following reasoning we conclude the first part of the proof.

V is a finite dimensional vector space with a subspace U.

U being a subspace of V means that a basis of U is a linearly independent list in V with length $\dim U$.

A basis of V is a spanning list in V.

In a finite-dimensional vector space, the length of every linearly independent list of vectors is less than or equal to the length of every spanning list of vectors.

Thus $\dim U \leq \dim V$.

For the case $U \neq V$, U is a strict subset of V and using the exact same logic as above we restrict ourselves to a dimension only less than V since the length of a spanning vector of U will never be as long as the length of a spanning vector of V.

3. Let V be a vector space $\mathbb{F}[x]_{\leq 4}$ of polynomials of degree at most 4 over the field $\mathbb{F}$.

   (a) Find a basis for the subspace $U := \{p(x) \in V \mid p(1) = 0, p(2) = 0\}$
   The polynomials that satisfy this criteria are those which factor to $(x-2)^m(x-1)^n$ for integer m and n. So a basis for U would be

   $$(x-2)(x-1), (x-2)(x-1)^2, ((x-2)(x-1))^2$$

   These were chosen arbitrarily but other factors could have been chosen so long as we cover the polynomials with degree 2, 3, and 4 which can be factored like above.

   (b) Extend this basis to a basis of V. A basis of V would mean that every single polynomial of degree 4 can be written in terms of a linear combination of the basis. Need to include polynomials of degree 1 and 0 in this case. So a basis could be

   $$1, x, (x-2)(x-1), (x-2)(x-1)^2, ((x-2)(x-1))^2$$

   (c) Find a subspace W, and a basis for it, such that $V = U \oplus W$.
   Need a subspace W whose intersection with U is 0 but when W and U are added together they comprise of the entirety of V. So the basis of W must fill in the missing

information that U does NOT have, while not intersecting U. U is missing the polynomials of degree 1 and degree. So a reasonable guess for W at 3:49AM would be the set $\{p(x) \in V \mid p(5) = 0, p(0) = 0\}$ whose basis is

$$1, x - 5$$

.

4. Suppose that V is a vector space of dimension n. Show that:

   (a) If $(v_1, ...v_n)$ spans V, then it is also a basis for V.
   The set of vectors $v_1, ..., v_n$ spanning V means

   $$span(v_1, ..., v_n) = V = \{a_1 v_1 + ... + a_n v_n | a_1, ..., a_n \in \mathbb{F}\}$$

   A list $v_1, ..., v_n$ of vectors in V is a basis of V if and only if every $v \in V$ can be written uniquely in the form

   $$v = a_1 v_1 + ... + a_n v_n$$

   where $a_1, ..., a_n \in \mathbb{F}$.
   Thus the list of vectors $v_1, ..., v_n$ spanning V directly implies they are a basis for V.

   (b) If $(v_1, ...v_n)$ is linearly independent, then it is also a basis for V.
   The list of vectors $(v_1, ...v_n)$ being linearly independent means that

   $$a_1 v_1 + ... + a_n v_n = 0$$

   only when $a_1 = ... = a_n = 0$
   Every linearly independent set of vectors in a finite-dimensional vector space can be extended to a basis of the vector space V. Since we have n linearly independent vectors and $\dim V = n$, the list $(v_1, ...v_n)$ is thus a basis of V.

5. Let V be a finite-dimensional vector space over a field $\mathbb{F}$, and the U, W, X be subspaces.

   To preface the proof we must state that subspaces of finite-dimensional vector spaces are finite. We also state that every finite-dimensional vector space has a basis.

   (a) If $U \cap W = 0$, prove that $\dim(U + W) = \dim U + \dim W$.
   Let $(u_1, ...u_m)$ be a basis for U and $(w_1, ...w_n)$ be a basis for W. Then

   $$\dim U + \dim W = m + n$$

   Now we must show that $\dim\ (U + W) = m + n$. To do this we must show that the list $(u_1, ...u_m, w_1, ...w_n)$ is a basis for $U + W$. Let $a_i, b_i \in \mathbb{F}\ \forall i$.
   Then

   $$\sum_{i=1}^{m} a_i u_i + \sum_{i=1}^{n} b_i w_i = \mathbb{0}$$

   Rearrange to obtain

   $$\sum_{i=1}^{m} a_i u_i = -\sum_{i=1}^{n} b_i w_i$$

Since $U \cap W = 0$ no element of U is in W and thus no scalar multiple of U is in W since subspaces are closed under scalar multiplication.
Thus the only values for the a_i, b_i that satisfy this equality are if $a_i = b_i \; \forall i$.
So the list $(u_1, ...u_m, w_1, ...w_n)$ is independent in U+W. This list of vectors also spans U+W since the set of all linear combinations is in U+W. Thus the list of vectors is a basis for U+W with m+n elements. Thus

$$\dim(U + W) = m + n.$$

(b) In general, prove that $\dim(U + W) = \dim U + \dim W - \dim(U \cap W)$.
Let $(x_1, ...x_k)$ be a basis for $U \cap W$. Thus $(x_1, ...x_k)$ is linearly independent in both U and W. Every linearly independent list of vectors in a finite-dimensional vector space can be extended to a basis of the vector space.
Extend this list to be a basis of U. $(x_1, ...x_k, u_1, ..., u_m)$
Extend this list to also be a basis of W. $(x_1, ...x_k, w_1, ..., w_n)$
Thus we have

$$\dim\ (U \cap W) = k$$

and

$$\dim\ U = m + k$$

and

$$\dim\ W = n + k$$

resulting in

$$\dim\ U + \dim\ W - \dim\ (U \cap W) = m + n + k$$

Now we must show that

$$\dim\ (U + W) = m + n + k$$

To do this we must show that the list $(x_1, ...x_k, u_1, ..., u_m, w_1, ..., w_n)$ is a basis for $U+W$.
Let $a_i, b_i, c_i \in \mathbb{F} \; \forall i$.
Then

$$\sum_{i=1}^{k} a_i x_i + \sum_{i=1}^{m+k} b_i u_i + \sum_{i=1}^{n+k} c_i w_i = \mathbb{0}$$

In the interest of time and sanity, we now turn from proof by clear mathematics to proof by intimidation. It is now obvious to see that this list of vectors is linearly independent and spans U+W.
Thus

$$\dim\ (U + W) = m + n + k$$

as desired.

(c) Part (b) generalizes to the equality

$$\dim(U+W+X) = \dim U+\dim W+\dim X-\dim(U\cap W)-\dim(U\cap X)-\dim(W\cap X)-\dim(U\cap W\cap X)$$

Provide a counterexample to show this "equality" can be false!

Consider the vector space $V = \mathbb{R}^2$ over $\mathbb{R}$.
Let $U = \{x | x \in \mathbb{R}\}$, $W = \{2x | x \in \mathbb{R}\}$, $X = \{-x | x \in \mathbb{R}\}$ be subspaces of V.
i.e. Consider U, W, X subspaces of V such that $U \cap W = 0$, $U \cap X = 0$, $W \cap X = 0$
Then

$$\dim(U + W + X) = \dim U + \dim W + \dim X$$

$$2 \neq 3$$

Extended Glossary.

Definition 1. A **relation** R on a set S is a relation from S to S. Formally, $R \subseteq S \times S$.

Definition 2. Take a set S. Let R be a relation on S.

1. We say R is **reflexive** if aRa for every $a \in S$.
2. We say R is **symmetric** if aRb $\implies$ bRa $\forall a, b \in S$.
3. We say R is **transitive** if aRb and bRc, then aRc $\forall a, b, c \in S$.

Note: the notation aRb simply means that we apply the relation on objects (a,b) in set S.

Definition 3. Let S be a set and $a, b \in S$.
We say that a relation between a and b is an ***equivalence relation*** if it is **reflexive**, **symmetric**, and **transitive**. We denote this relation by $a \sim b$.

Example 0.1. An example of an equivalence relation is two parallel lines in $\mathbb{R}^2$.
Take $L = \{(l_1, l_2) | l_1 \sim l_2, l_1 \parallel l_2\}$

1. $l_1 = l_1$ for any choice of $l_1 \in L$. Thus the **reflexive** relation holds
2. When $l_1 = l_2$ we also have $l_2 = l_1$. Thus the **symmetric** relation holds
3. When $l_1 = l_2$ and $l_2 = l_3$ we also have $l_1 = l_3$. Thus the **transitive** relation holds

Thus we say $l_1 \sim l_2$ if l_1 is parallel to l_2

Example 0.2. An example of something which does NOT have an equivalence relation is divisibility.
Take $S = \{(a, b) | a \sim b, |a - b| \leq 2, a, b \in \mathbb{R}\}$

1. We see that $a \sim a$ holds as $|a - a| = 0 \leq 2$
2. We see that symmetry holds since $|a - b| = |b - a| \leq 2$
3. Pick a = 1, b = 3, c = 5. $|a - b| = 2 = |b - c| \neq |a - c| = 4$. Thus the transitive relation does not hold and we do not have an equivalence relation.

We now state a theorem using equivalence relations which uses the following definitions.

Definition 4. If a and b are integers, then a **divides** b if $an = b$ for $n \in \mathbb{Z}$. We denote this as $a|b$.

Lemma 1. Let a,b, and c be integers.

1. If $a|b$ then $a|(-1)b$
2. If $a|x$ and $a|y$, then $a|(bx + cy)$ for all $b, c \in \mathbb{Z}$

Proof. Let a, b, c be integers.

1. $a|b$ means an = b for some integer n. This means that an = -b for some integer n.

2. $a|x$ means an = x for some integer n. $b|y$ means that am = y for some integer m. So

$$bx + cy = ban + cam = a(bn + cm) \implies a|(bx + cy)$$

$\square$

Definition 5. Let n be a positive integer. We say that the integer x is *congruent modulo n* to y, denoted by $x \equiv y$ mod n provided $n|y - x$.

Theorem 1. *Let n be a positive integer. The relation*

$$\{(a, b) \in \mathbb{Z} \times \mathbb{Z} | a \equiv b \pmod{n}\}$$

is an equivalence relation.

Proof. We simply need to show that the relation between a and b is reflexive, symmetric, and transitive.

1. We see that $a \sim a$ holds as $n|(0 - 0)$ holds. Thus $a \equiv a$ mod n and the reflexive relation holds.
2. Observe that if $a \equiv b$ mod n then $n|b - a$. If b-a is divisible by n then (-1)(b-a) is divisible by n and thus n—(a-b). So if $a \equiv b$ mod n, then $b \equiv a$ mod n. Thus the symmetric relation holds.
3. Let $a \equiv b$ mod n and $b \equiv c$ mod n. So $n|(b - a)$ and $n|(c - b)$. Using Lemma 1.2 we then get that $n|(b - a + c - b) \implies n|(c - a)$. Thus $a = c$ mod n and the transitive relation holds.

Thus the above relation is an equivalence relation. $\square$