OH Exam Week

M 1:30 - 3:30

T 3 - 5

Th 1:30 - 3:30

Problem 4 HW 5

(a), (b) - you should NOT need finite dimensional

(c), (d) - Let V be finite dimensional $S, T \in \mathcal{L}(V)$

Dual Vector Spaces

Recall: ① a LT $\varphi: V \to \mathbb{F}$ is called a <u>linear function</u> on V

<u>example</u>

$V = C^\infty(\mathbb{R})$ choose $n \in \mathbb{Z}, n \geq 0$

Define $\varphi_n: V \to \mathbb{R}$ by $\varphi_n(f) = \int_0^{2\pi} f(x)\cos(n\pi x)\,dx$

② $V^* = \mathcal{L}(V, \mathbb{F})$ dual vector space to V

Proposition

If $\dim V = n$

then $\dim V^* = n = \dim V$

V, V^* isomorphic b/c same dimension

<u>Proof</u>

Know $\dim \mathcal{L}(V, W) = (\dim V)(\dim W)$

$\therefore \mathcal{L}(V, \mathbb{F})$ has $\dim = (\dim V)(\dim \mathbb{F}) = \dim V$

Basis of V^*

Suppose $a = (\vec{v}_1, \dots, \vec{v}_n)$ is a basis of V.

To give a $\varphi: V \to \mathbb{F}$ LT, it suffices to give n numbers

$a_1, \dots, a_n \in \mathbb{F}$ then $\exists! \ \varphi: V \to \mathbb{F}$ s.t. $\varphi(\vec{v}_i) = a_i \ \forall i$

Let's get some elements of V^*

Let $1 \leq i \leq n$.

Define $\varphi_i : V \xrightarrow{LT} \mathbb{F}$

$v_i \mapsto 1$

$v_{j \neq i} \mapsto 0$

$$\varphi_i(a_1 v_1 + \dots + a_n v_n)$$
$$= a_i \varphi(\vec{v}_i) = a_i$$

We have $\varphi_1, \dots, \varphi_n \in V^*$ as just defined

Is $\varphi_1, \dots, \varphi_n$ a basis for V^*?

"easy" lemma: Suppose $\varphi \in V^*$, then $(\varphi: V \to \mathbb{F})$

$$\varphi = \sum_{i=1}^{n} \underbrace{\varphi(v_i)}_{\in \mathbb{F}} \underbrace{\varphi_i}_{\in V^*} \in V^*$$

Proof: To show LHS (φ) = RHS it suffices to show that for each $\vec{v}_j$ $1 \le j \le n$ that $\varphi(\vec{v}_j) = \text{RHS}(\vec{v}_j)$

$$\varphi(\vec{v}_j) = \sum_{i=1}^{n} \left[\varphi(\vec{v}_i)\varphi_i\right](\vec{v}_j)$$

$$= \sum_{i=1}^{n} \varphi(\vec{v}_i)\varphi_i(\vec{v}_j) = \begin{cases} 1 \cdot \varphi(\vec{v}_j) & , i=j \\ 0 & , i \ne j \end{cases}$$

Proposition: Let $a^* = (\varphi_1, \dots, \varphi_n)$

Then a^* is a basis of V^* (assuming a is a basis of V)

One way: Prove $\varphi_1, \dots, \varphi_n$ is LI, than since it has same number of elements as dimension it must span

Another: Have already shown $(\varphi_1, \dots, \varphi_n)$ spans V^*, since $n = \dim V^*$ it is also LI

Third: Show $\varphi_1, \dots, \varphi_n$ LI

Show $\varphi_1, \dots, \varphi_n$ spans V^*

$\boxed{\text{Proof}}$ $a_1\varphi_1 + a_2\varphi_2 + \cdots + a_n\varphi_n = 0$ as functions $V \to \mathbb{F}$

$a_i \in \mathbb{F}$

Show $a_1 = a_2 \cdots = a_n = 0$

Apply $a_1\varphi_1 + a_2\varphi_2 + \cdots + a_n\varphi_n$ to $\vec{v}_i$

$$a_1\varphi_1(\vec{v}_i) + \cdots + a_n\varphi_n(\vec{v}_i) \stackrel{?}{=} 0$$

$$a_i \ne 0$$

example: $V = \mathbb{R}^2$, $V^* = (\mathbb{R}^2)^* \simeq \mathbb{R}^2$ isomorphic to

Definition: Let $T: V \to W$ be a LT. Define the <u>transpose of T</u>

$$T^t : W^* \to V^*$$
$$f \mapsto T^t(f)$$

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[dr, "g"'] & W \arrow[d, "f"] \\ & \mathbb{F} \end{tikzcd}$$

$$T^t(f) = g = fT$$

$S: V \to W$
$T: W \to U$
$TS: V \to U$

Basic Shopping List of Facts

① T^t is a linear transformation

② $(TS)^t(f) = S^t T^t$