① Poisson Process

Suppose we have two samples of Polonium 210. Each sample emits alpha particles at the times of a Poisson process.
Let $\{N_1(t)\}_{t\geq 0}$ and $\{N_2(t)\}_{t\geq 0}$ denote the two independent Poisson processes with rates λ_1 and λ_2 that describe the emissions from the first and second samples, respectively.

A Geiger counter is used to count the emitted alpha particles. Let $N(t)$ denote the number of alpha particles detected by the Geiger counter during the interval $[0,t]$.

(a) Suppose that initially the second Polonium sample is shielded so that only alpha particles from the first sample reach the Geiger counter, i.e.,

$$N(t) = N_1(t).$$

For a given τ, what is the expected number of particles counted by the Geiger counter during $[0,\tau]$? What is the variance?

In $[0,\tau]$, $N_1(t) \sim \text{Pois}(\lambda_1(\tau - 0))$

Thus in this interval we have expected amount

$$\lambda_1\tau$$

and variance

$$\lambda_1\tau$$

by property of Poisson random variables.

(b) Suppose that at the fixed time τ, the shield is removed from the second sample. From that time onward, the Geiger counter can then count particles emitted from both samples, i.e.

$$N(t) = \begin{cases} N_1(t) & \text{if } 0 \le t < \tau \\ N_1(t) + N_2(t) - N_2(\tau) & \text{if } t \geq \tau \end{cases}$$

i. Compute the mean function $\mu_N(t)$ and the autocovariance function $C_N(s,t)$ of $N(t)$.

$$\mu_N(t) = \begin{cases} \mathbb{E}[N_1(t)] & \text{if } 0 \le t < \tau \\ \mathbb{E}[N_1(t) + N_2(t) - N_2(\tau)] & \text{if } t \geq \tau \end{cases}$$

$$= \begin{cases} \lambda_1 t & \text{if } 0 \le t < \tau \\ \lambda_1 t + \lambda_2 t - \lambda_2 \tau & \text{if } t \geq \tau \end{cases}$$

$$= \begin{cases} \lambda_1 t & \text{if } 0 \le t < \tau \\ \lambda_1 t + \lambda_2 (t - \tau) & \text{if } t \geq \tau \end{cases}$$

Let $s < t$.

$$C_N(s,t) \triangleq \mathbb{E}[(N(t) - \mathbb{E}[N(t)])(N(s) - \mathbb{E}[N(s)])]$$

$$= \mathbb{E}[N(t)N(s)] - \mathbb{E}[N(t)\mathbb{E}[N(s)]] - \mathbb{E}[N(s)\mathbb{E}[N(t)]] + \mathbb{E}[N(t)]\mathbb{E}[N(s)]$$

Three cases (again, $s < t$)

① $0 \leq s < t < \tau$

$$= \underline{\mathbb{E}[N(t)N(s)]} - \lambda_1 t \lambda_1 s - \lambda_1 t \lambda_1 s + \lambda_1 t \lambda_1 s$$

$$\lambda_1 s + \lambda_1^2 s^2 + \lambda_1^2 st - \lambda_1^2 s^2 - 2\lambda_1^2 ts + \lambda_1^2 ts$$

$$\mathbb{E}[N(t)N(s)] = \mathbb{E}[N(s)(N(t)-N(s)+N(s))]$$
$$= \mathbb{E}[N^2(s)] + \mathbb{E}[(N(t)-N(s))N(s)]$$
$$= \lambda_1 s + \lambda_1^2 s^2 + \lambda_1 s(\lambda_1(t-s))$$
$$= \lambda_1 s + \lambda_1^2 s^2 + \lambda_1^2 st - \lambda_1^2 s^2$$

$$= \lambda_1 s$$

② $0 < \tau \leq s < t$

$$C_N(s,t) \triangleq \mathrm{Cov}(N(s), N(t))$$
$$= \mathrm{Cov}(N_1(s)+N_2(s)-N_2(\tau),\ N_1(t)+N_2(t)-N_2(\tau))$$
$$= \mathrm{Cov}(N_1(s), N_1(t))$$
$$+ \mathrm{Cov}(N_1(s), N_2(t)-N_2(\tau))$$
$$+ \mathrm{Cov}(N_2(s)-N_2(\tau), N_1(t))$$
$$+ \mathrm{Cov}(N_2(s)-N_2(\tau), N_2(t)-N_2(\tau))$$

$$= \lambda_1 s$$
$$+ 0 \quad (N_1 \perp\!\!\!\perp N_2)$$
$$+ 0 \quad (N_1 \perp\!\!\!\perp N_2)$$
$$+ \mathrm{Cov}(N_2(s)-N_2(\tau),\ N_2(t)-N_2(s)+N_2(s)-N_2(\tau))$$

$= \lambda_1 s$

$+ \mathrm{Cov}(N_2(s)-N_2(\tau), N_2(t)-N_2(s)) \longrightarrow 0$

$+ \mathrm{Cov}(N_2(s)-N_2(\tau), N_2(s)-N_2(\tau))$

$= \lambda_1 s$

$+ 0$ (independent increments)

$+ \mathrm{Var}(N_2(s)-N_2(\tau))$

$= \lambda_1 s + \lambda_2(s-\tau)$

③ $s < \tau < t$

$$
\begin{aligned}
C_N(s,t) &= \mathrm{Cov}(N(s), N(t)) \\
&= \mathrm{Cov}(N_1(s), N_1(t) + N_2(t) - N_2(\tau)) \\
&= \mathrm{Cov}(N_1(s), N_1(t)) + \mathrm{Cov}(N_1(s), N_2(t)-N_2(\tau)) \\
&= \mathrm{Cov}(N_1(s), N_1(t)) + 0 \text{ (independent increments)} \\
&= \lambda_1 s
\end{aligned}
$$

In conclusion,

$$
C_N(s,t) = \begin{cases} \lambda_1 \min(s,t) & , \text{ if } \min(s,t) < \tau \\ \lambda_1 \min(s,t) + \lambda_2(\min(s,t)-\tau), & \text{if } \min(s,t) \geq \tau \end{cases}
$$

ii. Given that there are total 2 particles counted by the Geiger counter during the interval $[\tau-1, \tau+1]$, what is the expected number of particles emitted from the second sample during this interval?

Let X denote the number of particles emitted from the FIRST sample during $[\tau-1, \tau+1]$

Let Y denote the number of particles emitted from the SECOND sample during $[\tau, \tau+1]$

Note that

$$X \sim \text{Pois}(2\lambda_1),\ Y \sim \text{Pois}(\lambda_2)$$

$$X \perp\!\!\!\perp Y$$

Want

$$\mathbb{E}[Y \mid X+Y=2]$$

Y can take values $0, 1, 2$

Have

(by independence)

$$\begin{aligned}\Pr[X+Y=2] &= \Pr[X=0, Y=2] &&= \Pr[X=0]\Pr[Y=2] \\ &\quad + \Pr[X=1, Y=1] &&\quad + \Pr[X=1]\Pr[Y=1] \\ &\quad + \Pr[X=2, Y=0] &&\quad + \Pr[X=2]\Pr[Y=0]\end{aligned}$$

$$= \left(e^{-2\lambda_1}\right)\left(\frac{\lambda_2^2 e^{-\lambda_2}}{2}\right) + \left(2\lambda_1 e^{-2\lambda_1}\right)\left(\lambda_2 e^{-\lambda_2}\right) + \left(\frac{4\lambda_1^2 e^{-2\lambda_1}}{2}\right)\left(e^{-\lambda_2}\right)$$

Then

$$\Pr[Y=2 \mid X+Y=2] = \frac{\Pr[Y=2,\ X+Y=2]}{\Pr[X+Y=2]} = \frac{\Pr[Y=2,\ X=0]}{\Pr[X+Y=2]}$$

$$= \frac{\dfrac{\lambda_2^2 e^{-\lambda_2}}{2} e^{-2\lambda_1}}{\left(e^{-2\lambda_1}\right)\left(\dfrac{\lambda_2^2 e^{-\lambda_2}}{2}\right) + \left(2\lambda_1 e^{-2\lambda_1}\right)\left(\lambda_2 e^{-\lambda_2}\right) + \left(\dfrac{4\lambda_1^2 e^{-2\lambda_1}}{2}\right)\left(e^{-\lambda_2}\right)}$$

$$= \frac{\lambda_2^2}{\lambda_2^2 + 4\lambda_1\lambda_2 + 4\lambda_1^2}$$

$$\Pr[Y=1 \mid X+Y=2] = \frac{\Pr[Y=1,\ X+Y=1]}{\Pr[X+Y=2]} = \frac{\Pr[Y=1,\ X=1]}{\Pr[X+Y=2]}$$

$$= \frac{2\lambda_1\lambda_2 e^{-2\lambda_1} e^{-\lambda_2}}{\left(e^{-2\lambda_1}\right)\left(\dfrac{\lambda_2^2 e^{-\lambda_2}}{2}\right) + \left(2\lambda_1 e^{-2\lambda_1}\right)\left(\lambda_2 e^{-\lambda_2}\right) + \left(\dfrac{4\lambda_1^2 e^{-2\lambda_1}}{2}\right)\left(e^{-\lambda_2}\right)}$$

$$= \frac{4\lambda_1\lambda_2}{\lambda_2^2 + 4\lambda_1\lambda_2 + 4\lambda_1^2}$$

$$\mathbb{E}[Y \mid X+Y=2] = \sum y \Pr[Y=y \mid X+Y=2]$$

$$= 0 \cdot \Pr[Y=0 \mid X+Y=2] + 1 \cdot \Pr[Y=1 \mid X+Y=2] + 2 \cdot \Pr[Y=2 \mid X+Y=2]$$

$$= \frac{\lambda_2^2 + 4\lambda_1\lambda_2}{\lambda_2^2 + 4\lambda_1\lambda_2 + 4\lambda_1^2}$$

② Differentiating Brownian Motion

Let $\{X(t)\}_{t\geq 0}$ be a Brownian motion with parameter σ^2. We have shown that the mean function and the autocorrelation function of Brownian motion are given by

$$\mu_X(t) \triangleq \mathbb{E}[X(t)] = 0$$

$$R_X(s,t) \triangleq \mathbb{E}[X(s)X(t)] = \sigma^2 \min(s,t)$$

and Brownian motion is a Gaussian process.

For a fixed $\varepsilon > 0$, let $\{Y_\varepsilon(t)\}_{t\geq\varepsilon}$ denote the normalized increment of $\{X(t)\}_{t\geq 0}$ over an interval of length ε:

$$Y_\varepsilon(t) = \frac{X(t) - X(t-\varepsilon)}{\varepsilon}, \quad t \geq \varepsilon$$

(a) Is $\{Y_\varepsilon(t)\}_{t\geq\varepsilon}$ a Gaussian Process?

$\forall n$, $\varepsilon \leq t_1 < t_2 < \cdots < t_n$

$$Y = a_1 Y_\varepsilon(t_1) + a_2 Y_\varepsilon(t_2) \cdots + a_n Y_\varepsilon(t_n)$$

$$= \frac{1}{\varepsilon}\Big(a_1\big(X(t_1) - X(t_1-\varepsilon)\big) + a_2\big(X(t_2) - X(t_2-\varepsilon)\big) + \cdots + a_n\big(X(t_n) - X(t_n-\varepsilon)\big)\Big)$$

$\{X(t)\}_{t\geq 0}$ a Brownian motion $\Rightarrow X(t_1) - X(t_1-\varepsilon) \sim \mathcal{N}(0, \sigma^2\varepsilon)$

Since each increment is independent, Y is a linear combination of independent Gaussians; which is Gaussian.

This is true $\forall n$. Thus

$$\{Y_\varepsilon(t)\}_{t\geq\varepsilon} \text{ is a Gaussian process.}$$

(b) Compute the mean function $\mu_Y(t)$ and the autocorrelation function $R_Y(s,t)$ of $\{Y_\varepsilon(s,t)\}_{t\geq\varepsilon}$.

$$\mu_Y(t) \triangleq \mathbb{E}[Y_\varepsilon(t)] = \mathbb{E}\left[\frac{X(t)-X(t-\varepsilon)}{\varepsilon}\right]$$

$$= \frac{1}{\varepsilon}\left(\mathbb{E}[X(t)] - \mathbb{E}[X(t-\varepsilon)]\right)$$

$$= \frac{1}{\varepsilon}(0-0)$$

$$= 0$$

Assume $s<t$

$R_Y(s,t) = C_Y(s,t)$ due to zero mean.

$$C_Y(s,t) = \mathrm{Cov}(Y_\varepsilon(s), Y_\varepsilon(t))$$

$$= \mathrm{Cov}\left(\frac{X(s)-X(s-\varepsilon)}{\varepsilon}, \frac{X(t)-X(t-\varepsilon)}{\varepsilon}\right)$$

If $|t-s|>\varepsilon$ then have independent increments and thus

$$\mathrm{Cov}\left(\frac{X(s)-X(s-\varepsilon)}{\varepsilon}, \frac{X(t)-X(t-\varepsilon)}{\varepsilon}\right) = 0$$

If $|t-s|\leq\varepsilon$, Assume $(t-\varepsilon<s<t)$

$$\mathrm{Cov}\left(\tfrac{1}{\varepsilon}(X(s)-X(s-\varepsilon)), \tfrac{1}{\varepsilon}(X(t)-X(t-\varepsilon))\right)$$

$$= \frac{1}{\varepsilon^2}\Big(\mathrm{Cov}(X(s),X(t)) - \mathrm{Cov}(X(s),X(t-\varepsilon)) - \mathrm{Cov}(X(t),X(s-\varepsilon)) + \mathrm{Cov}(X(s-\varepsilon),X(t-\varepsilon))\Big)$$

$$= \frac{1}{\varepsilon^2}\Big(\sigma^2\min(s,t) - \sigma^2\min(s,t-\varepsilon) - \sigma^2\min(t,s-\varepsilon) + \sigma^2\min(s-\varepsilon,t-\varepsilon)\Big)$$

$$= \frac{\sigma^2}{\varepsilon^2}(\varepsilon-(t-s))$$

In general,

$$R_Y(s,t) = \begin{cases} 0 & , \quad \text{if } |t-s| > \varepsilon \\ \frac{\sigma^2}{\varepsilon^2}\left(\varepsilon - |t-s|\right) & , \quad \text{if } |t-s| \leq \varepsilon \end{cases}$$

(c) Is $\{Y_\varepsilon(t)\}_{t \geq \varepsilon}$ wide-sense stationary? Is it stationary?

$$\mu_Y(t) = 0 \quad \forall \quad t$$

$$R_Y(s,t) = \begin{cases} 0 & , \quad |\tau| > \varepsilon \\ \frac{\sigma^2}{\varepsilon^2}\left(\varepsilon - |\tau|\right) & , \quad |\tau| \leq \varepsilon \end{cases}$$

where $\tau = t - s$

Thus WSS ✓

Since $\{Y_\varepsilon(t)\}_{t \geq \varepsilon}$ a Gaussian process and WSS it is also Stationary.

③ Markov Chain

A fair coin is flipped successively. Each of the following random processes are defined based on the outcomes of this sequence of coin flips. For each of the processes, determine whether it is a Markov Chain. If yes, show why and write out the transition probability matrix. If no, give a counter-example.

(a) Define a process $\{X_n\}_{n\geq 1}$ where

$$X_n = \begin{cases} 1 & \text{if } n^{th} \text{ outcome is "head"} \\ 2 & \text{if } n^{th} \text{ outcome is "tail"} \end{cases}$$

$\{X_n\}_{n\geq 1}$ an i.i.d sequence of Bernoulli$(p=\frac{1}{2})$ random variables. Trivially a Markov process.

$$\Pr[X_{n+1}=x_{n+1} \mid X_n = x_n, \ldots, X_1 = x_1] = \Pr[X_{n+1} = x_{n+1}]$$
$$\Pr[X_{n+1} = x_{n+1} \mid X_n = x_n] = \Pr[X_{n+1} = x_{n+1}]$$

Markov process by definition

Transition matrix given by

$$\mathbb{P} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

(b) Define a process $\{Y_n\}_{n\geq 1}$ where

$$Y_n = \begin{cases} 1 & \text{if } n^{th} \text{ and } (n+1)\text{th outcomes are both "head"} \\ 2 & \text{if } n^{th} \text{ and } (n+1)\text{th outcomes are "head" and "tail"} \\ 3 & \text{if } n^{th} \text{ and } (n+1)\text{th outcomes are "tail" and "head"} \\ 4 & \text{if } n^{th} \text{ and } (n+1)\text{th outcomes are both "tail"} \end{cases}$$

$$\Pr[Y_{n+1} = y_n \mid Y_n = y_n, \ldots, Y_1 = y_1]$$
$$= \Pr[X_{n+2} = x_{n+2}, X_{n+1} = x_{n+1} \mid X_{n+1} = x_{n+1}, \ldots, X_1 = x_1] \quad \text{by iid } \{X\}_{n\geq 1}$$
$$= \Pr[X_{n+2} = x_{n+2}, X_{n+1} = x_{n+1} \mid X_{n+1} = x_{n+1}, X_n = x_n]$$

equal

$$\Pr[Y_{n+1} = y_n \mid Y_n = y_n]$$
$$= \Pr[X_{n+2} = x_{n+2}, X_{n+1} = x_{n+1} \mid X_{n+1} = x_{n+1}, X_n = x_n]$$

Thus a Markov Process

Transition matrix given by

$$\mathbb{P} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

(c) Define a process $\{Z_n\}_{n\geq 1}$ where

$$Z_n = \begin{cases} 1 & \text{if } n\text{th and } (n+1)\text{th outcomes are the same} \\ 2 & \text{if } n\text{th and } (n+1)\text{th outcomes are different} \end{cases}$$

$$\Pr[Z_{n+1} = z_{n+1} \mid Z_n = z_n, \ldots, Z_1 = z_1]$$

$$= \Pr[X_{n+1} = 1] \Pr[Z_{n+1} = z_{n+1} \mid X_{n+1} = 1, Z_n = z_n, \ldots, Z_1 = z_1]$$
$$+ \Pr[X_{n+1} = 2] \Pr[Z_{n+1} = z_{n+1} \mid X_{n+1} = 2, Z_n = z_n, \ldots, Z_1 = z_1]$$

$$= \Pr[X_{n+1} = 1] \Pr[Z_{n+1} = z_{n+1} \mid X_{n+1} = 1] + \Pr[X_{n+1} = 2] \Pr[Z_{n+1} = z_{n+1} \mid X_{n+1} = 2]$$

$$= \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2}$$

$$= \frac{1}{2}$$

$$\Pr[Z_{n+1} = z_{n+1} \mid Z_n = z_n]$$

$$= \Pr[X_{n+1} = 1] \Pr[Z_{n+1} = z_{n+1} \mid X_{n+1} = 1, Z_n = z_n]$$
$$+ \Pr[X_{n+1} = 2] \Pr[Z_{n+1} = z_{n+1} \mid X_{n+1} = 1, Z_n = z_n]$$

$$= \Pr[X_{n+1} = 1] \Pr[Z_{n+1} = z_{n+1} \mid X_{n+1} = 1] + \Pr[X_{n+1} = 2] \Pr[Z_{n+1} = z_{n+1} \mid X_{n+1} = 2]$$

$$= \frac{1}{2}$$

thus a Markov process

The transition matrix is

$$\mathbb{P} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$