Recap

Gaussian rv: $X \sim \mathcal{N}(\mu, \sigma^2)$

Jointly Gaussian: X, Y two rvs

JG if $aX + bY$ is G $\forall a, b \in \mathbb{R}$

JG $\underset{\not\Leftarrow}{\Rightarrow}$ marginally Gaussian

Properties

① If JG, $\underset{\text{Cov}(X,Y)=0}{\text{Uncorrelated}} \Rightarrow$ independence NOT generally true

joint distribution factors into product of marginal distributions

② If you have two rvs which are marginally Gaussian **AND** they're independent, can conclude they are jointly Gaussian

③ If X, Y jointly Gaussian

then $X|Y$ is Gaussian **USEFUL**

with mean $\mathbb{E}[X|Y] = \hat{X}_{LMMSE} = \mathbb{E}[X] + \frac{\text{Cov}(X,Y)}{\text{Var}(Y)}(Y - \mathbb{E}[Y])$

Variance $\text{Var}(X|Y) = \text{Var}(X) - \frac{\text{Cov}^2(X,Y)}{\text{Var}(Y)}$ ← MMSE

in general, a rV w.r.t variable being conditioned on

$= \underbrace{\text{Var}(X)(1 - \rho_{XY})}_{\text{a CONSTANT}}$

For every realization of Y, have the same result

Random Vectors

$$\underline{X} = \begin{pmatrix} X_1 \\ \vdots \\ X_m \end{pmatrix}, \quad \mathbb{E}[\underline{X}] = \begin{bmatrix} \mathbb{E}[X_1] \\ \vdots \\ \mathbb{E}[X_m] \end{bmatrix}$$

Correlation (b/t a random vector w/ itself)

Notice Symmetry

$$\mathbb{E}[\underbrace{\underset{m\times 1}{\underline{X}}\ \underset{1\times m}{\underline{X}^T}}_{m\times m}] = \begin{bmatrix} \mathbb{E}[X_1^2] & \mathbb{E}[X_1X_2] & \cdots & \mathbb{E}[X_1X_m] \\ \mathbb{E}[X_2X_1] & \mathbb{E}[X_2^2] & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{E}[X_mX_1] & \mathbb{E}[X_mX_2] & \cdots & \mathbb{E}[X_m^2] \end{bmatrix}$$

Variance of $\underline{X}_{m\times 1}$

$$\mathbb{E}\left[(\underline{X}-\mathbb{E}[\underline{X}])(\underline{X}-\mathbb{E}[\underline{X}])^T\right] = \begin{bmatrix} Var(X_1) & Cov(X_1X_2) & \cdots & Cov(X_1X_m) \\ Cov(X_2X_1) & Var(X_2) & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ Cov(X_mX_1) & Cov(X_mX_2 & \cdots & Var(X_m) \end{bmatrix}_{m\times m}$$

Covariance matrix K

Cross-Correlation (b/t two random vectors)

$$\mathbb{E}[\underbrace{\underset{m\times 1}{\underline{X}}\ \underset{1\times n}{\underline{Y}^T}}_{m\times n}] = \left\{ \mathbb{E}[X_iY_j] \right\}_{\substack{1\le i\le m \\ 1\le j\le n}}$$

i rows
j columns

$$Cov(\underline{X},\underline{Y}) = \left\{ Cov(X_i,Y_j) \right\}_{\substack{1\le i\le m \\ 1\le j\le n}}$$

Gaussian Random Vectors

X is a Gaussian random vector if its coordinates are JOINTLY GAUSSIAN

$X \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$ a Gaussian random vector $\Leftrightarrow a_1 X_1 + \cdots + a_m X_m$ is Gaussian $\forall a_i \in \mathbb{R}, 1 \le i \le m$.

Use $X \sim \mathcal{N}(\mu, K)$ to denote Gaussian random vector

If $X \sim \mathcal{N}(\mu, K)$, then

1. Any subvector of X is a Gaussian random vector

   - $X_i \sim \mathcal{N}(\mu_i, K_{i,i})$

   - $\begin{bmatrix} X_i \\ X_j \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \mu_i \\ \mu_j \end{bmatrix}, \begin{bmatrix} K_{i,i} & K_{i,j} \\ K_{j,i} & K_{j,j} \end{bmatrix} \right)$

2. $\forall A, b$

$$\underset{n\times 1}{Y} = \underset{n \times m}{A} \underset{m\times 1}{X} + \underset{n\times 1}{b}$$

is a Gaussian random vector

$$Y \sim \mathcal{N}(A\mu + b, \underbrace{AKA^T}_{\mathbb{E}[(Y - \mathbb{E}[Y])(Y - \mathbb{E}[Y])^T]})$$

3. If K is diagonal, every rv in X is independent of every other rv in X

(4) If $\underline{X}_{m\times 1}$ and $\underline{Y}_{n\times 1}$ are jointly Gaussian then they are independent iff $Cov(\underline{X}, \underline{Y}) = 0$

MMSE Estimate of $\underline{X}$ using $\underline{Y}$

MSE $\mathbb{E}[\|\underline{X} - \hat{\underline{X}}\|^2] = \mathbb{E}[\underbrace{(\underline{X} - \hat{\underline{X}})^T}_{1\times m} \underbrace{(\underline{X} - \hat{\underline{X}})}_{m\times 1}]$

$$= \sum_{i=1}^{n} \mathbb{E}[X_i - \hat{X}_i]$$

Need to "design"

$$\hat{\underline{X}} = g(\underline{Y}) = \begin{bmatrix} g_1(Y) \\ g_2(Y) \\ \vdots \\ g_m(Y) \end{bmatrix}$$

Problem decoupled into m independent MMSE estimation problems, one for each X_i.

$$\hat{X}_i = g_i(Y)$$

Our best MMSE estimator for X_i is

$$\mathbb{E}[X_i | \underline{Y}]$$

Need conditional distribution

$$f_{X_i|\underline{Y}}(x|y) = \frac{f_{x_i, \underline{Y}}(x_i, y_1, y_2, \ldots, y_m)}{f_{\underline{Y}}(y_1, \ldots, y_n)}$$

LMMSE Estimator of $\underline{X}$

Similar to rv case,

$$\underset{m\times 1}{\hat{\underline{X}}} = \underset{m\times n}{A}\,\underset{n\times 1}{\underline{Y}} + \underset{m\times 1}{b}$$

$$= \underset{m\times 1}{\mathbb{E}[\underline{X}]} + \underset{m\times n}{\mathrm{Cov}(\underline{X},\underline{Y})}\,\underset{n\times n}{\mathrm{Cov}^{-1}(\underline{Y})}\,\underset{n\times 1}{(\underline{Y}-\mathbb{E}[\underline{Y}])}$$

If $\underline{X}, \underline{Y}$ are jointly Gaussian, (similar to rv case)

$$\mathbb{E}[\underline{X}\mid\underline{Y}] = \underset{m\times 1}{\mathbb{E}[\underline{X}]} + \underset{m\times n}{\mathrm{Cov}(\underline{X},\underline{Y})}\,\underset{n\times n}{\mathrm{Cov}^{-1}(\underline{Y})}\,\underset{n\times 1}{(\underline{Y}-\mathbb{E}[\underline{Y}])}$$

Example: Estimating a Gaussian Signal w/ Gaussian Noise

$$X \sim N(\mu_x, \sigma_x^2)$$

$$Y_i = X + W_i \qquad i = 1, \ldots, n$$

$$W_i \sim N(0, \sigma_w^2)$$

$$\{W_i\}_{i=1,\ldots,n} \text{ and } X \text{ are independent}$$

Thus all Y_i are jointly Gaussian and

$$\underline{Y} = \begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix} \sim N\left(\underbrace{\begin{bmatrix} \mu_x \\ \vdots \\ \mu_x \end{bmatrix}_{n\times 1}}_{\mu_x \vec{1}_{n\times 1}}, K_{\underline{Y}} \right)$$

$$K_{\underline{Y}} = \begin{bmatrix} \sigma_x^2+\sigma_w^2 & \sigma_x^2 & \cdots & \sigma_x^2 \\ \sigma_x^2 & \sigma_x^2+\sigma_w^2 & & \vdots \\ \vdots & & \ddots & \sigma_x^2+\sigma_w^2 \\ \sigma_x^2 & \sigma_x^2 & \cdots & \end{bmatrix} = \sigma_w^2 I + \sigma_x^2 \vec{1}\vec{1}^T$$