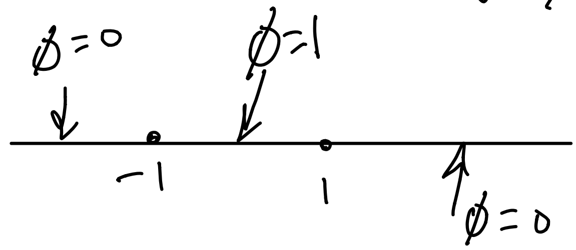
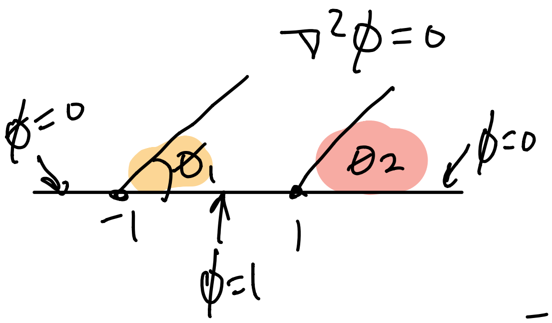
Ex 3: "Wall"

$\nabla^2 \phi = 0$ here



Sol'n

Define two angles



Seek soln of

$$\phi = A_1\theta_1 + A_2\theta_2 + B$$

$$= A_1 \operatorname{Arg}(z+1) + A_2 \arg(z-1) + B$$

$$-\pi < \theta_1 \leq \pi$$

$$-\pi < \theta_2 \leq \pi$$

$\boxed{\text{Impose B.C.}}$

For

$z = x > 1$, have $\theta_1 = \theta_2 = 0$

So

$$\phi = A_1(0) + A_2(0) + B$$

$$\Rightarrow B = 0$$

For

$-1 < x < 1$: $\theta_1 = 0$, $\theta_2 = \pi$

So

$$\phi = A_1(0) + A_2\pi = 1$$

$$A_2 = \frac{1}{\pi}$$

For $z = x < -1$: $\theta_1 = \theta_2 = \pi$

$$\phi = 0 = A_1\theta_1 + \frac{1}{\pi}\theta_2$$

$$0 = A_1\pi + \frac{1}{\pi}\pi$$

$$A_1 = -\frac{1}{\pi}$$

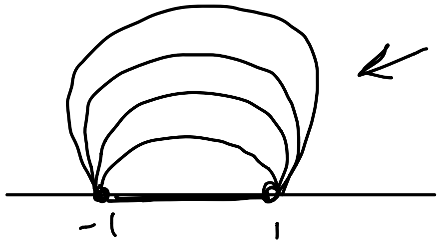
Therefore,

$$\phi = \frac{1}{\pi}\left[\text{Arg}(z-1) - \text{Arg}(z+1)\right]$$

$$= \frac{1}{\pi}\left[\theta_2 - \theta_1\right]$$

Beautiful Geometry Hidden in the Formula!

The isotherms look like

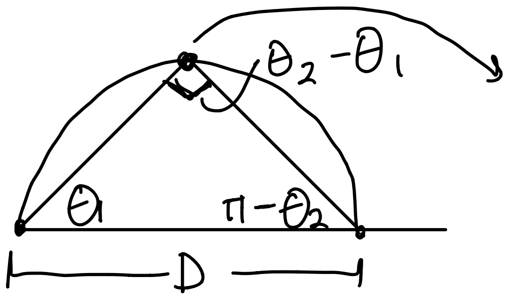


all arcs of circles w/ a chord given by $-1 < x < 1$, $y = 0$

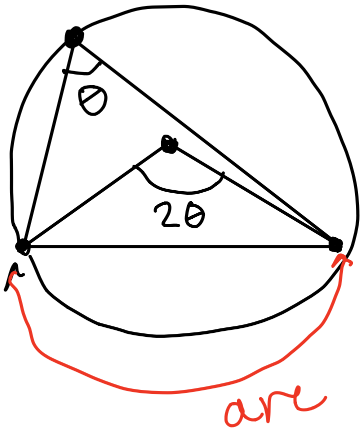
Contours of constant ϕ are arcs of circles through ± 1.

Ex: When $\phi = 1/2$, then

$$\frac{1}{2} = \frac{1}{\pi}(\theta_2 - \theta_1) \Rightarrow (\theta_2 - \theta_1) = \frac{\pi}{2}$$



whenever we move this point we have an angle of $\pi/2$. Geometry theorem

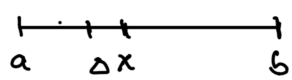


Note: not discussing 3.5
→ Should still look at it!

Complex Integration (Chapter 4)

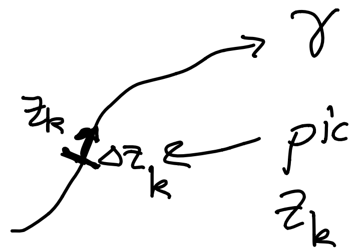
In Calculus, we had

$$\int_a^b f(x)\,dx$$



Let γ = oriented piecewise smooth curve in the complex plane.

Define $\int_\gamma f(z)\,dz$ } kinda like a line integral from multi



pick some point z_k and compute f, f is given

Take

$$\sum_{k=1}^{N} f(z_k)\Delta z_k = \int_\gamma f(z)\,dz$$

"won't be proven"

The limiting complex # is called the integral of $f(z)$ along γ.

"Contour Integral" ; γ = "contour"

The definition is not too useful for calculations. But it does give us an important estimate.

The ML bound

$$\left|\int_\gamma f(z)\,dz\right| \le ML$$

where $M = \max_{z \in \gamma} |f(z)|$

and L = length of γ

'Proof'

$$\left|\sum_{k=1}^{N} f(z_k)\,\Delta z_k\right| \le \left|\sum_{k=1}^{N} f(z_k)\right| \left|\Delta z_k\right|$$

"Triangle Inequality"

$$\leq M \sum_{k=1}^{N} |\Delta z_k| \leq ML$$

since $|f(z_k)| \leq M$

Properties of $\int f(z)\, dz$

- $\int_{-\gamma} f(z)\, dz = -\int_{\gamma} f(z)\, dz$

  opposite orientation

- $\int_{\gamma_1 + \gamma_2} f(z)\, dz = \int_{\gamma_1} f(z)\, dz + \int_{\gamma_2} f(z)\, dz$

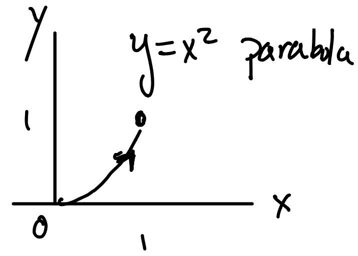
To <u>calculate</u>, we need a <u>Fundamental Theorem</u> (connects integrals to derivitives)

In Real variables, this says

$$\int_a^b f(x)\, dx = F(b) - F(a) \ ; \text{ where } F' = f$$

Some complex integrals reduce to this case.

Example: find $\int_{\gamma_1} \bar{z}\, dz$ where



$f(z) = \bar{z}$ = Complex conjugate

γ_1 = arc of parabola from $(0,0)$ to $(1,i)$

Write parabola in parametric form as

$$x(t) = t \qquad 0 \le t \le 1$$
$$y(t) = t^2$$

$$z = x+iy = t+it^2$$
$$\bar{z} = x-iy = t-it^2$$
$$dz = dt + i2t\,dt$$

$$\bar{z}\,dz = (t-it^2)(dt+i2t\,dt)$$

$$\int_{\gamma_1} \bar{z}\,dz = \int_0^1 (t-it^2)(1+2it)\,dt$$

Note the real range of integration.

$$= \int_0^1 (t - it^2 + 2it^2 + 2t^3)\,dt$$

$$= \frac{t^2}{2} + \frac{it^3}{3} + \frac{2t^4}{4}\Big|_0^1 = \boxed{1+\frac{1}{3}i}$$

Note: $\bar{z}$ is not analytic & therefore no path independence!

Ex: γ_2 = straight line from $(0,0)$ to $(1,1)$

$x(t) = t$

$y(t) = t$

$\bar{z} = t - it$

$dz = dt(1-i)$

$$\int_0^1 (t-it)(1-i)\,dt = 1 \neq 1 + \frac{i}{3} \; !$$