

Let

$$f = u + iv$$

be an analytic function. Then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

and

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

These are the **Cauchy-Riemann** equations. They link the real + imaginary part of an analytic function.

Example: Suppose $f(z) = u(x,y) + iv(x,y)$ is analytic. Find $v(x,y)$.

$$\frac{\partial u}{\partial x} = 1 + y = \frac{\partial v}{\partial y}, \quad -\frac{\partial u}{\partial y} = -x = \frac{\partial v}{\partial x}$$

$$v(x,y) = \int \frac{\partial v}{\partial y} dy = y + \frac{1}{2}y^2 + h(x)$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \Rightarrow h'(x) = -x$$

$$h(x) = \int -x dx = -\frac{1}{2}x^2 + C$$

$$v(x,y) = y + \frac{1}{2}(y^2 - x^2) + C$$

$$f(z) = x + xy + i\left[y + \frac{1}{2}(y^2 - x^2) + C\right]$$

$$= x + iy + xy + i\left(\frac{y^2}{2} - \frac{x^2}{2}\right) + \text{constant}$$

$$= z - i\frac{z^2}{2}$$

only a function of z , NOT \bar{z} !

Q: If u, v satisfy C-R equations, is $f = u + iv$ analytic?

A: Yes, **IF** u, v are smooth enough. - i.e. they have continuous partial derivatives (at least first order)

Two important consequences of C-R equations.

① Real + Imaginary parts of an analytic function are "harmonic" functions.

→ they satisfy Laplace's equation $\nabla^2 \phi = 0$

where

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

Proof: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \xrightarrow{\partial/\partial x} \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial v}{\partial y} \right)$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \xrightarrow{\partial/\partial y} \frac{\partial^2 u}{\partial y^2} = -\frac{\partial}{\partial y} \left(\frac{\partial v}{\partial x} \right)$$

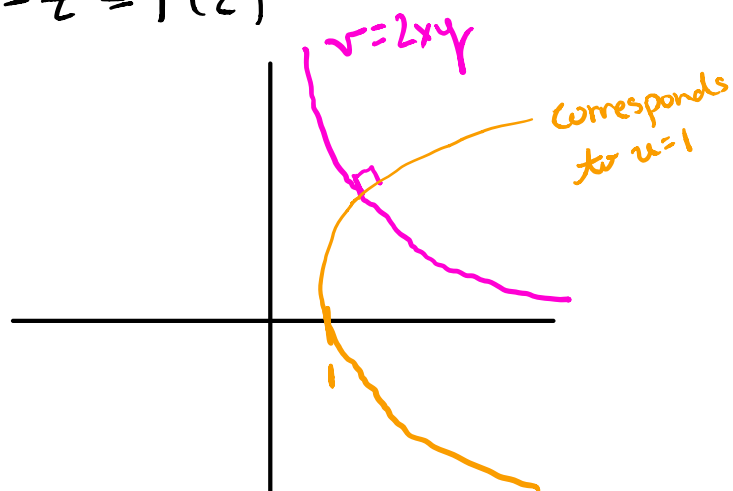
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x}$$

Note: If v has continuous 2^{nd} derivatives,
AND IT DOES, order of mixed partial
 derivatives is irrelevant.

Therefore, $\nabla^2 u = 0$ $\left\{ \begin{array}{l} \text{- governs flow of fluid} \\ \text{- elastic equilibrium} \\ \text{- thermal equilibrium} \end{array} \right.$

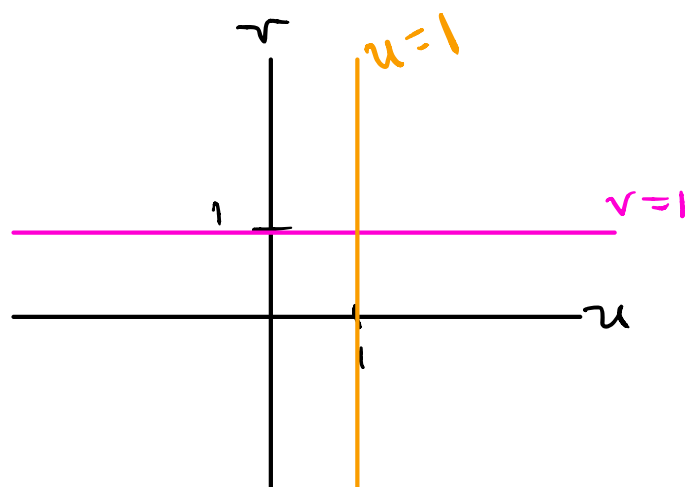
(2) Contours of u, v (in z -plane) intersect at right
 angles. [∂ pts where $f'(z) \neq 0$]

$$w = z^2 = f(z)$$



z -plane
 intersect @ right angles.

$$w = u + iv = x^2 - y^2 + i2xy$$



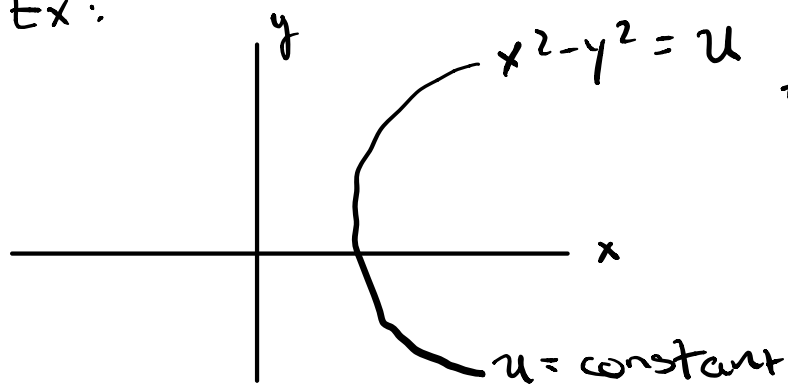
w -plane
 $u = x^2 - y^2$, $v = 2xy$

Physically: Think of contours as electrostatic field lines

Lets see why they meet at right angles!

Recall: ∇u is a vector \perp to contours of u . It
 points towards increasing u .

Ex:



$$(f(z) = z^2)$$

$$\nabla u = \left\langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right\rangle$$

$$\nabla u = \langle 2x, -2y \rangle$$

"a way of calculating
normal vectors"

Angle between contours of u, v = angle between their normal vectors.

$$\nabla v = \langle 2y, 2x \rangle$$

$$\nabla u \cdot \nabla v = \langle 2x, -2y \rangle \cdot \langle 2y, 2x \rangle$$

$$= 4xy - 4xy = 0 \quad \therefore \text{intersect at } 90^\circ \text{ angles!}$$

More generally,

$$\nabla u \cdot \nabla v = \left\langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right\rangle \cdot \left\langle \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \right\rangle$$

$$= \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} \quad \leftarrow \text{C-R Substitution}$$

$$= \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} = 0$$

\therefore either u, v contours are \perp or $\nabla u = 0$ (in which case $\nabla v = 0$ by C-R eqns and $f'(z) = 0$!)

Exponential Function, e^z

Define e^z as $e^z = e^{(x+iy)} = e^x e^{iy}$

- e^x is usual exponential function

- $e^{iy} = \cos(y) + i \sin(y)$

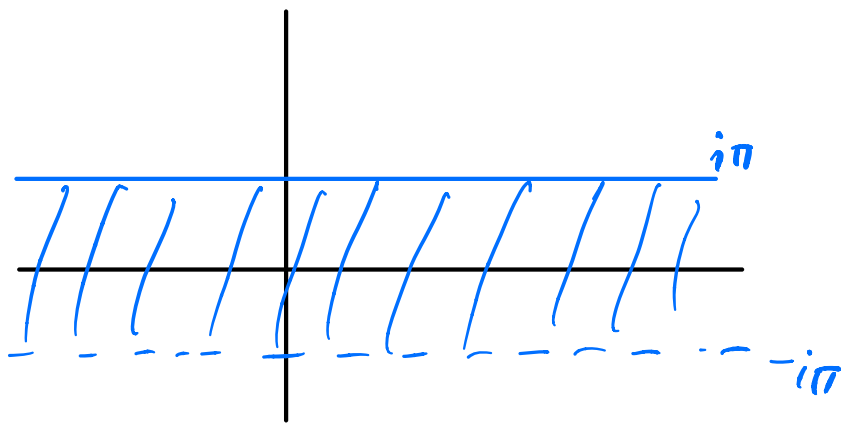


$$e^{iy} = \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \dots \right) + i \left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \dots \right)$$

$$= 1 + iy - \frac{y^2}{2!} - i \frac{y^3}{3!} \dots = e^{iy}$$

e^z is periodic with period $2\pi i$.

$$e^{z+2\pi i} = e^z e^{2\pi i} = e^z, \quad e^{2\pi i} = 1!$$



e^z is injective in this strip (one-to-one)

Its behavior outside the strip is a repeat of what it does in there!

Series:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

analytic everywhere \Rightarrow "entire"
 \rightarrow converges everywhere in complex plane

Also

$$\frac{d}{dz} e^z = e^z$$

$$|e^z| = |e^x e^{iy}| = e^x \quad \left. \begin{array}{l} e^x > 0 \quad \forall x \\ |e^{iy}| = 1 \quad \forall y \end{array} \right\}$$

Define

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}, \quad \cos(z) = \frac{e^{iz} + e^{-iz}}{2i}$$

Example: solve $e^z = 1$

$$e^x e^{iy} = 1$$

$$e^x = 1, \quad x = 0$$

$$e^{iy} = 1, \quad y = 2\pi k, \quad k \in \mathbb{Z}$$