ECE4110: Random Signals in Communications and Signal Processing

Spectral Analysis and Linear Filtering of Random Processes

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Outline

- Autocorrelation function and power spectrum density.
- Joint WSS, crosscorrelation function, and cross power spectrum density.
- White noise.
- Linear filtering of random processes.
- Optimal linear filtering and Wiener filter.

WSS and Correlation Functions

Wide Sense Stationarity (WSS):

A random process is wide sense stationary (WSS) if

$$\begin{aligned} (i) \quad & \mathbb{E}[X(t_i)] = m \\ (ii) \quad & R_X(t_1, t_2) \stackrel{\Delta}{=} \mathbb{E}[X(t_1)X(t_2)] \\ & = R_X(t_1 - t_2) \\ & = R_X(\tau), \quad \text{where } \tau = t_1 - t_2 \end{aligned}$$

Properties of the Autocorrelation function $R_X(\tau)$:

Let $R_X(\tau)$ be the autocorrelation function of a zero-mean WSS random process. Then

1. $R_X(0) = \mathbb{E}[X^2(t)]$ is the average power of $X(t)$.
2. $R_X(\tau)$ is even: $R_X(-\tau) = R_X(\tau)$.
3. $R_X(0) \geq |R_X(\tau)|$.

Joint WSS:

Two random processes $\{X(t)\}_{t=-\infty}^{\infty}$ and $\{Y(t)\}_{t=-\infty}^{\infty}$ are jointly wide sense stationary (WSS) if

$$\begin{aligned} (i) \quad & \text{both } \{X(t)\}_{t=-\infty}^{\infty} \text{ and } \{Y(t)\}_{t=-\infty}^{\infty} \text{ are WSS;} \\ (ii) \quad & R_{X,Y}(t_1, t_2) \stackrel{\Delta}{=} \mathbb{E}[X(t_1)Y(t_2)] \\ & = R_{X,Y}(t_1 - t_2) \\ & = R_{X,Y}(\tau), \quad \text{where } \tau = t_1 - t_2 \end{aligned}$$

Power Spectrum Density

Power Spectrum Density:

□ The power spectrum density $S_X(f)$ of a discrete-time WSS random process $\{X_n\}_{n=-\infty}^{\infty}$ is the discrete-time Fourier transform of the autocorrelation function $R_X(k)$:

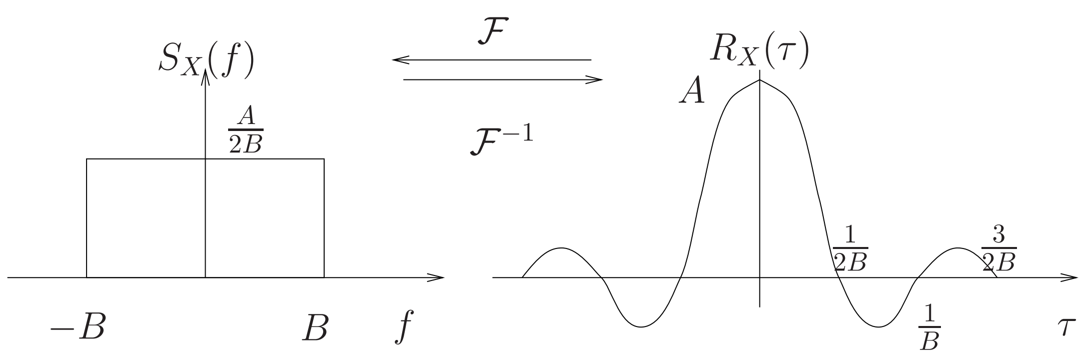
$$S_X(f) \triangleq \sum_{k=-\infty}^{\infty} R_X(k)e^{-j2\pi fk} \qquad (-\frac{1}{2} < f \leq \frac{1}{2})$$

$$R_X(k) = \int_{-\frac{1}{2}}^{\frac{1}{2}} S_X(f)e^{j2\pi fk}df$$

□ The power spectrum density $S_X(f)$ of a continuous-time WSS random process $\{X(t)\}_{t=-\infty}^{\infty}$ is the Fourier transform of the autocorrelation function $R_X(\tau)$:

$$S_X(f) \triangleq \int_{-\infty}^{\infty} R_X(\tau)e^{-j2\pi f\tau}d\tau$$

$$R_X(\tau) = \int_{-\infty}^{\infty} S_X(f)e^{j2\pi f\tau}df$$



$\frac{A}{2B}rect(\frac{f}{2B})$ $\qquad$ $Asinc(2B\tau)$

Power Spectrum Density

Properties of Power Spectrum Density:

For a real-valued random process $\{X(t)\}_{t=-\infty}^{\infty}$,

1. $S_X(f) \geq 0$ for all f;
2. $S_X(f)$ is real and even;
3. Average power:

$$R_X(0) = \int_{-\frac{1}{2}}^{\frac{1}{2}} S_X(f) df \quad \text{(discrete time)}$$

$$R_X(0) = \int_{-\infty}^{\infty} S_X(f) df \quad \text{(continuous time)}$$

Cross Power Spectrum Density:

For jointly WSS random processes $\{X(t)\}_{t=-\infty}^{\infty}$ and $\{Y(t)\}_{t=-\infty}^{\infty}$, the cross power spectrum density $S_{X,Y}(f)$ is the Fourier transform of the crosscorrelation function $R_{X,Y}(\tau)$.

Linear Filtering of Random Processes

Discrete Time:

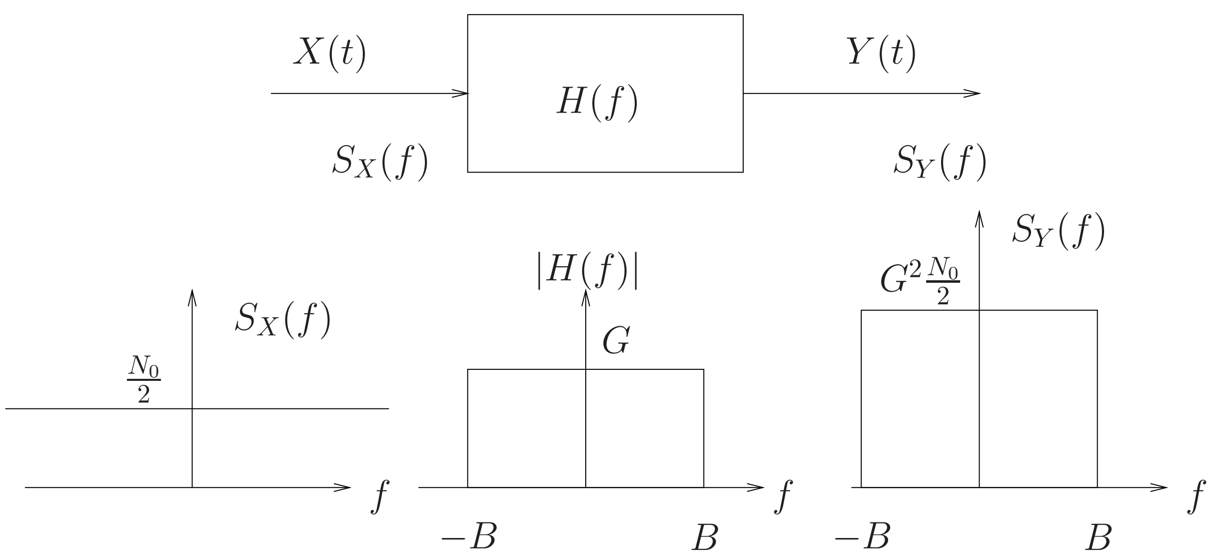
$$Y_n = \sum_{l=-\infty}^{\infty} h_l X_{n-l}$$

$$S_Y(f) = |H(f)|^2 S_X(f) \quad (-\frac{1}{2} < f \leq \frac{1}{2}) \tag{1}$$

Continuous Time:

$$Y(t) = \int_{-\infty}^{\infty} h(s)X(t-s)ds$$

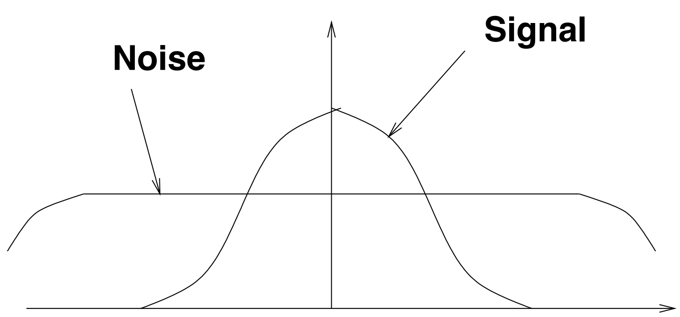
$$S_Y(f) = |H(f)|^2 S_X(f) \tag{2}$$



White Noise and Gaussian Processes

White Noise:

- Continuous-time white noise $\{X(t)\}$: a WSS process with zero mean and PSD $S_X(f) = N_0/2$ within the frequency range of interest $f \in [-W, W]$.
- Discrete-time white noise $\{X_n\}$: a sequence of zero-mean and uncorrelated random variables each with variance $\mathrm{Var}(X_n) = \sigma^2$, i.e., $R_X(k) = \sigma^2 \delta_k$ and $S_X(f) = \sigma^2$ $(-\frac{1}{2} < f \leq \frac{1}{2})$.



Gaussian Processes

- Definition: A random process $X(t)$ is Gaussian if for all n and $t_1, \cdots, t_n$, random variables $X(t_1), \cdots, X(t_n)$ are jointly Gaussian.
- Properties:
  - The output of a linear filter driven by a Gaussian process is Gaussian.
  - Wide sense stationarity implies strict stationarity.

Optimal Filtering of Random Processes

Optimal Filtering:

$\{X_n\}_{n=-\infty}^{\infty}$ and $\{Z_n\}_{n=-\infty}^{\infty}$ are zero-mean jointly WSS random processes. Estimate Z_n using a linear function of $\{X_{n-a}, \ldots, X_{n+b}\}$:

$$\hat{Z}_n = \sum_{j=-b}^{a} h_j X_{n-j}$$

Choose the linear filter response $\{h_n\}_{n=-b}^{a}$ to minimize MSE $\mathbb{E}[(Z_n - \hat{Z}_n)^2]$.

The Optimality Equations:

Based on the orthogonality principle, the optimal linear filter $\{h_n\}_{n=-b}^{a}$ satisfies

$$R_{Z,X}(m) = \sum_{j=-b}^{a} h_j R_X(m-j), \qquad \forall m = -b, \ldots, a.$$

The MMSE is given by

$$R_Z(0) - \sum_{j=-b}^{a} h_j R_{Z,X}(j)$$

Wiener Filter:

For infinite smoothing with $a = \infty$ and $b = \infty$, we have

$$R_{Z,X}(m) = \sum_{j=-\infty}^{\infty} h_j R_X(m-j) = h(m) * R_X(m)$$

Taking Fourier transform leads to

$$H(f) = \frac{S_{Z,X}(f)}{S_X(f)}$$