

Theorem: $S = (\vec{v}_1, \dots, \vec{v}_k)$ orthogonal set in V
 Spanning $W \subseteq V$, then for $\vec{w} \in W$

$$\vec{w} = \sum_{i=1}^k \frac{\langle \vec{w}, \vec{v}_i \rangle}{\langle \vec{v}_i, \vec{v}_i \rangle} \vec{v}_i$$

Corollary If S is orthogonal set in V ^{inner product space} then S is LI

Proof Suppose $\vec{v}_1, \dots, \vec{v}_k \in S$

$$\text{If } c_1 \vec{v}_1 + \dots + c_k \vec{v}_k = \vec{0}$$

$$\text{dot w/ } \vec{v}_1 \quad 0 + \dots + 0 + c_1 \langle \vec{v}_1, \vec{v}_1 \rangle + 0 + \dots + 0 = 0$$

$$\Rightarrow c_1 = 0 \quad \forall i$$

$$\therefore (\vec{v}_1, \dots, \vec{v}_k) \text{ LI}$$

Remark Cool fact

if $B = (\vec{v}_1, \dots, \vec{v}_k)$ is an orthonormal basis of V and $\vec{v} \in V$

then

$$[\vec{v}]_{\beta} = \begin{bmatrix} \langle \vec{v}_1, \vec{v} \rangle \\ \vdots \\ \langle \vec{v}_k, \vec{v} \rangle \end{bmatrix}$$

Gram-Schmidt

Let V be an inner product space.

Question: Can we find an orthonormal basis for V ? ($\dim V = n < \infty$)

Example: Consider 2 vectors $\vec{v}, \vec{w} \in V$ inner product space.

Suppose (\vec{v}, \vec{w}) are LI.

Consider $W = \text{span}(\vec{v}, \vec{w}) \subseteq V$

(a) Find an orthonormal basis of W .

Let $p_v(\vec{w}) = \text{projection of } w \text{ onto span } \{v\}$

then $\vec{v} \perp (\vec{w} - p_v(\vec{w}))$

How?

$$p_v(\vec{w}) = c\vec{v} \quad \text{some } c \in \mathbb{R}$$

$$\vec{v} \perp (w - c\vec{v}) \quad \text{means } \langle v, w - cv \rangle = 0$$

i.e.

$$\langle v, w \rangle - c \langle v, v \rangle = 0$$

$$c = \frac{\langle v, w \rangle}{\langle v, v \rangle}$$

then $\langle \vec{v}, \vec{w} - \frac{\langle v, w \rangle}{\langle v, v \rangle} \vec{v} \rangle$ is an orthogonal basis of W

non-zero, LI, spans W

Theorem (Gram-Schmidt)

Let V be an inner product space.

Define

$$S' = (\vec{v}_1, \dots, \vec{v}_n) \text{ by}$$

$$\vec{v}_1 = \vec{w}_1$$

$$\vec{v}_2 = \vec{w}_2 - \frac{\langle \vec{w}_2, \vec{v}_1 \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle} \vec{v}_1$$

⋮

$$\vec{v}_k = \vec{w}_k - \frac{\langle \vec{w}_k, \vec{v}_1 \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle} \vec{v}_1 - \frac{\langle \vec{w}_k, \vec{v}_2 \rangle}{\langle \vec{v}_2, \vec{v}_2 \rangle} \vec{v}_2 - \dots - \frac{\langle \vec{w}_k, \vec{v}_{k-1} \rangle}{\langle \vec{v}_{k-1}, \vec{v}_{k-1} \rangle} \vec{v}_{k-1}$$

then

① S' is an orthonormal set

② $\text{span}(S') = \text{span}(S)$

in fact,

③ $\text{span}(\vec{v}_1, \dots, \vec{v}_k) = \text{span}(\vec{w}_1, \dots, \vec{w}_k) \quad \forall k \in 1, \dots, n$

④ $(\vec{v}_1, \dots, \vec{v}_k)$ is an orthogonal basis of $\text{span}(\vec{w}_1, \dots, \vec{w}_k)$

idea of proof

Construct

$$\vec{v}_1 = \vec{w}_1$$

$$\vec{v}_2 \in \text{span}(\vec{w}_1, \vec{w}_2), \quad \vec{v}_2 \perp \vec{v}_1$$

$$\vec{v}_3 \in \text{span}(\vec{w}_1, \vec{w}_2, \vec{w}_3), \quad \vec{v}_3 \perp \vec{v}_1, \vec{v}_2$$

⋮

Proof: induct on $\#S = n$

Let

$$S_k = (\vec{w}_1, \dots, \vec{w}_k)$$

$$S'_k = (\vec{v}_1, \dots, \vec{v}_k)$$

show (given true for $k-1$) that

(a) S'_k, S_k have same span

(b) S'_k is orthogonal

if $k=1$

$$S_1 = \{\vec{w}_1\}$$

$$S'_1 = \{\vec{w}_1\}$$

Note: $\vec{w}_1 \neq \vec{0}$ since S is LI

clearly $\text{span}(S_1) = \text{span}(S'_1)$

Using induction,

$$S'_k = (\vec{v}_1, \dots, \vec{v}_k)$$

know

$$\langle \vec{v}_i, \vec{v}_j \rangle = 0 \text{ for } i \neq j \quad i, j \leq k-1$$

Need to show: $\langle \vec{v}_i, \vec{v}_k \rangle = 0$ for $1 \leq i \leq k-1$

$$\vec{v}_k \neq 0$$

$$\text{span}(\vec{v}_1, \dots, \vec{v}_k) = \text{span}(\underbrace{\vec{v}_1, \dots, \vec{v}_{k-1}}_{\text{span}(\vec{w}_1, \dots, \vec{w}_{k-1})}, \vec{w}_k)$$

so

$$\langle \vec{v}_k, \vec{v}_i \rangle = \langle \vec{w}_k, \vec{v}_i \rangle - \frac{\langle \vec{w}_k, \vec{v}_i \rangle}{\langle \vec{v}_i, \vec{v}_i \rangle} \langle \vec{v}_i, \vec{v}_i \rangle$$

$\vec{v}_k \neq \vec{0}!$ If it is

then $w_k \in \text{span}(v_1, \dots, v_{k-1}) = \text{span}(w_1, \dots, w_{k-1})$
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$$\begin{aligned} \text{span}(\vec{v}_1, \dots, \vec{v}_{k-1}, \vec{v}_k) &= \text{span}(\vec{w}_1, \dots, \vec{w}_{k-1}, \vec{v}_k) \quad (\text{induction}) \\ &\parallel \\ &= \text{span}(\vec{v}_1, \dots, \vec{v}_{k-1}, \vec{w}_k) \end{aligned}$$

$$\therefore \text{span}(S_k) = \text{span}(S'_k), \quad S'_k \text{ is orthogonal}$$

Example Find an orthonormal basis for

$$W = \text{span} \left(\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right)$$

Solution:

$$(a) \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$(b) \vec{v}'_2 = \vec{w}_2 - \frac{\langle \vec{w}_2, \vec{v}_1 \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle} \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\vec{v}_2 = 2\vec{v}'_2 = \begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \end{pmatrix}$$