Exercises.

Solution to Question 1.

(a) The rank of A is the same as the row rank of A, which is the same as the number of non-zero rows of B.
The number of non-zero rows of B is that of non-zero pivot columns of B. So it is the same as the non-zero pivot columns of A.

(b) Let P be the subset of $\mathbf{a}_1, \ldots, \mathbf{a}_n$ consisting of those $\mathbf{a}_i$, where i is a pivot column. So P is linearly independent and $\text{span}(P)$ is a subspace of $\text{Im}(A)$.
By part (a), $|P| = \text{rank}(A)$. Therefore $\dim \text{span}(P) = \dim \text{Im}(A)$. So $\text{span}(P) = \text{Im}(A)$. P is a basis for $\text{im}(A)$.

(c) The row span of A is the same as the row span of B, because B is the reduced row echelon form of A.
The non-zero rows of B are linearly independent and span the row space of B, so they form a basis for the row space of B.
By part (a), the number of non-zero rows in B equals that of pivot columns of A, hence it is the rank of A.

(d) Let the matrix be
$$K = \begin{bmatrix} -C \\ I_{n-r} \end{bmatrix}.$$
We want to prove the columns of K form a basis for the kernel of A.
First of all, notice that $\ker A = \ker B$, because B is obtained by doing invertible row operations to A.
Next, $BK = 0$. So $\text{Im}(K) \subseteq \ker(B)$. Because
$$\text{rank}(K) = \text{rank} \begin{bmatrix} -C \\ I_{n-r} \end{bmatrix} = n - r,$$
so $\dim \text{Im}(K) = n - r$. And $\dim \ker(B) = n - \dim \text{Im}(K) = n - r$. Therefore, $\dim \text{Im}(K) = \dim \ker(B)$. Thus $\text{Im}(K) = \ker(B)$ and columns of K is a basis for $\ker A$.

Solution to Question 2. Let

$$\begin{aligned}\mathbf{e}_1 &= (1,0,0),\\ \mathbf{e}_2 &= (0,1,0),\\ \mathbf{e}_3 &= (0,0,1).\end{aligned}$$

T is invertible, because the matrix L_T associated to T with respect to the basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is

$$\begin{bmatrix} 3 & 0 & -2 \\ 0 & 1 & 0 \\ 3 & 4 & 0 \end{bmatrix},$$

which is an invertible matrix.
The inverse of this matrix is

$$\begin{bmatrix} 0 & -4/3 & 1/3 \\ 0 & 1 & 0 \\ -1/2 & -2 & 1/2 \end{bmatrix}.$$

So the inverse of T is

$$T^{-1}(a,b,c) = \left(-\frac{4}{3}b + \frac{1}{3}c, b, -\frac{1}{2}a - 2b + \frac{1}{2}c\right).$$

Solution to Question 3.

(a) Note that

$$\begin{aligned}
T(E_{11}) &= E_{11}B - BE_{11} \\
&= \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = E_{21}, \\
T(E_{12}) &= E_{12}B - BE_{12} \\
&= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 0, \\
T(E_{21}) &= E_{21}B - BE_{21} \\
&= \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = -E_{11} + E_{22}, \\
T(E_{22}) &= E_{22}B - BE_{22} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = -E_{21}.
\end{aligned}$$

This implies

$$T(E_{11}, E_{12}, E_{21}, E_{22}) = (E_{11}, E_{12}, E_{21}, E_{22}) \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

So $L_T = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ is the matrix of T with respect to $\mathcal{B}$.

(b) The first and third column form a basis for the image of L_T. Therefore, $(E_{21}, -E_{11} + E_{22})$ is a basis of $\mathrm{im}(T)$.

A basis for the kernel of L_T is $\left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}\right)$. So a basis for $\ker(T)$ is $(E_{11} + E_{22}, E_{12})$.

Solution to Question 4.

(a) Note that if $v \in V$ such that $Tv = 0$, then $STv = S(Tv) = 0$. Therefore $\ker(T) \subseteq \ker(ST)$. So ST is injective $\implies \ker(ST) = 0 \implies \ker(T) = 0 \implies T$ is injective.

(b) Because

$$\operatorname{Im}(ST) \subseteq \operatorname{Im}(S) \subseteq U, \tag{1}$$

so $\operatorname{Im}(ST) = U$ forces every "$\subseteq$" in (1) to be "$=$". Therefore, $\operatorname{Im}(ST) = U$ implies $\operatorname{Im}(S) = U$.

(c) Note that

$$\operatorname{rank}(T) = \dim \operatorname{Im}(T),$$
$$\operatorname{rank}(ST) = \dim \operatorname{Im}(ST).$$

Because $\operatorname{Im}(ST) \subseteq \operatorname{Im}(T)$, so $\operatorname{rank}(ST) \leq \operatorname{rank}(T)$.
Now because S is an isomorphism, so $T = S^{-1}ST$. Therefore, by the same reasoning, $\operatorname{rank}(T) = \operatorname{rank}(S^{-1}ST) \leq \operatorname{rank}(ST)$.
Putting together, $\operatorname{rank}(T) = \operatorname{rank}(ST)$.

(d)

- ($\Longleftarrow$) First of all, because S and T are both linear transformations, so ST must be a linear transformation.
  We know that S and T are both injective. So $STx = 0 \implies Tx = 0 \implies x = 0$. We may conclude that ST is injective.
  S and T are surjective. So $\operatorname{Im}(T) = W$ implies $\operatorname{Im}(ST) = \operatorname{Im}(S)$. And we know $\operatorname{Im}(S) = U$. Therefore ST is surjective.
  Hence ST is an isomorphism.
- ($\implies$) Assume ST is an isomorphism. By part (a) and part (b), T is injective and S is surjective. Because both S and T are in $\mathcal{L}$, so $U = W = V$. Therefore, T is injective implies it is also surjective and S is surjective implies it is also injective. Hence, both T and S are isomorphic linear transformations.

Solution to Question 5.

(a) The matrix of $[L_A]_{\text{std}}$ is A.

(b) $S = [\text{id}]_{\mathcal{B}\leftarrow\text{std}}$ means

$$\mathcal{B} = \text{id}(\mathcal{B}) = \text{std} \cdot S.$$

Therefore,

$$\text{id}(\text{std}) = \text{std} = \mathcal{B} \cdot S^{-1},$$

which means $[L_A]_{\text{std}\leftarrow\mathcal{B}} = S^{-1}$.

(c) Because

$$[ST]_{\mathcal{C}\leftarrow\mathcal{A}} = [S]_{\mathcal{C}\leftarrow\mathcal{B}}[T]_{\mathcal{B}\leftarrow\mathcal{A}},$$

so

$$B = [L_A]_{\mathcal{B}\leftarrow\mathcal{B}} = [\text{id}]_{\mathcal{B}\leftarrow\text{std}}[L_A]_{\text{std}\leftarrow\text{std}}[\text{id}]_{\text{id}\leftarrow\mathcal{B}} = S^{-1}AS.$$

(d) i. Apply A to $\mathcal{B}$:

$$A\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}3\\3\\2\end{pmatrix} = 2\begin{pmatrix}1\\1\\1\end{pmatrix} + 1\begin{pmatrix}1\\0\\1\end{pmatrix} + (-1)\begin{pmatrix}0\\-1\\1\end{pmatrix},$$
$$A\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}2\\2\\2\end{pmatrix} = 2\begin{pmatrix}1\\1\\1\end{pmatrix} + 0\begin{pmatrix}1\\0\\1\end{pmatrix} + 0\begin{pmatrix}0\\-1\\1\end{pmatrix},$$
$$A\begin{pmatrix}0\\-1\\1\end{pmatrix} = \begin{pmatrix}0\\1\\1\end{pmatrix} = 2\begin{pmatrix}1\\1\\1\end{pmatrix} + (-2)\begin{pmatrix}1\\0\\1\end{pmatrix} + 1\begin{pmatrix}0\\-1\\1\end{pmatrix}.$$

So $B = \begin{bmatrix}2 & 2 & 2\\ 1 & 0 & -2\\ -1 & 0 & 1\end{bmatrix}$.

ii. $S = [\text{id}]_{\text{std}\leftarrow\mathcal{B}}$ is a matrix such that

$$\mathcal{B} = \text{std} \cdot S.$$

So

$$S = \begin{bmatrix}1 & 1 & 0\\ 1 & 0 & -1\\ 1 & 1 & 1\end{bmatrix},$$

and

$$S^{-1} = \begin{bmatrix}-1 & 1 & 1\\ 2 & -1 & -1\\ -1 & 0 & 1\end{bmatrix}.$$

iii. By part (c),

$$\begin{aligned}
B &= S^{-1}AS \\
&= \begin{bmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 2 \\ 1 & 1 & -1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 2 & 2 \\ 1 & 0 & -2 \\ -1 & 0 & 1 \end{bmatrix}.
\end{aligned}$$

So we get the same result as in part i.