ECE 4110 Homework 4

Instructor: Qing Zhao
Office: 325 Rhodes Hall
Email: qz16@cornell.edu

Due by 5pm on October 3

1 Reading Material

- Jointly Gaussian random variables (Chapter 6.4).
- MMSE and LMMSE estimators (Chapter 6.5.3).

2 Assignment

1. **Conditional Variance**

   You have seen in Problem 2 of Homework 3 the *conditional variance* of X given Y. It is defined as
   $$\text{Var}(X|Y) \triangleq \mathbb{E}((X - \mathbb{E}(X|Y))^2|Y).$$

   (a) Show that the expectation of the conditional variance is the mean square error of the MMSE estimator of X given Y:
   $$\mathbb{E}[\text{Var}(X|Y)] = \mathbb{E}[(X - \mathbb{E}[X|Y])^2].$$

   (b) Show that the variance of X can be decomposed into the variance of the MMSE estimator plus the mean square error of that estimator:
   $$\text{Var}(X) = \text{Var}(\mathbb{E}[X|Y]) + \mathbb{E}[\text{Var}(X|Y)].$$

   Hint: You have already shown this in Problem 2 of Homework 3. This time, try to prove it by using the orthogonality principle. First write
   $$\textit{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X|Y] + \mathbb{E}[X|Y] - \mathbb{E}[X])^2]$$
   and then expand the square (and then use the orthogonality principle).

   (c) Based on (b), show that the MMSE of estimating X based on observations of Y is smaller than or equal to that of estimating X without any observations (i.e., estimating X using a constant). When do these two types of estimators offer the same MMSE?

2. **MMSE and LMMSE Estimator**

   Let X and Y be jointly uniformly distributed over the triangular region in the x–y plan with corners $(0,0)$, $(0,1)$, and $(1,2)$.

(a) What is the MMSE estimator of X using a constant? What is the resulting MSE?

(b) What is the LMMSE estimator of X given Y? What is the resulting MSE? Compare it with the MSE in (a).

(c) What is the best MMSE estimator of X given Y? What is the resulting MSE? Compare it with the MSE in (a) and (b).

3. **Marginally Gaussian but not Jointly Gaussian**

   Consider independent Gaussian random variables $X, Y \sim \mathcal{N}(0, 1)$. Let $Z \in \pm 1$ be a binary random variable with $\Pr(Z = 1) = \frac{1}{2}$ and independent of X and Y.

   (a) Show that $W = ZX + Y$ is also Gaussian.
   (*Hint: first show that ZX is Gaussian by writing out its PDF through conditioning on Z, then conclude W is Gaussian based on the independence of ZX and Y.*)

   (b) Show that X and W are not jointly Gaussian by finding the joint PDF of X and W.
   (*Hint: note that conditioning on $Z = z$, the random vector $[X, W]^T$ is a linear transform of the random vector $[X, Y]^T$.*)

4. **MMSE Estimator for Jointly Gaussian Random Variables**

   Let $\mathbf{X} = [X_1, X_2, X_3]^T$ be a Gaussian random vector with

$$\mathbb{E}(\mathbf{X}) = \begin{pmatrix} 1 \\ 4 \\ 6 \end{pmatrix} \qquad \mathbf{K} = \mathsf{Cov}(\mathbf{X}) = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

   (a) Compute the MMSE estimate of X_1 given X_2 and the resulting MSE.

   (b) Compute the MMSE estimate of X_1 given X_3 and the resulting MSE.

   (c) Compute the MMSE estimate of X_1 given both X_2 and X_3 and the resulting MSE.

   (d) Note that X_1 and X_3 are uncorrelated, and hence independent. Yet the MMSE estimate of X_1 given X_2 and X_3 as obtained in (c) is a function of both X_2 and X_3. Why is that?

① Conditional Variance

(a) Show that the expectation of the conditional variance is the mean square error of the MMSE estimator of X given Y:

$$\overset{\text{LHS}}{\mathbb{E}[\mathsf{Var}(X|Y)]} = \overset{\text{RHS}}{\mathbb{E}[(X - \mathbb{E}[X|Y])^2]}.$$

Know

$$\mathrm{Var}(X|Y) \triangleq \mathbb{E}\left[(X - \mathbb{E}[X|Y])^2 \,|\, Y\right]$$

Thus the LHS is

$$\mathbb{E}[\mathrm{Var}(X|Y)] = \mathbb{E}\left[\mathbb{E}\left[(X - \mathbb{E}[X|Y])^2 \,|\, Y\right]\right]$$

The rhs is

$$\mathbb{E}_Y[(X - \mathbb{E}[X|Y])^2] = \mathbb{E}_Y\left[\mathbb{E}_{X|Y}[(X - \mathbb{E}[X|Y])^2 | Y]\right]$$

by law of iterated expectations.

Thus,

$$\mathbb{E}[\mathrm{Var}(X|Y)] = \mathbb{E}[(X - \mathbb{E}[X|Y])^2 | Y]$$

(b) Show that the variance of X can be decomposed into the variance of the MMSE estimator plus the mean square error of that estimator:

$$\mathsf{Var}(X) = \mathsf{Var}(\mathbb{E}[X|Y]) + \mathbb{E}[\mathsf{Var}(X|Y)].$$

Could also use

$$\text{Var}(X) = \mathbb{E}\left[\left(X - \mathbb{E}[X|Y] + \mathbb{E}[X|Y] - \mathbb{E}[X]\right)^2\right]$$

and orthogonality principle!

Let $X = X - \mathbb{E}[X|Y] + \mathbb{E}[X|Y]$

Then

$$\text{Var}(X) = \text{Var}(X - \mathbb{E}[X|Y] + \mathbb{E}[X|Y])$$

Since $\mathbb{E}[X|Y]$ is our BEST estimator $\hat{X}^*$ which minimizes our error $W = X - \hat{X}$ over ALL possible functions $g(Y)$, the orthogonality principle states

$$W \perp g(Y) \text{ for ALL possible functions } g(Y).$$

Visualizaltion

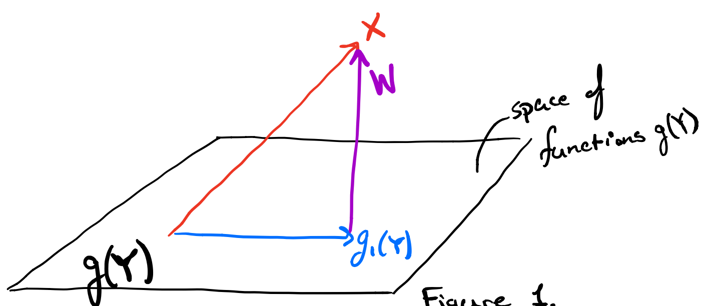


Figure 1.

$W = X - \hat{X}^* = X - \mathbb{E}[X|Y]$

The $X - \mathbb{E}[X|Y] \perp \mathbb{E}[X|Y]$

$\Rightarrow \mathbb{E}[(X - \mathbb{E}[X|Y])\,\mathbb{E}[X|Y]] = 0$

$\Rightarrow$ uncorrelatedness

If X, Y two uncorrelated random variables,

$$\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y)$$

Since W is uncorrelated with $\mathbb{E}[X|Y]$,

$$\text{Var}(W + \mathbb{E}[X|Y]) = \text{Var}(W) + \text{Var}(\mathbb{E}[X|Y])$$

Thus

$$\text{Var}(X) = \text{Var}(X - \mathbb{E}[X|Y]) + \text{Var}(\mathbb{E}[X|Y])$$

By Law of Iterated Expectations,

$$\mathrm{Var}(X - \mathbb{E}[X|Y]) = \mathbb{E}\left[\mathbb{E}\left[(X - \mathbb{E}[X|Y])^2 | Y\right]\right]$$

$$= \mathbb{E}[\mathrm{Var}(X|Y)] \quad \text{Shown in part (a)}$$

Thus

$$\mathrm{Var}(X) = \mathbb{E}[\mathrm{Var}(X|Y)] + \mathrm{Var}(\mathbb{E}[X|Y])$$

(c) Based on (b), show that the MMSE of estimating X based on observations of Y is smaller than or equal to that of estimating X without any observations (i.e., estimating X using a constant). When do these two types of estimators offer the same MMSE?

MMSE using NO observations of Y yields $\mathrm{Var}(X)$

MMSE using observations of Y yields $\mathbb{E}_Y[\mathrm{Var}(X|Y)]$

} From Lecture Notes, Time crunch prevented rederivation

From part b

$$\mathrm{Var}(X) = \mathbb{E}[\mathrm{Var}(X|Y)] + \mathrm{Var}(\mathbb{E}[X|Y])$$

Rearrange to get

$$\mathbb{E}[\mathrm{Var}(X|Y)] = \mathrm{Var}(X) - \mathrm{Var}(\mathbb{E}[X|Y])$$

From HW3, $\mathrm{Var}(\mathbb{E}[X|Y]) \geq 0$

Thus

$$\mathbb{E}[\mathrm{Var}(X|Y)] \leq \mathrm{Var}(X)$$

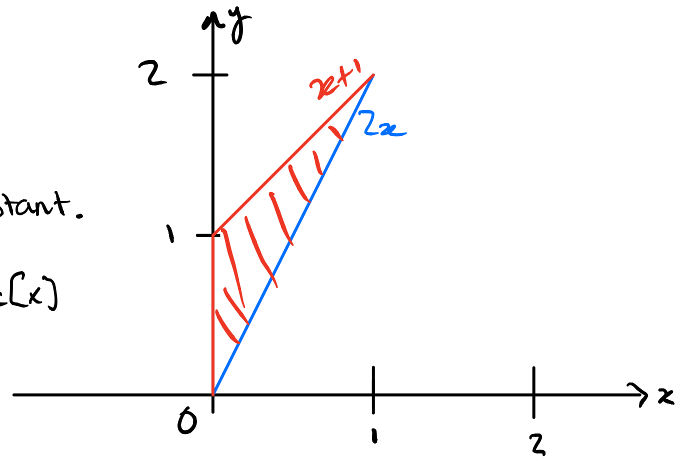
also proved in HW3

With equality when <u>X is independent of Y.</u>

X, Y independent $\Rightarrow$ $\mathbb{E}[X|Y] = \mathbb{E}[X]$

$$\mathrm{Var}(\mathbb{E}[X|Y]) = \mathrm{Var}(\underbrace{\mathbb{E}[X]}_{\text{constant}}) = 0$$

(2) X, Y jointly uniform over



(a) MMSE estimator of X using a constant.

Know that best estimator $\hat{x}$ is $\mathbb{E}[x]$

$$f_{X,Y}(x,y) = \frac{1}{\text{area}}, \quad \begin{matrix} 0 \le x \le 1 \\ 2x \le y \le x+1 \\ x \le \frac{y}{2} \le \frac{x+1}{2} \end{matrix}$$

$$\text{area} = \int_{x=0}^{1} x+1 \, dx - \int_{x=0}^{1} 2x \, dx = \frac{3}{2} - 1 = \frac{1}{2}$$

$$f_{X,Y}(x,y) = 2, \quad \begin{matrix} 0 \le x \le 1 \\ 2x \le y \le x+1 \end{matrix}$$

$$f_X(x) = \int_{y=2x}^{y=x+1} 2 \, dy = 2(x+1) - 2(2x) = 2(1-x), \quad 0 \le x \le 1$$

$$\mathbb{E}[X] = \int_{x=0}^{1} x(2(1-x)) \, dx = 2\int_{x=0}^{1} x - x^2 dx = 2\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{3}$$

The MSE using $\mathbb{E}[x]$ is $\mathrm{Var}(X)$

$$\mathbb{E}[X^2] = 2\int_{x=0}^{1} x^2(1-x) \, dx = 2\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{1}{6}$$

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$
$$= \frac{1}{6} - \frac{1}{9} = \frac{1}{18}$$

(b) LMMSE estimator of X given Y?

$$\hat{X}_{LMMSE} = aY + b$$

where

$$a = \frac{Cov(X,Y)}{Var(Y)} \quad , \quad b = \mathbb{E}[X] - a\,\mathbb{E}[Y]$$

So

$$\hat{X}_{LMMSE} = E[X] + \frac{Cov(X,Y)}{Var(Y)}(Y - \mathbb{E}[Y])$$

$$\boxed{\mathbb{E}[X] = \frac{1}{3}}$$

$\mathbb{E}[Y]$?

$$f_{Y|X=x}(y|x=x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{2}{2(1-x)} = \frac{1}{1-x} \quad , \quad 2x \le y \le x+1$$

$$\mathbb{E}[Y|X=x] = \int_{y=2x}^{y=x+1} \frac{1}{1-x}\, y \, dy = \frac{1}{(1-x)}\,\frac{1}{2}y^2 \Big|_{y=2x}^{y=x+1} = \frac{1}{2(1-x)}\left[(x+1)^2 - (2x)^2\right]$$

$$\mathbb{E}[Y|X] = \frac{1}{2(1-X)}\left[2X + 1 - 3X^2\right]$$

$$\boxed{\mathbb{E}[Y] = \mathbb{E}_X[\mathbb{E}[Y|X]] = \int_{x=0}^{1} \mathbb{E}[Y|x]\, f_X(x)\, dx = \int_{x=0}^{1} 2x + 1 - 3x^2\, dx = 1}$$

Var(Y)? $\quad Var(Y) = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2$

$$\mathbb{E}[Y^2] = \mathbb{E}_X[\mathbb{E}[Y^2|X]]$$

$$\int_y y^2 f_{Y|X}(y|x)\,dy \;-\; \mathbb{E}[Y^2|X=x] = \int_{y=2x}^{y=x+1} \frac{1}{1-x}\, y^2\, dy = \frac{1}{3(1-x)}\left[(x+1)^3 - (2x)^3\right]$$

$$\mathbb{E}[Y^2 | X = x] = \frac{x^3 + 3x^2 + 3x + 1 - 8x^3}{3(1-x)}$$

$$= \frac{3x^2 + 3x + 1 - 7x^3}{3(1-x)}$$

$$\mathbb{E}[Y^2 | X] = \frac{3X^2 + 3X + 1 - 7X^3}{3(1-X)}$$

$$\mathbb{E}[Y^2] = \mathbb{E}_X[\mathbb{E}[Y^2|X]] = \int_{x=0}^{x=1} \frac{2(1-x)}{3(1-x)} (3x^2 + 3x + 1 - 7x^3)\, dx$$

$$\int_x \mathbb{E}[Y^2|X] f_X(x)\, dx$$

$$= \frac{2}{3}\left[1 + \frac{3}{2} + 1 - \frac{7}{4}\right] = \frac{7}{6}$$

$$\boxed{Var(Y) = \frac{7}{6} - 1 = 1/6}$$

$Cov(X,Y)$?

$$\mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$$

$$\mathbb{E}[XY] = \int_y \int_x xy\, f_{X,Y}(x,y)\, dx\, dy = \int_{y=0}^{1} \int_{x=0}^{y/2} 2xy\, dx\, dy + \int_{y=0}^{1} \int_{x=y-1}^{y/2} 2xy\, dx\, dy$$

$$= \int_{y=0}^{1} \frac{y^3}{4}\, dy + \int_{y=0}^{1} \left[\left(\frac{y}{2}\right)^2 - (y-1)^2\right] y\, dy$$

$$= 5/12$$

$$Cov(X,Y) = \frac{5}{12} - 1\left(\frac{1}{3}\right) = \frac{1}{12}$$

Thus

$$\boxed{\hat{X}_{LMMSE} = \frac{1}{3} + \frac{1/12}{1/6}(Y - 1)}$$

$$MSE_{LMMSE} = Var(X) - \frac{Cov^2(X,Y)}{Var(Y)} = \frac{1}{18} - \frac{1/144}{1/6} = \frac{1}{18} - \frac{1}{24}$$
$$= 1/72$$

This value is LESS THAN previous value of $1/18$!

(c) MMSE estimator of X given Y?

This is $\mathbb{E}[X|Y]$.

$$f_Y(y) = \begin{cases} \int_0^{y/2} f_{X,Y}(x,y)\,dx = y, & 0 \le y \le 1 \\ \int_{y-1}^{y/2} f_{X,Y}(x,y)\,dx \quad 2-y, & 1 \le y \le 2 \\ 0, & o/w \end{cases}$$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} \frac{2}{y}, & 0 \le y \le 1,\ 0 \le x \le y/2 \\ \frac{2}{2-y}, & 1 \le y \le 2,\ y-1 \le x \le y/2 \end{cases}$$

$$\mathbb{E}[X|Y] = \begin{cases} \int_{x=0}^{y/2} f_{X|Y}(x|y)\cdot x\,dx = \frac{Y}{4}, & 0 \le Y \le 1 \\ \int_{x=y-1}^{x=y/2} f_{X|Y}(x|y)\cdot x\,dx = \frac{3Y-2}{4}, & 1 \le Y \le 2 \end{cases}$$

MSE is

$Var(X|Y=y) = \mathbb{E}[X^2|Y=y] - (\mathbb{E}[X|Y=y])^2$

$$\mathbb{E}[Var(X|Y)] = \int_y Var(X|Y=y)\, f_Y(y)\,dy = \frac{1}{96}$$

③ $X, Y \sim \mathcal{N}(0,1)$, X independent of Y

Z is bin ± 1, w/ $P_Z(z) = \begin{cases} 1, & \text{w.p } 1/2 \\ -1, & \text{w.p } 1/2 \end{cases}$

Let $V = ZX$

$$F_V(v) = \Pr(V \leq v) = \Pr(ZX \leq v) = \Pr(Z=1)\Pr(X \leq v) + \Pr(Z=-1)\Pr(-X \leq v)$$
$$= F_X(v)$$

Note the symmetry of the Gaussian used!

$$\Pr(X \leq v) = \Pr(X \geq -v) = \Pr(-X \leq v)$$

Thus

$$V = ZX \sim \mathcal{N}(0,1)$$

Since X, Y, Z jointly independent, V is independent of Y.

Therefore V and Y jointly Gaussian.

Thus

$$W = V + Y \sim \mathcal{N}(0,2)$$

(b) Show that X and W are not jointly Gaussian by finding the joint PDF of X and W. (*Hint: note that conditioning on $Z = z$, the random vector $[X, W]^T$ is a linear transform of the random vector $[X, Y]^T$.*)

By definition we have

$$\begin{pmatrix} X \\ W \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ Z & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$$

Here, $\binom{X}{Y}$ is a Gaussian random vector since X, Y jointly Gaussian.

Conditioning on $Z=z$, $\binom{X}{W}$ is a linear transform of $\binom{X}{Y}$ and thus a Gaussian random vector with zero mean and

$$\mathbb{K}_z = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}^T = \begin{pmatrix} 1 & z \\ z & 1+z^2 \end{pmatrix}$$

We thus have

$$f_{X,W}(x,w) = \Pr(Z=1) f_{X,W|Z=1}(x,w|Z=1) + \Pr(Z=-1) f_{X,W|Z=-1}(x,w|Z=-1)$$

$$= \frac{1}{2} \cdot \frac{1}{\sqrt{2\pi}\,|K|} e^{-\frac{1}{2}\binom{X}{W}\mathbb{K}_{Z=1}^{-1}\binom{X}{W}^T} + \frac{1}{2} \cdot \frac{1}{\sqrt{2\pi}\,|K|} e^{-\frac{1}{2}\binom{X}{W}\mathbb{K}_{Z=-1}^{-1}\binom{X}{W}^T}$$

$$= \frac{e^{-\frac{2x\omega + \omega^2}{2}}}{4\pi}\left(e^{2x\omega} + e^{-2x\omega}\right)$$

NOT Gaussian!

④

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}, \quad \mathbb{E}[\mathbf{X}] = \begin{bmatrix} \mathbb{E}[X_1] \\ \mathbb{E}[X_2] \\ \mathbb{E}[X_3] \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix}$$

$$\mathbb{K} = \mathrm{Cov}(\mathbf{X}) = \begin{bmatrix} \mathrm{Cov}(X_1,X_1) & \mathrm{Cov}(X_1,X_2) & \mathrm{Cov}(X_1,X_3) \\ \mathrm{Cov}(X_2,X_1) & \mathrm{Cov}(X_2,X_2) & \mathrm{Cov}(X_2,X_3) \\ \mathrm{Cov}(X_3,X_1) & \mathrm{Cov}(X_3,X_2) & \mathrm{Cov}(X_3,X_3) \end{bmatrix} = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

(a) MMSE estimate of X_1 given X_2 and resulting MSE.

$$\hat{X}_{MMSE} = \mathbb{E}[X_1 | X_2]$$

Since X_1, X_2 jointly Gaussian (any subset of a Gaussian random vector is a Gaussian random vector)

$$\hat{X}_{MMSE} = \hat{X}_{LMMSE} = \mathbb{E}[X_1] + \frac{\mathrm{Cov}(X_1,X_2)}{\mathrm{Var}(X_2)}(X_2 - \mathbb{E}[X_2])$$

$$= 1 + \frac{1}{2}(X_2 - 4)$$

$$\boxed{= \frac{X_2}{2} - 1}$$

$$MSE = \mathrm{Var}(X_1) - \frac{\mathrm{Cov}^2(X_1,X_2)}{\mathrm{Var}(X_2)}$$

$$= 3 - \frac{1}{2} = \boxed{\frac{5}{2}}$$

Expected a constant!

(b) MMSE estimate of X_1 given X_3 and resulting MSE.

$$\hat{X}_{MMSE} = \mathbb{E}[X_1 | X_3]$$

Since X_1, X_3 jointly Gaussian

$$\hat{X}_{MMSE} = \hat{X}_{LMMSE} = \mathbb{E}[X_1] + \frac{Cov(X_1, X_3)}{Var(X_3)} (X_2 - \mathbb{E}[X_3])$$

$$= 1 + 0$$

$$= 1$$

$$MSE = Var(X_1) - \frac{Cov^2(X_1, X_3)}{Var(X_3)}$$

$$= 3 - 0 = 3$$

(c) MMSE estimate of X_1 given X_2, X_3?

$$\hat{X}_{MMSE} = \mathbb{E}[X_1 | X_2, X_3]$$

Gaussian random vectors

Let $\mathbb{Z}_1 = \begin{bmatrix} X_2 \\ X_3 \end{bmatrix} \subset \mathbb{X}$, $\mathbb{Y} = [X_1] \subset X$

Then

$$\hat{X}_{MMSE} = \mathbb{E}[\mathbb{Y} | \mathbb{Z}_1]$$

One Gaussian Vector Conditioned on Another

from lecture notes

$$= \mathbb{E}[\mathbb{Y}] + Cov(\mathbb{Y}_1, \mathbb{Z}_1) Cov^{-1}(\mathbb{Z}_1)(\mathbb{Z}_1 - \mathbb{E}[\mathbb{Z}_1])$$

$$\text{Cov}({}^{1}Z_1) = \begin{bmatrix} \text{Var}(X_2) & \text{Cov}(X_2 X_3) \\ \text{Cov}(X_3 X_2) & \text{Var}(X_3) \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\text{Cov}^{-1}({}^{1}Z_1) = \frac{1}{2-1}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

A a 2×2 w/ $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Then $A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$

$$\mathbb{E}[{}^{1}Z_1] = \begin{bmatrix} \mathbb{E}[X_2] \\ \mathbb{E}[X_3] \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

$$\mathbb{E}[Y] = [\mathbb{E}[X_1]] = [1]$$

$$\text{Cov}(Y, Z_1) = [\text{Cov}(X_1, X_2) \quad \text{Cov}(X_1, X_3)] = [1 \quad 0]$$

So

$$\hat{X}_{MMSE} = [1] + [1 \quad 0]\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\left(\begin{bmatrix} X_2 \\ X_3 \end{bmatrix} - \begin{bmatrix} 4 \\ 6 \end{bmatrix}\right)$$

$$= [1] + [1 \quad -1]\begin{bmatrix} (X_2 - 4) \\ (X_3 - 6) \end{bmatrix}$$

$$= [1 + (X_2 - 4) - (X_3 - 6)]$$

$$= [1 + X_2 - 4 - X_3 + 6]$$

$$= [X_2 - X_3 + 3]_{1\times1}$$

(d) The MMSE of X_1 given X_2, X_3 is a function of X_2 AND X_3 (even though X_1 independent of X_3) because X_2 is **NOT** independent of X_3. i.e X_2 occurring has dependence on X_3. So, $X_1 \mid X_2, X_3$ is dependent on BOTH.