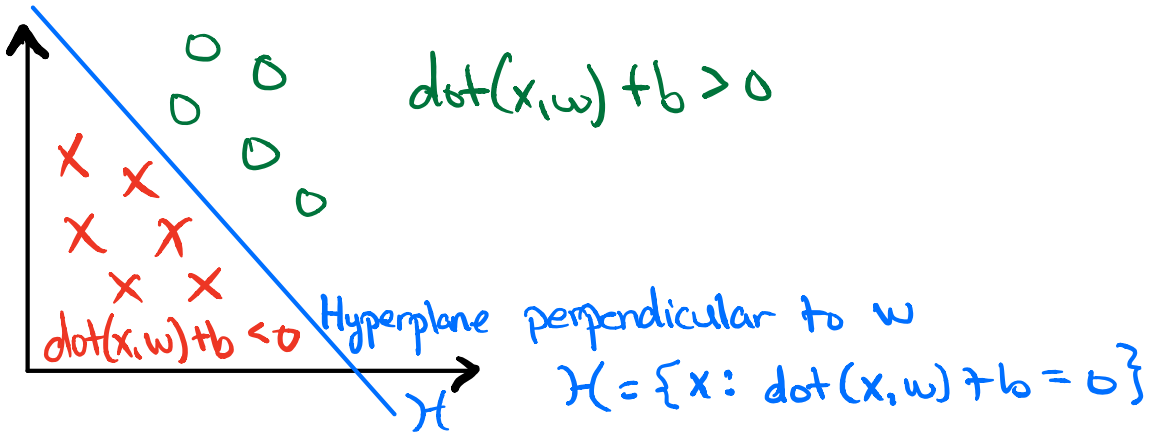
Assumptions:

- Binary classification (i.e $y_i \in \{-1, 1\}$)
- Data is linearly seperable

Classifier

$$h(x_i) = \text{sign}(\underline{w}^T \underline{x}_i + b)$$



b is the bias term (w/o it the hyperplane that $\underline{w}$ defines would always have to go through the origin).

Dealing w/ b can be a **PAIN!**

So instead we 'absorb' it into the feature vector $\underline{w}$ by adding one additional constant dimension.

Under this convention,

$\bar{x}_i$ — $\underline{x}_i$ becomes $\begin{bmatrix} \underline{x}_i \\ 1 \end{bmatrix}$

$\bar{w}$ — $\underline{w}$ becomes $\begin{bmatrix} \underline{w} \\ b \end{bmatrix}$

what happened? No bias term ⇒ hyperplane always through origin! (see figure)

Verify that

$$\begin{bmatrix} \underline{x}_i \\ 1 \end{bmatrix}^T \begin{bmatrix} \underline{w} \\ b \end{bmatrix} = \underline{w}^T x_i + b$$

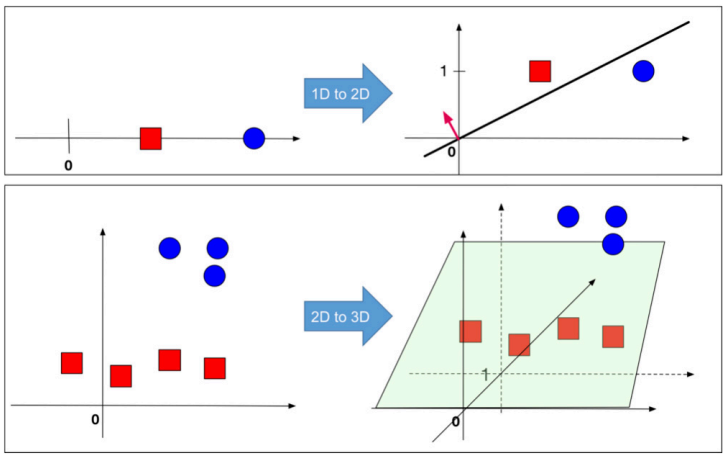
Using this, we can simplify the above formulation of $h(\underline{x}_i)$ to

$$h(\underline{x}_i) = \text{sign}(\underline{w}^T \underline{x})$$

think! This is intuitive!

if >0 classify point as one label

if <0 classify point as other label



(Left:) The original data is 1-dimensional (top row) or 2-dimensional (bottom row). There is no hyper-plane that passes through the origin and separates the red and blue points. (Right:) After a constant dimension was added to all data points such a hyperplane exists.

Observation: $y_i(\underline{w}^T \underline{x}_i) > 0 \iff \underline{x}_i$ is classified correctly

where 'classified correctly' means that $\underline{x}_i$ is on the correct side of the hyperplane defined by $\underline{w}$.

Also, note that the left side depends on $y_i \in \{-1, +1\}$

(would **NOT** work if $y_i \in \{0, +1\}$)

An aside: in higher dimensional spaces where points are very far apart (as seen in k-NN) this almost always holds

Perceptron Algorithm

Let's look at how we can get a hyperplane $\underline{w}$ which seperates the data

Initialize $\vec{w} = \vec{0}$ ← misclassifies EVERYTHING

while True do

  $m = 0$ ← # misclassifications

  for $(x_i, y_i) \in \mathcal{D}$ do ← for (data, label) pair in dataset

    if $y_i(\vec{w}_T \cdot \vec{x}_i) \leq 0$ then ← If pair is misclassified

$\begin{bmatrix} w \\ b \end{bmatrix} + y \begin{bmatrix} x_i \\ 1 \end{bmatrix}$ $\vec{w} \leftarrow \vec{w} + y\vec{x}$ ← update weight vector $\vec{w}$

$w + yx_i + b$ $m \leftarrow m + 1$ ← count misclassifications

    end if

  end for

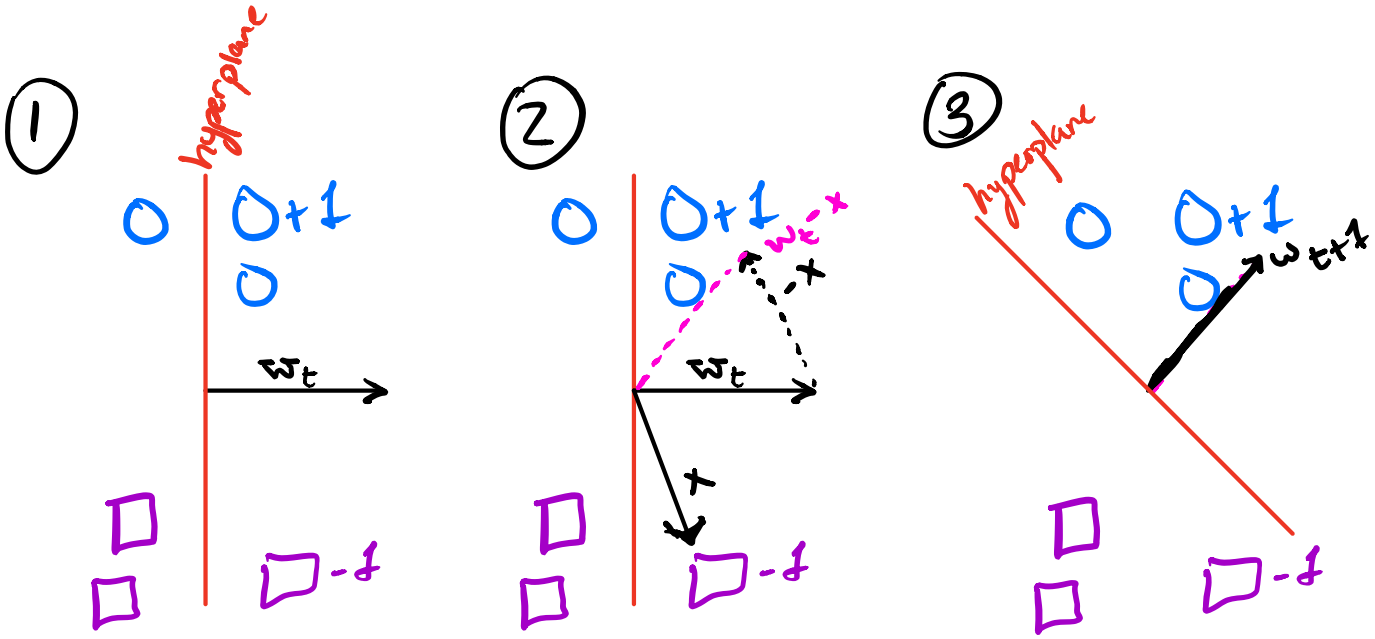
  if $m = 0$ then ← if no misclassifications, sufficient $\vec{w}$

    break ← break out of while loop

  end if

end while

Geometric Intuition



① Initially: hyperplane defined by w_t misclassifies one □ and one ○

② The misclassified □ (x) is chosen and used for an update. Because its label is -1 we **subtract x from** w_t.

③ $w_{t+1} = w_t - x$ is updated as the new hyperplane; our algorithm has converged.

Perceptron Convergence

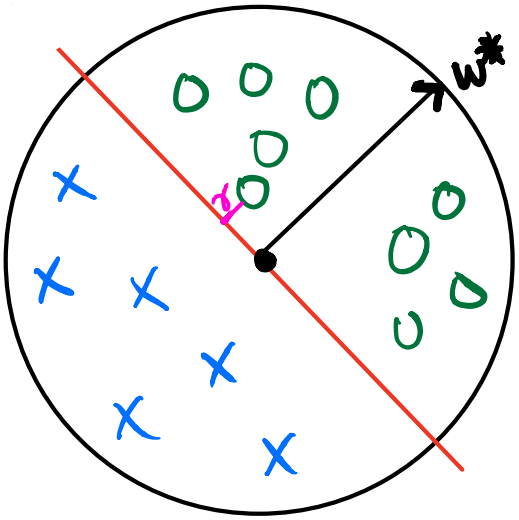
Suppose $\exists \underline{w}^*$ such that $y_i(\underline{x}^T \underline{w}^*) > 0 \; \forall (x_i, y_i) \in D$

Now suppose we rescale each data point and $\underline{w}^*$ such that

$$\|\underline{w}^*\| = 1 \quad \text{and} \quad \|x_i\| \leq 1 \; \forall \; x_i \in D$$

Define the margin γ of the hyperplane $\underline{w}^*$ as

$$\gamma = \min_{(x_i, y_i) \in D} \left(|\underline{x}_i^T \underline{w}^*| \right)$$



Setup:

- All inputs $\underline{x}_i$ live within the unit sphere
- $\exists \; \underline{w}^*$ w/ $\|w^*\| = 1$ (i.e $\underline{w}^*$ lies exactly on unit sphere)
- γ is the distance from this hyperplane to the closest data point

Theorem: If all the above holds, then the perceptron make at **MOST** $1/\gamma^2$ mistakes.

Proof:

Consider the effect of an update ($\underline{w}$ becomes $\underline{w} + y\underline{x}$) on the two terms $\underline{w}^T\underline{w}^*$ and $\underline{w}^T\underline{w}$

- $y(\underline{x}^T\underline{w}) \leq 0$: holds bc $\underline{x}$ is misclassified by $\underline{w}$
  (otherwise we wouldn't make the update)
- $y(\underline{x}^T\underline{w}^*) > 0$: Holds bc $\underline{w}^*$ is a seperating hyperplane and classifies all points correctly

① Consider the impact of an update on $\underline{w}^T\underline{w}^*$:

$$(\underline{w} + y\underline{x})^T\underline{w}^* = \underline{w}^T\underline{w}^* + y(\underline{x}^T\underline{w}^*) \geq \underline{w}^T\underline{w}^* + \gamma$$

γ follows from the fact that, for $\underline{w}^*$, the distance from the hyperplane defined by $\underline{w}^*$ to $\underline{x}$ must be AT LEAST γ (i.e $y(\underline{x}^T\underline{w}^*) = |\underline{x}^T\underline{w}^*| \geq \gamma$)

So, for each update, $\underline{w}^T\underline{w}^*$ grows by AT LEAST γ

② Consider the impact of an update on $\underline{w}^T\underline{w}$:

$$(\underline{w} + y\underline{x})^T(\underline{w} + y\underline{x}) = \underline{w}^T\underline{w} + \underbrace{y\,\underline{w}^T\underline{x} + y\,\underline{x}^T\underline{w}}_{2y(\underline{w}^T\underline{x}) < 0} + \underbrace{y^2\underline{x}^T\underline{x}}_{0 \leq \;\leq 1}$$

$$\leq \underline{w}^T\underline{w} + 1$$

$\leq$ follows from

(i) $2y(\underline{w}^T\underline{x}) < 0$ as we had to make an update $\Rightarrow$ x misclassified

(ii) $0 \leq y^2(\underline{x}^T\underline{x}) \leq 1$ as $y^2 = 1$ and $\underline{x}^T\underline{x} \leq 1$ $(\|x\| \leq 1)$

So, for each update $\underline{w}^T\underline{w}$ grows by **AT MOST** 1

③ Thus, after M updates, the following must hold:

(i) $\underline{w}_T\underline{w}^* \geq M\gamma$

(ii) $\underline{w}_T\underline{w} \leq M$

The proof follows

$M\gamma \leq \underline{w}_T\underline{w}^*$ from 3i

$= \|w\| \cos(\theta)$ definition of inner product

$\leq \|w\|$ $\cos\theta \leq 1$

$= \sqrt{\underline{w}^T\underline{w}}$ definition of $\|w\|$

$\leq \sqrt{M}$ from 3ii

$\Rightarrow M\gamma \leq \sqrt{M}$

$M^2\gamma^2 \leq M$

$M \leq \frac{1}{\gamma^2}$ ← M is bounded above by a constant