Laurent Series

Suppose a is the pole of an analytic function $f(z)$

i.e $\lim_{z \to a} f(z) = \infty$

An aside on essential singularities

If $f(z)$ approaches ∞ as z approaches a from any direction then a is a pole of f. Since f is analytic, it follows that

$$F(z) \equiv \frac{1}{f(z)}$$

is also analytic and has a root at a. If this root has multiplicity m, then the factorization of F is

$$F(z) = (z-a)^m \Omega(z)$$

where Ω is analytic and nonzero at a; in fact we know that $\Omega(a) = \frac{F^{(m)}(a)}{m!}$

The local behavior of f near a is therefore given by

$$f(z) = \frac{\tilde{\Omega}(z)}{(z-a)^m}$$

where $\tilde{\Omega}(z) = \dfrac{1}{\Omega(z)}$ is analytic and nonzero at a.

This expression brings out the analogy with rational functions and enables us to identify m as the algebraic multiplicity or order of the pole at a.

End Aside

We can express $f(z)$ as

$$f(z) = \frac{\phi(z)}{(z-a)^m}$$

where $\phi(z)$ is analytic and $\phi(a) \neq 0$. m is the order of the pole, and the greater the order the faster we approach infinity!

$\phi(z)$ can be expressed as a Taylor series centered at a:

$$\phi(z) = \sum_{n=0}^{\infty} c_n (z-a)^n \quad , \quad c_n = \frac{\phi^{(n)}(a)}{n!}$$

Hence we deduce that

> If an analytic function $f(z)$ has a pole of order m at a, then in the vicinity of this pole, $f(z)$ possesses a Laurent series of the form

$$f(z) = \frac{c_0}{(z-a)^m} + \frac{c_1}{(z-a)^{m-1}} + \cdots + \frac{c_{m-1}}{(z-a)} + c_m + c_{m+1}(z-a) + \cdots$$

Recall that $\frac{1}{z-a}$ is the **RESIDUE** of $f(z)$ at a, denoted $\mathrm{Res}[f, a]$.

Also recall the crucial significance of the residue in evaluating integrals: if C is a simple loop containing a but no other singularities of f

$$\oint_C f(z)\,dz = 2\pi i \,\mathrm{Res}[f, a]$$

Annular Laurent Series

We just saw that the Laurent series is the natural generalization of the Taylor series when the centre of the expansion is a pole rather than a non-singular point.

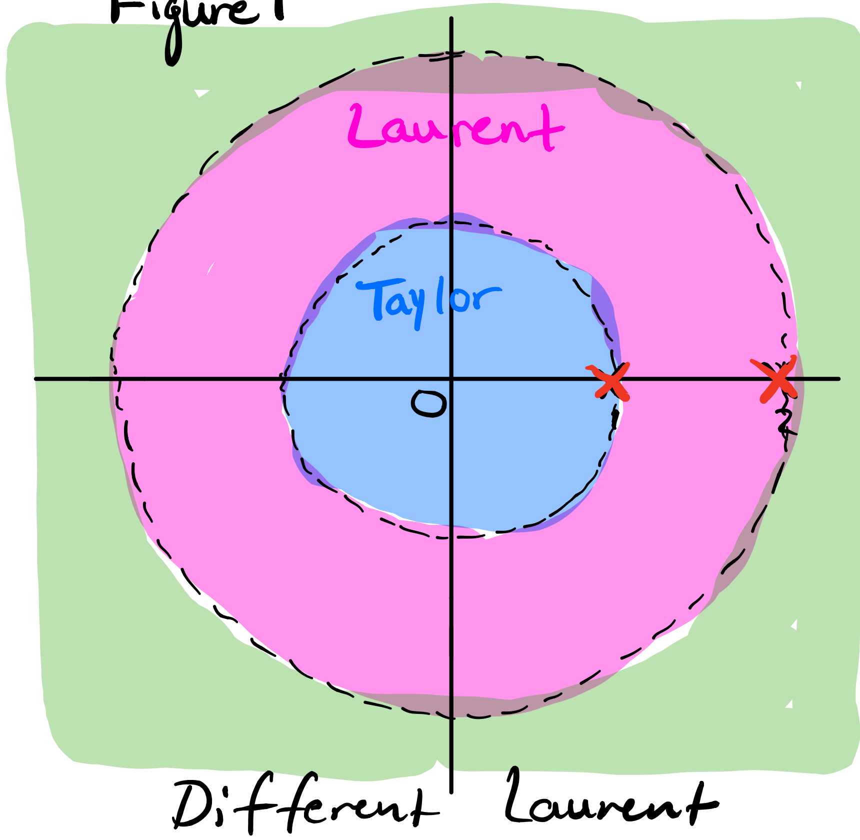
This is **NOT** the only situation in which Laurent series are needed.

For example, consider

$$F(z) = \frac{1}{(z-1)(z-2)}$$

whose poles can be seen to the right.

Figure 1



Let's look at the three shaded regions of Fig 1

$|z| < 1$, $1 < |z| < 2$, $|z| > 2$

- Since F is analytic within the unit disc, it possesses a Taylor series in powers of z.

$$F(z) = \frac{1}{1-z} - \frac{1}{2-z} \quad \text{(via partial fractions)}$$

$$= \frac{1}{(1-z)} - \frac{1}{2[1-\frac{z}{2}]} = \underbrace{\sum_{n=0}^{\infty} z^n}_{\text{for } |z|<1} - \underbrace{\frac{1}{2}\sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n}_{\text{for } |z|<2}$$

$$= \frac{1}{2} + \frac{3}{4}z + \cdots + \left[1 - \left(\frac{1}{2}\right)^{n+1}\right] z^n + \cdots, \quad \text{for } |z|<1$$

The pole at $z=1$ means that outside the unit disc F cannot be expressed as a power series in z. However in the shaded annulus $1<|z|<2$ it can be expressed as a Laurent series in z:

$$F(z) = \frac{-1}{z[1-\frac{1}{z}]} - \frac{1}{2[1-z/2]}$$

$$= \underbrace{-\sum_{n=0}^{\infty} \left[\frac{1}{z}\right]^{n+1}}_{|z|>1} - \underbrace{\frac{1}{2}\sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n}_{|z|<2}$$

$$= \cdots \frac{-1}{z^3} - \frac{1}{z^2} - \frac{1}{z} - \frac{1}{2} - \frac{z}{4} - \frac{z^2}{8} - \frac{z^3}{16} - \cdots \quad 1<|z|<2$$

Finally, in the region $|z|>2$ beyond the annulus we obtain a DIFFERENT Laurent series

This was NOT worked out in book so verify result.

$$F(z) = \frac{1}{(1-z)} - \frac{1}{(2-z)}, \qquad |z|>2$$

$$u = \frac{2}{z}$$

$$= \frac{1}{1-z} - \frac{1}{2(1-2/z)}, \qquad \frac{1}{u} = \frac{z}{2}$$

$$= \frac{1}{1-\frac{2}{u}} - \frac{1}{2(1-\frac{1}{u})} \qquad |u|<1$$

$$= \frac{u}{u-2} - \frac{u}{2(u-1)}$$

$$= \frac{u}{2(\frac{u}{2}-1)} + \frac{u}{2}\cdot\frac{1}{1-u}$$

$$= \left[-\frac{u}{2}\cdot\frac{1}{1-\frac{u}{2}}\right] + \left[\frac{u}{2}\cdot\frac{1}{1-u}\right]$$

$$= -\frac{u}{2}\left(1+\left(\frac{u}{2}\right)+\left(\frac{u}{2}\right)^2+\left(\frac{u}{2}\right)^3+\cdots\right) + \frac{u}{2}\left(1+u+u^2+u^3+\cdots\right)$$

$$= -\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right) + \frac{1}{z}\left(1+\left(\frac{2}{z}\right)+\left(\frac{2}{z}\right)^2+\left(\frac{2}{z}\right)^3+\cdots\right)$$

$$= \left(-\frac{1}{z}-\frac{1}{z^2}-\frac{1}{z^3}-\frac{1}{z^4}-\cdots\right) + \left(\frac{1}{z}+\frac{2}{z^2}+\frac{4}{z^3}+\frac{8}{z^4}+\cdots\right)$$

$$-\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n + \frac{1}{2z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n$$

$$= \underset{n=0}{0} + \underset{n=1}{\left(\frac{2}{z^2} - \frac{1}{z^2}\right)} + \underset{n=2}{\left(\frac{4}{z^3} - \frac{1}{z^3}\right)} + \underset{n=3}{\left(\frac{8}{z^4} - \frac{1}{z^4}\right)} + \cdots$$

$$= 0 + \frac{1}{z^2} + \frac{3}{z^3} + \frac{7}{z^4}$$

$$= \sum_{n=0}^{\infty} \frac{2^n - 1}{z^{n+1}}$$

We have just seen the illustration of a general phenomenon!

LAURENT'S THEOREM

If $f(z)$ is analytic everywhere within an annulus A centered at a, then $f(z)$ can be expressed as a Laurent series within A. In fact, if K is any simple loop winding once round a,

$$f(z) = \sum_{n=-\infty}^{+\infty} c_n (z-a), \quad c_n = \frac{1}{2\pi i}\oint_K \frac{f(z)}{(z-a)^{n+1}}\,dz$$

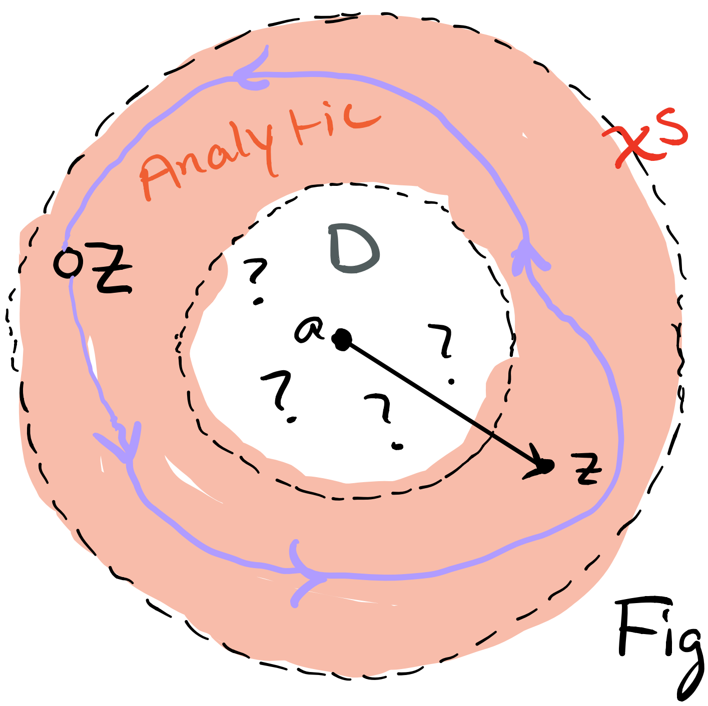


Figure 2

Before establishing Laurent's Theorem, note the following observations regarding its significance

① The surprising thing about the result is the existence of Laurent series, not the fact that it converges in an annulus. Since we know that a power series in $(z-a)$ will converge inside a disk centered at a, it follows that a power series in $1/(z-a)$ will converge outside a disc centered at a.

Since a Laurent series is the sum of a power series in $(z-a)$ and a power series in $1/(z-a)$, it follows that it will converge in an annulus.

(2) Previously we could only deduce the existence of a Laurent series in the vicinity of a pole. The present result is much more powerful: as indicated in figure 7, we make no assumptions at all concerning the behavior of $f(z)$ in the disc D bounded by the inner edge of the annulus.
In practice the outer edge of the annulus may be expanded until it hits a singularity s of $f(z)$. Likewise the inner edge may be contracted until it hits the outermost singularity lying in D.

(3) If there are no singularities in D, than the inner edge may be completely collapsed. In this case our c_n do not contain any negative integers! For if n is negative then $\dfrac{f(z)}{(z-a)^{n+1}}$ is analytic everywhere in K an $c_n = 0$. In this way we recover the existence of Taylor series as a special case of Laurent's Theorem.

(4) Suppose that a is a singularity and that for sufficiently small ε, there are no other singularities within a distance ε of a. In this case we say that a is an **isolated singularity** of $f(z)$. Applying Laurent's Theorem to $0 < |z-a| < \varepsilon$, we find there are just two fundamentally different possibilities:

i) The principle part of the Laurent series has finitely many terms

OR

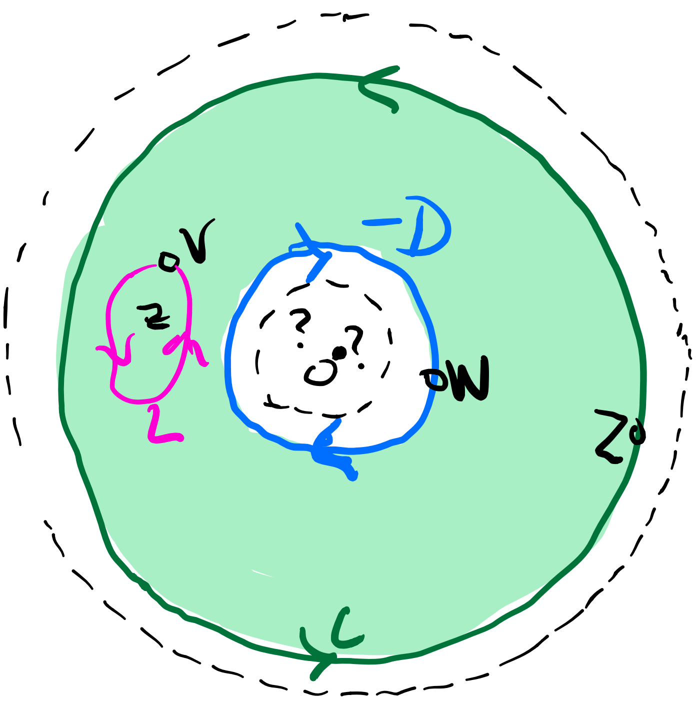
ii) infinitely many terms.

To sum up,

An isolated singularity of an analytic function is either a pole or an essential singularity.

Establishing Laurent Series

Consider the case wher $a=0$. In the figure z is a general point in the annulus, C and D are CCW circles such that z lies between them and L is a simple loop around z, lying in the annulus.



First, by Cauchy's Formula

$$f(z) = \frac{1}{2\pi i}\oint_L \frac{f(V)}{V-z}\,dV = \underbrace{\frac{1}{2\pi i}\oint_C \frac{f(Z)}{Z-z}\,dZ - \frac{1}{2\pi i}\oint_D \frac{f(W)}{W-z}\,dW}$$

Follows from the fact that L may be deformed into $C + -D$

Next we rewrite the above equation as

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(Z)}{Z}\left[\frac{1}{1-\frac{z}{Z}}\right]dZ + \frac{1}{2\pi i}\oint_D \frac{f(W)}{z}\left[\frac{1}{1-\frac{W}{z}}\right]dW$$

The significance of this is that

$$\left|\frac{z}{Z}\right| < 1 \quad \text{and} \quad \left|\frac{W}{z}\right| < 1$$

so both integrands can be expanded into a geometric series.

The integral around C can be expressed as

$$\frac{1}{2\pi i}\oint_C \frac{f(Z)}{Z}\left[\frac{1}{1-\frac{z}{Z}}\right]dZ = \sum_{n=0}^{\infty}\left[\frac{1}{2\pi i}\oint_C \frac{f(Z)}{Z^{n+1}}dZ\right]z^n$$

Essentially identical reasoning also justifies term by term integration in the case of the integral around D?

$$\frac{1}{2\pi i}\oint_D \frac{f(W)}{z}\left[\frac{1}{1-\left(\frac{W}{z}\right)}\right]dW = \frac{1}{2\pi i}\oint_D \frac{f(W)}{z}\sum_{n=0}^{\infty}\frac{W^n}{z^n}dW$$

$$= \sum_{n=1}^{\infty}\left[\frac{1}{2\pi i}\oint_D W^{n-1}f(W)\,dW\right]\frac{1}{z^n}$$

<u>Note</u>

$$\sum_{n=0}^{\infty}\frac{W^n}{z^{n+1}} = \sum_{n=1}^{\infty}\frac{W^{n-1}}{z^n}$$

Thus the existence of Laurent series is established!

$$f(z) \equiv \cdots + \frac{d_3}{z^3} + \frac{d_2}{z^2} + \frac{d_1}{z} + c_0 + c_1 z + c_2 z^2 + \cdots$$

where

$$d_m = \frac{1}{2\pi i} \oint_D w^{m-1} f(w)\, dw \quad \text{and} \quad c_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z^{n+1}}\, dz$$

By the Deformation Theorem,

Deformation Theorem:

If a contour sweeps only through analytic points as it is deformed, the value of the integral does **NOT** change!

the integrals defining d_m and c_n do NOT change their values if we allow C to contract and D to expand till they coalesce into the same circles!

Indeed, we may replace both C and D by any simple loop K contained in the annulus and winding around it once.

Now let $m = -n$, then d_{-n} of z^n has integrand

$$w^{-(n+1)} f(w) = \frac{f(w)}{w^{n+1}}$$

which is the same integrand for the c_n!

Thus, as was to be shown, the Laurent series may be expressed as

$$f(z) = \sum_{n=-\infty}^{+\infty} c_n z^n \quad \text{where} \quad c_n = \frac{1}{2\pi i} \oint_K \frac{f(z)}{z^{n+1}} dz$$