Recall

Multiple discrete rvs

- Joint pmfs $p_{X,Y}(x,y) = \mathbb{P}(\{X=x\} \cap \{Y=y\})$ etc.
- Marginals in terms of joint
- >2 rvs

Started Conditional probability story for discrete rvs
$p_{X|A}(x)$ for a discrete rv and event A

Total probability rule: If $A_1, \dots, A_n$ partition Ω, then

$$p_X(x) = \sum_{k=1}^{n} p_{X|A_k}(x)\, \mathbb{P}(A_k)$$

Given two rvs on same $\Omega, \mathbb{P}$ - say X and Y - define

$$p_{X|Y}(x|y) = \frac{\mathbb{P}(\{X=x\} \cap \{Y=y\})}{\mathbb{P}(\{Y=y\})} = \frac{p_{X,Y}(x,y)}{p_Y(y)} \; ; \quad \forall x, y \text{ w/ } p_Y(y) > 0$$

For fixed y, $p_{X|Y}(x|y)$ defines a pmf "over x-values" -
i.e

$$p_{X|Y}(x|y) \geq 0 \;\; \forall x \quad \text{and} \quad \sum_{x \in X} p_{X|Y}(x|y) = 1$$

"$p_{X|Y}(x|y)$ = conditional pmf of X given $Y=y$"

Like the conditional "event-centered story", have a product rule of sorts

$$p_{X,Y}(x,y) = p_Y(y)\,p_{X|Y}(x|y) \quad \forall \begin{matrix} x \in X \\ y \in Y \end{matrix}$$

OR

$$p_{X,Y}(x,y) = p_X(x)\,p_{Y|X}(y|x) \quad \forall \begin{matrix} x \in X \\ y \in Y \end{matrix}$$

This expresses joint in terms of marginals + conditional(s).

Also have a total-probability rule of sorts:

$$p_X(x) = \sum_{y \in Y} p_Y(y)\,p_{X|Y}(x|y)$$

$$p_Y(y) = \sum_{x \in X} p_X(x)\,p_{Y|X}(y|x)$$

Comment: These constructions generalize to >2 r.v.s. i.e. X, Y, Z defined on same $\Omega, \mathbb{P}$

Like

$$p_{X|Y,Z}(x|y,z) = \frac{\mathbb{P}(\{X=x\} \cap \{Y=y\} \cap \{Z=z\})}{\mathbb{P}(\{Y=y\} \cap \{Z=z\})} = \frac{p_{X,Y,Z}(x,y,z)}{p_{Y,Z}(y,z)}$$

$$p_{X,Y|Z}(x,y|z) = \frac{p_{X,Y,Z}(x,y,z)}{p_Z(z)}$$

⋮ etc

Get more involved product rules – $p_{X,Y,Z}(x,y,z) = p_Z(z)\,p_{Y|Z}(y|z)\,p_{X|Y,Z}(x|y,z)$

... etc

Example- Spinner + Hockey

Spinner has 4 equal wedges numbered 0,1,2,3 (that's X).

- Take X shots on goal when spinner lands
- Shots you take are mutually independent
- Score with probability p each

Let $Y = \#(\text{goals you score})$

Here, $p_{Y|X}(y|x)$ and $p_X(x)$ is easy to figure out.

$$p_X(x) = \frac{1}{4} \text{ for } x = 0,1,2,3$$

$$p_{Y|X}(y|0) = \begin{cases} 1, & y=0 \\ 0, & \text{otherwise} \end{cases}$$

$$p_{Y|X}(y|1) = \begin{cases} p, & y=1 \quad \leftarrow \text{make your one shot} \\ 1-p, & y=0 \quad \leftarrow \text{miss your one shot} \end{cases}$$

$$p_{Y|X}(y|2) = \begin{cases} (1-p)^2, & y=0 \quad \leftarrow \text{Miss both} \\ p(1-p)+p(1-p), & y=1 \quad \leftarrow \text{Two ways to score 1 goal!} \\ p^2, & y=2 \quad \leftarrow \text{Make both} \end{cases}$$

$$p_{Y|X}(y|3) = \begin{cases} \binom{3}{k} p^k (1-p)^{3-k}, & 0 \le k \le 3 \\ 0, & \text{else} \end{cases}$$

From these get joint pmf $p_{X,Y}(x,y) = p_X(x)\, p_{Y|X}(y|x)$.

Also can get marginal $p_Y(y)$ via

$$p_Y(y) = \sum_{x \in X} p_X(x)\, p_{Y|X}(y|x)$$

use this to answer "what is $\mathbb{P}$(score 2 goals)?"

$$p_Y(y=2) = 0 + 0 + \underbrace{\tfrac{1}{4}}_{p_X(2)} \underbrace{p^2}_{p_{Y|X}(2|2)} + \underbrace{\tfrac{1}{4}}_{p_X(3)} \underbrace{\binom{3}{2} p^2 (1-p)}_{p_{Y|X}(2|3)}$$

can't score 2 goals in 0 or 1 shots

Next, Conditional expectation (or conditional expected value)

Given $\Omega, \mathbb{P}$; discrete rv X; event A:

$$\underbrace{\mathbb{E}(X|A)}_{\text{conditional expectation of } X \text{ given event } A} = \sum_{x \in X} x\, p_{X|A}(x)$$

just plug the pmf $p_{X|A}(x)$ into standard $\mathbb{E}$-formula

note how it's NOT $p_{X|A}(x|A)$. This is b/c A is an event & NOT a rv.

If $A_1, A_2, \ldots, A_n$ partition Ω, then we know that

$$p_X(x) = \sum_k p_{X|A_k}(x|A_k)\, \mathbb{P}(A_k)$$

Hence,

$$\mathbb{E}(X) = \sum_{x\in X} x\, p_X(x) = \sum_x \sum_k x\, p_{X|A_k}(x)\, \mathbb{P}(A_k)$$

$$= \sum_k \underbrace{\left(\sum_x x\, p_{X|A_k}(x)\right)}_{\mathbb{E}(X|A_k)} \mathbb{P}(A_k)$$

Given $\Omega, \mathbb{P}$, rvs X, Y:

$$\mathbb{E}(X \mid Y=y) = \mathbb{E}(X|A) \;;\; A \text{ is the event } \{Y=y\}$$

"conditional expectation of X given $Y=y$"

So

$$\mathbb{E}(X|A) = \sum_{x\in X} x\, p_{X|A}(x) = \sum_{x\in X} x\, p_{X|Y}(x|y)$$

Since all events $\{Y=y\}$ partition Ω, get

$$\mathbb{E}(X) = \sum_{y\in Y} \mathbb{E}(X|Y=y)\, \mathbb{P}(\{Y=y\})$$

That's the second instance of

total expectation = avg of conditional expectations

Example - Spinner & Shots on Goal... continued

Easy to find

$$\mathbb{E}(Y|X=0)=0 \qquad \mathbb{E}(Y|X=2)=2p$$

$$\mathbb{E}(Y|X=1)=p \qquad \mathbb{E}(Y|X=3)=3p$$

Thus

expected number of goals scored

$$\mathbb{E}(Y)=\frac{1}{4}p+\frac{1}{4}2p+\frac{1}{4}3p=\frac{3}{2}p$$

Example - 2 envelopes

Have 2 envelopes. One has twice the amount of money as the other.

i.e

one has m dollars

one has $2m$ dollars

Call the envelopes X and Y.

let Z = total amount of money between the two

Say you find

$$X=m$$

Get to keep m if you want. Should you?

Figure out $\mathbb{E}(Y|X=m)$ using following NAIVE approach:

$$\boxed{\mathbb{P}(\{Y=2m\}|\{X=m\})=\mathbb{P}\{Y=\tfrac{m}{2}|\{X=m\})=1/2}$$

This line of reasoning is incorrect

Thus

This leads you to believe you should switch.

$$\mathbb{E}(Y|X=m)=2m\left(\frac{1}{2}\right)+\frac{m}{2}\left(\frac{1}{2}\right)=\frac{5}{4}m>m$$

The reasoning is specious because it is based on a woefully incomplete probability model.

How's it incomplete?

- Are X, Y integers? Continuous valued?
  - Eg if integers, if X is 19 you should definitely switch
- What's $p_Z(z)$? Any complete model would enable you to find this...

Let us make it more complete.

- Dealer picks z "at random" - a multiple of 3 dollars
- Divides into two piles. $z/3$; $2z/3$.
- Flip a fair coin to decide how to allot piles to 2 envelopes, X, Y.

Say you open X and find m dollars.

$$Z = \begin{cases} 3m, & \text{if } X \text{ is smaller envelope} \\ \frac{3}{2}m, & \text{if } X \text{ is larger envelope} \end{cases}$$

If m is odd, can't have $Z = \frac{3}{2}m$ - but say m is even.

Fact: It's <u>impossible</u> to have

$$p_{Z|X}(3m|m) = p_{Z|X}\left(\tfrac{3}{2}m \,\middle|\, m\right) = \frac{1}{2}$$