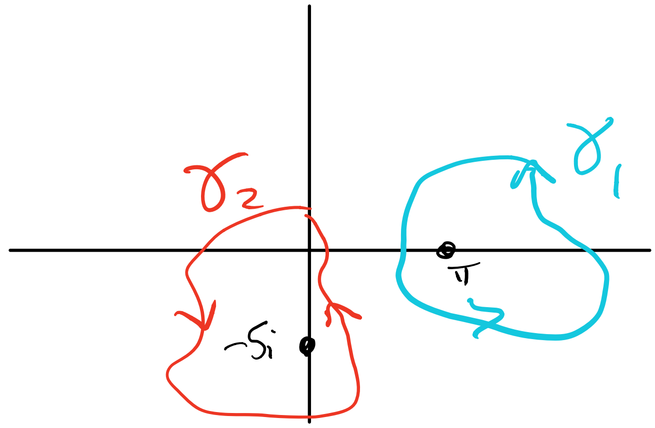
Evaluate

$$\oint \frac{\cos(z)}{(z-\pi)(z+5i)}\,dz$$

along $\gamma_1 + \gamma_2$




On $\gamma_1$, use

$$2\pi i f(\pi) = \oint_{\gamma_1} \frac{\cos(z)/(z+5i)}{(z-\pi)}\,dz \quad \Bigg\} \quad f(z) = \frac{\cos(z)}{(z+5i)}$$

On $\gamma_2$, use

$$2\pi i f(5i) = \oint_{\gamma_2} \frac{\cos(z)/(z-\pi)}{z+5i}\,dz \quad \Bigg\} \quad f(z) = \frac{\cos(z)}{(z-\pi)}$$

Need appropriate choices of $f$ and $a$!
Need $f$ to be analytic **ON AND INSIDE** $\gamma$ to apply Cauchy's Integral Formula

In conclusion,

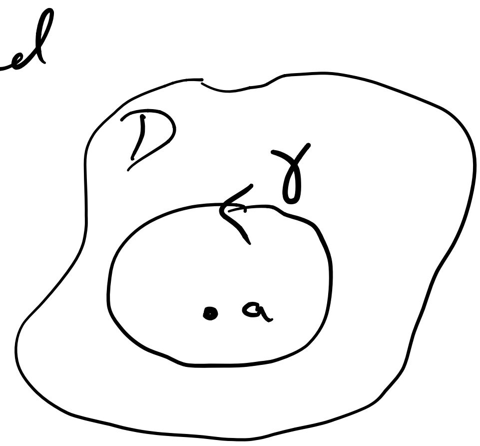
$$\oint_{\gamma_1+\gamma_2} \frac{\cos z}{(z-\pi)(z+5i)}\,dz = \oint_{\gamma_1} \frac{f(z)}{z-\pi}\,dz + \oint_{\gamma_2} \frac{f(z)}{z+5i}$$

$$= 2\pi i \frac{\cos(\pi)}{\pi+5i} + 2\pi i \frac{\cos(5i)}{5i-\pi}$$

# Collaries of Cauchy's Integral Formula

**Derivatives:** Suppose $f(z)$ is analytic on and inside a closed, positively oriented $\gamma$.

Let $a$ be a point inside $\gamma$ and let $f$ be analytic in $D$.




Then $$f(a) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z-a}\,dz$$

Regard $f(a)$ as a function of $a$.
Then

$$f'(a) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{(z-a)^2}\,dz$$

Differentiating Order the Integral Sign

Similarly,

$$f''(a) = \frac{2}{2\pi i} \oint_\gamma \frac{f(z)}{(z-a)^3}\,dz$$

In General, $$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_\gamma \frac{f(z)}{(z-a)^{n+1}}\,dz$$

**AMAZING!** $f(z)$ once differentiable (analytic) $\Rightarrow$ $f(z)$ infinitely differentiable

Example 1: Let $f(z) = \cos z$

Evaluate $\oint_\gamma \frac{f(z)}{(z-i)^3} dz$ at $z = i$

$$f^{n}(i) = \frac{n!}{2\pi i} \oint \frac{f(z)}{(z-i)^3} dz \quad \Big\} \quad \begin{matrix} 3 = n+1 \\ n = 2 \end{matrix}$$

$$\frac{2\pi i}{2!} f^2(i) = \oint \frac{f(z)}{(z-i)^3} dz$$

$$\pi i \left[ \frac{d^2}{dz^2}[\cos(z)] \Big|_{z=i} \right] = -i\pi \cos(i)$$

**Bounds on Derivatives:** Cauchy's Estimate.

Suppose $f(z)$ analytic **ON AND INSIDE** circle of radius $R$ about $a$.

Suppose $M = \max_{z \in C_R} |f(z)|$

Then

$$|f^{(n)}(a)| \leq \frac{n!\, M}{R^n}$$

Proof: $$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_{C_R} \frac{f(z)}{(z-a)^{n+1}} dz$$

ML Estimate $\Rightarrow$ $$\left|f^{(n)}(a)\right| \leq \underbrace{\max_{z \in C_R} \left|\frac{f(z)}{(z-a)^{n+1}}\right| \frac{n!}{2\pi}}_{M} \cdot \underbrace{2\pi R}_{L}$$

$$= \max_{z \in C_R} |f(z)| \frac{n!}{R^{n+1}} R$$

Note: $|z-a| = R$

$$= M \frac{n!}{R^n}$$

So, $$\left|f^{(n)}(a)\right| \leq \frac{M n!}{R^n}$$

Applications of this estimate are seen in Louisville's Theorem. Suppose $f(z)$ is <u>entire</u> and is bounded by some constant: $|f(z)| \leq M \ \forall z$. Then $f(z)$ has to be constant.

Proof: $|f'(a)| \leq \frac{M}{R} \ \forall R$ (since $M$ is independent of $R$ by boundness)

As $R \to \infty$, $|f'(a)| \leq 0$

$$\Rightarrow f(a) = C \text{ (constant)}$$

# Fundamental Theorem of Algebra

Every non-constant polynomial $P(z)$ has a root in $\mathbb{C}$.

Let $P(z) = c_0 + c_1 z + \cdots + c_n z^n$

then $P(z) = 0$ for some $z \in \mathbb{C}$

Proof: Suppose $P(z) \neq 0 \ \forall z$.

Consider $f(z) = \dfrac{1}{P(z)}$

$f(z)$ is now analytic since $P(z) \neq 0$ anywhere.

i.e

f(z) is entire, and also bounded

As $z \to \infty$ $\quad f(z) = \dfrac{1}{P(z)} \simeq \dfrac{1}{c_n z^n}$

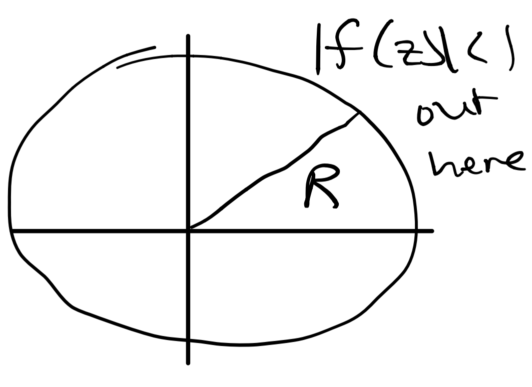
So outside some big circle $|z| = R$

we have

$$|z| = R$$

we have

$$|f(z)| < 1$$



[a continuous function on a compact set achieves its maximum and minimum value]

from real analysis.

So inside circle $|f(z)|$ is bounded by $M$.

Then $|f(z)| \leq \max(M, 1)$. So $f$ is bounded and entire which implies $f(z)$

So then $P(z)$ must be a constant by Liouville's Theorem!