

Last Time

Division algorithm

$$\begin{aligned} p, s &\in \mathbb{F}[x], s \neq 0 \\ \Rightarrow \exists q, r &\in \mathbb{F}[x] \\ \text{s.t. } p &= qs + r \\ \deg r &< \deg s \end{aligned}$$

i.e. Not identically 0

Proposition: Suppose $p(x) \in \mathbb{F}[x]$, $\lambda \in \mathbb{F}$ assume $p(x) \not\equiv 0$
then $p(\lambda) = 0 \Leftrightarrow p(x) = (x - \lambda)q(x)$

Proof $(\Leftarrow) \checkmark$

(\Rightarrow) by division algorithm

$$(a) p(x) = (x - \lambda)q(x) + r(x)$$

$$(b) \deg r < \deg(x - \lambda) = 1$$

i.e. $r(x) = r \in \mathbb{F}$ is a constant

(c) plug in λ

$$p(\lambda) = 0 \cdot q(\lambda) + r(\lambda) = r \in \mathbb{F} \Rightarrow r = 0$$

$$\therefore p(x) = (x - \lambda)q(x)$$

Corollary: For any field \mathbb{F} , if $m = \deg p(x) \geq 0$ then

$$\# \text{zeros of } p(x) = \# \{ \lambda \in \mathbb{F} \mid p(\lambda) = 0 \} \leq m = \deg p(x)$$

Proof Use induction on $\deg p(x) = m$

if $m = 0$, $\# \text{zeros} = 0 = m$

Suppose statement is true for $m-1$, show it for m

Consider $p(x)$ $\deg m$

- if no roots, statement is true

Let $\lambda \in \mathbb{F}$
be a root

- By prop. $p(x) = (x - \lambda)q(x)$, $q(x)$ has $\deg m-1$

\therefore

$$\# \text{roots of } p(x) = 1 + \# \text{roots of } q(x) \leq 1 + m-1 \leq m$$

\therefore

by induction the corollary holds

Fundamental Theorem of Algebra

Every non-constant polynomial with non-constant coefficients has a zero

Proof/Take Analysis

Corollary: If $p(x) \in \mathbb{C}[x]$, \exists unique factorization $m = \deg p(x)$
$$p(x) = a(x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_m)$$

where $a, \lambda_1, \dots, \lambda_m \in \mathbb{C}$

NEXT

Applying a polynomial to a matrix A ($n \times n$) over \mathbb{F} or a linear operator $T: V \rightarrow V$ ($T \in \mathcal{L}(V)$), \forall a v.s. over \mathbb{F} ,

Definition: If $p(x) \in \mathbb{F}[x]$, $p(x) = a_0 + a_1x + \dots + a_mx^m$, $a_i \in \mathbb{F}$
 then define

often $\text{id}_V = I$

$$p(A) = a_0 I_{n \times n} + a_1 A + a_2 A^2 + \dots + a_m A^m$$

$$p(T) = a_0 \text{id}_V + a_1 T + a_2 T^2 + \dots + a_m T^m$$

Proposition: if $f(x) = p(x)q(x)$, $p, q \in \mathbb{F}[x]$
 then $f(T) = p(T)q(T) = q(T)f(T)$

Proof

Suppose $p(x) = \sum_{j=0}^m a_j x^j$

$$q(x) = \sum_{k=0}^n b_k x^k$$

$$p(x)q(x) = \sum_{j=0}^m a_j x^j \sum_{k=0}^n b_k x^k = \sum_{j=0}^m \sum_{k=0}^n a_j b_k x^{j+k}$$

$$p(T)q(T) = \left(\sum_{j=0}^m a_j T^j \right) \left(\sum_{k=0}^n b_k T^k \right) = \sum_{j=0}^m \sum_{k=0}^n (a_j b_k T^j T^k)$$

$$f(T) = p(T)q(T) \rightarrow \text{other side works too}$$

Example

Consider

$$x''(t) + 3x'(t) + 2x(t) = 0$$

Let

$$V = \{ x(t) \in C^\infty(\mathbb{R}) \mid x'' + 3x' + 2x = 0 \}$$

DE class $\leadsto V$ is a vector space of dimension 2

$$\text{Let } D: C^\infty(\mathbb{R}) \rightarrow C^\infty(\mathbb{R})$$

$$x(t) \mapsto x'(t)$$

then

$$V = \ker(D^2 + 3D + 2I)$$

$$\text{and if } p(x) = x^2 + 3x + 2$$

$$\text{Note: } p(x) = (x+2)(x+1)$$

$$\text{then } V = \ker(p(D))$$

$$V = \ker((D+2I)(D+I))$$

$$= \ker((D+I)(D+2I))$$

Note: V contains $\ker(D+I)$

i.e. if $x' + x = 0$ then $x \in V$

$$x(t) = A e^{-t}, \quad A \in \mathbb{R}$$

$$\text{i.e. } \text{span}(e^{-t}) \subseteq V$$

Similarly, for $\ker(D+2I)$

$$\text{span}(e^{-2t}) \subseteq V$$

$$\therefore \text{span}(e^{-t}, e^{-2t}) \subseteq V$$

but these are LI \Rightarrow basis for V .