Recall

Started discrete random variables (r.v.'s).

Given $\Omega, \mathbb{P}$; a r.v X is a function $X: \Omega \to \mathbb{R}$ that takes on a finite or countably infinite number of valuables.

Associated with a discrete r.v. is its pmf p_X defined for each possible value of x for X

$$p_X(x) = \mathbb{P}(\{s \in \Omega : X(s) = x\}) \text{"="} \mathbb{P}(\{X = x\}) \text{"="} \mathbb{P}(X = x)$$

On a given Ω w/ a given $\mathbb{P}$, many discrete r.v's are defined

For a given pmf there are many Ω's, $\mathbb{P}$'s, $X: \Omega \to \mathbb{R}$ that give rise to p_X = that pmf.

Examples - constant, Bernoulli p, Binomial (n,p) (from last time)
disrete uniform on $0 \le k \le b$

Geometric Random Variable

Geometric(p):

$$p_X(k) = p(1-p)^{k-1}, \quad 1 \leq k < \infty$$

A possible $\Omega, \mathbb{P}$? You have a coin where $\mathbb{P}(H) = p$;
Ω = set of all semi-infinite $(1 \leq k < \infty)$ of H, T; $X(s)$ = index of first head for all $s \in \Omega$

Reality check: recall $\sum_{x \in X} p_X(x) = 1$ for any discrete r.v X.

Verify this for geometric r.v.'s.

$$\sum_{x \in X} p_X(k) = \sum_{k=1}^{\infty} p(1-p)^{k-1} = p\sum_{m=0}^{\infty}(1-p)^m = p\frac{1}{1-(1-p)} = 1 \quad \checkmark$$

Poisson Random Variable

Poisson (λ):

$$p_X(k) = e^{-\lambda}\frac{\lambda^k}{k!} \quad 0 \leq k < \infty \quad (k \in \mathbb{N})$$

$$\left(e^{1} = \sum_{n=0}^{\infty}\frac{1}{n!}\right)$$

$$\sum_{x \in X} p_X(k) = \sum_{k=0}^{\infty}\frac{\lambda^k}{k!}e^{-\lambda} = e^{\lambda}e^{-\lambda} = 1 \quad \checkmark$$

Given a random variable X and any function $g : \mathbb{R} \to \mathbb{R}$, can define another r.v $Y = g(X)$

$$\begin{tikzcd} \Omega \arrow[r, "X"] \arrow[rr, bend right, "Y"'] & \mathbb{R} \arrow[r, "g"] & \mathbb{R} \end{tikzcd}$$

$\forall\, s \in \Omega,\ Y(s) = g(X(s))$

possible X values being finite
$\Rightarrow$ possible Y values are finite.

"Function of a discrete r.v. is another discrete r.v."
Generally non-trivial to get pmf of $Y = g(X)$ from pmf of X.

BUT, sometimes easy.

Example - p_X discrete uniform on say $-3 \le k \le 3$

$$p_X(k) = \begin{cases} 1/7, & -3 \le k \le 3 \\ 0, & \text{else} \end{cases}$$

Say

$$g(x) = x^2 \text{ for all } x \in \mathbb{R} \text{ - set } Y = g(X) = X^2$$

what is p_Y?

Possible values of Y are $0, 1, 4, 9$.

Easy to figure out probability!

i.e

$$p_Y(0) = \mathbb{P}(\{Y=0\}) = \mathbb{P}(\{X=0\}) = 1/7$$

$p_Y(4) = \mathbb{P}(\{Y=4\}) = \mathbb{P}(\{X=-2\}) \cup \mathbb{P}(\{X=2\}) = 2/7$

disjoint! X can't be -2 and 2 at once.

Aside

A frequently encountered function: affine:
given a r.v X, define

$$Y = \alpha X + \beta \quad , \quad \alpha, \beta \in \mathbb{R} \text{ given.}$$

Expected Value

Given a discrecte r.v X w/ $p_X(x)$ pmf, define the expected value (or expectation)

$$\mathbb{E}(X) = \sum_{x \in X} x \, p_X(x)$$

Motivation? Say Ω is set of outcomes of a spinner – say outcomes are ①, ②, ..., ⓝ - say $\mathbb{P}(ⓚ) = p_k$, $k \in \mathbb{N} \setminus \{0\}$

Define $X : \Omega \to \mathbb{R}$ by

$$X(k) = x_k \quad , \quad k \in \mathbb{N} \setminus \{0\}$$

think of this as payment you get when spinner lands on k.

pmf of X:

$$p_X(x_k) = p_k \quad , \quad k \in \mathbb{N} \setminus \{0\}$$

Suppose you spin again and again repeatedly - collect payments each time - over long haul, what is avg win per spin.

If $L > 0$ is large integer, winnings/spin over L spins is

$$\frac{x_1 \#(1) + x_2 \#(2) + \cdots + x_n \#(n)}{L}$$

Intuition: for L large, $\frac{\#(k)}{L} \sim p_k$ (as $L \to \infty$)

this $\rightarrow$ $\sum_k k \underbrace{p_X(x_k)}_{p_k} = \mathbb{E}(X)$

i.e $\mathbb{E}(X)$ = "average winnings per spin over long haul"

Terms (synonymous: expected value, expectation, mean, average, ...

Example - X is Bernoulli(p)

$$p_X(x) = \begin{cases} p & \text{if } x=1 \\ 1-p & \text{if } x=0 \end{cases}$$

$$\mathbb{E}(X) = \sum_{x\in\mathcal{X}} x\, p_X(x) = 1\cdot p + 0\cdot(1-p) = p$$

when $p \neq 0, 1$, $\mathbb{E}(X)$ is "not a possible value of X" ∎

Given X, $Y = g(X)$, what is $\mathbb{E}(Y)$? One way to find $\mathbb{E}(Y)$: figure out $p_Y(y)$ for all possible values $y \in \mathcal{Y}$ - then

$$\mathbb{E}(Y) = \sum_{y\in\mathcal{Y}} y\, p_Y(y)$$

and can get p_Y from p_X - not trivial in general....

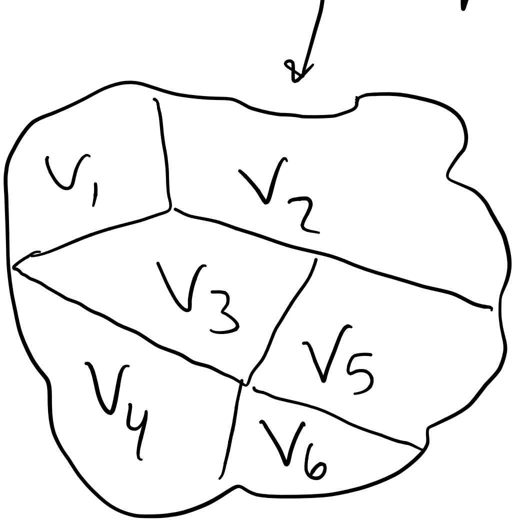
Another easier way: **Expected Value Rule**

Given X, p_X, and $Y = g(X)$,

$$\mathbb{E}(Y) = \sum_{x\in\mathcal{X}} g(x)\, p_X(x)$$

Why?

set of all possible X-values - say



$y_1, \dots, y_6$ are possible Y values

V_k = set of x-values for which $g(x) = y_k$

$$p_Y(y_k) = \sum_{x \in V_k} p_X(x)$$

Thus,

$$\mathbb{E}(Y) = \sum_{y_k} y_k \, p_Y(y_k)$$

$$= \sum_{y_k} y_k \sum_{x \in V_k} p_X(x)$$

for all $x \in V_k$, $g(x) = y_k$
(for $1 \le k \le 6$)

$$= \sum_{y_k} \sum_{x \in V_k} g(x) \, p_X(x)$$

$$= \sum_{x \in X} g(x) \, p_X(x)$$

Special Case: $Y = \alpha X + \beta$

Use expected value rule to get $\mathbb{E}(Y)$.

$$\mathbb{E}(Y) = \sum_{x \in X} g(x) \, p_X(x)$$

$$= \sum_{x \in X} (\alpha x + \beta) \, p_X(x) = \alpha \underbrace{\sum_{x \in X} x \, p_X(x)}_{\mathbb{E}(X)} + \beta \underbrace{\sum_{x \in X} p_X(x)}_{1}$$

So,

$$\mathbb{E}(Y) = \alpha \mathbb{E}(X) + \beta$$

$\boxtimes$

This is the one case Proffessor Delchamps knows of where

$$\mathbb{E}(g(X)) = g(\mathbb{E}(X))$$

Generally, knowing $\mathbb{E}(X)$ doesn't suffice for finding $\mathbb{E}(g(X))$.

VARIANCE

Variance of a r.v X w/ pmf p_X

Think of this as $\mathbb{E}(g(X))$ where $g(X) = (X - \mathbb{E}(X))^2$

measures spread of p_X about $\mathbb{E}(X)$ $\rightarrow$

$$\mathrm{Var}(X) = \mathbb{E}\left((X - \mathbb{E}(X))^2\right)$$

we define

$$\sigma_X = \sqrt{\mathrm{Var}\, X}$$

to be the Standard deviation of X!

Example - Calculations

$$p_X(k) = \begin{cases} \frac{1}{7}, & -3 \le k \le 3 \\ 0, & 0 \end{cases}$$

$$\mathbb{E}(X) = 0$$

$$\mathrm{Var}(X) = \mathbb{E}((X - \mathbb{E}(X))^2) = \mathbb{E}(X^2)$$

$$\mathbb{E}(X^2) = \sum_{x \in X} x^2 p_X(x)$$

$$= 9 \cdot \frac{1}{7} + 4 \cdot \frac{1}{7} + 1 \cdot \frac{1}{7} + 0 \cdot \frac{1}{7} + 1 \cdot \frac{1}{7} + 4 \cdot \frac{1}{7} + 9 \cdot \frac{1}{7}$$

$$= \frac{28}{7} = 4$$

So, $\mathrm{Var}(X) = 4$

$\sigma_X = 2$

Fact: If $Y = \alpha X + \beta$, than $\mathrm{Var}(Y) = \alpha^2 \mathrm{Var}(X)$

So,

$$\sigma_Y = \sqrt{\mathrm{Var}\, Y} = |\alpha| \sigma_X$$

easy to show