Let

$$f = u + iv$$

be an analytic function. Then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

These are the Cauchy-Riemann equations. They link the real + imaginary part of an analytic function.

Example: Suppose $f(z) = x + xy + iv(x,y)$ is analytic. Find $v(x,y)$.

$$\frac{\partial u}{\partial x} = 1 + y = \frac{\partial v}{\partial y} \quad , \quad -\frac{\partial u}{\partial y} = -x = \frac{\partial v}{\partial x}$$

$$v(x,y) = \int \frac{\partial v}{\partial y}\, \partial y = y + \frac{1}{2}y^2 + h(x)$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \Rightarrow h'(x) = -x$$

$$h(x) = \int -x \, dx = -\frac{1}{2}x^2 + C$$

$$v(x,y) = y + \frac{1}{2}(y^2 - x^2) + C$$

$$f(z) = x + xy + i\left[y + \frac{1}{2}(y^2 - x^2) + C\right]$$

$$= x + iy + xy + i\left(\frac{y^2}{2} - \frac{x^2}{2}\right) + \text{constant}$$

$$= z - i\frac{z^2}{2}$$

only a function of z, NOT $\bar{z}$!

Q: If u, v satisfy C-R equations, is $f = u + iv$ analytic?

A: Yes, IF u, v are smooth enough. - i.e they have continuous partial derivatives (at least first order)

Two important consequences of C-R equations.

① Real + Imaginary parts of an analytic function are "harmonic" functions.

→ they satisfy Laplace's equation $\nabla^2\phi = 0$

where

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

Proof: $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \xRightarrow{\partial/\partial x} \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right)$$

$$\frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x} \xRightarrow{\partial/\partial y} \frac{\partial^2 u}{\partial y^2} = \frac{-\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x}$$

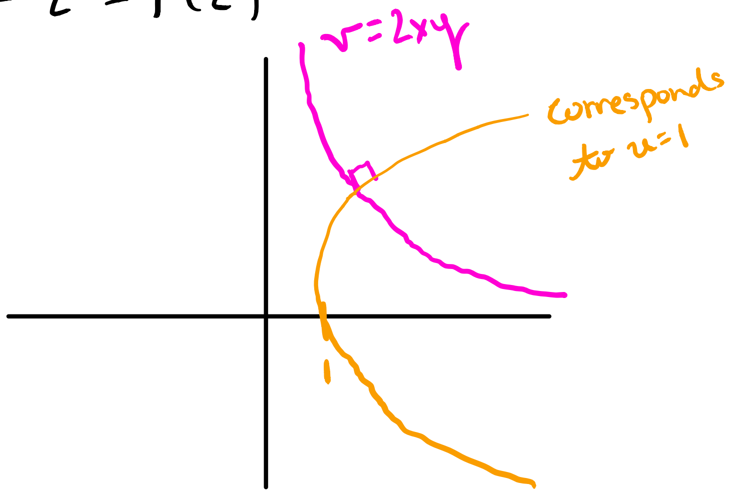
Note: If v has continuous 2^{nd} derivatives,
AND IT DOES, order of mixed partial derivatives is irrelevant.

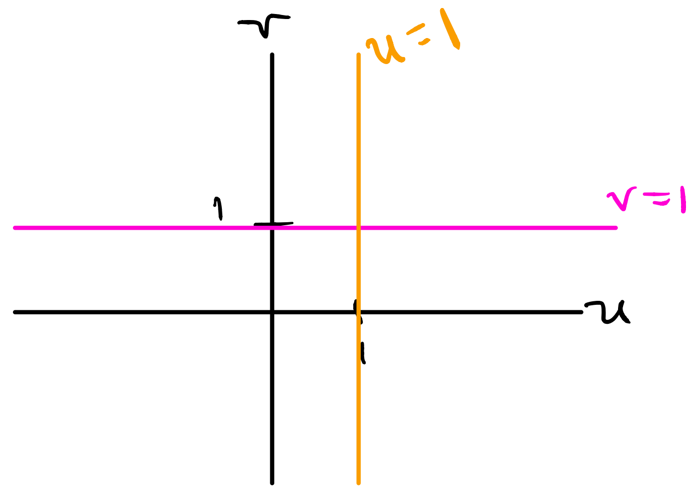
Therefore,

$$\nabla^2 u = 0 \quad \left.\begin{array}{l} \text{- governs flow of fluid} \\ \text{- elastic equilibrium} \\ \text{- thermal equilibrium} \end{array}\right.$$

② Contours of u, v (in z-plane) intersect at right angles. [@ pts where $f'(z) \neq 0$]

$\omega = z^2 = f(z)$ $\qquad w = u + iv = x^2 - y^2 + i2xy$





z-plane $\qquad$ w-plane

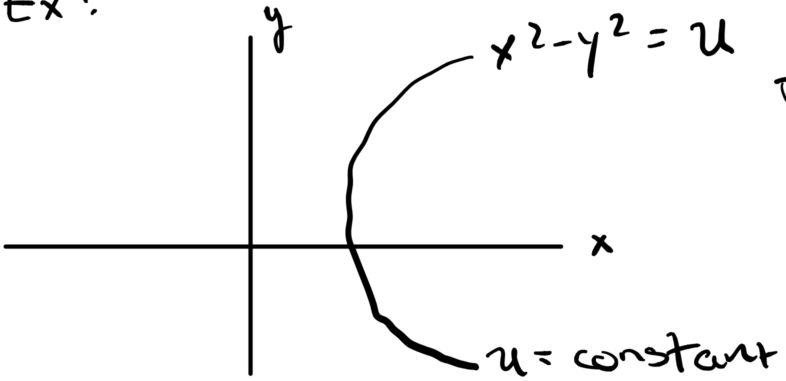
intersect @ right angles. $\qquad u = x^2 - y^2, \quad v = 2xy$

Physically: Think of contours as electrostatic field lines

Lets see why they meet at right angles!

Recall: ∇u is a vector $\perp$ to contours of u. It points towards increasing u.

Ex: $\quad (f(z) = z^2)$



$$\nabla u = \left\langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right\rangle$$

$$\nabla u = \langle 2x, -2y \rangle$$

"a way of calculating normal vectors"

Angle between contours of u, v = angle between their normal vectors.

$$\nabla v = \langle 2y, 2x \rangle$$

$$\nabla u \cdot \nabla v = \langle 2x, -2y \rangle \cdot \langle 2y, 2x \rangle$$

$$= 4xy - 4xy = 0$$

$\therefore$ intersect @ $90°$ angles!

More generally,

$$\nabla u \cdot \nabla v = \left\langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right\rangle \cdot \left\langle \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \right\rangle$$

$$= \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}$$

$\Leftarrow$ C-R substitution

$$= \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} = 0$$

So either u, v contours are $\perp$ or $\nabla u = 0$ (in which case $\nabla v = 0$ by C-R eqns and $f'(z) = 0$!)

Exponential Function, e^z

Define e^z as $e^z = e^{(x+iy)} = e^x e^{iy}$

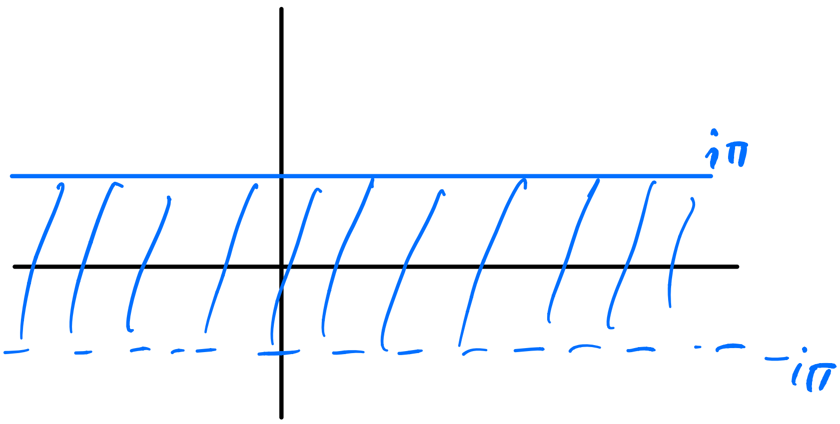
- e^x is usual exponential function
- $e^{iy} = \cos(y) + i\sin(y)$

$$e^{iy} = \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \cdots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \cdots\right)$$

$$= 1 + iy - \frac{y^2}{2!} - i\frac{y^3}{3!} \cdots = e^{iy}$$

e^z is periodic with period $2\pi i$.

$$e^{z+2\pi i} = e^z e^{2\pi i} = e^z, \quad e^{2\pi i} = 1!$$



e^z is injective in this strip (one-to-one)

Its behavior outside the strip is a repeat of what it does in there!

Series:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!},$$

analytic everywhere $\Rightarrow$ "entire"

$\rightarrow$ converges everywhere in complex plane

Also

$$\frac{d}{dz} e^z = e^z$$

$$|e^z| = |e^x e^{iy}| = e^x \quad \left.\right\} \begin{matrix} e^x > 0 \quad \forall x \\ |e^{iy}| = 1 \quad \forall y \end{matrix}$$

Define

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}, \qquad \cos(z) = \frac{e^{iz} + e^{-iz}}{2i}$$

Example: solve $e^z = 1$

$$e^x e^{iy} = 1$$

$$e^x = 1, \quad x = 0$$

$$e^{iy} = 1, \quad y = 2\pi k, \quad k \in \mathbb{Z}$$