

Let  $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$ .

We can then obtain

$$z^2 = r^2 e^{i2\theta}$$

Since  $z^3 = z z^2$ ,

$$z^3 = r^3 e^{i3\theta}$$

Continuing in this manner we arrive at the formula for the  $n^{\text{th}}$  power of  $z$ :

$$z^n = r^n e^{in\theta} = r^n (\cos(n\theta) + i\sin(n\theta)) \quad (1)$$

Eqn (1) is an appealing formula for raising a complex number to a positive integer power. It is easy to see that the identity is also valid for negative integers  $n$ . The question arises whether the formula will work for  $n = \frac{1}{m}$ , so that  $\zeta = z^{\frac{1}{m}}$  is an  $m^{\text{th}}$  root of  $z$  satisfying

$$\zeta^m = z$$

(2)

Certainly if we define

$$\zeta = \sqrt[m]{r} e^{i\theta/m}$$

(3)

(where  $\sqrt[m]{r}$  denotes the customary, positive,  $m^{\text{th}}$  root, we compute a complex number  $\zeta$  satisfying Eq(2) [as is easily seen by applying Eq(1)].

It's more complicated than this.

$1$  for example has two square roots:  $1, -1$ : And each of these in turn has two square roots - generating four fourth roots of  $1$ , namely  $1, -1, i, -i$ .

To see how the additional roots fit into the scheme of things, let's work out polar descriptions of the equation  $\zeta^4 = 1$  for each of these numbers:

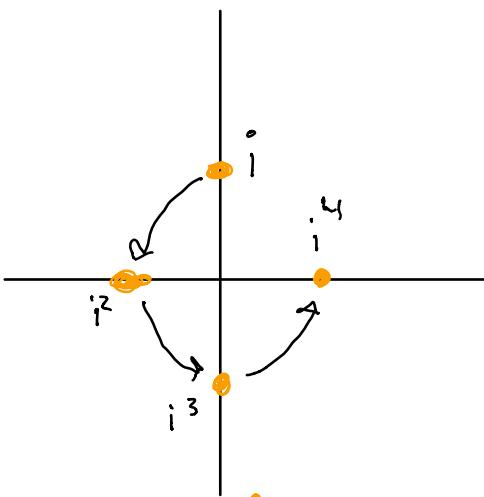
$$1^4 = (1e^{i0})^4 = 1^4 e^{i0} = 1$$

$$i^4 = (1e^{i\pi/2})^4 = 1^4 e^{i2\pi} = 1$$

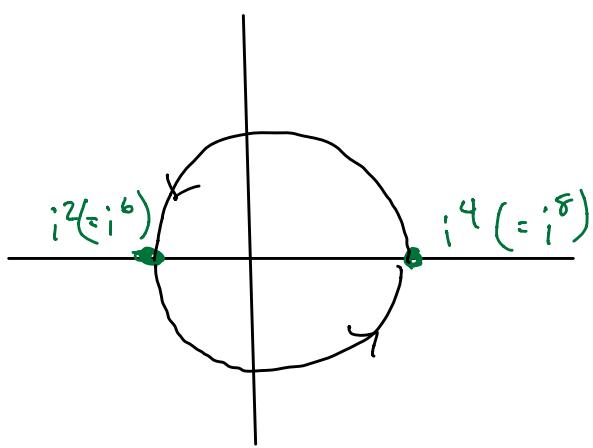
$$(-1)^4 = (1e^{i\pi})^4 = 1^4 e^{i4\pi} = 1$$

$$(-i)^4 = (1e^{i3\pi/2})^4 = 1^4 e^{i6\pi} = 1$$

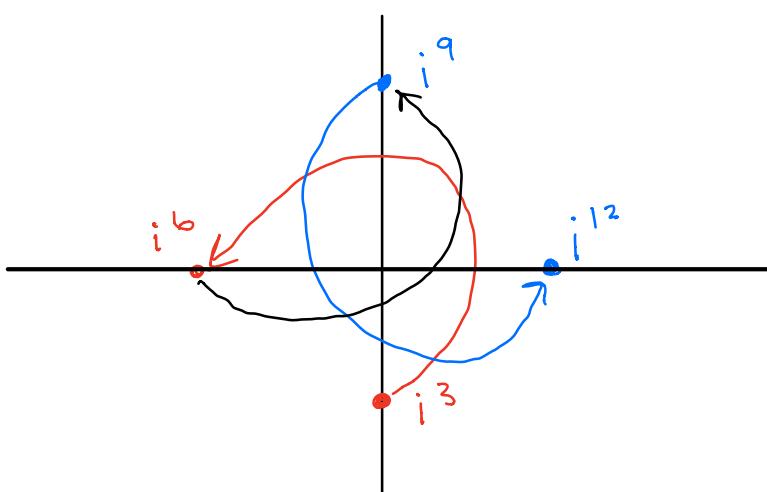
The powers of these roots are explored in the Argand diagram, (below)



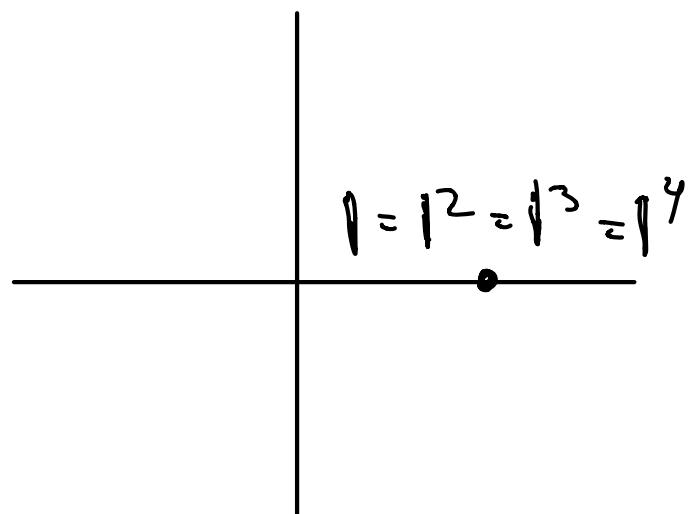
Powers of  $i$



Powers of  $i^2 = -1$



Powers of  $i^3 = -i$



Powers of  $i^4 = 1$

The multiplicity of roots is tied to the ambiguity in representing 1, in polar form as  $e^{i\theta}$ ,  $e^{i(2\pi)}$ ,  $e^{i(4\pi)}$ , etc. Thus to compute all the  $n$ th roots of a number  $z$ , we must apply formula (3) to **EVERY** polar representation of  $z$ .

For the cube roots of unity, for example, we would compute as follows:

## Polar Representation of 1

$$\begin{aligned}1 &= e^{-i6\pi} \\1 &= e^{-i4\pi} \\1 &= e^{-i2\pi} \\1 &= e^{i0} \\1 &= e^{i2\pi} \\1 &= e^{i4\pi} \\1 &= e^{i6\pi}\end{aligned}$$

⋮

## Application of (3)

$$\begin{aligned}1^{1/3} &= e^{-i6\pi/3} = 1 \\1^{1/3} &= e^{-i4\pi/3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2} \\1^{1/3} &= e^{-i2\pi/3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2} \\1^{1/3} &= e^{i0/3} = 1 \\1^{1/3} &= e^{i2\pi/3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2} \\1^{1/3} &= e^{i4\pi/3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2} \\1^{1/3} &= e^{i6\pi/3} = 1\end{aligned}$$

⋮

Obviously the roots recur in sets of three,  
since  $e^{i2\pi m_1/3} = e^{i2\pi m_2/3}$  whenever  $m_1 - m_2 = 3$ .

Generalizing, we can see that there are exactly  $m$  distinct  $m^{\text{th}}$  roots of unity, denoted by  $1^{1/m}$ , and they are given by

$$1^{1/m} = e^{\frac{i2\pi k}{m}} = \cos\left(\frac{2\pi k}{m}\right) + i \sin\left(\frac{2\pi k}{m}\right), \quad k = 0, 1, 2, \dots, m-1 \quad (4)$$

The arguments of these roots are  $\frac{2\pi}{m}$  radians apart, and the roots themselves form vertices of a regular polygon.

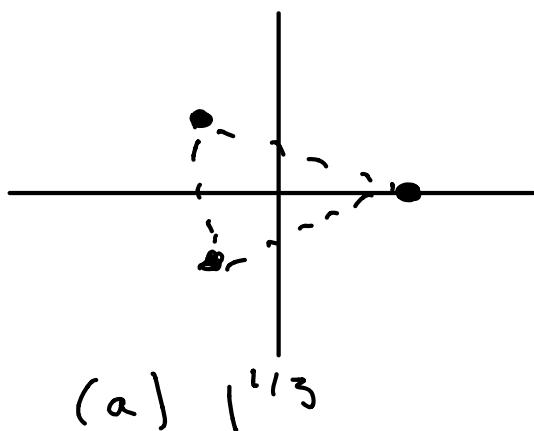
Taking  $k=1$  in Eq(4), we obtain the root

$$\omega_m := e^{i \frac{2\pi}{m}} = \cos\left(\frac{2\pi}{m}\right) + i \sin\left(\frac{2\pi}{m}\right)$$

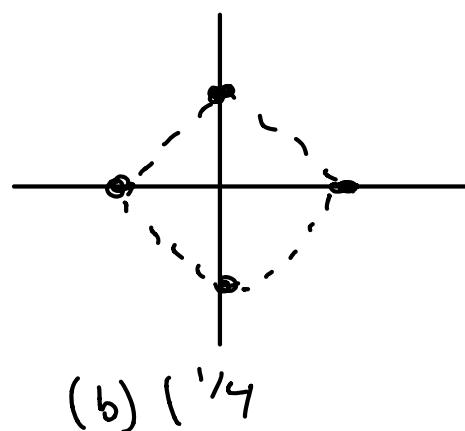
and it is easy to see that the complete set of roots can be displayed as

$$1, \omega_m, \omega_m^2, \dots, \omega_m^{m-1}$$

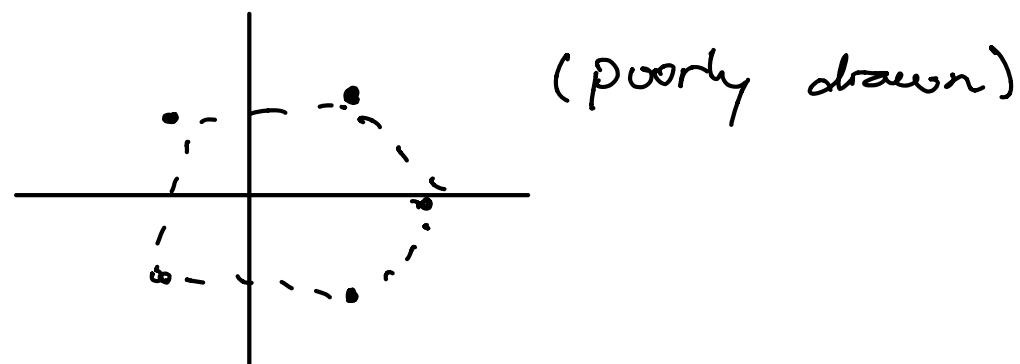
Aside (Polygons formed by roots of unity)



(a)  $1^{1/3}$



(b)  $1^{1/4}$



(poorly drawn)

**Example 1:** Prove that

$$1 + \omega_m + \omega_m^2 + \dots + \omega_m^{m-1} = 0 \quad (5)$$

From a physical point of view, since, by symmetry, the center of mass  $(1 + \omega_m + \omega_m^2 + \dots + \omega_m^{m-1})/m$  of the system of  $m$  unit masses located at  $m^{\text{th}}$  roots of unity must be at the origin.

Algebraically,

$$(\omega_m - 1)(1 + \omega_m + \omega_m^2 + \dots + \omega_m^{m-1}) = \omega_m^m - 1 = 0$$

Since  $\omega_m \neq 1$ , Eqn (5) follows.

To obtain the  $m^{\text{th}}$  roots of an arbitrary (nonzero) complex number  $z = r e^{i\theta}$ , we generalize the idea displayed by Eqn (4) and, reasoning similarly, conclude that  $m$  distinct  $m^{\text{th}}$  roots of  $z$  are given by

$$z^{\frac{1}{m}} = \sqrt[m]{|z|} e^{i(\theta + 2\pi k)/m} \quad (k = 0, 1, 2, \dots, m-1) \quad (6)$$

**Example 2:** Find all cube roots of  $\sqrt{2} + i\sqrt{2}$

The polar form is  $\sqrt{2} + i\sqrt{2} = 2e^{i\pi/4}$

Putting  $|z|=2$ ,  $\theta=\pi/4$ ,  $m=3$  into Eqn (6), we obtain

$$(\sqrt{2} + i\sqrt{2})^{1/3} = \sqrt[3]{2} e^{i\left(\frac{\pi}{12} + \frac{2k\pi}{3}\right)}, \quad k=0,1,2$$

Therefore the three cube roots of  $\sqrt{2} + i\sqrt{2}$  are

$$\sqrt[3]{2} \cos\left(\frac{\pi}{12} + i \sin \frac{\pi}{12}\right)$$

$$\sqrt[3]{2} \left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}\right)$$

$$\sqrt[3]{2} \left(\cos \frac{17\pi}{12} + i \sin \frac{17\pi}{12}\right)$$

### Example 3:

Let  $a, b, c$  be complex constants with  $a \neq 0$ . Prove the solutions of the equation

$$az^2 + bz + c = 0 \quad (7)$$

are given by the (usual) quadratic formula

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad (8)$$

where  $\sqrt{b^2 - 4ac}$  denotes one of the values of  $(b^2 - 4ac)^{1/2}$

Multiplying (7) by 4a,

$$4a^2z^2 + 4abz + 4ac = 0$$

$$4a^2z^2 + 4abz + b^2 = b^2 - 4ac$$

$$(2az + b)^2 = b^2 - 4ac$$

$$2az + b = (b^2 - 4ac)^{1/2} = \pm\sqrt{b^2 - 4ac}$$

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \checkmark$$