Bases of Linear Transformations

$T : \mathbb{R}^2 \to \mathbb{R}^2$;

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{(x+y)}{2} \\ \left(\frac{x+y}{2}\right) \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \quad (*)$$

What's T? Want more clarity on T. Then (*) gives us

$$\vec{a} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Try

$$T(\vec{a}_1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0\vec{a}_1 + 0\vec{a}_2 = [a_1 \; a_2]\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{pmatrix} 3/2 \\ 3/2 \end{pmatrix} = \alpha\begin{pmatrix} -1 \\ 1 \end{pmatrix} + \beta\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\downarrow$$

$$T(\vec{a}_2) = \begin{pmatrix} 3/2 \\ 3/2 \end{pmatrix} = -\frac{1}{2}\vec{\alpha}_1 + 1\vec{a}_2 = [a_1 \; a_2]\begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$$

Def

$[T]$ w.r.t $\{\vec{a}_1, \vec{a}_2]$ is $\begin{bmatrix} 0 & -1/2 \\ 0 & 1 \end{bmatrix}$

Use $\vec{b}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $b_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$T(b_1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 \cdot b_1 + 0 \cdot b_2 = [b_1 \; b_2]\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$T(b_2) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = [b_1 \; b_2]\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$[T]$ w.r.t $\underbrace{\{\vec{b}_1, \vec{b}_2\}}$ the basis

is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

projection onto b_1

Given a LT $T: V \to V$, we often need a basis B s.t. $[T]$ w.r.t B is nice.

If $\dim(V) = n$, have to solve n different systems of n equations in n vars to find $[T]$ w.r.t the basis B.

Is this COST worth the GAIN?

Right perspective in our example to understand T is $B = \{\vec{b}_1, \vec{b}_2\}$, it becomes evident T is a projection onto $\vec{b}_1$.

Q) What does $[T]$ w.r.t the basis $\{\vec{b}_1, \vec{b}_2\}$ represent?

$$[T] \to \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

What is $T\begin{pmatrix} 17 \\ 19 \end{pmatrix}$? $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} 18 \\ 18 \end{pmatrix} = 18\,\vec{b}_1$$

Aside: On the $[T]_{\beta \leftarrow \beta}$ "=" $[T]$ w.r.t B

$$\vec{w}_1 = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

B_1: $\vec{u}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\vec{u}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

B_2: $\vec{w}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, $\vec{w}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$B = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$$

← What's this represent

← Built by writing out coords of B_2 in B_1

$$\vec{x} = \begin{pmatrix} 10 \\ 20 \end{pmatrix} = [\vec{w}_1 \ \vec{w}_2] \begin{bmatrix} 0 \\ 10 \end{bmatrix} = [\vec{u}_1 \ \vec{u}_2] \begin{bmatrix} 20 \\ -10 \end{bmatrix}$$

Notice

$$B \begin{bmatrix} 0 \\ 10 \end{bmatrix} = \begin{bmatrix} 20 \\ -10 \end{bmatrix}$$

$\left(\begin{array}{l} \text{Matrix for} \\ \text{writing out} \\ \beta_2 \text{ in terms} \\ \text{of } \beta_1 \end{array}\right) \cdot \beta_2$ coords of $\vec{x}$ = β_1 coords of $\vec{x}$

<u>**FACT**</u> $[T]$ w.r.t the basis $\beta_1 = B^{-1} \left([T]_{\substack{\text{w.r.t the} \\ \text{basis of } \beta_2}} \right) B$

LHS, input β_1 coords of $\vec{x}$

$\downarrow$

outputs β_1 coords of $T(x)$

RHS input β_1 coords of $\vec{x}$. B takes those β_2 coords of $\vec{x}$.

$[T]_{\text{w.r.t}}$ β_2 takes it to β_2 coords of $T(\vec{x})$, B^{-1} gives B_2 coords of $T(\vec{x})$.

Example

$$\vec{y} = \begin{pmatrix} 6 \\ 0 \end{pmatrix} = [\vec{w}_1 \ \vec{w}_2] \begin{bmatrix} -4 \\ 2 \end{bmatrix} \leftarrow \beta_2 \text{ coords of } \vec{y}$$

$$= [\vec{u}_1 \ \vec{u}_2] \begin{bmatrix} 0 \\ 6 \end{bmatrix} \leftarrow \beta_1 \text{ coords of } \vec{y}$$