Discrete Time Markov Chain

t : $1, 2, 3, \dots$

State space $\mathcal{X}$: $1, 2, 3, \dots, m, \dots$

finite-state could be infinite

$$\Pr[X_{n+1} = x_{n+1} \mid X_n = x_n, \dots, X_0 = x_0]$$
$$= \Pr[X_{n+1} = x_{n+1} \mid X_n = x_n]$$

Initial distribution X_0

$$\mathbb{P}_0 = [\Pr[X_0 = i]]_{i \in \mathcal{X}} = \pi$$

$$\mathbb{P} = [P_{i,j}]$$

$$\mathbb{P}(n) \triangleq [\Pr[X_n = i]]_{i \in \mathcal{X}}$$

$$= \mathbb{P}(0)\,\mathbb{P}^n$$

Stationary Distribution

- π is a row vector with dim $|\mathcal{X}|$.

s.t.

$$\pi \mathbb{P} = \pi$$

$$\begin{array}{ccccccc} n=0 & & n=1 & & n=2 & \dots & \\ \mathbb{P}(0) & \rightarrow & \mathbb{P}(1) & \rightarrow & \mathbb{P}(2) & \rightarrow & \dots \\ \pi & \rightarrow & \mathbb{P}\pi = \pi & \rightarrow & \pi & \rightarrow & \dots \end{array}$$

Solve for π?

$$\bar{\pi} = [\pi_1, \pi_2, \ldots, \pi_m]$$

Use conditions

$$\bar{\pi}\mathbb{P} = \bar{\pi}$$

$$\sum_i \pi_i = 1$$

Example: Two Coins A and B.

A: Fair; $Pr[H] = \frac{1}{2}$

B: $Pr[H] = \frac{1}{4}$

Let

$H = 1$

$T = 2$

$X_0 \leftarrow A$

$$X_{n+1} \leftarrow \begin{cases} A & \text{if } X_n = 2 \\ B & \text{if } X_n = 1 \end{cases}$$

$$\mathbb{P}(0) = [Pr[X_0 = 1] \quad Pr[X_0 = 2]] = [\tfrac{1}{2} \quad \tfrac{1}{2}]$$

$$\mathbb{P}(1) = [Pr[X_1 = 1] \quad Pr[X_1 = 2]] = \mathbb{P}(0)\mathbb{P}$$

$$\mathbb{P} = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$[\pi_1 \quad \pi_2] \begin{bmatrix} \frac{1}{4} & 3/4 \\ 1/2 & 1/2 \end{bmatrix} = [\pi_1 \quad \pi_2]$$

$$\begin{cases} \frac{1}{4}\pi_1 + \frac{1}{2}\pi_2 = \pi_1 \\ \frac{3}{4}\pi_1 + \frac{1}{2}\pi_2 = \pi_2 \end{cases}$$

and $\pi_1 + \pi_2 = 1$ $\quad\quad \bar{\pi} = [\frac{2}{5}, \frac{3}{5}]$

Stationary Markov Chain

If X_0 is distributed at a stationary distribution π, then $\{X_n\}_{n\geq 1}$ is stationary.

$$\Pr[X_n, X_{n-1}, \ldots X_0 = x_0] = \Pr[X_{n+k} = x_n, X_{n-1+k} = x_{n-1}, \ldots, X_k = x_0]$$

why?

$$\downarrow$$

$$\Pr[X_0 = x_0]\Pr[X_1 = x_1 | X_0 = x_0] \cdots \Pr[X_n = x_n | X_{n-1} = x_{n-1}]$$

Stationary Distribution as an Equilibrium

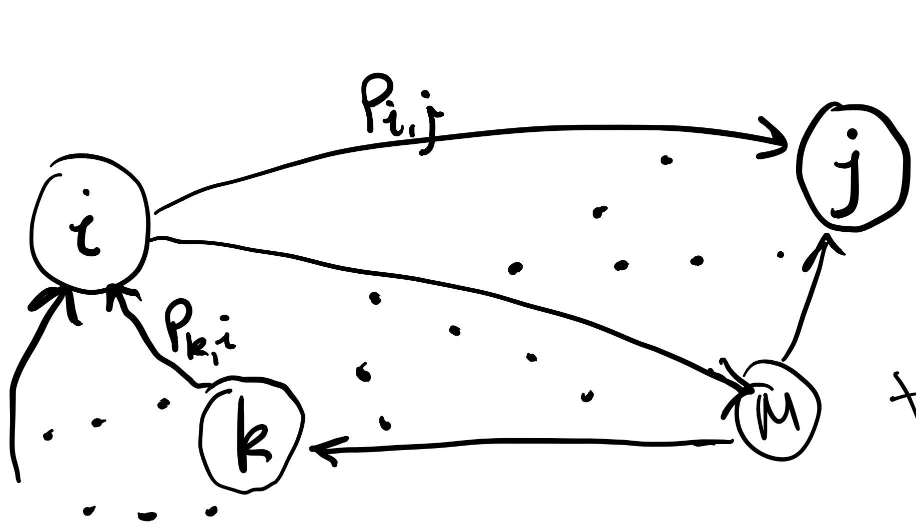


Diagram of Possibilities, don't take to seriously

1 gallon of water

X_0 — total probability of 1

Total amount of water given out by i at n?

$$\sum_{j \neq i} p_i(n) P_{i,j}$$

Water in:

$$\sum_{k \neq i} P_k(n) P_{k,i}$$

If $p_i(0) = \pi_i$

At time $n=1$

Water out of i :

$$\sum_{j \neq i} \pi_i P_{i,j} = \pi_{i,j} \sum_{j \neq i} P_{i,j}$$

$$= \pi_i (1 - P_{i,i})$$

Water in:

$$\sum_{k \neq i} \pi_k P_{k,i} = \pi_k (1 - P_{i,i})$$

We know

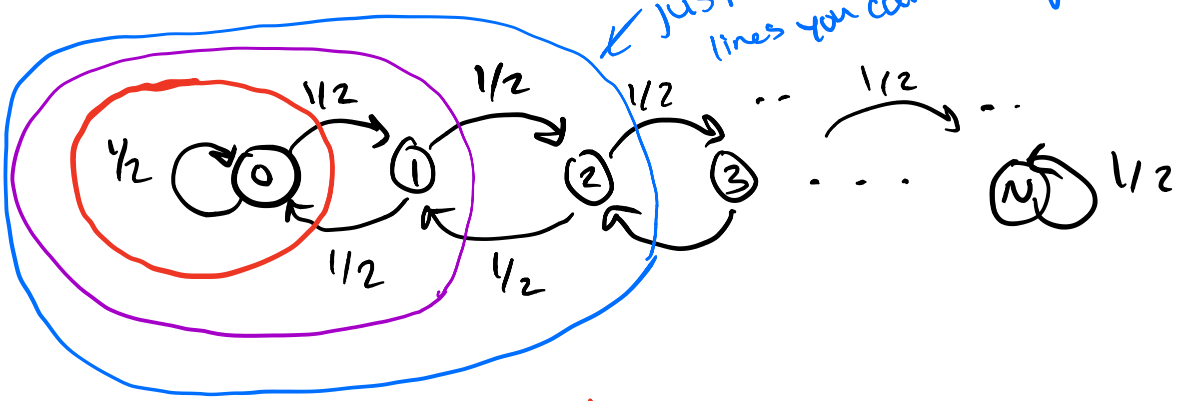
$$\pi = \pi \mathbb{P}$$

$$\pi_i = \sum_{k \in \mathcal{X}} \pi_k \mathbb{P}_{k,i} = \sum_{k \neq i} \pi_k P_{k,i} + \pi_i P_{ii}$$

$$\pi_i (1 - P_{i,i}) = \sum_{k \neq i} \pi_k P_{k,i} \qquad !!$$

Generalize, states $0, \dots, M$, ANY arbitrary closed loop, have

just care about lines you cut through



$$\frac{1}{2}\pi_0 = \frac{1}{2}\pi_1 \Rightarrow \pi_0 = \pi_1$$

$$\frac{1}{2}\pi_1 = \frac{1}{2}\pi_2 \Rightarrow \pi_1 = \pi_2$$

$$\frac{1}{2}\pi_2 = \frac{1}{2}\pi_3 \Rightarrow \pi_2 = \pi_3$$

$$\vdots$$

$$\pi_{N-1} = \pi_N$$

$$\Rightarrow \pi_i = \frac{1}{N+1} \quad \forall i$$

Accessible: A state j is accessible from state i if $P_{i,j}^{(n)} > 0$ for some $n \geq 0$

probability of going from $i \to j$ in n steps

By definition: $i \to i$ (i is alway accessible from itself)

We say i communicates with j if $i \to j$ and $j \to i$ (denote $i \leftrightarrow j$).

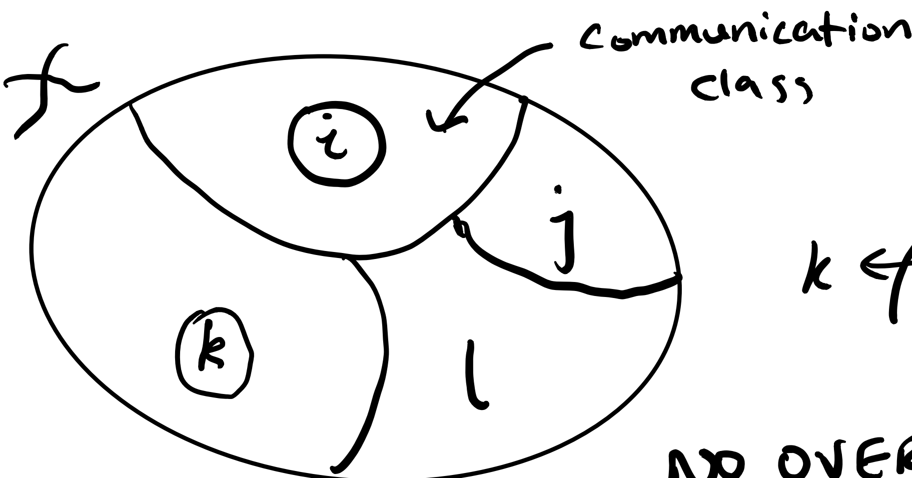
"Communication" is an equivalence relation

① Reflexive: $i \leftrightarrow i$

② Symmetric: if $i \leftrightarrow j$ then $j \leftrightarrow i$

③ Transitive: $i \leftrightarrow j$ and $j \leftrightarrow k$ then $i \leftrightarrow k$

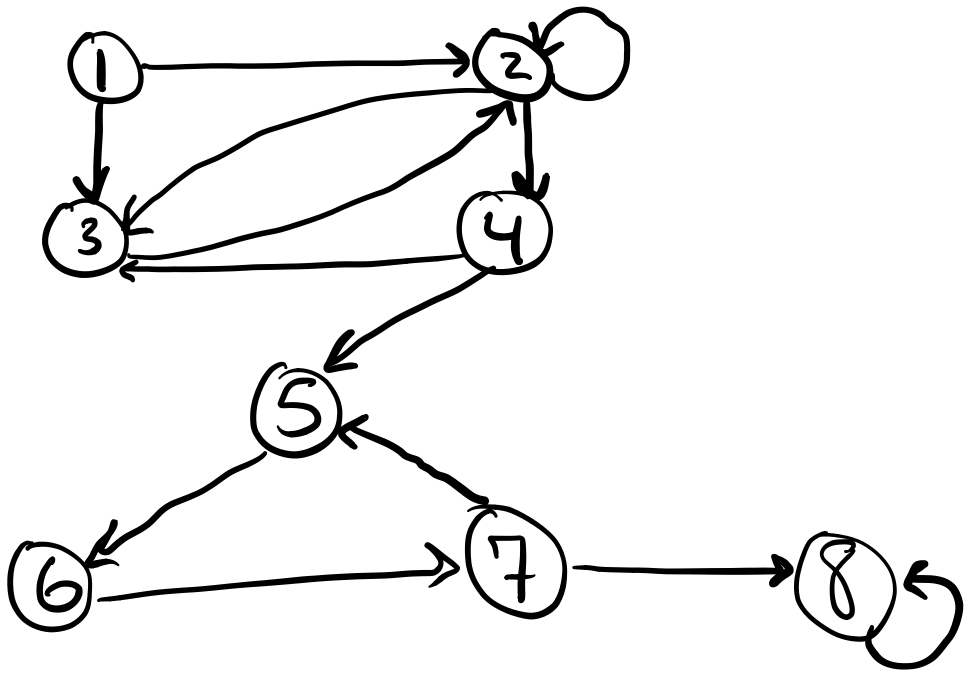
Note that communication partitions our state space.



$k \not\leftrightarrow i$

NO OVERLAP b/t two different classes by definition

Example



Step 1: Pick 1

$C_1 = \{1\}$ ← no return state

Pick 2

$C_2 = \{2, 3, 4\}$

Pick 5

$C_3 = \{5, 6, 7\}$

$C_4 = \{8\}$ ← called an absorbing state