Rational numbers are ratios of integers, written in the form m/n, $n \neq 0$, with the stipulation that all rationals of the form $n/n = 1$.

Commutative Law

Addition $\quad a+b = b+a$

Multiplication $\quad ab = ba$

Associative Law

Addition $\quad a+(b+c) = (a+b)+c$

Multiplication $\quad a(bc) = (ab)c$

Distributive law

$$(a+b)c = ac+bc$$

above laws apply for any rationals a, b, c.

Rationals are the only numbers needed to solve equations of the form

$$ax+b=0$$

The solution for nonzero a is $x = -b/a$.

Since this is the ratio of two rationals it is itself rational.

However, attempting to solve quadratics with the rational system, we find that some of them have no solution.
Consider
$$x^2 = 2 \qquad (1)$$
which can not be satisfied by any rational number.
We therefore extend the concept of "number" by appending to the rationals a new symbol, written as $\sqrt{2}$.
$\sqrt{2}$ is the solution to (1).

Our revised concept of a number is now an expression in the standard form
$$a + b\sqrt{2} \qquad (2)$$
where a and b are rationals.
Addition and Subtraction are performed according to
$$(a + b\sqrt{2}) \pm (c + d\sqrt{2}) = (a \pm c) + (b \pm d)\sqrt{2} \qquad (3)$$
Multiplication is defined via the distributive law w/ the provision that the square of the symbol $\sqrt{2}$ can be denoted 2.

Thus we have

$$(a+b\sqrt{2})(c+d\sqrt{2}) = (ac+2bd) + (bc+ad)\sqrt{2} \quad (4)$$

Using 'rationalizing the denominator' we can put the quotient of any two of these numbers into the standard form

$$\frac{a+b\sqrt{2}}{c+d\sqrt{2}} = \frac{a+b\sqrt{2}}{c+d\sqrt{2}} \cdot \frac{c-d\sqrt{2}}{c-d\sqrt{2}} = \frac{ac-2bd}{c^2-2d^2} + \frac{bc-ad}{c^2-2d^2}\sqrt{2} \quad (5)$$

Above procedure w/ rationals at this point should be familiar.

Now observe that we still cannot solve the equation

$$x^2 = -1 \quad (6)$$

Prior experience suggests we expand our number system again by appending a symbol for a solution to (6). Instead of $\sqrt{-1}$, it is customary to use the symbol i. Next, we imitate the model of expressions (2)-(5) [pertaining to $\sqrt{2}$] and thereby generalize our concept of number as follows:

Definition 1: A complex number is an expression of the form $a+bi$, where a and b are real numbers. Two complex numbers $a+bi$ and $c+di$ are said to be equal if and only if $a=c$ and $b=d$.

The addition/subtraction of complex numbers are given by $(a+bi)\pm(c+di)\equiv(a\pm c)+(b\pm d)i$.
With the provision that $i^2=-1$ and in accordance w/ the distributive law, we postulate the following:
The multiplication of two complex numbers is defined by

$$(a+bi)(c+di)\equiv(ac-bd)+(bc+ad)i$$

To compute the quotient of two complex numbers, we again "rationalize the denominator."

$$\frac{a+bi}{c+di}=\frac{a+bi}{c+di}\cdot\frac{c-di}{c-di}=\frac{ac+bd}{c^2+d^2}+\frac{bc-ad}{c^2+d^2}i$$

gives the division of complex numbers if $c^2+d^2\neq 0$

Example 1

Find the quotient

$$\frac{(6+2i)-(1+3i)}{(-1+i)-2}$$

$$\frac{5-i}{-3+i} = \frac{(5-i)(-3-i)}{(-3+i)(-3-i)} = \frac{-15-1-5i+3i}{9+1}$$

$$= -\frac{8}{5} - \frac{1}{5}i$$

Definition 2: The <u>real part</u> of a complex number $a+bi$ is the (real) number a; its <u>imaginary part</u> is the (real) number b. If a is zero, the number is is said to be a <u>pure imaginary number</u>

For convenience we use z to denote a complex number.

Its real part is $\mathrm{Re}\{z\}$

Its imaginary part is $\mathrm{Im}\{z\}$

With this notation we have

$$z = \mathrm{Re}\, z + i\, \mathrm{Im}\, z$$

So when we say $z_1 = z_2$,

$$\mathrm{Re}\, z_1 = \mathrm{Re}\, z_2 \text{ and } \mathrm{Im}\, z_1 = \mathrm{Im}\, z_2.$$

The set of all complex numbers is sometimes denoted as $\mathbb{C}$. There is no natural ordering of the elements of $\mathbb{C}$.