Polynomial functions of z are of the form

$$p_n(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \qquad (1)$$

and rational functions are ratios of polynomials:

$$R_{m,n}(z) = \frac{a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m}{b_0 + b_1 z + b_2 z^2 + \cdots + b_n z^n} \qquad (2)$$

- The degree of polynomial (1) is n if complex constant $a_n \neq 0$.
- The rational function (2) has $\begin{cases} \text{numerator degree } m \\ \text{denominator degree } n \end{cases}$

provided $a_m, b_n \neq 0$.

Analyticity:
- polynomials are entire,
- rational functions are analytic everywhere the denominator is nonzero

Take

$$p_3(z) = 12 + 10z - 4z^2 - 2z^3 \qquad (3)$$

as our example.

We want to know how $p_3(z)$ is characterized by its zeros and by the value of its derivatives at a single point.

Zeros:

$$-2z^3 - 4z^2 + 10z + 12 = 0 \qquad \frac{p}{q} = \frac{-6}{1} \rightarrow \frac{\pm 1, \pm 2, \pm 3, \pm 6}{\pm 1}$$

$$-2(z^3 + 2z^2 - 5z - 6) = 0$$

Zeros $\rightarrow 2, -1, -3$

We can thus express $p_3(z)$ in factored form

$$p_3(z) = -2(z-2)(z+1)(z+3) \qquad (4)$$

One can always divide a "dividend" polynomial by a "divisor" polynomial to obtain a "quotient" polynomial and a "remainder" polynomial whose degree is less than that of the divisor:

$$\text{dividend} = \text{divisor} \times \text{quotient} + \text{remainder} \qquad (5)$$

If z_1 is any arbitrary complex number, then division of $p_n(z)$ in (1) by the degree one polynomial $z - z_1$ must result in a remainder of a lower degree.

$$p_n(z) = (z - z_1)\, p_{n-1}(z) + \text{constant} \qquad (6)$$

where the quotient polynomial $p_{n-1}(z)$ has degree $n-1$.

But suppose z_1 happens to be a zero of $p_n(z)$.

Then by setting $z = z_1$ in (6) we deduce that the remainder is zero. Thus (6) displays how $(z-z_1)$ has been factored out from $p_n(z)$: we say $p_n(z)$ has been "deflated".

In the case of $p_3(z)$, factoring out our first zero $z_1 = 2$ results in

$$-2z^3 - 4z^2 + 10z + 12 = (z-2)\left[-2z^2 - 8z - 6\right]$$

Now if z_2 is a zero of the quotient $p_{n-1}(z)$ we can deflate further by factoring out $(z-z_2)$, and so on until we run out of zeros, leaving us with the factorization

$$p_n(z) = (z-z_1)(z-z_2)\cdots(z-z_k)\,p_{n-k}(z) \qquad (7)$$

Since we know 3 zeros for $p_3(z)$, Eq (4) displays its "complete" deflation down to factors of degree one (and the degree-zero factor -2.)

Example 1: Carry out the deflation of the polynomial
$$z^3 + (2-i)z^2 - 2iz$$

$$z^3 + (2-i)z^2 - 2iz = z\left(z^2 + (2-i)z - 2i\right) = 0$$

$z_1 = 0$ is a zero.

For the other two zeros $\rightarrow$ quadratic formula

$$z_2, z_3 = \frac{-(2-i) \pm \sqrt{(2-i)^2 - 4(1)(-2i)}}{2} = -2, i \qquad \sqrt{\ } = -2+i$$

And the factored form is

$$z^3 + (2-i)z^2 - 2z = z(z+2)(z-i)$$

Contemplate the deflation of $p_6(z) = z^6 + z^4 - 4iz^2 - 4z + 4 - 3i$.
How do we know $p_6(z)$ has any zeros?

Thanks to Gauss we have the Fundamental Theorem of Algebra

Theorem 1. Every nonconstant polynomial with complex coefficients has at least one zero in $\mathbb{C}$

Proof in Section 4.6

We immediately conclude that a polynomial of degree n has n zeros, since we can continue to factor out zeros in the deflation process until we reach the final constant quotient.

Repeated zeros are counted according to their multiplicities. i.e

$$z^4 + 2z^2 + 1 = (z-i)^2 (z+i)^2$$

has zeros in two points, each of multiplicity 2, and we count them as 4 zeros.

Regarding the issue of the existence of zeros, in the deflation process, for the quotients settled we have a complete factorization of any polynomial

$$p_n(z) = a_n (z - z_1)(z - z_2) \cdots (z - z_n) \qquad (8)$$

(8) conveys a lot of information.

- $p_n(z)$ of degree n has n zeros no more, no less
- $p_n(z)$ is completely determined by its zeros, up to a constant multiple (a_n).

→ If two polynomials of degree n have the same zeros, then they are constant multiples of each other.

Furthermore, the factorization shows that z_0 is a zero of $p(z)$ of multiplicity precisely k iff

$$p_n(z) = (z - z_0)^k q(z)$$

where $q(z)$ is a polynomial with $q(z_0) \neq 0$.

Example 2:

Show that if polynomial $p(z)$ has real coefficients, it can be expressed as a product of linear and quadratic factors, each having real coeffients.

The nonreal zeros of a polynomial w/ real coefficients occur in complex conjugate pairs. Say z_2 is $\bar{z}_1$.

We can combine

$$(z - z_1)(z - z_2) \text{ in } (8)$$

to get

$$p_n(z) = a_n \left(z^2 - (z_1 + \bar{z}_1) z + z_1 \bar{z}_1\right) \cdots (z - z_n)$$

$$= a_n \left(z^2 - 2(\operatorname{Re} z_1) z + |z_1|^2\right) \cdots (z - z_n)$$