Recall

Started conditional stuff for continuous rvs.

Given continuous rv X, event A, conditional pdf of X given A is defined as the function $f_{X|A}(x)$ satisfying

$$\mathbb{P}(X \in V \mid A) = \int_V f_{X|A}(x)\,dx \quad \forall\, V \subset \mathbb{R}$$

Special Case: If A is an event of the form $\{X \in W\}$ for some $W \subset \mathbb{R}$,

$$f_{X|A}(x) = \begin{cases} \dfrac{f_X(x)}{\mathbb{P}(\{X \in W\})}, & x \in W \\ 0, & x \notin W \end{cases}$$

Example - "Light BulB"

Suppose T is exponential(λ) rv

$$f_T(t) = \begin{cases} \lambda e^{-\lambda t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Pick $t_0 > 0$; let

$$X = T - t_0,$$
$$A = \{T \geq t_0\};$$

find $f_{X|A}(x)$.

Note: $A = \{T \geq t_0\} = \{X \geq 0\}$!

Hence,

$$f_{X|A}(x) = \begin{cases} \dfrac{f_X(x)}{\mathbb{P}(\{T \geq t_0\})} & , x \geq 0 \\ 0 & , x < 0 \end{cases}$$

where

$$\mathbb{P}(\{T \geq t_0\}) = \int_{t_0}^{\infty} f_T(t)\,dt = e^{-\lambda t_0}$$

and

$$f_X(x) = \begin{cases} \lambda e^{-\lambda(t_0+x)} & , x \geq -t_0 \\ 0 & , x < -t_0 \end{cases}$$

How'd we get this? First find $F_X(x)$.

$$F_X(x) = \mathbb{P}(\{X \leq x\}) = \mathbb{P}(\{T \leq t_0 + x\})$$
$$= \int_{-\infty}^{t_0+x} f_T(t)\,dt$$

Now take $\frac{d}{dx}$

$$f_X(x) = \frac{d}{dx}\left(\int_{-\infty}^{t_0+x} f_T(t)\,dt\right) = f_T(t_0+x)$$

where

$$f_T(t_0+x) = \begin{cases} \lambda e^{-\lambda(t_0+x)} & , x \geq -t_0 \\ 0 & , x < -t_0 \end{cases}$$

Therefore,

$$f_{X|A}(x) = \begin{cases} \dfrac{\lambda e^{-\lambda(t_0+x)}}{e^{-\lambda t_0}}, & x \geq 0 \\ 0, & x < 0 \end{cases} = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

In light-bulb terms, "find bulb still on at time t_0, remaining bulb lifetime is still exponential (λ), as 'brand-new' lifetime"

This is the "resetting"/"regeneration" property of exponential pdfs — similar to geometric "resetting" in discrete world

Total Probability Theorem in context of $f_{X|A}$:

If X is a continuous rv and $A_1, \ldots, A_n$ are events of positive probability that partition Ω, then

$$f_X(x) = \sum_{k=1}^{n} f_{X|A_k}(x)\, \mathbb{P}(A_k)$$

To see this: go via cdfs.

$$F_{X|A_k} = \frac{\mathbb{P}(\{X \leq x\} \cap A_k)}{\mathbb{P}(A_k)}$$

$$\frac{d}{dx} F_{X|A_k} = f_{X|A_k}(x)$$

By Total Probability Theorem,

$$F_X(x) = \mathbb{P}(\{X \le x\}) = \sum_{k=1}^{n} F_{X|A_k} \mathbb{P}(A_k) \xrightarrow{\frac{d}{dx}} \sum_{k=1}^{n} f_{X|A_k} \mathbb{P}(A_k) = f_X(x)$$

Comment: this holds when A_k aren't of the special form $\{X \in W_k\}$!

Example - Walking To Class

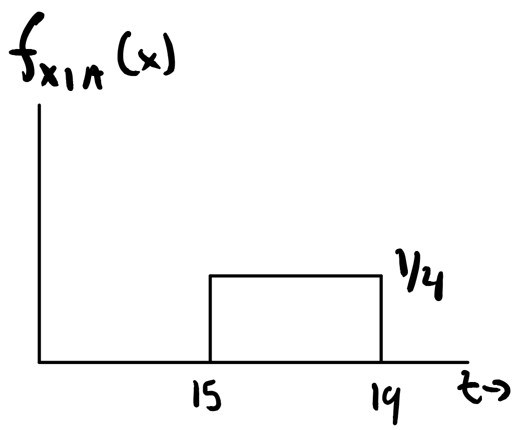
Before leaving for campus, spin wheel; w/ prob $2/3$ I walk via the gorge, $1/3$ via collegetown.

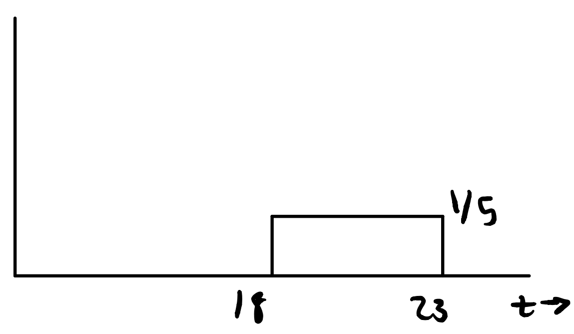
- Walking via gorge, travel time uniform on $[15, 19]$ minutes
- Walking via ctown, travel time uniform on $[18, 23]$ minutes

Let X = travel time.

$f_X(x)$?

Let A = walk via gorge $\Rightarrow A^c$ = walk via ctown



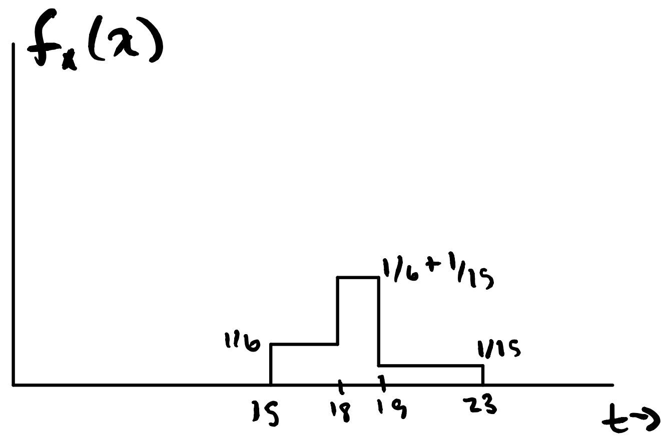


Thus

$$f_X(x) = f_{X|A}(x)\cdot\left(\frac{2}{3}\right) + f_{X|A^c}(x)\cdot\left(\frac{1}{3}\right)$$

$$f_X(x) = \begin{cases} 0 & , \quad x < 15 \\ \frac{1}{6} & , \quad 15 \le x < 18 \\ \frac{1}{6} + \frac{1}{15} & , \quad 18 \le x < 19 \\ 1/15 & , \quad 19 \le x < 23 \\ 0 & , \quad x \geq 23 \end{cases}$$

Visualization



Next, discuss conditional pdf of a continuous rv X given $Y=y$ for some other continuous rv Y

Naive approach: Let $A = \{Y=y\}$. But Y is continuous! So $\mathbb{P}(\{Y=y\}) = 0 \Rightarrow$ it is **NOT** a suitable A for $f_{X|A}(x)$

Instead we proceed as follows:

1) Given V, look at $\mathbb{P}(\{X\in V\}\mid\{Y\in[y-\delta,y+\delta]\})$

2) Take $\lim \delta\to 0$

3) Find that it equals

$$\int_V (\quad)\,dx$$

this is our goal

$$\mathbb{P}(\{X\in V\}\mid\{Y\in[y-\delta,y+\delta]\}) = \frac{\mathbb{P}(\{X\in V\}\cap\{Y\in[y-\delta,y+\delta]\})}{\mathbb{P}(\{Y\in[y-\delta,y+\delta]\})}$$

$$= \frac{\int_V dx \int_{y-\delta}^{y+\delta} dt\,\left(f_{X,Y}(x,t)\right)}{f_Y(t)\,dt} = \frac{\int_V dx\left(\frac{1}{2\delta}\int_{y-\delta}^{y+\delta} dt\, f_{X,Y}(x,t)\right)}{\left(\frac{1}{2\delta}\int_{y-\delta}^{y+\delta} dt\, f_Y(t)\right)}$$

Multiply by $1=\frac{1/2\delta}{1/2\delta}$

$$\frac{\int_V dx\left(\frac{1}{2\delta}\int_{y-\delta}^{y+\delta} dt\, f_{X,Y}(x,t)\right)}{\left(\frac{1}{2\delta}\int_{y-\delta}^{y+\delta} dt\, f_Y(t)\right)} \xrightarrow[\delta\to 0]{\lim} \frac{\int_V f_{X,Y}(x,y)\,dx}{f_Y(y)}$$

$$\frac{\int_V f_{X,Y}(x,y)\,dx}{f_Y(y)} = \int_V \left(\frac{f_{X,Y}(x,y)}{f_Y(y)}\right)dx$$

Bottom line: conditional pdf of X given $Y=y$ is

$$f_{X|Y}(x,y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

What you integrate over for any $x \in U$ to get $\mathbb{P}(\{X \in U\} | \{Y=y\})$

Note: for fixed y, this as a function of x is a legit pdf

$$\int_{-\infty}^{+\infty} f_{X|Y}(x|y)\,dx = \int_{-\infty}^{+\infty} \frac{f_{X,Y}(x,y)}{f_Y(y)}\,dx = \frac{1}{f_Y(y)} \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dx = \frac{f_Y(y)}{f_Y(y)} = 1 \checkmark$$

As for conditional pmfs in discrete-world, can use conditional pdfs to compute joints, marginals, etc, in situations most naturally expressed in conditional terms.

e.g.

$$f_{X,Y}(x,y) = f_{X|Y}(x|y) f_Y(y)$$

Integrate over x or y to get

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X|Y}(x|y)\,dy \quad \text{OR} \quad f_Y(y) = \int_{-\infty}^{+\infty} f_{Y|X}(y|x) f_X(x)\,dx$$

This generalizes "obviously" to > 2 rvs.

""What mathematicians say when they know something that isn't entirely obvious to anybody else"
- Some kid in a math class"

– Rami Pellumbi

Take X, Y, Z for example.

$$f_{X,Y|Z}(x,y|z) = \frac{f_{X,Y,Z}(x,y,z)}{f_Z(z)} \quad \text{OR} \quad f_{X|Y,Z}(x|y,z) = \frac{f_{X,Y,Z}(x,y,z)}{f_{Y,Z}(y,z)}$$

etc, etc, etc

And we have chain rules such as

Need not memorize

$$f_X(x) = f_{X|Y,Z}(x|y,z)\, f_{Y|Z}(y|z)\, f_Z(z)$$

Example - Radar Gun

Speed of passenger vehicle - X - exponential w/ $\lambda = 50$.
Y = radar gun measurement of X, given $X = x$, is Gaussian; mean x, variance $\sigma = x/10$

$$f_X(x) = \begin{cases} 50e^{-50x} & , x \geq 0 \\ 0 & , x < 0 \end{cases}$$

$$f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi\left(\frac{x^2}{100}\right)}}\, e^{-\frac{(y-x)^2}{(x^2/50)}}$$

Hence $f_{X,Y}(x,y) = f_{Y|X}(y|x)\, f_X(x)$ and $f_Y(y) = \int_{-\infty}^{+\infty} (\quad)\, dx$

One loose end:

Expected value rule for joints.

$$\mathbb{E}(g(X,Y)) = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \; g(x,y) f_{X,Y}(x,y)$$