Last time: $A \in \mathbb{R}^{n \times n}$ symmetric, $A^T = A$

1. Every (complex) eigenvalue of A is in $\mathbb{R}$

   $\therefore$ A has real eigenvalues

2. $\vec{v}_1, \vec{v}_2$ eigenvectors for different eigenvalues

$$\Rightarrow \vec{v}_1 \cdot \vec{v}_2 = 0$$

3. W is A-invariant $\Rightarrow W^\perp$ is A-invariant

4. β orthonormal basis of V

$$[A]_\beta = B$$

   is also symmetric

5. If $\mathcal{U} \subseteq V$ is an A-invariant subspace,

$$\begin{aligned} A|_{\mathcal{U}} : \mathcal{U} &\to \mathcal{U} \\ u &\mapsto A\vec{u} \end{aligned} \quad \text{is a LT}$$

   and if q is an orthonormal basis of $\mathcal{U}$, then

$$[A|_{\mathcal{U}}]_q = B_2$$

   is also symmetric.

   This implies that every eigenvalue of B_2 is also an eigenvalue of $A|_{\mathcal{U}}$, and also of A. (requires proof)

(6) If W is A-invariant, and if $(u_1,\dots,u_r)$ an orthonormal basis of W and $(u_{r+1},\dots,u_n)$ is an orthonormal basis of $W^\perp$ and $\beta = (u_1,\dots,u_r)$

then
$$[A]_\beta = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}$$

and
$$\{\text{eigenvalues of } A\} = \{\text{eigenvals } B_1\} \cup \{\text{eigenvals } B_2\}$$

Proposition: If $A = A^T$, $\lambda_1,\dots,\lambda_m \in \mathbb{R}$ distinct eigenvalues of A, then
$$V = E_{\lambda_1}(A) \oplus \cdots \oplus E_{\lambda_m}(A)$$

Proof: Showed $E_{\lambda_1}(A) + \cdots + E_{\lambda_m}(A) = W$

Satisfies $\quad W = E_{\lambda_1}(A) \oplus \cdots \oplus E_{\lambda_m}(A)$

need to show $\quad W = V = \mathbb{R}^n$

i.e $\quad W^\perp = 0$

Show $W^\perp = 0$

Assume otherwise, $W^\perp \neq 0$ $\therefore$ $W^\perp$ is A-invariant

Consider
$$A|_{W^\perp} : W^\perp \to W^\perp$$

and by (5), $A|_{W^\perp}$ has an eigenvalue $\therefore \exists\, v \neq 0 \in W^\perp$ an eigenvalue of $W^\perp$. But $v \in W$ too, and $W \cap W^\perp = 0$

!C $\quad \therefore W^\perp = 0$

Spectral Theorem

Let $A = A^T$, $V = \mathbb{R}^n$ $(A \in \mathbb{R}^{n \times n})$ (V an inner product space)

then

ⓐ $\exists$ orthonormal basis of V consisting of eigenvectors of A (i.e. A is orthonormally diagonalizable)

ⓑ $\exists$ orthogonal matrix Q $(n \times n)$ and a diagonal matrix Σ $(n \times n)$ s.t. $A = Q \Sigma Q^T$

Proof Basically all done

ⓐ Take an orthonormal basis for each $E_{\lambda_i}(A)$ (all eigenvalues) put together.

This is an orthonormal basis of eigenvectors, say $u_1, \ldots, u_n$

ⓑ $Q = u_1, \ldots, u_n$

then $\quad \Sigma = Q^{-1} A Q$

however $\quad Q^T Q = I$

$\therefore \quad Q^T = Q^{-1}$

and $\quad \Sigma = Q^T A Q$

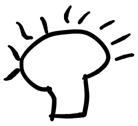
Q.E.D

Remark: $Q^{-1} = Q^T$, $Q Q^T = I$

$\Rightarrow$ the rows of A all orthonormal

Singular Value Decomposition

Let A be any $n \times n$ matrix over $\mathbb{R}$ of rank r.
We showed that $A^T A$ also has rank r.

 $A^T A$ $(n \times n)$ is a symmetric. Use this

$A^T A$ has n real eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$
with multiplicity
Note: only r of theze non-zero.

Claim: All eigenvalues of $A^T A \geq 0$
Proof: If $\vec{v}$ is s.t.

$$A^T A \vec{v} = \lambda \vec{v} \quad , \quad \vec{v} \neq 0$$

$$\left((Av)^T Av =\right) \vec{v}^T A^T A \vec{v} = \lambda \vec{v}^T \vec{v} = \lambda \langle v, v \rangle \quad , \quad \langle \vec{v}, \vec{v} \rangle > 0$$

$$0 \leq \langle A\vec{v}, A\vec{v} \rangle$$

So $\lambda \geq 0$

So the non-zero eigenvalues of $A^T A$

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r > 0$$

Definition: Let $\sigma_i = \sqrt{\lambda_i}$, then $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > 0$
are called the **singular values** of A.

Let $V = (\vec{v}_1, \dots, \vec{v}_n)$ be a matrix whose columns form an orthonormal basis of $V = \mathbb{R}^n$ of eigenvectors of $A^T A$ such that

$$A^T A v_i = \lambda v_i \quad , \quad i = 1, \dots, r$$

$$A^T A v_j = 0 \quad , \quad j > r$$

For $i = 1, \dots, r$

Let

$$u_i \triangleq \frac{1}{\sigma_i} A v_i$$

Claim: $u_1, \dots, u_r$ are orthonormal basis

$$u_j^T u_i = \frac{1}{\sigma_i \sigma_j} v_j^T \underbrace{A^T A v_i}_{\lambda_i v_i \;\rightarrow\; \lambda_i = \sigma_i^2}$$

$$= \frac{\sigma_i^2}{\sigma_i \sigma_j} v_j^T v_i$$

So

$$\langle u_j, u_i \rangle = \begin{cases} 0 \, , & i \neq j \\ 1 \, , & i = j \end{cases}$$

Extend $u_1, \dots, u_r$ to an orthonormal basis $u_1, \dots, u_r, u_{r+1}, \dots, u_m$ of $\mathbb{R}^m$.

Let $U = (u_1, \dots, u_m)$

Have

1. $\sigma_1, \dots, \sigma_r$ s.t. $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > 0$
2. V $n \times n$ matrix, orthog. matrix
3. U $m \times m$ matrix, orthog. matrix

Note: $[Av_1 \ Av_2 \ \cdots \ Av_n] = [\sigma_1 u_1 \ \sigma_2 u_2 \cdots \sigma_r u_r \ 0 \cdots 0]$

Let

$$\Sigma = \begin{bmatrix} \sigma_1 & & & 0 & \\ & \sigma_2 & & & \\ & & \ddots & & \\ 0 & & & \sigma_r & \\ & & & & 0 \end{bmatrix}$$

an example

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 \\ 0 & 0 & \sigma_3 & 0 \end{bmatrix}$$

So

$$AV = U\Sigma \qquad (V^{-1} = V^T)$$

$$A = U\Sigma V^T$$