ECE 4110 Homework 6

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Due by 5pm on October 31

1 Reading Material

- Gaussian process (Chapter 9.5.1).
- Wiener process and Brownian motion (Chapter 9.5.2 and 9.6.2).
- Markov process: definition and one-step transition matrix (Chapter 11.1 and 11.2).

2 Assignment

1. **Gaussian Process**

   Let $X \sim \mathcal{N}(0,1)$ and $Y \sim \mathcal{N}(0,1)$ be independent Gaussian random variables. Define the random process $\{Z(t)\}_{t=-\infty}^{\infty}$ as

   $$Z(t) = X\cos(2\pi ft) + Y\sin(2\pi ft),$$

   where $f > 0$.

   (a) Is $\{Z(t)\}$ a Gaussian process? Explain.
   (b) Compute the mean and autocorrelation of $\{Z(t)\}$.
   (c) Is $\{Z(t)\}$ wide-sense stationary? Is it stationary?

2. **MMSE Prediction for a Gaussian Process**

   Let $X(t)$ be a stationary Gaussian process with

   $$\mu_X(t) \triangleq \mathbb{E}[X(t)] = 0, \quad R_X(\tau) \triangleq \mathbb{E}[X(t)X(t+\tau)] = 5\cos(\frac{\pi\tau}{2})3^{-|\tau|}$$

   (a) Find the covaraince matrix of $[X(2), X(3), X(4)]^T$.
   (b) Find $\mathbb{E}[X(4)|X(2)]$, the MMSE prediction of $X(4)$ given $X(2)$.
   (c) Find $\mathbb{E}[X(4)|X(2), X(3)]$, the MMSE prediction of $X(4)$ given both $X(2)$ and $X(3)$.

3. **Gaussian Moving Average Process**

   Let $\{X_n\}_{n=-\infty}^{\infty}$ be a sequence of i.i.d. Gaussian random variables with zero mean and unit variance. Define

   $$Y_n = \frac{1}{3}(X_n + X_{n-1} + X_{n-2})$$

   as the moving average of three consecutive X_n's.

(a) Is $\{Y_n\}_{n=-\infty}^{\infty}$ a Gaussian random process? Explain.

(b) Determine the mean function and the autocorrelation function of $\{Y_n\}_{n=-\infty}^{\infty}$.

(c) Is $\{Y_n\}_{n=-\infty}^{\infty}$ (strictly) stationary? Explain.

(d) Does $\{Y_n\}_{n=-\infty}^{\infty}$ have independent increments? Explain.

4. **The Brownian Bridge**

Let $\{X(t)\}_{t\geq 0}$ be a Brownian motion with parameter σ^2. Define the following random process $\{Y(t)\}_{0\leq t\leq 1}$ called a *Brownian bridge* as follows:

$$Y(t) = X(t) - tX(1), \quad 0 \leq t \leq 1.$$

(a) Draw several sample paths of the brownian bridge. Are the sample paths continuous?

(b) Is the Brownian bridge a Gaussian process? Explain.

(c) Compute the mean function and the autocorrelation function of $\{Y(t)\}_{0\leq t\leq 1}$.

(d) Is the Brownian bridge (strictly) stationary? Explain.

(e) Recall that a Brownian motion has independent increments. Does the Brownian bridge have independent increments? Explain.

5. **Markov Chains**

A die is rolled repeatedly. Which of the following sequences $\{X_n\}_{n=0}^{\infty}$ are Markov chains? Prove your statement. For those that are, give the transition matrices.

(a) X_n is the largest number shown up to the nth roll.

(b) X_n is the number of 6's up to the nth roll.

(c) X_n is the number of rolls since the most recent 6.

(d) X_n is the number of rolls until the next 6 (when you see a 6, you reset your count to 0).

6. **A Markov Process with Dependent Increments**

Consider the following first-order Gaussian autoregression process:

$$\begin{aligned} X_0 &= 0 \\ X_{n+1} &= \alpha X_n + W_{n+1} \end{aligned}$$

where $|\alpha| \leq 1$, $\{W_n\}$ are i.i.d. Gaussian with zero mean.

(a) Is $\{X_n\}_{n\geq 0}$ a Gaussian process? Justify your answer.

(b) Show that $\{X_n\}_{n\geq 0}$ is a Markov process (a random process that is Gaussian and Markovian is called a Gaussian-Markov process).

(c) Does $\{X_n\}_{n\geq 0}$ have independent increments? Justify your answer.

① Gaussian Process

Let $X \sim \mathcal{N}(0,1)$, $Y \sim \mathcal{N}(0,1)$ be independent Gaussian random variables.
Define random process $\{Z(t)\}_{t=-\infty}^{+\infty}$ as

$$Z(t) = X\cos(2\pi f t) + Y\sin(2\pi f t)$$

where $f>0$.

(a) Is $Z(t)$ a Gaussian process?

Sample $Z(t)$ at 2 times $t_m < t_n$, m,n arbitrary.
Then

$$Z(t_m) = X\cos(2\pi f t_m) + Y\sin(2\pi f t_m)$$

and

$$Z(t_n) = X\cos(2\pi f t_n) + Y\sin(2\pi f t_n)$$

We note that $\cos(2\pi f t)$, $\sin(2\pi f t)$ are just constants in $\mathbb{R}$ $\forall\ t \in \mathbb{R}$ (fixed f).
Since X, Y independent and Gaussian any linear combination of them is Gaussian.
Thus $Z(t_m), Z(t_n)$ Gaussian.
To show $Z(t)$ a Gaussian random process, need to show that $\forall\ n$ and $t_1, \dots, t_n$ the random variables $Z(t_1), \dots, Z(t_n)$ are jointly Gaussian.

Let $\qquad, a, b, \dots, n \in \mathbb{R}$

$$\mathcal{Z} = aZ(t_1) + bZ(t_2) + \cdots + nZ(t_n)$$
$$= X[a\cos(2\pi f t_1) + b\cos(2\pi f t_2) + \cdots + n\cos(2\pi f t_n)]$$
$$+ Y[a\sin(2\pi f t_1) + b\sin(2\pi f t_2) + \cdots + n\sin(2\pi f t_n)]$$

$\forall n$, this is just a linear combination of X, Y which are two independent Gaussians. Thus $\mathcal{Z}$ is jointly Gaussian.

Thus $\{Z(t)\}_{t=-\infty}^{+\infty}$ a Gaussian Random Process.

(b) Mean and Autocorrelation Function of $\{Z(t)\}$?

$\mu_x(t)$

$$\mathbb{E}[Z(t)] = \mathbb{E}[X\cos(2\pi f t) + Y\sin(2\pi f t)]$$
$$= \cos(2\pi f t)\mathbb{E}[X] + \sin(2\pi f t)\mathbb{E}[Y]$$
$$= 0 + 0 \qquad \forall t \in \mathbb{R}$$
$$= 0$$

$R_Z(t_1, t_2)$

$$R_Z(t_1, t_2) \triangleq \mathbb{E}[Z(t_1)Z(t_2)]$$

$$= \mathbb{E}[\; X^2\cos(2\pi f t_1)\cos(2\pi f t_2)$$
$$+ XY\cos(2\pi f t_1)\sin(2\pi f t_2)$$
$$+ XY\cos(2\pi f t_2)\sin(2\pi f t_1)$$
$$+ Y^2\sin(2\pi f t_1)\sin(2\pi f t_2)]$$

$$
\begin{aligned}
&= \mathbb{E}[X^2 \cos(2\pi f t_1)\cos(2\pi f t_2)] \\
&\quad + \cancel{\mathbb{E}[XY\cos(2\pi f t_1)\sin(2\pi f t_2)]} \to 0,\ X,Y \text{ uncorrelated due to J.G. + independence} \\
&\quad + \cancel{\mathbb{E}[XY\cos(2\pi f t_2)\sin(2\pi f t_1)]} \to 0 \\
&\quad + \mathbb{E}[Y^2 \sin(2\pi f t_1)\sin(2\pi f t_2)]
\end{aligned}
$$

Var(X)+ $(\mathbb{E}[X])^2$

$$
\begin{aligned}
&= \cos(2\pi f t_1)\cos(2\pi f t_2)\,\mathbb{E}[X^2] + \sin(2\pi f t_1)\sin(2\pi f t_2)\,\mathbb{E}[Y^2] \\
&= \cos(2\pi f t_1)\cos(2\pi f t_2) + \sin(2\pi f t_1)\sin(2\pi f t_2) \\
&= \cos(2\pi f t_1 - 2\pi f t_2) \\
&= \cos(2\pi f(t_1 - t_2)) \\
&= \cos(2\pi f(t_2 - t_1)) \quad \text{since cos even}
\end{aligned}
$$

(c) Is $\{Z(t)\}$ Wide-Sense Stationary? Stationary?

WSS

Note $\mathbb{E}[Z(t)] = 0 \quad \forall\, t$

and $R_X(t_1, t_2) = \cos(2\pi f \tau)$, $\tau = t_2 - t_1$

depends only on τ.

Thus $\{Z(t)\}$ is wide sense stationary.

Stationary?

Since $\{Z(t)\}_{t=-\infty}^{+\infty}$ a Gaussian random process and wide-sense stationary, it is strictly stationary.

② MMSE Prediction for a Gaussian Process

Let $X(t)$ be a stationary Gaussian process with

$$\mu_X(t) \triangleq \mathbb{E}[X(t)] = 0 \qquad R_X(\tau) \triangleq \mathbb{E}[X(t)X(t+\tau)] = 5\cos\left(\frac{\pi\tau}{2}\right)3^{-|\tau|}$$

(a) Find the Covariance Matrix of $[X(2) \;\; X(3) \;\; X(4)]^T$

$$\mathbb{K} = \{R_X(t_i, t_j) - \mu_X(t_i)\mu_X(t_j)\}_{n\times n} = \{C_X(t_i, t_j)\}_{n\times n}$$

$X(2), X(3), X(4) \Rightarrow$ Sample at time 2, 3, 4. $\mu_X(t) = 0 \; \forall t.$

$$\mathbb{K} = \begin{bmatrix} \mathbb{E}[X(2)X(2)] & \mathbb{E}[X(2)X(3)] & \mathbb{E}[X(2)X(4)] \\ E[X(3)X(2)] & \mathbb{E}[X(3)X(3)] & \mathbb{E}[X(3)X(4)] \\ E[X(4)X(2)] & \mathbb{E}[X(4)X(3)] & \mathbb{E}[X(4)X(4)] \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 & -5/9 \\ 0 & 5 & 0 \\ -5/9 & 0 & 5 \end{bmatrix}$$

(b) Find $\mathbb{E}[X(4) \mid X(2)]$, the MMSE Prediction of $X(4)$ given $X(2)$.

Since $X(4), X(2)$ jointly Gaussian,

$$\mathbb{E}[X(4) \mid X(2)] = \mathbb{E}[X(4)] + \frac{\mathrm{Cov}(X(4), X(2))}{\mathrm{Var}(X(2))}\left(X(2) - \mathbb{E}[X(2)]\right)$$

$$= 0 + \frac{-5}{9}\cdot\frac{1}{5}(X(2) - 0) = -\frac{1}{9}X(2)$$

This estimator results in following MSE:

$$\mathrm{Var}(X(4) \mid X(1)) = \mathrm{Var}(X(4)) - \frac{\mathrm{Cov}^2(X(4), X(1))}{\mathrm{Var}(X(1))}$$

$$= 5 - \frac{25}{81} \cdot \frac{1}{5} = 5 - \frac{5}{81}$$

(c) Find $\mathbb{E}[X(4) \mid X(2), X(3)]$, the MMSE prediction of $X(4)$ given both $X(2), X(3)$.

Note: $\begin{bmatrix} X(2) \\ X(3) \end{bmatrix}$ a Gaussian random vector w/

since $X(2), X(3)$ zero mean

$$\mu = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathbb{K} = \begin{bmatrix} \mathbb{E}[X(2)X(2)] & \mathbb{E}[X(2)X(3)] \\ \mathbb{E}[X(3)X(2)] & \mathbb{E}[X(3)X(3)] \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

Let

$$\mathbb{X} = [X(4)] \sim \mathcal{N}([0], [5])$$

$$\mathbb{Y} = \begin{bmatrix} X(2) \\ X(3) \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}\right)$$

Then

$$\mathbb{E}[\mathbb{X} \mid \mathbb{Y}] = \underbrace{\mathbb{E}[\mathbb{X}]}_{1\times1} + \underbrace{\mathrm{Cov}(\mathbb{X}, \mathbb{Y})}_{1\times2} \underbrace{\mathrm{Cov}^{-1}(\mathbb{Y})}_{2\times2} \underbrace{(\mathbb{Y} - \mathbb{E}[\mathbb{Y}])}_{2\times1}$$

$$= 0 + \begin{bmatrix} -5/9 & 0 \end{bmatrix} \begin{bmatrix} 1/5 & 0 \\ 0 & 1/5 \end{bmatrix} \begin{bmatrix} X(2) \\ X(3) \end{bmatrix}$$

$$= 0 + \begin{bmatrix} -1/9 & 0 \end{bmatrix} \begin{bmatrix} X(2) \\ X(3) \end{bmatrix}$$

$$= 0 + \frac{-1}{9} X(2) + 0$$

$$= \frac{-1}{9} X(2).$$

Makes sense! $X(4)$ uncorrelated w/ $X(3)$ AND $X(3)$ uncorrelated w/ $X(2)$.

③ Gaussian Moving Average

Let $\{X_n\}_{n=-\infty}^{+\infty}$ be a sequence of iid Gaussian random variables w/ zero mean and unit variance.

Define

$$Y_n = \frac{1}{3}(X_n + X_{n-1} + X_{n-2})$$

as the moving average of X_n's.

(a) Is $\{Y_n\}_{n=-\infty}^{+\infty}$ a Gaussian Random Process?

For arbitrary m, n w/ $m < n$

$$Y_m = \frac{1}{3}(X_m + X_{m-1} + X_{m-2}) \sim \text{Gaussian}$$

$$Y_n = \frac{1}{3}(X_n + X_{n-1} + X_{n-2}) \sim \text{Gaussian}$$

and

$$aY_m + bY_n \qquad a, b \in \mathbb{R}$$

$$= \frac{a}{3}(X_m + X_{m-1} + X_{m-2}) + \frac{b}{3}(X_n + X_{n-1} + X_{n-2}) \sim \text{Gaussian}$$

Since X_i iid $\forall i$ (overlap case is also included above).

(b) Determine the mean function and the autocorrelation function of $\{Y_n\}_{n=-\infty}^{+\infty}$

$$\mathbb{E}[Y_n] = \frac{1}{3}\mathbb{E}[X_n + X_{n-1} + X_{n-2}]$$

$$= \frac{1}{3}\left(\mathbb{E}[X_n] + \mathbb{E}[X_{n-1}] + \mathbb{E}[X_{n-2}]\right)$$

$$= \frac{1}{3}(0 + 0 + 0)$$

$$= 0$$

Assume $m<n$

$$R_Y(m,n) \hat{=} \mathbb{E}[Y_m Y_n]$$

$$= \frac{1}{9}\mathbb{E}\left[(X_n + X_{n-1} + X_{n-2})(X_m + X_{m-1} + X_{m-2})\right]$$

4 cases.

① $m < n-2$

then $X_i \perp\!\!\!\perp X_j \ \forall\, i \neq j$ and $\mathbb{E}[Y_m Y_n] = 0$

② $m = n-2$

$$R_Y(m,n) = \frac{1}{9}\mathbb{E}\left[(X_n + X_{n-1} + X_{n-2})(X_{n-2} + X_{n-3} + X_{n-4})\right]$$

$$= \frac{1}{9}\mathbb{E}[X_{n-2}^2] = \frac{1}{9}$$

③ $m = n-1$

$$R_Y(m,n) = \frac{1}{9}\mathbb{E}\left[(X_n + X_{n-1} + X_{n-2})(X_{n-1} + X_{n-2} + X_{n-3})\right]$$

$$= \frac{1}{9}\left(\mathbb{E}[X_{n-1}^2] + \mathbb{E}[X_{n-2}^2]\right) = \frac{2}{9}$$

④ $m = n$

$$R_Y(m,n) = \frac{1}{9}\left(\mathbb{E}[X_n^2] + \mathbb{E}[X_{n-1}^2] + \mathbb{E}[X_{n-2}^2]\right) = \frac{3}{9} = \frac{1}{3}$$

Thus

$$R_Y(m,n) = \begin{cases} 0, & |n-m| > 2 \\ \frac{1}{9}, & |n-m| = 2 \\ \frac{2}{9}, & |n-m| = 1 \\ \frac{1}{3}, & |n-m| = 0 \end{cases}$$

(c) Is $\{Y_n\}_{n=-\infty}^{+\infty}$ (strictly) stationary?

Y_n is Gaussian $\forall n$ - which is specified completely by mean (0) and variance ($1/3$). The mean and variance do NOT change over time, thus $\{Y_n\}_{n=-\infty}^{+\infty}$ is strictly stationary.

(d) Does $\{Y_n\}_{n=-\infty}^{+\infty}$ have independent increments?

No! For all s and for all $n_1, n_2, \ldots, n_s$

$Y_{n_2} - Y_{n_1}, \ldots, Y_{n_s} - Y_{n_{s-1}}$ not ALWAYS mutually independent.

Easy to see this. Take $2 < 3 < 5 < 8$

$$Y_8 - Y_5 = X_8 + X_7 + X_6 - X_5 - X_4 - X_3$$
$$Y_3 - Y_2 = X_3$$

both increments have X_3 in them.

(4) The Brownian Bridge

Let $\{X(t)\}_{t \geq 0}$ be a Brownian motion with parameter σ^2. Define the following random process $\{Y(t)\}_{0 \leq t \leq 1}$ called a Brownian bridge as follows:

$$Y(t) = X(t) - tX(1), \quad 0 \leq t \leq 1.$$

(a) Draw several sample paths of the brownian bridge. Are the sample paths continuous?

(b) Is the Brownian bridge a Gaussian process? Explain.

Take $0 \leq t_1 < t_2 \leq 1$

Then $\quad Y(t_1) + Y(t_2) = X(t_1) + X(t_2) - X(1)(t_1 + t_2)$

Since X a Gaussian process then any linear combination of X is Gaussian and thus $Y(t_1) + Y(t_2)$ is Gaussian.

Extend to general n: $0 \leq t_1 < t_2 < \cdots < t_n \leq 1$

$$Y(t_1) + Y(t_2) + \cdots + Y(t_n) = X(t_1) + X(t_2) + \cdots + X(t_n) - X(1)(t_1 + t_2 + \cdots + t_n)$$

which is also Gaussian - and introduction of a scalar does not change this.

Thus $\{Y\}_{0 \leq t \leq 1}$ is a Gaussian process.

(c) Mean function and autocorrelation function of $\{Y(t)\}_{0\le t\le 1}$.

$$\mu_Y(t) \triangleq \mathbb{E}[Y(t)] = \mathbb{E}[X(t) - tX(1)]$$
$$= \mathbb{E}[X(t)] - \mathbb{E}[tX(1)]$$
$$= \mathbb{E}[X(t)] - t\,\mathbb{E}[X(1)]$$
$$= 0 - t\cdot 0$$
$$= 0$$

Take $t_1 < t_2$

$$C_Y(t_1,t_2) \triangleq \mathbb{E}\Big[\big(Y(t_1) - \mathbb{E}[Y(t_1)]\big)\big(Y(t_2) - \mathbb{E}[Y(t_2)]\big)\Big]$$

$R_Y(t_1,t_2)$"

$$= \mathbb{E}[Y(t_1)Y(t_2)]$$
$$= \mathbb{E}\big[(X(t_1) - t_1X(1))(X(t_2) - t_2X(1))\big]$$
$$= \mathbb{E}\big[X(t_1)X(t_2) - X(1)(t_1+t_2) + t_1t_2X^2(1)\big]$$
$$= \mathbb{E}[X(t_1)X(t_2)] - (t_1+t_2)\mathbb{E}[X(1)] + t_1t_2\,\mathbb{E}[X^2(1)]$$
$$= \sigma^2 t_1 - 0 + t_1t_2\sigma^2$$

In general,

$$R_Y(t_1,t_2) = \min(t_1,t_2)\sigma^2 + t_1t_2\sigma^2$$

(d) Is the Brownian bridge strictly stationary?

(e) Does the Brownian bridge have independent increments?

⑤ Markov Chains

A die is rolled repeatedly. Which of the following sequences $\{X_n\}_{n=0}^{\infty}$ are Markov chains? Prove your statement. For those that are, give transition matrices.

(a) X_n is the largest number shown up to the n^{th} roll.

(b) X_n is the number of 6's up to the n^{th} roll.

(c) X_n is the number of rolls since the most recent 6.

(d) X_n is the number of rolls until the next 6.

⑥ A Markov Process with Dependent Increments

Consider the following first-order Gaussian autoregression process:

$$X_0 = 0$$

$$X_{n+1} = \alpha X_n + W_{n+1}$$

where $|\alpha| \le 1$, $\{W_n\}$ are iid Gaussian with zero mean.

(a) Is $\{X_n\}_{n \geq 0}$ a Gaussian process?

(b) Show that $\{X_n\}_{n \geq 0}$ is a Markov process.

(c) Does $\{X_n\}_{n \geq 0}$ have independent increments?