Recall

Started continuous random variables.

Given $\Omega, \mathbb{P}$, $X : \Omega \to \mathbb{R}$ is a continuous rv when there's a "reasonable function $f_X(x)$ such that for every $V \subset \mathbb{R}$, we have

$$\mathbb{P}(\{X \in V\}) = \int_V f_X(x)\,dx$$

Call $f_X(x)$ the probability density function (pdf) of X.

Special case of V: $V = [a,b]$ or $[a,b)$, $(a,b]$, (a,b) we have

$$\mathbb{P}(\{X \in V\}) = \int_a^b f_X(x)\,dx$$

Some properties of $f_X(x)$:

- $f_X(x) \geq 0 \;\; \forall x$ (need this to ensure $\mathbb{P}(\{X \in V\}) \geq 0 \;\; \forall V \subset \mathbb{R}$)

- $\displaystyle\lim_{R \to \infty} \int_{-R}^{+R} f_X(x)\,dx = 1 \longrightarrow \int_{-\infty}^{\infty} f_X(x)\,dx = \mathbb{P}(\{X \in (-\infty, \infty)\}) = 1$

- Given $x \in \mathbb{R}$, $f_X(x)$ is **NOT** $\mathbb{P}$(some event) – in particular,

$$f_X(x) \neq \mathbb{P}(\{X = x\})$$

Turns out $\mathbb{P}(\{X=x\})=0 \quad \forall x\in\mathbb{R}$ when X is a continuous random variable

Since $f_X(x)$ isn't $\mathbb{P}$(some event), need not have $f_X(x)\leq 1$!
In fact, $f_X(x)$ can take on arbitrarily large values!

Interpretation of $f_X(x)$: probability "mass" per unit "length"

Idea: Given $x_0\in\mathbb{R}$, look at

$$\int_{x_0-\delta}^{x_0+\delta} f_X(x)\,dx = \mathbb{P}(\{x_0-\delta\leq X\leq x_0+\delta\})$$

interval length of 2δ; want center

Take $\frac{1}{2\delta}$; let $\delta\to 0$; get $f_X(x_0)$ - units $\dfrac{\text{probability mass}}{\text{length}}$

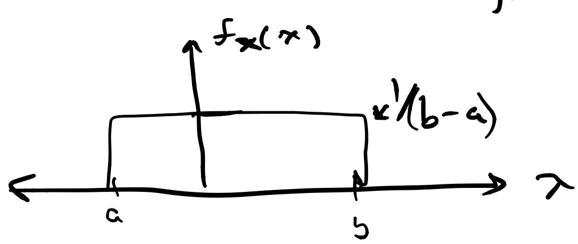
Example - Uniform

← analogous to discrete uniform

DOESN'T MATTER if open or closed or half-open

Given $a,b\in\mathbb{R}$; $a<b$; let $f_X(x)=\begin{cases}\frac{1}{b-a}, & \text{when } x\in[a,b]\\ 0, & \text{else}\end{cases}$

Call the associated rv X "uniform on $[a,b]$"



Example - Exponential

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & ; \; x \geq 0 \quad \forall \, \lambda > 0 \\ 0 & ; \; x < 0 \end{cases}$$

Given a function $g : \mathbb{R} \to \mathbb{R}$ and a continuous rv X, $Y = g(X)$ is another rv - turns out Y might be a discrete rv or a continuous rv. It'll have a pmf $p_Y(y)$ OR pdf $f_Y(y)$ depending on what type of rv it is. Generally, these are NOT easy to find.

But, consider X is exponential(λ) - i.e

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & ; \; x \geq 0 \\ 0 & ; \; x < 0 \end{cases}$$

Let $Y = \lceil X \rceil$ - i.e $Y = k$ when $X \in (k-1, k] \; \forall \, k > 0$.
Y is a discrete rv! Let's find its pmf.

$$p_Y(k) = \mathbb{P}(\{Y = k\}) = \mathbb{P}(\{X \in (k-1, k]\}) \quad \text{when } k > 0, \; k \in \mathbb{Z}$$

$$= \int_{k-1}^{k} \lambda e^{-\lambda x} \, dx$$

$$= \lambda \cdot -\frac{1}{\lambda} e^{-\lambda x} \Big|_{k-1}^{k} = -e^{-\lambda x} \Big|_{k-1}^{k} = -e^{-\lambda(k)} + e^{-\lambda(k-1)} = (1 - e^{-\lambda}) e^{-\lambda(k-1)}$$

Note: $(1-e^{-\lambda})e^{-\lambda(k-1)} = p(1-p)^{k-1}$, where $p = 1-e^{-\lambda}$!

i.e

$$P_Y(k) = \begin{cases} p(1-p)^{k-1} & , k > 0 \\ 0 & , k < 0 \end{cases}$$

– Y is geometric w/ $p = 1-e^{-\lambda}$

This is instance 1 of connection between exponential continuous rvs and geometric discrete rvs.

Expected Value

The expected value of a continuous rv X w/ pdf $f_X(x)$:

$$\mathbb{E}[X] = \int_{-\infty}^{+\infty} x f_X(x)\,dx$$

caution: NOT always defined - integral might fail to exist

Example - Non Existance

$$f_X(x) = \begin{cases} \frac{1}{x^2} & \text{when } x \geq 1 \\ 0 & \text{when } x < 1 \end{cases}$$

$$\mathbb{E}[X] = \int_1^{\infty} \frac{1}{x}\,dx \rightarrow \text{nonexistent!}$$

As for discrete rvs, have

Expected Value Rule

Given X w/ pdf $f_X(x)$ and $Y = g(X)$, we have

$$\mathbb{E}[Y] = \int_{-\infty}^{+\infty} g(x) f_X(x)\, dx$$

← enables $\mathbb{E}[Y]$ computation w/out finding $f_Y(y)$ or $p_Y(y)$

Proof is a "little" harder than proof for discrete rvs.

Special case: $g(x) = \alpha X + \beta$, then

$$\mathbb{E}[g(X)] = \alpha \mathbb{E}[X] + \beta$$

Variance

Variance of continuous rv:

$$\mathrm{Var}(X) = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right]$$

By expected value rule, we have

$$\mathrm{Var}(X) = \int_{-\infty}^{+\infty} (X - \mathbb{E}[X])^2 f_X(x)\, dx$$

Also, as before,

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

Interpretations of $\mathbb{E}(X)$, $Var(X)$?

center of "probability mass"

spread of the "probability mass" about the center

Also, for $m > 0$, the m^{th} moment of X is $\mathbb{E}[X^m]$

Example - $\mathbb{E}[X]$, $Var(X)$ for Uniform

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & \text{when } x \in [a,b] \\ 0, & \text{else} \end{cases}$$

$\mathbb{E}[x] = \frac{b+a}{2}$; Proof: $\mathbb{E}[x] = \int_a^b \frac{x}{b-a}\,dx = \frac{1}{2}\left(\frac{b^2-a^2}{b-a}\right)$

$$= \frac{b+a}{2}$$

$\mathbb{E}[x^2]$

$$Var(x) = \int_a^b \frac{x^2}{b-a}\,dx - \left(\int_a^b \frac{x}{b-a}\,dx\right)^2$$

$$= \frac{\frac{1}{3}(b^3-a^3)}{(b-a)} - \left(\frac{b+a}{2}\right)^2$$

$$= \frac{1}{3}(a^2+ab+b^2) - \frac{b^2+2ab+a^2}{4} = \frac{(b-a)^2}{12}$$

Example - $\mathbb{E}[X]$, $Var(X)$ for Exponential

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & , x \geq 0 \\ 0 & , x < 0 \end{cases}$$

Integration by parts!

$$\mathbb{E}[X] = \int_{-\infty}^{+\infty} x f_X(x) = \int_0^{\infty} \lambda x e^{-\lambda x} dx$$

OR Tabular Method

u	dv
x	$\lambda e^{-\lambda x}$
-1	$-e^{-\lambda x}$
0	$\frac{1}{\lambda} e^{-\lambda x}$

$u = x \quad du = dx$

$dv = \lambda e^{-\lambda x} \quad v = -e^{-\lambda x}$

$$= -xe^{-\lambda x} \Big|_0^{\infty} + \int_0^{\infty} e^{-\lambda x} dx = \frac{1}{\lambda}$$

$$Var(X) = \int_0^{\infty} x^2 \lambda e^{-\lambda x} dx - \left(\frac{1}{\lambda}\right)^2$$

u	dv
x^2	$\lambda e^{-\lambda x}$
$-2x$	$-e^{-\lambda x}$
2	$1/\lambda \, e^{-\lambda x}$
0	$\frac{1}{\lambda^2} e^{-\lambda x}$

$$= -x^2 e^{-\lambda x} \Big|_0^{\infty} + 2 \underbrace{\int_0^{\infty} x e^{-\lambda x} dx}_{\frac{\mathbb{E}[X]}{\lambda}!} = \frac{2}{\lambda^2}$$

So,

$$Var(X) = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$