Math 4310

Homework 4

Due 10/2/19

Name: ____________________

Collaborators: ____________________

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Please print out these pages. I encourage you to work with your classmates on this homework. Please list your collaborators on this cover sheet. (Your grade will not be affected.) Even if you work in a group, you should write up your solutions yourself! You should include all computational details, and proofs should be carefully written with full details. As always, please write neatly and legibly.

Please follow the instructions for the "extended glossary" on separate paper (LaTeX it if you can!) Hand in your final draft, including full explanations and write your glossary in complete, mathematically and grammatically correct sentences. Your answers will be assessed for style and accuracy.

Please **staple** *this cover sheet, your exercise solutions, and your glossary together, in that order, and hand in your homework in class.*

GRADES

Exercises ____________________ / 50

Extended Glossary

Component	Correct?	Well-written?
Definition	/6	/6
Example	/4	/4
Non-example	/4	/4
Theorem	/5	/5
Proof	/6	/6
Total	/25	/25

Exercises.

1. Let $T \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^3)$ and suppose

$$T\left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \; T\left(\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \text{ and } T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

   Let $\mathcal{S}$ denote the standard basis e_1, e_2, e_3. Determine $[T]_{\mathcal{S}\leftarrow\mathcal{S}}$.

   For the basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$, determine $[T]_{\mathcal{B}\leftarrow\mathcal{B}}$.

2. Let V be a finite dimensional vector space over a field $\mathbb{F}$, let X be a non-empty finite set, and let $\mathcal{F}un(X, V)$ denote the vector space of functions from the set X to the vector space V. (You do not need to prove that this is a vector space over $\mathbb{F}$.) Fix an element $a \in X$, and define a function

$$\begin{aligned} T_a : \mathcal{F}un(X, V) &\to V \\ f &\mapsto f(a). \end{aligned}$$

   Prove that T_a is a linear transformation, and find $\ker(T_a)$ and $\mathrm{Im}(T_a)$. What are the dimensions of $\mathcal{F}un(X, V)$, $\ker(T_a)$ and $\mathrm{Im}(T_a)$?

3. A **chain complex C** is a sequence of vector spaces V_i, and linear transformations $d_i : V_i \longrightarrow V_{i-1}$:
$$0 \longrightarrow V_p \xrightarrow{d_p} V_{p-1} \xrightarrow{d_{p-1}} \cdots \longrightarrow V_1 \xrightarrow{d_1} V_0 \longrightarrow 0.$$
which satisfies that $d_i \circ d_{i+1} = 0$, for all i, with the convention that $d_i = 0$ for all $i > p$ and $i \leq 0$. In this problem we assume that all of the vector spaces appearing (V_i, V, W, U) are finite dimensional vector spaces over a field $\mathbb{F}$. The chain complex C is called **exact**, or an **exact sequence**, if $\mathrm{im}(d_{i+1}) = \ker(d_i)$, for all i.

   (a) Show that if **C** is a chain complex, then $\mathrm{im}(d_{i+1}) \subseteq \ker(d_i)$, for all i.

   (b) Suppose that $T : V \longrightarrow W$ is a linear transformation, and consider
   $$0 \longrightarrow \ker(T) \xrightarrow{f} V \xrightarrow{g} \mathrm{im}(T) \longrightarrow 0.$$
   Here f and g are the "natural maps". First, define these maps, then show that this is an exact chain complex. (An exact chain complex with three nonzero vector spaces is called a **short exact sequence**).

   (c) Show that if the chain complex
   $$0 \longrightarrow V \xrightarrow{T} W \xrightarrow{S} U \longrightarrow 0$$
   is an exact sequence, then $\dim W = \dim V + \dim U$.

   (d) Show that if **C** is an exact sequence,
   $$\sum_{i=0}^{p} (-1)^i \dim V_i = 0.$$
   hint: Use induction on p, find a related exact sequence of length one less, use the induction hypothesis.

4. We will let $\mathcal{L}(V)$ denote all linear transformations from a vector space V to itself (sometimes called **linear operators**.). Let $T \in \mathcal{L}(V)$, and let T^2 denote the composition $T \circ T$.

   (a) Prove that if $T^2 = T$, then $V = \ker(T) \oplus \ker(T - \mathbb{1}_V)$.

   (b) Prove that if $V = \ker(T) + \ker(T - \mathbb{1}_V)$, then $T^2 = T$.

   (c) Give an example of a vector space V and $T \in \mathcal{L}(V)$ such that $T^2 = -\mathbb{1}_V$.

   (d) Prove that if $T^2 = \mathbb{0}_{V \to V}$ is the zero transformation, then $\mathrm{rank}(T) \leq \frac{\dim(V)}{2}$.

Extended Glossary. For any vector spaces V and W (over a field $\mathbb{F}$), define the **external direct product** $V \times W$. This is also called the external direct sum $V \oplus W$. Define the vector space operations (but you do not need to prove that it is in fact a vector space). When V and W are finite dimensional, compute the dimension of the external direct sum, by finding a basis for it. Finally, state and prove a theorem about external direct sums.

You may work in groups, but please write up your solutions **yourself**. If you do work together, your group should come up with at least two examples, two non-examples, and two theorems. Each one (example/non-example/theorem) should be included in some group member's extended glossary. Your solutions should be written formally, so that we could cut and paste them into a textbook.