Last time

Example

$$V = \operatorname{span}\left(\begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\1\end{pmatrix}\right) \subseteq \mathbb{R}^4$$

Find orthogonal basis $\vec{w}_1, \vec{w}_2, \vec{w}_3$

So, find $\vec{v}_1, \vec{v}_2, \vec{v}_3$ such that

① $\operatorname{span}(v_1, v_2, v_3) = \operatorname{span}(w_1, w_2, w_3)$

② $\vec{v}_1, \vec{v}_2, \vec{v}_3$ is an orthogonal set

Solution

① $\vec{v}_1 = \begin{pmatrix}1\\1\\0\\0\end{pmatrix} = \vec{w}_1$

$$\vec{v}_2' = \vec{w}_2 - \frac{\langle \vec{w}_2, \vec{v}_1\rangle}{\langle \vec{v}_1, \vec{v}_1\rangle}\vec{v}_1 = pr_{\vec{v}_1}(\vec{w}_2) = \begin{pmatrix}1/2\\-1/2\\1\\0\end{pmatrix}$$

(where $\vec{v}_2 = \begin{pmatrix}1\\-1\\2\\0\end{pmatrix}$)

$$\vec{v}_3' = \vec{w}_3 - \frac{\langle \vec{w}_3, \vec{v}_1\rangle}{\langle \vec{v}_1, \vec{v}_1\rangle}\vec{v}_1 - \frac{\langle \vec{w}_3, \vec{v}_2\rangle}{\langle \vec{v}_2, \vec{v}_2\rangle}\vec{v}_2 = \vec{w}_3 - \frac{1}{2}\vec{v}_1 - \frac{1}{6}\vec{v}_2$$

where $\vec{v}_3 = \begin{pmatrix}1\\-1\\-1\\3\end{pmatrix}$

$$\vec{u}_1 = \frac{1}{\sqrt{2}}\vec{v}_1, \quad \vec{u}_2 = \frac{1}{\sqrt{6}}\vec{v}_2, \quad \vec{u}_3 = \frac{1}{\sqrt{12}}\vec{v}_3$$

$(\vec{v}_1, \vec{v}_2, \vec{v}_3)$ orthogonal basis of V

$(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ orthonormal basis

Question

Given A $n \times r$ $A = [\vec{a}_1 \cdots \vec{a}_r]$

B $n \times s$ $B = [\vec{b}_1 \cdots \vec{b}_s]$

What is $A^T B$? $r \times s$ matrix

$$(A^T B)_{i,j} = a_i^T b_j = a_i \cdot b_j \qquad \begin{matrix} 1 \le i \le r \\ 1 \le j \le s \end{matrix}$$

Example

If $A = [\vec{a}_1 \cdots \vec{a}_r]$ $n \times r$, $(\vec{a}_1, \ldots, \vec{a}_r)$ orthonormal set of vectors in $\mathbb{R}^n$ then

$$A^T A = \begin{bmatrix} a_1 \cdot a_1 & a_1 \cdot a_2 & \cdots & a_1 \cdot a_r \\ a_2 \cdot a_1 & \ddots & & \vdots \\ \vdots & & & \\ a_r \cdot a_1 & \cdots & & a_r \cdot a_r \end{bmatrix}$$

Definition:

ⓐ If $A^T A$ is a diagonal matrix (w/ no zero entries on diagonal) we say we say A has <u>orthogonal</u> columns

ⓑ If $A^T A = I_{r \times r}$, then we say A has <u>**orthonormal**</u> columns

ⓒ If A is $n \times n$, and $A^T A = I_{n \times n}$, then we say A is **orthogonal**

Example

$$R_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$R_\theta^T R_\theta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

dim im A

<u>Proposition</u>: Let $A \in \mathbb{R}^{m \times n}$. Then $\operatorname{rank}(A^T A) = \operatorname{rank}(A)$

<u>Proof</u> (

$$A : \mathbb{R}^n \to \mathbb{R}^m$$
$$A^T A : \mathbb{R}^n \to \mathbb{R}^n$$

Use rank-nullity

Show

$$\dim \ker(A) = \dim \ker(A^T A)$$

actually, can show $\ker(A) = \ker(A^T A)$

If $A\vec{v} = \vec{0}$ $(\vec{v} \in \ker A)$

then $A^T A \vec{v} = \vec{0}$ $(\vec{v} \in \ker A^T A)$

So

$$\ker A \subseteq \ker A^T A$$

Now need $\ker(A^T A) \subseteq \ker A$

Let $\vec{v} \in \ker(A^T A)$

$$A^T A \vec{v} = \vec{0}$$

Need

$$A\vec{v} = \vec{0}$$

Take

$$\vec{v}^T A^T A \vec{v} = \vec{v}^T \vec{0} = 0$$

$$\langle A\vec{v}, A\vec{v} \rangle = 0$$

Thus $A\vec{v} = 0$!

So $\vec{v} \in \ker(A)$, and $\ker A^TA \subseteq \ker A$

Thus

$$\ker A = \ker A^TA$$

Corollary

If the columns of $A_{m\times n}$ are LI then A^TA is invertible.

Columns of A LI $\Rightarrow m \geq n$, rank $A = n$

$\therefore$ rank $A^TA = n$

$\therefore$ A^TA invertible

Example: Consider Graham Schmidt Example

$$A = [\vec{w}_1 \quad \vec{w}_2 \quad \vec{w}_3] \quad \text{original}$$

$$Q = [\vec{u}_1 \quad \vec{u}_2 \quad \vec{u}_3] \quad \text{output orthonormal basis}$$

$$\therefore \quad Q^TQ = I_{3\times 3}$$

Note also: $\exists$ $R_{3\times 3}$, $S_{3\times 3}$ s.t.

$$A = QR$$

$$Q = AS$$

What is R, S?

$$A = QR$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = Q \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}$$

some upper triangular matrix

Theorem: Given $A \in \mathbb{R}^{m \times n}$ with rank n then $\exists$ matrices Q, R $\begin{pmatrix} Q & m \times n \\ R & n \times n \end{pmatrix}$ s.t.

(a) $A = QR$

(b) $Q^T Q = I_n$

(c) R is upper triangular + interval

Can find R by

$$Q^T A = R$$

if $A = [\vec{w}_1 \ \vec{w}_2 \ \vec{w}_3]$

$Q = [\vec{u}_1 \ \vec{u}_2 \ \vec{u}_3]$

$$Q^T A = \begin{bmatrix} u_1 \cdot w_1 & u_1 \cdot w_2 & u_1 \cdot w_3 \\ u_2 \cdot w_1 & u_2 \cdot w_2 & u_2 \cdot w_3 \\ u_3 \cdot w_1 & u_3 \cdot w_2 & u_3 \cdot w_3 \end{bmatrix}$$