① MMSE Estimation

X is a continuous random variable with PDF given by

$$f_X(x) = \begin{cases} 1/x & , \quad 1 \le x \le e \\ 0 & , \quad \text{elsewhere} \end{cases}$$

Given $X = x$, Y has the following conditional pdf:

$$f_{Y|X=x} = \frac{x}{2} e^{-x|y|} \quad , \quad -\infty < y < \infty$$

Find the MMSE estimate of X given Y.

Want

$\mathbb{E}[X|Y]$.

$$f_{Y|X=x}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

$$f_{X,Y}(x,y) = f_{Y|X=x}(y|x)\, f_X(x)$$

$$= \begin{cases} \frac{1}{2} e^{-x|y|} & , \quad \begin{array}{l} 1 \le x \le e \\ -\infty < y < \infty \end{array} \\ 0 & , \quad o/w \end{cases}$$

Want

$$f_{X|Y=y}(x|y) \rightarrow \int_{-\infty}^{+\infty} x f_{X|Y}(x|y)\, dx$$

$$f_Y(y) = \int_{x=1}^{e} \frac{1}{2} e^{-x|y|}\, dx$$

$$= \frac{-1}{2|y|} e^{-x|y|} \Big|_1^e$$

$$= \frac{1}{2|y|}\left(e^{-|y|} - e^{-e|y|}\right)$$

$$f_{X|Y=y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{\frac{1}{2} e^{-x|y|}}{\frac{1}{2|y|}\left(e^{-|y|} - e^{-e|y|}\right)} = \begin{cases} \dfrac{|y| e^{-x|y|}}{e^{-|y|} - e^{-e|y|}}, & 1 \le x \le e \\ 0, & \text{o/w} \end{cases}$$

$$\mathbb{E}[X|Y=y] = \int_{x=1}^{e} \frac{|y|}{e^{-|y|} - e^{-e|y|}}\, x\, e^{-x|y|}\, dx$$

$$\begin{array}{cc} u & dv \\ x & e^{-x|y|} \\ -1 & -\frac{1}{|y|} e^{-x|y|} \\ 0 & \frac{1}{|y|^2} e^{-x|y|} \end{array}$$

$$= \frac{|y|}{e^{-|y|} - e^{-e|y|}} \left(\frac{-x}{|y|} e^{-x|y|} - \frac{1}{|y|^2} e^{-x|y|} \Big|_{x=1}^{e} \right)$$

$$\frac{|y|}{e^{-|y|} - e^{-e|y|}} \left(\frac{-e e^{-e|y|} + e^{-|y|}}{|y|} - \frac{1}{|y|^2}\left(e^{-e|y|} - e^{-|y|}\right) \right)$$

$$= \boxed{\frac{e^{-|y|} - e^{(1-e)|y|}}{e^{-|y|} - e^{-e|y|}} + \frac{1}{|y|}}$$

② Poisson Processes

Let $\{N_1(t)\}_{t\geq 0}$ and $\{N_2(t)\}_{t\geq 0}$ be two independent Poisson processes.

Determine whether each of the following statements is true or false.

(a) $\{N_1(t)\}_{t\geq 0}$ is NOT stationary, but it is wide-sense stationary.

A WSS Poisson process $\Rightarrow$ Stationarity. FALSE

(b) $\{N_1(t)\}_{t\geq 0}$ has independent and stationary increments.

TRUE

(c) $\{N_1(t)\}_{t\geq 0}$ is a Markov Process

TRUE

(d) If it is known $\mathbb{E}[N_1(1)] = \alpha$, then $\forall\, t > 0$,

$$\mathbb{E}[N_1(t)] = Var(N_1(t)) = \alpha t$$

TRUE

(e) Let $X(t) = N_1(t) + N_2(t)$. Then $\{X(t)\}_{t\geq 0}$ is a Poisson Process.

TRUE

$N_1(t) - N_1(s) + N_2(t) - N_2(s)$

$\lambda_1(t-s) + \lambda_2(t-s)$

$\sim (\lambda_1 + \lambda_2)(t-s)$

(f) Let $Y(t) = N_1(t) - N_2(t)$. Then $\{Y(t)\}_{t\geq 0}$ is a Poisson Process.

$$Y(t) - Y(s) = N_1(t) - N_2(t) - (N_1(s) - N_2(s))$$
$$= \underbrace{N_1(t) - N_1(s)}_{\sim \lambda_1(t-s)} - \underbrace{N_2(t) + N_2(s)}_{\sim \lambda_2(t+s)}$$

FALSE

(g) Let $Z(t) = N_1(t+3) - N_1(t)$. Then $\{Z(t)\}_{t\geq 0}$ is wide-sense stationary.

$$\mathbb{E}[Z(t)] = \mathbb{E}[N_1(t+3) - N_1(t)]$$
$$= \lambda \cdot 3 \rightarrow \text{constant}$$

$$R_Z(t_1, t_2) \stackrel{\Delta}{=} \mathbb{E}[Z(t_1)Z(t_2)]$$

$$= \mathbb{E}[(N_1(t_1+3) - N_1(t))(N_1(t_2+3) - N_1(t_2))]$$

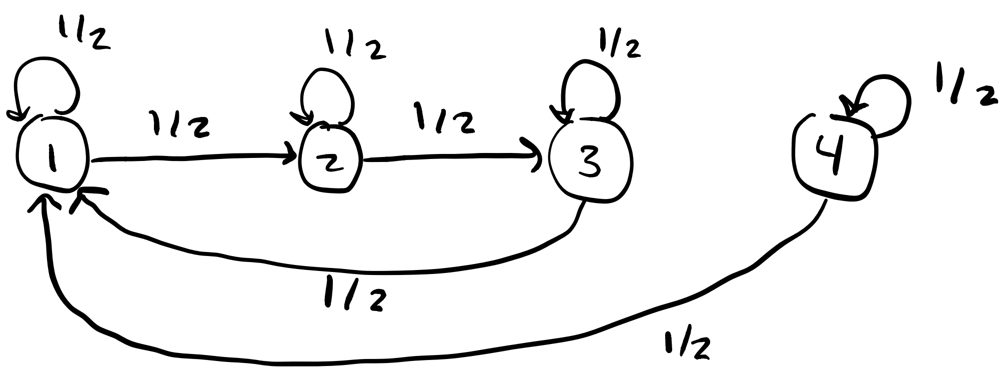
TRUE

③ Markov Chain: Basic Concepts

Consider a discrete-time Markov chain with four states $\{1,2,3,4\}$ and the following one-step transition matrix

$$\mathbb{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 \end{bmatrix} \end{array}$$

(a) Draw the state transition diagram. Identify all communication classes. Find the period of each class. Is the chain irreducible?



$1 \leftrightarrow 2, \quad 1 \leftrightarrow 3, \quad 2 \leftrightarrow 3 \qquad 4 \leftrightarrow 4$

Communication Classes

$\{1,2,3\}, \quad \{4\}$ ← Chain **NOT** irreducible

Period of each class = 1

(b) How many stationary distributions does the chain have?
Find all stationary distributions.

Just one stationary distribution

$$\pi \mathbb{P} = \pi$$

$$[\pi_1 \ \pi_2 \ \pi_3 \ \pi_4] \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 \end{bmatrix} = [\pi_1 \ \pi_2 \ \pi_3 \ \pi_4]$$

$$\tfrac{1}{2}\pi_1 + \tfrac{1}{2}\pi_3 + \tfrac{1}{2}\pi_4 = \pi_1 \rightarrow \pi_1 = \pi_3 + \pi_4$$

$$\tfrac{1}{2}\pi_1 + \tfrac{1}{2}\pi_2 = \pi_2 \longrightarrow \pi_2 = \pi_1$$

$$\tfrac{1}{2}\pi_2 + \tfrac{1}{2}\pi_3 = \pi_3 \longrightarrow \pi_3 = \pi_2$$

$$\tfrac{1}{2}\pi_4 = \pi_4 \rightarrow \pi_4 = 0$$

Normalize

$$\sum_{i=1}^{4} \pi_i = 1$$

$$\pi_1 + \pi_1 + \pi_1 + 0 = 1$$

$$\pi_1 = 1/3$$

$$\pi = [1/3 \ \ 1/3 \ \ 1/3 \ \ 0]$$

(c) Determine the long-run fraction of time the chain spends in State 1 given that it starts in state 4.

State 4 transient $\rightarrow$ eventually moves out to $\{1, 2, 3\}$

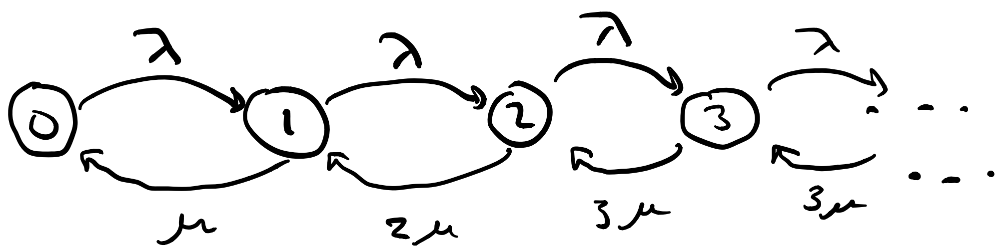
$\Rightarrow$ $\pi_1 = 1/3$ time spent in 1 long-run

(4) Continuous-Time Markov Chain: M/M/s queuing model

Consider the following M/M/s queuing model.

- There are a total $s=3$ servers.
- Customers arrive according to a Poisson process with rate λ
  - → arriving customers wait in the queue and go to the first free server in order of arrival
- Time required to service a customer by any server is exponentially distributed with rate μ.

(a) Construct a continuous-time Markov chain for this queuing model with the number of customers waiting in the queue as a state.



That is,

$$q_{i,i+1} = \lambda \quad \forall\, i \geq 0$$

$$q_{i,i-1} = \begin{cases} \mu, & i=1 \\ 2\mu, & i=2 \\ 3\mu, & i=3 \end{cases}$$

$\mathcal{X} = \{0,1,2,\ldots\}$ is our state space.

(b) Identify the condition under which the chain has a stationary distribution. Compute the stationary distribution.

For existence of stationary distribution, arrival rate must be less then service rate for large i.

That is,

$$\lambda < 3\mu$$

$$\Rightarrow \frac{\lambda}{3\mu} < 1$$

$$Q = \begin{bmatrix} -\lambda & \lambda & & & & \\ \mu & -(\lambda+\mu) & \lambda & & & \\ & 2\mu & -(\lambda+2\mu) & \lambda & & \\ & & 3\mu & -(\lambda+3\mu) & \lambda & \\ & & & \ddots & \ddots & \ddots \end{bmatrix} ; \quad \bar{\pi} Q = 0$$

$-\lambda\pi_0 + \mu\pi_1 = 0 \rightarrow$ $\quad \pi_0\lambda = \pi_1\mu$

$\pi_0\lambda - (\lambda+\mu)\pi_1 + 2\mu\pi_2 = 0$

$-\pi_1\lambda + 2\mu\pi_2 = 0 \rightarrow$ $\quad \pi_1\lambda = \pi_2(2\mu)$

$\pi_1\lambda - (2\mu+\lambda)\pi_2 + \pi_3 3\mu \rightarrow$

$2\mu\pi_2 \quad \lambda\pi_2 = \pi_3 3\mu$ $\quad \pi_2\lambda = \pi_3(3\mu)$

$$\vdots$$

$$\pi_{k\geq 2}\lambda = \pi_{k+1}(3\mu)$$

Thus,

$$\pi_i = \begin{cases} \frac{\lambda}{\mu}\pi_0 & , \quad i=1 \\ \frac{\lambda^2}{2\mu^2}\pi_0 & , \quad i=2 \\ \frac{9}{2}\left(\frac{\lambda}{3\mu}\right)^i \pi_0 & , \quad i>2 \end{cases}$$

Normalizing gives

$$\pi_0\left(1 + \frac{\lambda}{\mu} + \frac{\lambda^2}{2\mu^2} + \frac{3}{2}\sum_{i=2}^{\infty}\left(\frac{\lambda}{3\mu}\right)^i\right) = 1$$

Since $\lambda/3\mu < 1$ required for existence of $\bar{\pi}$,

$$\pi_0\left(1 + \frac{\lambda}{\mu} + \frac{\lambda^2}{2\mu^2} + \frac{3}{2}\left(\frac{1}{1-\frac{\lambda}{3\mu}} - 1 - \frac{\lambda}{3\mu} - \frac{\lambda^2}{9\mu^2}\right)\right) = 1$$

$$\pi_0 = \left(1 + \frac{\lambda}{\mu} + \frac{1}{2}\left(\frac{\lambda}{\mu}\right)^2 + \frac{3}{2}\,\frac{(\lambda/3\mu)^3}{1-\lambda/3\mu}\right)^{-1}$$

(c) What is the percentage of time at least one server is idle?

Idle server if <3 customers.

$$\rightarrow \pi_0 + \pi_1 + \pi_2$$

(d) What is the <u>average</u> queue length in terms of the stationary distribution?

$$\sum_{i=1}^{\infty} i\,\pi_i$$

⑤ Spectral Analysis and Linear Filtering of Random Processes

Let $\{X_n\}_{n=-\infty}^{+\infty}$ be a wide-sense stationary random process with zero mean and autocorrelation

$$R_X(k) = \begin{cases} 1/2, & \text{if } |k| = 1 \\ 1, & \text{if } k = 0 \\ 0, & \text{o/w} \end{cases}$$

Suppose that we sample this process to form two new processes.

The process $\{Y_n\}$ consists of even-numbered samples. That is,

$$Y_n = X_{2n}, \quad -\infty < n < \infty$$

The process $\{Z_n\}$ consists of odd-numbered samples. That is,

$$Z_n = X_{2n+1}, \quad -\infty < n < \infty$$

(a) Compute the autocorrelation of $\{Y_n\}$. Is this process wide-sense stationary? If so, compute its power spectral density.

$$R_Y(n,s) \triangleq \mathbb{E}[Y_n Y_s]$$
$$= \mathbb{E}[X_{2n} X_{2s}] = \begin{cases} 1, & n = s \\ 0, & \text{o/w} \end{cases}$$

$\mathbb{E}[Y_n] = \mathbb{E}[X_{2n}] = 0$

$$R_Y(n, n+k) \triangleq \mathbb{E}[Y_n Y_{n+k}]$$
$$= \mathbb{E}[X_{2n} X_{2n+2k}]$$
$$= R_X(2n, 2n+2k)$$
$$= R_X(2k) \quad (X \text{ WSS}) \quad \therefore Y \text{ WSS}$$

$$S_Y(f) = \mathcal{F}\{R_X(2k)\} = \mathcal{F}\{\delta(k)\} = 1$$

(b) Compute the autocorrelation of $\{Z_n\}$. Is this process wide-sense stationary? If so, compute its power spectral density.

$$R_Z(n,s) \triangleq \mathbb{E}[Z_n Z_s]$$
$$= \mathbb{E}[X_{2n+1} X_{2s+1}]$$
$$= \begin{cases} 1, & n=s \\ 0, & o/w \end{cases}$$

$2s+1-2n-1$

$2s-2n$

$R_X(\underbrace{2s-2n}_{\in \mathbb{Z}})$

$\mathbb{E}[Z_n] = \mathbb{E}[X_{2n+1}] = 0$

$$R_Z(n, n+k) \triangleq \mathbb{E}[Z_n, Z_{n+k}]$$
$$= \mathbb{E}[X_{2n+1}, X_{2n+2k+1}]$$
$$= R_X(2k) = \delta(k) \longrightarrow Z \text{ WSS}$$

$S_Z(f) = 1$

(c) Are $\{Y_n\}$ and $\{Z_n\}$ jointly wide-sense stationary? If so, compute their cross power spectral density.

- Know Y_n, Z_n both WSS from (a), (b).

Need

$$R_{Y,Z}(n, n+k) = R_{Y,Z}(k)$$

$$R_{Y,Z}(n, n+k) \triangleq \mathbb{E}[Y_n Z_{n+k}]$$

$$= \mathbb{E}[X_{2n} X_{2n+2k+1}]$$

$$= R_X(2n+2k+1-2n) \qquad k=0 \rightarrow R_x(1)$$

$$= R_X(2k+1) \qquad k=-1 \rightarrow R_x(-1)$$

$$= \begin{cases} 1/2, & k=0 \\ 1/2, & k=-1 \\ 0, & \text{o/w} \end{cases} = \frac{1}{2}\delta(k) + \frac{1}{2}\delta(k+1)$$

$$S_{Y,Z}(f) = \mathcal{F}\{R_x(2k+1)\} = \frac{1}{2} + \frac{1}{2}e^{+j2\pi f}$$

(d) If we pass $\{Y_n\}$ through a discrete-time low-pass filter with

$$H(f) = \begin{cases} 1, & -\frac{1}{4} < f < \frac{1}{4} \\ 0, & o/w \end{cases}$$

Compute the average power of the output process.

$$\text{Output} = S_o(f) = |H(f)|^2 S_Y(f) = 1, \quad -\frac{1}{4} < f < \frac{1}{4}$$

$$R_o(0) = \int_{-1/2}^{+1/2} S_o(f)\, df = \int_{-1/4}^{+1/4} 1\, df = \frac{1}{2}$$