6c) $\displaystyle\int_{\Gamma} \bar{z}\, dz$, Γ is $|z| = 2$ traversed 3 times clockwise

$z(t) = Re^{it}$, $dz = -iRe^{it}\, dt$, $0 \le t \le 6\pi$
$\bar{z}(t) = Re^{-it}$ $R = 2$

$$\int_{\Gamma} \bar{z}\, dz = \int_0^{6\pi} -iR^2\, dt = -R^2 6\pi i \overset{R=2}{\Longrightarrow} \boxed{-24\pi i}$$

9) $\displaystyle\int_{\Gamma} (x - 2xyi)\, dz$ over $\Gamma: z = t + it^2$
$0 \le t \le 1$

where $x = \mathrm{Re}(z)$
$y = \mathrm{Im}(z)$

$dz = 1 + i2t\, dt$

$$\int_{\Gamma} (x - 2xyi)\, dz = \int_{t=0}^{t=1} (t - 2(t)(t^2)i)(1 + i2t)\, dt$$

$$= \int_0^1 t + i(2t^2 - 2t^3) + 4t^4\, dt$$

$$= \frac{1}{2}t^2 + i\left(\frac{2}{3}t^3 - \frac{1}{2}t^4\right) + \frac{4}{5}t^5 \Big|_0^1 = \boxed{\frac{13}{10} + \frac{i}{6}}$$

(12)

True or False:

$$\oint_{|z|=1} \bar{z}\, dz = \oint_{|z|=1} \frac{1}{z}\, dz$$

$$\left.\begin{array}{l} z(t) = Re^{it}, \quad 0 \le t \le 2\pi \\ \bar{z}(t) = Re^{-it}, \quad 0 \le t \le 2\pi \\ dz = iRe^{it} \end{array}\right\} R = 1$$

$$\int_0^{2\pi} R^2 i\, dt = \oint_0^{2\pi} \frac{i}{\cancel{R}} \cancel{R}\, dt$$

Wow! For $|z|=1$ this is TRUE!
But it's kinda weird that

$$\oint_0^{2\pi} \frac{1}{z}\, dz = 2\pi i \quad \text{FOR ANY } R!$$

But

$\int_0^{2\pi} \bar{z}\, dz$ completely depends on R.

14) a) C is $|z|=3$, traversed once ccw

$$\left| \int_C \frac{dz}{z^2-i} \right| \leq \frac{3\pi}{4}$$

$$\left| \int_C \frac{dz}{z^2-i} \right| \leq \max_{z \in |z|=3} \left| \frac{1}{z^2-i} \right| \cdot \text{length}(C)$$

$$\leq \frac{1}{\min\limits_{z \in C} |z^2-i|} \cdot 2\pi(3)$$

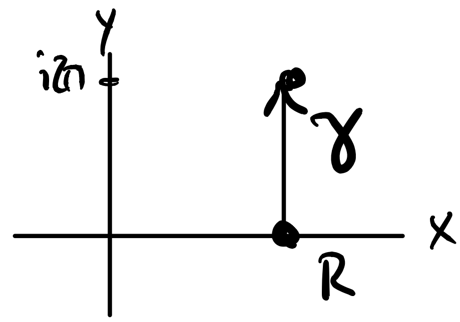
$$|z^2-i| \geq |z^2|-1$$
$$\geq 9-1=8$$

So

$$\left| \int \frac{dz}{z^2-i} \right| \leq \frac{1}{8} \cdot 2\pi(3) = \frac{3\pi}{4}$$

b) γ is the vertical line from $z = R\ (>0)$ to $z = R + 2\pi i$, then

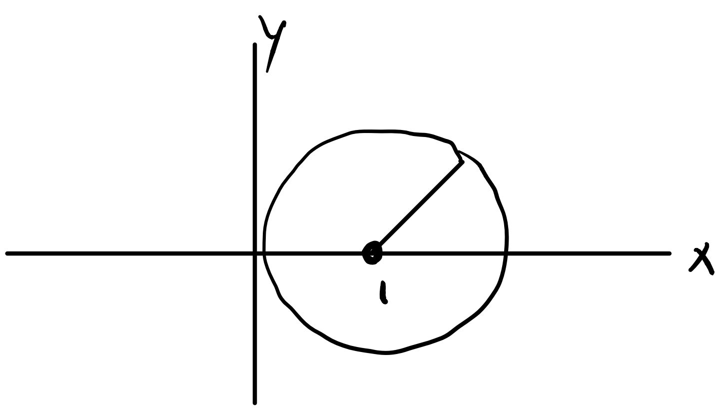
$$\left| \int_\gamma \frac{e^{3z}}{1+e^z}\, dz \right| \leq \frac{2\pi e^{3R}}{e^R - 1}$$



$$\left| \int_\gamma \frac{e^{3z}}{1+e^z}\, dz \right| \leq \frac{\max_{z \in \gamma} |e^{3z}|}{\min_{z \in \gamma} |1+e^z|} \cdot \underbrace{\text{length}(\gamma)}_{2\pi}$$

$|e^{3z}| = |e^{3x} e^{i3y}| = |e^{3x}|$, $x = R$ and only moves along that line so $|e^{3z}| = |e^{3R}|$.

$e^z + 1 = e^x e^{iy} + 1 \leftarrow$ some circle of radius e^x shifted right 1



So the point closest to the origin here is $e^x - 1$!

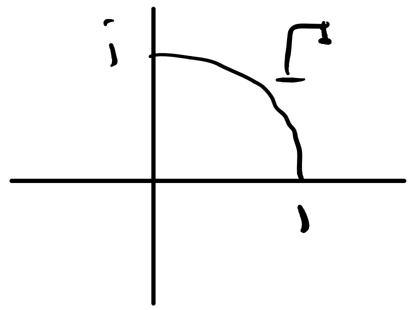
Since $x = R$

$|1+e^z| = e^R - 1$

Simplifying

$$\left| \int_\gamma \frac{e^{3z}}{1+e^z} dz \right| \leq \frac{e^{3R}}{e^R - 1} 2\pi$$

c) Γ is



$$\left| \int_\Gamma \operatorname{Log} z \, dz \right| \leq \frac{\pi^2}{4}$$

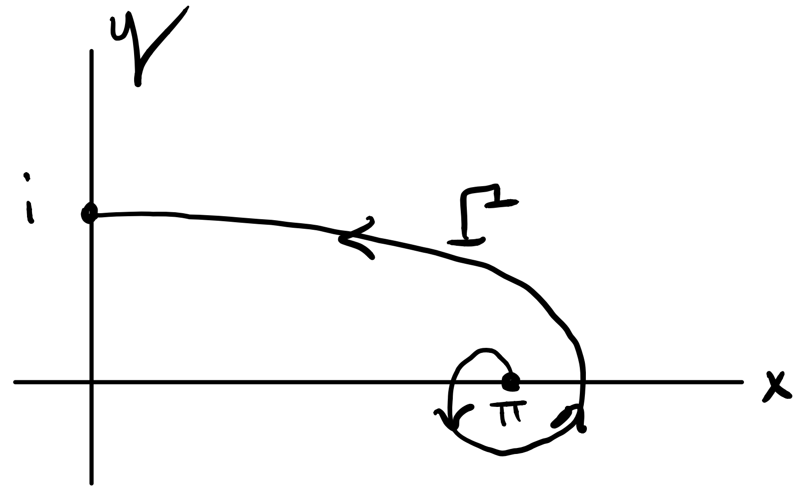
Our path length is $\dfrac{2\pi R}{4} = \dfrac{\pi}{2}$

$\max\limits_{z \in \Gamma} |\operatorname{Log} z| \longrightarrow$ Log z (for $r=1$) is maximized at the max angle, in this case $\pi/2$.

So

$$\left| \int_\Gamma \operatorname{Log} z \, dz \right| \leq \frac{\pi}{2} \cdot \frac{\pi}{2} = \frac{\pi^2}{4}$$

4.3



1e) $\int_{\Gamma} \sin^2 z \cos z \, dz$

$u = \sin(z)$

$du = \cos(z) dz$

$$\int_{\sin(\pi)}^{\sin(i)} u^2 \, du = \frac{1}{3} u^3 \Big|_0^{\sin(i)}$$

$\sin(i) = i \sinh(1)$

$$= \frac{1}{3} \sin^3(i)$$

$$= \frac{i^3}{3} \sinh^3(1)$$

$$\boxed{= \frac{-i}{3} \sinh^3(1)}$$

1g) $\int_{\Gamma} z^{1/2}\, dz = \frac{2}{3} z^{3/2} \Big|_{\pi}^{i} = \frac{2}{3}\left(i^{3/2} - \pi^{3/2}\right)$

$$i^{3/2} = -\frac{\sqrt{2}}{3} + i\frac{\sqrt{2}}{3}$$

$$i = e^{i\frac{\pi}{2}}$$
$$i^{3/2} = e^{i\frac{3\pi}{4}} = -\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} = \frac{-\sqrt{2}}{2} + i\frac{\sqrt{2}}{3}$$

So,

$$\int_{\Gamma} z^{1/2}\, dz = \frac{2}{3}\left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} - \pi^{3/2}\right)$$

$$= -\frac{\sqrt{2}}{3} - \frac{2}{3}\pi^{3/2} + i\frac{\sqrt{2}}{3}$$

4) If f is analytic at each point of a closed contour Γ

Then

$$\int_{\Gamma} f(z)\, dz = 0$$

FALSE: Region enclosed must be analytic as well

7) Show that if C is a positively oriented circle and z_0 lies outside C, then

$$\int_C \frac{1}{z-z_0}\,dz = 0$$

Since z_0 lies outside C then $\frac{1}{z-z_0}$ is analytic on and in C so the integral goes to zero by Cauchy's Theorem.