Recall

Derived Distributions: given $f_X(x)$, find $f_Y(y)$ when $Y = g(X)$.

Not easy in general - looked at a few easy cases.

Next, suppose X, Y jointly continuous w/ joint pdf $f_{X,Y}(x,y)$

<u>Goal</u>: find $f_Z(z)$ where $Z = g(X,Y)$

- Find $F_Z(z)$
- $f_Z(z) = \frac{d}{dz} F_Z(z)$

Not easy in general, but...

Example - $X \sim Y \sim \text{Uniform}[0,1]$

$X, Y \sim \text{Uniform}[0,1]$; X, Y independent.

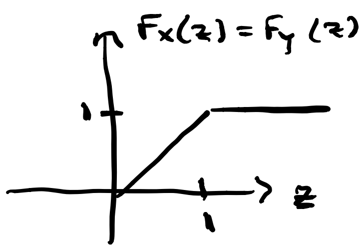
Let $Z = \max\{X, Y\}$

Then

$$F_Z(z) = \mathbb{P}(\{Z \leq z\}) = \mathbb{P}(\{X \leq z\} \cap \{Y \leq z\}) = \mathbb{P}(\{X \leq z\})\,\mathbb{P}(\{Y \leq z\})$$

So

$$\mathbb{P}(\{X \leq z\}) = \mathbb{P}(\{Y \leq z\}) = \begin{cases} z, & z \in [0,1] \\ 0, & \text{else} \end{cases}$$



Hence

$$F_Z(z) = \begin{cases} 0, & z < 0 \\ z^2, & 0 \leq z \leq 1 \\ 1, & 1 < z \end{cases} \Rightarrow f_Z(z) = \begin{cases} 0, & z < 0 \\ 2z, & z \in [0,1] \\ 0, & z > 1 \end{cases}$$

Example – $X, Y \sim \text{exponential}(\lambda)$

X, Y independent.

$Z = g(X,Y) = X - Y$

want $f_Z(z)$.

$$F_Z(z) = \mathbb{P}(\{Z \le z\}) = \mathbb{P}(\{X - Y \le z\})$$

$$\text{independence} \Rightarrow f_{X,Y}(x,y) = \begin{cases} \lambda^2 e^{-(x+y)} & , \quad x>0, y>0 \\ 0 & , \quad \text{else} \end{cases}$$

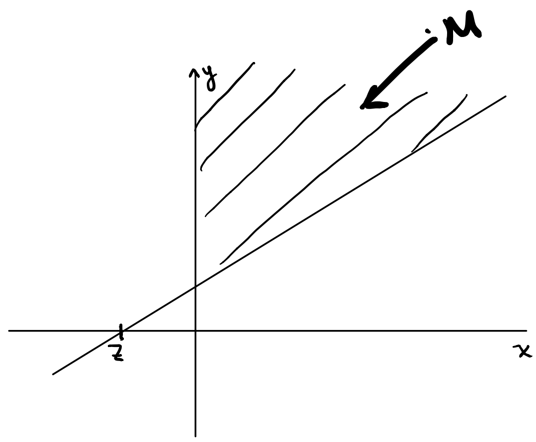
The set where

$$x - y \le z$$

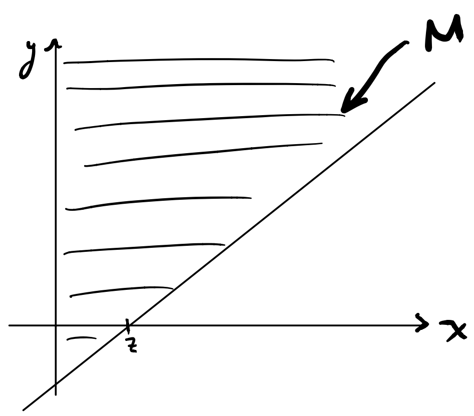
is the same as

$$y \geq x - z \qquad , \quad x = \text{anything}$$

when $z < 0$



when $z > 0$



Thus

$$F_Z(z) = \iint_{\substack{\text{shaded} \\ \text{region}}} f_{X,Y}(x,y)\, dx\, dy$$

When $z<0$,

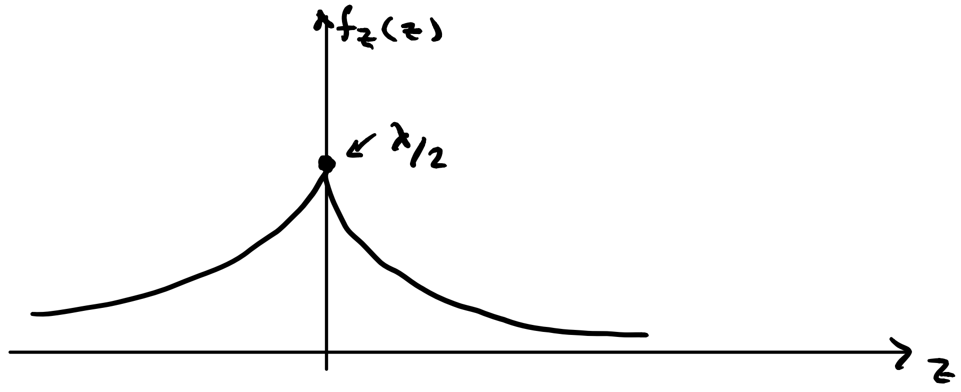
$$F_Z(z) = \int_0^{\infty} dx \int_{x-z}^{\infty} dy \left(\lambda^2 e^{-\lambda(x+y)}\right) = \frac{1}{2} e^{\lambda z}$$

Similarly, when $z \geq 0$,

$$F_Z(z) = \frac{1}{2} - \frac{1}{2} e^{-\lambda z}$$

In conclusion,

$$f_Z(z) = \begin{cases} \frac{\lambda}{2} e^{\lambda z} & , \ z<0 \\ \frac{\lambda}{2} e^{-\lambda z} & , \ z \geq 0 \end{cases} = \frac{\lambda}{2} e^{-\lambda|z|}$$



In both of these examples, X, Y independent - consider X, Y independent; respective pdfs $f_X(x)$, $f_Y(y)$; find $f_Z(z)$ when $Z = X+Y$.

Instead of following recipe, recall that (saw on problem 1, HW IX) even when X, Y not independent,

$$f_Z(z) = \int_{-\infty}^{+\infty} f_{X,Y}(x, z-x)\, dx$$

Independent $X, Y \Rightarrow f_{X,Y}(x,y) = f_X(x) f_Y(y) \Rightarrow f_{X,Y}(x, z-x) = f_X(x) f_Y(z-x)$

Conclude: when X, Y independent and $Z = X+Y$,

$$f_Z(z) = \int_{-\infty}^{+\infty} f_X(x) f_Y(z-x)\, dx$$ ← **always** works

Could redo difference of exponentials example using this!

$Z = X - Y = X + [-Y]$; X, Y independent.

Suppose however X, Y independent $N(\mu_1, \sigma_1^2)$, $N(\mu_2, \sigma_2^2)$ random variables respectively.

Gaussian

Consider $Z = X+Y$.

Know immediately

$$\mathbb{E}(Z) = \mu_1 + \mu_2 \qquad Var(Z) = \sigma_1^2 + \sigma_2^2$$

But what is $f_Z(z)$?

$f_Z(z)$ = combo of $f_X(x)$, $f_Y(y)$ = some **HIDEOUS** integral

Miraculously, (tedious to show), Z is also Gaussian!

Thus

$$f_Z(z) = \frac{1}{\sqrt{2\pi(\sigma_1^2 + \sigma_2^2)}} e^{-\frac{(z-(\mu_1+\mu_2))^2}{2(\sigma_1^2+\sigma_2^2)}}$$

Similarly, for any α, β not both 0, $Z = \alpha X + \beta Y$ is Gaussian, mean $\mu = \alpha\mu_1 + \beta\mu_2$ and variance $\sigma^2 = \alpha^2\sigma_1^2 + \beta^2\sigma_2^2$

Mantra: Any nontrivial linear combo of independent Gaussians is also Gaussian.

Turns out there's a discrete version of convolution thing.

Suppose X, Y are integer-valued random variables w/ a joint pmf $p_{X,Y}(m,n)$.

Let $Z = X + Y$.

Turns out

$$p_Z(k) = \sum_{m=-\infty}^{+\infty} p_{X,Y}(m, k-m) \qquad \forall k$$

If in addition X, Y independent, so that $p_{X,Y}(x,y) = p_X(x)\, p_Y(y)$

$$p_Z(k) = \sum_{m=-\infty}^{+\infty} p_X(m)\, p_Y(k-m)$$ ← convolution of marginal pmfs