A series is a formal expression of the form $c_0 + c_1 + c_2 + \cdots$ ($\sum_{j=0}^{\infty} c_j$) where the terms c_j are complex numbers.

The n^{th} partial sum of the series, S_n, is

$$S_n := \sum_{j=0}^{n} c_j \qquad (\text{sums first } n+1 \text{ terms})$$

If the sequence of partial sums $\{S_n\}_{n=0}^{\infty}$ has a limit S, the series is said to converge to S, and we write

$$S = \sum_{j=0}^{\infty} c_j .$$

If it doesn't converge we say it diverges.

The series

$$\sum_{j=0}^{\infty} c^j$$

converges to

$$\frac{1}{1-c}$$

if $|c|<1$;

that is,

$$1 + c + c^2 + c^3 + \cdots = \frac{1}{1-c} \quad \text{if } |c| < 1$$

PROOF: Observe that

$$(1-c)(1 + c + c^2 + \cdots + c^{n-1} + c^n)$$

$$= 1 + c + c^2 + \cdots + c^{n-1} + c^n$$
$$\quad - c - c^2 - \cdots - c^{n-1} - c^n - c^{n+1}$$

$$= 1 - c^{n+1}$$

Rearrangin this yields

$$\frac{1}{1-c} - (1 + c + c^2 + \cdots + c^{n-1} + c^n) = \frac{c^{n+1}}{1-c}$$

Since $|c| < 1$ the series converges!

The "remainder"

$$\frac{c^{n+1}}{1-c}$$

approaches zero as $n \to \infty$

COMPARISON TEST

If the terms satisfy the inequality

$$|c_j| \leq M_j$$

for all integers j larger than some number J.

Then if the series

$$\sum_{j=0}^{\infty} M_j \quad \text{converges so does} \quad \sum_{j=0}^{\infty} c_j$$

Example 1: Show $\sum_{j=0}^{\infty} \frac{(3+2i)}{(j+1)^j}$ converges.

$$\sum_{j=0}^{\infty} \frac{3+2i}{(j+1)^j} = (3+2i) + \frac{(3+2i)}{2} + \frac{(3+2i)}{9} + \frac{(3+2i)}{64} + \cdots$$

Compare this with the convergent geometric series

$$\sum_{j=0}^{\infty} \frac{1}{2^j} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

Since $|3+2i| = \sqrt{13} < 4$ it's easy to verify that

$$\left|\frac{3+2i}{(j+1)^j}\right| < \frac{4}{(j+1)^j}$$

and that this is less then $\frac{1}{2^j}$ for $j \geq 3$.

thus the series converges.

RATIO TEST

Suppose that the terms of the series

$\sum_{j=0}^{\infty} c_j$ have the property that the

ratios $\left|\frac{c_{j+1}}{c_j}\right|$ approach a limit L as

$j \to \infty$. If $L<1$ the seris converges
If $L>1$ the series diverges

Example 2: Show $\sum_{j=0}^{\infty} \frac{4^j}{j!}$ converges

We have

$$\left|\frac{c_{j+1}}{c_j}\right| = \frac{4^{j+1}}{(j+1)!} \cdot \frac{j!}{4^j} = \frac{4}{j+1}$$

$\lim_{j \to \infty} \frac{4}{j+1} = 0 < 1$ $\therefore$ the series converges

A series $\sum_{j=0}^{\infty}$ is said to be absolutely convergent if the series $\sum_{j=0}^{\infty} |c_j|$ converges.

Any absolutely convergent series is convergent by the comparison test.

Example 3: If $z_0 \neq 0$ is fixed, show that

$\sum_{j=0}^{\infty} \left(\frac{z}{z_0}\right)^j$ converges for $|z| < |z_0|$

if $|z| < z_0$
then $\left|\frac{z}{z_0}\right| < 1$

So,

$$\sum_{j=0}^{\infty} \left(\frac{z}{z_0}\right)^j = \frac{1}{1-\frac{z}{z_0}} \quad !$$

The sequence
$$\{F_n(z)\}_{n=1}^{\infty}$$

is said to converge uniformally to $F(z)$ on the set T if for any $\varepsilon > 0$ there exists an integer N such that when $n > N$,

$$|F(z) - F_n(z)| < \varepsilon \text{ for all } z \text{ in } T$$

Accordingly, the series $\sum_{j=0}^{n} f_j(z)$ converges uniformally to $f(z)$ on T if the sequence of its partial sums converges uniformly to $f(z)$ there.

Example 4: Show that the series

$$\sum_{j=0}^{\infty} \left(\frac{z}{z_0}\right)^j \text{ is uniformally convergent}$$

in every closed disk $|z| \leq r$, if $r < |z_0|$

Given $\varepsilon > 0$, we have to show that the remainder after $n+1$ terms will be less than ε for all z in the disk, when n is large enough.

$$\left| \frac{(z/z_0)^{n+1}}{1-(z/z_0)} \right| \le \frac{(r/|z_0|)^{n+1}}{1-r/|z_0} \quad \text{for } |z| \le r$$

This can be made arbitrarily small since $r < z_0$.