ECE 4110 Homework 7

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Due by 5pm on November 14

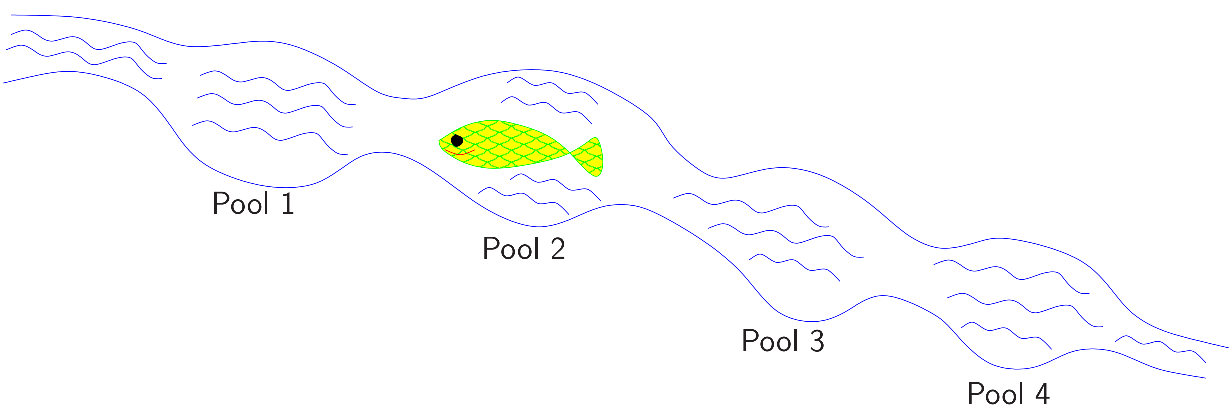
1 Reading Material

- The n-step transition probabilities and the state probabilities (Chapter 11.2.1 and 11.2.2).
- Stationary distribution (Chapter 11.2.3).
- Communication classes (Chapter 11.3.1).

2 Assignment

1. **A Fish Called Wanda**

   A fish called Wanda is swimming among 4 pools (see the figure below). When Wanda changes pools (at discrete time instants), she goes downstream with probability p and upstream with probability $1-p$ when she has a choice. At pool 1, she stays with probability $1-p$ and moves to pool 2 with probability p. At pool 4, she stays with probability p and moves to pool 3 with probability $1-p$.



   (a) Suppose that Wanda is initially in Pool 1. Find out the distribution of Wanda's location after two time instants.

   (b) Find the stationary distribution of Wanda's location.

2. **Ferris Bueller's Day Off**

   Three out of every four trucks on the road are followed by a car, while one out of every five cars is followed by a truck. What fraction of vehicles on the road are trucks?

   Hint: Imagine sitting on the side of the road watching vehicles go by (sounds more interesting than going to a class on random processes!). Set the flow of the vehicles as a Markov chain.

3. **The Ehrenfest Model of Diffusion**

   A cubical volume is partitioned into two equal-sized chambers, a left chamber and a right chamber, by a barrier containing a small hole. In this volume there are M gas molecules. Suppose that at time 0, all of the molecules are in the left chamber. At each subsequent discrete time, one of the molecules, uniformly chosen from among the M, switches chambers. Let X_n denote the number of molecules in the left chamber at time n.

   (a) Argue that $\{X_n\}_{n\geq 0}$ is Markov and draw its state transition diagram.

   (b) Show that $\pi = [\pi_0, \dots, \pi_M]$ with

   $$\pi_i = 2^{-M}\binom{M}{i}, \qquad i = 0, \dots, M$$

   is a stationary distribution of the chain.

4. **Binomial Counting Process as a Markov Chain**

   Consider the Binomial counting process with parameter p.

   (a) Show that it is a Markov chain and write out the transition matrix.

   (b) What are the communication classes? Is the chain irreducible?

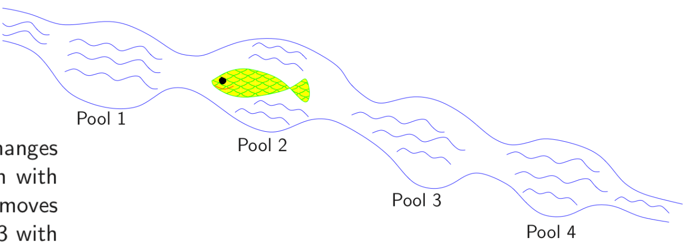
5. **Communication Class**

   For each of the Markov chains with transition probability matrices given below, draw the state transition diagrams and identify all communication classes. Is the Markov chain irreducible?

$$\mathbf{P}_1 = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 0 & 0 \end{pmatrix} \quad \mathbf{P}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \mathbf{P}_3 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

$$\mathbf{P}_4 = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \mathbf{P}_5 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix} \quad \mathbf{P}_6 == \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

① A Fish Called Wanda



A fish called Wanda is swimming among 4 pools (see the figure below). When Wanda changes pools (at discrete time instants), she goes downstream with probability p and upstream with probability $1-p$ when she has a choice. At pool 1, she stays with probability $1-p$ and moves to pool 2 with probability p. At pool 4, she stays with probability p and moves to pool 3 with probability $1-p$.

(a) Suppose that Wanda is initially in pool 1. Find out the distribution of Wanda's location after two time instants

Scenarios

$$\text{pool } 1 \xrightarrow{1-p} \text{pool } 1 \xrightarrow{1-p} \text{pool } 1$$

$$\text{pool } 1 \xrightarrow{p} \text{pool } 2 \xrightarrow{1-p} \text{pool } 1$$

$$\text{pool } 1 \xrightarrow{1-p} \text{pool } 1 \xrightarrow{p} \text{pool } 2$$

$$\text{pool } 1 \xrightarrow{p} \text{pool } 2 \xrightarrow{p} \text{pool } 3$$

$$\text{Wanda is at } \begin{cases} \text{pool } 1 \text{ w.p. } (1-p)^2 + p(1-p) \\ \text{pool } 2 \text{ w.p. } (1-p)p \\ \text{pool } 3 \text{ w.p. } p^2 \end{cases}$$

We verify these probabilities sum to 1, since Wanda can only be at these 3 locations.

$$\rightarrow (1-p)^2 + p(1-p) + (1-p)p + p^2$$
$$= 1 - 2p + p^2 + p - p^2 + p - p^2 + p^2 = 1$$

✓ Nice

It's nice going through cases and all, but now we do this more cleverly.

$$p(2) = p(0)\mathbb{P}^2$$

where

$$\mathbb{P} = \begin{bmatrix} 1-p & p & 0 & 0 \\ 1-p & 0 & p & 0 \\ 0 & 1-p & 0 & p \\ 0 & 0 & 1-p & p \end{bmatrix}$$

is the transition matrix

$$\mathbb{P}^2 = \begin{bmatrix} (1-p)^2 + p(1-p) & (1-p)p & p^2 & 0 \\ (1-p)^2 & 2p(1-p) & 0 & p^2 \\ (1-p)^2 & 0 & 2p(1-p) & p^2 \\ 0 & (1-p)^2 & p(1-p) & p(1-p) + p^2 \end{bmatrix}$$

$$p(0) = [1 \quad 0 \quad 0 \quad 0]$$

$$p(0)\mathbb{P}^2 = (1-p)^2 + p(1-p) + p(1-p) + p^2$$

the pmf at time 2 s obtained by multiplying initial state by $\mathbb{P}^2$.

(b) Find the Stationary distribution of Wanda's location

Need

Know $\pi \mathbb{P} = \pi$

and thus

$$\mathbb{P}^T \pi^T = \pi^T$$

$$\mathbb{P} = \begin{bmatrix} 1-p & p & 0 & 0 \\ 1-p & 0 & p & 0 \\ 0 & 1-p & 0 & p \\ 0 & 0 & 1-p & p \end{bmatrix}$$

i.e $\mathbb{P}^T$ has eigenvalue 1 w/ eigenvector π^T.

$$(\mathbb{P}^T - \lambda I) = 0 \qquad \lambda = 1$$

$$\mathbb{P}^T - I = 0$$

i.e want $E_{\lambda=1}(\mathbb{P}^T) = \ker(\mathbb{P}^T - I)$

$$= \pi^T$$

$$\mathbb{P}^T - I = \begin{bmatrix} -p & 1-p & 0 & 0 \\ p & -1 & 1-p & 0 \\ 0 & p & -1 & 1-p \\ 0 & 0 & p & p-1 \end{bmatrix} \underbrace{\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}}_{\pi^T} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\mathbb{P}^T \pi = 0$ Want non-trivial solution

$$\pi = [a \quad b \quad c \quad d]$$

$$-pa + (1-p)b = 0$$
$$pa - b + (1-p)c = 0$$
$$pb - c + (1-p)d = 0$$
$$pc + (p-1)d = 0$$

$$(1-p)\,b = pa \qquad a = \frac{1-p}{p}\,b$$

$$p\left(\frac{1-p}{p}\right)b - b + (1-p)\,c = 0$$

$$(1-p)\,b - b + (1-p)\,c = 0$$

$$b(1-p-1) + c(1-p) = 0$$

$$c = b\,\frac{p}{1-p}$$

$$p\left(\frac{p}{1-p}\right)b + (p-1)\,d = 0$$

$$d = \frac{-p^2 b}{(1-p)(p-1)} = \frac{p^2}{(1-p)^2}\,b$$

$$\therefore \quad \boxed{\bar{\pi} = \left[\frac{1-p}{p}b \quad b \quad \frac{p}{1-p}b \quad \frac{p^2}{(1-p)^2}b\right]}$$

Now

$$\frac{1-p}{p}b + b + \frac{p}{1-p}b + \frac{p^2}{(1-p)^2} = 1$$

$$b\left(1 + \frac{1-p}{p} + \frac{p}{1-p} + \frac{p^2}{(1-p)^2}\right) = 1$$

$$\boxed{b = \frac{1}{1 + \frac{1-p}{p} + \frac{p}{1-p} + \frac{p^2}{(1-p)^2}}}$$

2. **Ferris Bueller's Day Off**

Three out of every four trucks on the road are followed by a car, while one out of every five cars is followed by a truck. What fraction of vehicles on the road are trucks?

Hint: Imagine sitting on the side of the road watching vehicles go by (sounds more interesting than going to a class on random processes!). Set the flow of the vehicles as a Markov chain.

Think of sampling infinite flow at some random time to get initial state

$$\mathbb{P} = \begin{array}{c c} & \begin{array}{cc} T & C \end{array} \\ \begin{array}{c} T \\ C \end{array} & \begin{bmatrix} 1/4 & 3/4 \\ 1/5 & 4/5 \end{bmatrix} \end{array}$$

Macaulay2

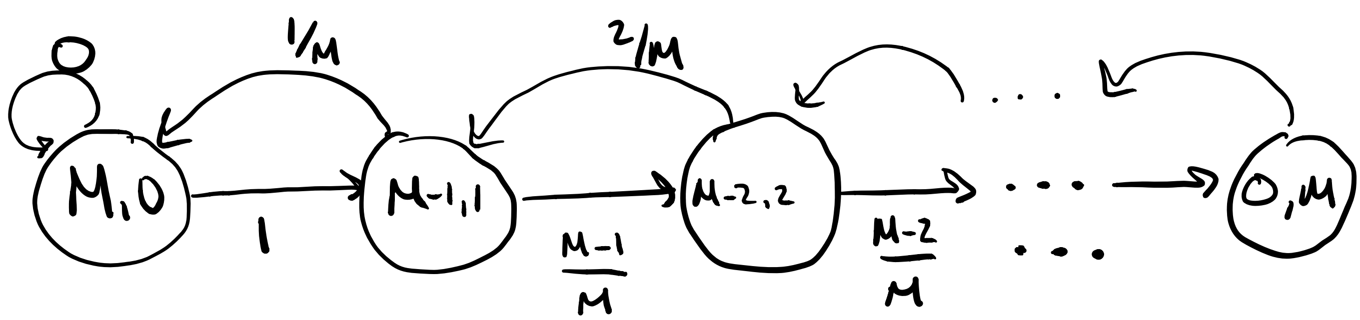
$$\lim_{n\to\infty} \mathbb{P}^n = \begin{bmatrix} .210526 & .789474 \\ .210526 & .789474 \end{bmatrix}$$

3. **The Ehrenfest Model of Diffusion**

A cubical volume is partitioned into two equal-sized chambers, a left chamber and a right chamber, by a barrier containing a small hole. In this volume there are M gas molecules. Suppose that at time 0, all of the molecules are in the left chamber. At each subsequent discrete time, one of the molecules, uniformly chosen from among the M, switches chambers. Let X_n denote the number of molecules in the left chamber at time n.

(a) Argue that $\{X_n\}_{n\geq 0}$ is Markov and draw a state-transition diagram.

The state diagram is as follows



$\{X_n\}_{n\geq 0}$ is intuitively Markov.

Let $\{L_n\}$ denote amount in left half

$\{R_n\}$ denote amount in right half

Then

$$X_n = (L_n, R_n)$$

$$X_{n+1} = \begin{cases} (L_n - 1, R_n + 1) & \text{w.p. } \frac{L_n}{M} \\ (L_n + 1, R_n - 1) & \text{w.p. } \frac{R_n}{M} \end{cases}$$

independence of uniform selection $\Rightarrow X_n \perp\!\!\!\perp X_{n+1}$.

(b) Show that $\pi = [\pi_0, \dots, \pi_m]$ with

$$\pi_i = 2^{-M} \binom{M}{i} \quad i = 0, \dots, M$$

is a stationary distribution of the chain.

$$\mathbb{P} = \begin{bmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ \frac{1}{M} & 0 & \frac{M-1}{M} & \cdots & \cdots & 0 \\ 0 & \frac{2}{M} & 0 & \frac{M-2}{M} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$(m+1) \times (m+1)$ ~~matrix~~ matrix

Need $\bar{\pi} \mathbb{P} = \bar{\pi}$ $\qquad 2^{-M} \binom{M}{i}$

Base case, $M = 1$

$$\mathbb{P} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$\pi_1 = \frac{1}{2}$, $\pi_2 = 0$ $\qquad \bar{\pi}$ is length $M+1$

$$\bar{\pi} \mathbb{P} \stackrel{?}{=} \bar{\pi} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} \checkmark$$

Assume true for $M = k \geq 1$

Need to show true for $M = k+1$.

True for $m = k$

$$\mathbb{P}_{k+1,k+1} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \frac{1}{k} & 0 & \frac{k-1}{k} & \cdots & 0 \\ 0 & \frac{2}{k} & 0 & \frac{k-2}{k} \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots 1 & 0 \end{bmatrix}$$

$$\bar{\pi}\mathbb{P} = \bar{\pi} \quad \text{true by assumption}$$

thus

$$\bar{\pi} = \left[2^{-M}\binom{k}{1} \quad 2^{-M}\binom{k}{2} \quad \cdots \quad 2^{-M}\binom{k}{k+1}\right]$$

For $M = k+1$,

$$\bar{\pi} = \left[2^{-M}\binom{k}{1} \quad \cdots \quad 2^{-M}\binom{k}{k+1} \quad 2^{-M}\binom{k+1}{k+2}\right]$$

$\mathbb{P}_{k+2,k+2}$ has same structure as $\mathbb{P}_{k,k}$ but one more row and column since one more molecule.

It follows that $\bar{\pi}_{k+2} = \bar{\pi}_{k+2}\mathbb{P}_{k+2,k+2}$.

(Could have really done nitty gritty but quite intuitive to see).

4. **Binomial Counting Process as a Markov Chain**

Consider the Binomial counting process with parameter p.

(a) Show that this is a Markov chain and write out the transition matrix.

$$X_{n+1} \mid X_n, X_{n-1}, \dots, X_1 \sim \text{Bernoulli}(p)$$

$$X_{n+1} \mid X_n \sim \text{Bernoulli}(p)$$

Thus $\Pr[X_{n+1} = x_{n+1} \mid X_n = x_n, \dots, X_1 = x_1] = \Pr[X_{n+1} = x_{n+1}] = \Pr[X_{n+1} = x_{n+1} \mid X_n = x_n]$

$$\mathbb{P} = \begin{bmatrix} 1-p & p & 0 & \cdots & & 0 & \cdots \\ 0 & 1-p & p & 0 & \cdots & 0 & \cdots \\ 0 & 0 & \ddots & \ddots & \cdots & & \\ 0 & 0 & 0 & 1-p & p & \cdots & \\ 0 & 0 & 0 & 0 & \ddots & \ddots & \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \end{bmatrix}$$

(b) What are the communication classes? Is the chain irreducible?

Communication classes are

$$\{0\}, \{1\}, \{2\}, \dots$$

thus the chain is **NOT** reducible.

5. **Communication Class**

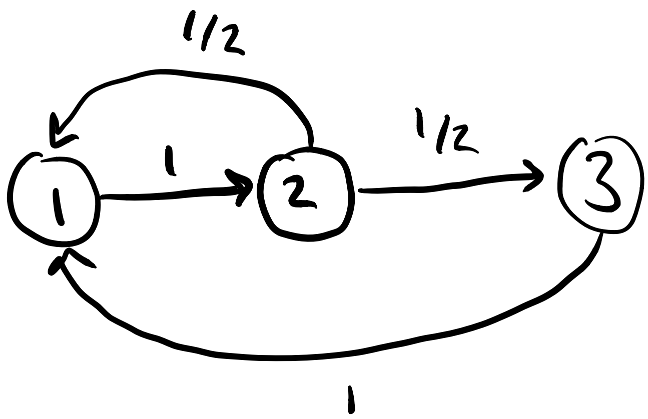
For each of the Markov chains with transition probability matrices given below, draw the state transition diagrams and identify all communication classes. Is the Markov chain irreducible?

$$\mathbf{P}_1 = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 0 & 0 \end{pmatrix} \quad \mathbf{P}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \mathbf{P}_3 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

$$\mathbf{P}_4 = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \mathbf{P}_5 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix} \quad \mathbf{P}_6 == \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

Communication Classes

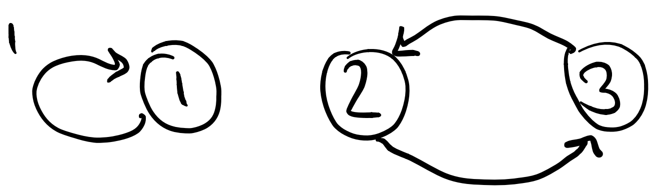
$\underline{P_1}$



$\{1, 2, 3\}$

$\therefore$ irreducible

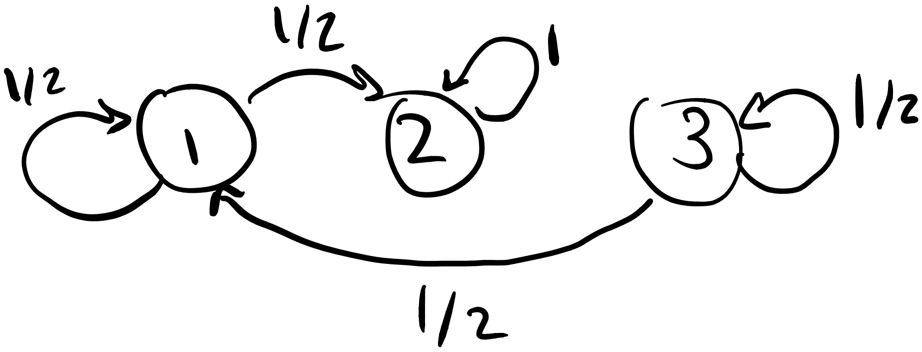
$\underline{P_2}$



$\{1\}, \{2, 3\}$

$\therefore$ not irreducible

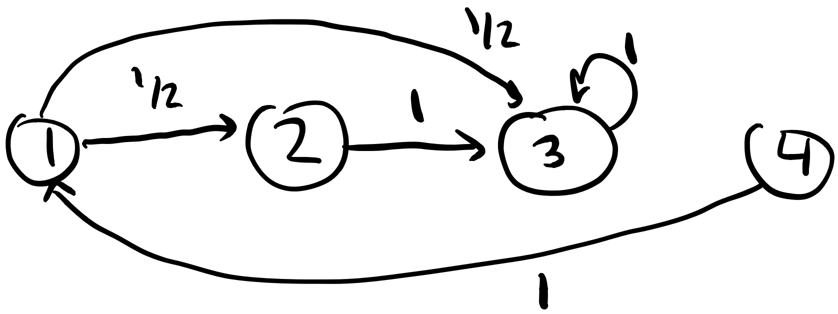
P_3



$\{1\}, \{2\}, \{3\}$

$\therefore$ not irreducible

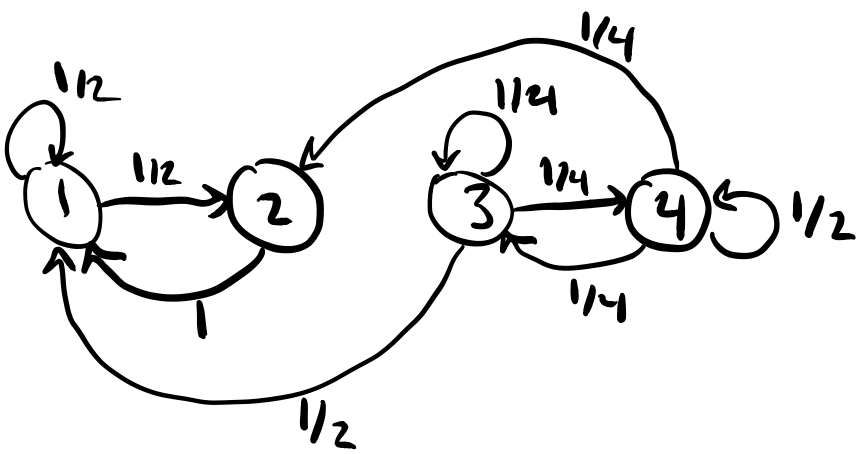
P_4



$\{1\}, \{2\}, \{3\}, \{4\}$

$\therefore$ not irreducible

P_5

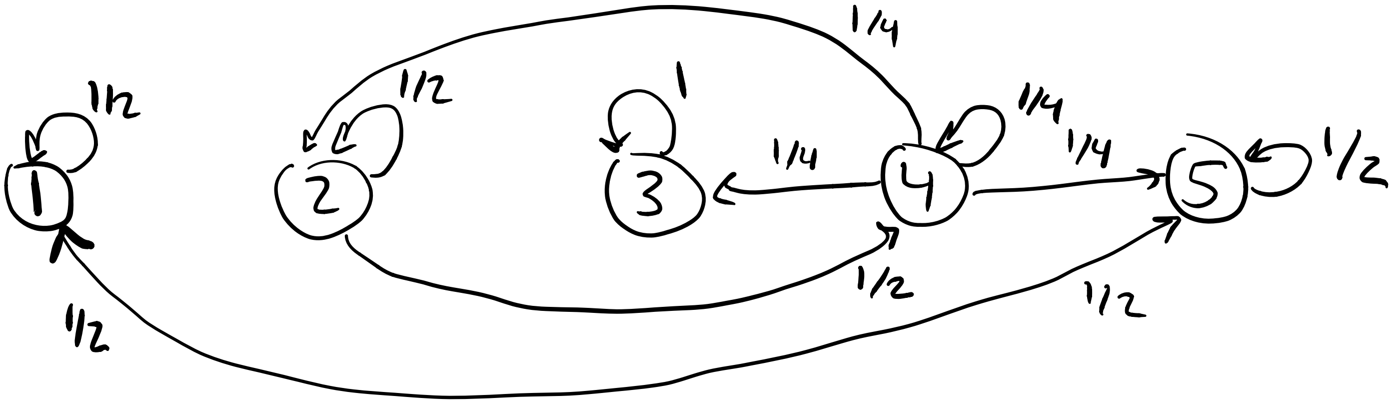


$\{1,2\}$

$\{3,4\}$

$\therefore$ not irreducible

P_6



$\{1\}, \{2,4\}, \{3\}, \{5\}$

$\therefore$ not reducible