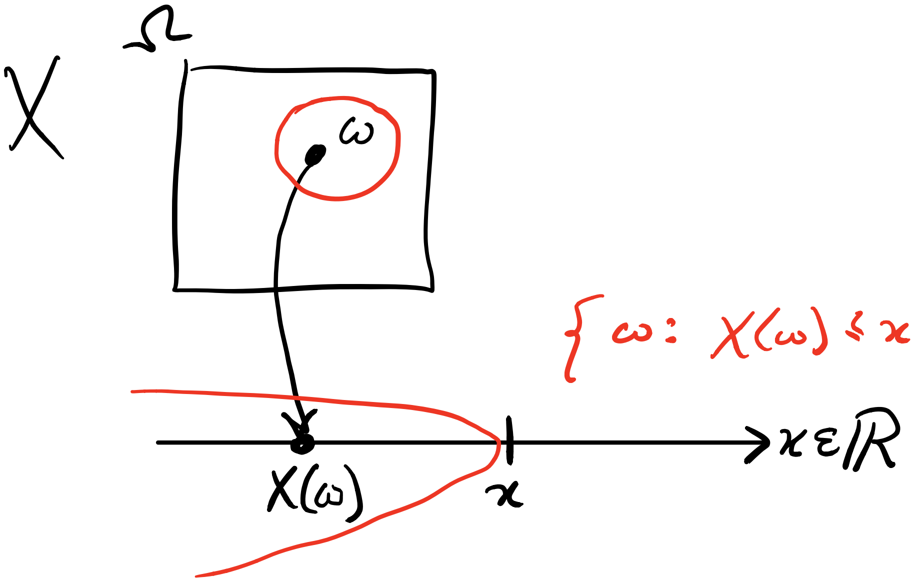
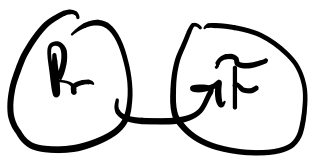
A r.v. $X : \Omega \to \mathbb{R}$



$\{\omega : X(\omega) \leq x\} \subseteq \Omega$

Example: Throwing a Die

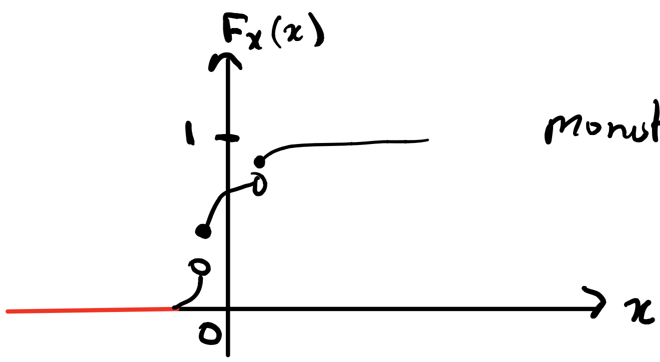
$$\begin{array}{cccccccc} & & \tfrac{1}{6} & \tfrac{1}{6} & \tfrac{1}{6} & \tfrac{1}{6} & \tfrac{1}{6} & \tfrac{1}{6} \\ \Omega & = & \{1, & 2, & 3, & 4, & 5, & 6\} \\ X & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ & & -5 & 5 & -5 & 5 & -5 & 5 \end{array}$$



$$X = \begin{cases} 5 & Pr(5) = 1/2 \\ -5 & Pr(-5) = \frac{1}{2} \end{cases}$$

CDF

$F_X(x) = Pr[X \leq x] \quad \forall\, x \in \mathbb{R}$



monotonically increasing ; $\forall\, x < y$ $\quad F_X(x) \leq F_Y(y)$

like die CDF example for equality (last lecture)

Properties of CDF

① $\lim_{x \to \infty} F_X(x) = 0$

$\lim_{x \to -\infty} F_X(x) = 1$

② $\forall\, x < y \quad F_X(x) \le F_X(y)$

③ F is right continuous

$$\lim_{x \to x_0^+} F_X(x) = F_X(x_0)$$

④ $Pr[x \le X \le y]$

$= F_X(y) - F_X(x)$

PMF for Discrete Random Variables

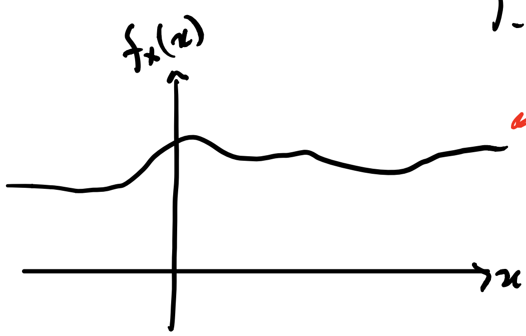
$$P_X(x) = Pr[X = x]$$

$$F_X(x) = \sum_{u \le x} P_X(u)$$

PDF for Continuous Random Variables

$$f_X(x) = \frac{dF_X(x)}{dx}$$ } measures how **fast** we accumulate probability

Thus $F_X(x) = \int_{-\infty}^{x} f_X(u)\, du$



Must be non-negative

Properties of PDF

① $f_X(x) \geq 0 \quad \forall x$

② $\int_{-\infty}^{+\infty} f_X(x) = 1 = F_X(\infty)$

③ $\Pr[x \leq X \leq y] = \int_x^y f_X(u)\, du$

Expectation

$$\mathbb{E}[X] = \begin{cases} \sum\limits_k k \Pr[X=k] , & X \text{ discrete} \\ \int_{-\infty}^{+\infty} x f_X(x)\, dx , & X \text{ continuous} \end{cases}$$

Properties

LOTUS Rule

If

$$Y = g(X)$$

$f_X(X) \xrightarrow{g}$

then

$$\mathbb{E}[Y] = \int_{-\infty}^{+\infty} x f_Y(y)\, dy$$

$$\mathbb{E}[g(X)] = \int_{-\infty}^{+\infty} g(x) f_X(x)\, dx$$

Linearity of Expectation

$$\mathbb{E}[\alpha X + \beta Y] = \alpha \mathbb{E}[X] + \beta \mathbb{E}[Y], \quad \alpha, \beta \in \mathbb{R}$$

Preservation of Order

If

$$\Pr[X \geq Y] = 1$$

then

$$\mathbb{E}[X] \geq \mathbb{E}[Y]$$

Integration by Parts

Simple Case

A discrete, non-negative r.v. $X = 0, 1, 2, 3, \ldots$

$$\mathbb{E}[X] = \sum_{i=0}^{\infty} i \Pr(X=i)$$

$$= \sum_{i=0}^{\infty} \underbrace{\Pr(X > i)}_{1 - F_X(i)} \quad \leftarrow \text{Tail Probability}$$

To see this, observe

$$\sum_{i=0}^{\infty} i \Pr(X=i) = 0 \cdot \Pr(X=0) + 1 \cdot \Pr(X=1) + 2 \cdot \Pr(X=2) + \cdots$$

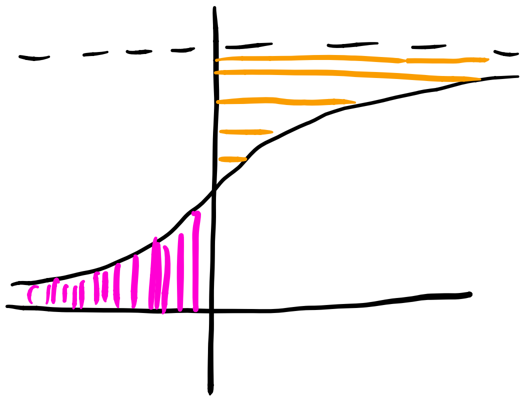
$$\sum_{i=0}^{\infty} \Pr(X > i) \qquad \begin{array}{l} i=0 \quad \checkmark \quad \checkmark \\ i=1 \quad \checkmark \\ \vdots \\ \text{etc} \end{array}$$

Thus

$$\boxed{\mathbb{E}[X] = \int_0^{\infty} (1 - F_X(x))\,dx - \int_{-\infty}^{0} F_X(x)\,dx}$$

To see this, observe

$$\mathbb{E}[X] = \int_{-\infty}^{+\infty} x \, dF_X(x)$$



$$= \boxed{\int_0^{\infty} x \, dF_X(x)} - \boxed{\int_{-\infty}^{0} (-x) \, dF_X(x)}$$

$$= \int_0^{\infty} \left(1 - F_X(x)\right) dx - \int_{-\infty}^{0} F_X(x) \, dx$$

Variance

How much a r.v varies from its expectation

$$\mathrm{Var}(X) = \mathbb{E}\Big[\overbrace{(X - \mathbb{E}[X])^2}^{g(X)}\Big]$$

$$= \mathbb{E}\left[X^2 - 2\,\mathbb{E}[X] + (\mathbb{E}[X])^2\right]$$

$$= \mathbb{E}[X^2] + (\mathbb{E}[X])^2$$

$$\mathrm{Var}(X) = \int_{-\infty}^{+\infty} g(x) f_X(x) \, dx = \int_{-\infty}^{+\infty} (X - \mathbb{E}[X])^2 f_X(x) \, dx$$

M^{th} Moment

$$\mathbb{E}[X^n] = \int_{-\infty}^{+\infty} x^n f_X(x)\, dx$$

M^{th} Central Moment

$$\mathbb{E}[(X - \mathbb{E}[X])^n]$$

Multiple Random Variables

Joint Distribution of $(X_1, X_2, \ldots, X_n)$

Joint CDF

$$F_{X_1,\ldots,X_n}(x_1,\ldots,x_n) = \Pr[X_1 \leq x_1, \ldots, X_n \leq x_n]$$

Joint PMF

$$p_{X_1,\ldots,X_n}(x_1,\ldots,x_n) = \Pr[X_1 = x_1, \ldots, X_n = x_n]$$

Joint PDF

$$f_{X_1,\ldots,X_n}(x_1,\ldots,x_n) = \frac{\partial^m F_{X_1,\ldots,X_n}(x_1,\ldots,x_n)}{\partial x_1 \cdots \partial x_n}$$

Marginal CDF of X_1

$$F_{X_1} = F_{X_1,\ldots,X_n}(x, \infty, \infty, \ldots, \infty)$$

$$f_{X_1} = \int_{-\infty}^{+\infty} \overset{(n-1)\ \text{integrals}}{\cdots} \int_{-\infty}^{+\infty} f_{X_1,\ldots,X_n}(x_1, x_2, \ldots, x_n)\, dx_2\, dx_3 \cdots dx_n$$

Independence

$$F_{X_1,\dots,X_n}(x_1,\dots,x_n) = F_{X_1}(x_1)\cdots F_{X_n}(x_n)$$

$$\Pr[X_1 \le x_1,\dots, X_n \le x_n] = \Pr[X_1 \le x_1]\cdots\Pr[X_n \le x_n]$$

Conditioning on Random Variables

Suppose X and Y have joint pmf $p_{XY}(x,y)$

The conditional pmf of X given $\{Y=y\}$ is

$$p_{X|Y}(x|y) \triangleq \Pr[\overbrace{X=x}^{A} \mid \overbrace{Y=y}^{B}]$$

$$= \frac{\Pr(\{X=x\}\cap\{Y=y\})}{\Pr(\{Y=y\})} = \frac{p_{X,Y}(x,y)}{p_Y(y)}$$

Similarly for continuous r.v.'s

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} \quad \text{if } f_Y(y)\neq 0 \ \forall y$$

Conditional Expectation

$$\mathbb{E}[X] = \int_{-\infty}^{+\infty} x f_X(x)\, dx$$

$$\mathbb{E}[X \mid Y = y] = \int_{-\infty}^{+\infty} x f_{X \mid \{Y = y\}}(x \mid y)\, dx$$

$\mathbb{E}[X \mid Y]$ is a r.v w/ respect to Y!

i.e a function of Y