

Recall

Continuous r.v's.

r.v X ; pdf $f_x(x)$; $P\{X \in V\} = \int_V f_x(x) dx \quad \forall V \subset \mathbb{R}$

X can be uniform (a,b) ; exponential (λ) , etc.

$$\mathbb{E}(X) = \int_{-\infty}^{+\infty} x f_x(x) dx$$

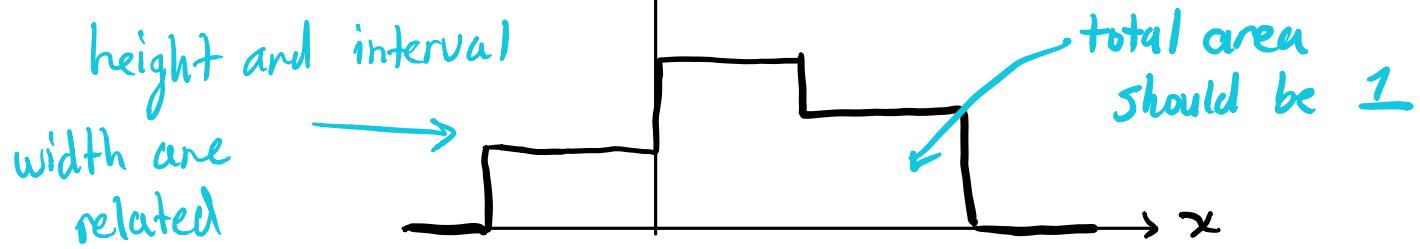
Expected Value Rule:

$$\mathbb{E}(g(X)) = \int_{-\infty}^{+\infty} g(x) f_x(x) dx$$

$$\text{Var}(X) = \mathbb{E}((X - \mathbb{E}(X))^2)$$

Another type of pdf we encounter frequently (especially when doing conditional stuff): **piecewise uniform**

i.e.



Next, define - for ANY rv X (discrete OR continuous) -
the cumulative distribution function (cdf) by

$$F_X(x) = P(\{X \leq x\}) \quad \forall x \in \mathbb{R}$$

Observation: If X is a continuous rv w/ pdf $f_X(x)$, then since

$$P(\{X \leq x\}) = \int_{-\infty}^x f_X(t) dt$$

we have

$$F_X(x) = \int_{-\infty}^x f_X(t) dt \quad \text{and} \quad f_X(x) = \frac{d}{dx} F_X(x)$$

Discrete version: If X is a discrete rv w/ pmf $p_X(x)$ we have

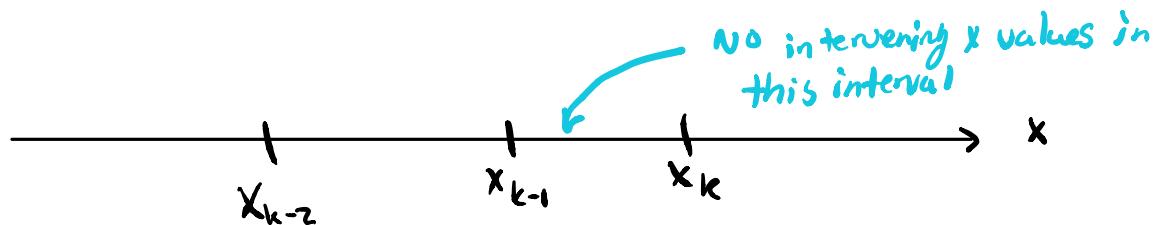
$$F_X(x) = \sum_{\{x_k : x_k \leq x\}} p_X(x_k)$$

set of all possible X -values
that don't exceed x

Can invert this formula to get $p_X(x)$ in terms of $F_X(x)$:

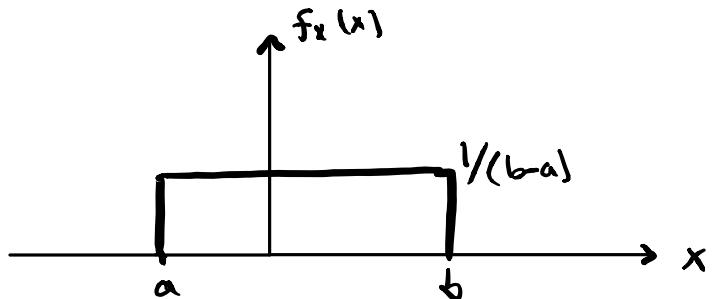
$$p_X(x_k) = F_X(x_k) - F_X(x_{k-1})$$

where x_{k-1} is the "NEXT largest value" of X below x_k .



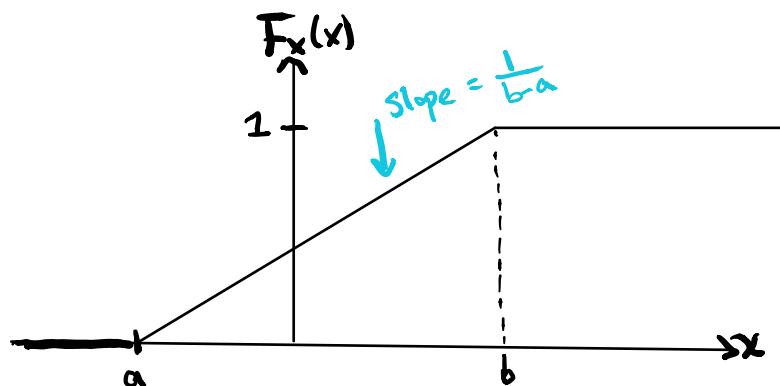
Example - $X \sim \text{Uniform}(a,b)$

$X \sim \text{Uniform}(a,b) \Rightarrow$



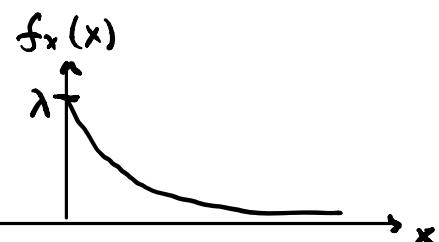
$$F_x(x) = \int_{-\infty}^x f_x(t) dt = \begin{cases} 0, & x < a \\ \frac{x-a}{b-a}, & a \leq x < b \\ 1, & b \leq x \end{cases}$$

inequalities
NOT too strict

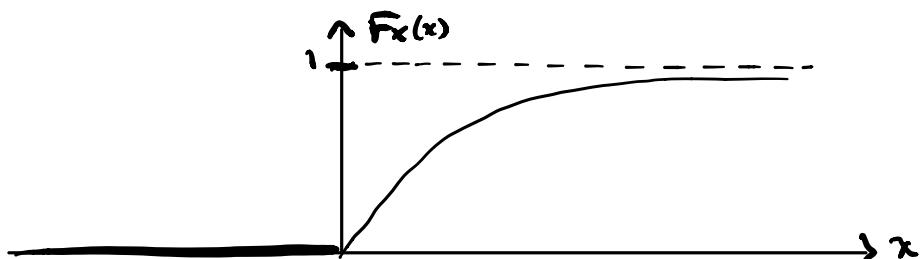


Example - $X \sim \text{exponential}(\lambda)$

$$f_x(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{else} \end{cases} \Rightarrow$$



$$F_x(x) = \int_{-\infty}^x \lambda e^{-\lambda t} dt = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0 \\ 0, & \text{else} \end{cases}$$



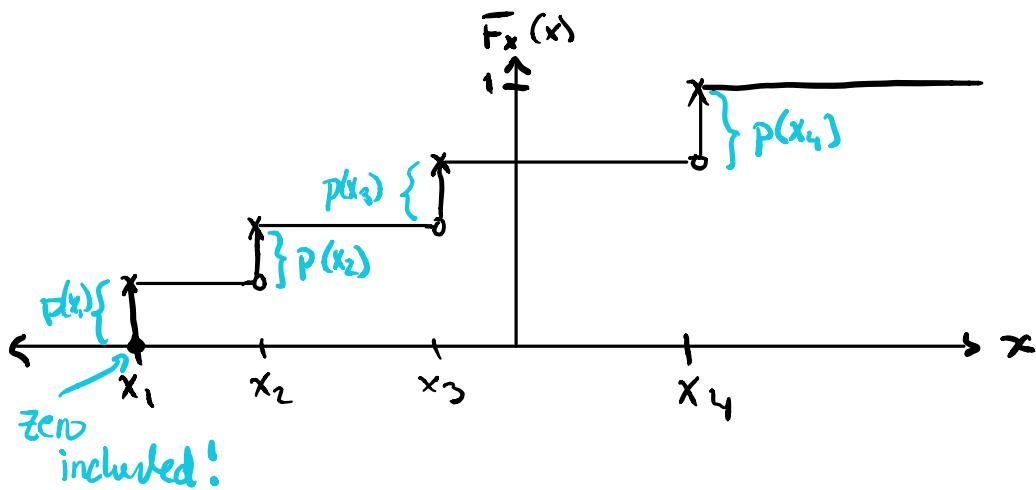
General Properties of CDFs

① $\lim_{x \rightarrow -\infty} F_X(x) = 0$ and $\lim_{x \rightarrow \infty} F_X(x) = 1$

② When X is a continuous RV, $F_X(x)$ is continuous in x and differentiable "almost everywhere" (corners in $F_X(x)$ correspond to jumps in $f_X(x)$)

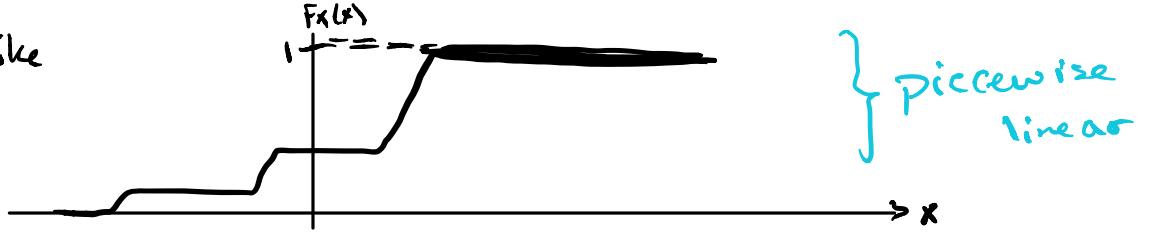
③ X is a discrete iff $F_X(x)$ is a piecewise constant.

Therefore $F_X(x)$ looks like



Caution: don't conflate piecewise constant CDF for any discrete RV w/ piecewise constant for a continuous piecewise uniform RV.

The latter looks like



④ $F_X(x)$ is monotonically increasing in x .

$$x_1 \leq x_2 \Rightarrow F_X(x_1) \leq F_X(x_2)$$

CDFs are useful for a variety of reasons, not least of which is getting a pdf of X by first computing $F_X(x)$, then taking d/dx .

Example - Discrete rv Version of this Phenomenon

Have 3 prelims - scores X_1, X_2, X_3 - on each you get a score of 0 to 10 (integer) distributed uniformly.

$$\text{i.e. } P_{X_i}(k) = \frac{1}{11}, \quad 0 \leq k \leq 10$$

The events

$$\{X_1 = k\}, \{X_2 = l\}, \{X_3 = m\}$$

are independent of each other

$$\text{"Prelim score" at the end} = \max(X_1, X_2, X_3) = X$$

Find pmf of X .

$$\text{Given } k, \quad P(\{X \leq k\}) = P(\{X_1 \leq k\} \cap \{X_2 \leq k\} \cap \{X_3 \leq k\})$$

$$= F_{X_1}(k) F_{X_2}(k) F_{X_3}(k) \quad \leftarrow \text{by independence and definition of cdfs}$$

$$\text{Note: } F_{X_i}(k) = P(\{X_i \leq k\}) = \frac{(k+1)}{11} = F_{X_2}(k) = F_{X_3}(k)$$

$$\text{Conclusion: } P(\{X \leq k\}) = F_X(k) = \left(\frac{k+1}{11} \right)^3, \quad 0 \leq k \leq 10$$

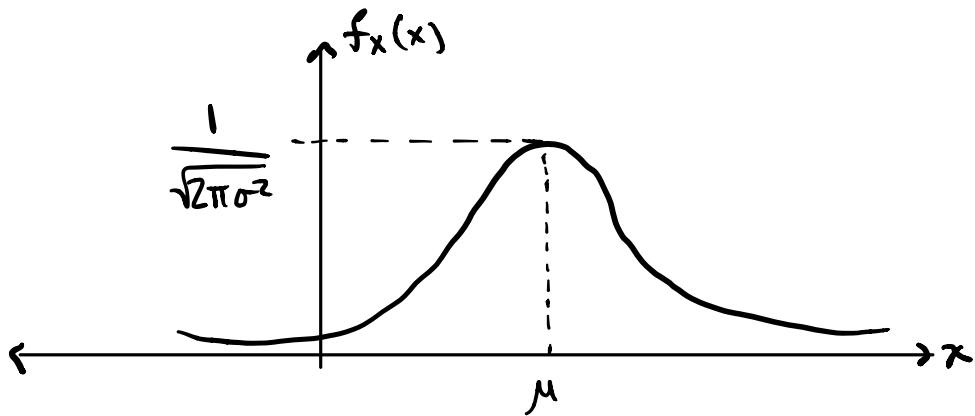
PMF is thus

$$p_X(k) = F_X(k) - F_X(k-1) = \left(\frac{k+1}{11} \right)^3 - \left(\frac{k}{11} \right)^3, \quad 1 \leq k \leq 10$$

Another important continuous rv: Gaussian

The pdf of such a rv is:

$$f_x = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$



Check that:

$$\int_{-\infty}^{+\infty} f_x(x) dx = 1 \quad (\text{by "standard" math trick})$$

"Standard" Math Trick

$$I = \int_{-\infty}^{+\infty} e^{-x^2} dx$$

$$I^2 = \left(\int_{-\infty}^{+\infty} e^{-x^2} dx \right) \left(\int_{-\infty}^{+\infty} e^{-y^2} dy \right)$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-(x^2+y^2)} dx dy \stackrel{\text{polarize}}{=} \int_0^{2\pi} d\theta \int_0^{\infty} r e^{-r^2} dr = \pi$$

$$\mathbb{E}(X) = \int_{-\infty}^{+\infty} x f_x(x) dx = \mu$$

$$\text{Var}(X) = \mathbb{E}((X-\mu)^2)$$

$$= \int_{-\infty}^{+\infty} (x-\mu)^2 f_x(x) dx = \sigma^2 \quad (\text{integration by parts})$$

Why are Gaussian rvs so persuasive?

Turns out that the sum of a bunch of independent random variables, all of which have same pdf (not necessarily Gaussian) "converges" to a Gaussian rv.

CDF of a Gaussian X :

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^X e^{-\frac{(t-\mu)^2}{2\sigma^2}} dt$$

← Can't calculate in closed form

Standard Normal (synonym for Gaussian) pdf is

$$\frac{1}{\sqrt{2\pi}} e^{-x^2/2} \quad \therefore \mu=0, \sigma=1$$

CDF of this

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2} dt$$

← book has a table of values for this

Given a Gaussian w/ mean μ and var σ^2 ,

$$Y = \frac{X - \mu}{\sigma}$$

has the standard normal pdf

For example, say you want $P(X > 17)$

$$\begin{aligned} P(X > 17) &= P(\sigma Y + \mu > 17) \\ &= P(Y > \frac{17 - \mu}{\sigma}) \quad \text{Get from table!} \\ &= 1 - \Phi\left(\frac{17 - \mu}{\sigma}\right) = P(Y \leq \frac{17 - \mu}{\sigma}) \end{aligned}$$

$$\begin{array}{c} f_{X,Y}(x,y) \\ \downarrow \\ \frac{\partial}{\partial x} \frac{\partial}{\partial y} \\ F_{X,Y}(x,y) \end{array}$$