Math 4310

Homework 6

Due 10/23/19

Name: ______________________

Collaborators: ______________________

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Please print out these pages. I encourage you to work with your classmates on this homework. Please list your collaborators on this cover sheet. (Your grade will not be affected.) Even if you work in a group, you should write up your solutions yourself! You should include all computational details, and proofs should be carefully written with full details. As always, please write neatly and legibly.

Please follow the instructions for the "extended glossary" on separate paper (LaTeX it if you can!) Hand in your final draft, including full explanations and write your glossary in complete, mathematically and grammatically correct sentences. Your answers will be assessed for style and accuracy.

Please **staple** *this cover sheet, your exercise solutions, and your glossary together, in that order, and hand in your homework in class.*

GRADES

Exercises ______________________ / 50

Extended Glossary

Component	Correct?	Well-written?
Definition	/6	/6
Example	/4	/4
Non-example	/4	/4
Theorem	/5	/5
Proof	/6	/6
Total	/25	/25

Exercises.

1. Let $\mathbb{R}^\infty = \{(\alpha_1, \alpha_2, \dots) \mid \alpha_i \in \mathbb{R}\}$ be the set of sequences of real numbers. Let $U \subset \mathbb{R}^\infty$ be the subspace of sequences

$$U = \left\{ (\alpha_1, \alpha_2, \dots) \in \mathbb{R}^\infty \;\middle|\; \alpha_{i+2} = \alpha_i + \alpha_{i+1} \text{ for all } i \right\}.$$

(You do not need to prove that U is as subspace for this HW, but you should check it for yourself!)

(a) Prove that U is finite-dimensional and compute its dimension.

(b) Find a **complementary** subspace V so that $U \oplus V = \mathbb{R}^\infty$. (Note: V will not be finite-dimensional!!)

(c) It follows from (a) and (b) that $\dim(\mathbb{R}^\infty/V) = 2$ and $\dim(\mathbb{R}^\infty/U) = \infty$. (You do not need to verify those dimensions for this HW, but you should check them for yourself!) Find a third subspace W of $\mathbb{R}^\infty$ and a complement X so that both $\mathbb{R}^\infty/W$ and $\mathbb{R}^\infty/X$ are infinite dimensional.

2. Let V be a vector space over the field $\mathbb{F}$, and let W be a subspace. If U_1 and U_2 are both complements of W in V:

(a) Show that U_1 and U_2 are isomorphic.

(b) Is $U_1 = U_2$? Either prove this, or give a counter-example.

3. Let $O(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) \mid f(-x) = -f(x)\}$ be the set of **odd** smooth functions and let $E(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) \mid f(-x) = f(x)\}$ be the set of **even** smooth functions. Prove that $O(\mathbb{R})$ and $E(\mathbb{R})$ are complementary. Prove that $C^\infty(\mathbb{R})/O(\mathbb{R}) \cong E(\mathbb{R})$.

4. Let U and V be vector spaces with respective subspaces X and Y. Prove that there is an isomorphism $(U \times V)/(X \times Y) \cong (U/X) \times (V/Y)$.

5. Let $C^\infty(\mathbb{R})$ denote the vector space (over $\mathbb{R}$) of infinitely-differentiable real-valued functions $f : \mathbb{R} \to \mathbb{R}$. (Note: $C^\infty(\mathbb{R})$ is **very** infinite-dimensional!)

   Let W denote the subspace of $C^\infty(\mathbb{R})$ consisting of those functions which "vanish to n^{th} order at 0":
$$W = \left\{ f \in C^\infty(\mathbb{R}) \;\middle|\; f(0) = 0,\ \frac{df}{dx}(0) = 0, \ldots,\ \text{and } \frac{d^n f}{dx^n}(0) = 0 \right\}.$$
   Prove that the quotient vector space $C^\infty(\mathbb{R})/W$ is finite-dimensional and find a basis.

6. Suppose that m is a positive integer, and let $V = \mathbb{R}[x]_{\leq m}$ be the vector space of polynomials of degree at most m. Consider the basis $\mathcal{A} = (1, x, x^2, \ldots, x^m)$ of V.

   (a) Show that the dual basis to $\mathcal{A}$ is $\mathcal{A}^* = (\phi_0, \phi_1, \ldots, \phi_m)$, where
$$\phi_j(p(x)) = \frac{p^{(j)}(0)}{j!}.$$
   (b) Show that $\mathcal{B} = (1, x-3, (x-3)^2, \ldots, (x-3)^m)$ is a basis of V.

   (c) Find the dual basis $\mathcal{B}^*$.

Extended Glossary. There is no extended glossary this week.