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Homework 9

Please print out these pages. I encourage you to work with your classmates on this homework. Please list your collaborators on this cover sheet. (Your grade will not be affected.) Even if you work in a group, you should write up your solutions yourself! You should include all computational details, and proofs should be carefully written with full details. As always, please write neatly and legibly.

Please follow the instructions for the "extended glossary" on separate paper (LaTeX it if you can!) Hand in your final draft, including full explanations and write your glossary in complete, mathematically and grammatically correct sentences. Your answers will be assessed for style and accuracy.

Please **staple** *this cover sheet, your exercise solutions, and your glossary together, in that order, and hand in your homework in class.*

GRADES

Exercises ______________ / 50

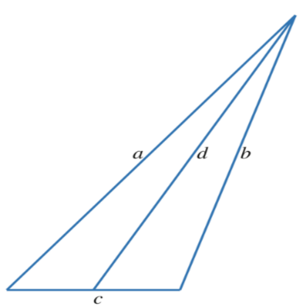
Extended Glossary

Component	Correct?	Well-written?
Definition	/6	/6
Example	/4	/4
Non-example	/4	/4
Theorem	/5	/5
Proof	/6	/6
Total	/25	/25

Exercises.

1. In a triangle with sides of lengths a,b, and c, let d be the length of the line segment from the midpoint of the side with length c to the opposite vertex. Show that

$$a^2 + b^2 = \frac{1}{2}c^2 + 2d^2$$



After a lot of meaningless calculations it was realized that this problem is cleaned up nicely by realizing the parallelogram inequality.

Theorem 1. *Parallelogram Equality*

Suppose $u, v \in V$ *(V and inner product space over* $\mathbb{F}$*). Then*

$$\|u+v\|^2 + \|u-v\|^2 = 2(\|u\|^2 + \|v\|^2)$$

We represent the figure via vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ in place of the magnitudes. Then

$$\vec{a} + \vec{b} = 2\vec{d}$$

and

$$\vec{a} - \vec{b} = \vec{c}.$$

The proof now follows by the parallelogram equality.

$$\left\|\vec{a} + \vec{b}\right\|^2 + \left\|\vec{a} - \vec{b}\right\|^2 = 2(\|\vec{a}\|^2 + \left\|\vec{b}\right\|^2)$$

$$4d^2 + c^2 = 2(a^2 + b^2)$$

$$a^2 + b^2 = \frac{1}{2}c^2 + 2d^2$$

2. Apply the Gram-Schmidt process to find an orthonormal basis of the space $\mathbb{R}[x]_{\leq 2}$ of polynomials of degree at most 2, where the inner product is given by

$$\langle f, g\rangle = \int_0^1 f(x)g(x)dx$$

A basis of $\mathbb{R}[x]_{\leq 2}$ is $\{1, x, x^2\}$. We know apply the Gram-Schmidt process to this basis (let $w_1 = 1, w_2 = x, w_3 = x^2$).

Take

$$v_1 = w_1 = 1.$$

Then

$$\begin{aligned} v_2 &= w_2 - \frac{\langle w_2, v_1\rangle}{\langle v_1, v_1\rangle}v_1 \\ &= x - \int_0^1 x dx \\ &= x - \frac{1}{2} \end{aligned}$$

Finally,

$$\begin{aligned} v_3 &= w_3 - \frac{\langle w_3, v_1\rangle}{\langle v_1, v_1\rangle}v_1 - \frac{\langle w_3, v_2\rangle}{\langle v_2, v_2\rangle}v_2 \\ &= x^2 - \int_0^1 x^2 dx - \frac{\int_0^1 x^2(x - \frac{1}{2})dx}{\int_0^1 (x - \frac{1}{2})^2 dx}(x - \frac{1}{2}) \\ &= x^2 - \frac{1}{3} - \frac{\frac{1}{12}}{\frac{1}{12}}(x - \frac{1}{2}) \\ &= x^2 - \frac{1}{3} - (x - \frac{1}{2}) \\ &= x^2 - x + \frac{1}{6} \end{aligned}$$

We now have an orthogonal basis of $\mathbb{R}[x]_{\leq 2}$ in $(1, x - \frac{1}{2}, x^2 - x + \frac{1}{6})$.

This basis is now made orthonormal by dividing each element of the orthogonal basis by its magnitude.

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{1} = 1$$

$$u_2 = \frac{v_2}{\|v_2\|} = \frac{x - \frac{1}{2}}{\sqrt{\int_0^1 (x - \frac{1}{2})^2 dx}} = \frac{x - \frac{1}{2}}{\frac{1}{\sqrt{12}}} = \sqrt{12}(x - \frac{1}{2})$$

$$u_3 = \frac{v_3}{\|v_3\|} = \frac{x^2 - x + \frac{1}{6}}{\sqrt{\int_0^1 (x^2 - x + \frac{1}{6})^2 dx}} = \frac{x^2 - x + \frac{1}{6}}{\frac{1}{\sqrt{180}}} = \sqrt{180}(x^2 - x + \frac{1}{6})$$

In conclusion, an orthonormal basis of $\mathbb{R}[x]_{\leq 2}$ is

$$\mathcal{B} = \{1, \sqrt{12}(x - \frac{1}{2}), \sqrt{180}(x^2 - x + \frac{1}{6})\}$$

3. Prove the Cauchy-Schwarz formula: if V is an inner product space (not necessarily of finite dimension), and $\vec{v}$ and $\vec{w}$ are elements of V, then

$$|\langle \vec{v}, \vec{w} \rangle| \leq \|\vec{v}\| \|\vec{w}\|$$

Part 1: This proof was initially done using techniques learned from analysis. Another proof of the Linear Algebra way to do it is included below.

Proof. Since $f(x) = x^2$ is a monotonically increasing function for $x > 0$, we note that proving

$$|\langle \vec{v}, \vec{w} \rangle|^2 \leq (\|\vec{v}\| \|\vec{w}\|)^2$$

will also prove what is desired.

The left hand side becomes

$$|\langle \vec{v}, \vec{w} \rangle|^2 = \left(\sum_{i=1}^{n} v_i w_i \right)^2$$

where $\vec{v} = (v_1, ..., v_n)$, and $\vec{w} = (w_1, ..., w_n)$.

The right hand side becomes

$$(\|\vec{v}\| \|\vec{w}\|)^2 = \left(\sum_{i=1}^{n} v_i^2 \right) \left(\sum_{i=1}^{n} w_i^2 \right)$$

Our statement to prove is now

$$\left(\sum_{i=1}^{n} v_i w_i \right)^2 \leq \left(\sum_{i=1}^{n} v_i^2 \right) \left(\sum_{i=1}^{n} w_i^2 \right).$$

Now let us evaluate the left hand side first and be clever with indices.

$$\left(\sum_{i=1}^{n} v_i w_i\right)^2 = \sum_{i=1}^{n} v_i w_i \left(\sum_{j=1}^{n} v_j w_j\right) = \sum_{i=j} (v_i w_i)^2 + 2\sum_{i\neq j} v_i w_i v_j w_j$$

Now we evaluate the right hand side.

$$\left(\sum_{i=1}^{n} v_i^2\right)\left(\sum_{i=1}^{n} w_i^2\right) = \sum_{i=1}^{n}\left(\sum_{j=1}^{n} v_i v_j w_i w_j\right) = \sum_{i=j}(v_i w_i)^2 + \sum_{i\neq j}(v_i^2 w_j^2 + v_j^2 w_i^2)$$

To recap, we have left hand side $\leq$ right hand side as follows:

$$\sum_{i=j}(v_i w_i)^2 + 2\sum_{i\neq j} v_i w_i v_j w_j \leq \sum_{i=j}(v_i w_i)^2 + \sum_{i\neq j}(v_i^2 w_j^2 + v_j^2 w_i^2)$$

It now suffices to show that

$$2\sum_{i\neq j} v_i w_i v_j w_j \leq \sum_{i\neq j}(v_i^2 w_j^2 + v_j^2 w_i^2)$$

$$2v_i w_i v_j w_j \leq v_i^2 w_j^2 + v_j^2 w_i^2$$
$$v_i^2 w_j^2 + v_j^2 w_i^2 - 2v_i w_i v_j w_j \geq 0$$
$$(v_i w_j - v_j w_i)^2 \geq 0$$

Any number not equal to zero squared is greater than zero by field axioms! Thus, not only have we proved the Cauchy Schwarz inequality, but we have also shown that equality holds only when

$$v_i w_j = v_j w_i.$$

□

Part 2: This proof was added after I read the corresponding proof in the book and is also included to better aid in my understanding of inner products.

First, we define an orthogonal decomposition.

Definition 1. Suppose $u, v \in V$ (V an inner product space over $\mathbb{F}$), with $v \neq 0$. Set $c = \frac{\langle u,v\rangle}{\|v\|^2}$ and $w = u - \frac{\langle u,v\rangle}{\|v\|^2}v$. Then

$$\langle w, v\rangle = 0$$

and

$$u = cv + w.$$

We also state the Pythagorean Theorem for later use.

Theorem 2. *Suppose* u *and* v *are orthogonal vectors in an inner product space* V *over* $\mathbb{F}$*. Then*

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2$$

Proof. Next we aim to prove that for $u, v \in V$ (V an inner product space over $\mathbb{F}$),

$$|\langle \vec{v}, \vec{w} \rangle| \leq \|\vec{v}\| \|\vec{w}\|$$

If $v = 0$, then both sides of the desired inequality equal 0. Thus we can assume that $v \neq 0$. Consider the orthogonal decomposition

$$u = \frac{\langle u, v \rangle}{\|v\|^2} v + w$$

where w is orthogonal to v.

Take the norm of both sides and apply the Pythagorean theorem.

$$\|u\|^2 = \left\| \frac{\langle u, v \rangle}{\|v\|^2} v \right\|^2 + \|w\|^2$$

$$\|u\|^2 = \frac{|\langle u, v \rangle|}{\|v\|^2} + \left\| w^2 \right\|$$

$$\|u\|^2 \geq \frac{|\langle u, v \rangle|^2}{\|v\|^2}$$

$$\|u\|^2 \|v\|^2 \geq \| \langle u, v \rangle|^2$$

and thus

$$|\langle u, v \rangle| \leq \|u\| \|v\|$$

This method the condition for equality doesn't jump out at you like in method 1.

Here we observe that the Cauchy–Schwarz Inequality is an equality if and only if $\|u\| = \frac{|\langle u,v \rangle|^2}{\|v\|^2}$ is an equality. This happens if and only if $w = 0$. But $w = 0$ if and only if u is a multiple of v. Thus the Cauchy–Schwarz Inequality is an equality if and only if u is a scalar multiple of v or v is a scalar multiple of u.

This is certainly a clearer condition for equality then dissecting the indices from the above notion of equality. $\square$

4. Prove the following statements (written as a separate question to the Cauchy-Schwarz question for better organization).

(a) Prove that for all positive real numbers a, b, c, d,

$$16 \leq (a+b+c+d)(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}).$$

$$16 \leq (a+b+c+d)(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}) = 4+\frac{b+c+d}{a}+\frac{a+c+d}{b}+\frac{a+b+d}{c}+\frac{a+b+c}{d}$$

$$12 \leq \frac{b+c+d}{a}+\frac{a+c+d}{b}+\frac{a+b+d}{c}+\frac{a+b+c}{d}$$

$$12 \leq \frac{a^2+b^2}{ab}+\frac{a^2+c^2}{ac}+\frac{a^2+d^2}{ad}+\frac{b^2+c^2}{bc}+\frac{b^2+d^2}{bd}+\frac{c^2+d^2}{cd}$$

Look at just the first fraction. I claim that

$$\frac{a^2+b^2}{ab} \geq 2$$

For the case $a = b$, this holds true since

$$\frac{a^2+b^2}{ab} = \frac{2a^2}{a^2} = 2.$$

Now consider $a > b$,

$$\frac{a^2+b^2}{ab} > 2$$

$$a^2+b^2 > 2ab$$

$$a^2+b^2-2ab > 0$$

$$(a-b)^2 > 0$$

This is clearly true since any number squared is greater than 0 by the field axioms. The case for $b > a$ is identical.
Thus,

$$\frac{a^2+b^2}{ab} \geq 2.$$

The analysis for the other five fraction is identical and is thus omitted. Thus we have proved

$$16 \leq (a+b+c+d)(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d})$$

as desired.

(b) Prove that for all positive integers n and all real numbers $a_1, ..., a_n$, then

$$(a_1+...+a_n)^2 \leq n(a_1^2+...+a_n^2).$$

Let $P(n)$ be the statement

$$(a_1+...+a_n)^2 \leq n(a_1^2+...+a_n^2).$$

Let our base case be $n = 1$, it is easy to verify this is true

$$a_1^2 \leq 1 \cdot a_1^2 = a_1^2.$$

For all $k \geq 1$ we now show that if $P(k)$ is true then $P(k+1)$ is true.
Assume $P(k)$ is true, that is

$$(a_1 + ... + a_k)^2 \leq k(a_1^2 + ... + a_k^2)$$

We want to show $P(k+1)$ is true, that is,

$$(a_1 + ... + a_k + a_{k+1})^2 \leq (k+1)(a_1^2 + ... + a_k^2 + a_{k+1}^2)$$

To see this, observe that we can rearrange the left hand side as follows

$$(a_1 + ... + a_k + a_{k+1})^2 = ((a_1 + ... + a_k) + a_{k+1})^2$$

$$((a_1 + ... + a_k) + a_{k+1})^2 = (a_1 + ... + a_k)^2 + 2(a_1 + ... + a_k)a_{k+1} + a_{k+1}^2.$$

The statement to prove has now become

$$(a_1 + ... + a_k)^2 + 2(a_1 + ... + a_k)a_{k+1} + a_{k+1}^2 \leq (k+1)(a_1^2 + ... + a_k^2 + a_{k+1}^2)$$

Massage the right hand side a little to obtain

$$(a_1+...+a_k)^2+2(a_1+...+a_k)a_{k+1}+a_{k+1}^2 \leq k(a_1^2+...+a_k^2)+(k+1)a_{k+1}^2+(a_1^2+...+a_k^2)$$

NOW the proof begins.
We note that

$$a_{k+1}^2 \leq (k+1)a_{k+1}^2$$

for $k > 0$, and

$$(a_1 + ... + a_k)^2 \leq k(a_1^2 + ... + a_k^2)$$

by the induction hypothesis.
If $2(a_1 + ... + a_k)a_{k+1} < 0$, then we are done since

$$2(a_1 + ... + a_k)a_{k+1} < (a_1^2 + ... + a_k^2)$$

and the right hand side of the above expression is never less than zero.
If $2(a_1 + ... + a_k)a_{k+1} \geq 0$, then we note that the left hand side of the proof

$$(a_1 + ... + a_k)^2 + 2(a_1 + ... + a_k)a_{k+1} + a_{k+1}^2 \geq (a_1 + ... + a_k)^2 + a_{k+1}^2$$

and the statement to prove becomes

$$(a_1 + ... + a_k)^2 + a_{k+1}^2 \leq k(a_1^2 + ... + a_k^2) + (k+1)a_{k+1}^2 + (a_1^2 + ... + a_k^2).$$

This is clearly the case since

$$a_{k+1}^2 \leq (k+1)a_{k+1}^2$$

for $k > 0$, and

$$(a_1 + ... + a_k)^2 \leq k(a_1^2 + ... + a_k^2)$$

by the induction hypothesis.

We have just shown $P(k+1)$ is true when $P(k)$ is true for $k \geq 1$. Thus, by induction

$$(a_1 + ... + a_n)^2 \leq n(a_1^2 + ... + a_n^2).$$

This proof could have really been a lot easier had I just applied Cauchy-Schwarz, but it's always good to practice induction I guess :).

(c) Prove that for continuous functions f and g on the interval $[0,1]$, that

$$\left|\int_0^1 f(x)g(x)dx\right|^2 \leq \left|\int_0^1 f^2(x)dx\right|\left|\int_0^1 g^2(x)dx\right|.$$

This is essentially saying for continuous functions f and g, show that

$$|\langle f,g\rangle|^2 \leq |\langle f,f\rangle||\langle g,g\rangle|$$

This is clearly true. Observe that by the Cauchy-Schwartz inequality

$$|\langle f,g\rangle| \leq \|f\|\|g\|$$

and since $f(x) = x^2$ is monotonically increasing for $x > 0$ this inequality squared on both sides also holds.

$$|\langle f,g\rangle|^2 \leq (\|f\|\|g\|)^2$$

which is equivalent with what is to be proved since

$$\|f\|^2 = \left|\int_0^1 (f(x))^2 dx\right|$$

Note that in this question it was assumed that $\langle f,g\rangle$ was an inner product space in $P(\mathcal{R})$.

5. Let $V = \mathbb{R}^n$. As usual, we think of each $\vec{v} \in \mathbb{R}^n$ as a column vector, that is, as a $n \times 1$ matrix. We also equate scalars and 1×1 matrices.

(a) Suppose that A is a symmetric $n \times n$ matrix, and we define

$$\langle \vec{v}, \vec{w}\rangle = \vec{v}^T A\vec{w},$$

show that this is an inner product if and only if A satisfies: for $\vec{x} \neq \vec{0} \in \mathbb{R}^n$ then $\vec{x}^T A\vec{x} > 0$ (such a symmetric matrix is called positive definite).

Proof. ($\implies$) Assume that $\langle \vec{v}, \vec{w}\rangle = \vec{v}^T A\vec{w}$ is an inner product on V.
Then

$$\langle \vec{x}, \vec{x}\rangle = \vec{x}^T A\vec{x} > 0$$

for $\vec{x} \neq \vec{0}$ by property of inner products.

($\Longleftarrow$) Assume that for $\vec{x} \neq \vec{0} \in \mathbb{R}^n$ that $\vec{x}^T A\vec{x} > 0$. Need to show $\vec{v}^T A\vec{w}$ is an inner product.

Positivity

Follows immediately by assumption.

Definiteness

$\vec{x}^T A\vec{x} = 0$ if and only if $\vec{x} = \vec{0}$ follows near immediately from assumption.

Additivity in the First Slot

$$\langle \vec{v} + \vec{u}, \vec{w}\rangle = (\vec{v} + \vec{u})^T A\vec{w} = (\vec{v}^T + \vec{u}^T)A\vec{w} = \vec{v}^T A\vec{w} + \vec{u}^T A\vec{w} = \langle \vec{v}, \vec{w}\rangle + \langle \vec{u}, \vec{w}\rangle$$

as desired.

Homogeneity in the First Slot

$$\langle \lambda\vec{v}, \vec{w}\rangle = (\lambda\vec{v}^T)A\vec{w} = \lambda(\vec{v}^T A\vec{w}) = \lambda\langle \vec{v}, \vec{w}\rangle$$

as desired.

Conjugate Symmetry

Need to define some notion of symmetry on this definition of inner product. I claim

$$\langle v, w\rangle = (\langle w, v\rangle)^T$$

will give us symmetry as desired.

We observe this by taking

$$\langle \vec{v}, \vec{w}\rangle = \vec{v}^T A\vec{w} = (\vec{w}^T A^T \vec{v})^T = (\langle \vec{w}, \vec{v}\rangle)^T$$

which is true since A is symmetric (i.e. $A = A^T$).

Thus, $\vec{v}^T A\vec{w}$ is an inner product on V. $\square$

(b) Suppose that $\langle \cdot, \cdot\rangle$ is an inner product on $\mathbb{R}^n$. Show that there exists a positive definite matrix A such that

$$\langle \vec{v}, \vec{w}\rangle = \vec{v}^T A\vec{w}.$$

Existence proofs have been the bane of my mathematical career. I digress, suppose that $(\vec{e}_1, ..., \vec{e}_n)$ is the standard basis of $\mathbb{R}^n$.

Also suppose that $\vec{v} = (v_1, ..., v_n)$ and $\vec{w} = (w_1, ..., w_n)$. Then

$$\vec{v} = v_1\vec{e_1} + ... + v_n\vec{e}_n$$

and

$$\vec{w} = w_1\vec{e_1} + ... + w_n\vec{e}_n.$$

Take $\langle \vec{v}, \vec{w}\rangle$ to obtain

$$\langle \vec{v}, \vec{w}\rangle = \langle v_1\vec{e_1} + ... + v_n\vec{e}_n, w_1\vec{e_1} + ... + w_n\vec{e}_n\rangle = \sum_i \sum_j v_i w_j \langle e_i, e_j\rangle.$$

This is precisely the form that our defined dot product takes. Observe that

$$\vec{v}^{\mathsf{T}} A \vec{w} = \sum_{i=1}^{n} \sum_{j=1}^{n} v_j a_{ij} w_i$$

and that a_{ij} has taken the place of the scalar $\langle e_i, e_j \rangle$.
Thus the matrix A is such that each element at the i^{th} row and j^{th} column is $\langle e_i, e_j \rangle$. To see this is positive definite we observe that

$$\langle v, v \rangle = \sum_i \sum_j v_i v_j \langle e_i, e_j \rangle > 0.$$

It's not quite so clear why.

(c) If

$$A = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix},$$

is the function defined by

$$\langle \vec{v}, \vec{w} \rangle = \vec{v}^{\mathsf{T}} A \vec{w}$$

($\forall\ \vec{v}, \vec{w} \in \mathbb{R}^2$) an inner product on $\mathbb{R}^2$?
Need to show that A is positive definite.

$$\vec{x}^{\mathsf{T}} A \vec{x} = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 5x_1 + 2x_2 \\ 2x_1 + x_2 \end{pmatrix} = 3x_1^2 + 2x_2 x_1 + 2x_1 x_2 + x_2^2$$

$$= 3x_1^{+}4x_1 x_2 + x_2^2$$
$$= 3x_1^2 + 2x_2 x_1 + 2x_1 x_2 + x_2^2$$
$$= 3x_1^2 + 4x_1 x_2 + x_2^2 + x_1^2 - x_1^2$$
$$= (2x_1 + x_2)^2 - x_1^2.$$

which is less than zero for $x_1 = -2x_2$. Therefore is A is not positive definite and therefore $\vec{v}^{\mathsf{T}} A \vec{w}$ is **NOT** an inner product on $\mathbb{R}^2$.

6. Recall that for a subspace $W \subset V$ of a real inner product space V,

$$W^{\perp} = \{\vec{v} \in V \mid \langle \vec{v}, \vec{w} \rangle = 0\ \ \forall\ \vec{w} \in W\}.$$

Note: V may be infinite dimensional, W is finite dimensional.

Show that

(a) $V = W \oplus W^{\perp}$
First, we show that
$$W \cap W^{\perp} = \{0\}.$$
Let $\vec{v} \in W \cap W^{\perp}$. Then $\vec{v} \in W$, and $\vec{v} \in W^{\perp}$. This means that
$$\langle \vec{v}, \vec{v} \rangle = \vec{0} \to \vec{v} = \vec{0}.$$

Next, we show that
$$W + W^{\perp} = V.$$
Let $e_1, ..., e_m$ be an orthonormal basis of W. Suppose that $v \in V$.
As per usual, we add and subtract an awfully convenient thing to get what is desired. Note that
$$v = v + \langle v, e_1 \rangle e_1 + ... + \langle v, e_m \rangle e_m - \langle v, e_1 \rangle e_1 - ... - \langle v, e_m \rangle e_m$$
We rearrange this to get a more convenient expression towards our proof,
$$v = \underbrace{\langle v, e_1 \rangle e_1 + ... + \langle v, e_m \rangle e_m}_{w} + \underbrace{v - \langle v, e_1 \rangle e_1 - ... - \langle v, e_m \rangle e_m}_{v-w}.$$

Now it must be shown that $w \in W$ and $v - w \in W^{\perp}$.
To see that $w \in W$ we just realize that $w \in span(e_1, ..., e_m)$.
To see that $v - w \in W^{\perp}$ we note that for each $j = 1, ..., m$
$$\langle v - w, e_j \rangle = \langle v, e_j \rangle - \langle w, e_j \rangle$$
$$= \langle v, e_j \rangle - \langle v, e_j \rangle = 0$$
since $e_1, ..., e_m$ is an orthonormonal basis for W. Thus, $v - w$ is orthogonal to every vector in the span of $(e_1, ..., e_m)$ and $v - w \in W^{\perp}$.
In conclusion, every element of V can be written as the sum of an element in W and an element in $W^{\perp}$, and $W \cap W^{\perp} = \{0\}$. Thus,
$$V = W \oplus W^{\perp}.$$

(b) If A is an $m \times n$ matrix, and $V = \mathbb{R}^n$ is equipped with the standard inner product, then
$$ker(A)^{\perp} = image(A^T)$$
and therefore that
$$V = ker(A) \oplus image(A^T).$$
Before the proofs begin we note that
$$A : \mathbb{R}^n \to \mathbb{R}^m$$
$$A^T : \mathbb{R}^m \to \mathbb{R}^n$$
for better clarity.

i. We begin with proving the first statement.
This proof will use the following corollary (theorem?, proposition?) whose proof will also be shown (basically copy paste from the textbook).

Corollary 1. Suppose U is a finite-dimensional subspace of V. Then

$$U = (U^{\perp})^{\perp}.$$

Proof. First it will be shown that

$$U \subset (U^{\perp})^{\perp}.$$

Suppose $u \in U$. Then $\langle u, v \rangle = 0$ for every $v \in U^{\perp}$.
Since u is orthogonal to every vector in $(U^{\perp})^{\perp}$, we have $u \in (U^{\perp})^{\perp}$ and $U \subset (U^{\perp})^{\perp}$ as desired.
Now it will be shown that

$$(U^{\perp})^{\perp} \subset U.$$

Suppose that $v \in (U^{\perp})^{\perp} \subseteq V$.
Can write $v \in V$ as

$$v = u + w$$

where $u \in U$ and $w \in U^{\perp}$.
We have

$$v - u = w \in U^{\perp}.$$

Since

$$v \in (U^{\perp})^{\perp}$$

and

$$u \in U \implies u \in (U^{\perp})^{\perp}$$

from first part of the proof. Thus

$$v - u \in U^{\perp} \cap (U^{\perp})^{\perp}$$

implying $v - u$ is orthogonal to itself. Thus

$$v - u = 0 \implies v = u \implies v \in U.$$

Thus $(U^{\perp})^{\perp} \subset U$ and the proof is complete. □

We use the result of above to write

$$\ker(A)^{\perp} = image(A^{T})$$

as

$$(\ker(A)^{\perp})^{\perp} = image(A^{T})^{\perp}$$

and thus

$$\ker(A) = image(A^{T})^{\perp}$$

We prove this statement instead for simplicity.

Proof. Now we actually start this problem.
First it is shown that

$$\mathtt{ker}(A) \subset \mathtt{image}(A^T)^\perp.$$

Suppose $w \in \mathtt{ker}(A)$. Also suppose that $v \in \mathtt{image}(A^T)$. This means that $\exists y \in \mathbb{R}^m$ such that $v = A^T y$. Take

$$\langle v, w \rangle = v^T w = y^T A w = 0.$$

Thus $w \in \mathtt{image}(A^T)^\perp$ and this portion of the proof is complete.
Next we show that

$$\mathtt{image}(A^T)^\perp \subset \mathtt{ker}(A).$$

Suppose $w \in \mathtt{image}(A^T)^\perp$. Also suppose that $v \in \mathtt{image}(A^T)$. This means that $\exists y \in \mathbb{R}^m$ such that $v = A^T y$. Then

$$\langle v, w \rangle = 0 = v^T w = x^T A w \implies Aw = 0.$$

Thus $w \in \mathtt{ker}(A)$ and the proof is now complete.
To conclude,

$$\mathtt{ker}(A) = \mathtt{image}(A^T)^\perp$$

and after taking the orthogonal complement of both sides,

$$\mathtt{ker}(A)^\perp = (\mathtt{image}(A^T)^\perp)^\perp \to \mathtt{ker}(A)^\perp = \mathtt{image}(A^T)^\perp$$

as desired. $\square$

ii. The kernel of A is a finite dimensional subset of V (in this case, this is immediate since V is finite dimensional). Therefore the proof for the second statement was proved in part (a) and is identical.