

$\log z = \ln z$ (i.e. "base e ")

Want $\log(z)$ to be the inverse of e^z

Let $z = re^{i\theta}$.

Hope: $\log(z) = \log(r) + i\theta$

Problems arise because $\arg(z)$ is multivalued!

(in the sense that $\theta, \theta+2\pi, \theta+4\pi \dots$ all correspond to the same point z , but give different values of $\log(z)$).

Example:

$$\log(1) = \log(e^{i2\pi k}) = i2\pi k, \quad k \in \mathbb{Z}$$

$$\log(-2) = \log(2) + i2\pi k + i\pi$$

$\log(0)$ is still VERY nasty!

* Not allowed, $\log(r) \quad r \rightarrow 0 = -\infty$

BUT $i\theta$ could be anything...

Need to make $\log(z)$ into an analytic single valued function.

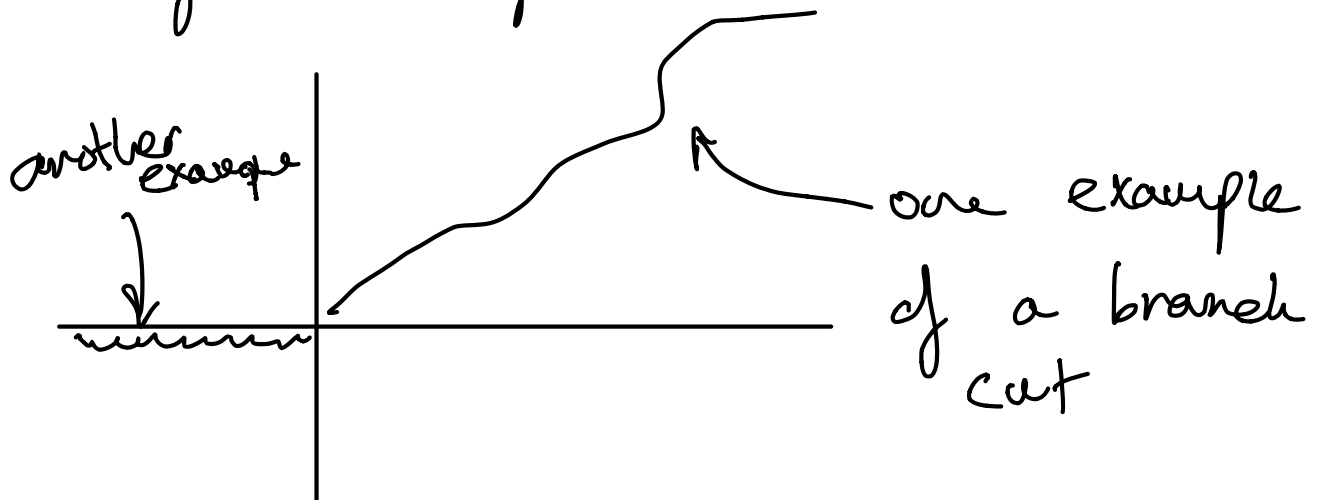
Restrict θ to be single valued. $-\pi < \theta \leq \pi$

NOT GOOD ENOUGH - still not analytic.

Such a definition would give us a discontinuous function.

If we want $\log(z)$ to be analytic, we must forbid encirclement of the origin. Otherwise θ jumps by $2\pi i$ when we cross the negative real axis.

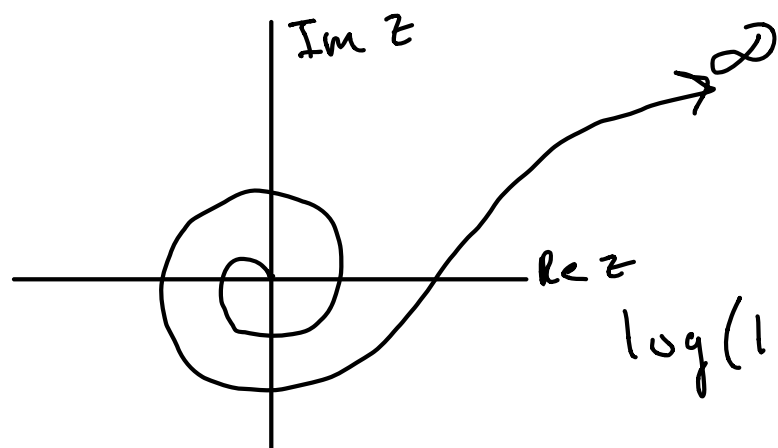
We must cut the plane to prevent paths encircling the origin.



Branch point: for a function $f(z)$ there is a point z_0 such that on any path encircling z_0 , $f(z)$ is discontinuous at some point on that path.

A **branch** is a single-valued, continuous function whose values agree with one choice of the original function's multiple values.

Example: Spiral branch cut of $\log(z)$



look at branch of $\log(z)$ where

$\log(1) = -8\pi i$. What's $\log(4)$?

$$-8\pi i + \underset{\substack{\uparrow \\ \text{rotating} \\ \text{around cut}}}{4\pi i} + \log(4) = \log(4) - 4\pi i$$

Other multivalued functions

$$f(z) = z^\alpha, \quad \alpha \neq \text{integer}$$

$$z^\alpha = e^{\log z^\alpha} = e^{\alpha \log z} \quad \left. \begin{array}{l} \log \text{ has branch cuts} \\ \text{so } z^\alpha \text{ has branch cuts} \end{array} \right\}$$

With $\alpha = 1/2$

$$z^{1/2} = r^{1/2} e^{i(\theta + 2\pi k)/2} = \left[r^{1/2} e^{i\theta/2} \right] e^{i\pi k}$$

two values

Slit plane \nrightarrow plane - negative real axis