Recall

All r.v's $(\Omega, \mathcal{F}, Pr)$

Linear Space: addition $X \cdot Y$

scalar multiplication aX bY

Hilbert space is the same as a linear space w/ one more operation.

<u>inner product</u> $\langle X, Y \rangle = \mathbb{E}[XY]$

inner product definition allows for ("induces")

length/norm: $||X|| = \langle X, X \rangle$

θ : $\cos\theta = \dfrac{\langle X, Y \rangle}{||X|| \, ||Y||}$

distance : $d(X,Y) = ||X - Y|| = \sqrt{\langle X-Y, X-Y \rangle}$

orthogonality: $X \perp Y \iff \cos\theta = 0$

$\Rightarrow \langle X, Y \rangle = 0$

$\Rightarrow \mathbb{E}[XY] = 0$

Have rv X, estimate $\hat{X}$.

Define error $W = X - \hat{X}$

Want error^2 since $-1, +1$ should have same error

Want to minimize average error.

Thus, want

$$\min_{\hat{X}} \mathbb{E}[W^2] = \min_{\hat{X}} \mathbb{E}[(X-\hat{X})^2]$$

① Restrict $\hat{X}$ to be a constant.

So,

$$a^* = \underset{a \in \mathbb{R}}{\operatorname{argmin}} \, \mathbb{E}[(X-a)^2]$$

$$MSE_{a^*} = Var(X)$$

② Have rv Y whose observations determine X.

Restrict $\hat{X}$ to be a function of Y, $g(Y)$.

i.e

$$\hat{X} = g(Y)$$

want

$$g^*(Y) = \underset{g}{\operatorname{argmin}} \, \mathbb{E}[(X-g(Y))^2]$$

Given $Y=y$, estimate $\hat{X} = g^*(y)$ a constant

So,

$$\hat{X} = \mathbb{E}[X \mid Y=y]$$

How we use Y?

$$f_X(x) \longrightarrow f_{X|Y=y}(X|Y=y)$$

So g^* is

$$g^*(Y) = \mathbb{E}[X|Y]$$ ← statistic completely determined by variable you condition on

and

$$MMSE = \mathbb{E}_Y[Var(X|Y)]$$

$g(Y)$

The error

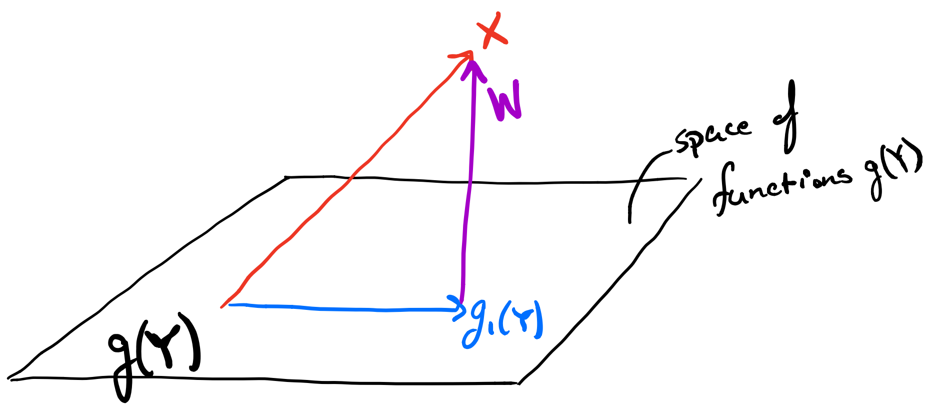
$$W = X - \mathbb{E}[X|Y]$$

has

$$\mathbb{E}[W] = \mathbb{E}[X - \mathbb{E}[X|Y]] = \mathbb{E}[X] - \mathbb{E}[X] = 0$$

we call this type of estimator UNBIASED

VISUALIZATION



$X, Y, \hat{X}$

$W = X - \hat{X}^*$

want $\min_{\hat{X}} \mathbb{E}[W^2]$

- Orthogonality principle:

$$W \perp g(Y) \quad \forall g$$ } necessary AND sufficient condition

W induced by $\hat{X}^*$ (best estimator)

Claim $X - \mathbb{E}[X|Y] \perp g(Y) \ \forall g$

inner product $\langle X - \mathbb{E}[X|Y], g(Y) \rangle$

Proof $\mathbb{E}\left[(X - \mathbb{E}[X|Y]) g(Y) \right] = 0$

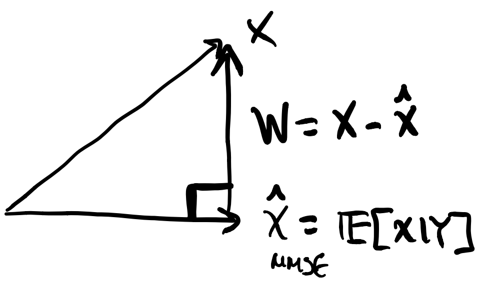
LHS

$$\mathbb{E}_{XY}[X g(Y)] - \mathbb{E}_{Y}\left[(\mathbb{E}_{X|Y}[X|Y]) g(Y) \right]$$

$$\mathbb{E}_Y\left[\mathbb{E}_{X|Y}[X g(Y) | Y] \right] - \mathbb{E}_Y\left[\mathbb{E}_{X|Y}[X|Y] \, g(Y) \right]$$

$$\mathbb{E}_Y\left[g(Y) \mathbb{E}_{X|Y}[X|Y] \right] - \mathbb{E}_Y\left[g(Y) \mathbb{E}_{X|Y}[X|Y] \right] = 0 \ \checkmark$$

MSE Using Pythagorean Theory



$\mathbb{E}[X] = \mathbb{E}[\hat{X}]$

$$\underset{\|}{\|W\|^2} = \langle W, W \rangle \text{ in Hilbert Space}$$

$$\text{MSE} = \mathbb{E}[W^2] = \mathbb{E}[X^2] - \mathbb{E}[\hat{X}^2]$$

$$= \text{Var}(X) + (\mathbb{E}(X))^2 - \left(\text{Var}(\hat{X}_{MMSE}) + (\mathbb{E}[\hat{X}_{MMSE}])^2 \right)$$

$$= \text{Var}(X) - \text{Var}(\hat{X}_{MMSE})$$

Example (Like Prelim 1):

$$f_{X,Y}(x,y) = \begin{cases} x+y & , 0 \le x,y \le 1 \\ 0 & , o/w \end{cases}$$

$\hat{X}_{MMSE}$ using Y ? MSE ?

Know $\hat{X}_{MMSE}$ is $\mathbb{E}[X|Y]$ and MSE is $\mathbb{E}_Y[Var(X|Y)]$

Step 1: Find $f_Y(y)$. $f_Y(y) = \int_x f_{X,Y}(x,y)\,dx = \begin{cases} \frac{1}{2}+y & , 0 \le y \le 1 \\ 0 & , o/w \end{cases}$

Step 2: Find $f_{X|Y}(x|y)$. $f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} \frac{x+y}{\frac{1}{2}+y} & , 0 \le x,y \le 1 \\ 0 & , o/w \end{cases}$

Step 3: $\mathbb{E}[X|Y=y] = \int_{x=0}^{1} x f_{X|Y}(x|y)\,dx$

$$= \int_{x=0}^{1} \frac{x+y}{\frac{1}{2}+y} x \, dx = \int_{x=0}^{1} \frac{x^2+xy}{\frac{1}{2}+y} dx$$

$$= \frac{2}{1+y}\left(\frac{1}{3}x^3 + \frac{1}{2}x^2 y \Big|_0^1 \right)$$

$$= \frac{2}{1+2y}\left(\frac{1}{3} + \frac{1}{2}y\right)$$

$$= \frac{2\left(\frac{1}{3}+\frac{1}{2}y\right)}{1+2y} = \frac{2+3y}{3+6y}$$

Thus

$$\mathbb{E}[X|Y] = \frac{2+3Y}{3+6Y}$$

Step 4 : $\mathrm{Var}(X \mid Y=y) = \mathbb{E}[X^2 \mid Y=y] - (\mathbb{E}[X \mid Y=y])^2$

$$= \int_{x=0}^{1} x^2 f_{X|Y}(x|y)\, dx - \left(\frac{2+3y}{3+6y}\right)^2$$

$$= \frac{3+4y}{6+12y} - \left(\frac{2+3y}{3+6y}\right)^2$$

Then

$$\mathrm{Var}(X \mid Y) = \frac{3+4Y}{6+12Y} - \left(\frac{2+3Y}{3+6Y}\right)^2$$

Step 5: $\mathbb{E}_Y[\mathrm{Var}(X \mid Y)] = \int_y \mathrm{Var}(X \mid Y=y)\, f_Y(y)\, dy$