Cayley Hamilton Theorem
Minimal-Characteristic Polynomials
Applications to Jordan Canonical Form

Situation:
- $T \in \mathcal{L}(V)$
- V a v.s. over $\mathbb{C}$
- $\dim V = n < \infty$

Definition: The characteristic polynomial $q_T(x) \in \mathbb{C}[x]$

is

$$q_T(x) = (x-\lambda_1)^{a_1}(x-\lambda_2)^{a_2}\cdots(x-\lambda_m)^{a_m}$$

where $\lambda_1, \dots, \lambda_m$ are the distinct eigenvalues of $a_i = \dim G_{\lambda_i}(T) = \dim(\ker(T-\lambda_i)^n)$

Thus, $\deg q_T(x) = a_1 + \cdots + a_m = n$

(note: $q_A(x) = \det(xI_n - A)$)

Definition: The <u>minimal polynomial</u> of T is the unique monic polynomial $m_T(x) \in \mathbb{C}[x]$ of smallest degree s.t. $m_T(T) = 0$.

"recall"

A polynomial is said to be monic if its lead coefficient is 1.

Recall: The annihilator of T — zero transformation

$$\operatorname{ann}(T) = \{ f(x) \in \mathbb{C}[x] \mid f(T) = 0 \}$$

"this is an ideal in $\mathbb{C}[x]$!"

Proposition: Let I be a non-zero ideal.

Then

(a) $I = \langle f(x) \rangle$ for $f(x)$ the unique monic polynomial of lowest degree in I

(b) If $g(x) \in I$, then $f(x) \mid g(x)$

So, m_T is the generator of (a).

One way to compute the minimal polynomial

Let $A \in \mathbb{C}^{n \times n}$

Consider $I, A, A^2, \ldots, A^{n^2}, \ldots$ all in $\mathbb{C}^{n \times n}$

Choose smallest m s.t. $I, A, A^2, \ldots, A^m$ of dim n^2 are LD so

$$a_0 I + a_1 A + \cdots + a_m A^m = 0.$$

Take $a_m = 1$ and

$$I, A, \ldots, A^{m-1}$$

are LI.

Then

$$m_A(x) = a_m x^m + a_{m-1} x^{m-1} + \cdots + a_0$$

So

$$\deg m_A \leq n^2 \text{ and } m_A \text{ exists.}$$

Examples

$$A = J(\lambda, 3) = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$$

$$q_A(x) = (x-\lambda)^{3} \leftarrow \dim G_\lambda(T)$$

$$m_A(x) =$$

$\boxed{\text{Cayley-Hamilton Theorem}}$

If $T \in \mathcal{L}(V)$, $\dim V = n$ (over $\mathbb{C}$), then $q_T(T) = 0$.

Notes: this means $m_T(T) \mid q_T$

Proof

Let $\lambda_1, \dots, \lambda_m$ be the distinct eigenvalues of T. Know

$$V = G_{\lambda_1}(T) \oplus G_{\lambda_2} \oplus \cdots \oplus G_{\lambda_m}(T)$$

and

$$a_i = \dim G_{\lambda_i}(T).$$

Also,

$$q_T = (x-\lambda_1)^{a_1} \cdots (x-\lambda_m)^{a_m}$$

Need to show

$$q_T(T) = (T-\lambda_1 I)^{a_1} \cdots (T-\lambda_m I)^{a_m}$$

it suffices to show, if $v_i \in G_{\lambda_i}(T)$, then $q_T(T)v_i = 0$.

$$\left.\begin{array}{l} \vec{v} \in V \text{ can be written as } v_1 + \cdots + v_m \\ \rightarrow q_T(T)v = 0 + \cdots + 0 = 0 \end{array}\right\} \text{form we want}$$

Let $v_i \in G_{\lambda_i}(t) \Rightarrow v_i \in \ker(T-\lambda I)^{\text{big power}}$

$$\therefore \quad v_i \in \ker(T-\lambda I)^{a_1}$$

take, and rearrange q_T, (proved $p(T)q(T) = q(T)p(T)$)

$$q_T(T)\, v_i = (T-\lambda_1 I)^{a_1} \cdots (T-\lambda_m I)^{a_m} (T-\lambda_i I)\, v_i = 0$$

Q.E.D

Know $m_T(x) \mid q_T(x) = (x-\lambda_1)^{a_1} \cdots (x-\lambda_m)^{a_m}$

so

$$m_T(x) = (x-\lambda_1)^{b_1} \cdots (x-\lambda_m)^{b_m} \qquad 0 \le b_i \le a_i$$

Proposition : If λ = eigenvalue of T, then $m_T(\lambda)=0$.

$$\Rightarrow 1 \le b_i \le a_i \quad \forall i$$

<u>Proof</u>

Let $\vec{v}$ be an eigenvector for T, $\vec{v} \neq 0$,

$$T(\vec{v}) = \lambda \vec{v} .$$

$Iv = v$

$Tv = \lambda v$

$T^2 v = \lambda^2 v$

$\vdots$

$$0 = m_T(T)(v)$$
$$= c_0 + c_1 x + c_2 x^2 + \cdots + c_r x^r$$
$$= c_0 I + c_1 T + c_2 T^2 + \cdots + c_r T^r$$

Apply $\vec{v}$

$$c_0 I v + c_1 T v + c_2 T^2 v + \cdots + c_r T^r v \qquad (v \neq 0)$$
$$= c_0 v + c_1 \lambda v + \cdots + c_r \lambda^r v = m_T(\lambda) v \Rightarrow \underline{m_T(\lambda) = 0}$$

Examples

① $A = J(\lambda, n) = \begin{pmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix}$

$$\chi_A(x) = (x-\lambda)^n$$

$$m_A(x) = (x-\lambda)^a, \quad 1 \le a \le n$$

a??

e.g.

$n = 3$

$$(x-\lambda)^a$$

$$(A-\lambda I)^a = 0$$

$$A - \lambda I = N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$N^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$N^3 = 0$$

$a = 3$ (in general $a = n$ if $A = J(\lambda, n)$)

② Suppose

$$A = A_1 \oplus A_2 = \begin{bmatrix} A_1 & \\ & A_2 \end{bmatrix}$$

then

$$\chi_A(x) = \chi_{A_1}(x)\, \chi_{A_2}(x)$$

$$m_A(x) = \mathrm{LCM}(m_{A_1}(x), m_{A_2}(x))$$

(e.g. LCM $(x-1)^2(x-2)(x-3)$, $(x+1)(x-2)^4(x-7)^2$ ← take highest degree in both

$\longrightarrow (x-1)^2(x-2)^4(x-3)(x-7)^2$)

then

$$f(A) = \begin{bmatrix} f(A_1) & 0 \\ 0 & f(A_2) \end{bmatrix}$$

example $\quad J(\lambda, \underline{k}) \quad , \quad \underline{k} = (k_1, \ldots, k_r)$

$k_1 \geq k_2 \geq \cdots k_r$

then

$$q_A(x) = q_{J(\lambda,k_1)} \cdots q_{J(\lambda,k_r)}$$

$$= (x-\lambda)^n$$

$$m_A(x) = LCM\left((x-\lambda)^{k_1} (x-\lambda)^{k_2} \cdots (x-\lambda)^{k_m} \right)$$
$$= (x-\lambda)^{k_1}$$

Example

A is 6×6

$q_A = (x-3)^3 (x-7)^3$

$m_A = (x-3)^2 (x-7)$

Possible JCF's of A?