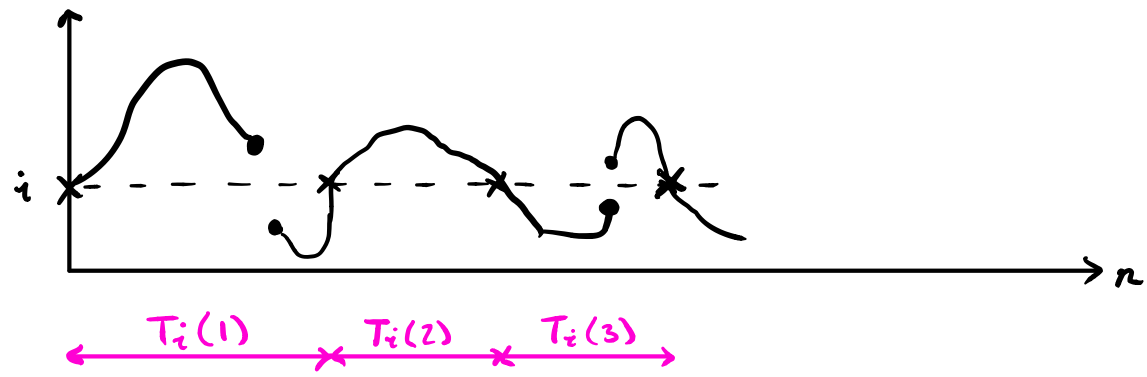
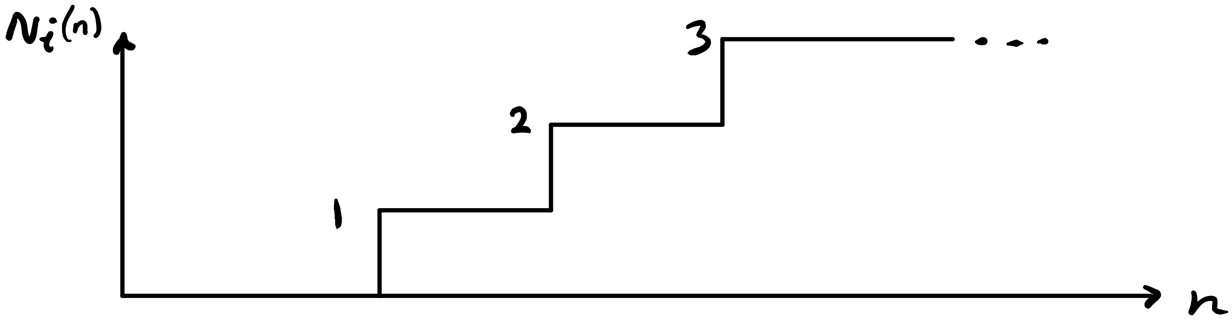


regenerative cycle



i transient : $\Pr[T_i < \infty] < 1$

i recurrent : $\Pr[T_i < \infty] = 1$

Two notions of recurrence

Null

$$\lim_{n\to\infty} \frac{N_i(n)}{n} = 0$$

Positive

$$\lim_{n\to\infty} \frac{N_i(n)}{n} > 0$$

equivalent statement

$$\mathbb{E}[T_i] = \infty$$

$$\mathbb{E}[T_i] < \infty$$

Occupancy rate of i.

$$r_i = \frac{\lim_{n\to\infty} \sum_{m=1}^{n} \mathbb{1}[X_m = i \mid X_0 = i]}{n}$$

(kinda like arrival rate)

If i transient,

$$r_i = 0$$

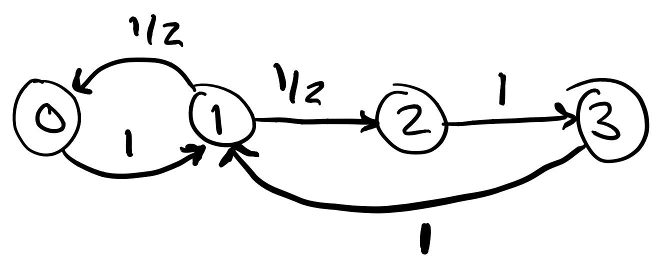
If i is NULL recurrent,

$$r_i = 0$$

If i positive recurrent,

$$r_i = \frac{1}{\mathbb{E}[T_i]}$$

Example



State 0:

$$T_0 = \begin{cases} 2 & \text{w.p. } 1/2 \\ 4 & \text{w.p. } 1/2 \end{cases}$$

$$\mathbb{E}[T_0] = 2 \cdot \frac{1}{2} + 4 \cdot \frac{1}{2} = 3$$

$$r_0 = \frac{1}{3}$$

"$\frac{1}{3}$ of the time, the state is @ state 0"

Recurrent Class

A communication class that is recurrent (NULL or POSITIVE)

$$\boxed{\text{Every recurrent class is closed.}}$$

Cononical Decomposition

$$\mathcal{X} = T \cup C_1 \cup C_2 \cup C_3 \cup \cdots$$

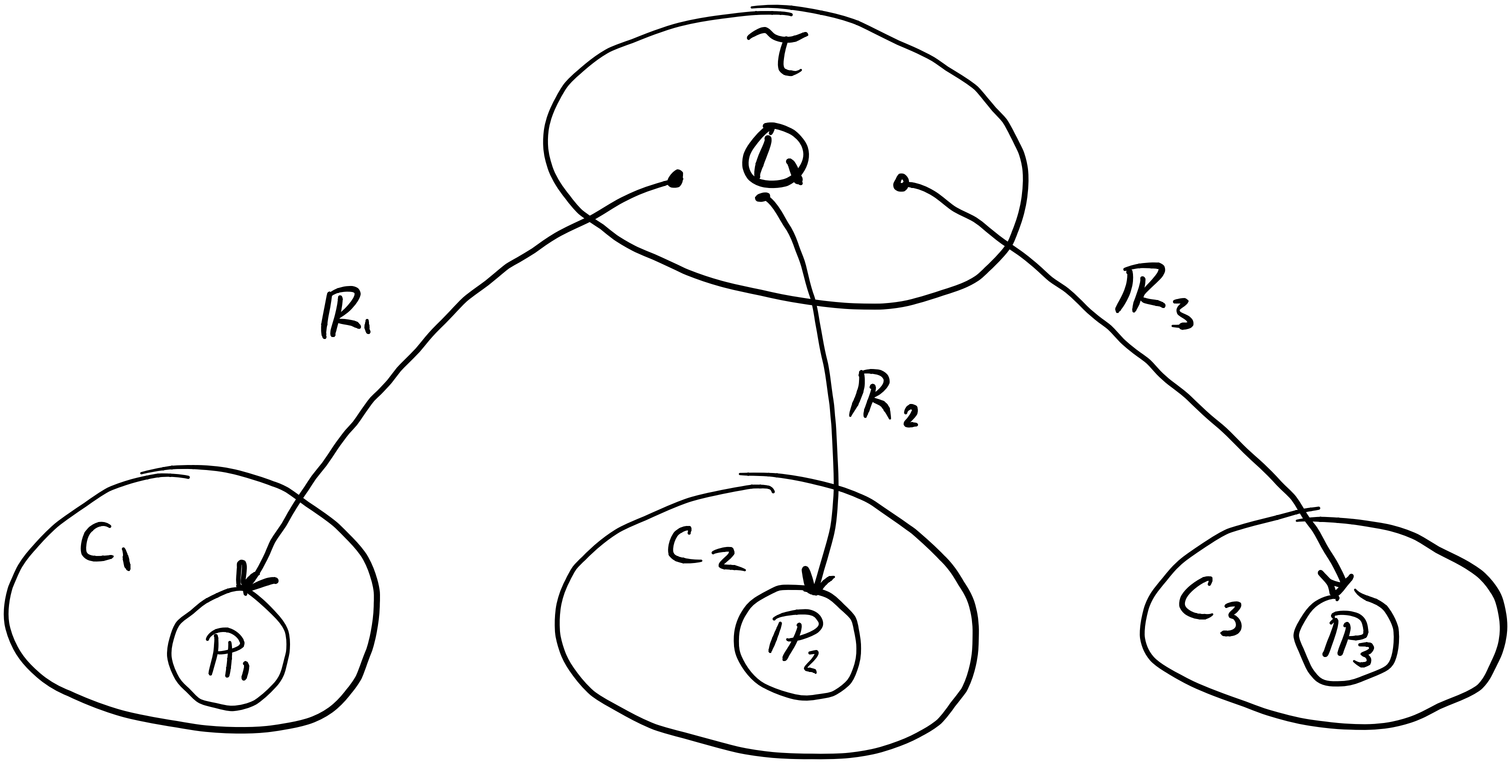
last labels

States $\{1, 2, \dots, m_1\}$ States $\{m_1+1, \dots, m_1+m_2\}$

Expeet (for C_1, C_2, C_3)

$$\mathbb{P} = \begin{array}{cc} & \begin{array}{cccc} 1 \cdots m_1 & & & \end{array} \\ \begin{array}{c} 1 \\ \vdots \\ m_1 \\ m_1+1 \\ \vdots \\ m_1+m_2 \\ m_1+m_2+1 \\ \vdots \\ m_1+m_2+m_3 \\ \tau \end{array} & \begin{bmatrix} \begin{array}{c} \text{NON ZERO} \\ \mathbb{P}_1 \end{array} & 0 & 0 & 0 \\ 0 & \begin{array}{c} \text{Non-zero} \\ \mathbb{P}_2 \\ \text{transition matrix} \\ \text{b/t states of } C_2 \end{array} & 0 & 0 \\ 0 & 0 & \begin{array}{c} \mathbb{P}_3 \\ \text{transition matrix} \\ \text{b/t states of } C_3 \end{array} & 0 \\ \mathbb{R}_1 & \mathbb{R}_2 & \mathbb{R}_3 & Q \end{bmatrix} \end{array}$$

transition matrix b/t states of C_1



Existence of Stationary Distribution

$$\underbrace{\begin{bmatrix} \pi_1 & \pi_2 & \cdots & \pi_n \end{bmatrix}}_{\bar{\pi}} \; \boxed{\mathbb{P}} \; = \; \underbrace{\begin{bmatrix} \pi_1 & \pi_2 & \cdots & \pi_n \end{bmatrix}}_{\bar{\pi}}$$

$$\sum_{i \in \mathcal{X}} \pi_i = 1$$

- π_i : occupancy rate $= r_i = \dfrac{1}{\mathbb{E}[T_i]}$

① If all states transient OR null recurrent,

$$r_i = 0 \quad \forall i \in \mathcal{X}$$

then this chain has 0 stationary distribution.

② If the chain has a single positive recurrent classes (no NULL recurrent) then $\exists$ unique stationary distribution

③ If the chain has multiple positive recurrent classes, then $\exists$ infinitely many stationary distributions

use

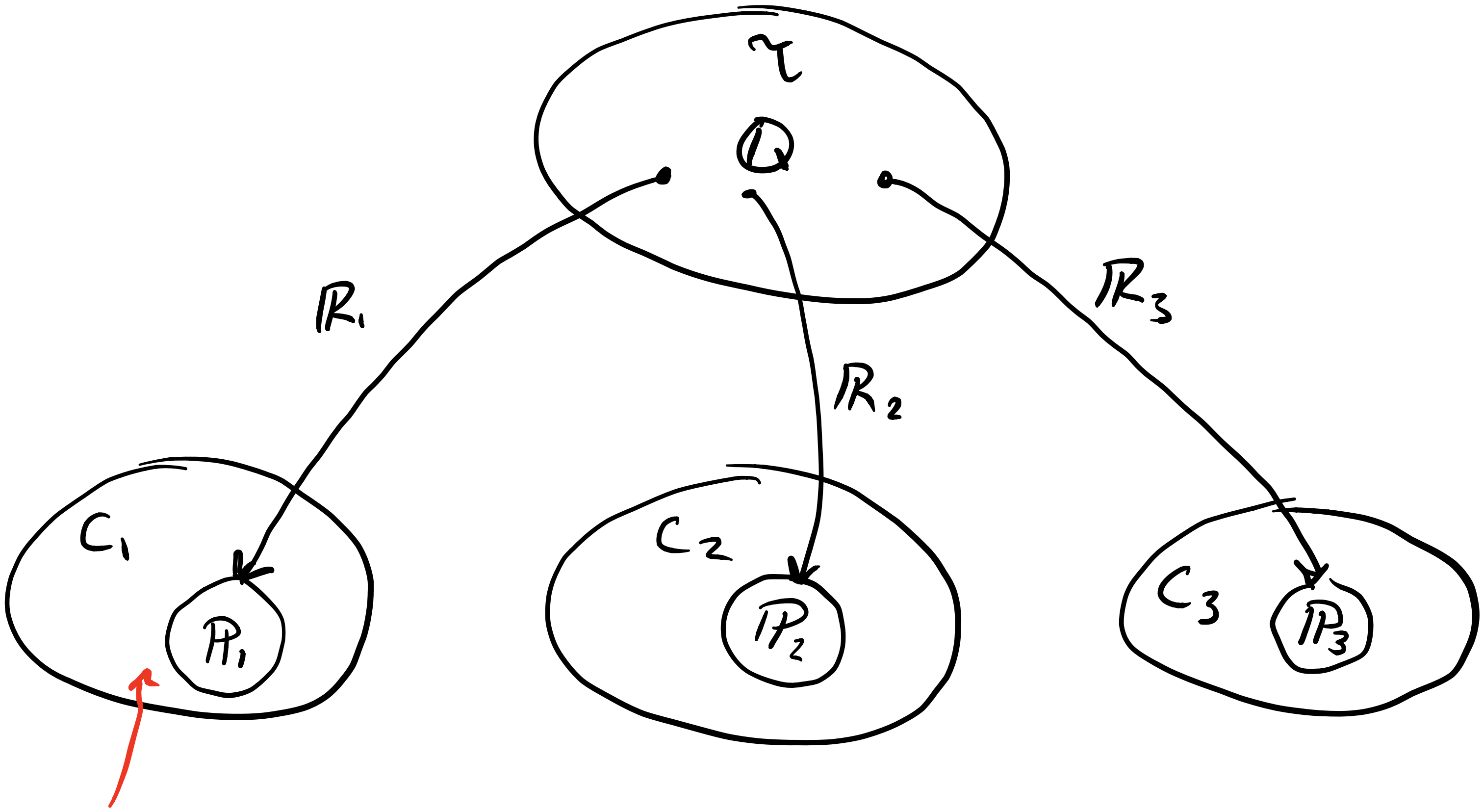
$$\begin{bmatrix} \pi_1 & \pi_2 & \cdots & \pi_{m_1} \end{bmatrix} \underbrace{\boxed{\mathbb{P}_1}}_{m_1 \times m_1} = \begin{bmatrix} \pi_1 & \pi_2 & \cdots & \pi_n \end{bmatrix}$$

For $\mathbb{P}$:

$$\begin{array}{lll} & \alpha_1 \begin{bmatrix} \pi_1 & \pi_2 & \cdots & \pi_{m_1} & 0 & \cdots & 0 & \cdots & 0 \end{bmatrix} & (1) \\ + & \alpha_2 \begin{bmatrix} 0 & 0 & \cdots & 0 & \pi_{m_1+1} & \cdots & \pi_{m_1+m_2} & 0 \cdots & 0 \end{bmatrix} & (2) \\ + & \alpha_3 \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \pi_{m_1+m_2+1} \cdots & \pi_{m_1+m_2+m_3} \end{bmatrix} & (3) \end{array}$$

infinite b/c can choose $\alpha_1(1) + \alpha_2(2) + \alpha_3(3)$ s.t. $\alpha_1 + \alpha_2 + \alpha_3 = 1$ where $\alpha_i \in \mathbb{R}$ and the linear combination of (1), (2), (3) is a valid $\bar{\pi}$ satisfying our criteria

"like a change of basi ! "



Probability of being absorbed here?