2.1

1f) $G(z) = e^{z} + e^{-z}$

$$\omega = e^{x}e^{iy} + e^{-x}e^{-iy}$$

$$\omega = e^{x}[\cos(y) + i\sin(y)] + e^{-x}[\cos(y) - i\sin(y)]$$

$$\omega = e^{x}\cos(y) + e^{-x}\cos(y) + i[e^{x}\sin(y) - e^{-x}\sin(y)]$$

$$\omega = u(x,y) + iv(x,y)$$

4) $\omega = f(z) = 1/z$ maps

(a) $|z| = r$ onto $|\omega| = 1/r$

$$\omega = 1/z = \frac{1}{x+iy} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$

Note
$|z| = \sqrt{x^2+y^2} = r$

$$u(x,y) = \frac{x}{x^2+y^2} \qquad v(x,y) = \frac{-y}{x^2+y^2}$$

$$|\omega| = \sqrt{\frac{x^2+y^2}{(x^2+y^2)^2}} = \sqrt{\frac{1}{x^2+y^2}} = \frac{1}{r}$$

(b) The ray $\operatorname{Arg} z = \theta_0$, $-\pi < \theta_0 < \pi$, onto the ray $\operatorname{Arg} \omega = -\theta_0$

$$\operatorname{Arg} z = \theta_0 = \operatorname{Arg}\left(\frac{y}{x}\right)$$

$$\operatorname{Arg} w = \operatorname{Arg}\left(\frac{v(x,y)}{u(x,y)}\right) = \operatorname{Arg}\left(\frac{-y}{x}\right) = -\operatorname{Arg}\left(\frac{y}{x}\right)$$

(c) The circle $|z-1|=1$ onto the vertical line $x=1/2$

$$w = \frac{1}{z} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^3}$$

$$|z-1| = |x+iy-1| = 1$$

$$(x-1)^2 + y^2 = 1$$

$$x^2+y^2-2x=0$$

$$x^2+y^2 = 2x$$

$$\Rightarrow w = \frac{x}{2x} - i\frac{y}{2x}$$

$$= \frac{1}{2} - i\frac{y}{2x}$$

Now we must show that $\frac{y}{x}$ spans from $-\infty$ to $+\infty$ as we traverse our circle.

Let

$x = 1+\cos\phi$

$y = \sin\phi$

Then

$$\frac{y}{x} = \frac{\sin\phi}{1+\cos\phi} = \tan\left(\frac{\phi}{2}\right)$$

As we traverse the circle y/x spans from $-\infty$ to $+\infty$ and therefore the circle $|z-1|=1$ is mapped onto the line $x=1/2$.

6) $w = J(z) = \frac{1}{2}\left(z+\frac{1}{z}\right)$ } Joukowski Mapping

(a) Show

$$J(z) = J(1/z)$$

$$J(z) = \frac{1}{2}(z + 1/z)$$

$$J(1/z) = \frac{1}{2}\left(1/z + 1/(1/z)\right) = \frac{1}{2}\left(z + 1/z\right) = J(z)$$

(b) J maps the unit circle $|z|=1$ onto real interval $[-1,1]$

$$x^2+y^2=1$$

$$x^2 = 1-y^2$$

$$x = \sqrt{1-y^2}$$

$$J(z) = \tfrac{1}{2}(z + 1/z)$$

$$w = \tfrac{1}{2}\left(x+iy + \frac{1}{x+iy}\right)$$

$$= \frac{1}{2}\left(x+iy + \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}\right)$$

$$= \frac{1}{2}\left(x+iy + \frac{x}{1} - i\frac{y}{1}\right)$$

$$= \frac{1}{2}x + \frac{1}{2}x + \frac{1}{2}iy - \frac{1}{2}iy$$

$$= x$$

By the restriction $|z|=1$, it is automatically known that the bounds on x, y are

$$-1 \leq x \leq 1$$
$$-1 \leq y \leq 1$$

Since $w = J(z) = x$ we therefore have a mapping ont the real axis from -1 to 1.

(c) J maps $|z|=r$ $(r>0, r\neq 1)$ onto the ellipse

$$\frac{u^2}{\left[\frac{1}{2}\left(r+\frac{1}{r}\right)\right]^2} + \frac{v^2}{\left[\frac{1}{2}\left(r-\frac{1}{r}\right)\right]^2} = 1,$$ which has foci at ± 1.

$$x^2+y^2 = r^2$$

$$w = J(z) = \tfrac{1}{2}\left(z + \tfrac{1}{z}\right), \quad z = x+iy$$

$$w = \frac{1}{2}\left(x+iy + \frac{1}{x+iy}\right) = \frac{1}{2}\left(x+iy + \frac{x-iy}{x^2+y^2}\right) = \frac{1}{2}\left(x + \frac{x}{r^2}\right) + \frac{1}{2}i\left(y - \frac{y}{r^2}\right)$$

$$w = \frac{1}{2}\cdot\frac{x}{r}\left(r + \frac{1}{r}\right) + \frac{i}{2}\cdot\frac{y}{r}\left(r - \frac{1}{r}\right) \quad \Big\} \quad \frac{x}{r} = \cos\theta, \quad \frac{y}{r} = \sin\theta$$

$$w = \underbrace{\frac{\cos\theta}{2}\left(r + \frac{1}{r}\right)}_{u(r,\theta)} + i\underbrace{\frac{\sin\theta}{2}\left(r - \frac{1}{r}\right)}_{v(r,\theta)}$$

$$u^2 = \frac{\cos^2\theta}{4}\left(r+\frac{1}{r}\right)^2 \quad , \quad v^2 = \frac{\sin^2\theta}{4}\left(r-\frac{1}{r}\right)^2$$

$$\frac{u^2}{\left[\frac{1}{2}\left(r+\frac{1}{r}\right)\right]^2} = \cos^2\theta \qquad \frac{v^2}{\left(\frac{1}{2}\left(r-\frac{1}{r}\right)\right)^2} = \sin^2\theta$$

$$\cos^2\theta + \sin^2\theta = 1 \checkmark$$ Everything works!

7a) $F(z) = z + c$, where $c \in \mathbb{C}$, generates a translation mapping

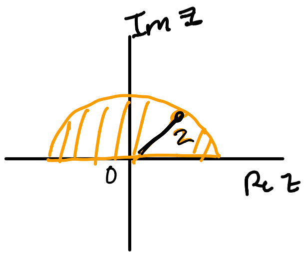
Sketch image of $|z| \leq 2$, $\operatorname{Im} z \geq 0$ under F when $c = 3$.

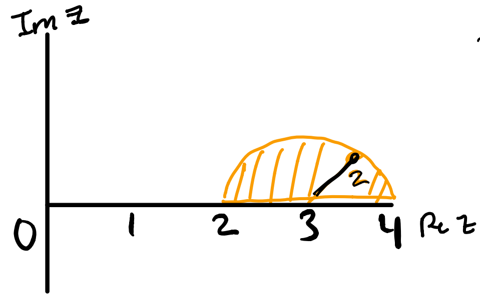
$x^2 + y^2 \leq 4$

$y \geq 0$

$F(z) = z + 3$

$= (x+3) + iy$

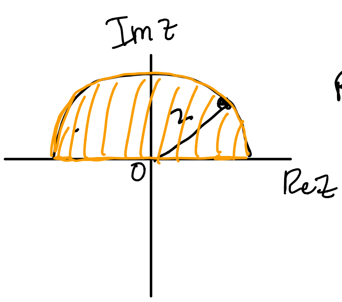




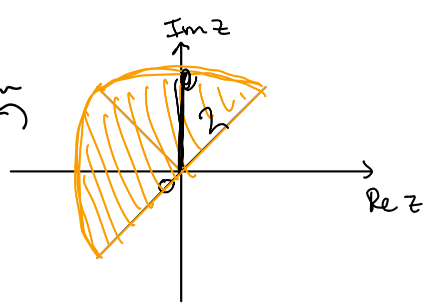
every point gets shifted on the real axis by 3

8a) $G(z) = e^{i\phi} z$, $\phi \in \mathbb{R}$, generates a ROTATION MAPPING.

Sketch $|z| \leq 2$, $\operatorname{Im} z \geq 0$ under G when $\phi = \pi/4$



Rotation by $\pi/4$



every point $z = re^{i\theta}$ gets rotated by ϕ

10b) Let $F(z) = z + i$, $G(z) = e^{i\pi/4} z$, and $H(z) = z/2$.

Sketch $|z| \leq 2$, $\operatorname{Im} z \geq 0$ under

$G(H(z))$

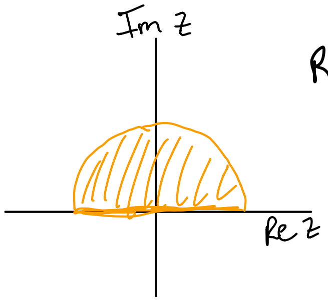
$G(H(z)) = G(z/2) = e^{i\pi/4} \dfrac{z}{2}$ } rotation by $\pi/4$ and $\frac{1}{2}\cdot$ Magnitude

$$\left|\frac{z}{2}\right| \leq 2$$

$$\sqrt{\left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2} \leq 2$$

$$\frac{1}{2}\sqrt{x^2+y^2} \leq 2$$

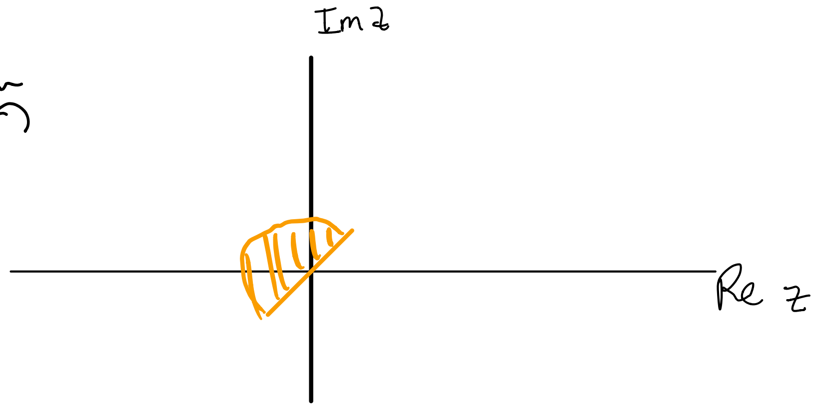
$$\sqrt{x^2+y^2} \leq 4$$



Rotation by $\pi/4$ ↺

⟶

$\frac{1}{2}$Mag



$\boxed{2.3}$

4a) Show Re z is nowhere differentiable

$$\frac{df}{dz}(\text{Re } z) \equiv f'(\text{Re } z) := \lim_{\Delta z \to 0} \frac{\text{Re}(z+\Delta z) - \text{Re}(z)}{\Delta z} = \frac{\text{Re}(\Delta z)}{\Delta z}$$

if $\Delta z \to 0$ through real values, then $\Delta z = \Delta x$ and

$$\frac{\text{Re}(\Delta z)}{\Delta z} = \frac{\Delta x}{\Delta x} = 1$$

However, if $\Delta z \to 0$ through imaginary values, then $\Delta z = i\Delta y$ and

$$\frac{\text{Re}(\Delta z)}{\Delta z} = 0$$

Consequently, there is no way of assigning a unique value to the derivative of Re(z) at any point.

9b) Determine the points at which the function is not analytic

$$\frac{iz^3 + 2z}{z^2 + 1}$$

The function is not analytic at $z^2 + 1 = 0 \Rightarrow z = i, -i$ as the denominator is zero at those two points and therefore is not differentiable.

10) $f(z) = |z|^2$. Show f is differentiable at $z=0$ but not at any other point.

$$f'(z) := \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{|z+\Delta z|^2 - |z|^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(z+\Delta z)\overline{(z+\Delta z)} - z\bar{z}}{\Delta z} = \lim_{\Delta z \to 0} \frac{z\overline{\Delta z} + \bar{z}\Delta z + \Delta z \overline{\Delta z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{z\overline{\Delta z}}{\Delta z} + \bar{z} + \overline{\Delta z}$$

At $z = 0$

$$(0 + 0 + 0) = 0$$

At $z \neq 0$ we would imply the differentiability of $\bar{z}$

$\boxed{2.4}$

2) Show $h(z) = x^3 + 3xy^2 - 3x + i(y^3 + 3x^2y - 3y)$ is differentiable on the coordinate axes but nowhere analytic.

To be differentiable on the coordinate axes means to be be differentiable at each x, y where $xy = 0$.

$$\frac{\partial u}{\partial x} = 3x^2 + 3y^2 - 3 \qquad \frac{\partial v}{\partial x} = 6xy$$

$$\frac{\partial u}{\partial y} = 6xy \qquad \frac{\partial v}{\partial y} = 3y^2 + 3x^2 - 3$$

Since the first partial derivatives are continuous $h(z)$ is differentiable on the coordinate axes.

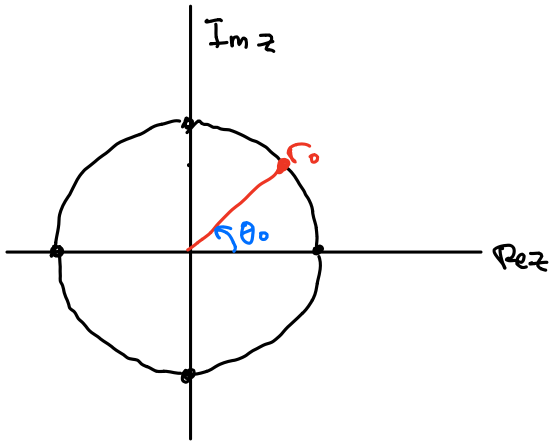
$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$ however, and therefore

$h(z)$ is nowhere analytic.

6) Show

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad , \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

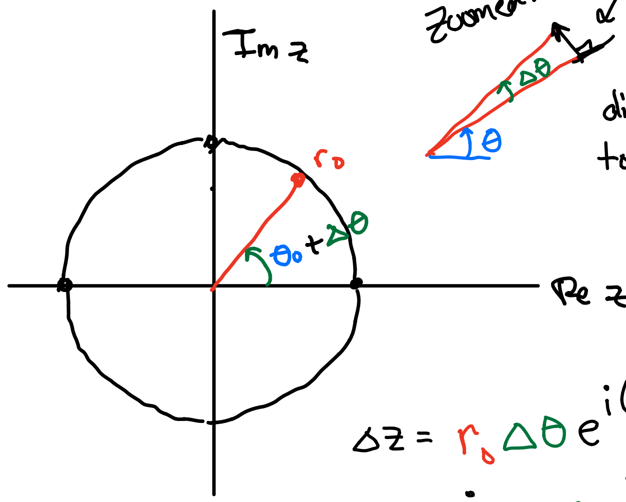
Using the hint,

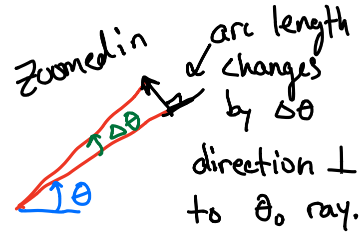


First lets change θ while keeping r constant.

$$f(z) = u(r,\theta) + i\,v(r,\theta)$$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{\Delta f}{\Delta z}$$





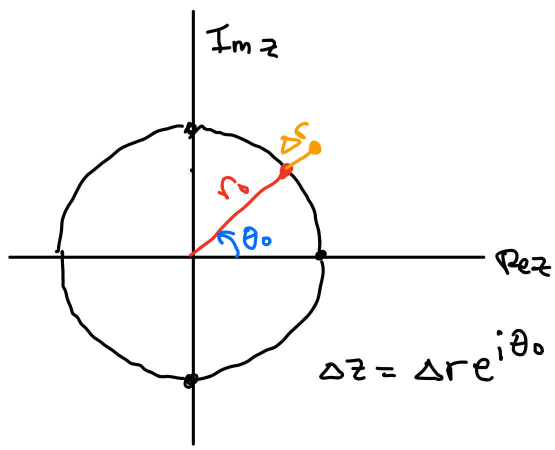
$$\Delta z = r_0 \Delta\theta\, e^{i(\theta_0 + \frac{\pi}{2})}$$

$$\Delta z = i r_0 \Delta\theta\, e^{i\theta_0}$$

$$f'(z) = \frac{\Delta f}{\Delta z} = \frac{\Delta f}{i r_0 \Delta\theta\, e^{i\theta_0}} = \frac{\Delta f}{\Delta\theta} \cdot \frac{1}{i r_0 e^{i\theta_0}} = \frac{\partial f}{\partial\theta} \cdot \frac{1}{i r_0 e^{i\theta_0}}$$

Now lets change r while keeping θ constant.

$$f'(z) = \frac{\Delta f}{\Delta z} = \frac{\Delta f}{\Delta r\, e^{i\theta_0}} = \frac{\partial f}{\partial r} \cdot \frac{1}{e^{i\theta_0}}$$



$$\Delta z = \Delta r\, e^{i\theta_0}$$

Equating our $f'(z)$,

$$f'(z) = \frac{\partial f}{\partial\theta} \cdot \frac{1}{i r_0 \cancel{e^{i\theta_0}}} = \frac{\partial f}{\partial r} \cdot \frac{1}{\cancel{e^{i\theta_0}}}$$

$$\frac{\partial f}{\partial\theta} = i r_0 \frac{\partial f}{\partial r}$$

$$\frac{\partial f}{\partial\theta} = \frac{\partial u}{\partial\theta} + i\frac{\partial v}{\partial\theta} \quad ; \quad \frac{\partial f}{\partial r} = \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}$$

$$\Rightarrow \frac{\partial u}{\partial\theta} + i\frac{\partial v}{\partial\theta} = i r_0\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right] \Rightarrow \boxed{\frac{\partial u}{\partial\theta} = -r_0\frac{\partial v}{\partial r} \quad , \quad \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial\theta}}$$

10) Show if $f(z) \in \mathbb{R}$ + analytic, then $f(z)=c$

If $f(z)$ is real that means

$$f(z) = u(x,y) + iv(x,y) = u(x,y) + i(0) = u(x,y).$$

If $f(z)$ is analytic,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Since $v(x,y) = 0$,

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$$

Therefore $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$

By Theorem 6, if $f(z)$ is analytic and $f'(z) = 0 \ \forall z$, $f(z)$ must be constant.