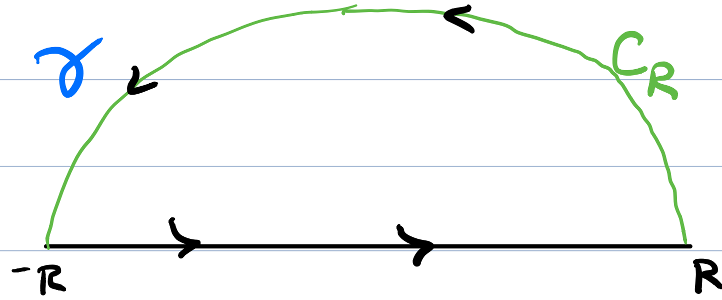
$$\int_{-\infty}^{+\infty} \frac{1}{x^6+1}\,dx$$



$$\int_{\gamma} \frac{1}{z^6+1}\,dz = \lim_{R\to\infty} \int_{-R}^{+R} \frac{1}{x^6+1}\,dx + \int_{C_R} \frac{1}{z^6+1}\,dz$$

$$\int_{\gamma} \frac{1}{z^6+1}\,dz = 2\pi i \sum_{k} \operatorname{Res}\left(\frac{1}{z^6+1} ; z_k\right)$$

$$z^6+1 = 0$$

$$z^6 = -1 = e^{i\pi + 2\pi k}$$

$$z = e^{i\frac{\pi}{6} + \frac{\pi}{3}k}, \quad k = 0,1,2,3,4,5, \dots$$

$$\left.\begin{aligned} z_0 &= e^{i\pi/6} \\ z_1 &= e^{i\pi/2} \\ z_2 &= e^{i\,5\pi/6} \end{aligned}\right\} \text{poles in our contour } \gamma$$

$$\left| \int_{C_R} \frac{1}{z^6+1}\,dz \right| \leq \pi R \cdot \max_{z\in C_R} \left| \frac{1}{z^6+1} \right| \leq \pi R \cdot \frac{1}{1-R^6}$$

as $R \to \infty$ this integral $\to 0$

So,

$$\lim_{R\to\infty} \int_{-R}^{+R} \frac{1}{x^6+1}\,dx = \int_{\gamma} \frac{1}{z^6+1}\,dz = 2\pi i \sum_{k=0}^{2} \operatorname{Res}(f;\, z_k)$$

Where $f = f(z) = \dfrac{1}{z^6+1}$

$$\operatorname{Res}(f;\, z_0) = \lim_{z\to z_0} (z-z_0) f(z)$$

$$= \lim_{z\to z_0} \frac{1}{6z^5} = \frac{1}{6z_0^5}$$

$$\operatorname{Res}(f;\, z_1) = \lim_{z\to i} \frac{1}{6(z)^5} = \frac{1}{6(i)^5}$$

$$\operatorname{Res}(f;\, z_2) = \frac{1}{6(z_2)^5}$$

$$2\pi i \left(\frac{1}{6z_0^5} + \frac{1}{6(i)^5} + \frac{1}{6(z_2)^5} \right)$$

$$= 2\pi i \left(\frac{-4}{12} i \right) = -\frac{8\pi}{12} (i)^2 = \frac{2\pi}{3}$$

w/o Complex analysis?

$$\int_{-\infty}^{+\infty} \frac{1}{x^6+1}\,dx = \int_{-\infty}^{+\infty} \frac{1}{(x^2+1)(x^4-x^2+1)}$$

$$= \int_{-\infty}^{+\infty} \frac{x^2+1-x^2}{(x^2+1)(x^4-x^2+1)}\,dx$$

$$= \frac{1}{2}\int_{-\infty}^{+\infty} \frac{x^2+1-(x^2-1)}{x^4-x^2+1}\,dx - \int_{-\infty}^{+\infty} \frac{x^2}{(x^3)^2+1}\,dx$$

$$= \frac{1}{2}\int_{-\infty}^{+\infty} \frac{1+\frac{1}{x^2}}{x^2-1+\frac{1}{x^2}}\,dx - \frac{1}{2}\int_{-\infty}^{+\infty} \frac{1-\frac{1}{x^2}}{x^2-1+1/x^2}\,dx$$

$$= \frac{1}{2}\int_{-\infty}^{+\infty} \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+1} - \frac{1}{2}\int \frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)^2-(\sqrt{3})^2}\,dx$$

$$= \lim_{b\to\infty} \left(\frac{1}{2}\tan^{-1}\left(x-\frac{1}{x}\right) + \frac{1}{2\sqrt{3}}\tanh^{-1}\left(\frac{1}{\sqrt{3}}\left(x+\frac{1}{x}\right)\right)\right)\Bigg|_{-b}^{+b}$$

$$= L\ \frac{1}{2}\left[\tan^{-1}\left(b-\frac{1}{b}\right) - \tan^{-1}\left(-b+\frac{1}{b}\right)\right]$$

$$= \frac{1}{2}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = \pi$$

$$= L \frac{1}{2\sqrt{3}}\left(\tanh^{-1}\left(b + \frac{1}{b}\right) - \tanh^{-1}\left(-b - \frac{1}{b}\right)\right)$$

$$= 0$$

Back to last integral!

So far

$$\int_{-\infty}^{+\infty} \frac{1}{x^6+1}\,dx = \pi + \int_{-\infty}^{+\infty} \frac{x^2}{(x^3)^2+1}\,dx$$

$$= \pi - \frac{1}{3}\tan^{-1}(x^3)\Big|_{-b}^{b}$$

$$= \pi - \lim_{b\to\infty} \frac{1}{3}\left[\tan^{-1}(b^3) - \tan^{-1}((-b)^3)\right]$$

$$= \pi - \frac{\pi}{3} = \boxed{\frac{2\pi}{3}}$$

$f: \mathbb{R} \to \mathbb{R} \implies f(x) = x(\text{stuff} + c)$

$g: \mathbb{R}^2 \to \mathbb{R} \implies f(x, y) = xy(\text{stuff} + c)$

$h: \mathbb{R}^2 \to \mathbb{R}^2 \implies f(x, y) =$