Recall

Independence - Ω, $\mathbb{P}$

A, B - events - independent when $\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B)$

Same as

$$\mathbb{P}(A|B) = \mathbb{P}(A) \quad \text{when} \quad \mathbb{P}(B) = 0$$
$$\mathbb{P}(B|A) = \mathbb{P}(B) \quad \text{when} \quad \mathbb{P}(A) = 0$$

Looked at examples - one was 2 coin flips, all 4 outcomes equally probable

$A \to$ first head

$B \to$ second hed

$C \to$ two flips different

Any pair of these is independent

Generalization: Say events $A_1, A_2, \ldots, A_n$ are independent when

$\mathbb{P}$(conjunction of any nonempty subsets of these events)

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product of $\mathbb{P}$(events in the conjunction)

i.e A_1, A_2, A_3 are indep. when

$\mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1)\mathbb{P}(A_2)$ and $\mathbb{P}(A_1 \cap A_3) = \mathbb{P}(A_1)\mathbb{P}(A_3)$ and

$\mathbb{P}(A_2 \cap A_3) = \mathbb{P}(A_2)\mathbb{P}(A_3)$ and $\mathbb{P}(A_1 \cap A_2 \cap A_3) = \mathbb{P}(A_1)\mathbb{P}(A_2)\mathbb{P}(A_3)$

CAUTION: Independence doesn't follow from either

- para
- $\mathbb{P}(A_1 \cap \cdots \cap A_n) = \mathbb{P}(A_1) \cdots \mathbb{P}(A_n)$

Conditional Independence: $\Omega, \mathbb{P}$; say events A and B are <u>conditionally</u> dependent given (event) C when

$$\mathbb{P}(A \cap B \mid C) = \mathbb{P}(A \mid C)\,\mathbb{P}(B \mid C)$$

$$\mathbb{P}(A \mid B \cap C) = \mathbb{P}(A \mid C) \;;\quad \mathbb{P}(B \mid A \cap C) = \mathbb{P}(B \mid C)$$

Knowledge of B gives no function info about probability of A on top of knowledge of C

To see this just play w/ formulas

$$\mathbb{P}(A \mid B \cap C) = \frac{\mathbb{P}(A \cap B \cap C)}{\mathbb{P}(B \cap C)} = \frac{\mathbb{P}(A \cap B \mid C)\,\mathbb{P}(B \mid C)}{\mathbb{P}(B \mid C)} = \mathbb{P}(A \mid C)$$

Suppose A and B are independent; are they <u>conditionally</u> dependent given some random variable C?

NO! - eg - flip one fair coin twice - all outcomes equally likely.

A - first flip head
B - second flip head
C - two flips different

$\left.\begin{array}{l} \\ \\ \\ \end{array}\right\}$ independent

A, B $\to$ independent
A, C $\to$ independent
B, C $\to$ independent

$$\mathbb{P}(A \cap B \mid C) = 0$$ ← Can't have both be heads if they're different!

If they were
$\mathbb{P}(A \cap B \mid C) = \mathbb{P}(A \mid C)\mathbb{P}(B \mid C) = 1/4$

So we conclude, A + B are independent, but NOT **conditionally** dependent given C.

Similarly, even when A and B are conditionally independent given C, A and B need not be independent.

Example - Coins

Have two unfair coins

Coin 1: $\mathbb{P}(H) = \frac{2}{3}$ Coin 2: $\mathbb{P}(H) = \frac{1}{3}$

Pick one coin and flip it twice.

Let

$$C = \text{event "pick coin 1"}$$

Assume

$$\mathbb{P}(C) = 1/2$$

Let

$$A \to \text{head on first flip}$$
$$B \to \text{head on second flip}$$

Claim: A, B are conditionally dependent given C.

$$\mathbb{P}(A \cap B \mid C) = \mathbb{P}(A \mid C)\,\mathbb{P}(B \mid C)$$

If you know what coin you're flipping

$\Rightarrow \mathbb{P}(2 \text{ heads} \mid C)$, then $\frac{2}{3} \cdot \frac{2}{3}$ or $\frac{1}{3} \cdot \frac{1}{3}$, etc

Aside: why is $\mathbb{P}(A \cap B \mid C) = \frac{2}{3} \cdot \frac{2}{3}$? Because of independence of flips knowing you're using coin 1

$\Rightarrow \mathbb{P}(\text{Head second} \cap \text{Head first})$

$$= \underbrace{\mathbb{P}(H \text{ on } 2^{nd} \mid H \text{ on } 1^{st})}_{2/3}\ \underbrace{\mathbb{P}(H \text{ on first})}_{2/3}$$

However, can easily show that A and B not independent. Do this by computing $\mathbb{P}(A)$, $\mathbb{P}(B)$ and $\mathbb{P}(A \cap B)$ using Total Probability Theorem; find $\mathbb{P}(A \cap B) \neq \mathbb{P}(A)\mathbb{P}(B)$.

Eg

$$\mathbb{P}(A) = \mathbb{P}(A \mid C)\mathbb{P}(C) + \mathbb{P}(A \mid C^c)\mathbb{P}(C^c)$$
$$= \frac{2}{3} \cdot \frac{1}{2} + \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{2}, \text{ etc}$$

Last "foundational" topic: Counting / Combinatorics

Useful especially in contexts where "all outcomes in Ω are equally probable"

Example - Counting (Elementary)

Given $\Omega = \{s_1, s_2, \ldots, s_n\}$, how many events are there? I.e how many subsets of Ω are there?

Claim: Set of subsets of Ω is in 1-to-1 correspondence / set of all binary n-strings

Idea: Given $A \subset \Omega$, each s_i claim is or isn't in A. Define in binary string.

$\mathrm{Str}(A) =$ string w/ $\begin{cases} \text{zero in position } k \text{ when } s_k \notin A \\ 1 \text{ in position } k \text{ when } s_k \in A \end{cases}$

Similarly, every binary n string "defines" a subset of Ω, and "everybody knows" there are 2^n binary n-strings. Thus,

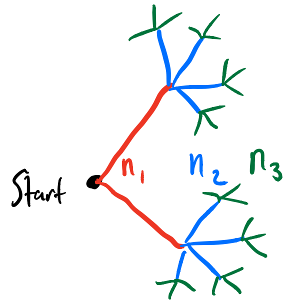
$$\#(\text{Subsets of } \Omega) = 2^n$$

Generally, interested in counting up objects that are new as arising from a sequential buildup ________ of the following type:

Stage 1: Make a choice from among n_1 options
Stage 2: Make a choice from among n_2 options no matter what choice you made at stage 1
⋮
Stage r: Make a choice from among n_r options no matter what choice you made at stage $(r-1)$

Example – Tree Building

Say you have $r=3$. $n_1=2$, $n_2=4$, $n_3=3$



each object you build by this process corresponds w/ a terminal node on the tree. # terminal nodes $= n_1 n_2 \cdots n_r$

Counting Principle:

Re-count # of subsets of $\Omega = \{s_1, \ldots, s_n\}$.

We can view each subset as arising from a multi stage building process.

Stage 1: Choose whether to put s_1 in the subset
- either yes or no $(n_1 = 2)$

Stage 2: Choose whether to put s_2 in the subset
- either yes or no $(n_2 = 2)$

⋮

Stage n: Choose whether to put s_n in the subset
- either yes or no $(n_n = 2)$

$$\#(\text{Subsets}) = n_1 n_2 \cdots n_n = 2^n$$