

Recall

Definition: $\{X(t)\}_{t \geq 0}$ is a Markov process if $X(t_n) = x_n$ is all that matters for determining the value at $X(t_{n+1})$

i.e.

$$\forall n, t_1 < t_2 < \dots < t_n < t_{n+1}$$

$$Pr[X(t_{n+1}) = x_{n+1} \mid X(t_n) = x_n, X(t_{n-1}) = x_{n-1}, \dots, X(t_1) = x_1] = Pr[X_{n+1} = x_{n+1} \mid X_n = x_n]$$

State: $X(t)$ is called the state at time t .

All the values $X(t)$ can take is called the state-space \mathcal{X}

If \mathcal{X} is discrete, then we call this a Markov chain.

Example: Random Walk, $\mathcal{X} = \mathbb{Z}$

is a discrete state-space \Rightarrow Markov Chain

Example: Brownian motion: $\mathcal{X} = \mathbb{R}$

is a continuous state-space \Rightarrow Markov process

Example: Binomial Counting: $\mathcal{X} = \mathbb{N}$

Discrete state-space
Markov chain

Example: Poisson Process: $\mathcal{X} = \mathbb{N}$

continuous state-space
Markov chain

Discrete-Time Markov Chain

$\{X_n\}_{n=0}^\infty$ is a random process

$k, \{n_i\}_{i=1}^k$

$X_{n_1}, \dots, X_{n_k} \rightarrow$ need joint pmf to describe it

$$\Pr[X_0=x_0, X_1=x_1, \dots, X_k=x_k]$$

$$= \Pr[X_0=x_0] \Pr[X_1=x_1 | X_0=x_0] \Pr[X_2=x_2 | X_1=x_1, X_0=x_0] \dots \Pr[X_k=x_k | X_{k-1}=x_{k-1}, \dots, X_0=x_0]$$

$$= \Pr[X_0=x_0] \Pr[X_1=x_1 | X_0=x_0] \Pr[X_2=x_2 | X_1=x_1] \dots \Pr[X_k=x_k | X_{k-1}=x_{k-1}]$$

one-step transition probability

i.e. for Markov process only need the marginal distribution of the initial state and the transition probabilities from one state to the next.

~~i.e.~~ Need $P_{X_0}(x_0)$ and $\{\Pr[X_{n+1}=j | X_n=i]\}_{i,j \in \mathcal{X}}$

Homogeneous Markov Chain \rightarrow Time INVARIANT

$$\Pr[X_{n+1}=j | X_n=i] \triangleq P_{i,j} \quad \forall n$$

Transition Matrix:

$$P = \begin{bmatrix} P_{1,1} & P_{1,2} & \dots & P_{1,M} \\ P_{2,1} & P_{2,2} & \dots & P_{2,M} \\ \vdots & \vdots & \ddots & \vdots \\ P_{M,1} & P_{M,2} & \dots & P_{M,M} \end{bmatrix}$$

$$\mathcal{X} = \{1, 2, \dots, M\}$$

$$0 \leq P_{i,j} \leq 1$$

$$\sum_j P_{i,j} = 1 \quad \forall i$$

Example:

Coin A and Coin B

A: fair

B: biased, $\Pr[H] = \frac{1}{4}$

X_0 : flip A $\begin{pmatrix} H=1 \\ T=2 \end{pmatrix}$

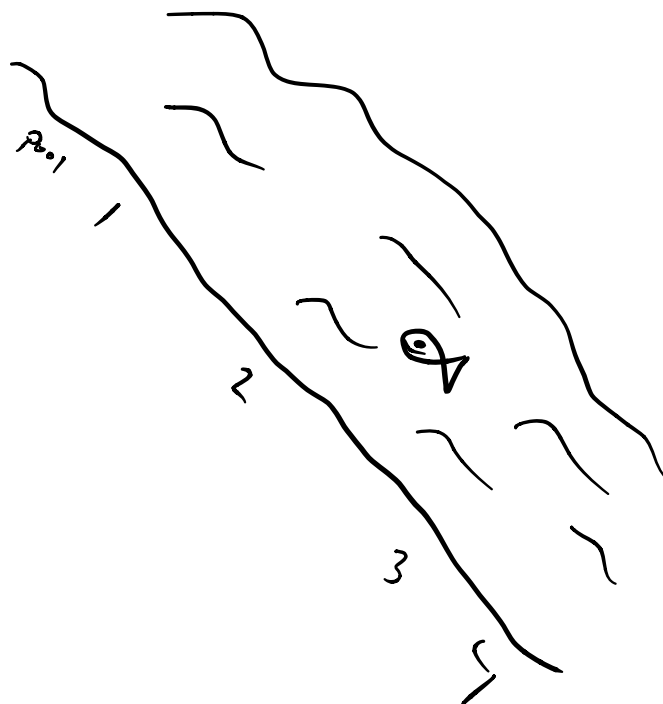
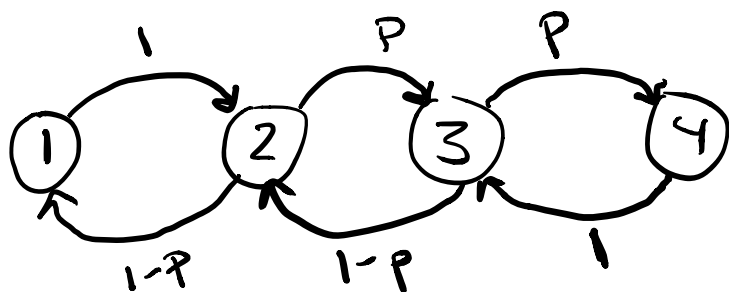
X_{n+1} : outcome of flip B if $X_n = 1$

outcome of flip A if $X_n = 2$

$$\Pr[X_0 = 1] = \Pr[X_0 = 2] = \frac{1}{2}$$

$$P = \begin{matrix} & \begin{matrix} 1 & 2 \end{matrix} \\ \begin{matrix} 1 \\ 2 \end{matrix} & \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \end{matrix}$$

Example: A Fish Called Wanda



$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1-p & 0 & p & 0 \\ 0 & 1-p & 0 & p \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

n-Step Transition Matrix

$$P^{(n)} = \left\{ P_{i,j} \triangleq \Pr[X_n = j | X_0 = i] \right\}$$

Chapman-Korogorov Equation

$$P_{i,j}^{(m+n)} = \sum_{k \in \mathcal{X}} \underbrace{P_{i,k}^{(m)}}_{i \times k} \underbrace{P_{k,j}^{(n)}}_{k \times j}$$

$\downarrow i \times j$

$$P^{(m+n)} = P^{(m)} P^{(n)}$$

Can use this to obtain

$$\begin{aligned} P^{(n)} &= P P^{(n-1)} \\ &= P P P^{(n-2)} \\ &\vdots \\ &= P^n \end{aligned}$$

$$\{X_n\}_{n \geq 0} : \vec{p}(0) = \underbrace{\Pr[X_0=1, \dots, X_0=m]}_{\substack{\text{pmf of } X_0 \\ P}}$$

row vector
by definition

$$\vec{p}(n) \triangleq \Pr[X_n=1, \dots, X_n=m]$$

$$\begin{aligned} \Pr[X_n = j] &= \sum_{i=1}^m \Pr[X_0 = i] \Pr[X_n = j | X_0 = i] \\ &= \vec{p}(0) P^{(n)} \end{aligned}$$

State Probability

This is the distribution at time n .

$$p(n) = [Pr[x_n=1] \quad \dots \quad Pr[x_n=m]] \leftarrow \text{vector}$$

$$p(0) = [Pr[x_0=0] \quad \dots \quad Pr[x_0=m]]$$

$$p(1) = p(0)P$$

:

$$p(n) = p(0)P^n$$

Stationary Distribution

$$\pi = \pi P, \pi \text{ is stationary}$$