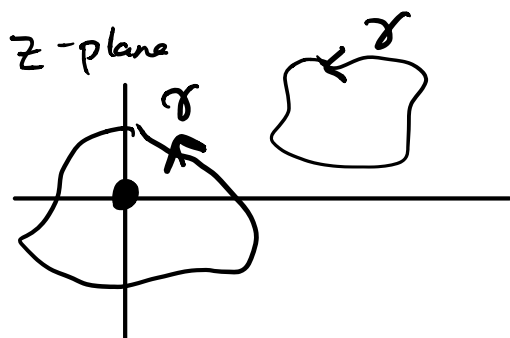


Winding Numbers and Rouché's Theorem

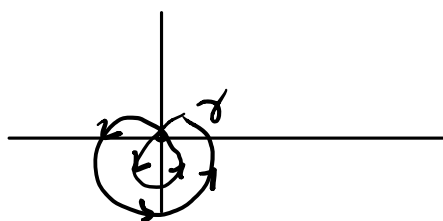
Recall

$$\frac{1}{2\pi i} \oint_{\gamma} \frac{1}{w} dw = \begin{cases} 1, & \gamma \text{ encloses origin} \\ 0, & \gamma \text{ does NOT enclose origin} \end{cases}$$



But say we looped over the origin more than once.

Then



$$\frac{1}{2\pi i} \oint_{\gamma} \frac{1}{w} dw = \begin{cases} 2, & \gamma \text{ encloses origin} \\ 0, & \gamma \text{ does NOT enclose origin} \end{cases}$$

We can therefore define the winding number

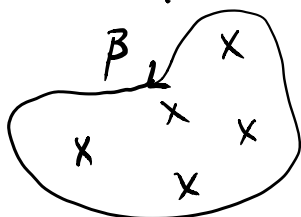
$$w(\gamma; \overset{\text{singularity}}{\downarrow} 0) = \frac{1}{2\pi i} \oint_{\gamma} \frac{1}{w} dw$$

which is the amount of times γ encircles our singularity.

This is a **device for counting zeros of an analytic function**.

Say $f(z)$ is analytic in and inside β

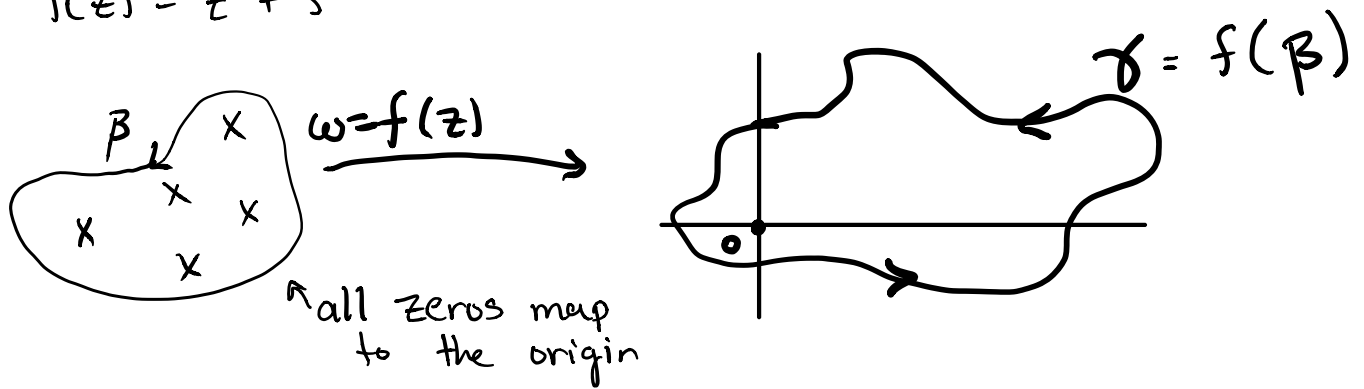
where β is



Assumption, $f'(z) \neq 0$ on β .

Map β onto another region using f .

For $f(z) = z^2 + 3$



Note: Simple $\beta \Rightarrow$ simple γ (0 is not on γ)

$$W(\gamma; 0) = \frac{1}{2\pi i} \oint_{\gamma} \frac{1}{w} dw$$

$$f(z) = 0 \text{ when } z = \pm \sqrt{3}i$$

$$\begin{aligned} w &= f(z) & \rightarrow & \frac{1}{2\pi i} \oint_{\beta} \frac{1}{f(z)} f'(z) dz = \# \text{ zeros} \\ dw &= f'(z) dz & & \text{inside } \beta, \text{ count} \\ & & & \text{multiplicities} \end{aligned}$$

Suppose f is analytic inside and on β , and $\neq 0$ on β .

Then

$$\# \text{Zeros of } f \text{ inside } \beta, \text{ count multiplicities} = \frac{1}{2\pi i} \oint_{\beta} \frac{f'(z)}{f(z)} dz$$

Proof

Suppose a_1, a_2, \dots, a_m are zeros of f inside β
with n_1, n_2, \dots, n_m multiplicities.

Then

$$f(z) = (z - a_1)^{n_1} h(z)$$

where

$h(z)$ does NOT have a root of a_1 .

Continuing,

$$f(z) = (z - a_1)^{n_1} (z - a_2)^{n_2} \dots (z - a_m)^{n_m} h(z)$$

$\Rightarrow h(z)$ doesn't have any zeros inside or on β

Aside:

$$\text{If } f = f_1 f_2, \quad f' = f_1' f_2 + f_1 f_2'$$

$$\frac{f'}{f} = \frac{f_1' f_2 + f_1 f_2'}{f_1 f_2} = \frac{f_1'}{f_1} + \frac{f_2'}{f_2}$$

$$\begin{aligned} \text{So } \frac{f'(z)}{f(z)} &= \frac{n_1 (z - a_1)^{n_1-1}}{(z - a_1)^{n_1}} + \frac{n_2 (z - a_2)^{n_2-1}}{(z - a_2)^{n_2}} + \dots + \frac{n_m}{(z - a_m)} + \frac{h'(z)}{h(z)} \\ &= \frac{n_1}{(z - a_1)} + \frac{n_2}{(z - a_2)} + \dots + \frac{n_m}{(z - a_m)} + \frac{h'(z)}{h(z)} \end{aligned}$$

$\frac{h'}{h}$ is analytic int on β

$$S_o, \quad \frac{1}{2\pi i} \left[\oint_{\Gamma} \frac{n_1}{(z-a_1)} + \frac{n_2}{(z-a_2)} + \dots + \frac{n_m}{(z-a_m)} + \frac{h'(z)}{h(z)} dz \right]$$

$$= \frac{1}{2\pi i} \left[2\pi i n_1 + 2\pi i n_2 + \dots + 2\pi i n_m + 0 \right]$$

$$= n_1 + n_2 + \dots + n_m$$

Back to example, $f(z) = z^2 + 3$; count zeros inside circle of radius R at 0

$$\frac{1}{2\pi i} \oint_{|z|=R} \frac{2z}{z^2+3} dz$$

if $R > \sqrt{3}$

$$= \frac{1}{2\pi i} \left[\text{Res} \left\{ \frac{f'}{f}, \sqrt{3}i \right\} + \text{Res} \left\{ \frac{f'}{f}, -\sqrt{3}i \right\} \right]$$

$$= \frac{1}{2\pi i} [2\pi i + 2\pi i] = 4$$

OR

$$\frac{1}{2\pi i} \int_0^{2\pi} \frac{2Re^{it}}{(Re^{it})^2 + 3} Rie^{it} dt$$

In practice!
Don't always know root location.

n?? Trapezoid??

Rouche's Theorem

Baby version: Given two closed curves γ_1, γ_2 . Then provided that $|\gamma_2(t)| \leq |\gamma_1(t)|$ for all t , and γ_1, γ_2 do not intersect 0,

$$\text{Then } W(\gamma_1; 0) = W(\gamma_1 + \gamma_2; 0)$$

Let $f(z)$, $g(z)$ be analytic functions on and inside a simple closed curve β , and $|f(z)| > |g(z)|$.

Then

$$\# \text{ zeros of } f = \# \text{ of zeros of } f+g$$

$$\frac{1}{2\pi i} \int_{\beta} \frac{f'}{f} dz = W(f(\beta); 0)$$

$$= \frac{1}{2\pi i} \int_{\beta} \frac{(f+g)'}{f+g} dz = W(f(\beta)+g(\beta); 0)$$

Example: Determine number of zeros of $z^4 - 2z^3 + 9z^2 + z - 1$
inside $|z| = 2$

Pick

$$f(z) = 9z^2$$

$$g(z) = z^4 - 2z^3 + z - 1$$

$$|f(z)| = 36$$

$$|g(z)| \leq 16 + 16 + 2 + 1 = 35$$

\Rightarrow 2 zeros inside circle