ECE4110: Random Signals in Communications and Signal Processing

Discrete-Time Markov Chain

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Outline

Concepts

- Markov processes: definition and examples.
- Discrete-time Markov chain:
  - Homogeneity and transition matrix.
  - n-step transition matrix, Chapman-Kolmogorov equations, and state probabilities.
  - Stationary distribution.
  - Communicating class and irreducibility.
  - Period and aperiodicity.
  - Recurrence and transience.
  - Existence and uniqueness of stationary distribution.
  - Absorption time and absorption probability.
  - Limiting distribution.

Markov Process

Conditional Independent Events:

Events A and B are independent conditioning on C if

$$\Pr(AB|C) = \Pr(A|C)\Pr(B|C),$$

or equivalently,

$$\Pr(A|BC) = \Pr(A|C)$$

Conditional Independent Random Variables:

(Continuous) random variables X and Y are independent conditioning on Z if

$$f_{XY|Z}(x, y|z) = f_{X|Z}(x|z) f_{Y|Z}(y|z)$$

Markov Process

A random process $\{X(t)\}_{t\geq 0}$ is said to be a Markov process if, for all n and for all $t_1 < t_2 < \ldots < t_{n+1}$, $X(t_{n+1})$ is independent of $X(t_{n-1}), \ldots, X(t_1)$ conditioning on $X(t_n)$, i.e.,

$$\begin{aligned} & \Pr\{X(t_{n+1}) = x_{n+1} | X(t_n) = x_n, X(t_{n-1}) = x_{n-1}, \cdots, X(t_1) = x_1\} \\ = \ & \Pr\{X(t_{n+1}) = x_{n+1} | X(t_n) = x_n\} \end{aligned}$$

for all x_i (assuming $X(t)$ taking discrete values).

State and State Space

If $\{X(t)\}_{t\geq 0}$ is a Markov process, then $X(t)$ is called the state at time t. The set $\mathcal{X}$ of all possible values $X(t)$ can take is called the state space. A Markov process with a discrete state space is called a Markov chain.

Markov Process and Independent Increments

Markov Process and Independent Increments:

- If $\{X(t)\}_{t\geq 0}$ has independent increments and $X(0) = c$ (a constant), then $\{X(t)\}_{t\geq 0}$ is a Markov process.
- The converse is not true: a Markov process may have dependent increments.

Examples:

- Random walk is a discrete-time Markov chain with $\mathcal{X} = \mathbb{Z}$.
- Brownian motion is a continuous-time Markov process with $\mathcal{X} = \mathbb{R}$.
- Poisson process is a continuous-time Markov chain with $\mathcal{X} = \mathbb{Z}^+$.

Discrete-Time Markov Chain

Characterization of Discrete-Time Markov Chain:

A discrete-time Markov chain $\{X_n\}_{n\geq 0}$ is characterized by

- the PMF $\mathbf{p}(0)$ of its initial state X_0:

$$\mathbf{p}(0) \triangleq \left[\underbrace{\Pr[X_0 = i]}_{p_i(0)}\right]_{i\in\mathcal{X}}$$

- and one-step transition probabilities:

$$\left\{\Pr[X_{n+1} = j | X_n = i]\right\}_{n\geq 0, i,j\in\mathcal{X}}$$

Remark: the joint PMF is given by

$$\Pr[X_0 = i_0, \cdots, X_n = i_n] = \Pr[X_0 = i_0\} \Pr[X_1 = i_1 | X_0 = i_0] \ldots \Pr[X_n = i_n | X_{n-1} = i_{n-1}]$$

Homogeneous Markov Process

A Markov chain is said to be homogeneous (time invariant) if the transition probability $\Pr[X_{n+1} = j | X_n = i]$ does not depend on n:

$$\Pr[X_{n+1} = j | X_n = i] = \Pr[X_1 = j | X_0 = i] \triangleq p_{i,j}.$$

Probability Transition Matrix

A homogeneous Markov chain is characterized by its initial state PMF $\mathbf{p}(0)$ and a probability transition matrix $\mathbf{P} = \left[p_{i,j}\right]_{i,j\in\mathcal{X}}$ satisfying

$$\bullet\ p_{i,j} \geq 0; \qquad \bullet\ \sum_{j\in\mathcal{X}} p_{i,j} = 1,\ \forall i$$

Two Examples

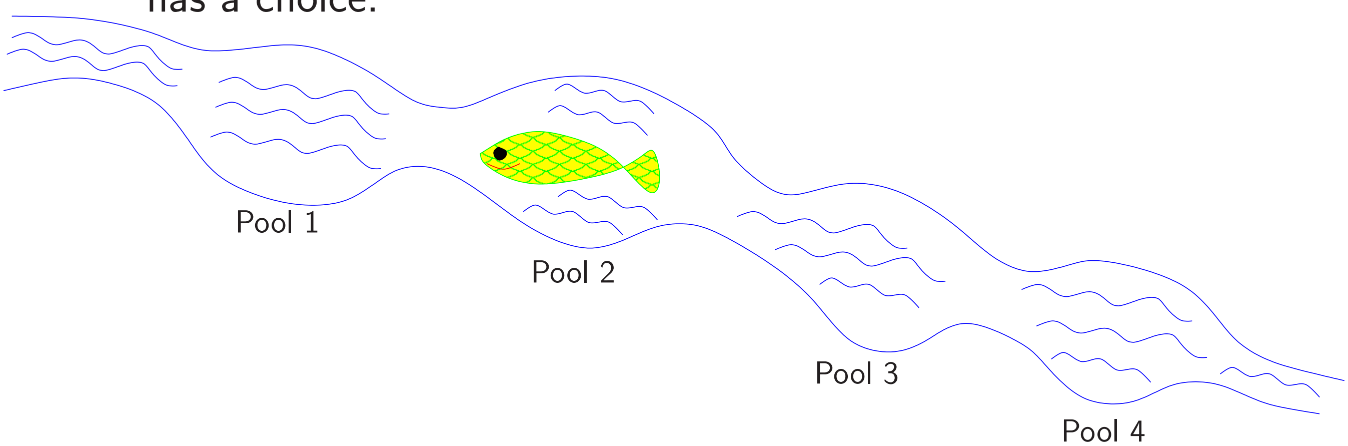
Examples:

- Coin A and Coin B

  Two coins, A is fair, B has a "head" probability of $1/4$. Consider the Markov chain given by i) X_0 is the outcome of flipping coin A; ii) X_{n+1} is the outcome of flipping the biased one if $X_n = 1$ ("head") and the outcome of flipping the fair one if $X_n = 2$ ("tail").

- A Fish Called Wanda

  Wanda changes pool at every discrete time. She goes downstream with probability p and upstream with probability $1 - p$ when she has a choice.



Remarks:

Both examples lead to a homogeneous Markov chain with transition matrix

$$\mathbf{P} = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \qquad \mathbf{P} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1-p & 0 & p & 0 \\ 0 & 1-p & 0 & p \\ 0 & 0 & 1 & 0 \end{pmatrix}$$