

# The theory of analytic functions.

So far we have viewed a complex function of a complex variable,  $f(z)$ , as an arbitrary mapping from the  $xy$ -plane to the  $uv$ -plane.

Notice there is something special about the pair

$$u_1(x,y) = x^2 - y^2, \quad v_1(x,y) = 2xy,$$

as opposed to,

$$u_2(x,y) = x^2 - y^2, \quad v_2(x,y) = 3xy:$$

namely, the complex function

$$u_1 + i v_1$$

treats

$$z = x + iy$$

as a single "unit", because it equals

$$x^2 - y^2 + i2xy = (x + iy)^2$$

and thus it respects the complex structure of  
 $z = x + iy$ .

However, the formulation  $u_2 + i v_2$  requires us to break apart the real + imaginary parts of  $z$ .

- In (real) calculus we don't deal with functions that look at a number  $3+4\sqrt{2}$  and square the 3 but cube the 4.
- The interesting calculus functions treat the number as an individual module.
- We seek to classify the complex functions that behave this same way with regard to their complex argument.

Thus we want to admit functions such as

$$\begin{aligned} z &= x+iy && \text{(admissible)} \\ z^2 &= x^2 - y^2 + i2xy && \text{(admissible)} \\ z^3 &= x^3 - 3xy^2 + i(3x^2y - y^3) && \text{(admissible)} \\ \frac{1}{z} &= \frac{x}{x^2+y^2} - i \frac{y}{x^2+y^2} && \text{(admissible)} \end{aligned}$$

and their basic arithmetic combinations (sums, products, quotients, powers, and roots) but ban functions such as

$$\begin{aligned} \operatorname{Re} z &= x && \text{(inadmissible)} \\ \operatorname{Im} z &= y && \text{(inadmissible)} \\ x^2 - y^2 + i3xy & && \text{(inadmissible)} \end{aligned}$$

Notice we must ban the conjugate function  $\bar{z}$ , because if we admit it we open the gate to

$$x = \frac{(z+\bar{z})}{2}, \quad y = \frac{(z-\bar{z})}{2}$$

$$z = x - iy \quad (\text{inadmissible})$$

Similarly, admitting  $|z|$  would be a mistake as well, since  $\bar{z} = |z|^2/z$

$$|z| \quad (\text{inadmissible})$$

One could criticize our "inadmissible" classifications of

$$u_2 + iv_2 = x^2 - y^2 + i3xy$$

because we have not yet proved that it could be written in terms of  $z$  alone.

The following computation is instructive: we set

$$x = \left( \frac{z + \bar{z}}{2} \right) \quad , \quad y = \left( \frac{z - \bar{z}}{2i} \right) \quad (1)$$

in  $u_2 + iv_2$  and obtain,

$$\begin{aligned} u_2 + iv_2 &= x^2 - y^2 + i3xy \\ &= \frac{(z + \bar{z})^2}{4} + i \frac{3(z + \bar{z})}{2} \frac{(z - \bar{z})}{2i} \end{aligned}$$

$$= \frac{5}{4}z^2 - \frac{1}{4}\bar{z}^2$$

Now we see that if we admit  $u_2 + i v_2$ , we would have to admit  $\bar{z}^2$  and its undesirable root  $\bar{z}$ .

**Example 1** Express the following in terms of  $z, \bar{z}$ :

$$f_1(z) = \frac{x - 1 - iy}{(x-1)^2 + y^2}$$

$$f_2(z) = x^2 + y^2 + 3x + 1 + i3y$$

$$f_1(z) = \frac{\frac{z+\bar{z}}{2} - 1 - i\left(\frac{z-\bar{z}}{2i}\right)}{\left(\frac{z+\bar{z}}{2} - 1\right)^2 + \left(\frac{z-\bar{z}}{2i}\right)^2}$$

$$= \frac{\bar{z}-1}{z\bar{z}-z-\bar{z}+1} = \frac{1}{z-1}$$

$$\begin{aligned} f_2(z) &= \frac{(z+\bar{z})^2}{4} + \frac{(z-\bar{z})^2}{4i^2} + 3\left(\frac{z+\bar{z}}{2}\right) + 1 + i3\left(\frac{z-\bar{z}}{2i}\right) \\ &= z\bar{z} + 3z + 1 \end{aligned}$$

The process of disqualifying functions with  $\bar{z}$  in their formulas doesn't lead to a workable criterion.

The criterion we are seeking (the test that will distinguish admissible functions from others) can be expressed simply in terms of differentiability.

**Definition 4.** Let  $f$  be a complex-valued function defined in a neighbourhood of  $z_0$ . Then the derivative of  $f$  at  $z_0$  is given by

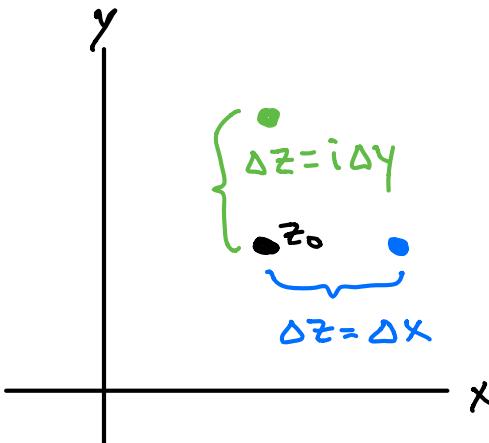
$$\frac{df}{dz}(z_0) \equiv f'(z_0) := \lim_{z \rightarrow z_0} \frac{f(z+z_0) - f(z_0)}{\Delta z}$$

provided this limit exists. (Such an  $f$  is said to be **differentiable** at  $z_0$ .)

The catch here is that  $\Delta z$  is a complex number, so it can approach zero in many different ways; but the difference quotient must tend to a **unique** limit  $f'(z_0)$  independent of the manner in which  $\Delta z \rightarrow 0$ .

Let's see why this notion disqualifies  $\bar{z}$ .

**Example 2:** Show that  $f(z) = \bar{z}$  is nowhere differentiable.



← Horizontal + Vertical approach  
to zero of  $\Delta z$ .

The difference quotient for this function takes the form:

$$\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{\overline{(z_0 + \Delta z)} - \bar{z}_0}{\Delta z} = \frac{\overline{\Delta z}}{\Delta z}$$

If  $\Delta z \rightarrow 0$  through real values, then  $\Delta z = \Delta x$  and  $\overline{\Delta z} = \Delta z$ , so the difference quotient is 1.

If  $\Delta z \rightarrow 0$  from above, then  $\Delta z = i \Delta y$  and  $\overline{\Delta z} = -\Delta z$ , so the quotient is -1.

Consequently, there's no way of assigning a unique value to the derivative of  $\bar{z}$  at any point.

Hence  $\bar{z}$  is NOT differentiable.

**Example 3:** Show that for any positive integer  $n$ ,

$$\frac{d}{dz} z^n = nz^{n-1} \quad (f(z) = z^n) \quad (4)$$

$$\begin{aligned} \frac{d}{dz} z^n &= \lim_{\Delta z \rightarrow 0} \frac{(z + \Delta z)^n - z^n}{\Delta z} \\ &= \lim_{\Delta z \rightarrow 0} \frac{n z^{n-1} \Delta z + \frac{n(n-1)}{2} z^{n-2} (\Delta z)^2 + \cdots + (\Delta z)^n}{\Delta z} \\ &= nz^{n-1} \end{aligned}$$

Binomial Formula

Note the similarity to the real-variable case

**Theorem 3:** If  $f$  and  $g$  are differentiable at  $z$ , then

$$(f \pm g)'(z) = f'(z) \pm g'(z) \quad (5)$$

$$(cf)'(z) = cf'(z) \quad (6)$$

$$(fg)'(z) = f(z)g'(z) + f'(z)g(z) \quad (7)$$

$$(f/g)'(z) = \frac{g(z)f'(z) - f'(z)g(z)}{g(z)^2}, \quad g(z) \neq 0 \quad (8)$$

If  $g$  is differentiable at  $z$  and  $f$  is differentiable at  $g(z)$ , then the chain rule holds

$$\frac{d}{dz} f(g(z)) = f'(g(z)) g'(z) \quad (9)$$

As in the real variable case, differentiability implies continuity.

It follows that any polynomial in  $z$ ,

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$$

is differentiable in the whole plane and its derivative is given by

$$P'(z) = n a_n z^{n-1} + (n-1) a_{n-1} z^{n-2} + \dots + a_1,$$

**Example 4:** Compute the derivative of

$$f(z) = \left( \frac{z^2 - 1}{z^2 + 1} \right)^{100}$$

Unless  $z = \pm i$ ,

$$\begin{aligned} f'(z) &= 100 \left( \frac{z^2 - 1}{z^2 + 1} \right)^{99} \frac{(z^2 + 1) 2z - (z^2 - 1) 2z}{(z^2 + 1)^2} \\ &= 400z \frac{(z^2 - 1)^{99}}{(z^2 + 1)^{101}} \end{aligned}$$

**Definition 5.** A complex-valued function  $f(z)$  is said to be **analytic** on an open set  $G$  if it has a derivative at every point  $b$ .

We **EMPHASIZE** that analyticity is a property defined over open sets, while differentiability could conceivably hold at one point only.

We use " $f(z)$  is analytic at the point  $z_0$ " to mean that  $f(z)$  is analytic in some neighbourhood of  $z_0$ .

A point where  $f$  is NOT analytic but which is the limit of points where  $f$  is analytic is known as a **Singularity**.

Thus we can say a rational function  $z$  is analytic at every point for which its denominator is nonzero, and the zeros of the denominators are singularities.

If  $f(z)$  is analytic on the whole complex plane, it is said to be **entire**.