① Differentiating a Poisson Process

Let $\{N(t)\}_{t\geq 0}$ be a Poisson process with rate λ. We know that the mean function and the autocovariance process of a Poisson process are given by

$$\mathbb{E}[N(t)] = \lambda t$$
$$C_N(s,t) \triangleq \mathbb{E}[(N(s)-\mathbb{E}[N(s)])(N(t)-\mathbb{E}[N(t)])] = \lambda \min(s,t)$$

(a) Is $\{N(t)\}_{t=0}^{\infty}$ wide sense stationary?

$$\mathbb{E}[N(t)] = \lambda t$$
$$R_N(s,t) = C_N(s,t) + \mu_N(s)\mu_N(t) = \lambda \min(s,t) + \lambda^2 st$$

__No__! Nonconstant mean. Also autocorrelation depends on both s,t.

(b) For a constant $\varepsilon > 0$, define a random process $Y(t)$ as

$$Y(t) = \frac{N(t+\varepsilon) - N(t)}{\varepsilon}.$$

Compute $\mu_Y(t)$ and $C_Y(s,t)$ of $\{Y(t)\}_{t\geq \varepsilon}$

$$\mu_Y(t) \triangleq \mathbb{E}[Y(t)] = \frac{\mathbb{E}[N(t+\varepsilon) - N(t)]}{\varepsilon}$$
$$= \frac{1}{\varepsilon}(\lambda t + \lambda\varepsilon - \lambda t) = \lambda$$

$$C_Y(s,t) = \operatorname{Cov}(Y(s), Y(t))$$
$$= \operatorname{Cov}\left(\tfrac{1}{\varepsilon}(N(s+\varepsilon)-N(s)),\ \tfrac{1}{\varepsilon}(N(t+\varepsilon)-N(t))\right)$$
$$= \frac{1}{\varepsilon^2}\operatorname{Cov}\left(N(s+\varepsilon)-N(s),\ N(t+\varepsilon)-N(t)\right)$$

Two cases. Assume $s \leq t$.

① $|s-t| > \varepsilon \rightarrow$ no overlap

② $|s-t| \leq \varepsilon \rightarrow$ overlapping intervals

Case 1: $C_Y(s,t) = 0$ since independent increments ($\{N(t)\}$ Poisson)

Case 2: $\frac{1}{\varepsilon^2}\left[\lambda \min(s+\varepsilon, t+\varepsilon) - \lambda \min(s+\varepsilon, t) - \lambda \min(t, s+\varepsilon) - \lambda \min(s,t)\right]$

$$= \frac{\lambda}{\varepsilon^2}(\varepsilon - |t-s|)$$

(c) Is $\{Y(t)\}$ WSS?

Yes! Observable from work in b.

② A modified Brownian Motion

Let $\{X(t)\}_{t\geq 0}$ be a Brownian motion with parameter σ^2. We have shown that the mean function and the autocorrelation function of a Brownian motion are given by

$$\mu_X(t) \triangleq \mathbb{E}[X(t)] = 0$$
$$R_X(s,t) \triangleq \mathbb{E}[X(s)X(t)] = \sigma^2 \min(s,t)$$

and Brownian motion is a Gaussian process.

For a fixed $\tau \in (0,1)$, define a random process $\{Y(t)\}_{0\leq t\leq 1}$ as

$$Y(t) = \begin{cases} X(t) & , \ t \leq \tau \\ 2X(\tau) - X(t) & , \ \tau < t \leq 1 \end{cases}$$

(a) Compute the mean function of $\{Y(t)\}_{0\leq t\leq 1}$

$$\mu_Y(t) \triangleq \mathbb{E}[Y(t)] = \Pr[t\leq\tau]\,\mathbb{E}[Y(t)\mid t\leq\tau] + \Pr(t>\tau)\,\mathbb{E}[Y(t)\mid t>\tau]$$
$$= 0 + 0 = 0$$

(b) Compute the autocorrelation function $R_Y(s,t)$ of $\{Y(t)\}_{0\leq t\leq 1}$

$R_Y(s,t) = C_Y(s,t)$ since $\mu_Y(t) = 0 \ \forall t$

$$C_Y(s,t) = \mathrm{Cov}(Y(s), Y(t))$$

Cases

1 $s < t < \tau$

$$\mathrm{Cov}(Y(s), Y(t)) = \mathrm{Cov}(X(s), X(t))$$
$$= \sigma^2 s$$

2 $\tau < s < t$

$$\mathrm{Cov}(Y(s), Y(t)) = \mathrm{Cov}(2X(\tau) - X(s), 2X(\tau) - X(t))$$
$$= 4\sigma^2\tau - 2\sigma^2\tau - 2\sigma\tau + \sigma^2 s = \sigma^2 s$$

3 $s < \tau < t$

$$\mathrm{Cov}(Y(s), Y(t)) = \mathrm{Cov}(X(s), 2X(\tau) - X(t))$$
$$= 2\sigma^2 s - \sigma^2 s = \sigma^2 s$$

Thus

$$C_Y(s,t) = \sigma^2 \min(s,t)$$

(c) Is $\{Y(t)\}_{0 \le t \le 1}$ a Gaussian process?

Need $\forall n$, $t_1 < t_2 < \cdots < t_n$, that

$$a_1 Y(t_1) + \cdots + a_n Y(t_n)$$

is Gaussian.

Say $t_1 < t_2 < \cdots < t_{\tau'} < t_{\tau'+1} < \cdots < t_n$.

Then

$$a_1 X(t_1) + a_2 X(t_2) + \cdots + a_{\tau'} X(\tau) + a_{\tau'+1}\left(2X(\tau) - X(\tau'+1)\right) + \cdots + a_n\left(2X(\tau) - X(t_n)\right)$$

is Gaussian since $\{X(t)\}$ a Brownian motion making it a linear combination of jointly Gaussian r.v's. Since n arbitrary, $\{Y(t)\}_{0 \le t \le 1}$ a Gaussian process.

(d) Find the MMSE prediction of $Y\left(\frac{\tau+1}{2}\right)$ given $Y(\tau)$.

First note $\frac{\tau+1}{2} > \tau \quad \forall \ \tau \in (0,1)$.

Then, since Y a Gaussian random process,

$$\mathbb{E}\left[Y\left(\tfrac{\tau+1}{2}\right) \mid Y(\tau)\right] = \mathbb{E}\left[Y\left(\tfrac{\tau+1}{2}\right)\right] + \frac{\mathrm{Cov}\left(Y\left(\frac{\tau+1}{2}\right), Y(\tau)\right)}{\mathrm{Var}(Y(\tau))}\left(Y(\tau) - \mathbb{E}(Y(\tau))\right)$$

$$= 0 + \frac{\sigma^2 \tau}{\sigma^2 \tau}\left(Y(\tau) - 0\right)$$

$$= Y(\tau)$$

③ Modified Random Walk

Too much text, Markov chain question.