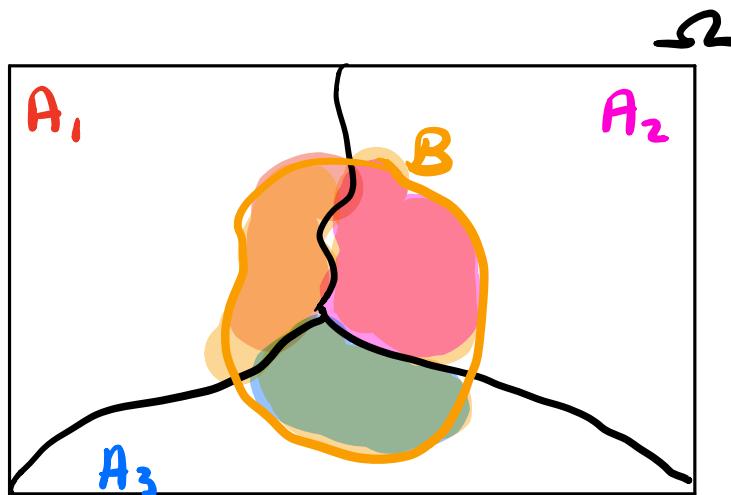


Recall: Conditional Probability

Total Probability Theorem: Given Ω, P . if A_1, A_2, \dots, A_n are events that partition Ω , then for any event B ,

$$P(B) = P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + \dots + P(B|A_n)P(A_n)$$

Picture when $n=3$



$$P(B|A_1)P(A_1) =$$



$$P(B|A_2)P(A_2) =$$



$$P(B|A_3)P(A_3) =$$



$P(B)$ is sum these!

Next BIG idea: Bayes Rule

Given Ω , P , if A_1, \dots, A_n are events that partition Ω and have nonzero probability, then for any event B , and for any $k: 1 \leq k \leq n$,

$$P(A_k | B) = \frac{P(B | A_k) P(A_k)}{P(B | A_1) P(A_1) + \dots + P(B | A_n) P(A_n)}$$

"PROOF": Numerator = $P(B \cap A_k)$ by definition of conditional probability

Denominator = $P(B)$ by total probability theorem

$$\frac{P(A_k \cap B)}{P(B)} = P(A_k | B) !$$

Background: Think of B as an "effect" that can have exactly one of n possible "causes" - these are the A_1, \dots, A_n .

Know:

- prior probabilities of A_1, \dots, A_n - $P(A_k)$, $1 \leq k \leq n$
- For each k , $P(B|A_k)$ is the probability that the "effect" arises when "cause" k occurs.

So, for each k , we want the probability that "cause" k gave rise to "effect" B given "effect" B .

Example - Coins:

Two coins, one fair ($P(\{H\}) = P(\{T\}) = \frac{1}{2}$)

one unfair ($P(\{H\}) = \frac{2}{3}$, $P(\{T\}) = \frac{1}{3}$)

$U \rightarrow$ Unfair

$F \rightarrow$ Fair

Pick a coin uniformly at random

$$\text{i.e. } P(\{U\}) = P(\{F\}) = \frac{1}{2}$$

and start flipping.

Let event $B =$ Heads on first flip.

Intuition: $P(U|B) > \frac{1}{2}$ and $P(F|B) < \frac{1}{2}$

By Bayes' Rule

$$P(U|B) = \frac{P(B|U) P(U)}{P(B|U) P(U) + P(B|F) P(F)} = \frac{\frac{2}{3} \cdot \frac{1}{2}}{\frac{2}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{2}} = \frac{4}{7}$$

intuition
was right!

Let event $C =$ two heads in a row to start

$$P(U|C) = \frac{P(C|U)P(U)}{P(C|U)P(U) + P(C|F)P(F)} = \frac{\left(\frac{2}{3}\right)^2 \cdot \frac{1}{2}}{\left(\frac{2}{3}\right)^2 \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{2}}$$

↑
expect to be
bigger than $\frac{1}{2}$

Note however, we can never be **COMPLETELY SURE** of what coin we are flipping.

Example - Aircraft (alien detection) revisited:

A = Spacecraft - present

B = see a blip on radar

Given: $P(B|A) = 0.99 \Rightarrow P(B^c|A) = 1 - 0.99 = 0.01$

$$P(B|A^c) = 0.10$$

$$P(A) = 0.05$$

Want $P(A|B)$

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)} \approx 0.3426$$

Example - Medical Testing:

Some disease out there.

Let

A = event person from some given population has a disease.

We have a test (an imperfect test) for the disease.

Let

B = event person drawn from population tests positive.

Think of population as Ω

Question: Given a positive test result, what is the likelihood the person has the disease?

what does this mean?

Given Info:

Test is "95% accurate"

$$\Rightarrow P(B|A) = 0.95$$

$$P(B|A^c) = 0.05$$

① If you have the disease,

95% of the time it shows positive

② If you don't have the disease,

95% of the time it shows negative

Assume $P(A) = 0.001$

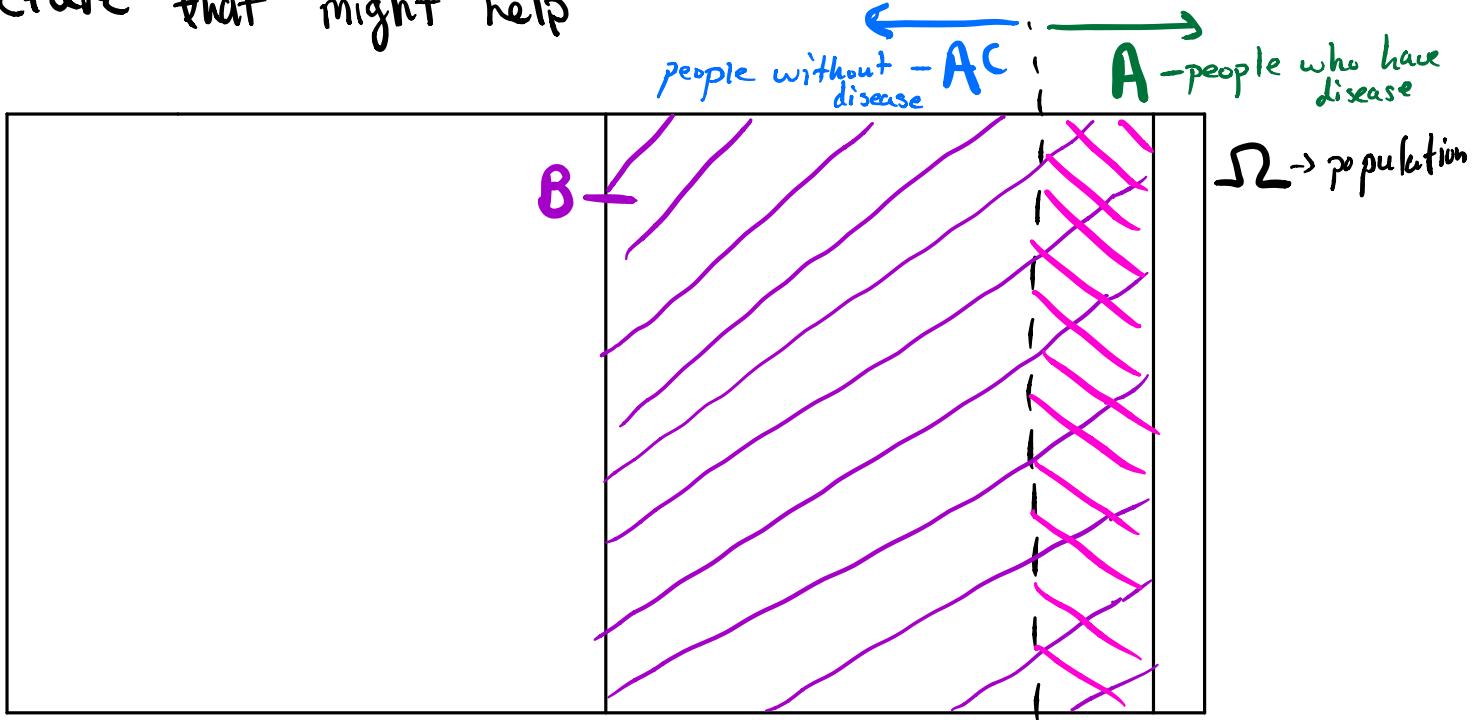
disease is rare in population

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)} = \frac{(0.95)(0.001)}{(0.95)(0.001) + (0.05)(0.999)}$$

$$P(A|B) = 0.0189 \approx 0.19\%$$

even though test was very accurate, bc we knew rarity of disease the likelihood of person having disease is small.

Picture that might help



$$P(B|A) = \frac{2}{3}$$

$$P(B|A^C) = \frac{1}{3}$$

$$P(A|B) = \frac{P(\text{X})}{P(\text{---})} \ll \frac{1}{2}$$

Lesson here
is to be careful
with conditional
probability!
Intuition can lead
you astray!!

Comment: In chess example, could use Bayes to compute.
eg $P(\text{played a master} | \text{won the game})$ — ~~WON'T DO IN CLASS~~

Next Circle of Ideas: Independence

Given, Ω , P . Say two events, $A, B \subset \Omega$ are independent when $P(A \cap B) = P(A)P(B)$

Same as $P(A|B) = P(A)$ when $P(B) > 0$ Knowing B occurred says nothing more about whether A occurred

$$P(B|A) = P(B) \text{ when } P(A) > 0$$

Knowing that A occurred says nothing more about whether B occurred.

CAUTION: Independence is "about" not only the events and how they sit in Ω , but also (more crucially) about P !!

Common error: if " $A \cap B = \emptyset$, then A and B are independent"

WRONG whenever $P(A) > 0$ and $P(B) > 0$

Example - Dice:

Have two standard six-sided dice. Assume every outcome is equally likely. i.e each has prob $\frac{1}{36}$

\uparrow
pairs of rolls (rolls of first die, roll of second die)

Consider some events

A = first die shows 3

B = second die shows 5

$$A = \{(3,1), (3,2), (3,3), (3,4), (3,5), (3,6)\} \Rightarrow P(A) = 6 \cdot \frac{1}{36} = \frac{1}{6}$$

$$B = \{(1,5), (2,5), (3,5), (4,5), (5,5), (6,5)\} \Rightarrow P(B) = 6 \cdot \frac{1}{36} = \frac{1}{6}$$

$$P(A \cap B) = \frac{1}{36} = P(A)P(B)$$

The two events are therefore independent

Let

A = same as above A. C = 2 dice are different.

$$A = \{(3,1), (3,2), (3,3), (3,4), (3,5), (3,6)\} \Rightarrow P(A) = 6 \cdot \frac{1}{36} = \frac{1}{6}$$

C is the same as (rolling doubles)^c

$$P(C) = \frac{30}{36} = \frac{5}{6}$$

Therefore, A, C are independent!

$$P(A \cap C) = \frac{5}{36} = \left(\frac{1}{6}\right) \cdot \left(\frac{5}{6}\right) = P(A)P(C)$$

More generally, say A_1, A_2, \dots, A_n are independent when ?

2 that DO NOT work

$$\textcircled{1} \quad \overline{P}(A_1 \cap A_2 \cap \dots \cap A_n) = \overline{P}(A_1) \overline{P}(A_2) \dots \overline{P}(A_n)$$

$$\textcircled{2} \quad \overline{P}(A_k \cap A_l) = \overline{P}(A_k) \overline{P}(A_l) \quad \text{for all } \begin{matrix} 1 \leq k, l \leq n \\ k \neq l \end{matrix}$$

What works? Find out next lecture on Dragon Ball 2