Markov Chains (part 2)

Stochastic Matrix $A_{n\times n}$

$\lim A^m = L$

If A diagonalizable, then this limit exists

$\Leftrightarrow$ all complex eigenvalues λ satisfy

$|\lambda| < 1$ or $\lambda = 1$

Note:

$$\lim_{m\to\infty} AA^m = L = AL$$

So, all columns of L are eigenvectors w/ $\lambda = 1$.

Note: if $\vec{u} = (1, \dots, 1)$

then $\lim_{m\to\infty} (\vec{u}A^m) = \lim_{m\to\infty} \vec{u} = \vec{u}$

$= \vec{u} \lim_{m\to\infty} A^m$

$= \vec{u} L \Rightarrow L$ is a stochastic matrix

Gershgorin Disks

$$C_i = \{ z \in \mathbb{C} \mid |z - A_{ii}| \leq r_i \}$$

where

$$r_i = \rho_i(A) - |A_{ii}|$$

Theorem: If $A \in \mathbb{C}^{n\times n}$, if λ = eigenvalues of A, then $\lambda \in C_i$ for some i.

Note: if $z, w \in \mathbb{C}$,

$$|z+w| \le |z| + |w|$$

$$|z-w| \le |z| + |w|$$

Proof: Suppose $A\vec{v} = \lambda\vec{v}$, $\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \neq \vec{0}$

So

$$\bigstar \quad \sum_{j=1}^{n} A_{ij} v_j = \lambda v_i \quad \text{for } i = 1, \dots, n$$

Suppose

v_k = entry s.t. $|v_k| \geq |v_1|, \dots, |v_n| > 0$

Show $\lambda \in \mathbb{C}_k$ use $i = k$ in $\bigstar$

$$\sum_{j=1}^{n} A_{kj} v_j = \lambda_k$$

$$|\lambda v_k - A_{kk} v_k| = \Big|\sum_{j\neq k} A_{kj} v_j\Big| \leq \sum_{j\neq k} |A_{kj}||v_j|$$

$$\leq \sum_{j\neq k} |A_{kj}||v_k| = r_k |v_k|$$

So $|\lambda - A_{kk}||v_k| \leq r_k |v_k| \rightarrow |\lambda - A_{kk}| \leq r_k$

Corollary: If A is stochastic, then any eigenvalue λ lies in $|\lambda| \leq 1$

Proof: Know $0 \leq A_{ii} \leq 1$

Know $\lambda \in C_i = \{ z \in \mathbb{C} \mid |z - A_{ii}| \leq r_i \}$

for some $i = 1, \dots, n$, $r_i = 1 - A_i$

So $C_1 = \{ z : |z| < 1 \}$

Fact (requires proof)

$$\dim E_\lambda(A) = 1$$

$AL = L \Rightarrow$ each column of L identical.