Cauchy's Theorem

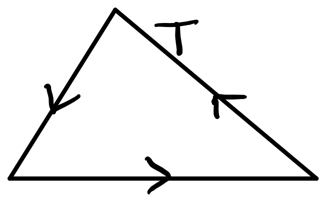
Suppose $f(z)$ is analytic in a simply connected region R, γ is any closed curve in R.

Then

$$\int_\gamma f(z)\,dz = 0$$

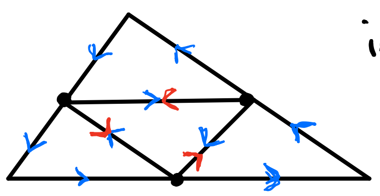
Proof: (Goursat)

Do it for a triangle first. Then geralize to any polygonal curve and then to a smooth curve.



Suppose T has perimeter length L, diameter D. D is the farthest distance between two points. (i.e $|z-w| \leq D \;\forall\, z,w$ on T).

1) Bisect sides of triangle, making 4 new triangles.

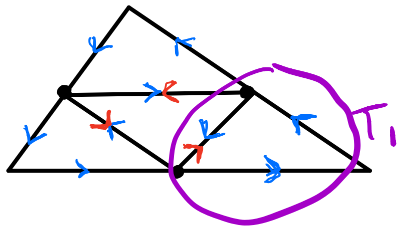


inner edges are traversed twice

contribution from inner edges cancel out.

$$\int_T f(z)\,dz = \int_{\substack{4 \text{ little} \\ \text{triangles}}} f(z)\,dz$$

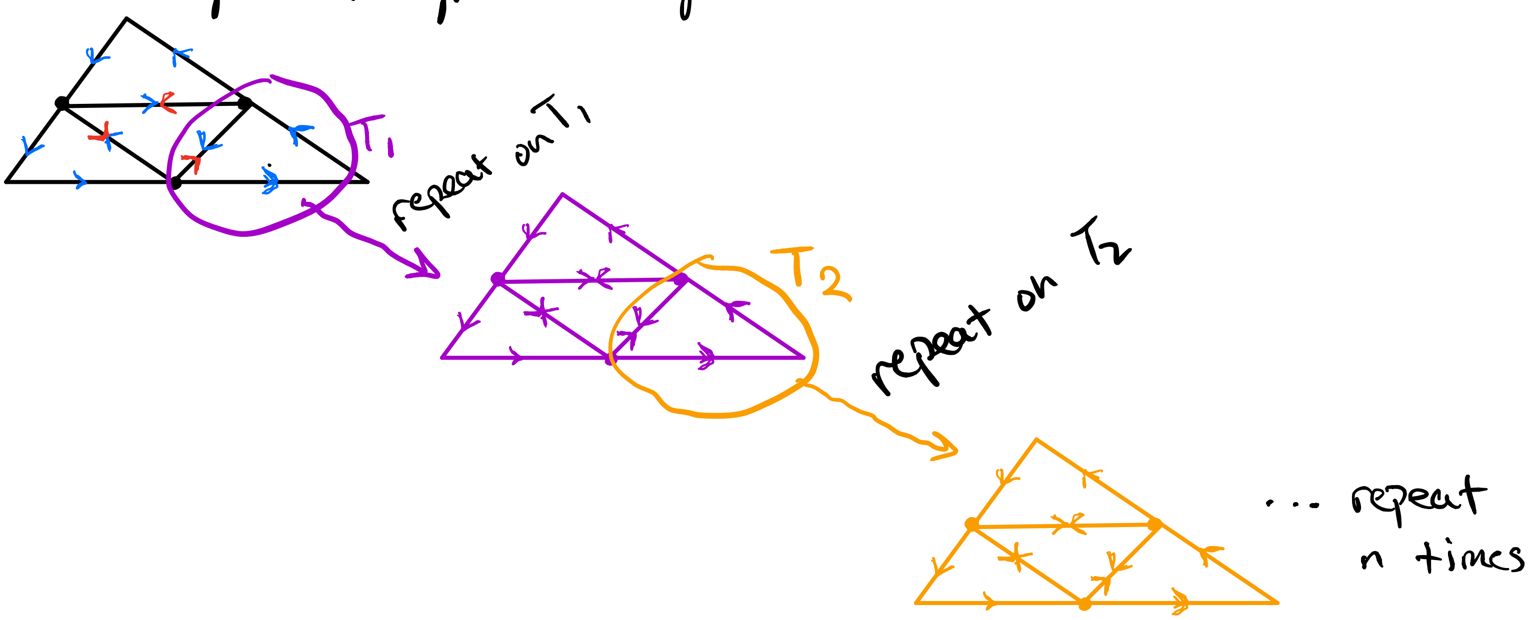
2) Choose little triangle with largest magnitude integral. Call it T_1.



By triangle inequality,

$$\left| \int f(z)\,dz \right| \le 4 \left| \int_{T_1} f(z)\,dz \right|$$

3) Play the same game again and again and again. At the n^{th} stage of doing this, let T_n be the triangle with the largest magnitude integral.



$$\left| \int_T f(z)\,dz \right| \le 4^n \left| \int_{T_n} f(z)\,dz \right|$$

4) There exists a point, a, trapped in the infinite nested intersection of all the interiors of the T_k

$$a \subset \cdots \subset T_2 \subset T_1 \subset T$$

not obvious, requires proof

uses "compactness" of closed triangle T

5) Estimate $\int_{T_n} f(z)\,dz$. On triangle interior T_n, we can write $f(z)$ as

$$f(z) = f(a) + \underbrace{f'(a)(z-a)}_{\text{linear}} + \underbrace{\varepsilon(z)(z-a)}_{\text{small correction}}$$

Here $\varepsilon(z)$ is a function defined by

$$\varepsilon(z) = \frac{f(z) - f(a)}{z-a} - f'(a)$$

equivalent to saying f is analytic at a!

i.e

$\lim_{z \to 0} \varepsilon(z) = 0$ because f is analytic throughout R and here at a.

antiderivatives exist, integrated on closed loop

So,

$$\int_{T_n} f(z)\,dz = \overset{0}{\cancel{\int_{T_n} f(a)\,dz}} + \overset{0}{\cancel{\int_{T_n} f'(z)(z-a)}} + \int_{T_n} \varepsilon(z)(z-a)\,dz$$

So,

$$\left| \int_{T_n} f(z)\,dz \right| = \left| \int_{T_n} \varepsilon(z)(z-a)\,dz \right|$$

Recall $\left| \int_T f(z)\,dz \right| \leq 4^n \left| \int_{T_n} f(z)\,dz \right| = 4^n \left| \int_{T_n} \varepsilon(z)(z-a)\,dz \right|$

6) Use ML estimate.

$$\left| \int_{T_n} \varepsilon(z)(z-a)\,dz \right| \leq \max_{z \in T_n} |\varepsilon(z)| \max_{z \in T_n} |z-a| \text{ length}(T_n)$$

length $T_n = \dfrac{L}{2^n}$ $\Big\}$ each bisect yields a simalar triangle w/ perimeter $\frac{L}{2}$. n reductions

$\max |z-a| = \dfrac{D}{2^n}$ $\Big\}$ hypotenuse halves in length w/ each reduction, $\left(\frac{D}{2}\right)$, n times.

realized by T_n being constructed by repeated bisection of T.

7) Summarize.

$$\left| \int_T f(z)\,dz \right| \leq 4^n \left| \int_{T_n} f(z)\,dz \right| = 4^n \left| \int_{T_n} \varepsilon(z)(z-a)\,dz \right|$$

$$= 4^n \frac{DL}{2^n 2^n} \max_{z \in T_n} |\varepsilon(z)| = DL \max_{z \in T_n} |\varepsilon(z)|$$

Know that

$\varepsilon(z) \to 0$ as $z \to a$ $\Big\}$ as $n \to \infty$, $T_n \to a$

As $n \to \infty$, $\max_{z \in T_n} |\varepsilon(z)|$ goes to zero by analyticity of f.

So,

$$\left| \int_T f(z)\,dz \right| \leq 4^n \left| \int_{T_n} \varepsilon(z)(z-a)\,dz \right| \leq 0$$

LHS is independent of n, so LHS estimate is exactly 0!