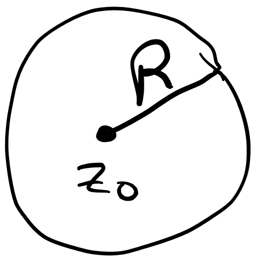
Taylor Series

Suppose $f(z)$ is analytic on and inside a circle of radius R, C_R, about a certain point z_0.



Then $f(z)$ can be written as

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$

"Any analytic function can be written as a convergent series"

NOTES: 1) This Taylor series converges for all z in the open disk $|z-z_0| < R$, and it converges to the correct value, $f(z)$

2) The maximum R that will ensure convergence is called the "Radius of convergence" for f about z_0.

Formula for R:

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

where

$$a_n = \frac{f^{(n)}(z_0)}{n!}$$ (assuming limit exists)

3) R = distance from z_0 to the nearest "singularity" of f

"singularity": a point where f is **NOT** analytic

4) If $z_0 = 0$, Taylor series $\rightarrow$ Maclaurin Series

Examples

1) $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ $\quad a_n = \frac{1}{n!}$ so $R = \lim_{n\to\infty} \left| \frac{\frac{1}{n!}}{\frac{1}{(n+1)!}} \right| \rightarrow \infty$

So f is entire [converges $\forall z$]

2) $\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} \cdots$

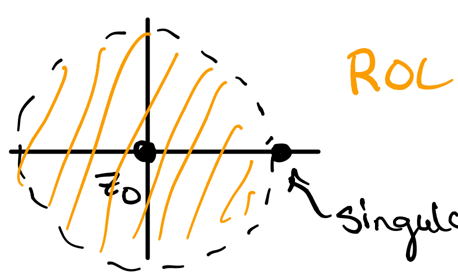
3) $\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} \cdots$

Maclaurin Series

Both entire

4) $\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots$ only converges for $|z| < 1$

We're expanding around $z_0 = 0$



singularity of $\frac{1}{1-z}$ @ $z = 1$

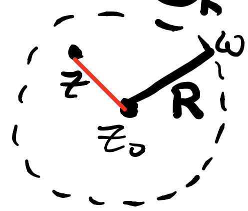
Proof of Taylor Series – uses Cauchy's Integral Formula

Cauchy's Integral Formula

C_R - open disk

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(w)}{(w-z)}\,dw$$

simple change of variable



$\dfrac{z-z_0}{R} < 1$!

Want to rewrite this as an infinite series

Step 1: Write $\dfrac{1}{w-z}$ as a convergent geometric series inside C_R [KEY TRICK]

Need some variable $|u|<1$, and we want to use

$$\frac{1}{1-u} = 1 + u + u^2 + \cdots$$

What should u be?

IDEA: $|w-z_0| > |z-z_0|$ since z is inside our circle and w is on the circle.

So, $\left|\dfrac{z-z_0}{w-z_0}\right| < 1$!

So let that be u!

WATCH: $$\frac{1}{w-z} = \frac{1}{(w-z_0)-(z-z_0)} = \frac{1}{w-z_0}\left[\frac{1}{1-\dfrac{z-z_0}{w-z_0}}\right]$$

$$= \frac{1}{w-z_0}\left[\frac{1}{1-u}\right] = \frac{1}{w-z_0}\left[1+u+u^2+u^3+\cdots\right]$$

$$= \frac{1}{w-z_0}\left[\sum_{n=0}^{\infty} u^n\right] = \frac{1}{w-z_0}\sum_{n=0}^{\infty}\left[\frac{z-z_0}{w-z_0}\right]^n$$

$$= \frac{1}{w-z_0}\sum_{n=0}^{\infty}\frac{(z-z_0)^n}{(w-z_0)^n} = \sum_{n=0}^{\infty}\frac{(z-z_0)^n}{(w-z_0)^{n+1}}$$

Thus $\displaystyle\frac{1}{w-z} = \sum_{n=0}^{\infty}\frac{(z-z_0)}{(w-z_0)^{n+1}}$; converges inside C

Step 2: Substitute for $\displaystyle\frac{1}{w-z}$ and use Cauchy's formula for derivatives.

From step 1

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(w)}{w-z}\,dz = \frac{1}{2\pi i}\oint_C f(w)\sum_{n=0}^{\infty}\frac{(z-z_0)^n}{(w-z_0)^{n+1}}\,dw$$

Interchange integral and sum (not obvious... requires proof)

→ geometric series converges uniformally inside $|z-z_0| \le r < 1$ ←

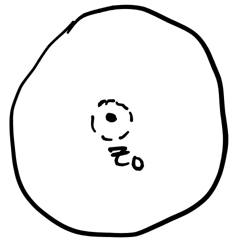
$$f(z) = \sum_{n=0}^{\infty}\left[\frac{1}{2\pi i}\oint_C \frac{f(w)}{(w-z_0)^{n+1}}\,dw\right](z-z_0)^n$$

This is just $\displaystyle\frac{f^{(n)}(z_0)}{n!}$ from Cauchy's formula for derivative

$$\Rightarrow \quad f(z) = \sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$

Isolated Singularities

We say $f(z)$ has an isolated singularity at z_0 if $f(z)$ is analytic in a punctured disk $0 < |z - z_0| < R$, for some $R > 0$



Note: Branch points are **NOT** isolated because $f(z)$ is not analytic in any punctured disk about z_0