SVD

A $m \times n$ matrix over $\mathbb{R}$, rank r

$A^T A$ $n \times n$ symmetric, rank r.

Eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_r \geq \lambda_{r+1} = 0 = \cdots = \lambda_n$

Singular values of A:

① $\sigma_1 = \sqrt{\lambda_1} \geq \sigma_2 = \sqrt{\lambda_2} \geq \cdots \geq \sigma_r = \sqrt{\lambda_r}$

② Let $\vec{v}_1, \ldots, \vec{v}_n$ be an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of $A^T A$.

So

$$A^T A \vec{v}_i = \lambda_i \vec{v}_i \quad , \quad i = 1, \ldots, n$$

③ Let

$$\vec{u}_i = \frac{1}{\sigma_i} A \vec{v}_i \quad , \quad i = 1, \ldots, r$$

NOTE

(note $\vec{u}_i \in \mathbb{R}^m$)

These are orthonormal, and extend to orthonormal basis $\vec{u}_1, \ldots, \vec{u}_m$ of $\mathbb{R}^m$.

Let

$$U = (\vec{u}_1 \ \ \vec{u}_2 \ \cdots \ \vec{u}_m)$$ $m \times m$ orthogonal matrix

$$V = (\vec{v}_1 \ \ \vec{v}_2 \ \cdots \ \vec{v}_n)$$ $n \times n$ orthogonal matrix

$$\Sigma = \begin{pmatrix} \sigma_1 & & & & & \\ & \sigma_2 & & & \mathbf{0} & \\ & & \ddots & & & \\ & & & \sigma_r & & \\ & \mathbf{0} & & & 0 & \\ & & & & & \ddots \; 0 \end{pmatrix}$$ $m \times n$ matrix

SVD of A:

$$A = U \Sigma V^T \qquad (V^T = V^{-1})$$

since V orthogonal

<u>Proof</u>

check

$$AV = U\Sigma$$

$$[A\vec{v}_1 \ \ A\vec{v}_2 \ \cdots \ A\vec{v}_n] = [\sigma_1 \vec{u}_1 \cdots \sigma_r \vec{u}_r \ 0 \cdots 0] \ \checkmark$$

Recap

An SVD of A is a factorization

$$A = U \Sigma V^T$$

such that

① U, V are orthogonal matrices

② $\Sigma = \begin{pmatrix} \sigma_1 & & & & & \\ & \sigma_2 & & & & \\ & & \ddots & & & \\ & & & \sigma_r & & \\ & & & & 0 & \\ & & & & & \ddots \; 0 \end{pmatrix}$, $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$

Example:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad \text{rank} = 2$$

find SVD.

$$A^T A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

eigenvalues of A^TA,

$$0 \quad \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$1 \quad \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$3 \quad \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

So,

$$\sigma_1 = \sqrt{3}$$

$$\sigma_2 = 1$$

and

$$\vec{v}_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad \vec{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \vec{v}_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$\vec{u}_1 = \frac{1}{\sqrt{3}} A \vec{v}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \vec{u}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

So,

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \end{pmatrix}$$

Proposition: Every $A \in \mathbb{R}^{m \times n}$ has an SVD (Prove Yourself)

Note:

$$A = (\vec{u}_1 \cdots \vec{u}_r \ \vec{u}_{r+1} \cdots \vec{u}_m)\begin{pmatrix} \sigma_1 & & & & 0 \\ & \ddots & & & \\ & & \sigma_r & & \\ & & & 0 & \\ 0 & & & & \ddots \ 0 \end{pmatrix}\begin{pmatrix} \vec{v}_1^T \\ \vec{v}_2^T \\ \vdots \\ \vec{v}_r^T \\ \vec{v}_{r+1}^T \\ \vdots \\ \vec{v}_n^T \end{pmatrix}$$

$$= (\sigma_1 \vec{u}_1 \cdots \sigma_r \vec{u}_r \ 0 \cdots 0) V^T$$

$$= \sigma_1 \vec{u}_1 \vec{v}_1^T + \sigma_2 \vec{u}_2 \vec{v}_2^T + \cdots + \sigma_r \vec{u}_r \vec{v}_r^T$$

"slim, or compact, pretty fine" SVD

$$= (\vec{u}_1 + \cdots + \vec{u}_r)\begin{pmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_r \end{pmatrix}\begin{pmatrix} \vec{v}_1^T \\ \vdots \\ \vec{v}_r^T \end{pmatrix}$$

$\boxed{\text{Slim version of SVD}}$

$$A = U \Sigma V^T$$

but U is $m \times r$, $U^T U = I_r$ ($r = \text{rank } A$)
V is $n \times r$, $V^T V = I_r$

$$\Sigma = \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{pmatrix}, \quad \sigma_1 \geq \cdots \geq \sigma_r > 0$$

$\boxed{\text{NOTE}}$ if $A = uv^T$, $m \times n$

$u = m \times 1$
$v = n \times 1$

rank $A = 1$

<u>Example</u> A $m \times n$, 1000×1000

each $A_{ij} \in [0, 1]$

A is a gray image

often rank $A \approx 1000$

$$\sigma_1 \geq \sigma_2 \geq \underbrace{\cdots \geq \sigma_{1000}}_{\text{but MANY very close to } 0}$$

Suppose all but 50 very small

$$A = \sigma_1 u_1 v_1^T + \cdots + \sigma_{50} u_{50} v_{50}^T$$

Storage for A: 1000 000 numbers

BUT w/ ignoring small values, $50 + 50000 + 50000$ <u>Pretty Good</u>