Recall

Counting – mop up some details

Have n objects and r labeled bins; how many ways to allocate objects to bins so that n_1 in Bin 1, n_2 in Bin 2, ..., n_r in Bin r?

Answer: Counting Principle!

$$\underbrace{\binom{n}{n_1}}_{\text{ways to pick } n_1 \text{ objects from } n \text{ objects}}\underbrace{\binom{n-n_1}{n_2}}_{\text{ways to pick } n_2 \text{ objects from } n-n_1 \text{ objects}}\binom{n-n_1-n_2}{n_3}\cdots\underbrace{\binom{n-n_1-\cdots-n_{r-1}}{n_r}}_{\text{Cleverly look at this to realize its } \binom{n_r}{n_r}=1}$$

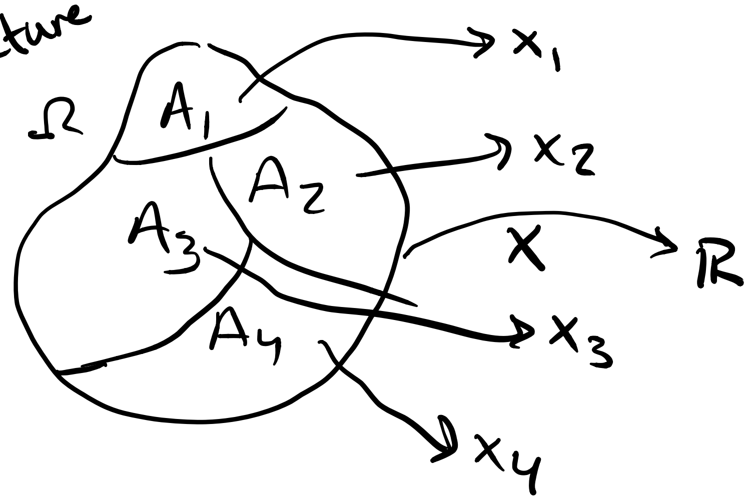
$$= \frac{n!}{n_1!\,n_2!\cdots n_r!}$$

Next BIG TOPIC: DISCRETE RANDOM VARIABLES

Start with Ω and $\mathbb{P}$: a discrete random variable (r.v.) is a real valued function with domain Ω that takes on only finite or countably infinite number of different values.

i.e $X : \Omega \to \mathbb{R}$ "X is a mapping from Ω to real's"

Picture



i.e $X(s) = x_1 \ \forall s \in A_1$, etc

Example – Constant random variable

$\Omega, \mathbb{P}$ anything. $X : \Omega \to \mathbb{R}$ defined by $X(s) = 19 \ \forall s \in \Omega$

Example – Coins

$\Omega = \{H, T\}$; $\mathbb{P}(H) = \mathbb{P}(T) = 1/2$

$X : \Omega \to \mathbb{R}$ is defined by
$$X(H) = 1$$
$$X(T) = 0$$

Example - Dice

$\Omega = \{$ordered pair of rolls of 2 standard 6-sided dice$)$
$\mathbb{P}$ defined so all 36 outcomes have equal probability $1/36$.
A couple associated r.v.'s
- $X(s)$ = the number that comes up on first die - possible values are $1,2,3,4,5,6$.
- $X(s)$ = sum of two rolls - possible values $2,3,4,5,6,7,8,9,10,11,12$

Example - Coin again

Ω = set of all H,T-sequences of length 7; $\mathbb{P}(\text{any}) = 2^{-7}$
X could be $X(s) = \#(H\text{'s})$ in sequence

What if we wanted index of first head?

$$X(s) = \begin{cases} \text{index of first Head in sequence, if } s \text{ contains } H \\ \text{Anything, when } s = \text{all - Tail sequence} \end{cases}$$

case of no heads

Must be a bipartite definition for this example because X must assign a # to **every** $s \in \Omega$

So, given $\Omega, \mathbb{P}$; associated w/ a discrete r.v. $X : \Omega \to \mathbb{R}$ is X's probability mass function (p.m.f)

Notation for pmf will be

p_X ← lowercase things are <u>deterministic</u> quantities

– upper case random variables ALWAYS

p_X is defined as follows: for every possible value of $x \in X$,

$$p_X(x) = \mathbb{P}(A_x) \text{ where } A_x = \{s \in \Omega : X(s) = x\}$$

i.e $p_X(x) =$ the probability that the r.v. X takes on the specific value x

Book uses

$$\mathbb{P}(\{X = x\}) \text{ or } \mathbb{P}(X = x)$$

← misuse of notation compared to above

to refer to $\mathbb{P}(A_x)$, where $A_x = \{s \in \Omega : X(s) = x\}$

Things to note about p_X:

- $p_X(x) \geq 0$ for all possible values of X

  (why? cause for any x, $p_X(x)$ is $\mathbb{P}(\text{an event}) \geq 0$!)

- If V is <u>any</u> finite or countably infinite set of possible values of X, then if we set

  $B =$ the event "$X \in V$"

  i.e $B = \{s \in \Omega : X(s) \in V\}$

then

$$\mathbb{P}(B) = \sum_{x \in V} p_X(x)$$

Idea: events $A_x = \{s \in \Omega . X(s) = x\}$ as x runs over V are disjoint subsets of B whose union is B.

Thus,

$$\mathbb{P}(B) = \sum_{x \in V}' \mathbb{P}(A_x) = \sum_{x \in V}' P_X(x)$$

"Probability that X is in V = sum over $x \in V$ of the probability that $X = x$"

Special Case: V = set of all possible values of X

$$\sum_{x \in V}' P_X(x) = 1$$

Some common pmf's that arise in applications

- Discrete uniform pmf on the interval $a \le k \le b$ $(a, b \in \mathbb{Z})$

$$P_X(k) = \begin{cases} \frac{1}{b-a+1}, & a \le k \le b \\ 0, & \text{all other } k \end{cases}$$

- Let $p \in [0,1]$, the Bernoulli p pmf defined by

$$P_X(k) = \begin{cases} p, & k = 1 \\ 1-p, & k = 0 \end{cases}$$

Where does X come from? What is Ω? What is $\mathbb{P}$?

Turns out, *Cue dramatic music DOESN'T MATTER

Just as a given $\Omega, \mathbb{P}$ give rise to many different rv's, a given "legal" pmf has many different background $\Omega, \mathbb{P}$, $X : \Omega \to \mathbb{R}$.

What's important about a discrete r.v. is its pmf. Where the r.v. came from doesn't really matter.

That said, we could build an $\Omega, \mathbb{P}$ if we wanted too.

Example - Bernoulli

$$\Omega = \{H, T\} \quad \mathbb{P}(\{H\}) = p \,; \quad \mathbb{P}(\{T\}) = 1-p$$

$$X(H) = 1 \qquad X(T) = 0$$

That's just one of many ways to do it!

Example - Discrete uniform on interval $7 \leq k \leq 12$

Ω = set of outcomes of one fair die roll

$= \{1, 2, 3, 4, 5, 6\}$

$\mathbb{P}$ = discrete uniform probability law on Ω

$X(s) = s + 6$ for all $s \in \Omega$

So we can do this, but we NEVER do. What's important is the r.v w/ that pmf not where it came from.

Some more pmf's below

Binomial RV

Given positive integer n for some $p \in [0,1]$, the **Binomial (n,p) pmf** defined as follows:

$$P_X(k) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k}, & 0 \le k \le n \\ 0, & \text{all other } k \end{cases}$$

This could arise from

$\Omega = \{\text{all H-T sequences of length } n\}$

$\mathbb{P}(\text{any sequence}) = p^{\#H}(1-p)^{\#T}$

like the outcome of n-independent flips of a possibly unfair coin.

$X(\text{any sequence}) = \#(\text{Heads})$ in sequence ← saw this last time in counting example!

Geometric R.V

Given $p \in (0,1)$, geometric pmf defined by

$$P_X(k) = p(1-p)^{k-1} \quad \text{for all } 1 \le k < \infty \text{ (positive integers)}$$

arises from $\mathbb{P}(\text{it takes } k \text{ flips to flip a heads})$