

$$5) \quad I = \int_{-\infty}^{\infty} \frac{x \sin(3x)}{x^4 + 4} dx$$

There are simple poles at $z_k = \exp\left(\frac{\pi}{4} + \frac{\pi}{2}k\right)$, $k = 0, 1, 2, 3$

It can be noted that

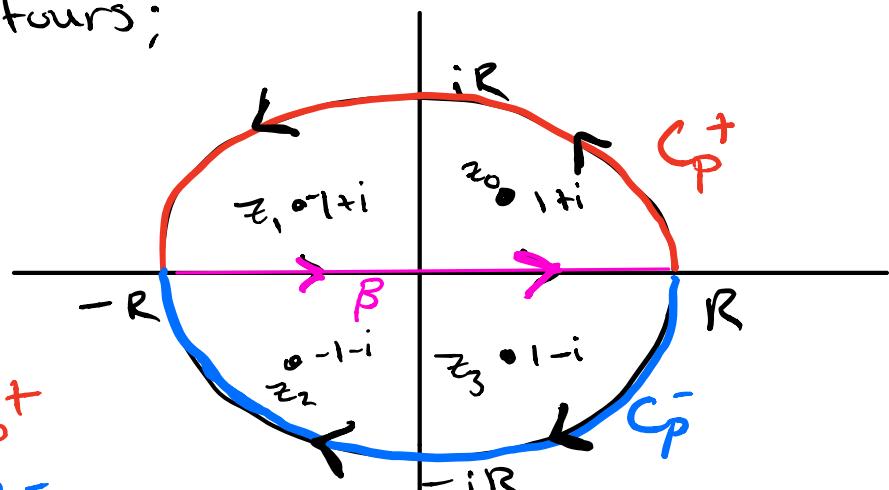
$$\begin{aligned} \left| \frac{z \sin(3z)}{z^4 + 4} \right| &= \left| \frac{z(e^{i3z} - e^{-i3z})}{2(z^4 + 4)} \right| \\ &= \frac{(e^{-3R} + e^{3R}) |z|}{2 |z^4 + 4|} \end{aligned}$$

where $R \rightarrow \infty$. This means

- e^{i3z} is bounded in the upper half plane
- e^{-i3z} is bounded in the lower half plane

Thus, I shall use 2 contours;

- 1) C_p^+
- 2) C_p^-



where z_0, z_1 lie in C_p^+
and z_2, z_3 lie in C_p^-

$$\begin{aligned} \text{Where } \gamma_1 &= \beta + C_p^+ \\ \gamma_2 &= \beta + C_p^- \end{aligned}$$

Let

$$I = \int_{-\infty}^{+\infty} \frac{x(e^{i3x} - e^{-i3x})}{2(x^4 + 4)} dx = I_1 + I_2$$

$$= \frac{1}{i} \int_{-\infty}^{+\infty} \frac{xe^{i3x}}{2(x^4 + 4)} dx - \frac{1}{i} \int_{-\infty}^{+\infty} \frac{xe^{-i3x}}{2(x^4 + 4)} dx$$

Now define

$$f_i(z) := \frac{ze^{i3z}}{i2(z^4 + 4)}$$

which has singularities at

$$z_0 = 1+i, z_1 = -1+i$$

Then note

$$|f_i(z)| = |f_i(x+iy)| = \frac{|e^{i3x}e^{-3y}|}{2|z^4+4|} ||z||$$

In the upper half plane $y > 0$, therefore

$$|f_i(z)| \leq \frac{|z|}{2|z^4+4|}$$

Thus for $R > \sqrt{2}$, the integral over C_p^+ is bounded by

$$\left| \int_{C_p^+} f_i(z) dz \right| \leq \max_{z \in C_p^+} \left| \frac{z}{2|z^4+4|} \right| \quad \begin{cases} \text{as } R \rightarrow \infty \\ \text{essentially } \frac{1}{z^3} \end{cases}$$

So, for $R > \sqrt{2}$

$$\int_{\gamma_1} f_1(z) dz = \lim_{R \rightarrow \infty} \int_{-R}^R f_1(x) dx + \int_{C_R^+} f(z) dz$$

$$2\pi i \sum \text{Res} = \lim_{R \rightarrow \infty} \int_{-R}^R \frac{xe^{izx}}{i2(x^4+4)} dx = I_1$$

$$\text{Res}(f_1(z); z_1) = \lim_{z \rightarrow z_1} \frac{ze^{iz}}{2(z-z_1)(z-z_2)(z-z_3)(z-z_4)i} \\ \frac{z_1 e^{iz_1}}{2(z_1-z_2)(z_1-z_3)(z_1-z_4)} \quad e^{i3(1+i)} = e^{i3} e^{-3}$$

$$= \frac{e^{i3} (1+i)}{2e^3 (z)(2i)(2+2i)2} = \frac{e^{i3}}{e^3 |6i|^2}$$

$$\text{Res}(f_1(z); z_2) = \frac{z_2 e^{iz_2}}{2(z_2-z_1)(z_2-z_3)(z_2-z_4)i}$$

$$= \cancel{(-1+i)} \frac{e^{-i3}}{e^3 (-2)(-2i)(-2+2i)2}$$

$$= -\frac{e^{-i3}}{e^3 |6i|^2}$$

Result of I_1

$$2\pi i \sum \text{Res} = \frac{2\pi i}{48ie^3} \left(\cancel{\frac{e^{i3}-e^{-i3}}{2i}} \right) = \boxed{\frac{\pi \sin(3)}{4e^3}}$$

Now for the I_2 ! (expect same result.)

let

$$f_2(z) := -\frac{z}{2i} \frac{e^{-3iz}}{(z^4+4)}$$

$$|f_2(z)| = \frac{|e^{-3ix} e^{3y}| |z|}{2|z^4+4|}$$

In the lower half plane $y \leq 0$, therefore

$$|f_2(z)| \leq \frac{|z|}{2|z^4+4|}$$

so for $R > \sqrt{2}$

$$\left| \int_{C_-} f_2(z) dz \right| \rightarrow 0 \text{ as } R \rightarrow \infty \quad (\text{essentially } \gamma_{z^3})$$

So

$$\int_{\gamma_2} f_2(z) dz = -\lim_{R \rightarrow \infty} \int_{-R}^R \frac{-ze^{-iz}}{2i(z^4+4)} dz + \int_{C_-} \frac{-ze^{-iz}}{2i(z^4+4)} dz$$

γ_2 is negatively oriented

$$2\pi i \operatorname{Res} = -I_2$$

$$I_2 = -2\pi i (\operatorname{Res}(f_2(z); z_2) + \operatorname{Res}(f_2(z); z_3))$$

$$I_2 = -\frac{2\pi i}{e^{38}i} \left[\frac{-e^{i3} + e^{-i3}}{2i} \right] = \frac{\pi \sin(3)}{4e^3}$$

So $I = I_1 + I_2 = \boxed{\frac{\pi \sin(3)}{2e^3}}$

12) Given

$$I_G = \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

Integrate e^{iz^2} around

$$S_p : \{z = re^{i\theta} : 0 \leq \theta \leq \pi/4, 0 \leq r \leq R\}$$

and let $R \rightarrow +\infty$ to prove that

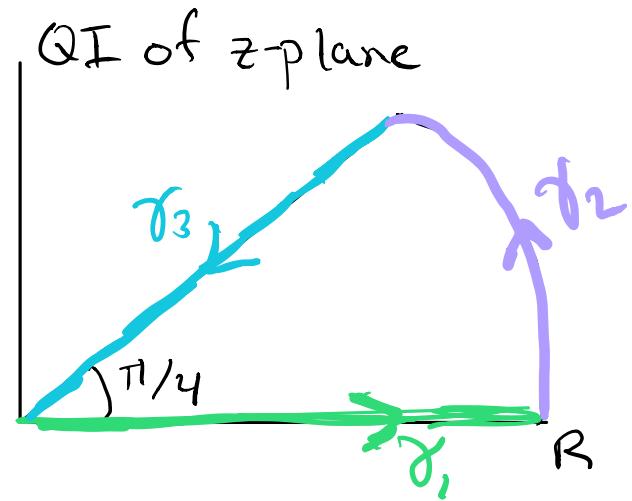
$$I = \int_0^\infty e^{ix^2} dx = \frac{\sqrt{2\pi}}{4} (1+i) = \frac{\sqrt{\pi}}{2} \cdot \underbrace{\frac{\sqrt{2}}{2} (1+i)}_{\text{multiple of } I_G}$$

let $\gamma = \gamma_1 + \gamma_2 + \gamma_3$ (shown below)

let $f(z) = e^{iz^2}$. Note that $f(z)$ is entire so an integral around γ must sum to zero.

$$\int_{\gamma} f(z) dz$$

Expect one of curves to be zero, one to be I , and one to be multiple of I_G



$$\int_{\gamma} f(z) dz = \int_{\gamma_1} f(z) dz + \int_{\gamma_2} f(z) dz + \int_{\gamma_3} f(z) dz = 0$$

On γ_1 , $z=x$, $dz=dx$

$$\int_{\gamma_1} f(z) dz = \lim_{R \rightarrow \infty} \int_0^R e^{ix^2} dx = I$$

On γ_2 ,

$$z = Re^{i\theta} \quad (0 \leq \theta \leq \pi/4)$$

$$z^2 = R^2 e^{i2\theta}$$

$$f(z) = e^{iz^2}$$

$$\left| \int_{\gamma_2} f(z) dz \right| \leq \left| \int_0^{\pi/4} e^{i(p^2 e^{i2\theta})} i p e^{i\theta} d\theta \right|$$

$$\leq \int_0^{\pi/4} p e^{-p^2 \sin(2\theta)} d\theta \leq 0$$

decays faster than \exists increases
as $p \rightarrow \infty$

On γ_3 ,

$$z = xe^{i\pi/4}, z^2 = x^2 e^{i\pi/2} = x^2 i$$

$$dz = e^{i\pi/4} dx$$

$$\int_{\gamma_3} e^{iz^2} dz = \lim_{R \rightarrow \infty} \int_R^0 e^{i(x^2 i)} e^{i\pi/4} dx = -\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i\right) \int_0^R e^{-x^2} dx = -\frac{\sqrt{\pi}}{2} \cdot \frac{\sqrt{2}}{2} (1+i)$$

In conclusion,

$$\int_{\gamma} f(z) dz = \underbrace{\int_{\gamma_1} f(z) dz}_{I} + \underbrace{\int_{\gamma_2} f(z) dz}_{0} + \underbrace{\int_{\gamma_3} f(z) dz}_{-\frac{\sqrt{2}\pi}{4}(1+i)} = 0$$

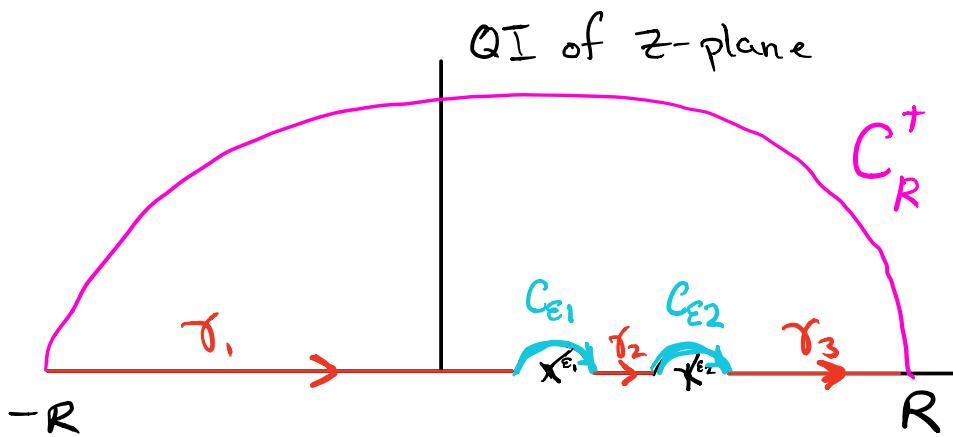
$$I = \lim_{R \rightarrow \infty} \int_0^R e^{ix^2} dx = \frac{\sqrt{2\pi}}{4} (1+i)$$

16.5

$$3) \lim_{R \rightarrow \infty} \int_{-R}^R \frac{e^{ix}}{(x-1)(x-2)} dx = \pi i (e^{2i} - e^i)$$

$f(x) = \frac{e^{ix}}{(x-1)(x-2)}$ has simple poles at $x=1, 2$

So, our contour becomes



Let $f(z) := \frac{e^{iz}}{(z-1)(z-2)}$ we are analytic

So, for $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + C_{\epsilon_1} + C_{\epsilon_2} + C_R^+$

$$\int_{\gamma} f(z) dz = \left(\int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} + \int_{C_{\epsilon_1}} + \int_{C_{\epsilon_2}} + \int_{C_R^+} \right) f(z) dz = 0$$

By Jordan's lemma,

$$\lim_{R \rightarrow \infty} \int_{C_R^+} f(z) dz = 0$$

By Lemma 4 (Page 340)

$$\begin{aligned} \lim_{\epsilon \rightarrow 0^+} \int_{C_{\epsilon 1}} f(z) dz &= i(0 - \pi) \operatorname{Res}(f(z); 1) \\ &= -i\pi \lim_{z \rightarrow 1} \frac{e^{iz}}{(z-1)} = \frac{e^i}{-1}(-i\pi) = \pi e^i \end{aligned}$$

$$\begin{aligned} \lim_{\epsilon \rightarrow 0^+} \int_{C_{\epsilon 2}} f(z) dz &= i(0 - \pi) \operatorname{Res}(f(z); 2) \\ &= -i\pi \lim_{z \rightarrow 2} \frac{e^{iz}}{(z-2)} = -e^{i2}i\pi \end{aligned}$$

Then we note

$$\left(\int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} \right) f(z) dz = \lim_{\substack{R \rightarrow \infty \\ \epsilon \rightarrow 0}} \left[\left(\int_{-R}^{1-\epsilon} + \int_{1+\epsilon}^{2-\epsilon} + \int_{2+\epsilon}^R \right) f(x) dx \right]$$

where

$$\lim_{\substack{R \rightarrow \infty \\ \epsilon \rightarrow 0}} \left[\left(\int_{-R}^{1-\epsilon} + \int_{1+\epsilon}^{2-\epsilon} + \int_{2+\epsilon}^R \right) f(x) dx \right] = I$$

In conclusion,

$$\int_{\gamma} f(z) dz = \left(\int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} + \int_{C_{\epsilon 1}} + \int_{C_{\epsilon 2}} + \int_{C_R^+} \right) f(z) dz = 0$$

$$\Rightarrow I = (e^{i2} - e^i)i\pi$$

$$5) I = \lim_{R \rightarrow \infty} \int_0^R \frac{\cos x - 1}{x^2} dx = -\frac{\pi}{2}$$

Note: $I = \frac{1}{2} \lim_{R \rightarrow \infty} \int_{-R}^R \frac{\cos x - 1}{x^2} dx$

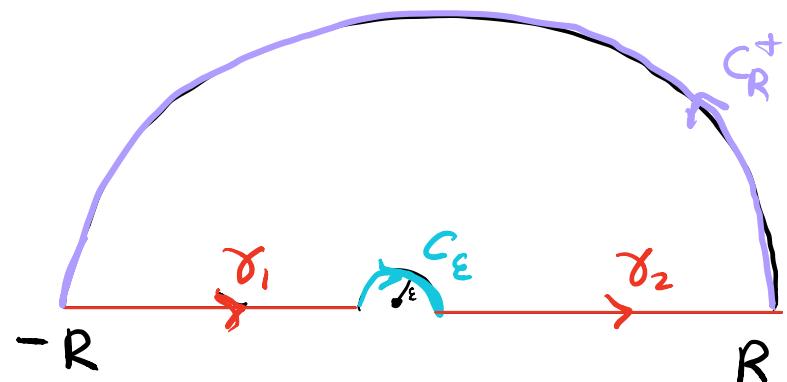
DON'T WORK WITH COSINE

let $\cos(x) = e^{ix}$

$$\lim_{R \rightarrow \infty} \int_0^R \frac{e^{ix} - 1}{x^2} dx$$

} pole of order 2
at $x=0$
Also note a zero at $x=0$!

$$\gamma = \gamma_1 + \gamma_2 + C_\epsilon + C_R^+$$



Let

$$f(z) := \frac{e^{iz} - 1}{z^2}$$

$$\int_{\gamma} f(z) dz = \left(\int_{\gamma_1} + \int_{\gamma_2} + \int_{C_\epsilon} + \int_{C_R^+} \right) f(z) dz = 0$$

By Jordan's lemma,

$$\lim_{R \rightarrow \infty} \int_{C_R^+} f(z) dz = 0$$

By Lemma 4 (Page 340)

$$\lim_{\epsilon \rightarrow 0^+} \int_{C_{\epsilon 1}} f(z) dz = i(0 - \pi) \operatorname{Res}(f(z); 1)$$

$$= -i\pi \left(\lim_{z \rightarrow 0} \frac{d}{dz} \left[\frac{e^{iz} - 1}{z^2} \right] \right)$$

$$= \pi$$

comes out to i

Then we note

$$\left(\int_{\gamma_1} + \int_{\gamma_2} \right) f(z) dz = \lim_{\substack{R \rightarrow \infty \\ \epsilon \rightarrow 0}} \left[\left(\int_{-R}^{-\epsilon} + \int_{\epsilon}^R \right) f(x) dx \right] = 2I$$

So,

$$\int_{\gamma} f(z) dz = \left(\underbrace{\int_{\gamma_1} + \int_{\gamma_2}}_{2I} + \int_{C_{\epsilon}} + \int_{C_R^+} \right) f(z) dz = 0$$

$$2I = -\pi$$

$$I = \lim_{R \rightarrow \infty} \int_0^R \frac{\cos x - 1}{x^2} dx = -\frac{\pi}{2}$$

1)

$$I = \int_0^\infty \frac{\sqrt{x}}{x^2 + 1} dx = \frac{\pi}{\sqrt{2}}$$

6.6

Let $z = re^{i\theta}$; Restrict θ ! $0 \leq \theta \leq 2\pi$

Just "above" x-axis

$$\begin{aligned} z^{1/2} &= r^{1/2} e^{i\theta/2}, \quad \theta = 0, r > 0 \\ &= r^{1/2} \end{aligned}$$

Just "below" x-axis

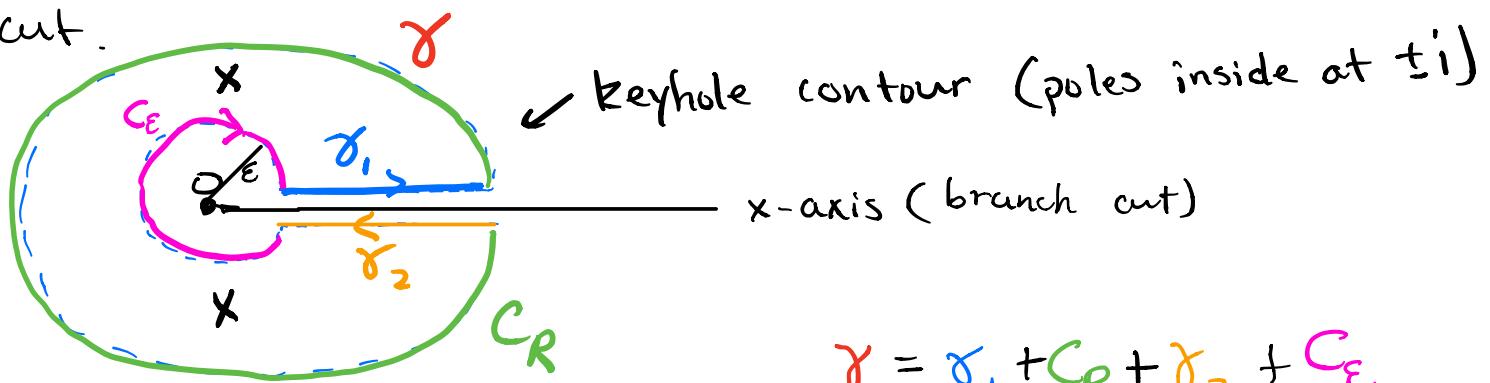
$$\begin{aligned} z^{1/2} &= r^{1/2} e^{i\theta/2}, \quad \theta = 2\pi, r > 0 \\ &= r^{1/2} e^{i\pi} = -r^{1/2} \end{aligned}$$

We have an argument change of $e^{i\pi}$ as a result of our branch cut on the positive x-axis.

Let

$$f(z) := \frac{z^{1/2}}{z^2 + 1}$$

Need a contour that doesn't lie on our branch cut.



$$\int_{\gamma} f(z) dz = 2\pi i \left[\operatorname{Res}(f(z); i) + \operatorname{Res}(f(z); -i) \right]$$

$$= 2\pi i \left[\lim_{z \rightarrow i} \frac{z^{1/2}}{z+i} + \lim_{z \rightarrow -i} \frac{z^{1/2}}{z-i} \right]$$

$$= 2\pi i \left[\frac{(i)^{1/2}}{2i} + \frac{(-i)^{1/2}}{-2i} \right]$$

KEY

$$i = e^{i\pi/2}$$

$$-i = e^{-i\pi/2}$$

$$= 2\pi \left[\frac{(e^{i\pi/2})^{1/2}}{2} + \frac{(e^{-i\pi/2})^{1/2}}{2} \right] = 2\pi \cos\left(\frac{\pi}{4}\right)$$

$$\int_{\gamma} f(z) dz = \int_{\tau_1} f(z) dz + \int_{C_R} f(z) dz + \int_{\tau_2} f(z) dz + \int_{C_\epsilon} f(z) dz$$

$$\int_{\tau_1} f(z) dz = \lim_{\epsilon \rightarrow 0} \int_{R \rightarrow \infty}^R \frac{x^{1/2}}{x^2+1} dx = I \quad \left. \begin{array}{l} \tau_1 \text{ is "just above" case} \\ \tau_2 \text{ is "just below" case} \end{array} \right\}$$

$$\int_{\tau_2} f(z) dz = \lim_{\epsilon \rightarrow 0} \int_{R \rightarrow \infty}^{\epsilon} -\frac{x^{1/2}}{x^2+1} dx = I \quad \left. \begin{array}{l} \tau_2 \text{ is "just below" case} \\ \tau_1 \text{ is "just above" case} \end{array} \right\}$$

$$\int_{C_R} f(z) dz = 0, \quad \int_{C_\epsilon} f(z) dz = 0 \quad \left. \begin{array}{l} \text{expect but} \\ \text{will verify} \end{array} \right\}$$

$$\left| \int_{C_R} f(z) dz = 0 \right| \leq \lim_{R \rightarrow \infty} 2\pi R \max_{z \in C_R} \left| \frac{z^{1/2}}{z^2 + 1} \right|$$

$$\Rightarrow \lim_{R \rightarrow \infty} \frac{2\pi R (R^{1/2})}{R^2 + 1} = 0$$

$$\left| \int_{C_\epsilon} f(z) dz \right| \leq \max_{z \in C_\epsilon} \left| \frac{z^{1/2}}{z^2 + 1} \right| 2\pi \epsilon$$

$$\leq \max_{z \in C_\epsilon} \left| \frac{z^{1/2}}{1} \right| 2\pi \epsilon$$

$$\Rightarrow \lim_{\epsilon \rightarrow 0} \epsilon^{3/2} 2\pi = 0$$

In conclusion,

$$\int_{\gamma} f(z) dz = \int_{C_1} f(z) dz + \int_{C_R} f(z) dz + \int_{C_\epsilon} f(z) dz$$

$$2 \lim_{R \rightarrow \infty} \int_{C_R} \frac{x^{1/2}}{x^2 + 1} dx = \frac{2\pi}{\sqrt{2}}$$

$$I = \lim_{R \rightarrow \infty} \int_{C_R} \frac{x^{1/2}}{x^2 + 1} dx = \frac{\pi}{\sqrt{2}}$$

$$4) \int_0^\infty \frac{x^\alpha}{(x^2+1)^2} dx = \frac{\pi(1-\alpha)}{4\cos(\alpha\frac{\pi}{2})} \quad -1 < \alpha < 3$$

$\alpha \neq 1$

let $z = re^{i\theta}$; Restrict θ ! $-\pi \leq \theta \leq \pi$

Just "above" x-axis

$$\begin{aligned} z^\alpha &= r^\alpha e^{i\alpha\theta}, \quad \theta = 0, r > 0 \\ &= r^\alpha \end{aligned}$$

Just "below" x-axis

$$\begin{aligned} z^\alpha &= r^\alpha e^{i\alpha\theta}, \quad \theta = 2\pi, r > 0 \\ &= r^\alpha e^{i2\pi\alpha} \end{aligned}$$

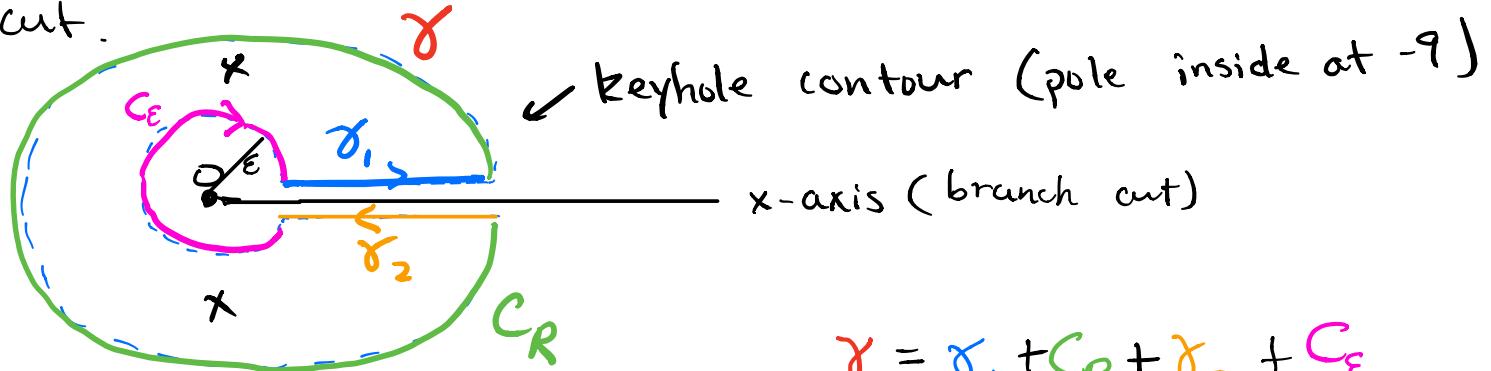
We have an argument change of $e^{i2\pi\alpha}$ as a result of our branch cut on the positive x-axis.

Let

$$f(z) := \frac{z^\alpha}{((z-i)(z+i))^2}$$

which has two poles of order 2 at $x = \pm i$ ($n=2$)

Need a contour that doesn't lie on our branch cut.



$$\int_{\gamma} f(z) dz = 2\pi i \left[\text{Res}(f(z); i) + \text{Res}(f(z); -i) \right]$$

$$\lim_{z \rightarrow i} \frac{d}{dz} \left[\frac{z^\alpha}{(z+i)^2} \right] = \frac{(z+i)^2 \alpha z^{\alpha-1} - z^\alpha 2(z+i)}{(z+i)^4}$$

$$= \frac{+4\alpha \left(e^{i\frac{\pi\alpha}{2}}\right)i - e^{i\frac{\pi\alpha}{2}} 4i}{16} = \frac{-4ie^{i\frac{\pi\alpha}{2}} [1-\alpha]}{16}$$

$$\lim_{z \rightarrow -i} \frac{d}{dz} \left[\frac{z^\alpha}{(z-i)^2} \right] = \frac{(z-i)^2 \alpha z^{\alpha-1} - 2z^\alpha (z-i)}{(z-i)^4}$$

$$= \frac{-4\alpha e^{-i\frac{\pi\alpha}{2}} i + 2(+2i) e^{-i\frac{\pi\alpha}{2}}}{16} = \frac{4ie^{-i\frac{\pi\alpha}{2}} [1-\alpha]}{16}$$

$$2\pi i \left[\text{Res}(f(z); i) + \text{Res}(f(z); -i) \right]$$

$$= \frac{2\pi \left[4e^{i\frac{\pi\alpha}{2}} [1-\alpha] - 4e^{-i\frac{\pi\alpha}{2}} [1-\alpha] \right]}{16}$$

$$= \frac{8\pi (1-\alpha)}{16} \left[4e^{i\frac{\pi}{2}} - 4e^{-i\frac{\pi}{2}} \right]$$

$$\int_{\gamma_1} f(z) dz = \lim_{\substack{\epsilon \rightarrow 0 \\ R \rightarrow \infty}} \int_{\epsilon}^R \frac{z^\alpha}{(z^2+1)^2} dx = I \quad \left. \begin{array}{l} \text{γ_1 is "just above" case} \end{array} \right\}$$

$$\int_{\gamma_2} f(z) dz = \lim_{\substack{\epsilon \rightarrow 0 \\ R \rightarrow \infty}} \int_R^\epsilon \frac{z^\alpha e^{i2\pi\alpha}}{(z^2+1)^2} dx = -I e^{i2\pi\alpha} \quad \left. \begin{array}{l} \text{γ_2 is "just below" case} \end{array} \right\}$$

$$\int_{C_R} f(z) dz = 0, \quad \int_{C_\epsilon} f(z) dz = 0 \quad \left. \begin{array}{l} \text{expect but} \\ \text{will verify} \end{array} \right\}$$

$$\left| \int_{C_R} f(z) dz = 0 \right| \leq \lim_{R \rightarrow \infty} 2\pi R \max_{z \in C_R} \left| \frac{z^\alpha}{(z^2+1)^2} \right|$$

$$\Rightarrow \lim_{R \rightarrow \infty} \frac{2\pi R^{\alpha+1}}{R^4} = 0 \quad \left. \begin{array}{l} \text{by restrictions} \\ \text{on } \alpha \end{array} \right\}$$

$$\left| \int_{C_\epsilon} f(z) dz \right| \leq \max_{z \in C_\epsilon} \left| \frac{z^\alpha}{(z^2+1)^2} \right| 2\pi\epsilon$$

$$\Rightarrow \lim_{\epsilon \rightarrow 0} \frac{(\epsilon)^{\alpha+1}}{\epsilon} 2\pi = 0$$

In conclusion,

$$\int_{\gamma} f(z) dz = \int_{\gamma_1} f(z) dz + \int_{C_R} f(z) dz + \int_{\gamma_2} f(z) dz + \int_{C_\epsilon} f(z) dz$$

$$\int_{\gamma} f(z) dz = I(1 - e^{i2\pi})$$

$$I = \frac{2\pi i \left[\text{Res}(f(z); i) + \text{Res}(f(z); -i) \right]}{(1 - e^{i2\pi\alpha})}$$

$$= 8\pi \frac{\left[e^{i\pi \frac{\alpha}{2}} - e^{-i\frac{\pi\alpha}{2}} \right] (1-\alpha)}{16 (1 - e^{i2\pi\alpha})}$$

Intense Messaging



$$\boxed{\frac{\pi(1-\alpha)}{4 \cos(\alpha\pi/2)}}$$