

Real Integrals from $-\infty$ to $+\infty$

How to calculate $\int_{-\infty}^{+\infty} f(x) dx$, given $f(x)$.

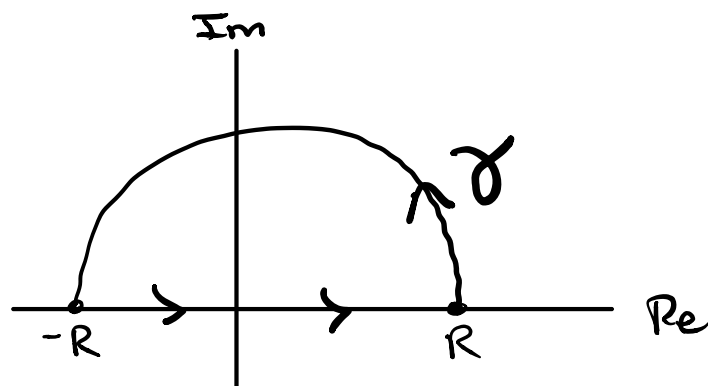
Cauchy principle value $\Rightarrow \lim_{R \rightarrow \infty} \int_{-R}^R f(x) dx$

Example 1)

$$I = \int_{-\infty}^{+\infty} \frac{dx}{1+x^4}$$

1. Choose a clever contour.

\rightarrow Consider a large semicircle as part of contour (common trick)



Put a semicircle
in complex plane connecting
points

\leftarrow what we're interested
in

Now look at $\oint_{\gamma} \frac{dz}{1+z^4} = \int_{-R}^R \frac{dx}{1+x^4} + \int_{\text{semi circle}} \frac{dz}{1+z^4}$

Strategy

calculate w/
residues

This is what we want
as $R \rightarrow \infty$

semi
circle
 \nearrow goes to zero
as $R \rightarrow \infty$

2. Show why $\int_{\text{semi circle}} \frac{dz}{1+z^4}$ goes to zero as $R \rightarrow \infty$

Use ML estimate:

$$L = \pi R \quad \text{on semicircle}$$

$$M = \max_{z \in \text{semi circle}} \left| \frac{1}{1+z^4} \right|$$

Sloppy Way (Real Life): $\left| \frac{1}{1+z^4} \right| \approx \left| \frac{1}{R^4} \right|$ for $R \gg 1$

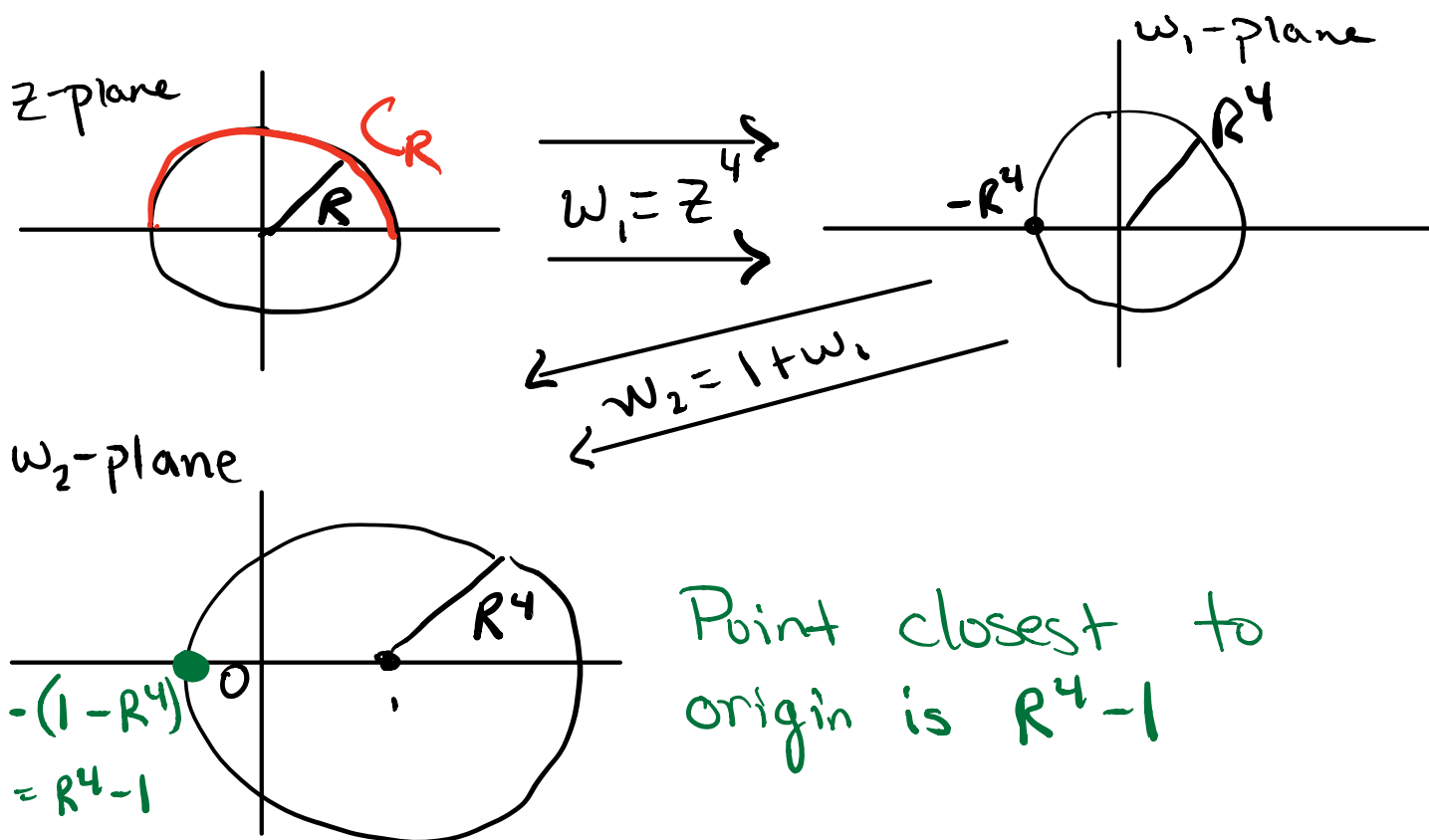
$$\text{So expect } \left| \int_{\text{semi circle}} \frac{1}{1+z^4} dz \right| \approx \frac{\pi R}{R^4} = \frac{\pi}{R^3} \rightarrow 0 \text{ as } R \rightarrow \infty$$

Careful Way (Papers, Tests):

$$\left| \int_{\text{semi-circle}} \frac{1}{1+z^4} \right| \leq \frac{\pi R}{R^4 - 1} \quad \forall R > 1$$

Let C_R denote semicircle

$$\max_{z \in C_R} \left| \frac{1}{1+z^4} \right| = \frac{1}{\min_{z \in C_R} |1+z^4|}$$



$$\text{So, } \min_{z \in C_R} |1 + z^4| = R^4 - 1$$

And

$$\left| \int_{C_R} \frac{dz}{1 + z^4} \right| \leq \frac{\pi R}{R^4 - 1}$$

Hence,
$$\oint \frac{dz}{1+z^4} = 2\pi i \sum_{\substack{\text{poles inside} \\ \gamma}} \text{Res} \left(\frac{1}{1+z^4} \right) \text{ as } R \rightarrow \infty$$

$$= \lim_{R \rightarrow \infty} \int_{-R}^R \frac{dx}{1+x^4}$$

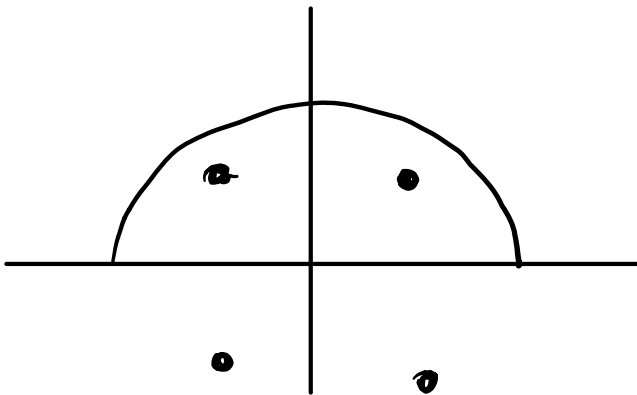
3. Find Residues

Poles occur when $z^4 = -1$

$$z^4 = e^{i\pi + i2\pi k}$$

$$z = e^{i\pi/4 + i\pi/2 k}$$

$$, k = 0, 1, 2, 3$$



only 2 of them lie
in the contour
($k=0, k=1$)

$$\text{Res} \left(f(z); e^{i\pi/4} \right) = \lim_{z \rightarrow e^{i\pi/4}} \left[(z - e^{i\pi/4}) \frac{1}{z^4 + 1} \right]$$

Use L'Hopital's

$$\lim_{z \rightarrow e^{i\pi/4}} \frac{1}{4z^3} = \frac{1}{4e^{i3\pi/4}}$$

Similarly,

$$\text{Res}(f(z); e^{i\frac{3\pi}{4}}) = \frac{1}{4e^{i9\pi/4}} = \frac{e^{-i\pi/4}}{4}$$

Now,

$2\pi i \sum \text{Residues}$

$$= \frac{2\pi i}{4} \left(e^{-i\frac{3\pi}{4}} + e^{-i\pi/4} \right)$$

$$= \frac{2\pi i}{4} \left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i + \frac{-\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i \right)$$

$$= \frac{2\pi i}{4} (-\sqrt{2}i) = \frac{2\sqrt{2}\pi}{4} = \boxed{\frac{\sqrt{2}}{2}\pi}$$

So,

$$\lim_{R \rightarrow \infty} \int_{-R}^R \frac{dx}{1+x^4} = \frac{\sqrt{2}}{2}\pi \quad \left. \vphantom{\int_{-R}^R} \right\} \begin{array}{l} \text{SO MUCH} \\ \text{WORK} \\ \text{SO SIMPLE} \end{array}$$

Example 2

$$I = \lim_{R \rightarrow \infty} \int_{-R}^{+R} \frac{\cos(ax)}{1+x^2} dx, \quad a > 0$$

Can't use the same contour + strategy.
 $\cos(az)$ is exponentially large on semi-circle
as $R \rightarrow \infty$.

For instance,

$$\cos(aiR) = \cosh(aR) = e^{aR/2} \text{ as } R \rightarrow \infty$$

Don't wanna deal with cosine!

$$\begin{aligned} I &= \int_{-R}^R \frac{\cos(ax)}{1+x^2} dx = \int_{-R}^R \frac{e^{iax}}{1+x^2} dx \\ &= \int_{-R}^R \frac{\cos(ax)}{1+x^2} dx + i \underbrace{\int_{-R}^R \frac{\sin(ax)}{1+x^2} dx}_{\text{goes to 0! odd function over symmetric interval.}} \end{aligned}$$

We're led to consider

$$\oint_{\gamma} \frac{e^{iaz}}{1+z^2} dz$$

The point? Unlike $\cos(az)$, e^{iaz} is bounded ON semicircle C_R .

$$|e^{iaz}| = |e^{ia(x+iy)}| = \underbrace{|e^{iax}|}_1 \underbrace{|e^{-ay}|}_{\leq 1} \quad \swarrow a, y > 0$$

So,

$$\left| \oint_{C_R} \frac{e^{iaz}}{1+z^2} \right| \leq \pi R \cdot \max_{z \in C_R} \left| \frac{e^{iaz}}{1+z^2} \right|$$

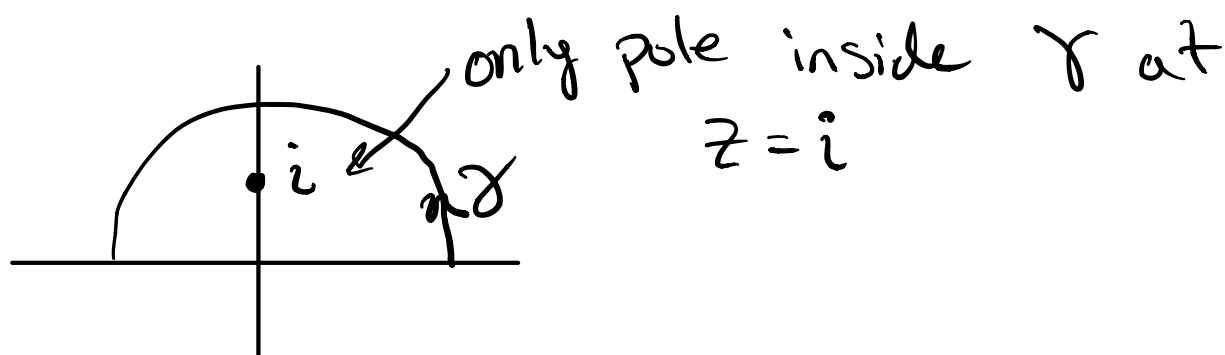
$$\leq \frac{\pi R}{R^2 - 1}$$

$$\text{So } ML \leq \frac{\pi R}{R^2 - 1} \rightarrow 0 \text{ as } R \rightarrow \infty$$

* If $a < 0$, use lower semicircle! *

Thus

$$\oint_{\gamma} \frac{e^{iaz}}{1+z^2} dz = 2\pi i \sum \text{Residues}$$



$$\text{Res}(f; i) = \lim_{z \rightarrow i} (z-i) \frac{e^{iaz}}{z^2+1} = \frac{e^{-a}}{2i}$$

$$= \frac{2\pi i e^{-a}}{2i} = \pi e^{-a}$$

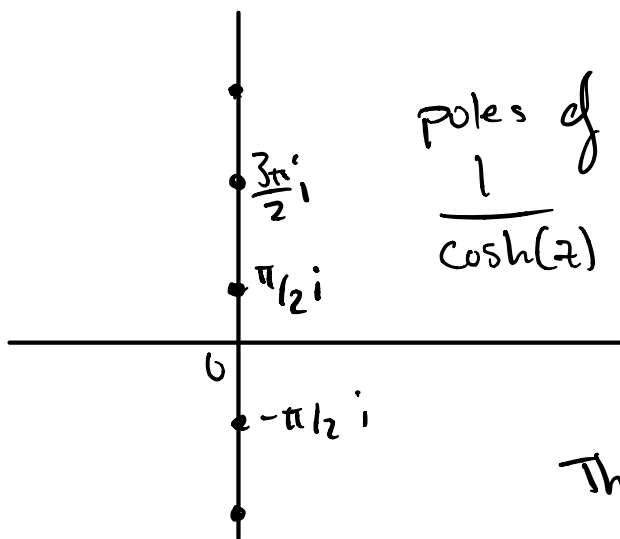
$$\text{So, } \lim_{R \rightarrow \infty} \int_{-R}^{+R} \frac{\cos(ax)}{1+x^2} dx = \pi e^{-a}, \quad a > 0$$

Example 3.

$$I = \int_{-\infty}^{+\infty} \frac{dx}{\cosh(x)} \quad \left. \vphantom{\int_{-\infty}^{+\infty}} \right\} \begin{array}{l} \text{use a} \\ \text{rectangular} \\ \text{contour} \end{array}$$

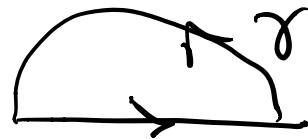
$$\cosh(z) = 0 \text{ at } z = i\frac{\pi}{2} + i\pi k, \quad k \in \mathbb{Z}$$

z -Plane



poles of
 $\frac{1}{\cosh(z)}$

If we use the usual
contour



we get an infinite series
when we do sum of residues.

This is OK BUT there's an
easier way.