

Correlation

$$\text{IE}[XY] = \sum_{x_i} \sum_{y_j} x_i y_j p_{X,Y}(x_i, y_j) \quad (\text{integral for continuous } N)$$

Covariance

$$\begin{aligned}\text{Cov}(X,Y) &\triangleq \text{IE} [(X - \text{IE}[X])(Y - \text{IE}[Y])] \\ &= \text{IE}[XY] - \text{IE}[X]\text{IE}[Y]\end{aligned}$$

If $\text{IE}[XY] = 0$, we say X is orthogonal to Y .

If $\text{Cov}(X,Y) = 0$, we say X is uncorrelated to Y .

Properties

Independence of r.v's X, Y

$$f_{X,Y}(x,y) = f_X(x)f_Y(y) \Rightarrow \text{uncorrelatedness}$$

ONE DIRECTIONAL STATEMENT

X, Y, Z r.v.'s

$$\text{Cov}(X, aY + bZ) = a\text{Cov}(X, Y) + b\text{Cov}(X, Z)$$

$$\text{Cov}(X - \text{IE}[X], Y - \text{IE}[Y]) = \text{Cov}(X, Y)$$

Cauchy-Schwartz Inequality

$$\text{IE}[XY] \leq \sqrt{\text{IE}[X^2]\text{IE}[Y^2]}$$

Correlation Coefficient

$$P_{X,Y} = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}}$$

normalization comes
from Cauchy-Schwarz
inequality

$$(|P_{X,Y}| \leq 1)$$

If X_1, \dots, X_m are pairwise uncorrelated, then

$$\text{Var}(X_1 + X_2 + \dots + X_m) = \text{Var}(X_1) + \text{Var}(X_2) + \dots + \text{Var}(X_m)$$

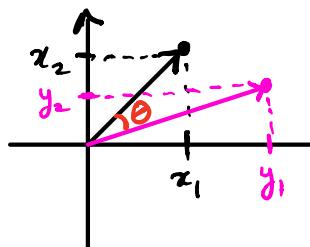
i.e.

$$\text{Var}\left(\sum_{i=1}^m X_i\right) = \sum_{i=1}^m \text{Var}(X_i)$$

Hilbert Space

Example: 2-dimensional Euclidean space.

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$



$$\vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$\langle \vec{x}, \vec{y} \rangle = x_1 y_1 + x_2 y_2 \quad (\text{inner product})$$

$$\text{length/norm } (\vec{x}) = \|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x} \rangle}$$

$$\text{angle b/t } (\vec{x}, \vec{y}) \quad \cos \theta = \frac{\langle \vec{x}, \vec{y} \rangle}{\|\vec{x}\| \|\vec{y}\|} \quad (\text{via law of cosines})$$

Cauchy-Schwarz Inequality

$$|\langle \vec{x}, \vec{y} \rangle| \leq \|\vec{x}\| \|\vec{y}\|$$

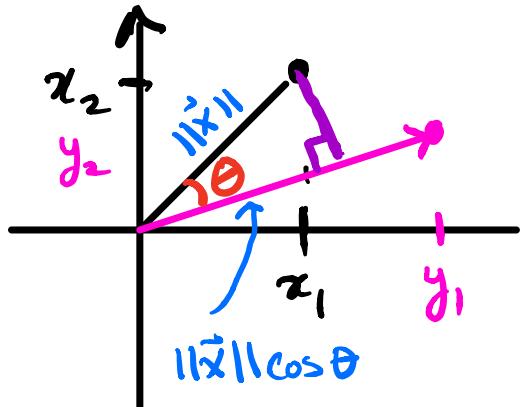
magnitude norm

Projection of \vec{x} onto \vec{y}

$$\Pi_{\vec{y}}(\vec{x}) = \|\vec{x}\| \cos(\theta) \frac{\vec{y}}{\|\vec{y}\|}$$

notation
for projection

$$= \frac{\langle \vec{x}, \vec{y} \rangle}{\|\vec{y}\|^2} \vec{y}$$



$\|\vec{x}\| \cos \theta$
along \vec{y} , normalized to unit length

Orthogonality

If $\vec{x} \perp \vec{y}$ $\Leftrightarrow \cos \theta = 0$
then $(\|\vec{x} + \vec{y}\|)^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2$

$$\Pi_{\vec{y}}(\vec{x}) = 0$$

Vector Space (Linear Space)

* addition defined

* scalar multiplication by $a \in \mathbb{R}$ defined

Inner Product: An inner product on a vector space V is a mapping $V \times V \rightarrow \mathbb{R}$ such that $\forall \vec{x}, \vec{y} \in V$ and $a, b \in \mathbb{R}$

$$\textcircled{1} \quad \langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle$$

$$\textcircled{2} \quad \langle \vec{x}, \vec{x} \rangle \geq 0 \text{ w/ equality iff } \vec{x} = \vec{0}$$

$$\textcircled{3} \quad \langle x, a\vec{y} + b\vec{z} \rangle = a\langle x, \vec{y} \rangle + b\langle x, \vec{z} \rangle$$

Hilbert Space

A Hilbert space is a vector space that

\textcircled{1} Has an inner product $\langle \cdot, \cdot \rangle$ defined

\textcircled{2} Is complete w/ respect to the norm $\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x} \rangle}$
induced by $\langle \cdot, \cdot \rangle$

→ complete in the sense every Cauchy sequence
converges

Hilbert Space of Random Variables

All random variables w/ finite 2nd moments form a Hilbert space

w/ inner product

$$\langle X, Y \rangle \stackrel{\text{r.v's NOT vectors}}{\Rightarrow} \mathbb{E}[XY]$$

Inner Product \Rightarrow norm/length \Rightarrow distance metric

$$\text{length/norm of } X = \sqrt{\langle X, X \rangle} = \sqrt{\mathbb{E}[X^2]} \quad \begin{matrix} \text{Cauchy-Schwarz} \\ \Rightarrow \text{this} \leq 1 \end{matrix}$$

$$\text{Angle b/t } X, Y; \theta_{XY}: \cos(\theta_{XY}) = \frac{\langle X, Y \rangle}{\|X\| \|Y\|} = \frac{\mathbb{E}[XY]}{\sqrt{\mathbb{E}[X^2] \mathbb{E}[Y^2]}}$$

Projection of X onto Y

$$\Pi_Y(X) = \frac{\langle X, Y \rangle}{\|Y\|^2} Y = \frac{\text{IE}[XY]}{\text{IE}[Y^2]} Y$$

a random variable

Orthogonality

$$\langle X, Y \rangle \triangleq \text{IE}[XY] = 0$$

$$\Pi_Y(X) = 0$$

Pythagorean

$$\text{IE}\left[\left(\sum_{i=1}^m X_i\right)^2\right] = \sum_{i=1}^m \text{IE}[X_i^2]$$

all X_i pairwise orthogonal

Mean Squared Error (MSE)

Have r.v. X which can't be directly observed.

Want to estimate \hat{X} .

The estimation error

$$W = X - \hat{X}$$

The MSE of \hat{X} is

$$\text{IE}[(X - \hat{X})^2] = \text{IE}[W^2] = \langle W, W \rangle = \|W\|^2 = (\text{d}(X, \hat{X}))^2$$

Note: we are in a Hilbert space of random variables

Minimum Mean Squared Error

$$\hat{X}_{\text{MMSE}} = \underset{x}{\operatorname{argmin}} \mathbb{E}[(x - \hat{x})^2]$$

MMSE of X using a constant

$$\hat{X}_{\text{MMSE}} = a^*$$

where

$$a^* = \underset{a \in \mathbb{R}}{\operatorname{argmin}} \mathbb{E}[X - a]^2$$

} it's $\mathbb{E}[x]$!

$$\text{MSE} = \mathbb{E}[(X - a)^2]$$

$$= \mathbb{E}[(X - \mathbb{E}[x] + \mathbb{E}[x] - a)^2]$$

$$= \mathbb{E}[(X - \mathbb{E}[x])^2] + 2\mathbb{E}[(X - \mathbb{E}[x])(\mathbb{E}[x] - a)] + \mathbb{E}[(\mathbb{E}[x] - a)^2]$$

Want to minimize

$$\mathbb{E}[(X - \mathbb{E}[x])^2] + \mathbb{E}[(\mathbb{E}[x] - a)^2]$$

$$\rightarrow \text{Var}(X) + (\mathbb{E}[x] - a)^2$$

$$\text{Take } a = \mathbb{E}[x]$$

MMSE of X given $Y=y$

similar proof

$$\hat{X}_{\text{MMSE}} = \mathbb{E}[X | Y=y]$$

$$\text{MSE}(\hat{X}_{\text{MMSE}}) = \text{Var}(X | Y=y)$$