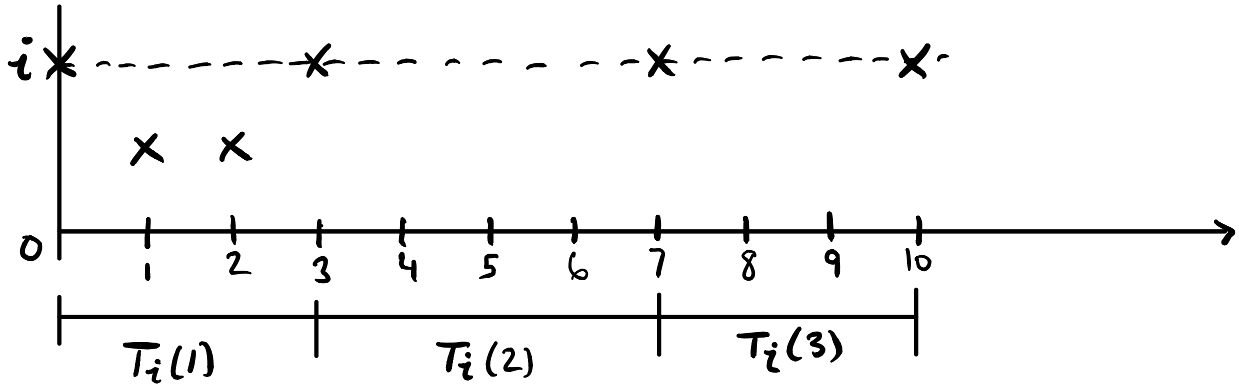
Return Time T_i



T_i iid.

Return Time

$$T_i \triangleq \min\{n \geq 1 : X_n = i \mid X_0 = i\}$$

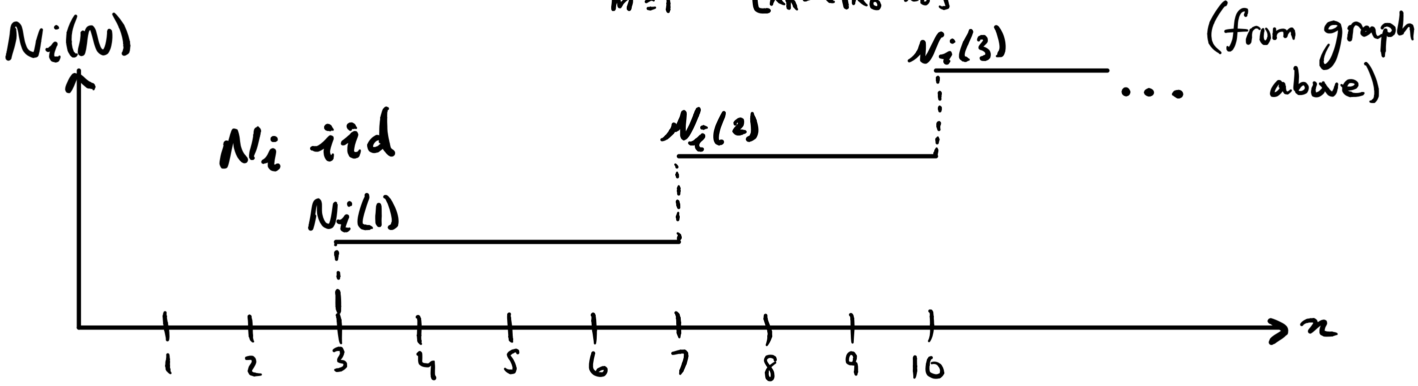
PMF of T_i

$$f_i^{(n)} = \Pr[T_i = n] \qquad n = 1, 2, \ldots$$

$$f_i = \sum_{n=1}^{\infty} f_i^{(n)}$$

Number of Returns (up to time n)

$$N_i(n) \triangleq \sum_{m=1}^{n} \mathbb{1}_{[X_n = i \mid X_0 = x_0]}$$



$$N_i \triangleq N_i(\infty) \triangleq \sum_{m=1}^{\infty} \mathbb{1}_{[X_m = i \mid X_0 = i]}$$

PMF of Ni

$$\Pr[N_i = k] = f_i^k(1-f_i) \qquad k=0,1,2,\dots$$

Say that i is transient if $f_i < 1$

$$\begin{cases} \Pr[T_i < \infty] = f_i < 1 \\ \Pr[T_i = \infty] = 1 - f_i > 0 \end{cases}$$

Note T_i is not properl distributed

$$\Pr[N_i < \infty] \triangleq \sum_{k=0}^{\infty} \Pr[N_i = k] = 1$$

$$\mathbb{E}[N_i] = \frac{f_i}{1-f_i} < \infty$$

$$\sum_{n=1}^{\infty} P_{i,i}^{n} < \infty$$ → necessary & sufficient condition

Say that i is recurrent if $f_i = 1$

$$\begin{cases} \Pr[T_i < \infty] = 1 \\ \Pr[T_i = \infty] = 0 \end{cases}$$

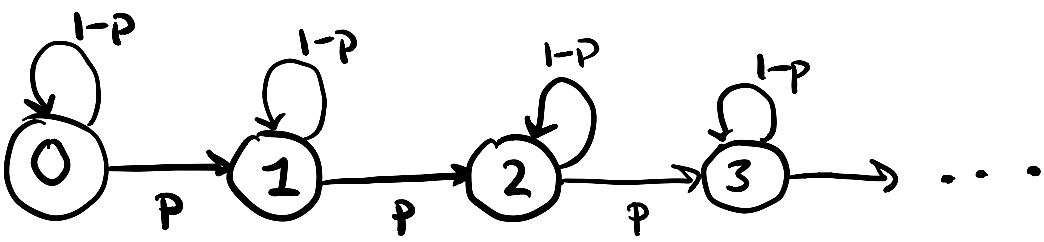
$$\begin{cases} \Pr[N_i < \infty] \triangleq \sum_{k=0}^{\infty} \Pr[N_i = k] = 0 \\ \mathbb{E}[N_i] = \infty \end{cases}$$

$$\mathbb{E}[N_i] = \sum_{n=1}^{\infty} p_{i,i}^{(n)} = \infty$$

Note

Finite-State Irreducible Markov Chains are Recurrent

Example: Binomial Counting

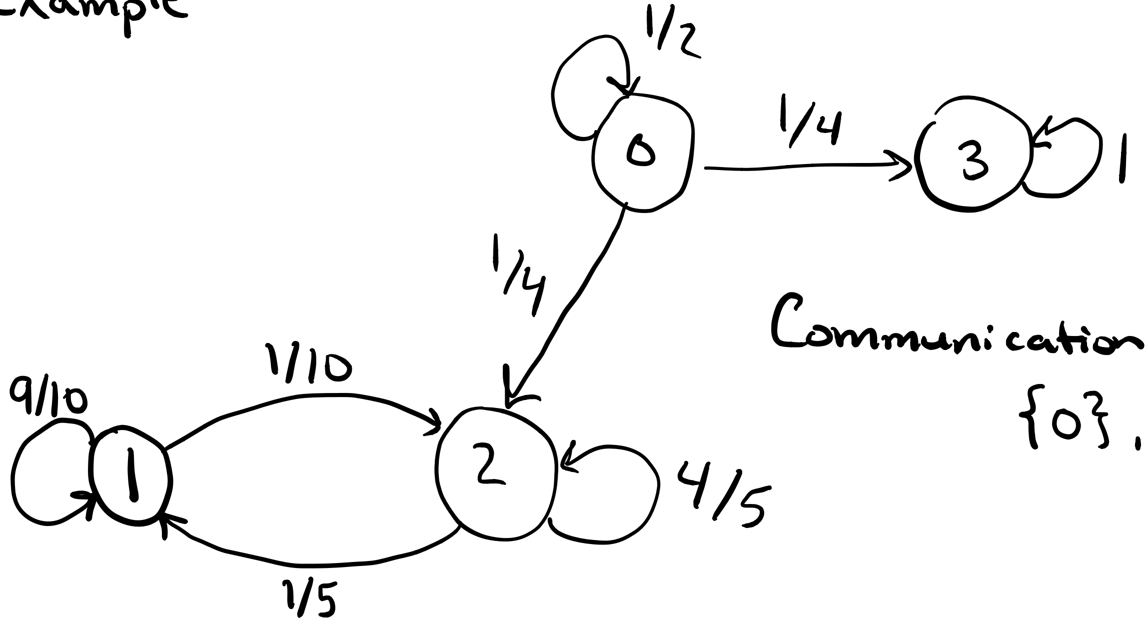


Communication Classes

$$\{0\}, \{1\}, \{2\}, \ldots$$

$$p_{i,i}^{(n)} = (1-p)^n \rightarrow \sum_{n=1}^{\infty} p_{i,i}^{(n)} = \frac{1-p}{p}$$

Example



Communication Classes

$\{0\}, \{1,2\}, \{3\}$

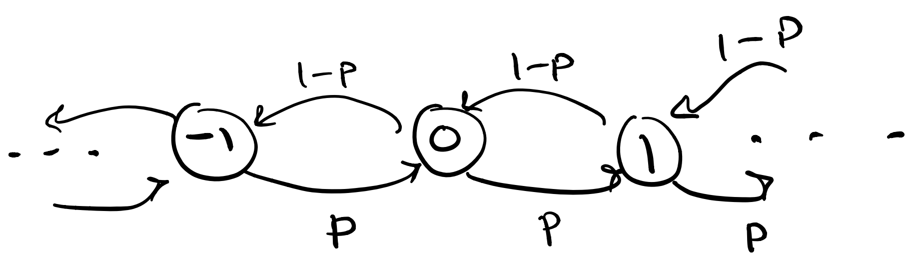
$$p_{0,0}^{(n)} = \left(\tfrac{1}{2}\right)^n$$

$$\sum_{n=1}^{\infty} \left(\tfrac{1}{2}\right)^n = 1 \Rightarrow 0 \text{ transient}$$

$p_{1,1}^{(n)} = \left(\frac{9}{10}\right)^n \Rightarrow 1$ transient, similarly 2 transient

$p_{3,3}^{(n)} = 1^n \Rightarrow 3$ recurrent

Example:



$p = \frac{1}{2} \Rightarrow$ every state recurrent

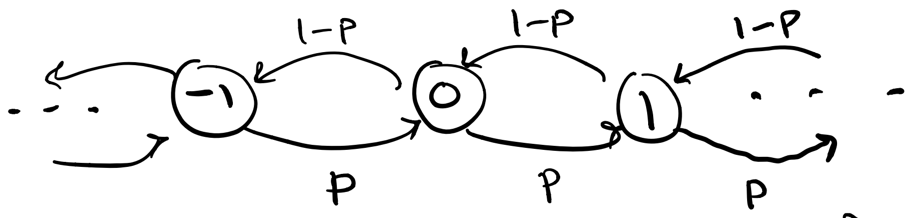
$p < \frac{1}{2}$ or $p > \frac{1}{2} \Rightarrow$ transient states

Positive Recurrent

- i is positive recurrent if $\mathbb{E}[T_i] < \infty$.
- i is null recurrent if $\mathbb{E}[T_i] = \infty$.

How recurrent classification is split up

Observe the random walk



Period 2n

This chain is Null recurrent

To see this, observe

$$\Pr[T_i = 2n] \sim \frac{c}{n^{3/2}}$$

$$\sum_{n=1}^{\infty} \Pr[T_i = 2n] = 1$$

$$\mathbb{E}[T_i] = \sum_{n=1}^{\infty} 2n \frac{c}{n^{3/2}} \longrightarrow \infty$$

Positive Recurrent

$$N_i(n) \propto n \quad \nearrow$$

Null Recurrent

$$N_i(n) \propto \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}} \quad \searrow$$