Math 4310 Name: Rami Pellumbi (rp534)

Homework 6

Please print out these pages. I encourage you to work with your classmates on this homework. Please list your collaborators on this cover sheet. (Your grade will not be affected.) Even if you work in a group, you should write up your solutions yourself! You should include all computational details, and proofs should be carefully written with full details. As always, please write neatly and legibly.

Please follow the instructions for the "extended glossary" on separate paper (LaTeX it if you can!) Hand in your final draft, including full explanations and write your glossary in complete, mathematically and grammatically correct sentences. Your answers will be assessed for style and accuracy.

Please **staple** *this cover sheet, your exercise solutions, and your glossary together, in that order, and hand in your homework in class.*

GRADES

Exercises ______________________ / 50

Extended Glossary

Component	Correct?	Well-written?
Definition	/6	/6
Example	/4	/4
Non-example	/4	/4
Theorem	/5	/5
Proof	/6	/6
Total	/25	/25

Exercises.

1. Let $\mathbb{R}^\infty = \{(\alpha_1, \alpha_2, ...) \in \mathbb{R}^\infty \mid \alpha_i \in \mathbb{R}\}$ be the set of sequences of real numbers. Let $U \subset \mathbb{R}^\infty$ be the subspace of sequences

$$U = \{(\alpha_1, \alpha_2, ...) \in \mathbb{R}^\infty \mid \alpha_{i+2} = \alpha_i + \alpha_{i+1} \ \forall i\}$$

(a) Prove that U is finite-dimensional and compute its dimension.

Writing out the sequence contained in U,

$$U = (\alpha_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, 2\alpha_1 + 3\alpha_2, ...)$$

we see that every single element of U can be written as a linear combination of α_1 and α_2.

Thus,

$$U = span(\alpha_1, \alpha_2)$$

and

$$dim\ U = 2.$$

(b) Find a **complementary** subspace V so that $U \oplus V = \mathbb{R}^\infty$. (Note: V will not be finite dimensional!).

A complementary subspace would be the set of all sequences which are linear combinations of every other element. i.e.

$$V = span(\alpha_3, \alpha_4, \alpha_5, ...)$$

which is infinite dimensional.

(c) It follows from (a) and (b) that $\mathrm{dim}(\mathbb{R}^\infty/V) = 2$ and $\mathrm{dim}(\mathbb{R}^\infty/U) = \infty$ Find a third subspace W of $\mathbb{R}^\infty$ and a complement X so that both $\mathbb{R}^\infty/W$ and $\mathbb{R}^\infty/X$ are infinite dimensional.
Let $W = \mathrm{span}(\alpha_1, \alpha_3, \alpha_5, ...)$ and $X = \mathrm{span}(\alpha_2, \alpha_4, \alpha_6, ...)$.
We verify first that W is the complement of X by realizing they only intersect at the all zero sequence. We then verify their sum is equivalent to R^∞ which is obvious since it is the span of every α.
Next, we realize that

$$\mathrm{dim}(\mathbb{R}^\infty/W) = \infty$$

and

$$\mathrm{dim}(\mathbb{R}^\infty/X) = \infty$$

as desired.

2. Let V be a vector space over the field $\mathbb{F}$, and let W be a subspace. If U_1 and U_2 are both complements of W in V:

(a) Show that U_1 and U_2 are isomorphic.
First assume that V is finite dimensional. Then U_1, U_2 are finite dimensional as proved in earlier homework.
Two finite dimensional vector spaces over $\mathbb{F}$ are isomorphic if and only if they have the same dimension.
U_1 a complement of W means that

$$U_1 \cap W = \{0\}$$

and

$$U_1 + W = \{u + w \mid u \in U_1,\ w \in W\} = V.$$

Thus we can conclude

$$\mathrm{dim}(V) = \mathrm{dim}(U_1 + W) = \mathrm{dim}(U_1) + \mathrm{dim}(W) - \mathrm{dim}(U_1 \cap W).$$

Rearranging and using the fact $\mathrm{dim}(U_1 \cap W) = 0$

$$\mathrm{dim}(U_1) = \mathrm{dim}(V) - \mathrm{dim}(W).$$

In a similar fashion, we can conclude

$$\mathrm{dim}(U_2) = \mathrm{dim}(V) - \mathrm{dim}(W).$$

Since U_1 and U_2 have the same dimension they are thus isomorphic.
BUT, it is not necessary V be finite dimensional. We use the fact that if

$$V = W \oplus U_1$$

then

$$V/W \cong U_1$$

as shown in class. Similarly,

$$V/W \cong U_2.$$

Thus it follows that $U_1 \cong U_2$.

(b) Is $U_1 = U_2$? Prove this or give a counter example.
This question is essentially asking if complements are unique.
Assume towards a contradiction that $U_1 = U_2$. Then

$$U_1 + W = \{u + w \mid u \in U_1,\ w \in W\} = V$$

$$U_2 + W = \{u^* + w \mid u^* \in U_2,\ w \in W\} = V$$

where $u = u^*$ by assumption.
This would mean that V can only take the above form for a complement of W.
Next we attempt to see if this would hold on a vector space we are familiar with, R^2. Take the line $y = x$. A complement of this like would be $y = 3x + 1$ and it can easily be verified their intersection is only 0 and they span the whole space.
In a similar fashion we note that we can also take any other non scalar multiple of $y = x$ and obtain a complement - which in a similar fashion would span all of R^2 and only intersect at the origin.
Thus we have contradicted our "proof" above. **It is not necessary** that $U_1 = U_2$.

3. Let $O(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) \mid f(-x) = -f(x)\}$ be the set of **odd** smooth functions and let $E(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) \mid f(-x) = f(x)\}$ be the set of **even** smooth functions.

(a) Prove that $O(\mathbb{R})$ and $E(\mathbb{R})$ are complementary.

Proof. Need to show that $O(\mathbb{R}) \cap E(\mathbb{R}) = 0_f$ and that their sum is $C^\infty(\mathbb{R})$.
Without intimidation intended, the first point is obvious and does not require proof - since the zero function is the only function which is both odd and even.
Next we show that $O(\mathbb{R}) + E(\mathbb{R}) = C^\infty(\mathbb{R})$
To see this we observe that

$$O(\mathbb{R}) + E(\mathbb{R}) = \{f_e + f_o \mid f_e \in E(\mathbb{R}),\ f_o \in O(\mathbb{R})\}$$

and make the observation that when $f_o = 0_f$ we have all the elements in $E(\mathbb{R})$, and when $f_e = 0_f$ we have all the elements in $O(\mathbb{R})$.
When neither are zero we have the set of all functions which are *neither* even or odd (since the sum of an odd function with an even is neither even nor odd).
Thus the addition of every element of each set with every element of the other encompasses all possible smooth functions and we get

$$O(\mathbb{R}) + E(\mathbb{R}) = C^\infty(\mathbb{R})$$

as desired. □

(b) Prove that $C^\infty(\mathbb{R})/O(\mathbb{R}) \cong E(\mathbb{R})$.

Proof. Let
$$T : C^\infty(\mathbb{R}) \to E(\mathbb{R})$$
be the linear transformation defined by
$$f_{C^\infty(\mathbb{R})} \mapsto f_{E(\mathbb{R})}.$$
To see this is a linear transform, observe that
$$T(cf_1 + df_2) = cf_{e1} + df_{e2} = cT(f_1) + dT(f_2)$$
where $f_1, f_2 \in C^\infty(\mathbb{R})$, $c, d \in R$.
Every function can be written as the sum of an even and odd component thus a function which is neither even or odd can be mapped to an even function by offsetting the odd component.
A function which is odd must be sent to the zero function. Thus it can be seen that the ker(T) must be all of the **odd functions**, $O(\mathbb{R})$.
Next we define the linear transformation
$$\tilde{T} : C^\infty(\mathbb{R})/ker(T) \to E(\mathbb{R})$$
by
$$f_{C^\infty(\mathbb{R})} + O(\mathbb{R}) \mapsto T(C^\infty(\mathbb{R})).$$
We now note that T as defined above is surjective since every even function is mapped onto by the domain. Thus $\tilde{T}$ is also surjective. It can also be seen that $ker(\tilde{T})$ is $\{0\}$ and thus the map is injective.
Since $\tilde{T}$ is bijective there is an isomorphism from $C^\infty(\mathbb{R})/O(\mathbb{R})$ to $E(\mathbb{R})$.
NOTE: The same fact from question (2) could be used but I wanted some practice using this type of reasoning. If it happens to be wrong I would like to switch my answer to the fact that since O and E are complementary the isomorphism desired exists. □

4. Let U and V be vector spaces (over $\mathbb{F}$ assumed) with respective subspaces X and Y. Prove there is an isomorphism
$$(U \times V)/(X \times Y) \cong (U/X) \times (V/Y)$$

First, we begin by recalling the definition of external direct sum.

$$U \times V = \{(u, v) \mid u \in U,\ v \in V\}$$

$$X \times Y = \{(x, y) \mid x \in X,\ y \in Y\}$$

Next, we recall the definition of a quotient space

$$U/X = \{u + X \mid u \in U\}$$

$$V/Y = \{v + Y \mid v \in V\}$$

Finally, we write the left hand side and right hand side out in their set notation to better understand what they represent.

$$(U \times V)/(X \times Y) = \{(u, v) + (X \times Y) \mid (u, v) \in U \times V\}$$

$$(U/X) \times (V/Y) = \{(u + X, v + Y) \mid u \in U,\ v \in V\}$$

Now we begin the proof.

Proof. Define T to be the linear map

$$T : U \times V \to (U/X) \times (V/Y)$$

by
$$(u, v) \mapsto (u + X, v + Y).$$

We first verify that T is indeed a linear map by verifying

$$T[(u_1, v_1) + (u_2, v_2)] = T[u_1, v_1] + T[(u_2, v_2)]$$

$$(u_1 + u_2 + X, v_1 + v_2 + Y) = (u_1 + X, v_1 + Y) + (u_2 + X, v_2 + Y)$$
$$(u_1 + u_2 + X, v_1 + v_2 + Y) = (u_1 + u_2 + X, v_1 + v_2 + Y)$$

and
$$T[c \cdot (u_1, v_1)] = c \cdot T[(u_1, v_1]$$

$$(cu_1 + X, cv_1 + Y) = c \cdot (u_1 + X, v_1 + Y)$$

$$(cu_1 + X, cv_1 + Y) = (cu_1 + X, cv_1 + Y)$$

as desired - where $u_1, u_2 \in U$, $v_1, v_2 \in V$, and $c \in \mathbb{F}$.

Next, we note that as defined the kernel of T is precisely $X \times Y$.

Finally, we will note that T as defined is a surjective map. It is easy to observe this as every element in the range is clearly mapped onto.

We now define the linear map induced by above, $\tilde{T}$, as

$$\tilde{T} : (U \times V)/(X \times Y) \to (U/X) \times (V/Y)$$

by

$$((u, v) + X \times Y) \mapsto (u + X, v + Y)$$

.

We know that $\tilde{T}$ is a linear map since T is a linear map.

We now note that $\tilde{T}$ as defined is bijective.

Since T was surjective, $\tilde{T}$ is surjective.

The kernel of $\tilde{T}$ is precisely the zero element so the map is injective.

Thus, we can conclude $(U \times V)/(X \times Y) \cong (U/X) \times (V/Y)$. $\square$

5. Let $C^\infty(\mathbb{R})$ denote the vector space (over $\mathbb{R}$) of infinitely-differentiable real-valued functions $f : \mathbb{R} \to \mathbb{R}$ (Note: $C^\infty(\mathbb{R})$ is **very** infinite-dimensional!)

   Let W denote the subspace $C^\infty(\mathbb{R})$ consisting of those functions which "vanish to the n^{th} order at 0":
$$W = \left\{ f \in C^\infty(\mathbb{R}) \mid f(0) = 0, \frac{df}{dx} = 0, ..., \frac{d^n f}{dx^n} = 0 \right\}$$

   Prove that $C^\infty(\mathbb{R})/W$ is finite-dimensional and find a basis.

   Before beginning the proof we first look at a special case (n = 0) where
$$W = \{f \in C^\infty(\mathbb{R}) \mid f(0) = 0\}.$$

   Then
$$C^\infty(\mathbb{R})/W = \{f + W \mid f \in C^\infty(\mathbb{R})\}.$$

   Define T to be the linear map
$$T : C^\infty(\mathbb{R}) \to \mathbb{R}$$
   by
$$f(x) \mapsto f(0).$$

   We first verify T is linear:
$$T(af_1(x) + bf_2(x)) = af(0) + bf(0) = aT(f_1(x)) + bT(f_2(x)).$$

   Next we realize that the kernel of T is W - since W is the elements which vanish to 0 at the 0^{th} order. We also realize that T is a surjective mapping of $C^\infty(\mathbb{R})$ onto $\mathbb{R}$ as every value of $\mathbb{R}$ can be taken by the mapping (i.e if we had all the possible constant functions in $\mathbb{R}$).

   Thus the induced map
$$\tilde{T} : C^\infty(\mathbb{R})/W \to \mathbb{R}$$
   is a linear transform and its domain is isomorphic to its range. Thus the dimension of $C^\infty(\mathbb{R})/W$ is 1, and a basis is (f(x)).

   Now we begin the proof.

   Proof. Fix $n \in \mathbb{N}$
$$W = \left\{ f \in C^\infty(\mathbb{R}) \mid f(0) = 0, \frac{df}{dx} = 0, ..., \frac{d^n f}{dx^n} = 0 \right\}$$

   Define T to be the linear map
$$T : C^\infty(\mathbb{R}) \to \mathbb{R}^{n+1}$$
   by
$$(f(x), \frac{df}{dx}(x), ..., \frac{d^n f}{dx^n}(x)) \mapsto (f(0), \frac{df}{dx}(0), ..., \frac{d^n f}{dx}(0))$$

   To see that T is a linear transform we use the same proof as the special case but instead on the tuple of consecutive derivatives of functions.

Similar to the special scenario above, it is clear to see that the kernel of T is W, and that T is surjective.

Thus, we have the induced map

$$\tilde{T} : C^{\infty}(\mathbb{R})/W \to \mathbb{R}^{n+1}$$

and we can conclude there is an isomorphism from $C^{\infty}(\mathbb{R})/W$ onto $\mathbb{R}^{n+1}$.

Since there is an isomorphism onto $\mathbb{R}^{n+1}$ we know that $C^{\infty}(\mathbb{R})/W$ has the same dimension as it - n+1.

A basis for $C^{\infty}(\mathbb{R})$ is an extension of the basis found above, but now including all of the derivative terms. i.e. $(f(x), \frac{df}{dx}(x), ..., \frac{d^n f}{dx^n}(x))$

$\square$

6. Suppose that m is a positive integer, and let $V = \mathbb{R}[x]_{\leq m}$ be the vector space of polynomial of degree at most m.

   Consider the basis $\mathcal{A} = (1, x, x^2, ..., x^m)$ of V.

   (a) Show that the dual basis to $\mathcal{A}$ is $\mathcal{A}^* = (\phi_0, \phi_1, ..., \phi_m)$, where

   $$\phi_j(p(x)) = \frac{p^{(j)}(0)}{j!}.$$

   The dual basis to A is by definition

   $$\phi_j(b_k) = \begin{cases} a_j & k = j \\ 0 & k \neq j \end{cases}$$

   where $b_1, ..., b_m$ correspond to $(1, x, ..., x^m)$, a_j is the coefficient in front of the x^j term, and $1 \leq k \in \mathbb{N} \leq m$.

   We can now see that (for $p(x) = a_0 + a_1 x + ... + a_m x^m$)

   $$\phi_0(p(x)) = p(0) = a_0$$

   $$\phi_1(p(x)) = \frac{dp}{dx}(0) \cdot \frac{1}{1!} = a_1$$

   $$\phi_2(p(x)) = \frac{d^2p}{dx^2}(0) \cdot \frac{1}{2!} = a_2$$

   ...

   $$\phi_m(p(x)) = \frac{d^mp}{dx^m}(0) \cdot \frac{1}{m!} = a_m$$

   follows the structure of the definition of duel basis as each ϕ_j returns the coefficient in front of x^j.

   (b) Show that $\mathcal{B} = (1, x-3, (x-3)^2, ..., (x-3)^m)$ is a basis of V.

   We first show that every element of the proposed basis is linearly independent. Observe

   $$a_0 + a_1(x-3) + a_2(x-3)^2 + ... + a_m(x-3)^m = 0$$

   only if x = 3 and $a_0 = 0$, else it is required that $a_0 = ... = a_m = 0$ and thus the list is linearly independent.

   Since dim $\mathbb{R}[x]_{\leq m} = m + 1$ and we have a linearly independent list of length m+1 it is a basis of $\mathbb{R}[x]_{\leq m}$. (It would be equally simple to observe every polynomial of degree at most m could be written as a linear combination of the polynomials in the proposed basis).

   (c) Find the dual basis $\mathcal{B}^*$.

   In a similar fashion to the duel basis of A we just have to make sure the only lasting term for each ϕ_j is the coefficient of the $(x-3)^j$ term. This would mean the others vanish, so we must evaluate it at 3 - instead of 0. Thus,

   $$B^* = (\phi_0, \phi_1, ..., \phi_m)$$

   where

   $$\phi_j(p(x)) = \frac{p^{(j)}(3)}{j!}$$