Recall

- Covariance

$$\mathrm{Cov}(X,Y) = \mathbb{E}\big((X-\mathbb{E}(X))(Y-\mathbb{E}(Y))\big)$$

- X, Y uncorrelated $\iff \mathrm{Cov}(X,Y) = 0$
- X, Y uncorrelated $\Rightarrow \mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$

- "Conditional Expectation Revisited"

Given X, Y define a new rv $\mathbb{E}(X \mid Y)$ as follows

- For each Y, compute $\mathbb{E}(X \mid Y = y)$
  - this defines a function $g(y)$
- Define $\mathbb{E}(X \mid Y) = g(Y)$ $\xrightarrow{\text{i.e}}$ $Y \to \boxed{g} \to \mathbb{E}(X \mid Y)$

Did a few examples...

Law of Iterated Expectations: $\mathbb{E}(\mathbb{E}(X \mid Y)) = \mathbb{E}(X)$

For any function $h(Y)$,

- $\mathbb{E}(h(Y) \mid Y) = h(Y)$
- $\mathbb{E}(h(Y)X \mid Y) = h(Y)\mathbb{E}(X \mid Y)$

Can think of $\mathbb{E}(X \mid Y)$ as an <u>estimator of X</u> given Y.

In what sense does it "act like an estimator?"

- $\mathbb{E}(X \mid Y) = \mathbb{E}(X)$ by law of iterated expectations
- The estimation error $X - \mathbb{E}(X \mid Y)$ is uncorrelated w/ the estimate $\mathbb{E}(X \mid Y)$ - in fact, $X - \mathbb{E}(X \mid Y)$ is uncorrelated with Y - More generally, w/ any function $h(Y)$

Note: in class we went straight from (1)→(2)→(3). The work inbetween was done by me in an attempt to organize things/understand; so if it is incorrect I apologize.

To see this,

(1) $$Cov(\underbrace{X-\mathbb{E}(X|Y)}_{\text{a rv-}A}, \underbrace{\mathbb{E}(X|Y)}_{\text{a rv-}B}) = \mathbb{E}\left[(A-\mathbb{E}(A))(B-\mathbb{E}(B))\right]$$

$$= \mathbb{E}\left[\left(X-\mathbb{E}(X|Y) - \underbrace{\mathbb{E}[X-\mathbb{E}(X|Y)]}_{0}\right)\left(\mathbb{E}(X|Y) - \mathbb{E}(\underbrace{\mathbb{E}(X|Y)}_{X})\right)\right]$$

(2) $$= \mathbb{E}\left[\overbrace{\underbrace{\left(X-\mathbb{E}(X|Y)\right)}_{=\mathbb{E}(X)-\mathbb{E}(\mathbb{E}(X|Y)) = 0}}^{\text{has zero mean}}\left(\mathbb{E}(X|Y) - \overbrace{\mathbb{E}(X)}^{\text{mean }\mathbb{E}(X)}\right)\right]$$

$$= \mathbb{E}\left[\left(X-\mathbb{E}(X|Y)\right)\mathbb{E}(X|Y) - \left(X-\mathbb{E}(X|Y)\right)\mathbb{E}(X)\right]$$

(3) $$= \underbrace{\mathbb{E}\left[\left(X-\mathbb{E}(X|Y)\right)\mathbb{E}(X|Y)\right]}_{=0 \text{ (shown below)}} - \underbrace{\mathbb{E}\left[\left(X-\mathbb{E}(X|Y)\right)\mathbb{E}(X)\right]}_{=0 \text{ (similar proof as green)}} = 0$$

To see this, observe

$$\mathbb{E}\left[\left(X-\mathbb{E}(X|Y)\right)\mathbb{E}(X|Y)\right] = \mathbb{E}\left[\mathbb{E}\left(\left(X-\mathbb{E}(X|Y)\right)\underline{\mathbb{E}(X|Y)} \,\middle|\, Y\right)\right]$$ ↙ law of iterated expectation

$$= \mathbb{E}\left[\mathbb{E}\left(\left(X-\mathbb{E}(X|Y)\right)g(Y) \,\middle|\, Y\right)\right]$$

$$= g(Y)\,\mathbb{E}\left[\left(X-\underline{\mathbb{E}(X|Y)}\right) \,\middle|\, Y\right]$$ (next page)

$$= g(Y)\,\mathbb{E}[(X - g(Y)) \mid Y]$$

$$= g(Y)\left(\underline{\mathbb{E}[X \mid Y]} - \underline{\mathbb{E}[g(Y) \mid Y]}\right)$$

$$= g(Y)\,(g(Y) - g(Y)) = 0$$

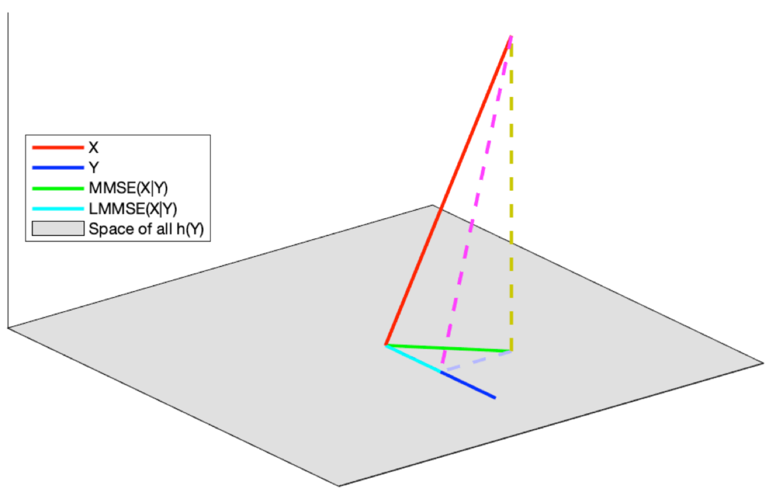
Bottom Line: $X - \mathbb{E}(X \mid Y)$ is <u>uncorrelated</u> with $\mathbb{E}(X \mid Y)$

Visualizing this:

Random Variable Space

(Diagram Courtesy of Alex Coy)

(sorry colors don't match diagram - read legend)



Think of Covariance $\approx$ a dot product

Think of Uncorrelated $\approx$ orthogonal

Taking $\mathbb{E}(X \mid Y) \approx$ orthogonally projecting X onto "space" of functions of Y

Further justification of this geometric world-view:

- orthogonal projections of X onto space of $h(Y)$-functions should be the thing in that space "closest to X"

Standard notion of "closeness": mean-squared difference

Fact (Major): $\mathbb{E}(X \mid Y)$ is the function of Y that minimizes $\mathbb{E}((X-h(Y))^2)$ over ALL functions $h(Y)$

I.e.

$\mathbb{E}(X \mid Y)$ is the minimum mean-square estimator (MMSE) of X given Y

Idea/Proof: Given <u>ANY</u> $h(Y)$

you'd be surprised how many proofs become easier by adding and subtracting "1"

$$\mathbb{E}\left[(X-h(Y))^2\right] = \mathbb{E}\left[\left((X-\mathbb{E}(X \mid Y)) + (\mathbb{E}(X \mid Y) - h(Y))\right)^2\right]$$

$$= \mathbb{E}\left[(X-\mathbb{E}(X \mid Y))^2\right] + \underbrace{2\,\mathbb{E}\left[(X-\mathbb{E}(X \mid Y))(\mathbb{E}(X \mid Y)-h(Y))\right]}_{=0} + \mathbb{E}\left[(\mathbb{E}(X \mid Y)-h(Y))^2\right]$$

Middle part $= 2\,\mathbb{E}\left[\underline{\mathbb{E}\left(((X-\mathbb{E}(X \mid Y))(\mathbb{E}(X \mid Y)-h(Y))) \mid Y\right)}\right]$

$= \left((\mathbb{E}(X \mid Y)-h(Y))\,\mathbb{E}(X-\mathbb{E}(X \mid Y)) \mid Y\right) \Rightarrow$ middle term $= 0$

"obvious" choice of $h(Y)$ to minimize LHS is $h(Y) = \mathbb{E}(X \mid Y)$

Bottom Line: for any function $h(Y)$

$$\mathbb{E}\left[(X-h(Y))^2\right] = \mathbb{E}\left[(X-\mathbb{E}(X \mid Y))^2\right] + \mathbb{E}\left[(\mathbb{E}(X \mid Y)-h(Y))^2\right]$$

Next topic.

Conditional Variance

Given X, Y conditional variance of X given Y is the random variable

$$\mathrm{Var}(X|Y) = \mathbb{E}\left((X - \mathbb{E}(X|Y))^2 \,|\, Y\right)$$

A recipe similar to the "g-thing" for computing $\mathrm{Var}(X|Y)$

- Given y, compute

  $$\mathrm{Var}(X|Y=y) = \mathbb{E}\left[(X - \mathbb{E}(X|Y=y))^2 \,|\, Y=y\right]$$

  - Do this by finding conditional pmf $p_{X|Y}(x|y)$ or pdf $f_{X|Y}(x|y)$ and then computing variance of it

- This yields a function of $y - \gamma(y)$ - plug Y in for y; that yields

  $$\mathrm{Var}(X|Y) = \gamma(Y)$$

Example - See This In Action

$Y \sim \text{Uniform}[0,1]$

$X = \#(\text{Heads})$ in n flips of coin w/ $\mathbb{P}(\{H\}) = Y$

Saw last time

$\mathbb{E}(X|Y=y) = ny \Rightarrow \mathbb{E}(X|Y) = nY$

$\mathrm{Var}(X|Y=y) =$ sum of variances of n-independent Bernoulli(y) trials

$= ny(1-y)$

Thus

$$\mathrm{Var}(X|Y) = \gamma(Y) = nY(1-Y)$$

Since $X = \underbrace{(X - \mathbb{E}(X|Y)) + \mathbb{E}(X|Y)}$ uncorrelated ⇒ variances add

... finish next time