1. This problem pertains to a finite sample space Ω, a probability law $\mathbb{P}$ on Ω, events in Ω, and random variables defined on Ω.

(a) (6 points) Prove or give a counterexample: if A is independent of B, and A is independent of C, and B and C are disjoint, then A is independent of $B \cup C$.

(b) (6 points) Prove or give a counterexample: if $\mathbb{P}(B) > 0$ and $\mathbb{P}(A_1) > \mathbb{P}(A_2) > 0$, then $\mathbb{P}(A_1 \mid B) > \mathbb{P}(A_2 \mid B)$.

(c) (6 points) Prove or give a counterexample: if $\mathbb{P}(B) = 1$, then any event A is independent of B.

(d) (6 points) X, Y, and Z are integer-valued random variables on Ω.

- If $3X + 7Y - 7 = Z$, express $\mathbb{E}(X)$ in terms of $\mathbb{E}(Y)$ and $\mathbb{E}(Z)$.
- If $3X + 7Y - 7 = 0$, find $\mathrm{Var}(X)$ in terms of $\mathrm{Var}(Y)$ given that $\mathbb{E}(Y) = 7$.

2. This problem pertains to a standard deck of 52 playing cards containing 13 cards of each of the four suits hearts, diamonds, clubs, and spades.

(a) (6 points) You shuffle the full deck and then lay out the top seven cards face-up in a row. How many ways can this play out?

(b) (6 points) You shuffle the full deck, draw a card, then replace the card and repeat the shuffle-and-draw process. All shuffle-and-draw operations are independent. Let X be the number of draws it takes to get a spade. What is $p_X(k)$ for $k > 0$? What is $\mathbb{E}(X)$?

(c) (6 points) You shuffle the full deck and then draw cards one by one independently without replacing and re-shuffling. Let Y be the number of draws it takes to get your first heart. What is $p_Y(5)$?

(d) (6 points) A 13-card hand is just a selection of 13 distinct cards from the deck. How many 13-card hands exist that consist of six hearts and seven spades?

(e) (6 points) Four bridge players A, B, C, and D sit around a table. Player A is the dealer. She deals out all the cards, cycling through the players, so in the end each player has a 13-card hand. What is the probability that Player B's hand contains all the spades?

3. Legolas has ten arrows in his quiver. One arrow is magic and the others are ordinary. If he shoots the magic arrow, he always hits the bulls-eye. If he shoots an ordinary arrow, he hits the bulls-eye with probability p. He takes ten independent shots, one with each arrow, drawing the arrows independently and uniformly at random without replacement.

(a) (6 points) Find $\mathbb{P}(A)$, where A is the event that he shoots a bulls-eye on his first shot.

(b) (6 points) With A defined as in part (a), what is the probability given A that he shot the magic arrow?

(c) (8 points) Let B be the event that he shoots a bulls-eye on at least one of his first two shots. Find the probability given B that on one of those two shots he shot the magic arrow.

4. A manufacturer of artificial heart valves has a small fraction p of defective valves in its output. Interpret this as follows: if D is the event that a valve chosen uniformly at random from the output is defective, then $\mathbb{P}(D) = p$. The manufacturer has a quality-control inspector who is 90% accurate in the sense that

$$\begin{array}{ll} \mathbb{P}(\text{rejects} \mid D) = 0.9 & \mathbb{P}\left(\text{rejects} \mid D^c\right) = 0.1 \\ \mathbb{P}(\text{accepts} \mid D) = 0.1 & \mathbb{P}\left(\text{accepts} \mid D^c\right) = 0.9 \end{array}$$

(a) (6 points) Suppose we draw n valves uniformly at random from the output, replacing each valve before the next draw. What is the probability that exactly k of them are defective?

(b) (6 points) What is the probability that the inspector rejects a valve drawn uniformly at random from the output?

The company president, unhappy at the large number of rejected valves, hires a second inspector with the same 90% accuracy rate as the first. Given event D or D^c, the inspectors' determinations are conditionally independent. A valve is trashed if and only if both inspectors reject it.

(c) (6 points) What is the probability that a valve drawn uniformly at random from the output gets trashed? Compare to the answer to (b).

(d) (6 points) What is the probability that a valve doesn't get trashed given that it's defective?

① Pertaining to a finite sample space Ω, a probability law $\mathbb{P}$ on Ω, events in Ω and random variables defined on Ω.

(a) Prove or give a counterexample: if A is independent of B and A is independent of C, and B and C are disjoint, then A is independent of BUC.

A independent of B $\Leftrightarrow$ $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$

A independent of C $\Leftrightarrow$ $\mathbb{P}(A \cap C) = \mathbb{P}(A)\mathbb{P}(C)$

B disjoint from C $\Leftrightarrow$ $\mathbb{P}(B \cap C) = \emptyset$; $\mathbb{P}(B \cup C) = \mathbb{P}(B) + \mathbb{P}(C)$

Need to use givens to prove $\mathbb{P}(A \cap (B \cup C)) = \mathbb{P}(A)\mathbb{P}(B \cup C)$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$$\begin{aligned}
\mathbb{P}(A \cap (B \cup C)) &= \mathbb{P}((A \cap B) \cup (A \cap C)) \\
&= \mathbb{P}(A \cap B) + \mathbb{P}(A \cap C) - \mathbb{P}((A \cap B) \cap (A \cap C)) \\
&= \mathbb{P}(A)\mathbb{P}(B) + \mathbb{P}(A)\mathbb{P}(C) - \mathbb{P}[(A \cap B) \cap \emptyset] \\
&= \mathbb{P}(A)\mathbb{P}(B \cup C)
\end{aligned}$$

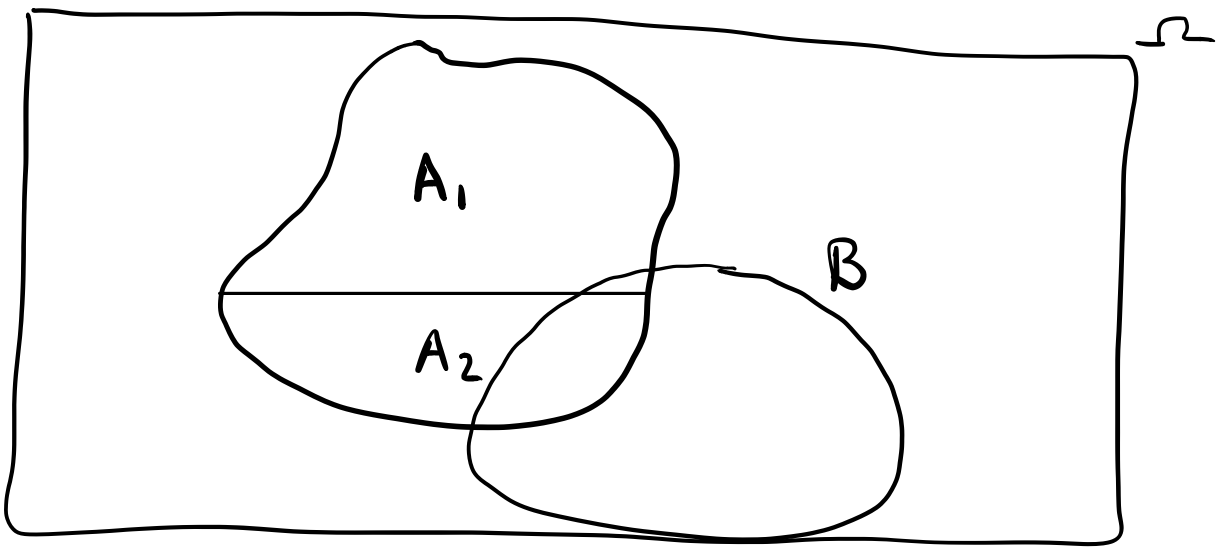
True

(b) Prove or give a counterexample: if $\mathbb{P}(B) > 0$ and $\mathbb{P}(A_1) > \mathbb{P}(A_2) > 0$, then $\mathbb{P}(A_1 | B) > \mathbb{P}(A_2 | B)$.

$\mathbb{P}(A_1|B) > \mathbb{P}(A_2|B)$

$$\Rightarrow \frac{\mathbb{P}(A_1 \cap B)}{\mathbb{P}(B)} > \frac{\mathbb{P}(A_2 \cap B)}{\mathbb{P}(B)} \Rightarrow \mathbb{P}(A_1 \cap B) > \mathbb{P}(A_2 \cap B)$$

Not necessarily true! Observe



(c) Prove or give a counterexample: if $\mathbb{P}(B) = 1$, then any event A is independent of B.

For any event A,

$$\mathbb{P}(A) = \mathbb{P}(A \cap B) + \mathbb{P}(A \cap B^c) \leftarrow \text{by total probability theorem}$$

$\mathbb{P}(B) = 1 \Rightarrow \mathbb{P}(B^c) = 0$

$A \cap B^c \subset B^c \Rightarrow \mathbb{P}(A \cap B^c) \leq \mathbb{P}(B^c) \rightarrow \mathbb{P}(A \cap B^c) = 0$

Thus

$$\mathbb{P}(A) = \mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot 1 = \mathbb{P}(A) \cdot \mathbb{P}(B)$$

So A, B independent

(d) (6 points) X, Y, and Z are integer-valued random variables on Ω.

- If $3X + 7Y - 7 = Z$, express $\mathbb{E}(X)$ in terms of $\mathbb{E}(Y)$ and $\mathbb{E}(Z)$.
- If $3X + 7Y - 7 = 0$, find $\operatorname{Var}(X)$ in terms of $\operatorname{Var}(Y)$ given that $\mathbb{E}(Y) = 7$.

- $\mathbb{E}[3X+7Y-7] = \mathbb{E}[Z]$

$$3\mathbb{E}[X] + 7\mathbb{E}[Y] - 7 = \mathbb{E}[Z]$$

$$\mathbb{E}[X] = \frac{\mathbb{E}[Z] - 7\mathbb{E}[Y]}{3} + \frac{7}{3}$$

- $3X = 7 - 7Y$

$$9\operatorname{Var}(X) = 49\operatorname{Var}(1-Y)$$

$$\operatorname{Var}(X) = \frac{49}{9}\operatorname{Var}(Y)$$

2. This problem pertains to a standard deck of 52 playing cards containing 13 cards of each of the four suits hearts, diamonds, clubs, and spades.

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(c) (6 points) You shuffle the full deck and then draw cards one by one independently without replacing and re-shuffling. Let Y be the number of draws it takes to get your first heart. What is $p_Y(5)$?

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(e) (6 points) Four bridge players A, B, C, and D sit around a table. Player A is the dealer. She deals out all the cards, cycling through the players, so in the end each player has a 13-card hand. What is the probability that Player B's hand contains all the spades?

\# of permutations

number of ways to chose 7 cards from 52

(a) $7!\binom{52}{7}$

(b) X = number of draws to get a spade.

First draw → $\frac{13}{52}$ chance to draw spade, $\frac{39}{52}$ chance NOT to

Second draw $\frac{13}{52}$ chance to draw spade, $\frac{39}{52}$ chance NOT to

⋮

Let k^{th} draw $(k \in \mathbb{N}\setminus\{0\})$ be the draw you draw a spade.

$$p_X(k) = \begin{cases} \left(\frac{13}{52}\right)\left(\frac{39}{52}\right)^{k-1} & , k > 0 \\ 0 & , \text{else} \end{cases}$$

(c) Y = number of draws it takes to draw first heart.

$p_Y(5)$?

$H^{\to \text{heart}}$, $NH^{\to \text{not heart}}$

This is the same as asking

"probability of NOT drawing a heart first 4 draws then drawing a heart fifth draw".

first draw $\rightarrow$ $\frac{13}{52}$ H, $\frac{39}{52}$ NH

second draw $\rightarrow$ $\frac{13}{51}$ H, $\frac{38}{51}$ NH

third draw $\rightarrow$ $\frac{13}{50}$ H, $\frac{37}{50}$ NH

fourth draw $\rightarrow$ $\frac{13}{49}$ H, $\frac{36}{49}$ NH

fifth draw $\rightarrow$ $\frac{13}{48}$ H, $\frac{35}{48}$ NH

So, $Pr(5) = \frac{39}{52} \cdot \frac{38}{51} \cdot \frac{37}{50} \cdot \frac{36}{49} \cdot \frac{13}{48}$

ways to choose them ← ways to pick 6H ← ways to pick 7 spades

(d) $13!\binom{13}{6}\binom{13}{7}$ distinct hands

$\boxed{\binom{13}{6}\binom{13}{7} \quad \text{total hands}}$

(e) $\mathbb{P}(\{\text{Player B's hand contains all spades}\})$

$= \frac{\text{One player getting all spades}}{\text{all ways of getting cards}}$

One player getting all spades $= \dfrac{39!}{13!\,13!\,13!}$

all ways of distributing cards $= \left(\dfrac{52!}{13!\,13!\,13!\,13!}\right)$

thus,

$\mathbb{P}(\{\text{Player B's hand contains all spades}\})$

$$= \frac{1}{\binom{52}{13}}$$

3. Legolas has ten arrows in his quiver. One arrow is magic and the others are ordinary. If he shoots the magic arrow, he always hits the bulls-eye. If he shoots an ordinary arrow, he hits the bulls-eye with probability p. He takes ten independent shots, one with each arrow, drawing the arrows independently and uniformly at random without replacement.

(a) (6 points) Find $\mathbb{P}(A)$, where A is the event that he shoots a bulls-eye on his first shot.

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(c) (8 points) Let B be the event that he shoots a bulls-eye on at least one of his first two shots. Find the probability given B that on one of those two shots he shot the magic arrow.

(a) $A = \{\text{shoots bulls-eye first shot}\})$

$$\mathbb{P}(A) = \mathbb{P}(\{\text{draw magic arrow}\})\,\mathbb{P}(\{\text{magic arrow hits bulls-eye}\}) + \mathbb{P}(\{\text{draw regular arrow}\})\,\mathbb{P}(\{\text{regular arrow hits bulls-eye}\})$$

$$= \frac{1}{10}(1) + \frac{9}{10}p = \frac{1+9p}{10}$$

(b) Let $S = \{\text{he shot magic arrow}\})$

$$\mathbb{P}(\{S\}|\{A\}) = \frac{\mathbb{P}(\{A\}\,|\,\{S\})\,\mathbb{P}(\{S\})}{\mathbb{P}(\{A\}\,|\,\{S\})\,\mathbb{P}(\{S\}) + \mathbb{P}(\{A\}\,|\{S^c\})\,\mathbb{P}(\{S^c\})}$$

$$= \frac{(1)\,1/10}{(1)\,1/10 + p\,(9/10)} = \frac{\frac{1}{10}}{\frac{1}{10} + \frac{9p}{10}} = \frac{1}{9p+1}$$

(c) Let $B = \{\text{shoots a bulls-eye in at least one of first two shots}\}$

$$\mathbb{P}(\{S\}|\{B\}) = \frac{\mathbb{P}(B\,|\,S)\,\mathbb{P}(S)}{\mathbb{P}(B\,|\,S)\,\mathbb{P}(S) + \mathbb{P}(B\,|\,S^c)\,\mathbb{P}(S^c)}$$

$\mathbb{P}(S^c) = \mathbb{P}(\text{shot ordinary arrow twice}) = \frac{9}{10} \cdot \frac{8}{9} = 4/5$

Thus $\mathbb{P}(S) = 1/5$

$\mathbb{P}(B|S) = 1$

$\mathbb{P}(B|S^c) = 1-(1-p)^2$ ← $1-p$ is $\mathbb{P}$(he misses)

$(1-p)^2$ is $\mathbb{P}$(he misses twice)

$1-(1-p)^2$ is $\mathbb{P}$(he hits at least once

$$\boxed{\mathbb{P}(S|B) = \frac{1 \cdot 1/5}{1/5 + (1-(1-p)^2)4/5}}$$

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(c) (6 points) What is the probability that a valve drawn uniformly at random from the output gets trashed? Compare to the answer to (b).

(d) (6 points) What is the probability that a valve doesn't get trashed given that it's defective?

(a) $\mathbb{P}(k \text{ are defective}) = \binom{n}{k} p^k (1-p)^{n-k}$ (Binomial rv)

(b) $\mathbb{P}(\text{rejects}) = \mathbb{P}(\text{rejects} \mid D)\mathbb{P}(D) + \mathbb{P}(\text{rejects} \mid D^c)\,\mathbb{P}(D^c)$

$= 0.9p + 0.1(1-p)$ (Total Probability Theorem)

(c)

$\mathbb{P}(\text{first rejects} \cap \text{second rejects} \mid D) = \mathbb{P}(\text{first rejects} \mid D)\,\mathbb{P}(\text{second rejects} \mid D)$

$= (0.9)(0.9)$

$\mathbb{P}(\text{first rejects} \cap \text{second rejects} \mid D^c) = \mathbb{P}(\text{first rejects} \mid D^c)\,\mathbb{P}(\text{second rejects} \mid D^c)$

$= (0.1)(0.1)$

Thus

$\mathbb{P}(\text{Both reject}) = \mathbb{P}(\text{first rejects} \cap \text{second rejects} \mid D)\,\mathbb{P}(D)$

$+ \mathbb{P}(\text{first rejects} \mid D^c)\,\mathbb{P}(\text{second rejects} \mid D^c)\,\mathbb{P}(D^c)$

$= (0.9)^2 p + (0.1)^2 (1-p)$

(d) $\mathbb{P}(\text{Not rejected} \mid D) = 1 - \mathbb{P}(\text{Both Reject} \mid D)$

$= 1 - 0.81 = 0.19 > 0.1$