

$$\text{cis } \theta = \cos \theta + i \sin \theta = e^{i\theta}$$

Useful Properties

$$\begin{aligned} e^{i\theta_1} e^{i\theta_2} &= \text{cis}(\theta_1 + \theta_2) \quad // \text{Can prove using trig} \\ &= [\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2] + i [\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2] \\ &= \cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2) \\ &= \text{cis}(\theta_1 + \theta_2) \end{aligned}$$

More generally,

$$\begin{aligned} z_1 z_2 &= (r_1 \text{cis } \theta_1)(r_2 \text{cis } \theta_2) \\ &= r_1 r_2 \text{cis}(\theta_1 + \theta_2) \end{aligned}$$

If $z_1 = z_2$,

$$z^2 = r^2 \text{cis}(2\theta)$$

More generally,

$$z^n = r^n \text{cis}(n\theta), \text{ for any positive integer } n$$

De Moivre's Theorem

actually ($n \in \mathbb{R}$)

Example

$$\sqrt[3]{-8} = \sqrt[3]{8} e^{i \frac{2\pi k + \theta}{3}}$$

$$= 2 e^{i \left(\frac{\pi}{3} + \frac{2\pi k}{3} \right)}, k = 0, 1, 2$$

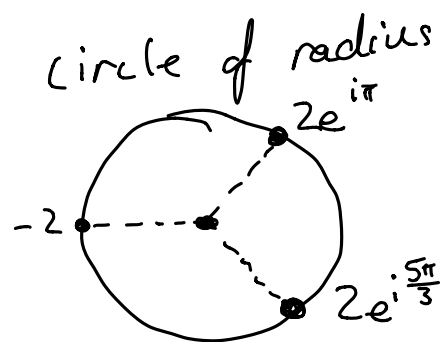
$$z = -8$$

2

$$k=0, \quad 2 e^{i \pi/3} = 1 + \sqrt{3} i$$

$$k=1, \quad 2 e^{i \pi} = -2$$

$$k=2, \quad 2 e^{i 5\pi/3} = 1 - \sqrt{3} i$$



Example: Trig identities via De Moivre's Theorem

$$(\text{cis } \theta)^2 = \cos 2\theta + i \sin 2\theta = \text{cis } 2\theta$$

$$(\cos \theta + i \sin \theta)^2 = (\cos^2 \theta - \sin^2 \theta) + i(2 \sin \theta \cos \theta)$$

Example: Show

$$\frac{\pi}{4} = 2 \tan^{-1} \left(\frac{1}{3} \right) + \tan^{-1} \left(\frac{1}{7} \right)$$

Get this by expanding $(3-i)^2(1+i)$!

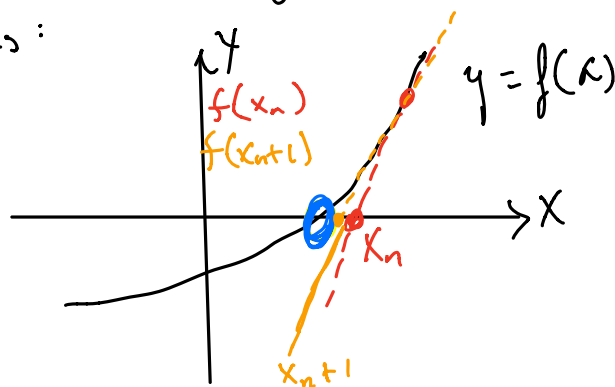
$$\arg[(3-i)^2(1+i)] = \arg[(3-i)(3-i)] + \arg(1+i) = \arg[14+2i]$$

$$-2 \tan^{-1} \left(\frac{1}{3} \right) + \frac{\pi}{4} = \tan^{-1} \left(\frac{1}{7} \right)$$

$$\frac{\pi}{4} = 2 \tan^{-1} \left(\frac{1}{3} \right) + \tan^{-1} \left(\frac{1}{7} \right)$$

Newton's Method: finding solutions to $f(z)=0$

Recall from Calculus:



Basically draw tangent lines to the curve until you approach the zero

From picture:

$$\frac{f(x_n) - 0}{x_n - x_{n+1}} = f'(x_n)$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Complex Version:

$$z_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)} \quad \text{"iterative mapping"}$$

A solution to $f(z)=0$ gives a fixed point ($z_{n+1}=z_n$) for Newton iteration.

Example: Find a soln of $z^2=3+4i$ in first quadrant
 $x \geq 0, y \geq 0$

Define $f(z) = z^2 - (3+4i)$

Assuming derivatives stay the same,
 $f'(z) = 2z$

$$z_{n+1} = z_n - \frac{z_n^2 - (3+4i)}{2z_n} = \frac{z_n^2 + 3 + 4i}{2z_n}$$

Start at some reasonable guess, say, $x_0=1, y_0=1$

just a neat way to simplify

$$z_{n+1} = \frac{(x_n^2 - y_n^2 + 3) + i(4 + 2x_n y_n)}{2(x_n + i y_n)}$$

$$x_0 = 1, \quad y_0 = 1$$

$$x_1 = 1.25, \quad y_1 = 0.75$$

$$x_2 = 1.991667, \quad y_2 = 0.975 \quad \leftarrow 2 \text{ correct}$$

$$x_3 = 1.99993, \quad y_3 = 1.00014 \quad \leftarrow 4 \text{ correct}$$

$$x_4 = 1.99999995, \quad y_4 = 0.99999997 \quad \leftarrow 8 \text{ correct}$$

$$x_5 = 2, \quad y_5 = 1 \quad \leftarrow 16 \text{ correct}$$

typically doubles # of correct digits
 Number of correct digits asymptotically approaches the right answer

How do we choose initial guess?

What if \exists 2 or more solutions - which one does it find?

Example

$$f(z) = z^2 - 1 = 0 \quad \text{Sol'n, } \pm 1$$

$$\text{Newton: } z_{n+1} = z_n - \frac{(z_n^2 - 1)}{2z_n} = \frac{z_n^2 + 1}{2z_n}$$

If we start on an equidistant point (Im axis) we stay there forever

