1. Let $\mathbb{N}$ be the natural numbers, and for each $n \in \mathbb{N}$ let

$$A_n = \{m \in \mathbb{N} : m \geq n\} = \{n, n+1, n+2, \ldots\} .$$

Show that

$$A_0 \cap A_1 \cap A_2 \cap \cdots = \cap_{n=0}^{\infty} A_n = \phi .$$

2. (Problem 1.2 in the book) Let A and B be subsets of another set Ω. Let A^c and B^c be the complements of A and B respectively in Ω.

(a) Show that

$$A^c = (A^c \cap B) \cup (A^c \cap B^c)$$

and

$$B^c = (A \cap B^c) \cup (A^c \cap B^c) .$$

(b) Show that

$$(A \cap B)^c = (A^c \cap B) \cup (A^c \cap B^c) \cup (A \cap B^c) .$$

(c) You roll a six-sided die once. Let A be the set of outcomes where the roll yields an odd number and B the set of outcomes where the roll yields a number less than 4. Verify that the equality in (b) holds.

3. Let A and B be events, and suppose that the probability that neither A nor B occurs is 1/3. What is the probability that one or both of A or B occurs?

4. (Problem 1.7 in the book) Consider the following experiment. You roll a four-sided die with faces numbered 1 through 4 repeatedly until an even number comes up, at which point you stop. If an even number never comes up, you keep rolling forever. The outcome of the experiment is the sequence of rolls you make. What is a good sample space Ω for this experiment? What event in Ω corresponds to "first even number appears on roll n"? What event in Ω corresponds to the situation when you continue rolling forever?

5. Given a point (x, y) in the plane $\mathbb{R}^2$, the *Manhattan distance* between (x, y) and $(0, 0)$ is $|x| + |y|$. Consider an experiment where you select a point from the unit square $[0, 1] \times [0, 1]$ "at random" in the sense that every point in the unit square is equally likely to be selected. What is a good sample space Ω for this experiment? What is an appropriate probability law $\mathbb{P}$? Given $0 \leq a \leq 2$, using your probability law find the probability that the Manhattan distance between the chosen point and $(0, 0)$ is at most a.

6. A dart player throws two darts at a circular board of radius R (and the darts stick). Each dart is equally likely to land anywhere on the board. Define the outcome of the experiment to be the pair of positions where the two darts land.

(a) Construct a good sample space Ω and probability law $\mathbb{P}$ for this experiment.

(b) Suppose the radius of the bullseye is 1/10 of the radius of the dartboard. Using the probability law from (a), find the probability of the event that the first dart hits the bullseye and the second dart doesn't.

Properties of Sets

Identity: $A \cap \Omega = A$, $A \cup \Omega = A$

$A \cup \emptyset = A$

Commutativity: $A \cup B = B \cup A$

$A \cap B = B \cap A$

Associativity: $(A \cup B) \cup C = A \cup (B \cup C)$

$(A \cap B) \cap C = A \cap (B \cap C)$

Distributivity: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

De Morgan's: $\left(\bigcup_n S_n\right)^c = \bigcap_n S_n^c$

$\left(\bigcap_n S_n\right)^c = \bigcup_n S_n^c$

2a) Show that

$$A^c = (A^c \cup B) \cup (A^c \cap B^c)$$

$$= A^c \cap (B \cup B^c)$$

$$= A^c \cap \Omega$$

$$= A^c$$

$$B^c = (A \cap B^c) \cup (A^c \cap B^c)$$

$$= B^c \cap (A \cup B^c)$$

$$= B^c \cap \Omega$$

$$= B^c$$

2b) $(A \cap B)^c = (A^c \cap B) \cup (A^c \cap B^c) \cup (A \cap B^c)$

$= (A^c \cap B) \cup (A^c \cap B^c) \cup (A^c \cap B^c) \cup (A \cap B^c)$

$= A^c \cup B^c$

$= (A \cap B)^c$

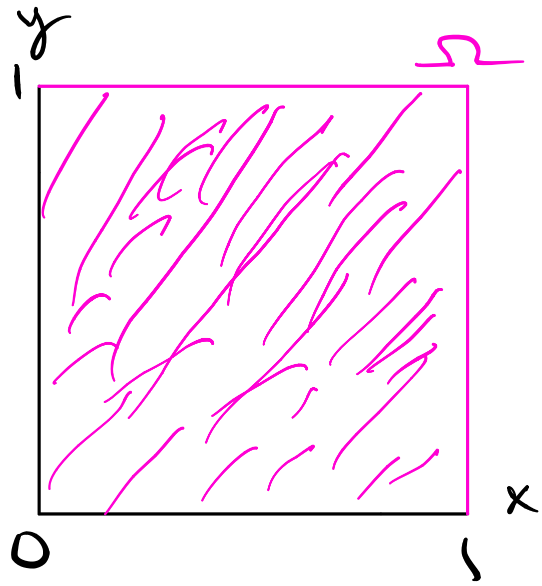
5) Let

$$\Omega = \{(x,y) : 0 \le x, y \le 1\}$$

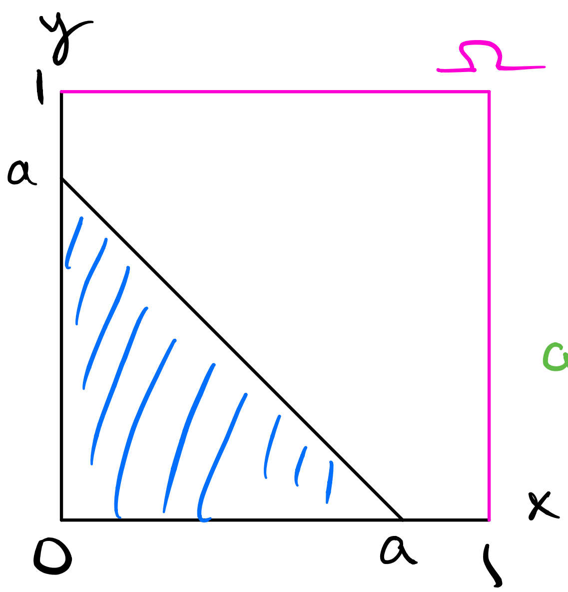
$$\mathbb{P}(A) = \frac{\#\text{ elements in } A}{\text{total \# elements}}$$

$$A = \{(x,y) : |x| + |y| \le a\} \quad , \quad 0 \le a \le 2$$

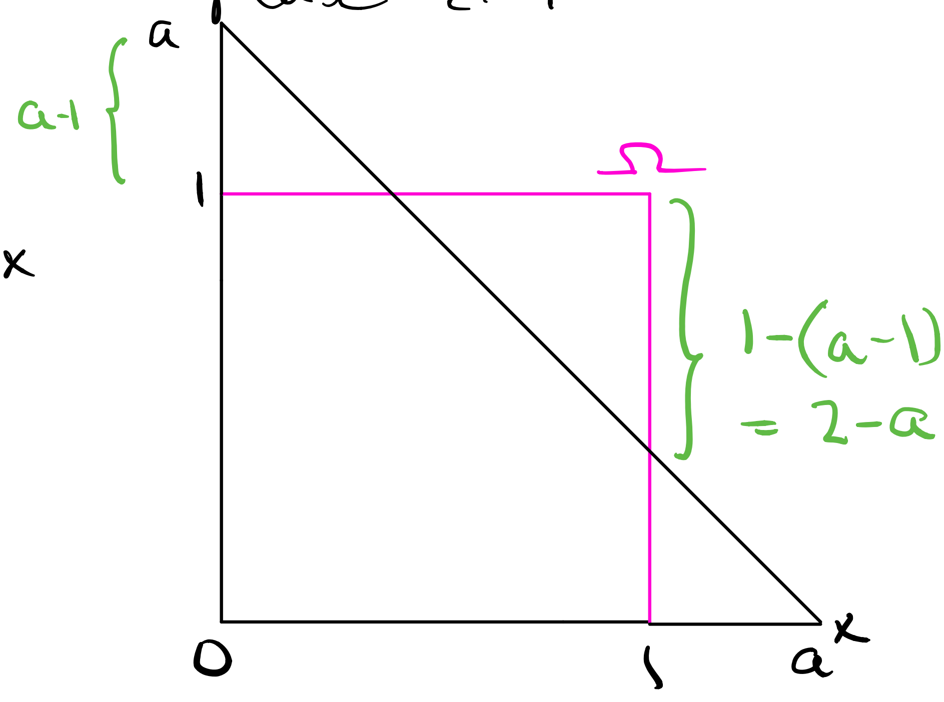
$$\mathbb{P}(A) = \frac{\text{area of } A}{\text{total area}}$$



Case 1: $0 \le a \le 1$



Case 2: $1 < a \le 2$



$\underbrace{}_{a-1}$

For case 1,

$$\text{Area} = \frac{1}{2}a^2$$

$$\boxed{\mathbb{P}(A) = \frac{a^2}{2}}$$

$$\text{Area}_{\text{Total}} = 1$$

For case 2,

$$\text{Area} = 1 - \frac{(2-a)^2}{2} \qquad \mathbb{P}(A) = \frac{1-(2-a)^2}{2}$$

So,

$$\mathbb{P}(A) = \begin{cases} a^2/2, & 0 \leq a \leq 1 \\ \dfrac{1-(2-a)^2}{2}, & 1 < a \leq 2 \end{cases}$$