

- 1.** Frodo tosses a coin repeatedly and independently. The probability that the coin comes up heads on any toss is  $p$ . Every time Frodo tosses a head, Sam pays him \$1. Let  $X$  be the total amount of money Frodo has received after  $n$  tosses.
- Find  $\mathbb{E}(X)$ .
  - Find  $\text{Var}(X)$ .
- 2.** You roll a pair of fair six-sided dice. Let  $X$  be the total of the two numbers facing up. After rolling the dice, you flip a fair coin repeatedly and independently until  $X$  heads come up. Let  $Y$  be the index of the flip after which you stop.
- Find  $\mathbb{E}(Y | X = x)$  for each possible value  $x$  of  $X$ .
  - Find a formula for  $\mathbb{E}(Y)$ .
- 3.** For the random variables  $X$  and  $Y$  defined in Problem 3 on Homework VI, find
- the conditional pmf  $p_{X|Y}(x | y)$  for  $y = 1$  and  $y = 2$  and all values of  $x$ .
  - $\mathbb{E}(X | Y = y)$  for  $y = 1$  and  $y = 2$ .
  - $\mathbb{E}(X)$  from your answer to (b) — note that you solved already for  $\mathbb{E}(X)$  on Homework VI.
- 4.** A discrete random variable takes on seven possible values  $x_1, x_2, \dots, x_7$  and has pmf
- $$p_X(x) = \begin{cases} 1/4 & \text{when } x \in \{x_1, x_2\} \\ 1/8 & \text{when } x \in \{x_3, x_4, x_5\} \\ 1/16 & \text{when } x \in \{x_6, x_7\}. \end{cases}$$
- Consider a naive Yes/No questioning scheme where you order the outcomes from most to least probable and ask in order of decreasing probability
    - “Is  $X = x_1$ ?”
    - “Is  $X = x_2$ ?”
    - ...
    - “Is  $X = x_6$ ?”
 Such a scheme always allows you to determine the value of  $X$ . Define the discrete random variable  $L$  as the number of questions you need to ask.  $L$  depends on  $X$  since, for example, you'll need to ask only one question if  $X = x_1$  but three questions when  $X = x_3$ . Find the pmf of  $L$  and also  $\mathbb{E}(L)$ .
  - Find the entropy of the random variable  $X$  using the defining formula
- $$H(X) = - \sum_{k=1}^7 p_X(x_k) \log_2(p_X(x_k)).$$
- Recall that  $H(X)$  is a lower bound for  $\mathbb{E}(L)$  in any successful Yes/No questioning scheme for determining the value of  $X$ .
- Devise a Yes/No questioning scheme for which  $\mathbb{E}(L) = H(X)$ . Describe the rationale behind your scheme and display it as a binary tree.
- 5.** Let  $X$  be the random variable you get by rolling three fair six-sided dice simultaneously and summing up the numbers facing up.

- (a) Find the entropy  $H(X)$ .
- (b) Devise a Yes/No questioning scheme, exhibited as a binary tree, for determining the value of  $X$  using the Huffman approach we discussed in class. Compare  $\mathbb{E}(L)$  for your scheme to  $H(X)$ . Recall that in class nobody managed to determine  $X$  with fewer than five questions.

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HW7 ECE3100

① Frodo tosses a coin repeatedly + independently.

$$P(\{\text{coin comes up heads on any toss}\}) = p$$

Every time Frodo tosses a head, Sam pays \$1.

Let

$X$  = total amount of money after  $n$  tosses.

(a)  $E(X)$ ? let's think.

$$X = \begin{cases} \$1 & \text{if } n=1 \text{ AND tosses head, no tails} \\ \$1 & \text{if } n=2 \text{ AND tosses one head, one tail} \\ \vdots \\ \$1 & \text{if } n=n \text{ AND tosses one head, } (n-1) \text{ tails} \end{cases}$$

$$X = \begin{cases} \$2 & \text{if } n=2 \text{ AND tosses two heads, no tails} \\ \$2 & \text{if } n=3 \text{ AND tosses two heads, one tail} \\ \vdots \\ \$2 & \text{if } n=n \text{ AND tosses two heads, } (n-2) \text{ tails} \end{cases}$$

$$X = \begin{cases} \$3 & \text{if } n=3 \text{ AND tosses 3 heads, no tails} \\ \$3 & \text{if } n=4 \text{ AND tosses three heads, one tail} \\ \vdots \\ \$3 & \text{if } n=n \text{ AND tosses three heads, } (n-3) \text{ tails} \end{cases}$$

total amount  
↓

$X = \$M$  if you toss  $M$  heads in  $n$  tosses  $\Rightarrow (n-M)$  tails

$$P_X(x) = \begin{cases} P^M (1-p)^{n-M}, & x=M \\ P^{M-1} (1-p)^{n-(M-1)}, & x=M-1 \\ \vdots \\ P^0 (1-p)^{n-0}, & x=0 \end{cases}$$

in my notation,  
 $M=n$

multiplied by  
 $\binom{n}{x}$ ,  $1$  is how many ways to flip  $H$

i.e.  $P_X(x) = \binom{n}{x} p^x (1-p)^{n-x}, 0 \leq x \leq M, x \in \mathbb{Z}$

Note: multiple ways to toss those heads

So,

$$\text{IE}(X) = \sum_{x \in X} x p_X(x) = \sum_{x=0}^n \binom{n}{x} p^x (1-p)^{n-x}$$

Define

$$Z_x = \begin{cases} 1, & x \text{ is H} \\ 0, & x \text{ is T} \end{cases}$$

Note:  $X = Z_1 + \dots + Z_n$

$$\begin{aligned} \text{IE}(Z_x) &= \text{IE}(Z_1) + \dots + \text{IE}(Z_n) \\ &= p + \dots + p \\ &= np = \text{IE}(X) \end{aligned}$$

$$(b) \text{Var}(X) = \text{IE}(X^2) - (\text{IE}(X))^2$$

$$\text{Var}(Z_x) = p(1-p) \approx x$$

$$\text{Var}(X) = np(1-p) \text{ by independence}$$

(2) Roll two six-sided die.

Let

$X = \text{sum of two rolls.}$

Flip a fair coin repeatedly & independently until  $X$  heads come up.

Let

$Y = \text{index of the flip after which you stop}$

(a)  $\mathbb{E}(Y|X=x)$  if  $x \in X$ ?

$$X = \{x \in \mathbb{N} : 2 \leq x \leq 12\}$$
$$Y = \{y \in \mathbb{N} : 2 \leq y \leq n\}; \quad \left. \begin{array}{l} n \text{ is the last flip needed such that you} \\ \text{flipped } X \text{ heads total. i.e. } n \geq 12 \end{array} \right\}$$

Let the random variable  $Z_i$  be index of first head. For  $i \geq 2$ , let  $Z_i$  be the number of flips it takes to see the  $i^{\text{th}}$  head after seeing  $(i-1)^{\text{th}}$  head.

Thus,  $Z_i$  is a Geometric rv with parameter  $p$ .

$$\mathbb{E}(Y|X=x) = \mathbb{E}\left(\sum_{i=1}^x Z_i\right) = \sum_{i=1}^x \mathbb{E}(Z_i) = \frac{x}{p}$$

(b)  $\mathbb{E}(Y)$ ?

$$\mathbb{E}(Y) = \sum_x \mathbb{E}(Y|X=x) P(X=x)$$

$$P_X(x) = \begin{cases} \frac{6 - |x-7|}{36}, & 2 \leq x \leq 12 \\ 0, & \text{else} \end{cases}$$

fancy way  
to do pmf  
of  $\sim$  describing  
sum of  $\sim$  rolls

Thus

$$IE(Y) = \sum_{x=2}^{12} \frac{x}{p} \cdot \frac{6 - |x-7|}{36} = \frac{252}{36p}$$

③  $X, Y$  have joint pmf

$$P_{X,Y} = \begin{cases} xy^2, & x \in \{1, 3, 4\}, y \in \{1, 2\} \\ 0, & \text{otherwise} \end{cases}$$

(a)  $P_{X|Y}(x|y)$  for  $y=1, y=2$  and all values of  $x$ ?

$$P_{X|Y}(1|1) = \frac{1/40}{8/40} = 1/8$$

$$P_{X|Y}(1|2) = 1/8$$

$$P_{X|Y}(3|1) = 3/8$$

$$P_{X|Y}(3|2) = 3/8$$

$$P_{X|Y}(4|1) = P_{X|Y}(4|2) = 4/8$$

$$P_{X|Y}(x|y) = \begin{cases} \frac{1}{8}, & x=1, \forall y \\ \frac{3}{8}, & x=3, \forall y \\ \frac{1}{2}, & x=4, \forall y \end{cases}$$

(b)  $E(X|Y=y)$  for  $y=1, y=2$ ?

$$E(X|Y=1) = \sum_{x \in X} P(X=x | Y=1) x$$

$$= 1\left(\frac{1}{8}\right) + 3\left(\frac{3}{8}\right) + 4\left(\frac{4}{8}\right) = 13/4$$

$$E(X|Y=2) = \sum_{x \in X} P(X=x | Y=2) x = 1\left(\frac{1}{8}\right) + 3\left(\frac{3}{8}\right) + 4\left(\frac{4}{8}\right) = 13/4$$

(c)  $E(X)$  from answer to (b)?

$$\begin{aligned}E(X) &= E(x|Y=1)P(Y=1) + E(x|Y=2)P(Y=2) \\&= \frac{13}{4} \left(\frac{1}{5}\right) + \frac{13}{4} \left(\frac{4}{5}\right) \\&= \boxed{\frac{13}{4}}\end{aligned}$$

(4)

$$P_x(x) = \begin{cases} 1/4, & x \in \{x_1, x_2\} \\ 1/8, & x \in \{x_3, x_4, x_5\} \\ 1/16, & x \in \{x_6, x_7\} \end{cases}$$

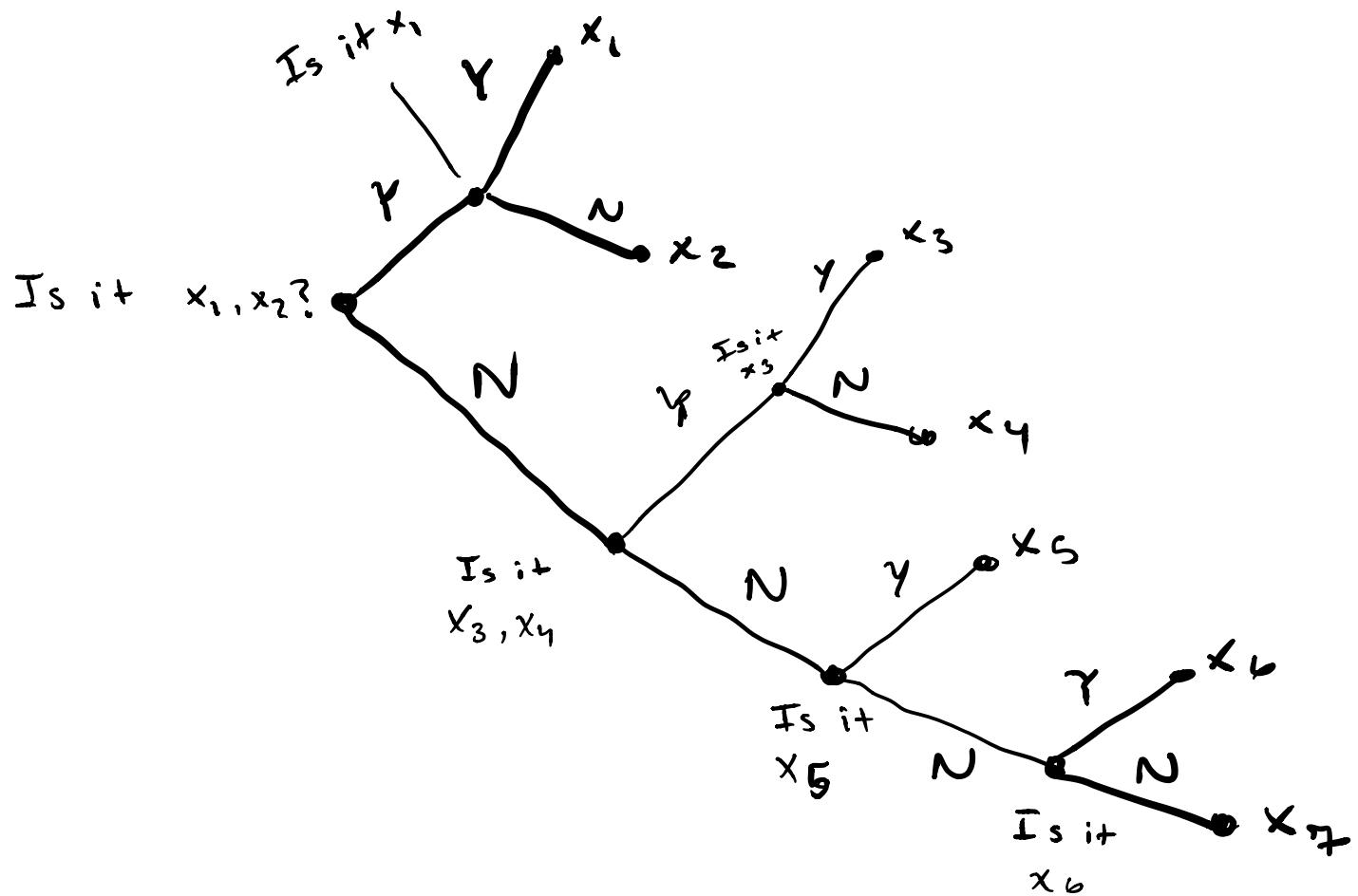
$$P_L(l) = \begin{cases} 1/4, & l = 1, 2 \\ 1/8, & l = 3, 4, 5 \\ 1/16, & l = 6, 7 \end{cases}$$

$$\begin{aligned} \text{IE}(L) &= \sum_{l \in L} l P_L(l) = (1+2) \frac{1}{4} + (3+4+5) \frac{1}{8} + (6+7) \frac{1}{16} \\ &= 48/16 = 3 \end{aligned}$$

$$(b) H(x) = - \sum_{i=1}^7 P_x(x_i) \log_2(P_x(x_i))$$

$$\begin{aligned} &= - \left( \frac{1}{4} \cdot \left( \log_2\left(\frac{1}{4}\right) + \log_2\left(\frac{1}{4}\right) \right) + \frac{1}{8} \left( 3 \log_2\left(\frac{1}{8}\right) \right) + \frac{1}{16} \left( 2 \cdot \log_2\left(\frac{1}{16}\right) \right) \right) \\ &\stackrel{\text{distribute negative}}{=} \frac{1}{4} \cdot (2 \log_2(4)) + \frac{1}{8} (3 \cdot \log_2(8)) + \frac{1}{16} (2 \cdot \log_2(16)) = 2.63 \end{aligned}$$

(c)



⑤  $X$  = rv from rolling 3 six-sided dice simultaneously and summing up the numbers facing up.

(a) Entropy  $H(X)$ ?

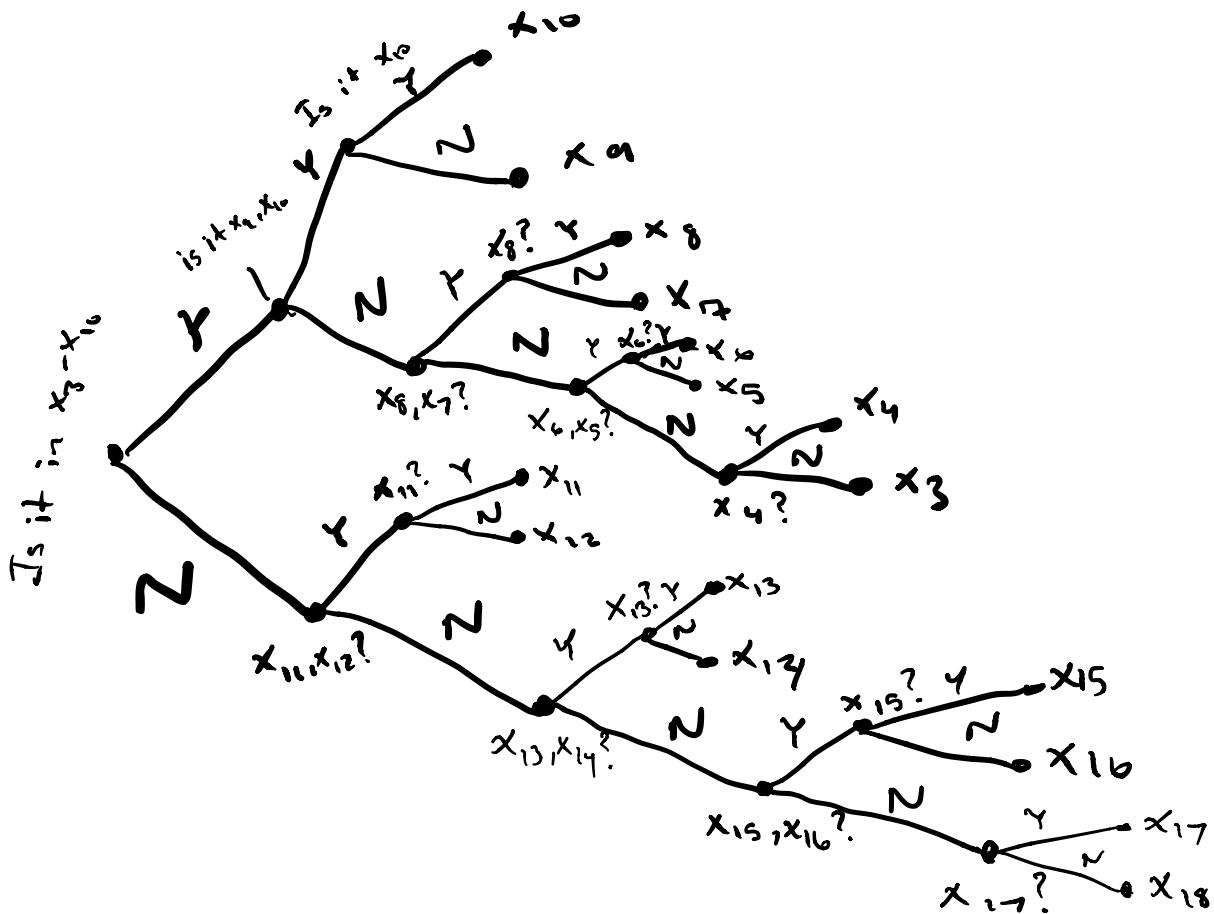
$$H(X) = - \sum_{n=3}^{18} p_x(x_n) \log_2(p_x(x_n)) ?$$

First need  $p_x(x)$ :

$$p_x(x) = \begin{cases} \frac{1}{216} & x = 3 \\ \frac{3}{216} & x = 4 \\ \frac{6}{216} & x = 5 \\ \frac{10}{216} & x = 6 \\ \frac{15}{216} & x = 7 \\ \frac{21}{216} & x = 8 \\ \frac{25}{216} & x = 9 \\ \frac{27}{216} & x = 10 \\ \frac{27}{216} & x = 11 \\ \frac{25}{216} & x = 12 \\ \frac{21}{216} & x = 13 \\ \frac{15}{216} & x = 14 \\ \frac{10}{216} & x = 15 \\ \frac{6}{216} & x = 16 \\ \frac{3}{216} & x = 17 \\ \frac{1}{216} & x = 18 \end{cases}$$

$$H(x) = -\sum_{n=3}^{18} p_x(x_n) \log_2(p_x(x_n))$$

$$\approx 3.6$$



$$IE(L) = \sum_{l \in L} l p_l(l)$$

$$= 2 \left( \frac{1}{216} \left( 27 \cdot 3 + 25 \cdot 3 + 21 \cdot 4 + 15 \cdot 4 + 10 \cdot 5 + 6 \cdot 5 \right) \right) + 3 \cdot 5 + 1 \cdot 5$$

$$\approx 3.7$$

$IE(L) > H(L)$  which is expected due to structure of the tree.