External Direct Sum

Product of Vector Spaces

$$V_1 \times V_2 \times \cdots \times V_m \triangleq \{(v_1, \ldots, v_m) \mid v_1 \in V_1, \ldots, v_m \in V_m\}$$

addition, scalar mult. as expected

Product of Vector Spaces is a Vector Space

$V_1, \ldots, V_m$ vector spaces over $\mathbb{F}$. Then $V_1 \times \cdots \times V_m$ is a vector space over $\mathbb{F}$.

Proof → just show V.S. axioms

Dimension of a Product is a Sum of Dimensions

$V_1, \ldots, V_m$ finite dimensional vector spaces over $\mathbb{F}$

$$\dim(V_1 \times \cdots \times V_m) = \sum_{i=1}^{m} \dim(V_i)$$

Proof → Choose a basis of each V_j $(j = 1, \ldots, m)$.

For each basis vector of EACH V_j, consider the elements of $V_1 \times \cdots \times V_m$ that equals the basis vector in the j^{th} slot and 0 in the other slots.

The list of all such vectors is LI and spans $V_1 \times \cdots \times V_m$ ⇒ basis for $V_1 \times \cdots \times V_m$.

The length of this basis is $\dim V_1 + \cdots + \dim V_m$ as desired.

Products and Direct Sums

$U_1, \ldots, U_m$ subspaces of V. Define a linear map

$$\Gamma : U_1 \times \cdots \times U_m \longrightarrow U_1 + \cdots + U_m$$

by

$$(u_1, \ldots, u_m) \longmapsto u_1 + \cdots + u_m$$

Then $U_1 + \cdots + U_m$ is a direct sum iff Γ is injective.

Quotients of Vector Spaces

$v + U$

Suppose $v \in V$ and $U \subseteq V$. Then $v + U$ is the subset of V defined by

$$v + U = \{v + u \mid u \in U\}$$

Quotient Space, V/U

Suppose U is a subspace of V. Then the **quotient space** V/U is the set of all affine subsets of V parallel to U.

i.e.

$$V/U = \{v + U \mid v \in V\}$$

Two Affine Subsets Parallel to U are Equal or Disjoint

Suppose U is a subspace of V and $v, w \in V$. Then the following are equivalent:

(a) $v - w \in U$

(b) $v + U = w + U$

(c) $(v + U) \cap (w + U) \neq \emptyset$

Proof → (a) ⇒ (b)

Suppose $v - w \in U$.

If $u \in U$, then

$$v + u = w + ((v - w) + u) \in w + U$$

Thus $v + U \subset w + U$

If $u \in U$, then

$$w + u = v + ((w - v) + u) \in v + U$$

Thus $w + U \subset v + U$ and

$$v + U = w + U$$

(b) ⇒ (c)

$v + U = w + U \Rightarrow$ NOT disjoint. QED

(c) ⇒ (a)

Suppose $(v + U) \cap (w + U) \neq \emptyset$

Then $\exists\, u_1, u_2 \in U$ s.t. $v + u_1 = w + u_2$

and thus $v - w = u_2 - u_2 \Rightarrow v - w \in U$.

Addition and Scalar Multiplication on V/U

$$(v+U) + (w+U) = (v+w) + U$$
$$\lambda(v+U) = \lambda v + U$$

for $v, w \in V$, $\lambda \in \mathbb{F}$

Quotient Space is a Vector Space

Quotient Map, π

Suppose U a subspace of V. The **quotient map** π is the **linear** map

$$\pi : V \to V/U$$

defined by

$$v \to v + U$$

for $v \in V$

Dimension of a Quotient Space

Suppose V is finite dimensional and U is a subspace of V. Then

$$\dim V/U = \dim V - \dim U$$

Proof → let $\pi : V \to V/U$

$$\ker(\pi) = U, \quad \operatorname{im}(\pi) = V/U$$

$\therefore$

$$\dim V = \dim \ker(\pi) + \dim \operatorname{im}(\pi)$$
$$\dim V = \dim U + \dim V/U$$

$\tilde{T}$

Suppose $T \in \mathcal{L}(V, W)$. Define

$$\tilde{T} : V/\text{Null}(T) \to W$$

by

$$v + \text{Null}(T) \mapsto Tv$$

Null Space and Range of $\tilde{T}$

Suppose $T \in \mathcal{L}(V, W)$. Then

(a) $\tilde{T}$ is a linear map from $V/\text{Null}(T) \to W$

(b) $\tilde{T}$ is injective (i.e. $\ker \tilde{T} = \{0\}$)

(c) range $\tilde{T}$ = range T

(d) $V/\text{Null}(T)$ is isomorphic to range T

Proof $\to$ (a)

$$\begin{aligned}
&\tilde{T}(v_1 + \text{Null } T + v_2 + \text{Null } T) \\
&= \tilde{T}((v_1 + v_2) + \text{Null}(T)) \\
&= T(v_1 + v_2) \\
&= T(v_1) + T(v_2) \\
&= \tilde{T}(v_1 + \text{Null}(T)) + \tilde{T}(v_2 + \text{Null}(T))
\end{aligned}$$

$$\begin{aligned}
&\tilde{T}(\lambda(v_1 + \text{Null}(T))) \\
&= \tilde{T}(\lambda v_1 + \text{Null}(T)) \\
&= T(\lambda v_1) \\
&= \lambda T(v_1) \\
&= \lambda \tilde{T}(v_1 + \text{Null}(T))
\end{aligned}$$

(b) Suppose $v \in V$ and $\tilde{T}(v + \text{Null } T) = 0$

Then $Tv = 0 \Rightarrow v = 0 \Rightarrow \tilde{T}$ injective

(c)

range T = range $\tilde{T}$ by definition

(d) View $\tilde{T} : V/\ker(T) \to \text{range}(T)$

$\Rightarrow \tilde{T}$ an isomorphism from $V/\ker(T)$ onto range T

The Dual Space and Dual Map

Linear Functional

A linear functional on V is a linear map from V to $\mathbb{F}$.
i.e an element of $\mathcal{L}(V, \mathbb{F})$.

Dual Space, V'

The dual space of V, denoted V', is the vector space of all linear functionals on V. In other words, $V' = \mathcal{L}(V, \mathbb{F})$

$\dim V' = \dim V$

$$\dim \mathcal{L}(V, \mathbb{F}) = (\dim V)(\dim \mathbb{F})$$
$$\dim V' = (\dim V)(1)$$

Dual Basis

If $v_1, \dots, v_n$ is a basis of V, then the **dual** basis of $v_1, \dots, v_n$ is the list $\varphi_1, \dots, \varphi_n$ of elements of V', where each φ_j is the linear functional on V such that

$$\varphi_j(v_k) = \begin{cases} 1, & k = j \\ 0, & k \neq j \end{cases}$$

Note: φ_j well defined since a unique linear map exists for a basis of domain

Dual Basis is a Basis of the Dual Space

Suppose V is finite dimensional. Then the dual basis of a basis of V is a basis of V'

Proof → Suppose $v_1, \dots, v_n$ is a basis of V.

Let $\varphi_1, \dots, \varphi_n$ denote dual basis

Suppose $a_1, \dots, a_n \in \mathbb{F}$ such that

$$a_1\varphi_1 + \cdots + a_n\varphi_n = 0$$

Now $(a_1\varphi_1 + \cdots + a_n\varphi_n)(v_j) = a_j$ for $j = 1, \dots, n$.

Thus $a_1 = \cdots = a_n = 0$ and $\varphi_1, \dots, \varphi_n$ is LI.

A LI list of of V' with length $\dim V'$ is a basis of V'.

Dual Map, T'

If $T \in \mathcal{L}(V, W)$, then the **dual map** of T is the linear map $T' \in \mathcal{L}(W', V')$ defined by

$$T'(\varphi) = \varphi \circ T$$

for $\varphi \in W'$

U^0 Basis, $\varphi_{m+1}, \dots, \varphi_n$

V f.d.

$W^0 = \{\varphi \in V' \mid \varphi(w) = 0 \ \forall w \in W\}$

$W^\perp = \{v \in V \mid \langle v, w \rangle = 0 \ \forall w \in W\}$

$$V = W \oplus W^\perp$$
$$= W \oplus W^0$$

$\ker T' = (\operatorname{im} T)^0$

$\dim \operatorname{null} T' = \dim \operatorname{null} T + \dim W - \dim V$

Algebraic Properties of Dual Maps

- $(S+T)' = S' + T' \quad \forall\, S, T \in \mathcal{L}(V,W)$
- $(\lambda T)' = \lambda T' \quad \forall\, \lambda \in \mathbb{F},\ \forall\, T \in \mathcal{L}(V,W)$
- $(ST)' = T'S' \quad \forall\, T \in \mathcal{L}(U,V),\ \forall\, S \in \mathcal{L}(V,W)$

Proof $\rightarrow$ (a) $(S+T)'(\varphi) = \varphi \circ (S+T) = \varphi \circ S + \varphi \circ T = S' + T'$

(b) $(\lambda T)'(\varphi) = \varphi \circ (\lambda T) = \lambda\, \varphi \circ T = \lambda T'$

(c) Suppose $\varphi \in W'$

$$(ST)'(\varphi) = \varphi \circ (ST) = (\varphi \circ S) \circ T = T'(\varphi \circ S) = T'(S'(\varphi)) = (T'S')(\varphi)$$

Transpose, A^t

The transpose of a matrix A, denoted A^t, is the matrix obtained from A by interchanging rows and columns.

$$(A^t)_{k,j} = A_{j,k}$$

Transpose of Products of Matrices

$$(AC)^t = C^t A^t$$

The matrix of T' is the transpose of the matrix of T

Suppose $T \in \mathcal{L}(V,W)$. Then $\mathcal{M}(T') = (\mathcal{M}(T))^t$

Proof → Let $A = \mathcal{M}(T)$ and $C = \mathcal{M}(T')$

Suppose $1 \le j \le m$ and $1 \le k \le n$.

From definition of $\mathcal{M}(T)$

$$T'(\varphi_j) = \sum_{r=1}^{n} C_{r,j}\varphi_r$$

$T'(\varphi_j) = \varphi_j \circ T$ so

$$(\varphi_j \circ T)v_k = \sum_{r=1}^{n} C_{r,j}\varphi_r v_k$$

$$= C_{k,j}$$

Could also write

$$\varphi_j \circ (Tv_k) = \varphi_j\left(\sum_{r=1}^{m} A_{r,k} w_r\right)$$

$$= \sum_{r=1}^{m} A_{r,k}\varphi_j(w_r)$$

$$= A_{j,k}$$

i.e. $C_{k,j} = A_{j,k} \Rightarrow C = A^t$

Row Rank, Column Rank

Suppose A is an $\boxed{m\text{-by-}n}$ matrix with entries in $\mathbb{F}$.

- The row rank of A is the dimension of the span of the rows of A is $\mathbb{F}^{1,n}$
- The column rank of A is the dimension of the span of the columns of A is $\mathbb{F}^{m,1}$

Dimension of range T equals column rank of $\mathcal{M}(T)$.

Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V,W)$. Then dim range T = column rank of $\mathcal{M}(T)$.

Row rank equals column rank

Suppose $A \in \mathbb{F}^{m \times n}$. Then the row rank of A equals the column rank of A.

Proof → Define $T: \mathbb{F}^{n,1} \to \mathbb{F}^{m,1}$ by $Tx = Ax$.

Thus $\mathcal{M}(T) = A$, where $\mathcal{M}(T)$ is computed w/ respect to the standard bases of $\mathbb{F}^{n,1}, \mathbb{F}^{m,1}$.

Then

$$
\begin{aligned}
\text{column rank } A &= \text{column rank } \mathcal{M}(T) \\
&= \dim \text{range } T \\
&= \dim \text{range } T' \\
&= \text{column rank of } \mathcal{M}(T') \\
&= \text{column rank of } A^t \\
&= \text{row rank of } A
\end{aligned}
$$

The rank of a matrix $A \in \mathbb{F}^{m,n}$ is the column rank of A.

Polynomials

$p: \mathbb{F} \to \mathbb{F}$ is called a polynomial with coefficients in $\mathbb{F}$ if there exist $a_0, \ldots, a_m \in \mathbb{F}$ such that

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m \quad \forall z \in \mathbb{F}$$

If a polynomial is the zero function, then all coefficients are 0

Suppose $a_0, \ldots, a_m \in \mathbb{F}$. If

$$a_0 + a_1 z + \cdots + a_m z^m = 0$$

for every $z \in \mathbb{F}$, then $a_0 = \cdots = a_m = 0$

Uniqueness of Coefficients for Polynomials

This is implied from result above.

Division Algorithm for Polynomials

Suppose $p, s \in \mathcal{P}(\mathbb{F})$, with $s \neq 0$. Then there exist unique polynomials $q, r \in \mathcal{P}(\mathbb{F})$ such that

$$p = sq + r$$

and $\deg r < \deg s$.

$n \geq m$

$$\frac{p}{s} = q + \frac{r}{s}$$

$$\frac{x}{x^2} = 0 + \frac{x}{x^2}$$

Proof → Let $n = \deg p$ and $m = \deg s$.

If $n < m$, then take $q = 0$ and $r = p$ to get desired result.

Thus we can assume $n \geq m$.

Define

$$T : P_{n-m}(\mathbb{F}) \times P_{m-1}(\mathbb{F}) \to P_n(\mathbb{F})$$ ← linear map

by

$$(q, r) \longmapsto sq + r$$

If $(q, r) \in \ker(T)$, then $qs + r = 0$.

Thus $(q = 0, r = 0)$ $\dim \ker(T) = 0$ and we have proved uniqueness.

Have

$$\dim\left(P_{n-m}(\mathbb{F}) \times P_{m-1}(\mathbb{F})\right) = (n-m+1) + (m-1+1) = n+1$$

Thus

$$\dim P_n(\mathbb{F}) = \dim \operatorname{im} T = n+1$$

$\Rightarrow \operatorname{im} T = P_n(\mathbb{F})$, and hence $\exists\, q \in P_{n-m}(\mathbb{F})$ and $r \in P_{m-1}(\mathbb{F})$ s.t. $p = sq + r = T(q, r)$.

Zero of a Polynomial

$\lambda \in \mathbb{F}$ is called a **root** of $p \in \mathcal{P}(\mathbb{F})$ if $p(\lambda) = 0$

Factor

A polynomial $s \in \mathcal{P}(\mathbb{F})$ is called a **factor** of $p \in \mathcal{P}(\mathbb{F})$ if there exists a polynomial $q \in \mathcal{P}(\mathbb{F})$ such that $p = sq$.

Each Zero of a Polynomial Corresponds to a Degree-1 Factor

Suppose $p \in \mathcal{P}(\mathbb{F})$ and $\lambda \in \mathbb{F}$. Then $p(\lambda) = 0$ iff $\exists\, q \in \mathcal{P}(\mathbb{F})$ s.t.

$$p(z) = (z - \lambda)\, q(z) \qquad \forall\, z \in \mathbb{F}$$

Proof $\rightarrow$ $(\Rightarrow)$ $p(\lambda) = 0$

$\deg(z - \lambda) = 1$.

division algorithm $\Rightarrow \exists\, q \in \mathcal{P}(\mathbb{F})$ s.t.

$$p(z) = (z - \lambda)\, q(z) + r \qquad \forall\, z \in \mathbb{F}.$$

$$p(\lambda) = 0 \Rightarrow r = 0.$$

$$p(z) = (z - \lambda)\, q(z) \qquad \forall\, z \in \mathbb{F}.$$

$(\Leftarrow)$ $p(z) = (z - \lambda)\, q(z)$

$$p(\lambda) = (\lambda - \lambda)\, q(\lambda) = 0$$

A polynomial has at most as many zeros as its degree.

Suppose $p \in \mathcal{P}(\mathbb{F})$ is a polynomial with degree $m \geq 0$. Then p has at most m distinct zeros in $\mathbb{F}$.

Proof $\rightarrow$ induction on m using above

Fundamental Theorem of Algebra

Every nonconstant polynomial w/ complex coefficients has a zero.

Factorization of a polynomial over $\mathbb{C}$

If $p \in \mathcal{P}(\mathbb{C})$ is a nonconstant polynomial, then p has a unique factorization of the form

$$p(z) = c(z-\lambda_1)\cdots(z-\lambda_m)$$

where $c, \lambda_1, \ldots, \lambda_m \in \mathbb{C}$

Polynomials with real coefficients have zeros in pairs

Suppose $p \in \mathcal{P}(\mathbb{C})$ is a polynomial with real coefficients. If $\lambda \in \mathbb{C}$ is a zero of p, then so is $\bar{\lambda}$.

Factorization of a polynomial over $\mathbb{R}$

If $p \in \mathcal{P}(\mathbb{R})$ is a nonconstant polynomial, then p has a unique factorization of the form

$$p(x) = c(x-\lambda_1)\cdots(x-\lambda_m)(x^2+b_1x+c_1)\cdots(x^2+b_M x_m+c_M)$$

where $c, \lambda_1, \ldots, \lambda_m, b_1, \ldots, b_M, c_1, \ldots, c_M \in \mathbb{R}$ with $b_j^2 < 4c_j$ for each c_j.

Invariant Subspaces

Suppose $T \in \mathcal{L}(V)$. $U \subseteq V$ is called **invariant** under T if $u \in U \Rightarrow Tu \in U$.

Take any $v \in V$ with $v \neq 0$ and let U = scalar multiples of v:

$$U = \{\lambda v : \lambda \in \mathbb{F}\} = \operatorname{span}(v)$$

Then $\dim U = 1$.

If U is invariant under $T \in \mathcal{L}(V)$, then $Tv \in U$ and hence $\exists \lambda \in \mathbb{F}$ such that

$$Tv = \lambda v.$$

Conversely, if $Tv = \lambda v$ for some $\lambda \in \mathbb{F}$, then $\operatorname{span}(v)$ is a 1-dimensional subspace of V invariant under T.

Eigenvalue

Suppose $T \in \mathcal{L}(V)$. A number $\lambda \in \mathbb{F}$ is called an **eigenvalue** of T if $\exists v \in V$ s.t. $v \neq 0$ and $Tv = \lambda v$.

Equivalent Conditions to Be An Eigenvalue

Suppose V is finite dimensional, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbb{F}$. Then the following are equivalent:

(a) λ has an eigenvalue of T

(b) $T - \lambda I$ is NOT injective

(c) $T - \lambda I$ is NOT surjective

(d) $T - \lambda I$ is NOT invertible

Eigenvector

Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$ is an eigenvalue of T.
A vector $v \in V$ is called an **eigenvector** of T corresponding to λ if $v \neq 0$ and $Tv = \lambda v$.

Note: $v \in \ker(T - \lambda I)$

Linearly Independent Eigenvectors

Let $T \in \mathcal{L}(V)$. Suppose $\lambda_1, \dots, \lambda_m$ are distinct eigenvalues of T and $v_1, \dots, v_m$ are corresponding eigenvectors. Then $v_1, \dots, v_m$ are linearly independent.

Proof → Assume $v_1, \dots, v_m$ linearly dependent.

Then

$v_k \in \operatorname{span}(v_1, \dots, v_{k-1})$ for **smallest** k

thus

$$v_k = a_1 v_1 + \dots + a_{k-1} v_{k-1} \qquad (1)$$

Apply T

$$T v_k = T(a_1 v_1 + \dots + a_{k-1} v_{k-1})$$

$$\lambda_k v_k = a_1 \lambda_1 v_1 + \dots + a_{k-1} \lambda_{k-1} v_{k-1} \qquad (2)$$

Multiply (1) by λ_k

$$\lambda_k v_k = a_1 v_1 \lambda_k + \dots + a_{k-1} v_{k-1} \lambda_k$$

Subtract (2)

$$- \lambda_k v_k = a_1 \lambda_1 v_1 + \dots + a_{k-1} v_{k-1} \lambda_{k-1}$$

$$\Rightarrow 0 = a_1 v_1 (\lambda_k - \lambda_1) + \dots + a_{k-1} v_{k-1} (\lambda_k - \lambda_{k-1})$$

But $v_1, \dots, v_{k-1}$ LI. $\Rightarrow \lambda_k = \lambda_1, \dots, \lambda_k = \lambda_{k-1}$. λ_i distinct. !(

Number of Eigenvalues

Suppose V is finite-dimensional. Then each operator on V has at most $\dim V$ distinct eigenvalues.

Restriction and Quotient Operators

Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V invariant under T.

- The restriction operator $T|_U \in \mathcal{L}(U)$ is defined by

$$T|_U(u) = Tu \qquad \text{for } u \in U$$

- The quotient operator $T/U \in \mathcal{L}(V/U)$ is defined by

$$(T/U)(v+U) = Tv + U \qquad \text{for } v \in V$$

✓ Makes sense

$$v + U = w + U \Rightarrow v - w \in U$$

$$T(v-w) \in U$$

$$Tv - Tw \in U$$

$$\Rightarrow Tv + U = Tw + U$$

$p(T)$

Suppose $T \in \mathcal{L}(V)$ and $p \in \mathcal{P}(\mathbb{F})$ is a polynomial given by

$$p(z) = a_0 + a_1 z + \cdots + a_m z^m \qquad \text{for } z \in \mathbb{F}.$$

Then $p(T)$ is the operator defined by

$$p(T) = a_0 I + a_1 T + \cdots + a_m T^m$$

Product of Polynomials

If $p, q \in \mathcal{P}(\mathbb{F})$, then $pq \in \mathcal{P}(\mathbb{F})$ is the polynomial defined by

$$(pq)(z) = p(z)q(z) \qquad \text{for } z \in \mathbb{F}$$

Multiplicative Properties

Suppose $p, q \in \mathcal{P}(\mathbb{F})$, $T \in \mathcal{L}(V)$.

Then

(a) $(pq)(T) = p(T)q(T)$

(b) $p(T)q(T) = q(T)p(T)$

Operators on Complex Vector Spaces Have an Eigenvalue

Every operator on a finite-dimensional, non-zero, complex vector space has an eigenvalue.

Diagonal Matrix

A **diagonal matrix** is a square matrix that is 0 everywhere except possibly along the diagonal

Eigenspace - $E(\lambda, T)$

Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$. The eigenspace of T corresponding to λ, denoted $E(\lambda, T)$, is defined by

$$E(\lambda, T) = \ker(T - \lambda I)$$

In other words, $E(\lambda, T)$ is the set of all eigenvectors of T corresponding to λ, along w/ zero vector.

Sum of Eigenspaces is a Direct Sum

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Suppose also that $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of T. Then

$$E(\lambda_1, T) + \cdots + E(\lambda_m, T)$$

is a direct sum. Furthermore,

$$\dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T) \leq V$$

Diagonalizable

An operator $T \in \mathcal{L}(V)$ is called **diagonalizable** if the operator has a diagonal matrix with respect to some basis of V.

Conditions Equivalent to Diagonalizability

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Let $\lambda_1, \ldots, \lambda_m$ denote the distinct eigenvalues of T. Then the following are equivalent.

(a) T is diagonalizable

(b) V has a basis consisting of eigenvectors of T

(c) $\exists$ 1-dimensional subspaces $U_1, \ldots, U_n$ of V, each invariant under T, such that

$$V = U_1 \oplus \cdots \oplus U_n$$

(d) $V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$

(e) $\dim V = \dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T)$

Enough Eigenvalues Implies Diagonalizability

If $T \in \mathcal{L}(V)$ has $\dim V$ distinct eigenvalues, then T is diagonalizable.

Inner Products + Norms

Inner Product

An inner product on V is a function that takes each ordered pair (u,v) of elements of V to a number $\langle u,v\rangle \in \mathbb{F}$ and has the following properties:

positivity

$$\langle v,v\rangle \geq 0 \quad \forall\ v \in V$$

definiteness

$$\langle v,v\rangle = 0 \text{ iff } v = 0$$

additivity in the second slot

$$\langle u+v, w\rangle = \langle u+w\rangle + \langle v,w\rangle \quad \forall\ u,v,w \in V$$

homogeneity in the first slot

$$\langle \lambda u, v\rangle = \lambda\langle u,v\rangle \quad \forall\ \lambda \in \mathbb{F},\ u,v \in V$$

conjugate symmetry

$$\langle u,v\rangle = \overline{\langle v,u\rangle} \quad \forall\ u,v \in V$$

Inner Product Space

An inner product space is a vector space V along with an inner product on V.

Basic Properties of Inner Product

(a) For each fixed $u \in V$, the function that takes v to $\langle v, u \rangle$ is a linear map from V to $\mathbb{F}$

(b) $\langle 0, u \rangle = 0 \quad \forall \; u \in V$

(c) $\langle u, 0 \rangle = 0 \quad \forall \; u \in V$

(d) $\langle u, v+w \rangle = \langle u, v \rangle + \langle u, w \rangle \quad \forall \; u, v, w \in V$

(e) $\langle u, \lambda v \rangle = \bar{\lambda} \langle u, v \rangle \quad \forall \; \lambda \in \mathbb{F}$ and $u, v \in \mathbb{F}$

Norm, $\|v\|$

For every $v \in V$, the **norm** of v, denoted $\|v\|$, is defined by

$$\|v\| = \sqrt{\langle v, v \rangle}$$

Basic Properties of Norm

Suppose $v \in V$.

(a) $\|v\| = 0$ iff $v = 0$

(b) $\|\lambda v\| = |\lambda| \, \|v\| \quad \forall \; \lambda \in \mathbb{F}$

Orthogonal

Two vectors u, v are called orthogonal if $\langle u, v \rangle = 0$.

Orthogonality and 0

- 0 is orthogonal to every vector in V
- 0 is the only vector in V orthogonal to itself

Pythagorean Theorem

u, v orthogonal vectors in V. Then

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2$$

Proof →

$$\begin{aligned}\|u+v\|^2 &= \langle u+v, u+v\rangle \\ &= \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle \\ &= \|u\|^2 + \|v\|^2\end{aligned}$$

An Orthogonal Decomposition

$u, v \in V$, $v \neq 0$. Set

$$c = \frac{\langle u,v\rangle}{\|v\|^2}$$

and

$$w = u - \frac{\langle u,v\rangle}{\|v\|^2} v.$$

Then

$$\langle w,v\rangle = 0 \quad \text{and} \quad u = cv + w$$

Cauchy-Schwarz Inequality

Suppose $u, v \in V$. Then

equality iff $u = cv$, $c \in \mathbb{F}$

$$|\langle u, v\rangle| \leq \|u\| \|v\|$$

Proof → Use orthogonal decomposition

$$u = \frac{\langle u, v\rangle}{\|v\|^2} v + w \quad , \text{ where } \quad \langle w, v\rangle = 0$$

$$\|u\|^2 = \left\| \frac{\langle u, v\rangle}{\|v\|^2} v \right\|^2 + \|w\|^2 \quad \text{by Pythagorean Theorem}$$

$$= \frac{|u, v|^2}{\|v\|^2} + \|w\|^2$$

$$\geq \frac{|\langle u, v\rangle|^2}{\|v\|^2}$$

$$|u, v|^2 \leq \|u\|^2 \|v\|^2$$

$$|u, v| \leq \|u\| \|v\| \quad \text{as desired}$$

Triangle Inequality

$u, v \in V$

equality iff $u = cv$, $c > 0$

$$\|u + v\| \leq \|u\| + \|v\|$$

$$\|u + v\|^2 = \langle u + v, u + v\rangle$$

$$= \langle u, u\rangle + \langle v, v\rangle + \langle u, v\rangle + \langle v, u\rangle$$

$$\leq \|u\|^2 + \|v\|^2 + 2\|u\| \|v\|$$

$$= (\|u\| + \|v\|)^2 \quad \text{as desired}$$

Parallelogram Equality

Suppose $u, v \in V$. Then

$$\|u+v\|^2 + \|u-v\|^2 = 2(\|u\|^2 + \|v\|^2)$$

$\langle u+v, u+v\rangle + \langle u-v, u-v\rangle$

$= \langle u,u\rangle + \langle v,v\rangle + \langle v,u\rangle + \langle u,v\rangle + \langle u,u\rangle + \langle v,v\rangle + \langle u,-v\rangle + \langle -v,u\rangle$

$= 2\langle u,u\rangle + 2\langle v,v\rangle = 2\|u\|^2 + 2\|v\|^2$

Orthonormal

A list of vectors is called orthonormal if each vector in the list has norm 1 and is orthogonal to all other vectors in the list.

The norm of an orthonormal linear combination

If $e_1, \dots, e_m$ is an orthonormal list of vectors in V, then

$$\|a_1 e_1 + \cdots + a_m e_m\|^2 = |a_1|^2 + \cdots + |a_m|^2$$

$\forall\, a_1, \dots, a_m \in \mathbb{F}$

An orthonormal list is linearly independent

Proof $\rightarrow$ $e_1, \dots, e_m$ orthonormal list of vectors in V

$a_1, \dots, a_m \in \mathbb{F}$ s.t.

$$a_1 e_1 + \cdots + a_n e_m = 0$$

then $|a_1|^2 + \cdots + |a_m|^2 = 0 \Rightarrow$ all $a_j = 0$

$\therefore e_1, \dots, e_m$ LI

Orthonormal Basis

An orthonormal basis of V is an orthonormal list of vectors that is also a basis of V.

An orthonormal list of the right length is an orthonormal basis

Writing a vector as a linear combination of orthonormal basis

Suppose $e_1, \ldots, e_n$ is an orthonormal basis of V and $v \in V$. Then

$$v = \langle v, e_1 \rangle e_1 + \cdots + \langle v, e_n \rangle e_n$$

and

$$\|v\|^2 = |\langle v, e_1 \rangle|^2 + \cdots + |\langle v, e_n \rangle|^2$$

Gram-Schmidt Procedure

Suppose $v_1, \ldots, v_m$ is LI list in V. Let $e_1 = \frac{v_1}{\|v_1\|}$.

For $j = 2, \ldots, m$, define e_j inductively by

$$e_j = \frac{v_j - \langle v_{j-1}, e_1\rangle e_1 - \cdots - \langle v_j, e_{j-1}\rangle e_{j-1}}{\|v_j - \langle v_{j-1}, e_1\rangle e_1 - \cdots - \langle v_j, e_{j-1}\rangle e_{j-1}\|}$$

Then $e_1, \ldots, e_m$ is an orthonormal list of vectors in V s.t.

$$\operatorname{span}(v_1, \ldots, v_j) = \operatorname{span}(e_1, \ldots, e_j)$$

for $j = 1, \ldots, m$.

Every finite-dimensional vector space has an orthonormal basis

Orthonormal list extends to orthonormal basis