**1.** This is a variant of the last problem on the previous homework assignment. The input $X$ to a binary communication channel is a continuous random variable distributed uniformly on the interval $[-c, c]$, Here, $c > 0$ is given. The channel output is

$$Y = X + N\ ,$$

where $N$ is Gaussian with zero mean and variance 1.

(a) Find the marginal pdf $f_Y(y)$. Please provide a closed-form solution for your answer in terms of the standard normal cdf $\Phi$.
(b) Plot $f_Y(y)$ on the interval $-10 \le y \le 10$ for $c = 0.1$ and $c = 5$.

**2.** Two jointly continuous random variables $X$ and $Y$ have joint pdf

$$f_{X,Y}(x,y) = \begin{cases} 12xy(1-x) & \text{when } x \in [0,1] \text{ and } y \in [0,1] \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find $\mathbb{E}(X)$
(b) Find $\mathbb{E}(Y)$.
(c) Find $\mathrm{Var}(X)$.
(d) Find $\mathrm{Var}(Y)$.
(e) Are $X$ and $Y$ independent?

**3.** While an excellent archer, Legolas falls short as a dart thrower. When he throws a dart at a circular dart board of radius $R$, it lands at a point $(X, Y)$ distributed uniformly over the board, where the center of the board lies at $X = Y = 0$. That is, the dart's landing coordinates $X$ and $Y$ have a uniform joint pdf

$$f_{X,Y}(x,y) = \begin{cases} c & \text{when } x^2 + y^2 \le R^2 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the constant $c$.
(b) Find the marginal pdfs $f_X(x)$ and $f_Y(y)$.
(c) Let $D$ be the distance the dart lands from the center of the board. Find $\mathbb{P}(\{D \le d\})$ as a function of $d$ for $d \in (0, R]$.
(d) Find $\mathbb{E}(D)$.

**4.** Let $X_1,\ X_2,\ \dots\ ,\ X_n$ be independent random variables with respective cdfs $F_{X_k}(x)$ for $1 \le k \le n$.

(a) Find the cdf of the random variable $Y = \max\{X_1, \dots, X_n\}$ in therms of the $F_{X_k}$'s.
(b) Find the cdf of the random variable $Z = \min\{X_1, \dots, X_n\}$ in therms of the $F_{X_k}$'s.

**5.** Computers are good at generating values of a random variable $X$ distributed uniformly on the interval $[0, 1]$, and we can exploit that capability to simulate other random variables. Suppose, for example, that $G(y)$ takes values in the interval $[0, 1]$ and is strictly increasing in $y$ on some interval $a < y < b$ with $G(a) = 0$ and $G(b) = 1$ — $a = -\infty$ and/or $b = \infty$ are allowed. Define a random variable $Y$ as follows: $Y = h(X)$, where for each $x \in [0, 1]$ $h(x)$ is the unique value of $y$ such $G(y) = x$.

(a) Show that the cdf $F_Y(y)$ is $G(y)$. Explain how to use the computer to generate a sequence of numbers that looks like a sequence of independent draws of the random variable $Y$.
(b) Give the details of implementing the procedure when the random variable you want to simulate is exponential with rate parameter $\lambda$.
(c) How might you use the computer's uniform-$X$-generation capability to simulate a discrete integer-valued random variable? For starters, consider a Bernoulli random variable with parameter $p$.

**6.** Let $X_1$, $X_2$, $X_3$, and $X_4$ be independent continuous random variables with a common pdf $f(x)$ and cdf $F(x)$. Define $Y_1$, $Y_2$, $Y_3$, and $Y_4$ as

$$\begin{aligned} Y_1 &= \text{largest of } X_1, \ldots, X_4 \\ Y_2 &= \text{second largest of } X_1, \ldots, X_4 \\ Y_3 &= \text{third largest of } X_1, \ldots, X_4 \\ Y_4 &= \text{smallest of } X_1, \ldots, X_4 \,. \end{aligned}$$

(a) Find $f_{Y_k}(y)$, the marginal pdf of $Y_k$, for $1 \le k \le 4$
(b) Find $\mathbb{E}\,(Y_2)$.

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-rp534-

HWX 4/22/19

① $X \sim \text{Uniform}[-c, +c]$ , $c > 0$ given.

$N \sim \text{Gaussian}(\mu = 0, \sigma^2 = 1)$

$$Y = X + N$$

$$Y = x + N \sim \text{Gaussian}(x, 1)$$

(a) $f_Y(y)$?

For each $x \in X$ we can find $f_{Y|X}(y|x)$.

It is

$$f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-x)^2}{2}}$$

Thus by Total Probability Theorem,

$$f_Y(y) = \int_{-\infty}^{+\infty} f_{Y|X}(y|x) f_X(x) \, dx$$

$$= \int_{x=-c}^{x=+c} \frac{1}{2c} \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-x)^2}{2}} dx \qquad \begin{aligned} t &= y - x \\ dt &= -dx \end{aligned}$$

$$= -\int_{t=y+c}^{t=y-c} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt \cdot \frac{1}{2c}$$

$$= \int_{y-c}^{y+c} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt \cdot \frac{1}{2}$$

$$= \frac{\Phi(y+c) - \Phi(y-c)}{2c}$$

(b) Plots

② $$f_{X,Y}(x,y) = \begin{cases} 12xy(1-x), & (x,y) \in [0,1] \times [0,1] \\ 0, & \text{else} \end{cases}$$

(a) $\mathbb{E}(X)$?

$$\mathbb{E}(X) = \int_{-\infty}^{+\infty} x f_X(x)\, dx = \int_{x=-\infty}^{x=+\infty} x \left( \int_{y=-\infty}^{y=+\infty} f_{X,Y}(x,y)\, dy \right) dx$$

$$= \int_{y=0}^{y=1} 12xy - 12x^2y \, dy = 12x\left(\tfrac{1}{2}y^2\right) - 12x^2\left(\tfrac{1}{2}y^2\right) \Big|_{y=0}^{y=1} = 6x - 6x^2 = 6x(1-x), \ x \in [0,1]$$

$$= \int_{x=0}^{x=1} x \cdot 6x(1-x)\, dx = \int_{x=0}^{x=1} 6x^2 - 6x^3\, dx = 6\left(\tfrac{1}{3}x^3\right) - 6\left(\tfrac{1}{4}x^4\right) \Big|_{x=0}^{x=1}$$

$$\boxed{\mathbb{E}(X) = \frac{6}{3} - \frac{6}{4} = \frac{1}{2}}$$

(b) $\mathbb{E}(Y)$?

$$\mathbb{E}(Y) = \int_{y=-\infty}^{y=+\infty} y f_Y(y)\, dy = \int_{y=-\infty}^{y=+\infty} \left( \int_{x=-\infty}^{x=+\infty} f_{X,Y}(x,y)\, dx \right) dy$$

$$= \int_{x=0}^{x=1} 12xy - 12x^2y \, dx = 12y\left(\tfrac{1}{2}x^2\right) - 12y\left(\tfrac{1}{3}x^3\right) \Big|_{x=0}^{x=1} = \frac{12y}{2} - \frac{12y}{3} = 2y, \ y \in [0,1]$$

$$\mathbb{E}(Y) = \int_{y=0}^{y=1} y \cdot 2y\, dy = \frac{2}{3}y^3 \Big|_{y=0}^{y=1} = \frac{2}{3}$$

(c) $Var(X)$?

$$\mathbb{E}(X^2) = \int_{x=-\infty}^{x=+\infty} x^2 f_X(x)\,dx = \int_{x=0}^{x=1} x^2 \left(6x(1-x)\right) dx = \int_{x=0}^{x=1} 6x^3 - 6x^4 \, dx$$

$$= \frac{6}{4}x^4 - \frac{6}{5}x^5 \Big|_{x=0}^{x=1} = \frac{6}{4} - \frac{6}{5} = \frac{6}{20} = \frac{3}{10}$$

$$Var(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \frac{6}{20} - \frac{1}{4} = \frac{6}{20} - \frac{5}{20} = \boxed{\frac{1}{20}}$$

(d) $Var(Y)$?

$$\mathbb{E}(Y^2) = \int_{y=-\infty}^{y=+\infty} y^2 f_Y(y)\,dy = \int_{y=0}^{y=1} y^2 (2y)\,dy = \frac{2}{4}y^4 \Big|_{y=0}^{y=1} = \frac{1}{2}$$

$$Var(Y) = \mathbb{E}(Y^2) - (\mathbb{E}(Y))^2 = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{2} - \frac{4}{9} = \frac{9}{18} - \frac{8}{18} = \boxed{\frac{1}{18}}$$

(e) $X, Y$ independent $\iff f_{X,Y}(x,y) = f_X(x) f_Y(y)$

$$12xy(1-x) \stackrel{?}{=} 6x(1-x)\,2y \quad , \quad x \in [0,1],\ y \in [0,1]$$

$$12xy(1-x) = 12xy(1-x)$$

$\therefore$ $X, Y$ independent

(3) Throw a dart at circular dart board of radius R. It lands at a point (X,Y) distributed uniformally over the board. Center $\Rightarrow (X,Y) = (0,0)$

That is, the dart's landing coordinates X and Y have a uniform joint pdf

$$f_{X,Y}(x,y) = \begin{cases} c, & x^2+y^2 \le R^2 \\ 0, & \text{else} \end{cases}$$

$x^2+y^2 \le R^2$

$x^2 \le R^2 - y^2$ | $y^2 \le R^2 - x^2$

$x \le +\sqrt{R^2-y^2}$ | $y \le +\sqrt{R^2-x^2}$

$x \geq -\sqrt{R^2-y^2}$ | $y \geq -\sqrt{R^2-x^2}$

(a) Find constant c.

$$\int_{y=-\infty}^{+\infty}\int_{x=-\infty}^{+\infty} f_{X,Y}(x,y)\,dx\,dy = 1$$

$r \in [0, R]$

$\theta \in [0, 2\pi]$

$$= \iint_{\{(x,y):\, x^2+y^2 \le R^2\}} c\,r\,dr\,d\theta \Rightarrow c\cdot \pi R^2 = 1$$

$0 \le \theta = \tan^{-1}\left(\frac{y}{x}\right) \le 2\pi$

$0 < r = \pm\sqrt{x^2+y^2} \le R$

$$c = \frac{1}{\pi R^2}$$

(b) $f_X(x)$, $f_Y(y)$?

$$f_X(x) = \int_{y=-\infty}^{y=+\infty} f_{X,Y}(x,y)\,dy$$

$$f_Y(y) = \int_{x=-\infty}^{x=+\infty} f_{X,Y}(x,y)\,dx$$

$$= \int_{y=-\sqrt{R^2-x^2}}^{y=+\sqrt{R^2-x^2}} \frac{1}{2\pi R}\,dy = \boxed{\frac{2\sqrt{R^2-x^2}}{\pi R^2}}$$

$$= \int_{x=-\sqrt{R^2-y^2}}^{x=\sqrt{R^2-y^2}} \frac{1}{2\pi R}\,dx = \boxed{\frac{2\sqrt{R^2-y^2}}{\pi R^2}}$$

(c) Let D be the distance the dart lands from the center of the board. $\mathbb{P}(\{D \le d\})$, as a function of $d$, for $d \in (0, R]$

$$\mathbb{P}(\{D \le d\}) = \int_0^{2\pi}\int_0^{d} c\, r\, dr\, d\theta = \frac{2\pi}{\pi R^2} \cdot \frac{1}{2} d^2 = \boxed{\frac{d^2}{R^2}}$$

(d) $$\mathbb{E}(d) = \int_{-\infty}^{+\infty} d \cdot f_D(d)\, dd = \int_0^R d \cdot \frac{2d}{R^2}\, dd$$

$$= \frac{2}{R^2} \cdot \frac{1}{3} d^3 \Big|_0^R = \frac{2}{3}\frac{R^3}{R^2} \quad \boxed{\frac{2R}{3}}$$

④ $X_1, X_2, \ldots, X_n$ independent random variables with respective cumulative distribution functions $F_{X_k}(x)$ for $1 \le k \le n$.

(a) cdf of rv $Y = \max\{X_1, \ldots, X_n\}$ $\swarrow Y = g(X)$ in terms of $F_{X_k}(x)$'s, i.e $F_Y(y)$?

$X_1, \ldots, X_n$ independent $\Longleftrightarrow F_{X_1}(x) F_{X_2}(x) \cdots F_{X_n}(x) = F_{X_1, X_2, \ldots, X_n}(x)$

$$F_Y(y) = \mathbb{P}(\{Y \le y\}) = \mathbb{P}(\{\max\{X_1, \ldots, X_n\} \le y\}) = \mathbb{P}(\{X_1 \le y\} \cap \cdots \cap \{X_n \le y\})$$

$$= \mathbb{P}(\{X_1 \le y\}) \cdots \mathbb{P}(\{X_n \le y\})$$

$$\boxed{= \prod_{k=1}^{n} F_{X_k}(y)}$$

(b) cdf of rv $Z = \min\{X_1, \ldots, X_n\}$ in terms of $F_{X_k}(x)$'s.

$$\mathbb{P}(\{Z > z\}) = \mathbb{P}(\{X_1 > z\}) \cdots \mathbb{P}(\{X_n > z\})$$

$$= (1 - F_{X_1}(z)) \cdots (1 - F_{X_n}(z))$$

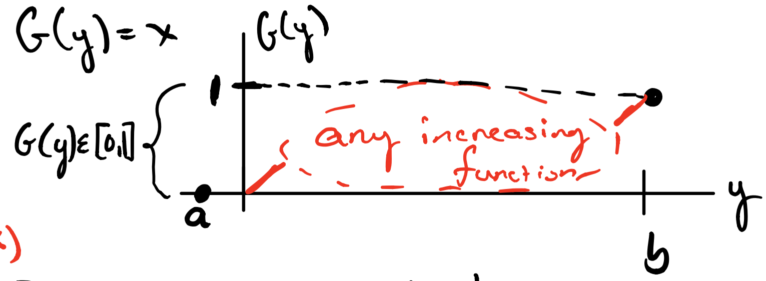
$$= \prod_{i=1}^{n} (1 - F_{X_i}(z))$$

⑤ $X \sim \text{Uniform}[0,1]$ generated by computer

$G(y) \in [0,1]$, and is strictly increasing in $y$ on some interval $a < y < b$ with $G(a) = 0$ and $G(b) = 1$ ($a = -\infty$ and/or $b = \infty$ allowed)

Define rv $Y$ as follows:

$Y = h(X)$, where for each $x \in [0,1]$ $h(x)$ is the unique value of $y$ such that $G(y) = x$




(a) Show cdf $F_Y(y)$ is $G(y)$

$Y = h(x)$. Computer takes an $x \in [0,1]$ (say $x^*$). For this $x^* \in X$, have $y^* \in Y$ such that **THIS** $y^*$ plugged into $G(y)$ (i.e $G(y^*)$) equals $x^*$.

So $\mathbb{P}(\{Y \leq y\}) = \mathbb{P}(\{h(x) \leq y\}) = \mathbb{P}(\{G(h(x)) \leq G(y)\}) = \mathbb{P}(X \leq G(y))$
$= G(y)$

So computer generates $Y$ by mapping each $X$ onto $Y$.

(b) $X \sim \text{Exponential}(\lambda)$?

Sample enough points to have enough data to simulate exponential

(c) Use computer to generate Bernoulli(p)

```
if (!Bernoulli_p())
    generate_Bernoulli_p();
```

⑥ Let $X_1, X_2, X_3, X_4$ be independent continuous random variables with a common pdf $f_X(x)$ and cdf $F(x)$.

Define $Y_1, Y_2, Y_3, Y_4$ as:

$$Y_1 = \text{largest of } X_1, X_2, X_3, X_4$$
$$Y_2 = \text{second largest of } X_1, X_2, X_3, X_4$$
$$Y_3 = \text{third largest of } X_1, X_2, X_3, X_4$$
$$Y_4 = \text{Smallest of } X_1, X_2, X_3, X_4$$

(a) $f_{Y_k}(y)$ for $1 \leq k \leq 4$ ?

$Y_1 = \max\{X_1, X_2, X_3, X_4\}$

$$F_{Y_1} = \mathbb{P}(\{Y_1 \leq y\})$$

$$\begin{aligned} F_{Y_1} &= \mathbb{P}(\{\max\{X_1, X_2, X_3, X_4\} \leq y) \\ &= \mathbb{P}(\{X_1 \leq y\} \cap \dots \cap \{X_4 \leq y\}) \\ &= \mathbb{P}(\{X_1 \leq y\}) \cdots \mathbb{P}(\{X_4 \leq y\}) \\ &= F(y) \cdot F(y) \cdot F(y) \cdot F(y) \\ &= (F(y))^4 \end{aligned}$$

Thus

$$\boxed{\begin{aligned} f_{Y_1} = \frac{d}{dy} F_{Y_1} &= \frac{d}{dy}(F(y))^4 \\ &= 4[F(y)]^3 \cdot \frac{d}{dy} F(y) \\ &= 4(F(y))^3 f(y) \end{aligned}}$$

$Y_2 = \max\left(\underbrace{\{X_1, X_2, X_3, X_4\} - \{Y_1\}}\right)$ the set which consist of $X_1, X_2, X_3, X_4$ MINUS result of $Y_1$ i.e set has 3 elements

$Y_3 = \max(\{X_1, X_2, X_3, X_4\} - \{Y_1, Y_2\})$

$Y_4 = \min(\{X_1, X_2, X_3, X_4\})$

$Y_4$ also simple.

$$f_{Y_4} = \frac{d}{dy} F_{Y_4} = \frac{d}{dy}\left(1 - (1-F(y))^4\right)$$

$$\boxed{= +4(1-F(y))^3 f(y)}$$

$Y_2, Y_3$ not so simple.

$$(b)\ \mathbb{E}(Y_2) = \int_{-\infty}^{+\infty} y_2 f_{Y_2}(y)\, dy$$