

①  $X \sim Y \sim \text{Uniform}[0, 1]$ ;  $X, Y$  independent

(a)  $\text{Cov}(X, Y) = \mathbb{E}((X - E(X))(Y - E(Y)))$

$$= \mathbb{E}\left((X - \frac{1}{2})(Y - \frac{1}{2})\right)$$
$$= \mathbb{E}\left[XY - \frac{1}{2}X - \frac{1}{2}Y + \frac{1}{4}\right]$$
$$= \mathbb{E}[XY] - \frac{1}{2}\mathbb{E}(X) - \frac{1}{2}\mathbb{E}(Y) + \frac{1}{4}$$
$$= \mathbb{E}(XY) - \frac{1}{4} - \frac{1}{4} + \frac{1}{4}$$
$$= \mathbb{E}(XY) - \frac{1}{4}$$

Note: this is  $\mathbb{E}(X)\mathbb{E}(Y)$

$$= \mathbb{E}(XY) - \frac{1}{4}$$

↓ independence

$$= \mathbb{E}(X)\mathbb{E}(Y) - \frac{1}{4}$$
$$= \frac{1}{4} - \frac{1}{4} = 0$$

Thus

$$\text{Cov}(X, Y) = 0$$

Define event  $A = \{X \leq Y\}$

(b)  $X, Y$  independent  $\Leftrightarrow f_{x,y}(x, y) = f_x(x)f_y(y)$

$$f_{x,y}(x, y) = \begin{cases} 1, & 0 < x, y < 1 \\ 0, & \text{o/w} \end{cases}$$

$$f_{X,Y|A}(x) = \begin{cases} \frac{f_{X,Y}(x,y)}{P(A)} & , 0 \leq x \leq y \leq 1 \\ 0 & , \text{o/w} \end{cases}$$

$$P(A) = \int_{y=0}^{y=1} \int_{x=0}^{x=y} f_{X,Y}(x,y) dx dy = \frac{1}{2}$$

Thus

$$f_{X|A}(x) = \int_{y=x}^1 f_{X,Y|A}(x,y) dy = \int_x^1 2 dy = 2(1-x)$$

$$(c) E[X|A] = \int_{-\infty}^{+\infty} x f_{X|A}(x) dx = \int_0^1 x [2(1-x)] dx = 1 - \frac{2}{3} = \frac{1}{3}$$

(d) Var(Z)?

$$\begin{aligned} \text{Var}(Z) &= E((XY)^2) - (E(XY))^2 \\ &= E(X^2)E(Y^2) - (E(X)E(Y))^2 \\ &= \left( \int_{-\infty}^{+\infty} x^2 f_X(x) dx \right) \left( \int_{-\infty}^{+\infty} y^2 f_Y(y) dy \right) - \left( \frac{1}{4} \right)^2 \\ &= \left( \int_0^1 x^2 dx \right) \left( \int_0^1 y^2 dy \right) - \left( \frac{1}{4} \right)^2 \\ &= \left( \frac{1}{3} \right) \left( \frac{1}{3} \right) - \frac{1}{16} = \frac{1}{9} - \frac{1}{16} = \frac{7}{144} \end{aligned}$$

(e) CDF  $F_Z(z)$  and PDF  $f_Z(z)$ ?

$$F_Z(z) = P(Z \leq z) = P(XY \leq z) = \int_0^1 P(XY \leq z | Y=y) f_Y(y) dy$$

$$= \int_0^1 P(X \leq \frac{z}{y}) dy$$

$$P\left(X \leq \frac{z}{y}\right) = \begin{cases} 0 & , 0/y \\ \frac{\frac{z}{y} - 0}{1 - 0} & , 0 < (\frac{z}{y}) < 1 \\ 1 & , (\frac{z}{y}) > 1 \end{cases}$$

So

$$F_Z(z) = \int_0^z dy + \int_z^1 \frac{z}{y} dy$$

$$= z + z \ln(1) - z \ln(z)$$

$$= z(1 - \ln(z))$$

$$f_Z(z) = \frac{d}{dz} F_Z(z) = z - \ln(z) \cdot z \quad 0 < z < 1$$

$$= 1 - \ln(z) + -\frac{z}{z} = \boxed{-\ln(z)}$$

②  $X \sim \text{Uniform}[0, 4]$

$Y \sim \text{exponential}(\lambda=2)$

$X, Y$  independent

(a) Mean and variance of  $2X - 3Y$

$$\begin{aligned}\mathbb{E}(2X - 3Y) &= 2\mathbb{E}(X) - 3\mathbb{E}(Y) \\ &= 2(2) - \frac{3}{2} = 4 - \frac{3}{2} = \boxed{\frac{5}{2}}\end{aligned}$$

$$\begin{aligned}\text{independence} \\ \text{Var}(2X - 3Y) &\stackrel{!}{=} \text{Var}(2X) + \text{Var}(3Y) \\ &= 4\text{Var}(X) + 9\text{Var}(Y) \\ &= 4 \cdot \frac{(4)^2}{12} + \frac{9}{2^2} \\ &= \frac{64}{12} + \frac{9}{4} = \frac{91}{12}\end{aligned}$$

(b)  $\text{Cov}(X, Y) = 0$  ( $X, Y$  independent)

Why?

$$\text{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$$

$$\text{independence} \rightarrow = (\mathbb{E}(X)(\mathbb{E}(Y)) - \mathbb{E}(X)\mathbb{E}(Y) = 0$$

(c)  $P(X \leq Y)$ ?

$$P(X \leq Y) = F_X(x) \rightarrow P(X \leq Y)$$

$$P(X \leq Y) = \int_y P(X \leq Y | Y=y) f_Y(y) dy$$

$$= \int_y P(X \leq y) f_Y(y) dy$$

$$P(X \leq y) = \begin{cases} 0 & ; y < 0 \\ y/4 & ; 0 \leq x < y \leq 4 \\ 1 & ; x \leq y < 4 \end{cases}$$

Thus

$$\begin{aligned}
 P(X \leq y) &= \int_0^y \frac{y}{4} \lambda e^{-\lambda y} dy + \int_y^\infty \lambda e^{-\lambda y} dy \\
 &= -\frac{\lambda}{4} \left[ \frac{y}{\lambda} e^{-\lambda y} + \frac{1}{\lambda^2} e^{-\lambda y} \right]_0^y - e^{-\lambda y} \Big|_y^\infty \\
 &= -\frac{4}{\lambda} e^{-\lambda y} - \frac{1}{\lambda^2} e^{-\lambda y} + \frac{1}{2} \cdot \frac{1}{\lambda} - (0 - e^{-\lambda y}) \\
 &= -e^{-\lambda y} + \frac{1}{\lambda} + e^{-\lambda y} - \frac{1}{\lambda} e^{-\lambda y} \\
 &= \boxed{-\frac{1}{\lambda} e^{-\lambda y}}
 \end{aligned}$$

(d)

$$f_{X,Y|A}(x,y) = \begin{cases} \frac{f_{X,Y}(x,y)}{P(A)} &= \frac{\frac{1}{4} \lambda e^{-\lambda y}}{c} = \frac{e^{-2y}}{2c}, \quad x \in [0,4] \\ &0 \quad , \quad \text{o/w} \end{cases}$$

(e) Define  $\sim W$  as follows:

- Toss a fair coin independent of  $Y$
- If H ; let  $W=Y$
- If T ; let  $W=2+Y$

$$P(\{H\} | \{W=3\}) = \frac{P(\{W=3\} | \{H\}) P(\{H\})}{P(\{W=3\})}$$

$$P(W=3) = P(W=3 | H) P(H) + P(W=3 | T) P(T) \quad \} \text{Law of Total Probability}$$

Thus

$$P(\{H\} | \{W=3\}) = \frac{P(\{W=3\} | \{H\}) P(\{H\})}{P(W=3 | H) P(H) + P(W=3 | T) P(T)}$$

$$P(H) = P(T) = \frac{1}{2}$$

$$P(W=3 | H) = P(Y=3 | H) = \frac{P(Y=3) P(H)}{P(H)} = P(Y=3)$$

$$P(W=3|T) = P(Z+Y=3|T) = \frac{P(Y=1)P(T)}{P(T)} = P(Y=1)$$

Thus

$$\begin{aligned} P(H|W=3) &= \frac{f_Y(3) \cdot \frac{1}{2}}{f_Y(3) \cdot \frac{1}{2} + f_Y(1) \cdot \frac{1}{2}} \\ &= \frac{2e^{-6} \cdot \frac{1}{2}}{2e^{-6} \cdot \frac{1}{2} + 2e^{-2} \cdot \frac{1}{2}} = \frac{e^{-6}}{e^{-6} + e^{-2}} \end{aligned}$$

③  $N \sim \text{geometric}(p=1/2)$ ; number of rounds

Each round  $i$ , win  $X_i$ .

Assume

$$\mathbb{E}(X_i) = \mu > 0 \text{ and } \text{Var}(X_i) = \sigma^2 \quad \forall i$$

All random variables are independent.

Define

$$Y = \sum_{i=1}^N X_i$$

as total amount won.

$$(a) \mathbb{E}(Y|N=n) = \sum_{i=1}^n \mathbb{E}(X_i) = \mu n \Rightarrow \mathbb{E}(Y) = \mathbb{E}(\mathbb{E}(Y|N)) = \mu N$$

$$(b) \text{Var}(Y|N=n) = \sum_{i=1}^n \text{Var}(X_i) = n\sigma^2$$

By independence

$$\text{Var}(Y|N) = N\sigma^2$$

$$\text{Var}(N) = \frac{1-p}{p^2} = \frac{\frac{1}{2}}{\frac{1}{4}} = 2$$

$$\begin{aligned}\text{Var}(Y) &= \text{Var}(\mathbb{E}(Y|N)) + \mathbb{E}(\text{Var}(Y|N)) \quad \mathbb{E}(N) = \frac{1}{p} = 2 \\ &= \text{Var}(\mu N) + \mathbb{E}(N\sigma^2) \\ &= \mu^2 \text{Var}(N) + \sigma^2 \mathbb{E}(N) \\ &= 2\mu^2 + 2\sigma^2\end{aligned}$$

$$\begin{aligned}
 (c) \text{Cov}(Y, N) &= \mathbb{E}((Y - \mathbb{E}(Y))(N - \mathbb{E}(N))) \\
 &= \mathbb{E}[YN] - \mathbb{E}[Y]\mathbb{E}[N] \\
 &= \mu\mathbb{E}[N^2] - 4\mu \\
 &= \mu[\text{Var}(N) + (\mathbb{E}(N))^2] - 4\mu \\
 &= 6\mu - 4\mu = \boxed{2\mu}
 \end{aligned}$$

$$\begin{aligned}
 \mathbb{E}[YN] &= \mathbb{E}[\mathbb{E}[YN|N]] \\
 &= \mathbb{E}[N\mathbb{E}[Y|N]] \\
 &= \mathbb{E}[N^2\mu] \\
 &= \mu\mathbb{E}[N^2]
 \end{aligned}$$

(d)

$$P(Y, N) = \frac{\text{Cov}(Y, N)}{\sigma_Y \sigma_N} = \frac{2\mu}{\sqrt{(2\mu^2 + 2\sigma^2)(2)}} = \frac{\mu}{\sqrt{\mu^2 + \sigma^2}}$$

$$\textcircled{4} \quad \begin{aligned} \Theta &\sim \text{Uniform}(0, l > 0) \\ W &\sim \text{Uniform}[0, \pi/2] \end{aligned} \quad \xrightarrow{\text{independent}}$$

$$X = \Theta \cos W$$

$$(a) \text{ LMSE } \text{IE}[\Theta | X]$$

$$\text{IE}(\Theta | X=x) = \int_{\Theta} \Theta f_{\Theta | X=x}(\Theta) d\Theta$$

$$f_{\Theta | X}(\Theta | x) = \frac{f_{X|\Theta}(x|\Theta) f_{\Theta}(\Theta)}{f_X(x)} = \frac{f_{X|\Theta}(x|\Theta) f_{\Theta}(\Theta)}{\int_{\Theta=-\infty}^{+\infty} f_{X|\Theta}(x|\Theta) f_{\Theta}(\Theta) d\Theta}$$

$$\begin{aligned} F_{X|\Theta}(x|\Theta) &= P(X \leq x | \Theta = \theta) \\ &= P(\Theta \cos W \leq x | \Theta = \theta) \\ &= P(\cos W \leq \frac{x}{\theta}) \\ &= 1 - P(W \leq \cos^{-1}(\frac{x}{\theta})) \\ &= 1 - F_W(\cos^{-1}(\frac{x}{\theta})) \\ &= 1 - \frac{2}{\pi} \cos^{-1}(\frac{x}{\theta}) \end{aligned}$$

$$\text{Thus } F_{X|\Theta}(x|\Theta) = \begin{cases} 0 & , 0 < x \leq \theta < l \\ 1 - \frac{2}{\pi} \cos^{-1}(\frac{x}{\theta}) & , 0 < x \leq \theta < l \\ 1 & , x > \theta \end{cases}$$

So

$$f_{X|\Theta}(x|\theta) = \frac{d}{dx} F_{X|\Theta}(x|\theta) = \begin{cases} 0 & , \theta/\omega \\ \frac{2}{\pi\theta} \cdot \frac{1}{\sqrt{1-\left(\frac{x}{\theta}\right)^2}} & , 0 < x < \theta \end{cases}$$

$$f_{X|\Theta}(x|\theta) = \begin{cases} 0 & , \theta/\omega \\ \frac{2}{\pi\sqrt{\theta^2-x^2}} & , 0 < x \leq \theta < 1 \end{cases}$$

$$\begin{aligned} f_x(x) &= \int_0^l \frac{2}{\pi\sqrt{\theta^2-x^2}} \cdot \frac{1}{l} d\theta \\ &= \frac{2}{\pi l} \ln\left(\frac{l+\sqrt{l^2-x^2}}{x}\right) \sqrt{\theta^2-x^2} \end{aligned}$$

So.

$$f_{\Theta|X}(\theta|x) = \begin{cases} \frac{\frac{2}{l\pi\sqrt{\theta^2-x^2}}}{\frac{2}{\pi l} \ln\left(\frac{l+\sqrt{l^2-x^2}}{x}\right)} & , 0 < x \leq \theta < l \\ 0 & , \theta/\omega \end{cases}$$

$$= \begin{cases} \frac{1}{\sqrt{\theta^2-x^2} \ln\left(\frac{l+\sqrt{l^2-x^2}}{x}\right)} & , 0 < x \leq \theta < l \\ 0 & , \theta/\omega \end{cases}$$

Thus

$$\begin{aligned} \mathbb{E}[\theta | x=x] &= \frac{1}{\ln\left(\frac{l+\sqrt{l^2-x^2}}{x}\right)} \int_0^L \frac{1}{\sqrt{\theta^2-x^2}} d\theta \\ &= \frac{1}{\ln\left(\frac{l+\sqrt{l^2-x^2}}{x}\right)} \sqrt{l^2-x^2} \end{aligned}$$

(5)  $X, Y$  jointly continuous rv.

$$f_{X,Y}(x,y) = \begin{cases} 1, & -\frac{1}{2} < y < \frac{1}{2}, \\ & y - \frac{1}{2} < x < y + \frac{1}{2} \\ 0, & \text{else} \end{cases}$$

(a) MMSE estimator of  $X$  given observation  $Y$ . i.e  $\text{IE}(X|Y)$

$$\text{IE}[X|Y=y] = \int_x x f_{X|Y}(x|y) dx$$

where

$$f_{X|Y}(x|y) = \begin{cases} \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{1}{\int_{y-\frac{1}{2}}^{y+\frac{1}{2}} f_{X,Y}(x,y) dx} = 1, & |y| < \frac{1}{2}, \\ & y - \frac{1}{2} < x < y + \frac{1}{2} \\ 0, & \text{else} \end{cases}$$

$$\text{IE}[X|Y=y] = \int_{y-\frac{1}{2}}^{y+\frac{1}{2}} x dx = \left. \frac{1}{2} x^2 \right|_{y-\frac{1}{2}}^{y+\frac{1}{2}} = y$$

Thus

$$\text{IE}[X|Y] = Y$$

$$\text{IE}[X] = \text{IE}[\text{IE}[X|Y]] = \text{IE}[Y]$$

$$(b) \quad \hat{a} = \frac{\text{Cov}(X, Y)}{\text{Var}(Y)}, \quad \hat{b} = |\mathbb{E}(x) - \hat{a}| \mathbb{E}(Y)$$

$$\text{Cov}(X, Y) = |\mathbb{E}(XY) - |\mathbb{E}(X)|\mathbb{E}(Y)|$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} xy f_{X,Y}(x,y) dx dy - \int_{-\infty}^{+\infty} xf_x(x) dx \int_{-\infty}^{+\infty} y f_y(y) dy$$

$$= \int_{y=-1/2}^{+1/2} \int_{x=y-1/2}^{x=y+1/2} xy dx dy = \int_{y=-1/2}^{+1/2} \left( \frac{1}{2}x^2 \Big|_{y-1/2}^{y+1/2} \right) y dy$$

$$= \frac{1}{3} y^3 \Big|_{-1/2}^{1/2} = \frac{1}{3} \left( \frac{1}{8} + \frac{1}{8} \right) = \frac{1}{12}$$

$$\text{Var}(Y) = \frac{b-a}{12} = 1/12$$

$$\text{Thus } \hat{a} = \frac{1/12}{1/12} = 1, \quad \hat{b} = |\mathbb{E}[x] - \hat{a}| \mathbb{E}(Y) \\ = |\mathbb{E}[x] - 1| \mathbb{E}(Y) = 0!$$

So  $\hat{x}_{\text{opt}} = Y$  as well!

(c) QMSE of  $x$  given observation  $y$

$$\text{IE} \left[ (x - (ax^2 + by + c))^2 \right]$$

Since MMSE is  $\text{IE}[x|y] = y$  is linear there does NOT exist a nonlinear estimator w/ smaller mean so

$$\text{IE} \left[ (x - (ax^2 + by + c))^2 \right] = y$$