1. Suppose Ω is a sample space, $\mathbb{P}$ is a probability law on Ω, and $A_1, A_2, \ldots, A_n$ are independent events with respect to $\mathbb{P}$. Show that

$$\mathbb{P}\left(A_1^c \cup A_2^c \cup \cdots \cup A_n^c\right) = 1 - \mathbb{P}(A_1)\mathbb{P}(A_2)\cdots\mathbb{P}(A_n)\ .$$

2. The NY Rangers and Montréal Canadiens play a best-of-seven series in the NHL Eastern Conference finals. The first team to win four games takes the series. The outcomes of the games are independent, and in a given game the Canadiens win with probability p. What is the probability that the Rangers win the series?

3. Consider a communication system with n transmitters. In each time slot, each of the transmitters sends a message with probability p, where p is the same for all transmitters and the transmitters act independently of each other. Successful transmission occurs in a given time slot when exactly one transmitter sends a message during that time slot, thus avoiding collisions between messages.

(a) Find the probability of successful transmission in a given time slot.

(b) What value of p maximizes the probability you computed in (a)? What is the maximum value of the successful-transmission probability. Discuss your result.

(c) Find the limit as $n \to \infty$ of the probability you computed in (a). Discuss your result.

(d) Suppose the system adds transmitters one by one. Every time a new transmitter joins, every transmitter, including the new one, adjusts its transmission probability so as to maximize the probability of successful transmission for the new expanded system. What is the limit as $n \to \infty$ of the probability of successful transmission in this case? You may want to use the fact that

$$\lim_{n\to\infty}\left(1+\frac{\alpha}{n}\right)^{n-1} = \lim_{n\to\infty}\left(1+\frac{\alpha}{n}\right)^{n} = e^{\alpha}$$

for every real number α.

4. Suppose the communication system in the previous problem operates for T time slots. The transmitters' actions are independent of each other in each time slot, and what happens in a given time slot is independent of what happens in all the other time slots.

(a) What is the probability that successful transmission occurs in exactly k time slots, where $1 \le k \le T$?

(b) What is the probability that successful transmission occurs in at least one time slot? (Suggestion: think complements.)

(c) Suppose $0 < \alpha < 1$. As a function of α, p, and n, what is the minimum number of time slots T for which the probability you computed in (b) is at least α? Intuition tells us that more time slots would make for a higher probability of at least one successful transmission.

(d) Find the limit as $\alpha \to 1$ of the lower bound on T that you calculated in part (c).

5. Your online banking site asks you to create a password subject to the following constraints:

- The password contains 8 to 10 characters and no spaces

- The password includes at least one of the special characters (?,!,#,$,&).
- The password may contain numbers (0-9) and lower- and upper-case English letters (a-z, A-Z) and is case-sensitive.

(a) How many passwords are legal, i.e. satisfy the constraints?

(b) If an automated hacker can try a given password every microsecond, about how many years would the hacker take to try all legal passwords?

Rami Pellumbi

ECE3100

HW3

Did NOT turn in

(1) $A_1, \dots, A_n$ independent events with respect to $\mathbb{P}$ (a probability law on a sample space Ω)

$$\mathbb{P}(A_1^c \cup \cdots \cup A_n^c) = \mathbb{P}\left([A_1 \cap \cdots \cap A_n]^c\right)$$
$$= 1 - \mathbb{P}(A_1 \cap \cdots \cap A_n)$$
$$= 1 - \mathbb{P}(A_1) \cdots \mathbb{P}(A_n)$$

(2) Best of 7 series. First to win 4 games wins. Outcomes independent. NY Rangers vs. Canadians. Canadians win with probability p.

$\mathbb{P}(\{\text{Rangers win Series}\})$?

WIN IN

4 Games Win 4 games in a row $\rightarrow (1-p)^4$

5 Games Win 3 games, lose one, win fifth $\rightarrow \binom{4}{3} p (1-p)^4$

6 Games Win 3 games, lose two, win sixth $\rightarrow \binom{5}{3} p^2 (1-p)^4$

7 Games Win 3 games, lose three, win seventh $\rightarrow \binom{6}{3} p^3 (1-p)^4$

So,

$$\mathbb{P}(\{\text{Rangers Win Series}\}) = \sum_{m=3}^{6} \binom{m}{3} p^{m-3} (1-p)^4$$

(3) n transmitters, act independently of eachother

Each transmitter sends message with probability p.

Successful transmission occurs in a given time slot when exactly one transmitter sends a message during that time slot, thus avoiding collisions b/t messages

(a) $\mathbb{P}(\{\text{successful transmission in a given time slot}\})$

So, one transmitter sends a message and other $n-1$ don't.

So, $\binom{n}{1} p(1-p)^{n-1}$.

(b) What p maximizes value in (a)?

$$\frac{d}{dp}\left[np(1-p)^{n-1}\right] = -np(n-1)(1-p)^{n-2} + (1-p)^{n-1}n = 0$$

$$\Rightarrow 1-p-(n-1)p = 0$$

$$p = 1/n$$

So, the successful transmission probability with this p is

$$\frac{n}{n}\left(1-\frac{1}{n}\right)^{n-1}$$

(c)

$$\lim_{n\to\infty} np(1-p)^{n-1} \to \infty$$

(d) $$\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{n-1} = e^{-1}$$

④ Transmitter operates for T time slots. Lots of independence.

(a) $\mathbb{P}(\{\text{successful transmission occurs in exactly } k \text{ time slots}\})$

$0 \le k \le T$

$$\underbrace{\binom{T}{k}}_{\text{ways to get } k \text{ from } T \text{ total}} \underbrace{\left[np(1-p)^{n-1}\right]^k}_{\text{successful transmission}} \underbrace{\left[1 - \left(np(1-p)^{n-1}\right)\right]^{T-k}}_{\text{unsuccessful transmission}}$$

(b) Successful in at least one time slot is the complement of

NOT successful in all slots

$$\left[1 - \left(np(1-p)^{n-1}\right)\right]^T = 1 - \left[1 - \left(np(1-p)^{n-1}\right)\right]^T$$

(c) $0 < \alpha < 1$. $f(\alpha, p, n)$ - what is minimum number of time slots T for which the probability in (b) is at least α.

$$1 - \left[1 - \left(np(1-p)^{n-1}\right)\right]^T \geq \alpha$$

$$\left[1 - \left(np(1-p)^{n-1}\right)\right]^T < 1 - \alpha$$

$$T \ln\left[1 - \left(np(1-p)^{n-1}\right)\right] < \ln(1-\alpha)$$

$$T \geq \frac{\ln(1-\alpha)}{\ln\left(1-np(1-p)^{n-1}\right)}$$

Minimum integer-T required is thus

$$T = \left\lceil \frac{\ln(1-\alpha)}{\ln\left(1-np(1-p)^{n-1}\right)} \right\rceil$$

(d) $\lim_{\alpha \to 1} \frac{\ln(1-\alpha)}{\ln\left(1-np(1-p)^{n-1}\right)} \to \infty$

5 (a) $\sum_{n=8}^{16} \left(67^n - 62^n\right) \simeq 9.87 \times 10^{17}$ total passwords

(b) 10^6 checks / second. So 9.87×10^{11} seconds needed (roughly)

$\simeq 31{,}600$ years