

10) Simplify

$$\left[\frac{2+i}{6i - (1-2i)} \right]^2$$

$$= \left[\frac{2+i}{8i-1} \right]^2 = \left[\frac{2+i}{8i-1} \cdot \frac{-8i-1}{-8i-1} \right] = \left[\frac{6-17i}{65} \right]^2$$

$$= \frac{(6-17i)^2}{65^2} = \frac{-253}{4225} - i \frac{204}{4225}$$

$$21) \quad (1-i)z_1 + 3z_2 = 2-3i$$

$$iz_1 + (1+2i)z_2 = 1$$

$$z_1 = \frac{1 - (1+2i)z_2}{i} = \frac{1 - z_2 - 2iz_2}{i}$$

$$(1-i) \left[\frac{1 - (1+2i)z_2}{i} \right] + 3z_2 = 2-3i$$

$$(1-i) - (1-i)(1+2i)z_2 + 3iz_2 = 3+2i$$

$$(1-i) - [1 + \cancel{2i} \cancel{i} + 2]z_2 + 3iz_2 = 3+2i$$

$$1-i - [3+i]z_2 + 3iz_2 = 3+2i$$

$$-3z_2 - iz_2 + 3iz_2 = 2+3i$$

$$2iz_2 - 3z_2 = 2 + 3i$$

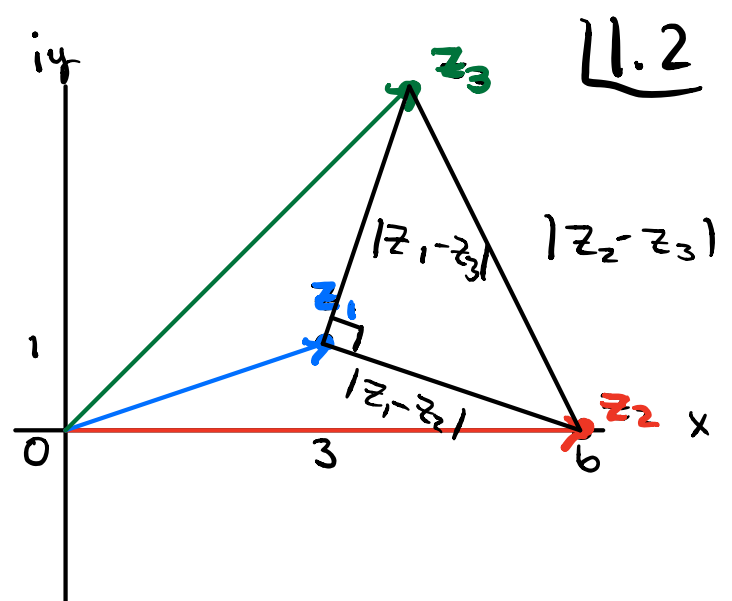
$$z_2[2i - 3] = 2 + 3i$$

$$z_2 = \frac{2+3i}{2i-3} \cdot \frac{-2i-3}{-2i-3} \rightarrow \boxed{z_2 = -i}$$

$$iz_1 + (1+2i)(-i) = 1$$

$$z_1 = \frac{-1-i}{i} = \boxed{1+i}$$

6) $z_1 = 3+i$
 $z_2 = 6$
 $z_3 = 4+4i$



$$|z_1 - z_2| = \sqrt{3^2 + 1^2} = \sqrt{10}$$

evidently a right triangle

$$|z_1 - z_3| = \sqrt{3^2 + 1^2} = \sqrt{10}$$

$$|z_2 - z_3| = \sqrt{2^2 + 4^2} = \sqrt{20}$$

Edges satisfy Pythagorean Theorem!

$$|z_1 - z_2|^2 + |z_1 - z_3|^2 = |z_2 - z_3|^2$$

7) Describe the set of points

$$|z - 1| + |z + 1| = 7$$

The equality here describes a set of complex points z whose distance from $(-1, 0)$ and $(1, 0)$ is constant!

This is the definition of foci on an ellipse.

Therefore the set of points described is an ellipse in the complex plane with its foci at $(-1, 0)$, $(1, 0)$.

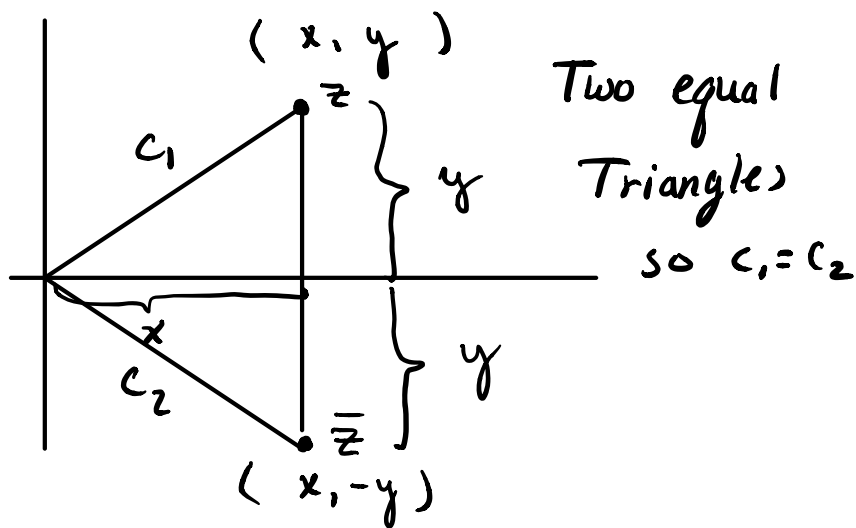
8) Show

$$|z-1| = |\bar{z}-1|$$

Algebraic

$$\sqrt{(x-1)^2 + 1^2} = \sqrt{(x-1)^2 + (-1)^2} \quad \checkmark$$

Geometrically



16) Prove that if

$$|z|=1, (z \neq 1)$$

then

$$\operatorname{Re}\left(\frac{1}{1-z}\right) = \frac{1}{2}$$

$$z = x+iy \neq 1$$

$$|z| = \sqrt{x^2+y^2} = 1 \Rightarrow y^2 = 1-x^2$$

$$\begin{aligned} \operatorname{Re}\left(\frac{1}{1-z}\right) &= \operatorname{Re}\left(\frac{1}{1-(x+iy)}\right) = \operatorname{Re}\left(\frac{(1-x)+iy}{(1-x)^2+y^2}\right) \\ &= \frac{(1-x)}{2(1-x)} = \frac{1}{2} \end{aligned}$$

7h) Find the argument of $\frac{-\sqrt{7}(1+i)}{\sqrt{3}+i}$

Let

$$z = \frac{-\sqrt{7}(1+i)}{\sqrt{3}+i} \cdot \frac{\sqrt{3}-i}{\sqrt{3}-i} = -\frac{(\sqrt{21}+\sqrt{7})}{4} + i \frac{(\sqrt{7}-\sqrt{21})}{4}$$

$$\arg(z) = \arctan\left(\frac{(\sqrt{7}-\sqrt{21})}{-(\sqrt{21}+\sqrt{7})}\right) = \frac{13\pi}{12} + 2\pi k, \quad k \in \mathbb{Z}$$

$$|z| = \sqrt{\left(-\frac{(\sqrt{21}+\sqrt{7})}{4}\right)^2 + \left(\frac{(\sqrt{7}-\sqrt{21})}{4}\right)^2} = \frac{\sqrt{14}}{2}$$

Our answer in polar form is thus

$$\frac{\sqrt{14}}{2} e^{i \frac{13\pi}{12}}$$

ii) Use $(1+i)(5-i)^4$ to derive

$$\frac{\pi}{4} = 4 \arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right)$$

$$\arg(z^n) = \arg(z) + \arg(z) + \dots + \arg(z) = n \cdot \arg(z) \quad n \in \mathbb{Z}$$

$$(1+i)(5-i)^4 = 956-4i$$

$$\arg[(1+i)(5-i)^4] = \arg(956-4i)$$

$$\Rightarrow \arg(1+i) + 4\arg(5-i) = \arg(956-4i)$$

$$\frac{\pi}{4} + 4\arctan\left(-\frac{1}{5}\right) = \arctan\left(-\frac{1}{239}\right)$$

$$\arctan(-u) = -\arctan(u)$$

$$\frac{\pi}{4} = 4\arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right)$$

5a) Find $(-16)^{1/4}$

$$z = -16$$

$$|z| = 16$$

$$\arg(z) = \pi + 2\pi k, k \in \mathbb{Z}$$

$$z = 16 e^{i\pi + 2\pi k}$$

$$z^{1/4} = (16)^{1/4} e^{i\frac{\pi}{4} + \frac{\pi}{2}k}, k = 0, 1, 2, 3$$

11) Solve $(z+1)^5 = z^5$

$$\frac{(z+1)^5}{z^5} = 1$$

$$\frac{z+1}{z} = 1^{1/5} = e^{i\left(\frac{2\pi k}{5}\right)}, k \neq 5n, n \in \mathbb{Z}$$

$$z(e^{i\frac{2\pi k}{5}} - 1) = 1$$

$$z = \frac{1}{e^{i\frac{2\pi k}{5}} - 1}, k \neq 5n, n \in \mathbb{Z}$$