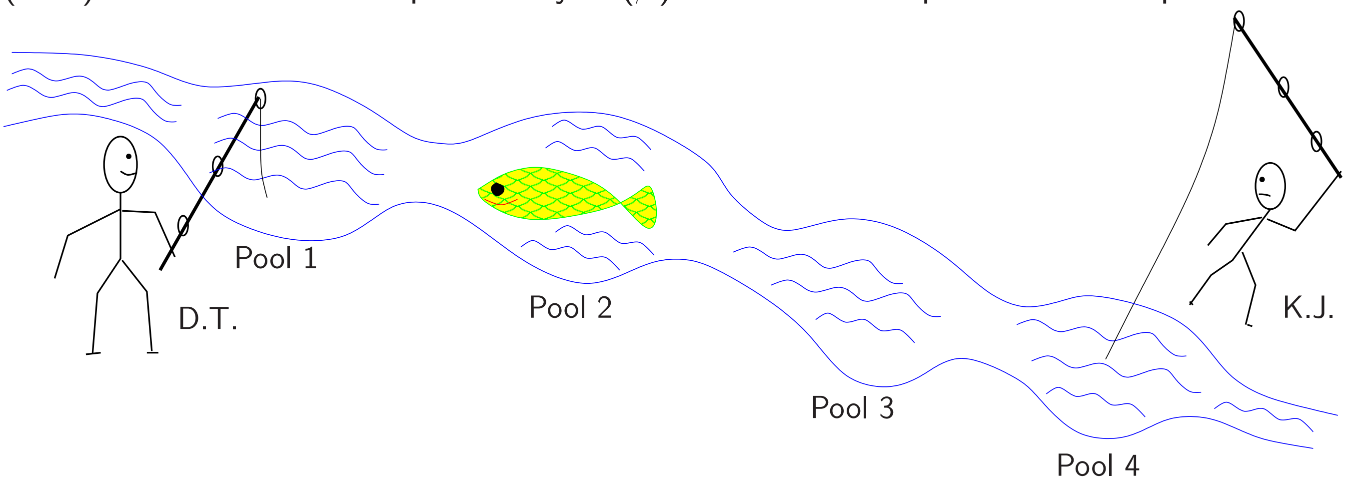
Absorption Probability and Absorption Time

Example: Wanda, Donald Trump, and Kim Jong-un:

D.T./K.J: "Mirror mirror on the wall, who has the fairest hair of them all?"
Mirror: "Whoever catches Wanda, wins the title."

So you can be sure that D.T. and K.J. will be fishing until one of them catches Wanda. D.T., rocking the finest toupee, fishes in pool 1, while K.J., rocking a butcher block brush, fishes in pool 4. Each time Wanda enters pool 1 (pool 4), D.T. (K.J.) will catch her with probability α (β) before she escapes to another pool.



1. Who is more likely to win the title of The Fairest Hair, assuming that Wanda has been swimming in these pools ever since the invention of hair gel and has reached her steady state?
2. What is the expected time until Wanda gets caught?

Absorption Probability:

☐ Modify the original Markov chain by adding two absorbing states 0 and 5 corresponding to, respectively, that Wanda gets caught by D.T. and Wanda gets caught by K.J..

☐ Define the conditional absorption probability

$$g_i \triangleq \Pr\left[\text{Obscorbed at state } 0 \,\middle|\, \text{currently at state } i\right]$$

Based on the Markovian property, we have

$$\begin{aligned} g_0 &= 1 \\ g_1 &= \alpha g_0 + (1-\alpha) g_2 \\ g_2 &= (1-p) g_1 + p g_3 \\ g_3 &= (1-p) g_2 + p g_4 \\ g_4 &= (1-\beta) g_3 + \beta g_5 \\ g_5 &= 0 \end{aligned}$$

☐ The probability that D.T. catches Wanda is $\sum_{i=1}^{4} \pi_i g_i$, and the probability that K.J. catches Wanda is $1 - \sum_{i=1}^{4} \pi_i g_i$.

Absorption Time:

☐ Define the conditional expected absorption time

$$e_i \triangleq \mathbb{E}\left[\text{Remaining time until absorption} \,\middle|\, \text{currently at state } i\right]$$

Based on the Markovian property, we have

$$\begin{aligned} e_1 &= \alpha \times 1 + (1-\alpha)(1+e_2) \\ e_2 &= (1-p)(1+e_1) + p(1+e_3) \\ e_3 &= (1-p)(1+e_2) + p(1+e_4) \\ e_4 &= \beta \times 1 + (1-\beta)(1+e_3) \end{aligned}$$

Solving these linear equations gives e_i $(i = 1, 2, 3, 4)$.

☐ The expected absorption time is given by $\sum_{i=1}^{4} \pi_i e_i$.

Absorption Time

Example: Coin A and Coin B:

What is the expected waiting time until you see three consecutive heads for the first time?

- ☐ Construct a 4-state Markov chain whose state at time n is (X_{n-1}, X_n), the outcomes of the last two consecutive coin flips.
- ☐ Modify the above 4-state Markov chain by adding an absorbing state of $(1,1,1)$ (three consecutive heads).
- ☐ Define the conditional expected absorption time

$$e_i \triangleq \mathbb{E}\Big[\text{Remaining time until absorption} \,\Big|\, \text{currently at state } i\Big]$$

Based on the Markovian property, we have

$$\begin{aligned}
e_{(1,1)} &= \frac{1}{4} \times 1 + \frac{3}{4}(1 + e_{(1,2)}) \\
e_{(1,2)} &= \frac{1}{2}(1 + e_{(2,1)}) + \frac{1}{2}(1 + e_{(2,2)}) \\
e_{(2,1)} &= \frac{1}{4}(1 + e_{(1,1)}) + \frac{3}{4}(1 + e_{(1,2)}) \\
e_{(2,2)} &= \frac{1}{2}(1 + e_{(2,1)}) + \frac{1}{2}(1 + e_{(2,2)})
\end{aligned}$$

which leads to

$$e_{(1,1)} = 40,\ e_{(1,2)} = 52,\ e_{(2,1)} = 50,\ e_{(2,2)} = 52.$$

- ☐ Average over the distribution of the initial state of

$$p_{(1,1)} = \frac{1}{8},\ p_{(1,2)} = \frac{3}{8},\ p_{(2,1)} = \frac{1}{4},\ p_{(2,2)} = \frac{1}{4}$$

The answer is

$$2 + p_{(1,1)}e_{(1,1)} + p_{(1,2)}e_{(1,2)} + p_{(2,1)}e_{(2,1)} + p_{(2,2)}e_{(2,2)} = 52$$

Limiting Distribution

Limiting Distribution

π is a limiting distribution of a Markov chain with transition probabilities $\{p_{i,j}\}$ if

$$\begin{cases} \lim_{n\to\infty} p_{i,j}^{(n)} = \pi_j, & \forall i, j \in \mathcal{X} \\ \pi_j \geq 0, & \sum_{j\in\mathcal{X}} \pi_j = 1 \end{cases}$$

Remark: Since $\lim_{n\to\infty} \mathbf{p}(n) = \lim_{n\to\infty} \mathbf{p}(0)\mathbf{P}^n = \pi$ for all initial condition $\mathbf{p}(0)$, a Markov chain with a limiting distribution eventually forgets its initial conditions in the sense that $\mathbf{p}(n) \to \pi$ as $n \to \infty$ regardless of $\mathbf{p}(0)$.

Limiting Distribution and Stationary Distribution

- □ A limiting distribution is a stationary distribution.
- □ If a limiting distribution exists, it is the unique stationary distribution.

Existence of Limiting Distribution:

An irreducible, aperiodic, and positive recurrent Markov chain has a limiting distribution, given by its unique stationary distribution.