Last Time

Defined eigenvalues + eigenvectors

- if $\vec{v}_1, \ldots, \vec{v}_m$ are eigenvectors, w/ distinct eigenvalues $\lambda_1, \ldots, \lambda_m$ then $(\vec{v}_1, \ldots, \vec{v}_m)$ is LI.

Corollary: If $\dim V = n < \infty$, then $T \in \mathcal{L}(V)$ has at MOST n **distinct** eigenvalues

Today

Does $\exists$ an eigenvalue?

Is T diagonalizable

Next Goal: $T \in \mathcal{L}(V)$, $\dim V = n < \infty$, vector space over $\mathbb{C}$.

Show: $\exists$ eigenvalue of T

Idea: Apply polynomials to T

Given $p(x) \in \mathbb{F}[x]$

1. Already have defined $p(T)$ (also $p(A)$)
2. if $p(x) = f(x)g(x)$ then $p(T) = f(T)g(T)$

③ Suppose $T(\vec{v}) = \lambda\vec{v}$, $p(x) \in \mathbb{F}[x]$.

What is $p(T)(\vec{v})$?

if $p(x) = a_0 + a_1x + \cdots + a_mx^m$, $a_i \in \mathbb{F}\ \forall i$

$$p(T) = a_0I + a_1T + \cdots + a_mT^m$$
$$= a_0\vec{v} + a_1\lambda\vec{v} + \cdots + a_m\lambda^m\vec{v}$$
$$= \vec{v}(a_0 + a_1\lambda + a_2\lambda^2 + \cdots + a_m\lambda^m)$$
$$= \vec{v}\,p(\lambda)$$

④

If $B = Q^{-1}AQ$ are similar and $p(x) \in \mathbb{F}[x]$

we have $p(A)$, $p(B)$ $n \times n$ matrices, how are they related?

$$B = Q^{-1}AQ$$
$$B^2 = Q^{-1}AQQ^{-1}AQ = Q^{-1}A^2Q$$
$$\vdots$$
$$B^n = Q^{-1}A^nQ$$

So

$$p(B) = a_0I + a_1B + \cdots + a_mB^m$$
$$= Q^{-1}(a_0I + a_1A + \cdots + a_mA^m)Q$$
$$= Q^{-1}p(A)Q$$

Aside on direct sums

Suppose $V_1, V_2, \ldots, V_r \subseteq V$ are subspaces

suppose $W = V_1 + V_2 + \cdots + V_r$

we say

$$W = V_1 + \cdots + V_r$$

is a direct sum written

$$W = V_1 \oplus \cdots \oplus V_r$$

if

① $W = V_1 + \cdots + V_r$

② Whenever $\vec{u}_1, \ldots, \vec{u}_r$, $\vec{u}_i \in V$ and

$$\vec{u}_1 + \cdots + \vec{u}_r = 0$$

then $\vec{u}_1 = \cdots \vec{u}_r = 0$

Example: if V has a basis $(\vec{v}_1, \ldots, \vec{v}_n)$ and if we let $V_i = \text{span}(\vec{v}_i)$, then $V = V_1 \oplus V_2 \oplus \cdots \oplus V_n$

also: if $(v_1, \ldots, v_n)$ spans V

then $(v_1, \ldots, v_n)$ is LI $\Leftrightarrow V = V_1 \oplus \cdots \oplus V_n$

Remark: $\vec{v} \neq 0$. Then $\vec{v}$ is an eigenvector of $T \in \mathcal{L}(V)$ w/ eigenvalue λ

$\Leftrightarrow T\vec{v} = \lambda \vec{v}$

$\Leftrightarrow T\vec{v} = \lambda I_V \vec{v}$

$\Leftrightarrow (T - \lambda I)\vec{v} = 0$

$\Leftrightarrow \vec{v} \in \ker(T - \lambda I)$

So λ is an eigenvalue $\iff \ker(T-\lambda I) \neq 0$

Bonus

$\iff T-\lambda I$ invertible, NOT

If $\dim V = n < \infty$

Definition: Given a vector space V, $T \in \mathcal{L}(V)$ or $A_{n \times n}$,

let $E_\lambda(T) = E_\lambda \triangleq \ker(T-\lambda I) \subseteq V$

be the **λ-eigenspace** of T.

Also

$$E_\lambda(A) = \ker(A-\lambda I) \subseteq \mathbb{F}^n$$

Theorem: Let V be a finite dimensional vector space over $\mathbb{C}$ and let $T \in \mathcal{L}(V)$.

Then $\exists \ \lambda \in \mathbb{C}$ s.t. λ = eigenvalue of T.

Proof: Suppose $\dim V = n > 0$

Choose a vector $\vec{v} \in V$, $\vec{v} \neq 0$.

Consider

$$\vec{v}, T(\vec{v}), T^2(\vec{v}), \ldots, T^n(\vec{v}) \leftarrow n+1 \text{ vecs}$$

$\therefore \ \exists$ LD on above list.

i.e $\exists \ a_0, \ldots, a_n \in \mathbb{C}$ s.t.

$$\vec{0} = a_0\vec{v} + a_1\vec{v} + \cdots + a_n T^n(\vec{v}) = p(T)\vec{v}$$

if $p(x) = a_0 + a_1 x + \cdots + a_m x^m$

suppose $\deg p(x) = m \leq n$

Now use fundamental theorem of algebra

$p(x) = c(x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_m)$

$p(T) = c(T-\lambda_1 I)(T-\lambda_2 I)\cdots(T-\lambda_m I)$

Know $\vec{v} \in \ker p(T)$

$$c(T-\lambda_1 I)\cdots(T-\lambda_m I) = 0$$

So if $\ker(T-\lambda_i I) \neq 0$ for some i

then we have λ_i = eigenvalue.

Other: $T-\lambda_i I$ is invertible $\forall i = 1, \ldots, m$

$\therefore \prod_i (T-\lambda_i I)$ invertible.

But $\vec{v} \in \ker T$.

Contradiction!

$\therefore$ Some $T-\lambda_i I$ is NOT invertible

- that is λ_i is an eigenvalue