Last Time

$\text{span}(v_1, \ldots, v_m) = \{ a_1 v_1 + a_2 v_2 + \cdots + a_m v_m \subseteq V \mid a_1, \ldots, a_m \in \mathbb{F} \}$

$\text{Span}(\{0_V\}) = 0$

Proposition: Let $v_1, \ldots, v_m \in V$, $m \geq 0$. Then the

$$\overbrace{\text{span}(v_1, \ldots, v_m)}^{\text{LHS}}$$

is $\overbrace{\text{the}}^{\text{RHS}}$ smallest subspace of V containing $v_1, \ldots, v_m$

PROOF

- If $m = 0$, LHS $= 0$

  RHS = smallest subspace of $V = 0$

- If $m > 0$, want to show $S = \text{span}(v_1, \ldots, v_m)$ is a subspace
  - $0 \in S$? $0 = 0v_1 + 0v_2 + \cdots + 0v_m = 0$ ✓
  - closed under addition?

    Let

    both elements of S

    $v = a_1 v_1 + \cdots + a_m v_m \quad w = b_1 v_1 + \cdots + b_m v_m$

    then

    $v + w = (a_1 + b_1) v_1 + \cdots + (a_m + b_m) v_m \in S$
  - closed under multiplication?

    $cv = (ca_1) v_1 + \cdots + (ca_m) v_m \in S$ ✓

- Also need to show it is the <u>smallest</u> subspace

If U is a subspace of V containing $v_1, \dots, v_m$ then show $W \subseteq U$.

Given $w = a_1 v_1 + \cdots + a_m v_m \in U$

Since U contains $v_1, \dots, v_m$ then any linear combination is also in U.

Definition Central

Dependence

Def: ⓐ The list $(v_1, v_2, \dots, v_m)$ of vectors of V is called linearly dependent (LD) if $\exists\, a_1, \dots, a_m \in \mathbb{F}$ NOT ALL ZERO such that $a_1 v_1 + \cdots + a_m v_m = 0$

Def: ⓑ The list $(v_1, v_2, \dots, v_m)$ of vectors of V is called linearly independent (LI) if it is NOT linearly dependent.

i.e Whenever $a_1 v_1 + \cdots + a_m v_m = 0$

then $a_1 = a_2 = \cdots = a_m = 0$

Def: ⓒ $(\ )$ is linearly independent

Def: ⓐ A vector span V is called finite dimensional (i.e finitely operated) if $\exists v_1, \dots, v_m$ $(m \geq 0)$ such that

$$V = \text{Span}(v_1, \dots, v_m)$$

Def: ⓑ $(v_1, \dots, v_m)$ is called a basis of V if

(i) $V = \text{span}(v_1, \dots, v_m)$

(ii) $(v_1, \dots, v_m)$ is linearly INDEPENDENT

Examples

(Zero vector in

① (v_1) is LI $\Leftrightarrow v_1 \neq 0$

② (v_1, v_2) is LD $\Leftrightarrow$ $v_1 = 0$, or $v_2 = cv_1$ for some $c \in \mathbb{F}$

③ $\left(\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}\right) \in \mathbb{R}^3$?

Linearly Dependent!

$$-2\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

④ $(1, x, x^2, x^3) \in \mathbb{F}[x]$?

<u>LI</u> Since $a_0 + a_1 x + a_1 x^2 + a_3 x^3 = 0$ ← 0 polynomial

(5) In $Fun(\mathbb{R}, \mathbb{R})$, is $(\sin x, \sin(2x))$ LD?

i.e are there $a, b \in \mathbb{R}$ $a, b \neq 0$ s.t. $\quad \sin 2x = 2 \sin x \cos x$

$$a \sin x + b \sin 2x = 0$$ ← Zero function

Assume $\exists$ a, b, one NOT zero, s.t.

$$a \sin x + b \sin 2x = 0$$

Try $x = \frac{\pi}{2}$ (since it must hold $\forall x$) then

$$a \cdot 1 + b \cdot 0 = 0 \; ; \; a = 0$$

Try $x = \frac{\pi}{4}$, then

$$b \cdot 1 = 0 \Rightarrow b = 0 \quad \therefore \text{LI}$$

Lemma (LI Lemma)

Suppose $(v_1, \ldots, v_m)$ is LD in V, then $\exists$ $j \in 1, \ldots, m$ such that

(a) $v_j \in \text{span}(v_1, \ldots, v_{j-1})$

(b) if the j^{th} element v_j is removed from $(v_1, \ldots, v_m)$ then span of resulting list is $= \text{span}(v_1, \ldots, v_m)$

i.e v_j isn't needed to get the span

PROOF

$(v_1, \ldots, v_m)$ is LD.

$\exists a_1, \ldots, a_m$, not all zero, such that $a_1 v_1 + \cdots + a_m v_m = 0$.

Let j be the largest index in $1, \ldots, m$ s.t. $a_j \neq 0$.

Get $a_1 v_1 + \cdots + a_j v_j = 0$, then

$$v_j = \frac{a_1 v_1}{a_j} + \frac{a_2 v_2}{a_j} + \cdots + \frac{a_{j-1} v_{j-1}}{a_j}$$

Proves (a), (b) follows.