

Recap

if $a \in \mathbb{F}$, $0 \cdot a = 0$?

$$0 \cdot a = 0 \quad \xleftarrow{\text{multiply by } 1} = (1+0) \cdot a$$

$$a = a + 0 \cdot a$$

$$0 = 0 \cdot a$$

Vector Spaces

① \mathbb{F}^n : recall \mathbb{R}^n = "vector space over \mathbb{R} "

- has 0 element, written as 0 or $\vec{0}$ or $0_{\mathbb{R}^n}$
- $(+)$
- scalar multiplication

Definition: Suppose n is a non-negative integer and \mathbb{F} is any set.

An n -tuple (or list of length n) of \mathbb{F} is an ordered collection of n elements of \mathbb{F} .

Notation: $x = (x_1, \dots, x_n)$ $x_i \in \mathbb{F}$

$$(a_1, a_2, \dots, a_n) = (b_1, b_2, \dots, b_m) \iff \begin{matrix} m=n \\ a_i = b_i \forall i \text{ in } [1, m] \end{matrix}$$

Definition: $\mathbb{F}^n = \{(a_1, a_2, \dots, a_n) \mid a_i \in \mathbb{F}\}$ = set of n -tuples of \mathbb{F}

If $x = (x_1, x_2, \dots, x_n)$ then x_i is called the " i th entry"

Definition: Given a field \mathbb{F} , $x = (x_1, \dots, x_n) \in \mathbb{F}^n$, $y = (y_1, \dots, y_n) \in \mathbb{F}^n$, $a \in \mathbb{F}$

(a) the zero element $0 = \vec{0} = 0_{\mathbb{F}^n} = (0, 0, \dots, 0)$

(b) $x + y = (x_1 + y_1, \dots, x_n + y_n)$

(c) $ax = (ax_1, \dots, ax_n)$

\mathbb{F}^n w/ these operations is
our friendly vector space

General Vector Spaces Over a Field \mathbb{F}

Definition: a set V equipped with

a) an element $0 = 0_V \in V$

b) an addition operation $\alpha: V \times V \rightarrow V$

$$(x, y) \mapsto \alpha(x, y) = x + y$$

c) a scalar multiplication operation $\mu: \mathbb{F} \times V \rightarrow V$

$$(a, x) \mapsto \mu(a, x) = a \cdot x = ax$$

is called a vector space over \mathbb{F} if the following 8

properties hold:

(VS1) Commutativity of $+$: $\forall x, y \in V \quad x + y = y + x$

(VS2) Associativity of $+$: $\forall x, y, z \in V \quad x + (y + z) = (x + y) + z$

(VS3) Additive Identity: $\forall x \in V \quad x + 0_V = x$

(VS4) Additive Inverse: $\forall x \in V \exists y \in V$ such that $x + y = 0_V$

(VS5) Multiplication Identity: $\forall x \in V \quad 1 \cdot x = x$ Note: $1 \in \mathbb{F}$

(VS6) Associativity of \cdot : $\forall a, b \in \mathbb{F}, x \in V \quad a(bx) = (ab)x$

(VS7) Distributivity #1: $\forall a \in \mathbb{F}, x, y \in V \quad a \cdot (x + y) = a \cdot x + a \cdot y$

(VS8) Distributivity #2: $\forall a, b \in \mathbb{F}, x \in V \quad (a + b) \cdot x = a \cdot x + b \cdot x$

Examples of Vector Spaces Over \mathbb{F}

① \mathbb{F}^n , $0_{\mathbb{F}^n} = (0, 0, \dots, 0)$, '+' , '.' as above

Some notes: a) $n=1$, $\mathbb{F}^1 = \mathbb{F}$

b) often write $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{F}^n$ ↙ "column vector"

or $(x_1, \dots, x_n) \in \mathbb{F}^n$ ↘ "row vector"

$$\mathbb{F} = \mathbb{Z}_7 = \{0, 1, 2, 3, 4, 5, 6\}$$

$$\mathbb{F}^3 \quad 2 \cdot \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 6 \\ 3 \end{pmatrix} + \begin{pmatrix} 6 \\ 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

proposition: \mathbb{F}^n is a vector space over \mathbb{F}

more notes: c) \mathbb{F}^1 is a vector space

$$d) \mathbb{F}^0 = 0 \text{ vector space} \quad V = \{0_V\} ; \quad x+y = 0_V \\ a \cdot 0_V = 0_V \quad \forall a \in \mathbb{F}$$

② $m \times n$ matrices over \mathbb{F}

$$A = (A_{ij}) = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & & & \\ \vdots & & & \\ A_{m1} & A_{m2} & \dots & A_{mn} \end{pmatrix} \left. \vphantom{\begin{pmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & & & \\ \vdots & & & \\ A_{m1} & A_{m2} & \dots & A_{mn} \end{pmatrix}} \right\} \begin{matrix} n \text{ columns} \\ m \text{ rows} \end{matrix}$$

where $A_{ij} \in \mathbb{F}$

let $\mathbb{F}^{m \times n}$ denote the set of ALL $m \times n$ matrices over \mathbb{F} .

$$\textcircled{a} \quad 0_{\mathbb{F}^{m \times n}} = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{pmatrix}$$

$$\textcircled{b} \quad A+B = (A_{ij} + B_{ij}), \quad A, B \in \mathbb{F}^{m \times n}, \quad 1 \leq i \leq m \quad 1 \leq j \leq n$$