

To find a suitable definition for  $e^z$  when  $z = x + iy$ , we want to preserve the basic identities satisfied by the real function  $e^x$ .

We therefore postulate that the multiplicative property persists.

$$e^{z_1} e^{z_2} = e^{z_1 + z_2} \quad (1)$$

This enables the decomposition

$$e^z = e^{x+iy} = e^x e^{iy} \quad \leftarrow \text{How do we deal with } e^{iy} ? \quad (2)$$

Next we propose that the differentiation law be preserved:

$$\frac{d}{dz} (e^z) = e^z \quad (3)$$

Consider

$$\frac{d}{d(iy)} e^{(iy)} = e^{iy}$$

Or, equivalently, by the chain rule

$$\frac{d}{dy} e^{iy} = ie^{iy} \quad (4)$$

If we differentiate again,

$$\frac{d^2}{dy^2} e^{iy} = \frac{d}{dy} ie^{iy} = i^2 e^{iy}$$

$$= -e^i$$

In other words, the function  $g(y) := e^{iy}$  satisfies the differential equation

$$\frac{d^2}{dy^2} g = -g$$

(5)

Now observe that any function of the form

$$A \cos(y) + B \sin(y) \quad (A, B \rightarrow \text{constants})$$

From the theory of differential equations it is known that every solution of Eq. 5 must have this form.

Hence we can write

$$g(y) = A \cos(y) + B \sin(y)$$

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To evaluate A and B we use the conditions that

$$g(0) = e^{i0} = e^0 = 1 = A \cos(0) + B \sin(0)$$

and

$$\frac{d}{dy} g(0) = ig(0) = i = -A \sin(0) + B \cos(0)$$

Thus  $A = 1$  and  $B = i$ , leading us to the identification

$$e^{iy} = \cos(y) + i \sin(y)$$

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Eq. 7 is known as Euler's equation.

**Definition 5:** If  $z = x + iy$ , then  $e^z$  is defined to be the complex number  $e^z := e^x (\cos y + i \sin y)$  (8)

It can then be directly verified that  $e^z$ , as defined above, satisfied the associated division rule with multiplicative identity (1).

$$\frac{e^{z_1}}{e^{z_2}} = e^{z_1 - z_2} \quad (9)$$

**Example 1:** Show that Euler's equation is consistent with the usual Taylor series expansions

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots,$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots,$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots,$$

For now we ignore the questions of convergence, etc, and simply substitute  $x = iy$  into the exponential series.

$$\begin{aligned}
 e^{iy} &= 1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \frac{(iy)^5}{5!} + \dots \\
 &= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \dots\right) + i \left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \dots\right) \\
 &= \cos(y) + i \sin(y) \quad \text{WOAH!}
 \end{aligned}$$

Euler's Eqn 7 allows us to write the polar form of a complex number as

$$z = r \operatorname{cis}(\theta) = r(\cos \theta + i \sin \theta) = r e^{i\theta}$$

Thus we can drop the stupid 'cis' artefact and use

$$z = r e^{i\theta} = |z| e^{i \arg(z)} \quad (10)$$

In particular, notice the following identities

$$e^{i0} = e^{2\pi i} = e^{-2\pi i} = e^{4\pi i} = e^{-4\pi i} = \dots = 1$$

$$e^{(\pi/2)i} = i, \quad e^{(-\pi/2)i} = -i, \quad e^{\pi i} = -1 \quad \leftarrow \text{Amazing}$$

Observe that  $|e^{i\arg z}| = 1$  and that Euler's equation leads to the following representations of the customary trigonometric functions:

$$\cos(\theta) = \operatorname{Re}(e^{i\theta}) = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad (11)$$

$$\sin(\theta) = \operatorname{Im}(e^{i\theta}) = \frac{e^{i\theta} - e^{-i\theta}}{2i} \quad (12)$$

The rules derived in 1.3 for multiplying/dividing complex numbers in polar form now find very natural expressions:

$$z_1 z_2 = (r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = (r_1 r_2) e^{i(\theta_1 + \theta_2)} \quad (13)$$

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \left( \frac{r_1}{r_2} \right) e^{i(\theta_1 - \theta_2)} \quad (14)$$

and complex conjugation of  $z = re^{i\theta}$  is accomplished by changing the sign of  $i$  in the exponent:

$$\bar{z} = r e^{-i\theta} \quad (15)$$

**Example 2:** Compute (a)  $\frac{1+i}{\sqrt{3}-i}$  (b)  $(1+i)^{24}$

(a)  $1+i = \sqrt{2} e^{i\pi/4}$        $\sqrt{3}-i = 2e^{-i\pi/6}$

$$\frac{1+i}{\sqrt{3}-i} = \frac{\sqrt{2}}{2} e^{i\frac{5\pi}{12}}$$

(b)

$$(1+i)^{24} = (\sqrt{2} e^{i\pi/4})^{24} = (\sqrt{2})^{24} e^{i\frac{24\pi}{4}}$$
$$= 2^{12} e^{i6\pi} = 2^{12}$$

**Example 3:**

Prove De Moivre's formula:

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$$(\cos \theta + i \sin \theta)^n = \cos(n\theta) + i \sin(n\theta), \quad n=1, 2, 3, \dots$$

$$(e^{i\theta})^n = \underbrace{e^{i\theta} e^{i\theta} \dots e^{i\theta}}_{n \text{ times}} = e^{i\theta + i\theta + \dots + i\theta} = e^{in\theta}$$

$$e^{in\theta} = \cos(n\theta) + i \sin(n\theta)$$

**Example 4:** Express  $\cos(3\theta)$  in terms of  $\cos(\theta), \sin(\theta)$

$$\cos(3\theta) = \operatorname{Re}[\cos(3\theta) + i \sin(3\theta)] = \operatorname{Re}(\cos \theta + i \sin \theta)^3$$

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Thus,

$$\begin{aligned}\cos(3\theta) &= \operatorname{Re}[\cos^3 \theta + 3\cos^2 \theta (i \sin \theta) + 3\cos \theta (-\sin^2 \theta) - i \sin^3 (\theta)] \\ &= \cos^3 \theta - 3\cos \theta \sin^2 \theta\end{aligned}$$

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**Example 5:** Compute the integral

$$\int_0^{2\pi} \cos^4 \theta \, d\theta$$

Note that

$$\cos^4 \theta = \left( \frac{e^{i\theta} + e^{-i\theta}}{2} \right)^4 = \frac{1}{2^4} (e^{i\theta} + e^{-i\theta})^4$$

and expanding via binomial formula gives

$$\begin{aligned}\cos^4(\theta) &= \frac{1}{16} (e^{4i\theta} + 4e^{i3\theta} e^{-i\theta} + 6e^{i2\theta} e^{-2i\theta} + 4e^{i\theta} e^{-3i\theta} + e^{-4i\theta}) \\ &= \frac{1}{16} (e^{4i\theta} + 4e^{i2\theta} + 6 + 4e^{-2i\theta} + e^{-4i\theta}) \\ &= \frac{1}{16} (6 + 8\cos(2\theta) + 2\cos(4\theta))\end{aligned}$$

Thus,

$$\int_0^{2\pi} \cos^4 \theta \, d\theta = \int_0^{2\pi} \frac{1}{16} (6 + 8\cos(2\theta) + 2\cos(4\theta)) \, d\theta$$

$$= \frac{1}{16} \left[ 6\theta + 4\sin(2\theta) + \frac{1}{2} \sin(4\theta) \right]_0^{2\pi}$$

$= \frac{3}{4}\pi$