The iteration of $f(x) = x^2$ is very interesting when performed with complex numbers; the sequence of points

$$z_0,\ f(z_0),\ f(f(z_0)),\ f(f(f(z_0))),\ \ldots$$

becomes an orbit in the complex plane.

When the starting point z_0 lies in the unit circle the orbit stays bounded and converges to $z=0$. If $|z_0|>1$ the orbit is unbounded.

Example 1: Show that if

(i) $f(z)$ is analytic in a neighborhood of $z=\xi$.

(ii) $f(z) = \xi$, and

(iii) $|f'(z)| < 1$

then there is a disk around ξ with the property that all orbits launched from inside the disk remain confined to the disk and converge to ξ.

Since

$$\lim_{z\to\xi} \left| \frac{f(z)-f(\xi)}{z-\xi} \right| = |f'(\xi)| < 1$$

and $f(\xi) = \xi$, we can pick a real number ρ lying between $|f'(\xi)|$ and 1 such that

$$|f(z)-f(\xi)| = |f(z)-\xi| \le \rho|z-\xi| \qquad (1)$$
for all z in a sufficiently small disk around ξ.
Such a disk meets the specifications; indeed, if any point z_0 in this disk is the seed for an orbit $z_1 = f(z_0)$, $z_2 = f(z_1)$, ..., then by (1) we have

$$|z_n-\xi| \le \rho|z_{n-1}-\xi| \le \cdots \le \rho^n|z_0-\xi|.$$

Since $\rho<1$, the point z_n lies closer to ξ than z_{n-1} and, in fact, $\lim_{n\to\infty} z_n = \xi$

If $f(\xi)=\xi$, then ξ is called a fixed point of the function f. A fixed point meeting the conditions of Example 1 is called an attractor, and the set of seed points whose orbits converge to ξ is called its basis of attraction.
Thus $\xi=0$ is an attractor for $f(z)=z^2$ (since

$$0=0^2=f(0) \text{ and } |f'(0)|=0<1)$$

whose basin is the open disk $|z|<1$.
Example 1 shows that every attractor has a basin containing, at least, a small disk.

The other fixed point for $f(z)=z^2$, $\xi=1$, is a repellor. (Prob 2)

For the function

$$f(z) = z^2$$
$$f(z) = x^2 - y^2 + i2xy,$$

if z_0 lies on the unit circle $|z_0| = 1$, so does the entire orbit launched from z_0.

In fact if $z_0 = 1$ or -1 the orbit quickly settles down to the fixed point $z = 1$. If $z_0 = e^{i2\pi/3}$ (a primitive cube root of unity) the orbit oscillates between two points $e^{i\pm 2\pi/3}$ and is called a 2-cycle with period 2.

It can be shown that the seed choice $z_0 = e^{i\alpha 2\pi}$, for irrational α, generates an orbit whose points never repeat and, in fact, permeate the unit circle densely.

So the unit circle, which seperates the seeds of orbits converging to zero from those of unbounded orbits, contains a variety of orbits itself.

Definition 7. The **filled Julia Set** for a polynomial function $f(z)$ is defined to be the set of points that launch bounded orbits through iterations of f; the **Julia Set** is the boundary of the filled Julia set.

So the Julia set for z^2 is the unit circle, and the filled Julia set is the closed unit disk.