Exercises.

Solution to Question 1.

A. If
$$a_1f_1 + a_2f_2 + a_3f_3 = 0,$$
then
$$\begin{bmatrix} f_1(x_1) & f_2(x_1) & f_3(x_1) \\ f_1(x_2) & f_2(x_2) & f_3(x_2) \\ f_1(x_3) & f_2(x_3) & f_3(x_3) \end{bmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$
If the columns of the matrix $[f_j(x_i)]$ are linearly independent in $\mathbb{R}^3$, then
$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$
So f_1, f_2, f_3 are linearly independent in $\mathrm{Fun}(\mathbb{R}, \mathbb{R})$.

B. We may take, for example $x_1 = 0$, $x_2 = 1$ and $x_3 = 2$, then
$$\det \begin{bmatrix} f_1(x_1) & f_2(x_1) & f_3(x_1) \\ f_1(x_2) & f_2(x_2) & f_3(x_2) \\ f_1(x_3) & f_2(x_3) & f_3(x_3) \end{bmatrix} = \det \begin{bmatrix} 1 & 0 & 1 \\ e^{-1} & 1 & e \\ e^{-2} & 2 & e^2 \end{bmatrix} = e^2 - 2e + 2e^{-1} - e^{-2} = (e - e^{-1})(e - 2 + e^{-1}) \neq 0.$$
By part A., f_1, f_2, f_3 are linearly independent in $\mathrm{Fun}(\mathbb{R}, \mathbb{R})$.

C. We may take, for example $x_1 = 0$, $x_2 = \pi/2$ and $x_3 = 2\pi$, then
$$\det \begin{bmatrix} f_1(x_1) & f_2(x_1) & f_3(x_1) \\ f_1(x_2) & f_2(x_2) & f_3(x_2) \\ f_1(x_3) & f_2(x_3) & f_3(x_3) \end{bmatrix} = \det \begin{bmatrix} 1 & 0 & 1 \\ e^{-\pi/2} & 1 & 0 \\ e^{-\pi} & 0 & 1 \end{bmatrix} = 1 - e^{-\pi} \neq 0.$$
By part A., f_1, f_2, f_3 are linearly independent in $\mathrm{Fun}(\mathbb{R}, \mathbb{R})$.

Answer to Question 2.
Assume that $\dim U = m$, and that $(u_1, \ldots, u_m)$ is a basis for U, then $u_i \in V$ for all $i = 1, \ldots, m$ and $u_1, \ldots, u_m$ are linearly independent. So $\dim V \geq m$.
If $(u_1, \ldots, u_m)$ spans V, then $V = U$. So $\dim V = \dim U$.
If $U \neq V$, then exists a vector $w \in V - U$. Let $U' = \operatorname{span}\{u_1, \ldots, u_m, w\}$. Then $\dim U' = m + 1$ and $U' \subseteq V$. So $\dim V \geq m + 1 > m$.

Solution to Question 3.

(a) If $p(x) \in V$ satisfies $p(1) = p(2) = 0$, then $p(x)$ is divisible by $q(x) := (x-1)(x-2)$. So

$$\begin{aligned} U &= \{p(x) \in V \mid p(1) = p(2) = 0\} \\ &= \{(ax^2 + bx + c)q(x) \mid a, b, c \in \mathbb{R}\}. \end{aligned}$$

We may choose $(u_1, u_2, u_3) := (q(x), xq(x), x^2q(x))$ as a basis for U.

(b) We know that $(1, x, x^2, x^3, x^4)$ is a basis for V, so $\dim V = 5$. By part (a), $\dim U = 3$. Therefore, if we want to extend this basis to a basis for V, we need 2 more vectors.
Let $(x_1, x_2, \ldots, x_5) = (1, 2, 3, 4, 5)$. If $(w_1, w_2, u_1, u_2, u_3)$ is a basis for V, then by Problem 1 A., the columns of

$$\begin{bmatrix} w_1(x_1) & w_2(x_1) & u_1(x_1) & u_2(x_1) & u_3(x_1) \\ w_1(x_2) & w_2(x_2) & u_1(x_2) & u_2(x_2) & u_3(x_2) \\ w_1(x_3) & w_2(x_3) & u_1(x_3) & u_2(x_3) & u_3(x_3) \\ w_1(x_4) & w_2(x_4) & u_1(x_4) & u_2(x_4) & u_3(x_4) \\ w_1(x_5) & w_2(x_5) & u_1(x_5) & u_2(x_5) & u_3(x_5) \end{bmatrix} = \begin{bmatrix} w_1(x_1) & w_2(x_1) & 0 & 0 & 0 \\ w_1(x_2) & w_2(x_2) & 0 & 0 & 0 \\ w_1(x_3) & w_2(x_3) & u_1(x_3) & u_2(x_3) & u_3(x_3) \\ w_1(x_4) & w_2(x_4) & u_1(x_4) & u_2(x_4) & u_3(x_4) \\ w_1(x_5) & w_2(x_5) & u_1(x_5) & u_2(x_5) & u_3(x_5) \end{bmatrix}$$

must be linearly independent. Therefore, the left top 2×2 square

$$\begin{bmatrix} w_1(x_1) & w_2(x_1) \\ w_1(x_2) & w_2(x_2) \end{bmatrix} = \begin{bmatrix} w_1(1) & w_2(1) \\ w_1(2) & w_2(2) \end{bmatrix}$$

must be invertible. If we take $w_1 = x - 1$ and $w_2 = x - 2$, then

$$\begin{bmatrix} w_1(1) & w_2(1) \\ w_1(2) & w_2(2) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

is invertible. Hence $(w_1, w_2, u_1, u_2, u_3)$ is a basis for V if we take $w_1 = x - 1$ and $w_2 = x - 2$.

(c) Take $w_1 = x - 1$ and $w_2 = x - 2$ as in part (b) and let $W = \operatorname{span}\{w_1, w_2\}$. Then $W \cap U = \{0\}$ and $W + U = V$. Hence $V = U \oplus W$.

Solution to Question 4.

(a) We only need to show $v_1, \dots, v_n$ are linearly independent. Because $\dim V = n$, so we may assume that $(u_1, \dots, u_n)$ is a basis for V. Because $(v_1, \dots, v_n)$ spans V, so there exists a $n \times n$ matrix A such that

$$(u_1, \dots, u_n) = (v_1, \dots, v_n)A.$$

On the other hand, $(u_1, \dots, u_n)$ is a basis of V, so there exists a $n \times n$ matrix B such that

$$(v_1, \dots, v_n) = (u_1, \dots, u_n)B.$$

Therefore,

$$(u_1, \dots, u_n) = (u_1, \dots, u_n)BA,$$

which implies $BA = I_n$. Hence B is invertible. If there exists $a_1, \dots, a_n \in \mathbb{F}$ such that

$$(v_1, \dots, v_n)\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = 0,$$

then it is equivalent to

$$(u_1, \dots, u_n)B\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = 0 \iff B\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \iff \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Hence $v_1, \dots, v_n$ are linearly independent.

(b) We only need to show $(v_1, \dots, v_n)$ spans V. Because $\dim V = n$, so we may assume that $(u_1, \dots, u_n)$ is a basis for V, then there exists a $n \times n$ matrix B such that

$$(v_1, \dots, v_n) = (u_1, \dots, u_n)B.$$

Because $(v_1, \dots, v_n)$ is linearly independent, so

$$(v_1, \dots, v_n)\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = 0 \implies \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix},$$

which is equivalent to say

$$B\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \implies \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Therefore, B is invertible and

$$(u_1, \dots, u_n) = (v_1, \dots, v_n)B^{-1}.$$

For any $v \in V$, it can be written as

$$v = (u_1, \ldots, u_n)\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = (v_1, \ldots, v_n)B^{-1}\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = (v_1, \ldots, v_n)\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}.$$

Hence $(v_1, \ldots, v_n)$ spans V.

Solution to Question 5.

(a) Because $U \cap W = 0$, so $U + W = U \oplus W$.
If $(u_1, \dots, u_n)$ is a basis for U and $(w_1, \dots, w_m)$ is a basis for W, then $(u_1, \dots, u_n, w_1, \dots, w_m)$ is a basis for $U \oplus W$.
So $\dim U + W = \dim U \oplus W = \dim U + \dim W$ if $U \cap W = 0$.

(b) Let $Y := U \cap W$. Then $Y \subseteq U$ is a subspace. So we may find a complement U' such that $U = Y \oplus U'$. Similarly, we may find a complement W' such that $W = Y \oplus W'$. Now assume that $(y_1, \dots, y_s)$ is a basis for Y, $(u_1, \dots, u_r)$ is that for U' and $(w_1, \dots, w_t)$ for W'. It is that $(y_1, \dots, y_s, u_1, \dots, u_r, w_1, \dots, w_t)$ spans $U + W$. We need to prove that it is linearly independent. Assume that

$$\mathbf{ay} + \mathbf{bu} + \mathbf{cw} = a_1 y_1 + \cdots + a_s y_s + b_1 u_1 + \cdots + b_r u_r + c_1 w_1 + \cdots + c_t w_t = 0.$$

Then $\mathbf{bu} + \mathbf{cw} = -\mathbf{ay} \in U \cap W \implies \mathbf{bu}, \mathbf{cw} \in U \cap W$. But $\mathbf{bu} \in U'$ and $\mathbf{cw} \in W'$, hence $\mathbf{bu} = \mathbf{cw} = 0$. Because $\mathbf{u}$ is a basis for U' and $\mathbf{w}$ is a basis for W', so $\mathbf{b} = \mathbf{c} = 0$.
Therefore, $(y_1, \dots, y_s, u_1, \dots, u_r, w_1, \dots, w_t)$ is a basis for $U + W$ and

$$\dim(U + W) = \dim U' + \dim Y + \dim W' = \dim U + \dim W - \dim Y.$$

(c) By part (b),

$$\begin{aligned}\dim(U + W + X) &= \dim U + W + \dim X - \dim(U + W) \cap X \\ &= \dim U + \dim W + \dim X - \dim U \cap W - \dim(U + W) \cap X.\end{aligned}$$

So we only need to find an example such that

$$\dim(U + W) \cap X \neq \dim U \cap X + \dim W \cap X - \dim U \cap W \cap X.$$

Take $\mathbb{F} = \mathbb{R}$ and $V = \mathbb{R}^2$. Let $U = \operatorname{span}\{\begin{pmatrix}1\\0\end{pmatrix}\}$, $W = \operatorname{span}\{\begin{pmatrix}0\\1\end{pmatrix}\}$ and $X = \operatorname{span}\{\begin{pmatrix}1\\1\end{pmatrix}\}$. Then $U + W = V$, so $(U + W) \cap X = X$. And $U \cap X = W \cap X = U \cap W \cap X = 0$. Therefore,

$$1 = \dim(U + W) \cap X \neq \dim U \cap X + \dim W \cap X - \dim U \cap W \cap X = 0.$$

Extended Glossary.

For any set X, a **relation** on X is a set R consisting of some ordered pairs in X, in other words, a subset of $X \times X$. We may say aRb, or $a \sim_R b$, if $(a, b) \in R$. For simplicity, sometimes we just write $a \sim b$ if there is no ambiguity.

Definition 1. An **equivalence equation** R is a relation on a set X satisfying the following conditions:

(a) For all $x \in X$, $(x, x) \in R$.

(b) For any $x, y \in X$, if $(x, y) \in R$, then $(y, x) \in R$.

(c) For any $x, y, z \in X$, if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$.

If R is an equivalence relation, and $(x, y) \in R$, we may also say x and y are equivalent with respect to R.
We give an example of equivalence relation:

Example 1. The $=$ is an equivalence relation on $\mathbb{R}$. We may check the 3 conditions:

(a) For all $x \in \mathbb{R}$, $x = x$.

(b) For any $x, y \in \mathbb{R}$, $x = y$ implies $y = x$.

(c) For any $x, y, z \in \mathbb{R}$, $x = y$ and $y = z$ implies $x = z$.

In this example, $R = \{(x, x) \in X \times X \mid x \in X\}$.

Not all relations are equivalence relations. Here is an example of a relation that is not equivalence relation:

Example 2. The $\neq$ is NOT an equivalence relation on $\mathbb{R}$, because $x \neq x$ is false.
Moreover, pay attention in this example that $x \neq y \neq z$ does NOT imply $x \neq z$.

For equivalence relations, there is an important theorem:

Theorem 1. *Let X be a set and R be a equivalent relation on X. Then X may be divided into disjoint union of subsets, each of which consists of equivalent elements with respect to R.*

Proof. We only need to show that any two such subsets are either disjoint or identical. For any $u, v \in X$, let

$$U := \{x \in X : \ x \sim u\},$$

and

$$V := \{y \in X : \ y \sim v\}.$$

If $U \cap V = \emptyset$, then they are disjoint.
Otherwise, take $w \in U \cap V$. Then by definition, $w \sim u$ and $w \sim v$, so $u \sim v$. Hence $U = V$. $\square$