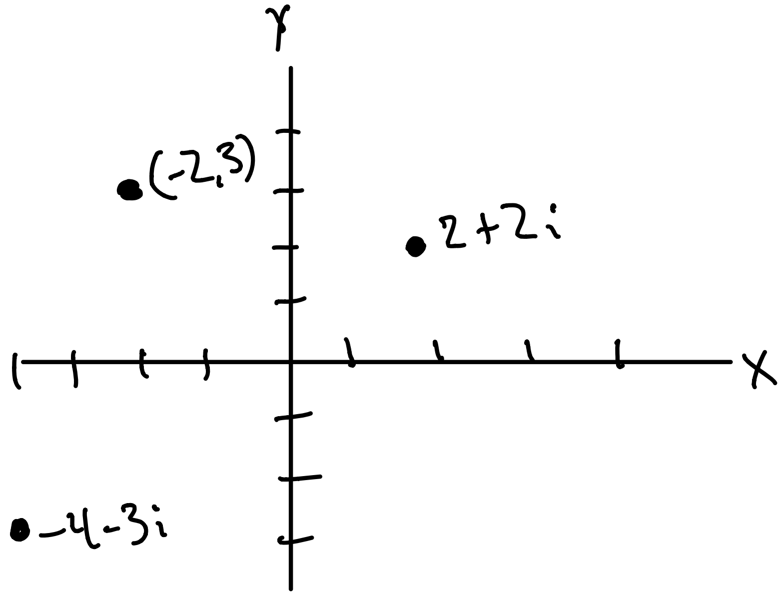


The cartesian coordinate system suggests a convenient way to represent complex numbers as points in the xy plane.

We denote an xy-plane that is used for the purpose of describing complex numbers as the z-plane.

x-axis $\rightarrow$ real axis

y-axis $\rightarrow$ imaginary axis

The point $z = (a+bi)$ is the point w/ cordinates (a,b).

Example 1:

Suppose that n particles with masses $m_1, m_2, \ldots m_n$ are located at the respective points $z_1, z_2, \ldots z_n$ in the complex planes. Show the center of mass of the system is the point

$$\hat{z} = \frac{m_1 z_1 + m_2 z_2 + \cdots + m_n z_n}{m_1 + m_2 + \cdots + m_n}$$

Write $z_1 = x_1 + y_1 i$, $z_2 = x_2 + y_2 i$, $\ldots$ $z_n = x_n + y_n i$ and let M be the total mass:

$$M = \sum_{k=1}^{n} m_k$$

Recall that the center of mass of a given system is the point $(\hat{x}, \hat{y})$, where

$$\hat{x} = \frac{\sum_{k=1}^{n} m_k x_k}{M}, \quad \hat{y} = \frac{\sum_{k=1}^{n} m_k y_k}{M}$$

But clearly $\hat{x}$ and $\hat{y}$ are, respectively, the real + imaginary parts of the complex number

$$\sum_{k=1}^{n} \frac{m_k z_k}{M} = \hat{z}$$

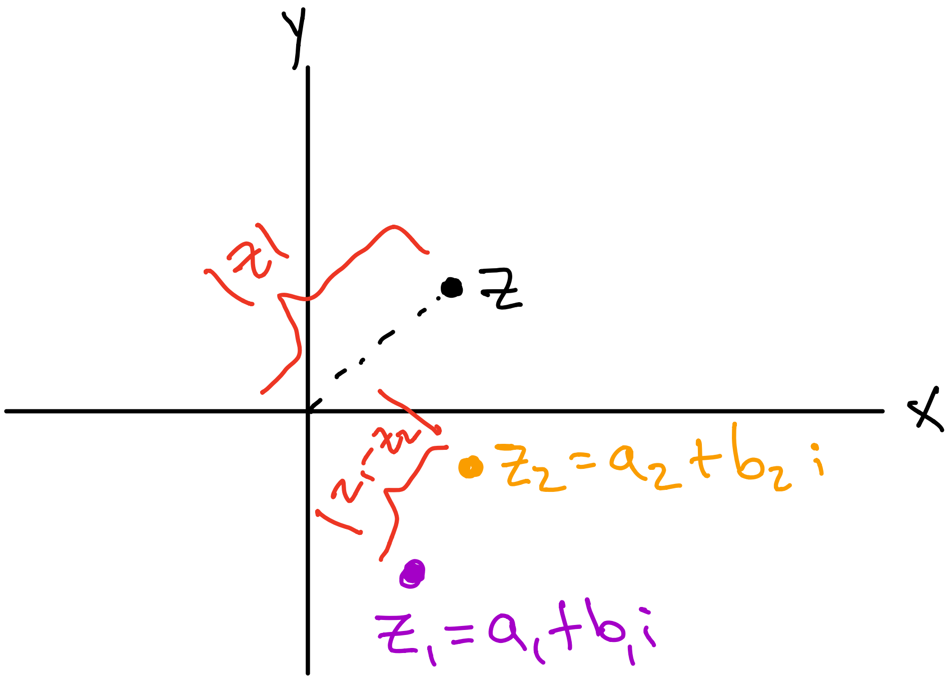
The distance from the point $z = a + bi$ to the origin is given by $\sqrt{a^2 + b^2}$.

Definition 3: The <u>absolute value</u> of the number $z = a + bi$ is denoted by $|z|$ and is given by

$$|z| = \sqrt{a^2 + b^2}$$

In particular,

$$|0| = 0, \quad \left|\frac{i}{2}\right| = \frac{1}{2}, \quad |3-4i| = \sqrt{9+16} = 5$$



$$|z_1 - z_2| = \sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2}$$

which is the distance b/t points with coordinates (a_1, b_1), (a_2, b_2)

Example 2:

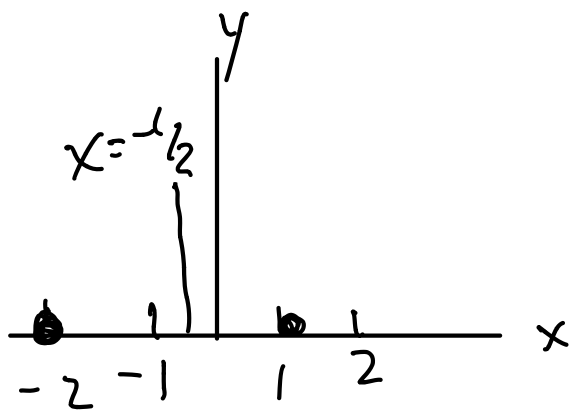
Describe the set of points z that satisfy the equations

(a) $|z+2| = |z-1|$ (b) $|z-1| = \operatorname{Re} z + 1$

a) z must be equidistant from the points -2 and 1.
Hence eqn a is is the eqn of the perpendicular bisector of the line segment joining -2 and 1.
That is: $\underline{x = -1/2}$

A more routine method is to set $z = x+iy$

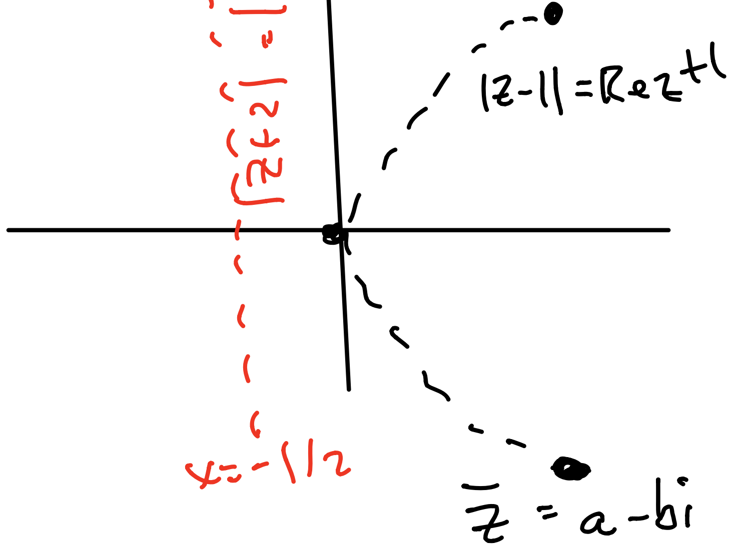
$$|z+2| = |z-1|$$
$$|x+iy+2| = |x+iy-1|$$
$$(x+2)^2 + y^2 = (x-1)^2 + y^2$$
$$4x+4 = -2x+1$$
$$\underline{x = -1/2}$$



b) $\sqrt{(x-1)^2+y^2} = x+1$

$$y^2 + (x-1)^2 = (x+1)^2$$
$$y^2 + x^2 - 2x + 1 = x^2 + 2x + 1$$
$$y^2 = 4x$$
$$y = \sqrt{4x}$$

$z = a+bi$



Definition 4: The <u>complex conjugate</u> of the number $z = a + bi$ is denoted by $\bar{z}$ and is given by

$$\bar{z} = a - bi$$

Thus,

$$\overline{-1+5i} = -1-5i \qquad \overline{\pi - i} = \pi + i$$

Note: $z = \bar{z}$ if z is a real number

$$\left.\begin{aligned} \overline{z_1 + z_2} &= \bar{z}_1 + \bar{z}_2 \\ \overline{z_1 - z_2} &= \bar{z}_1 - \bar{z}_2 \end{aligned}\right\} \text{obvious to prove}$$

Not so obvious to prove is the analogous property for multiplication.

Example 3: Verify that $\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$

$$\overline{z_1 z_2} = \overline{a_1 a_2 - b_1 b_2 + (a_1 b_2 + a_2 b_1) i}$$
$$= a_1 a_2 - b_1 b_2 - (a_1 b_2 + a_2 b_1) i$$

$$\overline{z_1}\,\overline{z_2} = (a_1 - b_1 i)(a_2 - b_2 i)$$
$$= a_1 a_2 - b_1 b_2 - a_1 b_2 i - a_2 b_1 i$$
$$= a_1 a_2 - b_1 b_2 - (a_1 b_2 + a_2 b_1) i$$

✓

In addition to above,

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}} \qquad (z_2 \neq 0)$$

can also be seen

$$\left.\begin{aligned} \operatorname{Re} z &= \frac{z + \overline{z}}{2} \\ \operatorname{Im} z &= \frac{z - \overline{z}}{2} \end{aligned}\right\} \text{\underline{USEFUL}!}$$

Aside: $\overline{\overline{z}} = z$ $\qquad z\overline{z} = a^2 + b^2 = |z|^2$

$|z| = |\overline{z}|$

Looking back at 1.1,

$$\frac{z_1}{z_2} = \frac{z_1 \overline{z_2}}{z_2 \overline{z_2}} = \frac{z_1 \overline{z_2}}{|z_2|^2}$$

In particular,

$$\frac{1}{z} = \frac{\overline{z}}{|z|^2}$$