Recall

Bayes rule stuff involving continuous rvs.

Example – Bit Through Noisy Channel

Consider : - $S = \pm 1$; $\mathbb{P}(\{S=1\}) = p$, $p \in (0,1)$

- N Gaussian, 0 mean, variance σ^2
- $Y = S + N$ $\quad S \longrightarrow$ Noisy Channel $\longrightarrow Y$

Want: $\mathbb{P}(\{S=1\} \mid Y=y)$

Have a grip on: $\mathbb{P}(\{S=1\})$, $\mathbb{P}(\{S=-1\})$, $f_{Y|S}(y|1)$, $f_{Y|S}(y|-1)$

$f_{Y|S}(y|1) =$ Gaussian, mean 1, var σ^2

$f_{Y|S}(y|-1) =$ Gaussian, mean -1, var σ^2

By Baye's Rule:

$$\mathbb{P}(\{S=1 \mid Y=y\}) = \frac{f_{Y|S}(y|1)\,\mathbb{P}(\{S=1\})}{f_{Y|S}(y|1)\,\mathbb{P}(\{S=1\}) + f_{Y|S}(y|-1)\,\mathbb{P}(\{S=-1\})}$$

$$= \frac{p\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(y-1)^2}{2\sigma^2}\right)}{p\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(y-1)^2}{2\sigma^2}\right) + (1-p)\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(y+1)^2}{2\sigma^2}\right)}$$

$$= \frac{pe^{y}}{pe^{y} + (1-p)e^{-y}} \quad \begin{cases} \xrightarrow{y\to\infty} 1 \\ \xrightarrow{y\to-\infty} 0 \end{cases}$$

This example is inferring about a discrete event from a continuous observation.

The **reverse** situation also arises. Observe some event A; infer about conditional rv Y - specifically $f_{Y|A}(y)$ - know $f_Y(y)$ and $\mathbb{P}(A|Y=y)$, $\mathbb{P}(A^c|Y=y)$.

Flip Baye's Rule.

$$\mathbb{P}(A|Y=y) = \frac{f_{Y|A}(y)\,\mathbb{P}(A)}{f_Y(y)} \rightsquigarrow f_{Y|A}(y) = \frac{\mathbb{P}(A|Y=y)\,f_Y(y)}{\mathbb{P}(A)}$$

Rewriting $\mathbb{P}(A)$ using Total Probability Theorem yields

$$f_{Y|A}(y) = \frac{\mathbb{P}(A|Y=y)\,f_Y(y)}{\int_{-\infty}^{+\infty} \mathbb{P}(A|Y=y)\,f_Y(y)\,dy}$$

NEXT TOPIC

Derived Distributions

Setup: X continuous rv; pdf $f_X(x)$; $Y=g(X)$; want $f_Y(y)$.

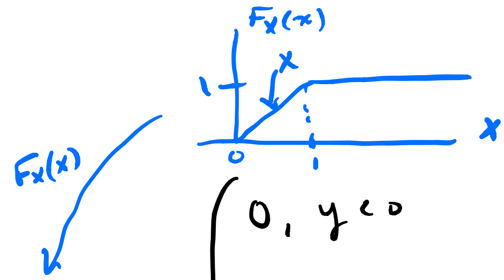
NOT easy in general - look at some special situations where it is.

In these situations, get $f_Y(y)$ via

- find $F_Y(y)$
- $f_Y(y) = dF_Y(y)/dy$

Example

X uniform on $[0,1]$. $Y = g(X) = \sqrt{X}$.



$$F_Y(y) = \mathbb{P}(\{Y \le y\}) = \mathbb{P}(\{\sqrt{X} \le y\}) = \mathbb{P}(\{X \le y^2\}) = F_X(y^2) = \begin{cases} 0, & y < 0 \\ y^2, & y \in [0,1) \\ 1, & y > 1 \end{cases}$$

Thus

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \begin{cases} 2y, & y \in [0,1] \\ 0, & \text{else} \end{cases}$$

Example

X has pdf $f_X(x)$, cdf $F_X(x)$; $Y = X^2$.

$$F_Y(y) = \mathbb{P}(\{Y \le y\}) = \mathbb{P}(\{X^2 \le y\}) = \begin{cases} 0, & y < 0 \\ \mathbb{P}(-\sqrt{Y} \le X \le \sqrt{Y}), & y \geq 0 \end{cases} = \begin{cases} 0, & y < 0 \\ F_X(\sqrt{Y}) - F_X(-\sqrt{Y}), & y \geq 0 \end{cases}$$

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \begin{cases} 0, & y < 0 \\ \frac{1}{2\sqrt{y}} f_X(\sqrt{y}) + \frac{1}{2\sqrt{y}} f_X(-\sqrt{y}), & y \geq 0 \end{cases}$$

Example – More General

X has $f_X(x)$, $F_X(x)$; $Y = \alpha X + b$, $\alpha \neq 0$

$$F_Y(y) = \mathbb{P}(\alpha X + b \le y) = \begin{cases} \mathbb{P}\left(\left\{X \le \frac{y-b}{\alpha}\right\}\right), & \alpha > 0 \\ \mathbb{P}\left(\left\{X \geq \frac{y-b}{\alpha}\right\}\right), & \alpha < 0 \end{cases}$$

Thus when $\alpha > 0$,

$$F_Y(y) = F_X\left(\frac{y-b}{\alpha}\right) \Rightarrow f_Y(y) = \frac{1}{\alpha} f_X\left(\frac{y-b}{\alpha}\right)$$

and when $\alpha < 0$

$$F_Y(y) = 1 - F_X\left(\frac{y-b}{\alpha}\right) \Rightarrow f_Y(y) = -\frac{1}{\alpha} f_X\left(\frac{y-b}{\alpha}\right)$$

Examples - Affine Situation

① X is exponential (λ); $Y = \alpha X + \beta$.

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases} \rightarrow f_Y(y) = \frac{1}{|\alpha|} f_X\left(\frac{y-\beta}{\alpha}\right) = \begin{cases} \frac{\lambda}{|\alpha|} e^{-\lambda\left(\frac{y-\beta}{\alpha}\right)}, & \frac{y-\beta}{\alpha} \geq 0 \\ 0, & \frac{y-\beta}{\alpha} < 0 \end{cases}$$

Note: when $\beta = 0$, and $\alpha > 0$, Y is also exponential w/ rate parameter λ/α

② X is Gaussian μ, σ^2, $Y = \alpha X + \beta$.

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \rightarrow f_Y(y) = \frac{1}{\alpha\sqrt{2\pi\sigma^2}} e^{-\frac{(y-a\mu-b)^2}{2a^2\sigma^2}}$$

$\frac{1}{\alpha} = \frac{1}{\sqrt{\alpha^2}}$!

Note: Y also Gaussian! mean $= \alpha\mu + \beta$. Variance $= \alpha^2\sigma^2$

Comment: $\mathbb{E}(Y) = \alpha\mu + \beta$, $Var(Y) = \alpha^2\sigma^2$, but until we do the calculation, **NOT** obvious that Y is also Gaussian

All these examples have something in common! g is strictly monotonic and differentiable

Here's a careful-ish statement of a general result of which all examples so far are special cases.

- X is a continuous rv whose density $f_X(x)$ is concentrated on a single interval $a<x<b$. $a=-\infty$ and/or $b=+\infty$ allowed.
- $Y=g(X)$, g strictly monotonic and differentiable, implying that $f_Y(y)$ is concentrated on $(g(a),g(b))$ ← g increasing OR $(g(b),g(a))$ ← g decreasing
- Let h be the inverse function of g - defined only on $(g(a),g(b))$ or $(g(b),g(a))$ — h is also strictly monotonic and differentiable on its domain of definition

Then

$$f_Y(y)=\begin{cases} \left|\frac{dh(y)}{dy}\right| f_X(h(y)) , & y\in(g(a),g(b)) \text{ (increasing) OR } y\in(g(b),g(a)) \text{ (decreasing)} \\ 0 , & \text{else} \end{cases}$$

Example -

X has pdf $f_X(x)$; $Y=e^{3X}$

$(a,b)=(-\infty,\infty)$

$(g(a),g(b))=(0,\infty)$

$h(y)=\frac{1}{3}\ln(y)$ on $(0,\infty)$

$$\Rightarrow f_Y(y)=\begin{cases} \frac{1}{3y} f_X(h(y)) , & y>0 \\ 0 , & y<0 \end{cases}$$

$f_X(\frac{1}{3}\ln(y))$

Example - g strictly decreasing

- Drive from NYC to Boston (≈ 180 miles) at constant speed X.
- X uniform on $[30, 60] = [a, b]$
- Y = travel time in hours $= \frac{180}{X} = g(X)$

$(g(a), g(b)) = (6, 3) \rightarrow f_Y(y)$ concentrated on $3 < y < 6$

$$h(y) = \frac{180}{y} \xrightarrow{d/dy} \frac{-180}{y^2}$$

$$f_Y(y) = \begin{cases} \frac{180}{y^2} f_X\left(\frac{180}{y}\right), & y \in [3,6] \\ 0, & y \notin [3,6] \end{cases}$$