Quotients of Vector Spaces

$v+U$

Suppose $v \in V$ and $U \subseteq V$. Then $v+U$ is the subset of V defined by

$$v+U = \{v+u \mid u \in U\}$$

Quotient Space, V/U

Suppose U is a subspace of V. Then the **quotient space** V/U is the set of all affine subsets of V parallel to U. i.e.

$$V/U = \{v+U \mid v \in V\}$$

Two Affine Subsets Parallel to U are Equal or Disjoint

Suppose U is a subspace of V and $v,w \in V$. Then the following are equivalent:

(a) $v-w \in U$
(b) $v+U = w+U$
(c) $(v+U) \cap (w+U) \neq \emptyset$

Proof → (a) ⇒ (b)
Suppose $v-w \in U$.
If $u \in U$, then
$v+u = w+((v-w)+u) \in w+U$
Thus $v+U \subset w+U$
If $u \in U$, then
$w+u = v+((w-v)+u) \in v+U$
Thus $w+U \subset v+U$ and $v+U = w+U$

(b) ⇒ (c)
$v+U = w+U \Rightarrow$ NOT disjoint. QED

(c) ⇒ (a)
Suppose $(v+U)\cap(w+U) \neq \emptyset$
Then $\exists u_1,u_2 \in U$ s.t. $v+u_1 = w+u_2$
and thus $v-w = u_2-u_1 \Rightarrow v-w \in U$.

Addition and Scalar Multiplication on V/U

$$(v+U)+(w+U) = (v+w)+U$$
$$\lambda(v+U) = \lambda v + U$$

for $v,w \in V$, $\lambda \in \mathbb{F}$

Quotient Space is a Vector Space

Quotient Map, π

Suppose U a subspace of V. The **quotient map** π is the **linear** map

$$\pi: V \to V/U$$

defined by

$$v \mapsto v+U$$

for $v \in V$

Dimension of a Quotient Space

Suppose V is finite dimensional and U is a subspace of V. Then

$$\dim V/U = \dim V - \dim U$$

Proof → Let $\pi: V \to V/U$
$\ker(\pi) = U$, $\operatorname{im}(\pi) = V/U$
∴ $\dim V = \dim \ker(\pi) + \dim \operatorname{im}(\pi)$
$\dim V = \dim U + \dim V/U$

$\tilde{T}$

Suppose $T \in \mathcal{L}(V,W)$. Define

$$\tilde{T}: V/\operatorname{Null}(T) \to W$$

by

$$v+\operatorname{Null}(T) \mapsto Tv$$

Null Space and Range of $\tilde{T}$

Suppose $T \in \mathcal{L}(V,W)$. Then

(a) $\tilde{T}$ is a linear map from $V/\operatorname{Null}(T) \to W$
(b) $\tilde{T}$ is injective (i.e. $\ker \tilde{T} = \{0\}$)
(c) $\operatorname{range} \tilde{T} = \operatorname{range} T$
(d) $V/\operatorname{Null}(T)$ is isomorphic to $\operatorname{range} T$

Proof → (a)
$\tilde{T}(v_1+\operatorname{Null}T + v_2+\operatorname{Null}T)$
$= \tilde{T}((v_1+v_2)+\operatorname{Null}(T))$
$= T(v_1+v_2)$
$= T(v_1)+T(v_2)$
$= \tilde{T}(v_1+\operatorname{Null}(T)) + \tilde{T}(v_2+\operatorname{Null}(T))$

$\tilde{T}(\lambda(v_1+\operatorname{Null}(T)))$
$= \tilde{T}(\lambda v_1 + \operatorname{Null}(T))$
$= T(\lambda v_1)$
$= \lambda T(v_1)$
$= \lambda\tilde{T}(v_1+\operatorname{Null}(T))$

(b) Suppose $v \in V$ and $\tilde{T}(v+\operatorname{Null}T) = 0$
Then $Tv = 0 \Rightarrow v = 0 \Rightarrow \tilde{T}$ injective

(c) $\operatorname{range} T = \operatorname{range} \tilde{T}$ by definition

(d) View $\tilde{T}: V/\ker(T) \to \operatorname{range}(T)$
→ $\tilde{T}$ an isomorphism from $V/\ker(T)$ onto $\operatorname{range} T$

Linear Functional

A **linear functional** on V is a linear map from V to $\mathbb{F}$. i.e. an element of $\mathcal{L}(V,\mathbb{F})$.

Dual Space, V'

The dual space of V, denoted V', is the vector space of all linear functionals on V. In other words, $V' = \mathcal{L}(V,\mathbb{F})$

$\dim V' = \dim V$

$\dim \mathcal{L}(V,\mathbb{F}) = (\dim V)(\dim \mathbb{F})$
$\dim V' = (\dim V)(1)$

Dual Basis

If $v_1,\dots,v_n$ is a basis of V, then the **dual basis** of $v_1,\dots,v_n$ is the list $\varphi_1,\dots,\varphi_n$ of elements of V', where each φ_j is the linear functional on V such that

$$\varphi_j(v_k) = \begin{cases} 1 & k=j \\ 0 & k\neq j \end{cases}$$

Note: φ_j well defined since a unique linear map exists for a basis of domain

Dual Basis is a Basis of the Dual Space

Suppose V is finite dimensional. Then the dual basis of a basis of V is a basis of V'

Proof → Suppose $v_1,\dots,v_n$ is a basis of V.
Let $\varphi_1,\dots,\varphi_n$ denote dual basis
Suppose $a_1,\dots,a_n \in \mathbb{F}$ such that
$a_1\varphi_1 + \dots + a_n\varphi_n = 0$
Now $(a_1\varphi_1+\dots+a_n\varphi_n)(v_j) = a_j$ for $j=1,\dots,n$.
Thus $a_1 = \dots = a_n = 0$ and $\varphi_1,\dots,\varphi_n$ is LI.
A LI list of of V' with length $\dim V'$ is a basis of V'.

Transpose, A^t

The **transpose** of a matrix A, denoted A^t, is the matrix obtained from A by interchanging rows and columns.

$$(A^t)_{k,j} = A_{j,k}$$

Transpose of Products of Matrices

$$(AC)^t = C^t A^t$$

Row Rank, Column Rank

Suppose A is an $\boxed{m\text{-by-}n}$ matrix with entries in $\mathbb{F}$.

- The row rank of A is the dimension of the span of the rows of A is $\mathbb{F}^{1,n}$
- The column rank of A is the dimension of the span of the columns of A is $\mathbb{F}^{m,1}$

Dimension of range T equals column rank of $\mathcal{M}(T)$.

Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V,W)$. Then $\dim \operatorname{range} T$ = column rank of $\mathcal{M}(T)$.

Row rank equals column rank

Suppose $A \in \mathbb{F}^{m\times n}$. Then the row rank of A equals the column rank of A.

Proof → Define $T: \mathbb{F}^{n,1} \to \mathbb{F}^{m,1}$ by $Tx = Ax$.
Thus $\mathcal{M}(T) = A$, where $\mathcal{M}(T)$ is computed w/ respect to the standard bases of $\mathbb{F}^{n,1}, \mathbb{F}^{m,1}$.
Then
column rank A = column rank $\mathcal{M}(T)$
$= \dim \operatorname{range} T$
$= \dim \operatorname{range} T'$
= column rank of $\mathcal{M}(T')$
= column rank of A^t
= row rank of A

The rank of a matrix $A \in \mathbb{F}^{m,n}$ is the column rank of A.

Polynomials

$p: \mathbb{F} \to \mathbb{F}$ is called a polynomial with coefficients in $\mathbb{F}$ if there exist $a_0,\dots,a_m \in \mathbb{F}$ such that

$$p(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_m z^m \quad \forall z \in \mathbb{F}$$

If a polynomial is the zero function, then all coefficients are 0

Suppose $a_0,\dots,a_m \in \mathbb{F}$. If
$$a_0 + a_1 z + \dots + a_m z^m = 0$$
for every $z \in \mathbb{F}$, then $a_0 = \dots = a_m = 0$

Uniqueness of Coefficients for Polynomials

This is implied from result above.

Division Algorithm for Polynomials

Suppose $p,s \in \mathcal{P}(\mathbb{F})$, with $s \neq 0$. Then there exist unique polynomials $q,r \in \mathcal{P}(\mathbb{F})$ such that

$$p = sq + r$$

and $\deg r < \deg s$.

$n \, m$
$\frac{p}{s} = q + \frac{r}{s}$
$\frac{x}{x^2} = 0 + \frac{x}{x^2}$

Proof → Let $n = \deg p$ and $m = \deg s$.
If $n < m$, then take $q = 0$ and $r = p$ to get desired result.
Thus we can assume $n \geq m$.
Define (linear map)
$$T: \mathcal{P}_{n-m}(\mathbb{F}) \times \mathcal{P}_{m-1}(\mathbb{F}) \to \mathcal{P}_n(\mathbb{F})$$
by
$$(q,r) \mapsto sq + r$$
If $(q,r) \in \ker(T)$, then $sq + r = 0$.
Thus $(q=0, r=0)$ $\dim \ker(T) = 0$ and we have proved uniqueness.
Have
$\dim(\mathcal{P}_{n-m}(\mathbb{F}) \times \mathcal{P}_{m-1}(\mathbb{F})) = (n-m+1)+(m-1+1) = n+1$
Thus
$\dim \mathcal{P}_n(\mathbb{F}) = \dim \operatorname{im} T = n+1$

→ $\operatorname{im} T = \mathcal{P}_n(\mathbb{F})$, and hence $\exists q \in \mathcal{P}_{n-m}(\mathbb{F})$ and $r \in \mathcal{P}_{m-1}(\mathbb{F})$ s.t. $p = sq + r = T(q,r)$.

Zero of a Polynomial

$\lambda \in \mathbb{F}$ is called a **root** of $p \in \mathcal{P}(\mathbb{F})$ if $p(\lambda) = 0$

Factor

A polynomial $s \in \mathcal{P}(\mathbb{F})$ is called a **factor** of $p \in \mathcal{P}(\mathbb{F})$ if there exists a polynomial $q \in \mathcal{P}(\mathbb{F})$ such that $p = sq$.

Each Zero of a Polynomial Corresponds to a Degree-1 Factor

Suppose $p \in \mathcal{P}(\mathbb{F})$ and $\lambda \in \mathbb{F}$. Then $p(\lambda) = 0$ iff $\exists q \in \mathcal{P}(\mathbb{F})$ s.t.

$$p(z) = (z-\lambda)q(z) \quad \forall z \in \mathbb{F}$$

Proof → (⇒) $p(\lambda) = 0$
$\deg(z-\lambda) = 1$.
division algorithm → $\exists q \in \mathcal{P}(\mathbb{F})$ s.t.
$p(z) = (z-\lambda)q(z) + r \quad \forall z \in \mathbb{F}$.
$p(\lambda) = 0 \Rightarrow r = 0$.
$p(z) = (z-\lambda)q(z) \quad \forall z \in \mathbb{F}$.
(⇐) $p(z) = (z-\lambda)q(z)$
$p(\lambda) = (\lambda-\lambda)q(\lambda) = 0$

A polynomial has at most as many zeros as its degree.

Suppose $p \in \mathcal{P}(\mathbb{F})$ is a polynomial with degree $m \geq 0$. Then p has at most m distinct zeros in $\mathbb{F}$.

Proof → induction on m using last result.

Fundamental Theorem of Algebra

Every nonconstant polynomial w/ complex coefficients has a zero.

Factorization of a polynomial over $\mathbb{C}$

If $p \in \mathcal{P}(\mathbb{C})$ is a nonconstant polynomial, then p has a unique factorization of the form

$$p(z) = c(z-\lambda_1)\cdots(z-\lambda_m)$$

where $c, \lambda_1,\dots,\lambda_m \in \mathbb{C}$

Polynomials with real coefficients have zeros in pairs

Suppose $p \in \mathcal{P}(\mathbb{C})$ is a polynomial with real coefficients. If $\lambda \in \mathbb{C}$ is a zero of p, then so is $\bar{\lambda}$.

Factorization of a polynomial over $\mathbb{R}$

If $p \in \mathcal{P}(\mathbb{R})$ is a nonconstant polynomial, then p has a unique factorization of the form

$$p(x) = c(x-\lambda_1)\cdots(x-\lambda_m)(x^2+b_1x+c_1)\cdots(x^2+b_Mx+c_M)$$

where $c, \lambda_1,\dots,\lambda_m, b_1,\dots,b_M, c_1,\dots,c_M \in \mathbb{R}$ with $b_j^2 < 4c_j$ for each c_j.

Invariant Subspaces

Suppose $T \in \mathcal{L}(V)$. $U \subseteq V$ is called **invariant** under T if $u \in U \Rightarrow Tu \in U$.

Take any $v \in V$ with $v \neq 0$ and let U = scalar multiples of v:
$$U = \{\lambda v : \lambda \in \mathbb{F}\} = \operatorname{span}(v)$$
Then $\dim U = 1$.
If U is invariant under $T \in \mathcal{L}(V)$, then $Tv \in U$ and hence $\exists \lambda \in \mathbb{F}$ such that
$$Tv = \lambda v.$$
Conversely, if $Tv = \lambda v$ for some $\lambda \in \mathbb{F}$, then $\operatorname{span}(v)$ is a 1-dimensional subspace of V invariant under T.

Eigenvalue

Suppose $T \in \mathcal{L}(V)$. A number $\lambda \in \mathbb{F}$ is called an eigenvalue of T if $\exists v \in V$ s.t. $v \neq 0$ and $Tv = \lambda v$.

Equivalent Conditions to Be An Eigenvalue

Suppose V is finite dimensional, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbb{F}$. Then the following are equivalent:

(a) λ has an eigenvalue of T
(b) $T-\lambda I$ is NOT injective
(c) $T-\lambda I$ is NOT surjective
(d) $T-\lambda I$ is NOT invertible

Eigenvector

Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$ is an eigenvalue of T. A vector $v \in V$ is called an **eigenvector** of T corresponding to λ if $v \neq 0$ and $Tv = \lambda v$.

Note: $v \in \ker(T-\lambda I)$

Linearly Independent Eigenvectors

Let $T \in \mathcal{L}(V)$. Suppose $\lambda_1,\dots,\lambda_m$ are distinct eigenvalues of T and $v_1,\dots,v_m$ are corresponding eigenvectors. Then $v_1,\dots,v_m$ are linearly independent.

Proof → Assume $v_1,\dots,v_m$ linearly dependent.
Then
$v_k \in \operatorname{span}(v_1,\dots,v_{k-1})$ for **smallest** k
thus
$$v_k = a_1v_1 + \dots + a_{k-1}v_{k-1} \quad (1)$$
Apply T
$$Tv_k = T(a_1v_1 + \dots + a_{k-1}v_{k-1})$$
$$\lambda_k v_k = a_1\lambda_1 v_1 + \dots + a_{k-1}\lambda_{k-1}v_{k-1} \quad (2)$$
Multiply (1) by λ_k
$$\lambda_k v_k = a_1 v_1 \lambda_k + \dots + a_{k-1}v_{k-1}\lambda_k$$
Subtract (2)
$$-\lambda_k v_k = a_1\lambda_1 v_1 + \dots + a_{k-1}v_{k-1}\lambda_{k-1}$$
$$\Rightarrow 0 = a_1v_1(\lambda_k-\lambda_1) + \dots + a_{k-1}v_{k-1}(\lambda_k-\lambda_{k-1})$$
But $v_1,\dots,v_{k-1}$ LI. $\Rightarrow \lambda_k = \lambda_1, \dots, \lambda_k = \lambda_{k-1}$. λ_i distinct. !<

Number of Eigenvalues

Suppose V is finite-dimensional. Then each operator on V has at most $\dim V$ distinct eigenvalues.

Restriction and Quotient Operators

Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V invariant under T.

- The restriction operator $T|_U \in \mathcal{L}(U)$ is defined by
$$T|_U(u) = Tu \quad \text{for } u \in U$$
- The quotient operator $T/U \in \mathcal{L}(V/U)$ is defined by
$$(T/U)(v+U) = Tv + U \quad \text{for } v \in V$$

✓ Makes sense
$v+U = w+U \Rightarrow v-w \in U$
$T(v-w) \in U$
$Tv - Tw \in U$
$\Rightarrow Tv + U = Tw + U$

$p(T)$

Suppose $T \in \mathcal{L}(V)$ and $p \in \mathcal{P}(\mathbb{F})$ is a polynomial given by
$$p(z) = a_0 + a_1z + \dots + a_mz^m \quad \text{for } z \in \mathbb{F}.$$
Then $p(T)$ is the operator defined by
$$p(T) = a_0I + a_1T + \dots + a_mT^m$$

Product of Polynomials

If $p,q \in \mathcal{P}(\mathbb{F})$, then $pq \in \mathcal{P}(\mathbb{F})$ is the polynomial defined by
$$(pq)(z) = p(z)q(z) \quad \text{for } z \in \mathbb{F}$$

Multiplicative Properties

Suppose $p,q \in \mathcal{P}(\mathbb{F})$, $T \in \mathcal{L}(V)$.
Then
(a) $(pq)(T) = p(T)q(T)$
(b) $p(T)q(T) = q(T)p(T)$

Operators on Complex Vector Spaces Have an Eigenvalue

Every operator on a finite-dimensional, non-zero, complex vector space has an eigenvalue.

Diagonal Matrix

A **diagonal matrix** is a square matrix that is 0 everywhere except possibly along the diagonal

Orthonormal Basis

An orthonormal basis of V is an orthonormal list of vectors that is also a basis of V.

Eigenspace - $E(\lambda,T)$

Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$. The eigenspace of T corresponding to λ, denoted $E(\lambda,T)$, is defined by

$$E(\lambda,T) = \ker(T-\lambda I)$$

In other words, $E(\lambda,T)$ is the set of all eigenvectors of T corresponding to λ, along w/ zero vector.

Sum of Eigenspaces is a Direct Sum

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Suppose also that $\lambda_1,\dots,\lambda_m$ are distinct eigenvalues of T. Then

$$E(\lambda_1,T) + \dots + E(\lambda_m,T)$$

is a direct sum. Furthermore,

$$\dim E(\lambda_1,T) + \dots + \dim E(\lambda_m,T) \leq V$$

Diagonalizable

An operator $T \in \mathcal{L}(V)$ is called **diagonalizable** if the operator has a diagonal matrix with respect to some basis of V.

Conditions Equivalent to Diagonalizability

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Let $\lambda_1,\dots,\lambda_m$ denote the distinct eigenvalues of T. Then the following are equivalent.

(a) T is diagonalizable
(b) V has a basis consisting of eigenvectors of T
(c) $\exists$ 1-dimensional subspaces $U_1,\dots,U_n$ of V, each invariant under T, such that
$$V = U_1 \oplus \dots \oplus U_n$$
(d) $V = E(\lambda_1,T) \oplus \dots \oplus E(\lambda_m,T)$
(e) $\dim V = \dim E(\lambda_1,T) + \dots + \dim E(\lambda_m,T)$

Enough Eigenvalues Implies Diagonalizability

If $T \in \mathcal{L}(V)$ has $\dim V$ distinct eigenvalues, then T is diagonalizable.

Inner Product

An inner product on V is a function that takes each ordered pair (u,v) of elements of V to a number $\langle u,v\rangle \in \mathbb{F}$ and has the following properties:

positivity
$$\langle v,v\rangle \geq 0 \quad \forall v \in V$$
definiteness
$$\langle v,v\rangle = 0 \text{ iff } v = 0$$
additivity in the second slot
$$\langle u+v,w\rangle = \langle u,w\rangle + \langle v,w\rangle \quad \forall u,v,w \in V$$
homogeneity in the first slot
$$\langle \lambda u,v\rangle = \lambda\langle u,v\rangle \quad \forall \lambda \in \mathbb{F},\ u,v \in V$$
conjugate symmetry
$$\langle u,v\rangle = \overline{\langle v,u\rangle} \quad \forall u,v \in V$$

Inner Product Space

An inner product space is a vector space V along with an inner product on V.

Basic Properties of Inner Product

(a) For each fixed $u \in V$, the function that takes v to $\langle v,u\rangle$ is a linear map from V to $\mathbb{F}$
(b) $\langle 0,u\rangle = 0 \quad \forall u \in V$
(c) $\langle u,0\rangle = 0 \quad \forall u \in V$
(d) $\langle u,v+w\rangle = \langle u,v\rangle + \langle u,w\rangle \quad \forall u,v,w \in V$
(e) $\langle u,\lambda v\rangle = \bar{\lambda}\langle u,v\rangle \quad \forall \lambda \in \mathbb{F}$ and $u,v \in \mathbb{F}$

Norm, $\|v\|$

For every $v \in V$, the **norm** of v, denoted $\|v\|$, is defined by

$$\|v\| = \sqrt{\langle v,v\rangle}$$

Basic Properties of Norm

Suppose $v \in V$.
(a) $\|v\| = 0$ iff $v = 0$
(b) $\|\lambda v\| = |\lambda|\|v\| \quad \forall \lambda \in \mathbb{F}$

Orthogonal

Two vectors u,v are called orthogonal if $\langle u,v\rangle = 0$.

Orthogonality and 0

- 0 is orthogonal to every vector in V
- 0 is the only vector in V orthogonal to itself

Pythagorean Theorem

u,v orthogonal vectors in V. Then
$$\|u+v\|^2 = \|u\|^2 + \|v\|^2$$

An Orthogonal Decomposition

$u,v \in V$, $v \neq 0$. Set
$$c = \frac{\langle u,v\rangle}{\|v\|^2}$$
and
$$w = u - \frac{\langle u,v\rangle}{\|v\|^2}v.$$
Then $\langle w,v\rangle = 0$ and $u = cv + w$

Cauchy-Schwarz Inequality

Suppose $u,v \in V$. Then
$$|\langle u,v\rangle| \leq \|u\|\|v\|$$
equality iff $u = cv$, $c \in \mathbb{F}$

Triangle Inequality

$u,v \in V$
$$\|u+v\| \leq \|u\| + \|v\|$$
equality iff $u = cv$, $c > 0$

Parallelogram Equality

Suppose $u,v \in V$. Then
$$\|u+v\|^2 + \|u-v\|^2 = 2(\|u\|^2 + \|v\|^2)$$

Orthonormal

A list of vectors is called orthonormal if each vector in the list has norm 1 and is orthogonal to all other vectors in the list.

The norm of an orthonormal linear combination

If $e_1,\dots,e_m$ is an orthonormal list of vectors in V, then

$$\|a_1e_1 + \dots + a_me_m\|^2 = |a_1|^2 + \dots + |a_m|^2$$

$\forall a_1,\dots,a_m \in \mathbb{F}$

An orthonormal list is linearly independent

Proof → $e_1,\dots,e_m$ orthonormal list of vectors in V
$a_1,\dots,a_m \in \mathbb{F}$ s.t.
$$a_1e_1 + \dots + a_ne_m = 0$$
then $|a_1|^2 + \dots + |a_m|^2 = 0 \Rightarrow$ all $a_j = 0$
∴ $e_1,\dots,e_m$ LI

$U \subset V$ a subspace if
(i) $0_V \in U$
(ii) $u_1+u_2 \in U \ \forall u_1,u_2 \in U$
(iii) $\lambda u_1 \in U$, $\lambda \in \mathbb{F}$

$T: V \to W$
$\dim V > \dim W \Rightarrow$ NOT inj.
$\dim V < \dim W \Rightarrow$ NOT surj.

injective iff $\ker = \{0\}$
surjective iff $W = \operatorname{range} T$

$\dim(\mathcal{L}(V,W)) = (\dim V)(\dim W)$

$U_1,U_2 \subset V \Rightarrow \dim(U_1+U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2)$

A real positive matrix is positive definite if for all $x \neq 0$ $x^TAx > 0$

Fundamental Theorem of Linear Maps
$T: V \to W$ $\dim V = \dim \operatorname{im}(T) + \dim \ker(T)$

An orthonormal list of the right length is an orthonormal basis

Writing a vector as a linear combination of orthonormal basis

Suppose $e_1,\dots,e_n$ is an orthonormal basis of V and $v\in V$. Then
$$v = \langle v,e_1\rangle e_1 + \cdots + \langle v,e_n\rangle e_n$$
and
$$\|v\|^2 = |\langle v,e_1\rangle|^2 + \cdots + |\langle v,e_n\rangle|^2$$

Gram-Schmidt Procedure

Suppose $v_1,\dots,v_m$ is LI list in V. Let $e_1 = \frac{v_1}{\|v_1\|}$. For $j=2,\dots,m$, define e_j inductively by
$$e_j = \frac{v_j - \langle v_j,e_1\rangle e_1 - \cdots - \langle v_j,e_{j-1}\rangle e_{j-1}}{\|v_j - \langle v_j,e_1\rangle e_1 - \cdots - \langle v_j,e_{j-1}\rangle e_{j-1}\|}$$
Then $e_1,\dots,e_m$ is an orthonormal list of vectors in V s.t.
$$\operatorname{span}(v_1,\dots,v_j) = \operatorname{span}(e_1,\dots,e_j)$$
for $j=1,\dots,m$.

Every finite-dimensional vector space has an orthonormal basis

Orthonormal list extends to orthonormal basis

Riesz Representation Theorem

Suppose V is finite dimensional and $\varphi\in\mathcal{L}(V)$. Then $\exists$ a unique vector $u\in V$ such that
$$\varphi(v) = \langle v,u\rangle \qquad \forall v\in V$$

Proof

Let $e_1,\dots,e_n$ be an orthonormal basis for V.

Then $\exists$
$$\varphi(v) = \varphi(\langle v,e_1\rangle e_1 + \cdots + \langle v,e_n\rangle e_n)$$
$$= \varphi(e_1)\langle v,e_1\rangle + \cdots + \varphi(e_n)\langle v,e_n\rangle$$
$$= \langle v, \overline{\varphi(e_1)}e_1\rangle + \cdots + \langle v, \overline{\varphi(e_n)}e_n\rangle$$
$$= \langle v, \underbrace{\overline{\varphi(e_1)}e_1 + \cdots + \overline{\varphi(e_n)}e_n}_{u}\rangle$$

! Suppose $u_1,u_2\in V$ such that
$$\varphi(v) = \langle v,u_1\rangle = \langle v,u_2\rangle \qquad \forall v\in V.$$
Then
$$0 = \langle v, u_1-u_2\rangle \to u_1-u_2 = 0 \to u_1 = u_2$$

Orthogonal Complement, $U^\perp$

If $U\subseteq V$, then the orthogonal complement of U, denoted $U^\perp$, is set of all vectors in V that are orthogonal to every vector in U
$$U^\perp = \{v\in V \mid \langle v,u\rangle = 0 \ \forall u\in U\}$$

Properties of Orthogonal Complement

(a) If $U\subseteq V$, then $U^\perp\subseteq V$
(b) $\{0\}^\perp = V$
(c) $V^\perp = \{0\}$
(d) If $U\subseteq V$, then $U\cap U^\perp \subset \{0\}$
(e) If U,W subsets of V and $U\subseteq W$, then $W^\perp\subseteq U^\perp$

Direct Sum of Subspace and Orthogonal Complement

Suppose U is a finite-dimensional subspace of V. Then
$$V = U\oplus U^\perp \Rightarrow \dim U^\perp = \dim V - \dim U$$
Proof $v = u + v - u$ $\quad u = \langle v,e_1\rangle e_1 + \cdots + \langle v,e_m\rangle e_m$

Orthogonal Complement of Orthogonal Complement

$U\subseteq V$, $\dim U = m < \infty$
$$U = (U^\perp)^\perp$$

Orthogonal Projection, P_U

Suppose U is a finite-dimensional subspace of V. The orthogonal projection of V onto U is the operator $P_U\in\mathcal{L}(V)$ defined as follows: For $v\in V$, write $v = u+w$, where $u\in U$ and $w\in U^\perp$. Then $P_U v = u$

Properties of the Orthogonal Projection, P_U

Suppose U is a finite-dimensional subspace of V and $v\in V$. Then
(a) $P_U\in\mathcal{L}(V)$ (c) $P_U w = 0 \ \forall w\in U^\perp$
(b) $P_U u = u \ \forall u\in U$ (d) range $P_U = U$
(e) null $P_U = U^\perp$ (g) $P_U^2 = P_U$
(f) $v - P_U v \in U^\perp$ (h) $\|P_U v\| \le \|v\|$
(i) For every orthonormal basis $e_1,\dots,e_m$ of U,
$$P_U v = \langle v,e_1\rangle e_1 + \cdots + \langle v,e_m\rangle e_m$$

Minimizing the Distance to a Subspace

Suppose U is finite-dimensional subspace of V, $v\in V$, $u\in U$. Then
$$\|v - P_U v\| \le \|v-u\|$$
Furthermore, the inequality above is an equality iff $u = P_U v$.

Proof
$$\|v - P_U v\|^2 \le \|v - P_U v\|^2 + \|P_U v - u\|^2$$
$$= \|v - P_U v + P_U v - u\|^2$$
$$= \|v-u\|^2$$

Adjoint

Suppose $T\in\mathcal{L}(V,W)$. The adjoint of T is the function $T^*: W\to V$ such that
$$\langle Tv,w\rangle = \langle v,T^*w\rangle$$
$\forall v\in V, w\in W$.

Example

Define $T:\mathbb{R}^3\to\mathbb{R}^2$ by $(x_1,x_2,x_3)\mapsto(x_2+3x_3, 2x_1)$.
Find a formula for $T^*:\mathbb{R}^2\to\mathbb{R}^3$.
Fix $(y_1,y_2)\in\mathbb{R}^2$. Then for every $(x_1,x_2,x_3)\in\mathbb{R}^3$ we have
$$\langle (x_1,x_2,x_3), T^*(y_1,y_2)\rangle = \langle T(x_1,x_2,x_3),(y_1,y_2)\rangle$$
$$= \langle (x_1,x_2,x_3),(2y_2,y_1,3y_1)\rangle$$
Thus $T^*(y_1,y_2) = (2y_2,y_1,3y_1)$

The Adjoint is a Linear Map

If $T\in\mathcal{L}(V,W)$, then $T^*\in\mathcal{L}(W,V)$.

Proof

Suppose $T\in\mathcal{L}(V,W)$, $w_1,w_2\in W$, $\lambda_1,\lambda_2\in\mathbb{R}$.
Then $\forall v\in V$,
$$\langle v, T^*(\lambda_1w_1+\lambda_2w_2)\rangle = \langle Tv, \lambda_1w_1+\lambda_2w_2\rangle$$
$$= \langle Tv,\lambda_1w_1\rangle + \langle Tv,\lambda_2w_2\rangle = \bar\lambda_1\langle Tv,w_1\rangle + \bar\lambda_2\langle Tv,w_2\rangle$$
$$= \bar\lambda_1\langle v,T^*w_1\rangle + \bar\lambda_2\langle v,T^*w_2\rangle = \langle v, \lambda_1T^*w_1 + \lambda_2T^*w_2\rangle$$

Properties of the Adjoint

(a) $(S+T)^* = S^*+T^*$ $\forall S,T\in\mathcal{L}(V,W)$
(b) $(\lambda T)^* = \bar\lambda T^*$ $\forall \lambda\in\mathbb{F}$, $T\in\mathcal{L}(V,W)$
(c) $(T^*)^* = T$ $\forall T\in\mathcal{L}(V,W)$
(d) $I^* = I$, I is the identity operator on V
(e) $(ST)^* = T^*S^*$ $\forall T\in\mathcal{L}(V,W)$ and $S\in\mathcal{L}(W,U)$
(here, U an inner product space over $\mathbb{F}$)

Proof (all similar)

(e) For $T\in\mathcal{L}(V,W)$, $S\in\mathcal{L}(W,U)$; if $v\in V$, $u\in U$
$$\langle v,(ST)^*u\rangle = \langle STv,u\rangle = \langle S(Tv),u\rangle = \langle Tv,S^*u\rangle = \langle v,T^*S^*u\rangle$$

Null Space and Range of T^*

Suppose $T\in\mathcal{L}(V,W)$. Then
(a) $\ker T^* = (\text{range } T)^\perp$
(b) $\text{range } T^* = (\ker T)^\perp$
(c) $\ker T = (\text{range } T^*)^\perp$
(d) $\text{range } T = (\ker T^*)^\perp$

Matrix of T^*

Suppose $\mathcal{A}$ is an orthonormal basis for V and β is an orthonormal basis for W. Then if $A = [T]_{\beta\leftarrow\mathcal{A}} \to [T^*]_{\mathcal{A}\leftarrow\beta} = A^T$

[illegible]

An operator $T\in\mathcal{L}(V)$ is called self-adjoint if $T=T^*$. In other words, $T\in\mathcal{L}(V)$ is self-adjoint if and only if
$$\langle Tv,w\rangle = \langle v,Tw\rangle \qquad \forall v,w\in\mathbb{F}$$

Eigenvalues of Self-Adjoint Operators are Real

Every eigenvalue of a self-adjoint operator is real.

Proof
$$\lambda\|v\|^2 = \langle\lambda v,v\rangle = \langle Tv,v\rangle = \langle v,Tv\rangle = \langle v,\lambda v\rangle = \bar\lambda\|v\|^2$$
$$\Rightarrow \lambda = \bar\lambda \Rightarrow \lambda\in\mathbb{R}$$

Self-adjoint Operators Have Eigenvalues

Suppose $V\neq\{0\}$ and $T\in\mathcal{L}(V)$ is a self-adjoint operator. Then T has an eigenvalue.

Self-adjoint Operators and Invariant Subspaces

Suppose $T\in\mathcal{L}(V)$ is self-adjoint and U is a subspace of V that is invariant under T. Then
(a) $U^\perp$ is invariant under T
(b) $T|_U$ is self-adjoint
(c) $T|_{U^\perp}$ is self-adjoint

Real Spectral Theorem

Suppose $\mathbb{F}=\mathbb{R}$ and $T\in\mathcal{L}(V)$. Then the following are equivalent:
(a) T is self-adjoint
(b) V has an orthonormal basis consisting of eigenvectors of T
(c) T has a diagonal matrix with respect to some orthonormal basis of V
($\exists$ orthogonal matrix Q, diagonal matrix Σ s.t. $[T] = Q\Sigma Q^T$)

Sequence of Increasing Null Spaces

Suppose $T\in\mathcal{L}(V)$. Then
$$\{0\} = \ker T^0 \subset \ker T^1 \subset \cdots \subset \ker T^k \subset \ker T^{k+1} \subset \cdots$$

Proof

Suppose $k\in\mathbb{Z}_{\ge1}$ and $v\in\ker T^k$. Then $T(T^kv) = T^{k+1}v = 0$
$\to v\in\ker T^{k+1} \to \ker T^k \subset \ker T^{k+1}$

Equality in the Sequence of Null Spaces

Suppose $T\in\mathcal{L}(V)$. Suppose m is a nonnegative integer such that $\ker T^m = \ker T^{m+1}$. Then
$$\ker T^m = \ker T^{m+1} = \ker T^{m+2} = \cdots$$

Proof

Let $k\in\mathbb{Z}_{\ge1}$. Want
$$\ker T^{m+k} = \ker T^{m+k+1}$$
Know $\ker T^{m+k} \subset \ker T^{m+k+1}$
Need $\ker T^{m+k+1} \subset \ker T^{m+k}$
Suppose $v\in\ker T^{m+k+1}$.
$$T^{m+1}(T^kv) = T^{m+k+1}v = 0$$
$\to T^kv \in \ker T^{m+1} = \ker T^m$
$\Rightarrow v\in\ker T^{m+k}$
So $\ker T^{m+k+1} \subset \ker T^{m+k}$ as desired.

Null Spaces Stop Growing

Suppose $T\in\mathcal{L}(V)$. Let $n=\dim V$. Then
$$\ker T^n = \ker T^{n+1} = \ker T^{n+2} = \ker T^{n+3} = \cdots$$

Proof

Only need $\ker T^n = \ker T^{n+1}$.
Suppose $\ker T^n \neq \ker T^{n+1}$
Then $\{0\} = \ker T^0 \subsetneq \ker T^1 \subsetneq \cdots \subsetneq \ker T^n \subsetneq \ker T^{n+1}$
At each strict inclusion, dim ker increases by at least one. This gives $\dim\ker T^{n+1} \ge n+1$. !

V is the Direct Sum of $\ker T^{\dim V}$ and $\operatorname{im} T^{\dim V}$

Suppose $T\in\mathcal{L}(V)$. Let $n=\dim V$. Then
$$V = \ker T^n \oplus \operatorname{im} T^n.$$

Proof

First show $\ker T^n \cap \operatorname{im} T^n = \{0\}$
Suppose $v\in\ker T^n\cap\operatorname{im}T^n$
then $T^nv = 0$ and $\exists u\in V$ s.t. $T^nu = v$
Applying T^n, $T^{2n}u = T^nv$
$T^{2n} = T^n \to T^{2n}v = 0 \to T^nu = 0 \to v = 0$

Generalized Eigenvector

Suppose $T\in\mathcal{L}(V)$ and λ an eigenvalue of T. A vector $v\in V$ is called a generalized eigenvector of T corresponding to λ if $v\neq0$ and
$$(T-\lambda I)^jv = 0$$
for some positive integer j.

Generalized Eigenspace

Suppose $T\in\mathcal{L}(V)$ and $\lambda\in\mathbb{F}$. The generalized eigenspace of T corresponding to λ, denoted $G(\lambda,T)$, is defined to be the set of all generalized eigenvectors of T corresponding to λ, along with the zero vector.

Note: $E(\lambda,T)\subset G(\lambda,T)$

Description of Generalized Eigenspaces

Suppose $T\in\mathcal{L}(V)$ and $\lambda\in\mathbb{F}$. Then $G(\lambda,T) = \ker(T-\lambda I)^{\dim V}$

Proof

Suppose $v\in\ker(T-\lambda I)^{\dim V}$. Then $v\in G(\lambda,T) \to \ker(T-\lambda I)^{\dim V}\subset G(\lambda,T)$
Conversely, suppose $v\in G(\lambda,T)$. Then $\exists j\in\mathbb{Z}_+$ s.t.
$$v\in\ker(T-\lambda I)^j \subset \ker(T-\lambda I)^{\dim V}$$
$$\Rightarrow G(\lambda,T)\subset\ker(T-\lambda I)^{\dim V}$$

Nilpotent

An operator is called **nilpotent** if some power of it equals 0.

Nilpotent Operator Raised to Dimension of Domain is 0

Suppose $N\in\mathcal{L}(V)$ is nilpotent. Then $N^{\dim V} = 0$.

Proof

Because N is nilpotent, $G(0,N) = V$.
Then $\ker N^{\dim V} = V$

Matrix of a Nilpotent Operator

Suppose N is a nilpotent operator on V. Then there is a basis of V with respect to which the matrix N has the form
$$\begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix}$$

The Null Space and Range of $p(T)$ are Invariant Under T

Suppose $T\in\mathcal{L}(V)$ and $p\in\mathcal{P}(\mathbb{F})$. Then $\ker p(T)$ and range $p(T)$ are invariant under T.

Description of Operators on Complex Vector Spaces

Suppose V is a complex vector space and $T\in\mathcal{L}(V)$. Let $\lambda_1,\dots,\lambda_m$ be the distinct eigenvalues of T. Then
(a) $V = G(\lambda_1,T)\oplus\cdots\oplus G(\lambda_m,T)$
(b) each $G(\lambda_j,T)$ is invariant under T
(c) each $(T-\lambda_jI)|_{G(\lambda_j,T)}$ is nilpotent

A Basis of Generalized Eigenvectors

Suppose V is a complex vector space and $T\in\mathcal{L}(V)$. Then there is a basis of V consisting of generalized eigenvectors of T.

Multiplicity

Suppose $T\in\mathcal{L}(V)$. The multiplicity of an eigenvalue λ of T is defined to be the dimension of the corresponding generalized eigenspace $G(\lambda,T)$.

Sum of Multiplicities Equals dim V

Suppose V is a complex vector space and $T\in\mathcal{L}(V)$. Then the sum of all the eigenvalues of T equals $\dim V$.
Note: Geometric Multiplicity $= \dim E(\lambda,T)$
Algebraic Multiplicity $= \dim G(\lambda,T)$

Characteristic Polynomial

Suppose V is a complex vector space and $T\in\mathcal{L}(V)$. Let $\lambda_1,\dots,\lambda_m$ denote the distinct eigenvalues of T, with multiplicities $d_1,\dots,d_m$. The polynomial
$$(z-\lambda_1)^{d_1}\cdots(z-\lambda_m)^{d_m}$$
is called the characteristic polynomial of T.

Degrees and Zeros of Characteristic Polynomial

Suppose V is a complex vector space and $T\in\mathcal{L}(V)$. Then
(a) the characteristic polynomial of T has degree $\dim V$
(b) the zeros of the characteristic polynomial of T are the eigenvalues of T

Cayley Hamilton Theorem

Suppose V is a complex vector space and $T\in\mathcal{L}(V)$. Let q denote the characteristic polynomial of T. Then
$$q(T) = 0$$

Proof

Let $\lambda_1,\dots,\lambda_m$ be the distinct eigenvalues of T with $d_1,\dots,d_m$ algebraic multiplicity each.
For each $j\in\{1,\dots,m\}$ we have $(T-\lambda_jI)|_{G(\lambda_j,T)}$ is nilpotent.
Thus $(T-\lambda_jI)^{d_j}|_{G(\lambda_j,T)} = 0$
Every $v\in V$ can be written as
$$v = g_1+\cdots+g_m \ ; \ g_j\in G(\lambda_j,T)$$
Suffices to show
$$q(T)|_{G(\lambda_j,T)} = 0 \quad \forall j$$
Fix $j\in\{1,\dots,m\}$. Then
$$q(T) = (T-\lambda_1I)^{d_1}\cdots(T-\lambda_mI)^{d_m}$$
All operators on left-hand side commute.
Thus can rearrange to
$$q(T) = (T-\lambda_1I)\cdots(T-\lambda_mI)^{d_m}(T-\lambda_jI)^{d_j}$$
Since
$$(T-\lambda_jI)|_{G(\lambda_j,T)} = 0 \quad \forall j$$
then
$$q(T)|_{G(\lambda_j,T)} = 0 \quad \forall j$$

Monic Polynomial

A monic polynomial is a polynomial whose highest-degree coefficient equals 1.

Minimal Polynomial

Suppose $T\in\mathcal{L}(V)$. Then the minimal polynomial of T is the unique monic polynomial p of smallest degree such that $p(T)=0$.
Note: has at MOST degree $\dim V$

Characteristic Polynomial is a Multiple of Minimal Polynomial

Eigenvalues are the Zeros of the Minimal Polynomial

Let $T\in\mathcal{L}(V)$. Then the zeros of the minimal polynomial of T are precisely the eigenvalues of T.

Proof

Let
$$p(z) = a_0 + a_1z + \cdots + a_{m-1}z^{m-1} + z^m$$
be the minimal polynomial of T.
Suppose λ is a zero of p. Then
$$p(z) = (z-\lambda)q(z)$$
where q is a monic polynomial with coefficients in $\mathbb{F}$ and $\deg q = \deg p - 1$. Since $p(T)=0$, we have
$$0 = (T-\lambda I)(q(T)v)$$
Then $\exists v\in V$ such that $q(T)v\neq0$, and
$$T(q(T)v) = \lambda(q(T)v)$$
This gives λ is an eigenvalue of T.
Suppose now $\lambda\in\mathbb{F}$ is an eigenvalue of T.
Then $\exists v\neq0\in V$ such that
$$Tv = \lambda v.$$
Applying T to both sides repeatedly gives
$$T^jv = \lambda^jv \qquad j = 1,2,\dots$$
Thus
$$0 = p(T)v = (a_0I + a_1T + \cdots + a_{m-1}T^{m-1} + T^m)v$$
$$= (a_0 + a_1\lambda + \cdots + a_{m-1}\lambda^{m-1} + \lambda^m)v$$
$$= p(\lambda)v \Rightarrow p(\lambda) = 0$$
Q.E.D

SVD

A $m\times n$ matrix over $\mathbb{R}$, rank r
A^TA $n\times n$ symmetric, rank r.
Eigenvalues $\lambda_1\ge\lambda_2\ge\cdots\lambda_r > \lambda_{r+1} = 0 = \cdots = \lambda_n$
Singular values of A:
① $\sigma_1 = \sqrt{\lambda_1} \ge \sigma_2 = \sqrt{\lambda_2} \ge \cdots \ge \sigma_r = \sqrt{\lambda_r}$
② Let $\vec v_1,\dots,\vec v_n$ be an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of A^TA.
So
$$A^TA\vec v_i = \lambda_i\vec v_i, \quad i = 1,\dots,n$$
③ Let $\vec u_i = \frac{1}{\sigma_i}A\vec v_i$, $\quad i = 1,\dots,r$ NOTE (note $\vec u_i\in\mathbb{R}^m$)

These are orthonormal, and extend to orthonormal basis $\vec u_1,\dots,\vec u_m$ of $\mathbb{R}^m$.

An SVD of A is a factorization
$$A = U\Sigma V^T$$
such that ① U,V are orthogonal matrices
② $\Sigma = \begin{pmatrix}\sigma_1 & & & & \\ & \sigma_2 & & & \\ & & \ddots & & \\ & & & \sigma_r & \\ & & & & 0 \ddots 0\end{pmatrix}$,
$$\sigma_1\ge\sigma_2\ge\cdots\ge\sigma_r>0$$

Jordan Stuff

$N: J(0;k_1,\dots,k_r) = J(0,k_1)\oplus\cdots\oplus J(0,k_r)$
$k_1\ge k_2\ge\cdots\ge k_r>0$

$\dim\ker N = r$, $\dim\operatorname{im} N = \sum k_i - r$
index of Nilpotence, k_1
For each Jordan block $J(0,k_i)$,
$$\operatorname{rank}(J(0,k_i)^m) = \max\{0,k_i-m\}$$
Thus $\operatorname{rank} N^m = \sum_{i=1}^r \max\{0,k_i-m\}$
and
$$\dim\ker N = n - \operatorname{rank} N^m$$
Knowing $\dim\ker N^m$ $\forall m$ determines $(k_1,\dots,k_r)$
Assume $d_m = \dim\ker N^m$.
Then $\Delta_m = d_m - d_{m-1}$ is #$J(0,k)$ w/ $k\ge m$
So
$$\Delta_m - \Delta_{m+1} = 2d_m - d_{m+1} - d_{m-1}$$
is number of $J(0,k)$ in N with $k=m$.

definition: Let $\lambda\in\mathbb{C}$, $n\in\mathbb{Z}_+$.
Define
$$J(\lambda,n) = \begin{pmatrix}\lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda\end{pmatrix} \quad (n\times n)$$
$J(\lambda,n)$ is called a Jordan Block.
$$J(\lambda,4) = \begin{pmatrix}\lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda\end{pmatrix}$$

Note: if $A = J(\lambda,n)$, then λ is the only eigenvalue of A, and
$$V = \mathbb{C}^n = \ker(A-\lambda I)^n$$
so $N = A-\lambda I$ is nilpotent

Theorem: If $\dim V = n < \infty$, V a v.s over $\mathbb{C}$, and $T\in\mathcal{L}(V)$.
Then $\exists$ Jordan basis of T.

Corollary: Every $A\in\mathbb{C}^{n\times n}$ is similar to a direct sum of Jordan blocks.

$A = \begin{bmatrix}A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_r\end{bmatrix}$ A_i $n_i\times n_i$, A $n\times n$, $n = n_1+\cdots+n_r$

Remark: If $A = A_1\oplus\cdots\oplus A_r$ and π is a permutation of $1,\dots,r$
then $B = A_{\pi(1)}\oplus A_{\pi(2)}\oplus\cdots\oplus A_{\pi(r)}$ is similar to A.

Definition: $(k_1,\dots,k_r)$ is called a partition of $n\in\mathbb{Z}_{\ge0}$ if
ⓐ $k_1+\cdots+k_r = n$
ⓑ $k_1\ge k_2\ge\cdots\ge k_r>0$

Definition: If $\underline{k} = (k_1,\dots,k_r)$ is a partition of n, let $J(\lambda,\underline{k}) = J(\lambda,k_1)\oplus\cdots\oplus J(\lambda,k_r)$

Example: Suppose A is 8×8 with eigenvalues

Always $\leq$

$\lambda_1 = 7$, $\dim E_{\lambda_1} = 1$, $\dim G_{\lambda_1} = 3$

$\lambda_2 = 15$, $\dim E_{\lambda_2} = 1$, $\dim G_{\lambda_2} = 1$

$\lambda_3 = -1$, $\dim E_{\lambda_3} = 3$, $\dim G_{\lambda_3} = 4$

Find all possible JCF's of A

<u>Sol'n</u>

3x3 only has eigenvalue $\lambda = \lambda_1 = 7$

$$A \sim A_1 \oplus \overset{1\times 1}{A_2} \oplus \overset{4\times 4}{A_3}$$

where

$$A \sim \overset{A_1}{J(\lambda, \underline{k}^{(1)})} \oplus \overset{A_2}{J(\lambda, \underline{k}^{(2)})} \oplus \overset{A_3}{J(\lambda, k^{(3)})}$$

$\underline{k}^{(1)}$ is a partition of 3 $\to$ 111, 21, 3

$\underline{k}^{(2)}$ is a partition of 1 $\to$ 1

$\underline{k}^{(3)}$ is a partition of 4 $\to$ 1111, 211, 22, 31, 4

NOTE: 4, 2, 2, 3, 1 $\} \dim E_{\lambda_3}$

minimal polynomials is

$$m_T(x) = (x-7)(x-15)(x+1)^3$$