1. Suppose that 13% of Cornell engineering students like deadmau5 and 37% like Miley Cyrus.

(a) Find the lowest and highest possible probabilities that a Cornell engineering student selected at random likes either deadmau5 or Miley Cyrus or both.
(b) Find the lowest and highest possible probabilities that a Cornell engineering student selected at random likes both deadmau5 and Miley Cyrus.
(c) In (a) and (b), do you expect the actual probability of the indicated event to be closer to its minimum possible value or to its maximum possible value? Why?

2. Let events A and B have nonzero probabilities, and suppose that $\mathbb{P}(A \mid B) > \mathbb{P}(A)$, i.e. that occurrence of B increases the likelihood of the occurrence of A. Answer each of the following questions and justify your answers rigorously.

(a) Does occurrence of A increase the likelihood of the occurrence of B?
(b) Is it possible for A and B to be independent?
(c) Is it possible for A and B to be disjoint?

3. You roll a fair six-sided die twice with faces numbered 1 through 6, where the rolls are independent of each other. Define events

- $A = \{$ sum of the two rolls is 12 $\}$
- $B = \{$ at least one roll was a 6 $\}$
- $C = \{$ at least one roll was a 2 $\}$

(a) Is event A independent of event B? Justify your answer.
(b) Is event A independent of event C? Justify your answer.

4. Galadriel has two six-sided dice, one with the standard faces numbered 1 through 6 and one with three faces numbered 3 and three numbered 6. Assume the dice are fair in the sense that on a single roll of either die each of the six physical faces is equally likely to come up. Galadriel selects one of the dice at random, with probability $p \in (0, 1)$ that she selects the standard die. Assume that all subsequent rolls of the selected die are mutually independent.

(a) She rolls the die once and a 3 comes up. What is the probability that the die is the standard one?
(b) She rolls the die again and this time a 6 comes up. Now what is the probability that the die is the standard one?
(c) She rolls the die a third time and a 5 comes up. Now what is the probability that the die is the standard one?

5. We transmit a bit pair through a noisy channel depicted schematically in the accompanying diagram, which is supposed to represent the following model for the channel: the probability of transmission error given any transmitted bit pair is ϵ, and when an error occurs it's always of the form 00 sent and 01 received, 01 sent and 10 received, 10 sent and 11 received, or 11 sent and 00 received. Assume that the input is 00 with probability 3/8, 01 with probability 1/8, 10 with probability 1/3, and 11 with probability 1/6.

(a) For each bit pair find the probability that it is received at the channel output.
(b) Given we receive 00 at the output, what is the probability that the transmitted pair was 00? Or 01? Or 10? Or 11?

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ECE 3100 HW2

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① Given

- 13% of Cornell engineering students like deadmau5
- 37% of Cornell engineering students like Miley Cyrus

(a) Find the lowest **AND** highest probabilities that a Cornell engineering student selected at random likes either deadmau5 <u>or</u> Miley Cyrus <u>or</u> both.

Let

Ω be the population of all Cornell Engineering students
A be the event a student likes deadmau5
B be the event a student likes Miley Cyrus

Then,

$$\mathbb{P}(A) = 0.13 \Rightarrow \mathbb{P}(A^c) = 0.87$$
$$\mathbb{P}(B) = 0.37 \Rightarrow \mathbb{P}(B^c) = 0.63$$

The probability a Cornell engineering student selected at random likes either deadmau5 <u>or</u> Miley Cyrus <u>or</u> both is the same as the event a Cornell engineering student likes **<u>AT LEAST</u>** one artist.

i.e

$$A \cup B \cup (A \cap B)$$

Lowest Probability occurs when the student likes both!

$$\min\{\mathbb{P}(A \cap B)\} = \operatorname{Max}\{\mathbb{P}(A), \mathbb{P}(B)\}$$

$$\boxed{=0.37}$$

Highest Probability occurs when the events are disjoint!

$$\mathbb{P}(A \cup B \cup (A \cap B)) = 0.13 + 0.37 + 0 = \boxed{0.50}$$

(b) Find the lowest and highest probabilities that a Cornell engineering student <u>likes both</u> artists.

Let

Ω be the population of all Cornell Engineering students

A be the event a student likes deadmau5

B be the event a student likes Miley Cyrus

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

The <u>highest</u> probability occurs when $\mathbb{P}(A \cup B)$ is <u>minimized</u>.

The <u>lowest</u> probability occurs when $\mathbb{P}(A \cup B)$ is <u>maximized</u>.

Right off the bat, we know the maximum possible value for $\mathbb{P}(A \cap B) = \min(\mathbb{P}(A), \mathbb{P}(B))$.
Thus the highest possible probability is

$$\mathbb{P}(A \cap B) = 0.13 \quad \left.\right\} \text{Proof not shown, but occurs when } A \subset B$$

The minimum value occurs when $\mathbb{P}(A \cup B)$ is maximized.
This is when $A \cap B = \emptyset$
Thus the lowest possible probability is 0.0

(c) I would expect them to be closer to the maximum value because to be close to the minimum value would require the events to be completely disjoint which isn't very likely.

② Let A and B be events with nonzero probabilities, and suppose $\mathbb{P}(A|B) > \mathbb{P}(A)$

i.e that occurence of B increases the likelihood of occurrence of A.

(a) Does occurrence of A increase the likelihood of occurrence of B?

Know,

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} > \mathbb{P}(A)$$

Also know A, B are **NOT** independent

Want to know if,

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} > \mathbb{P}(B)$$

From the information given, occurrence of A **DOES** increase the likelihood of occurrence of B.

PROOF:

Given $\frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} > \mathbb{P}(B)$ we know that

$\mathbb{P}(A \cap B) > \mathbb{P}(B)\mathbb{P}(A) > 0 \Rightarrow$ NOT disjoint events

$\mathbb{P}(B|A)$ is then $\frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} > \frac{\mathbb{P}(B)\mathbb{P}(A)}{\mathbb{P}(A)} > \mathbb{P}(B)$

Q.E.D

(b) Is it possible for A, B to be independent?

No!

$\mathbb{P}(A \cap B)$ is strictly greater than $\mathbb{P}(A)\mathbb{P}(B)$. If it were equal to (implying independence), we would have to have

$$\mathbb{P}(A \mid B) = \mathbb{P}(A)$$

but we don't, so it must be they are independent.

$\boxtimes$

(c) Is it possible for A and B to be disjoint?

No!

$$\mathbb{P}(A \cap B) = 0 \quad \text{for} \quad A \cap B = \emptyset$$

Given $\mathbb{P}(A \mid B) > \mathbb{P}(A)$ this is NOT possible!
This was also realized in part (a).

$\boxtimes$

③ Roll a fair six-sided die twice with faces numbered 1 through 6, where the rolls are independent of eachother.

Define events

$A = \{\text{sum of the two rolls is } 12\}$

$B = \{\text{at least one roll was } 6\}$

$C = \{\text{at least one roll was } 2\}$

(a) Is event A independent of B?

$A = \{66\}$

$B = \{61, 62, 63, 64, 65, 66, 56, 46, 36, 26, 16\}$

$\mathbb{P}(A) = \frac{1}{36}$

$\mathbb{P}(B) = \frac{11}{36}$

$\mathbb{P}(A \cap B) = \frac{1}{36} \neq \mathbb{P}(A)\,\mathbb{P}(B)$

So A is **NOT** independent of B

(b) Is event A independent of C?

$\mathbb{P}(A) = \frac{1}{36}$

$\mathbb{P}(A \cap C) = 0 \neq \mathbb{P}(A)\mathbb{P}(B)$

$\mathbb{P}(C) = \frac{11}{36}$

So A is NOT independent of C.

4. Galadriel has two six-sided dice, one with the standard faces numbered 1 through 6 and one with three faces numbered 3 and three numbered 6. Assume the dice are fair in the sense that on a single roll of either die each of the six physical faces is equally likely to come up. Galadriel selects one of the dice at random, with probability $p \in (0, 1)$ that she selects the standard die. Assume that all subsequent rolls of the selected die are mutually independent.

(a) She rolls the die and a 3 comes up. What is the probability the die is the standard one?

Define events

A = rolling a 3

B = choosing standard die

C = choosing non standard die

$\mathbb{P}(B) = p$, $\mathbb{P}(C) = 1-p$, $\mathbb{P}(A \mid B) = \frac{1}{6}$, $\mathbb{P}(A \mid C) = \frac{1}{2}$

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A|B)\mathbb{P}(B)}{\mathbb{P}(A|B)\mathbb{P}(B) + \mathbb{P}(A|C)\mathbb{P}(C)} = \frac{\frac{1}{6}p}{\frac{1}{6}p + \frac{1}{2} - \frac{1}{2}p} = \frac{\frac{1}{6}p}{\frac{1}{2} - \frac{2}{6}p} = \frac{p}{3-2p}$$

(b) rolls die again, 6 comes up. $\mathbb{P}(B)$?

Let D = event a 6 is rolled

$\mathbb{P}(D|B) = \frac{1}{6}$, $\mathbb{P}(D|C) = \frac{1}{2}$

$A, D \rightarrow$ independent events!

$$\mathbb{P}(B|(A\cap D)) = \frac{\mathbb{P}(A\cap D|B)\mathbb{P}(B)}{\mathbb{P}(A\cap D|B)\mathbb{P}(B) + \mathbb{P}(A\cap D|C)\mathbb{P}(C)}$$

$$= \frac{\frac{1}{6}\cdot\frac{1}{6}\cdot p}{\frac{1}{6}\cdot\frac{1}{6}p + \frac{1}{2}\cdot\frac{1}{2}(1-p)} = \frac{\frac{1}{36}p}{\frac{1}{36}p + \frac{1}{4} - \frac{1}{4}p}$$

$$= \frac{\frac{1}{36}p}{\frac{1}{4} - \frac{8}{36}p} = \boxed{\frac{p}{9-8p}}$$

(c) Rolls the die a third time and a 5 comes up. Probability the die is standard?

$E = \{6, 3, 5\}$ ← rolls 1,2,3

$$\mathbb{P}(B|C) = \frac{\mathbb{P}(E|B)\mathbb{P}(B)}{\mathbb{P}(E|B)\mathbb{P}(B) + \mathbb{P}(E|C)\mathbb{P}(C)} = \frac{(1/6)^3 p}{(1/6)^3 p + 0} = 1$$

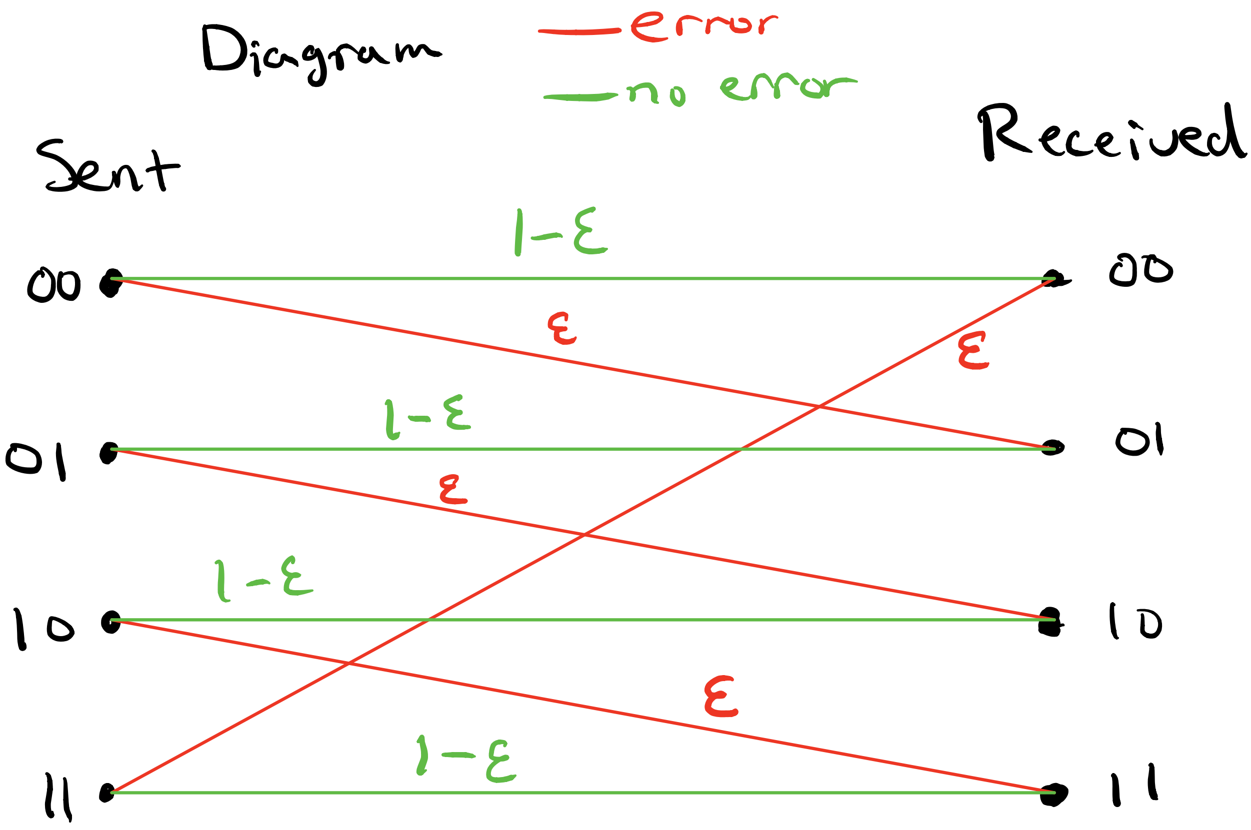
(5) Transmit a bit through a noisy channel. The probability of transmission error any given transmitted pair is ε, and when an error occurs its always of the form

00 sent, 01 received
01 sent, 10 received,
10 sent, 11 received
11 sent, 00 received.

Assume that input is

00, w/ $\mathbb{P}(\{00\}) = 3/8$
01, w/ $\mathbb{P}(\{01\}) = 1/8$
10, w/ $\mathbb{P}(\{10\}) = 1/3$
11, w/ $\mathbb{P}(\{11\}) - 1/6$

Diagram



(a) For each bit pair find the probability that it is received at the channel output!

*A bit is received when the input bit is successfully passed through **OR** there was an error as depicted above.

00 Received

$\mathbb{P}(00 \text{ Received}) = \mathbb{P}(00 \text{ Transmitted} \cap 00 \text{ Received}) + \mathbb{P}(11 \text{ transmitted} \cap 00 \text{ received})$

Claim: XX Transmitted and YY Received are independent events! This is because the probability of an error occurring or not is not dependent on what is sent!

So,

$$\mathbb{P}(00 \text{ Received}) = \mathbb{P}(00 \text{ Transmitted})\,\mathbb{P}(00 \text{ Received}) + \mathbb{P}(11 \text{ transmitted})\,\mathbb{P}(00 \text{ received})$$

$$= \frac{3}{8}(1-\varepsilon) + \frac{1}{6}\varepsilon$$

$$= \frac{3}{8} - \left(\frac{3}{8} - \frac{1}{6}\right)\varepsilon$$

$$\boxed{= \frac{3}{8} - \frac{5}{24}\varepsilon}$$

Work will not be shown for other 3 scenarios.

$\mathbb{P}(01 \text{ received}) = \frac{1}{8} + \frac{1}{4}\varepsilon$

$\mathbb{P}(10 \text{ received}) = \frac{1}{3} - \frac{5}{24}\varepsilon$

$\mathbb{P}(11 \text{ received}) = \frac{1}{6} + \frac{1}{6}\varepsilon$

(b) Given we receive 00 at output, what is the probability that the transmitted pair was

(i) 00 (ii) 01 (iii) 10 (iv) 11

(i) $\mathbb{P}(00 \text{ transmitted} \mid 00 \text{ received}) = \dfrac{\mathbb{P}(00 \text{ Sent} \cap 00 \text{ received})}{\mathbb{P}(00 \text{ received})}$

$$= \frac{\mathbb{P}(00 \text{ sent})\mathbb{P}(\text{No Error})}{\mathbb{P}(00 \text{ received})} = \boxed{\frac{\frac{3}{8}(1-\varepsilon)}{\frac{3}{8} - \frac{5}{24}\varepsilon}}$$

(ii) $\mathbb{P}(01 \text{ transmitted} \mid 00 \text{ received}) = 0$
(iii) $\mathbb{P}(10 \text{ transmitted} \mid 00 \text{ received}) = 0$
$\Big\}$ Just think a little

(iv) $\mathbb{P}(11 \text{ transmitted} \mid 00 \text{ received}) = \dfrac{\mathbb{P}(11 \text{ Sent} \cap 00 \text{ received})}{\mathbb{P}(00 \text{ received})}$

$$= \boxed{\frac{\frac{1}{6}\varepsilon}{\frac{3}{8} - \frac{5}{24}\varepsilon}}$$