① Let $X(t)$ and $Y(t)$ be independent wide-sense stationary random processes, and define

$$Z(t) = X(t)Y(t)$$

(a) Show that $Z(t)$ is wide-sense stationary.

$$\begin{aligned} \mathbb{E}[Z(t)] &= \mathbb{E}[X(t)Y(t)] \\ &= \mathbb{E}[X(t)]\,\mathbb{E}[Y(t)] \\ &= \mu_X \mu_Y \qquad (X, Y \text{ WSS}) \end{aligned}$$

$$\begin{aligned} R_Z(t_1, t_2) &\triangleq \mathbb{E}[Z(t_1)Z(t_2)] \\ &= \mathbb{E}[X(t_1)X(t_2)Y(t_1)Y(t_2)] \\ &= R_X(t_1 - t_2)\,R_Y(t_1 - t_2) \\ &= R_X(\tau)\,R_Y(\tau) \end{aligned}$$

$\Rightarrow Z(t)$ WSS

(b) Find $R_Z(\tau)$ and $S_Z(f)$ in terms of $R_X(\tau)$, $R_Y(\tau)$, and $S_X(f)$, $S_Y(f)$.

$$R_Z(\tau) = R_X(\tau)R_Y(\tau)$$

$$S_Z(f) = \mathcal{F}\{R_Z(t)\} = \mathcal{F}\{R_X(\tau)R_Y(\tau)\} = S_X(f) * S_Y(f)$$

② Let $\{X_n\}_{n=-\infty}^{+\infty}$ be a stationary Gaussian process with zero mean and autocorrelation function

$$R_X(k) = \begin{cases} 1, & \text{if } k=0 \\ 1/2, & \text{if } |k|=1 \\ 0, & \text{if } |k|>1 \end{cases}$$

(a) Compute the power spectral density of $\{X_n\}$.

$$S_X(f) \triangleq \sum_{k=-\infty}^{+\infty} R_X(k)\, e^{-j2\pi f k} \qquad \left(-\tfrac{1}{2} < f \leq \tfrac{1}{2}\right)$$

$$= 1 + \tfrac{1}{2} e^{-j2\pi f} + \tfrac{1}{2} e^{j2\pi f}$$

$$= 1 + \cos(2\pi f)$$

(b) Compute $\mathbb{E}[X_{n+1} \mid X_n, X_{n-1}]$

Let $\underline{X} = \underline{X_{n+1}}$, $\underline{Y} = \begin{bmatrix} X_n \\ X_{n-1} \end{bmatrix}$

$$\mathbb{E}[\underline{X} \mid \underline{Y}] = \mathbb{E}[\underline{X}] + \mathrm{Cov}(\underline{X}, \underline{Y})\, \mathrm{Cov}^{-1}(\underline{Y})\, (\underline{Y} - \mathbb{E}[\underline{Y}])$$

$$\mathrm{Cov}(\underline{Y}) = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$$

$\mathbb{E}[\underline{X}] = [0]$

$\mathbb{E}[\underline{Y}] = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\mathrm{Cov}^{-1}(\underline{Y}) = \frac{4}{3}\begin{pmatrix} 1 & -1/2 \\ -1/2 & 1 \end{pmatrix}$$

$$\mathrm{Cov}(\underline{X}, \underline{Y}) = \begin{pmatrix} 1/2 & 0 \end{pmatrix}$$

$$\rightarrow \mathbb{E}[\underline{X} \mid \underline{Y}] = \begin{pmatrix} 1/2 & 0 \end{pmatrix} \begin{pmatrix} 4/3 & -2/3 \\ -2/3 & 4/3 \end{pmatrix} \begin{pmatrix} X_n \\ X_{n-1} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} X_n \\ X_{n-1} \end{pmatrix} = \frac{2}{3} X_n - \frac{1}{3} X_{n-1}$$

(c) Find the impulse response of a discrete-time LTI filter with the property that if the input to the filter is $\{X_n\}_{n=-\infty}^{+\infty}$, then the output Y_n at time n is $\mathbb{E}[X_{n+1} \mid X_n, X_{n-1}]$.

$$Y_n = \frac{2}{3} X_n - \frac{1}{3} X_{n-1} \text{ desired}$$

$$\xrightarrow{X_n} \boxed{h_n} \longrightarrow \frac{2}{3} X_n - \frac{1}{3} X_{n-1} \qquad Y_n = \sum_{l=-\infty}^{+\infty} h_n X_{n-l}$$

$$h_n = \frac{2}{3} \delta_n - \frac{1}{3} \delta_{n-1}$$

(d) Compute the transfer function of the filter found in (c).

$$H_n = \mathcal{F}\left\{ \frac{2}{3} \delta_n - \frac{1}{3} \delta_{n-1} \right\} = \frac{2}{3} - \frac{1}{3} e^{-j2\pi f}$$

(e) Consider the process

$$Z_n = X_{n+1} - Y_n.$$

Show that this process is wide-sense stationary and determine its autocorrelation. What is the mean square error?

$$Z_n = X_{n+1} - \frac{2}{3}X_n + \frac{1}{3}X_{n-1}$$

$$\mathbb{E}[Z_n] = \mathbb{E}[X_{n+1}] - \frac{2}{3}\mathbb{E}[X_n] + \frac{1}{3}\mathbb{E}[X_n] = 0 - 0 + 0 = 0$$

$$R_Z(n_1, n_2) \triangleq \mathbb{E}[Z_{n_1} Z_{n_2}]$$

$$= \mathbb{E}\left[\left(X_{n_1+1} - \frac{2}{3}X_{n_1} + \frac{1}{3}X_{n_1-1}\right)\left(X_{n_2+1} - \frac{2}{3}X_{n_2} + \frac{1}{3}X_{n_2-1}\right)\right]$$

$$= R_X(n_2-n_1) - \frac{2}{3}R_X(n_2-n_1-1) + \frac{1}{3}R_X(n_2-n_1-2) - \frac{2}{3}R_X(n_2-n_1+1) + \frac{1}{3}R_X(n_2-n_1+2)$$
$$+\frac{4}{9}R_X(n_2-n_1) - \frac{2}{9}R_X(n_2-n_1-1) \qquad - \frac{2}{9}R_X(n_2-n_1+1)$$
$$+\frac{1}{9}R_X(n_2-n_1)$$

Let $k = n_2 - n_1$

$$= \frac{14}{9}R_X(k) - \frac{8}{9}\left(R_X(k-1) + R_X(k+1)\right) + \frac{1}{3}\left(R_X(k-2) + R_X(k+2)\right)$$

$\Rightarrow Z_n$ WSS

$$MSE = R_Z(0) = \frac{14}{9}R_X(0) - \frac{8}{9}\left(R_X(-1) + R_X(1)\right) + \frac{1}{3}\left(R_X(-2) + R_X(2)\right)$$

$$= \frac{14}{9} - \frac{8}{9} = \frac{6}{9} = \frac{2}{3}$$

③ Let X_n be a zero-mean, first order autoregressive process with autocorrelation function

$$R_X(k) = \alpha^{|k|},$$

where $|\alpha| < 1$. Let

$$Y_n = X_n + \beta X_{n-1}$$

(a) Compute $R_{Y,X}(k)$, and $S_{Y,X}(f)$.

$$R_{Y,X}(k) \triangleq \mathbb{E}[X_n Y_{n+k}]$$

$$= \mathbb{E}[X_n(X_{n+k} + \beta X_{n+k-1})$$

$$= \mathbb{E}[X_n X_{n+k} + \beta X_n X_{n+k-1}]$$

$$= \alpha^{|k|} + \alpha^{|k-1|}\beta$$

$$S_{Y,X}(f) = \mathcal{F}\{\alpha^{|k|} + \alpha^{|k-1|}\beta\}$$

$\alpha^{|k|}$

$$= (1 + \beta e^{-j2\pi f}) \frac{1 - \alpha^2}{1 - 2\alpha\cos(2\pi f) + \alpha^2}$$

(b) Compute $S_Y(f)$, $R_Y(k)$, and $\mathbb{E}[Y_n^2]$.

$$
\begin{aligned}
R_Y(k) &\triangleq \mathbb{E}[Y_n Y_{n+k}] \\
&= \mathbb{E}[(X_n + \beta X_{n-1})(X_{n+k} + \beta X_{n+k-1})] \\
&= \mathbb{E}[X_n X_{n+k} + \beta X_n X_{n+k-1} + \beta X_{n+k} X_{n-1} + \beta^2 X_{n+k-1} X_{n-1}] \\
&= \alpha^{|k|} + \beta \alpha^{|k-1|} + \beta \alpha^{|k+1|} + \beta^2 \alpha^{|k|}
\end{aligned}
$$

$$
\begin{aligned}
S_Y(f) &= (1 + \beta^2 + \beta e^{-j2\pi f} + \beta e^{j2\pi f}) \frac{1-\alpha^2}{1 - 2\alpha\cos(2\pi f) + \alpha^2} \\
&= (1 + \beta^2 + 2\beta\cos(2\pi f)) \frac{1-\alpha^2}{1 - 2\alpha\cos(2\pi f) + \alpha^2}
\end{aligned}
$$

(c) For what value of β is $\{Y_n\}$ a white noise process?

Need to get rid of oscillating term.
If $\beta = -\alpha$, then

$S_Y(f) = 1 - \alpha^2$ is constant as desired.

$$
\begin{aligned}
\mathbb{E}[Y_n^2] = R_Y(0) &= 1 + \beta\alpha + \beta\alpha + \beta^2 \\
&= 1 + \beta^2 + 2\beta\alpha
\end{aligned}
$$