

Problem 4 HW 5

(a), (b) - you should NOT need finite dimensional

(c), (d) - Let V be finite dimensional
 $S, T \in \mathcal{L}(V)$

al 1:30-3:30

T 3-5

Th 1:30-3:30

Dual Vector Spaces

Recall: ① a LT $\phi: V \rightarrow \mathbb{F}$ is called a linear function on V example

$$V = C^\infty(\mathbb{R}) \quad \text{choose } n \in \mathbb{Z}, n \neq 0$$

$$\text{Define } \varphi_n: V \rightarrow \mathbb{R} \text{ by } \varphi_n(f) = \int_0^{2\pi} f(x) \cos(n\pi x) dx$$

$$\textcircled{2} V^* = \mathcal{L}(V, \mathbb{F}) \text{ dual vector space to } V$$

Proposition

$$\text{If } \dim V = n$$

$$\text{then } \dim V^* = n = \dim V$$

 V, V^* isomorphic
b/c same dimensionProof

$$\text{Know } \dim \mathcal{L}(V, W) = (\dim V)(\dim W)$$

$$\therefore \mathcal{L}(V, \mathbb{F}) \text{ has } \dim = (\dim V)(\dim \mathbb{F}) = \dim V$$

Basis of V^*

Suppose $a = (\vec{v}_1, \dots, \vec{v}_n)$ is a basis of V .

To give a $\varphi: V \rightarrow \mathbb{F}$ LT, it suffices to give n numbers

$a_1, \dots, a_n \in \mathbb{F}$ then $\exists!$ $\varphi: V \rightarrow \mathbb{F}$ s.t. $\varphi(\vec{v}_i) = a_i \forall i$

Let's get some elements of V^*

Let $1 \leq i \leq n$.

Define $\varphi_i: V \rightarrow \mathbb{F}$ LT

$$v_i \mapsto 1$$

$$v_{j \neq i} \mapsto 0$$

$$\begin{aligned} \varphi_i(a_1 v_1 + \dots + a_n v_n) \\ = a_i \varphi_i(v_i) = a_i \end{aligned}$$

We have $\varphi_1, \dots, \varphi_n \in V^*$ as just defined

Is $\varphi_1, \dots, \varphi_n$ a basis for V^* ?

"easy" lemma: Suppose $\varphi \in V^*$, then $(\varphi: V \rightarrow \mathbb{F})$

$$\varphi = \sum_{i=1}^n \underbrace{\varphi(v_i)}_{\in \mathbb{F}} \underbrace{\varphi_i}_{\in V^*} \in V^*$$

Proof: To show LHS $(\varphi) = \text{RHS}$ it suffices to show that for each \vec{v}_j $1 \leq j \leq n$ that $\varphi(\vec{v}_j) = \text{RHS}(\vec{v}_j)$

$$\varphi(\vec{v}_j) = \sum_{i=1}^n [\varphi(\vec{v}_i) \varphi_i](\vec{v}_j)$$

$$= \sum_{i=1}^n \varphi(\vec{v}_i) \varphi_i(\vec{v}_j) = \begin{cases} 1 \cdot \varphi(\vec{v}_j) & , i=j \\ 0 & , i \neq j \end{cases}$$

Proposition: Let $a^* = (\varphi_1, \dots, \varphi_n)$

Then a^* is a basis of V^* (assuming a is a basis of V)

One way: Prove $\varphi_1, \dots, \varphi_n$ is LI, then since it has same number of elements as dimension it must span

Another: Have already shown $(\varphi_1, \dots, \varphi_n)$ spans V^* since $n = \dim V^*$ it is also LI

Third: Show $\varphi_1, \dots, \varphi_n$ LI

Show $\varphi_1, \dots, \varphi_n$ spans V^*

Proof $a_1 \varphi_1 + a_2 \varphi_2 + \dots + a_n \varphi_n = 0$ ^{as functions $V \rightarrow \mathbb{F}$} $a_i \in \mathbb{F}$

Show $a_1 = a_2 = \dots = a_n = 0$

Apply $a_1 \varphi_1 + a_2 \varphi_2 + \dots + a_n \varphi_n$ to \vec{v}_i

$$a_1 \varphi_1(\vec{v}_i) + \dots + a_n \varphi_n(\vec{v}_i) \stackrel{?}{=} 0$$

$$a_i \neq 0$$

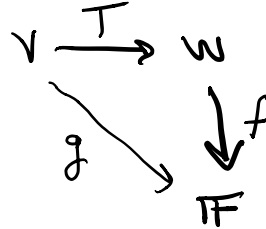
example: $V = \mathbb{R}^2$, $V^* = (\mathbb{R}^2)^* \simeq \mathbb{R}^2$ isomorphic to

Definition: Let $T: V \rightarrow W$ be a LT. Define the transpose of T

$$T^t: W^* \rightarrow V^*$$

$$f \mapsto T^t(f)$$

$$T^t(f) = g = fT$$



Basic Shopping List of Facts

① T^t is a linear transformation

$$\textcircled{2} (TS)^t(f) = S^t T^t$$

$$S: V \rightarrow W$$

$$T: W \rightarrow U$$

$$TS: V \rightarrow U$$