ECE 4110 Homework 9

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Due 5pm on December 5

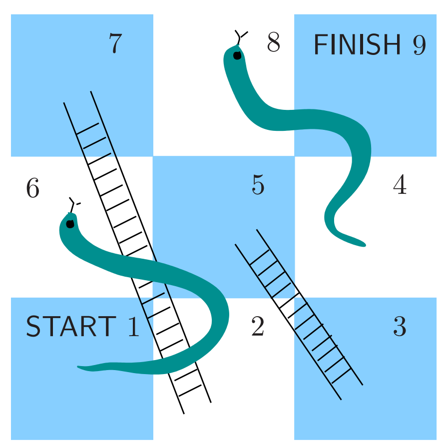
1 Reading Material

- Absorption time and absorption probability.
- Continuous-time Markov chain (Chapter 11.4).

2 Assignment

1. **Absorption Time and Absorption Probability**

   A simple game of "snakes and ladders" is played on a board of nine squares. At each turn a player tosses a fair coin and advances one or two places according to whether the coin shows head or tail. If you land at the foot of a ladder, you climb to the top; if you land at the head of a snake, you slide down to the tail.

   (a) Define a Markov chain with the state X_n being the square that the player is in at time n. Give the transition matrix.

   (b) How many turns on average does it take to complete the game?

   (c) What is the probability that a player who has reached the middle square will complete the game without slipping back to square 1?

2. **A Dinner Treat**

   You are playing a game with your friend John who is majoring in Literature and thinks "Markov Chain" is a new franchise of fancy Russian cuisine.

   The game is based on the outcomes of flipping a fair coin in succession. Each of you chooses a pattern consisting of three consecutive coin flip outcomes. Whoever's pattern appears first will be the winner and will be treated to a lavish dinner. To show your generosity, you tell your friend: " I will let you choose first. You can choose any pattern from the 8 possible ones. I will then choose a pattern from the remaining 7." Your friend is grateful, not knowing that for any of the eight patterns that he chooses, you can find a pattern that is at least twice better (in terms of the chance to win) than his pattern.

(a) Show that if John chooses HHH, you can have a winning chance 7 times better than John by choosing THH.

(b) Show that if John chooses HTT, you can have a winning chance twice better by choosing HHT.

(c) (Optional) If you do want to play this game with a friend, you might want to fill out the table below. You will see that there is a general rule for finding the best winning pattern against every pattern your friend may choose.

John's Pattern	Your Pattern	$\Pr[\text{you win}]$
HHH		
HHT		
HTH		
HTT		
THH		
THT		
TTH		
TTT		

3. **Properties of Exponential Random Variables**

Let $T_1, \ldots, T_n$ be independent random variables with T_i being exponentially distributed with parameter λ_i. Let

$$T = \min(T_1, \ldots, T_n)$$

denote the minimum of these n random variables and let K denote the index of the random variable that equals the minimum

$$K = \arg\min(T_1, \ldots, T_n).$$

(a) Show that the probability of T_k being the minimum is given by

$$\Pr[K = k] = \frac{\lambda_k}{\sum_{i=1}^{n} \lambda_i}$$

(b) Show that T is exponentially distributed with parameter $\sum_{i=1}^{n} \lambda_i$.

(c) Show that T and K are independent.

4. **Erlang's Formula**

At the times of a Poisson process with rate λ, someone in Acahti places a call to the Big City. The phone network can handle k such calls. If there is spare capacity when a call is placed, the call is connected and lasts an exponentially-distributed amount of time with mean $1/\mu$, independent of the other calls. If someone attempts to place a call and k calls are currently in progress, the person hears a fast busy signal and gives up.

(a) Argue that the number of connected calls is a Markov chain and draw its transition rate diagram. Is the chain irreducible?

(b) Find all stationary distributions of the chain.

(c) When the state is k, the network cannot handle any additional calls. What is the long-term fraction of time that the chain is in this state?

5. **Inventory Control**

The Elppa Store sells chic Orange Watch Series 5 for a tidy sum. Customers enter the store at the times of a Poisson process with rate λ. If an Orange Watch is available when a customer arrives, the customer purchases it. Otherwise, the customer makes a scene and then leaves. When the stock of Orange Watch in the store reaches one, the store orders 4 more, which arrive after an exponentially-distributed amount of time with mean $1/\mu$. Suppose that initially the store has 5 Orange Watch in stock.

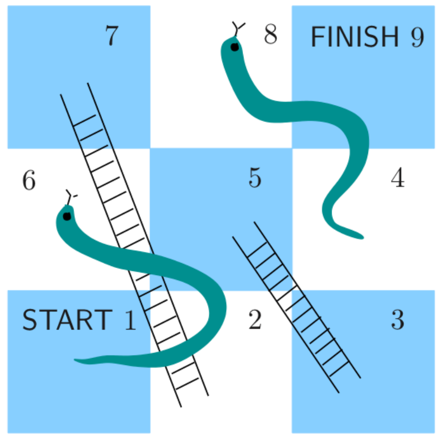
(a) Argue that the number of Orange Watch in the store is a Markov chain and draw its transition rate diagram. Is the chain irreducible?

(b) Compute all stationary distributions for the chain.

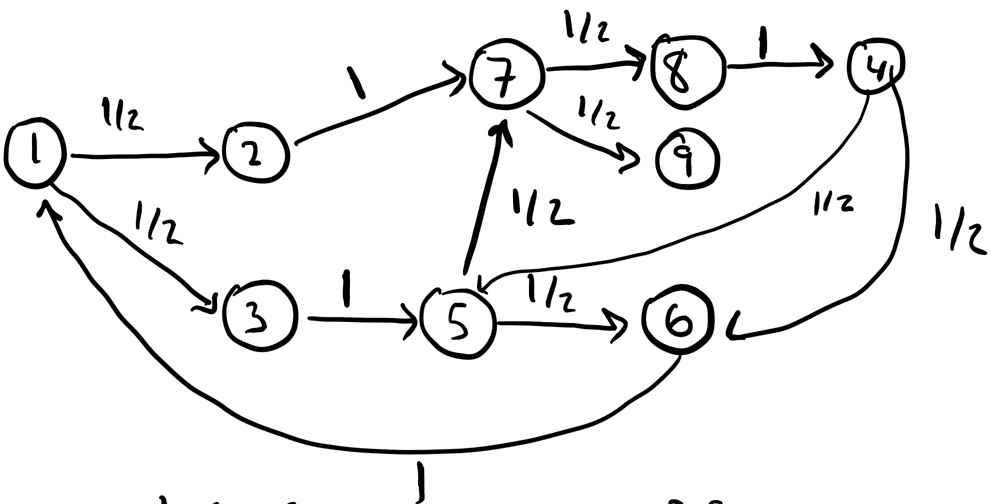
(c) What is the long-run fraction of time that the store is out of stock of Orange Watch?

① Absorption Time and Absorption Probability

A simple game of "snakes and ladders" is played on a board of nine squares. At each turn a player tosses a fair coin and advances one or two places according to whether the coin shows head or tail. If you land at the foot of a ladder, you climb to the top; if you land at the head of a snake, you slide down to the tail.



(a) Define a Markov chain with the state X_n being the square that a player is in at time n. Give the transition matrix.



$$
\mathbb{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array}
\begin{array}{c}
\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\
\begin{bmatrix}
0 & 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1/2 & 1/2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1/2 & 1/2 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 1/2 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\end{array}
$$

$X = \{1, 4, 5, 7, 9\}$

↑

$(2,7)$ $(3,5)$ $(6,1)$ $(8,4)$ can be combined to single states

Note ↗

Thus $\mathbb{P}$ simplifies to

$$\mathbb{P} = \begin{array}{c|c} & \begin{array}{ccccc} 1 & 4 & 5 & 7 & 9 \end{array} \\ \begin{array}{c} 1 \\ 4 \\ 5 \\ 7 \\ 9 \end{array} & \begin{bmatrix} 0 & 0 & 1/2 & 1/2 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

(b) How many turns on average does it take to complete the game?

Define

$$e_i \triangleq \mathbb{E}\left[\begin{array}{c} \text{remaining time} \\ \text{until absorption} \end{array} \middle| \text{current state is } i \right]$$

then

$$e_1 = \tfrac{1}{2}(e_5+1) + \tfrac{1}{2}(e_7+1)$$
$$e_4 = \tfrac{1}{2}(e_1+1) + 1/2(e_5+1)$$
$$e_5 = 1/2(e_1+1) + 1/2(e_7+1)$$
$$e_7 = 1/2(e_4+1) + 1/2(e_9+1)$$
$$e_9 = 0$$

Solving yields

$$e_1 = 7, \quad e_4 = 8, \; e_5 = 7, \; e_7 = 5$$

So, on average it takes $e_1 = 7$ moves to complete the game.

(c) What is the probability that a player who has reached the middle square will complete the game without slipping back to square 1?

Define

$$g_i = \Pr\left[\begin{matrix}\text{Winning the game w/o} \\ \text{going back to 1}\end{matrix} \,\middle|\, \text{started at state } i\right]$$

then

$$g_1 = \tfrac{1}{2} g_7 + \tfrac{1}{2} g_5$$
$$g_4 = \tfrac{1}{2} g_5$$
$$g_5 = \tfrac{1}{2} g_7$$
$$g_7 = \tfrac{1}{2} g_9 + \tfrac{1}{2} g_4$$
$$g_9 = 1$$

Solving gives

$$g_1 = 3/7,\ g_4 = 1/7,\ g_5 = 2/7,\ g_7 = 4/7$$

Middle square is 5, thus answer is $g_5 = 2/7$

② A Dinner Treat

You are playing a game with your friend John who is majoring in Literature and thinks "Markov Chain" is a new franchise of fancy Russian cuisine.

The game is based on the outcomes of flipping a fair coin in succession. Each of you chooses a pattern consisting of three consecutive coin flip outcomes. Whoever's pattern appears first will be the winner and will be treated to a lavish dinner. To show your generosity, you tell your friend: " I will let you choose first. You can choose any pattern from the 8 possible ones. I will then choose a pattern from the remaining 7." Your friend is grateful, not knowing that for any of the eight patterns that he chooses, you can find a pattern that is at least twice better (in terms of the chance to win) than his pattern.

(a) Show that if John chooses HHH, you can have a winning chance of 7 times better than John by choosing THH.

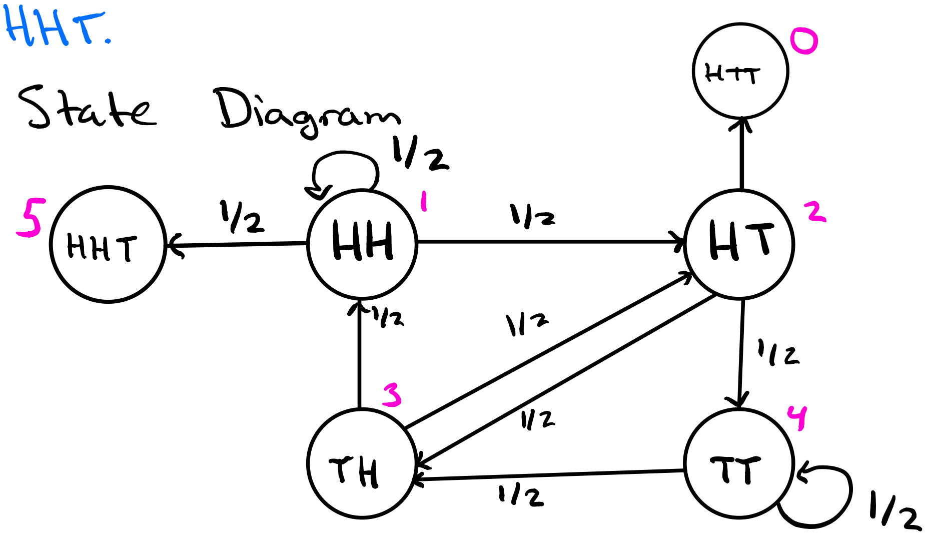
If first flip is T, John never wins. Thus he only wins if first flip is H and following two flips are H.

$$\Pr[HHH] = 1/8$$

thus I win w/ probability $1 - 1/8 = 7/8$

(b) Show that if John chooses HTT, you can have a winning chance twice better by choosing HHT.

State Diagram



H → 0

T → 1

Define

$$g_i \triangleq Pr[\text{HTT appears first} \mid \text{current state is } i]$$

$g_0 = 1$

$g_1 = \frac{1}{2} g_5 + \frac{1}{2} g_1 \rightarrow g_1 = 0$

$g_2 = \frac{1}{2} + \frac{1}{2} g_3 \rightarrow g_2 = 2/3$

$g_3 = \frac{1}{2} g_1 + 1/2\, g_2 \rightarrow g_3 = 1/3$

$g_4 = 1/2\, g_3 + \frac{1}{2} g_4 \rightarrow g_4 = 1/3$

$g_5 = 0$

$Pr[HH] = Pr[HT] = Pr[TH] = Pr[TT] = 1/4$

$Pr[A \text{ wins}] = \frac{1}{4} g_1 + \frac{1}{4} g_2 + \frac{1}{4} g_3 + \frac{1}{4} g_4 = 1/3$

$Pr[B \text{ wins}] = 1 - Pr[A \text{ wins}] = 2/3$

③ Properties of Exponential Random Variables

Let $T_1, \ldots, T_n$ be iid $\exp(\lambda_i)$. (λ_i corresponds to T_i)

Let

$$T = \min(T_1, \ldots, T_n)$$

denote the minimum of these n random variables and let K denote the index of the random variable that equals the minimum

$$K = \operatorname{argmin}(T_1, \ldots, T_n)$$

(a) Show that the probability of T_k being the minimum is given by

$$\Pr[K=k] = \frac{\lambda_k}{\sum_{i=1}^{n} \lambda_i}$$

Conditioning on T_k and using the total probability theorem yields

$$\Pr[K=k] = \int_{t_k=0}^{\infty} \Pr[T_1 > t, \ldots, T_{k-1} > t, T_{k+1} > t, \ldots, T_n > t \mid T_k = t] f_{T_k}(t)\, dt$$

$$= \int_{t_k=0}^{\infty} \prod_{i \neq k} \Pr[T_i > t] f_{T_k}(t)\, dt$$

$$= \int_{t_k=0}^{\infty} \prod_{i \neq k} e^{-\lambda_i t} \lambda_k e^{-\lambda_k} \, dt$$

$$= \int_{t_k=0}^{\infty} \lambda_k \, e^{-t\sum_{i=1}^{n}\lambda_i} \, dt$$

$$= \lambda_k \left(\frac{1}{\sum_{i=1}^{n}\lambda_i} e^{-t\sum_{i=1}^{n}\lambda_i} \Big|_{t=0}^{\infty} \right)$$

$$= \frac{\lambda_k}{\sum_{i=1}^{n}\lambda_i}$$

(b) Show that T is exponentially distributed with parameter $\sum_{i=1}^{n}\lambda_i$

$$\Pr[T>t] = \Pr[T_1>t, \ldots, T_n>t]$$

$$= \prod_{i=1}^{n} \Pr[T_i>t]$$

$$= \prod_{i=1}^{n} e^{-\lambda_i t}$$

$$= e^{-t\sum_{i=1}^{n}\lambda_i} \Rightarrow \text{parameter} = \sum_{i=1}^{n}\lambda_i$$

(c) Show that T and K are independent.

Need

$$\Pr[T>t, K=k] = \Pr[T>t]\Pr[K=k]$$

$$\Pr[T>t, K=k] = \int_t^\infty \Pr[T>t, K=k \mid T_k = s] f_{T_k}(s)\, ds$$

$$= \int_t^\infty \Pr[T_1>s, \ldots, T_{k-1}>s, T_{k+1}>s, \ldots, T_n>s] f_{T_k}(s)\, ds$$

$$= \int_t^\infty \prod_{i\neq k} \Pr[T_i>s] f_{T_k}(s)\, ds = \int_t^\infty \prod_{i\neq k} e^{-\lambda_i s} \lambda_k e^{-\lambda_k s}\, ds$$

$$= \lambda_k \int_t^\infty e^{-s\sum_{i\neq k}\lambda_i} e^{-\lambda_k s}\, ds$$

$$= \lambda_k \cdot \frac{1}{\sum_{i=1}^n \lambda_i} e^{-s\sum_{i=1}^n \lambda_i} \Big|_t^\infty$$

$$= \frac{\lambda_k}{\sum_{i=1}^n \lambda_i} e^{-s\sum_{i=1}^n \lambda_i} = \Pr[K=k]\Pr[T>t]$$

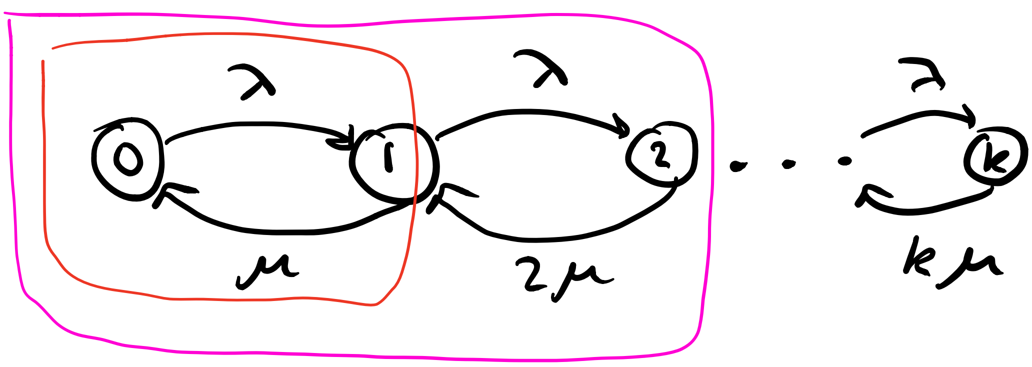
④ Erlang's Formula

At the times of a Poisson process with rate λ, someone in Acahti places a call to the Big City. The phone network can handle k such calls. If there is spare capacity when a call is placed, the call is connected and lasts an exponentially-distributed amount of time with mean $1/\mu$, independent of the other calls. If someone attempts to place a call and k calls are currently in progress, the person hears a fast busy signal and gives up.

(a) Argue that the number of connected calls is a Markov chain and draw its transition rate diagram. Is the chain irreducible?

$$q_{i,i+1} = \lambda \quad , \quad i = 0, 1, \dots, k-1$$

$$q_{i,i-1} = i\mu \quad , \quad i = 1, \dots, k$$



the chain is irreducible.

(b) Find all stationary distributions of the chain.

$$\begin{cases} \pi Q = 0 \\ \sum_{i \in \mathcal{X}} \pi_i = 0 \end{cases}$$

$$Q = \begin{bmatrix} -\lambda & \lambda & & & \\ \mu & -(\lambda+\mu) & \lambda & & \\ & 2\mu & -(\lambda+2\mu) & \lambda & \\ & & & \ddots & \end{bmatrix}$$

$$\lambda \pi_0 = \mu \pi_1$$

$$\lambda \pi_1 = 2\mu \pi_2 \qquad \Rightarrow \quad \pi_i = \frac{1}{i!}\left(\frac{\lambda}{\mu}\right)^i \pi_0$$

$$\vdots$$

$$\lambda \pi_{k-1} = k\mu \pi_k$$

$$\sum_{i=1}^{k} \pi_i = 1 \longrightarrow \quad \pi_0 = \frac{1}{\sum_{i=0}^{k} \frac{1}{i!}\left(\frac{\lambda}{\mu}\right)^i}$$

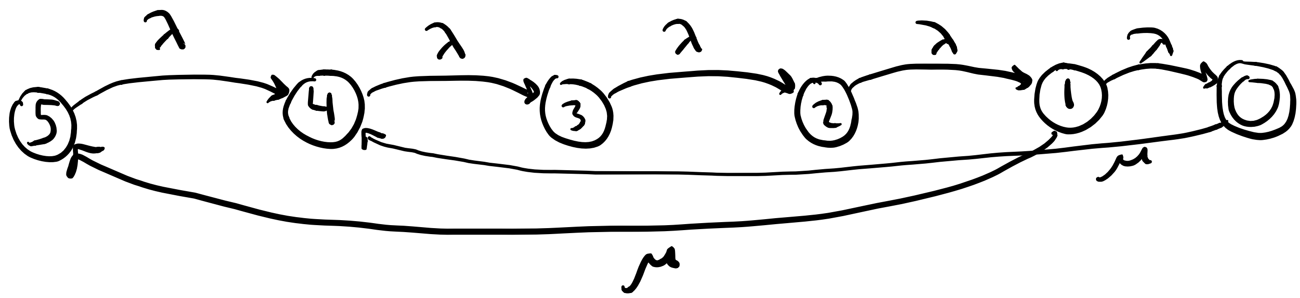
(c) When the state is k, the network cannot handle any additional calls. What is the long-term fraction of time that the chain is in this state?

$$\pi_k = \frac{1}{k!}\left(\frac{\lambda}{\mu}\right)^k \frac{1}{\sum_{i=0}^{k} \frac{1}{i!}\left(\frac{\lambda}{\mu}\right)^i}$$

⑤ Inventory Control

The Elppa Store sells chic Orange Watch Series 5 for a tidy sum. Customers enter the store at the times of a Poisson process with rate λ. If an Orange Watch is available when a customer arrives, the customer purchases it. Otherwise, the customer makes a scene and then leaves. When the stock of Orange Watch in the store reaches one, the store orders 4 more, which arrive after an exponentially-distributed amount of time with mean $1/\mu$. Suppose that initially the store has 5 Orange Watch in stock.

(a) Argue that the number of Orange Watches in the store is a Markov chain and draw its transition rate diagram. Is the chain irreducible?



$$q_{i,i-1} = \lambda \quad , \quad i = 5,4,3,2,1$$

$$q_{i,i+4} = \mu \quad , \quad i = 0,1$$

The chain is irreducible.

$$Q = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} -\mu & 0 & 0 & 0 & \mu & 0 \\ \lambda & -(\mu+\lambda) & 0 & 0 & 0 & \mu \\ 0 & \lambda & -\lambda & 0 & 0 & 0 \\ 0 & 0 & \lambda & -\lambda & 0 & 0 \\ 0 & 0 & 0 & \lambda & -\lambda & 0 \\ 0 & 0 & 0 & 0 & \lambda & -\lambda \end{bmatrix} \end{array}$$

(b) Compute all stationary distributions of the chain.

$$\pi_1 = \frac{\mu}{\lambda}\pi_0$$

$$\pi_2 = \frac{(\lambda+\mu)}{\lambda}\pi_1 = \frac{(\lambda+\mu)\mu}{\lambda^2}\pi_0$$

$$\pi_3 = \pi_2$$

$$\pi_4 = \pi_3$$

$$\pi_5 = \frac{\mu^2}{\lambda^2}\pi_0$$

$$\pi_0\left(1 + \frac{\mu}{\lambda} + \frac{3(\lambda+\mu)\mu}{\lambda^2} + \frac{\mu^2}{\lambda^2}\right) = 1$$

$$\pi_0 = \frac{\lambda^2}{(\lambda+2\mu)^2}$$

(c) What is the long-run fraction of time the store is out of stock?

$$\pi_0 = \left(\frac{\lambda}{\lambda+2\mu}\right)^2$$