Recall

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}, \quad \mathbb{P}(B) \neq 0$$

Product Rule: If $D_1 \supset D_2 \supset D_3 \supset \cdots \supset D_n$ is a nested decreasing sequence of positive-probability events, then

$$\mathbb{P}(D_n) = \mathbb{P}(D_1)\mathbb{P}(D_2|D_1)\mathbb{P}(D_3|D_2)\cdots\mathbb{P}(D_n|D_{n-1})$$

Example: Aircraft (alien detection):

A = Spacecraft-present

B = see a blip on radar

Given: $\mathbb{P}(B|A) = 0.99 \Rightarrow \mathbb{P}(B^c|A) = 1 - 0.99 = 0.01$

$\mathbb{P}(B|A^c) = 0.10$

$\mathbb{P}(A) = 0.05$

Find: $\mathbb{P}(A \cap B^c)$ = Probability of failed detection

$\mathbb{P}(A^c \cap B)$ = Probability of false alarm

via product rule

Note: $A \supset (A \cap B^c) \Rightarrow \mathbb{P}(A \cap B^c) = \mathbb{P}(A)\mathbb{P}(A \cap B^c | A)$

$$\mathbb{P}(A \cap B^c | A) = \frac{\mathbb{P}(A \cap B^c \cap A)}{\mathbb{P}(A)} = \frac{\mathbb{P}(A \cap B^c)}{\mathbb{P}(A)} = \mathbb{P}(B^c|A)$$

$= \mathbb{P}(A)\mathbb{P}(B^c|A)$

$= (0.05)(0.01) = 0.005$

What about $\mathbb{P}(A^c \cap B)$?

$$A^c \supset A^c \cap B \Rightarrow \mathbb{P}(A^c \cap B) = \mathbb{P}(A^c)\mathbb{P}(A^c \cap B \mid A^c)$$
$$= \mathbb{P}(A^c)\mathbb{P}(B \mid A^c)$$
$$= (0.95)(0.10) = 0.095$$

Example: Team Formation

12 agents: Frodo, Sam, Gandalf, Legolas and 8 miscellaneous hobbits to be assigned randomly to four teams of three via:

- hat containing twelve slips – three 1's, three 2's, etc.
- draw a slip "at random", give to Frodo, then S, G, L
- then do it for rest of the hobbits.

Find: $\mathbb{P}$(each team has one of the "leaders")
= $\mathbb{P}$(all four "leaders" are assigned to different teams)

Let D_1 = Frodo + Sam on different teams
D_2 = F, S, G on different teams
D_3 = F, S, G, L all on different teams.

Notice! $D_1 \supset D_2 \supset D_3$

$$\mathbb{P}(D_3) = \mathbb{P}(D_1)\mathbb{P}(D_2 \mid D_1)\mathbb{P}(D_3 \mid D_2)$$

$\mathbb{P}(D_1) = \dfrac{9}{11}$ ← Number of non-Frodo-team slips left when assigning Sam
← Total number of slips left.

Similarly,

$$\mathbb{P}(D_2 | D_1) = \frac{6}{10} = \frac{3}{5}$$

$$\mathbb{P}(D_3 | D_2) = \frac{3}{9} = \frac{1}{3}$$

$$\mathbb{P}(D_3) = \frac{9}{11} \cdot \frac{\cancel{3}}{5} \cdot \frac{1}{\cancel{3}} = \boxed{\frac{9}{55}}$$

Example: Birthday (a classic)

Have a population of n-people. Want (as a function of n)

$$\mathbb{P}(\text{at least two share a birthday})$$

Go through people one-by-one, ask birthday.
Assuming that every day is equally probable as birthday,

D_1 = first two people have different birthdays

$$\mathbb{P}(D_1) = \frac{364}{365}$$

D_2 = first three people have different birthdays.

$$\mathbb{P}(D_3 | D_2) = \frac{363}{365}$$

Generally, for any n,

$$\mathbb{P}(D_n) = \mathbb{P}(\text{first } (n+1) \text{ people have different birthdays})$$
$$= \frac{364}{365} \cdot \frac{363}{365} \cdot \frac{362}{365} \cdots \frac{365-n}{365}$$

$n=30 \Rightarrow 0.29 \longrightarrow \mathbb{P}(\text{two have same birthday}) = 0.71$

$n=53 \Rightarrow 0.01 \longrightarrow \mathbb{P}(\text{two have same birthday}) = 0.99$

Comment: Attacking this combinatorically would be hard!

Next, if you're given an event B, say, $C_1, C_2, \ldots, C_n$ (events) is a partition of B when

- $B = C_1 \cup C_2 \cup \cdots \cup C_n$
- c's all disjoint

If $A_1, A_2, \ldots, A_n$ is a partition of Ω, then $C_1, C_2, \ldots, C_n$ partitions B, where

$$C_k = B \cap A_k \quad 1 \le k \le n$$

Idea: $B = B \cap \Omega = B \cap (A_1 \cup \cdots \cup A_n)$

$= (B \cap A_1) \cup (B \cap A_2) \cup \cdots \cup (B \cap A_n)$

By additivity,

$$\mathbb{P}(B) = \mathbb{P}(C_1) + \cdots + \mathbb{P}(C_n)$$

LAW OF TOTAL PROBABILITY

$$\mathbb{P}(B) = \mathbb{P}(B \cap A_1) + \cdots + \mathbb{P}(B \cap A_n)$$

$$\boxed{\mathbb{P}(B) = \mathbb{P}(B|A_1)\mathbb{P}(A_1) + \mathbb{P}(B|A_2)\mathbb{P}(A_2) + \cdots + \mathbb{P}(B|A_n)\mathbb{P}(A_n)}$$

Example - Chess

Pick a chess opponent uniformally from a group where half are novice, one-fourth are decent, and one-fourth are masters.

Let,

B = event that you win

Suppose

$$\mathbb{P}(B|\text{Novice}) = 50\%$$
$$\mathbb{P}(B|\text{Decent}) = 40\%$$
$$\mathbb{P}(B|\text{Master}) = 10\%$$

Let $\left.\begin{array}{l} A_1 = \text{play a novice} \\ A_2 = \text{play a decent} \\ A_3 = \text{Play a master} \end{array}\right\}$ Partition of Ω

Thus, by the law of total probability,

$$\mathbb{P}(B) = \mathbb{P}(B|A_1)\mathbb{P}(A_1) + \mathbb{P}(B|A_2)\mathbb{P}(A_2) + \mathbb{P}(B|A_3)\mathbb{P}(A_3)$$

that we win

$$= \frac{1}{2}\cdot\frac{1}{2} + \frac{4}{10}\cdot\frac{1}{4} + \frac{1}{10}\cdot\frac{1}{4}$$

$$= \frac{1}{4} + \frac{1}{10} + \frac{1}{40}$$

$$= \frac{10}{40} + \frac{4}{40} + \frac{1}{40}$$

$$= \frac{15}{40} = \boxed{\frac{3}{8}}$$

Chess example elaborated below.

In class today we did an example from the book about playing chess against a random opponent drawn from a roomful of opponents of different abilities. Students' questions after class made apparent that the treatment in the book, which I pretty much replicated in class, confuses people — like, what's Ω here? What's random? Is it just what opponent you choose or whether you win or lose against a specific opponent? When you say, "The probability that you beat a master is 0.1," what does that mean? Does that mean that you'll certainly beat 10% of the masters in the room, or that if you play a given master 100 times you'll win 10 times on average over the long haul?

A "toy" model that makes the math work out

Here's a way to think about the example that makes clearer (to me, at least) how the Total Probability Theorem applies. First of all, you pick an opponent uniformly at random from the finite set of opponents in the room. Let Ω be the set of all opponents and let $\mathbb{P}$ be the discrete uniform probability law on Ω. For each opponent in Ω, think of it as predetermined whether you will beat that opponent or not. Thus once you pick an opponent, there's nothing random about the outcome of the game — it's pre-ordained to be a win or a loss for you. A reasonable way to view the opponents is as follows. First number the opponents 1 through n, where n is the total number of opponents. Opponent k is the 3-tuple

- (N, W, k) if opponent k is a novice you'll beat
- (N, L, k) if opponent k is a novice you'll lose to
- (D, W, k) if opponent k is a decent player you'll beat
- (D, L, k) if opponent k is a decent player you'll lose to
- (M, W, k) if opponent k is a master you'll beat
- (M, L, k) if opponent k is a master you'll lose to

When you enter the room of potential opponents, you can't see the opponents' "labels." Here are some events along with probabilities for a few of them.

(1) A_1 = set of all novices in Ω; i.e. A_1 is the set of 3-tuples in Ω with N as first letter. $\mathbb{P}(A_1) = 0.5$, meaning that half the 3-tuples in Ω have N as first letter.
(2) A_2 = set of all decent players in Ω; i.e. A_2 is the set of 3-tuples in Ω with D as first letter. $\mathbb{P}(A_2) = 0.25$, meaning that a quarter of the 3-tuples in Ω have first letter D.
(3) A_3 = set of all masters in Ω; i.e. A_3 is the set of all 3-tuples in Ω with M as first letter. $\mathbb{P}(A_3) = 0.25$, meaning that a quarter of the 3-tuples in Ω have first letter M.
(4) B = set of all players in Ω whom you'll beat; i.e. B is the set of all 3-tuples in Ω with W as second letter. We want to find $\mathbb{P}(B)$.

Note that A_1, A_2, and A_3 partition Ω. We're given that $\mathbb{P}(B \mid A_1) = 0.5$, meaning that half of the 3-tuples in A_1 have W as second letter. Similarly, 40% of the 3-tuples in A_2 have second letter W, as do 10% of the 3-tuples in A_3. By the Total Probability Theorem,

$$\mathbb{P}(B) = \mathbb{P}(B \mid A_1)\mathbb{P}(A_1) + \mathbb{P}(B \mid A_2)\mathbb{P}(A_2) + \mathbb{P}(B \mid A_3)\mathbb{P}(A_3) = 3/8 ,$$

meaning that 3/8 of the 3-tuples in Ω have W as second letter.

It might be fun to let $n = 40$ (you can make the fractions work in that case) and make Ω out of slips of paper, each labeled with a 3-tuple, and draw repeatedly with replacement and see what fraction of draws have W as second letter. Or maybe write a Python program with $n = 40$ that draws repeatedly uniformly randomly over k from the appropriate set of 40 3-tuples, etc. You might have to number the opponents 0 through 39 in that case.

A better model

Again, you number the opponents 1 through n. The sample space Ω contains, for every k, the 3-tuples (type, W, k) and (type, L, k). These represent the respective outcomes "you choose opponent k, who is of the given type, and you win" and "you choose opponent k, who is of the given type, and you lose." How do we assign probabilities? If opponent k is a novice, we set

$$\mathbb{P}(\{(N, W, k)\}) = 0.5/n \text{ and } \mathbb{P}(\{N, L, k)\} = 0.5/n\ .$$

If opponent k is decent, we set

$$\mathbb{P}(\{(D, W, k)\}) = 0.4/n \text{ and } \mathbb{P}(\{D, L, k)\} = 0.6/n\ .$$

If opponent k is a master, we set

$$\mathbb{P}(\{(M, W, k)\}) = 0.1/n \text{ and } \mathbb{P}(\{M, L, k)\} = 0.9/n\ .$$

You can check that the probabilities of the singleton events in Ω sum to 1 as required. Note that Ω contains $2n$ outcomes, a win and a loss for each opponent. Now for some events.

(1) Given k, the event that you choose opponent k to play against is

$$\{(\text{type}, W, k), (\text{type}, L, k)\}\ ,$$

where type is N, D, or L depending on opponent k's type. Observe that the probability that you pick opponent k is $1/n$ for every k — you're equally likely to choose any of the opponents to play.

(2) A_1 = set of all outcomes in Ω corresponding to novices; i.e. A_1 is the set of 3-tuples in Ω with N as first letter. A_1 is the event "you choose a novice to play," $\mathbb{P}(A_1) = 0.5$, and half the 3-tuples in Ω line in A_1. Thus A_1 contains n outcomes, a win and a loss against each of the novices, which number $n/2$.

(3) A_2 = set of all outcomes in Ω corresponding to decent players; i.e. A_2 is the set of 3-tuples in Ω with D as first letter. A_2 is the event "you choose a decent opponent," $\mathbb{P}(A_2) = 0.25$, and a quarter of the 3-tuples in Ω lie in A_2. Thus A_2 contains $n/2$ outcomes, a win and a loss against each of the $n/4$ decent players.

(4) A_3 = set of all outcomes in Ω corresponding to masters; i.e. A_3 is the set of all 3-tuples in Ω with M as first letter. A_3 is the event "you choose a master opponent," $\mathbb{P}(A_3) = 0.25$, and a quarter of the 3-tuples in Ω lie in A_3. Thus A_3 contains $n/2$ outcomes, a win and a loss against each of the $n/4$ masters.

(5) B = set of all outcomes in Ω corresponding with wins; i.e. B is the set of all 3-tuples in Ω with W as second letter. We want to find $\mathbb{P}(B)$.

From our definition of $\mathbb{P}$ it follows that $\mathbb{P}(B \mid A_1) = 0.5$, $\mathbb{P}(B \mid A_2) = 0.4$, and $\mathbb{P}(B \mid A_3) = 0.1$, as required. To see why, look at A_2. Every outcome in A_2 corresponding to a win has probability $0.4/n$. Thus $\mathbb{P}(B \cap A_2)$, being the sum of the probabilities of all the winning outcomes in A_2, is $0.4m/n$, where m is the number of winning outcomes in A_2. But m is exactly half the total number of outcomes in A_2, so $m = n/4$. Thus $\mathbb{P}(B \cap A_2) = 0.1$. Since $\mathbb{P}(A_2) = 0.25$,

$$\mathbb{P}(B \mid A_2) = \frac{\mathbb{P}(B \cap A_2)}{\mathbb{P}(A_2)} = \frac{0.1}{0.25} = 0.4\ .$$

The same type of calculation works for $\mathbb{P}(B \mid A_1)$ and $\mathbb{P}(B \mid A_3)$.

In any event, the Total Probability Theorem works as in the simple model, since all the numbers are the same. One of the TAs mentioned to me the utility of Venn diagrams in understanding the theorem, which made me realize that I didn't draw a Venn diagram when introducing the theorem. I'll make sure to do that in class, but for now see the accompanying diagram, in which A_1, A_2, and A_3 partition Ω. You can see that $B \cap A_1$, $B \cap A_2$, and $B \cap A_3$ partition B. Accordingly,

$$\mathbb{P}(B) = \mathbb{P}(B \cap A_1) + \mathbb{P}(B \cap A_2) + \mathbb{P}(B \cap A_3) .$$

Assuming $\mathbb{P}(A_k) > 0$ for each k, we have for each k

$$\mathbb{P}(B \cap A_k) = \mathbb{P}(B \mid A_k)\mathbb{P}(A_k) ,$$

from which follows

$$\mathbb{P}(B) = \mathbb{P}(B \mid A_1)\mathbb{P}(A_1) + \mathbb{P}(B \mid A_2)\mathbb{P}(A_2) + \mathbb{P}(B \mid A_3)\mathbb{P}(A_3) .$$