ECE4110: Random Signals in Communications and Signal Processing

Random Processes

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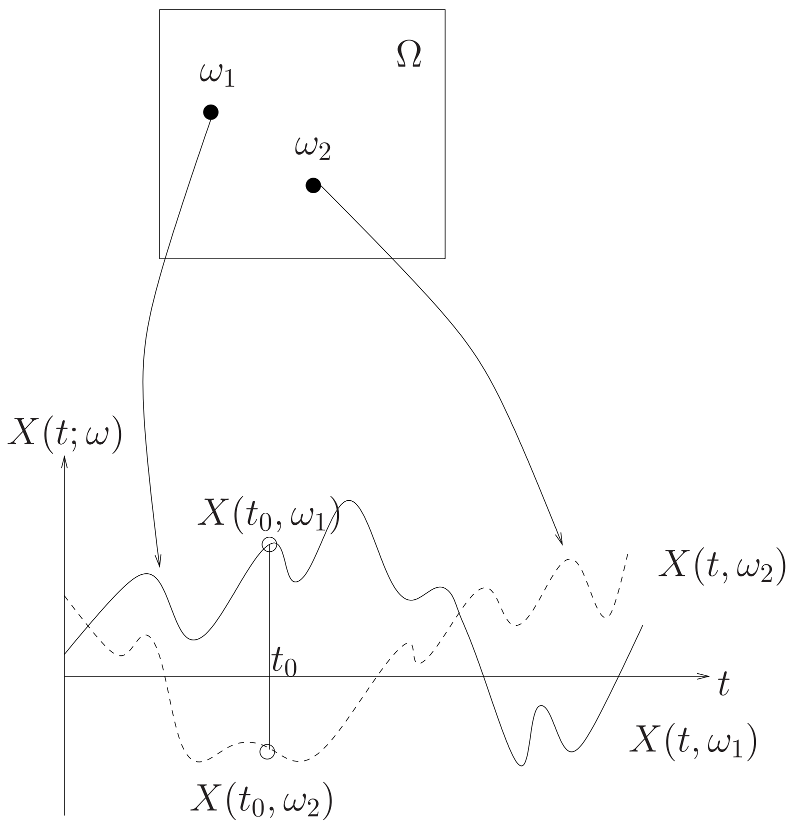
Outline

- Definition of random processes
- Description of a random process:
  - Complete characterization
  - Partial characterization: moments of a random process
- Important properties of a random process:
  - Stationarity and wide-sense stationarity
  - Independent increments
- Examples of widely used random processes:
  - Discrete-time processes:
    i.i.d. sequence, the sum process, random walk, and Binomial counting process
  - Continuous-time processes:
    Poisson process, Gaussian process, Wiener process and Brownian motion

Random Processes

Random Process:

Each sample $\omega \in \Omega$ in the sample space is mapped to a time function $X(t,\omega)$.



- For a fixed time $t = t_0$, $X(t_0;\omega)$ is a random variable.
- For a fixed ω, $X(t;\omega)$ is a time function referred to as a realization or a sample path of the random process.
- $X(t)$ denotes a random process (without explicitly including ω); it is an indexed (by t) family of random variables.
- $x(t)$ denote a sample path of $X(t)$.

Examples of Random Processes

Examples of Discrete-Time Random Processes:

- An i.i.d. Sequence of Discrete Random Variables:

$$X_1,\ X_2,\ X_3,\ \ldots$$

where $X_i \sim B(p)$ are i.i.d. Bernoulli random variables.

- An i.i.d. Sequence of Continuous Random Variables:

$$X_1,\ X_2,\ X_3,\ \ldots$$

where $X_i \sim \mathcal{N}(\mu, \sigma^2)$ are i.i.d. Gaussian.

- Binomial Counting Process (counting the number of "heads" in a sequence of coin flips):

$$S_n = \sum_{i=1}^{n} X_i$$

where $X_i \sim B(p)$ are i.i.d. Bernoulli random variables.

- Random Walk:

$$S_n = \sum_{i=1}^{n} Y_i$$

where $\{Y_i\}_{i \geq 1}$ are i.i.d. with PMF given by $\Pr(Y_i = 1) = 1 - \Pr(Y_i = -1) = \frac{1}{2}$.

[0] We use subscript n or i rather than (t) for discrete-time random processes.

Examples of Random Processes

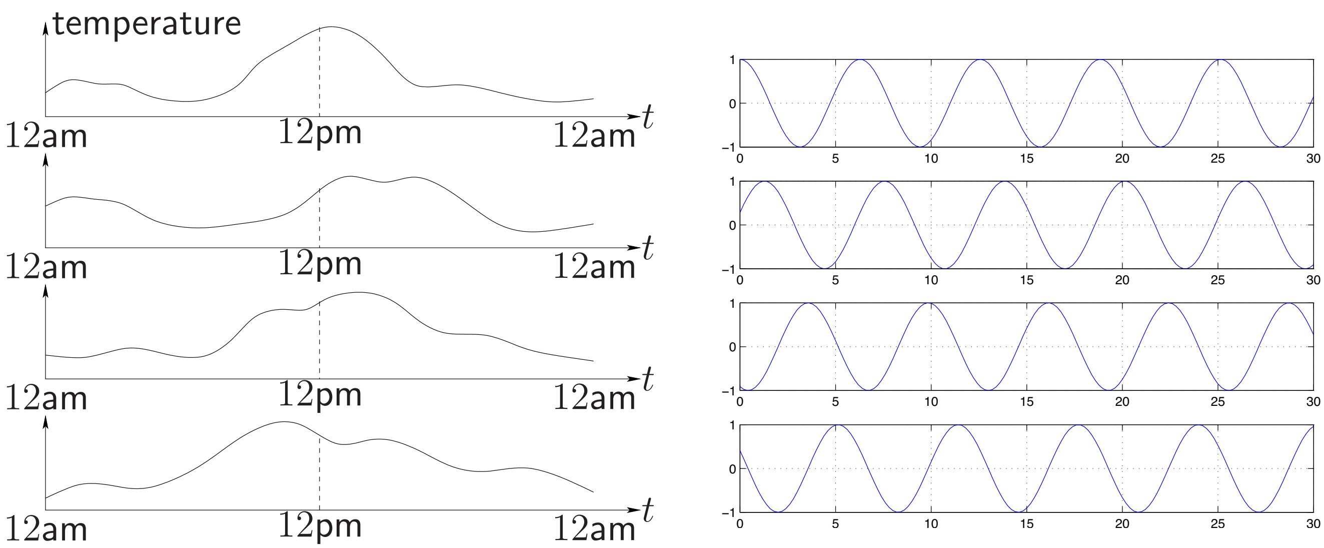
Examples of Continuous-Time Random Processes:

- Temperature in Ithaca on Jan 1.

- Sinusoid with Random Phase:

$$X(t) = \cos(2\pi t + \Theta)$$

where $\Theta \sim \mathcal{U}(0, 2\pi)$ is uniformly distributed in $(0, 2\pi)$.



- Random Parabolas:

$$X_t = A + Bt + t^2$$

where $A \sim \mathcal{N}(0,1)$ and $B \sim \mathcal{N}(0,1)$ are independent.

Descriptions of Random Processes

Description of a Random Process:

A random process is fully specified by the joint CDF

$$F_{X(t_1),\cdots,X(t_k)}(x_1,\cdots,x_k)$$

for all k and all sets of $t_1,\cdots,t_k$.

Moments of a Random Process

- The mean function:

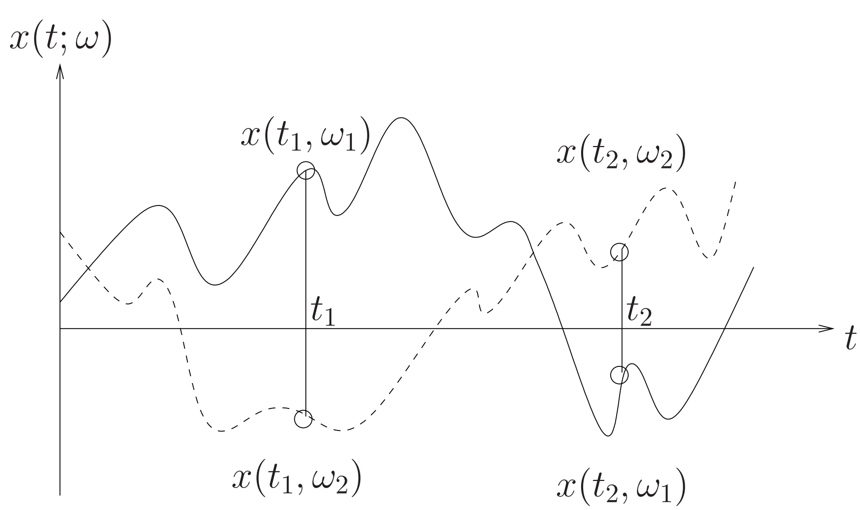
$$\mu_X(t)=\mathbb{E}[X(t)]$$

- The autocorrelation function:

$$R_X(t_1,t_2)\triangleq\mathbb{E}[X(t_1)X(t_2)]$$

- The autocovariance function:

$$\begin{aligned}C_X(t_1,t_2) &\triangleq \mathbb{E}[(X(t_1)-\mu_X(t_1))(X(t_2)-\mu_X(t_2))]\\ &= R_X(t_1,t_2)-\mu_X(t_1)\mu_X(t_2)\end{aligned}$$



Examples

- The example of i.i.d. sequence of Bernoulli random variables:

$$\mu_X(n) = p, \quad R_X(m,n) = \begin{cases} p & \text{if } m = n \\ p^2 & \text{if } m \neq n \end{cases}$$

- The example of sinusoid with random phase:

$$\begin{aligned} \mu_X(t) &= \mathbb{E}[\cos(2\pi t + \Theta)] \\ &= \int_0^{2\pi} \cos(2\pi t + \theta) f_\Theta(\theta) d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \cos(2\pi t + \theta) d\theta \\ &= 0 \end{aligned}$$

$$\begin{aligned} R_X(t_1, t_2) &\stackrel{\Delta}{=} \mathbb{E}[X(t_1)X(t_2)] = \mathbb{E}[\cos(2\pi t_1 + \Theta)\cos(2\pi t_2 + \Theta)] \\ &= \frac{1}{2}\mathbb{E}[\cos(2\pi(t_2 - t_1)) + \cos(2\pi(t_2 + t_1) + 2\Theta)] \\ &= \frac{1}{2}\left\{\cos(2\pi(t_2 - t_1)) + \frac{1}{2\pi}\int_0^{2\pi} \cos(2\pi(t_2 + t_1) + 2\theta) d\theta\right\} \\ &= \frac{1}{2}\cos(2\pi(t_2 - t_1)) \end{aligned}$$

- The example of random parabolas:

$$\begin{aligned} \mu_X(t) &= t^2 \\ R_X(t_1, t_2) &= 1 + t_1 t_2 + t_1^2 t_2^2 \\ C_X(t_1, t_2) &= R_X(t_1, t_2) - \mu_X(t_1)\mu_X(t_2) = 1 + t_1 t_2 \end{aligned}$$

Stationarity of Random Processes

Stationarity:

A random process is strictly stationary if its statistical characteristics do not change with time, or in another word, a shift of time origin is impossible to detect. Mathematically,

$$F_{X(t_1),\cdots,X(t_n)}(x_1,\cdots,x_n) = F_{X(t_1+\Delta),\cdots,X(t_n+\Delta)}(x_1,\cdots,x_n)$$

for all n, t_i, and Δ.

Wide Sense Stationarity (WSS):

A random process is wide sense stationary (WSS) if

$$(i) \quad \mathbb{E}[X(t)] = \mu \quad \text{(a constant not changing with } t\text{)}$$
$$(ii) \quad R_X(t_1,t_2) \triangleq \mathbb{E}[X(t_1)X(t_2)] = R(\tau), \quad \text{where } \tau = t_2 - t_1$$

Examples:

- i.i.d. sequences: strictly stationary.
- Binomial counting process: not WSS.
- Random walk: not WSS.
- Temperature in Ithaca on Jan 1: not WSS.
- Sinusoid with random phase: strictly stationary thus also WSS.
- Random parabolas: not WSS.

Independent Increments

Increment:

The increment of a random process $\{X(t)\}$ over an interval $[a, b]$ is the random variable $X(b) - X(a)$.

Independent Increments:

A random process has independent increments if for all n and for all $t_0 < t_1 < \ldots < t_n$, the increments $X(t_1) - X(t_0)$, $X(t_2) - X(t_1)$, $\ldots$, $X(t_n) - X(t_{n-1})$ over these n non-overlapping intervals are mutually independent.

Stationary Increments:

A random process has stationary increments if the distribution of the increment $X(t + \tau) - X(t)$ depends only on τ, not t. In other words, the increments in intervals of the same length have the same distribution regardless of when the interval begins.

Examples:

- i.i.d. sequence of Bernoulli random variables: stationary but not independent increments.
- Binomial counting process: independent and stationary increments: $S_n - S_m \sim \text{Binomial}(n - m, p)$ for all $n \geq m$.
- Random walk: independent and stationary increments.

Processes with Independent Increments

Characterizing Processes with Independent Increment:

Key idea: express the random variables of the process in terms of increments over non-overlapping intervals.

Example: Binomial Counting Process:

□ Joint PMF of $S_{n_1}, S_{n_2}, S_{n_3}$ $(n_1 < n_2 < n_3)$:

$$
\begin{aligned}
&\Pr\left[S_{n_1} = s_1, S_{n_2} = s_2, S_{n_3} = s_3\right] \\
&= \Pr\left[S_{n_1} = s_1, S_{n_2} - S_{n_1} = s_2 - s_1, S_{n_3} - S_{n_2} = s_3 - s_2\right] \\
&= \Pr\left[S_{n_1} = s_1\right]\Pr\left[S_{n_2} - S_{n_1} = s_2 - s_1\right]\Pr\left[S_{n_3} - S_{n_2} = s_3 - s_2\right] \\
&= \Pr\left[S_{n_1} = s_1\right]\Pr\left[S_{n_2 - n_1} = s_2 - s_1\right]\Pr\left[S_{n_3 - n_2} = s_3 - s_2\right] \\
&= \binom{n_1}{s_1} p^{s_1}(1-p)^{n_1 - s_1}\binom{n_2 - n_1}{s_2 - s_1} p^{s_2 - s_1}(1-p)^{n_2 - n_1 - (s_2 - s_1)} \\
&\quad \binom{n_3 - n_2}{s_3 - s_2} p^{s_3 - s_2}(1-p)^{n_3 - n_2 - (s_3 - s_2)} \\
&= \binom{n_1}{s_1}\binom{n_2 - n_1}{s_2 - s_1}\binom{n_3 - n_2}{s_3 - s_2} p^{s_3}(1-p)^{n_3 - s_3}
\end{aligned}
$$

□ Moments:

$$
\begin{aligned}
\mu_S(n) &= np \\
R_S(m,n) &= \mathbb{E}[S_m S_n] \quad \text{(assume } m < n\text{)} \\
&= \mathbb{E}[S_m(S_n - S_m + S_m)] \\
&= \mathbb{E}[S_m^2] + \mathbb{E}[S_m(S_n - S_m)] \\
&= \mathbb{E}[S_m^2] + \mathbb{E}[S_m]\mathbb{E}[S_n - S_m] \\
&= ((mp)^2 + mp(1-p)) + mp(n-m)p \\
&= mp(np + 1 - p)
\end{aligned}
$$

Summary of Binomial Counting

Binomial Counting: A Discrete-Time Counting Process:

Counting the number of "heads" in a sequence of coin flips:

$$S_n = \sum_{i=1}^{n} X_i$$

where $X_i \sim B(p)$ are i.i.d. Bernoulli random variables.

- $\{S_n\}_{n\geq 1}$ is not stationary or wide-sense stationary.
- $\{S_n\}_{n\geq 1}$ has independent and stationary increments with $S_n - S_m \sim$ Binomial$(n-m, p)$ for all $n \geq m$.
- The inter-arrival times are i.i.d. with a geometric distribution with parameter p.
- The joint PMF of $S_{n_1}, S_{n_2}, \ldots, S_{n_k}$ $(n_1 < n_2 < \ldots < n_k)$:

$$\begin{aligned} &\Pr\left[S_{n_1} = s_1, S_{n_2} = s_2, \ldots, S_{n_k} = s_k\right] \\ &= \binom{n_1}{s_1}\binom{n_2 - n_1}{s_2 - s_1}\cdots\binom{n_k - n_{k-1}}{s_k - s_{k-1}} p^{s_k}(1-p)^{n_k - s_k} \end{aligned}$$

- Moment Functions:

$$\begin{aligned} \mu_S(n) &= np \\ \text{Var}(S_n) &= np(1-p) \\ R_S(m,n) &= mp(np+1-p) \end{aligned}$$

Poisson Process

Three Equivalent Definitions of Poisson Process:

1. $\{N(t)\}_{t\geq 0}$ is a Poisson process with rate λ if $\{N(t)\}_{t\geq 0}$ is a counting process with independent increments and $N(t) - N(s) \sim$ Poisson$(\lambda(t-s))$ for all $t \geq s$.

2. $\{N(t)\}_{t\geq 0}$ is a Poisson process with rate λ if $\{N(t)\}_{t\geq 0}$ is a counting process with i.i.d. inter-arrival times that have an exponential distribution with parameter λ.

3. $\{N(t)\}_{t\geq 0}$ is a Poisson process with rate λ if $\{N(t)\}_{t\geq 0}$ is a counting process such that for all $\tau > 0$, $N(\tau) \sim$ Poisson$(\lambda\tau)$ and given $N(\tau) = n$, these n arrival times are i.i.d. with distribution $\mathcal{U}[0, \tau]$.

Poisson Process as a Limit of Discrete-Time Binomial Counting:

- Consider a continuous-time arrival process with rate λ.
- Partition $[0, t]$ into equal-length intervals with length δ.
- For δ sufficiently small, assume that the probability of having more than one arrival in the same interval is negligible and the number of arrivals is independent across intervals.
- Let p denote the probability of having an arrival in an interval. Since the expected number of arrivals in $[0, t]$ is λt, we have

$$\frac{t}{\delta}p = \lambda t \implies p = \lambda\delta$$

- The resulting discrete-time counting process $S_\delta(t)$ has independent increments with

$$S_\delta(t_2) - S_\delta(t_1) \sim \text{Binomial}\left(\frac{t_2 - t_1}{\delta}, \lambda\delta\right) \xrightarrow{\delta\to 0} \text{Poisson}(\lambda(t_2 - t_1))$$

- Geometric inter-arrival time approaches to exponential inter-arrival time:

$$\Pr[T_1 > t] = (1-p)^{\frac{t}{\delta}} = (1-\lambda\delta)^{\frac{t}{\delta}} \xrightarrow{\delta\to 0} e^{-\lambda t}$$

Poisson Process

Properties of Poisson Processes:

- Moments of a Poisson process $\{N(t)\}_{t\geq 0}$ with rate λ:

$$\begin{aligned} \mu_N(t) &= \lambda t \\ \text{Var}(N(t)) &= \lambda t \\ C_N(t_1, t_2) &= \lambda \min(t_1, t_2) \end{aligned}$$

- Merging independent Poisson processes: Let $N_1(t), \ldots, N_k(t)$ be independent Poisson processes with rates $\lambda_1, \ldots, \lambda_k$, respectively. Then the sum (i.e., by counting arrivals in all k processes together)

$$N(t) = \sum_{i=1}^{k} N_i(t)$$

is a Poisson process with rate $\lambda = \sum_{i=1}^{k} \lambda_i$.

- Thinning a Poisson process: Let $N(t)$ be a Poisson process with rate λ. For each arrival in this process, label it to be of type i ($i = 1, \ldots, k$) with probability p_i where $\sum_{i=1}^{k} p_i = 1$. This labeling is independent across arrivals and independent of the arrival times. Then the type i arrivals form a Poisson process with rate $p_i\lambda$ for all $i = 1, \ldots, k$, and these k Poisson processes are independent.

Gaussian Processes

Gaussian Processes:

A random process $X(t)$ is Gaussian if for all n and $t_1, \ldots, t_n$, random variables $X(t_1), \ldots, X(t_n)$ are jointly Gaussian.

Properties of Gaussian Random Process:

- It is completely specified by its mean function and autocorrelation function:
  - For all n and $t_1, \ldots, t_n$, $[X(t_1), \ldots, X(t_n)]^T$ is a Gaussian random vector with mean

    $$[\mu_X(t_1), \ldots, \mu_X(t_n)]^T$$

    and covariance matrix

    $$\mathbf{K} = \Big\{R_X(t_i, t_j) - \mu_X(t_i)\mu_X(t_j)\Big\}_{n\times n} = \Big\{C_X(t_i, t_j)\Big\}_{n\times n}$$
- Wide sense stationarity implies strict stationarity.

The random parabolas process is a Gaussian process.

Brownian Motion

Brownian Motion:

A Brownian motion (also called a Wiener process) with parameter $\sigma^2 > 0$ is a random process $\{X(t)\}_{t\geq 0}$ such that

1. $X(0) = 0$.
2. $\{X(t)\}_{t\geq 0}$ has independent increments.
3. $X(t_2) - X(t_1) \sim \mathcal{N}(0, \sigma^2(t_2 - t_1))$ for all $t_2 \geq t_1$.
4. Every sample path is continuous.

Properties of Brownian Motion:

- $\mu_X(t) = 0$.
- $R_X(t_1, t_2) = \sigma^2 \min(t_1, t_2)$.
- $\{X(t)\}_{t\geq 0}$ is a Gaussian process.