

Recall

Counting – mop up some details

Have n objects and r labeled bins; how many ways to allocate objects to bins so that n_1 in Bin 1, n_2 in Bin 2, ..., n_r in Bin r ?

Answer: Counting Principle!

$$\binom{n}{n_1} \binom{n-n_1}{n_2} \binom{n-n_1-n_2}{n_3} \cdots \binom{n-n_1-\cdots-n_{r-1}}{n_r}$$

ways to pick n_1 objects
 from n objects ↑
 ways to pick n_2 objects
 from $n-n_1$ objects

Cleverly look at
 this to realize its
 $\binom{n_r}{n_r} = 1$

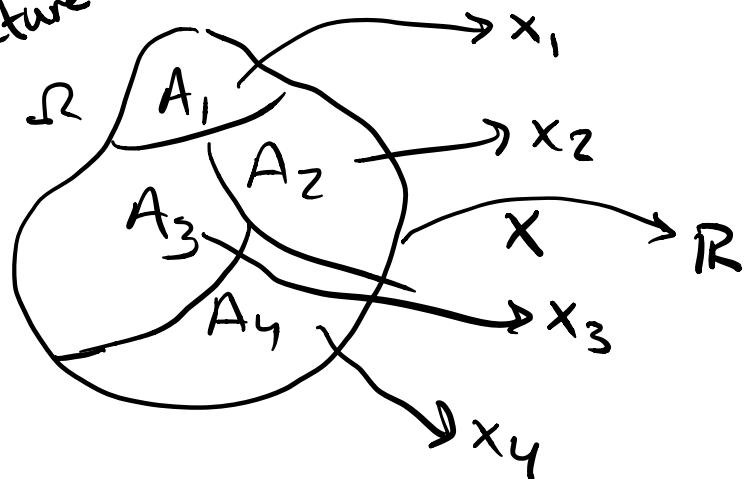
Next BIG TOPIC: DISCRETE RANDOM VARIABLES

Start with Ω and P ; a discrete random variable (r.v.) is a real valued function with domain Ω that takes on only finite or countably infinite number of different values.

i.e

$X: \Omega \rightarrow \mathbb{R}$ "X is a mapping from Ω to real's"

Picture



i.e $X(s) = x_i \nabla s \in A_i$, etc

Example - Constant random variable

Ω, P anything. $X: \Omega \rightarrow \mathbb{R}$ defined by $X(s) = 19 \nabla s \in \Omega$

Example - Coins

$$\Omega = \{H, T\}; P(H) = P(T) = 1/2$$

$X: \Omega \rightarrow \mathbb{R}$ is defined by $X(H) = 1$
 $X(T) = 0$

Example - Dice

$\Omega = \{\text{ordered pair of rolls of 2 standard 6-sided dice}\}$

IP defined so all 36 outcomes have equal probability $1/36$.

A couple associated r.v.'s

- $X(s) = \text{the number that comes up on first die} - \text{possible values are } 1, 2, 3, 4, 5, 6.$
- $X(s) = \text{sum of two rolls} - \text{possible values } 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$

Example - Coin again

$\Omega = \text{set of all H,T-sequences of length 7; } IP(\text{any}) = 2^7$

X could be $X(s) = \#\text{(H's)} \text{ in sequence}$

What if we wanted index of first head?

$$X(s) = \begin{cases} \text{index of first head in sequence, if } s \text{ contains H} \\ \text{Anything, when } s = \text{all-Tail sequence} \end{cases}$$

case of no heads

Must be a bipartite definition because X must assign a # to every $s \in \Omega$

So, given Ω, IP ; associated w/ a discrete r.v. $X: \Omega \rightarrow \mathbb{R}$
is X 's probability mass function (p.m.f)

Notation for prob will be

- upper case random variables **ALWAYS**

P_X lowercase things are deterministic quantities

P_X is defined as follows: for every possible value of $x \in X$,

$$P_X(x) = P(A_x) \text{ where } A_x = \{s \in \Omega : X(s) = x\}$$

i.e. $P_X(x)$ = the probability that the r.v. X takes on the specific value x

Book uses

$$P(\{X = x\}) \text{ or } P(X = x) \quad \begin{matrix} \text{misuse of notation} \\ \text{compared to above} \end{matrix}$$

to refer to $P(A_x)$, where $A_x = \{s \in \Omega : X(s) = x\}$

Things to note about P_X :

- $P_X(x) \geq 0$ for all possible values of X
(why? cause for any x , $P_X(x)$ is P (an event) ≥ 0 !)
- If V is any finite or countably infinite set of possible values of X , then if we set

$B = \text{the event } "X \in V"$

$$\text{i.e. } B = \{s \in \Omega : X(s) \in V\}$$

then

$$P(B) = \sum_{x \in V} P_X(x)$$

Idea: events $A_x = \{s \in \Omega : X(s) = x\}$ as X runs over V are disjoint subsets of B whose union is B .

Thus,

$$P(B) = \sum_{x \in V} P(A_x) = \sum_{x \in V} P_x(x)$$

"Probability that X is in V = sum over $x \in V$ of the probability that $X=x$ "

Special Case: $V = \text{set of all possible values of } X$

$$\sum_{x \in V} P_x(x) = 1$$

Some common pmf's that arise in applications

- Discrete uniform pmf on the interval $a \leq k \leq b$ ($a, b \in \mathbb{Z}$)

$$P_X(k) = \begin{cases} \frac{1}{b-a+1}, & a \leq k \leq b \\ 0, & \text{all other } k \end{cases}$$

- Let $p \in [0, 1]$, the Bernoulli p pmf defined by

$$P_X(k) = \begin{cases} p, & k=1 \\ 1-p, & k=0 \end{cases}$$

Where does X come from? What is Ω ? What is P ?
 Turns out, *cue dramatic music **DOESN'T MATTER**

Just as a given Ω, P give rise to many different rv's, a given "legal" pmf has many different background Ω, P , $X: \Omega \rightarrow \mathbb{R}$.

What's important about a discrete r.v. is its pmf. Where the r.v. came from doesn't really matter.

That said, we could build an Ω, \mathcal{P} if we wanted too.

Example - Bernoulli

$$\Omega = \{H, T\} \quad P(\{H\}) = p; \quad P(\{T\}) = 1-p \\ X(H) = 1 \quad X(T) = 0$$

That's just one of many ways to do it!

Example - Discrete uniform on interval $7 \leq k \leq 12$

$$\Omega = \text{set of outcomes of one fair die roll} \\ = \{1, 2, 3, 4, 5, 6\}$$

\mathcal{P} = discrete uniform probability law on Ω

$$X(s) = s \quad \text{for all } s \in \Omega$$

So we can do this, but we NEVER do. What's important is the r.v w/ that pmf not where it came from.

Some more pmf's below ↓

Binomial RV

Given positive integer n for some $p \in [0,1]$, the Binomial (n,p) pmf defined as follows:

$$P_X(k) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k}, & 0 \leq k \leq n \\ 0, & \text{all other } k \end{cases}$$

This could arise from

$$\Omega = \{\text{all H-T sequences of length } n\}$$

$$P(\text{any sequence}) = p^{\#H} (1-p)^{\#T}$$

like the outcome of n -independent flips of a possibly unfair coin.

$$X(\text{any sequence}) = \#\text{(Heads)} \text{ in sequence}$$

← saw this last time in counting example!

Geometric R.V

Given $p \in (0,1)$, geometric pmf defined by

$$P_X(k) = p(1-p)^{k-1} \text{ for all } 1 \leq k < \infty \text{ (positive integers)}$$



arises from $P(\text{it takes } k \text{ flips to flip a heads})$