

Last time we showed,

$$\int_{\gamma_1} \bar{z} dz \neq \int_{\gamma_2} \bar{z} dz$$

Theorem: (antiderivative)

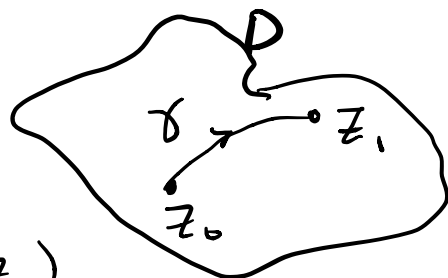
- Assuming f is continuous on an open, connected set D .

• γ is any contour in this set D

• $f(z)$ has an antiderivative $F(z)$.

where

$$F'(z) = f(z) \rightarrow F(z) \text{ is single valued in } D.$$



Then
$$\int_{\gamma} f(z) dz = F(z_1) - F(z_0)$$

depends on endpoints but not curve

\Rightarrow path independence if antiderivative exists

Proof: Parametrize γ by t .

Let $\gamma = z(t)$, $0 \leq t \leq 1$, where $z(0) = z_0$ and $z(1) = z_1$,

$$\int_{\gamma} f(z) dz = \int_{t=0}^{t=1} f(z(t)) \frac{dz}{dt} dt = \int_{t=0}^1 F'(z(t)) \frac{dz}{dt} dt$$

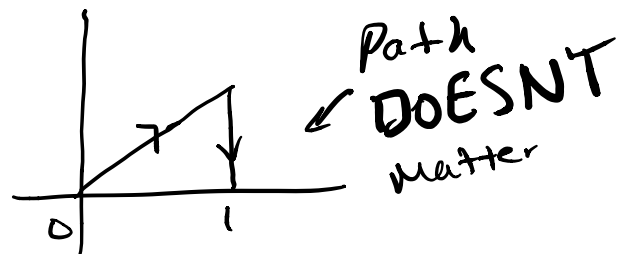
$$= \int_{t=0}^1 \frac{d}{dt} \left(F(z(t)) \right) dt \quad \swarrow \text{Fundamental THM of Calculus.}$$

$$= F(z(1)) - F(z(0))$$

$$= F(z_1) - F(z_0)$$

Example:

$$\int_{\gamma} z^2 dz \quad \text{where } \gamma \text{ is shown}$$



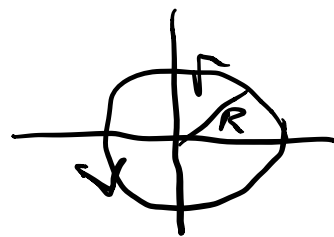
Since z^2 has an antiderivative $\frac{z^3}{3}$,

$$\int_0^1 z^2 dz = \frac{1}{3}$$

Example: Soln ①

$$\int_{\gamma} z dz \quad \text{where } \gamma = \text{circle of radius } R, \text{ centered at origin.} \quad \curvearrowright$$

$$\begin{aligned} x &= R \cos(t) \\ y &= R \sin(t) \end{aligned} \quad \left\{ \begin{aligned} z(t) &= R e^{it} \\ dz &= R e^{it} i dt \\ 0 &\leq t \leq 2\pi \end{aligned} \right.$$



$$\int_{\gamma} z dz = = \int_0^{2\pi} R^2 i e^{i2t} dt = R^2 i \left[e^{i4\pi} - e^0 \right]$$

$$= 0$$

Soln (2)

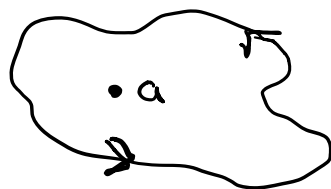
$$z_{2\pi} = z_0$$

$$\int_{\gamma} z \, dz = \int_{z_0}^{z_{2\pi}} z \, dz = \left. \frac{z^2}{2} \right|_{z_0}^{z_{2\pi}} = 0$$

"work around a closed loop is zero"

More General Result

Let γ = any closed curve that encircles the point $a \in \mathbb{C}$ once CCW.



THEN

$$\int_{\gamma} (z-a)^n \, dz = 0 \quad \text{for } n \neq -1$$

Proof: Antiderivative of $(z-a)^n$ is $\frac{1}{n+1} (z-a)^{n+1}$

"worry about a b/c we have negative powers"

single valued and analytic
in $\mathbb{C} - \{a\}$

open
connected
set

punctured plane

What goes wrong when $n = -1$?

$\int_{\gamma} \frac{dz}{z-a}$ } antiderivative would be $\log(z-a)$ BUT then it is **NOT** single valued in $\mathbb{C}-\{a\}$

Special Case: $\gamma =$ ^{ccw} positively oriented circle of radius R about a .

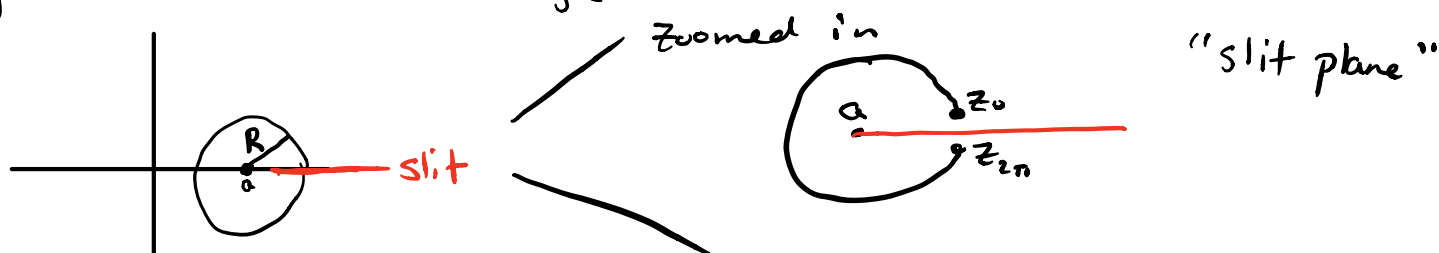
$$\int_{\gamma} \frac{dz}{z-a}, \quad z(t) = a + Re^{it} \rightarrow z-a = Re^{it}$$

$$dz = iRe^{it} dt \quad 0 \leq t \leq 2\pi$$

$$= \int_0^{2\pi} i dt = 2\pi i \neq 0!$$

TANGENT BEGINS

Say antiderivative is $\log(z-a)$ like real value case.



$$\log(z-a) \Big|_{z_0}^{z_{2\pi}} = \ln r + i\theta \Big|_{z_0}^{z_{2\pi}} = 2\pi i$$

$\log(z-a)$ is analytic and single valued in slit plane,
 \mathbb{C} -branch cut.

Note: slit plane doesn't include whole circle.

TANGENT ENDS

Cauchy's Theorem - One of the big theorems

Recall: $\int_{\gamma} (z-a)^n dz = 0$, $\gamma \rightarrow$ closed curve
 $n \neq -1$

So any polynomial $p(z) = c_0 + c_1 z + c_2 z^2 + \dots + c_n z^n$ also integrates to 0 around any closed curve.

i.e.
$$\oint_{\gamma} p(z) dz = 0$$

Cauchy's Theorem is a generalization of this:

$$\oint_{\gamma} f(z) dz = 0 \text{ if } f \text{ is analytic in a "simply connected" domain } D.$$

$\gamma \rightarrow$ any rectifiable closed curve in D .

"simply connected" - no holes.

Precisely: D is simply connected if every closed loop can be continuously deformed to a point while staying in D