Convolutions

$$(\mu * \nu)(A) = \int_{\mathcal{X}} \int_{\mathcal{X}} \mathbb{1}_{\{x+y \in A\}} \, d\mu(x) \, d\nu(x)$$

$$= \mathbb{E}_{\mu \otimes \nu} \left[\mathbb{1}_{\{x+y \in A\}} \right]$$

λ Lebesgue

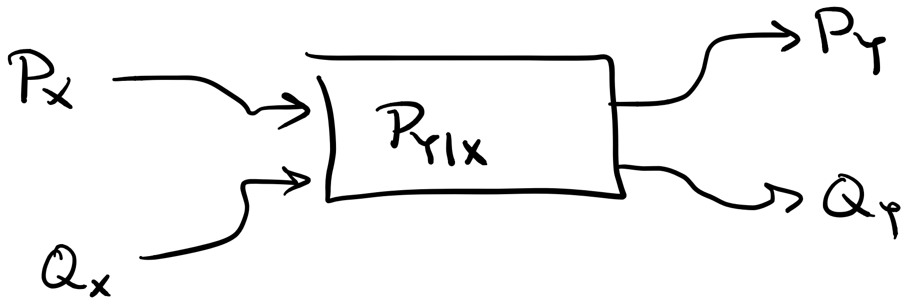
If μ and ν have densities $f = \frac{d\mu}{d\lambda}$ and $g = \frac{d\nu}{d\lambda}$:

$$(\mu * \nu)(A) = \int_{\mathcal{X}} \int_{\mathcal{X}} \mathbb{1}_{\{x+y \in A\}} f(x) \, g(y) \, dx \, dy$$

$$\overset{x+y=s}{=} \int_{\mathcal{X}} \mathbb{1}_{\{s \in A\}} \left(\int_{\mathcal{X}} f(s-y) \, g(y) \, ds \right)$$

$$= \int_A (f * g)(s) \, ds$$

Data Processing Inequality



$$D_f(P_X \| Q_X) \geq D_f(P_Y \| Q_Y)$$

Item (iv)

$$D_f(P_X \| Q_X) = D_f(\underset{\substack{\uparrow \\ P_X P_{Y|X}}}{P_{XY}} \| \underset{\substack{\uparrow \\ Q_X P_{Y|X}}}{Q_{XY}})$$

Projections

Let $\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2$

Let $X = (X_1, X_2) \sim P_{X_1,X_2} \in \mathcal{P}(\mathcal{X}_1 \times \mathcal{X}_2)$

$$X' = (X_1', X_2') \sim P_{X_1', X_2'} \in \mathcal{P}(\mathcal{X}_1 \times \mathcal{X}_2)$$

Let $\mathcal{Y} = \mathcal{X}_1$ and $P_{Y|X_1,X_2} = \delta_{X_1}$

namely,

$$Y = \mathrm{proj}_1(X_1, X_2) = X_1$$

DPI $\Rightarrow$ $D_f(P_{X_1,X_2} \| Q_{X_1,X_2}) \geq D_f(P_{X_1} \| Q_{X_1})$

Item (iv) $\Rightarrow$ $D_f(P_{X_1,X_2} \| Q_{X_1,X_2}) = D_f(P_{X_1,X_2,Y} \| Q_{X_1,X_2,Y})$

Note: $P_{X_1,X_2} = P_{X_1} P_{X_2|X_1}$; $Q_{X_1,X_2} = Q_{X_1} Q_{X_2|X_1}$

If $Q_{X_2|X_1} = P_{X_2|X_1}$, then

$$D_f(P_{X_1,X_2} \| Q_{X_1,X_2}) \underset{\substack{\uparrow \\ \text{item (iv) on DPI}}}{=} D_f(P_{X_1} \| Q_{X_1})$$

All in all: If $P_{X_2|X_1} = Q_{X_2|X_1}$ then

$$D_f(P_{X_1} \| Q_{X_1}) = D_f(P_{X_1,X_2} \| Q_{X_1,X_2}) = D_f(P_{X_1,X_2,Y} \| Q_{X_1,X_2,Y})$$

where

$$P_{X_1,X_2,Y} = P_{X_1} P_{X_2|X_1} P_{Y|X_1,X_2}$$

$$Q_{X_1,X_2,Y} = Q_{X_1} P_{X_2|X_1} P_{Y|X_1,X_2}$$

Examples of Transition Kernels

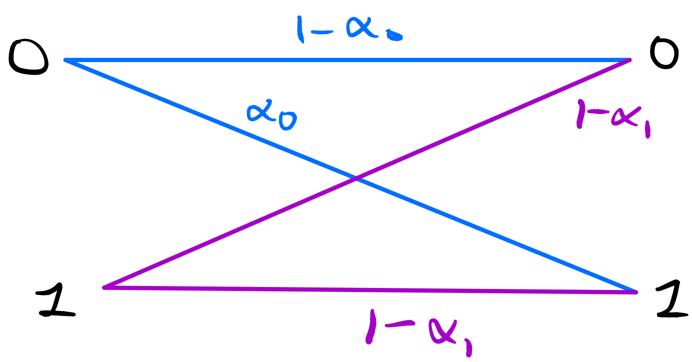
$$P_{Y|X}(\cdot|\cdot) : (\mathcal{X}, \mathcal{F}) \to (\mathcal{Y}, \mathcal{G})$$

(i) $P_{Y|X}(\cdot|x) \in \mathcal{P}(\mathcal{Y}) \quad \forall x \in \mathcal{X}$

(ii) $P_{Y|X}(B|\cdot)$ r.v. w.r.t $(\mathcal{X}, \mathcal{F}) \quad \forall B \in \mathcal{G}$

• Binary Channel

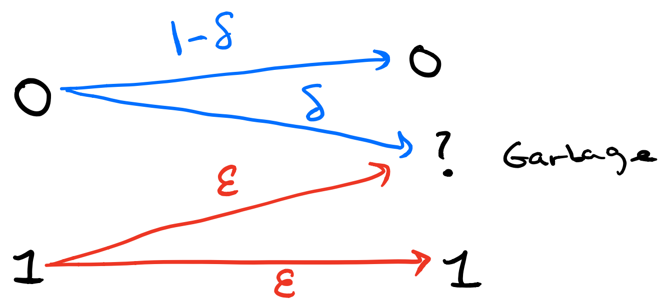
$\mathcal{X} = \{0, 1\} \qquad \mathcal{Y} = \{0, 1\}$



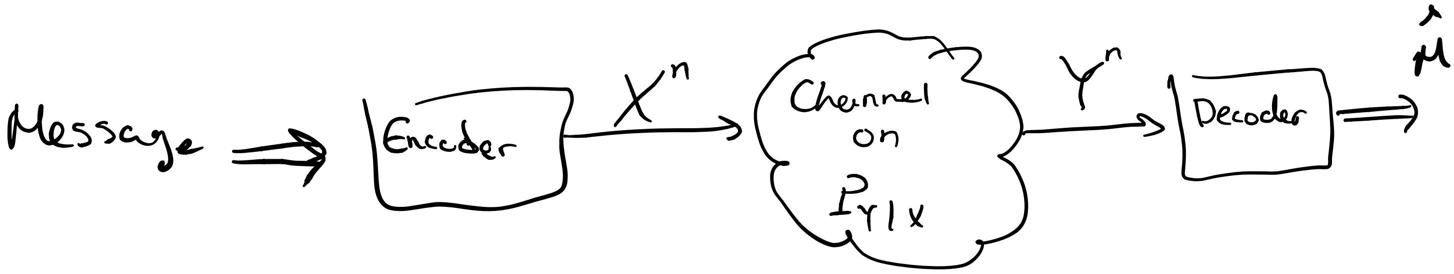
$P_{Y|X}(\cdot|0) = \mathrm{Ber}(\alpha_0)$

$P_{Y|X}(\cdot|1) = \mathrm{Ber}(1-\alpha_1)$

- Binary Erasure Channel



Shannon's Channel Capacity Result



code pair of encoding + decoding function

$$C_n = (f_n, g_n)$$

Fix $c_n = (f_n, g_n)$

$$\Rightarrow P^{(C_n)}_{MX^nY^n\hat{M}}$$