ECE 4110 Homework 3

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Due by 5pm on September 26

1 Key Concepts and Reading Material

- Correlation, independence, and covariance (Chapter 5.6).
- Conditional distribution and conditional expectation (Chapter 3.4, Chapter 5.7).
- MMSE estimator (Chapter 6.5.3).

2 Assignment

1. **Computing Expectations via Conditioning**

   The number of customers that arrive at a service station during a time t is a Poisson random variable with parameter βt. The time T required to service each customer is an exponential random variable with parameter α.

   Let N denote the number of customers arrived during the service time of a specific customer. Find the mean and variance of N.

   (*Hint: In many cases, it is easier to compute the expectation of X by first conditioning on another random variable Y and then taking expectation over Y. Specifically, for any function $h(\cdot)$ of X, we have*

   $$\mathbb{E}[(h(X)] = \mathbb{E}\Big[\mathbb{E}[h(X)|Y]\Big].$$

   Use this to find the mean and variance of N by conditioning on T.)

2. **Conditioning Reduces Variance**

   Given random variables X and Y, the conditional expectation $\mathbb{E}(X|Y)$ is a random variable itself with a well defined mean and variance. We have shown that the mean values of $\mathbb{E}(X|Y)$ and X are the same, i.e.,

   $$\mathbb{E}(\mathbb{E}(X|Y)) = \mathbb{E}(X).$$

   Show that the variance of $\mathbb{E}(X|Y)$ is smaller than or equal to the variance of X, i.e.,

   $$\mathsf{Var}(\mathbb{E}(X|Y)) \leq \mathsf{Var}(X)$$

   (*Hint: show that (by working from the right-hand side of the equation)*

   $$\mathsf{Var}(X) = \mathbb{E}(\mathsf{Var}(X|Y)) + \mathsf{Var}(\mathbb{E}(X|Y)),$$

where the conditional variance Var$(X|Y)$ *of* X *given* Y *is defined as*

$$\mathsf{Var}(X|Y) \triangleq \mathbb{E}((X - \mathbb{E}(X|Y))^2|Y).$$

The result thus follows since the conditional variance Var$(X|Y)$ *is a nonnegative random variable whose expectation must be nonnegative.*)

3. **Sum of Uncorrelated Random Variables**

   Show that if $X_1, \ldots, X_m$ are pairwise uncorrelated, then

$$\mathsf{Var}(\sum_{i=1}^{m} X_i) = \sum_{i=1}^{m} \mathsf{Var}(X_i).$$

4. **Independence, Correlation, and MMSE Estimation**

   Consider two random variables X and Y with the following joint PDF.

$$f_{XY}(x,y) = \begin{cases} ce^{-x}e^{-y} & 0 \le y \le x < \infty \\ 0 & \text{elsewhere} \end{cases}$$

   (a) Find the value of the constant c in the joint PDF.
   (b) Calculate $\Pr(X + Y \le 1)$.
   (c) Are X and Y independent (the joint PDF appears to factor)? Justify your answer.
   (d) Find the correlation, covariance, and correlation coefficient of X and Y.
   (e) Find the MMSE estimate of X using a constant and the resulting mean squared error.
   (f) Find the MMSE estimate of X based on Y (among all functions $g(Y)$ of Y).

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ECE 4110 HW3

9/26/19 5:00PM

1. **Computing Expectations via Conditioning**

The number of customers that arrive at a service station during a time t is a Poisson random variable with parameter βt. The time T required to service each customer is an exponential random variable with parameter α.

Let N denote the number of customers arrived during the service time of a specific customer. Find the mean and variance of N.

Arrival, $A \sim \text{Pois}(\beta t)$ $(\lambda = \beta t)$

Service Time, $T \sim \exp(\alpha)$

$N \rightarrow$ number of customers arrived during the service time of a specific customer

Want $\mathbb{E}[N]$, $\text{Var}(N)$.

Know

$$F_T(t) = \begin{cases} 1 - e^{-\alpha t} & , t \geq 0 \\ 0 & , t < 0 \end{cases} \qquad F_A(a) = \sum_{i=0}^{a} \frac{e^{-\beta t}(\beta t)^i}{i!}$$

Via Law of Iterated Expectation,

$$\mathbb{E}_N[N] = \mathbb{E}_T[\mathbb{E}_{N|T}[N|T]]$$

A customer is being serviced.

Want average number of arrivals of new customers during this service time.

So we have

$$\mathbb{E}[N \mid T] = \mathbb{E}[\text{Pois}(\beta T)] = \beta T$$

$$\mathbb{E}_T\left[\mathbb{E}_{N|T}[N \mid T]\right] = \mathbb{E}_T\left[\beta T\right] = \beta\, \mathbb{E}_T[T] = \frac{\beta}{\alpha}$$

Thus,

$$\mathbb{E}[N] = \beta/\alpha$$

By law of total variance, know

$$\text{Var}(N) = \mathbb{E}[\text{Var}(N \mid T)] + \text{Var}(\mathbb{E}[N \mid T])$$

$$\text{Var}(\mathbb{E}[N \mid T]) = \text{Var}(\beta T) = \beta^2 \text{Var}(T) = \frac{\beta^2}{\alpha^2}$$

$$\text{Var}(N \mid T) = \text{Var}(\text{Pois}(\beta T)) = \beta T$$

$$\mathbb{E}[\text{Var}(N \mid T)] = \mathbb{E}[\beta T] = \beta/\alpha$$

Thus

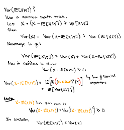
$$\text{Var}(N) = \frac{\beta}{\alpha} + \frac{\beta^2}{\alpha^2}$$

$$= \frac{\beta\alpha + \beta^2}{\alpha^2}$$

2. **Conditioning Reduces Variance**

Given random variables X and Y, the conditional expectation $\mathbb{E}(X|Y)$ is a random variable itself with a well defined mean and variance. We have shown that the mean values of $\mathbb{E}(X|Y)$ and X are the same, i.e.,

$$\mathbb{E}(\mathbb{E}(X|Y)) = \mathbb{E}(X).$$



Show that the variance of $\mathbb{E}(X|Y)$ is smaller than or equal to the variance of X, i.e.,

$$\text{Var}(\mathbb{E}(X|Y)) \leq \text{Var}(X)$$

We first show

$$\text{Var}(X) = \text{Var}(\mathbb{E}[X|Y]) + \mathbb{E}[\text{Var}(X|Y)]$$

where

$$\text{Var}(X|Y) \triangleq \mathbb{E}[(X - \mathbb{E}[X|Y])^2 | Y]$$

$$\begin{aligned}
\mathbb{E}[\text{Var}(X|Y)] &= \mathbb{E}[\mathbb{E}[(X - \mathbb{E}[X|Y])^2 | Y]] \\
&= \mathbb{E}[\mathbb{E}[\{X^2 - 2X\mathbb{E}[X|Y] + (\mathbb{E}[X|Y])^2\} | Y]] \\
&= \mathbb{E}[\mathbb{E}[X^2|Y] - 2(\mathbb{E}[X|Y])^2 + (\mathbb{E}[X|Y])^2] \\
&= \mathbb{E}[X^2] - \mathbb{E}[(\mathbb{E}[X|Y])^2]
\end{aligned}$$

$$\begin{aligned}
\text{Var}(\mathbb{E}[X|Y]) &= \mathbb{E}[(\mathbb{E}[X|Y])^2] - (\mathbb{E}[\mathbb{E}[X|Y]])^2 \\
&= \mathbb{E}[(\mathbb{E}[X|Y])^2] - (\mathbb{E}[X])^2
\end{aligned}$$

It follows that

$$\begin{aligned}
\text{Var}(X) &= \text{Var}(\mathbb{E}[X|Y]) + \mathbb{E}[\text{Var}(X|Y)] \\
&= \mathbb{E}[(\mathbb{E}[X|Y])^2] - (\mathbb{E}[X])^2 \quad \mathbb{E}[X^2] - \mathbb{E}[(\mathbb{E}[X|Y])^2] \\
&= \mathbb{E}[X^2] - (\mathbb{E}[X])^2
\end{aligned}$$

Thus, since $\text{Var}(X|Y)$ a positive rv, $\mathbb{E}[\text{Var}(X|Y)] \geq 0$ and

$$\text{Var}(\mathbb{E}[X|Y]) \leq \text{Var}(X)$$

3. **Sum of Uncorrelated Random Variables**

Show that if $X_1, \ldots, X_m$ are pairwise uncorrelated, then

$$\text{Var}(\sum_{i=1}^{m} X_i) = \sum_{i=1}^{m} \text{Var}(X_i).$$

X_1, X_2 uncorrelated $\Rightarrow \text{Cov}(X_1, X_2) = 0 \Rightarrow \mathbb{E}[X_1 X_2] = \mathbb{E}[X_1]\mathbb{E}[X_2]$

pairwise uncorrelated means <u>for any</u> X_i, X_j, $1 \le i, j \le m$, $i \ne j$, $\text{Cov}(X_i, X_j) = 0$.

Thus $\quad \mathbb{E}[X_j X_i] = \mathbb{E}[X_j]\,\mathbb{E}[X_i] \quad \forall\, i \ne j,\ 1 \le i, j \le m$

$$\begin{aligned}
\text{Var}\left(\sum_{i=1}^{m} X_i\right) &= \text{Var}(X_1 + X_2 + \cdots + X_m) \\
&= \mathbb{E}\left[\left((X_1 + \cdots + X_m) - \mathbb{E}[X_1 + \cdots + X_m]\right)^2\right] \\
&= \mathbb{E}\left[(X_1 + \cdots + X_m)^2\right] - \left(\mathbb{E}[X_1 + \cdots + X_m]\right)^2
\end{aligned}$$

First Term

$$\begin{aligned}
(X_1 + \cdots + X_m)^2 &= \sum_{i=1}^{m} X_i \left(\sum_{j=1}^{m} X_j\right) \\
&= \sum_{i=j} X_i^2 + \sum_{i<j} X_i X_j + \sum_{i>j} X_i X_j \qquad i, j \in [1, m] \\
&= \sum_{i=j} X_i^2 + 2 \sum_{i \ne j} X_i X_j
\end{aligned}$$

The same

Thus

$$\begin{aligned}
\mathbb{E}\left[(X_1 + \cdots + X_m)^2\right] &= \mathbb{E}\left[\sum_{i=j} X_i^2\right] + 2\,\mathbb{E}\left(\sum_{i \ne j} X_i X_j\right) \\
&= \sum_{i=j} \mathbb{E}[X_i^2] + 2 \sum_{i \ne j} \mathbb{E}[X_i X_j]
\end{aligned}$$

by linearity of expectation

Second Term

$$\mathbb{E}[X_1+\cdots+X_m] = \mathbb{E}[X_1]+\mathbb{E}[X_2]+\cdots+\mathbb{E}[X_m] = \sum_{i=1}^{m}\mathbb{E}[X_i]$$

$$\left(\mathbb{E}[X_1+\cdots+X_m]\right)^2 = \left(\sum_{i=1}^{m}\mathbb{E}[X_i]\right)^2 \qquad k,j \in [1,m]$$

$$= \sum_{i=1}^{m}\mathbb{E}[X_i]\left(\sum_{j=1}^{m}\mathbb{E}[X_j]\right)$$

$$= \sum_{i=j}\left(\mathbb{E}[X_i]\right)^2 + 2\sum_{\neq j}\mathbb{E}[X_i]\mathbb{E}[X_j]$$

$\mathbb{E}\left[(X_1+\cdots+X_m)^2\right] - \left(\mathbb{E}[X_1+\cdots+X_m]\right)^2$ is thus

$$\sum_{i=j}\mathbb{E}[X_i^2] + 2\sum_{i\neq j}\mathbb{E}[X_iX_j] - \left(\sum_{i=j}\left(\mathbb{E}[X_i]\right)^2 + 2\sum_{\neq j}\mathbb{E}[X_i]\mathbb{E}[X_j]\right)$$

$$= \sum_{i=j}\mathbb{E}[X_i^2] - \left(\mathbb{E}[X_i]\right)^2 + \cancel{2\sum_{i\neq j}\left(\mathbb{E}[X_iX_j] - \mathbb{E}[X_i]\mathbb{E}[X_j]\right)} \xrightarrow{} 0 \text{ by pairwise independence}$$

$$= \mathbb{E}[X_1^2] - \left(\mathbb{E}[X_1]\right)^2 + \mathbb{E}[X_2^2] - \left(\mathbb{E}[X_2]\right)^2 + \cdots + \mathbb{E}[X_m^2] - \left(\mathbb{E}[X_m]\right)^2$$

$$= \mathrm{Var}(X_1) + \mathrm{Var}(X_2) + \cdots + \mathrm{Var}(X_m)$$

$$= \sum_{i=1}^{m}\mathrm{Var}(X_i)$$

Thus,

$$\mathrm{Var}\left(\sum_{i=1}^{m}X_i\right) = \sum_{i=1}^{m}\mathrm{Var}(X_i)$$

if $X_1,\ldots,X_m$ are pairwise uncorrelated.

4. **Independence, Correlation, and MMSE Estimation**

Consider two random variables X and Y with the following joint PDF.

$$f_{XY}(x,y) = \begin{cases} ce^{-x}e^{-y} & 0 \le y \le x < \infty \\ 0 & \text{elsewhere} \end{cases}$$

(a) Find the value of the constant c in the joint PDF.

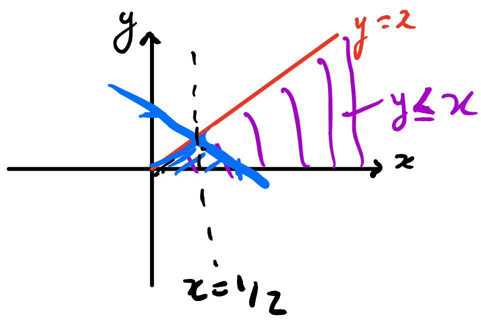
$$\int_{x=0}^{\infty}\int_{y=0}^{x} ce^{-y}e^{-x}\,dy\,dx = 1$$

$$= c\int_{x=0}^{\infty} e^{-x}\left(-e^{-y}\Big|_{y=0}^{x}\right)dx$$

$$= c\int_{x=0}^{\infty} e^{-x} - e^{-2x}\,dx = 1$$

$$= c\left[1 - \frac{1}{2}\right] \rightarrow \boxed{c = 2}$$

(b) Calculate $\Pr(X + Y \le 1)$.



$\Pr(X+Y \le 1) = \Pr(Y \le 1 - X)$

Integrate over blue region

$$\int_{x=0}^{1/2}\int_{y=0}^{x} f_{X,Y}(x,y)\,dy\,dx + \int_{x=1/2}^{1}\int_{y=0}^{1-x} f_{X,Y}(x,y)\,dy\,dx$$

Wolfram $\Longrightarrow$ $\boxed{0.264241}$

Now we do some precomputations

Marginal PDFs

$$f_Y(y) = \int_{x=y}^{\infty} f_{X,Y}(x,y)\,dx = \int_{x=y}^{\infty} 2e^{-x}e^{-y}\,dx = 2e^{-2y} \quad \text{for } 0 \le y \le x$$

$$f_X(x) = \int_{y=0}^{y=x} f_{X,Y}(x,y)\,dy = \int_{y=0}^{y=x} 2e^{-x}e^{-y}\,dy = 2e^{-x}(1-e^{-x}), \quad \text{for } y \le x \le \infty$$

Conditional PDFs

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{2e^{-x}e^{-y}}{2e^{-2y}} = e^{-x}e^{y}$$

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{2e^{-x}e^{-y}}{2(1-e^{-x})} = \frac{e^{-x}}{1-e^{-x}}e^{-y}$$

Expectations

$$\mathbb{E}[Y] = \int_{y=0}^{\infty} y\,2e^{-2y}\,dy = \frac{2}{4} = \frac{1}{2}$$

$$\mathbb{E}[X|Y=y] = \int_{x=y}^{\infty} x f_{X|Y=y}(x|y)\,dx = \int_{x=y}^{\infty} xe^{-x}e^{y}\,dx = y+1$$

$$\mathbb{E}[X|Y] = (Y+1)$$

$$\mathbb{E}[X] = \mathbb{E}_Y[\mathbb{E}_{X|Y}[X|Y]] = \mathbb{E}[Y]+1 = \frac{1}{2}+1 = \frac{3}{2}$$

Correlation b/t X,Y

$$\mathbb{E}[XY] = \int_{x=0}^{\infty}\int_{y=0}^{x} xy\, f_{X,Y}(x,y)\, dy\, dx = \int_{x=0}^{\infty}\int_{y=0}^{x} xy(2e^{-x}e^{-y})\, dy\, dx = 1$$

Covariance b/t X,Y

$$\mathrm{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y]$$
$$= 1 - \frac{3}{4} = \frac{1}{4}$$

(c) Are X and Y independent (the joint PDF appears to factor)? Justify your answer.

X,Y independent if

$$f_{X,Y}(x,y) = f_X(x) f_Y(y)$$

$$f_X(x) = \begin{cases} 2(e^{-x} - e^{-2x}), & y \le x < \infty \\ 0, & o/w \end{cases} \qquad f_Y(y) = \begin{cases} 2e^{-2y}, & 0 \le y \le x \\ 0, & o/w \end{cases}$$

$$f_X(x) f_Y(y) = 2(e^{-x} - e^{-2x})(e^{-2y}) \neq 2e^{-x}e^{-y} \; ; \; 0 \le y \le x < \infty$$

Thus X and Y are **NOT** independent

Also easy to see that x has y dependence for bounds.

(d) Find the correlation, covariance, and correlation coefficient of X and Y.

Correlation $= \mathbb{E}[XY] = 1$

$\mathrm{Cov}(X,Y) = 11/16$

$$\rho_{X,Y} = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}}$$

$$\mathrm{Var}(Y) = \int_{y=0}^{+\infty} y^2 f_Y(y)\,dy - \left(\int_{y=0}^{+\infty} y\, f_Y(y)\,dy\right)^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$$\mathrm{Var}(X) = \mathrm{Var}(\mathbb{E}[X|Y]) + \mathbb{E}[\mathrm{Var}(X|Y)]$$

$$\mathrm{Var}(X|Y=y) = \int_{x=y}^{\infty} x^2 f_{X|Y=y}(x|y)\,dx - \left(\int_{x=y}^{\infty} x\, f_{X|Y=y}(x|y)\,dx\right)^2$$

$$= y^2 + 2y + 2 - (y+1)^2$$

$$= y^2 + 2y + 2 - (y^2 + 2y + 1)$$

$$= y^2 + 2y + 2 - y^2 - 2y - 1 = 1$$

$\mathrm{Var}(X|Y) = 1$

$$\int_{y=0}^{\infty} (y+1)^2 f_Y(y)\,dy = \int_{y=0}^{\infty} (y+1)^2\, 2e^{-y}\,dy = \frac{5}{2}$$

Law of Total Variance

$$\mathrm{Var}(X) = \mathrm{Var}(Y+1) + 1$$

$$= \mathbb{E}[(Y+1)^2] - (\mathbb{E}[Y+1])^2 + 1$$

$$= \frac{5}{2} - \left(\frac{9}{4}\right) + 1$$

$$= \frac{10}{4} - \frac{9}{4} + \frac{4}{4}$$

$$= \frac{5}{4}$$

$$\rho_{X,Y} = \frac{1/4}{\sqrt{\left(\frac{1}{4}\right)\left(\frac{5}{4}\right)}} = \frac{1/4}{\sqrt{\frac{5}{16}}} = \frac{1}{\sqrt{5}} = 0.447$$

(e) MMSE estimate of X using a constant.

$$\hat{X}_{MMSE} = a^*$$

where

$$a^* = \underset{a \in \mathbb{R}}{\operatorname{argmin}}\ \mathbb{E}[(X-a)^2]$$

$$MSE = \mathbb{E}[(X-a)^2]$$
$$= \mathbb{E}\left[(X - \mathbb{E}[X] + \mathbb{E}[X] - a)^2\right]$$
$$= \mathbb{E}\left[(X-\mathbb{E}[X])^2\right] + \underbrace{\cancel{2\,\mathbb{E}[(X-\mathbb{E}[X])(\mathbb{E}[X]-a)]}}_{0} + \mathbb{E}\left[(\mathbb{E}[X]-a)^2\right]$$

Want to minimize

$$\mathbb{E}[(X-\mathbb{E}[X])^2] + \mathbb{E}[(\mathbb{E}[X]-a)^2]$$
$$\rightarrow \mathrm{Var}(X) + (\mathbb{E}[X]-a)^2$$

Take $a = \mathbb{E}[X]$

So,

$$\boxed{\hat{X}^*_{MMSE} = \mathbb{E}[X] = \frac{3}{2}}$$

Resulting MSE is

$$MSE = \mathrm{Var}(X) = \frac{5}{4}$$

(f) MMSE estimate of X based on Y

Want

$$\hat{X}_{MMSE} = Y^*$$

where

$$Y^* = \underset{Y \in h(Y)}{\operatorname{argmin}} \mathbb{E}\left[(X - h(Y))^2\right]$$

For ANY $h(Y)$,

$$\mathbb{E}\left[(X - h(Y))^2\right] = \mathbb{E}\left((X - \mathbb{E}(X|Y)) + (\mathbb{E}(X|Y) - h(Y))\right)^2$$

$$= \mathbb{E}\left[(X - \mathbb{E}(X|Y))^2\right] + \cancel{2\mathbb{E}\left[(X - \mathbb{E}(X|Y))(\mathbb{E}(X|Y) - h(Y))\right]}^{\,0} + \mathbb{E}\left[(\mathbb{E}(X|Y) - h(Y))^2\right]$$

So,

$$\mathbb{E}\left[(X - h(Y))^2\right] = \mathbb{E}\left[(X - \mathbb{E}(X|Y))^2\right] + \mathbb{E}\left[(\mathbb{E}(X|Y) - h(Y))^2\right]$$

The choice of $h(Y)$ which minimizes this is $h(Y) = \mathbb{E}[X|Y]$

Thus, $\mathbb{E}(X|Y)$ is the function of Y that minimizes $\mathbb{E}((X - h(Y))^2)$ over ALL functions $h(Y)$

Conclude

$$\boxed{\hat{X}^*_{MMSE} = \mathbb{E}[X|Y] = (Y+1)}$$