

Since last prelim

Dual Vector Spaces

define $f: V \rightarrow \mathbb{F}$ as . . .

$f \in V^*$ means f is a LT

$$- V^* = \text{Lin}(V, \mathbb{F})$$

$$- \dim V^* = \dim V$$

- given a basis V , dual basis for V^*

Direct Sums

$$V = W_1 \oplus W_2 \oplus \dots \oplus W_r$$

$$\text{if } V = W_1 + W_2 + \dots + W_r$$

and if $\vec{w}_1 + \vec{w}_2 + \dots + \vec{w}_r = 0$ then $\vec{w}_i \neq 0$

basis of $W_1 \oplus \dots \oplus W_r :=$ union of basis of each
and

$$\dim V = \sum_{i=1}^r \dim W_i$$

Quotient Vector Spaces

$W \subseteq V$, define V/W

2 Natural Maps

$$\textcircled{1} \quad T: V \rightarrow V/W$$

$$\ker T = W$$

$$\operatorname{im} T = V/W$$

$$\textcircled{2} \quad \text{if } T: V \rightarrow W \quad \ker \bar{T} = \{0\}$$

$$\text{define } \bar{T}: V/\ker(T) \rightarrow W \quad \operatorname{im} \bar{T} = \operatorname{im} T$$

$$- \dim V/W = \dim V - \dim W$$

$$- \text{if } V = W \oplus U_1, \text{ then } V/W \cong U_1$$

if $V = W \oplus U_2$, $U_1 \neq U_2$ often, but

$$U_1 \cong U_2 \cong V/W$$

Polynomials

division algorithm

roots

fund. Thm of Algebra

ideals, $I \subseteq \mathbb{F}[x]$, $\langle f(x) \rangle$

each ideal $I = \exists f(x)$ s.t. $I = \langle f(x) \rangle$

} From HW

$$D = \frac{d}{dx}$$

$p(D)$, $\ker(p(D))$ solution to DE's

Eigenvalues + Diagonalizable

Similar Matrices

$$B = Q^{-1}AQ$$

$[T]_\alpha, [T]_\beta$ similar

and this is how all similar matrices occur.

Diagonalizable

$$T: V \rightarrow V \quad (\dim V = n)$$

$$A_{n \times n}$$

Eigenvalues, Eigenvectors

$$- A\vec{v} = \lambda\vec{v}$$

Define

$$E_{\lambda_1}(A), E_{\lambda_2}(A), \dots, E_{\lambda_n}(A) ; E_{\lambda_i}(A) = \ker(A - \lambda_i I)$$

LI

v_1, \dots, v_k LI if v_i is an eigenvector w/ λ_i ,
 λ_i ALL distinct

$$E_{\lambda_1} \oplus \dots \oplus E_{\lambda_r}(T) \\ = E_{\lambda_1} + \dots + E_{\lambda_r}(T) \quad \begin{matrix} \lambda_i \neq \lambda_j \\ \text{for } i \neq j \end{matrix}$$

$\Leftrightarrow T$ diagonalizable \Leftrightarrow basis of eigenvectors

Key Important Fact

If $T \in \mathcal{L}(V)$, $\dim V = n < \infty$ over \mathbb{C} ,
then T has an eigenvalue

If $p(x) \in F[x]$, considered $p(T), p(A)$.

If $A^T =$ transpose, $(A \text{ } n \times n)$

A^T, A have the same eigenvalues

$$\text{rank } A^T = \text{rank } A$$

$$A^{n \times n} \dim \ker(A) = \dim \ker(A^T)$$

NOT true if $m \times n$

actually, if A is $\boxed{n \times n}$

$$\dim \ker(A - \lambda I) = \dim \ker(A^T - \lambda I)$$

$$\dim E_\lambda(A) = \dim E_\lambda(A^T)$$

Markov Chains

- Stochastic Matrices

- Regular Stochastic Matrices

$$\lim_{n \rightarrow \infty} A^n = L \quad \left(\lim_{n \rightarrow \infty} BA^n C = BLC \right)$$

(exists over \mathbb{R})

if A diagonalizable and all eigenvalues of A satisfy $|\lambda| < 1$ or $\lambda = 1$ then $\lim_{n \rightarrow \infty} A^n$ exists

Gershgorin Disks

A matrix in $\mathbb{C}^{n \times n}$.

$\lambda = \text{eigenvalue} \Rightarrow \lambda \in C_i \quad i = 1, \dots, n$

$$C_i = \left\{ |z - A_{ii}| \leq \text{Sum of all abs values of off diagonal entries in row } i \right\}$$

Main Theorem:

If A is diagonalizable, Stochastic, regular then

$$(a) \lim_{n \rightarrow \infty} A^n \text{ exists} \quad [\vec{p} \quad \vec{p} \quad \dots \quad \vec{p}] \quad \begin{matrix} \text{all cols} \\ \text{same} \end{matrix}$$

$$(b) \dim E_{\lambda_1}(A) = 1 \quad \text{and} \quad A\vec{p} = \vec{p}$$

$$(c) \lim_{n \rightarrow \infty} A^n \vec{q}_i = \vec{p}$$

Inner Products

def, norm, \perp

$$W^\perp \quad (\dim W < \infty)$$

$$W \subseteq V \quad \text{if } \dim W < \infty$$

$$W^\perp$$

$$W \oplus W^\perp = V$$

orthogonal sets, orthonormal sets \leadsto LI elements

Gram-Schmidt

Cor: If $W \subseteq V$ is finite dim'd

then W has an orthonormal basis

$A^T B =$ matrix of dot products