In some places you might want to use the Leibniz Integral Rule, which states that when everything is nice and differentiable we have

$$\frac{d}{dx}\int_{u(x)}^{v(x)} g(x,t)dt = g(x,v(x))\frac{dv}{dx} - g(x,u(x))\frac{du}{dx} + \int_{u(x)}^{v(x)} \frac{\partial}{\partial x} g(x,t)dt \ .$$

**1.** In class we noted that from the expected value rule for joint pdfs, which says

$$\mathbb{E}(g(X,Y)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)dxdy \ ,$$

it follows that

$$\mathbb{E}(X+Y) = \mathbb{E}(X) + \mathbb{E}(Y) \ .$$

(a) Show that expected value rule implies the last identity above.

(b) Let $Z = X+Y$. Given $z$, sketch the subset $W$ of the $x$-$y$ plane with the property that $Z \leq z$ if and only if $(X,Y) \in W$.

(c) Show that

$$f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}(x, z-x)dx \ .$$

(Suggestion: find the cdf of $Z$ first, then differentiate. The result of (b) might help.)

(d) Compute $\mathbb{E}(Z)$ using the result of (c) along with the definition

$$\mathbb{E}(Z) = \int_{-\infty}^{\infty} z f_Z(z)dz \ .$$

Verify that your answer agrees with what you know to be true, namely that $\mathbb{E}(Z) = \mathbb{E}(X) + \mathbb{E}(Y)$.

**2.** This problem is about marginal pdfs not determining joint pdfs.

(a) Suppose $X$ and $Y$ are jointly uniform on the unit square $[0,1] \times [0,1]$, i.e.

$$f_{X,Y}(x,y) = \begin{cases} 1 & \text{when } 0 \leq x, y \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find the marginal pdfs $f_X(x)$ and $f_Y(y)$.

(b) Now suppsoe instead that $X$ and $Y$ have joint pdf

$$f_{X,Y}(x,y) = \begin{cases} 2x(1-y) + 2y(1-x) & \text{when } 0 \leq x, y \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find the marginal pdfs $f_X(x)$ and $f_Y(y)$.

**3.** Sam breaks a stick of length 1 at a point $X$ distributed uniformly over the interval $(0,1)$. As a function of $p \in (0,1)$, find the expected length of the piece that contains the point $p$.

**4.** The time in minutes that it takes Maddy to commute to work is a continuous random variable $X$ with cdf $F_X(x)$. If she is $s$ minutes late to work, she incurs a cost $\alpha s$, whre $\alpha > 0$. If she is $s$ minutes early, she incurs a cost $\beta s$, where $\beta > 0$.

(a) Maddy would like to determine how many minutes before the start of work she should depart so as to minimize her expected cost. Show that the optimal solution $t^*$ satisfies

$$F_X\left(t^*\right) = \frac{\alpha}{\alpha + \beta}\ .$$

(b) How can we interpret the parameter $\gamma = \alpha/(\alpha + \beta\ )$?

Now suppose $X$ is uniformly distributed over the interval $[0, t_{\max}]$.

(c) Find the mean $\mu = \mathbb{E}(X)$ and standard deviation $\sigma = \sqrt{\mathrm{Var}(X)}$.

(d) Calculate $t^*$ from part (a) in terms of $\gamma$, $\mu$, and $\sigma$.

(d) Describe how the optimal solution $t^*$ depends on the mean and variance of $X$.

**5.** The input $X$ to a binary communication channel is either $+c$ or $-c$ with respective probabilities $1/4$ and $3/4$. Here, $c \geq 0$ is given. The channel output is

$$Y = X + N\ ,$$

where $N$ is Gaussian with zero mean and variance 1. In parts (b) and (c), please provide closed-form expressions for your answers. In part (c), you may express your answer in terms of the standard normal cdf $\Phi$.

(a) Find the conditional pdf $f_{Y|A}(y)$, where $A$ is the event $\{X = x\}$. What kind of pdf is this?

(b) Find the marginal pdf $f_Y(y)$. Plot $f_Y(y)$ on the interval $-10 \leq y \leq 10$ for $c = 0$ and $c = 5$.

(c) Find $\mathbb{P}(\{X = c\} \mid \{Y > 0\})$. What happens to this probability as $c$ gets larger? Why does this make intuitive sense?

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- rp534 -

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HW9 ECE3100

①

$$\mathbb{E}(g(X,Y)) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x,y)\, f_{X,Y}(x,y)\, dx\, dy$$

(a) $g(X,Y) = X + Y$

$$\mathbb{E}(X+Y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (x+y)\, f_{X,Y}(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x \underbrace{\left(f_{X,Y}(x,y)\, dy\right)}_{f_X(x)} dx + \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} y \underbrace{\left(f_{X,Y}(x,y)\, dx\right)}_{f_Y(y)} dy$$

$$= \mathbb{E}(X) + \mathbb{E}(Y)$$

(b) $Z = X + Y$ $\quad (Z = g(X,Y))$ $\quad g: \mathbb{R}^2 \to \mathbb{R}$

given $z^*$, sketch $W \subset \mathbb{R}^2$ w/ property $Z \le z$ iff $(X,Y) \in W$

$z = x + y$ where $x \in X$, $y \in Y$

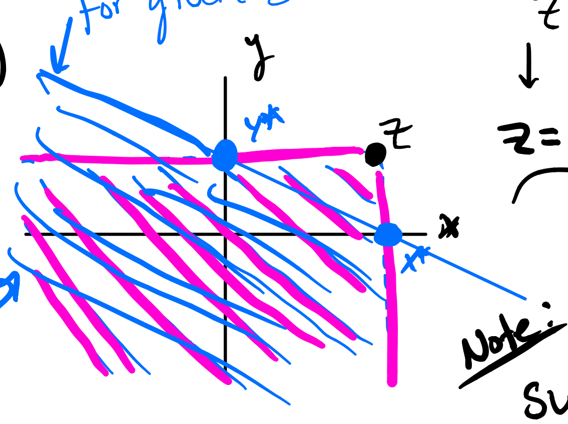
$Z \le z$

$(X + Y \le x + y)$

iff

$X, Y \in W$

for given $z^*$

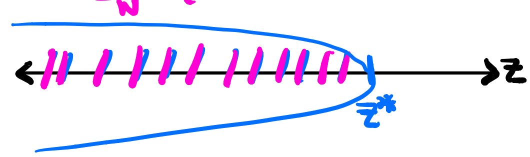



W

z is a mapping from $\mathbb{R}^2 \to \mathbb{R}$

$z = X + Y$

$Z_W = \{X + Y : (x,y) \in W\}$




Note:

$\sup Z_W = z^*$

(c) Show

$$f_Z(z) = \int_{-\infty}^{+\infty} f_{X,Y}(x, z-x)\, dx$$

$$F_Z(z) = \mathbb{P}(\{Z \leq z\}) = \int_{x=-\infty}^{+\infty} \int_{y=-\infty}^{z-x} f_{X,Y}(x,y)\, dy\, dx$$

$$f_Z(z) = \frac{d}{dz} F_Z(z) = \int_{x=-\infty}^{+\infty} \left( \frac{d}{dz} \int_{y=-\infty}^{z-x} f_{X,Y}(x,y)\, dy \right) dx$$

$$= \int_{x=-\infty}^{x=+\infty} f_{X,Y}(x, z-x)\, dx \quad \Bigg\} \text{ By Leibnez Rule}$$

(d) $$\mathbb{E}(Z) = \int_{z=-\infty}^{z=+\infty} z f_Z(z)\, dz = \int_{z=-\infty}^{z=+\infty} z \left( \int_{x=-\infty}^{x=+\infty} f_{X,Y}(x, z-x)\, dx \right) dz$$

$$= \int_{x=-\infty}^{x=+\infty} \left( \int_{z=-\infty}^{z=+\infty} z f_{X,Y}(x, z-x)\, dz \right) dx$$

Let $y = z - x$
$dy = dz$
$z = x + y$

$$= \int_{x=-\infty}^{y=+\infty} \int_{y=-\infty}^{y=+\infty} (x+y) f_{X,Y}(x,y)\, dy\, dx$$

$$= \mathbb{E}(X+Y) = \mathbb{E}(X) + \mathbb{E}(Y)$$

② 

(a) $X, Y$ are jointly uniform on $[0,1] \times [0,1]$.

i.e

$$f_{X,Y}(x,y) = \begin{cases} 1, & 0 \le x,y \le 1 \\ 0, & \text{else} \end{cases}$$

Thus

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dy = \begin{cases} \int_0^1 1\,dy = y\Big|_0^1 = 1, & 0 \le x \le 1 \\ 0, & \text{else} \end{cases}$$

$$f_Y(y) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dx = \begin{cases} \int_0^1 1\,dx = x\Big|_0^1 = 1, & 0 \le y \le 1 \\ 0, & \text{else} \end{cases}$$

Thus, $X \sim Y \sim \text{Uniform}[0,1]$

(b) $X, Y$ have joint pdf

$$f_{X,Y}(x,y) = \begin{cases} 2x(1-y) + 2y(1-x), & 0 \le x,y \le 1 \\ 0, & \text{else} \end{cases}$$

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dy = \int_0^1 2x(1-y) + 2y(1-x)\,dy$$

$$= 2x(y - \tfrac{1}{2}y^2) + (1-x)y^2 \Big|_0^1$$

$$= 2x(1-\tfrac{1}{2}) + (1-x) - (0+0)$$

$$= x + 1 - x$$

$$= 1 \quad \text{when } x \in [0,1]$$

$$f_Y(y) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dx = \int_0^1 2x(1-y) + 2y(1-x)\,dx$$

$$= x(1-y) + 2y\left(x - \tfrac{1}{2}x^2\right) \Big|_{x=0}^{x=1}$$

$$= 1-y + 2y\left(1-\tfrac{1}{2}\right) - (0+0)$$

$$= 1-y+y$$

$$= 1 \quad \text{when } y \in [0,1]$$

**Moral:** $X, Y$ can be individually uniform on $[0,1]$ but not jointly uniform on $[0,1] \times [0,1]$

③ Sam breaks a stick of length 1 at a point $X$ distributed uniformly over the interval $(0,1)$. As a function of $p \in (0,1)$, what's the expected length of the piece that contains the point $p$.

Know: - Uniform on $(0,1)$

$1/_{b-a}$, $b=1, a=0$

So

$$f_X(x) = \begin{cases} 1, & x \in (0,1) \\ 0, & \text{else} \end{cases}$$

$\frac{b+a}{2}$

$$\mathbb{E}(X) = \int_0^1 x \, dx = \frac{1}{2}$$

expected location to break stick

Let $Y$ be the length of the piece that contains the point $p$.
Then

$$Y = g(X) = \begin{cases} X, & X > p \\ 1-X, & X < p \end{cases}$$

So

$$\mathbb{E}(Y) = \int_0^p (1-x)\,dx + \int_p^1 x \, dx$$

$$= x - x^2 \Big|_0^p + \frac{1}{2}x^2 \Big|_p^1$$

$$= p(1-p) + 1/2$$

(4) Time it takes Maddy to get to work is a continuous rv $X$ w/ CDF $F_X(x)$.

- If $s$ minutes **LATE** to work, incurs cost $\alpha s$, $\alpha > 0$
- If $s$ minutes **EARLY** to work, incurs cost $\beta s$, $\beta > 0$.

(a) Time she should depart to **MINIMIZE** cost. Call this time $t^*$.

optimal solution satisfies $\longrightarrow$ $F_X(t^*) = \dfrac{\alpha}{\alpha + \beta}$

Let the time she arrives $= t_a$.
Let the time work starts $= t_s$.
Let the time she departs $= t_d$.
Put $x = t_s - t_d$.
$s = t_a - t_s$

If $s < 0$, cost is $\alpha s$
If $s > 0$, cost is $\beta s$

The cost is a function of commute time is thus

$$c(x) = \begin{cases} \alpha(x - t_a), & t_a \leq x \\ \beta(t_a - x), & t_a > x \end{cases} \quad \leftarrow x = t_s - t_d$$

Minimize cost.

$$\mathbb{E}(c(x)) = \int_0^\infty c(x) f_x(x)\, dx$$

$$= \int_0^{t_a} \beta(t-x) f_x(x)\, dx + \int_{t_u}^{\infty} \alpha(x-t) f_x(x)\, dx$$

$$= \int_0^{t_a} \beta t f_x(x)\, dx - \int_0^{t_a} \beta x f_x(x)\, dx + \int_{t_a}^{\infty} \alpha x f_x(x)\, dx - \int_{t_u}^{\infty} \alpha t f_x(x)\, dx$$

To minimize,

$$\frac{d}{dx}\left[\int_0^{t} \beta t f_x(x)\, dx - \int_0^{t} \beta x f_x(x)\, dx + \int_{t}^{\infty} \alpha x f_x(x)\, dx - \int_{t}^{\infty} \alpha t f_x(x)\, dx\right] = 0$$

$$\cancel{\beta t f_x(t)} + \beta F_x(t) - \cancel{\beta t f_x}(t) + \alpha F_x(t) - \alpha = 0$$

$$(\alpha+\beta) F_x(t) = \alpha$$

$$F_x(t) = \frac{\alpha}{\alpha+\beta}$$ ← call this $t$ $t^*$

(b) $\gamma = \dfrac{\alpha}{\alpha+\beta}$

(c) $X \sim \text{Uniform}[0, t_{max}]$

$$\mu = \frac{b+a}{2} = \frac{t_{max}}{2}, \quad \sigma^2 = \frac{(b-a)^2}{12} = \frac{t_{max}^2}{12} \Rightarrow \sigma = \frac{t_{max}}{2\sqrt{3}}$$

(d) $$F_X(x) = \frac{x-a}{b-a} = \frac{x}{t_{max}}$$

$$F_X(t^*) = \frac{t^*}{t_{max}} = \gamma$$

$$t^* = \gamma t_{max} = \sqrt{12}\sigma\gamma = 2\mu\gamma$$

(e) $t^*$ is linear in the mean of $X$ and also in the standard deviation of $X$.

⑤ Input $X$ to binary communication channel either $+c$ or $-c$ with probabilities $\frac{1}{4}$, $\frac{3}{4}$ respectively. Here, $c \geq 0$ is given.

Channel output is

$$Y = X + N$$

where $N$ is Gaussian$(0, 1)$

(a) $f_{Y|A}(y)$ where $A = \{X = x\}$ ?

$Y = c + N \sim \text{Gaussian}(\mu = c, \sigma^2 = 1)$

Thus

$$f_{Y|A}(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-c)^2}{2}}$$

and

$$f_{Y|A^c}(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(y+c)^2}{2}}$$ i.e $Y = N - c \sim \text{Gaussian}(-c, 1)$

(b)

$$f_Y(y) = f_{Y|A}(y) \mathbb{P}(A) + f_{Y|A^c} \mathbb{P}(A^c)$$

$$= \frac{1}{4\sqrt{2\pi}} e^{-\frac{(y-c)^2}{2}} + \frac{3}{4\sqrt{2\pi}} e^{-\frac{(y+c)^2}{2}}$$

(c) $$\mathbb{P}(\{X=c\} \mid \{Y>0\}) = \frac{\mathbb{P}(\{Y>0\} \mid \{X=c\})\,\mathbb{P}(\{X=c\})}{\mathbb{P}(\{Y>0\})}$$

$$\mathbb{P}(\{Y>0\} \mid \{X=c\})\,\mathbb{P}(\{X=c\}) = \frac{1}{4}\int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-c)^2}{2}}\,dy$$

$$= \frac{1}{4}\int_{y=-c}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\,dt$$

$$= \frac{1}{4}\left(1-\Phi(-c)\right)$$

$$\mathbb{P}(\{Y>0\}) = \int_0^\infty \frac{1}{4\sqrt{2\pi}} e^{-\frac{(y-c)^2}{2}} + \frac{3}{4\sqrt{2\pi}} e^{-\frac{(y+c)^2}{2}}\,dy$$

$$\frac{1}{4}\int_{y=-c}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\,dt + \frac{3}{4}\int_{y=+c}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\,dt$$

$$= \frac{1}{4}\left(1-\Phi(-c)\right) + \frac{3}{4}\left(1-\Phi(c)\right)$$

Thus

$$\mathbb{P}(\{X=c\} \mid \{Y>0\}) = \frac{\frac{1}{4}\left(1-\Phi(-c)\right)}{\frac{1}{4}\left(1-\Phi(-c)\right) + \frac{3}{4}\left(1-\Phi(c)\right)}$$