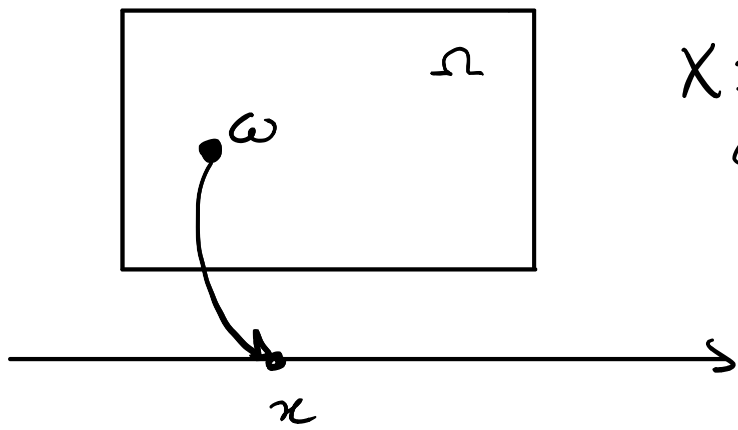
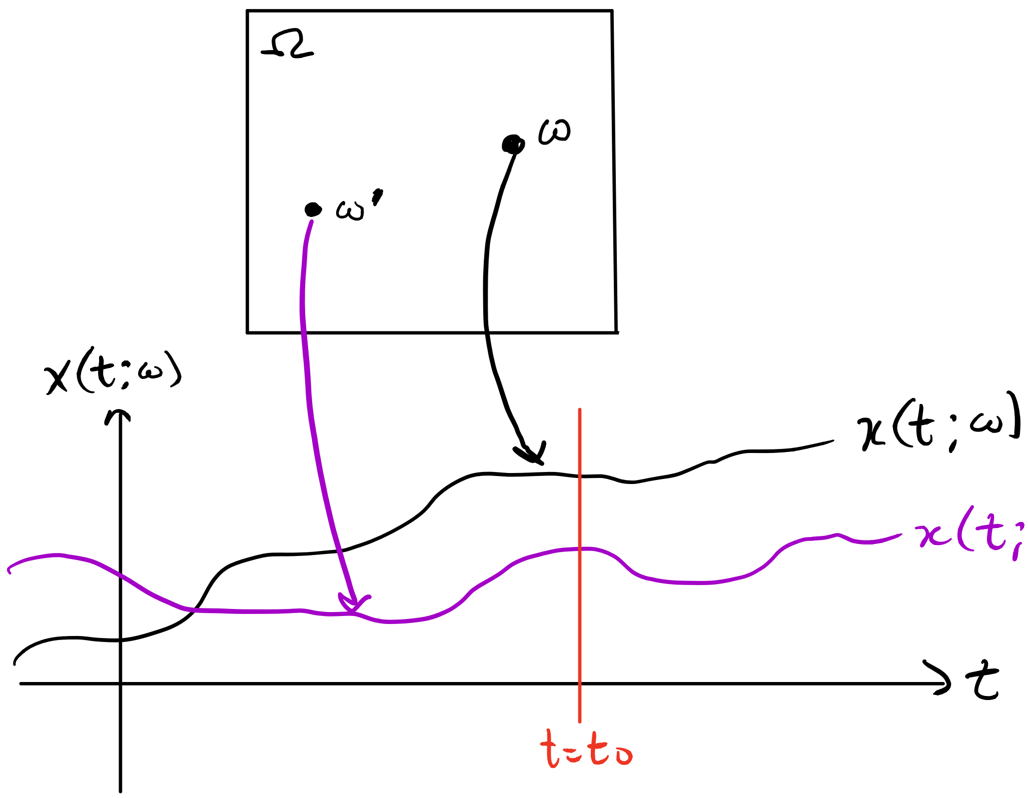
Have a r.v. X



$X: \Omega \to \mathbb{R}$
$\omega \mapsto r$

Random Process

Each sample $\omega \in \Omega$ in the sample space is mapped to a time function $X(t, \omega)$



Sampling a random process at a particular time t_0, you get a random variable

$X(t_0, \omega')$, $X(t_0, \omega)$ random variables

For a fixed ω,

$X(t;\omega)$ is a deterministic function of t, called a realization or a sample path of this random process

$X(t) \rightarrow$ r.p. $\quad x(t) \rightarrow$ sample path

Examples of Random Processes

Discrete-Time

1. An iid sequence of discrete random variables
$X_1, X_2, X_3, \ldots$
where $X_i \sim$ Bernoulli(p)

2. An i.i.d sequence of continuous random variables
X_1, X_2, X_3
where $X_i \sim \mathcal{N}(\mu, \sigma^2)$

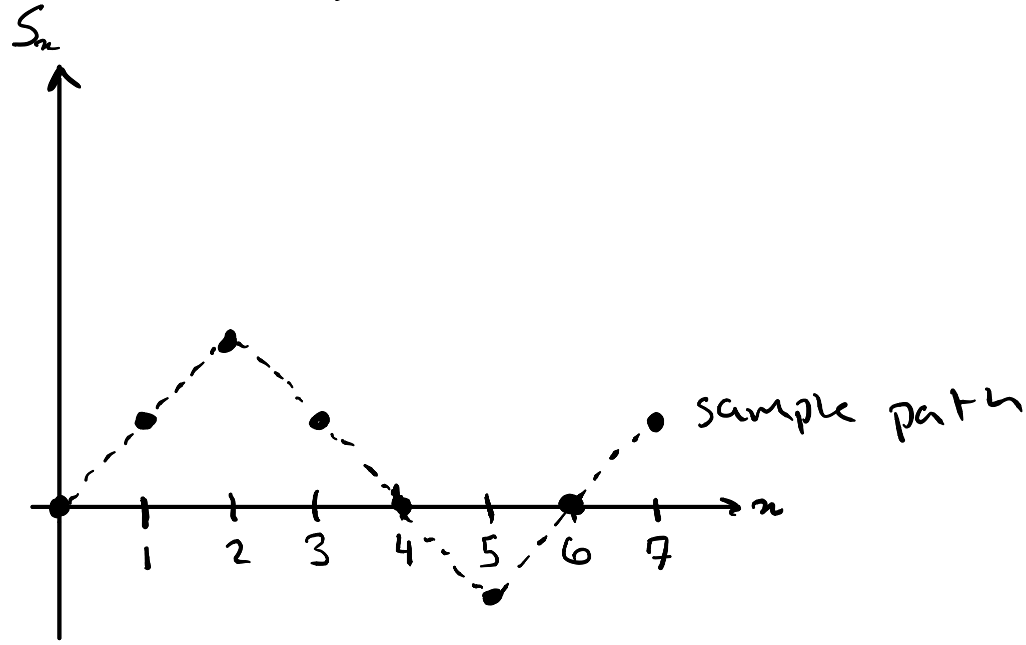
3. Binomial Counting Process
(counts # "heads" in a sequence of coin flips)
$$S_n = \sum_{i=1}^{n} X_i$$
X_i iid Bernoulli(p)

④ Random Walk:

$$S_n = \sum_{i=1}^{n} Y_i$$

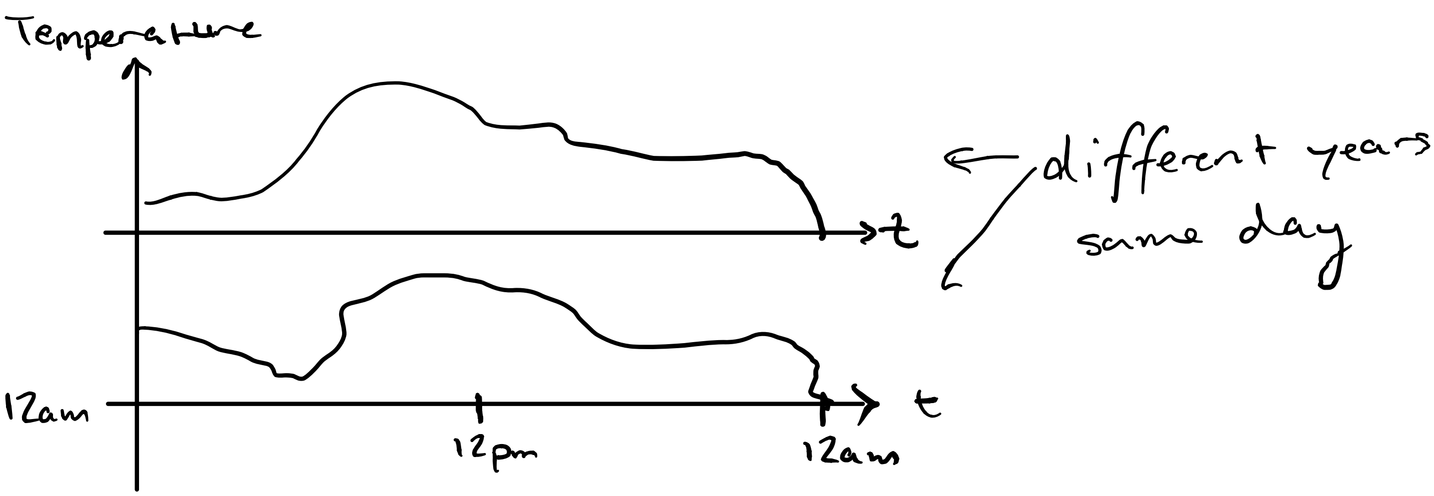
where $\{Y_i\}_{i \geq 1}$ i.i.d w/ pmf

$$\Pr[Y_i = 1] = 1 - \Pr[Y_i = -1] = 1/2$$



Continuous-Time

① Temperature in Ithaca on Jan. 1



② Sinusoid w/ a Random Phase

$$X(t) = \cos(2\pi t + \Theta)$$

where

$$\Theta \sim \text{Unif}[0, 2\pi)$$

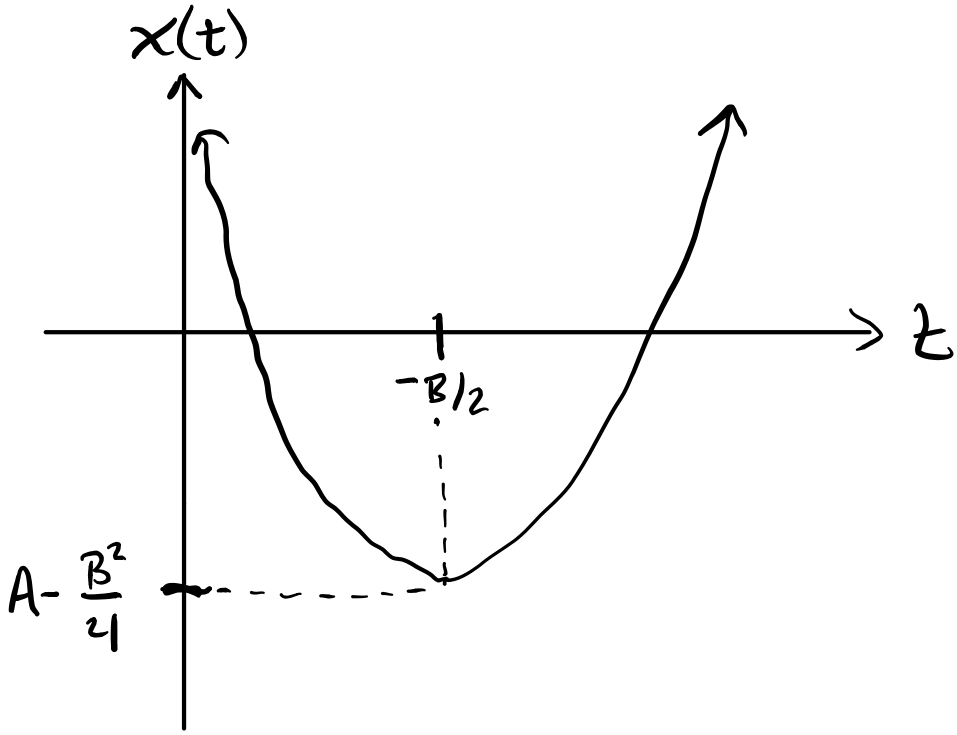
different sample paths for each realization of Θ

③ Random Parabolas

$$X(t) = A + Bt + t^2$$

where

$A, B \sim \mathcal{N}(0,1)$ and A independent B



Description of Random Processes

Complete Description: A random process is fully specified by the joint CDF

$$F_{X(t_1), X(t_2), \ldots, X(t_n)}(x_1, x_2, \ldots, x_n) \quad \forall\ n = 1, 2, 3, \ldots$$

and all sets of $t_1, t_2, \ldots, t_n$

Example: Bernoulli iid

① X_i iid , $X_i \sim \text{Bernoulli}(p)$

joint pmf $\forall n$ $k_1, k_2, \ldots, k_n$

$$P_{X_{k_1}, X_{k_2}, \ldots, X_{k_n}}(x_1, x_2, \ldots, x_n) \triangleq \Pr[X_{k_1} = x_1 \cap \cdots \cap X_{k_n} = x_n]$$

$$= p^{(x_1 + x_2 + \cdots + x_n)} (1-p)^{n - (x_1 + x_2 + \cdots + x_n)}$$

Example: Random Parabolas

$n=1$

$$X(t_1) = A + Bt_1 + t_1^2 \sim \mathcal{N}(t_1^2, 1+t_1^2)$$

Why? A, B independent Gaussian $\Rightarrow$ Jointly Gaussian

Any linear combination of J.G. r.v.'s is Gaussian.

$n=2$

$$X(t_1) = A + Bt_1 + t_1^2$$

$$X(t_2) = A + Bt_2 + t_2^2$$

$\{X(t_1), X(t_2)\}$ Jointly Gaussian!

$$\{X(t_1), X(t_2)\} \sim \mathcal{N}\left(\begin{bmatrix} t_1^2 \\ t_2^2 \end{bmatrix}, \begin{bmatrix} 1+t_1^2 & 1+t_1t_2 \\ 1+t_1t_2 & 1+t_2^2 \end{bmatrix} \right)$$

$$\mathbb{K} = \begin{bmatrix} \mathrm{Var}(X_1) & \mathrm{Cov}(X_1X_2) \\ \mathrm{Cov}(X_2X_1) & \mathrm{Var}(X_2) \end{bmatrix}$$

$$\begin{aligned} \mathrm{Cov}(X_1X_2) &= \mathbb{E}\left[(X(t_1) - \mathbb{E}[X(t_1)])(X(t_2) - \mathbb{E}[X(t_2)])\right] \\ &= \mathbb{E}[(A + Bt_1)(A + Bt_2)] \\ &= \mathbb{E}[A^2] + \mathbb{E}[ABt_2] + \mathbb{E}[ABt_1] + \mathbb{E}[t_1t_2] \\ &= 1 + t_1t_2 \end{aligned}$$

$\mathbb{E}[A^2] = \mathrm{Var}(A) + (\mathbb{E}[A])^2$

$n > 2$

$$f_{X_1(t),\dots,X_n(t)}(x_1, x_2, \dots, x_n) = \begin{cases} f_{AB}(a,b) & \text{if } a = x_1 - t_1^2 - \dfrac{x_2 - t_2^2 - t_1^2 - x_1}{t_2 - t_1} t_1 \\ & b = \dfrac{(x_2 - t_2^2 - t_1^2 - x_1)}{t_2 - t_1} \\ & x_i = a + b t_i + t_i^2 \quad i > 2 \\ 0 & \text{o/w} \end{cases}$$

Moments of a Random Process

The mean function

$$\mu(t) = \mathbb{E}[X(t)]$$

The auto-covariance function

$$C_X(t_1, t_2) \triangleq \mathbb{E}[(X(t_1) - \mu(t_1))(X(t_2) - \mu(t_2))]$$
$$= R_X(t_1, t_2) - \mu(t_1)\mu(t_2)$$

Example: iid $X_i \sim \text{Bernoulli}(p)$

$$\mu_X(i) = p$$

$$R_X(i_1, i_2) = \begin{cases} p, & i_1 = i_2 \\ p^2, & i_1 \neq i_2 \end{cases}$$

Example: $X(t) = \cos(2\pi t + \Theta)$, $\Theta \sim U[0, 2\pi)$

$$\mu_X(t) = \mathbb{E}[\cos(2\pi t + \Theta)]$$

$$= \int_0^{2\pi} \frac{1}{2\pi} \cos(2\pi t + \theta)\, d\theta = 0$$

$$R_X(t_1, t_2) = \mathbb{E}[X(t_1) X(t_2)]$$

$$= \int_0^{2\pi} \frac{1}{2\pi} \cos(2\pi t_1 + \theta) \cos(2\pi t_2 + \theta)\, d\theta$$