1. Saruman chooses a number of single dollar bills — call it Z — so that Z

$$\mathbb{P}(Z = n) = \begin{cases} p(1-p)^{(n/3)-1} & \text{when } n \text{ is a positive multiple of 3} \\ 0 & \text{otherwise,} \end{cases}$$

where $0 < p < 1$. (Austin Ellis-Mohr suggested parametrizing this problem in terms of Z, which Austin called "the total money in the system." The book does it differently.) Saruman divides the dollar bills into three equal piles and flips a fair coin to decide in which of two envelopes — call them X and Y — to put one third of the dollar bills and in which to put the other two thirds. He invites Gollum to open envelope X, and Gollum does so ravenously.

(a) What is the probability that Gollum finds 7 dollar bills in envelope X?
(b) What is the probability that Gollum finds 6 dollar bills in envelope X?
(c) Suppose Gollum can choose either to keep what he finds in X or instead take the contents of Y. What choice maximizes Gollum's expected payoff given that he finds an odd number dollars in X?
(d) Show that given Gollum finds an even number of dollars — call it m — in X, it is more probable that X contains more than Y than it is that Y contains more than X.
(e) As in part (c), suppose Gollum can choose either to keep what he finds in X or instead take the contents of Y. What choice maximizes Gollum's expected payoff given that he finds an even number of dollars — call it m — in X? Your answer might depend on m and p.

2. Two fair 3-sided dice with sides 1, 2, and 3 are labeled A and B. You roll them simultaneously. Let X be the outcome of die A and let Y be the outcome of die B.

(a) Find the joint pmf $p_{X,Y}(x, y)$.
(b) Let $Z = X + Y$. Find the pmf of Z.
(c) Find $\mathbb{E}(Z)$) and $\mathrm{Var}(Z)$.

3. Two random variables X and Y that take positive integer values have joint pmf

$$p_{X,Y}(x, y) = \begin{cases} cxy^2 & \text{when } x \in \{1, 3, 4\} \text{ and } y \in \{1, 2\} \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the constant c.
(b) Find $\mathbb{P}(\{Y = 2\})$.
(c) Find the marginal pmfs $p_X(x)$ and $p_Y(y)$.
(d) Find $\mathbb{E}(X)$, $\mathbb{E}(Y)$, and $\mathbb{E}(XY)$.
(e) Find $\mathrm{Var}(X)$ and $\mathrm{Var}(Y)$.

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HW6 ECE3100

3/15/19

① Saruman chooses a number of single dollar bills - call it Z - so that

$$\mathbb{P}(\{Z=n\}) = \begin{cases} p(1-p)^{(n/3)-1}, & n \text{ is a multiple of } 3 \\ 0, & \text{otherwize} \end{cases}$$

where $0<p<1$.

Saruman divides the dollars into 3 equal piles and flips a fair coin to decide in which of two envelopes - call them X and Y - to put $1/3$ of the dollar bills and in which to put the other $2/3$.

Gollum is invited to open envelope X.

(a) $\mathbb{P}(\{\text{Gollum finds 7 dollar bills in envelope } X\})$

This is the same as $\mathbb{P}(Z=21)$. Why? Because 7 dollars can be in envelope X iff $Z=21$

So,

$$\mathbb{P}(\{\text{Gollum finds 7 dollar bills in envelope } X\}) = \boxed{p(1-p)^6}$$

(b) $\mathbb{P}(\{\text{Gollum finds 6 dollar bills in envelope } X\})$

This is the same as $\mathbb{P}(Z=18)\cdot\frac{1}{2} + \mathbb{P}(Z=9)\cdot 1/2$

Why? Because 6 dollars can be in envelope X iff $Z=18$, and X is the envelope 1/3 of the total went into OR $Z=9$ and X is the envelope 2/3 of the total went into.

So,

$$\mathbb{P}(\{\text{Gollum finds 6 dollar bills in envelope } X\}) = \mathbb{P}(Z=18)\cdot\frac{1}{2} + \mathbb{P}(Z=9)\cdot 1/2$$
$$= \boxed{\frac{p(1-p)^5}{2} + \frac{p(1-p)^2}{2}}$$

(c) If there is an odd amount of dollars in X, Gollum should always switch since X couldn't possibly have twice as much money than Y in this scenario.

(d) Given X contains m-even-dollars $\Rightarrow X > Y$

X containing an even amount $\Rightarrow Z = \frac{3}{2}m$ OR $Z = 3m$.

If $X > Y$, then $Z = \frac{3}{2}m$. $\mathbb{P}(Z=\frac{3}{2}m) = p(1-p)^{\frac{m}{2}-1}$

If $X < Y$, then $Z = 3m$, $\mathbb{P}(Z=3m) = p(1-p)^{m-1}$

Claim: $\mathbb{P}(Z=\frac{3}{2}m) > \mathbb{P}(Z=3m)$

$$\cancel{p}(1-p)^{\frac{m}{2}-1} > \cancel{p}(1-p)^{m-1}$$
$$(1-p)^{1/2} > (1-p)$$
$$(1-p) > (1-p)^2 \quad \leftarrow \text{True for } 0<p<1,$$

So claim is correct

(e) Gollum can choose to keep what he finds in X or switch to Y. What choice maximizes Gollums expected payoff given that he finds an even number of dollars - m - in X?

From (d) we know if there are m dollars in X then X is likely to contain more money than y.

If X has more money our payoff is m dollars.
However if Y has more money our payoff is 2m dollars.

So we would choose X if

$$m\,\mathbb{P}(Z = \tfrac{3}{2}m) > 2m\,\mathbb{P}(Z = 3m)$$

and Y if

$$m\,\mathbb{P}(Z = \tfrac{3}{2}m) < 2m\,\mathbb{P}(Z = 3m)$$

(2) Two 3-sided fair die; A, B. Roll simultaneously.

Let

$$X = \text{outcome of } A$$
$$Y = \text{outcome of } B$$

(a) Find $p_{X,Y}(x,y)$

$\mathcal{X} = \{1, 2, 3\}$, $\mathcal{Y} = \{1, 2, 3\}$

$$p_{X,Y}(x,y) = \begin{cases} 1/9 \ ; & x \in \mathcal{X}, y \in \mathcal{Y} \\ 0 \ ; & \text{else} \end{cases}$$

(b) Let

$$Z = X + Y$$

$p_Z(z)$?

$\mathcal{Z} = \{2, 3, 4, 5, 6\}$

$$p_Z(z) = \begin{cases} \frac{1}{9} \ ; & z = 2 \\ 2/9 \ ; & z = 3 \\ 3/9 \ ; & z = 4 \\ 2/9 \ ; & z = 5 \\ 1/9 \ ; & z = 6 \end{cases}$$

Reality check: sums to 1!

(c) $\mathbb{E}(Z) = \sum_{z \in Z} z\, p_Z(z) = \frac{1}{9} \cdot 1 + \frac{2}{9} \cdot 3 + \frac{3}{9} \cdot 4 + \frac{2}{9} \cdot 5 + \frac{1}{9} \cdot 6 = 4$

$\mathrm{Var}(Z) = \mathbb{E}(Z^2) - (\mathbb{E}(Z))^2$

$\mathbb{E}(Z^2) = \sum_{z \in Z} z^2 p_Z(z) = \frac{1}{9}\left(1^2 + 3^2 \cdot 2 + 4^2 \cdot 3 + 5^2 \cdot 2 + 6^2 \cdot 1\right)$

$= \frac{1}{9}(1 + 18 + 48 + 50 + 36)$

$\mathrm{Var}(Z) = \frac{1}{9}(1 + 18 + 48 + 50 + 36) - 16$

③ X, Y have joint pmf

$$P_{X,Y} = \begin{cases} cxy^2, & x \in \{1,3,4\},\ y \in \{1,2\} \\ 0, & \text{otherwise} \end{cases}$$

(a)

$$\sum_{x,y} P_{X,Y}(x,y) = 1$$

$$c\left[\cancel{1(1)^2}^{1} + \cancel{1(2)^2}^{4} + \cancel{3(1)^2}^{3} + \cancel{3(2)^2}^{12} + \cancel{4(1)^2}^{4} + \cancel{4(2)^2}^{16}\right] = 1$$

$$\boxed{c = \frac{1}{40}}$$

(b) $\mathbb{P}(\{Y=2\}) = \mathbb{P}(Y=2 \cap X=1) + \mathbb{P}(Y=2 \cap X=3) + \mathbb{P}(Y=2 \cap X=4)$

$$= \frac{1}{40}\left[1(2)^2 + 3(2)^2 + 4(2)^2\right]$$

$$= \frac{1}{40}(4+12+16)$$

$$= 32/40$$

(c) $p_X(x) = \sum_{y \in Y} P_{X,Y}(x,y) = cx\left[(1)^2 + (2)^2\right] = \frac{5}{40}x$

$$p_Y(y) = \sum_{x \in X} p_{X,Y}(x,y) = \frac{1}{40} y^2 [8] = \frac{8}{40} y^2$$

(d) $$\mathbb{E}(X) = \sum_{x \in X} x p_X(x) = 1\left(\frac{5}{40}\right) + 3\left(\frac{15}{40}\right) + 4\left(\frac{20}{40}\right) = \frac{125}{40} = 3.125$$

$$\mathbb{E}(Y) = \sum_{y \in Y} y p_Y(y) = 1\left(\frac{8}{40}\right) + 2\left(\frac{32}{40}\right) = \frac{70}{40}$$

$$\mathbb{E}(XY) = \sum_{x,y} xy \, p_{X,Y}(x,y) = 1$$

(e) $$\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$$

$$\mathbb{E}(X^2) = \sum_{x \in X} x^2 p_X(x) = 1^2\left(\frac{5}{40}\right) + 3^2\left(\frac{15}{40}\right) + 4^2\left(\frac{20}{40}\right)$$

$$= \frac{5}{40} + \frac{9(15)}{40} + \frac{16(20)}{40}$$

$$\mathrm{Var}(X) = \frac{5}{40} + \frac{9(15)}{40} + \frac{16(20)}{40} - \left(\frac{125}{40}\right)^2$$

$$\mathbb{E}(Y^2) = \sum_{y \in Y} y^2 p_Y(y) = 1^2\left(\frac{8}{40}\right) + 2^2\left(\frac{32}{40}\right) \rightarrow \mathrm{Var}(Y) = \frac{8}{40} + \frac{4 \cdot 32}{40} - \left(\frac{70}{40}\right)^2$$