Inner Products

Only consider $\mathbb{F} = \mathbb{R}$
(if $\mathbb{F} = \mathbb{C}$, lots of this works, but needs modification)

Recall: if $V = \mathbb{R}^n$, the dot product of two vectors

$$\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}, \quad \vec{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$$

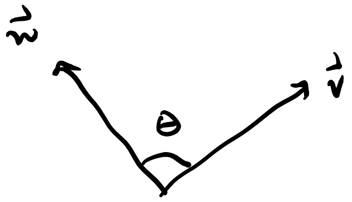
is

$$\vec{v} \cdot \vec{w} = \langle \vec{v}, \vec{w} \rangle = \sum_{i=1}^{n} v_i w_i \in \mathbb{R}$$

this gives us the notions:

length : $\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$

orthogonality : $\vec{v} \perp \vec{w}$ if $\vec{v} \cdot \vec{w} = 0$



$$\vec{v} \cdot \vec{w} = \|\vec{v}\| \|\vec{w}\| \cos \theta$$

Definition: Let V be an $\mathbb{R}$-vector space. An inner product on V is a function

$$\langle \, , \, \rangle : V \times V \longrightarrow \mathbb{R}$$
$$(\vec{v}, \vec{w}) \longmapsto \langle \vec{v}, \vec{w} \rangle$$

such that

$\forall \, \vec{u}, \vec{v}, \vec{w} \in V, \quad \alpha \in \mathbb{R}$, we have

(a) $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$

(b) $\langle c\vec{u}, \vec{v} \rangle = c \langle \vec{u}, \vec{v} \rangle$

(c) $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$

(d) $\langle \vec{v}, \vec{v} \rangle > 0$ if $\vec{v} \neq 0$ else $\langle \vec{v}, \vec{v} \rangle = 0$

examples

(1) Usual dot product on $\mathbb{R}^n$

"standard inner product" on $\mathbb{R}^n$

(2) on $\mathbb{R}[x]$, $f, g \in \mathbb{R}[x]$

$$\langle f, g \rangle = \int_{-1}^{1} f(x) g(x) \, dx$$

another one

$$\langle f, g \rangle = \int_{0}^{1} f(x) g(x) \, dx$$

i.e for other things you have to make some sort of choice

③ $A, B \in \mathbb{R}^{m \times n}$

$$\langle A, B \rangle = \sum_{j=1}^{n} \sum_{i=1}^{m} A_{ij} B_{ij} = tr(A^T B)$$

← trace → sum of diagonal entries

Definition: A vector space V (over $\mathbb{R}$), together with an inner product $\langle , \rangle$ is called an inner-product space.

Definition: Let V be an inner product space. Define

ⓐ the <u>length</u>, or <u>norm</u>, of $\vec{v} \in V$ by

$$\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$$

ⓑ if $\vec{v}, \vec{w} \in V$, then $\vec{v}, \vec{w}$ are orthogonal, written $\vec{v} \perp \vec{w}$ if $\langle \vec{v}, \vec{w} \rangle = 0$

<u>Key facts about the norm</u>

Theorem: Let V = inner product space. then

ⓐ $\|a\vec{v}\| = |a| \, \|\vec{v}\|$

ⓑ $\|\vec{v}\| = 0 \iff \vec{v} = \vec{0}$, in any case $\|\vec{v}\| \geq 0$

ⓒ Cauchy-Schwartz Inequality

$$|\langle \vec{v}, \vec{w} \rangle| \leq \|\vec{v}\| \, \|\vec{w}\|$$

ⓓ Triangle Inequality

$$\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$$

Remark: $\exists\ \Theta$ s.t. $\langle \vec{v}, \vec{w} \rangle = \|\vec{v}\| \|\vec{w}\| \cos\Theta$

This can define the angle b/t 2 vectors

Definition:

① $(\vec{v}_1, \dots, \vec{v}_n)$ is **orthogonal** if

$$\langle \vec{v}_i, \vec{v}_j \rangle = 0 \quad \text{if } i \neq j$$

and $\quad \vec{v}_i \neq \vec{0}\ \forall i$

② If $S \subseteq V$ is a set, S is orthogonal if $\forall\, \vec{v}, \vec{w} \in S$, $\vec{v} \neq \vec{w}$,

$$\langle \vec{v}, \vec{w} \rangle = 0$$

and $\vec{v} \in S \Rightarrow \vec{v} \neq 0$

③ $(\vec{v}_1, \dots, \vec{v}_n)$ is orthonormal if

$$\langle v_i, v_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Similarly define S is <u>orthonormal</u>

④ $(\vec{v}_1, \dots, \vec{v}_n)$ is an orthonormal basis of V if it is orthonormal and a basis examples

① $\mathbb{R}^4$ $(e_1, e_2, e_3, e_4) \to$ orthonormal basis

columns of
$$\begin{bmatrix} 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$$
$\to$ this is an orthogonal list of vectors

can make orthonormal via dividing each column by its length

② Let $S = \{\sin(nx)\}_{n \geq 1} \cup \{\cos(nx)\}_{n \geq 0}$

Let
$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) g(x)\, dx$$

$$\langle \sin(nx), \sin(mx) \rangle = \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases}$$

$$\langle \cos(nx), \sin(mx) \rangle = 0 \quad \forall m, n$$

$$\langle \cos(nx), \cos(mx) \rangle = \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases}$$

So S is an orthonormal set

Theorem: let V = inner product space and $(\vec{v}_1, \dots, \vec{v}_k)$ an orthonormal subset spanning $W \subseteq V$. Then if $\vec{w} \in W$,

$$\vec{w} = \sum_{i=1}^{k} \frac{\langle \vec{w}, \vec{v}_i \rangle}{\langle \vec{v}_i, \vec{v}_i \rangle} \vec{v}_i$$

Proof

$$\vec{w} = a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_k\vec{v}_k \quad (a_i \in \mathbb{R})$$

$$\langle \vec{w}, \vec{v}_i \rangle = a_i \langle \vec{v}_i, \vec{v}_i \rangle$$

$$a_i = \frac{\langle \vec{w}, \vec{v}_i \rangle}{\langle \vec{v}_i, \vec{v}_i \rangle}$$

Q.E.D

Note: If $(\vec{v}_1, \dots, \vec{v}_k)$ orthonormal this is simpler

$$\vec{w} = \sum_{i=1}^{k} \langle w, \vec{v}_i \rangle \vec{v}_i \qquad \text{(same proof)}$$