Frequency Analysis of Deterministic Signals

$$\underset{CT}{x(t)} \longrightarrow \boxed{\mathcal{F}} \longrightarrow X(f)$$

Fourier Transform

$$\begin{cases} X(f) = \displaystyle\int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi f t}\, dt \\ x(t) = \displaystyle\int_{-\infty}^{+\infty} X(f)\, e^{j2\pi f t}\, df \end{cases}$$

DTFT

$$x_n \longleftrightarrow X(f)$$

periodic, period 1

$$X(f) = \sum_{n=-\infty}^{+\infty} x_n e^{-j2\pi f n} \qquad -\frac{1}{2} \le f \le \frac{1}{2}$$

$$x(n) = \int_{-\frac{1}{2}}^{+\frac{1}{2}} X(f)\, e^{j2\pi f n}\, df$$

Elns

$\mathbb{X}(f)$ for given f:

Random process $X(t)$ WSS

$$\rightarrow R(\tau) = \mathbb{E}[X(t)X(t+\tau)]$$

$$S_X(f) = \mathcal{F}\{R_X(\tau)\}$$

$$= \int_{-\infty}^{+\infty} R_X(\tau) e^{-j2\pi f t}\, d\tau$$

$S_X(f)$ is called the power spectrum density of $X(t)$

$$S_X(f) = \lim_{T\to\infty} \frac{1}{T} \mathbb{E}\left[\|\mathbb{X}(f)\|^2\right] \Bigg\} \text{average power distributed over frequency } f$$

where

$$\mathbb{X}(f) = \mathcal{F}\left\{x(t)\,;\, t \in (-T/2, T/2)\right\}$$

Properties of $R_X(\tau)$

(i) $R_X(0) \triangleq \mathbb{E}[X^2(t)]$ is the average power of $X(t)$

(ii) $R_X(-\tau) = R_X(\tau)$; i.e $R_X(\tau)$ is even

(iii) $R_X(0) \geq |R_X(\tau)|$ (by Cauchy-Schwarz)

Properties of Power Spectrum Density

For a real-valued r.p. $\{X(t)\}_{t=-\infty}^{+\infty}$ w.s.s

① $S_X(f) \geq 0 \quad \forall f$

② $S_X(f)$ is real and even

③ $R_X(0) = \int_{-\infty}^{+\infty} S_X(f)\, df$ $\left(\text{in discrete time } R_X(0) = \int_{-1/2}^{1/2} S_X(f)\, df \right)$

$$R_X(\tau) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftarrows}} S_X(f)$$

PSD

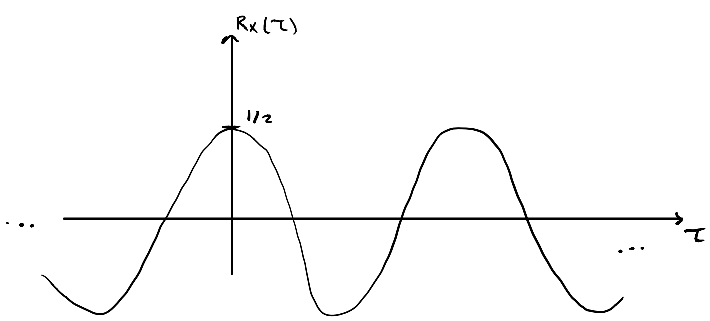
Example

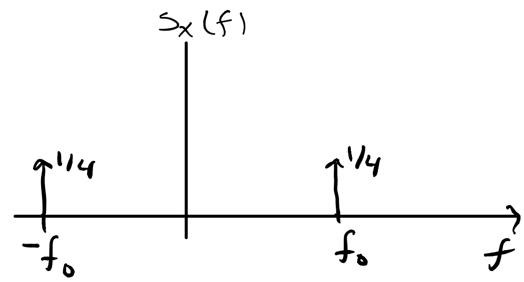
$$X(t) = \cos(2\pi f_0 t + \Theta)$$

$$\Theta \sim \text{Unif}[0,1]$$

$$R_X(\tau) = \frac{1}{2}\cos(2\pi f_0 \tau)$$

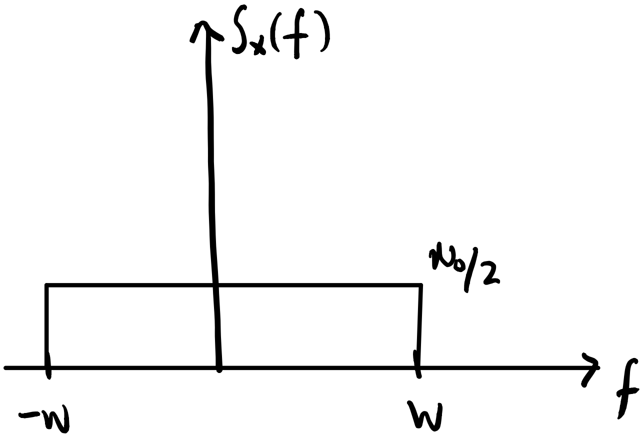
$$S_X(f) = \frac{1}{4}\left(\delta(f-f_0) + \delta(f+f_0)\right)$$

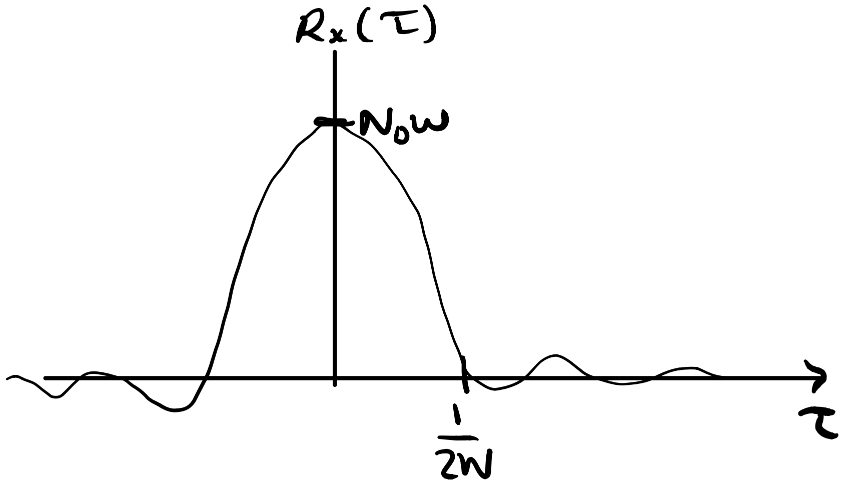




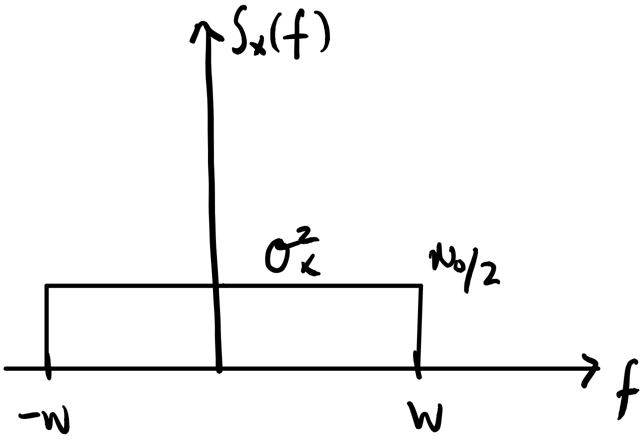
Example: CT White Noise

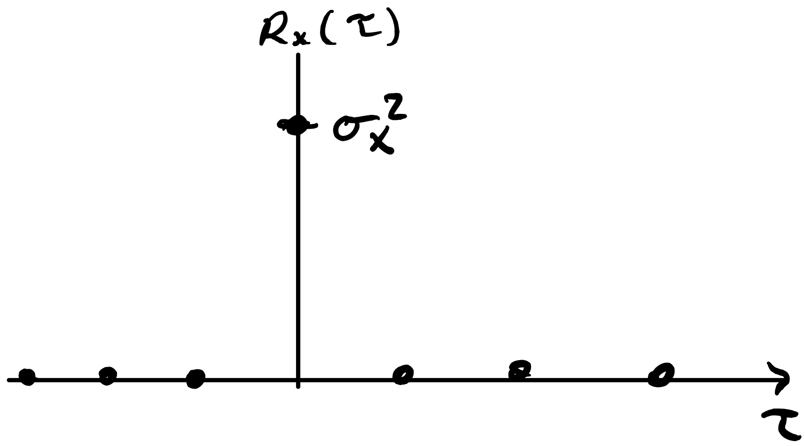
$$S_x(f) = \begin{cases} \frac{N_0}{2} & -W \le f \le W \\ 0 & \text{o/w} \end{cases}$$





Example: DT White Noise





Linear Transform of Deterministic Signals

$$x(s) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = x(t) * h(t)$$

$$y(t) = \int_{-\infty}^{+\infty} h(s)\, x(t-s)\, ds$$

$$X(f) \longrightarrow \boxed{H(f)} \longrightarrow Y(f) = X(f) H(f)$$

Have W.S.S r.p

$$X(t) \longrightarrow \boxed{h(t)} \longrightarrow Y(t)$$

also a random process

since for each realization $x(t)$ the system produces $y(t)$

Is $\{Y(t)\}_{t=-\infty}^{+\infty}$ W.S.S

$$\mathbb{E}[Y(t)] = \mathbb{E}\left[\int_{-\infty}^{+\infty} h(s) X(t-s)\, ds\right]$$

$$= \int_{-\infty}^{+\infty} \mathbb{E}[h(s) X(t-s)]\, ds$$

$$= \int_{-\infty}^{+\infty} h(s) \mu_X\, ds$$

$$= \mu_X \int_{-\infty}^{+\infty} h(s)\, ds$$

$$R_Y(t, t+\tau) \triangleq \mathbb{E}[Y(t)Y(t+\tau)]$$

$$= \mathbb{E}\left[\int_{-\infty}^{+\infty} h(s)\, X(t-s)\, ds \int_{-\infty}^{+\infty} h(r)\, X(t+\tau-r)\, dr\right]$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(s)h(r)\,ds\,dr\; R_X(\tau+s-r)$$

thus Y is ALSO W.S.S

$$S_Y(f) \triangleq \mathcal{F}\{R_Y(\tau)\}$$

$$= \int_{\tau=-\infty}^{+\infty} R_Y(\tau) e^{-j2\pi f\tau}\, d\tau$$

$$= \int_{\tau=-\infty}^{+\infty}\left(\int_{s=-\infty}^{+\infty}\int_{r=-\infty}^{+\infty} h(s)h(r)\,ds\,dr\right) R_X(\tau+s-r)\, e^{-j2\pi f\tau}\, d\tau$$

$$\tau = u-s+r \qquad d\tau = du$$

$$\int_{\tau=-\infty}^{+\infty}\left(\int_{s=-\infty}^{+\infty}\int_{r=-\infty}^{+\infty} h(s)h(r)\,ds\,dr\right) R_X(u)\, e^{-j2\pi f(u-s+r)}\, du$$

$$\boxed{= H^*(f)\,H(f)\,S_X(f) = |H(f)|^2\, S_X(f)}$$

What we have,

$$S_Y(f) = |H(f)|^2 S_X(f)$$

If $X(t)$ is a Gaussian random process, $h(t)$ a linear filter, $Y(t)$ ALSO a Gaussian random process

$$X(t) \longrightarrow \boxed{h(t)} \longrightarrow Y(t)$$

Consider arbitrary random processes

$$\{(X(t), Y(t))\}_t$$

Say X, Y jointly wide sense stationary if

1. Each of X, Y w.s.s
2. $R_{X,Y}(t_1, t_2) \triangleq \mathbb{E}[X(t_1)Y(t_2)] \equiv R_{X,Y}(\tau)$

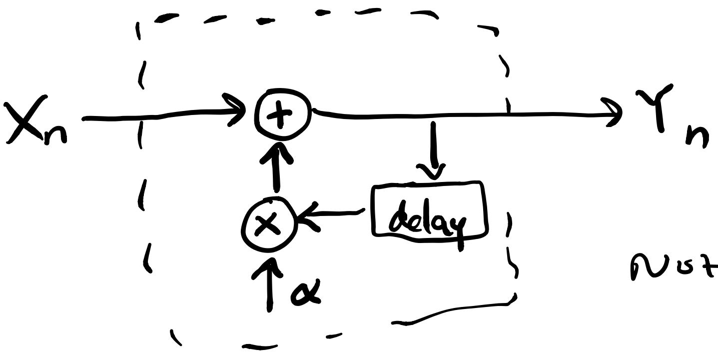
If $Y(t)$ is output of linear filter $h(t)$ driven by w.s.s $X(t)$,
Then

$X(t)$ and $Y(t)$ J.W.S.S

Also have a notion of cross-power spectrum density.

$S_{X,Y} = \mathcal{F}\{R_{X,Y}(\tau)\}$ is the cross PSD of $(X(t), Y(t))$

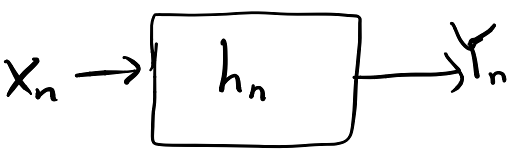
Example: First Order Auto Regression Process



Note: $0 < \alpha < 1$

$$Y_n = \alpha Y_{n-1} + X_n$$

$$= X_n + \sum_{l=1}^{\infty} \alpha^l X_{n-l}$$ can write as convolution



$$h_n = \begin{cases} 0, & n < 0 \\ 1, & n = 0 \\ \alpha^n, & n > 0 \end{cases}$$

$$H(f) = \sum_{n=0}^{\infty} \alpha^n e^{-j2\pi f n} = \frac{1}{1-\alpha e^{-j2\pi f}}$$

$$S_Y(f) = |H(f)|^2 S_X(f)$$

$$S_X(f) = \sigma_X^2$$