

12.5

12) Prove that if r, θ polar, then

$r^n \cos(n\theta)$ and $r^n \sin(n\theta)$, $n \in \mathbb{Z}$
are harmonic as functions of x, y

Let $z = r e^{i\theta}$

$$\begin{aligned}(re^{i\theta})^n &= (r \cos \theta + i r \sin \theta)^n = r^n e^{in\theta} \\ &= r^n \cos(n\theta) + i r^n \sin(n\theta)\end{aligned}$$

Show it is harmonic by showing it's the real part of something analytic.

$$\operatorname{Re}(z^n) = r^n \cos(n\theta)$$

$$\operatorname{Im}(z^n) = r^n \sin(n\theta)$$

\therefore
harmonic

$$r^n = (x^2 + y^2)^{n/2}$$

$$\theta = \tan^{-1} \left(\frac{y}{x} \right)$$

$$r^n \cos(n\theta) = (x^2 + y^2)^{n/2} \cos \left(n \tan^{-1} \left(\frac{y}{x} \right) \right)$$

$$r^n \sin(n\theta) = (x^2 + y^2)^{n/2} \sin \left(n \tan^{-1} \left(\frac{y}{x} \right) \right)$$

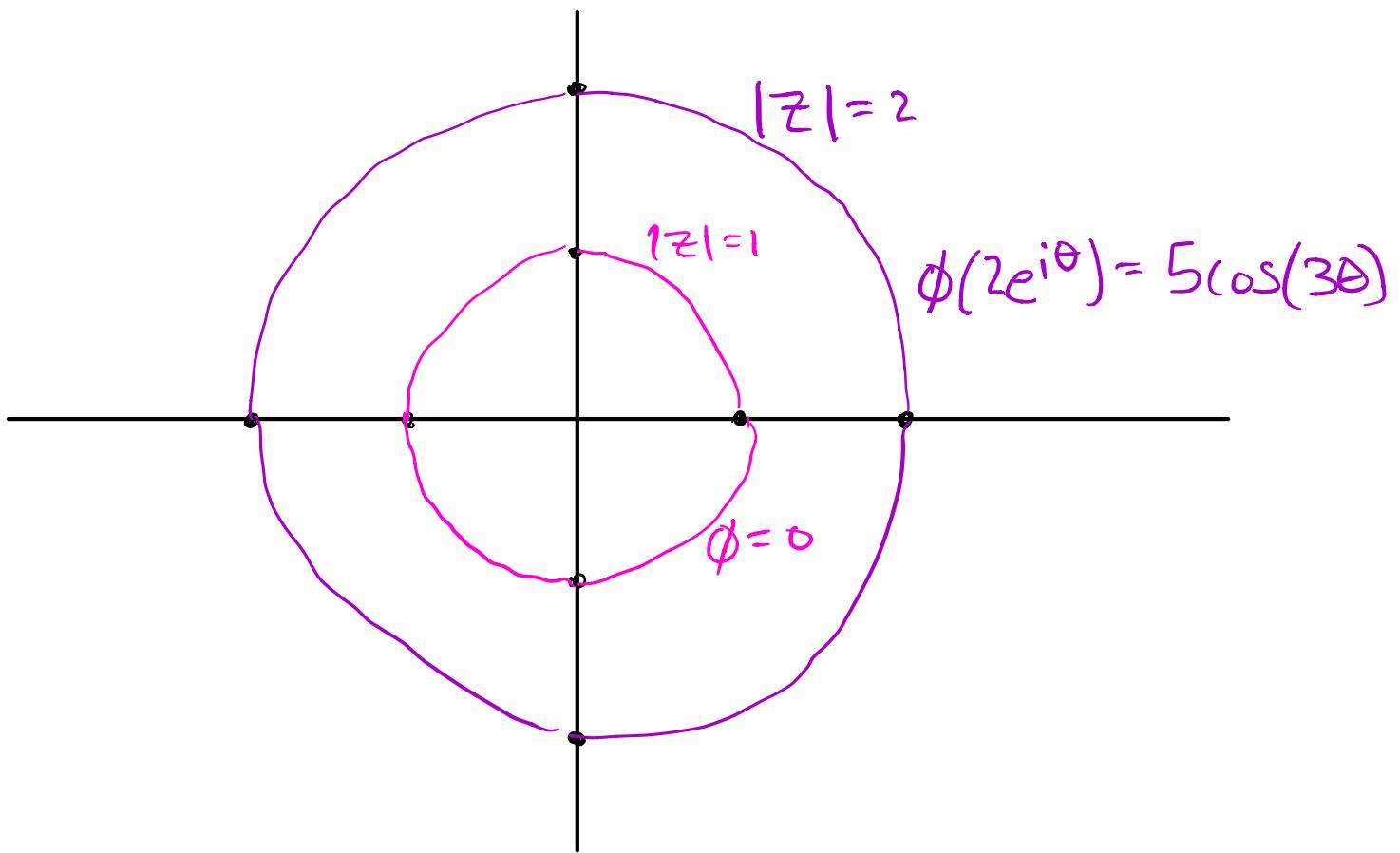
Show it is harmonic by showing it's the real part of something analytic.

$$z^n = r^n e^{in\theta} = r^n \cos(n\theta) + i r^n \sin(n\theta)$$

$$\operatorname{Re}(z^n) = r^n \cos(n\theta) \quad \therefore \text{harmonic}$$

$$\operatorname{Im}(z^n) = r^n \sin(n\theta)$$

(15) $\phi(z)$ such that



Hint suggests solution of the form

$$A(z^n + z^{-n})$$

$$z^3 = x^3 - 3xy^2 + y + i(3x^2y - y^2 - x)$$

$$z^3 = r^3 \cos(3\theta) + i r^3 \sin(3\theta) \text{ out!}$$

Our B.C. on $|z|=2$ look like $\operatorname{Re}|z^3|$

$$z^{-3} = r^{-3} \cos(-3\theta) + i r^{-3} \sin(-3\theta)$$

$$\operatorname{Re}(z^{-3}) = r^{-3} \cos(-3\theta)$$

So our solution is of the form

$$\begin{aligned}\phi(z) &= \operatorname{Re} \{ A(z^3 + z^{-3}) \} \\ &= A \left(r^3 \cos(3\theta) + r^{-3} \cos(-3\theta) \right)\end{aligned}$$

for $r=1, \phi=0$

$$0 = A \left(\cos(3\theta) + \cos(-3\theta) \right) \quad (1)$$

for $r=2, \phi=5\cos(3\theta)$

$$5\cos(3\theta) = A \left(8\cos(3\theta) + \frac{1}{8}\cos(3\theta) \right) \quad (2)$$

$$(2) - 8(1)$$

$$5\cos(3\theta) = A \left(8\cos(3\theta) + \frac{1}{8}\cos(3\theta) \right) \quad (2)$$

$$\underline{- \quad 0 = 8A \left(\cos(3\theta) + \cos(-3\theta) \right) \quad 8(1)}$$

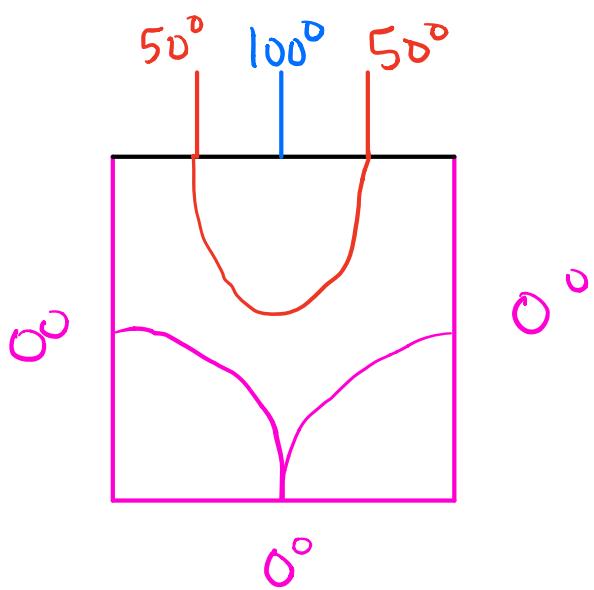
$$5\cos(3\theta) = 0 - A \frac{63}{8} \cos(-3\theta)$$

$$A = \frac{40}{63} !$$

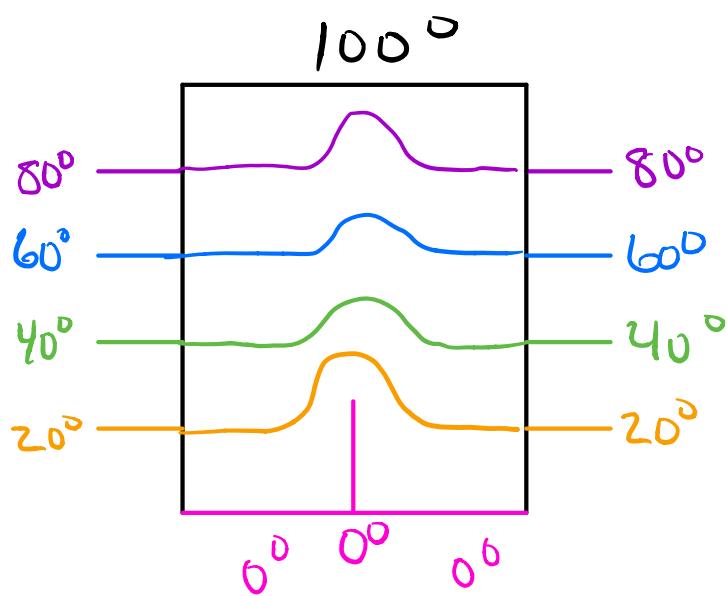
$$\boxed{\phi(z) = \operatorname{Re} \left\{ \frac{40}{63} (z^3 + z^{-3}) \right\}}$$

L2.6

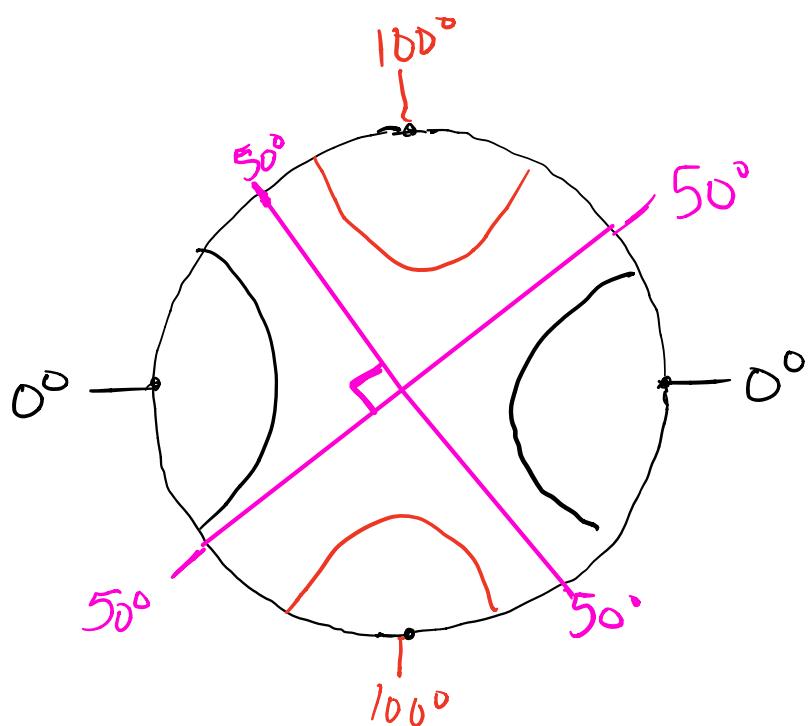
1d)



1e)



③



$$5a) e^{z+i\pi/4} = e^z e^{i\pi/4}$$

3.2

$$= e^z (\cos(\pi/4) + i \sin(\pi/4))$$

$$= e^z \left(-\frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \right)$$

$$5f) \cosh(i\frac{\pi}{2}) = \underbrace{\frac{e^{i\pi/2} + e^{-i\pi/2}}{2}}_{\cosh(x) = \frac{e^x + e^{-x}}{2}} = \cos\left(\frac{\pi}{2}\right) = 0$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

II) $\operatorname{Re}\left(\frac{\cos z}{e^z}\right)$ is harmonic in the whole

plane because it is the real part of an analytic function.

$$13a) \sin(x+iy) = \sin x \cosh(y) + i \cos x \sinh(y)$$

$$\sin(x+iy) = \operatorname{Im}\{e^{i(x+iy)}\}$$

$$= \operatorname{Im}\{e^{ix}e^{i(iy)}\} = \operatorname{Im}\{[(\cos(x)+i\sin(x)][\cos(iy)+i\sin(iy)]\}$$

$$= \cos x \sin(iy) + \sin(x) \cos(iy)$$

$$\sin(iy) = i \sinh(y)$$

$$\cos(iy) = \cosh(y)$$

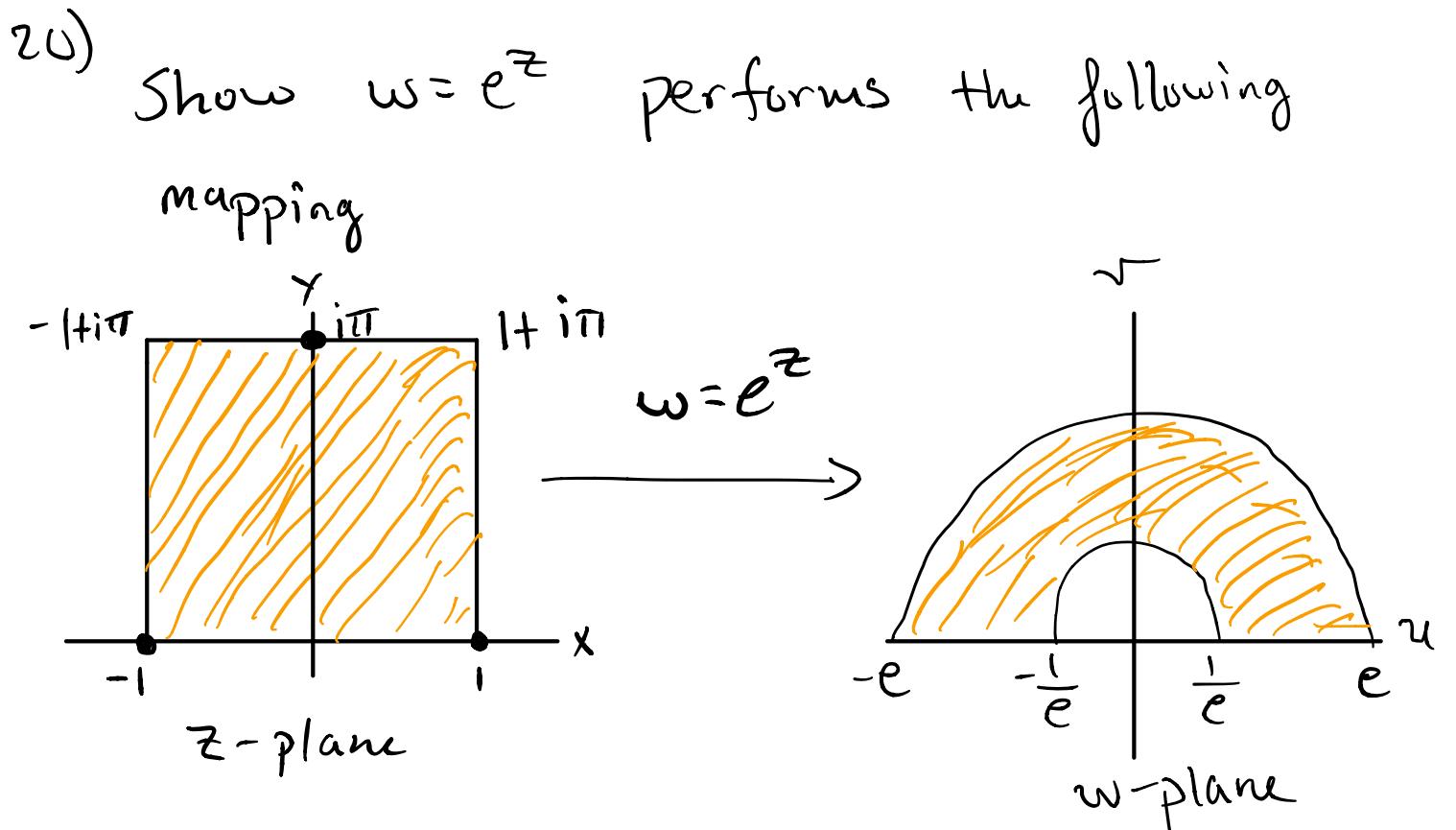
$$\Rightarrow \sin(x+iy) = \sin(x) \cosh(y) + i \cos(x) \sinh(y)$$

$$17b) e^{iz} = 3$$

$$\log_e(e^{iz}) = \log_e(3)$$

$$iz = \ln(3) + 2\pi n$$

$$z = \frac{\ln(3)}{i} = \boxed{i \ln(3) + 2\pi n, n \in \mathbb{Z}}$$



Considering the rectangular region in the z -plane in rectangular coordinates and the semi-annulus in the w -plane in polar coordinates.

The mapping is

$$\{x, y\} \mapsto (r = e^x, \theta = y)$$

It's easy to see that the image of the rectangle is contained in the semi-annulus under the exponential mapping as

$$-1 \leq x \leq 1$$

$$\Rightarrow \frac{1}{e} \leq r \leq e$$

A function is one-to-one and onto if it is invertible.

So we have

$$(x, y) \mapsto (r = e^x, \theta = y) \mapsto (\log e^x, y)$$

as the identity on the rectangle and

$$(r, \theta) \mapsto (x = \log r, y = \theta) \mapsto (e^{\log r}, \theta)$$

is the identity on our semi-annulus.

So the proposed inverse is correct.

3.3

$$1a) \log z = \ln(r) + i\theta + 2\pi n, n \in \mathbb{Z}$$

$$r=1, \theta = \frac{\pi}{2}$$

$$\log(z) = 0 + i\frac{\pi}{2} + 2\pi n$$

$$= i\frac{\pi}{2} + 2\pi n, n \in \mathbb{Z}$$

$$1c) \log(-i) = \ln(r) + i\theta$$

$$= 0 - i\frac{\pi}{2} = \boxed{-i\frac{\pi}{2}}$$

$$3) \text{ If } z_1 = i \text{ and } z_2 = i-1, \log \in [-\pi, \pi]$$

Then

$$\log z_1 z_2 \neq \log z_1 + \log z_2$$

$$\log(z_1 z_2) = \log(i^2 - i) = \log(-1 - i)$$

$$= \log|1-i| + i \operatorname{Arg}(-1-i)$$

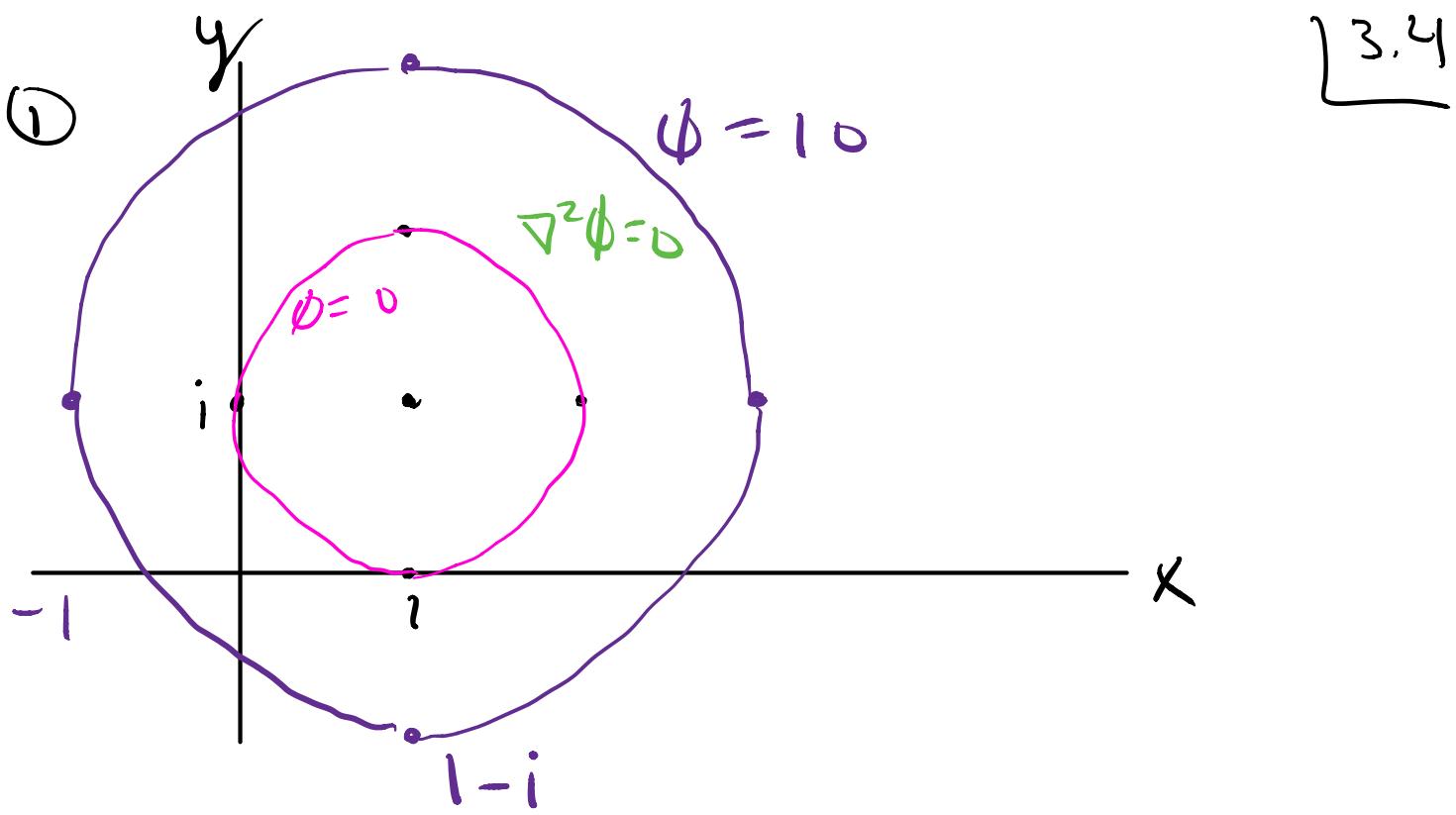
$$= \log(\sqrt{2}) - i \frac{3\pi}{4}$$

$$\log z_1 = i\frac{\pi}{2}$$

$$\log z_2 = \log(\sqrt{2}) + i \frac{3\pi}{4} \rightarrow \log z_1 + \log z_2$$

$$= \log \sqrt{2} + i \frac{5\pi}{4}$$

$$\neq \log(z_1 z_2)$$



Solution of the four washers is generally

$$\phi = A \log(r) + B$$

$$0 = A \log(1) + B \Rightarrow B = 0$$

$$10 = A \log(2) + B$$

$$A = \frac{10}{\log(2)}$$

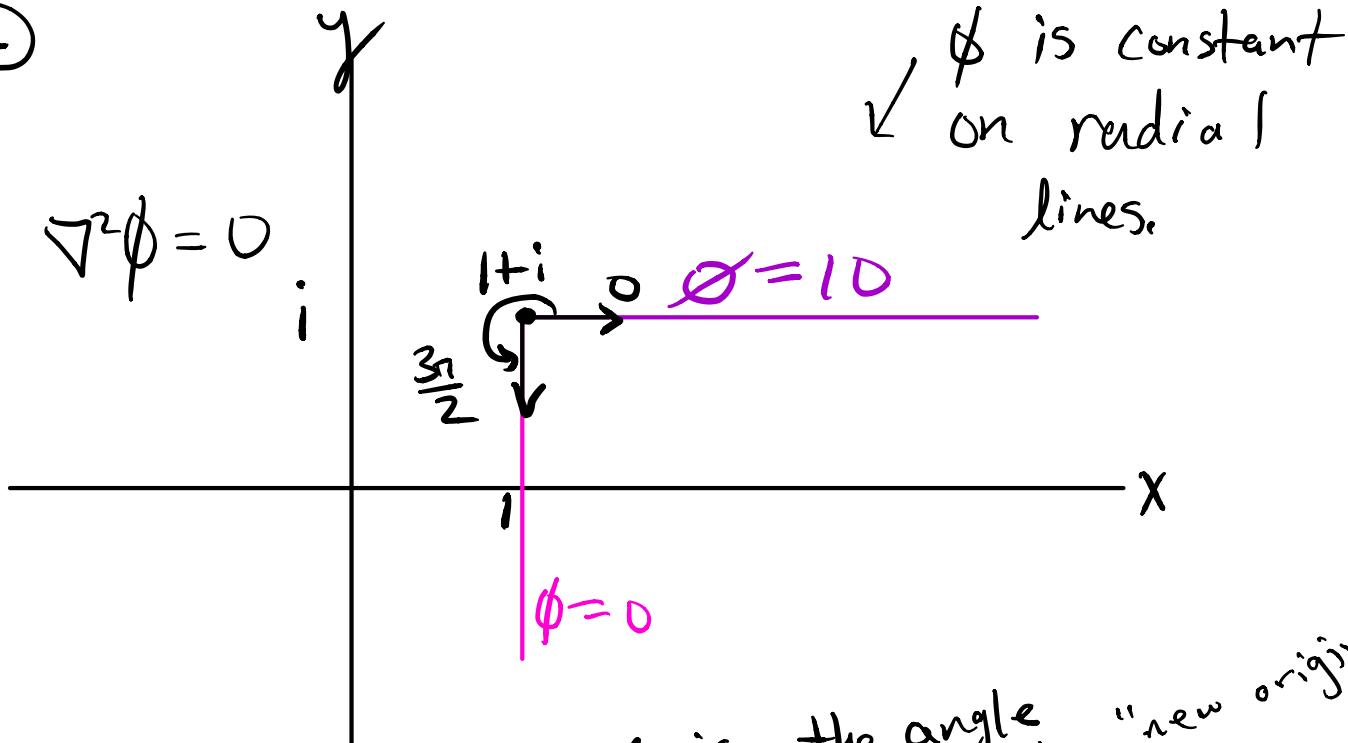
At $(0,0)$

$$\phi(z) = \frac{10}{\log(2)} \log |z - z_0|$$

$$= \frac{10}{\log(2)} \log |-i - 1|$$

$$= 10 \frac{\log \sqrt{2}}{\log 2} = 5$$

(2)



this θ is the angle from $1+i$, i.e. $1+i$ is the "new origin"

$$\phi(\theta) = A\theta + B$$

$$\phi(0) = 10 = B$$

$$\phi\left(\frac{3\pi}{2}\right) = 0 = 10 + A\frac{3\pi}{2}$$

$$\text{At } 1, \theta = 0$$

$$\text{At } i, \theta = 3\pi/2$$

$$A = -\frac{20}{3\pi}$$

$$\phi(\theta) = -\frac{20}{3\pi} \theta + 10$$

↓ as a function of z

$$\phi(z) = -\frac{20}{3\pi} \operatorname{Arg}(z - z_0) + 10$$