ECE 4110 Homework 5

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Due by 5pm on October 24

1 Reading Material

- Characterization of random processes (Chapter 9.1-9.2.2).
- Independent increments and Stationary increments (Chapter 9.3).
- Stationarity and wide-sense stationarity (Chapter 9.6-9.6.2).

2 Assignment

1. **Biased Random Walk**

   Let

   $$S_n = \sum_{i=1}^{n} Y_i$$

   where $\{Y_i\}_{i \geq 1}$ are i.i.d. with PMF given by $\Pr(Y_i = 1) = 1 - \Pr(Y_i = -1) = p$. For $p \neq \frac{1}{2}$, it is referred to as a *biased* random walk.

   (a) Find the mean function, autocorrelation function, and autocovariance function of $\{S_n\}_{n \geq 1}$.

   (b) Is $\{S_n\}_{n \geq 1}$ a stationary process? Is $\{S_n\}_{n \geq 1}$ wide-sense stationary? Do your answers change for a balanced random walk with $p = \frac{1}{2}$? Justify your answer.

   (c) Does $\{S_n\}_{n \geq 1}$ have independent increments? Does $\{S_n\}_{n \geq 1}$ have stationary increments? Justify your answer.

2. **A Process with an Abrupt Change at a Random Point**

   Let X_1 and X_2 be two random variables with zero mean, variance σ^2, and covariance satisfying $|\mathrm{Cov}(X_1, X_2)| < \sigma^2$. The random variable U is uniformly distributed over $[0, 1]$ and independent of X_1 and X_2. Define the following random process $\{X(t),\ 0 \leq t \leq 1\}$:

   $$X(t) = \begin{cases} X_1 & \text{if } 0 \leq t < U \\ X_2 & \text{if } U \leq t \leq 1 \end{cases} .$$

   (a) Obtain $\mu_X(t) \triangleq \mathbb{E}[X(t)]$ and $\mathrm{Var}(X(t))$ as functions of $t \in [0, 1]$.

   (b) Assume that $\tau \geq 0$. Calculate $\mathbb{E}[X(t)X(t + \tau)]$ (assume that both t and $t + \tau$ belong to $[0, 1]$).

(c) Assume that $\tau \leq 0$. Calculate $\mathbb{E}[X(t)X(t+\tau)]$ and combine the result with above into a single compact formula for $R_X(t, t+\tau) = \mathbb{E}[X(t)X(t+\tau)]$ valid for all t and $t+\tau$ that belong to $[0, 1]$.

(d) Is $\{X(t),\ 0 \leq t \leq 1\}$ wide-sense stationary?

3. **Random Arrival Times of a Poisson Process**

   Let $\{N(t)\}_{t\geq 0}$ be a Poisson process. Given that $N(\tau) = 1$ (i.e, a single arrival in the interval of $[0, \tau]$). Show that the arrival time T_1 of this given arrival is uniformly distributed in $[0, \tau]$.

   Hint: Show that $\Pr(T_1 \leq x | N(\tau) = 1) = \frac{x}{\tau}$ *for all* $0 \leq x \leq \tau$ *by noting that the event* $T_1 \leq x$ *is equivalent to* $N(x) = 1$.

4. **Poisson Process Probabilities**

   Consider a Poisson process with rate $\lambda > 0$.

   (a) Find the probability that there is exactly one arrival in each of the three intervals $[0, 1],\ [1, 2],\ [2, 3]$.

   (b) Find the probability that there are two arrivals in the interval $[0, 2]$ and two arrivals in the interval $[1, 3]$.
   Hint: write this event as a union of events defined by arrivals in non-overlapping intervals.

   (c) Find the probability that there are two arrivals in the interval $[1, 2]$, given that there are two arrivals in the interval $[0, 2]$ and two arrivals in the interval $[1, 3]$.

5. **Merging and Thinning of Poisson Processes**

   Poisson processes have the following two important properties:

   - *Merging independent Poisson processes:* Let $N_1(t), \ldots, N_k(t)$ be independent Poisson processes with rates $\lambda_1, \ldots, \lambda_k$, respectively. Then the sum (i.e., by counting arrivals in all k processes together)
   $$N(t) = \sum_{i=1}^{k} N_i(t)$$
   is a Poisson process with rate $\lambda = \sum_{i=1}^{k} \lambda_i$.
   - *Thinning a Poisson process:* Let $N(t)$ be a Poisson process with rate λ. For each arrival in this process, label it to be of type i $(i = 1, \ldots, k)$ with probability p_i where $\sum_{i=1}^{k} p_i = 1$. This labeling is independent across arrivals and independent of the arrival times. Then the type i arrivals form a Poisson process with rate $p_i\lambda$ for all $i = 1, \ldots, k$, and these k Poisson processes are independent.

   Use the above properties to solve the follow problem.

   In the little town of Acahti, there is a popular eatery called College Town Doughnuts (CTD). Customers arrive at CTD according to a Poisson process with rate λ (per hour). Each customer is of one of three types: the first type wants only a coffee, the second type wants only a doughnut, and the third type wants both a coffee and a doughnut. Each customer is of type i with probability p_i $(i = 1, 2, 3)$, independent of other customers and the arrival times.

   (a) What is the probability that the first customer to arrive wants a coffee?

   (b) What is the expected time between doughnut sales?

   (c) Are coffee and doughnut sales independent? Justify your answer.

(d) Given that no coffee is sold during the first hour, what is the expected number of doughnuts sold during the first two hours?

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10/24/19 rp534

ECE4110 HW5

① Biased Random Walk

Let

$$S_n = \sum_{i=1}^{n} Y_i$$

where $\{Y_i\}_{i\geq 1}$ are i.i.d. w/ pmf

$$\Pr(Y_i = 1) = 1 - \Pr(Y_i = -1) = p$$

For $p \neq 1/2$ we refer to this as a biased random walk.

(a) Find the mean function, auto correlation, and autocovariance function of $\{S_n\}_{n\geq 1}$

MEAN Function

$$\mathbb{E}[Y_i] = 1\cdot p + (-1)\cdot(1-p)$$

$$= 2p-1$$

Since biased over $\{-1,+1\}$
easier if $\{0,1\}$

$$\mu_S(n) = n(2p-1)$$

$$Var(Y_i) = (p + 1-p) - (2p-1)^2$$

$$= 1 - (2p-1)^2$$

AUTOCORRELATION Function

Assume $m < n$, $m,n \in \mathbb{N}$.

$$R_S(m,n) = \mathbb{E}[S_m S_n]$$

$$= \mathbb{E}[S_m(S_n - S_m + S_m)]$$

$$= \mathbb{E}[S_m^2] + \mathbb{E}[S_m(S_n - S_m)]$$

$$= Var(S_m) + \mathbb{E}^2[S_m] + \mathbb{E}[S_m]\mathbb{E}[S_n - S_m]$$

$$= mVar(Y_1) + m^2(2p-1)^2 + m(2p-1)(n-m)(2p-1)$$

$$= m - m(2p-1)^2 + m^2(2p-1)^2 + m(n-m)(2p-1)^2$$

$$= m + (mn - m)(2p-1)^2$$

Generalize to

$$R_S(m,n) = \min(m,n) + (mn - \min(m,n))(2p-1)^2$$

AUTOCOVARIANCE Function

$$C_s(m,n) \triangleq \mathbb{E}[(S_m-\mu_s(m))(S_n-\mu_s(n))]$$
$$= R_s(m,n) - \mu_s(m)\mu_s(n)$$

$$= \min(m,n) + (mn - \min(m,n))(2p-1)^2 - mn(2p-1)^2$$

$$= \min(m,n) - \min(m,n)(2p-1)^2$$

$$= \min(m,n)[1-(2p-1)^2]$$

$$= \min(m,n)\,4p(1-p)$$

(b) Is $\{S_n\}_{n\geq 1}$ a stationary process? Is it wide sense stationary?

Do these answers change for a balanced random walk ($p=1/2$)?

STRICTLY STATIONARY? **NO** Take k values of n WLOG

The joint pmf of $S_{n_1}, S_{n_2}, \ldots, S_{n_k}$ ($n_1 < n_2 < \cdots < n_k$) is

$$\Pr[S_{n_1}=s_1, S_{n_2}=s_2, \ldots, S_{n_k}=s_k]$$

$$= \Pr[S_{n_1}=s_1]\Pr[S_{n_2}=s_2 \mid S_{n_1}=s_1]\Pr[S_{n_3}=s_3 \mid S_{n_1}=s_1, S_{n_2}=s_2]\cdots\Pr[S_{n_k}=s_k \mid S_{n_1}=s_1, S_{n_2}=s_2, \ldots, S_{n_{k-1}}=s_{k-1}]$$

$$= \Pr[S_{n_1}=s_1]\Pr[S_{n_2}-S_{n_1}=s_2-s_1]\Pr[S_{n_3}-S_{n_2}=s_3-s_2]\cdots\Pr[S_{n_k}-S_{n_{k-1}}=s_k-s_{k-1}]$$

$$= \binom{n_1}{s_1}\binom{n_2-n_1}{s_2-s_1}\cdots\binom{n_k-n_{k-1}}{s_k-s_{k-1}} p^{s_k}(1-p)^{n_k-s_k}$$

From here it is easy to observe that 'shifting' the pmf does **NOT** yield the same answer!

$$\Pr[S_{n_1}=s_1+\Delta, S_{n_2}=s_2+\Delta, \ldots, S_{n_k}=s_k+\Delta]$$

$$= \binom{n_1+\Delta}{s_1}\binom{n_2-n_1}{s_2-s_1}\cdots\binom{n_k-n_k-1}{s_k-s_{k-1}} p^{s_k+\Delta}(1-p)^{n_k+\Delta-s_k}$$

$$\neq \binom{n_1}{s_1}\binom{n_2-n_1}{s_2-s_1}\cdots\binom{n_k-n_{k-1}}{s_k-s_{k-1}} p^{s_k}(1-p)^{n_k-s_k}$$

Thus **NOT** strictly stationary

WIDE SENSE STATIONARY? NO

$\mathbb{E}[\{S_n\}_{n\geq 1}] \neq \mu$ (a constant)! $R_X(m,n)$ doesn't depend on $m-n$

It is an increasing function in n.

In fact, $\mathbb{E}[\{S_n\}_{n\geq 1}] = np$

Thus $\{S_n\}_{n\geq 1}$ is **NOT** wide sense stationary.

WHAT IF $p=1/2$?

It is easy to see the choice of p does NOT impact the proof for strictly stationary above. For WSS however, the expectation is zero which is a constant BUT $R_X(m,n)$ doesn't depend only on $m-n$. So Not WSS.

(c) Does $\{S_n\}_{n\geq 1}$ have independent increments?

Does $\{S_n\}_{n\geq 1}$ have stationary increments?

Independent Increments? YES!

Take $n>m$. $S_n - S_m$ = $n-m$ Bernoulli(p) trials.

Since each Y_i i.i.d then $\forall n>m$ $S_n - S_m \sim \text{Binomial}(n-m, p)$

and they are independent. Thus every increment is independent.

STATIONARY Increments? YES!

$\forall k>0$

$S_{n+k} - S_n \sim \text{Bin}(n+k-n, p) \Rightarrow \text{Bin}(k,p)$ depends only on k not n thus all increments are stationary.

② A Process With an Abrupt Change at a RANDOM Point

Let X_1, X_2 be two random variables with mean 0, variance σ^2, and covariance satisfying $|Cov(X_1,X_2)| < \sigma^2$.

The random variable U is uniformly distributed on $[0,1]$ and is independent of X_1, X_2.

Define the random process $\{X(t)\}_{0\le t\le 1}$ by

$$X(t) = \begin{cases} X_1 & 0 \le t < U \\ X_2 & U \le t \le 1 \end{cases}$$

(a) Obtain $\mu_X(t) \triangleq \mathbb{E}[X(t)]$ and $Var(X(t))$ as functions of $t \in [0,1]$.

$\mu_X(t)$

$$\mu_X(t) = Pr(U \le t)\mathbb{E}[X_2] + Pr[U > t]\mathbb{E}[X_1]$$
$$= 0 + 0$$
$$= 0$$

$Var(X(t))$

$$Var(X(t)) = \mathbb{E}[X^2(t)] - \mu_X^2(t)$$
$$= Pr(U \le t)\mathbb{E}[X_2^2] + Pr(U > t)\mathbb{E}[X_1^2]$$
$$= \frac{t}{1}\sigma^2 + \frac{(1-t)}{1}\sigma^2$$
$$= \sigma^2$$

(b) Assume $\tau > 0$.

Calculate $\mathbb{E}[X(t)X(t+\tau)]$ - assuming $t, t+\tau \in [0,1]$

$$\mathbb{E}[X(t)X(t+\tau)] = \Pr(0 \le U \le t)\,\mathbb{E}[X_2^2]$$
$$+ \Pr(t < U \le t+\tau)\,Cov(X_1, X_2)$$
$$+ \Pr(t+\tau < U \le 1)\,\mathbb{E}[X_1^2]$$

$$= t\sigma^2 + (t+\tau-t)Cov(X_1,X_2) + (1-(t+\tau))\sigma^2$$

$$= t\sigma^2 + \sigma^2 - t\sigma^2 - \tau\sigma^2 + \tau Cov(X_1,X_2)$$

$$= \sigma^2(1-\tau) + \tau Cov(X_1,X_2)$$

(c) Assume $\tau \le 0$.

Calculate $\mathbb{E}[X(t)X(t+\tau)]$

$$\mathbb{E}[X(t)X(t+\tau)] = \Pr(t < U \le 1)\,\mathbb{E}[X_1^2]$$
$$+ \Pr(t+\tau < U \le t)\,Cov(X_1, X_2)$$
$$+ \Pr(0 \le U \le t+\tau)\,\mathbb{E}[X_2^2]$$

$$= (1-t)\sigma^2 + (t-(t+\tau))Cov(X_1,X_2) + (t+\tau)\sigma^2$$

$$= \sigma^2 - t\sigma^2 - \tau Cov(X_1,X_2) + t\sigma^2 + \tau\sigma^2$$

$$= (1+\tau)\sigma^2 - \tau Cov(X_1,X_2)$$

So, for general τ, have

$$\mathbb{E}[X(t)X(t+\tau)] = \sigma^2(1-|\tau|) + |\tau| \operatorname{Cov}(X_1, X_2)$$

(d) $\{X(t),\ 0 \le t \le 1\}$ is WSS.

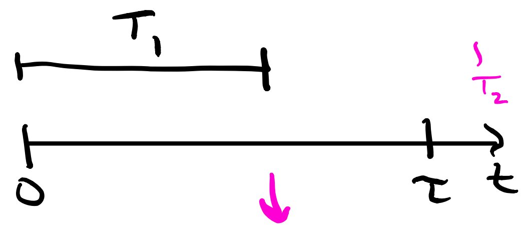
mean is zero for all $t \in [0,1]$.

$R_X(t, t+\tau)$ depends only on τ.

③ Random Arrival Times of a Poisson Process

Let $\{N(t)\}_{t \geq 0}$ be a Poisson process.

Given $N(\tau) = 1$, show that the arrival time T_1 of this given arrival is uniformly distributed in $[0, \tau]$.



$N(\tau) \to$ # arrivals in interval $[0, \tau]$.

all arrivals $Pois(\lambda\tau)$

$$Pr(T_1 \leq x \mid N(\tau) = 1) = Pr(N(x) = 1 \mid N(\tau) = 1)$$

$$= \frac{Pr(N(\tau) = 1 \mid N(x) = 1)\, Pr(N(x) = 1)}{Pr(N(\tau) = 1)}$$

$Pr($second arrival after $\tau)$

$Pr(T_2 > \tau)$

$$= \frac{e^{-\lambda(\tau - x)}\, \lambda x e^{-\lambda x}}{\lambda \tau e^{-\lambda \tau}} = \frac{x}{\tau} \quad \text{for } x \in [0, \tau)$$

Note that $x \geq \tau$ we always get one by definition.

Thus

$$Pr(T_1 \leq x \mid N(\tau) = 1) = \begin{cases} \frac{x}{\tau}, & 0 \leq x \leq \tau \\ 0, & x < 0 \\ 1, & x > 0 \end{cases}$$

i.e for every x we have shown we satisfy the uniform CDF when conditioning on $N(\tau)$, thus T_1 is uniform

(4) Poisson Process Probabilities

Consider a Poisson process w/ rate λ.

(a) Find the probability that there is exactly one arrival in each of the three intervals $[0,1], [1,2], [2,3]$.

non-overlapping increments.

$$\Pr(N(1) - N(0) = 1) = \lambda e^{-\lambda}$$

$$\Pr(N(2) - N(1) = 1) = \lambda e^{-\lambda(2-1)} = \lambda e^{-\lambda}$$

$$\Pr(N(3) - N(2) = 1) = \lambda e^{-\lambda}$$

Thus $\Pr[N(3)-N(2)=1, N(2)-N(1)=1, N(1)-N(0)=1]$

$$= (\lambda e^{-\lambda})^3 = \lambda^3 e^{-3\lambda}$$

(b) Find the probability that there are two arrivals in the interval $[0,2]$ and two arrivals in the interval $[1,3]$.

$= \Pr[N(2)-N(1)=2] \cdot \Pr[N(1)-N(0)=0] \cdot \Pr[N(3)-N(2)=0]$ Both in middle

$+ \Pr[N(1)-N(0)=2] \cdot \Pr[N(3)-N(2)=2] \cdot \Pr(N(2)-N(1)=0)$ in ends

$+ \Pr[N(1)-N(0)=1] \cdot \Pr[N(2)-N(1)=1] \cdot \Pr[N(3)-N(2)=1]$ one in each

$$= \frac{\lambda^2 e^{-\lambda}}{2}(e^{-\lambda})^2 + \left(\frac{\lambda^2 e^{-\lambda}}{2}\right)^2 e^{-\lambda} + (\lambda e^{-\lambda})^3$$

$$= e^{-3\lambda}\lambda^2\left(\frac{\lambda^2}{4} + \lambda + \frac{1}{2}\right)$$

(c) Find the probability that there are two arrivals in the interval [1,2], **given** that there are two arrivals in the interval [0,2] and two arrivals in the interval [1,3].

$$\frac{\lambda^2}{2} + \frac{\lambda}{2} + \frac{1}{4}$$

$$\Pr[N(2)-N(1)=2 \mid N(2)-N(0)=2,\ N(3)-N(1)=2]$$

$$\Pr[N(2)-N(0)=2,\ N(3)-N(1)=2] = e^{-3\lambda}\lambda^2\left(\frac{\lambda^2}{4}+\lambda+\frac{1}{2}\right)$$

$$\frac{\Pr[N(2)-N(0)=0,\ N(3)-N(1)=0,\ N(2)-N(1)=2]}{\Pr[N(2)-N(0)=2,\ N(3)-N(1)=2]}$$

$$= \frac{\frac{1}{2}e^{-3\lambda}\lambda^2}{e^{-3\lambda}\lambda^2\left(\frac{\lambda^2}{4}+\lambda+\frac{1}{2}\right)} = \frac{1}{\frac{\lambda^2}{2}+2\lambda+1}$$

$$\boxed{= \frac{2}{\lambda^2+4\lambda+2}}$$

③ Merging and Thinning of Poisson Process

Merging Independent Poisson Processes

Let $N_1(t), \dots, N_k(t)$ be independent Poisson processes w/ rates $\lambda_1, \dots, \lambda_k$ respectively.

Then

$$N(t) = \sum_{i=1}^{k} N_i(t)$$

is a Poisson process with rate parameter

$$\lambda = \sum_{i=1}^{k} \lambda_i$$

Thinning a Poisson Processes

Let $N(t)$ be a Poisson process w/ rate parameter λ.

For each arrival in this process, label it to be type i $(i = 1, \dots, k)$ with probability p_i, where $\sum_{i=1}^{k} p_i = 1$.

This labeling is independent across arrivals and independent of arrival times.

Then the type i arrivals form a Poisson process with rate $p_i \lambda$ $\forall\, i = 1, \dots, k$, and these k Poisson processes are independent.

In the little town of Acahti, there is a popular eatery called College Town Doughnuts (CTD). Customers arrive at CTD according to a Poisson process with rate λ (per hour). Each customer is of one of three types: the first type wants only a coffee, the second type wants only a doughnut, and the third type wants both a coffee and a doughnut. Each customer is of type i with probability p_i $(i = 1, 2, 3)$, independent of other customers and the arrival times.

(a) Probability first customer to arrive wants a coffee?

People who want coffee arrive at rates $p_1\lambda$ AND $p_3\lambda$

So the probability the first arrival wants coffee is contingent

it is customer of type 1 or 3. This is $p_1 + p_3$.

(b) Expected times between doughnut sales

Each interarrival of a person who wants a doughnut is

$$\text{exponential}(\lambda(p_2+p_3))$$

via merging of Poisson processes where people buy doughnuts.

Thus the expeted time between doughnut sales is

$$\frac{1}{\lambda(p_2+p_3)}$$

(c) Are coffee and doughnut sales independent?

No!

To see this observe that $p_1\lambda_1$ and $p_2\lambda_2$ are

not independent of $p_3\lambda_3$.

i.e the customer who wants both items prevents independence

of arrived customers wanting just one thing.

Also note, the expectation's don't factor.

i.e

Coffee $\sim$ Pois $(\lambda(p_1 + p_3))$

Doughnut $\sim$ Pois $(\lambda(p_2+p_3))$

Coffee + Doughnut $\sim$ Pois $(\lambda(p_1+p_2+p_3))$

$$\mathbb{E}[\text{coffee} + \text{Doughnut}] = \lambda(p_1+p_2+p_3)$$
$$\neq \mathbb{E}[\text{coffee}]\,\mathbb{E}[\text{Doughnut}]$$
$$= \lambda^2(p_1+p_3)(p_2+p_3)$$

(d) Given no coffee sold in first hour, what is the expected amount of doughnuts sold in first two hours?

So in first hour only customers of type 2 showed up (if any).

After, 2 or 3 showed up (if any).

$$\mathbb{E}[\text{Pois}(\lambda p_2)] + \mathbb{E}[\text{Pois}(\lambda(p_2+p_3))]$$

$$= \lambda p_2 + \lambda p_2 + \lambda p_3$$
$$= 2\lambda p_2 + \lambda p_3$$
$$= \lambda(2p_2 + p_3)$$