1. Let V be a vector space over $\mathbb{F}$. Prove from the axioms/properties:
   (a) (Cancellation) For all $u, v, w \in V$, if $u + v = u + w$, then $v = w$.
   (b) For all $a \in \mathbb{F}$, and $u, v \in V$, if $au = av$, then either $a = \mathbf{0}_{\mathbb{F}}$ or $u = v$.
   (c) For all $a, b \in \mathbb{F}$, and $u \in V$, if $au = bu$, then either $a = b$ or $u = \mathbf{0}_V$.
   (d) $\mathbf{0}_{\mathbb{F}} \cdot v = \mathbf{0}_V$.
   (e) If $c \in \mathbb{F}$, then $c \cdot \mathbf{0}_V = \mathbf{0}_V$.
   (f) $(-1)v = -v$, for all $v \in V$.

Answer to Question 1.

(a) If $u + v = u + w$, then we may add $-v$ on both sides and "=" still holds.

(b) If $a = 0_{\mathbb{F}}$ then we are done.
Otherwise, $a \neq 0_{\mathbb{F}}$. Then there exists $a^{-1} \in \mathbb{F}$. We may multiple a^{-1} on both sides of the equation and "=" still holds.

(c) To prove this part, we need part (e), which will be proved later.
By (VS 8), $au = bu \iff (a - b)u = 0_V$. If $a - b = 0_{\mathbb{F}}$, then we are done.
Otherwise, $c := (a - b) \neq 0$. We may multiple c^{-1} on both sides of the equation, so $u = c^{-1} \cdot 0_V$. And by part (e), $c^{-1} \cdot 0_V = 0_V$.

(d) By (VS 8), for any $a \in \mathbb{F}$,

$$av + 0_{\mathbb{F}}v = (a + 0_{\mathbb{F}})v = av = av + 0_V.$$

By part (a), av can be canceled from both sides of the equation.

(e) If $c = 0_{\mathbb{F}}$, then $c \cdot 0_V = 0_V$ by part (d).
If $c \neq 0_{\mathbb{F}}$, then by (VS 6) and (VS 7), for any $u \in V$, we have

$$c \cdot 0_V + u = c(0_V + c^{-1} \cdot u) = c(c^{-1}u) = u.$$

Again, by part (a), u can be canceled from both sides of the equation.

(f) By (VS 8) and part (d),

$$(-1)v = (-1)v + v - v = (1 - 1)v - v = 0_{\mathbb{F}} \cdot v - v = 0_V - v = -v.$$

2. Let

$$U = \left\{ \begin{pmatrix} x \\ y \\ x \\ y \end{pmatrix} \in \mathbb{F}^4 \mid x, y \in F \right\}.$$

(a) Show that $U \subset \mathbb{F}^4$ is a subspace.

(b) Find a list of vectors of U which spans U, and which is linearly independent (i.e. a basis of U).

(c) Find another subspace $W \subset \mathbb{F}^4$ such that $\mathbb{F}^4 = U \oplus W$.

Answer to Question 2.

(a) Assume that $v_1 = \begin{pmatrix} x_1 \\ y_1 \\ x_1 \\ y_1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} x_2 \\ y_2 \\ x_2 \\ y_2 \end{pmatrix}$ are two arbitrary vectors in U. Then for any $a, b \in \mathbb{F}$,

$$av_1 + bv_2 = \begin{pmatrix} ax_1 \\ ay_1 \\ ax_1 \\ ay_1 \end{pmatrix} + \begin{pmatrix} bx_2 \\ by_2 \\ bx_2 \\ by_2 \end{pmatrix} = \begin{pmatrix} ax_1 + bx_2 \\ ay_1 + by_2 \\ ax_1 + bx_2 \\ ay_1 + by_2 \end{pmatrix} \in U.$$

Hence, U is a subspace of $\mathbb{F}^4$.

(b) Notice that $u_1 := \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ and $u_2 := \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$ are two vectors in U. They are linearly independent.

Let $U' = \text{span}_{\mathbb{F}}\{u_1, u_2\}$, then clearly $U' \subseteq U$ is a subspace of U.

We also have $U' \supseteq U$. This is because for any $v = \begin{pmatrix} x \\ y \\ x \\ y \end{pmatrix} \in U$, v can be written as a linear combination of u_1 and u_2,

$$v = xu_1 + yu_2.$$

Therefore, $U' = U$.

(c) Let $w_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and $w_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, then u_1, u_2, w_1, w_2 are linearly independent and they span $\mathbb{F}^4$. This is because the determinant of $[u_1, u_2, w_1, w_2]$ is non-zero. In fact,

$$\det[u_1, u_2, w_1, w_2] = \det \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = 1.$$

If we take $W = \operatorname{span}_{\mathbb{F}}\{w_1, w_2\}$, then $W \cap U = \{0\}$ and $W + U = \mathbb{F}^4$. So $\mathbb{F}^4 = U \oplus W$. Note: the complement W is not unique. In fact, any two vectors w_1 and w_2 which make the matrix $[u_1, u_2, w_1, w_2]$ invertible would work.

3. Let $V = \mathbb{F}^{2\times 2}$ be the vector space of 2 by 2 matrices, with entries in $\mathbb{F}$.

   Determine if the following subsets are subspaces (justify your answer either way). For those that are subspaces, find a complement W: i.e. a subspace $W \subset V$, such that $V = U \oplus W$.

   (a) $U = \{A \in V \mid A^2 = A\}$.

   (b) $U = \{A \in V \mid AB = BA\}$, where $B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

   (c) $U = \{A \in V \mid A \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\}$.

Answer to Question 3.

(a) U is not a subspace:

- For $\mathbb{F} \neq \mathbb{Z}_2$, there exists $a \in \mathbb{F}$ such that $a^2 \neq a$. Notice that the identity matrix $I \in U$, but $(aI)^2 \neq aI$. So U is not a subspace.
- For $\mathbb{F} = \mathbb{Z}_2$, we may verify that

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in U,$$

but

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \notin U.$$

So U is not a subspace.

(b) U is a subspace:

- $0 \in U$, because $0 \cdot B = B \cdot 0$.
- If $A_1, A_2 \in U$, then for all $a, b \in \mathbb{F}$,

$$(aA_1 + bA_2)B = aA_1B + bA_2B = B(aA_1) + B(bA_2) = B(aA_1 + bA_2).$$

Next, we need to find a basis for U. By solving the equation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} B = B \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

for a, b, c, d, we get $b = 0$ and $a = d$. So

$$U = \{ \begin{bmatrix} a & 0 \\ c & a \end{bmatrix} | a, c \in \mathbb{F}\}.$$

We know that

$$E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

is a basis of $V = \mathbb{F}^{2\times 2}$. Let $u_1 = E_{11} + E_{22}$ and $u_2 = E_{21}$. It is clear that $U = \operatorname{span}_{\mathbb{F}}\{u_1, u_2\}$. Now we want to find a complement subspace W. Let $w_1 = E_{12}$ and $w_2 = E_{22}$. Then the coordinates of these vectors under the basis $(E_{11}, E_{12}, E_{21}, E_{22})$ are

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \; u_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \; w_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \; w_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Because the matrix

$$[u_1, u_2, w_1, w_2] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

is invertible, so $W = \operatorname{span}_{\mathbb{F}}\{w_1, w_2\}$ is a complement subspace.

(c) U is a subspace:

- $0 \in U$, because $0 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.
- If $A_1, A_2 \in U$, then for all $a, b \in \mathbb{F}$,

$$(aA_1 + bA_2)\begin{pmatrix} 1 \\ -1 \end{pmatrix} = aA_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + bA_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = a\begin{pmatrix} 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Next, we need to find a basis for U. By solving the equation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

for a, b, c, d, we get $b = -a$ and $c = d$. So

$$U = \{\begin{bmatrix} a & -a \\ c & c \end{bmatrix} | a, c \in \mathbb{F}\}.$$

We know that

$$E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \; E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \; E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

is a basis of $V = \mathbb{F}^{2\times 2}$. Let $u_1 = E_{11} - E_{12}$ and $u_2 = E_{21} + E_{22}$. It is clear that $U = \operatorname{span}_{\mathbb{F}}\{u_1, u_2\}$. Now we want to find a complement subspace W. Let $w_1 = E_{12}$ and $w_2 = E_{22}$. Then the coordinates of these vectors under the basis $(E_{11}, E_{12}, E_{21}, E_{22})$ are

$$u_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \; u_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \; w_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \; w_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Because the matrix

$$[u_1, u_2, w_1, w_2] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

is invertible, so $W = \text{span}_{\mathbb{F}}\{w_1, w_2\}$ is a complement subspace.

4. Find values $a, b \in \mathbb{Q}$ so that $\begin{pmatrix} 2 \\ a-b \\ 1 \end{pmatrix}$ and $\begin{pmatrix} a \\ b \\ 3 \end{pmatrix}$ are linearly dependent in $\mathbb{Q}^3$.

Answer to Question 4. $\begin{pmatrix} 2 \\ a-b \\ 1 \end{pmatrix}$ and $\begin{pmatrix} a \\ b \\ 3 \end{pmatrix}$ are linearly dependent if and only if there exists non-zero $c \in \mathbb{Q}$ such that

$$c \begin{pmatrix} 2 \\ a-b \\ 1 \end{pmatrix} + \begin{pmatrix} a \\ b \\ 3 \end{pmatrix} = 0_V.$$

To make this equation holds, c must be -3. Then we may solve

$$\begin{cases} -6 + a = 0, \\ -3(a-b) + b = 0, \end{cases}$$

for a and b. We get a solution $a = 6$ and $b = 9/2$.

5. Determine which of the following lists of vectors in $\mathsf{Fun}(\mathbb{R}, \mathbb{R})$ are linearly independent, and which are linearly dependent.

   (a) $(\sin^2 x, \cos^2 x)$.

   (b) $(1, \sin^2 x, \cos^2 x)$.

   (c) (e^x, e^{2x}).

Answer to Question 5. We know that

$$\begin{aligned} 0 : \mathbb{R} &\to \mathbb{R}, \\ x &\mapsto 0 \end{aligned}$$

is the 0 vector in the vector space $\mathsf{Fun}(\mathbb{R}, \mathbb{R})$.

(a) $\sin^2 x$ and $\cos^2 x$ are linearly independent. If not, there exists non-zero $a, b \in \mathbb{R}$ such that $a \sin^2 x + b \cos^2 x = 0$. In particular, this equation holds for $x = 0$ and $x = \pi/2$. Hence $a = b = 0$, which is a contradiction.

(b) $(1, \sin^2 x, \cos^2 x)$ are linearly dependent, because

$$1 \cdot 1 + (-1) \cdot \sin^2 x + (-1) \cdot \cos^2 x = 0.$$

(c) e^x and e^{2x} are linearly independent. If not, there exists non-zero $a, b \in \mathbb{R}$ such that $ae^x + be^{2x} = 0$. In particular, this equation holds for $x = 0$ and $x = 1$,

$$\begin{aligned} a + b &= 0, \\ ae + be^2 &= 0. \end{aligned}$$

So $a = b = 0$, a contradiction.

6. In this problem, assume that U_1, U_2 (and U_3 in the last two parts) are subspaces of V.

   (a) Is $U_1 \cap U_2$ a subspace? Either prove it, or give a counter-example.

   (b) Is $U_1 \cup U_2$ a subspace if neither contains the other? Either prove it, or give a counter-example.

   (c) (Optional) If $\mathbb{F} = \mathbb{R}$, show that $U_1 \cup U_2 \cup U_3$ is not a subspace of V, unless one of the subspaces contains the other two.

   (d) Suppose that $\mathbb{F} = \mathbb{F}_2$ is the field with two elements. Find an example of a vector space V, and subspaces U_1, U_2, U_3 with no one containing any other, such that $U_1 \cup U_2 \cup U_3$ is a subspace. (i.e. surprising things can happen sometimes with finite fields!)

Answer to Question 6.

(a) $U_1 \cap U_2$ is a subspace:
For any $u_1, u_2 \in U_1 \cap U_2$ and any $a, b \in \mathbb{F}$, $au_1 + bu_2 \in U_1$, because U_1 is a subspace. For the same reason, $au_1 + bu_2 \in U_2$. So $au_1 + bu_2 \in U_1 \cap U_2$.

(b) $U_1 \cup U_2$ is not a subspace if neither contains the other:
Because neither contains the other, we may pick $u_1 \in U_1 - U_2$ and $u_2 \in U_2 - U_1$. Then $u_1 + u_2 \notin U_1$, because $u_1 \in U_1$ and $u_2 \notin U_1$. For the same reason, $u_1 + u_2 \notin U_2$. So $u_1 + u_2 \notin U_1 \cup U_2$.

(c) Because none of the subspaces contains the other two, so none contains the sum of the other two, in particular,

$$U_1 \not\supseteq U_2 + U_3,$$
$$U_2 \not\supseteq U_1 + U_3.$$

We may pick

$$v_1 \in U_1 - (U_2 + U_3), \qquad v_2 \in U_2 - (U_1 + U_3).$$

Denote the set $\mathbb{R} - \{0\}$ by $\mathbb{R}^\times$. We may verify that for any $a, b \in \mathbb{R}^\times$, $av_1 \in U_1 - (U_2 + U_3)$ and $bv_2 \in U_2 - (U_1 + U_3)$. By part (b), we know that $av_1 + bv_2 \notin U_1 \cup U_2$ for all $a, b \in \mathbb{R}^\times$.

Claim: There exists $a, b \in \mathbb{R}^\times$ such that $av_1 + bv_2 \notin U_3$.
Reason: If not, then for all $a, b \in \mathbb{R}^\times$, $av_1 + bv_2 \in U_3$. In particular,

$$v_1 + v_2 \in U_3 \text{ and } v_1 + 2v_2 \in U_3.$$

This implies $v_1 \in U_3$ and $v_2 \in U_3$, which is a contradiction.

Therefore, we may pick $av_1 + bv_2 \notin U_3$ for some $a, b \in \mathbb{R}^\times$. And we know that $av_1 + bv_2 \notin U_1 \cup U_2$. Hence, $av_1 + bv_2 \notin U_1 \cup U_2 \cup U_3$ for some $v_1, v_2 \in U_1 \cup U_2 \cup U_3$ and some $a, b \in \mathbb{R}^\times$. So $U_1 \cup U_2 \cup U_3$ is not a vector space.

(d) Consider $V = \mathbb{F}_2^2$. Take

$$U_1 = \{0_V, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\},$$
$$U_2 = \{0_V, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\},$$
$$U_3 = \{0_V, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\}.$$

It is easy to verify that U_1, U_2 and U_3 are subspaces of V with no one containing each other and that

$$U_1 \cup U_2 \cup U_3 = V$$

is a subspace of V.