(9) $f(z) = z + f(z^2)$
$= z + z^2 + f(z^4)$
$= z + z^2 + z^4 + f(z^8) \cdots$

$$\Rightarrow f(z) = a_0 + \sum_{n=0}^{\infty} z^{2^n}$$

(10) $a_0 = a_1 = 1$
$a_n = a_{n-1} + a_{n-2} \quad (n \geq 2)$

Show
$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$$
defines
$$f(z) = 1 + z f(z) + z^2 f(z)$$

$$f(z)\left[1 - z - z^2\right] = 1$$

$$f(z) = \frac{-1}{z^2 + z - 1} = \frac{1}{\left(\frac{1}{2} + \frac{\sqrt{5}}{2} - z\right)\left(\frac{1-\sqrt{5}}{2} - z\right)}$$

$$= \frac{1}{\sqrt{5}}\left[\frac{1 + \sqrt{5}/2}{1 - \left(\frac{1+\sqrt{5}}{2}\right)z} - \frac{1 - \sqrt{5}/2}{1 - \left(\frac{1-\sqrt{5}}{2}\right)z}\right]$$

Therefore

$$f(z) = \frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} \left[\left(\frac{1+\sqrt{5}}{2} \right)^{n+1} - \left(\frac{1-\sqrt{5}}{2} \right)^{n+1} \right] z^{n}$$

15.5

① a. $\frac{1}{z+z^2}$, $0<|z|<1$

$$\frac{1}{z+z^2} = \frac{1}{z(1+z)} = \frac{1}{z(1-(-z))}, \quad |z|<1$$

$$= \frac{1}{z}\cdot\frac{1}{1-(-z)} = \frac{1}{z}\left[1+(-z)+(-z)^2+\cdots\right]$$

$$= \sum_{n=-1}^{\infty} z^n(-1)^{n+1} = \frac{1}{z} - 1 + z - z^3\cdots$$

b. $|z|>1$

$$\frac{1}{z+z^2} = \frac{1}{z^2}\cdot\frac{1}{\left(1+\frac{1}{z}\right)}$$

$$= \frac{1}{z^2}\cdot\frac{1}{1-\left(-\frac{1}{z}\right)} = \frac{1}{z^2}\left[1+\left(\frac{-1}{z}\right)+\left(\frac{-1}{z}\right)^2+\cdots\right]$$

$$= \frac{1}{z^2}\sum_{n=0}^{\infty}\left(-\frac{1}{z}\right)^n = \sum_{n=2}^{\infty}(-1)^n\left(\frac{1}{z}\right)^n$$

c. $0 < |z+1| < 1$

$$\frac{1}{z+z^2}$$

$$\left. f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n \right\}$$ By Laurent's Theorem

$$\frac{1}{z(1+z)} = \frac{1}{1+z}\left[\frac{1}{z}\right]$$

$$= \frac{1}{1+z} \cdot \frac{1}{(z+1)-1} = \frac{1}{1+z} \cdot \frac{-1}{1-(1+z)}$$

$$= \frac{-1}{1+z} \cdot \frac{1}{1-(1+z)} = -\frac{1}{1+z}\left(1+(1+z)+\cdots\right)$$

$$= -\sum_{n=-1}^{\infty} (1+z)^n$$

d. $|z+1| > 1$

From above

$$u = \frac{1}{1+z}$$

$$-u \cdot \frac{1}{1-1/u}$$

$$= -u \cdot -\frac{u}{1-u} = u^2 \cdot \frac{1}{1-u} = u^2\left[1+u+u^2+\cdots\right]$$

$= u^2 + u^3 + u^4 + \cdots$

$= \sum_{n=2}^{\infty} u^n = \sum_{n=2}^{\infty} \left(\frac{1}{1+z}\right)^n$

7b. $\dfrac{1}{e^z - 1}$, $0 < |z| < 2\pi$

want first few terms

$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$

$e^z - 1 = z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$

$\frac{1}{e^z - 1} = \frac{1}{z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots} = \frac{1}{z\left(1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots\right)}$

$= \frac{1}{z(1 + p(z))}$, $p(z) = \frac{z}{2!} + \frac{z^2}{3!} + \frac{z^3}{4!} + \cdots$

$= \frac{1}{z} - \frac{p(z)}{z} + \frac{(p(z))^2}{z} + \cdots$

$= \frac{1}{z} - \frac{1}{2} - \frac{z}{3!} + \left(\frac{z}{2!}\right)^2 - \frac{z^2}{4!} + \frac{2z^2}{6} - \frac{z^2}{8} + \cdots$

$= \boxed{\frac{1}{z} - \frac{1}{2} + \frac{z}{12}} + (\text{stuff})$

(9) $\displaystyle\sum_{j=-\infty}^{\infty} \frac{z^j}{2^{|j|}}$, annulus of convergence

$$= \underbrace{\sum_{j=-\infty}^{-1} \frac{z^j}{2^{|j|}}}_{\frac{1}{2z} + \left(\frac{1}{2z}\right)^2 + \left(\frac{1}{2z}\right)^3 + \cdots} + \underbrace{\sum_{j=0}^{\infty} \frac{z^j}{2^j}}_{1 + \left(\frac{z}{2}\right) + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \cdots}$$

$$\frac{1}{2z} + \left(\frac{1}{2z}\right)^2 + \left(\frac{1}{2z}\right)^3 + \cdots \;\to\; \frac{1}{1-\left(\frac{1}{2z}\right)}$$

implies $|z| > 1/2$

$$1 + \left(\frac{z}{2}\right) + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \cdots \;\to\; \frac{1}{1-\frac{1}{2}z}$$

implies $|z| < 2$

So annulus of convergence is

$$\frac{1}{2} < |z| < |2|$$

1a) $\dfrac{z^3+1}{z^2(z+1)}$

pole of order 2 at $z=0$
pole of order 1 at $z=-1$

1g) $\dfrac{\sin(3z)}{z^2} - \dfrac{3}{z}$

$$= \frac{1}{z}\cdot\frac{\sin(3z)}{z} - \frac{3}{z}$$

Let's look at $z=0$ case.

removable singularity at $z=0$

1h) $\cot\left(\dfrac{1}{z}\right)$

$\cot(z)$ has zeros at $z=n\pi,\ n\in\mathbb{Z}$
so $\cot(\frac{1}{z})$ has an essential singularity at

$$z = 1/n\pi,\ n \in \mathbb{Z}$$

12) $f(z)$ has a pole of order m at z_0, then

$$g(z) := \frac{f'(z)}{f(z)}$$

has a simple pole at z_0. (Show this)

Coefficient of $(z-z_0)^{-1}$ in laurent expansion?

Let $f(z) = \dfrac{f_1}{(z-z_0)^m}$

$$\frac{f'(z)}{f(z)} = \underbrace{\frac{(z-z_0)^m f_1' - f_1 m (z-z_0)^{m-1}}{(z-z_0)^{2m}}}_{f'(z)} \cdot \underbrace{\frac{\cancel{(z-z_0)^m}}{f_1}}_{\frac{1}{f(z)}}$$

$$= \frac{f_1'}{f_1} - \frac{f_1 m (z-z_0)^m}{f_1 (z-z_0)^{m+1}}$$

$$= \frac{f_1'}{f_1} - \frac{m}{z-z_0}$$ } So $g(z)$ has a simple pole at z_0!

$$= \frac{f_1'(z-z_0) - f_1 m}{f_1(z-z_0)} = \boxed{\frac{f_1'(z-z_0) - f_1 m}{f_1}} \cdot \frac{1}{z-z_0}$$

coefficient for $\frac{1}{z-z_0}$ in laurent expansion

1a) e^z, behavior at infinity?

$e^z = e^x e^{iy}$ at ∞ the e^{iy} term determines behavior so e^z has an essential singularity as $z \to \infty$ through **NONREAL** values.

1c) $\left.\dfrac{z-1}{z+1}\right\}$ analytic as $z \to \infty$ (L'hopitals)

1g) $\dfrac{\sin z}{z^2} = \dfrac{\sin z}{z} \cdot \dfrac{1}{z}$ as $z \to \infty$ we have an essential singularity!

⑦ Prove that if f is analytic on and outside the simple closed contour Γ and has a zero of order 2 or more at ∞, then

$$\int_\Gamma f(z)\, dz = 0.$$

For zeros of order two or more

$$\int_{\Gamma} f(z)\,dz = \int_{\Gamma} \frac{a_{-2}}{z^2}\,dz \quad \text{or} \quad \int_{\Gamma} \frac{a_{-3}}{z^3}\,dz \ \text{or} \ldots$$

So by the antiderivative property the integral goes to zero.

If we only had one zero as $z \to \infty$

$$\int_{\Gamma} \frac{dz}{z} = 2\pi i \ !$$

Our integral vanishes to this simple case.

$$\frac{1}{e^z - 1} = \frac{-1}{1 - e^z} \qquad u = \frac{1}{e^z}$$

$$= \frac{-1}{1 - 1/u}$$

$$= \frac{-u}{u-1} = \frac{u}{1-u}$$

$$= \frac{1}{e^z}\left[1 + \frac{1}{e^z} + \left(\frac{1}{e^z}\right)^2 + \cdots\right)$$