

Frequency Analysis of Deterministic Signals

$$\underset{\text{CT}}{x(t)} \longrightarrow \boxed{\mathcal{F}} \longrightarrow X(f)$$

Fourier Transform

$$\begin{cases} X(f) = \int_{-\infty}^{+\infty} x(t) e^{-j2\pi ft} dt \\ x(t) = \int_{-\infty}^{+\infty} X(f) e^{j2\pi ft} df \end{cases}$$

DTFT

$$x_n \longleftrightarrow X(f)$$

$$X(f) = \sum_{n=-\infty}^{+\infty} x_n e^{-j2\pi fn}$$

periodic, period 1

$$-\frac{1}{2} \leq f \leq \frac{1}{2}$$

$$x(n) = \int_{-\frac{1}{2}}^{+\frac{1}{2}} X(f) e^{j2\pi fn} df$$

E111s

$X(f)$ for given f :

Random process $X(t)$ WSS

$$\rightarrow R(\tau) = E[X(t)X(t+\tau)]$$

$$S_x(f) = \mathcal{F}\{R_x(\tau)\}$$

$$= \int_{-\infty}^{+\infty} R_x(\tau) e^{-j2\pi f\tau} d\tau$$

$S_x(f)$ is called the power spectrum density of $X(t)$

$$S_x(f) = \lim_{T \rightarrow \infty} \frac{1}{T} E[|X(f)|^2] \left. \vphantom{\lim_{T \rightarrow \infty}} \right\} \begin{array}{l} \text{average} \\ \text{power} \\ \text{distributed} \\ \text{over frequency } f \end{array}$$

where

$$X(f) = \mathcal{F}\{x(t); t \in (-T/2, T/2)\}$$

Properties of $R_x(\tau)$

- (i) $R_x(0) \triangleq E[X^2(t)]$ is the average power of $X(t)$
- (ii) $R_x(-\tau) = R_x(\tau)$; i.e. $R_x(\tau)$ is even
- (iii) $R_x(0) \geq |R_x(\tau)|$ (by Cauchy-Schwarz)

Properties of Power Spectrum Density

For a real-valued r.p. $\{X(t)\}_{t=-\infty}^{+\infty}$ w.s.s

- ① $S_X(f) \geq 0 \quad \forall f$
- ② $S_X(f)$ is real and even
- ③ $R_X(0) = \int_{-\infty}^{+\infty} S_X(f) df$ (in discrete time)
 $R_X(0) = \int_{-1/2}^{1/2} S_X(f) df$

$$R_X(\tau) \xLeftrightarrow[\mathcal{F}^{-1}]{\mathcal{F}} S_X(f) \quad \text{PSD}$$

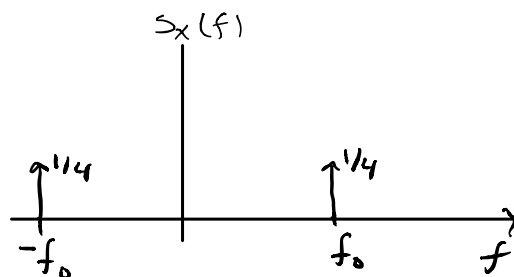
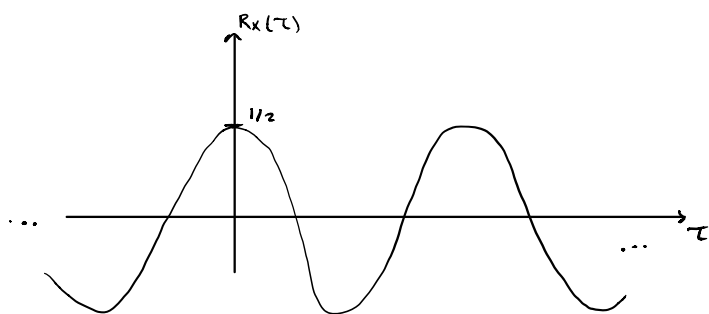
Example

$$X(t) = \cos(2\pi f_0 t + \mu)$$

$$\mu \sim \text{Unif}[0, 1]$$

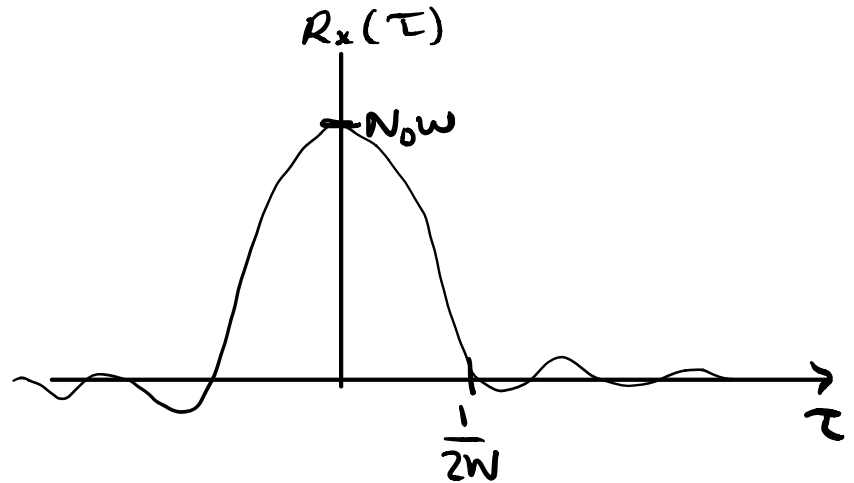
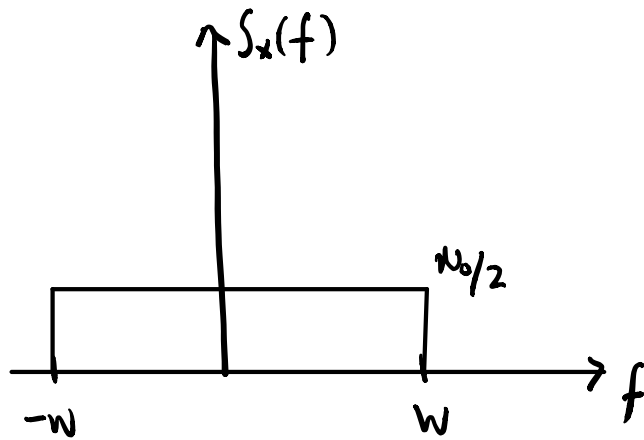
$$R_X(\tau) = \frac{1}{2} \cos(2\pi f_0 \tau)$$

$$S_X(f) = \frac{1}{4} (\delta(f - f_0) + \delta(f + f_0))$$

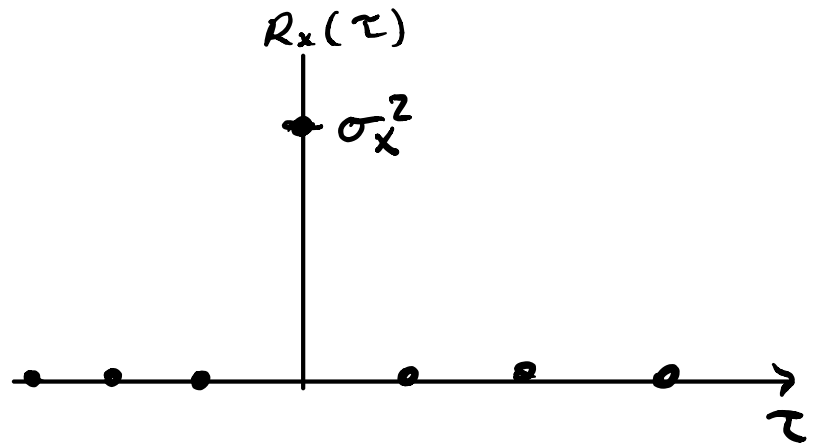
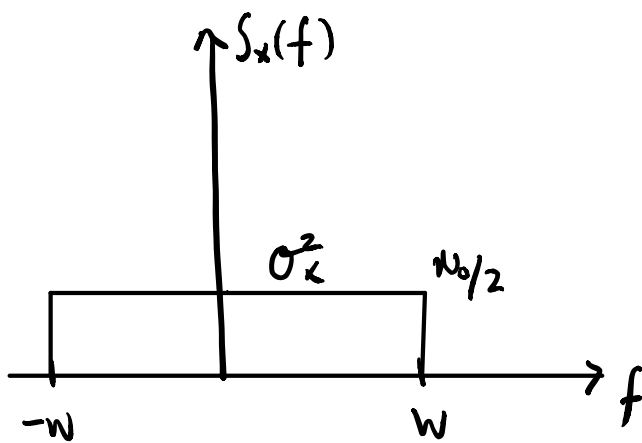


Example: ^{CT} White Noise

$$S_x(f) = \begin{cases} \frac{N_0}{2} & -W \leq f \leq W \\ 0 & \text{o/w} \end{cases}$$



Example: DT White Noise



Linear Transform of Deterministic Signals

$$x(s) \longrightarrow \boxed{h(t)} \xrightarrow{\begin{matrix} y(t) = x(t) * h(t) \\ y(t) = \int_{-\infty}^{+\infty} h(s)x(t-s) ds \end{matrix}}$$

$$X(f) \longrightarrow \boxed{\tilde{H}(f)} \xrightarrow{Y(f) = X(f)\tilde{H}(f)}$$

Have W.S.S r.p

$$X(t) \longrightarrow \boxed{h(t)} \longrightarrow Y(t) \quad \leftarrow \begin{matrix} \text{also a} \\ \text{random process} \end{matrix}$$

since for each realization $x(t)$ the system produces $y(t)$

Is $\{Y(t)\}_{t=-\infty}^{+\infty}$ W.S.S

$$\begin{aligned} \mathbb{E}[Y(t)] &= \mathbb{E}\left[\int_{-\infty}^{+\infty} h(s)x(t-s) ds\right] \\ &= \int_{-\infty}^{+\infty} \mathbb{E}[h(s)x(t-s)] ds \\ &= \int_{-\infty}^{+\infty} h(s)\mu_x ds \\ &= \mu_x \int_{-\infty}^{+\infty} h(s) ds \end{aligned}$$

$$R_Y(t, t+\tau) \triangleq \mathbb{E}[Y(t)Y(t+\tau)]$$

$$= \mathbb{E} \left[\int_{-\infty}^{+\infty} h(s) X(t-s) ds \int_{-\infty}^{+\infty} h(r) X(t+\tau-r) dr \right]$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h(s) h(r) ds dr R_X(\tau+s-r)$$

thus Y is ALSO W.S.S

$$S_Y(f) \triangleq \mathcal{F}\{R_Y(\tau)\}$$

$$= \int_{-\infty}^{+\infty} R_Y(\tau) e^{-j2\pi f\tau} d\tau$$

$$= \int_{\tau=-\infty}^{+\infty} \left(\int_{s=-\infty}^{+\infty} \int_{r=-\infty}^{+\infty} h(s) h(r) ds dr \right) R_X(\tau+s-r) e^{-j2\pi f\tau} d\tau$$

$$\tau = u - s + r \quad d\tau = du$$

$$\int_{\tau=-\infty}^{+\infty} \left(\int_{s=-\infty}^{+\infty} \int_{r=-\infty}^{+\infty} h(s) h(r) ds dr \right) R_X(u) e^{-j2\pi f(u-s+r)} du$$

$$= H^*(f) H(f) S_X(f) = |H(f)|^2 S_X(f)$$

What we have,

$$S_Y(f) = |H(f)|^2 S_X(f)$$

If $X(t)$ is a Gaussian random process, $h(t)$ a linear filter, $Y(t)$ ALSO a Gaussian random process

$$X(t) \rightarrow \boxed{h(t)} \rightarrow Y(t)$$

Consider arbitrary random processes

$$\{X(t), Y(t)\}_t$$

Say X, Y jointly wide sense stationary if

① Each of X, Y w.s.s

$$\textcircled{2} R_{X,Y}(t_1, t_2) \triangleq E[X(t_1)Y(t_2)] \equiv R_{X,Y}(\tau)$$

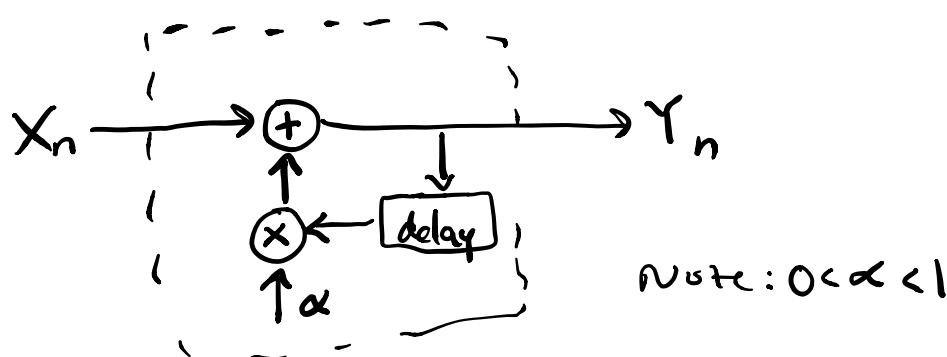
If $Y(t)$ is output of linear filter $h(t)$ driven by w.s.s $X(t)$,
Then

$$X(t) \text{ and } Y(t) \text{ J.w.s.s}$$

Also have a notion of cross-power spectrum density.

$S_{X,Y} = \mathcal{F} \{ R_{X,Y}(\tau) \}$ is the cross
PSD of $(X(t), Y(t))$

Example: First Order Auto Regression Process



$$Y_n = \alpha Y_{n-1} + X_n$$

$$= X_n + \sum_{l=1}^{\infty} \alpha^l X_{n-l} \quad \left. \vphantom{\sum_{l=1}^{\infty}} \right\} \text{can write as convolution}$$



$$h_n = \begin{cases} 0, & n < 0 \\ 1, & n = 0 \\ \alpha^n, & n > 0 \end{cases}$$

$$H(f) = \sum_{n=0}^{\infty} \alpha^n e^{-j2\pi f n} = \frac{1}{1 - \alpha e^{-j2\pi f}}$$

$$S_Y(f) = |H(f)|^2 S_X(f)$$

$$S_X(f) = \sigma_x^2$$