① $\frac{\pi}{4} = 4\tan^{-1}\left(\frac{1}{5}\right) - \tan^{-1}\left(\frac{1}{239}\right)$

$\arg(z^n) = \arg(z) + \arg(z) + \cdots + \arg(z) = n \cdot \arg(z)$

$n \in \mathbb{Z}$

$(1+i)(5-i)^4 = 956-4i$

$\arg\left[(1+i)(5-i)^4\right] = \arg(956-4i)$

$\Rightarrow \arg(1+i) + 4\arg(5-i) = \arg(956-4i)$

$\frac{\pi}{4} + 4\arctan\left(\frac{-1}{5}\right) = \arctan\left(\frac{-1}{239}\right)$

$\arctan(-u) = -\arctan(u)$

$\frac{\pi}{4} = 4\arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right)$

(2) $$\frac{d}{dz}\tan^{-1}(z) = \frac{1}{1+z^2}$$

Let

$$w = f(z) = \arctan(z)$$

So

$$\tan(w) = z$$

$$\frac{d}{dz}\left[\arctan(z)\right] = \frac{1}{\frac{d}{dw}(\tan(w))}$$

$$= \frac{1}{\sec^2 w}$$

$$= \frac{1}{1+\tan^2(w)}$$

$$= \frac{1}{1+z^2}$$

3 $\dfrac{1}{1+e^z} + \dfrac{A}{z^2+\pi^2}$

Humm

$e^z = -1$ $\quad z^2 = -\pi^2$

$z = \pm i\pi$

$z = \pm i\pi + 2\pi n, \; n \in \mathbb{Z}$

$$\int_0^\infty \frac{x^{\alpha-1}}{1+x} = \int_{-\infty}^{+\infty} \frac{e^{\alpha x}}{1+e^x}\,dx$$

$0 < \alpha < 1$

$\dfrac{\pi}{\sin(\pi\alpha)}$ $\qquad$ $\dfrac{\pi}{\sin(\pi\alpha)}$

⑥

$$\oint_{|z|=2} \tan(z)\, dz$$

$\cos(z) = 0$ @

$z = \frac{\pi}{2} + 2\pi n$

$$\tan(z) = \frac{\sin(z)}{\cos(z)} = z + \frac{z^3}{3} + \frac{2}{15} z^5 + \cdots$$

$f(z) =$

$\sin(z)$

$$\oint_{|z|=2} \frac{\sin(z)}{\cos(z)}\, dz$$

Simple poles at $z = \frac{\pi}{2} + 2\pi n, \ n \in \mathbb{Z}$

Two simple poles in our region @ $z_0 = \frac{\pi}{2}$, $z_1 = -\frac{\pi}{2}$

$$\oint_{|z|=2} \frac{\sin(z)}{\cos(z)}\, dz = 2\pi i \sum_{k=0}^{1} \mathrm{Res}(f; z_k)$$

$$\mathrm{Res}\left[\frac{\sin z}{\cos z};\ \frac{\pi}{2}\right] = \frac{\sin(\pi/2)}{\frac{d}{dz}\cos(z)\Big|_{\pi/2}} = \frac{1}{-1} = -1$$

$$\operatorname{Res}\left[\frac{\sin z}{\cos z}; -\frac{\pi}{2}\right] = \frac{\sin(-\pi/2)}{\frac{d}{dz}\cos(z)\Big|_{-\frac{\pi}{2}}} = \frac{-1}{1} = -1$$

$$= -4\pi i$$

$$2\pi i\, f(\pi/2) = 2\pi i$$

$$2\pi i\, f(-\pi/2) = -2\pi i$$

$$> 0$$

$$\frac{1}{(m-1)!}\left(\frac{d}{dz}\right)^{m-1} (z-z_0)^m f(z)$$

$$\lim_{z\to 0} \frac{1}{2} \frac{d^2}{dz^2} (z-0)^3 f(z)$$

$$= \frac{1}{2} \cdot \frac{d}{dz}\left[\frac{d}{dz}\left(1 + 2z + 4z^2 + 5z^3 + 6z^4\right)\right]$$

$$= \frac{1}{2} \cdot \frac{d}{dz}\left[2 + 8z + 15z^{2} + 24z^3\right]$$

$$= \frac{1}{2} \cdot \left[8 + 30z + 72z^2 \right]$$

$$= 4$$