Exercises.

Solution to Question 1.

(a) Because

$$\det(A - \lambda I) = (1 - \lambda)^2,$$

so eigenvalues of A are $\lambda_1 = \lambda_2 = 1$. The eigenvector corresponds to $\lambda = 1$ is $e = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

(b) A is diagnolizable if and only if the space spanned by eigenvectors is the whole space. But in this case, the whole space is 2-dimensional, while the space spanned by eigenvectors is 1-dimensional.

Solution to Question 2.

(a) Because

$$\det(A - \lambda I) = \lambda^2 + 1$$

so eigenvalues of A are $\lambda_1 = \sqrt{-1}$ and $\lambda_2 = -\sqrt{-1}$.
The eigenvector corresponds to $\lambda_1 = \sqrt{-1}$ is $e_1 = \begin{pmatrix} \sqrt{-1} \\ 1 \end{pmatrix}$.
The one corresponds to $\lambda_2 = -\sqrt{-1}$ is $e_2 = \begin{pmatrix} \sqrt{-1} \\ -1 \end{pmatrix}$

(b) Let $P = \begin{bmatrix} e_1 & e_2 \end{bmatrix}$ be the matrix with columns being e_1 and e_2. Then

$$P^{-1}AP = \begin{bmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{bmatrix}.$$

So A is diagonalizable over $\mathbb{C}$.
However, A is not diagonalizable over $\mathbb{R}$, because not all of its eigenvalues are in $\mathbb{R}$.

Solution to Question 3.

(a) We may check that
$$f(z) = (z-\lambda)(z-\bar{\lambda}) = z^2 - (\lambda+\bar{\lambda})z + \lambda\bar{\lambda},$$
so $f(z)$ is a quadratic polynomial. Both $(\lambda+\bar{\lambda}) = 2a$ and $\lambda\bar{\lambda} = a^2+b^2$ are real numbers.

(b) If we can write a quadratic polynomial $f(z) = \alpha z^2 + \beta z + \gamma$ as a product of 2 polynomials $f(z) = a(z-\lambda_1)(z-\lambda_2)$, then λ_1 and λ_2 must be the roots of $f(z)$. The roots are real if and only if the discriminant $\Delta = \beta^2 - 4\alpha\gamma \geq 0$. In our case, $\alpha = 1$, $\beta = -2a$ and $\gamma = a^2+b^2$. Remember that we assume $b > 0$ here, so
$$\Delta = 4a^2 - 4(a^2+b^2) < 0,$$
which means $f(z)$ is irreducible.

(c) By the fundamental theorem of algebra, $f(z)$ has a factorization of the form
$$f(z) = a(z-c_1)\dots(z-c_n),$$
where $c_1,\dots,c_n \in \mathbb{C}$. By reordering the factors, we can always assume that $c_1,\dots,c_r \in \mathbb{R}$ for some non-negative integer r, and the rest c_i's are not in $\mathbb{R}$.
Notice that if c satisfies $f(c) = 0$, then $f(\bar{c}) = 0$, too. This is because the coefficients of $f(z)$ are in $\mathbb{R}$. Therefore, if $c \notin \mathbb{R}$ and $(z-c)|f(z)$, then $(z-c)(z-\bar{c})|f(z)$. So by reordering the factors, we may assume
$$\begin{aligned}(z-c_{r+1})\dots(z-c_n) &= (z-c_{i_1})(z-\bar{c}_{i_1})\dots(z-c_{i_s})(z-\bar{c}_{i_s})\\ &= (z^2-p_1z+q_1)\dots(z^2-p_sz+q_s),\end{aligned}$$
where $p_j = c_j + \bar{c}_j$ and $q_j = c_j\bar{c}_j$ for $j = 1,\dots,s$. All of the p_j's and q_j's are real numbers.
We need to show that this factorization is unique up to reordering the factors.
We define an order $\leq$ on $\mathbb{R}^2$ by $(p,q) \leq (p',q')$ if $p < p'$, or if $p = p'$ and $q \leq q'$. This is an total order on $\mathbb{R}^2$. Then by reordering the factors, we may always assume $c_1 \leq c_2 \leq \dots \leq c_r$ and $(p_1,q_1) \leq \dots \leq (p_s,q_s)$.
Now assume there is another factorization
$$f(z) = (z-b_1)\dots(z-b_{r'})(z^2-d_1z+e_1)\dots(z^2-d_{s'}z+e_{s'}),$$
where $b_1 \leq \dots \leq b_{r'}$ and $(d_1,e_1) \leq \dots \leq (d_{s'},e_{s'})$.
First of all, notice that all the quadratic factors are irreducible. Therefore,
$$(z-b_1)\dots(z-b_{r'}) = (z-c_1)\dots(z-c_r)$$
and
$$(z^2-p_1z+q_1)\dots(z^2-p_sz+q_s) = (z^2-d_1z+e_1)\dots(z^2-d_{s'}z+e_{s'}).$$
In particular, by comparing the degrees, we can see $r = r'$ and $s = s'$.
Now, if there exist a number i such that $c_i \neq b_i$, then let
$$k := \min\{i \mid c_i \neq b_i\}.$$

So we have
$$(z - c_k) \dots (z - c_r) = (z - b_k) \dots (z - b_r).$$
WLOG, assume $c_k < b_k$. Then $c_k < b_j$ for all $j = k, \dots, r$. But
$$(c_k - b_k) \dots (c_k - b_r) = 0,$$
which is a contradiction. Hence $c_i = b_i$ for all $i = 1, \dots, r$.
Similarly, if there exist a number i such that $(p_i, q_i) \neq (d_i, e_i)$, then let
$$k := \min\{i \mid c_i \neq b_i\}.$$
So we have
$$(z^2 - p_k z + q_k) \dots (z^2 - p_r z + q_r) = (z^2 - d_k z + e_k) \dots (z^2 - d_r z + e_r).$$
WLOG, assume $(p_k, q_k) < (d_k, e_k)$. Then $(p_k, q_k) < (d_j, e_j)$ for all $j = k, \dots, r$. However, if $w \in \mathbb{C}$ is a root of $(z^2 - p_k z + q_k)$, then $(w^2 - d_j w + e_j) \neq 0$ for all $j = k, \dots, r$. So
$$(w^2 - d_k w + e_k) \dots (w^2 - d_s w + e_s) \neq 0,$$
which is a contradiction. Hence $(p_k, q_k) = (d_k, e_k)$ for all $k = 1, \dots, r$.

Solution to Question 4. Assume that

$$f(x) = \sum_{i=0}^{n} c_i x^i.$$

Because

$$A^k v = A^{k-1}(Av) = \lambda A^{k-1} v = \cdots = \lambda^k v,$$

so

$$f(A)v = (\sum_{i=0}^{n} c_i A^i)v = \sum_{i=0}^{n} c_i A^i v = (\sum_{i=0}^{n} c_i \lambda^i)v.$$

Therefore, v is also an eigenvector of $f(A)$. Its corresponding eigenvalue is $f(\lambda)$.

Solution to Question 5.

(a) This is because

$$D(f(t)e^{rt}) = f'(t)e^{rt} + rf(t)e^{rt}.$$

(b) Claim: $(e^{2t}, te^{2t}, t^2e^{2t})$ is a basis for $\ker(D-2I)^3$.
Reason: By repeatedly applying part (a), we can see that

$$(D-rI)^n(f(t)e^{rt}) = f^{(n)}(t)e^{rt}.$$

So $e^{2t}, te^{2t}, t^2e^{2t} \in \ker(D-2I)^3$ and they are linearly independent. Because $\dim\ker(D-2I)^3 = 3$, so we know $(e^{2t}, te^{2t}, t^2e^{2t})$ is a basis for $\ker(D-2I)^3$.

(c) Claim: (e^t, e^{2t}, te^{2t}) is a basis for $\ker(D-I)(D-2I)^2$.
Reason: Because $(D-I)(D-2I)^2 = (D-2I)^2(D-I)$, so $e^t \in \ker(D-I) \subset \ker(D-I)(D-2I)^2$ and $e^{2t}, te^{2t} \in \ker(D-2I)^2 \subset \ker(D-I)(D-2I)^2$.
It is easy to check (e^t, e^{2t}, te^{2t}) is linearly independent.

(d) The solution set of $y(t)$ satisfying

$$y''' + y'' - y' + y = 0$$

equals $\ker(D^3 + D^2 - D + I)$.
Let $f(x) = x^3 + x^2 - x + 1$. Because $\gcd(f(x), f'(x)) = 1$, so there is no multiple root of $f(x)$.
Assume the roots of $f(x)$ are λ_1, λ_2 and λ_3. Then

$$(e^{\lambda_1 t}, e^{\lambda_1 t}, e^{\lambda_1 t})$$

is a basis for $\ker(D^3 + D^2 - D + I)$.

(d′) The solution set of $y(t)$ satisfying

$$y''' - y'' - y' + y = 0$$

equals $\ker(D+I)(D-I)^2$. And (e^{-t}, e^t, te^t) is a basis of $\ker(D+I)(D-I)^2$.

Solution to Question 6.

(a) If A is the diagonal matrix $\operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$, then

$$A^k = \operatorname{diag}(\lambda_1^k, \lambda_2^k, \dots, \lambda_n^k).$$

So

$$\begin{aligned} e^A &= \sum_{k=0}^{\infty} \frac{A^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \operatorname{diag}(\lambda_1^k, \lambda_2^k, \dots, \lambda_n^k) \\ &= \operatorname{diag}(e^{\lambda_1}, \dots, e^{\lambda_n}). \end{aligned}$$

(b) Notice that

$$A^2 = -I.$$

Therefore,

$$e^A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

where

$$a = d = 1 - \frac{1}{2!} + \frac{1}{4!} + \dots$$

and

$$b = -c = \frac{1}{1!} - \frac{1}{3!} + \frac{1}{5!} + \dots.$$

Recall that

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

and

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}.$$

So $a = d = \cos 1$ and $b = -c = \sin 1$.

(c) Let $R = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then

$$A = I + R.$$

Notice that $R^2 = 0$. So

$$A^k = I^k + kRI^{k-1} = I + kR.$$

Therefore

$$
\begin{aligned}
e^A &= \sum_{k=0}^{\infty} \frac{1}{k!}(I + kR) \\
&= \begin{bmatrix} e & e \\ 0 & e \end{bmatrix}.
\end{aligned}
$$

(d) First of all notice that if

$$B = Q^{-1}AQ,$$

then

$$B^k = Q^{-1}A^kQ$$

for all $k = 1, 2, \ldots$. Hence,

$$
\begin{aligned}
e^B &= \sum_{k=0}^{\infty} \frac{1}{k!} Q^{-1}A^kQ \\
&= Q^{-1}\Big(\sum_{k=0}^{\infty} \frac{1}{k!} A^k\Big)Q \\
&= Q^{-1}e^AQ.
\end{aligned}
$$