

We want to find  $s \in \mathbb{C}$  which solves  $f(s) = 0$ , where  $f(s)$  is given by

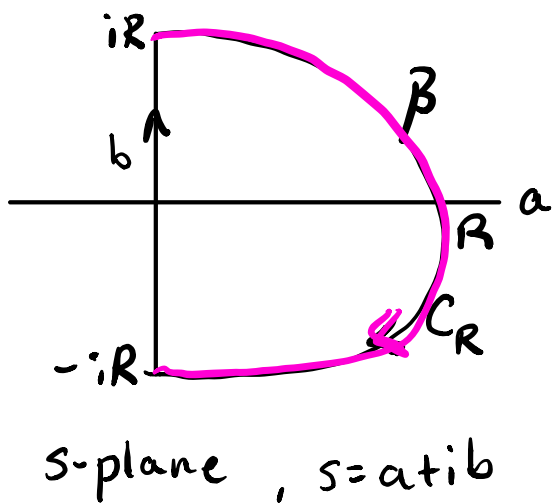
Too Ambitious.

$$f(s) = s + k e^{-\tau s} \quad \tau > 0 \text{ "delay"}$$

$$k > 0 \text{ "gain"}$$

Instead, settle for finding conditions on  $k, \tau$  st.  $\operatorname{Re}(s) < 0$  for all zeros of  $f(s)$ .

Stability condition.



For stability we want no zeros of  $f(s)$  inside  $\beta$  (as  $R \rightarrow \infty$ )

By the argument principle, this holds if  $\Delta J(\gamma; 0) = 0$  where  $\gamma = f(\beta)$

What does  $\gamma$  look like?

First ask what does  $f(C_R)$  look like?

On  $C_R$ ,

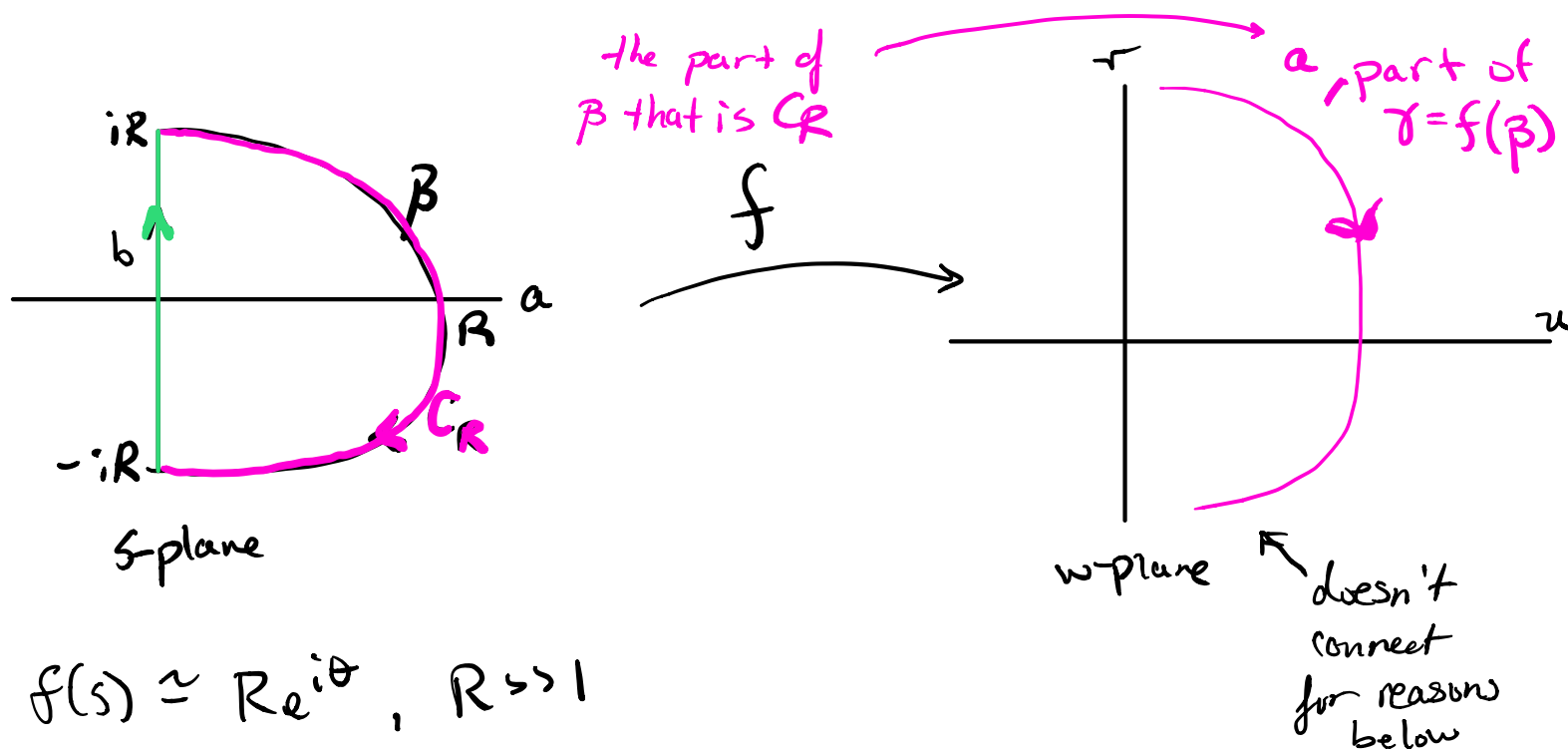
$$s = R e^{i\theta}, \quad \pi/2 \leq \theta \leq -\pi/2$$

$$\begin{aligned} \Rightarrow f(s) &= s + k e^{-\tau s} = R e^{i\theta} + k e^{-\tau R e^{i\theta}} \\ &= R e^{i\theta} + k e^{-\tau R (\cos\theta + i \sin\theta)} \\ &= R e^{i\theta} + k e^{-\tau R \cos\theta} e^{-i \tau R \sin\theta} \end{aligned}$$

We're thinking of arbitrarily large  $R$ .

$$Re^{i\theta} + \underbrace{ke^{-i\tau R \cos \theta}}_{\rightarrow 0 \text{ as } R \rightarrow \infty} \underbrace{e^{-i\tau R \sin(\theta)}}_{\text{"oscillates like crazy w/ negligible amplitude"}}$$

So the image of the semicircle is another big semicircle.



$$f(s) \approx Re^{i\theta}, R \gg 1$$

Need to map imaginary axis as well.

\* Look at  $f(ib)$  as  $b$  runs from  $-R$  to  $R$  \*

$$f(ib) = ib + ke^{-ib\tau}$$

$$= \underbrace{k \cos(\tau b)}_{u(b)} + i \underbrace{(b - k \sin(\tau b))}_{v(b)}$$

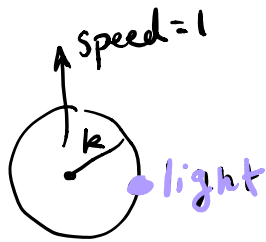
$w$  plane is  $u + iv$ !

What does the curve  $(u(b), v(b))$  look like?

"Think of  $b$  (the parameter in the parametric equations for a curve) as a kind of 'time'."

$$\left. \begin{aligned} u(b) &= k \cos(\tau b) \\ v(b) &= -k \sin(\tau b) + b \end{aligned} \right\} \begin{array}{l} \text{Circular motion} + \text{vertical translation} \\ \uparrow \qquad \qquad \uparrow \\ \text{frequency } \tau \quad \text{upward "speed" of 1} \\ \text{radius } k \end{array}$$

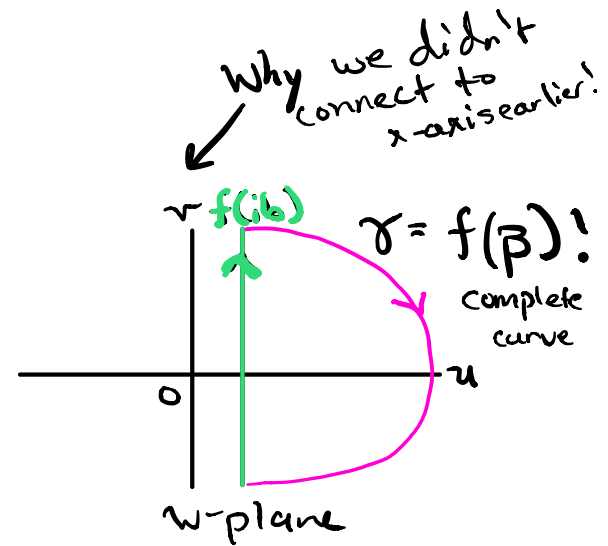
$f(ib) = \text{path traced by moving lightbulb.}$



If  $\tau = 0$  (warm up)

$$u(b) = k$$

$$v(b) = b$$



$W(\gamma; 0) = 0$  here, so no zeros of  $s$  in right half plane  
 $\Rightarrow$  control system stable (as expected, since no delay)

If  $\tau$  large,

$$u(b) = k \cos(\tau b), \quad -R \leq b \leq R; \quad R \rightarrow \infty$$

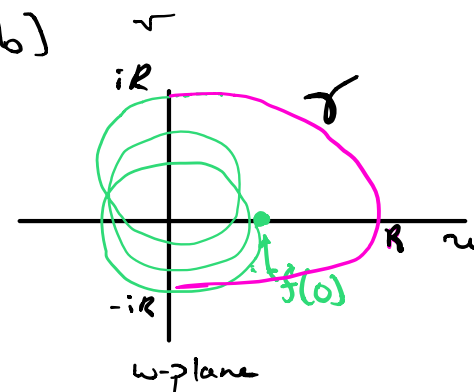
$$v(b) = -k \sin(\tau b) + b$$

when  $b=0$ ,  $f(ib) = k = u(b) + i v(b)$

as  $b$  increases the center of our circle changes.

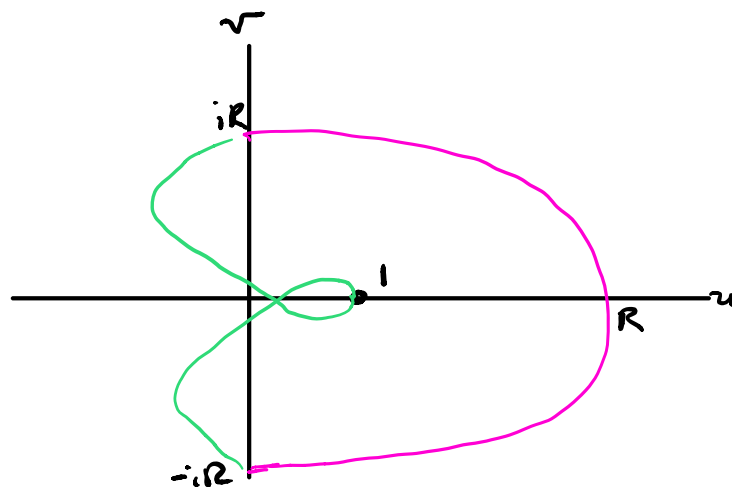
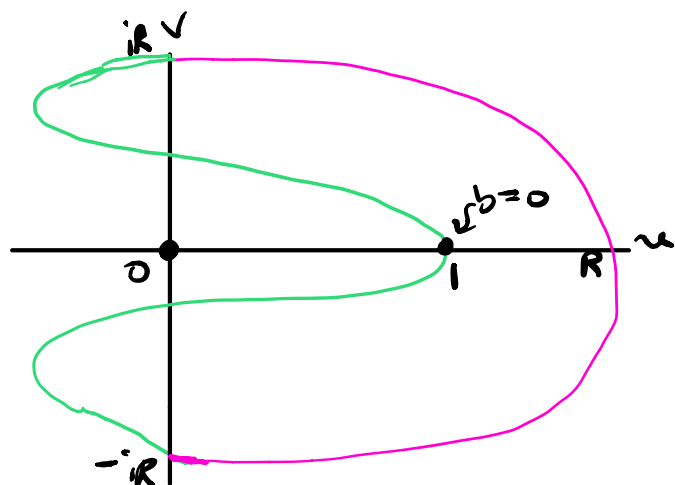
i.e. we encircle the origin many times

$\Rightarrow$  wickidly unstable



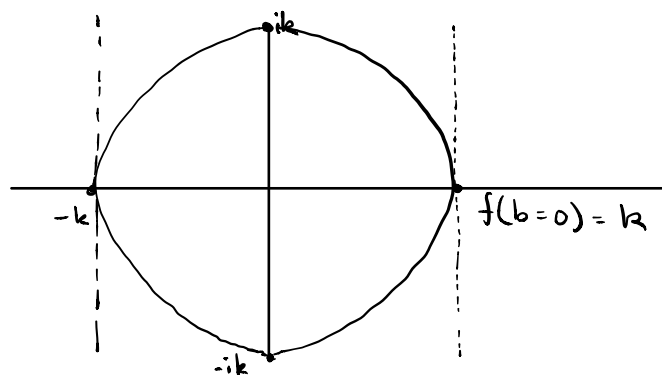
For small  $\tau > 0$ , (0.1)

(k=1) Slightly larger ( $\tau=1.3$ )



What's the critical point?

The critical case is when the lowest point due to circular motion is the same as the vertical distance traveled by translation.



$$u(b) = k \cos(\tau b)$$

$$v(b) = -k \sin(\tau b) + b$$

$\tau b = \pi/2 \Rightarrow$  lowest pt on circle

$$v(b) = b - k \leftarrow$$

$$b = k = \frac{\pi}{2\tau}$$

$S_0 \quad \tau_c = \frac{\pi}{2k}$