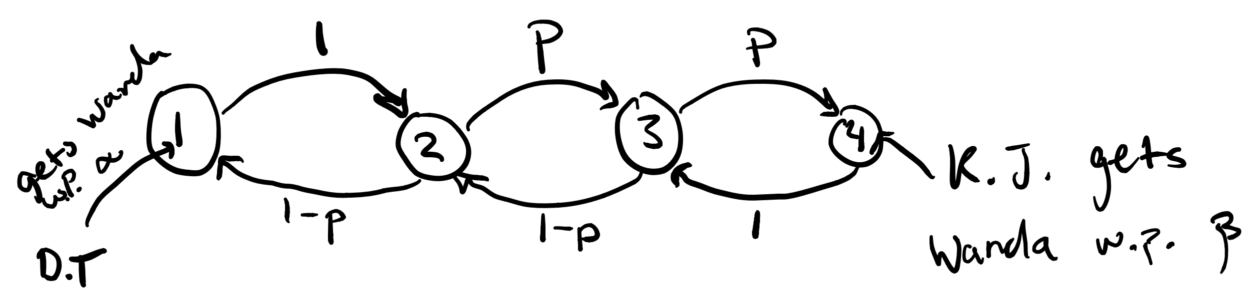
$$\mathcal{X} = \mathcal{T} \cup C_1 \cup C_2 \cup \cdots$$

$\mathcal{T}$: transient state

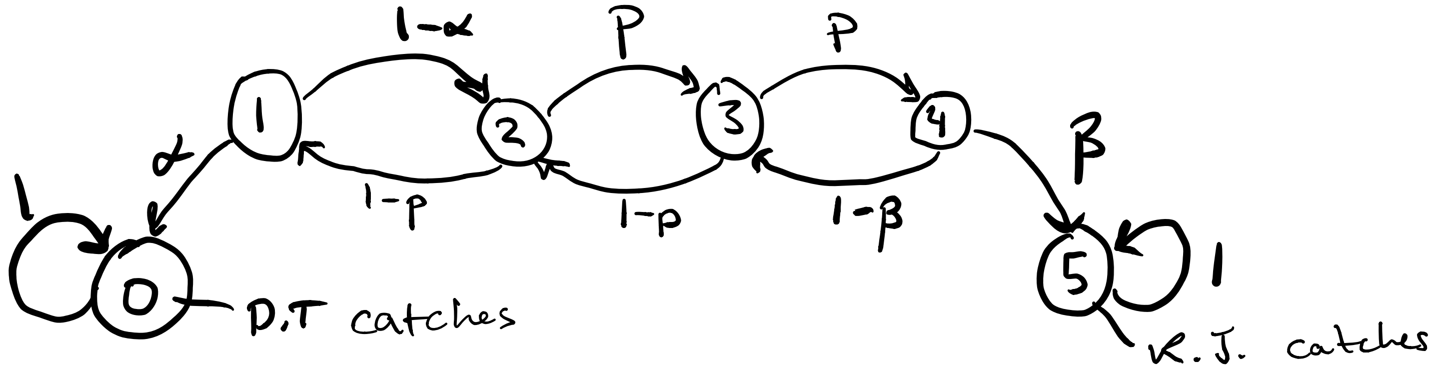
$C_1 = \{1 \cdots M_1\}$, $C_2 = \{M_1+1 \cdots M_1+M_2\}$, ...: recurrent classes

$$
\begin{array}{c}
\\
1 \\ \vdots \\ M_1 \\ M_1+1 \\ \vdots \\ M_1+M_2 \\ M_2+1 \\ \vdots \\ M_1+M_2+M_3 \\ \mathcal{T}\,\{
\end{array}
\mathbb{P} =
\begin{array}{cccc}
1 \cdots M_1 & M_1+1 \cdots M_2 & M_1+M_1+1 \cdots & M_1+M_2+M_3 \\
\left(\begin{array}{c|c|c|c}
\boxed{\mathbb{P}_1} & 0 & 0 & 0 \\
\hline
0 & \boxed{\mathbb{P}_2} & 0 & 0 \\
\hline
0 & 0 & \boxed{\mathbb{P}_3} & 0 \\
\hline
\mathbb{R}_1 & \mathbb{R}_2 & \mathbb{R}_3 & \mathbb{Q}
\end{array}\right)
\end{array}
$$

Absorption Problem



Add a state



once caught, 1, 2, 3, 4 become transient states.
0, 5 recurrent classes.

$\Pr[\text{absorbed at state } 0 \mid \text{current state } i] = g_i$

$$g_0 = 1$$
$$g_1 = \alpha + (1-\alpha) g_2$$
$$g_2 = (1-p) g_1 + p g_3$$
$$g_3 = (1-p) g_2 + p g_4$$
$$g_4 = (1-\beta) g_3 + \beta g_5$$
$$g_5 = 0$$

6 equations, 6 unknowns. Solve!

Need some initial conditions on Wanda's initial condition.

Use stationary distribution:

$$\bar{\pi} = [\pi_1 \ \pi_2 \ \pi_3 \ \pi_4]$$

and

$$g_1 = [g_1 \ g_2 \ g_3 \ g_4]$$

$$\Pr[\text{D.T wins}] = \langle \bar{\pi}, g_1 \rangle$$

$$= \pi_1 g_1 + \pi_2 g_2 + \pi_3 g_3 + \pi_4 g_4$$

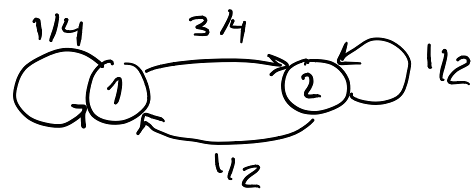
$$\Pr[\text{K.J. wins}] = 1 - \Pr[\text{D.T. wins}]$$

Absorption Time

Coin A, Coin B

$A: \Pr[H] = 1/2$

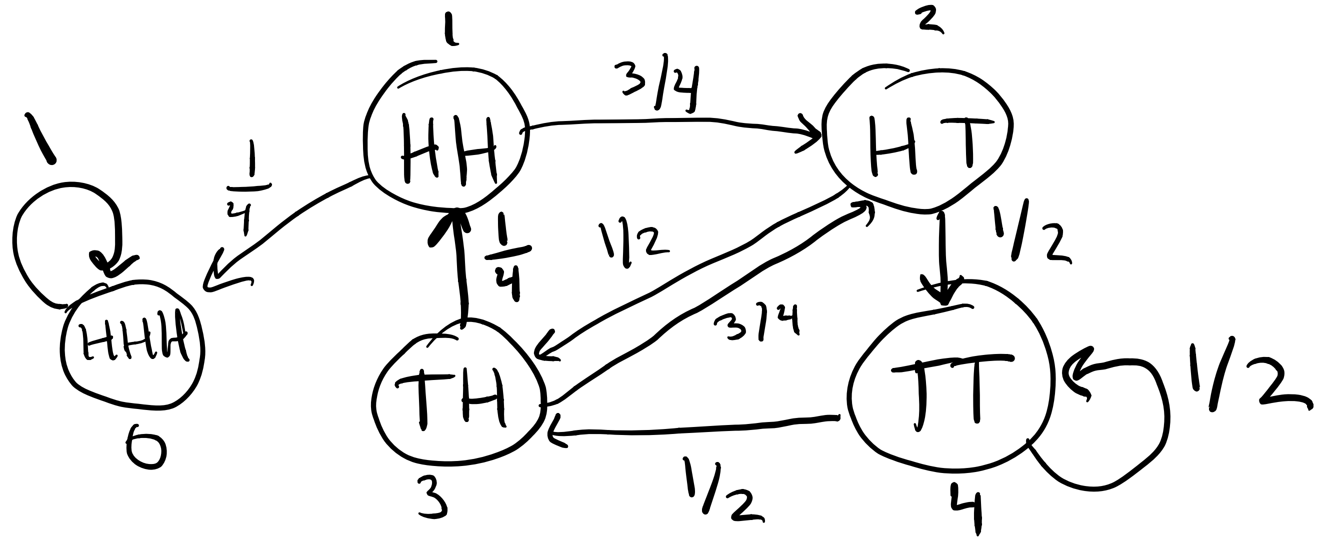
$B: \Pr[H] = 1/4$



$$X_0 \leftarrow A$$

$$X_{n+1} \leftarrow \begin{cases} B & \text{if } x_n = 1 \\ A & \text{if } x_n = 2 \end{cases}$$

Question: time until we see 3 consecutive 'H' for the first time.



Define

$$e_i \triangleq \mathbb{E}\left[\begin{matrix}\text{remaining time}\\ \text{until absorption}\end{matrix} \,\middle|\, \text{current state is } i\right]$$

$$e_0 = 0$$

$$e_1 = \frac{1}{4}\cdot \underset{}{\overset{(e_0+1)}{1}} + \frac{3}{4}\cdot(e_2+1)$$

$$e_2 = \frac{1}{2}(e_3+1) + \frac{1}{2}(e_4+1)$$

$$e_3 = \frac{1}{4}(e_1+1) + \frac{3}{4}(e_2+1)$$

$$e_4 = \frac{1}{2}(e_3+1) + \frac{1}{2}(e_4+1)$$

$Pr[①] = 1/8$, $Pr[②] = \frac{1}{2} \cdot \frac{3}{4} = \frac{3}{8}$

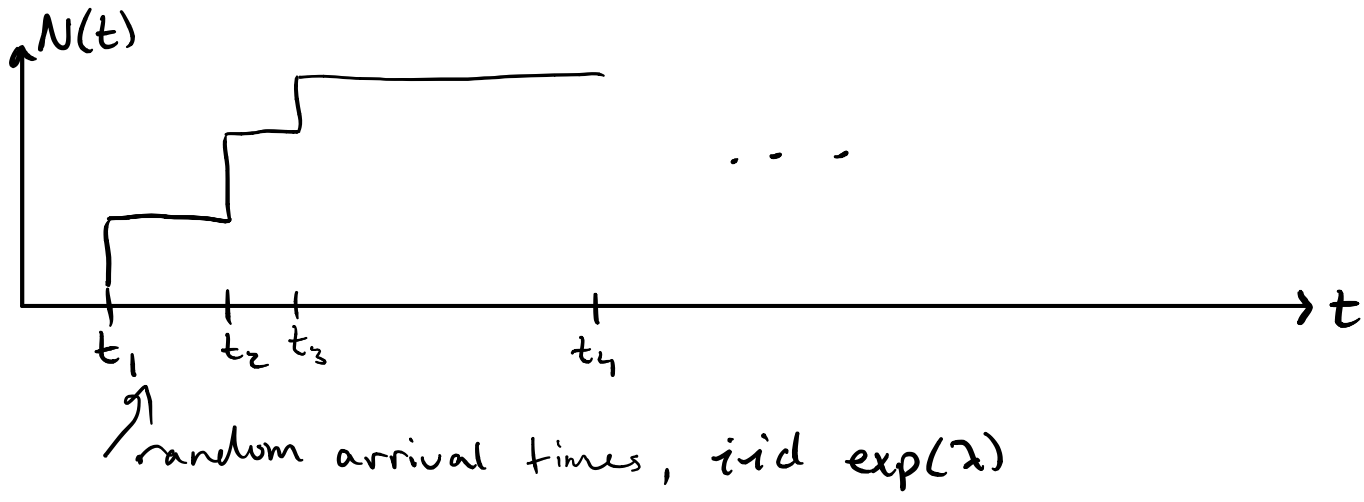
$Pr[③] = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$, $Pr[④] = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

final answer is

$$e_1 Pr[①] + e_2 Pr[②] + e_3 Pr[③] + e_4 Pr[④]$$

Continuous Time Markov Chains

Poisson Process



State transition times: $\{S_1, S_2, \ldots\}$

$$\{P_{i,j}\}_{i,j \in \mathcal{X}}$$

Recall: DTMC

$$\Pr[X_{n+1} = x_{n+1} \mid X_n = x_n, \ldots, X_0 = x_0]$$
$$= \Pr[X_{n+1} = x_{n+1} \mid X_n = x_n]$$

For CTMC: Pick $t_0 < t_1 < t_2 < t_3 < \cdots < t_{n+1}$

then

$$\Pr[X_{t_{n+1}} = x_{n+1} \mid X_{t_n} = x_n, \ldots, X_{t_0} = x_0]$$
$$= \Pr[X_{t_{n+1}} = x_{n+1} \mid X_{t_n} = x_n] \qquad \forall n, \forall \{t_k\}$$