Triangle Inequality

$$|z_1 + z_2| \le |z_1| + |z_2|$$
$$|z_2 + z_1| \ge |z_2| - |z_1|$$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

If f is differentiable in an open region we say f is analytic in that region

Cauchy-Riemann Equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$f'(z) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta y \to 0} \frac{f(z+i\Delta y) - f(z)}{i\Delta y}$$

$$\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$$

Example

$f(z) = x + xy + iv(x,y)$ analytic.

$$\frac{\partial u}{\partial x} = 1 + y = \frac{\partial v}{\partial y} \qquad -\frac{\partial u}{\partial y} = -x = \frac{\partial v}{\partial x}$$

$$v(x,y) = \int \frac{\partial v}{\partial y}\,\partial y = y + \frac{1}{2}y^2 + h(x)$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \Rightarrow \begin{array}{l} h'(x) = -x \\ h(x) = -\frac{1}{2}x^2 + C \end{array}$$

$$v(x,y) = y + \frac{1}{2}y^2 - \frac{1}{2}x^2 + C$$

Consequences

Real + Imaginary parts of an analytic function are harmonic!

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

Useful Stuff

- $\cos(z) = \cosh(y)\cos(x) + i\sinh(y)\sin(x)$
- $\sin(z) = \cosh(y)\sin(x) + i\sinh(y)\cos(x)$
- All polynomials are analytic

Complex Power

$f(z) = z^\alpha$

$z^\alpha = e^{\log z^\alpha} = e^{\alpha \log(z)}$

log has branch cuts so z^α has branch cuts

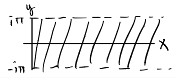
Newton's Method

$$z_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}$$

Exponential Function $f(z) = e^z$

$e^z = e^x e^{iy} = e^x[\cos(y) + i\sin(y)]$

periodic w/ period $2\pi i$



One-to-one in this strip!

Logarithms

$\log(z) = \log(r) + i\arg(z) \leftarrow$ Multivalued! Problem! Branch cut it!

$\text{Log}(z) = \text{Log}(|z|) + i\,\text{Arg}(z) \leftarrow \text{Arg}(z)$ is harmonic in $\mathbb{C} - \{\text{branch cut}\}$

Mean Value Property

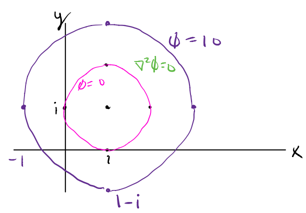
$$f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a + re^{i\theta})\,d\theta$$

f analytic in a neighborhood of pt a

Washers

$\phi(r) = A\ln(r) + B$ since it doesn't depend on θ

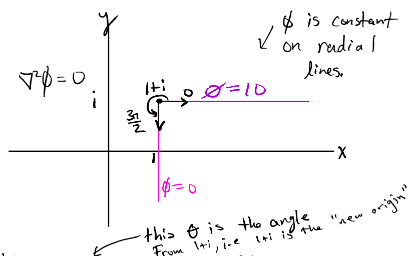
At end, $\phi(z) = A\ln(|z-a|) + B$; A,B solved earlier



Solution of the for washers is generally

$$\phi = A\,\text{Log}(r) + B$$
$$0 = A\,\cancel{\text{Log}(1)}^{0} + B \Rightarrow B = 0$$
$$10 = A\,\text{Log}(2) + B$$
$$A = \frac{10}{\text{Log}(2)}$$
$$\phi(z) = \frac{10}{\text{Log}(2)}\,\text{Log}\,|z - z_0|$$

Wedges



this θ is the angle from $1+i$, i.e. $1+i$ is the "new origin"

$\phi(\theta) = A\theta + B$ At 1, $\theta = 0$

$\phi(0) = 10 = B$ At i, $\theta = 3\pi/2$

$\phi\left(\frac{3\pi}{2}\right) = 0 = 10 + A\frac{3\pi}{2}$

$$A = -\frac{20}{3\pi}$$

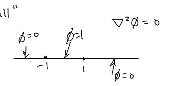
$$\phi(\theta) = -\frac{20}{3\pi}\theta + 10$$

as a function of z

$$\phi(z) = -\frac{20}{3\pi}\text{Arg}(z - z_0) + 10$$

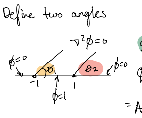
Wall

Ex 3: "Wall" $\nabla^2\phi = 0$ here



Sol'n

Define two angles



Seek soln of

$$\phi = A_1\theta_1 + A_2\theta_2 + B$$
$$= A_1\,\text{Arg}(z+1) + A_2\arg(z-1) + B$$

$-\pi < \theta_1 \le \pi$

$-\pi < \theta_2 \le \pi$

$\boxed{\text{Impose B.C.}}$

For $z = x > 1$, have $\theta_1 = \theta_2 = 0$

So $\phi = A_1(0) + A_2(0) + B$

$\Rightarrow B = 0$

For $-1 < x < 1$: $\theta_1 = 0$, $\theta_2 = \pi$

So $\phi = A_1(0) + A_2\pi = 1$

$A_2 = \frac{1}{\pi}$

For $z = x < -1$: $\theta_1 = \theta_2 = \pi$

$\phi = 0 = A_1\theta_1 + \frac{1}{\pi}\theta_2$

$0 = A_1\pi + \frac{1}{\pi}\pi$

$A_1 = -\frac{1}{\pi}$

Therefore,

$$\phi = \frac{1}{\pi}\left[\text{Arg}(z-1) - \text{Arg}(z+1)\right] = \frac{1}{\pi}\left[\theta_2 - \theta_1\right]$$

Cauchy's Theorem

$$\oint_\gamma f(z)\,dz = 0,$$

f analytic in a simply connected domain D, $\gamma \to$ curve in D

Consequences

Antiderivative Property

- Path dependent, let $F'(z) = f(z)$

f continuous on open, connected set

$$\int_{z_0}^{z_1} f(z)\,dz = F(z_1) - F(z_0)$$

'Proof'

$\gamma = z(t)$, $0 \le t \le 1$, $z(0) = z_0$, $z(1) = z_1$

$$\int_\gamma f(z)\,dz = \int_{t=0}^{t=1} f(z(t))\frac{dz}{dt}\,dt = \int_{t=0}^{t=1} F'(z(t))\frac{dz}{dt}\,dt$$

$$= \int_{t=0}^{1}\frac{d}{dt}\Big(F(z(t))\Big)\,dt = F(z(1)) - F(z(0)) = F(z_1) - F(z_0)$$

$\Rightarrow$

$$\oint_\gamma (z-a)^n = 0, \; n \ne -1$$

'Proof' $\frac{1}{n+1}(z-a)^{n+1}$ is single valued in $\mathbb{C} - \{a\}$ (punctured plane)

ML Estimate

$$\left|\int_\gamma f(z)\,dz\right| \le \max_{z\in\gamma}|f(z)| \cdot \text{length}(\gamma)$$

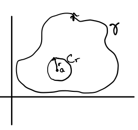
'Proof'

$$\left|\sum_{k=1}^{N} f(z_k)\Delta z_k\right| \le \left|\sum_{k=1}^{N} f(z_k)\right|\left|\Delta z_k\right| \le M\sum_{k=1}^{N}|\Delta z_k|$$

$\max|f(z_k)| \le M$, L

$$\oint_\gamma \frac{dz}{z-a}$$

where γ is any contour which encircles a once positively



$\frac{1}{z-a}$ is analytic everywhere EXCEPT $z = a$.

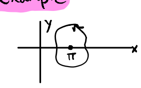
By last result, we can deform contour to a small circle C_r around a.

$$\int_\gamma \frac{dz}{z-a} = \int_{C_r}\frac{dz}{z-a} \qquad \begin{array}{l} z = re^{it} + a \quad 0 \le t \le 2\pi \\ dz = rie^{it}\,dt \end{array}$$

$$\Rightarrow \int_0^{2\pi}\frac{rie^{it}}{re^{it} + a - a}\,dt = 2\pi i$$

CIF Example

$$\oint_\gamma \frac{\cos(z)}{(z-\pi)(z+5i)}\,dz$$



$$f(z) = \cos(z)/(z+5i) \to \oint \frac{f(z)}{z+5i}\,dz = 2\pi i f(5i)$$

CIF For Derivative

$$f^{(n)}(a) = \frac{n!}{2\pi i}\oint_\gamma \frac{f(z)}{(z-a)^{n+1}}\,dz$$

$\Rightarrow$ Infinitely Differentiable

Cauchy's Integral Formula

$$f(a) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{(z-a)}\,dz$$

'Proof': Deform γ to C_r

$$\oint_\gamma \frac{f(z)}{z-a}\,dz = \oint_{C_r}\frac{f(z)}{z-a}\,dz$$

On C_r, $f(z) = f(a) + \varepsilon(z)$ $\varepsilon \to 0$ as $z \to a$

Sub f in, ML estimate on ε

Bounds on Derivative

$f(z)$ analytic on + inside circle of radius R about a.

$$M = \max_{z\in C_R}|f(z)| \xrightarrow{\text{Then}} \left|f^{(n)}(a)\right| \le \frac{n!\,f(z)}{R^n}$$

Proof: $f^{(n)}(a) = \frac{n!}{2\pi i}\oint_{C_R}\frac{f(z)}{(z-a)^{n+1}}\,dz$

ML Estimate $\Rightarrow$

$$\left|f^{(n)}(a)\right| \le \underbrace{\max_{z\in C_R}\left|\frac{f(z)}{(z-a)^{n+1}}\right|\left|\frac{n!}{2\pi}\right|}_{M}\cdot\underbrace{2\pi R}_{L}$$

$$= \max_{z\in C_R}|f(z)|\frac{n!}{R^{n+1}}R$$

Note: $|z-a| = R$

$$= M\frac{n!}{R^n}$$

So, $\left|f^{(n)}(a)\right| \le \frac{Mn!}{R^n}$

Louisville's Theorem

$f(z)$ is entire and bounded by some constant. $|f(z)| \le M \ \forall z$. Then $f(z)$ must be constant.

'Proof'

$|f'(a)| \le \frac{M}{R} \ \forall R$. (since M is independent of R by boundedness) As $R \to \infty$, $|f'(a)| \to 0$

Maximum Modulus Theorem

A function analytic in a bounded domain, continuous up to and including its boundary, attains its max or min on the boundary

Isolated Singularities

$f(z)$ has an isolated singularity at z_0 if $f(z)$ is analytic in punctured disk $0 < |z - z_0| < R$

1) Removable

$|f(z)|$ bounded as $z \to z_0$ (meaning finite / doesn't blow up)

2) pole

$|f(z)| \to \infty$ as $z \to z_0$

3) Essential

$f(z)$ has no limit as $z \to z_0$

(oscillates WILDLY as $z \to z_0$)

Deformation Theorem

If a contour sweeps only through analytic points as it is deformed, the value of the integral does NOT change

Residues

The a_{-1} term!

If f has a pole of order m at z_0, then

$$\text{Res}(f; z_0) = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[(z - z_0)^m f(z) \right]$$

If γ is a simple closed contour oriented CCW, and f analytic in and on γ except at the points $z_1, z_2, \dots, z_n$ inside γ, then

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{n=1}^{N} \text{Res}(f; z_n)$$

Taylor Series

Suppose $f(z)$ is analytic in and on C_R about z_0. Then

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

$$R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| \quad \text{RoC!}$$

"distance to next" singularity

- Any analytic function can be written as a convergent series

1) This Taylor series converges for all z in the open disk $|z - z_0| < R$, and it converges to the correct value, $f(z)$

2) The maximum R that will ensure convergence is called the "Radius of convergence" for f about z_0.

Laurent Series

If f has a pole of order m at a, then in the vicinity of this pole, $f(z)$ possesses a Laurent series of the form

$$f(z) = \frac{C_0}{(z-a)^m} + \frac{C_1}{(z-a)^{m-1}} + \dots + \frac{C_{m-1}}{z-a} + C_m + C_{m+1}(z-a) + \dots$$

If f analytic everywhere in an annulus A centered at a,

$$f(z) = \sum_{n=-\infty}^{+\infty} c_n (z - a_n)^n, \quad c_n = \oint_K \frac{f(z)}{(z-a)^{n+1}}\,dz$$

K is a simple loop winding once around a

$\frac{1}{1-z}$, $|z| < 1$

if $|z| > 1$

$z = \frac{1}{u}$

$u = \frac{1}{z}$ $|u| < 1$

Trigonometric Integrals

$$\int_0^{\pi} \frac{d\theta}{(3 + 2\cos\theta)^2} = \frac{3\pi\sqrt{5}}{25}$$

$z = e^{i\theta}$

$dz = ie^{i\theta} d\theta \Rightarrow d\theta = \frac{1}{i}\frac{dz}{z}$

$\cos\theta = \frac{z + 1/z}{2}$

Note: $\int_0^{\pi} \frac{d\theta}{3 + 2\cos\theta} = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{3 + 2\cos\theta}$

So our integral becomes

$$\frac{1}{2}\int_0^{2\pi} \frac{dz/zi}{\left[3 + 2\left(\frac{z + 1/z}{2}\right)\right]^2} = \frac{1}{2i}\int_0^{2\pi} \frac{z}{(3z + z^2 + 1)^2}\,dz$$

The roots of $z^2 + 3z + 1$ given by the quadratic formula are

$z_{1,2} = \frac{-3 \pm \sqrt{9-4}}{2} \Rightarrow z_1 = -\frac{3}{2} + \frac{\sqrt{5}}{2}$; $z_2 = -\frac{3}{2} - \frac{\sqrt{5}}{2}$

Therefore our integral becomes $\frac{1}{2i}\int_{|z|=1} \frac{dz}{[(z - z_1)(z - z_2)]^2}$

Only z_1 lies in our contour. Therefore z_1 is a pole of order 2.

$\Rightarrow \text{Res}(f(z); z_1) = \lim_{z \to z_1} \frac{1}{(m-1)!}\left(\frac{d}{dz}\right)^{m-1} (z - z_1)^m f(z)$

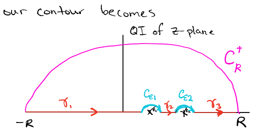
So $\frac{1}{2i}\oint_{|z|=1} \frac{dz}{(z-z_1)^2(z-z_2)^2} = \frac{2\pi i}{2i} \cdot \frac{3\sqrt{5}}{25} = \boxed{\frac{3\pi\sqrt{5}}{25}}$

Indented Contours

$$\lim_{R \to \infty} \int_{-R}^{R} \frac{e^{ix}}{(x-1)(x-2)}\,dx = \pi i \left(e^{2i} - e^{i}\right)$$

$f(x) = \frac{e^{ix}}{(x-1)(x-2)}$ has simple poles at $x = 1, 2$

So, our contour becomes



Let $f(z) := \frac{e^{iz}}{(z-1)(z-2)}$

So, for $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + C_{\epsilon_1} + C_{\epsilon_2} + C_R^+$ (we are analytic in our contour)

$$\int_\gamma f(z)\,dz = \left(\int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} + \int_{C_{\epsilon_1}} + \int_{C_{\epsilon_2}} + \int_{C_R^+}\right) f(z)\,dz = 0$$

By Jordan's lemma,

$$\lim_{R \to \infty} \int_{C_R^+} f(z)\,dz = 0$$

By Lemma 4 (Page 340)

$$\lim_{\epsilon \to 0^+} \int_{C_{\epsilon_1}} f(z)\,dz = i(0 - \pi)\,\text{Res}(f(z); 1)$$
$$= -i\pi \lim_{z \to 1} \frac{e^{iz}}{(z-2)} = \frac{e^{i}}{-1}(-i\pi) = \pi i e^{i}$$

$$\lim_{\epsilon \to 0^+} \int_{C_{\epsilon_2}} f(z)\,dz = i(0 - \pi)\,\text{Res}(f(z); 2)$$
$$= -i\pi \lim_{z \to 2} \frac{e^{iz}}{(z-1)} = -e^{i2} i\pi$$

Then we note

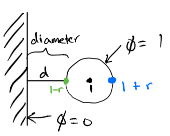
$$\left(\int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3}\right) f(z)\,dz = \lim_{\substack{R \to \infty \\ \epsilon \to 0}} \left[\left(\int_{-R}^{1-\epsilon} + \int_{1+\epsilon}^{2-\epsilon} + \int_{2+\epsilon}^{R}\right) f(x)dx\right] = I$$

In conclusion,

$$\int_\gamma f(z)\,dz = \left(\underbrace{\int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3}}_{I} + \underbrace{\int_{C_{\epsilon_1}}}_{e^{i}i\pi} + \underbrace{\int_{C_{\epsilon_2}}}_{-e^{i2}i\pi} + \underbrace{\int_{C_R^+}}_{0}\right) f(z)\,dz = 0$$

$$\Rightarrow \boxed{I = \left(e^{i2} - e^{i}\right) i\pi}$$

Example:



Given this in w-plane

in the w-plane.

We want diameter = distance from $\frac{1}{2}$ to edge

$$\frac{1}{1-r} - \frac{1}{1+r} = \frac{1}{1+r} - \frac{1}{2}$$

$\Rightarrow r^2 - 6r + 1 = 0$

$r = \frac{6 \pm \sqrt{32}}{2} = 3 \pm 2\sqrt{2}$

only want $r < 1$, so

$r = 3 - 2\sqrt{2}$

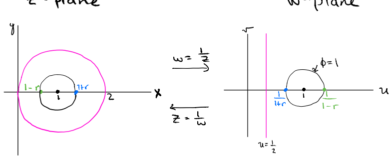
We know a geometry in the z-plane which maps to something similar.

Solve for $\phi(u,v)$ in region outside circle

Use inversion map

$w = 1/z = \frac{1}{x+iy} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$

z-plane w-plane

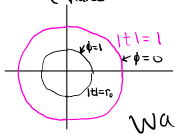


If we can find an $r < 1$ s.t. the circle on the z-plane maps to the given physical circle

We can simplify further by shifting circle to the origin.

Let $t = z - 1$, $t \in \mathbb{C}$

t-plane



Washer!

Argument Principle

$$\frac{1}{2\pi i}\oint \frac{f'(z)}{f(z)}\,dz = N_0(f) - N_p(f)$$

Rouche's Theorem

If f and h are each functions that are analytic in + on a simple closed contour C and if the strict inequality

$$|h(z)| < |f(z)|$$

holds at each point on C then f and $f + h$ must have the same total number of zeros inside C