

Recall

functions of >1 r.v - derived pdfs, pmfs, etc

Recipe: (1) $F_Z(z)$

$$(2) \frac{d}{dz}(F_Z(z)) = f_Z(z)$$

X, Y independent; $Z = X + Y \Rightarrow f_Z(z) = \text{convolution of } f_X(x) \text{ w/ } f_Y(y)$

discrete version
also holds

NEXT Family of Topics: Covariance, uncorrelatedness,
conditional expectation revisited (difficult)

Have Ω, \mathbb{P}

$X: \Omega \rightarrow \mathbb{R}$

$Y: \Omega \rightarrow \mathbb{R}$



Covariance

Given any X, Y rvs (discrete, continuous, whatever) defined on same probability space, the **covariance** of X and Y defined as

$$\text{Cov}(X, Y) = \mathbb{E}((X - \mathbb{E}(X))(Y - \mathbb{E}(Y)))$$

Note:

$$\text{Cov}(X, X) = \mathbb{E}((X - \mathbb{E}(X))^2) = \text{Var}(X)$$

$$\begin{aligned} \text{Cov}(X, Y) &= \mathbb{E}(XY) - \mathbb{E}(\mathbb{E}(X)Y) - \mathbb{E}(X\mathbb{E}(Y)) + \mathbb{E}(X)\mathbb{E}(Y) \\ &= \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) \end{aligned}$$

Terminology: When $\text{Cov}(X, Y) = 0$; say X and Y are uncorrelated.

Fact: If X, Y independent, then X, Y uncorrelated

If statement;
NOT iff

Independent $\Rightarrow \mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y) \Rightarrow \text{Cov}(X, Y) = 0$

Converse is NOT true

Example - Uncorrelated but NOT Independent

Consider (X, Y) that take on values

$$(3, 0), (0, 3), (-3, 0), (0, -3)$$

w/ equal probabilities $1/4$.

X, Y uncorrelated $\Rightarrow \text{IE}(XY) = \text{IE}(0) = 0$ and $\text{IE}(X) = \text{IE}(Y) = 0$

BUT, NOT independent

also saw this when
 X, Y independent

Fact: If X, Y uncorrelated, then $\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y)$

To see this, observe

$$\begin{aligned}\text{Var}(X+Y) &= \text{IE}((X - \text{IE}(X))(Y - \text{IE}(Y))^2) \\ &= \text{Var}(X) + 2 \underbrace{\text{IE}((X - \text{IE}(X))(Y - \text{IE}(Y)))}_{2 \text{Cov}(X, Y) = 0} + \text{Var}(Y)\end{aligned}$$

Terminology: Given X, Y , the correlation coefficient of X and Y is

$$\rho = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y}$$

Turns out, $|\rho| \leq 1$ - turns out this is a version of the Cauchy-Schwarz inequality.

Note: $\rho = 0 \iff X, Y$ uncorrelated and

$\rho = \pm 1 \iff X$ and Y are "aligned" in sense $X = \alpha Y$ for some $\alpha \neq 0$

Next,

Conditional Expectation Revisited

Have already seen: for any two X, Y on same probability space
 $\mathbb{E}(X|Y=y)$.

Get this as follows:

① Find $P_{X|Y}(x|y)$ ↗ Discrete

② $\mathbb{E}(X|Y=y) = \sum_x x P_{X|Y}(x|y)$

OR

① Find $f_{X|Y}(x|y)$

② $\mathbb{E}(X|Y=y) = \int_{-\infty}^{+\infty} x f_{X|Y}(x|y) dx$

↗ Continuous

i.e. someone tells you $Y=y$; can then figure out $\mathbb{E}(X|Y=y)$

Consider "automating" the computation - build a box that takes y ,
spits out $\mathbb{E}(X|Y=y)$

$$y \rightarrow \boxed{?} \longrightarrow \mathbb{E}(X|Y=y)$$

This implements a function $g: \mathbb{R} \rightarrow \mathbb{R}$

$$y \rightarrow \boxed{?} \rightarrow g(y) = \mathbb{E}(X|Y=y)$$

Finally, hook up Y at input - out comes $g(Y)$ - just a function
of a random variable. $g(Y)$ is also a random variable

We call $g(Y)$ " $\mathbb{E}(X|Y)$ " i.e. $g(Y) = \mathbb{E}(X|Y)$ here.

Terminology: $\text{IE}(X|Y)$ = conditional expectation of X given Y

Question: What is $\text{IE}(\text{IE}(X|Y))$?

Fact: Law of iterated expectations

$$\text{IE}(X) = \text{IE}(\text{IE}(X|Y))$$

Idea: $\text{IE}(X|Y) = g(Y)$ for some function Y .

Thus

$$\text{IE}(\text{IE}(X|Y)) = \text{IE}(g(Y)) \quad \begin{matrix} \leftarrow \\ \text{use expected value} \\ \text{rule to get this} \end{matrix}$$

$$\text{IE}(g(Y)) = \sum_{y \in Y} g(y) P_Y(y) \quad \text{Discrete}$$

$$\text{IE}(g(Y)) = \int_{-\infty}^{+\infty} g(y) f_Y(y) dy \quad \text{Continuous}$$

OR

$$= \sum_{y \in Y} \text{IE}(X|Y=y) P_Y(y)$$

$$= \int_{-\infty}^{+\infty} \text{IE}(X|Y=y) f_Y(y) dy$$

*law of total
expectation*

$$= \text{IE}(X)$$

$$= \text{IE}(X)$$

Example -

Y uniform on $[0, 1]$; when $Y=y$, X is Bernoulli(y).

i.e. $P(\{X=1\}) = y$; $P(\{X=0\}) = 1-y$

Get $\text{IE}(X|Y=y)$ as follows

- First get $\text{IE}(X|Y=y) = y = 1 \times y + 0 \times (1-y) = xy$
- Plug in Y for y . $\text{IE}(X|Y) = Y$

Example -

Y is uniform $[0,1]$, X binomial(n, y)

Given $Y=y$, $E(X|Y=y) = ny$; plug in Y for y

$$E(X|Y) = nY$$

↙ sum of n independent Bernoulli(y) rvs

Useful Fact: $E(h(Y)X|Y) = h(Y)E(X|Y)$

Idea: if y ,

$$E(h(Y)X|Y=y) = E(h(Y)X|Y=y) = h(y) \underbrace{E(X|Y=y)}_{g(Y)}$$

Plug in Y for y ,

$$\underbrace{h(Y)E(X|Y)}_{g(Y)}$$

Can view $E(X|Y)$ as an estimate of X given Y .

Sometimes we write

$$\hat{X} = E(X|Y)$$

and

$$\tilde{X} = X - \hat{X} = \text{"estimation error"}$$

Estimator-worthy properties of $E(X|Y) = \hat{X}$:

- $E(\hat{X}) = E(X)$ by law of iterated expectations

Finish next time