Last time: AER xxn symmetric, AT=A

① Every (complex) eigenvalue of A is in R
 ∴ A has real eigenvalues

② $\vec{v}_1 \cdot \vec{v}_2$ eigenvectors for different eigenvalues $\Rightarrow \vec{v}_1 \cdot \vec{v}_2 = 0$

3 W is A-invariant => W1 is A-invariant

 \mathcal{G} \mathcal{B} orthonormal basis of V $[A]_{\mathcal{B}}^{-}\mathcal{B}$ is also symmetric

5) If U∈V is an A-invariant subspace,

Alu: U→U

is a LT

u → Aū

and if q is an orthonormal basis of U, then

[Aju] q = Bz

is also symmetric.

This implies that every eigenvalue of Bz is also an eigenvalue of Aluz and also of A. (requires proof)

(6) If W is A-invariant, and if $(u_{1,...,Ur})$ an orthonormal basis of W and $(u_{rh_1,...,Ur})$ is an orthonormal basis of WI and $B = (u_{1,...,Ur})$

then $[A]_{B} = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}$

and

{eigenvalues of A} = {eigenvals B, }U {eigenvals Bz}

Proposition: If $A = A^T$, $\lambda_{1,...}$, $\lambda_{m} \in \mathbb{R}$ distinct eigenvalues of A, then $V = E_{\lambda_{1}}(A) \oplus \cdots \oplus E_{\lambda_{m}}(A)$

Proof: Showed Ex, (A) + ... + Fin (A) = W

Satisfies $W = E_{\lambda_1}(A) \oplus \cdots \oplus E_{\lambda_m}(A)$ Ned to show $W = V = \mathbb{R}^n$ i.e $w^{\perp} = 0$

Show WI = 5

Assume otherwise, W+ + 0 - W+ is A-invariant Consider

 $A_{|W^{\perp}}:W^{\perp}\to W^{\perp}$

and by \bigcirc , A_{lw1} has an eigenvalue $\therefore \exists v \neq v \in W^{\dagger}$ an eigenvalue of w^{\dagger} . But $v \in W + v \in W$ and $v \in W + v \in W$ $v \in W + v \in W$ $v \in W + v \in W$

Spectral Theorem Let $A = A^T$, $V = \mathbb{R}^n$ (AER^{nxn}) (Vanimer product space) then

- @ 3 orthonormal basis of V consisting of eigenvectors of A (i.e. A is orthonormally diagonalizable)

Proof Basically all done

- a Take an orthonormal basis for each $E_{\lambda_i}(A)$ (all eigenvalues) put together.

 This is an orthonormal basis of eigenvectors,

 Say u_1, \ldots, u_n
- (b) $Q = u_{1,---} u_{n}$ then $Z = Q^{-1}AQ$ however QTQ = T $\therefore QT = Q^{-1}$ and Z = QTAQQ. E.D

Remark:
$$Q^{-1} = Q^{T}$$
, $Q Q^{T} = I$
 \Rightarrow the rows of A all orthonormal

Singular Value Decomposition Let A be any nxn matrix over IR of rank r. We showed that ATA also has rank r.

ATA is a symmetric. Use this

ATA has nreal eigenvalues $\lambda > \lambda_2 > \dots > \lambda_n$ with multiplicity

Note: only r of theze non-zero.

Claim: All eigenvalues of ATA>O

Proof: If \vec{v} is s.t.

ATA $\vec{J} = \lambda \vec{v}$, $\vec{V} \neq \vec{b}$ $((Av)^T Av =) \vec{V}^T A^T A \vec{v} = \lambda \vec{v}^T \vec{v$

So the non-zero eigenvalues of ATA $\lambda_1 > \lambda_2 > \cdots > \lambda_r > 0$

Definition: Let $\sigma_i = \sqrt{\lambda_i}$, then $\sigma_i = \sqrt{\lambda_i} = \sqrt{\lambda$

Let $V = (\vec{v}_1, ..., \vec{v}_n)$ be a matrix whose columns form an orthonormal basis of $V = \mathbb{R}^n$ of eigenvectors of ATA such that

$$A^{T}Av_{i} = \lambda v_{i} \qquad i = 1, ..., r$$

$$A^{T}Av_{j} = 0 \qquad i \neq r$$

For i=1,...,r

Claim: U,,..., Ur are orthonormal basis

$$u_{j}^{T}u_{i} = \frac{1}{\sigma_{i}\sigma_{j}} \quad v_{j}^{T} \underbrace{A^{T}Av_{i}}_{\lambda_{i}v_{i}} \rightarrow \lambda_{i} = \sigma_{i}^{2}$$

$$= \frac{\sigma_{i}^{2}}{\sigma_{i}\sigma_{j}} \quad v_{j}^{T}v_{i}^{T}$$

So

$$\langle u_{j}, u_{i} \rangle = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

Extend $u_{1,...,}$ ur to an orthonormal basis $u_{1,...,}$ ur, $u_{r_{1},...,}$ un of \mathbb{R}^{m} .

Have

$$\Sigma = \begin{bmatrix} \sigma_1 & \sigma_2 & \sigma_3 \\ \sigma_2 & \sigma_4 & \sigma_5 \end{bmatrix}$$

on example

$$\sum = \begin{bmatrix}
\sigma_{1} & 0 & 0 & 0 \\
0 & \sigma_{2} & 0 & 0 \\
0 & 0 & \sigma_{3} & 0
\end{bmatrix}$$

$$AV = U\Sigma$$
 $(V^{-1} = V^{\dagger})$
 $A = U\Sigma V^{T}$