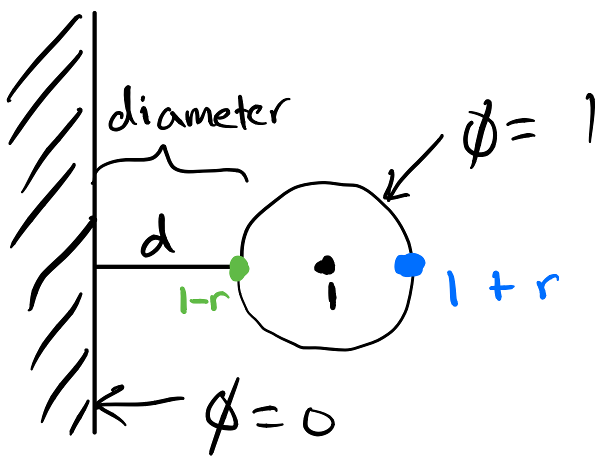
Use conformal mapping to solve $\nabla^2 \phi = 0$

Example:



Given this in w-plane

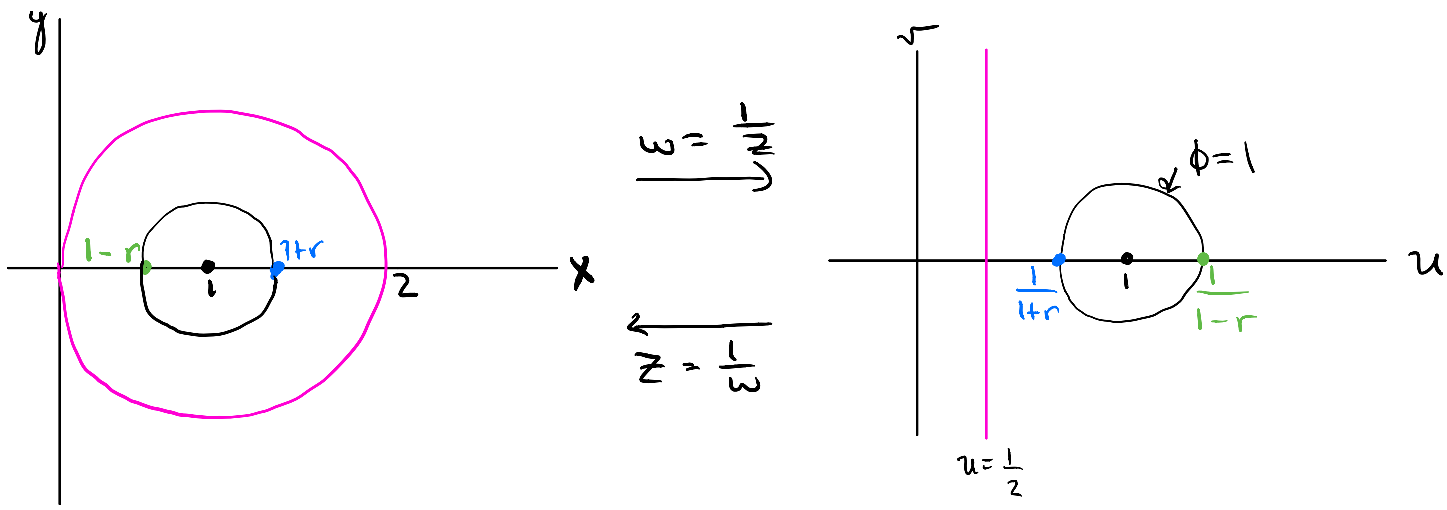
We know a geometry in the z-plane which maps to something similar.

Solve for $\phi(u,v)$ in region outside circle

Use inversion map

$$w = 1/z = \frac{1}{x+iy} = \frac{x}{x^2+y^2} - i\,\frac{y}{x^2+y^2}$$

z-plane w-plane



If we can find an $r<1$ s.t. the circle on the z-plane maps to the given physical circle in the w-plane.

We want diameter = distance from $\frac{1}{2}$ to edge

$$\frac{1}{1-r} - \frac{1}{1+r} = \frac{1}{1+r} - \frac{1}{2}$$

$$\Rightarrow r^2 - 6r + 1 = 0$$

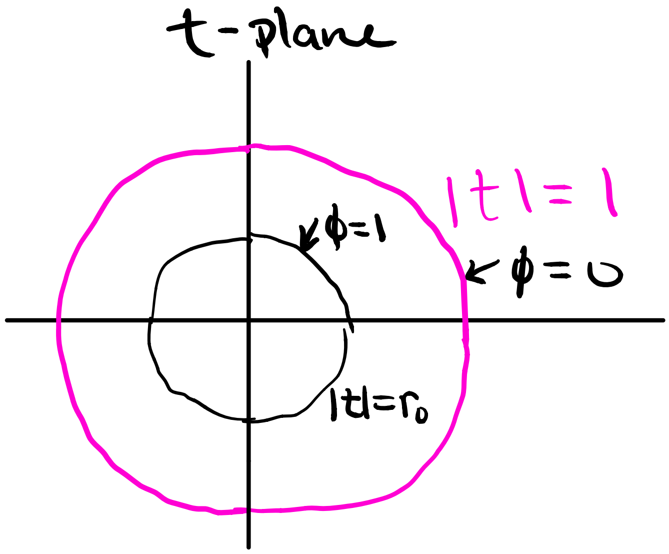
$$r = \frac{6 \pm \sqrt{32}}{2} = 3 \pm 2\sqrt{2}$$

only want $r < 1$, so

$$r = 3 - 2\sqrt{2}$$

We can simplify further by shifting circle to the origin.

Let $t = z - 1$, $t \in \mathbb{C}$



$$\phi = c_1 \log(c_2 R) \quad , \quad R = |t|$$

$$0 = c_1 \log(c_2(1))$$

$$\log(c_2) = 0$$

$$c_2 = 1$$

$$1 = c_1 \log(1 \cdot r_0)$$

$$c_1 = \frac{1}{\log(r_0)}$$

$$\phi = \frac{\log(R)}{\log(r_0)} = \frac{\log(|t|)}{\log(r_0)}$$

$$= \frac{\log(|z-1|)}{\log(r_0)} = \frac{\ln(|\frac{1}{w} - 1|)}{\log(r_0)}$$

$$w = u + iv$$

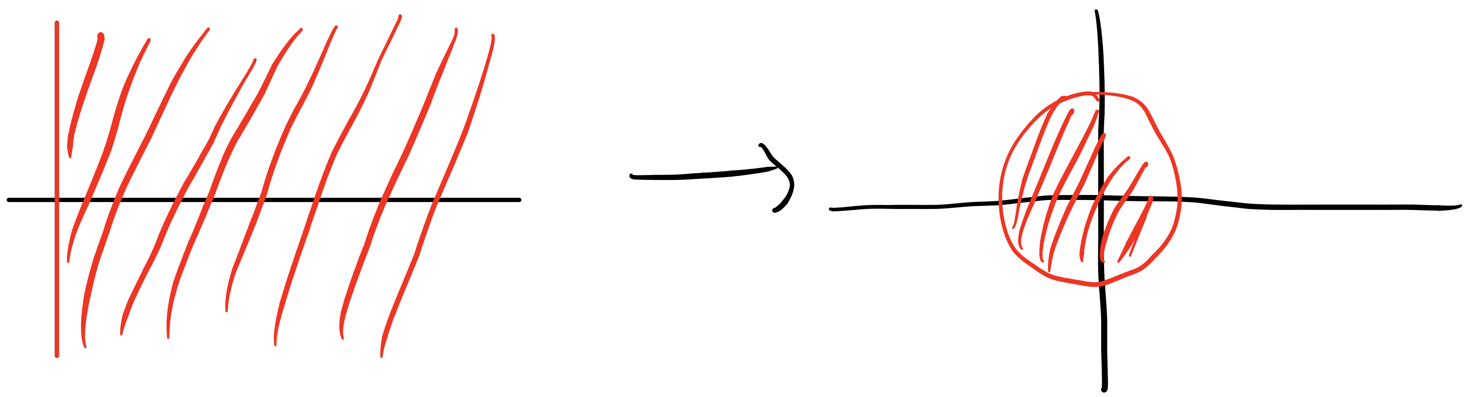
$$\left| \frac{1}{u+iv} - 1 \right| = \left| \frac{(1-u) - iv}{u+iv} \right| = \sqrt{\frac{(1-u)^2 + v^2}{u^2 + v^2}}$$

$$\phi(u,v) = \frac{\log\left[\sqrt{\dfrac{(1-u)^2+v^2}{u^2+v^2}}\right]}{\log(r_0)}$$

Mobius Transforms

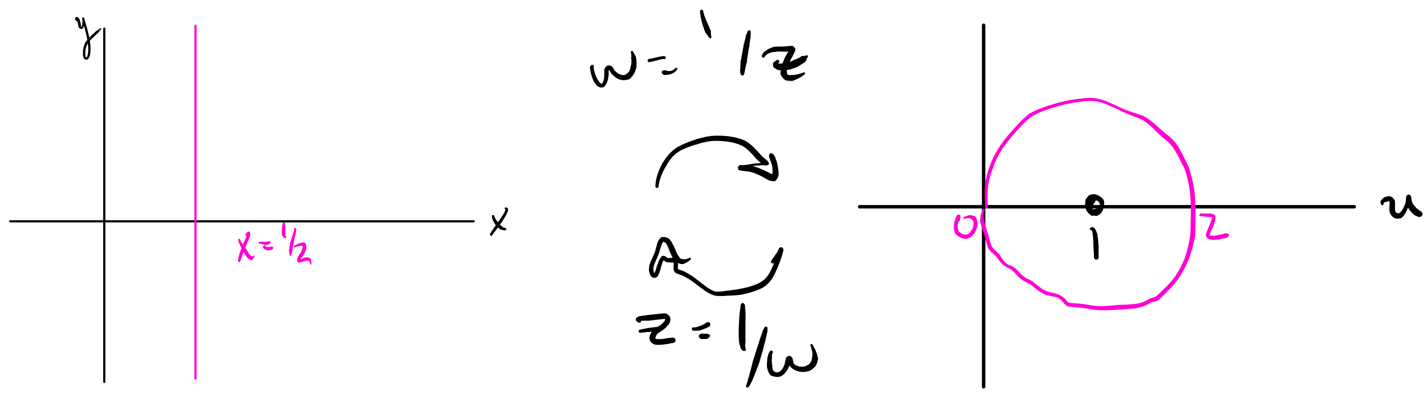
Q: Find a conformal map from the right half-plane into the unit disk.

So the imaginary axis maps to a unique circle.



Solution: convert to a $1/z$ problem.

We know



we can translate the first one and second one

$x \mapsto x + \frac{1}{2}$

$|w-1| \leq 1 \longmapsto |w| \leq 1$

so

$$\frac{1}{x+\frac{1}{2}} + 1 \to \boxed{\frac{1-2x}{1+2x}}$$

$$\frac{1}{w} + 1$$

$$f(z) = \frac{az+b}{cz+d}, \quad ad-bc \neq 0$$