ECE 4110 Homework 2

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Due 5pm on September 19

1 Key Concepts and Reading Material

- Random variables, distributions and densities (Chapter 3.1-3.2, Chapter 4.1-4.2).
- Expectation and variance (Chapter 3.3, Chapter 4.3).
- Jointly distributed random variables (Chapter 5.1-5.6).
- Conditional distribution and conditional expectation (Chapter 3.4, Chapter 5.7).
- Important random variables: geometric, Poisson, exponential (Chapter 3.5, Chapter 4.4).

2 Assignment

1. **The Exponential Distribution**

   (a) Let X be exponentially distributed with parameter λ. The CDF of X is given by

   $$F_X(x) = \begin{cases} 0 & x < 0 \\ 1 - \exp(-\lambda x) & x \geq 0 \end{cases}$$

   Use the integration by parts formula to determine the mean of X. Recall the integration by parts formula:

   $$\mathbb{E}(X) = \int_0^{\infty} (1 - F_X(x))dx - \int_{-\infty}^{0} F_X(x)dx$$

   (b) The exponential distribution is often a reasonable model for waiting times. Suppose that the time T that a transistor takes to fail is exponentially distributed. Show that given that the transistor is still working after t time units, the chance that it lasts an additional s time units is independent of t, i.e.,

   $$\Pr(T > t + s | T > t) = \Pr(T > s).$$

   This is called the *memoryless* property of the exponential distribution.

2. **The Poisson Distribution**

   (a) Show that the mean and variance of a Poisson random variable Y are equal.
   (*Hint: For the variance, compute* $\mathbb{E}[Y(Y-1)]$ *and then use the linearity of expectation.*)

(b) The Poisson distribution is a good model for manufacturing defects (or typos). Suppose a chip consists of n transistors, each of which is defective independently with probability p.

i. What is the distribution of the number of defective transistors on the chip? What is the mean number?

ii. Suppose n is large but p is small, so that the mean number of defects, λ, remains constant as n tends to infinity. Argue that the distribution of the number of defective transistors is close to Poisson.
You may need the following fact

$$\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n = \exp(-\lambda).$$

3. **Coupon Collector's Problem**

A die is thrown successively.

(a) What is the expected number of throws until you see "3" for the first time?

(b) What is the expected number of throws until you see every number at least once?

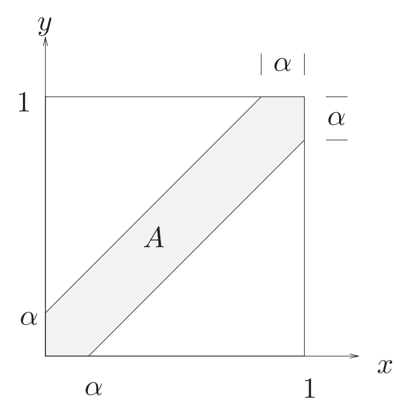
(c) In the coupon collector's problem, there are n different types of coupons distributed inside the cereal boxes of a particular brand. Each cereal box contains one coupon chosen randomly with equal probability from the n types. Once you collect all n types, you win a prize. What is the expected number of cereal boxes you need to buy for winning the prize?

4. **Conditioning in Mixed Probabilities**

Suppose that X and Y have joint density function

$$f_{X,Y}(x,y) = \begin{cases} 1 & 0 \le x \le 1, 0 \le y \le 1, \\ 0 & \text{otherwise} \end{cases}$$

Let event A be the shaded area in the figure below.



(a) Sketch the marginal PDF $f_X(x)$ and marginal CDF $F_X(x)$.

(b) Sketch the conditional CDF $F_{X|A}(x|A)$ and the conditional PMF $f_{X|A}(x|A)$ conditioned on event A, where

$$F_{X|A}(x|A) = \frac{\Pr(X \le x, A)}{\Pr(A)}, \quad f_{X|A}(x|A) = \frac{dF_{X|A}(x|A)}{dx}.$$

(c) Sketch the conditional probability $\Pr(A|X = x)$ of event A conditioned on $X = x$ as a function of $x \in [0, 1]$, where

$$\Pr(A|X = x) = \frac{f_{X|A}(x|A)\Pr(A)}{f_X(x)}.$$

5. **Conditional Expectation**

   There are two coins. Coin A is a fair coin. Coin B is biased with a "head" probability of $\frac{1}{4}$. Consider the following random experiment. First flip coin A. If a head shows up, flip coin A again. If it is a tail, flip coin B. Let X denote the outcome of the first coin flip with $X = 1$ for head and $X = 0$ for tail. Let Y denote the outcome of the second coin flip with a similar definition.

   (a) What is the PMF of $\mathbb{E}(Y|X)$? What is the expectation and the variance of $\mathbb{E}(Y|X)$? Compare them with the expectation and the variance of Y.

   (b) What is the PMF of $\mathbb{E}(X|Y)$? What is the expectation and the variance of $\mathbb{E}(X|Y)$? Compare them with the expectation and the variance of X.

Rami Pellumbi - rp534
ECE 4110 HW2
9/17/19 5:00PM

①

(a) Let X be exponentially distributed with parameter λ. The CDF of X is given by

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 1 - \exp(-\lambda x) & x \geq 0 \end{cases}$$

Use the integration by parts formula to determine the mean of X. Recall the integration by parts formula:

$$\mathbb{E}(X) = \int_0^\infty (1 - F_X(x))dx - \int_{-\infty}^0 F_X(x)dx$$

$$\mathbb{E}[X] = \begin{cases} \int_0^\infty e^{-\lambda x}\, dx - \int_{-\infty}^0 0\, dx & , x \geq 0 \\ 0 & , x < 0 \end{cases}$$

For $x \geq 0$ have

$$\int_0^\infty e^{-\lambda x}\, dx = -\frac{1}{\lambda} e^{-\lambda x} \Big|_0^\infty = 0 + \frac{1}{\lambda} = \frac{1}{\lambda}$$

(b) The exponential distribution is often a reasonable model for waiting times. Suppose that the time T that a transistor takes to fail is exponentially distributed. Show that given that the transistor is still working after t time units, the chance that it lasts an additional s time units is independent of t, i.e.,

$$\Pr(T > t + s | T > t) = \Pr(T > s).$$

$$\Pr(T > t+s \mid T > t) = \frac{\Pr(T > t+s \cap T > t)}{\Pr(T > t)}$$

$$\begin{cases} t+s \geq 0 \\ t \geq 0 \end{cases} \rightarrow s \geq 0$$

$$= \frac{e^{-\lambda(t+s)}}{e^{-\lambda(t)}} = e^{-\lambda s} = 1 - (1 - e^{-\lambda s})$$

$$= \Pr(T > s)$$

2. **The Poisson Distribution**

(a) Show that the mean and variance of a Poisson random variable Y are equal.
(*Hint: For the variance, compute* $\mathbb{E}[Y(Y-1)]$ *and then use the linearity of expectation.*)

Poisson RV $\Rightarrow Y \sim \text{Pois}(\lambda)$

$$P_Y(k) = \Pr(k \text{ events occur in an interval}) = e^{-\lambda}\frac{\lambda^k}{k!}, \quad k \in \mathbb{N}$$

$$\mathbb{E}[Y] = \sum_{k=0}^{\infty} k P_Y(k) = 0 + 1\frac{e^{-\lambda}\lambda}{1!} + 2\frac{e^{-\lambda}\lambda^2}{2!} + 3\frac{e^{-\lambda}\lambda^3}{3!} + 4\frac{e^{-\lambda}\lambda^4}{4!} + \cdots$$

$$\frac{x}{x!} = \frac{1}{(x-1)!} \qquad = \frac{e^{-\lambda}\lambda}{0!} + \frac{e^{-\lambda}\lambda^2}{1!} + \frac{e^{-\lambda}\lambda^3}{2!} + \frac{e^{-\lambda}\lambda^4}{3!} + \cdots$$

$$e^{\lambda} = 1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots \qquad = e^{-\lambda}\lambda\left(1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots\right)$$

$$= e^{-\lambda}\lambda e^{\lambda}$$

$$= \lambda$$

$$\text{Var}(Y) = \mathbb{E}[(Y - \mathbb{E}[Y])^2]$$

$$= \mathbb{E}[(Y-\lambda)^2] = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2$$

$$= \mathbb{E}[Y(Y-1) + Y] - (\mathbb{E}[Y])^2$$

$$= \mathbb{E}[Y(Y-1)] + \mathbb{E}[Y] - (\mathbb{E}[Y])^2$$

$(\mathbb{E}[Y])^2 = \lambda^2, \quad \mathbb{E}[Y] = \lambda$

Expect

$\mathbb{E}[Y(Y-1)] = \lambda$

$$\mathbb{E}[Y(Y-1)] = \sum_{k=2}^{\infty} k(k-1)\frac{e^{-\lambda}\lambda^k}{k!} = \sum_{k=2}^{\infty} \frac{e^{-\lambda}\lambda^k}{(k-2)!}$$

$$= \frac{e^{-\lambda}\lambda^2}{0!} + \frac{e^{-\lambda}\lambda^3}{1!} + \frac{e^{-\lambda}\lambda^4}{2!} + \cdots$$

$$= \lambda^2 e^{-\lambda}\left[1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \cdots\right] = \lambda^2 e^{-\lambda} e^{\lambda} = \lambda^2$$

Thus

$$Var(Y) = \lambda^2 + \lambda - \lambda^2 = \lambda$$

(b) The Poisson distribution is a good model for manufacturing defects (or typos). Suppose a chip consists of n transistors, each of which is defective independently with probability p.

i. What is the distribution of the number of defective transistors on the chip? What is the mean number?

The distribution of the number of defective transistors on the chip will be

$$X \sim p^k (1-p)^{n-k} \qquad k = 0, 1, 2, \ldots, n$$

where X is the rv which denotes k transistors are defective

The mean number of defective transistors will thus be

$$\mathbb{E}[X] = np \triangleq \lambda \quad \text{(i.e. 'population size' * 'probability')}$$

ii. Suppose n is large but p is small, so that the mean number of defects, λ, remains constant as n tends to infinity. Argue that the distribution of the number of defective transistors is close to Poisson.

being "fancy"

FACT $$\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n = e^{-\lambda}$$

$$\Pr(k \text{ defective transistors}) = \binom{n}{k}\left(1-\frac{np}{n}\right)^{n-k}\left(\frac{np}{n}\right)^k$$

$$\lim_{n\to\infty}\frac{n(n-1)(n-2)\cdots(n-k+1)}{n^k}=1 \qquad = \frac{n!}{k!(n-k)!}\left(\frac{n}{n}p\right)^k\left(1-\frac{n}{n}p\right)^k$$

Taking the limit as $n\to\infty$

$$\lim_{n\to\infty}\frac{n!}{k!(n-k)!}\left(1-\frac{np}{n}\right)^{n-k}\left(\frac{np}{n}\right)^k = \frac{e^{-np}(np)^k}{k!} = \frac{e^{-\lambda}\lambda^k}{k!}$$

$$\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{-k}=1$$

Thus $$\Pr(k \text{ defective transistors}) = \frac{e^{-\lambda}\lambda^k}{k!}$$

We see this is the Poisson distribution!

3. **Coupon Collector's Problem**

A die is thrown successively.

(a) What is the expected number of throws until you see "3" for the first time?

Have a discrete uniform distribution.

$$X = \{1,2,3,4,5,6\} \quad \text{where} \quad p_X(x_k) = \frac{1}{6}, \quad 1 \le k \le 6$$

Let Y be the number of tosses until you toss a 3.

prob of 3   prob of NOT 3

Then $Y = \frac{1}{6} + \frac{5}{6}\underbrace{(1+Y)}_{\text{spent a turn and now "restart"}}$

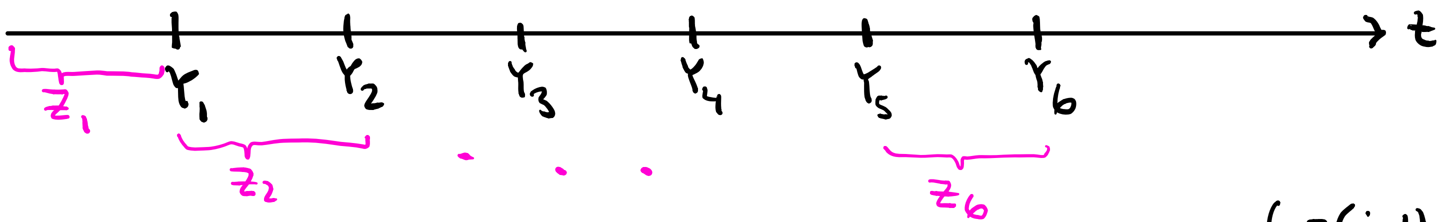
$\rightarrow Y = 6$

Could have also modeled as geometric r.v. $\mathbb{E}[X] = \frac{1}{p} = 6$

(b) What is the expected number of throws until you see every number at least once?

Let Y_i denote when you see a distinct element on the die. Let X be event you have seen 6 distinct values.

"stretchy timeline" approach.



All Z_i are random variables w/ probability $p_i = \frac{6-(i-1)}{6}$

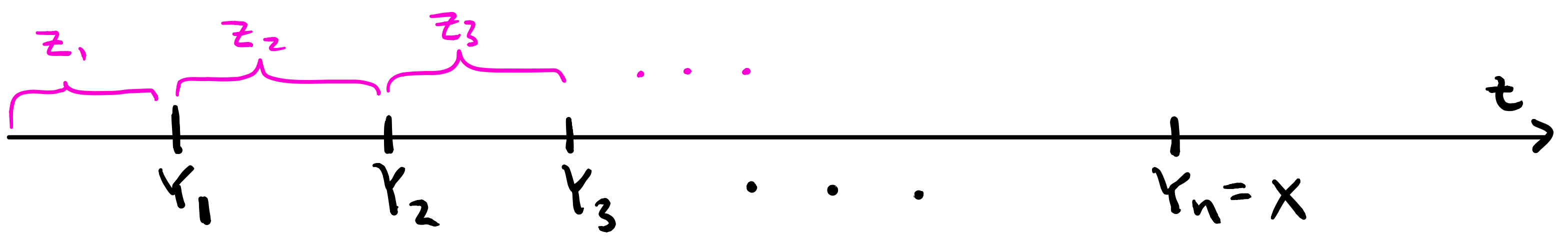
Why this p_i? Because the first value you roll is guaranteed to be on the die. The next distinct outcome has a $5/6$ chance, then $4/6$, etc.

Thus $\mathbb{E}[X] = \sum_{i=1}^{6} \frac{6}{6-(i-1)} = 6\left(\frac{1}{6} + \frac{1}{5} + \frac{1}{4} + \frac{1}{3} + \frac{1}{2} + 1\right)$

$\boxed{= 14.7}$

(c) In the coupon collector's problem, there are n different types of coupons distributed inside the cereal boxes of a particular brand. Each cereal box contains one cou~~pon chosen~~ randomly with equal probability from the n types. Once you collect all n types, you win a prize. What is the expected number of cereal boxes you need to buy for winning the prize?

"Stretchy timeline" approach



All these marked times – $Y_1, Y_2, Y_3, Y_4, \ldots Y_n$ – are random.
Here, each Y_i is the event you collect a distinct coupon.
So inbetween each Y_i are <u>failed</u> collections.
Let X = event you've collected all n.
write X as

$$X = \underset{Y_1}{Z_1} + \underset{Y_2 - Y_1}{Z_2} + \underset{Y_3 - Y_2}{Z_3} + \underset{Y_4 - Y_3}{Z_4} + \underset{Y_5 - Y_4}{Z_5} + \cdots + \underset{Y_n - Y_{n-1}}{Z_n}$$

All Z_i are geometric rvs w/ parameter $p_i = \dfrac{n-(i-1)}{n}$ } coupon 1 guaranteed! P_2 is pr(not c_1) ⋮ etc

AND they're independent

Thus,

$$\mathbb{E}[Z_i] = \frac{1}{p_i} \quad \forall i$$

And

$$\mathbb{E}(X) = \sum_{i=1}^{n} \mathbb{E}[Z_i] = \sum_{i=1}^{n} \frac{n}{n-(i-1)} = \frac{n}{n} + \frac{n}{n-1} + \frac{n}{n-2} + \cdots + \frac{n}{1}$$

$$\boxed{= n\left(\frac{1}{n} + \frac{1}{n-1} + \cdots + \frac{1}{2} + 1\right)}$$

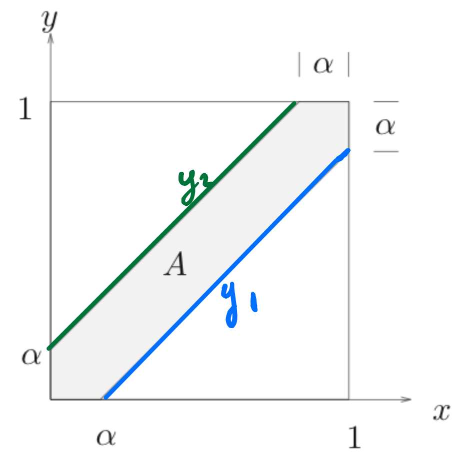
4. **Conditioning in Mixed Probabilities**

Suppose that X and Y have joint density function

$$f_{X,Y}(x,y) = \begin{cases} 1 & 0 \le x \le 1, 0 \le y \le 1, \\ 0 & \text{otherwise} \end{cases}$$

Let event A be the shaded area in the figure below.

$x-(1-\alpha)$



$y_2 = x+\alpha$

$y_1 = x-\alpha$

(a) Sketch the marginal PDF $f_X(x)$ and marginal CDF $F_X(x)$.

$$f_X(x) = \int_y f_{X,Y}(x,y)\,dy = \int_0^1 1\,dy = 1$$

i.e

$$f_X(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & o/w \end{cases}$$

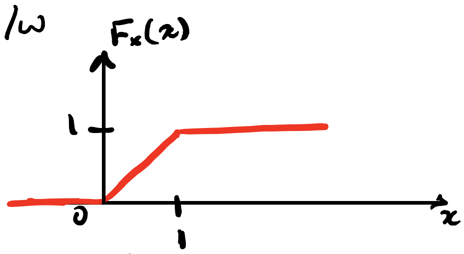
$$F_X(x) = \int_{-\infty}^{x} f_X(z)\,dz = \int_{z=0}^{x} 1\,dz = x, \quad 0 \le x \le 1$$

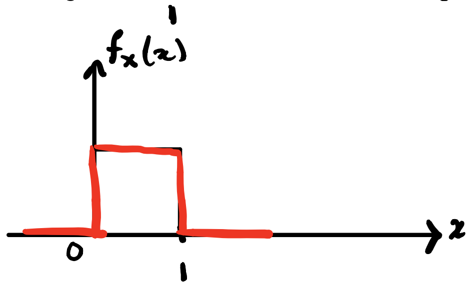
i.e

$$F_X(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \le x \le 1 \\ 1, & x > 1 \end{cases}$$

$\frac{1}{2}\alpha^2 + \frac{1}{2}\alpha^2 = \alpha^2$

$2\alpha($





(b) Sketch the conditional CDF $F_{X|A}(x|A)$ and the conditional PMF $f_{X|A}(x|A)$ conditioned on event A, where

shaded region above

$$\boxed{F_{X|A}(x|A) = \frac{\Pr(X \le x, A)}{\Pr(A)},} \quad f_{X|A}(x|A) = \frac{dF_{X|A}(x|A)}{dx}.$$

$$\Pr(A) = \frac{\text{area}(A)}{\text{Total area}} = 1 - \left(\frac{1}{2}(1-\alpha)^2 + \frac{1}{2}(1-\alpha)^2\right) = 1-(1-\alpha)^2 \;; \alpha \in (0,1)$$

$1-(1-\alpha)^2$

$= 1-(1-2\alpha+\alpha^2)$

$= 2\alpha - \alpha^2$

$$\Pr(\{X \le x\} \cap \{A\}) = \Pr(\{X \le x\} \cap \{0 \le x \le 1, 0 \le y \le 1\})$$
$$= \text{area}(A \text{ for } x \in (0,1)\}$$

For $0 < x \le \alpha$

$$\text{area}(A \text{ for } x (0,\alpha]\} = \frac{1}{2}\overset{h}{x}(\overset{b_1}{x+\alpha}+\overset{b_2}{\alpha}) \qquad \text{(Trapezoid)}$$
$$= \frac{1}{2}x(x+2\alpha)$$
$$= \frac{1}{2}x^2 + \alpha x \quad \leftarrow \text{Quadratic}$$

For $\alpha < x < 1-\alpha$

$$\text{area}(A \text{ for } x \in (\alpha, 1-\alpha)\} = \overset{\text{box}}{x} - \overset{\text{triangle}}{\frac{1}{2}(x-\alpha)^2} - \overset{\text{Trapezoid}}{\frac{x}{2}((1-\alpha)+(1-(x+\alpha)))}$$
$$= x - \frac{1}{2}(x^2 - 2\alpha x + \alpha^2) - \frac{x}{2}(1-\alpha+1-x-\alpha)$$
$$= x - \frac{x^2}{2} + \alpha x - \frac{\alpha^2}{2} - \frac{x}{2}(2-2\alpha-x)$$
$$= \cancel{x} - \cancel{\frac{x^2}{2}} + \alpha x - \frac{\alpha^2}{2} \cancel{- x} + \alpha x + \cancel{\frac{x^2}{2}}$$
$$= 2\alpha x - \frac{\alpha^2}{2} \quad \leftarrow \text{Linear}$$

For $1-\alpha < x < 1$

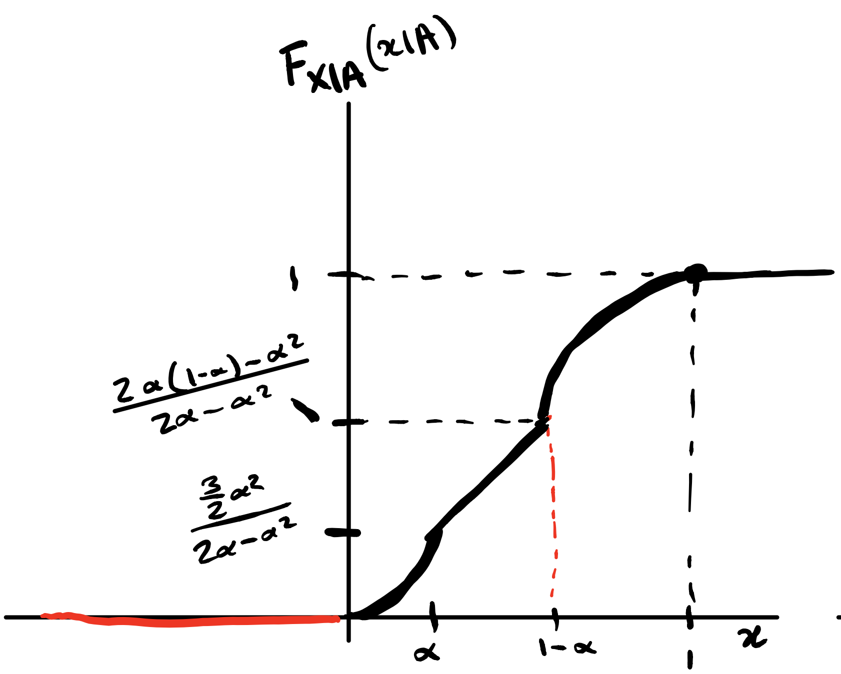
$$\text{area}(A \text{ for } x \in [1-\alpha, 1)\} = \overset{\text{box}}{x} - \overset{\text{Triangle}_1}{\frac{1}{2}(1-\alpha)^2} - \overset{\text{Triangle}_2}{\frac{1}{2}(x-\alpha)^2}$$
$$= x - \frac{1}{2}(1-2\alpha+\alpha^2) - \frac{1}{2}(x^2 - 2\alpha x + \alpha^2)$$
$$= x - \frac{1}{2} + \alpha - \frac{\alpha^2}{2} - \frac{1}{2}x^2 + \alpha x - \frac{\alpha^2}{2}$$
$$= -\frac{x^2}{2} + x(1+\alpha) - \alpha^2 + \alpha - \frac{1}{2} \quad \leftarrow \text{Quadratic}$$

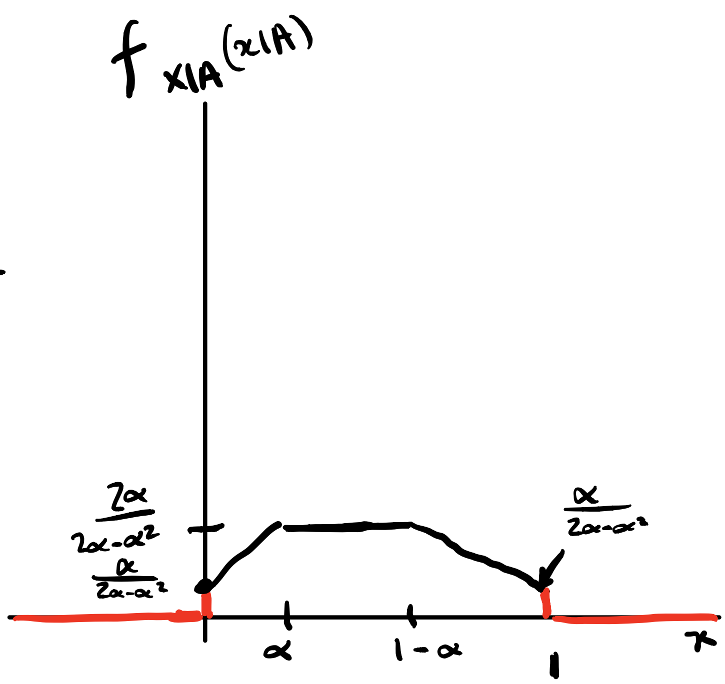
Thus,

$$F_{X|A}(x|A) = \begin{cases} 0 & , \quad x < 0 \\ \dfrac{\frac{1}{2}x^2 + \alpha x}{2\alpha - \alpha^2} & , \quad x \in [0, \alpha] \\ \dfrac{2\alpha x - \alpha^2}{2\alpha - \alpha^2} & , \quad x \in [\alpha, 1-\alpha] \\ \dfrac{-\frac{x^2}{2} + x(1+\alpha) - \alpha^2 + \alpha - \frac{1}{2}}{2\alpha - \alpha^2} & , \quad x \in [1-\alpha, 1] \\ 1 & , \quad x > 1 \end{cases}$$

and

$$f_{X|A}(x|A) = \begin{cases} \dfrac{x+\alpha}{2\alpha - \alpha^2} & \quad x \in [0, \alpha] \\ \dfrac{2\alpha}{2\alpha - \alpha^2} & \quad x \in [\alpha, 1-\alpha] \\ \dfrac{-x + (1+\alpha)}{2\alpha - \alpha^2} & \quad x \in [1-\alpha, 1) \\ 0 & , \quad o/w \end{cases}$$

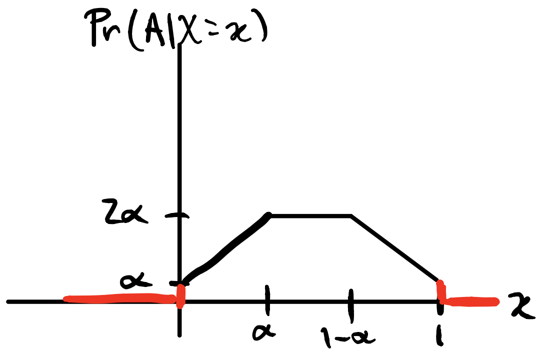




(c) Sketch the conditional probability $\Pr(A|X = x)$ of event A conditioned on $X = x$ as a function of $x \in [0, 1]$, where

$$\Pr(A|X = x) = \frac{f_{X|A}(x|A)\Pr(A)}{f_X(x)}.$$

$$\frac{f_{X|A}(x|A)\Pr(A)}{f_X(x)} = \begin{cases} x+\alpha & , x \in [0,\alpha] \\ 2\alpha & , x \in [\alpha, 1-\alpha] \\ -x+(1+\alpha) & , x \in [1-\alpha, 1] \\ 0 & , o/w \end{cases}$$ ← either 0 or undefined



5. **Conditional Expectation**

There are two coins. Coin A is a fair coin. Coin B is biased with a "head" probability of $\frac{1}{4}$. Consider the following random experiment. First flip coin A. If a head shows up, flip coin A again. If it is a tail, flip coin B. Let X denote the outcome of the first coin flip with $X = 1$ for head and $X = 0$ for tail. Let Y denote the outcome of the second coin flip with a similar definition.

(a) What is the PMF of $\mathbb{E}(Y|X)$? What is the expectation and the variance of $\mathbb{E}(Y|X)$? Compare them with the expectation and the variance of Y.

Random Experiment

① Flip Coin A

(i) If H, Flip A again

(ii) If T, flip B

Define

X: outcome of first flip

Y: outcome of second flip

Joint PMF

$$p_{X,Y}(1,1) = \Pr[X=1]\Pr[Y=1 \mid X=1] = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$$

$$p_{X,Y}(1,0) = \Pr[X=1]\Pr[Y=0 \mid X=1] = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$$

$$p_{X,Y}(0,1) = \Pr[X=0]\Pr[Y=1 \mid X=0] = \frac{1}{2}\cdot\frac{1}{4} = \frac{1}{8}$$

$$p_{X,Y}(0,0) = \Pr[X=0]\Pr[Y=0 \mid X=0] = \frac{1}{2}\cdot\frac{3}{4} = \frac{3}{8}$$

Sanity Check:

$$\sum_x \sum_y p_{X,Y}(x,y) = \frac{1}{4}+\frac{1}{4}+\frac{1}{8}+\frac{3}{8} = 1$$

Marginal PMF

$$p_X(1) = \sum_y p_{X,Y}(1,y) = 1/2$$

$$p_X(0) = \sum_y p_{X,Y}(0,y) = 1/2$$

$$p_Y(1) = \sum_x p_{X,Y}(x,1) = \frac{1}{4}+\frac{1}{8} = 3/8$$

$$p_Y(0) = \sum_x p_{X,Y}(x,0) = \frac{1}{4}+\frac{3}{8} = 5/8$$

Conditional PMF

$$P_{Y|X}(y=1 \mid x=1) = \frac{P_{X,Y}(1,1)}{P_X(1)} = \frac{1/4}{1/2} = 1/2$$

$$P_{Y|X}(y=0 \mid X=1) = \frac{P_{X,Y}(1,0)}{P_X(1)} = \frac{1/4}{1/2} = 1/2$$

$$P_{Y|X}(y=1 \mid X=0) = \frac{P_{X,Y}(0,1)}{P_X(0)} = \frac{1/8}{1/2} = 1/4$$

$$P_{Y|X}(y=0 \mid x=0) = \frac{P_{X,Y}(0,0)}{P_X(0)} = \frac{3/8}{1/2} = 3/4$$

Conditional Expectation

$$\mathbb{E}_{Y|X}[Y \mid X=1] = 1 \cdot Pr[Y=1 \mid X=1] + 0 \cdot Pr[Y=0 \mid X=1] = \frac{1}{2}$$

$$\mathbb{E}_{Y|X}[Y \mid X=0] = 1 \cdot Pr[Y=1 \mid X=0] + 0 \cdot Pr[Y=0 \mid X=0] = \frac{1}{4}$$

thus

$$\mathbb{E}_{Y|X}[Y \mid X] = \begin{cases} \frac{1}{2} & \text{w.p. } \frac{1}{2} \\ \frac{1}{4} & \text{w.p. } \frac{1}{2} \end{cases}$$

and

$$\mathbb{E}_X\left[\mathbb{E}_{Y|X}[Y \mid X]\right] = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{4} + \frac{1}{8} = \frac{3}{8}$$

$$\mathbb{E}[Y] = 1 \cdot \frac{3}{8} + 0 \cdot \frac{5}{8} = \frac{3}{8}$$

Thus $\boxed{\mathbb{E}_X[\mathbb{E}_{Y|X}[Y \mid X]] = \mathbb{E}[Y]}$

$$\mathrm{Var}_X(\mathbb{E}_{Y|X}[Y|X]) = \mathbb{E}_X\left[\left(\mathbb{E}_{Y|X}[Y|X] - \overbrace{\mathbb{E}_X[\mathbb{E}_{Y|X}[Y|X]]}^{\mathbb{E}[Y]}\right)^2\right]$$

$$\mathbb{E}_X\left[\left(\mathbb{E}_{Y|X}[Y|X] - \mathbb{E}[Y]\right)^2\right]$$

a constant

$$= \mathbb{E}_X\left[\left(\mathbb{E}_{Y|X}[Y|X]\right)^2\right] - 2\mathbb{E}_X\left[\mathbb{E}_{Y|X}[Y|X]\,\mathbb{E}[Y]\right] + \mathbb{E}_X\left[\left(\mathbb{E}[Y]\right)^2\right]$$

$$= \mathbb{E}_X\left[\left(\mathbb{E}_{Y|X}[Y|X]\right)^2\right] - 2\mathbb{E}[Y]\,\mathbb{E}_X\left[\mathbb{E}_{Y|X}[Y|X]\right] + \left(\mathbb{E}[Y]\right)^2$$

$$= \mathbb{E}_X\left[\left(\mathbb{E}_{Y|X}[Y|X]\right)^2\right] - 2\left(\mathbb{E}[Y]\right)^2 + \left(\mathbb{E}[Y]\right)^2$$

$$= \mathbb{E}_X\left[\left(\mathbb{E}_{Y|X}[Y|X]\right)^2\right] - \left(\mathbb{E}[Y]\right)^2$$

// I feel stupid for NOT just realizing we can skip to this formula

$$= \left(\tfrac{1}{2}\right)^2 \cdot \tfrac{1}{2} + \left(\tfrac{1}{4}\right)^2 \cdot \tfrac{1}{2} - \left(\tfrac{3}{8}\right)^2$$

$$= \frac{1}{8} + \frac{1}{32} - \frac{9}{64} = \frac{1}{64}$$

$$\boxed{\mathrm{Var}_X(\mathbb{E}_{Y|X}[Y|X]) = \frac{1}{64}}$$

$$\mathrm{Var}(Y) = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2$$

$$= \frac{3}{8} - \left(\frac{3}{8}\right)^2 = \frac{3}{8} - \frac{9}{64} = \frac{24}{64} - \frac{9}{64} = \frac{15}{64}$$

Note: $\mathrm{Var}(Y) = \mathrm{Var}_X(\mathbb{E}_{Y|X}[Y|X]) + \mathbb{E}_X[\mathrm{Var}(Y|X)]$

by law of total variance

(b) What is the PMF of $\mathbb{E}(X|Y)$? What is the expectation and the variance of $\mathbb{E}(X|Y)$? Compare them with the expectation and the variance of X.

Conditional PMF

$$P_{X|Y}(X=1 \mid Y=1) = \frac{P_{X,Y}(1,1)}{P_Y(1)} = \frac{1/4}{3/8} = 2/3$$

pairs

$$P_{X|Y}(X=1 \mid Y=0) = \frac{P_{X,Y}(1,0)}{P_Y(0)} = \frac{1/4}{5/8} = 2/5$$

pairs

$$P_{X|Y}(X=0 \mid Y=1) = \frac{P_{X,Y}(0,1)}{P_Y(1)} = \frac{1/8}{3/8} = = 1/3$$

$$P_{X|Y}(X=0 \mid Y=0) = \frac{P_{X,Y}(0,0)}{P_Y(0)} = \frac{3/8}{5/8} = 3/5$$

Conditional Expectation

$$\mathbb{E}_{X|Y}[X \mid Y=1] = 1 \cdot \Pr[X=1 \mid Y=1] + 0 \cdot \Pr[X=0 \mid Y=1]$$
$$= \frac{2}{3}$$

$$\mathbb{E}_{X|Y}[X \mid Y=0] = 1 \cdot \Pr[X=1 \mid Y=0] + 0 \cdot \Pr[X=0 \mid Y=0] = \frac{2}{5}$$

$$\mathbb{E}_{X|Y}[X|Y] = \begin{cases} \frac{2}{3} & \text{w.p. } 3/8 \\ \frac{2}{5} & \text{w.p. } 5/8 \end{cases}$$

$$\mathbb{E}_Y\left[\mathbb{E}_{X|Y}[X|Y]\right] = \frac{2}{3} \cdot \frac{3}{8} + \frac{2}{5} \cdot \frac{5}{8} = \frac{1}{2}$$

$$\mathbb{E}[X] = \frac{1}{2}$$

$\Big\}$ EQUAL

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$
$$= \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$$\mathrm{Var}_Y(\mathbb{E}_{X|Y}[X|Y]) = \mathbb{E}\left[(\mathbb{E}_{X|Y}[X|Y])^2\right] - \left(\mathbb{E}\left[\mathbb{E}_{X|Y}[X|Y]\right]\right)^2$$
$$= \left(\frac{2}{3}\right)^2 \cdot \frac{3}{8} + \left(\frac{2}{5}\right)^2 \cdot \frac{5}{8} - \left(\frac{1}{2}\right)^2$$
$$= \frac{4}{3\cancel{9}} \cdot \frac{\cancel{3}}{8} + \frac{4}{5} \cdot \frac{1}{8} - \frac{1}{4}$$
$$= \frac{\cancel{4}1}{\cancel{24}6} + \frac{\cancel{4}1}{\cancel{40}10} - \frac{1}{4} = \frac{5}{30} + \frac{3}{30} - \frac{1}{4} = \frac{4}{15} - \frac{1}{4}$$
$$= \frac{1}{60}$$

Variances NOT equal