

Recall

Have rv X w/ CDF $F_X(x) \triangleq \Pr[X \leq x]$

Discrete RV

$$p_X(x) = \Pr(X=x)$$

Continuous RV

$$f_X(x) = \frac{F_X(x+\Delta) - F_X(x)}{(x+\Delta) - x}$$

$$\lim_{\Delta \rightarrow 0} = \frac{F_X(x+\Delta) - F_X(x)}{\Delta}$$

$$= \frac{d}{dx} F_X(x)$$

Have rv X, Y w/ CDF $F_{X,Y}(x,y) \triangleq \Pr(\{X \leq x\} \cap \{Y \leq y\})$

Discrete RV

$$p_{X,Y}(x,y) = \Pr(X=x \cap Y=y)$$

Continuous RV

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y}$$

Gives us the definition of conditional pmf, pdf

$$p_{X|Y}(x|y) = \frac{\Pr(X=x \cap Y=y)}{\Pr(Y=y)} = \frac{p_{X,Y}(x,y)}{p_Y(y)}$$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

Conditional Expectation

$$E[X|Y=y] = \int_{-\infty}^{+\infty} x f_{X|Y}(x|y) dx$$

$E[X|Y]$ is a random variable which takes on value $E[X|Y=y]$ w/ density $f_Y(y)$.

Since $E_{X|Y}[X|Y]$ is r.v. can take its expectation.

$$E_Y[E_{X|Y}[X|Y]] = E[X]$$

$$\int_{-\infty}^{+\infty} y \int_{-\infty}^{+\infty} x f_{X|Y}(x|y) dx dy \quad \swarrow \frac{f_{X|Y}(x,y)}{f_Y(y)}$$

$$\int_{-\infty}^{+\infty} y f_{X,Y}(y) dy = f_X(x)$$

$$\int_{-\infty}^{+\infty} x f_X(x) dx = E[X]$$

Example: Coins

Coin A - fair

Coin B - two heads

Random Experiment

① First flip A

(i) If H, flip A again

(ii) If T, flip B

Define X : outcome of first flip i.e. $\begin{cases} 1 & H \\ 0 & T \end{cases}$
 Y : outcome of second flip

Joint PMF:

$$P_{X,Y}(1,1) = Pr(1,1) = Pr[X=1] Pr[Y=1|X=1] = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

$$P_{X,Y}(1,0) = Pr(1,0) = Pr[X=1] Pr[Y=0|X=1] = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

$$P_{X,Y}(0,1) = Pr(0,1) = Pr[X=0] Pr[Y=1|X=0] = \frac{1}{2} \cdot 1 = \frac{1}{2}$$

$$P_{X,Y}(0,0) = Pr(0,0) = Pr[X=0] Pr[Y=0|X=0] = \frac{1}{2} \cdot 0 = 0$$

Marginal PMF:

$$P_X(1) = 1/2$$
$$P_X(0) = 1/2 \quad \text{OR}$$

By Def

$$\sum_y P_{X,Y}(x,y) = P_{X,Y}(1,0) + P_{X,Y}(1,1)$$
$$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$\sum_y P_{X,Y}(x,y) = P_{X,Y}(0,0) + P_{X,Y}(0,1)$$
$$= 0 + \frac{1}{2} = 1/2$$

$$P_Y(1) = 3/4 = P_{X,Y}(0,1) + P_{X,Y}(1,1) = \sum_x P_{X,Y}(x,y) \text{ for } y=1$$

$$P_Y(0) = 1/4 = P_{X,Y}(0,0) + P_{X,Y}(1,0) = \sum_x P_{X,Y}(x,y) \text{ for } y=0$$

Conditional PMF $P_{Y|X}(y|x)$

$$P_{Y|X}(y=1|x=1) = \frac{1}{2} = \frac{P_{X,Y}(1,1)}{P_X(1)}$$

$$P_{Y|X}(y=0|x=1) = 1 - P_{Y|X}(y=1|x=1) = 1 - \frac{1}{2} = \frac{1}{2} = \frac{P_{X,Y}(1,0)}{P_X(1)}$$

$$P_{Y|X}(y=1|x=0) = 1 = \frac{P_{X,Y}(0,1)}{P_X(0)}$$

$$P_{Y|X}(y=0|x=0) = 1 - 1 = 0 = \frac{P_{X,Y}(0,0)}{P_X(0)}$$

Conditional Expectation

$$E_{Y|X}[Y|x=1] = 1 \cdot \frac{1}{2} + 0 \cdot \frac{1}{2} = 1 \cdot \Pr[Y=1|x=1] + 0 \cdot \Pr[Y=0|x=1] = \frac{1}{2}$$

$$E_{Y|X}[Y|x=0] = 1 \cdot \Pr[Y=1|x=0] + 0 \cdot \Pr[Y=0|x=0] = 1$$

$$E_{Y|X}[Y|x] = X \cdot \frac{1}{2} + (1-X) \cdot 1 = \begin{cases} \frac{1}{2} & \text{w.p. } \frac{1}{2} \\ 1 & \text{w.p. } \frac{1}{2} \end{cases}$$
$$= 1 - \frac{X}{2}$$

$$E_x[E_{Y|X}[Y|X]] = \frac{1}{2} \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{3}{4} = E[Y]!$$

MY WAY

$$\hookrightarrow = 1 - \frac{1}{2}E[X] = 1 - \frac{1}{4} = \frac{3}{4} = E[Y]!$$

Example - Gaussian Noise

$$Y = X + W \quad \text{Gaussian Noise} \quad f_W(w) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{w^2}{2\sigma^2}}$$

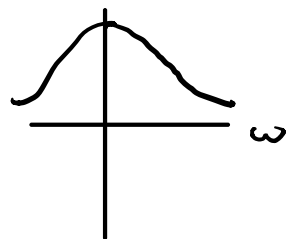
↑
signal in $\{+1, -1\}$

$$\Pr[X=1] = p$$

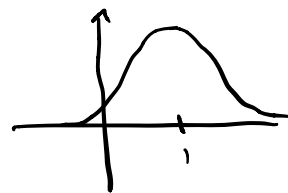
$$\Pr[X=0] = 1-p$$

i.e.

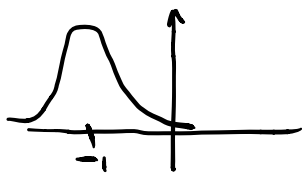
$$p_X(x) = \begin{cases} p & , x=1 \\ 1-p & , x=0 \end{cases}$$



$$f_{Y|X=1}(y|x=1) = 1 + W \quad (\text{just shifts the mean!})$$



$$f_{Y|X=0}(y|x=-1) = -1 + W$$



$$f_Y(y) = \Pr(X=1)f_{Y|X}(y|x=1) + \Pr(X=-1)f_{Y|X}(y|x=-1)$$

$$E[Y|X] = \begin{cases} 1 & \text{w.p. } p \\ -1 & \text{w.p. } 1-p \end{cases}$$

Want

$$\max_{\hat{x} \in \{+1, -1\}} \Pr[X = \hat{x} | Y=y]$$

$$\Pr[X=1 | Y=y] \stackrel{?}{>} \Pr[X=-1 | Y=y]$$

$$\Pr[X=1 | Y=y]$$

vs.

$$\Pr[X=-1 | Y=y]$$

$$\frac{\Pr[X=1] f_{Y|X}(y|x=1)}{f_Y(y)}$$

$$\frac{\Pr[X=-1] f_{Y|X}(y|x=-1)}{f_Y(y)}$$

$$p \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-1)^2}{2\sigma^2}}$$

$$(1-p) \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y+1)^2}{2\sigma^2}}$$

Derive This

$$y \sum_{\hat{x}=-1}^{\hat{x}=1} \tau^* = \frac{\sigma^2}{2} \log\left(\frac{p}{1-p}\right)$$

Correlation

Correlation between X and Y

$$E[XY] = \int_{y=-\infty}^{y=+\infty} \int_{x=-\infty}^{x=+\infty} xy f_{X,Y}(x,y) dx dy$$

Covariance between X and Y

$$\text{Cov}(X,Y) = E[(X - E[X])(Y - E[Y])]$$

$$= E[XY] - E[X]E[Y]$$