Stationarity: $F_{X(t_1+\Delta), X(t_2+\Delta), \ldots, X(t_n+\Delta)}(x_1, \ldots, x_n)$

$$= F_{X(t_1), \ldots, X(t_n)}(x_1, \ldots, x_n) \quad \forall n$$

$$\{t_i\}_{i=1}^{n} \quad \Delta$$

Wide-Sense Stationary:

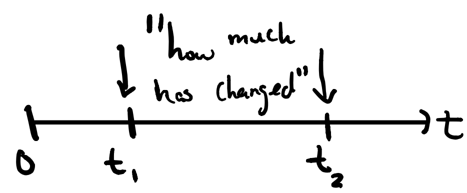
$$\mathbb{E}[X(t)] = \mu(t) \equiv \mu$$

$$R_X(t_1, t_2) \triangleq \mathbb{E}[X(t_1)X(t_2)]$$
$$= R_X(\underbrace{t_2 - t_1}_{\tau})$$

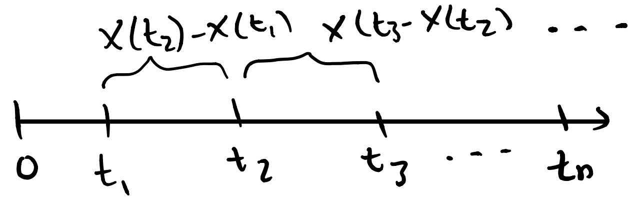
Note: $R(0) \triangleq \mathbb{E}[X^2(t_1)] \leftarrow$ will be <u>deterministic</u>

Increments:

$$X(t_2) - X(t_1)$$



Independent Increments



if all increments (which are rv) are independent we say we have independent increments.

Common Trick: Have

$$g(X(t_2), X(t_1)) \qquad t_2 > t_1$$

write $X(t_2)$ as

$$\underbrace{X(t_2) - X(t_1)}_{(t_1, t_2]} + \underbrace{X(t_1)}_{[0, t_1]}$$

to make analysis easier

Counting Process

$f(t)$, $t > 0$

$f(0) = 0$

f takes non-negative integers

f is nondecreasing

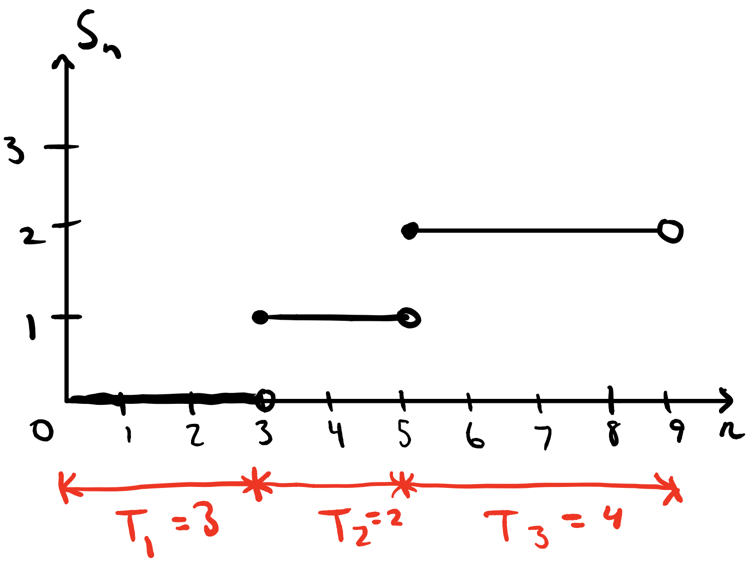
Binomial Counting Process

$$X_i \sim \text{Bernoulli}(p)$$

$$S_n = \sum_{i=1}^{n} X_i$$

$$S_n \sim \text{Binomial}(n, p)$$

$$\Pr[S_n = k] = \binom{n}{k} p^k (1-p)^{n-k}$$



T_1, T_2, T_3 are a sequence of inter-arrival times

Interarrival Time $\{T_i\}_{i=1}^{\infty}$

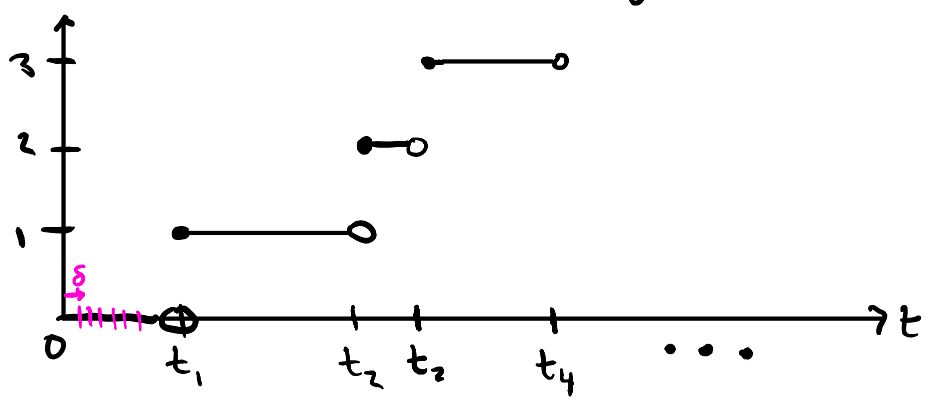
In above example,

$$T_1 \sim \text{Geometric}(p)$$

$$\Pr[T_1 = k] = (1-p)^{k-1} p$$

What about continuous counting processes?

Say events arrive continuously w/ rate λ ← avg arrival/unit time



Partition time into δ-length intervals.

Assumptions.

For $\delta \to 0$

① The probability of having more then one arrivals in δ-interval is negligible (i.e $\to 0$)

② Whether there is an arrival within a δ-interval is **independent** of arrivals in other δ-intervals

$$\boxed{p = \lambda\delta}$$

These assumptions make this similar to the binomial discrete case w/ $p = \lambda\delta$

$$\delta \to 0$$

$$p = \lambda\delta \to 0$$

$$n = \frac{t}{\delta} \to \infty$$

$$S_n \sim \text{Binomial}\left(\frac{t}{\delta}, \lambda\delta\right) \longrightarrow \text{Poisson}(\lambda t)$$

Poisson Process:

Definition: $\{N(t)\}_{t \geq 0}$ is Poisson process w/ rate λ if it is a counting process w/ independent increments and $N(t) - N(s) \sim \text{Pois}(\lambda(t-s))$ $\forall\, t \geq s$.

Interarrival Time (note: T_i iid $\forall\, i$)

Look at T_1

$$\Pr[T_1 > t] = (1-\lambda\delta)^{\frac{t}{\delta}}$$

$$\lim_{\delta \to 0} (1-\lambda\delta)^{t/\delta} = e^{-\lambda t}$$

i.e $T_1 \sim \exp(\lambda)$

$$\mathbb{E}[T_1] = \frac{1}{\lambda}$$

Leads to ANOTHER definition

Definition: $\{N(t)\}_{t \geq 0}$ is Poisson process w/ rate λ if it is a counting process w/ interarrival times iid exponentially distributed.

Also have a third definition (proved in HW5)

Definition: $\{N(t)\}_{t\geq 0}$ is a counting process w/ rate λ if it is a counting process such that $\forall\, \tau > 0$

any interval τ

$N(\tau) \sim \text{Pois}(\lambda\tau)$, and given $N(\tau)=n$ (n arrival times in interval τ), are iid $U\sim[0,\tau]$