Continuous-Time Markov Chain

Continuous-Time Markov Chain:

A continuous-time Markov chain $\{X(t)\}_{t\geq 0}$ satisfies

- the Markov property: $\forall t_0 < t_1 < \ldots < t_{n+1}$,

$$\begin{aligned} & \Pr[X(t_{n+1}) = j \,|\, X(t_n) = i, X(t_{n-1}) = i_{n-1}, \ldots, X(t_0) = i_0] \\ = \; & \Pr[X(t_{n+1}) = j \,|\, X(t_n) = i] \end{aligned}$$

- the state space $\mathcal{X}$ is discrete.

Homogeneity:

A continuous-time Markov chain $\{X(t)\}_{t\geq 0}$ is homogeneous (time invariant) if

$$\Pr[X(t_{n+1}) = j \,|\, X(t_n) = i] = p_{i,j}(t_{n+1} - t_n), \quad \forall t_{n+1}, t_n$$

or equivalently,

$$\Pr[X(s+t) = j \,|\, X(s) = i] = p_{i,j}(t), \quad \forall s, t$$

Transition Matrix and Stationary Distribution

Transition Matrix:

The transition matrix for a time period of length t:

$$\mathbf{P}(t) = \left\{ p_{i,j}(t) \stackrel{\Delta}{=} \Pr[X(t) = j \mid X(0) = i] \right\}_{i,j \in \mathcal{X}}$$

Chapman-Kolmogorov Equations:

$$\mathbf{P}(t+s) = \mathbf{P}(t)\mathbf{P}(s), \quad \text{equivalently,} \quad p_{i,j}(t+s) = \sum_{k \in \mathcal{X}} p_{i,k}(t) p_{k,j}(s)$$

State Probability at time t:

Let $\mathbf{p}(t) \stackrel{\Delta}{=} \left\{ \Pr[X(t) = i] \right\}_{i \in \mathcal{X}}$ be a row vector of the state probabilities at time t.

$$\mathbf{p}(t) = \mathbf{p}(0)\mathbf{P}(t)$$

Stationary Distribution:

A probability distribution π over $\mathcal{X}$ is a stationary distribution if

$$\pi = \pi \mathbf{P}(t), \quad \forall t$$

Question: How to characterize $\mathbf{P}(t)$ for all t? Is there a single matrix that fully characterizes $\{X(t)\}_{t \geq 0}$, similar to the one-step transition matrix for DTMC?

Characterizations of CTMC

Characterization of CTMC:

A CTMC can be characterized by

- the holding time $T_i \sim exp(\lambda_i)$ in each state i (the exponential distribution is due to the Markov property that dictates memoryless holding time);
- an embedded DTMC with transition probabilities $p_{i,j}$ ($p_{i,i} = 0$ for all i) that governs the state transition when the holding time in the current state is up.

Remarks: The dynamics of a CTMC can be viewed as follows. Each time a state, say i, is entered, a holding time $T_i \sim exp(\lambda_i)$ is selected. When the holding time is up, the next state j is selected according to the embedded DTMC $\{p_{i,j}\}$.

An Equivalent Characterization:

Based on properties of exponential distribution, an equivalent characterization of a CTMC with $\{\lambda_i\}_{i\in\mathcal{X}}$ and $\{p_{i,j}\}$ is as follows.

- Each time a state, say i, is entered, a transition time $T_{i,j} \sim exp(\lambda_i p_{i,j})$ is selected for each state $j \neq i$.
- When the minimum transition time $\min_j\{T_{i,j}\}$ is up, transit to the state k with the minimum transition time, i.e., $T_{i,k} = \min_j\{T_{i,j}\}$.

Transition Rate Matrix

Transition Rate Matrix $\mathbf{Q}$:

Based on the equivalent characterization, a CTMC is fully characterized by the transition rate matrix $\mathbf{Q} = \{q_{i,j}\}_{i,j\in\mathcal{X}}$ where

- ☐ $q_{i,j} = \lambda_i p_{i,j}$ for $i \neq j$ is the rate of transiting from state i to state j.
- ☐ $q_{i,i} = -\sum_{j\neq i} q_{i,j}$ is the rate of *leaving* state i (the negative sign indicates the direction of leaving); note that $-q_{i,i} = \lambda_i$ is the holding time parameter.
- ☐ each row of $\mathbf{Q}$ sums up to 0.

From $\mathbf{Q}$ to $\mathbf{P}(t)$:

$$\mathbf{P}'(t) = \mathbf{P}\mathbf{Q}$$

with initial condition $\mathbf{P}(0) = \mathbf{I}$. Solving this differential equation leads to

$$\mathbf{P}(t) = e^{t\mathbf{Q}} = \sum_{k=0}^{\infty} \frac{(t\mathbf{Q})^k}{k!}$$

Proof: to show that $\mathbf{P}$ satisfies the above differential equation, we first show that $\mathbf{Q} = \mathbf{P}'(0)$ and then apply Chapman-Kolmogorov equations to $\mathbf{P}(t+\delta)$ when evaluating $\mathbf{P}'(t)$.

Stationary Distribution

Obtaining Stationary Distribution:

The stationary distribution π can be obtained by solving the following linear equations:

$$\begin{cases} \pi \mathbf{Q} = 0 \\ \sum_{i \in \mathcal{X}} \pi_i = 1 \end{cases}$$

Proof: Take derivative on both sides of $\pi = \pi \mathbf{P}(t)$ and plug in the differential equation for $\mathbf{P}(t)$, we have

$$\pi \mathbf{P}'(t) = 0 \;\Rightarrow\; \underbrace{\pi \mathbf{P}(t)}_{=\pi} \mathbf{Q} = 0 \;\Rightarrow\; \pi \mathbf{Q} = 0$$

Global Balance Equations:

$\pi \mathbf{Q} = 0$ can be written as

$$\pi_i q_{i,i} = \sum_{j \neq i} \pi_j q_{j,i}, \quad \forall i$$

which are referred to as the global balance equations. They state that the rate of probability flow out of state i (the left-hand side) is equal to the rate of flow into state i (the right-hand side). More generally, the net flow through any closed loop must be zero for the chain to be in equilibrium. Based on this, you can create your own set of balance equations to solve for the stationary distribution.