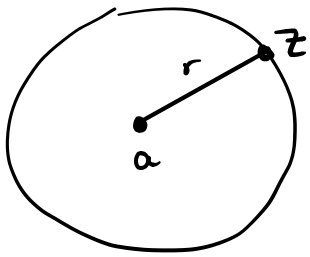
Mean value property

Suppose $f(z)$ is analytic in a neighborhood of point a. Then $f(a) = \text{avg } f(z)$ at any circle centered on a.



$$f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a+re^{i\theta})\,d\theta$$

Proof:

Let $z = a + re^{i\theta}$

Cauchy $\Rightarrow$ $f(a) = \frac{1}{2\pi i}\oint \frac{f(z)}{z-a}\,dz = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(a+re^{i\theta})\,ire^{i\theta}}{re^{i\theta}}\,d\theta$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(a+re^{i\theta})\,d\theta$$

The same mean-value property is true for real harmonic functions $u(x,y)$ – (u can be regarded as the real part of an analytic function

OR Gauss's mean value Theorem!

The average value of a harmonic function on a circle is equal to the value of the function at the center of the circle.

Dirichlet Problem: Find a function $\phi(x,y)$ continuous on a domain D and its boundary, harmonic in D, and taking specified values on the boundary of D.

Maximum Modulus Theorem:

If $f(z)$ is analytic on and inside a closed curve γ, then $|f(z)|$ attains its max and min on the boundary γ.

i.e interior max or min!

(So no stable equilibrium points)