Math 4310 Name: ____________________

Homework 9 Collaborators: ____________________

(Problem 3(c) corrected!) ____________________

Due 11/13/19

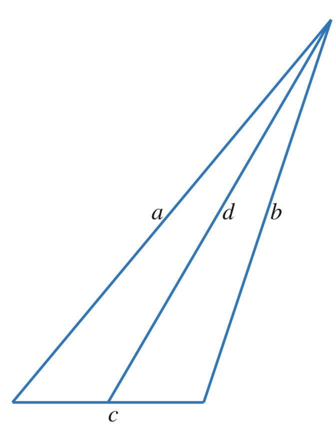
Please print out these pages. I encourage you to work with your classmates on this homework. Please list your collaborators on this cover sheet. (Your grade will not be affected.) Even if you work in a group, you should write up your solutions yourself! You should include all computational details, and proofs should be carefully written with full details. As always, please write neatly and legibly.

Please **staple** *this cover sheet, your exercise solutions, and your glossary together, in that order, and hand in your homework in class.*

GRADES	
Exercises ____________	/ 50

Exercises.

1. In a triangle with sides of lengths a, b, and c, let d be the length of the line segment from the midpoint of the side with length c to the oppositve vertex. Show that $a^2 + b^2 = \frac{1}{2}c^2 + 2d^2$.



2. Apply the Gram-Schmidt process to find an orthonormal basis of the space $\mathbb{R}[x]_{\leq 2}$ of polynomials of degree at most 2, where the inner product is given by

$$\langle f, g \rangle = \int_0^1 f(x)g(x)\,dx.$$

3. Prove the Cauchy-Schwarz formula: if V is an inner product space (not necessarily of finite dimension), and $\mathbf{v}$ and $\mathbf{w}$ are elements of V, then

$$|\langle \mathbf{v}, \mathbf{w} \rangle| \leq \|\mathbf{v}\|\,\|\mathbf{w}\|.$$

(*Hint:* Consider the projection of $\mathbf{w}$ onto the span of $\mathbf{v}$. Note that the length of the projection is no more than the length of $\mathbf{w}$ (why?)).

Prove the following statements.

(a) Prove that for all positive real numbers a, b, c, d,

$$16 \leq (a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right).$$

(b) Prove that for all positive integers n and all real numbers $a_1, \ldots, a_n$, then

$$(a_1 + \ldots + a_n)^2 \leq n(a_1^2 + \ldots + a_n^2).$$

(equivalently, the square of the average of a set of numbers is no more than the average of the squares of the numbers).

(c) Prove that for continuous functions f and g on the interval $[0, 1]$, that

$$\left|\int_0^1 f(x)g(x)dx\right|^2 \leq \left|\int_0^1 f(x)^2 dx\right| \left|\int_0^1 g(x)^2 dx\right|$$

4. Let $V = \mathbb{R}^n$. As usual, we think of each $\mathbf{v}$ in $\mathbb{R}^n$ as a column vector, that is, as a $n \times 1$ matrix. We also equate scalars and 1×1 matrices.

(a) Suppose that A is a symmetric $n \times n$ matrix, and we define

$$\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v}^T A \mathbf{w},$$

show that this is an inner product if and only if A satisfies: for $\mathbf{x} \neq \mathbf{0} \in \mathbb{R}^n$ then $\mathbf{x}^T A \mathbf{x} > 0$ (such a symmetric matrix is called **positive definite**).

(b) Suppose that $\langle \cdot, \cdot \rangle$ is an inner product on $\mathbb{R}^n$. Show that there exists a positive definite matrix A such that

$$\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v}^T A \mathbf{w},$$

(c) If

$$A = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix},$$

is the function defined by

$$\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v}^T A \mathbf{w}$$

(for all $\mathbf{v}, \mathbf{w} \in \mathbb{R}^2$) an inner product on $\mathbb{R}^2$?

5. Recall that for a subspace $W \subset V$ of a real inner product space V,

$$W^\perp = \{\mathbf{v} \in V \mid \langle \mathbf{v}, \mathbf{w} \rangle = 0, \text{ for all } \mathbf{w} \in W\}$$

Show that

(a) $V = W \oplus W^\perp$

(b) If A is an $m \times n$ matrix, and $V = \mathbb{R}^n$ is equipped with the standard inner product, then

$$(\ker A)^{\perp} = image(A^T)$$

and therefore that

$$V = \ker A \oplus image(A^T).$$

Extended Glossary. There is no extended glossary this week.