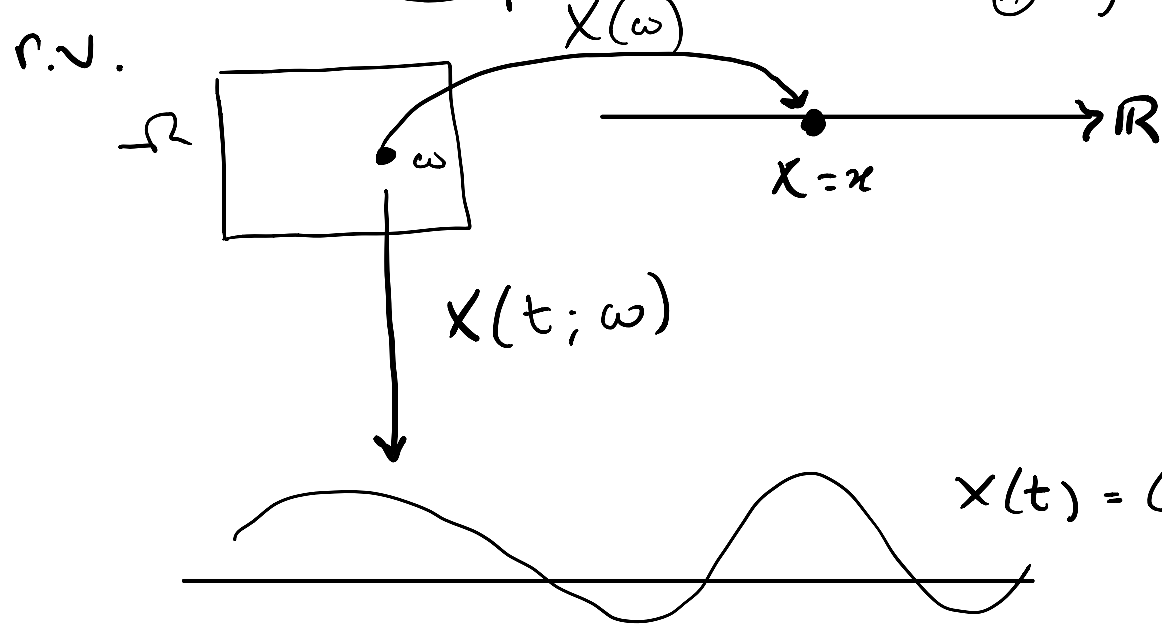
## Recap

$\Theta \sim \mathcal{U}[0, 2\pi]$




$X(t) = \cos(2\pi t + \Theta)$

$X(t = t_0) = \cos(2\pi t_0 + \Theta)$ // deterministic function of a r.v.

| | | |
|---|---|---|
| r.v. | PDF/PMF/CDF | $\mathbb{E}[X]$ |
| 2 r.v's | JOINT | $\mathbb{E}[XY]$ |
| r.p. | $\forall n$, $\{t_n\}_{i=1}^{n}$ | $\mathbb{E}[X(t)] = \mu(t)$ |
| | $\{X(t_i)\}_{i=1}^{n}$ | $\mathbb{E}[X(t_1)X(t_2)] = R_X(t_1, t_2)$ |
| | $F_{X(t_1), \dots, X(t_n)}(x_1, \dots, x_n)$ | |

**Stationarity:** A random process is stationary if its statistical characteristics do not change over time.

Specifically,

$$\forall \Delta \quad F_{X(t),\ldots,X_n(t)}(x_1,\ldots,x_n) = F_{X_1(t+\Delta),\ldots,X_n(t+\Delta)}(x_1,\ldots,x_n)$$

$$\forall n \quad \{t_i\}_{i=1}^{n} \quad \Delta.$$

**WIDE Sense Stationary:** A random process is WSS if

(i) $\mathbb{E}[X(t)] = \mu$

(ii) $R_X(t_1, t_2) \triangleq \mathbb{E}[X_1(t)X_2(t)]$

$= R(\tau)$

where $\tau = t_2 - t_1$

Examples

① i.i.d sequence of random variables. STRICTLY STATIONARY

$$F_{X(t_1)\cdots X(t_n)}(x_1,\ldots,x_n) = F_{X(t_1)}(x_1)\, F_{X(t_2)}(x_2) \cdots F_{X(t_n)}(x_n)$$

$$= F_{X(t_1+\Delta)}(x_1) \cdots F_{X(t_n+\Delta)} \qquad \text{by iid}$$

$$\varphi^{(1)} + \varphi^{(2)} + \varphi^{(3)} = \varphi^{(4)}$$

$$\varphi^{(4)} = \sum_{i=1}^{3} \varphi^{(i)} \quad ; \quad \varphi^{(0)}\,?$$

② Binomial Counting Process

$$X_i = \begin{cases} 1 & \text{w.p. } p \\ 0 & \text{w.p. } 1-p \end{cases}$$

$$S_n = \sum_{i=1}^{n} X_i$$

$\mathbb{E}[S_n] = np \neq$ Constant! **NOT STATIONARY**

③ Random Walk

$$X_i = \begin{cases} 1 & \text{w.p. } 1/2 \\ -1 & \text{w.p. } 1/2 \end{cases}$$

$$S_n = \sum_{i=1}^{n} X_i$$

$$\mathbb{E}[S_n] = \sum_{i=1}^{n} \mathbb{E}[X_i] = 0 \quad \boxed{\text{WSS}}$$

$$\mathrm{Var}(S_n) = \sum_{i=1}^{n} \mathrm{Var}(X_i) = n$$

④ $X(t) = \cos(2\pi t + \Theta)$, $\Theta \sim \mu[0, 2\pi)$

(see textbook for proof)

STRICTLY STATIONARY

**Increment:** The increment of a r.p. $\{X(t)\}$ over an interval $[a,b]$ is the r.v. $\underbrace{X(b)-X(a)}_{\text{another r.v.}}$

**Independent Increments:** $\{X(t)\}$ has independent increments $\boxed{\text{if } \forall n}$, and for all $t_0 < t_1 < \cdots < t_n$ the increments $X(t_1 - t_0), \ldots, X(t_n - t_{n-1})$ are independent

Examples

① i.i.d sequence of Bernoulli r.v's w/ 1/2

$$X_2 - X_1 = \begin{cases} 0 & \text{w.p. } 1/2 \\ -1 & \text{w.p. } 1/4 \\ 1 & \text{w.p. } 1/4 \end{cases}$$

$X_3 - X_2$ depends on realization of this increment

$$\Pr[X_3 - X_2 = 1 \mid X_2 - X_1 = 1] = 0 \neq \Pr[X_3 - X_2 = 1]$$

**NOT** independent increments

## ② Binomial Counting $S_n$

$$S_n = \sum_{i=1}^{n} X_i$$

YES

Independent Increments

$S(t_1) - S(t_0)$

$S(t_2) - S(t_1)$

$S(t_3) - S(t_2)$

$\vdots$

$S(t_n) - S(t_{n-1})$

## Stationary Increments

$\{X(t)\}$ has stationary increments if the distribution of $X(t+\tau) - X(t)$ depends only on $\tau$, not $t$.

Key Idea: when characterizing processes w/ independent increments over <u>non-overlapping</u> intervals

Example: Binomial Counting

$*$ Joint PMF of $S_{n_1}, S_{n_2}, S_{n_3}$ $(n_1 < n_2 < n_3)$

$$\Pr[S_{n_1} = s_1,\ S_{n_2} = s_2,\ S_{n_3} = s_3]$$

$$= \Pr[S_{n_1} = s_1] \Pr[S_{n_2} = s_2 \mid S_{n_1} = s_1] \Pr[S_{n_3} = s_3 \mid S_{n_1} = s_1, S_{n_2} = s_2]$$

$$= \Pr[S_{n_1} = s_1] \Pr[S_{n_2} - S_{n_1} = s_2 - s_1] \Pr[S_{n_3} - S_{n_2} = s_3 - s_2]$$

$$= \binom{n}{s_1} p^{s_1} (1-p)^{n_1 - s_1}$$

$$\binom{n_2 - n_1}{s_2 - s_1} p^{s_2 - s_1} (1-p)^{n_2 - n_1 - (s_2 - s_1)}$$

$$\binom{n_3 - n_2}{s_3 - s_2} p^{s_3 - s_2} (1-p)^{n_3 - n_2 - (s_3 - s_2)}$$

Example: Binomial Counting $(m < n)$

$$R_S(m,n) \triangleq \mathbb{E}[S_m, S_n]$$

by physical meaning of counting

$$= \mathbb{E}[S_m(S_m + (S_n - S_m))]$$

$$= \mathbb{E}[S_m^2 + S_m(S_n - S_m)]$$

$$= \mathbb{E}[S_m^2] + \mathbb{E}[S_m(S_n - S_m)]$$

independent increments

$$= \mathbb{E}[S_m^2] + \mathbb{E}[S_m]\,\mathbb{E}[S_n - S_m]$$

# Counting Function

$f(t)$ $(t \geq 0)$ is a counting function if $f(0)=0$, $f(t)$ takes non-negative integers, is non-decreasing, and is right continuous.

## Counting Process

$\{X(t)\}$ where every sample path is a counting function.