

Laurent Series

Suppose a is the pole of an analytic function $f(z)$

$$\text{i.e } \lim_{z \rightarrow a} f(z) = \infty$$

An aside on essential singularities

If $f(z)$ approaches ∞ as z approaches a from any direction then a is a pole of f .

Since f is analytic, it follows that

$$F(z) = \frac{1}{f(z)}$$

is also analytic and has a root at a .

If this root has multiplicity m , then the factorization of F is

$$F(z) = (z-a)^m \Omega(z)$$

where Ω is analytic and nonzero at a ; in fact we know that $\Omega(a) = \frac{F^{(m)}(a)}{m!}$

The local behavior of f near a is therefore given by

$$f(z) = \frac{\tilde{\Omega}(z)}{(z-a)^m}$$

where $\tilde{R}(z) = \frac{1}{R(z)}$ is analytic and nonzero at a .

This expression brings out the analogy with rational functions and enables us to identify m as the algebraic multiplicity or order of the pole at a .

End Aside

We can express $f(z)$ as

$$f(z) = \frac{\phi(z)}{(z-a)^m}$$

where $\phi(z)$ is analytic and $\phi(a) \neq 0$. m is the order of the pole, and the greater the order the faster we approach infinity!

$\phi(z)$ can be expressed as a Taylor series centered at a :

$$\phi(z) = \sum_{n=0}^{\infty} c_n (z-a)^n , \quad c_n = \frac{\phi^{(n)}(a)}{n!}$$

Hence we deduce that

If an analytic function $f(z)$ has a pole of order m at a , then in the vicinity of this pole, $f(z)$ possesses a Laurent series of the form

$$f(z) = \frac{c_0}{(z-a)^m} + \frac{c_1}{(z-a)^{m-1}} + \dots + \frac{c_{m-1}}{(z-a)} + c_m + c_{m+1}(z-a) + \dots$$

Recall that $\frac{1}{z-a}$ is the RESIDUE of $f(z)$ at a , denoted $\text{Res}[f, a]$.

Also recall the crucial significance of the residue in evaluating integrals: if C is a simple loop containing a but no other singularities of f

$$\oint_C f(z) dz = 2\pi i \text{Res}[f, a]$$

Annular Laurent Series

We just saw that the Laurent series is the natural generalization of the Taylor series when the centre of the expansion is a pole rather than a non-singular point.

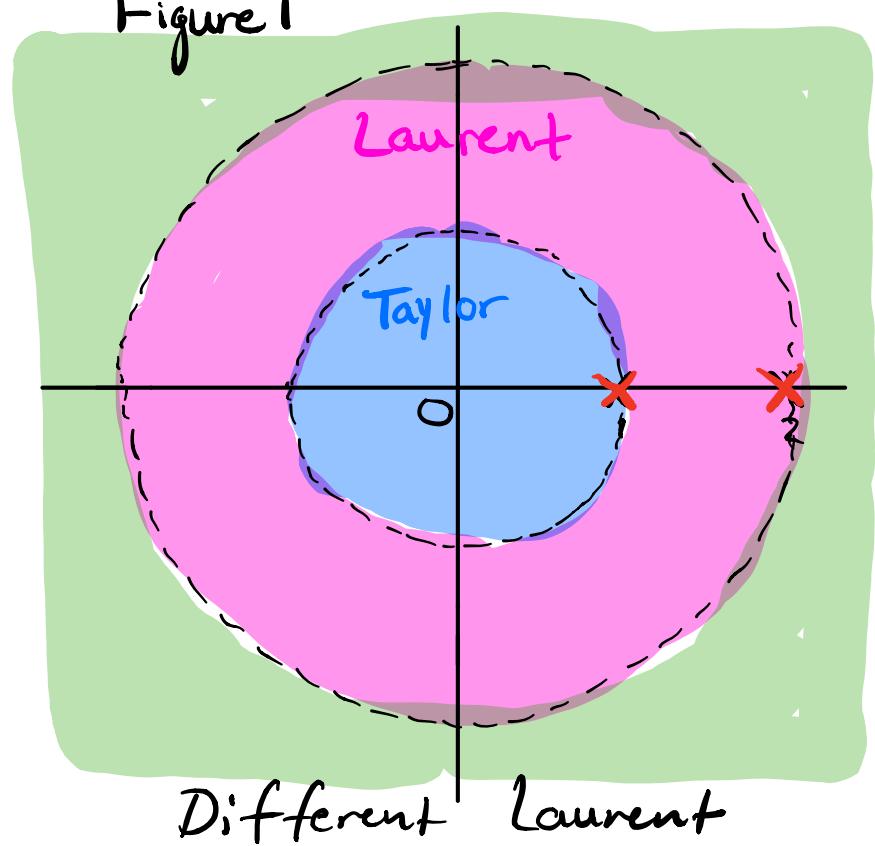
This is NOT the only situation in which Laurent series are needed.

Figure 1

For example, consider

$$F(z) = \frac{1}{(z-1)(z-2)}$$

whose poles can be seen to the right.



let's look at the three shaded regions of Fig 1

$$|z| < 1, \quad 1 < |z| < 2, \quad |z| > 2$$

- Since F is analytic within the unit disc, it possesses a Taylor series in powers of z .

$$F(z) = \frac{1}{1-z} - \frac{1}{2-z} \quad (\text{via partial fractions})$$

$$= \frac{1}{(1-z)} - \frac{1}{2[1-\frac{z}{2}]} = \underbrace{\sum_{n=0}^{\infty} z^n}_{\text{for } |z| < 1} - \frac{1}{2} \underbrace{\sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n}_{\text{for } |z| < 2}$$

$$= \frac{1}{2} + \frac{3}{4}z + \dots + \left[1 - \left(\frac{1}{2}\right)^{n+1}\right]z^n + \dots, \text{ for } |z| < 1$$

The pole at $z=1$ means that outside the unit disc F cannot be expressed as a power series in z . However in the shaded annulus $1 < |z| < 2$ it can be expressed as a Laurent series in z :

$$F(z) = \frac{-1}{z[1-\frac{1}{z}]} - \frac{1}{2[1-\frac{z}{2}]}$$

$$= - \underbrace{\sum_{n=0}^{\infty} \left[\frac{1}{z}\right]^{n+1}}_{|z| > 1} - \frac{1}{2} \underbrace{\sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n}_{|z| < 2}$$

$$= \dots - \frac{1}{z^3} - \frac{1}{z^2} - \frac{1}{z} - \frac{1}{2} - \frac{z}{4} - \frac{z^2}{8} - \frac{z^3}{16} - \dots \quad 1 < |z| < 2$$

Finally, in the region $|z| > 2$ beyond the annulus we obtain a **DIFFERENT** Laurent series

This was NOT worked out in book so verify result.

$$F(z) = \frac{1}{(1-z)} - \frac{1}{(z-2)}, \quad |z| > 2$$

$$= \frac{1}{1-z} - \frac{1}{z(1-\frac{2}{z})}, \quad u = \frac{2}{z}$$

$$= \frac{1}{1-\frac{2}{u}} - \frac{1}{z(1-\frac{1}{u})}, \quad \frac{1}{u} = \frac{z}{2} \\ |u| < 1$$

$$= \frac{u}{u-2} - \frac{u}{z(u-1)}$$

$$= \frac{u}{2(\frac{u}{2}-1)} + \frac{u}{2} \cdot \frac{1}{1-u}$$

$$= \left[-\frac{u}{2} \cdot \frac{1}{1-\frac{u}{2}} \right] + \left[\frac{u}{2} \cdot \frac{1}{1-u} \right]$$

$$= -\frac{u}{2} \left(1 + \left(\frac{u}{2}\right) + \left(\frac{u}{2}\right)^2 + \left(\frac{u}{2}\right)^3 + \dots \right) + \frac{u}{2} \left(1 + u + u^2 + u^3 + \dots \right)$$

$$= -\frac{1}{2} \left(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots \right) + \frac{1}{2} \left(1 + \left(\frac{2}{2}\right) + \left(\frac{2}{2}\right)^2 + \left(\frac{2}{2}\right)^3 + \dots \right)$$

$$= \left(-\frac{1}{2} - \frac{1}{2^2} - \frac{1}{2^3} - \frac{1}{2^4} - \dots \right) + \left(\frac{1}{2} + \frac{2}{2^2} + \frac{4}{2^3} + \frac{8}{2^4} + \dots \right)$$

$$-\frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n + \frac{1}{z^2} \sum_{n=0}^{\infty} \left(\frac{2}{z}\right)^n$$

$$= 0 + \left(\frac{2}{z^2} - \frac{1}{z^2} \right) + \left(\frac{4}{z^3} - \frac{1}{z^3} \right) + \left(\frac{8}{z^4} - \frac{1}{z^4} \right) + \dots$$

$$= 0 + \frac{1}{z^2} + \frac{3}{z^3} + \frac{7}{z^4}$$

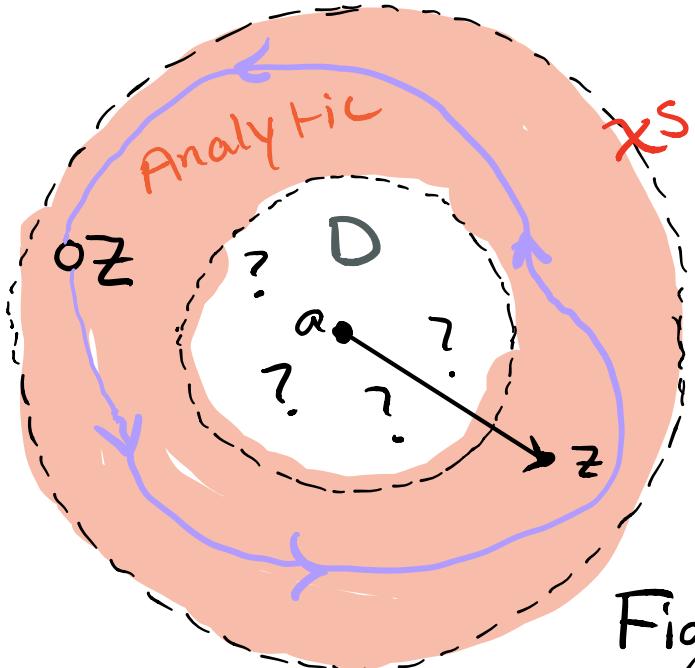
$$= \sum_{n=0}^{\infty} \frac{2^n - 1}{z^{n+1}}$$

We have just seen the illustration of a general phenomenon!

LAURENT'S THEOREM

If $f(z)$ is analytic everywhere within an annulus A centered at a , then $f(z)$ can be expressed as a Laurent series within A . In fact, if K is any simple loop winding once round a ,

$$f(z) = \sum_{n=-\infty}^{+\infty} c_n (z-a)^n, \quad c_n = \frac{1}{2\pi i} \oint_K \frac{f(z)}{(z-a)^{n+1}} dz$$



Before establishing Laurent's Theorem, note the following observations regarding its significance

Figure 2

- ① The surprising thing about the result is the existence of Laurent series, not the fact that it converges in an annulus. Since we know that a power series in $(z-a)$ will converge inside a disk centred at a , it follows that a power series in $1/(z-a)$ will converge outside a disc centered at a .

Since a Laurent series is the sum of a power series in $(z-a)$ and a power series in $1/(z-a)$, it follows that it will converge in an annulus.

② Previously we could only deduce the existence of a Laurent series in the vicinity of a pole. The present result is much more powerful: as indicated in Figure 2, we make no assumptions at all concerning the behavior of $f(z)$ in the disc D bounded by the inner edge of the annulus.

In practice the outer edge of the annulus may be expanded until it hits a singularity s of $f(z)$. Likewise the inner edge may be contracted until it hits the outermost singularity lying in D .

③ If there are no singularities in D , then the inner edge may be completely collapsed. In this case our c_n do not contain any negative integers! For if n is negative then $\frac{f(z)}{(z-a)^{n+1}}$ is analytic everywhere in K and $c_n = 0$. In this way we recover the existence of Taylor series as a special case of Laurent's Theorem.

④ Suppose that a is a singularity and that for sufficiently small ϵ , there are no other singularities within a distance ϵ of a . In this case we say that a is an **isolated singularity** of $f(z)$. Applying Laurent's Theorem to $0 < |z-a| < \epsilon$, we find there are just two fundamentally different possibilities:

i) The principle part of the Laurent series has finitely many terms

OR

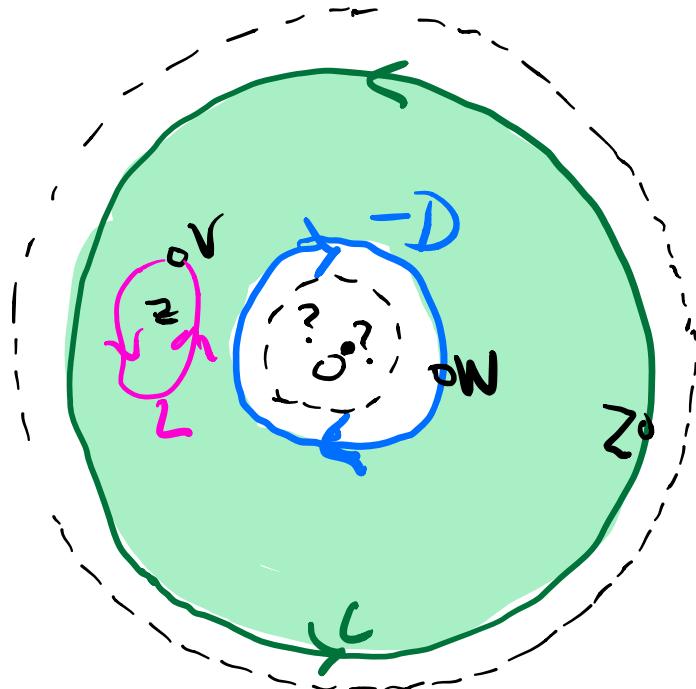
ii) infinitely many terms.

To sum up,

An isolated singularity of an analytic function is either a pole or an essential singularity.

Establishing Laurent Series

Consider the case where $a=0$. In the figure, z is a general point in the annulus, C and D are CCW circles such that z lies between them and L is a simple loop around z , lying in the annulus.



First, by Cauchy's Formula,

$$f(z) = \frac{1}{2\pi i} \oint_L \frac{f(v)}{v-z} dv = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-z} dz - \frac{1}{2\pi i} \oint_D \frac{f(w)}{w-z} dw$$

Follows from the fact that
 L may be deformed into
 $C + -D$

Next we rewrite the above equation as

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z} \left[\frac{1}{1 - \frac{z}{z}} \right] dz + \frac{1}{2\pi i} \oint_D \frac{f(w)}{w} \left[\frac{1}{1 - \frac{w}{z}} \right] dw$$

The significance of this is that

$$\left| \frac{z}{z} \right| < 1 \quad \text{and} \quad \left| \frac{w}{z} \right| < 1$$

so both integrands can be expanded into a geometric series.

The integral around C can be expressed as

$$\frac{1}{2\pi i} \oint_C \frac{f(z)}{z} \left[\frac{1}{1 - \frac{z}{z}} \right] dz = \sum_{n=0}^{\infty} \left[\frac{1}{2\pi i} \oint_C \frac{f(z)}{z^{n+1}} dz \right] z^n$$

Essentially identical reasoning also justifies term by term integration in the case of the integral around D ?

$$\frac{1}{2\pi i} \oint_D \frac{f(w)}{w} \left[\frac{1}{1 - \left(\frac{w}{z} \right)} \right] dw = \frac{1}{2\pi i} \oint_D \frac{f(w)}{w} \sum_{n=0}^{\infty} \frac{w^n}{z^n} dw$$

Note

$$\sum_{n=0}^{\infty} \frac{w^n}{z^{n+1}} = \sum_{n=1}^{\infty} \frac{w^{n-1}}{z^n} = \sum_{n=1}^{\infty} \left[\frac{1}{2\pi i} \oint_D w^{n-1} f(w) dw \right] \frac{1}{z^n}$$

Thus the existence of Laurent series is established!

$$f(z) = \dots + \frac{d_3}{z^3} + \frac{d_2}{z^2} + \frac{d_1}{z} + c_0 + c_1 z + c_2 z^2 + \dots$$

where

$$d_m = \frac{1}{2\pi i} \oint_D w^{m-1} f(w) dw \quad \text{and} \quad c_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z^{n+1}} dz$$

By the Deformation Theorem,

Deformation Theorem:

If a contour sweeps only through analytic points as it is deformed, the value of the integral does NOT change!

the integrals defining d_m and c_n do NOT change their values if we allow C to contract and D to expand till they coalesce into the same circles!

Indeed, we may replace both C and D by any simple loop K contained in the annulus and winding around it once.

Now let $m = -n$, then d_{-n} of z^n has integrand

$$w^{-(n+1)} f(w) = \frac{f(w)}{w^{n+1}}$$

which is the same integrand for the c_n !

Thus, as was to be shown, the Laurent series may be expressed as

$$f(z) = \sum_{n=-\infty}^{+\infty} c_n z^n \text{ where } c_n = \frac{1}{2\pi i} \oint_K \frac{f(z)}{z^{n+1}} dz$$