Similar Matrices

Situation

$$T: V \to V \quad (T \in \mathcal{L}(V))$$

Suppose

$$\dim V = n < \infty$$

and

$$\alpha, \beta \text{ two bases of } T$$

from HW:

$$[T]_\beta = [id]_{\beta \leftarrow \alpha} [T]_{\alpha \leftarrow \alpha} [id]_{\alpha \leftarrow \beta}$$

Let

$$[T]_\alpha = A$$

$$[T]_\beta = B$$

$$[id]_{\beta \leftarrow \alpha} = Q^{-1}$$

$$[id]_{\alpha \leftarrow \beta} = Q$$

Then

$$B = Q^{-1} A Q$$

Definition: We say A is **similar** to B $(A \sim B)$ if $\exists$ invertible $n \times n$ matrix Q s.t. $B = Q^{-1} A Q$

Proposition: $\sim$ is an equivalence relation on the set of $n \times n$ matrices

Aside

$\sim$ is an equivalence relation if $\forall x, y, z \in S$

① $x \sim x$

② $x \sim y \rightarrow y \sim x$

③ $x \sim y,\ y \sim z \rightarrow x \sim z$

Proof (of Proposition)

Let A, B, C be $n \times n$ matrices.

① $A \sim A \rightarrow$ choose $Q = I$

② $A \sim B$, $B = Q^{-1} A Q$

then $B \sim A$, $Q B Q^{-1} = A$

③ $A \sim B$ $\quad B = Q^{-1} A Q$, Q, S invertible

$B \sim C$ $\quad C = S^{-1} B S$

then $A \sim C$ $\quad C = S^{-1} Q^{-1} A Q S$

$= (QS)^{-1} A (QS)$

$A \sim C$

assuming $\dim V = n$

Problem: Find a nice form for $T \in \mathcal{L}(V)$, or $A_{n \times n}$

2 versions of problem

① find a basis β of V s.t. $[T]_\beta$ is "totally nice"

② find a matrix $\underbrace{B = Q^{-1} A Q}_{B \text{ similar to } A}$ s.t. B is "totally nice"

Definition: ① A matrix $A \in \mathbb{F}^{n \times n}$ is called diagonalizable if $\exists$ Q s.t. $B = Q^{-1} A Q$ is diagonal

i.e

$$B = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & \lambda_n \end{pmatrix}$$

② $T: V \to V$ LT

is called diagonalizable if $\exists$ basis β of V s.t. $[T]_\beta$ is diagonalizable

Example

Suppose $B = Q^{-1} A Q$ is diagonal $\begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}$

So Note: $QB = AQ$

Note: $[\vec{v}_1 \ldots \vec{v}_n]$ $\underbrace{\text{a basis of } V}$ which is $\mathbb{F}^n$ here

let $Q = [\vec{v}_1 \ldots \vec{v}_n]$, $\vec{v}_i \in \mathbb{F}^n \; \forall i$

$\Leftrightarrow$ $Q = [v_1 \ldots v_n]$ is invertible

$$[\vec{v}_1 \ldots \vec{v}_n] \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} = A[\vec{v}_1 \; \vec{v}_2 \; \ldots \; \vec{v}_n]$$

$$[\lambda_1 \vec{v}_1 \; \lambda_2 \vec{v}_2 \; \ldots \; \lambda_n \vec{v}_n] = [A\vec{v}_1 \; A\vec{v}_2 \; \ldots \; A\vec{v}_n]$$

i.e

$$A\vec{v}_1 = \lambda_1 \vec{v}_1$$
$$A\vec{v}_2 = \lambda_2 \vec{v}_2$$
$$\vdots$$
$$A\vec{v}_n = \lambda_n \vec{v}_n$$

Example (Part 2):

If T is diagonalizable, then what?

i.e. $\beta = [\vec{v}_1 \ \vec{v}_2 \cdots \vec{v}_n]$ basis of V $\quad [T]_\beta = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & \lambda_n \end{pmatrix}$

$$[T]_\beta = \begin{array}{c} \\ \vec{v}_1 \\ \vec{v}_2 \\ \vdots \\ \vec{v}_n \end{array} \begin{array}{c} \begin{array}{cccc} T(\vec{v}_1) & T(\vec{v}_2) & \cdots & T(\vec{v}_n) \end{array} \\ \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \end{array}$$

$T(\vec{v}_1) = \lambda_1 \vec{v}_1$

$T(\vec{v}_2) = \lambda_2 \vec{v}_2$

$\vdots$

$T(\vec{v}_n) = \lambda_n \vec{v}_n$

Definition: Let $T: V \to V$ be a LT ($T \in \mathcal{L}(V)$).

A vector $\vec{v} \in V$ is an eigenvector of T if

(a) $\vec{v} \neq 0$

(b) $T(\vec{v}) = \lambda \vec{v}$ for some $\lambda \in \mathbb{F}$

This λ is called the <u>eigenvalue associated to $\vec{v}$</u>.

$\lambda \in \mathbb{F}$ is an <u>eigenvalue of T</u> if $\exists$ eigenvector $\vec{v}$ of T w/ eigenvalue λ

Situation

Do NOT assume V finite dimensional until we say so

$$T \in \mathcal{L}(V)$$

Proposition: Suppose $\lambda_1, \lambda_2, \ldots, \lambda_m$ are **distinct** eigenvalues of T and $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m \in V$ are corresponding are corresponding eigenvectors ($\Rightarrow$ nonzero), then the set $(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m)$ are lineary INDEPENDENT

Proof

Suppose $(\vec{v}_1, \ldots, \vec{v}_m)$ is LD.

$$a_1\vec{v}_1 + \cdots + a_m\vec{v}_m = 0$$

$$T(a_1\vec{v}_1 + \cdots + a_m\vec{v}_m) = T(0)$$

$$a_1\lambda_1\vec{v}_1 + \cdots + a_m\lambda_m v_m = 0$$

Let k be the smallest integer s.t. $\vec{v}_k \in \operatorname{span}(v_1, \ldots \vec{v}_{k-1})$

i.e $\vec{v}_k = c_1\vec{v}_1 + \cdots + c_{k-1}\vec{v}_{k-1}$

then $T(\vec{v}_k) = \lambda_1\vec{v}_k = c_1\lambda_1\vec{v}_1 + \cdots + c_{k-1}\lambda_{k-1}\vec{v}_{k-1}$

Multiply by λ_k, subtract

$\therefore \vec{0} = (c_1\lambda_1 - c_1\lambda_k)\vec{v}_1 + \cdots + c_{k-1}(\lambda_{k-1} - \lambda_k)\vec{v}_{k-1}$

$= c_1(\lambda_1 - \lambda_k)\vec{v}_1 + \cdots + c_{k-1}(\lambda_{k-1} - \lambda_k)\vec{v}_{k-1}$

LI

$\Rightarrow c_1(\lambda_1 - \lambda_k) = 0 \quad \cdots \quad c_{k-1}(\lambda_{k-1} - \lambda_k) = 0$

Not all c_i can be zero b/c then $\vec{v}_k = \vec{0}$ but it is an eigenvector so that can't be the case.

$\Rightarrow \lambda_i = \lambda_k \quad i \in [1, k-1]$

Contradiction, λ_i distinct.