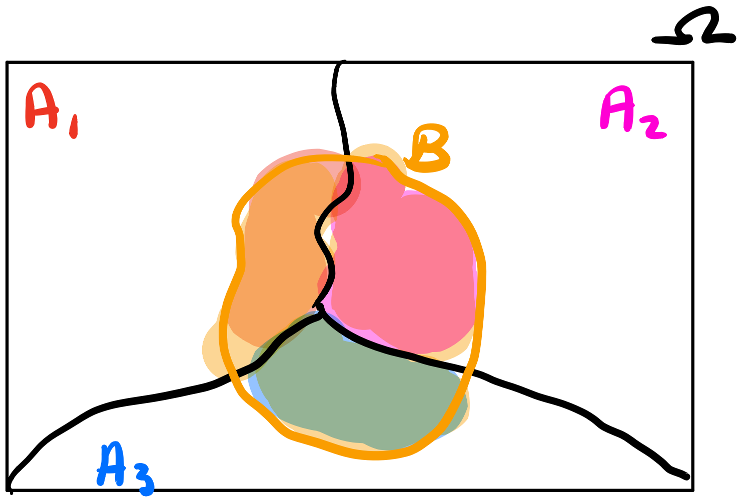
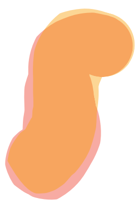
Recall: Conditional Probability

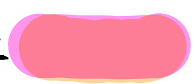
Total Probability Theorem: Given Ω, $\mathbb{P}$. if $A_1, A_2, \ldots, A_n$ are events that partition Ω, then for any event B,

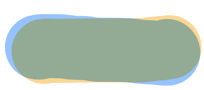
$$\mathbb{P}(B) = \mathbb{P}(B|A_1)\mathbb{P}(A_1) + \mathbb{P}(B|A_2)\mathbb{P}(A_2) + \cdots + \mathbb{P}(B|A_n)\mathbb{P}(A_n)$$

Picture when $n=3$



$\mathbb{P}(B|A_1)\mathbb{P}(A_1) =$ 

$\mathbb{P}(B|A_2)\mathbb{P}(A_2) =$ 

$\mathbb{P}(B|A_2)\mathbb{P}(A_2) =$ 

$\mathbb{P}(B)$ is sum these!

Next BIG idea: **Bayes Rule**

Given Ω, $\mathbb{P}$, if $A_1, \dots, A_n$ are events that partition Ω and have nonzero probability, then for any event B, and for any $k: 1 \le k \le n$,

$$\mathbb{P}(A_k \mid B) = \frac{\mathbb{P}(B \mid A_k)\,\mathbb{P}(A_k)}{\mathbb{P}(B \mid A_1)\,\mathbb{P}(A_1) + \cdots + \mathbb{P}(B \mid A_n)\,\mathbb{P}(A_n)}$$

"PROOF:" Numerator $= \mathbb{P}(B \cap A_k)$ by definition of conditional probability

Denominator $= \mathbb{P}(B)$ by total probability theorem

$$\frac{\mathbb{P}(A_k \cap B)}{\mathbb{P}(B)} = \mathbb{P}(A_k \mid B)\,!$$

Background: Think of B as an "effect" that can have exactly one of n possible "causes" – these are the $A_1, \dots, A_n$.

Know:

- prior probabilities of $A_1, \ldots, A_n$ $- \mathbb{P}(A_k)$, $1 \le k \le n$
- For each k, $\mathbb{P}(B|A_k)$ is the probability that the "effect" areses when "cause" k occurs.

So, for each k, we want the probability that "cause" k gave rise to "effect" B give "effect" B.

Example - Coins:

Two coins, one fair $\left(\mathbb{P}(\{H\}) = \mathbb{P}(\{T\}) = \frac{1}{2}\right)$
one unfair $\left(\mathbb{P}(\{H\}) = \frac{2}{3}, \mathbb{P}(\{T\}) = \frac{1}{3}\right)$

$U \to$ unfair
$F \to$ Fair

Pick a coin uniformally at random
i.e $\mathbb{P}(\{U\}) = \mathbb{P}(\{F\}) = 1/2$
and start flipping.

Let event B = Heads on first flip.

Intuition: $\mathbb{P}(U|B) > \frac{1}{2}$ and $\mathbb{P}(F|B) < \frac{1}{2}$

intuition was right!

By Bayes' Rule

$$\mathbb{P}(U|B) = \frac{\mathbb{P}(B|U)\,\mathbb{P}(U)}{\mathbb{P}(B|U)\,\mathbb{P}(U) + \mathbb{P}(B|F)\,\mathbb{P}(F)} = \frac{\frac{2}{3}\cdot\frac{1}{2}}{\frac{2}{3}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{2}} = \frac{4}{7}$$

Let event C = two heads in a row to start

$$\mathbb{P}(U \mid C) = \frac{\mathbb{P}(C \mid U)\,\mathbb{P}(U)}{\mathbb{P}(C \mid U)\,\mathbb{P}(U) + \mathbb{P}(C \mid F)\,\mathbb{P}(F)} = \frac{\left(\frac{2}{3}\right)^2 \cdot \frac{1}{2}}{\left(\frac{2}{3}\right)^2 \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{2}}$$

expect to be bigger than $4/7$

Note however, we can never be COMPLETELY SURE of what coin we are flipping.

Example - Aircraft (alien detection) revisited:

A = spacecraft-present

B = see a blip on radar

Given: $\mathbb{P}(B \mid A) = 0.99 \Rightarrow \mathbb{P}(B^c \mid A) = 1 - 0.99 = 0.01$

$\mathbb{P}(B \mid A^c) = 0.10$

$\mathbb{P}(A) = 0.05$

Want $\mathbb{P}(A \mid B)$

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(B \mid A)\,\mathbb{P}(A)}{\mathbb{P}(B \mid A)\,\mathbb{P}(A) + \mathbb{P}(B \mid A^c)\,\mathbb{P}(A^c)} \simeq 0.3426$$

Example – Medical Testing:

Some disease out there.

Think of population as Ω

Let

A = event person from some given population has a disease.

We have a test (an imperfect test) for the disease.

Let

B = event person drawn from population tests positive.

Question: Given a positive test result, what is the likelihood the person has the disease?

what does this mean?

① If you have the disease, 95% of the time it shows positive

② If you don't have the disease, 95% of the time it shows negative

Given Info:

Test is "95% accurate"

$\Rightarrow \mathbb{P}(B \mid A) = 0.95$

$\mathbb{P}(B \mid A^c) = 0.05$

Assume $\mathbb{P}(A) = 0.001$

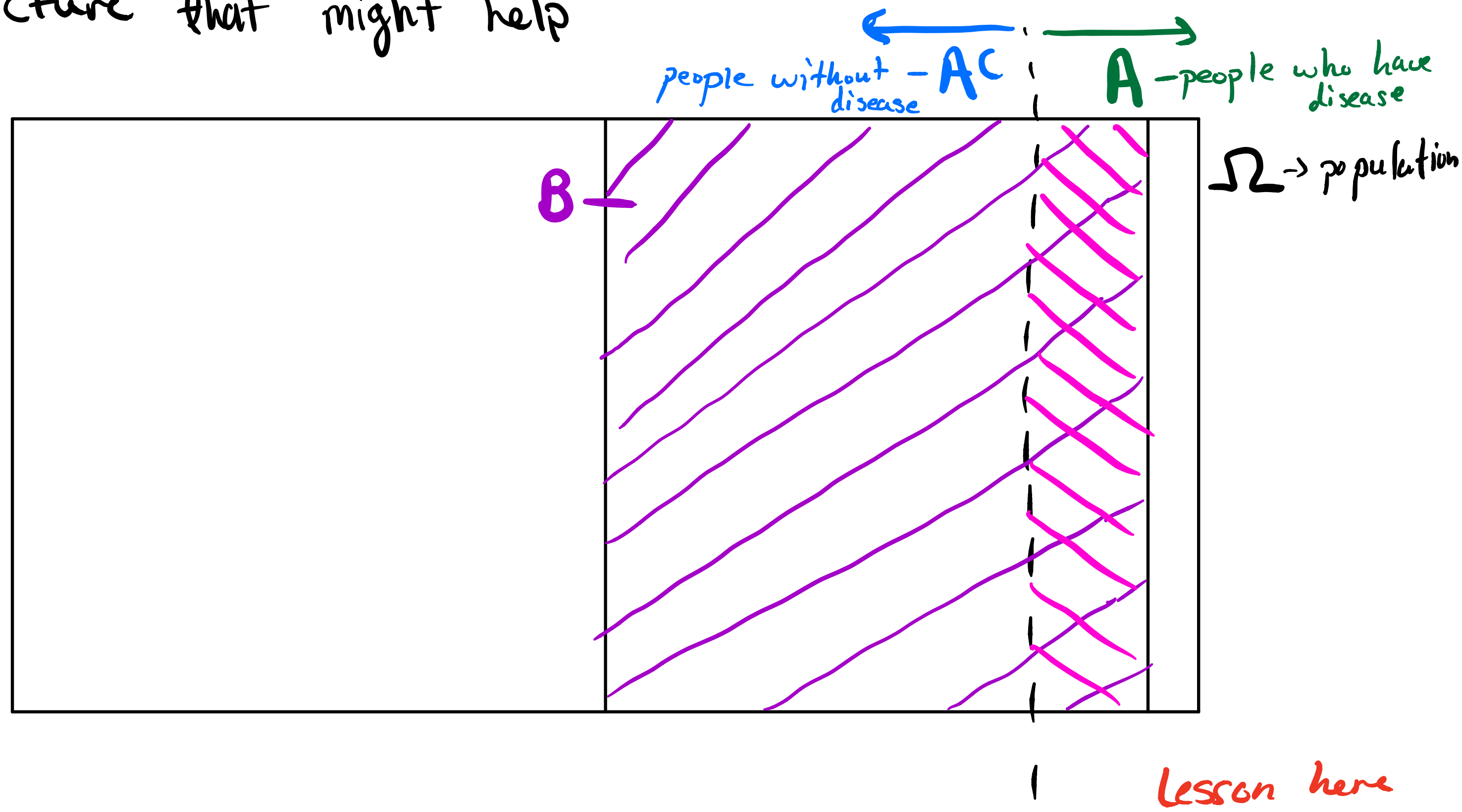
disease is rare in population

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(B \mid A)\,\mathbb{P}(A)}{\mathbb{P}(B \mid A)\,\mathbb{P}(A) + \mathbb{P}(B \mid A^c)\,\mathbb{P}(A^c)} = \frac{(0.95)(0.001)}{(0.95)(0.001) + (0.05)(0.999)}$$

$$\mathbb{P}(A \mid B) = 0.0189 \simeq 0.19\,\%$$

even though test was very accurate, bc we knew rarity of disease the likelihood of person having disease is small.

Picture that might help



$$\mathbb{P}(B \mid A) = \frac{2}{3}$$

$$\mathbb{P}(B \mid A^C) = \frac{1}{3}$$

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(\quad)}{\mathbb{P}(\quad)} \ll \frac{1}{2}$$

Lesson here is to be careful with conditional probability!

Intuition can lead you astray!!

Comment: In chess example, could use Bayes to compute, eg $\mathbb{P}(\text{played a master} \mid \text{won the game})$ — WON'T DO IN CLASS

Next Circle of Ideas: Independence

Given, Ω, $\mathbb{P}$. Say **two events**, $A, B \subset \Omega$ are **independent** when $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$

Same as $\mathbb{P}(A|B) = \mathbb{P}(A)$ when $\mathbb{P}(B) > 0$ ← Knowing B occurred says nothing more about whether A occurred

$\mathbb{P}(B|A) = \mathbb{P}(B)$ when $\mathbb{P}(A) > 0$

Knowing that A occurred says nothing more about whether B occurred.

CAUTION: Independence is "about" not only the events and how they sit in Ω, but also (more crucially) about $\mathbb{P}$!!

Common error: if "$A \cap B = \emptyset$, then A and B are independent"

WRONG whenever $\mathbb{P}(A) > 0$ and $\mathbb{P}(B) > 0$

Example - Dice:

Have two standard six-sided dice. Assume every outcome is equally likely. i.e each has prob $\frac{1}{36}$

pairs of rolls (rolls of first die, roll of second die)

Consider some events

$A =$ first die shows 3 $\qquad B =$ second die shows 5

$$A = \{(3,1),(3,2),(3,3),(3,4),(3,5),(3,6)\} \Rightarrow \mathbb{P}(A) = 6 \cdot \frac{1}{36} = \frac{1}{6}$$

$$B = \{(1,5),(2,5),(3,5),(4,5),(5,5),(6,5)\} \Rightarrow \mathbb{P}(B) = 6 \cdot \frac{1}{36} = \frac{1}{6}$$

$$\mathbb{P}(A \cap B) = \frac{1}{36} = \mathbb{P}(A)\,\mathbb{P}(B)$$

The two events are therefore independent

Let

$A =$ same as above A. $\qquad C = 2$ dice are different.

$$A = \{(3,1),(3,2),(3,3),(3,4),(3,5),(3,6)\} \Rightarrow \mathbb{P}(A) = 6 \cdot \frac{1}{36} = \frac{1}{6}$$

C is the same as $(\text{rolling doubles})^c$

$$\mathbb{P}(C) = \frac{30}{36} = \frac{5}{6}$$

Therefore, A, C are independent!

$$\mathbb{P}(A \cap C) = \frac{5}{36} = \left(\frac{1}{6}\right) \cdot \left(\frac{5}{6}\right) = \mathbb{P}(A)\,\mathbb{P}(C)$$

More generally, say $A_1, A_2, \ldots, A_n$ are independent when ?

2 that DO **NOT** work

① $\mathbb{P}(A_1 \cap A_2 \cap \cdots \cap A_n) = \mathbb{P}(A_1)\mathbb{P}(A_2)\cdots\mathbb{P}(A_n)$

② $\mathbb{P}(A_k \cap A_l) = \mathbb{P}(A_k)\mathbb{P}(A_l)$ for all $1 \le k, l \le n$, $k \neq l$

What works? Find out next lecture on Dragon Ball Z