

Last Time

Linear Maps

We had a Theorem: $T: V \rightarrow W$ a linear map is uniquely determined by where it takes a basis of V .

Example: $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$\begin{aligned} T\begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \begin{pmatrix} 4 \\ 3 \end{pmatrix} \\ T\begin{pmatrix} 0 \\ 1 \end{pmatrix} &= \begin{pmatrix} 3 \\ 4 \end{pmatrix} \end{aligned} \Rightarrow T(x, y) = \begin{pmatrix} 4x + 3y \\ 3x + 4y \end{pmatrix}$$

$\swarrow \quad \searrow$
 $xT\begin{pmatrix} 1 \\ 0 \end{pmatrix} + yT\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

But what about $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 + 3 \\ y^2 + 3 \end{pmatrix} \rightarrow \begin{matrix} f\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \\ f\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \end{matrix}$ NOT a Linear Map

Last Time: $\mathcal{L}(V, W) \leftarrow$ all linear maps from V to W
forms a vector space over \mathbb{F}

Form $\mathcal{L}(V, W)$ as it helps us understand linear transformations better.

What's 0 in $\mathcal{L}(V, W)$? $0: V \rightarrow W \Rightarrow S + 0 = 0 + S$
 $S: V \rightarrow W \quad 0\begin{pmatrix} \cdot \\ \cdot \end{pmatrix} = 0 \quad \forall \begin{pmatrix} \cdot \\ \cdot \end{pmatrix} \in V$

Check that if $S, T \in \mathcal{L}(V, W)$, $\lambda \in \mathbb{F}$, such that

$$(S + \lambda T)(v_1 + \alpha \vec{v}_2) \stackrel{??}{=} (S + \lambda T)(\vec{v}_1) + \alpha (S + \lambda T)(\vec{v}_2)$$

Special Case

$$S, T \in \mathcal{L}(V, W)$$

$$\begin{array}{c} V \xrightarrow{S} V \xrightarrow{T} V \\ \underbrace{\hspace{1.5cm}}_{T \circ S} \end{array}$$

$$S \circ T \stackrel{?}{=} T \circ S$$

$$\begin{array}{c} V \xrightarrow{T} V \xrightarrow{S} V \\ \underbrace{\hspace{1.5cm}}_{S \circ T} \end{array}$$

Example: $P: \mathbb{R}^3 \rightarrow \mathbb{R}^3$

$$P\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

← a projection

Definition: A Linear Transformation $P: V \rightarrow V$ is called a projection if $P^2 = P$

Fact: P a projection $\Rightarrow I - P$ a projection

$$\begin{aligned} (I - P)^2 &= I^2 - 2P + P^2 \leftarrow \begin{array}{l} P \text{ a projection} \\ \Rightarrow P^2 = P \end{array} \\ &= I^2 - 2P + P \leftarrow I^2 = I \\ &= I - P \end{aligned}$$

What is $I-P$ in our example?

"does the perpendicular thing"

$$(I-P) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} z \\ y \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix}$$

✓ projection to z-axis

$P: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ (same P as above)

P takes z -axis to $\vec{0} \in \mathbb{R}^3$

P takes xy -plane to itself

Definition: The null-space (kernel) of $T: V \rightarrow W$ is the set

$$\{ \vec{v} \in V \mid T(\vec{v}) = \vec{0} \} \subseteq V$$

Squished to 0

The image (=range) of T is

$$\{ \vec{w} \in W \mid \exists \vec{v} \in V \text{ with } T(\vec{v}) = \vec{w} \} \subseteq W$$

NOT proved in class

i.e. stuff in W actually hit by T

Stuff not squished to 0

Question: $T: \mathbb{R}^{50} \rightarrow W$

what happens to 50 dims of stuff we start with?

Answer: Squished to $\vec{0}$ OR NOT

Theorem: Let $T: V \rightarrow W$ be a linear transformation.

Then

$$\dim(V) = \dim \text{Null space}(T) + \dim \text{Image}(T)$$

Proof: Let $\{\vec{u}_1, \dots, \vec{u}_r\}$ be a basis of $\text{NS}(T)$.

Let $\{\vec{w}_1, \dots, \vec{w}_k\}$ be a basis of $\text{Image}(T)$.

$\exists \vec{v}_1, \dots, \vec{v}_k \in V$ such that $T(\vec{v}_i) = \vec{w}_i$ — \vec{v}_i NOT unique

'Hope' $B = \{\vec{v}_1, \dots, \vec{v}_k, \vec{u}_1, \dots, \vec{u}_r\}$ is a basis for V

B is independent would mean

$$\sum_{i=1}^k \alpha_i \vec{v}_i + \sum_{j=1}^r \beta_j \vec{u}_j = \vec{0}$$

Apply T

$$\sum_{i=1}^k \alpha_i \vec{w}_i + \vec{0} = \vec{0}$$

Basis \Rightarrow all $\alpha_i = 0$

All $\alpha_i = 0$ and basis $\Rightarrow \beta_j = 0$

Let $\vec{v} \in V$. We need to write \vec{v} as a linear combination of B .

$$T(\vec{v}) = \sum_{i=1}^k \gamma_i \vec{w}_i = \sum_{i=1}^k \gamma_i T(\vec{v}_i) = T\left(\sum_{i=1}^k \gamma_i \vec{v}_i\right)$$

in $\text{NS}(T)$ $\Rightarrow T\left(\vec{v} - \sum_{i=1}^k \gamma_i \vec{v}_i\right) = \vec{0}$

$$\Rightarrow \vec{v} - \sum_{i=1}^k \gamma_i \vec{v}_i = \sum_{j=1}^r \beta_j \vec{u}_j$$

$$\Rightarrow \vec{v} = \sum_{i=1}^k \gamma_i \vec{v}_i + \sum_{j=1}^r \beta_j \vec{u}_j$$

Q.E.D

Spanning