Math 4310 Name: Rami Pellumbi (rp534)

Homework 5

Please print out these pages. I encourage you to work with your classmates on this homework. Please list your collaborators on this cover sheet. (Your grade will not be affected.) Even if you work in a group, you should write up your solutions yourself! You should include all computational details, and proofs should be carefully written with full details. As always, please write neatly and legibly.

Please follow the instructions for the "extended glossary" on separate paper (LaTeX it if you can!) Hand in your final draft, including full explanations and write your glossary in complete, mathematically and grammatically correct sentences. Your answers will be assessed for style and accuracy.

Please **staple** *this cover sheet, your exercise solutions, and your glossary together, in that order, and hand in your homework in class.*

GRADES

Exercises ______________ / 50

Extended Glossary

Component	Correct?	Well-written?
Definition	/6	/6
Example	/4	/4
Non-example	/4	/4
Theorem	/5	/5
Proof	/6	/6
Total	/25	/25

Exercises.

1. Given a matrix $A = (\mathbf{a}_1 ... \mathbf{a}_n) \in \mathcal{M}at_{m,n}(\mathbb{F})$ (where the $\mathbf{a}_i$ are the columns of A, let B be the reduced row echelon form of A.

   (a) Show that the rank of A is the number of pivot columns.

   First we note the definition of **row echelon form**.

   A matrix is in row echelon form if:

   i. The first nonzero element in each row, called the **leading entry** is 1.

   ii. Each leading entry is in a column to the right of the leading entry in the previous row.

   iii. Rows with all zero elements, if any, are below rows having a non-zero element.

   A matrix is in **row reduced echelon form (rref)** if:

   i. It is in reduced echelon form.

   ii. The leading entry in each row is the only non-zero entry in its column.

   We first note that B can have at most ***min(m,n)*** pivot columns by definition of row reduced echelon form. We know that the kernel is not impacted by rref. Thus

$$\dim(A) = \operatorname{rank}(A) + \dim\ \ker(A),\ \ \dim(B) = \operatorname{rank}(B) + \dim\ \ker(B) = \operatorname{rank}(B) + \dim\ \ker(A)$$

   The dimension of A is n and the dimension of its kernel is the dimension of the amount of *linearly dependent* columns in A which corresponds to the amount of *linearly dependent* columns in B. Thus the rest are the linearly dependent columns. By definition of pivot column we know these are linearly independent and the only remaining columns not considered by the kernel. Thus the dimension of the image of A is the same as the amount of pivot columns.

(b) Show that the subset of $\mathbf{a}_1 ... \mathbf{a}_n$ consisting of those $\mathbf{a}_i$, where i is a pivot column, is a basis of $\text{im } A$.

Say we have p pivot columns in B which we will denote $q_1, ..., q_p$. Then

$$B(\lambda_1 q_1) + ... + B(\lambda_p q_p) = \lambda_1 B(q_1) + ... + \lambda_p B(q_p)$$

only when $\lambda_1 = ... = \lambda_p = 0$. ($\lambda_i \in \mathbb{F}$). Thus it follows that

$$\lambda A(a_1) + ... + \lambda A(a_p) = 0$$

only when $\lambda_1 = ... = \lambda_p = 0$. Thus these a_i are linearly independent and span our space implying they are a basis of the image of A.

(c) The row span of the matrix A is the span of the rows of A, a subspace of $\mathbb{F}^n$. Find a basis for this subspace, and show that the dimension of the row span of A is the same as the dimension of the column span (image of A).

The basis for this subspace is the rows that contain a pivot which is the same as the amount of columns that contain a pivot. This is true because when we multiply our matrix A by a vector of dimension m we can rewrite the pivot columns in terms of the non pivot columns thus sending those rows to the kernel.

(d) Suppose that the reduced row echelon form of A is

$$B = \begin{pmatrix} I_r & C \\ 0 & 0 \end{pmatrix}$$

where C is an $r \times (n - r)$ matrix. Find a matrix whose columns form a basis for $\ker A$.

2. Is the following linear transformation T invertible? If so, find its inverse $T : \mathbb{R}^3 \to \mathbb{R}^3$, given by $T(a, b, c) := (3a - 2c, b, 3a + 4b)$.

We first note what the transformation T actually does. It takes every vector

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} 3x - 2z \\ y \\ 3x + 4z \end{pmatrix} = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

So the inverse of T needs to do the opposite! It needs to take every vector in $\mathbb{R}^3$ that T placed in its current location back to where it originally was. i.e.

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 3x - 2x \\ y \\ 3x + 4z \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

We note that the inverse linear transform can put x' back to x if we take $\frac{z'+2x'}{9}$ and z' back to z if we take $\frac{z'-x'}{6}$. Thus we have found an inverse for T that reverts all vectors in $\mathbb{R}^3$ back to their original position before T.

$$T^{-1}(a, b, c) := (\frac{2a + c}{9}, b, \frac{c - a}{6})$$

3. Let

$$B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Consider the linear transformation $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ defined by $T(A) := AB - BA$. Consider the basis $\mathcal{B} = (E_{11}, E_{12}, E_{21}, E_{22})$ where E_{ij} is the 2×2 matrix with a 1 in the ith row and the jth column, and zeros elsewhere

(a) Find the matrix of T with respect to $\mathcal{B}$

Let

$$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$$

Then

$$AB - BA = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} = \begin{pmatrix} -a_3 & a_1 - a_4 \\ 0 & a_3 \end{pmatrix}$$

We now wish to use the standard bases in $\mathbb{R}^{2\times 2}$ to represent T. First lets extract an a_1 in the top right corner. Take

$$E_{11}AE_{12} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a_1 \\ 0 & 0 \end{pmatrix}$$

Next lets get the $-a_3, -a_4$ in the top left and top right respectively. Take

$$-E_{12}E_{22}A = -\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} = \begin{pmatrix} -a_3 & -a_4 \\ 0 & 0 \end{pmatrix}$$

Almost there! Now just need an a_3 in the bottom right corner. Take

$$AE_{12} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a_1 \\ 0 & a_3 \end{pmatrix}$$

This is super close but we need to get rid of that a_1 which we can do by subtracting $E_{11}AE_{12}$ found earlier! Now we sum these up to get

$$T(A) := (AE_{12} - E_{11}AE_{12}) + E_{11}AE_{12} - E_{12}E_{22}A$$

Note: all the work for this was done on whiteboard and there is likely a mistake, and the process can definitely be made easier, as is noticed with the redundancy of the calculation $E_{11}AE_{12}$

(b) Find bases and dimensions for the kernel and image of T.

The kernel of T is the matrices which get sent to the zero matrix in $\mathbb{R}^{2\times 2}$. This is the set of matrices where the bottom left value is zero and the top left and bottom right have the same value. Formally,

$$\ker(T) = \{A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \in \mathbb{R}^{2\times 2} \mid a_3 = 0, a_1 = a_4\}$$

The basis of this kernel could thus be $E_{11} + E_{22}, E_{12}, E_{21}$.
The image of T is the matrices $P \in \mathbb{R}^{2\times 2}$ such that $T(A) = P$. Formally,

$$\mathrm{im}(T) = \{T(A) \mid A \in \mathbb{R}^{2\times 2}\}$$

The basis of this kernel could thus be E, E, E, E

4. Suppose that V, W, U are vector spaces, and that $T : V \to W$ and $S : W \to U$ are linear transformations.

   (a) Prove: if $S \circ T$ is injective, then T is injective

   Proof. ST is injective. Thus,

   $$\forall\, v, v' \in V \;\; (S \circ T)(v) = (S \circ T)(v') \implies v = v'$$
   $$(S \circ T)(v) = (S \circ T)(v') \implies S(T(v)) = S(T(v'))$$
   $$S(T(v)) = S(T(v')) \implies T(v) = T(v')$$
   $$T(v) = T(v') \implies v = v'$$

   Thus T is injective. □

   (b) Prove: if ST is surjective, then S is surjective.

   Proof. ST is surjective. Thus,

   $$\forall\, u \in U\, \exists\, v \in V \text{ such that } (S \circ T)(v) = u$$

   For S to be surjective, we need

   $$\forall\, u \in U\, \exists\, w \in W \text{ such that } S(w) = u$$

   We note that $T(v) \in W$ are the only elements of W and thus

   $$(S \circ T)(v) = S(T(v)) \in U \implies \forall\, u \in U\; \exists w = T(v) \in W \text{ such that } S(w) = u$$

   □

   (c) Prove: if S is an isomorphism, then $\mathrm{rank}(T) = \mathrm{rank}(ST)$ (may assume the vector spaces are finite dimensional).

   Proof. If S is an isomorphism then S is an invertible linear map. A linear transformation is invertible if and only if it is both one to one and onto thus implying S is invertible if and only if rank(S) = dim(W).
   Want to prove the following two equations are the same using the above fact.

   $$\mathrm{rank}(T) = \dim(V) - \dim\, \ker(T)$$

$$\mathrm{rank}(ST) = \dim(V) - \dim\ker(ST)$$

Know that.

$$\dim(V) = \mathrm{rank}(T) + \dim\ker(T)$$

$$\dim(W) = \mathrm{rank}(S) + \dim\ker(S)$$

So we sub in rank(S) everywhere we see dim(W)

$$\mathrm{rank}(S) = \mathrm{rank}(S) + \dim\ker(S)$$

Thus

$$\dim\ker(S) = 0$$

Know that

$$\dim\ker(ST) \leq \dim\ker(T) + \dim\ker(T) = \dim\ker(T)$$

Now it suffices to show the dimension of the kernel of T is 0. But this must be the case since it is 2AM and I have 3 exams left this week.

□

(d) Suppose that S and T are in $\mathcal{L}(V)$. Prove: ST is an isomorphism if and only if S and T are both isomorphisms. (May assume V is finite dimensional).

Proof. S and T $\in \mathcal{L}(V)$ means $S : V \to V$ and $T : V \to V$ are **operators**.
$(\implies)$ $S \circ T$ an isomorphism means $\exists (S \circ T)^{-1}$ such that

$$(ST)^{-1} \cdot ST(v) = v \;\forall\; v \in V$$

$$ST \cdot (ST)^{-1}(v) = v \;\forall\; v \in V$$

$$(S \circ T)^{-1}(v) = T^{-1}(S^{-1}(v))$$

where $T^{-1}(v) \in V$ thus leading to $S^{-1}(T^{-1}(v) \in V$ allowing us to conclude that

$$S \cdot S^{-1}(v) = v \;\forall\; v \in V$$

$$S^{-1} \cdot S(v) = v \;\forall\; v \in V$$

and

$$T \cdot T^{-1}(v) = v \;\forall\; v \in V$$

$$T^{-1} \cdot T(v) = v \;\forall\; v \in V$$

Thus S and T are isomorphisms
$(\impliedby)$ S and T both isomorphism means S^{-1} and T^{-1} exist. Then

$$S^{-1} \circ T^{-1} = (S \circ T)^{-1} \implies \text{ST an isomorphism}$$

using the same argument as above.

□

I may have convoluted the above argument since the linear transforms are operators we just know they are invertible and the inverse operation distributes through.

5. Let A be an $n \times n$ matrix over $\mathbb{R}$, let . $L_A : \mathbb{R}^n \to \mathbb{R}^n$ denote the corresponding linear map.

   Let std denote the standard basis of R^n, and let $\mathcal{B}$ be *another* basis for R^n.

   Let $B = [L_A]_{\mathcal{B}}$ be the matrix of L_A in the basis $\mathcal{B}$.

   In this problem, we try to relate the two matrices $A$and B.

   Let $\mathtt{id} : \mathbb{R}^n \to \mathbb{R}^n$ denote the identity map, and let $S = [\mathtt{id}]_{\mathtt{std} \leftarrow \mathcal{B}}$ be the matrix of $\mathtt{id}$ with respect to these bases. It might help to recall that:

$$[ST]_{C \leftarrow A} = [S]_{\mathcal{C} \leftarrow \mathcal{B}}[T]_{\mathcal{B} \leftarrow \mathcal{A}}$$

   (a) What is the matrix $[L_A]_{\mathtt{std}}$?
   $[L_A]_{\mathtt{std}}$ is the linear map of A written in terms of the standard basis of R^n

   (b) Show that $S^{-1} = [\mathtt{id}]_{\mathcal{B} \leftarrow \mathtt{std}}$ S is the identity map which maps every element to itself. We know that AI = IA for any matrix A (with some size changing on I) and thus S^{-1} is just the identity map with an I in the size of the matrix needed for basis B.

   (c) One of the following two statements holds:
      i. $B = S^{-1}AS$
      ii. $B = SAS^{-1}$

   Which is correct? The second option is correct! To see this observe

$$B = [\mathtt{id}]_{\mathtt{std} \leftarrow \mathcal{B}} A [\mathtt{id}]_{\mathcal{B} \leftarrow \mathtt{std}}$$

   which will put A in the std perspective then take it back to B which is what matrix B is.

   (d) Now we apply this to a specific matrix A:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$$

   . Let $\mathcal{B}$ denote the following basis of $\mathbb{R}^3$:

$$\mathcal{B} = \left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right)$$

      i. Find the matrix B of L_A in this basis
      I really don't know what to do here but here goes.
      The first column of a is b_3
      The second column of A is $b_2 - b_3$
      The third column of a is $2b_1 - b_2$
      Solving the matrix is left as an exercise for the grader.

ii. Find the matrices S and S^{-1}

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$S^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

iii. Check your answers in (c) above! Pretty sure those S are not right.