Integrals Involving Multiple Valued Functions

1)

$$I = \int_0^\infty \frac{x^{\alpha-1}}{1+x}\,dx, \qquad 0<\alpha<1$$

Color protrait shows branch cut on positive real axis

How to define $z^{\alpha-1}$, $0<\alpha<1$

- "Just" Above $x=0$,

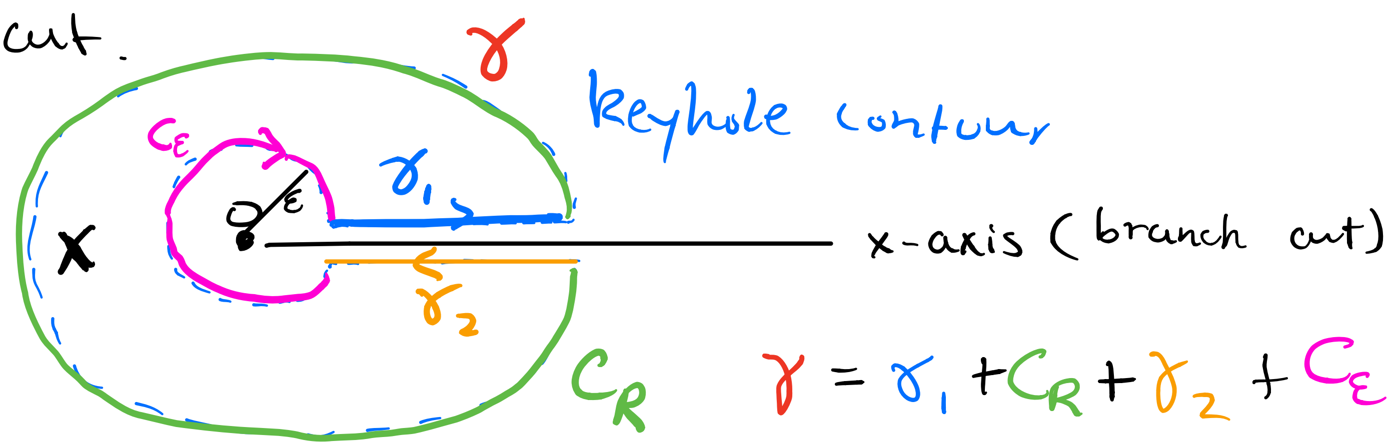
$$z^{\alpha-1} = (re^{i\theta})^{\alpha-1} = r^{\alpha-1}e^{i(\alpha-1)\theta} \; ; \; r>0,\ \theta=0$$
$$= r^{\alpha-1}$$

- "just" Below $x=0$

$$z^{\alpha-1} = (re^{i\theta})^{\alpha-1} = r^{\alpha-1}e^{i(\alpha-1)\theta} \; ; \; r>0,\ \theta=2\pi$$
$$= r^{\alpha-1}e^{i\alpha 2\pi}e^{-i2\pi}$$
$$= r^{\alpha-1}e^{i\alpha 2\pi}$$

When $\alpha = 1/2$ we have an argument change of $e^{i\pi}$

Need a contour that doesn't lie on our branch cut.



$$\gamma = \gamma_1 + C_R + \gamma_2 + C_\varepsilon$$

$$\int_{\gamma} f(z)\,dz = 2\pi i \operatorname{Res}(f(z); -1)$$

$$= 2\pi i \lim_{z \to -1} \left(z^{\alpha-1}\right)$$

$$= 2\pi i \,(-1)^{\alpha-1} \qquad e^{-i\pi} = \cos(\pi) = \cancel{\cos}(\pi) \overset{-1}{}$$

$$= 2\pi i \left(e^{i\pi}\right)^{\alpha-1}$$

$$= 2\pi i \; e^{i\pi\alpha} e^{i\pi}$$

$$= 2\pi i\left(-e^{i\pi\alpha}\right)$$

$$\int_{\gamma} f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{C_R} f(z)\,dz + \int_{\gamma_2} f(z)\,dz + \int_{C_\varepsilon} f(z)\,dz$$

$$\int_{\gamma_1} f(z)\,dz = \lim_{\substack{\varepsilon \to 0 \\ R \to \infty}} \int_{\varepsilon}^{R} \frac{x^{\alpha-1}}{x+1}\,dx = I \Bigg\} \gamma_1 \text{ is "just above" case}$$

$$\int_{\gamma_2} f(z)\,dz = \lim_{\substack{\varepsilon \to 0 \\ R \to \infty}} \int_{R}^{\varepsilon} \frac{x^{\alpha-1} e^{2\pi i(\alpha)}}{x+1} = -I e^{i2\pi\alpha} \Bigg\} \gamma_2 \text{ is "just below" case}$$

We expect the integral on C_R to be negligible.

technically less then but equal as $\varepsilon \to 0$

$$\left| \int_{C_R} f(z)\,dz \right| \le 2\pi R \max_{z \in C_R} \left| \frac{z^{\alpha-1}}{z+1} \right|$$

via restriction on α this is about

$$\max_{z \in C_R} \left| \frac{z^{\alpha}}{z^2} \right|$$ which $\to 0$ as $R \to \infty$

$$\left| \int_{C_\varepsilon} f(z)\,dz \right| \le \max_{z \in C_\varepsilon} \left| \frac{z^{\alpha-1}}{z+1} \right| 2\pi\varepsilon$$

$$\le \max_{z \in C_\varepsilon} \left| \frac{z^{\alpha-1}}{1} \right| 2\pi\varepsilon$$

$$\le \frac{\varepsilon^{\alpha}}{\varepsilon} 2\pi\varepsilon \le \varepsilon^{\alpha} 2\pi \to 0$$ as $\varepsilon \to 0$!

So,

$$\int_{\gamma} f(z)\,dz = \cancelto{I}{\int_{\gamma_1} f(z)\,dz} + \cancelto{0}{\int_{C_R} f(z)\,dz} + \cancelto{-e^{i2\pi\alpha} I}{\int_{\gamma_2} f(z)\,dz} + \cancelto{0}{\int_{C_\varepsilon} f(z)\,dz}$$

$$2\pi i \operatorname{Res}(f(z); -1) = I(1 - e^{i2\pi\alpha})$$

$$I = \frac{2\pi i(-e^{i\pi\alpha})}{1 - e^{i2\pi\alpha}} = \int_0^{\infty} \frac{x^{\alpha-1}}{1+x}\,dx$$

I is real! We had a real integral, so it should be real!

Massage it to get

$$\frac{2\pi i}{-e^{-i\pi\alpha}+e^{i\pi\alpha}} = \frac{\pi}{\sin(\pi\alpha)}$$

So,

$$I = \int_0^\infty \frac{x^{\alpha-1}}{1+x}\,dx = \frac{\pi}{\sin(\pi\alpha)}$$

2)

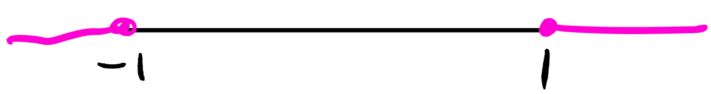
$$I = \int_{-1}^{1} \frac{\sqrt{1-x^2}}{\sqrt{1+x^2}}\, dx$$

$$= \int_{-1}^{1} \frac{\sqrt{1-x}\,\sqrt{1+x}}{1+x^2}\, dx$$

Want your branch cut on interval of integration.
Right now,

$$f(z) = \frac{\sqrt{1-z}\,\sqrt{1+z}}{1+z^2}$$

Branch cuts



the cuts don't lie on our interval of integration.

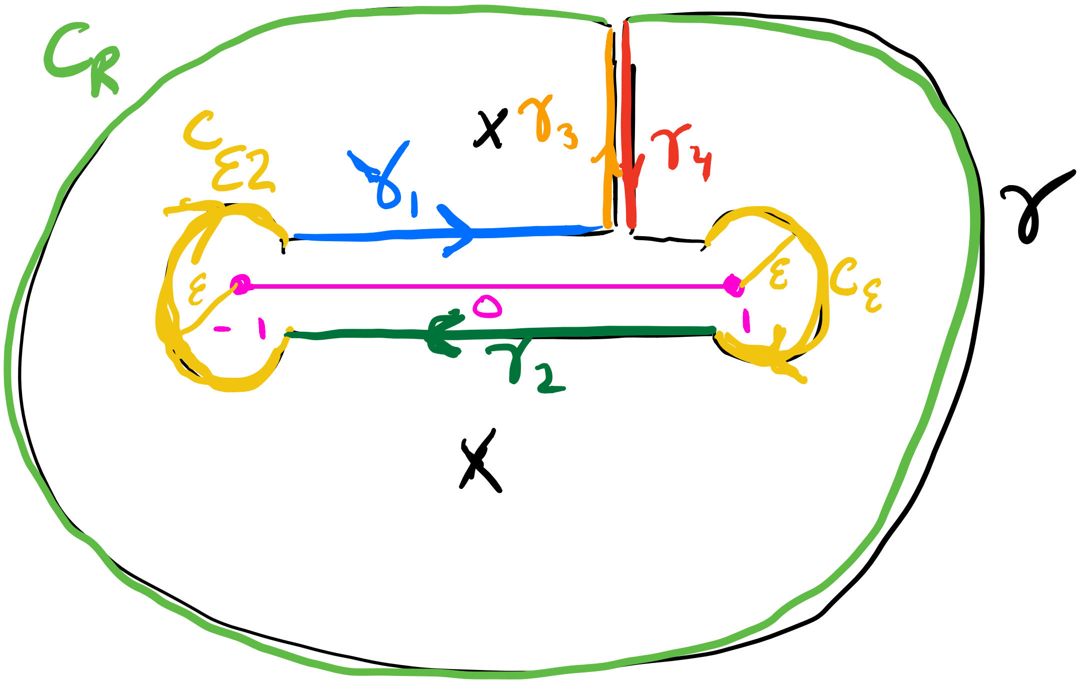
Rewrite the function so it does.

$$f(z) = i\,\frac{\sqrt{1-z}\,\sqrt{-1-z}}{1+z^2}$$

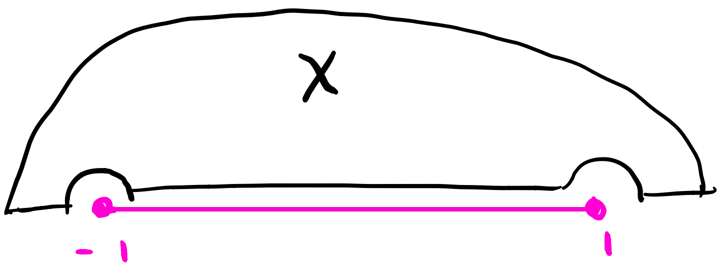
Now

branch cut

What's our contour? Need to encapsulate poles as well as avoid branch cut.



OR



A lot simpler

γ_1	γ_2	$C_{\varepsilon 2}$	C_ε	C_R	$\underbrace{\gamma_3+\gamma_4}$
I	$-e^{i\pi} I$	$\to 0$	$\to 0$	2π	0

-1

$$I(1-\cancel{e^{i\pi}}) = 2\pi$$

$$I = \pi$$