Math 4310

Homework 1

Due 9/9/19

Name: ____________________

Collaborators: ____________________

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Please print out these pages. I encourage you to work with your classmates on this homework. Please list your collaborators on this cover sheet. (Your grade will not be affected.) Even if you work in a group, you should write up your solutions yourself! You should include all computational details, and proofs should be carefully written with full details. As always, please write neatly and legibly.

Please follow the instructions for the "extended glossary" on separate paper (LaTeX it if you can!) Hand in your final draft, including full explanations and write your glossary in complete, mathematically and grammatically correct sentences. Your answers will be assessed for style and accuracy.

Please **staple** *this cover sheet, your exercise solutions, and your glossary together, in that order, and hand in your homework in class.*

GRADES

Exercises ____________________ / 50

Extended Glossary

Component	Correct?	Well-written?
Definition	/6	/6
Example	/4	/4
Non-example	/4	/4
Theorem	/5	/5
Proof	/6	/6
Total	/25	/25

Exercises.

1. Let $\mathbb{F}$ be the field $\mathbb{F} = \mathbb{F}_5 = \mathbb{Z}_5$.

   (a) Find the multiplicative inverse of the elements $1, 2, 3, 4$ of $\mathbb{F}$.

   (b) Compute the following in $\mathbb{F}^3$:

   $$\text{(a)} \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \quad \text{(b)} \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} + \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix} \quad \text{(c) } 4 \cdot \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} \quad \text{(d) } 3 \cdot \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$$

   (c) Find a nonzero vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{F}^3$ such that

   $$a \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} + b \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix} + c \begin{pmatrix} 3 \\ 4 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

   (Remember that in this entire problem, $\mathbb{F} = \mathbb{Z}_5$! You might want to mimic row reduction that you have done in an earlier linear algebra class).

2. Let $\mathbb{R}^{n \times n}$ denote the $n \times n$ matrices with coefficients in $\mathbb{R}$, for $n \geq 1$. This set has two natural operations, matrix addition $+$ and matrix multiplication $\cdot$. For which n is $\mathbb{R}^{n \times n}$ with these operations a field? (Remember to justify your answers! If it is a field, prove it, if not, give a reason).

3. Let $\mathbb{F}$ be a field. The **characteristic** of $\mathbb{F}$ is defined to be the smallest positive integer p such that $1 + 1 + \cdots + 1 = 0$, where there are p 1's in this formula. If no such sum is 0, then we say the characteristic of $\mathbb{F}$ is 0.
   (a) Find the characteristics of the fields $\mathbb{R}, \mathbb{C}, \mathbb{Z}_p$.
   (b) If $\mathbb{F}$ is a finite field, show that the characteristic p of $\mathbb{F}$ is not zero.
   (c) If $\mathbb{F}$ is a finite field, show that the characteristic p of $\mathbb{F}$ is a prime number.
4. In this problem we will investigate fields with 4 elements.
   (a) If $\mathbb{F}_4$ is a field with exactly 4 elements, what must the characteristic be? (justify your answer, of course! But you may use without proof statements from the previous problem).
   (b) Find a field $\mathbb{F}_4$ that has 4 elements. Write down the addition and multiplication tables of this field. Remember that two of your elements are 0 and 1!
   (c) Find all fields with 4 elements (i.e. write down all possible addition and multiplication tables. Your first two elements should be 0 and 1).
5. Show that $\mathbb{C}$ is a vector space over the field $\mathbb{R}$. More generally, if $\mathbb{F} \subset \mathbb{K}$ are both fields (with addition, multiplication, 0, 1, in $\mathbb{F}$ induced from the same operations/elements on $\mathbb{K}$), is $\mathbb{K}$ a vector space over $\mathbb{F}$? (for this one case, you should either provide a counter-example or a one or two line reason, no proof is required this time).

Extended Glossary. There is no extended glossary this week.

You may work in groups, but please write up your solutions **yourself**. If you do work together, your group should come up with at least two examples, two non-examples, and two theorems. Each one (example/non-example/theorem) should be included in some group member's extended glossary. Your solutions should be written formally, so that we could cut and paste them into a textbook.