Recap

if $a \in \mathbb{F}$, $0 \cdot a = 0$?

$$0 \cdot a = 0 \quad \xleftarrow{\text{multiply by } a} \quad 1 = (1+0)$$

$$a = a + 0 \cdot a$$

$$0 = 0 \cdot a$$

Vector Spaces

① $\mathbb{F}^n$: recall $\mathbb{R}^n$ = "vector space over $\mathbb{R}$"

- has 0 element, written as 0 or $\vec{0}$ or $0_{\mathbb{R}^n}$
- $(+)$
- scalar multiplication

Definition: Suppose n is a non-negative integer and $\mathbb{F}$ is any set. An n-tuple (or list of length n) of $\mathbb{F}$ is an ordered collection of n elements of $\mathbb{F}$.

Notation: $x = (x_1, \dots, x_n)$ $x_j \in \mathbb{F}$

$$(a_1, a_2, \dots, a_n) = (b_1, b_2, \dots, b_m) \Leftrightarrow \begin{matrix} m=n \\ a_i = b_i \ \forall i \text{ in } [1, m] \end{matrix}$$

Definition: $\mathbb{F}^n = \{(a_1, a_2, \dots, a_n) \mid a_i \in \mathbb{F}\}$ = set of n-tuples of $\mathbb{F}$

If $x = (x_1, x_2, \dots, x_n)$ then x_j is called the "j^{th} entry"

Definition: Given a field $\mathbb{F}$, $x = (x_1, \dots, x_n) \in \mathbb{F}^n$, $y = (y_1, \dots, y_n) \in \mathbb{F}^n$, $a \in \mathbb{F}$

ⓐ the zero element $0 = \vec{0} = 0_{\mathbb{F}^n} = (0, 0, \dots, 0)$

ⓑ $x + y = (x_1 + y_1, \dots, x_n + y_n)$

ⓒ $ax = (ax_1, \dots, ax_n)$

$\mathbb{F}^n$ w/ these operations is our friendly vector space

General Vector Spaces Over a Field $\mathbb{F}$

Definition: a set V equipped with

a) an element $0 = 0_V \in V$

b) an addition operation $\alpha : V \times V \to V$

$$(x,y) \mapsto \alpha(x,y) = x+y$$

c) a scalar multiplication operation $\mu : \mathbb{F} \times V \to V$

$$(a,x) \mapsto \mu(a,x) = a \cdot x = ax$$

is called a **vector space over $\mathbb{F}$** if the following 8 properties hold:

VS1 Commutativity of $+$: $\forall x,y \in V \quad x+y = y+x$

VS2 Associativity of $+$: $\forall x,y,z \in V \quad x+(y+z) = (x+y)+z$

VS3 Additive Identity: $\forall x \in V \quad x + 0_V = x$

VS4 Additive Inverse: $\forall x \in V \; \exists y \in V$ such that $x+y = 0_V$

VS5 Multiplication Identity: $\forall x \in V \quad 1 \cdot x = x$ Note: $1 \in \mathbb{F}$

VS6 Associativity of $\cdot$: $\forall a,b \in \mathbb{F},\, x \in V \quad a(bx) = (ab)x$

VS7 Distributivity #1: $\forall a \in \mathbb{F},\, x,y \in V \quad a \cdot (x+y) = a \cdot x + a \cdot y$

VS8 Distributivity #2: $\forall a,b \in \mathbb{F},\, x \in V \quad (a+b) \cdot x = a \cdot x + b \cdot x$

Examples of Vector Spaces Over $\mathbb{F}$

① $\mathbb{F}^n$, $0_{\mathbb{F}^n} = (0, 0, \dots, 0)$, '+', '·' as above

some notes: a) $n = 1$, $\mathbb{F}^1 = \mathbb{F}$

b) often write $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{F}^n$ "column vector"

or $(x_1, \dots, x_n) \in \mathbb{F}^n$ "row vector"

$\mathbb{F} = \mathbb{Z}_7 = \{0, 1, 2, 3, 4, 5, 6\}$

$\mathbb{F}^3$ $\quad 2 \cdot \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 6 \\ 3 \end{pmatrix} + \begin{pmatrix} 6 \\ 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

proposition: $\mathbb{F}^n$ is a vector space over $\mathbb{F}$

more notes: c) $\mathbb{F}^1$ is a vector space

d) $\mathbb{F}^0 = 0$ vector space $\quad V = \{0_V\}$; $x + y = 0_V$

$a 0_V = 0_V \quad \forall a \in \mathbb{F}$

(2) $m \times n$ matrices over $\mathbb{F}$

$$A = (A_{ij}) = \begin{array}{c} \overbrace{\hphantom{A_{11} \quad A_{12} \quad \cdots \quad A_{1n}}}^{n \text{ columns}} \\ \left(\begin{matrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{matrix} \right) \end{array} \left.\vphantom{\begin{matrix} A \\ A \\ A \\ A \end{matrix}}\right\} m \text{ rows}$$

where $A_{ij} \in \mathbb{F}$

let $\mathbb{F}^{m \times n}$ denote the set of ALL $m \times n$ matrices over $\mathbb{F}$.

(a) $0_{\mathbb{F}^{m \times n}} = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$

(b) $A + B = (A_{ij} + B_{ij})$, $A, B \in \mathbb{F}^{m \times n}$, $1 \leq i \leq m$ $1 \leq j \leq n$