

Recall

OH

$$VS1 \quad x+y = y+x$$

$$VS2 \quad x+(y+z) = (x+y)+z$$

$$VS3 \quad x+0_v = x$$

$$VS4 \quad \forall x \in V \exists y \in V \text{ s.t. } x+y = 0$$

$$VS5 \quad 1 \cdot x = x$$

$$VS6 \quad a(bx) = (ab)x$$

$$VS7 \quad a(x+y) = ax+ay$$

$$VS8 \quad (a+b)x = ax+bx$$

Prof: Tuesday 3-5PM MT 503

basic facts (will be on HW)

V is a vector space over \mathbb{F}

- basic {
- (a) the additive identity, 0_v is unique
 - (b) the additive inverse y of x is unique - denote it by $-x$
 - (c) Cancellation: if $x+z=y+z$, then $x=y$
 - (d) $0_{\mathbb{F}} \cdot x = 0_v \quad \forall x \in V$
 - (e) $(-1) \cdot x = -x$ (-1) always exists b/c 1 always exists
 - (f) $a \cdot 0_v = 0_v \quad \forall a \in \mathbb{F}$

Examples of Vector Spaces

① \mathbb{F}^n

② $m \times n$ matrices on \mathbb{F} , $\mathbb{F}^{m \times n}$

③ $\text{Fun}(X, \mathbb{F}) = \{f: X \rightarrow \mathbb{F}\}$

given a set X

function

← IMPORTANT

everything else can be written in terms of this

④ Polynomials: $\mathcal{P}(\mathbb{F})$, $\mathbb{F}[x]$

⑤ $\mathbb{F}^\infty = \{(a_0, a_1, \dots) \mid a_i \in \mathbb{F} \forall i\}$ infinite sequences

$\text{Fun}(X, \mathbb{F})$ Examples

$$X = [n] = \{1, 2, \dots, n\}$$

$$X = \mathbb{N} = \{0, 1, 2, \dots\}$$

$$X = [m] \times [n] = \{(i, j) \mid \begin{matrix} i \in [m] \\ j \in [n] \end{matrix}\}$$

We make $\text{Fun}(X, \mathbb{F})$ into a vector space over \mathbb{F}

(a) $0: X \rightarrow \mathbb{F}$
 $x \mapsto 0$

(b) If $f, g: X \rightarrow \mathbb{F}$, define $f+g: X \rightarrow \mathbb{F}$
 $(f+g)(x) = f(x) + g(x)$

elements of field; can be added

(c) If $a \in \mathbb{F}$, $f: X \rightarrow \mathbb{F}$, define
 $(af)(x) = af(x)$

Proposition: $\text{Fun}(X, \mathbb{F})$ with these operations (and 0) is a vector space over \mathbb{F}

PROOF

(v54) If $f, g: X \rightarrow \mathbb{F}$, show $f+g = g+f$ b/c its a field
for $x \in X$, $(f+g)(x) = f(x) + g(x) = g(x) + f(x) = (g+f)(x)$

Since x is arbitrary, applies to all x , thus

$$f+g = g+f \text{ in } \text{Fun}(X, \mathbb{F})$$

(v54) Given $f: X \rightarrow \mathbb{F}$, find $g: X \rightarrow \mathbb{F}$ such that $f+g = 0_{\text{Fun}(X, \mathbb{F})}$ Should always know which space 0 is located

Let $x \in X$ Define $g(x) = -f(x)$

Check $(f+g)(x)$

$$= f(x) + g(x) = 0$$

$$= f(x) + (-f(x)) = 0_{\mathbb{F}}$$

Because x is arbitrary, $f+g$ is thus $0_{\text{Fun}(X, \mathbb{F})} \forall x$

(v55) If $f: X \rightarrow \mathbb{F}$ is in $\text{Fun}(X, \mathbb{F})$ $1 \cdot f$ is the function
 $1 \cdot f(x) \Rightarrow x \mapsto (1 \cdot f(x)) = 1 \cdot f(x) = f(x)$

Rest of proof is similar

Examples

① If $X = [n] = \{1, 2, \dots, n\}$, what is $\text{Fun}(X, \mathbb{F})$?

To give $f: [n] \rightarrow \mathbb{F}$ is the same as giving
 $f(1), f(2), \dots, f(n) \in \mathbb{F}$ which is the same as an
element of \mathbb{F}^n !

Subspaces

Is

$$\left\{ \begin{pmatrix} a \\ b \\ a \end{pmatrix} : a, b \in \mathbb{F} \right\} \subseteq \mathbb{F}^3$$

a vector space (over \mathbb{F})?

(+), scalar multiplication come from \mathbb{F}^3 .

Definition: Let V be a vector space over \mathbb{F} .

A subset $U \subseteq V$ is called a subspace of V if it is a vector space under the operations $+$, scalar mult., and $0_V \in U$ is the additive identity.

Theorem: A subset U is a subspace of V if and only if

① $0_V \in U$

② U is closed under addition.

$$\forall x, y \in U ; x + y \in U \quad (\text{note } x + y \in V \text{ as well})$$

③ U is closed under scalar multiplication

$$\forall a \in \mathbb{F}, x \in U, ax \in U$$

Proof

VS1, VS2 follow immediately

VS3: Must show $(U, +, \cdot, 0_V)$ is a vector space over \mathbb{F}

0_V is the additive inverse! ✓

VS4: ?

The rest are automatic b/c they hold in V