(Continued from Lec 1)

3. Covariance

Given 2 rvs X, Y,

$$\mathrm{cov}(X,Y) = \mathbb{E}[(X-\mu_X)(Y-\mu_Y)] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (x-\mu_X)(y-\mu_Y) f_{X,Y}(x,y)\, dx\, dy$$

Covariance measures dependence of X and Y.

Definition: X, Y uncorrelated if $\mathrm{cov}(X,Y) = 0$

Independence

Two rvs X, Y are independent if

$$f_{X,Y}(x,y) = f_X(x)\, f_Y(y)$$

i.e. joint density = product of densities

If X, Y independent rvs, then

(i) $F_{X,Y}(x,y) = F_X(x)\, F_Y(y)$

(ii) $f_{X|Y}(x|y) = f_X(x)$

Remark: In data science it is typically assumed that noise + data are independent at each time

X, Y indpdt $\Rightarrow$ X, Y uncorrelated

Random Variables

1. Uniform RV $X \sim \mathcal{U}(a,b)$

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0, & o/w \end{cases}$$

$$\mathbb{E}[X] = \frac{b+a}{2}, \quad var(X) = \frac{1}{12}(b-a)^2$$

2. Gaussian RV, $X \sim \mathcal{N}(\mu, \sigma^2)$

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

$$F_X(\mu) = 1/2, \quad F_X(-\infty) = 0, \quad F_X(\infty) = 1$$

$$\mathbb{E}[X] = \mu, \quad Var(X) = \sigma^2$$

Result 1: Gaussian input to linear system yields Gaussian Output

If $Y = aX + b$, $X \sim \mathcal{N}(\mu, \sigma^2) \rightarrow Y \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$

To show Y is Gaussian,

$$F_Y(y) = Pr(Y \leq y) = Pr(aX + b \leq y)$$
$$= Pr\left(X \leq \frac{y-b}{a}\right), \quad a > 0$$
$$= F_X\left(\frac{y-b}{a}\right)$$

Prospect Theory for Human Decision Making

An expected utility maximizer chooses a decision which maximizes expected utility: Given two choices $a=1$, $a=2$, with utility $U_1(x)$ and $U_2(x)$, then

$$\mathbb{E}[U_1(x)] = \sum_x U_1(x_i)\, p_1(i)$$

$$\mathbb{E}[U_2(x)] = \sum_x U_2(x_i)\, p_2(i)$$

An expected utility maximizer chooses:

$$a = 1 \quad \text{if } \mathbb{E}[U_1(x)] > \mathbb{E}[U_2(x)]$$
$$a = 2 \quad \text{if } \mathbb{E}[U_2(x)] > \mathbb{E}[U_1(x)]$$

Example: Suppose there are two choices

1. Either: I give you 10 dollars. So

$$\mathbb{E}[U_1(x)] = 10$$

2. I flip a fair coin. Heads you get \$100. Tails: \$0

$$\mathbb{E}[U_2(x)] = 100 \cdot \frac{1}{2} + 0 \cdot \frac{1}{2} = 50$$

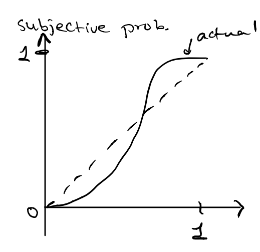
A (rational) utility maximizer chooses option

$$\underset{a}{\operatorname{argmax}}\ \mathbb{E}[U_a(x)] = 2$$

BUT MOST HUMANS WOULD RATHER CHOOSE OPTION ONE

Humans are not expected utility maximizers!

Humans are risk averse decision makers



Behavioral Economics Model

1. People have strong preference for certainty. They choose actions that yield lower expected utility with higher certainty.

    - I'd rather have $10 for certain than $50 w/ uncertainty
    - But risk-seeking for loss
        → I'd rather bet 50-50 on losing $100 or nothing than certaintly losing $20.

2. People give losses more weight than gains. Loss averse.

3. Relative positioning: People are more interested in relative gain/loss than absolute values

    - If you get a 10% raise and neighbor gets a 10% raise, you won't feel better off
    - If you get a 10% raise and neighbor gets NO raise, you feel amazing

4. People over-react to low probability events and under-react to high probability events

How do we model human decision making?

Prospect Theory

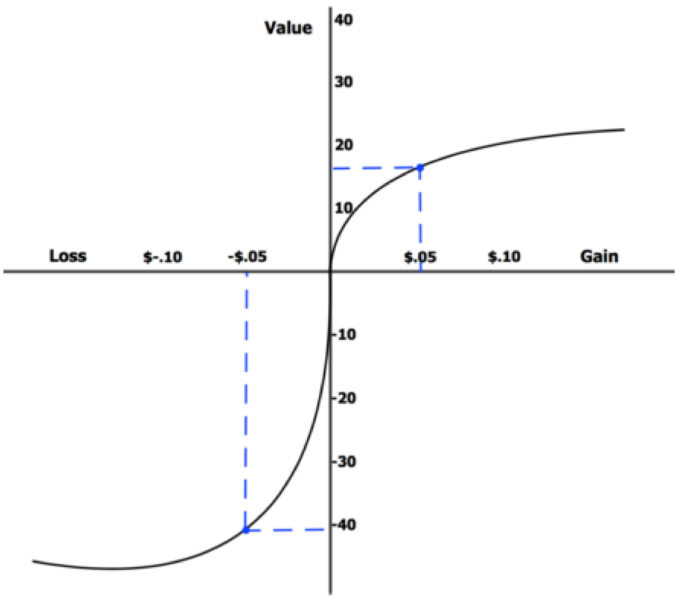
Utility for choice a is:

$$V_a = \sum_i v_a(x)\,\pi(p_a(i))$$

where $\pi(\cdot)$ is the subjective probability.

$\pi(\cdot)$ models the fact that people over-react to small probability events and under-react to large probability events.

v_a is the psychological value function for decision a.



(i) S-shaped curve, ASYMMETRICAL

(ii) Convex for losses, concave for gains.

i.e. losses HURT more than gains feel GOOD

→ Need \$150 to compensate \$100 loss

Compare prospect theory with expected utility maximizer for action a

$$\mathbb{E}[U_a(x)] = \sum_x U_a(x_i)\, p_a(i)$$

Example: Consider the following choices:

A. \$240 w/ certainty

B. \$1000 w.p. $1/4$, \$0 w.p. $3/4$

C. \$-700 w.p. 1

D. \$-1000 w.p. $3/4$, \$0 w.p. $1/4$

Classical expected utilities are

$\mathbb{E}[U_A] = 240$

$\mathbb{E}[U_B] = \frac{1}{4}(1000) + \frac{3}{4}(0) = 250$

$\mathbb{E}[U_C] = -700$

$\mathbb{E}[U_D] = -750$

from a plot like above

Prospect values are

$v(1000) = 3.5$

$v(240) = 1.5$

$v(-700) = -6$

$v(-1000) = -6.5$)

So, using prospect theory)

Correctly Models!

$\mathbb{E}[V_A] = 1.5$

Humans choose guaranteed amount!

$\mathbb{E}[V_B] = 3.5 \cdot \frac{1}{4} = 0.875$

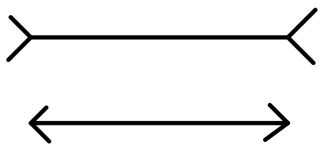
$\mathbb{E}[V_C] = -6$

$\mathbb{E}[V_D] = -6.5 \cdot \frac{3}{4} = -4.875$

Humans would rather choose the risky loss D than the certain loss C

Framing Effect

Cognitive bias where people decide on options based on whether they have positive or negative connotations.



i.e.
Visual framing

Effect of Framing

Suppose U.S. is preparing for outbreak of a disease expected to kill 600 people.
Consider two alternative programs to combat the disease:

- Program A: 200 ppl saved (72% favor)
- Program B: 600 ppl saved w.p. $\frac{1}{3}$, 0 w.p. $\frac{2}{3}$ (28%)

Same cover story, different framing

- Program A: 400 ppl die (22% favor)
- Program B: 0 ppl die w.p. $\frac{1}{3}$, 600 w.p. $\frac{2}{3}$ (78%)

Prospect theory explains these results!

Psychology has a profound effect on decision making.