

Residue Theory (Contour Integrals)

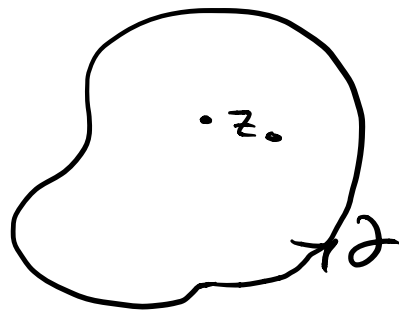
Suppose $f(z)$ has an isolated singularity at z_0 .
Write

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n (z-z_0)^n \quad \leftarrow \text{Laurent Series about } z_0$$

Have a contour γ around z_0
positively oriented

Consider

$$\oint_{\gamma} f(z) dz \quad \leftarrow \text{can integrate Laurent series term by term}$$



The point :

$$\oint_{\gamma} f(z) dz = \sum_{n=-\infty}^{+\infty} a_n \oint_{\gamma} (z-z_0)^n dz$$

$$\oint_{\gamma} (z-z_0)^n dz = \begin{cases} 0, & n \neq -1 \\ 2\pi i, & n = -1 \end{cases}$$

And the a_{-1} coefficient is called the residue of z at z_0 .

$$a_{-1} = \text{Res}(f(z); z_0) \quad \leftarrow \text{notation}$$

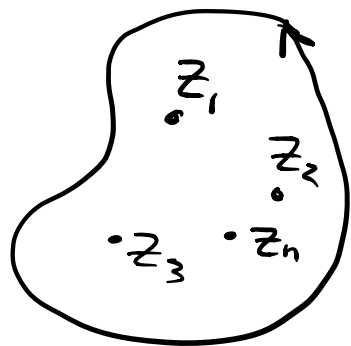
So calculating the contour boils down to finding the residue.

Suppose we had more than one singularity;

Say, z_1, \dots, z_n (finite amount)

Then

$$\oint_{\gamma} f(z) dz$$



$$\rightarrow = 2\pi i \sum_{k=1}^n \text{Res}(f(z); z_k) \quad \left. \vphantom{\sum_{k=1}^n} \right\} \begin{array}{l} \text{prove} \\ \text{by integrating sub contours} \\ \text{made by adding ridges} \end{array}$$

How to Calculate Residues

1) Laurent Series expansion.

Maclaurin Series for $e^{6/z}$

$$f(z) = z^3 e^{6/z} = z^3 \left[1 + \frac{6}{z} + \frac{1}{2!} \left(\frac{6}{z} \right)^2 + \frac{1}{3!} \left(\frac{6}{z} \right)^3 + \dots \right]$$

$$= \dots \left(\frac{6^4}{4!} \frac{1}{z} \right) \dots$$

don't care
b/c they integrate to
zero

The residue term

2) If a simple pole at z_0 ,

$$\text{Res}(f(z); z_0) = \lim_{z \rightarrow z_0} (z - z_0) f(z)$$

Proof:

$$f(z) = \dots \left(a_{-1} \frac{f(z)}{z-z_0} \right) + a_0 + a_1(z-z_0) \dots$$

All that's left is the residue!

Example:

$$\text{Res} \left(\frac{e^z}{\sin z} ; \pi \right)$$

1

$\sin z$ has a simple pole at $z = \pi$

$$\begin{aligned} \text{Res} &= \lim_{z \rightarrow \pi} (z - \pi) \frac{e^z}{\sin z} \stackrel{\text{L'Hopital's!}}{=} \lim_{z \rightarrow \pi} \frac{(z - \pi)e^z + e^z}{\cos z} \\ &= \frac{e^\pi}{-1} = -e^\pi \end{aligned}$$

The residue

$$a_{-1} = -e^\pi$$

3) If a pole of order m , then (slightly more complicated)

$$\text{Res}(f(z); z_0) = \lim_{z \rightarrow z_0} \frac{1}{(m-1)!} \left(\frac{d}{dz} \right)^{m-1} (z - z_0)^m f(z)$$

Proof: use Laurent series - Page 316

Using Residues to find Trigonometric Integrals on $[0, 2\pi]$

Example

$$I = \underset{\substack{\text{technically a function of } a \\ \rightarrow}}{\text{Integral}} = \int_0^{2\pi} \frac{d\theta}{a + \cos \theta} \quad , \quad \begin{array}{l} \text{where } a > 1 \\ \text{allows integral to be finite} \end{array}$$

$$\text{Let } z = e^{i\theta} \longrightarrow dz = e^{i\theta} i d\theta$$
$$d\theta = -i \frac{dz}{e^{i\theta}} = -i \frac{dz}{z}$$

Then

$$\cos \theta = \frac{z + 1/z}{2}$$

This converts I to an integral around unit circle

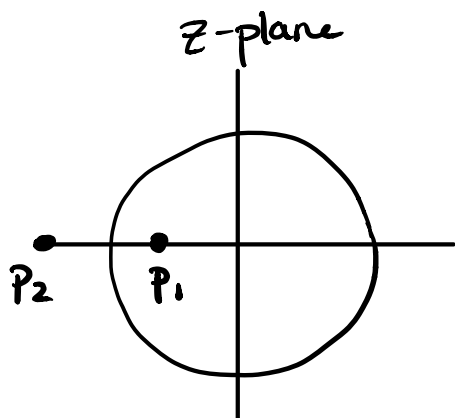
$$I = \oint_{|z|=1} \frac{-i \frac{dz}{z}}{a + \frac{z + 1/z}{2}} = \oint_{|z|=1} \frac{-2i}{z^2 + 2az + 1} dz$$

$$= -2i \oint_{|z|=1} \frac{dz}{z^2 + 2az + 1} \leftarrow \begin{array}{l} \text{any isolated singularities} \\ \text{inside unit circle?} \end{array}$$

quadratic has two roots \Rightarrow poles

$$-\frac{2a}{2} \pm \frac{\sqrt{4a^2 - 4}}{2} = -a \pm \sqrt{a^2 - 1}$$

$$\text{Poles: } P_1 = -a + \sqrt{a^2 - 1} \quad P_2 = -a - \sqrt{a^2 - 1}$$



So only residue from p_1 contributes to I .

$$I = 2\pi i \sum \text{Res}(\text{singularities inside } \gamma) \\ = 2\pi i \text{Res} \left[\frac{z}{i} \frac{1}{z^2 + 2az + 1} ; p_1 \right]$$

$$z^2 + 2az + 1 = (z - p_1)(z - p_2) \quad = 2\pi i \lim_{z \rightarrow p_1} \frac{z}{i} \frac{(z - p_1)}{(z - p_1)(z - p_2)}$$

$$p_1 - p_2 = 2\sqrt{a^2 - 1}$$

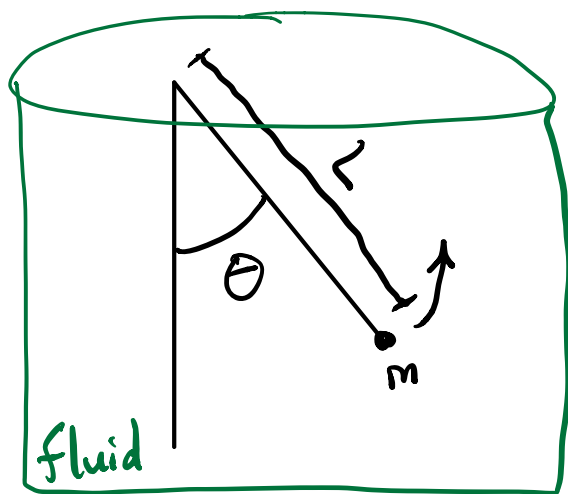
$$= \frac{4\pi}{p_1 - p_2} = \frac{2\pi}{\sqrt{a^2 - 1}}$$

Wowza!

$$I(a) = \int_0^{2\pi} \frac{d\theta}{a + \cos \theta} = \frac{2\pi}{\sqrt{a^2 - 1}} \quad \left. \vphantom{\int_0^{2\pi}} \right\} \text{This is a neat result}$$

Physics Application

Period of whirling, overdamped pendulum driven by a constant torque.



torque Γ (constant)

the torque will cause this to "whirl" in the sense that it will rotate all around.

$$\frac{d}{dt}(mL + L\dot{\theta}) + b\dot{\theta} + mgL\sin(\theta) = \tau$$

↗ 0 ↖ overdamped

damping so large, inertia term doesn't apply

$$T = \int dt = \int_0^{2\pi} \frac{d\theta}{a - \sin\theta}, \quad \frac{d\theta}{dt} + \sin\theta = a$$

$$dt = \left[\frac{a - \sin\theta}{d\theta} \right]^{-1}$$