OH Th. 1:30 - 3:30

Exam IN CLASS Friday

What Have We Done?

Fields

$\mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$, $\mathbb{F}_p = \mathbb{Z}_p$ (finite fields)

Vector Spaces over $\mathbb{F}$

"operations essentially done in $\mathbb{F}$"

- $\mathbb{F}^n$
- $\mathbb{F}^{m \times n}$
- $\mathbb{F}[x]$
- $C^\infty(\mathbb{R})$
- $\mathrm{Fun}(X, \mathbb{F})$
- $\mathrm{Fun}(X, V)$

Subspaces

$U \subseteq V$, $U_1, U_2 \subseteq V$ subspaces

$U_1 + U_2 \subseteq V$

$U_1 \oplus U_2 = V$, $U_1 \cap U_2 = 0$

Span, LI, LD, dim

- key result, dim V finite AND $v_1, \ldots, v_n$ spans V and $w_1, \ldots, w_m$ is LI $\Rightarrow m \leq n$
- any LI set $v_1, \ldots, v_r$ of V ($\dim V < \infty$) can be expanded to a basis $(v_1, \ldots, v_r, \ldots, v_m)$
- any spanning set $(v_1, \ldots, v_s)$ has a subset which is a basis

bases / $\beta = (\vec{v}_1, \dots, \vec{v}_n)$ basis $\Leftrightarrow$ any $\vec{v} \in V$ has a unique map

$v = a_1 v_1 + \dots + a_n v_n$, $a_i \in \mathbb{F}$

write $[v]_\beta = \left(\quad \right)$

- $U_1, U_2 \subseteq V$, $\dim V < \infty$

$\dim(U_1 + U_2) + \dim(U_1 \cap U_2) = \dim U_1 + \dim U_2$

Linear Transforms $T: V \to W$

def

$\mathcal{L}(V, W) = (\dim V)(\dim W)$

$\mathcal{L}(V) = (\dim V)^2$

Kernel, Image, Injective, Surjective

$T: V \to W$

$\dim(V) = \dim \operatorname{Ker}(T) + \dim \operatorname{Im}(T)$

rank $T = \dim \operatorname{im} T$

$L_A : \mathbb{F}^n \to \mathbb{F}^m$ if $A \in \mathbb{F}^{m \times n}$

$x \mapsto Ax$

$\ker L_A = \ker A$

$\operatorname{im} L_A = \operatorname{im} A$ rank L_A = rank A

Matrix of T

$[T]_{\beta \leftarrow \alpha}$

If $T \in \mathcal{L}(V)$, $[T]_\beta = [T]_{\beta \leftarrow \beta}$

<u>Isomorphisms</u>

$T : V \to W$

T isomorphism $\iff$ T injective AND T surjective

$\iff$ $\ker T = 0$, $\operatorname{im} T = W$

$\iff$ $\exists T^{-1} : W \to W$ s.t. $(T^{-1})T = id_V$

$T(T^{-1}) = id_W$

Suppose $\dim V = \dim W < \infty$

$V \underset{\uparrow}{\cong} W \iff \dim V = \dim W$

$\exists T : V \to W$ isom.

$T : V \to W$ T isom.

$(\dim V = \dim W < \infty)$

$\iff \ker T = 0$

$\iff \operatorname{im} T = W$

$\iff [T]_{\beta \leftarrow \alpha}$ $n \times n$ matrix has an inverse

Gaussian Elimination

- do it
- row operations are reversible
- $A \rightsquigarrow \mathrm{rref}(A) = B$
- get $\ker A = \ker B$ — dim = # cols − # pivots
- $\dim \operatorname{im} A = \dim \operatorname{im} B$
- rank A = # pivots
- get basis of im A

$\mathbb{F} = \mathbb{F}_2$

$$\dim(\mathbb{F}^2, \mathbb{F}) = 4$$

elem. × dim