Theorem: $S = (\vec{v}_1, \dots, \vec{v}_k)$ orthogonal set in V spanning $W \subseteq V$, then for $\vec{w} \in W$

$$\vec{w} = \sum_{i=1}^{k} \frac{\langle w, v_i \rangle}{\langle \vec{v}_i, \vec{v}_i \rangle} \vec{v}_i$$

Corollary If S is orthogonal set in V inner product space then S is LI

Proof Suppose $\vec{v}_1, \dots, \vec{v}_k \in S$

If $c_1\vec{v}_1 + \dots + c_k\vec{v}_k = 0$

dot w/ $\vec{v}_1$ $\quad 0 + \dots + 0 + c_i\langle \vec{v}_i \cdot \vec{v}_i \rangle + 0 + \dots + 0 = 0$

$$\Rightarrow c_i = 0 \ \forall i$$

$\therefore \ (\vec{v}_1, \dots, \vec{v}_k)$ LI

Remark Coool fact

if $B = (\vec{v}_1, \dots, \vec{v}_k)$ is an **orthonormal** basis of V and $\vec{v} \in V$

then

$$[\vec{v}]_\beta = \begin{bmatrix} \langle \vec{v}, \vec{v}_1 \rangle \\ \vdots \\ \langle v, v_k \rangle \end{bmatrix}$$

Gram - Schmidt

Let V be an inner product space.

Question: Can we find an orthonormal basis for V? ($\dim V = n < \infty$)

<u>Example</u>: Consider 2 vectors $\vec{v}, \vec{w} \in V$ inner product space.

Suppose $(\vec{v}, \vec{w})$ are LI.

Consider $W = \text{span}(\vec{v}, \vec{w}) \subseteq V$

(a) Find an orthonormal basis of W.

Let $p_v(\vec{w})$ = projection of w onto span (v)

then $\vec{v} \perp (\vec{w} - p_v(\vec{w}))$

How?

$$p_v(\vec{w}) = c\vec{v} \quad \text{some } c \in \mathbb{R}$$

$$\vec{v} \perp (w - c\vec{v}) \quad \text{means} \quad \langle v, w - cv \rangle = 0$$

i.e.

$$\langle v, w \rangle - c \langle v, v \rangle = 0$$

$$c = \frac{\langle v, w \rangle}{\langle v, v \rangle}$$

non-zero, LI, span W

then $\langle \vec{v}, \vec{w} - \frac{\langle v, \vec{w} \rangle}{\langle v, v \rangle} \vec{v} \rangle$ is an orthogonal basis of W

Theorem (Gram-Schmidt)

Let V be an inner product space.

Define

$S' = (\vec{v}_1, \ldots, \vec{v}_n)$ by

$$\vec{v}_1 = \vec{w}_1$$

$$\vec{v}_2 = \vec{w}_2 - \frac{\langle \vec{w}_2, \vec{v}_1 \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle} \vec{v}_1$$

$$\vdots$$

$$\vec{v}_k = \vec{w}_k - \frac{\langle \vec{w}_k, \vec{v}_1 \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle} \vec{v}_1 - \frac{\langle \vec{w}_k, \vec{v}_2 \rangle}{\langle \vec{v}_2, \vec{v}_2 \rangle} \vec{v}_2 - \cdots - \frac{\langle \vec{w}_k, \vec{v}_{k-1} \rangle}{\langle \vec{v}_{k-1}, \vec{v}_{k-1} \rangle} \vec{v}_{k-1}$$

then

1. S' is an orthonormal set
2. $\text{span}(S') = \text{Span}(S)$

in Fact,

3. $\text{span}(\vec{v}_1, \ldots, \vec{v}_k) = \text{span}(\vec{w}_1, \ldots, \vec{w}_k) \ \forall \ k \in 1, \ldots, n$
4. $(\vec{v}_1, \ldots, \vec{v}_k)$ is an orthogonal basis of $\text{span}(\vec{w}_1, \ldots, \vec{w}_k)$

<u>idea</u> of <u>proof</u>

Construct $\vec{v}_1 = \vec{w}_1$

$\vec{v}_2 \in \text{span}(\vec{w}_1, \vec{w}_2)$, $\vec{v}_2 \perp \vec{v}_1$

$\vec{v}_3 \in \text{span}(w_1, w_2, w_3)$, $v_3 \perp v_2, v_1$

$\vdots$

Proof: induct on $\#S = n$

Let

$$S_k = (\vec{w}_1, \dots, \vec{w}_k)$$
$$S_k' = (\vec{v}_1, \dots, \vec{v}_k)$$

show (given true for $k-1$) that

(a) S_k', S_k have same span

(b) S_k' is orthogonal

if $\underline{k=1}$

$$S_1 = \{\vec{w}_1\}$$
$$S_1' = \{\vec{w}_1\}$$

Note: $\vec{w}_1 \neq \vec{0}$ since S is LI

clearly $\text{span}(S_1) = \text{span}(S_1')$

Using induction,

$$S_k' = (\vec{v}_1, \dots, \vec{v}_k)$$

know

$$\langle \vec{v}_i, \vec{v}_j \rangle = 0 \quad \text{for } i \neq j \quad i, j \leq k-1$$

Need to show: $\langle \vec{v}_i, \vec{v}_k \rangle = 0$ for $1 \leq i \leq k-1$

$\vec{v}_k \neq 0$

$$\text{span}(\vec{v}_1, \dots, \vec{v}_k) = \text{span}(\underbrace{\vec{v}_1, \dots, \vec{v}_{k-1}}_{\text{span}(\vec{w}_1, \dots, \vec{w}_{k-1})}, \vec{w}_k)$$

So

$$\langle \vec{v}_k, \vec{v}_i \rangle = \langle \vec{w}_k, \vec{v}_i \rangle - \frac{\langle w_k, v_i \rangle}{\langle v_i, v_i \rangle} \langle v_i, v_i \rangle$$

$\vec{v}_k \neq \vec{0}$! If it is

then $w_k \in \text{span}(v_1, \dots, v_{k-1}) = \text{span}(w_1, \dots, w_{k-1})$

! C

$$\text{span}(\vec{v}_1, \dots, \vec{v}_{k-1}, \vec{v}_k) = \text{span}(\vec{w}_1, \dots, \vec{w}_{k-1}, \vec{v}_k) \quad \text{(induction)}$$

$$= \text{span}(\vec{v}_1, \dots, \vec{v}_{k-1}, \vec{w}_k)$$

$\therefore$ $\text{span}(S_k) = \text{span}(S_k')$, S_k' is orthogonal

<u>Example</u> Find an orthonormal basis for

$$W = \text{span}\left(\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right)$$

Solution:

(a) $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

(b) $\vec{v}_2' = \vec{w}_2 - \dfrac{\langle \vec{w}_2, \vec{v}_1 \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle} \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$\vec{v}_2 = 2\vec{v}_2' = \begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \end{pmatrix}$