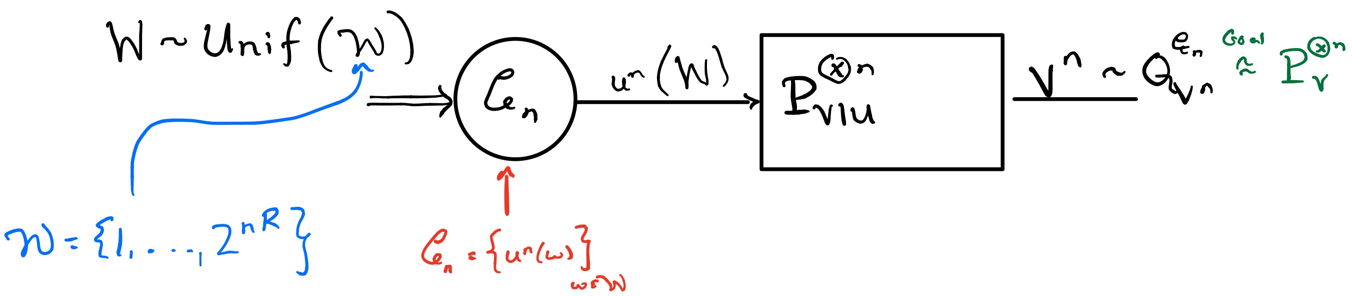
Recap:

We are considering approximate distribution simulation



Theorem (Approximate Distribution Simulation):

Fix $P_U \in \mathcal{P}(\mathcal{U})$, $P_V \in \mathcal{P}(\mathcal{V})$ and let $P_{V|U}$ be a transition kernel from $(\mathcal{U}, 2^{\mathcal{U}})$ to $(\mathcal{V}, 2^{\mathcal{V}})$ such that

$$\sum_{u \in \mathcal{U}} P_U(u) P_{V|U}(v|u) = P_V(v) \quad \forall v \in \mathcal{V}.$$

Let $C_n = \{U^n(\omega)\}_{\omega \in \mathcal{W}}$, $\mathcal{W} = \{1, \dots, 2^{nR}\}$, be an iid codebook distributed according to P_U, i.e. $U^n(\omega) \overset{iid}{\sim} P_U^{\otimes n}$ $\forall \omega \in \mathcal{W}$.
If $R > I_P(U;V)$ $\left(= I(P_U; P_{V|U})\right)$, then

$$\mathbb{E}_{C_n}\left[\delta_{TV}\left(Q_{V^n}^{(C_n)}, P_V^{\otimes n}\right)\right] \xrightarrow{n \to \infty} 0$$

where convergence happens exponentially fast and:

$$Q_{V^n}^{\mathcal{C}_n}(v^n) = \frac{1}{|\mathcal{W}|} \sum_{\omega \in \mathcal{W}} P_{V|U}^{\otimes n}(v^n | u^n(\omega))$$

for any <u>fixed</u> codebook $\mathcal{C}_n = \{u^n(\omega)\}_{\omega \in \mathcal{W}}$.

Proof

We want to seperate typical U^n's from atypical ones.

Define:

$$Q_1^{(\mathcal{C}_n)}(v^n) := \frac{1}{|\mathcal{W}|}\sum_{w\in\mathcal{W}} P_{V|U}^{\otimes n}(v^n|U^n(w))\,\mathbb{1}_{\{(v^n,U^n(w))\in\mathcal{T}_\varepsilon^{(n)}(P_{UV})\}}$$

$$Q_2^{(\mathcal{C}_n)}(v^n) := \frac{1}{|\mathcal{W}|}\sum_{w\in\mathcal{W}} P_{V|U}^{\otimes n}(v^n|U^n(w))\,\mathbb{1}_{\{(v^n,U^n(w))\notin\mathcal{T}_\varepsilon^{(n)}(P_{UV})\}}$$

where $\varepsilon>0$ is a fixed small number and $P_{UV} = P_U P_{V|U}$.

A few observations:

1. $Q_1^{(\mathcal{C}_n)}$, $Q_2^{(\mathcal{C}_n)}$ are **NOT** probability measures on $\mathcal{V}^n$ but their sum $Q_1^{(\mathcal{C}_n)} + Q_2^{(\mathcal{C}_n)} = Q_{V^n}^{(\mathcal{C}_n)}$ is a probability measure on $\mathcal{V}^n$.

2. The expected value of $Q_{V^n}^{(\mathcal{C}_n)}$ is in fact the target distribution $P_V^{\otimes n}$:

$$\mathbb{E}_{\mathcal{C}_n}\left[Q_{V^n}^{(\mathcal{C}_n)}(v^n)\right] = \mathbb{E}_{\mathcal{C}_n}\left[\frac{1}{|\mathcal{W}|}\sum_{w\in\mathcal{W}} P_{V|U}^{\otimes n}(v^n|U^n(w))\right]$$

$$= \frac{1}{|\mathcal{W}|}\sum_{w\in\mathcal{W}} \underbrace{\mathbb{E}_{\mathcal{C}_n}\left[P_{V|U}^{\otimes n}(v^n|U^n(w))\right]}$$

$$= \mathbb{E}_{U^n(1)}\left[P_{V|U}^{\otimes n}(v^n|U^n(1))\right]$$

$$= \sum_{u^n\in\mathcal{U}^n} P_U^{\otimes n}(u^n)\,P_{V|U}^{\otimes n}(v^n|u^n)$$

$$= P_V^{\otimes n}(v^n)$$

$\Longrightarrow$ We thus need to show that the TV distance b/t $Q^{(C_n)}_{V^n}$ and its expected value is small.

We split TV into 3 parts:

$$2 \cdot \mathbb{E}_{C_n}\left[\delta_{TV}\left(Q^{(C_n)}_{V^n}, P_V^{\otimes n}\right)\right] = \mathbb{E}_{C_n}\left[\sum_{v^n \in \mathcal{V}^n} \left| Q^{(C_n)}_{V^n}(v^n) - P_V^{\otimes n}(v^n)\right|\right]$$

Decompose

$$= \sum_{v^n \in \mathcal{T}_\varepsilon^{(n)}(P_V)} \mathbb{E}_{C_n}\left| Q^{(C_n)}_{V^n}(v^n) - P_V^{\otimes n}(v^n)\right| + \sum_{v^n \notin \mathcal{T}_\varepsilon^{(n)}(P_V)} \mathbb{E}_{C_n}\left| Q^{(C_n)}_{V^n}(v^n) - P_V^{\otimes n}(v^n)\right|$$

Triangle Inequality

$$\leq \sum_{v^n \in \mathcal{T}_\varepsilon^{(n)}(P_V)} \mathbb{E}_{C_n}\left| Q_1^{(C_1)}(v^n) - \mathbb{E}\, Q_1^{(C_n)}(v^n)\right|$$

$$+ \sum_{v^n \in \mathcal{T}_\varepsilon^{(n)}(P_V)} \mathbb{E}_{C_n}\left| Q_2^{(C_1)}(v^n) - \mathbb{E}\, Q_2^{(C_n)}(v^n)\right|$$

Can sum over all $v^n \in \mathcal{V}^n$ which adds more nonnegative terms

$$+ \sum_{v^n \notin \mathcal{T}_\varepsilon^{(n)}(P_V)} \mathbb{E}_{C_n}\left| Q^{(C_n)}_{V^n}(v^n) - \mathbb{E}_{C_n} Q^{(C_n)}_{V^n}(v^n)\right|$$

$$\leq \sum_{v^n \in \mathcal{T}_\varepsilon^{(n)}(P_V)} \mathbb{E}_{C_n}\left| Q_1^{(C_1)}(v^n) - \mathbb{E}\, Q_1^{(C_n)}(v^n)\right| \quad (1)$$

$$+ \underbrace{\sum_{v^n \in \mathcal{V}^n} \mathbb{E}_{C_n}\left| Q_2^{(C_1)}(v^n) - \mathbb{E}\, Q_2^{(C_n)}(v^n)\right|}_{(2)} + \underbrace{\sum_{v^n \notin \mathcal{T}_\varepsilon^{(n)}(P_V)} \mathbb{E}_{C_n}\left| Q^{(C_n)}_{V^n}(v^n) - \mathbb{E}_{C_n} Q^{(C_n)}_{V^n}(v^n)\right|}_{(3)}$$

Item 1 is the interesting one and is saved for last.

We start by showing 2 and 3 can be made small with n without further conditions.

$$② = \sum_{v^n \in V^n} \mathbb{E}_{C_n} \left| Q_2^{(C_n)}(v^n) - \mathbb{E}\, Q_2^{(C_n)}(v^n) \right| \qquad \left(\mathbb{E}\,|a-b| \leq \mathbb{E}[a] + \mathbb{E}[b] \right)$$

$$\leq 2 \cdot \sum_{v^n \in V^n} \mathbb{E}_{C_n} Q_2^{(C_n)}(v^n)$$

$$= 2 \cdot \sum_{v^n \in V^n} \frac{1}{|\mathcal{W}|} \sum_{w \in \mathcal{W}} \mathbb{E}_{C_n} \left[P_{V|U}^{\otimes n}(v^n | U^n(w)) \cdot \mathbb{1}_{\{(v^n, U^n(w)) \notin \mathcal{T}_\varepsilon^{(n)}(P_{UV})\}} \right]$$

$$= 2 \cdot \sum_{v^n \in V^n} \frac{1}{|\mathcal{W}|} \sum_{w \in \mathcal{W}} \mathbb{E}_{U^n(1)} \left[P_{V|U}^{\otimes n}(v^n | U^n(1)) \cdot \mathbb{1}_{\{(v^n, U^n(1)) \notin \mathcal{T}_\varepsilon^{(n)}(P_{UV})\}} \right]$$

$$= 2 \cdot \sum_{v^n \in V^n} \sum_{u^n \in U^n} P_U^{\otimes n}(u^n)\, P_{V|U}^{\otimes n}(v^n | u^n)\, \mathbb{1}_{\{(v^n, u^n) \notin \mathcal{T}_\varepsilon^{(n)}(P_{UV})\}}$$

$$= 2 \cdot \sum_{(u^n, v^n) \notin \mathcal{T}_\varepsilon^{(n)}(P_{UV})} P_{UV}^{\otimes n}(u^n, v^n)$$

$$= 2 \cdot \mathbb{P}_{P_{UV}^{\otimes n}} \left((U^n, V^n) \notin \mathcal{T}_\varepsilon^{(n)}(P_{UV}) \right)$$

$$\leq 2 \cdot 2 \cdot |\mathcal{U}||\mathcal{V}|\, e^{-2n\varepsilon^2 \mu_{uv}^2} \xrightarrow{n \to \infty} 0 \qquad \left\{ \text{where } \mu_{uv}^2 = \min_{(u,v) \in \operatorname{supp} P_{uv}} P_{uv}(u,v) > 0 \right.$$

$$③ = \sum_{v^n \notin \mathcal{T}_\varepsilon^{(n)}(P_v)} \mathbb{E}_{C_n} \left| Q_{V^n}^{(C_n)}(v^n) - \mathbb{E}_{C_n} Q_{V^n}^{(C_n)}(v^n) \right|$$

$$\leq 2 \cdot \sum_{v^n \notin \mathcal{T}_\varepsilon^{(n)}(P_v)} \mathbb{E}\, Q_{V^n}^{(C_n)}(v^n)$$

$$= 2 \cdot \sum_{v^n \notin \mathcal{T}_\varepsilon^{(n)}(P_v)} P_v^{\otimes n}(v^n)$$

$$= 2 \cdot \mathbb{P}_{P_v^{\otimes n}}\left(V^n \varepsilon \mathcal{T}_\varepsilon^{(n)}(P_v)\right)$$

$$\leq 2 \cdot 2 |\mathcal{V}| e^{-2n\varepsilon^2 \mu_v^2} \xrightarrow{n \to \infty} 0$$

We are thus left with bounding the first item, ①. Since now we are dealing with jointly typical U^n and V^n sequences we need a different analysis.

Let us derive an upper bound on each summand that is independent of $v^n \varepsilon \mathcal{T}_\varepsilon^{(n)}(P_v)$

$$\mathbb{E}_{C_n} \left| Q_1^{(C_n)}(v^n) - \mathbb{E}\, Q_1^{(C_n)}(v^n) \right| \leq \sqrt{\mathbb{E}_{C_n}\left[\left(Q_1^{(C_n)}(v^n) - \mathbb{E}\, Q_1^{(C_n)}(v^n)\right)^2\right]}$$

Jensen's Inequality

$\mathbb{E}[|\cdot|] = \mathbb{E}[\sqrt{(\cdot)^2}] \leq \sqrt{\mathbb{E}[(\cdot)^2]}$

$$= \sqrt{\mathrm{Var}_{C_n}\left(Q_1^{(C_n)}(v^n)\right)}$$

Since $C_n = \{U^n(\omega)\}_{\omega\in\mathcal{W}}$ are iid the variance tensorizes:

$$\mathrm{Var}_{C_n}\left(Q_Z^{(C_n)}(v^n)\right) = \mathrm{Var}_{C_n}\left(\frac{1}{|\mathcal{W}|}\sum_{\omega\in\mathcal{W}} P_{V|U}^{\otimes n}(v^n|U^n(\omega))\,\mathbb{1}_{\{(v^n,U^n(\omega))\in\mathcal{T}_\epsilon^{(n)}(P_{UV})\}}\right)$$

variance of sum of iid random variables

$$= \frac{1}{|\mathcal{W}|^2}\sum_{\omega\in\mathcal{W}} \underbrace{\mathrm{Var}_{C_n}\left(P_{V|U}^{\otimes n}(v^n|U^n(\omega))\,\mathbb{1}_{\{(v^n,U^n(\omega))\in\mathcal{T}_\epsilon^{(n)}(P_{UV})\}}\right)}$$

Due to symmetry this is independent of $\omega\in\mathcal{W}$ and $U^n(\omega)$ can be replaced with $U^n(1)$ throughout

$$= 2^{-nR}\,\mathrm{Var}_{U^n(1)}\left(P_{V|U}^{\otimes n}(v^n|U^n(1))\,\mathbb{1}_{\{(v^n,U^n(1))\in\mathcal{T}_\epsilon^{(n)}(P_{UV})\}}\right)$$

$$\leq 2^{-nR}\,\mathbb{E}_{U^n(1)}\left[\left(P_{V|U}^{\otimes n}(v^n|U^n(1))\,\mathbb{1}_{\{(v^n,U^n(1))\in\mathcal{T}_\epsilon^{(n)}(P_{UV})\}}\right)^2\right]$$

$$= 2^{-nR}\sum_{u^n\in\mathcal{U}^n} P_U^{\otimes n}(u^n)\left(P_{V|U}^{\otimes n}(v^n|u^n)\right)^2\,\mathbb{1}_{\{(v^n,U^n(1))\in\mathcal{T}_\epsilon^{(n)}(P_{UV})\}}$$

$$= 2^{-nR}\sum_{u^n\in\mathcal{T}_\epsilon^{(n)}(P_{UV}|v^n)} P_U^{\otimes n}(u^n)\,\underbrace{P_{V|U}^{\otimes n}(v^n|u^n)}_{\leq 2^{-n(H(V|U)-\delta_1(\epsilon))}}\,P_{V|U}^{\otimes n}(v^n|u^n)$$

$:= \{u^n \mid (u^n,v^n)\in\mathcal{T}_\epsilon^{(n)}(P_{UV})\}$

$$\leq 2^{-nR}\,2^{-n(H(V|U)-\delta_1(\epsilon))}\underbrace{\sum_{u^n\in\mathcal{U}^n} P_U^{\otimes n} P_{V|U}^{\otimes n}(v^n|u^n)}_{P_V^{\otimes n}(v^n)}$$

$$= 2^{-nR}\,2^{-n(H(V|U)-\delta_1(\epsilon))}\underbrace{P_V^{\otimes n}(v^n)}$$

$\leq 2^{-n(H(V)-\delta_2(\epsilon))}$
$\forall v^n\in\mathcal{T}_\epsilon^{(n)}(P_V)$
$(\delta_2(\epsilon)\xrightarrow{\epsilon\to 0} 0)$

$$\leq 2^{-nR}\,2^{-n(H(V|U)-\delta_1(\epsilon))}\,2^{-n(H(V)-\delta_2(\epsilon))}$$

$$= 2^{-n(R+H(V|U)+H(V)-\delta_1(\epsilon)-\delta_2(\epsilon))}$$

Recap

$$\Rightarrow \textcircled{1} = \sum_{v^n \in \mathcal{T}_\epsilon^{(n)}(P_v)} \mathbb{E}_{C_n} \left| Q_1^{(C_n)}(v^n) - \mathbb{E}\, Q_1^{(C_n)}(v^n) \right|$$

$$\leq \sum_{v^n \in \mathcal{T}_\epsilon^{(n)}(P_v)} 2^{-\frac{n}{2}\left(R + H(V|U) + H(V) - \delta_1(\epsilon) - \delta_2(\epsilon)\right)}$$

$$\leq 2^{n\left(H(V) + \delta_2(\epsilon)\right)} \, 2^{-\frac{n}{2}\left(R + H(V|U) + H(V) - \delta_1(\epsilon) - \delta_2(\epsilon)\right)}$$

$$= 2^{-\frac{n}{2}\left(R + H(V|U) - H(V) - \delta_1(\epsilon) - 3\delta_2(\epsilon)\right)}$$

where $\delta(\epsilon) = \delta_1(\epsilon) - 3\delta_2(\epsilon) \xrightarrow{\epsilon \to 0} 0$

$\Rightarrow$ Thus if $R > I_P(U;V)$ then we can take ϵ small enough such that the RHS above will go to 0 as $n \to \infty$, and converges exponentially fast

Lastly, we will show how the above expected value result can be strengthened into a concentration inequality.

<u>Definition (Functions w/ Bounded Differences)</u>: A function $f: \mathcal{X}^n \to \mathbb{R}$ is said to have $(c_i)_{i=1}^n$ bounded difference if

$$\left| f(x_1, \dots, x_n) - f(x_1, \dots, x_{i-1}, \tilde{x}, x_{i+1}, \dots, x_n) \right| \leq c_i \quad \forall x_1, \dots, x_n, \tilde{x} \in \mathcal{X}, \; i = 1, 2, \dots, n$$

Theorem (McDiarmid's Inequality): Let $X_1, \dots, X_n$ be $\mathcal{X}$-valued independent random variables and $f : \mathcal{X}^n \to \mathbb{R}$ have $(c_i)_{i=1}^n$-bounded differences. Then

$$\mathbb{P}\left(\left| f(X_1, \dots, X_n) - \mathbb{E}[f(X_1, \dots, X_n)] \right| \geq t \right) \leq 2 e^{-\frac{2t^2}{\sum_{i=1}^n c_i^2}}$$

Theorem (Strong Approximate Distribution Simulation):

Under the same setup of the expected value claim for approximate distribution simulation, if $R > I_P(U;V)$ then there exists $\gamma_1, \gamma_2 > 0$ such that

$$\mathbb{P}\left(\delta_{TV}\left(Q_{V^n}^{(C_n)}, P_V^{\otimes n} \right) > e^{-\gamma_1 n} \right) \leq e^{-e^{\gamma_2 n}}$$

for n large enough.

Proof Outline: Define $f(U_1, \dots, U_n) = \delta_{TV}\left(Q_{V^n}^{(C_n)}, P_V^{\otimes n} \right)$

$W := |\mathcal{W}|$

$$= \frac{1}{2} \mathbb{E}_{P_V^{\otimes n}} \left[\frac{Q_{V^n}^{(C_n)}(v^n)}{P_V^{\otimes n}(v^n)} - 1 \right]$$

$$= \frac{1}{2} \mathbb{E}_{P_V^{\otimes n}} \left[\frac{1}{W} \sum_{w=1}^{W} \frac{P_{V|U}^{\otimes n}(v^n | U_w)}{P_V^{\otimes n}(v^n)} - 1 \right]$$

$\Rightarrow$ We can show that $f(U_1, \dots, U_n)$ has $\left(\frac{1}{W}\right)_{i=1}^{W}$ bounded difference

Thus we can apply McDiormid's inequality & use the fact that in expectation

$$\mathbb{E}_{C_n} \delta_{TV}\left(Q_{V^n}^{(C_n)}, P_V^{\otimes n}\right) \leq e^{-\gamma n}$$

for some $\gamma > 0$ and n sufficiently large.

Taking t to be exponentially small with n $(t = e^{-\tilde{\gamma} n}, \tilde{\gamma} > 0)$ and plugging into the above will establish the claim.

Discussion Section

Proof of Strong Approximate Distribution Simulation

$\mathcal{W} = 2^{nR}$, $U_w = U^n(w)$

Fix $\mathcal{C}_n := \{U_w\}_{w \in \mathcal{W}}$ and define

$$f : \mathcal{C}_n \longmapsto \delta_{TV}\left(Q_{V^n}^{(\mathcal{C}_n)}, P_V^{\otimes n}\right)$$

i.e.

$$f(\mathcal{C}_n) = f(U_1, \ldots, U_{\mathcal{W}}) := \delta_{TV}\left(Q_{V^n}^{(\mathcal{C}_n)}, P_V^{\otimes n}\right)$$

$\delta_{TV} = D_{\frac{1}{2}|x-1|}$

$D_f(P \| Q) = \mathbb{E}_Q\left[f\left(\frac{dP}{dQ}\right)\right]$

ratio of pmf in discrete case

$$= \frac{1}{2} \mathbb{E}_{P_V^{\otimes n}} \left| \frac{Q_{V^n}^{(\mathcal{C}_n)}}{P_V^{\otimes n}} - 1 \right|$$

$$= \frac{1}{2} \mathbb{E}_{P_V^{\otimes n}} \left| \frac{1}{\mathcal{W}} \sum_{w=1}^{\mathcal{W}} \frac{P_{V|U}^{\otimes n}(V^n | u_w)}{P_V^{\otimes n}(V^n)} - 1 \right|$$

Bounded Differences Property:

Let $u_1,\dots,u_W,\tilde{u} \in \mathcal{U}^n$ and consider:

$$\circledast \quad f(u_1,\dots,u_{\ell-1},u_\ell,u_{\ell+1},\dots,u_W) - f(u_1,\dots,u_{\ell-1},\tilde{u},u_{\ell+1},\dots,u_W)$$

} absolute value omitted for proof simplicity. Other direction similar

$$= \underbrace{\frac{1}{2}\mathbb{E}_{P_V^{\otimes n}}\left|\frac{1}{W}\sum_{w=1}^{W}\frac{P^{\otimes n}_{V|U}(V^n|u_w)}{P^{\otimes n}_V(V^n)} - 1\right|}_{①} - \frac{1}{2}\mathbb{E}_{P_V^{\otimes n}}\left|\frac{1}{W}\left(\sum_{w\neq \ell}\frac{P^{\otimes n}_{V|U}(V^n|u_w)}{P^{\otimes n}_V(V^n)} + \frac{P^{\otimes n}_{V|U}(V^n|\tilde{u})}{P^{\otimes n}_V(V^n)}\right) - 1\right|$$

$$① = \frac{1}{2}\mathbb{E}_{P_V^{\otimes n}}\left|\frac{1}{W}\sum_{w\neq \ell}\frac{P^{\otimes n}_{V|U}(V^n|u_w)}{P^{\otimes n}_V(V^n)} + \frac{1}{W}\frac{P^{\otimes n}_{V|U}(V^n|u_\ell)}{P^{\otimes n}_V(V^n)} + \frac{1}{W}\frac{P^{\otimes n}_{V|U}(V^n|\tilde{u})}{P^{\otimes n}_V(V^n)} - \frac{1}{W}\frac{P^{\otimes n}_{V|U}(V^n|\tilde{u})}{P^{\otimes n}_V(V^n)} - 1\right|$$

(Triangle Ineq.)

$$\leq \frac{1}{2}\mathbb{E}_{P_V^{\otimes n}}\left|\frac{1}{W}\sum_{w\neq \ell}\frac{P^{\otimes n}_{V|U}(V^n|\tilde{u}_w)}{P^{\otimes n}_V(V^n)} + \frac{1}{W}\frac{P^{\otimes n}_{V|U}(V^n|\tilde{u})}{P^{\otimes n}_V(V^n)} - 1\right|$$

$$+ \frac{1}{2}\mathbb{E}_{P_V^{\otimes n}}\left|\frac{1}{W}\frac{P^{\otimes n}_{V|U}(V^n|u_\ell)}{P^{\otimes n}_V(V^n)} - \frac{1}{W}\frac{P^{\otimes n}_{V|U}(V^n|\tilde{u})}{P^{\otimes n}_V(V^n)}\right|$$

$$\Rightarrow \circledast \leq \frac{1}{2}\cdot\frac{1}{W}\,\mathbb{E}\left|\frac{P^{\otimes n}_{V|U}(V^n|u_\ell)}{P^{\otimes n}_V(V^n)} - \frac{P^{\otimes n}_{V|U}(V^n|\tilde{u})}{P^{\otimes n}_V(V^n)}\right|$$

$$\leq \frac{1}{2W}\left[\mathbb{E}_{P_V^{\otimes n}}\frac{P^{\otimes n}_{V|U}(V^n|u_\ell)}{P^{\otimes n}_V(V^n)} + \mathbb{E}_{P_V^{\otimes n}}\frac{P^{\otimes n}_{V|U}(V^n|\tilde{u})}{P^{\otimes n}_V(V^n)}\right]$$

$$= \frac{1}{W}$$

$\Rightarrow f(u_1,\dots,u_W) = \delta_{TV}\left(Q^{(C_n)}_{V^n}, P_V^{\otimes n}\right)$ is a $\left(\frac{1}{W}\right)_{w=1}^{W}$-bounded difference function.

Apply McDiarmids inequality

$$\mathbb{P}\left(f(U_1, \dots, U_{\overline{W}}) > \mathbb{E}[f(U_1, \dots, U_{\overline{W}}) + t \right) \leq e^{-\frac{2t^2}{\overline{W} \cdot \frac{1}{\overline{W}^2}}}$$

$$= e^{-2t^2 \overline{W}}$$

$$\overline{W} = 2^{nR} = e^{nR\ln(2)} \qquad\qquad = e^{-2t^2 \cdot e^{nR\ln(2)}}$$

Recall that if $R > I_P(U;V)$, then $\mathbb{E}[f(U_1, \dots, U_{\overline{W}})] \leq e^{\gamma n}$,
for some $\gamma > 0$, and large enough n.
Let $t = e^{-\tilde{\gamma} n}$; $\tilde{\gamma}$ to be chosen later.

First observe

$$\mathbb{P}\left(f(U_1, \dots, U_{\overline{W}}) > e^{-\gamma n} + e^{-\tilde{\gamma} n} \right) \leq e^{-2e^{-2\tilde{\gamma} n} \cdot e^{n \cdot R \cdot \ln(2)}}$$

$$= e^{-2e^{n}(R\ln(2) - 2\tilde{\gamma})}$$

Take $0 < \tilde{\gamma} < \frac{R\ln(2)}{2}$,

$$\gamma_1 = \min\{\gamma, \tilde{\gamma}\} > 0 \quad \text{and} \quad \gamma_2 = R\ln(2) - 2\tilde{\gamma} > 0$$