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Homework 7

Please print out these pages. I encourage you to work with your classmates on this homework. Please list your collaborators on this cover sheet. (Your grade will not be affected.) Even if you work in a group, you should write up your solutions yourself! You should include all computational details, and proofs should be carefully written with full details. As always, please write neatly and legibly.

Please follow the instructions for the "extended glossary" on separate paper (LaTeX it if you can!) Hand in your final draft, including full explanations and write your glossary in complete, mathematically and grammatically correct sentences. Your answers will be assessed for style and accuracy.

Please **staple** *this cover sheet, your exercise solutions, and your glossary together, in that order, and hand in your homework in class.*

GRADES

Exercises ____________________ / 50

Extended Glossary

Component	Correct?	Well-written?
Definition	/6	/6
Example	/4	/4
Non-example	/4	/4
Theorem	/5	/5
Proof	/6	/6
Total	/25	/25

Exercises.

1. Consider the matrix $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ over a field $\mathbb{F}$.

   (a) Find the eigenvalues and eigenvectors of A.
   The eigenvalues of A are λ which satisfy

$$A\vec{v} = \lambda\vec{v}$$

   for some $\vec{v} \neq 0$.
   Let $\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix}$. Then

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \lambda\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{pmatrix} a+b \\ b \end{pmatrix} = \begin{pmatrix} \lambda a \\ \lambda b \end{pmatrix}$$

   This gives

$$a + b = \lambda a$$

   and

$$b = \lambda b$$

   Let $b = 1$. Then $\lambda = 1$. It then follows that $a + 1 = 1 \rightarrow a = 0$. So there is only one distinct eigenvalue $\lambda = 1$, with corresponding eigenvector $\vec{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

(b) Show that A is not diagonalizable (similar to a diagonal matrix).
A matrix is diagonalizable if and only if for each eigenvalue the dimension of the eigenspace is equal to the multiplicity of that eigenvalue.

The eigenspace $E_{\lambda=1}(A) = \ker(A - I) = \{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\}$

Since we have eigenvalue 1 with multiplicity 2 and eigenspace of dimension 0. Thus A is not diagonalizable.

2. Consider the matrix $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

(a) Find the eigenvalues and eigenvectors of A over $\mathbb{C}$.
The eigenvalues of A are λ which satisfy

$$A\vec{v} = \lambda\vec{v}$$

for some $\vec{v} \neq 0$.
Let $\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix}$. Then

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{pmatrix} b \\ -a \end{pmatrix} = \begin{pmatrix} \lambda a \\ \lambda b \end{pmatrix}$$

This gives

$$b = \lambda a$$

and

$$-a = \lambda b$$

Let b = 1. Then

$$1 = -\lambda^2 \to \lambda_{1,2} = \pm i$$

and corresponding eigenvectors

$$\vec{v}_1 = \begin{pmatrix} -i \\ 1 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} i \\ 1 \end{pmatrix}$$

(b) Show that A is diagonalizable over $\mathbb{C}$, but not over $\mathbb{R}$.
Take $\lambda_1 = i$. The eigenspace for this eigenvalue is

$$\ker(A - iI) = \ker(\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix})$$

The vectors which multiply $A - iI$ to get the zero vector are vectors of the form

$$\vec{v} = \begin{pmatrix} \frac{b(1-i)^2}{2} \\ b \end{pmatrix}.$$

It is easy to see the dimension of this kernel is 1 which is equal to the multiplicity of this eigenvalue.
Similarly, Take $\lambda_1 = ii$. The eigenspace for this eigenvalue is

$$\ker(A - iI) = \ker(\begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix})$$

The vectors which multiply $A - iI$ to get the zero vector are vectors of the form

$$\vec{v} = \begin{pmatrix} \frac{-b(1+i)}{(i-1)} \\ b \end{pmatrix}.$$

It is easy to see the dimension of this kernel is 1 which is equal to the multiplicity of this eigenvalue.
Thus, since we have shown that each eigenvalues eigenspace has dimension equal to the multiplicity of that eigenvalue, A is diagonalizable.
It is easy to see that since the eigenvalues and eigenvectors exist only in $\mathbb{C}$ it is only diagonalizable over the complex field - and not the reals.

3. Suppose that $\lambda = a + bi$ is a complex number with $b \neq 0$. Let $\bar{\lambda} = a - bi$ denote the complex conjugate of λ. Consider the polynomial $f(z) = (z - \lambda)(z - \bar{\lambda})$.

   (a) Show that $f(z)$ is a quadratic polynomial, with real coefficients.

   $$f(z) = (z - \lambda)(z - \bar{\lambda}) = z^2 - z\lambda - z\bar{\lambda} + \lambda\bar{\lambda}$$
   $$= z^2 - z(a + bi) - z(a - bi) + (a^2 + b^2)$$
   $$= z^2 - 2az + (a^2 + b^2)$$

   where $a, b \in \mathbb{R}$ with $b \neq 0$. It is easy to now see that $f(z)$ is quadratic with respect to z with all real coefficients.

   (b) Show that $f(z)$ is irreducible, as a polynomial in $\mathbb{R}[z]$, that is, it cannot be written as a product of two polynomials - each of strictly smaller degree.
   To be irreducible as a polynomial in $\mathbb{R}[z]$ means that $f(z) = (z - c_1)(z - c^2)$ for some $c_1, c_2 \in \mathbb{R}$. However, we know that the zeros of $f(z)$ are specifically λ and $\bar{\lambda}$. Thus $f(z)$ is not reducible as a polynomial in $\mathbb{R}[z]$.

   (c) Show that if $p(z) \in \mathbb{R}[z]$ is a polynomial with real coefficients, then it has factorization of the form

   $$p(z) = a(z - c_1)...(z - c_r)(z^2 + p_1z + q_1)...(z^2 + p_sz + q_s),$$

   for some real numbers a, c_i, p_i, q_i, where each polynomial appearing is irreducible. Show that this is unique up to reordering the factors (but note, some factors may appear more than once).
   First we realize that this question is essentially asking us to show that every polynomial can be factored into the product of its factors, and that all complex factors come in conjugate pairs.

If $p \in \mathbb{C}[z]$ is a nonconstant polynomial, then p has a unique factorization (except for the order of the factors) of the form

$$p(z) = c(z - \lambda_1)...(z - \lambda_m)$$

where $c, \lambda_1, ..., \lambda_n \in \mathbb{C}$.

Now we note that for the case of our proof, if all the zeros of $p(z) \in \mathbb{C}[z]$ are real we are done.

However, if $p(z) \in \mathbb{C}[z]$ has a zero $\lambda \in C$, then $p(z)$ also has a zero $\bar{\lambda} \in \mathbb{C}$. As shown in part a, we can write

$$p(z) = (z - \lambda)(z - \bar{\lambda})q(z)$$
$$p(z) = (z^2 = 2Re(\lambda)z + |\lambda|^2)q(z)$$

for some $q(z)$ with degree two less then the degree of p. We now note that $p(z) \in \mathbb{R}[z]$ as written. If the zeros are real we factor out $(z - c_i)$, where $c_i \in \mathbb{R}$, and if the zeros are complex we note that each p_i is equal to the real part of the complex zero and that each q_i is the magnitude of the complex zero.

Now to show uniqueness we note that if a polynomial in $R[z]$ could be written in two different ways we break the fact that every polynomial in $\mathbb{C}$ is uniquely represented in the form stated above.

4. Suppose that the $n \times n$ matrix A has an eigenvector v with value λ. Suppose that $p \in \mathbb{F}[x]$ is a polynomial. Show that v is also an eigenvector of $p(A)$. Find its corresponding eigenvalue.

Have

$$A\vec{v} = \lambda\vec{v}$$

.

$\vec{v}$ being an eigenvector of $p(A)$ would mean $p(A)(\vec{v}) = \lambda^*\vec{v}$ for some λ^*.

$$p(A) = a_0 I + a_1 A + a_2 A^2 + ... + a_m A^m$$
$$p(A)\vec{v} = a_0 I\vec{v} + a_1 A\vec{v} + a_2 A^2\vec{v} + ... + a_m A^m\vec{v}$$
$$p(A)\vec{v} = a_0 I\vec{v} + a_1 \lambda\vec{v} + a_2 \lambda^2\vec{v} + ... + a_m \lambda^m\vec{v}$$
$$p(A)\vec{v} = \vec{v}(a_0 + a_1\lambda + a_2\lambda^2 + ... + a_m\lambda^m)$$
$$p(A)\vec{v} = p(\lambda)\vec{v}$$

Thus, $\vec{v}$ is an eigenvector of $p \in \mathbb{F}[x]$ with value $p(\lambda)$.

5. Let $D \in \mathcal{L}(C^\infty(\mathbb{R}))$ be the differentiation linear operator, that is $D(y(t)) = y'(t)$. In this problem, you may use the fact that if $p(x) \in \mathbb{R}[x]$ has degree d, then $\dim \ker p(D) = d$.

   (a) Show that $(D - rI)(f(t)e^{rt}) = f'(t)e^{rt}$.

$$\begin{aligned} (D - rI)(f(t)e^{rt}) &= D(f(t)e^{rt}) - rI(e^{rt}f(t)) \\ &= (f'(t)e^{rt} + re^{rt}f(t)) - re^{rt}f(t) \\ &= f'(t)e^{rt} \end{aligned}$$

(b) Find a basis for $\ker(D-2I)^3$.
We realize that $\ker((D-2I)^3)$ is a vector space of dimension 3.
Apply $D-2I$ to a function of the form $f(t)e^{rt}$ three times to get

$$(D-2I)(D-2I)(D-2I)e^{rt} = f'''(t)e^{2t}.$$

The kernel of this will be all the $f(t)$ which equal zero at their third derivative. These will be $f(t)$ of the form A, At, At^2, where $A \in \mathbb{R}$. All of the solutions are linearly independent since

$$Ae^{2t} + Bte^{2t} + Ct^2e^{2t} = 0$$
$$e^{2t}(A + Bt + Ct^2) = 0$$

only if $A = B = C = 0$. We also know that each of the solutions spans $\ker((D-2I)^3)$.
Thus,

$$e^{2t}, te^{2t}, t^2e^{2t}$$

is a basis for $\ker((D-2I)^3)$.

(c) Find a basis for $\ker((D-I)(D-2I)^2)$.
We note that $\ker((D-I)(D-2I)^2)$ contains $\ker((D-I)$ and $\ker((D-2I)^2)$. Thus we need f(t) of the form depicted below.

$$\ker(D-I) = \{f(t)e^t \mid f'(t)e^t = 0\}$$
$$\ker((D-2I)^2) = \{f(t)^{2t} \mid f''(t)e^{2t} = 0\}$$

f(t) satisfying the requirement in $\ker(D-I)$ are of the form $f(t) = A$, where $A \in \mathbb{R}$. f(t) satisfying the requirement in $\ker((D-2I)^2)$ are of the form $f(t) = A$ and $f(t) = At$, where $A \in \mathbb{R}$.
Each solution spans its set and so the linear combination of solutions spans the entirety of $\ker((D-I)(D-2I)^2)$. We next see that Ae^t, Ae^{2t}, Ate^{2t} are linearly independent since

$$Ae^t + Be^{2t} + Cte^{2t} = 0$$

only if $A = B = C = 0$.
Thus e^t, e^{2t}, te^{2t} is a basis for $\ker((D-I)(D-2I)^2)$.

(d) The solution set of the function y(t) which satisfy the differential equation

$$y''' - y'' - y' + y = 0$$

is a vector space.
Find a basis for this vector space.
This question is asking for the kernel of $D^3 - D^2 - D + I$. We note that this can alternatively be written as $(D-I)^2(D+I)$.
Thus this question can alternatively be written as find a basis for $\ker((D-I)^2(D+I))$.
Similar to above, we note that $\ker((D-I)^2(D+I))$ contains $\ker((D-I)^2)$ and $\ker(D+I)$.
Thus we need $y(t) = f(t)e^{rt}$ of the form

$$\ker(D+I) = \{f(t)e^{-t} \mid f'(t)e^{-t} = 0\}$$
$$\ker((D-I)^2) = \{f(t)e^{-t} \mid f''(t)e^{t} = 0\}$$

It is easy to see that $f(t) = A$ satisfies the requirement on $\ker(D+I)$ and so Ae^{-t} spans it.
Similarly, it is easy to see $f(t) = A$ and $f(t) = At$ satisfies the requirement on $\ker((D-I)^2)$ and so $\{Ae^t, Ate^t\}$ spans it.
Thus $\{e^{-t}, e^t, te^t\}$ spans $\ker((D-I)^2(D+I))$.

6. Given an $n \times n$ matrix A (over $\mathbb{R}$ or $\mathbb{C}$), we can define another $n \times n$ matrix by the formula

$$e^A = I + A + \frac{A^2}{2} + \frac{A^3}{3!} + ... + \frac{A^n}{n!} + ...$$

It turns out that this infinite sum of matrices converges for all matrices A.

(a) Compute e^A for the diagonal matrix with entries $\lambda_1, \lambda_2, ..., \lambda_n$.
We note that for a diagonal matrix with entries $\lambda_1, \lambda_2, ..., \lambda_n$,

$$A^n = \begin{pmatrix} \lambda_1^n & 0 & ... & 0 \\ 0 & \lambda_2^n & ... & 0 \\ . & . & . & . \\ 0 & 0 & ... & \lambda^n \end{pmatrix}$$

We can then see that

$$e^A = \begin{pmatrix} 1 + \lambda_1 + \frac{\lambda_1^2}{2!} + \frac{\lambda_1^3}{3!} + ... + \frac{\lambda_1^n}{n!} + ... & & \\ & \ddots & \\ & & 1 + \lambda_n + \frac{\lambda_n^2}{2!} + \frac{\lambda_n^3}{3!} + ... + \frac{\lambda_n^n}{n!} + ... \end{pmatrix}$$

which after simplifying yields

$$e^A = \begin{pmatrix} e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_n} \end{pmatrix}$$

(b) Compute e^A if $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.
First we note the pattern that A raised to any even power produces values in the main diagonal and A raised to any odd power produces values in the "off-diagonal".
The proof is by noting the transformation the matrix does.
The matrix A as defined is a 270-degree rotation about the rotation.

Thus A applied twice is a 270-degree rotation about the origin applied to a 270-degree rotation of a point about the origin which is a 180-degree rotation. This is seen in A^2 below.

$$A^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

Applying A again would now correspond to another 270-degree rotation of a point rotated 180-degree, which is a 90 degree rotation. This can be seen in A^3.

$$A^3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

A 270 degree rotation about a point rotated 90 degrees is a 360-degree rotation - which would bring you back to your starting point, so we would expect the identity matrix. This is indeed the case and can be seen in A^4.

$$A^4 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Now we rinse and repeat this process an infinite amount of times. Via the above logic we conclude that

$$A^{1+4i} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$A^{2+4i} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$A^{3+4i} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$A^{4+4i} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\forall i \in \mathbb{N}$.

Note: The proof could have been done by induction but the geometric intuition is much more pleasant to think about.

Now it is easy to see that

$$\sum_{n=1}^{\infty} \frac{1}{n!} A^n = \begin{pmatrix} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} & \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \\ \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!} & \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} \end{pmatrix}$$

and that

$$I + \sum_{n=1}^{\infty} \frac{1}{n!} A^n = \begin{pmatrix} \cos(1) & \sin(1) \\ \sin(-1) & \cos(1) \end{pmatrix}$$

(c) Compute e^A if $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

First we need to find some sort of pattern on A.

We note that A multiplied with itself will keep the bottom triangle the same and change the top right value to the sum of the top row. Since this multiplied version retains its form A multiplied with it again will produce the same result. Formally,

$$A^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

Then

$$e^A = \begin{pmatrix} \sum_0^\infty \frac{1}{n!} & \sum_1^\infty \frac{n}{n!} \\ 0 & \sum_0^\infty \frac{1}{n!} \end{pmatrix}$$

$$e^A = \begin{pmatrix} e & e \\ 0 & e \end{pmatrix}$$

(d) If $B = Q^{-1}AQ$, show that $e^B = Q^{-1}e^AQ$.

$$e^B = Q^{-1}IQ + Q^{-1}AQ + \frac{1}{2!}Q^{-1}AQQ^{-1}AQ + \frac{1}{3!}Q^{-1}AQQ^{-1}AQQ^{-1}AQ + ...$$

$$e^B = I + Q^{-1}AQ + \frac{Q^{-1}A^2Q}{2!} + \frac{Q^{-1}A^3Q}{3!} + ...Q^{-1}A^NQ + ...$$

$$e^B = Q^{-1}(I + A + \frac{A^2}{2!} + ... + \frac{A^n}{n!} + ...)Q$$

$$e^B = Q^{-1}e^AQ$$