Example

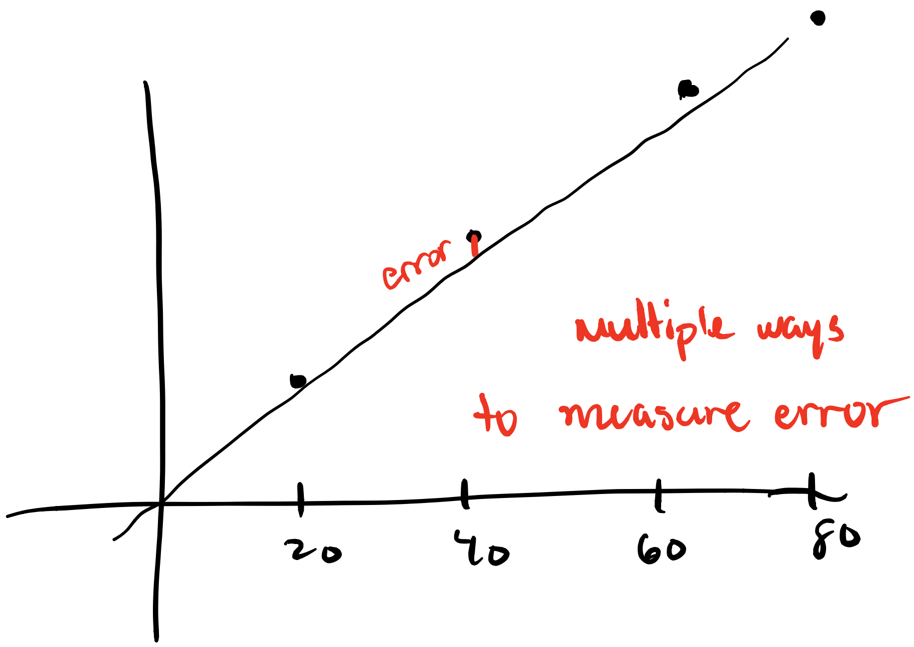
Car crash dummies

Speed of impact affects damage to persons

Measurements

speed	damage
20	2.7
40	5.0
66	7.5
76	8.6

$(\hat{x}_i, \hat{y}_i)$



Want

$$\sum_{i=1}^{N} \left((a+b\hat{x}_i)-\hat{y}_i\right)^2$$

s.t. is minimal

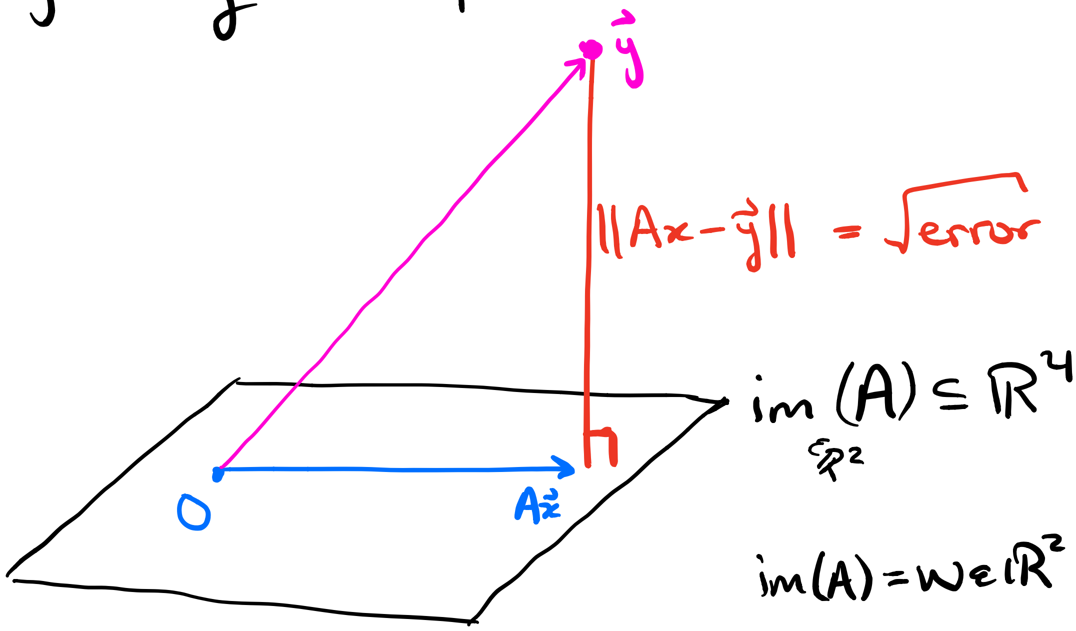
Want to find a, b s.t.

$$\underbrace{\begin{bmatrix} 1 & 20 \\ 1 & 40 \\ 1 & 60 \\ 1 & 80 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 2.7 \\ 5.0 \\ 7.5 \\ 8.6 \end{bmatrix}}_{\vec{y}}$$

NO SOL'N

i.e error never zero $\forall$ points

project y onto plane



Find the point $A\vec{x}$ in $\text{im}(A) = W$ s.t. $A\vec{x}$ is closest to $\vec{y}$.

Key point: $\|A\vec{x}-\vec{y}\|$ will be minimal at orthogonal projection $pr_W(\vec{y})$

Orthogonal Complements

If $W \subseteq V$ is a subspace (or subset) of inner product space V, let

$$W^{\perp} \triangleq \{ \vec{v} \in V \mid \langle \vec{v}, \vec{w} \rangle = \vec{0} \quad \forall \vec{w} \in W \}$$

Basic facts

① $W^\perp \subseteq V$ is a subspace

② $W \cap W^\perp = \{0\}$

Proof

① on your own

② if $\vec{v} \in W \cap W^\perp$, $\vec{v} \in W$, $\vec{v} \in W^\perp$.

so $\langle \vec{v}, \vec{v} \rangle = 0 \Rightarrow \vec{v} = 0$

Proposition

Suppose $W \subseteq V$ is a finite-dimensional subspace.

Then

$$V = W \oplus W^\perp$$

Proof

(a) $W \cap W^\perp = \{0\}$ as above

(b) $W + W^\perp = V$

Start w/ orthonormal basis $\vec{e}_1, \dots, \vec{e}_m$ of W.

Given $\vec{v} \in V$, want to write it as

$$\vec{v} = (\text{something in } W) + (\text{something in } W^\perp)$$

$$= \langle \vec{v}, \vec{e}_1 \rangle \vec{e}_1 + \langle \vec{v}, \vec{e}_2 \rangle \vec{e}_2 + \cdots + \langle \vec{v}, \vec{e}_m \rangle \vec{e}_m$$

So

$$\vec{v} = \underbrace{\vec{w}}_{\in W} + \underbrace{(\vec{v} - \vec{w})}_{\overset{?}{\in} W^\perp}$$

Need

$$\vec{v} - \vec{w} \perp \text{ all vectors in } W.$$

If

$$\langle \vec{v} - \vec{w}, \vec{e}_i \rangle = 0 \qquad i = 1, \ldots, m$$

then

$$\vec{v} - \vec{w} \in W^\perp$$

$$\langle \vec{v} - \vec{w}, \vec{e}_i \rangle = \langle \vec{v}, \vec{e}_i \rangle - \langle \vec{w}, \vec{e}_i \rangle$$
$$= \langle \vec{v}, \vec{e}_i \rangle - \langle \vec{v}, \vec{e}_i \rangle = 0$$

__Corollary__

$$\dim W^\perp = \dim V - \dim W$$

(in case $\dim W < \infty$)

__Proposition__

Let A be an $m \times n$ matrix, $V = \mathbb{R}^n$, $W = \operatorname{im} A^T$, then

$$A : \mathbb{R}^n \to \mathbb{R}^m$$
$$A^T : \mathbb{R}^m \to \mathbb{R}^n$$

$$(\operatorname{im} A^T) = \ker(A)^\perp$$

ASIDE

Recall If $\vec{x}, \vec{y} \in V$ (i.e $x, y \in \mathbb{R}^n$)

$$\langle \vec{x}, \vec{y} \rangle = \vec{x}^T \vec{y} = \vec{y}^T \vec{x}$$

"good to drop dot product things into this form and see what happens"

Proof

$\vec{v} \in \mathrm{im}(A^T)$

$\exists\, \vec{x} \in \mathbb{R}^m$ s.t. $\vec{v} = A^T \vec{x}$

If $\vec{w} \in \ker(A)$

then $\vec{v} \cdot \vec{w} = v^T w$

$= x^T A w = 0$

$\therefore \vec{v} \in \ker(A)^\perp$ and $\mathrm{im}\, A^T \subseteq \ker(A)^\perp$

Other half left for reader.

Orthogonal Projection

Suppose W is a finite-dimensional subspace of V.

Define

$$\mathrm{pr}_W : V \to V$$

by

$\mathrm{pr}_W(\vec{v}) = \vec{w}$ if $\vec{v} = \vec{w} + \vec{u}$ is the unique representation w/ $\vec{w} \in W$, $\vec{u} \in W^\perp$

Facts

- $pr_W \in \mathcal{L}(V)$
- $pr^2_W = pr_W$
- $\ker pr_W = W^{\perp}$
- $\operatorname{im} pr_W = W$