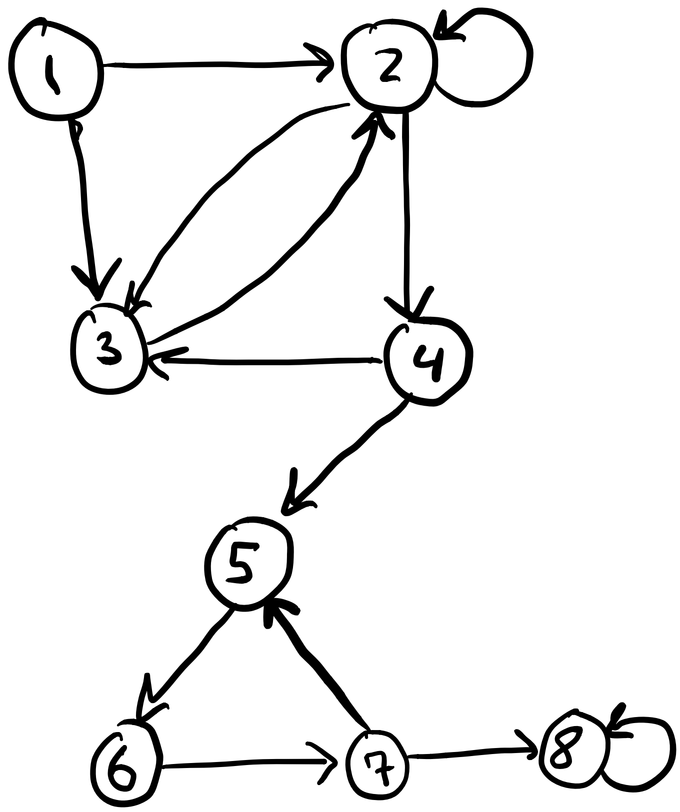
$i \rightarrow j$ if $P_{i,j}^{(n)} > 0$; i communicates w/ j

$i \leftrightarrow j$ they communicate w/ eachother
$\Rightarrow$ communication class

Example



Communication Classes

$C_1 = \{1\}$ "No Return"

$C_2 = \{2, 3, 4\}$

$C_3 = \{5, 6, 7\}$

$C_4 = \{8\}$ "Absorbant"

Period of a State

Let $\mathcal{N}_i \triangleq \{n \geq 1 \mid P_{i,i}^{(n)} > 0\}$

$\Downarrow$

$\Pr[X(n) = i \mid X(0) = i]$

$\mathbb{P}^{(n)} = \mathbb{P}^n$ by Chapman Komogorov

The period $d(i)$ of state i is defined as

$$d(i) \triangleq \begin{cases} \gcd\{\mathcal{N}_i\} & \text{if } \mathcal{N}_i \neq \phi \\ 1 & \text{if } \mathcal{N}_i = \phi \end{cases}$$

In example above:

① : $P_{1,1}^{(n)} = 0 \;, \quad n \geq 1$

$\mathcal{N}_1 = \phi$

$d(1) = 1$ ← call these aperiodic states

② : $\mathcal{N}_2 = \{1, \ldots, \}$ ↙ gcd of set w/ 1 is 1

$d(2) = 1$ also aperiodic

③ $\mathcal{N}_3 = \{2, 3, \ldots\}$

$d(3) = 1$

④ $\mathcal{N}_4 = \{3, 4, \ldots\} \rightarrow d(4) = 1$

Notice a pattern!

Every state in the same communication class have the same period.

Call this the period of a class / class period

$$⑤ : N_5 = \{3, 6, 9, \dots\}$$

$$d(5) = 3 \rightarrow d(6) = d(7) = 3$$

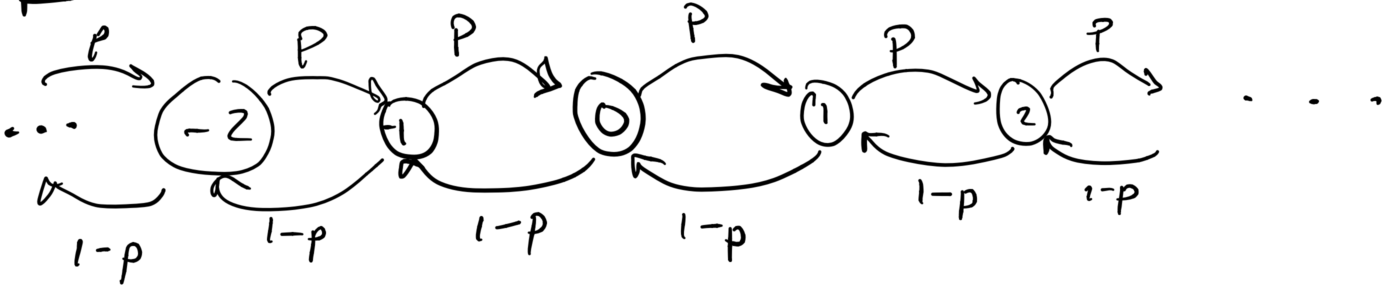
Same for ⑥, ⑦.

$$⑧ : N_8 = \{1, 2, \dots\}$$

$$d(8) = 1$$

Example: Random Walk

<u>$p > 0$</u>

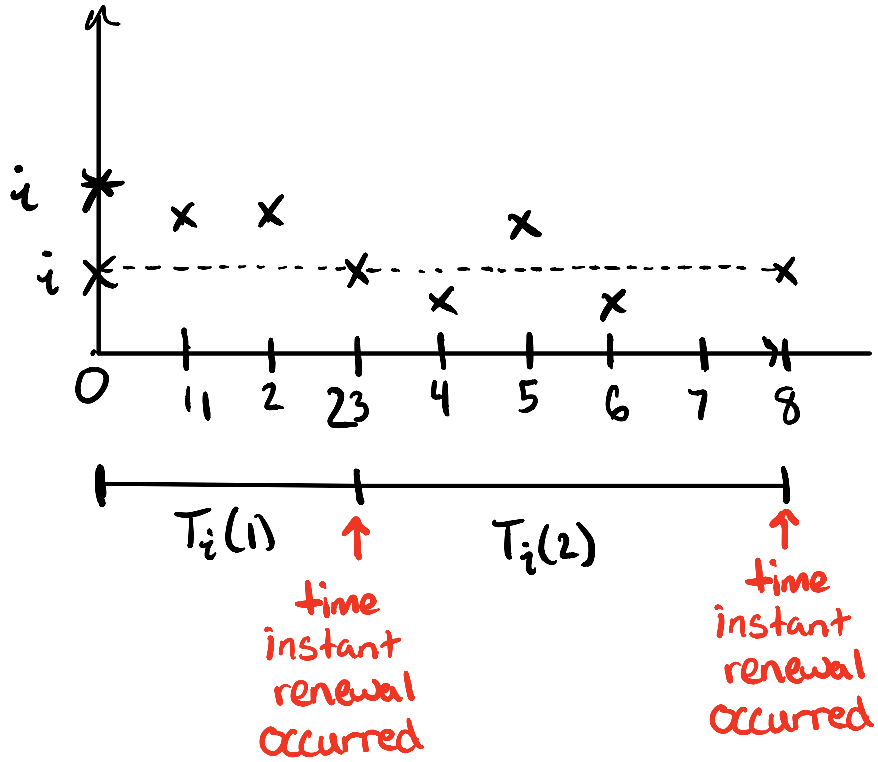


$$C = \{\dots -2, -1, 0, 1, 2, \dots\}$$

Markov chain <u>irreducible</u> if it only has ONE communication class.

$$d(i) = 2 \ \forall i$$

Return Time to state (i)



Once you return to state i the future evolution of the system is independent of the past.

$T_i(1) \perp\!\!\!\perp T_i(2) \perp\!\!\!\perp \cdots$

$\{T_i(k)\}$ iid

Probability of Return

$$f_i^{(n)} \triangleq \Pr[X_1 \neq i, X_2 \neq i, \ldots, X_{n-1} \neq i, X_n = i \mid X_0 = i]$$

NOTE

$$P_{i,i}^{(n)} \triangleq \Pr[X_n = i \mid X_0 = i]$$

NOT the same

pmf of T_i

$$f_i^{(n)} = \Pr[T_i(n)]$$

$$\left[f_i^{(1)}, f_i^{(2)}, f_i^{(3)}, \ldots\right]$$

$$f_i \triangleq \sum_{n=1}^{\infty} f_i^{(n)}$$ is the probability of ever returning to state i

Rearrange and Transience

- State is recurrent if $f_i = 1$
- State i is transient if $f_i < 1$

The # of returns

$$N_i \triangleq \sum_{n=1}^{\infty} \mathbb{1}_{[X_n = i \mid X_0 = i]}$$

where

$$\underbrace{\mathbb{1}_A}_{A \uparrow \text{ event}} = \begin{cases} 1 & \text{if } A \text{ occurs} \\ 0 & \text{otherwise} \end{cases}$$ i.e Bernoulli(p) w/ $p = Pr[A]$

$$\mathbb{E}[\mathbb{1}_A] = Pr[A]$$

Theorem:

stronger statement $Pr[N_i = \infty] = 1$! important

$$\begin{cases} \mathbb{E}[N_i] = \infty & \text{if } i \text{ is recurrent} \\ E[N_i] < \infty & \text{if } i \text{ is transient} \end{cases}$$

$\hookrightarrow Pr[N_i = \infty] = 0$

For i transient,

$$N_i \sim \text{geom}(p) \quad ; \quad p = (1 - f_i)$$

taking values

$$0, 1, 2, \ldots$$

$$\Pr[N_i = k] = p^k (1-p) = f_i^k (1 - f_i)$$

and

$$\mathbb{E}[N_i] = \frac{1-p}{p} = \frac{f_i}{1 - f_i}$$

Recurrence Criteria

- i is recurrent iff $f_i = 1$
- i is recurrent iff $\sum_{n=1}^{\infty} P_{i,i}^{(n)} = \infty$
- $i \leftrightarrow j$, then i is recurrent iff j is recurrent
- finite state Markov chain, irreducible, are recurrent