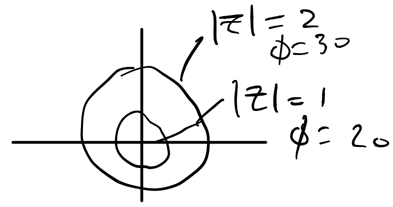
Example : "Washers"



By circular symmetry,
seek $\phi = \phi(r)$ only, not dependent on θ.
Guess:

$$\phi = A \ln r + B \quad \leftarrow \text{For Washer Probs}$$

Idea: Real and Imaginary part of an analytic $f(z)$ automatically satisfy $\nabla^2 u = 0$ or $\nabla^2 v = 0$

Recall $\log(z) = \ln(r) + i\theta$.

On $r=1$, $\phi = 20 = A \cancel{\ln(1)}^{0} + B \Rightarrow B = 20$

On $r=2$, $\phi = 30 = A \ln(2) + 20$

$\Rightarrow A = 10/\ln(2)$

$\therefore$

$$\phi(z) = \frac{10}{\ln(2)} \ln(|z|) + 20$$

①

$$\phi(z) = A \ln(r) + B$$

On $r=1$, $0 = A \ln(1) + B \Rightarrow B = 0$

On $r=2$, $10 = A \ln(2)$

$$A = \frac{10}{\ln(2)}$$

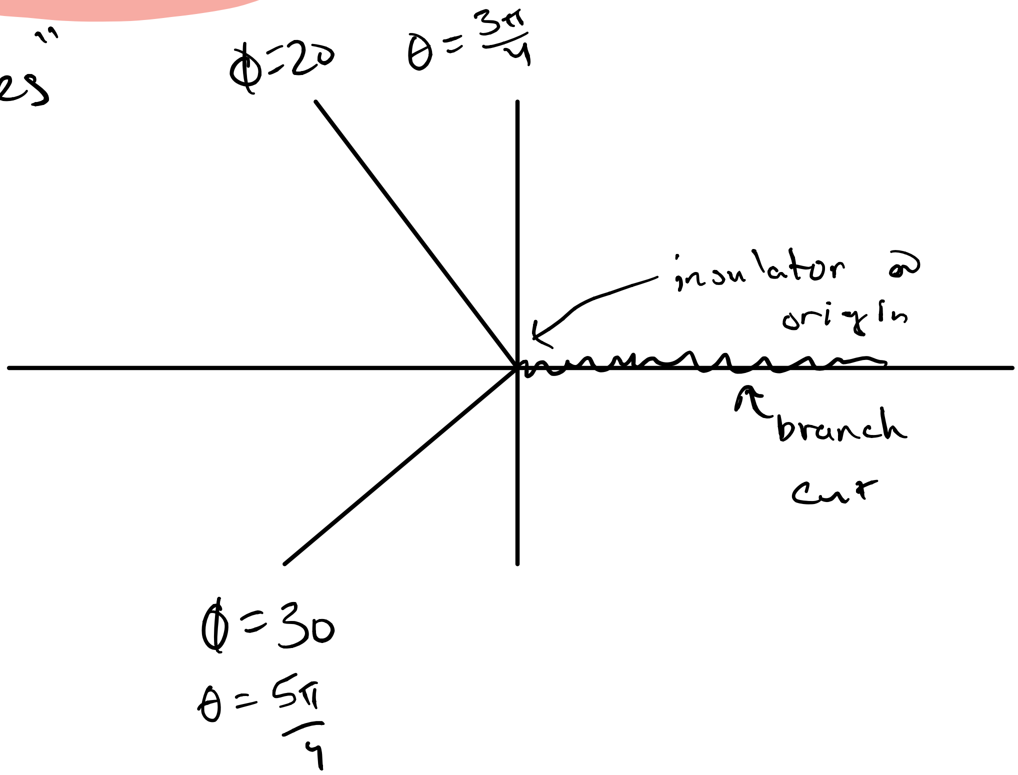
$$\phi(z) = \frac{10}{\ln(2)} \ln(|z - z_0|)$$

At $z=0$,

$$\phi(0) = \frac{10}{\ln(2)} \cdot \ln(|0-i-1|)$$

$$= \frac{10\ln(\sqrt{2})}{\ln(2)}$$

Example: "Weedges"



$\phi = A\theta + B$ (use when ϕ is constant on radial lines)

$20 = A\frac{3\pi}{4} + B$

$30 = A\frac{5\pi}{4} + B$

$-10 = -A\frac{\pi}{2}$ $\qquad B=5$

$A = \frac{20}{\pi}$ $\qquad \Rightarrow \phi = \frac{20}{\pi}\theta + 5$

(2) $\phi(z) = A\theta + B$ At $1, i$, angles $0, \frac{3\pi}{2}$

$0 = A\frac{3\pi}{2} + B$

$10 = A(0) + B \Rightarrow B = 10$

$A = -\frac{20}{3\pi}$

$\phi(z) = \frac{-20}{3\pi}\theta + 10$

At $z = 0$

$\phi(0) = \frac{-20}{3\pi} \operatorname{Arg}(0-1-i) + 10$

$= \frac{-20}{\cancel{3\pi}} \cdot \frac{5\cancel{\pi}}{4} + 10$

$= \frac{-20}{12} + 10$

(3) Solution has lines of slope 3

i.e

$$x + y = c$$

So

$$\phi(x,y) = A(x+y) + B$$

$$10 = A(3) + B$$

$$-50 = A(-3) + B$$

$$-40 = 2B$$

$$B = -20$$

$$10 = 3A - 20$$

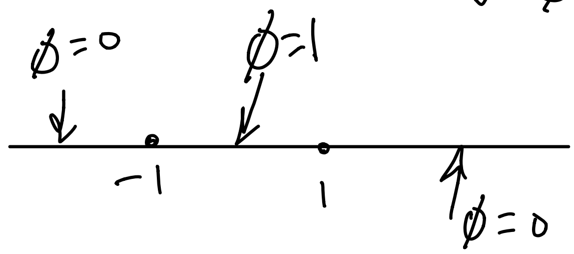
$$A = 10$$

$$\Rightarrow \phi(x,y) = 10(x+y) - 20$$

$$\phi(1,1) = 10(2) - 20 = 0$$

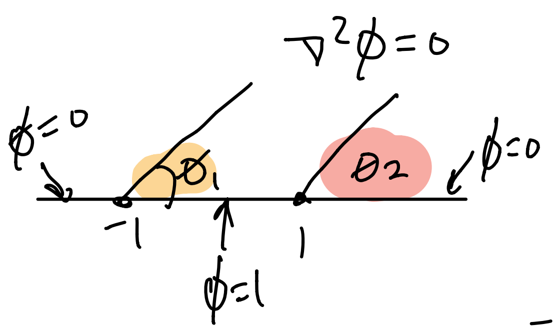
Ex 3: "Wall"

$\nabla^2 \phi = 0$ here



Sol'n

Define two angles



Seek soln of

$$\phi = A_1 \theta_1 + A_2 \theta_2 + B$$

$$= A_1 \operatorname{Arg}(z+1) + A_2 \arg(z-1) + B$$

$$-\pi < \theta_1 \leq \pi$$

$$-\pi < \theta_2 \leq \pi$$

$\boxed{\text{Impose B.C.}}$

For

$z = x > 1$, have $\theta_1 = \theta_2 = 0$

So

$$\phi = A_1(0) + A_2(0) + B$$

$$\Rightarrow B = 0$$

For

$-1 < x < 1$: $\theta_1 = 0$, $\theta_2 = \pi$

So

$$\phi = A_1(0) + A_2\pi = 1$$

$$A_2 = \frac{1}{\pi}$$

For $z = x < -1$: $\theta_1 = \theta_2 = \pi$

$$\phi = 0 = A_1\theta_1 + \frac{1}{\pi}\theta_2$$

$$0 = A_1\pi + \frac{1}{\pi}\pi$$

$$A_1 = -\frac{1}{\pi}$$

Therefore,

$$\phi = \frac{1}{\pi}\left[\text{Arg}(z-1) - \text{Arg}(z+1)\right]$$

$$= \frac{1}{\pi}\left[\theta_2 - \theta_1\right]$$

④ "Wall"

Soln of the form

$$\phi(z) = A_1 \text{Arg}|z+1| + A_2 \text{Arg}|z-2| + A_3$$

$$\phi(z>2) = A_1(0) + A_2(0) + A_3 = 0$$

$$\Rightarrow A_3 = 0$$

$$\phi(z<-1) = A_1\pi + A_2\pi = 0$$

$$A_1 = -A_2$$

$\phi(-1 < z < 2) = \pi = A_1(0) + A_2\pi$

$\Rightarrow A_2 = 1$

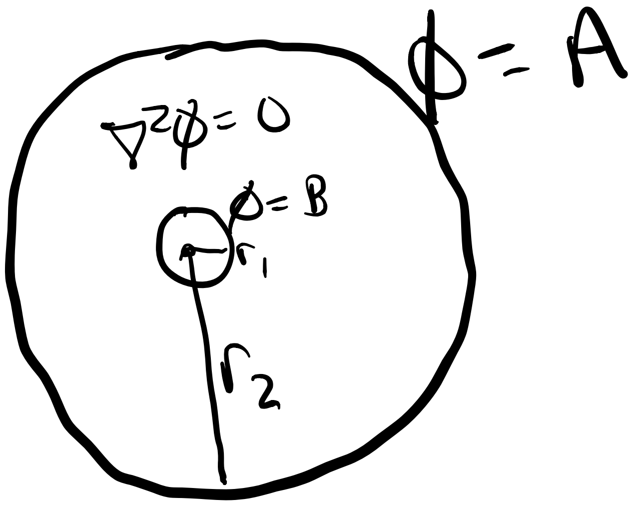
$A_1 = -1$

$\Rightarrow \phi(z) = \text{Arg}|z+1| - \text{Arg}|z-2|$

$\phi(2,3) = \text{Arg}|3+3i| - \text{Arg}|0+3i|$

$= \frac{\pi}{4} - \frac{\pi}{2} = -\frac{\pi}{2}$

⑤ General Soln To



$\phi(z) = C_1 \ln(r) + C_2$

$\phi(r_1) = C_1 \ln(r_1) + C_2 = B$

$$\phi(r_2) = C_1 \ln(r_2) + C_2 = A$$

$$C_1 \ln(r_1) - C_1 \ln(r_2) = B - A$$

$$C_1 = \frac{B-A}{\ln(r_1) - \ln(r_2)}$$

$$C_2 = B - \frac{(B-A)\ln(r_1)}{\ln(r_1) - \ln(r_2)}$$

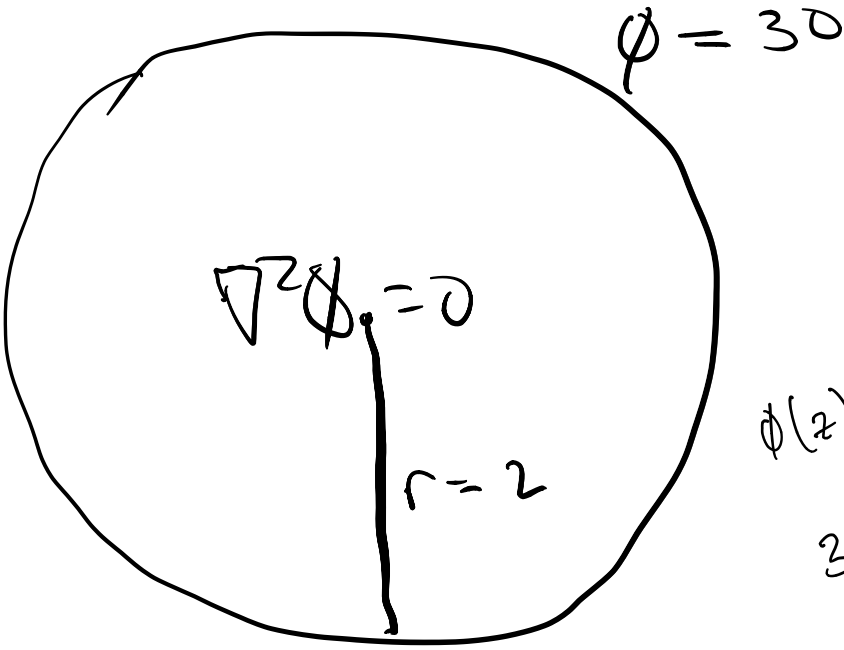
$$= \frac{B\ln(r_1) - B\ln(r_2) - B\ln(r_1) + A\ln(r_1)}{\ln(r_1) - \ln(r_2)}$$

$$= \frac{A\ln(r_1) - B\ln(r_2)}{\ln(r_1) - \ln(r_2)}$$

$$\phi(z) = \frac{B-A}{\ln(r_1) - \ln(r_2)} \ln(|z|) + \frac{A\ln(r_1) - B\ln(r_2)}{\ln(r_1) - \ln(r_2)}$$

$$\lim_{r_1 \to 0} \phi(z) = \frac{-A}{\ln(r_1) - \ln(r_2)} \ln(|z|) + \frac{A\ln(r_1)}{\ln(r_1) - \ln r_2)}$$

⑥ $\phi(x,y)$ for



$\phi(z) = A \ln r + B$

$30 = A \ln(2) + B$

$B = 30 - A\ln(2)$

$\phi(z) = A(\ln|z| - \ln(2)) + 30$

From above, let $A = 30$, $r \to 0$

$$\phi(z) = \frac{\ln(|z|) - +30}{\ln(2)} = \frac{30}{\ln|2|} \ln|z|$$

⑦ Prove that

$(\arg z)(\mathrm{Log}\,|z|)$ is harmonic

$(\arg z)(\mathrm{Log}\,|z|)$ is the imaginary part of

$\dfrac{(\log z)^2}{2}$ which is an analytic function

therefore it is harmonic!

$$\frac{(\log z)^2}{2} = \frac{(\ln r)^2}{2} + \frac{2i\theta \ln r}{2} - \frac{\theta^2}{2}$$

$$\operatorname{Im}\left(\frac{(\log z)^2}{2}\right) = \theta \ln r = (\arg z)(\operatorname{Log} |z|)\ !$$