(1)

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

avg of $\cos^4(\theta)$ on $0 \leq \theta \leq 2\pi$

$$\int_0^{2\pi} \cos^4\theta \, d\theta = \int_0^{2\pi} \frac{1}{16}\left(6 + 8\cos(2\theta) + 2\cos(4\theta)\right) d\theta$$

$$= \frac{1}{16}\left[6\theta + 4\sin(2\theta) + \frac{1}{2}\sin(4\theta)\Big|_0^{2\pi}\right]$$

$$\boxed{= \frac{3}{4}\pi}$$

The average is thus $\dfrac{\frac{3}{4}\pi}{2\pi} = \boxed{\dfrac{3}{8}}$

(2)

$f(z)$ and $\overline{f(z)}$ are analytic in domain D.

What can be concluded about $f(z)$?

$$f(z) = u(x,y) + i v(x,y)$$

$$\overline{f(z)} = u(x,y) - i v(x,y)$$

Cauchy-Riemann Equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

for $f(z)$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

for $\bar{f}(z)$

$$\frac{\partial u}{\partial x} = \frac{\partial(-v)}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial(-v)}{\partial x}$$

Since both of these must be satisfied it must be that $v(x,y) = 0$ and our function $f(z) = \bar{f}(z)$ is **REAL** valued.

③ a. $\left| \int_\gamma f(z)\, dz \right| \leq \max_{z \in \gamma} |f(z)| \cdot \text{Length}(\gamma)$

ML Estimate

b. γ denotes vertical line segment from $z = 0$ to $z = i$.

Use ML estimate to show that

$$\left| \int_\gamma e^{\sin(z)}\, dz \right| \leq 1$$

$\text{Length}(\gamma) = 1$

But what is $\sin(z)$?

Algebraically,

$$\begin{aligned} \sin(z) &= \sin(x+iy) \\ &= \sin(x)\cos(iy) + \cos(x)\sin(iy) \\ &= \sin(x)\cosh(y) + \cos(x)\sinh(y) \end{aligned}$$

Geometrically? No idea so lets keep doing Algebra sin(z) is also

$$\frac{e^{iz}-e^{-iz}}{2i} = \frac{e^{i(x+iy)}-e^{-i(x+iy)}}{2i}$$

$$= \frac{e^{ix}e^{-y}-e^{-ix}e^{y}}{2i}$$

Since γ is from $0 \to i$ along the imaginary axis
$x=0$, $0 \le y \le 1$

so our expression for sin(z) becomes

$$\frac{e^{-y}-e^{y}}{2i}$$ ← increases as y increases

$$e^{\left(\frac{e^{-y}-e^{y}}{2i}\right)} = e^{i\frac{(e^{y}-e^{-y})}{2}}$$

$$\left|e^{i\frac{(e^{y}-e^{-y})}{2}}\right| = 1$$

So,

$$\left|\int_\gamma e^{\sin z}\right| \le 1 \cdot 1 = 1$$

(4) a. Cauchy's Integral Formula for derivatives?

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_{\gamma} \frac{f(z)}{(z-a)^{n+1}} dz$$

b. $\oint_C \frac{e^{iz}}{z^3} dz$, C is $|z| = 2$

$(+)$ traversed once CCW

in $|z| = 2$ there's a singularity at $z = 0$.

$$\frac{2\pi i f''(0)}{2!} = \oint_{|z|=2} \frac{e^{iz}}{z^3} dz \quad , \quad f(z) = e^{iz}$$

$$\frac{2\pi i}{2!}\left[i^2 e^{iz}\right] = \frac{-2\pi i}{2} = \boxed{-i\pi}$$

⑤ $f(z) = \frac{1}{z-3}$, valid for $|z| > 3$

$$\frac{-1}{3-z} \to \frac{-1}{3(1-\frac{z}{3})} \qquad u = \frac{3}{z},\ |u|<1$$

$$\to \frac{-1}{3(1-\frac{1}{u})} = \frac{-u}{3(u-1)} = \frac{u}{3(1-u)}$$

$$= \frac{u}{3} \cdot \frac{1}{1-u}$$

$$= \frac{u}{3}\left[1 + u + u^2 + u^3 + \cdots\right]$$

First two terms?

$$\boxed{\frac{u}{3} + \frac{u^2}{3}} \qquad u = \frac{3}{z}$$

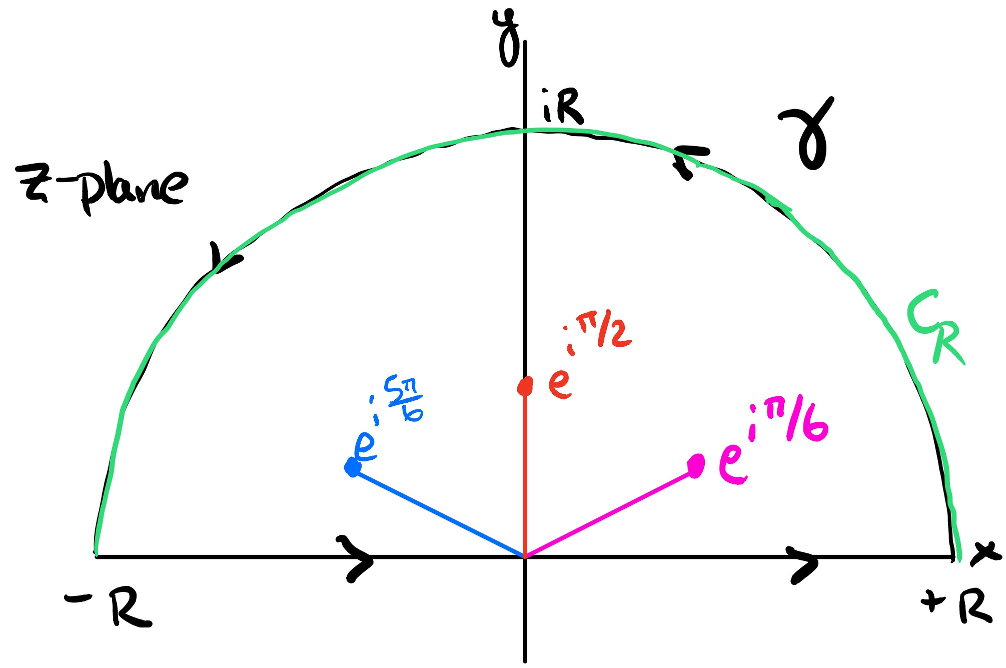
So

$$\frac{3}{3z} + \frac{3^2}{3z^2} = \boxed{\frac{1}{z} + \frac{3}{z^2}}$$

⑥ $$\int_0^\infty \frac{1}{x^6+1}\,dx = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{1}{x^6+1}\,dx$$

$$\int_\gamma \frac{1}{z^6+1}\,dz = \lim_{R\to\infty} \int_{-R}^{+R} \frac{1}{x^6+1}\,dx + \int_{\text{semi circle}} \frac{1}{z^6+1}\,dz$$

$$z^6+1=0 \Rightarrow z_k = e^{i\frac{\pi}{6}+\frac{2\pi}{3}k}$$



$k=0,1,2$ our singularities lie inside our contour

So z_0, z_1, z_2 lie in our contour

Need to find

$$\int_\gamma \frac{1}{z^6+1}\,dz = 2\pi i \sum_{k=0}^{2} \text{Res}\left(f(z); z_k\right)$$

$$\int_{C_R} \frac{1}{z^6+1}\,dz = ?$$

Intuition says this goes to zero. Let us verify.

$$\left| \int_{C_R} \frac{1}{z^6+1}\,dz \right| \le \max_{z \in C_R} \left| \frac{1}{z^6+1} \right| \cdot \underbrace{\pi R}_{\text{length of } C_R}$$

$$\le \lim_{R \to \infty} \frac{1}{R^6+1} \cdot \pi R \simeq \frac{1}{R^5} \to 0$$

$$\int_\gamma \frac{1}{z^6+1}\,dz = 2\pi i \sum_{k=0}^{2} \text{Res}\left(f(z); z_k\right)$$

$$\text{Res}(f; z_0) = \lim_{z \to e^{i\pi/6}} \frac{(z - e^{i\pi/6})}{z^6+1} \longrightarrow \lim_{z \to e^{i\pi/6}} \frac{1}{6z^5}$$

$$\text{Res}(f; z_1) = \lim_{z \to i} \frac{(z-i)}{z^6+1} \longrightarrow \lim_{z \to i} \frac{1}{6z^5}$$

$$\text{Res}(f; z_2) = \lim_{z \to e^{i5\pi/6}} \frac{(z - e^{i5\pi/6})}{z^6+1} \to \lim_{z \to e^{i5\pi/6}} \frac{1}{6z^5}$$

$$2\pi i \sum_{k=0}^{2} \mathrm{Res}(f; z_k) = 2\pi i \left[\frac{1}{6e^{i5\pi/6}} + \frac{1}{6e^{i\frac{5\pi}{2}}} + \frac{1}{6e^{i25\pi/6}}\right]$$

$$= 2\pi i \left[\frac{1}{6}e^{-i\frac{5\pi}{6}} + \frac{1}{6}e^{-i\frac{5\pi}{2}} + \frac{1}{6}e^{i\frac{-25\pi}{6}}\right]$$

$$= 2\pi i \left(\frac{1}{6}\left[\overset{\text{cancels out}}{\cancel{-\frac{\sqrt{3}}{2}}} - \frac{1}{2}i\right] + \frac{1}{6}\left[-i\right] + \frac{1}{6}\left[\overset{\text{cancels out}}{\cancel{\frac{\sqrt{3}}{2}}} - \frac{1}{2}i\right]\right)$$

$$= 2\pi i \left(-\frac{1}{12}i - \frac{1}{6}i - \frac{1}{12}i\right)$$

$$= 2\pi i \left(\frac{-4}{12}i\right) = \frac{-8}{12}\pi (i)^2 = \frac{2\pi}{3}$$

$$\int_\gamma \frac{1}{z^6+1}\,dz = \lim_{R\to\infty}\int_{-\infty}^{+\infty} \frac{1}{x^6+1}\,dx = \frac{2\pi}{3}$$

BUT NOT THE PROBLEMS ANSWER!

$$\lim_{R\to\infty}\int_{-\infty}^{+\infty} \frac{1}{x^6+1}\,dx = 2\lim_{R\to\infty}\int_{0}^{\infty} \frac{1}{x^6+1}\,dx$$

So, $\boxed{\int_0^\infty \frac{1}{x^6+1}\,dx = \frac{\pi}{3}}$

⑦

a. If f and h are each function that are analytic inside and on a simple closed contour C and if the strict inequality

$$|h(z)| < |f(z)|$$

holds at each point on C, then f and $f+h$ must have the same total number of zeros (counting multiplicities) inside C.

Show

b. $z^7 - 5z^3 + 12 = 0$ has zeros that lie in the annulus $1 \le |z| \le 2$

For $|z| \le 1$

Let

$$h = z^7 - 5z^3 \quad , \quad |h| \le 4 \quad \rightarrow \quad |h| < |f|$$
$$f = 12 \quad , \quad |f| = 12$$

f has no zeros on or in $|z| \le 1$ so $f+h$ has no zeros on or in $|z \le 1$

For $|z| \le 2$

let

$h = -5z^3 + 12$, $|h| \leq 28$, $|h| < |f|$

$f = z^7$, $|f| \leq 128$

f has 7 zeros on $|z| \leq 2$ so $f+h$ has 7 zeros on $|z| \leq 2$.

We showed

$$z^7 - 5z^3 + 12$$

has no zeros on $|z| \leq 1$ and 7 zeros on $|z| \leq 2$. So it must be that all the roots of

$$z^7 - 5z^3 + 12$$

lie between the circles

$$|z| = 1 \quad \text{and} \quad |z| = 2$$

⑧ Image under the mapping

$$w = \frac{z+i}{z-i}$$

of the closed first quadrant $\{(x,y) : x \geq 0, y \geq 0\}$
where $z = x+iy$

$$w = \frac{(x+iy)+i}{(x+iy)-i} = \frac{x+i(y+1)}{x+i(y-1)} \cdot \frac{x-i(y-1)}{x-i(y-1)}$$

$$= \frac{x^2 - ix(y-1) + ix(y+1) + (y+1)(y-1)}{x^2+(y-1)^2}$$

$$= \frac{x^2 + (y^2-1) - \cancel{ixy} + ix + \cancel{ixy} + ix}{x^2+(y-1)^2}$$

$$= \frac{x^2 + (y^2-1) + i2x}{x^2+(y-1)^2}$$

for $x=0, y=0$

$$w = \frac{-1}{1} = -1$$

for $x \to \infty$, $y=0$

$$w = 1$$

for $y \to \infty$, $x = 0$

$w = 1$

for $x \to \infty$, $y \to \infty$

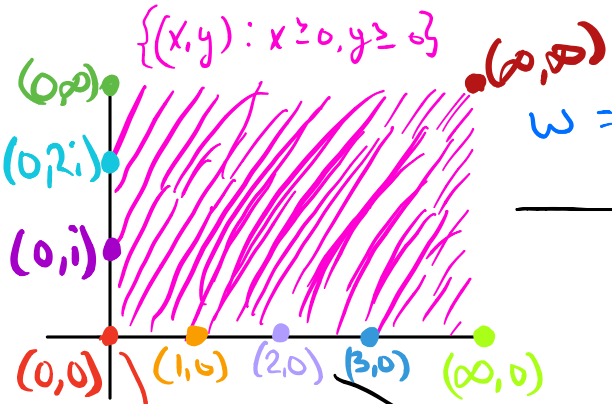
$= 1$

when $x = 0$, $y = i$

$w \to \infty$

our functio is

z-plane

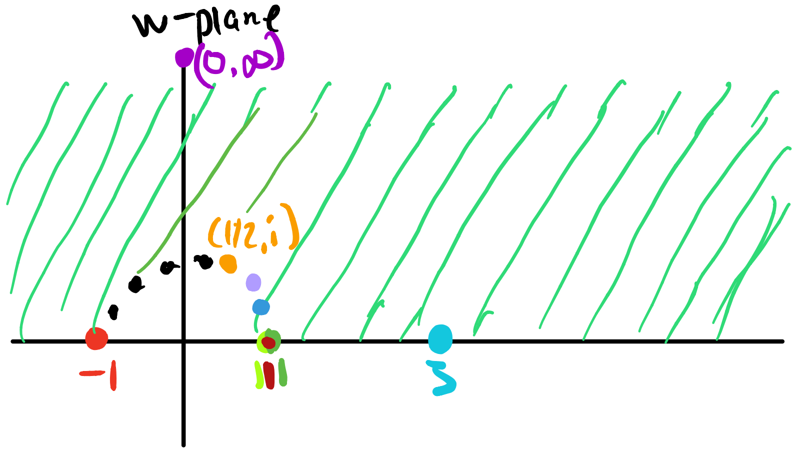


$$w = \frac{z+i}{z-i}$$

indicates our minimum

note mapping of individual points.

w-plane



$$\frac{x^2 + (y^2 - 1)}{x^2 + (y-1)^2} + \frac{i2x}{x^2 + (y-1)^2}$$

imaginary part of w always ≥ 0