

Gaussian Process

$X(t)$ Gaussian if $\{X_t\}_{t \in \mathbb{R}}$

$X(t_1), \dots, X(t_n)$ are \mathcal{G}

Process fully determined by

$$\mu_x(t), C_x(t_1, t_2)$$

and

WSS \Rightarrow Strictly Stationary

Example: Gaussian Process $X(t)$

$$\mu_x(t) = 3t$$

$$C_x(t_1, t_2) = 9 e^{-2|t_2 - t_1|}$$

$$\text{Find } \Pr[X(3) < 6]$$

Recall

$$C_x(t_1, t_2) \triangleq \mathbb{E}[X(t_1) - \mu_x(t_1)][X(t_2) - \mu_x(t_2)]$$

$$\Pr[X(1) + X(2) > 2]$$

①

$$X(3) \sim N(\mu_x(t=3), C_x(t_1=3, t_2=3)) \rightarrow N(9, 9)$$

$$\begin{aligned} \text{② } X(1) + X(2) &\sim N(\mu_x(1) + \mu_x(2), \sigma^2) \\ &\sim N(7, 9e^{-2}) \end{aligned}$$

$$\Pr[X(1) + X(2) > 2] = 1 - \Pr[X(1) + X(2) \leq 2]$$

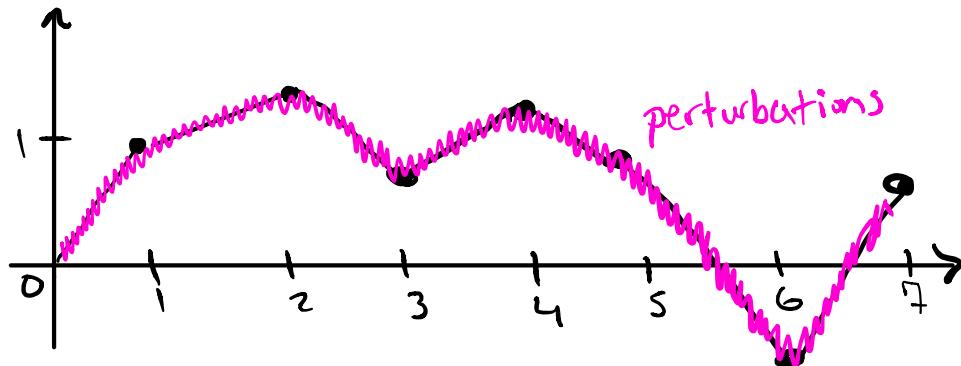
$$\begin{aligned}
 \sigma^2 &= \mathbb{E}[(X(1) + X(2) - \mathbb{E}[X(1) + X(2)])^2] \\
 &= \mathbb{E}[(X(1) - \mathbb{E}[X])^2 + (X(2) - \mathbb{E}[X])^2 + 2(X(1) - \mathbb{E}[X])(X(2) - \mathbb{E}[X])] \\
 &= C_x(1,1) + C_x(2,2) + 2C_x(1,2)
 \end{aligned}$$

Can also do MNE estimates.

Say ^{i.e.} $t=5$, $X(t=5)$ based on $X(t=4), X(t=3)$.

Brownian Motion

Discrete-time random walk



$$\delta \rightarrow 0$$

$$h \rightarrow 0$$

$$h \sim \sigma \sqrt{\delta} \Rightarrow RW \rightarrow BM$$

Definition: A Brownian Motion (also called a Wiener Process) with parameter σ^2 is a random process $\{X(t)\}_{t \geq 0}$ such that

- ① $X(0) = 0$
- ② $\{X(t)\}_{t \geq 0}$ has independent increments
- ③ $X(t_2) - X(t_1) \sim N(0, \sigma^2(t_2 - t_1))$
- ④ Every sample path is continuous

Properties of Brownian Motion

$$\textcircled{1} \quad \mu_x(t) \triangleq \mathbb{E}[X(t)] = \mathbb{E}[X(t) - X(0)] = 0$$

$$\textcircled{2} \quad R_{X(t_1, t_2)} = C_{X(t_1, t_2)} \quad t_2 > t_1$$

$$= \mathbb{E}[X(t_1)X(t_2)]$$

$$= \mathbb{E}[X(t_1)(X(t_1) + X(t_2) - X(t_1))]$$

$$= \mathbb{E}[X(t_1)^2] + \mathbb{E}[X(t_1)]\mathbb{E}[X(t_2) - X(t_1)]$$

just illustrates \rightarrow increments $= \mathbb{E}[(X(t_1) - X(0))^2] + \mathbb{E}[X(t_1) - X(0)]\mathbb{E}[X(t_2) - X(t_1)]$
 $= \sigma^2 t_1$,

In general,

$$C_{X(t_1, t_2)} = R_{X(t_1, t_2)} = \min(t_1, t_2) \sigma^2$$

$$\textcircled{3} \quad X(t) \sim N(0, \sigma^2 t)$$

Also have $X(t_1), X(t_2), \dots, X(t_n)$ i.i.d $\forall n$.
 i.e

$$\begin{bmatrix} X(t_1) \\ \vdots \\ X(t_n) \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} X(t_1) - X(0) \\ X(t_2) - X(t_1) \\ \vdots \\ X(t_n) - X(t_{n-1}) \end{bmatrix}$$

Independence Review

Independent Events: $A \perp\!\!\!\perp B$ if $\Pr[A \cap B] = \Pr[A]\Pr[B]$
or equivalently: $\Pr[A|B] = \Pr[A]$

Independent Random Variables:

$X \perp\!\!\!\perp Y$ if $f_{X,Y}(x,y) = f_X(x)f_Y(y)$

or equivalently,

$$f_{X|Y}(x|y) = f_X(x)$$

Conditional Independent Events

$A \perp\!\!\!\perp B$ conditioned on C if ie C is "new universe".

$$\Pr[A \cap B | C] = \Pr[A|C]\Pr[B|C]$$

$$\boxed{C} \quad C = \text{set}$$

or equivalently,

$$\Pr[A | B \cap C] = \Pr[A|C]$$

Conditional Independent Random Variables:

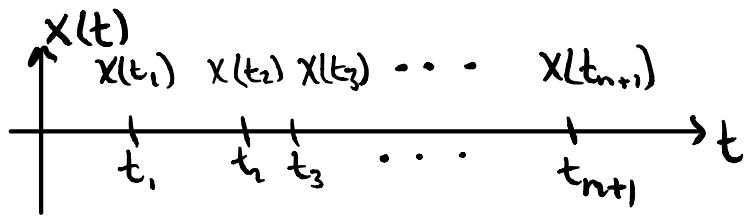
$X \perp\!\!\!\perp Y$ if $f_{X,Y|Z}(x,y|z) = f_X(x|z)f_Y(y|z)$

or equivalently,

$$f_{X|Y,Z}(x|y,z) = f_{X|Z}(x|z)$$

Markov Process

A random process $\{X(t)\}_{t \geq 0}$ is a Markov process if $\forall n$, and $t_1 < t_2 < \dots < t_{n+1}$



$$\Pr[X(t_{n+1}) = x_{n+1} \mid X(t_n) = x_n, X(t_{n-1}) = x_{n-1}, \dots, X(t_1) = x_1]$$

$$= \Pr[X(t_{n+1}) = x_{n+1} \mid X(t_n) = x_n]$$

i.e. conditioned on where you are now, the future is independent of the past.

We call the value we see the state.

Once you know the current state, future state $\perp\!\!\!\perp$ past states

Example: Random Walk

$$S_n = \sum_{i=1}^n X_i, \quad X_i \sim \text{Bernoulli}(p=\frac{1}{2})$$

Claim: $\{S_n\}_{n \geq 0}$ is Markov. Note, $S_0 = 0$

$$\text{LHS} = \Pr[S_{n+1} = s_{n+1} \mid S_n = s_n, \dots, S_1 = s_1]$$

$$\begin{aligned}
 &= \Pr [S_{n+1} - S_n = s_{n+1} - s_n \mid S_n = s_n, \dots, S_1 = s_1] \\
 &= \Pr [S_{n+1} - S_n = s_{n+1} - s_n \mid S_n = s_n] \quad \text{increments} \\
 &= \text{RHS}
 \end{aligned}$$

Example:

$$Z_n = \begin{cases} 1 & \text{if } S_n \geq 0 \\ -1 & \text{if } S_n < 0 \end{cases} \quad \begin{array}{l} \text{"called} \\ \text{Hidden} \\ \text{Markov"} \end{array}$$

Is $\{Z_n\}_{n \geq 0}$ Markov?

To show it is not, it suffices to show one case where it does not hold.

$$\star \Pr [Z_3 = 1 \mid Z_2 = 1, Z_1 = -1] \neq \star \Pr [Z_3 = 1 \mid Z_2 = 1, Z_1 = +1]$$

To see this, observe

