

Recall

$U \subseteq V$ subspace

defined $V/U = \{\vec{v} + U : \vec{v} \in V\}$

and $\pi: V \rightarrow V/U$ LT
 $\vec{v} \mapsto \vec{v} + U$

$$\ker \pi = U$$

$$\text{im } \pi = V/U \quad (\text{surjective})$$

rank-nullity formula $\Rightarrow \dim V/U = \dim V - \dim U$

Another induced map

Suppose we are given $T: V \rightarrow W$ LT.

Define a new LT

$$\bar{T}: V/\ker(T) \rightarrow W$$

by

$$\vec{v} + \ker(T) \mapsto T(\vec{v})$$

Proposition: In this setting,

① $\bar{T}: V/\ker(T) \rightarrow W$ is well-defined and is a LT

② $\ker \bar{T} = \ker T = 0$ ($\ker T / \ker T$)

③ $\text{im } \bar{T} = \text{im } T$

④ T Surjective ($\text{im } T = W$), then $\bar{T}: V/\ker(T) \rightarrow W$ is an isomorphism.

Proof of ②: Get to assume ①

Need $\ker \bar{T} = 0$

$$T(\vec{v} + \ker T) = 0_W$$

$$\text{then } \vec{v} + \ker T = \ker T$$

i.e. $\vec{v} \in \ker T$

NOT COMPLETED

Example: $F[x] = V$

$$U = \{x^2 h(x) \mid \text{all } h(x) \in F(x)\} \quad (\langle x^2 \rangle)$$

$$\text{basis: } U = \text{span}(x^2, x^3, x^4, \dots)$$

Find a vector space isomorphic to $V/U = F[x]/\langle x^2 \rangle$

$$\text{Solution: } F[x]/\langle x^2 \rangle \cong F^2$$

$$T: F[x] \rightarrow F^2$$

This is a LT

$$p(x) = a_0 + a_1 x + a_2 x^2 + \dots \mapsto \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}$$

$$\ker T = \langle x^2 \rangle \subseteq F[x]$$

T is surjective and $\therefore \bar{T}: F[x]/\ker(T) \rightarrow F^2$ is an isomorphism

Example: Suppose $V = U \oplus W$

i.e. $U \subseteq V$ subspace, $W = \text{complement of } U$

define the projection map

$$p: V \rightarrow W$$

$$u + w \mapsto w$$

check: p a LT

Note: $\ker p = U \quad \therefore \bar{p}: V/U \xrightarrow{\cong} W$ is an isomorphism

$$\text{im } p = W$$

Next Topics: Polynomials

Eigenvalues & Eigenvectors

Diagonalizability

Goal: $T: V \rightarrow W$ $\dim V = n < \infty$

Want a basis β s.t. $n \times n$ $[T]_{\beta}$ is "as nice" as possible.

Polynomials

① Division Algorithm

For integers, if p, s non-negative integers ($s \neq 0$)
then $\exists!$ non-neg integers q, r s.t.

$$(a) \quad p = qs + r$$

$$(b) \quad 0 \leq r < s$$

We want the same for polynomials

i.e. $p = x^3 + 1$, $s = x^2 + 1$

$$\begin{array}{r} x \\ x^2 + 1 \overline{) x^3 + 1} \\ \underline{- x^3 + x} \\ -x + 1 \end{array}$$

$$\text{so } \underbrace{x^3 + 1}_p = x \underbrace{(x^2 + 1)}_s + \underbrace{(-x + 1)}_r \quad \text{and } \deg r < \deg s$$

Theorem

Suppose $p, s \in \mathbb{F}[x]$, $s \neq 0$, then $\exists!$ polynomials $q, r \in \mathbb{F}[x]$ such that

$$\textcircled{a} \quad p = q \cdot s + r$$

$$\textcircled{b} \quad \deg r < \deg s \quad (\text{or } r=0)$$

Proof: Let $n = \deg p(x)$

$$m = \deg s(x)$$

if $m > n$, then $q(x) = 0$, $r(x) = p(x)$

So, assume $m \leq n$

Define $T: \mathbb{F}[x]_{\leq n-m} \times \mathbb{F}[x]_{\leq m-1} \longrightarrow \mathbb{F}[x]_{\leq n}$

by $(q(x), r(x)) \longmapsto q(x)s(x) + r(x)$

Note: T is a LT.

$$\textcircled{a} \quad \text{Ker } T = \left\{ \overset{\deg \leq n-m}{\underset{\downarrow}{q(x)}}, \overset{\deg \leq m-1}{\underset{\downarrow}{r(x)}} \mid q \cdot s + r = 0 \right\} \\ = \{ (0, 0) \}$$

$$\Rightarrow \dim \mathbb{F}[x]_{\leq n-m} \times \mathbb{F}[x]_{\leq m-1} = (n-m+1) + (m) \\ = n+1$$

$\therefore T$ an isomorphism \Rightarrow surjective