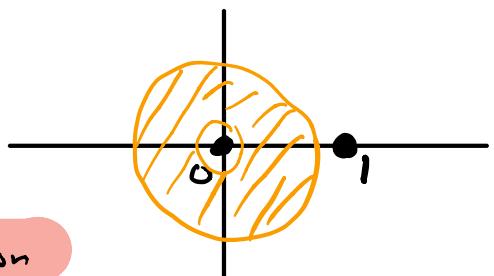


Laurent Series

Ex 1: Consider $f(z) = \frac{1}{z^2(1-z)}$. $f(z)$ has isolated singularities at $z=0, z=1$



Is it possible to find a series expansion for $f(z)$ valid in $0 < |z| < 1$? → Such a region is called an Annulus

Know That

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \dots \text{ valid for } |z| < 1$$

$$\Rightarrow f(z) = \frac{1}{z^2} (1 + z + z^2 + z^3 + \dots)$$

*Laurent Series for
f in this region*

$$= \frac{1}{z^2} + \frac{1}{z} + 1 + z + z^2 + \dots$$

"In different regions there will be different series"

More generally, suppose $f(z)$ analytic in annulus $r^2 < |z - z_0| < R$. Then $f(z)$ has a convergent Laurent series

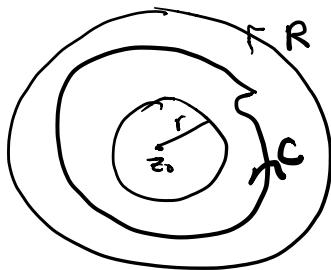
$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

note (-) powers!

where

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(\omega)}{(\omega - z_0)^{n+1}} d\omega$$

and



$C =$ Closed contour in annulus
that encircles z_0 once counterclockwise

Notes:

- ① Series may terminate at either end.
i.e. in example 1, the left terminated at $\frac{1}{z^2}$
and the right did NOT terminate
- ② For $n \geq 0$, the a_n would be the usual
Taylor coefficients if $f(z)$ were analytic
in the whole disk $|z - z_0| < R$
 \Rightarrow Laurent generalizes Taylor series and polynomials
- ③ SEE PROOF IN BOOK
- said to look at it like 10
times, similar to Taylor series
proof
- ④ To calculate a_n , can do integrals

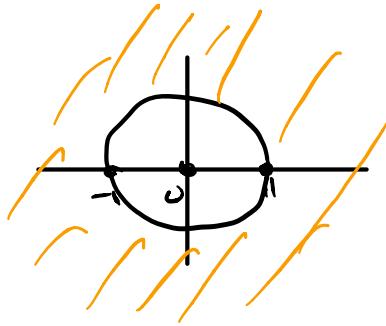
$$\frac{1}{2\pi i} \oint \frac{f(w)}{(w - z_0)^{n+1}} \text{ but}$$

easier ways exist

Example 2:

Expand

$$f(z) = \frac{1}{z^2} \left(\frac{1}{1-z} \right) \quad \text{for } |z| > 1$$



Do **NOT** use $\frac{1}{1-z} = 1+z+z^2+\dots$
Valid **ONLY** for $|z| < 1$!

What to do? Use a geometric series somehow. Seek a variable whose magnitude is less than 1.

Note for $z > 1$, $|z| > 1$. Define $u = \frac{1}{z}$. u has the desirable property $|u| < 1$.

Write $\frac{1}{1-z}$ in terms of u

$$\Rightarrow \frac{1}{1-z} = \frac{1}{1-\frac{1}{u}} = \frac{u}{u-1} = -u \left(\frac{1}{1-u} \right)$$

$$= -u \left(1 + u + u^2 + u^3 + \dots \right) \quad \left. \right\} \text{Correct because } |u| < 1$$

Hence for $|z| > 1$, we have

$$\frac{1}{1-z} = -\frac{1}{z} \left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \dots \right)$$

$$= -\frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} - \dots$$

so

$$f(z) = \frac{1}{z^2} \left(\frac{1}{1-z} \right) = -\frac{1}{z^3} - \frac{1}{z^4} - \frac{1}{z^5} - \frac{1}{z^6} - \dots$$

for $|z| > 1$ — specify region

Example 3:

Not going to do but LOOK AT IN BOOK

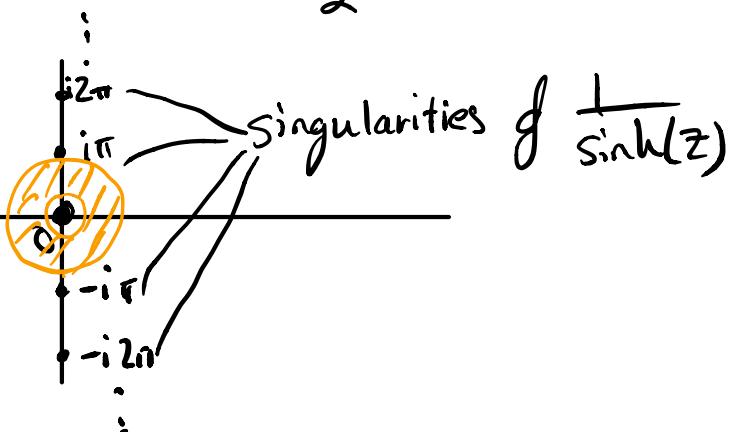
To do w/ partial fractions \rightarrow Example 2: Page 274-275

Example 4 (Long Division)

Laurent series for $f(z) = \frac{1}{\sinh(z)}$?

Recall $\sinh(z) = 0$ for $z = ik\pi$, $k \in \mathbb{Z}$

$$\sinh(z) = \frac{e^z - e^{-z}}{2} = -i\sin(iz)$$



Should be able to find a series in $0 < |z| < \pi$

For small z , $\sinh(z) \approx z$ (since $e^z = 1 + z + \dots$
 $e^{-z} = 1 - z - \dots$)

So we expect $\frac{1}{\sinh(z)}$ behaves like $\frac{1}{z} + \text{analytic}$ for $z \approx 0$
 stuff

saying $\frac{1}{\sinh(z)}$ is about this singular plus moderate stuff like an analytic function

This will be a MacLaurin Series

$$\sinh(z) = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \dots$$

$$= z + \frac{z^3}{6} + \frac{z^5}{120} + \dots$$

use long division to get analytic stuff

Then using long division, $\frac{1}{\sinh(z)}$ becomes

$$\begin{array}{r}
 \overline{1}z - \frac{z^3}{6} + \frac{7z^5}{360} \dots \\
 \hline
 z + \frac{z^3}{6} + \frac{z^5}{120} + \dots \\
 \hline
 - 1 + \frac{z^2}{6} + \frac{z^4}{120} + \dots \\
 \hline
 0 - \frac{z^2}{6} - \frac{z^4}{120} - \dots \\
 \hline
 - \frac{z^2}{6} - \frac{z^4}{36} - \dots \\
 \hline
 0 + \frac{7z^4}{360} - \dots
 \end{array}$$

The long division suggests that

$$\frac{1}{\sinh(z)} = \frac{1}{z} - \frac{z}{6} + \underbrace{\frac{7z^3}{360}}_{\substack{\text{over } \\ \text{Laurent}}} + \dots \text{ for } 0 < |z| < \pi$$

Isolated Singularities

Suppose z_0 is an isolated singularity of $f(z)$ $\left\{ \begin{array}{l} \text{meaning } f(z) \text{ is analytic} \\ \text{in some punctured} \\ \text{disk } 0 < |z - z_0| < R \\ \text{for some } R > 0 \end{array} \right\}$

1) Removable

$|f(z)|$ bounded as $z \rightarrow z_0$ meaning finite / doesn't blow up

2) pole

$|f(z)| \rightarrow \infty$ as $z \rightarrow z_0$

3) Essential

$f(z)$ has no limit as $z \rightarrow z_0$
(oscillates wildly as $z \rightarrow z_0$)

Examples

1) Removable : $f(z) = \frac{\sin z}{z}$ $\left\{ \begin{array}{l} \text{strictly speaking at} \\ z=0 \text{ we have a singularity} \\ \text{but} \end{array} \right\}$

$$\frac{\sin z}{z} = \frac{z + \frac{z^3}{3!} - \frac{z^5}{5!}}{z} = 1 + \frac{z^2}{3!} - \frac{z^4}{5!} \dots$$

So as $z \rightarrow 0$ $\frac{\sin z}{z} \rightarrow 1$. i.e by defining $f(0) = 1$

The singularity has been "removed"

A theoretical point:

At a removable singularity the Laurent series has no negative powers of $z - z_0$

2) POLES

$\frac{1}{z - z_0}$ = simple pole of order 1

$\frac{1}{(z - z_0)^m} \rightarrow$ pole of order m , $m > 0$

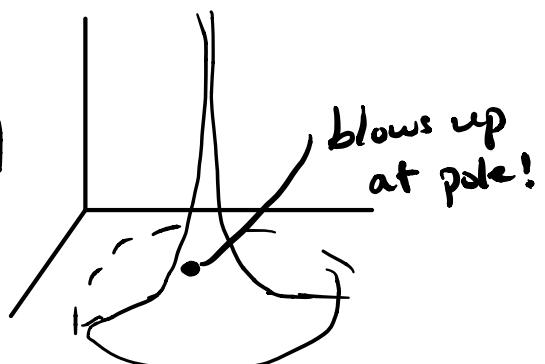
If $f(z)$ has a pole of order m ,
Laurent series begins w/ $-m$ power!

See book for
more detail

↔

$$f(z) = \frac{a_{-m}}{(z - z_0)^m} + \frac{a_{-m+1}}{(z - z_0)^{m+1}} + \dots + a_0 + \dots + a_1(z - z_0) \dots$$

Why called a pole. Visualize. $|f(z)|$



z_0

3) Essential $f(z)$ is neither bounded nor blows up as $z \rightarrow z_0$

i.e.

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \dots \quad \left. \begin{array}{l} \text{powers} \\ \text{of } z \text{ going} \\ \rightarrow \infty \end{array} \right\}$$

$z_0 = 0$ essential

let

$z \rightarrow 0$ along real values

$f(z) \rightarrow 0$ as $z \rightarrow 0$ from negative side

$f(z) \rightarrow \infty$ as $z \rightarrow 0$ from positive side

let

$z = iy$ along imaginary values

$$f(z) = e^{-\frac{i}{y}} = \cos\left(\frac{1}{y}\right) - i \sin\left(\frac{1}{y}\right) \quad \begin{array}{l} \text{graph,} \\ \text{crazy} \\ \text{oscillations} \end{array}$$