① X, Y uniformly distributed in [0,1] and [X,1] respectively.

(a) MMSE estimator of Y using a constant and resulting MSE.

(b) Linear MMSE estimator of X given Y and resulting MSE.

(c) MMSE estimator of X given Y.

(a) $X \sim U[0,1]$, $Y \sim U[x,1]$

$$f_X(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & o/w \end{cases} \qquad f_{Y|X=x}(y|x=x) = \begin{cases} \frac{1}{1-x}, & 0 \le x \le y \le 1 \\ 0, & o/w \end{cases}$$

MMSE estimator of y using a constant will be $\mathbb{E}[Y]$ and the resulting MSE will be $Var(Y)$

$$\mathbb{E}_Y[Y] = \mathbb{E}_X[\mathbb{E}_{Y|X}[Y|X]]$$

$f_{Y|x} = \frac{1}{1-x} \quad 0 \le x \le y \le 1$

$$\mathbb{E}_{Y|X}[Y|x] = \int_y y\, f_{Y|X}(y|x)\, dy$$

$$= \int_{y=x}^{1} y\left(\frac{1}{1-x}\right) dy$$

$$= \frac{\frac{1}{2} - \frac{x^2}{2}}{1-x} = \frac{1}{2}\,\frac{1-x^2}{1-x} = \frac{1+x}{2}$$

$$\mathbb{E}_Y[Y] = \int_x \mathbb{E}_{Y|X}[Y|X]\, f_X(x)\, dx = \int_{x=0}^{1} \left(\frac{1+x}{2}\right) dx = \frac{1}{2} + \frac{x^2}{4}\Big|_0^1 = \frac{3}{4}$$

So $\hat{Y}_{MMSE} = 5/12$

The resuting MSE is Var(Y)

$$\text{Var}(Y) = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2$$

$$\mathbb{E}[Y^2] = \mathbb{E}_X\left[\mathbb{E}[Y^2 \mid X]\right]$$

$$= \int_{x=0}^{1} \underbrace{\left(\int_{y=x}^{1} y^2 \left(\frac{1}{1-x}\right) dy\right)}_{\mathbb{E}[Y^2 \mid X]} f_X(x)\, dx$$

$$= \int_{x=0}^{1} \frac{1}{3}\left(\frac{1-x^3}{1-x}\right) dx$$

$$= \int_{x=0}^{1} \frac{1}{3}\left(1 + x + x^2\right) dx$$

$$= \frac{1}{3}x + \frac{x^2}{6} + \frac{x^3}{9}\Big|_0^1 = \frac{1}{3} + \frac{1}{6} + \frac{1}{9} = \frac{11}{18}$$

thus

$$\text{Var}(Y) = \frac{11}{18} - \left(\frac{3}{4}\right)^2 = \frac{7}{144}$$

(b) LMMSE Estimator of X given Y

$$\hat{X}_{LMMSE} = a^* Y + b^*$$

$$= \frac{\text{Cov}(X,Y)}{\text{Var}(Y)}\left(Y - \mathbb{E}[Y]\right) + \mathbb{E}[X]$$

$$\mathbb{E}[X] = \int_{x=0}^{1} x f_X(x)\, dx = \frac{1}{2}$$

$$\mathbb{E}[Y] = 3/4 \quad , \quad \text{Var}(Y) = 7/144$$

$$\mathrm{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y]$$

$$f_{X,Y}(x,y) = f_{Y|X}(y|x)\, f_X(x)$$

$$= \frac{1}{1-x} \qquad 0 \le x \le y \le 1$$

$$\mathbb{E}[xy] = \int_x \int_y xy\, f_{X,Y}(x,y)\, dy\, dx$$

$$= \int_{x=0}^{1} \int_{y=x}^{1} \frac{x}{1-x}\, y\, dy\, dx$$

$$= \frac{1}{2} \int_{x=0}^{1} \frac{1-x^2}{1-x}\, x\, dx \; = \frac{1}{2} \int_{x=0}^{1} x + x^2 \, dx$$

$$= \frac{1}{2}\left(\frac{1}{2} + \frac{1}{3}\right) = \frac{5}{12}$$

$$\mathrm{Cov}(X,Y) = \frac{5}{12} - \frac{3}{4}\left(\frac{1}{2}\right) = \frac{5}{12} - \frac{3}{8} = \frac{1}{24}$$

$$\hat{X}_{LLMSE} = \frac{\mathrm{Cov}(X,Y)}{\mathrm{Var}(Y)}\left(Y - \mathbb{E}[Y]\right) + \mathbb{E}[X]$$

$$= \frac{1/24}{7/144}\left(Y - \frac{3}{4}\right) + \frac{1}{2}$$

$$\frac{1}{24} \cdot \frac{144}{7} = \frac{6}{7}$$

$$= \frac{6}{7}\left(Y - \frac{3}{4}\right) + \frac{1}{2}$$

$$= \frac{6}{7} Y - \frac{18}{28} + \frac{14}{28} = \frac{6}{7} Y - \frac{1}{7}$$

Resulting MSE is

$$\text{Var}(X) - \frac{(\text{Cov}(X,Y))^2}{\text{Var}(Y)}$$

$$\text{Var}(X) = \frac{1}{12}$$

$$= \frac{1}{12} - \frac{(1/24)^2}{7/144}$$

(c) MMSE estimator of X given Y.

$$\hat{X}_{MMSE} = \mathbb{E}[X|Y] = \int_x f_{X|Y}(x|y)\, x \, dx$$

$$f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x) f_X(x)}{f_Y(y)}$$

$$f_Y(y) = \int_{x=0}^{y} f_{X,Y}(x,y)\, dx = \int_{x=0}^{1} \frac{1}{1-x}\, dx = -\ln|1-x| \Big|_0^y = -\ln|1-y| \qquad 0 \le x \le y \le 1$$

$$f_{X|Y}(x|y) = \frac{1}{1-x} \cdot \frac{-1}{\ln|1-y|} \qquad 0 \le x \le y \le 1$$

$u = 1-x$

$du = -dx$

$x = 1-u$

$$\mathbb{E}[X|Y] = -\frac{1}{\ln|1-y|} \int_{x=0}^{y} \frac{x}{1-x}\, dx$$

$$= +\frac{1}{\ln|1-y|} \int_{u=1}^{u=1-y} \frac{1-u}{u}\, du = \frac{1}{\ln|1-y|}\left(\ln|u| - u \Big|_1^{1-y} \right)$$

$$= \frac{\ln|1-y|}{\ln|1-y|} - \frac{1-y}{\ln|1-y|} - \left(0 - \frac{1}{\ln|1-y|}\right) = 1 + \frac{Y}{\ln|1-Y|}$$

② Let X, Y, Z be jointly Gaussian random variables with

$$\mathbb{E}[X] = \mathbb{E}[Y] = \mathbb{E}[Z] = 0$$

$$Var(X) = Var(Y) = Var(Z) = 2$$

Let $\rho_{XY}, \rho_{XZ}, \rho_{YZ}$ denote the correlation coefficients.

(a) MMSE estimator of X^2 given Z

(b) MMSE estimator of XY given Z

(a)

$$\mathbb{E}[X^2 | Z] = Var(X|Z) + (\mathbb{E}[X|Z])^2$$

Since X, Z jointly Gaussian,

$$\mathbb{E}[X|Z] = \hat{X}_{LMMSE} = \frac{Cov(X,Z)}{Var(Z)}(Z - \mathbb{E}[Z]) + \mathbb{E}[Y]$$

$$= \frac{\rho_{XZ}\sigma_X\sigma_Z}{\sigma_Z^2}(Z)$$

$$= Z\rho_{XZ} \qquad f(Z)$$

$$Var(X|Z) = Var(X)(1 - \rho_{XZ}^2)$$

$$= 2 - 2\rho_{XZ}^2 \qquad \text{constant}$$

$$\mathbb{E}[X^2|Z] = 2 - 2\rho_{XZ}^2 + Z^2\rho_{XZ}^2$$

$$= 2 + \rho_{XZ}^2(Z^2 - 2)$$

(b) $\mathbb{E}[XY|Z]$

$$2XY = (X+Y)^2 - X^2 - Y^2$$

$$\mathbb{E}[(X+Y)^2 - X^2 - Y^2 | Z] = \mathbb{E}[(X+Y)^2|Z] - \mathbb{E}[X^2|Z] - \mathbb{E}[Y^2|Z]$$

$$\mathbb{E}[X^2|Z] = 2 + \rho_{XZ}^2 (Z^2 - 2)$$

Similarly $\mathbb{E}[Y^2|Z] = 2 + \rho_{YZ}^2 (Z^2 - 2)$

Realize $X+Y$ is gaussian since X, Y jointly Gaussian!

$$\mathbb{E}[(X+Y)^2|Z] = \mathrm{Var}((X+Y)|Z) + (\mathbb{E}[(X+Y)|Z])^2$$

$$\begin{aligned} \mathrm{Cov}(X+Y, Z) &= \mathrm{Cov}(X,Z) + \mathrm{Cov}(Y,Z) \\ &= \sigma_X \sigma_Z \rho_{XZ} + \sigma_Y \sigma_Z \rho_{YZ} \\ &= 2\rho_{XZ} + 2\rho_{YZ} \end{aligned}$$

$$\begin{aligned} \mathbb{E}[X+Y|Z] &= \mathbb{E}[X|Z] + \mathbb{E}[Y|Z] \\ &= Z(\rho_{XZ} + \rho_{YZ}) \end{aligned}$$

$$\begin{aligned} \mathrm{Var}(X+Y) &= \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\mathrm{Cov}(X,Y) \\ &= 2 + 2 + 2\sigma_X\sigma_Y\rho_{XY} \\ &= 4 + 4\rho_{XY} \end{aligned}$$

$$\mathrm{Var}(X+Y \mid Z) = \mathrm{Var}(X+Y) - \frac{\mathrm{Cov}^2(X+Y, Z)}{\mathrm{Var}(Z)}$$

$$= 4 + 4\rho_{XY} - \frac{(2\rho_{XZ} + 2\rho_{YZ})^2}{2}$$

$$= 4 + 4\rho_{XY} - 2(\rho_{XZ} + \rho_{YZ})^2$$

Thus

$$\mathbb{E}[(X+Y)^2 \mid Z] = 4 + 4\rho_{XY} - 2(\rho_{XZ} + \rho_{YZ})^2 + Z^2(\rho_{XZ} + \rho_{YZ})^2$$

$$= 4 + 4\rho_{XY} + (Z^2 - 2)(\rho_{XZ} + \rho_{YZ})^2$$

FINALLY,

$$2\mathbb{E}[XY \mid Z] = 4 + 4\rho_{XY} + (Z^2-2)(\rho_{XZ} + \rho_{YZ})^2 - \left(2 + \rho_{XZ}^2(Z^2-2)\right) - \left(2 + \rho_{YZ}^2(Z^2-2)\right)$$

$$= 4\rho_{XY} + (Z^2-2)\left[-\rho_{YZ}^2 - \rho_{XZ}^2 + (\rho_{XZ} + \rho_{YZ})^2\right]$$

AND

$$\mathbb{E}[XY \mid Z] = 2\rho_{XY} + (Z^2-2)\rho_{XZ}\rho_{YZ}$$

HOLY CRAP...

③ A total of 11 people, including you, are invited to a party. The times at which people (including you) arrive at the party are independent and uniformly distributed in $(0,1)$.

(a) Find expected amount of people who arrive before you.

(b) Find the variance of the number of people who arrive before you.

(a)

10 people left not including me.

Let T be the time I arrive. Given $T=t$

Each person arriving before me is a Bernoulli event, with probability t.

Have 10 independent Bernoulli events ⇒ Binomial distribution!

Let

$$N \sim B(10, T)$$

Thus

$$\mathbb{E}[N \mid T=t] = 10t$$

$$\mathbb{E}[N] = \mathbb{E}_T[\mathbb{E}_{N|T}[N|T]]$$

$$= \mathbb{E}_T[10T]$$

$$= 10\,\mathbb{E}_T[T] = 5$$

$$f_T(t) = \begin{cases} 1 & 0 \le t \le 1 \\ 0 & \text{o/w} \end{cases}, \quad \mathbb{E}[T] = \frac{1}{2}$$

$$\mathbb{E}[T^2] = \int_0^1 t^2\,dt = \frac{1}{3}$$

(b) $Var(N)$?

$$Var(N|T=t) = nt(1-t)$$

$$Var(N|T) = 10T(1-T)$$

$$\begin{aligned}
Var(N) &= \mathbb{E}[Var(N|T)] + Var(\mathbb{E}[N|T]) \\
&= \mathbb{E}[10T(1-T)] + Var(10T) \\
&= 10\,\mathbb{E}[T] - 10\,\mathbb{E}[T^2] + 100\,Var(T) \\
&= 10(1/2) - 10(1/3) + 100(1/12) \\
&= 5 - \frac{10}{3} + \frac{100}{12} \\
&= \frac{5}{3} + \frac{25}{3} = \frac{30}{3} = 10
\end{aligned}$$

$$Var(N) = 10$$