Have a complex function $\omega = f(z)$.

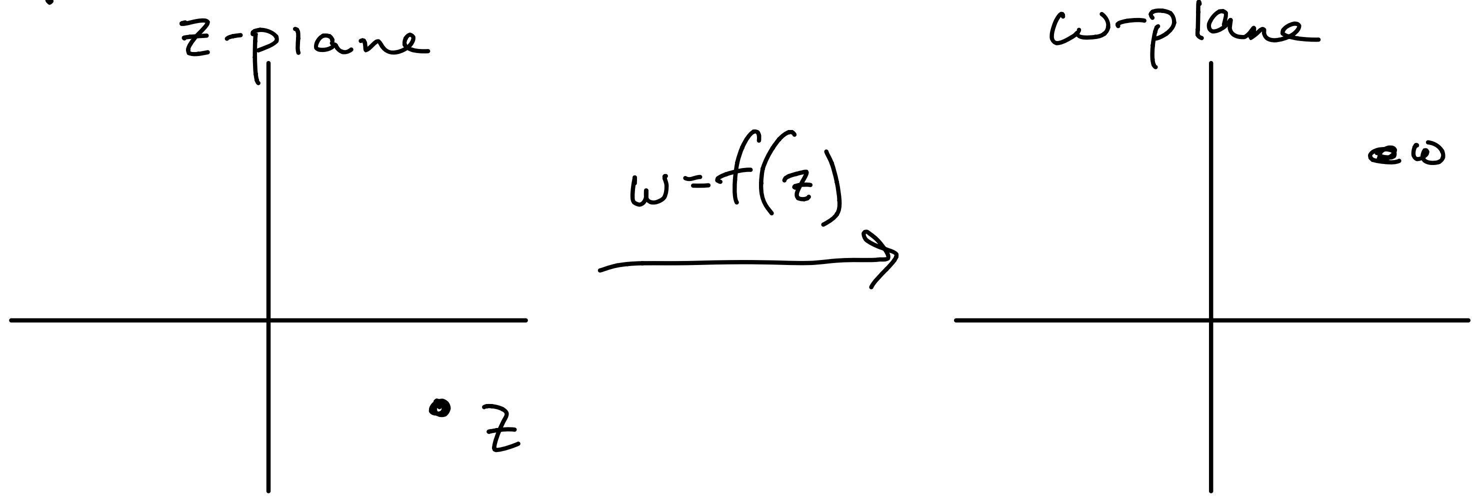
Input a complex z, output a complex ω.

For real variables, $y = f(x)$ can be visualized as graphs in the xy plane: 2 dimensional space.

How do we visualize $\omega = f(z)$?

→ Can't draw the graph - 2D surface in 4D space

Idea: Instead, regard f as a "mapping" from z-plane to ω-plane.



Example: $\omega = f(z) = z^2$

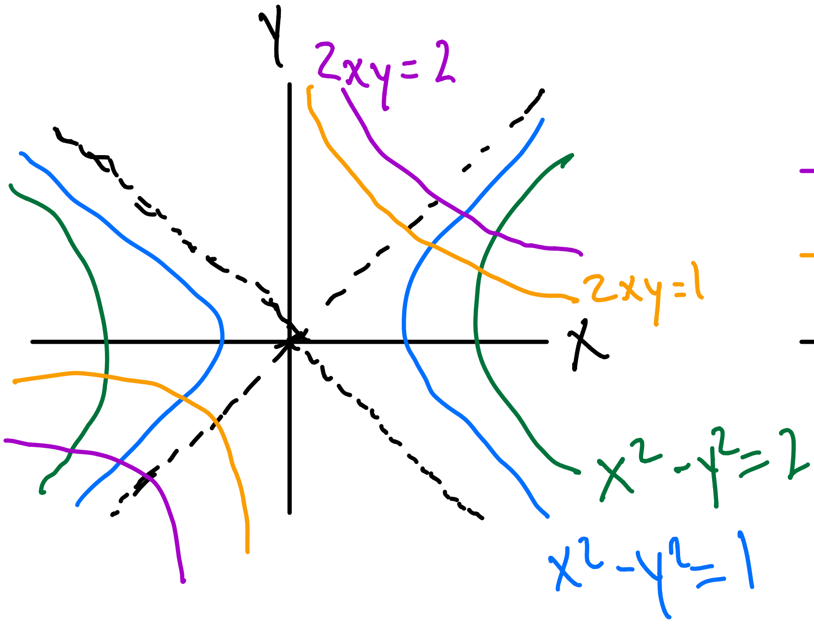
$z = x + iy$ $\qquad$ $w = u + iv$

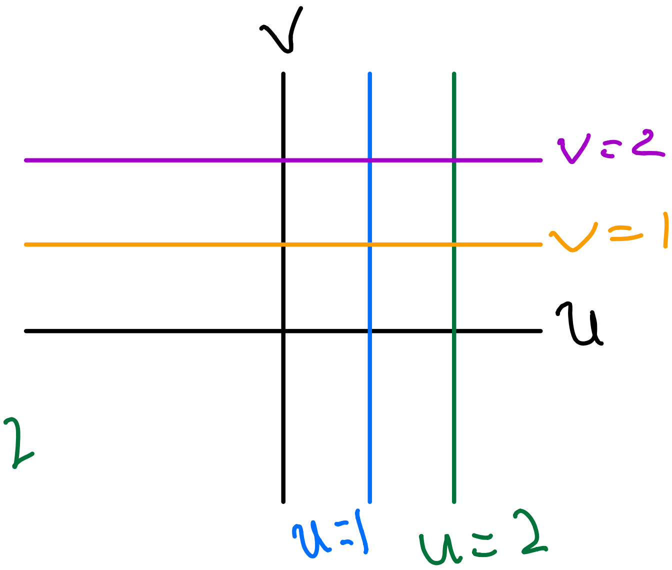
$$u + iv = (x+iy)^2 = x^2 - y^2 + i\,2xy$$

$$u(x,y) = x^2 - y^2$$

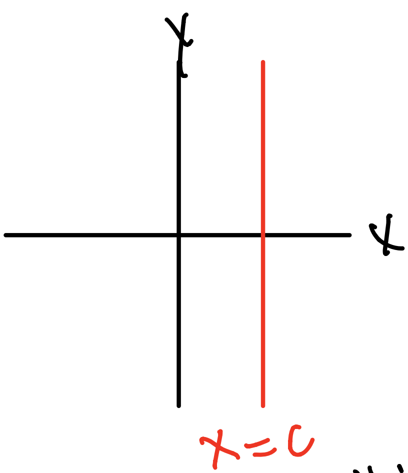
$$v(x,y) = 2xy$$

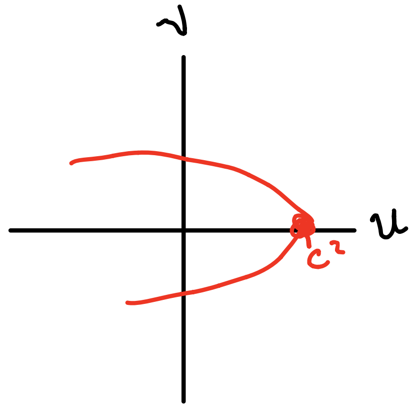
First look at pre-images of lines in (u,v) plane





Also, what is the image (in u-v plane) of the gradient lines in the xy-plane.





$y=$ anything

$u = c^2 - y^2$

$v = 2cy$

$u = c^2 - \dfrac{v^2}{4c^2}$

Time for Some Calculus!

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \quad \Big\} \text{ IF the limit exists}$$

In complex analysis, Δz can approach 0 from an infinite number of directions.

Definitions:

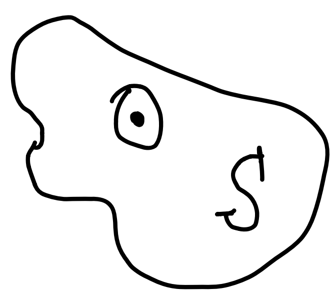
① f is "complex differentiable" at z if the limit exists and is independent of how $\Delta z \to 0$.

② If f is differentiable in an open region, we say f is analytic in that region.

③ If f is analytic in whole complex plane, f is "entire

Side Note: Open Set

Set S is an open set if any point in S is neighbored by points also in that set.



Example:

$$f(z) = z^2$$
$$f(z+\Delta z) = z^2 + 2z\Delta z + (\Delta z)^2$$
$$f(z) = z^2$$

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{z^2 + 2z\Delta z + (\Delta z)^2 - z^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} 2z + \Delta z = 2z$$

$$\frac{d}{dz}(z^2) = 2z \quad \checkmark$$

FACT: ALL POLYNOMIALS ARE ANALYTIC

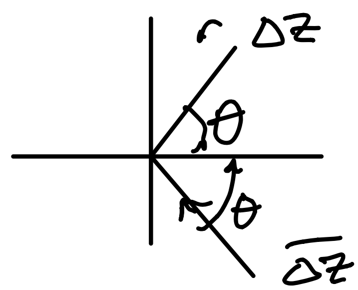
Example: Show $f(z) = \bar{z}$ is not differentiable and therefore not analytic anywhere.

$$f(z) = \bar{z}$$

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{\overline{z+\Delta z} - \bar{z}}{\Delta z} = \frac{\overline{\Delta z}}{\Delta z}$$

Let $\Delta z = r\,\mathrm{cis}(\theta)$

$$\frac{\overline{\Delta z}}{\Delta z} = \frac{r \cdot \mathrm{cis}(\theta)}{r \cdot \mathrm{cis}(-\theta)} = \mathrm{cis}(2\theta)$$



Limit is NOT independent of θ, so $\bar{z}$ is not differentiable anywhere.

Cauchy-Rieman Equations

Relate Re and Im parts of analytic functions.

Example: $w = f(z) = z^2 \Rightarrow u + iv = (x^2 - y^2) + i(2xy)$

$$\left|\frac{\partial u}{\partial x}\right| = 2x = \left|\frac{\partial v}{\partial y}\right| \quad , \quad \left|\frac{\partial u}{\partial y}\right| = -2y = \left|-\frac{\partial v}{\partial x}\right|$$

Analytic functions ALWAYS satisfy these equations.

Calculate $f'(z)$ in two ways $\begin{cases} 1)\ \Delta z = \Delta x \to 0 \\ 2)\ \Delta z = i\Delta y \to 0 \end{cases}$

$$f'(z) = \lim_{\Delta x \to 0} \frac{f(z+\Delta x) - f(z)}{\Delta x} = \lim_{\Delta y \to 0} \frac{f(z+i\Delta y) - f(z)}{i\Delta y}$$

$$\Rightarrow \frac{\partial f}{\partial x} = -i \frac{\partial f}{\partial y}$$