Recall: Discrete r.v's

Given X w/ pmf $p_X(x)$, expected value/mean of X is

$$\mathbb{E}(X) = \sum_{x \in X} x\, p_X(x)$$

Sum over all x in X

Expected Value Rule

If $Y = g(X)$, then

$$\mathbb{E}(Y) = \sum_{x} g(x) p_X(x)$$

Variance of X

$$\mathrm{Var}(X) = \mathbb{E}((X - \mathbb{E}(X))^2)$$

$$\sigma_X = +\sqrt{\mathrm{Var}(X)}$$ ← Standard deviation

Saw:
- X is Bernoulli(p) $\Rightarrow \mathbb{E}(X) = p$; $\mathrm{Var}(X) = p(1-p)$
- X is discrete uniform on $a \leq k \leq b \Rightarrow \mathbb{E}(X) = \frac{b+a}{2}$
- X is Poisson(λ) $\Rightarrow \mathbb{E}(X) = \lambda$; $\underbrace{\mathrm{Var}(X) = \lambda}_{\text{Proof NOT shown}}$

Proof

$$\mathbb{P}(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k \in \mathbb{N}$$

$$\mathbb{E}(X) = \sum_{k} k\, p_X(k) = \sum_{k=1}^{\infty} e^{-\lambda} \frac{\lambda^k}{(k-1)!} = \lambda \sum_{m=0}^{\infty} e^{-\lambda} \frac{\lambda^m}{m!} = \lambda \cdot 1 = \lambda$$

- X is Geometric(p) : $p_X(k) = p(1-p)^{k-1} \quad 1 \le k < \infty$

Compute using useful math tricks

$$\mathbb{E}(X) = \sum_k k\, p_X(k) = \sum_{k=1}^{\infty} k p (1-p)^{k-1}$$

$$= p \sum_{k=1}^{\infty} k(1-p)^{k-1} = p\left(-\frac{d}{dp}\left[\sum_{k=0}^{\infty}(1-p)^k\right]\right)$$

$$= p\left(-\frac{d}{dp}\left[\frac{1}{1-(1-p)}\right]\right) = p\left(-\frac{d}{dp}\left[\frac{1}{p}\right]\right) = \frac{1}{p}$$

"Forget about Variance for now"

Super Useful Formula:

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$$

Proof:

$$\mathrm{Var}(X) = \mathbb{E}\big(\underbrace{(X - \mathbb{E}(X))^2}_{g(X)}\big)$$

$$= \sum_x (x - \mathbb{E}(X))^2 p_X(x) \quad \text{by expected value rule.}$$

$$= \underbrace{\sum_x x^2 p_X(x)}_{\mathbb{E}(X^2)} + \underbrace{-2\mathbb{E}(X)\sum_x x\, p_X(x)}_{-2(\mathbb{E}(X))^2} + \underbrace{(\mathbb{E}(X))^2 \sum_x p_X(x)}_{(\mathbb{E}(X))^2}$$

So,

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$$

This is known as the "variance-in-terms-of-moments formula" because, $\forall k > 0$, define k^{th} moment of X as $\mathbb{E}(X^k)$

Expected Value comes up in **optimization problems** –
Making choices to maximize $\mathbb{E}\left(\substack{\text{gain}\\ \text{payoff}}\right)$ or minimize $\mathbb{E}(\text{loss})$

Example – Game Show

There are two questions Q_1, Q_2 available.

answer correctly!

$\mathbb{P}(\text{you answer } Q_1) = p_1$

$\mathbb{P}(\text{you answer } Q_2) = p_2$

Q_1 easy $\Rightarrow p_1$ high

Q_2 hard $\Rightarrow p_2$ low

Need to choose which question to address first. Get nothing if you answer first question wrong. If right you get a payoff – V_1 for Q_1, V_2 for Q_2; then you address the other question. Answering the two Q's independent.

Problem: Which Q to address first?

Let

choice of first Q

$$X = \text{payoff to given strategy}$$

If Q_1 is chosen first,

$X =$	0	V_1	$V_1 + V_2$	← possible values
	$(1-p_1)$	$p_1(1-p_2)$	$p_1 p_2$	← probabilities of possible values

Note probabilities add to 1

If Q_2 is chosen first,

$$X = 0 \qquad V_2 \qquad V_1 + V_2$$ ← possible values

$$(1-p_2) \qquad p_2(1-p_1) \qquad p_1 p_2$$ ← probabilities of possible values

$\mathbb{E}(X)$ when Q_1 is first is

$$\mathbb{E}(X) = V_1 p_1 (1-p_2) + (V_1 + V_2) p_1 p_2$$

$\mathbb{E}(X)$ when Q_2 is first is

$$\mathbb{E}(X) = V_2 p_2 (1-p_1) + (V_1 + V_2) p_1 p_2$$

Choosing Q_1 first is better when

$$\frac{p_1 V_1}{1-p_1} > \frac{p_2 V_2}{1-p_2}$$

Else Q_2 first is at least as good

Can also use expectation to answer "how much would you pay to play a game of chance"?

Example- Pay to Play

If you play, you collect \$k when face k comes **up** in fair die roll.
Let

$$X = \text{"outcome of die roll" dollars}$$

So

$$p_X(k) = 1/6, \quad 1 \le k \le 6$$

$$\mathbb{E}(X) = \frac{1}{6}(1+2+3+4+5+6) = 3.5$$

Bottom Line? A rational agent would play the game for any price less than \$3.50.

Example- Pay to Play Part 2 (St. Petersburg Paradox)

The game: flip a fair coin repeatedly. You get \$$2^k$ when k is the index of the first head.
Let

$$X = \text{payoff if you choose to play}$$

Then

$$p_X(2^k) = \underbrace{\left(\tfrac{1}{2}\right)^{k-1}}_{(k-1)T}\underbrace{\left(\tfrac{1}{2}\right)}_{(k)H}{}^{k} = 2^{-k}$$

$$\mathbb{E}(X) = \sum_{x \in X}' x\, p_X(x) = \sum_{x \in X} x\, p_X(x) = \sum_k' (2^{-k})(2^k) = \sum_k 1 = \infty$$

Thus a "rational agent" would pay an arbitrary large sum to play?!?

Daniel Bernoulli's proposed "resolution": instead of judging the worth of playing by $\mathbb{E}(X)$, use instead $\mathbb{E}(\log(X))$.

can use any base really

In the above case it turns out

$$\mathbb{E}(\log_2(X)) = \sum_{k=1}^{\infty} k 2^{-k} = 2$$

So Daniel Bernoulli reasons as follows: I will play if

$$\log_2(\text{price to play}) < 2$$

same as price to play $< \$4$

Makes sense that the minimum you will ever make is \$2 if you play - so acceptable price to play $> \$2$ is expected

Next BIG Topic:

Multiple Discrete Random Variables

Starting point: $\Omega, \mathbb{P}$; two discrete rv's X, Y defined on Ω.

For any pair (x,y) of possible (X,Y)-values, we have the event

$$\{X=x\} \cap \{Y=y\}$$

define for each (x,y) pair the joint pmf of X and Y at (x,y) as

$$p_{XY}(x,y) = \mathbb{P}(\{X=x\} \cap \{Y=y\})$$

$\mathbb{P}(X=x \text{ and } Y=y)$ or $\mathbb{P}(X=x, Y=y)$ when we're lazy