Recap

(a) Linear maps/ transformations

vector space

$$T: V \to W \qquad T \in \mathcal{L}(V, W)$$

examples: differentiation, projections, rotations,
if A is an $m \times n$ matrix over $\mathbb{F}$ (L_A) IMPORTANT

$$L_A : \mathbb{F}^n \to \mathbb{F}^m$$
$$\vec{x} \mapsto A\vec{x}$$

Key facts

- If $(v_1, \dots, v_n)$ is a basis of V and $(w_1, \dots, w_n)$ are ANY elements of W, then $\exists!\ T: V \to W$ s.t. $T(v_i) = w_i$ (LT)

(b) Kernel + Images $T: V \to W$

ker T (nullspace) $\subseteq V$ (all vectors $\vec{v}$ in V s.t. $T(\vec{v}) = 0$)

im T (range/image) $\subseteq W$

key facts: • ker $T \subseteq V$ is a subspace $\Rightarrow$ is a vector space

im $T \subseteq W$ is a subspace $\Rightarrow$ is a vector space

- $\dim V = \dim \ker T + \dim \operatorname{im} T$

(c) Matrix of $T: V \to W$ (LT)

If $A = (\vec{v}_1, \dots, \vec{v}_n)$ is a basis of V

$B = (\vec{w}_1, \dots, \vec{w}_m)$ is a basis of W

If $\vec{v} \in V$, $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n$ (unique, $\vec{v}_i$ form a basis)

$$[\vec{v}]_A := \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{F}^n$$

Matrix of T $M(T) = \underbrace{[T]_{B \leftarrow A}}_{m \times n \text{ matrix}}$

example: $V = [\mathbb{R}]_{\leq 3}$, basis $B = (1, x, x^2, x^3)$

$T: V \to V$ (differentiation)

find $[T]_B = [T]_{B \leftarrow B}$

$$[T]_B = \big[[T(1)]_B, [T(x)]_B, [T(x^2)], [T(x^3)] \big] \quad 4\times4$$

$T(1) = 0$

$T(x) = 1$

$T(x^2) = 2x$

$T(x^3) = 3x^2$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

In general (A, B as above)

$$[T]_{B \leftarrow A} = \big[[T(\vec{v}_1)]_B, \dots, [T(\vec{v}_n)]_B \big] \quad m\times n \text{ matrix over } \mathbb{F}$$

Key Properties

① $[T(\vec{v})]_B = [T]_{B\leftarrow A}[\vec{v}]_A$

② If also $S: W \to U$, $T: V \to W$, $C = (u_1, \dots, u_p)$ basis of U

$$ST : V \to U$$

$$[ST]_{C\leftarrow A} = [S]_{C\leftarrow B}[T]_{B\leftarrow A}$$

(d) Invertible T, inverse T^{-1}, isomorphism

- $T: V \to W$ is invertible if $\exists$ LT $S: W \to V$ s.t.

  $TS = I_W$, $ST = I_V$

- If T is invertible, S is unique, with $S = T^{-1}$

- T is an <u>isomorphism</u> if T is invertible

Key Facts

- T invertible $\Leftrightarrow$ T is injective, T is surjective $\Rightarrow$ T is bijective

  $\Leftrightarrow$ $\ker T = 0$ and $\operatorname{im} T = W$

- If V, W are both finite dimensional, then

  If $\dim V = \dim W \Leftrightarrow V \overset{\text{isomorphic}}{\cong} W$

- If $\dim V = n < \infty$

then $T: V \to V$ is an isomorphism $\iff$ T is injective

$\iff$ T is surjective

$\iff$ $\exists\, S: V \to V^{(LT)}$ s.t. $TS = I_V$

$\iff$ $S: V \to V^{(LT)}$ s.t. $ST = I_V$