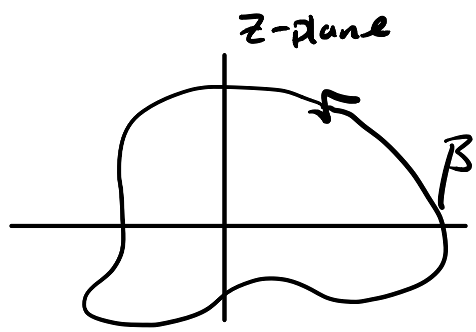
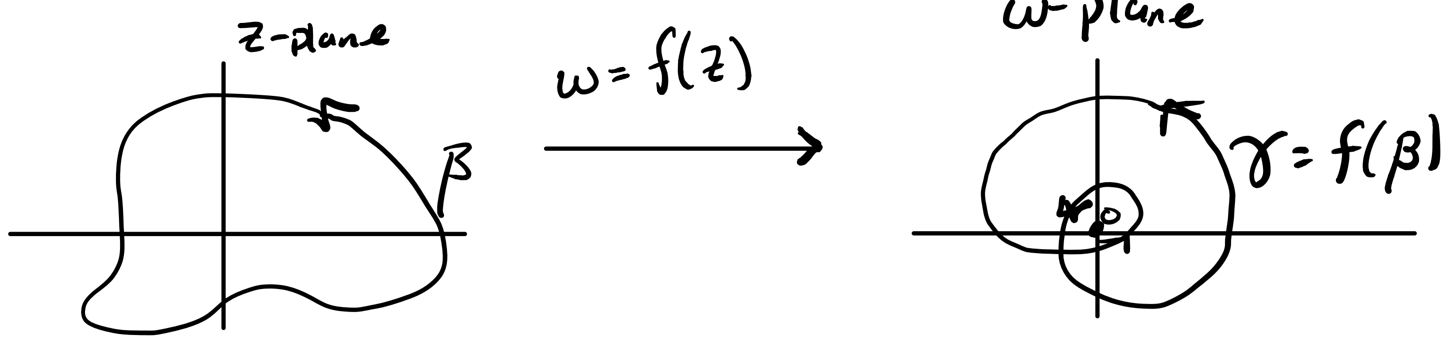
Argument Principle and Counting Zeros

$f \to$ analytic function (think of it as mapping)

$w = f(z)$



Q: Are there any zeros of f inside β?



$W = 2$

winding number of γ about 0

So,

$$\#\text{zeros inside } \beta = W(\gamma; 0)$$

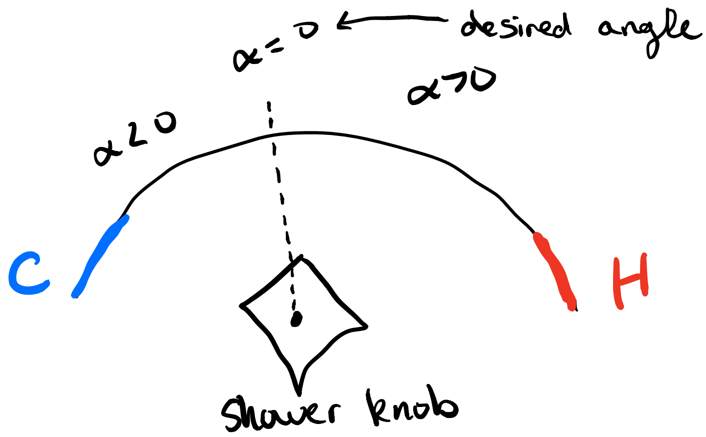
"essentially negative feedback control system"

Application of Argument Principle

Control of water temperature in shower, assuming delayed response of plumbing.

let $u(t)$ = difference between actual and desired water temp

"error"



α = angle on knob

τ = time delay (due to old pipes)

k_1 = Constant (converts angle to temp.)

$\tau \geq 0$ (zero if instantaneous response)

$$\boxed{1)\ u(t) = k_1 \alpha(t-\tau)}$$

2) Control Scheme:

If (water to hot)
    make colder
else if (water to cold)
    make hotter
else ← water perfect temp
    do nothing

So, change angle at a rate proportional to error signal.

$$\boxed{\frac{d\alpha}{dt} = -k_2 u(t)}$$

Note: $k_1, k_2 > 0$ ← Strength of attempted correction (gain)

Now have a "rough" model

Combine equations to get

$$\left.\frac{d\alpha}{dt} = -k_1 k_2 \alpha(t-\tau)\right\} \text{Let } k = k_1 k_2 > 0$$

$$\boxed{\frac{d\alpha}{dt} = -k\alpha(t-\tau)}$$ } infinite-dimensional dynamical systems
(time-delay differential equations)

Note how our differential equation is <u>linear in α</u>
(α and its derivatives appear to first power only)

"exponentials are the eigenfunctions of linear operators"
Guess eigensolution of form

$$\alpha(t) = \alpha_0 e^{st}$$ ← s is an eigenvalue

$$\frac{d\alpha(t)}{dt} = \alpha_0 s e^{st}$$

and

$$\frac{d\alpha}{dt} = -k\alpha(t-\tau) = -k\alpha_0 e^{s(t-\tau)} = -k\alpha_0 e^{st} e^{-s\tau}$$

$\frac{d\alpha}{dt} = -k\alpha(t-\tau)$ reduces to

$$\cancel{\alpha_0} s \cancel{e^{st}} = -k \cancel{\alpha_0} e^{-s\tau} \cancel{e^{st}}$$ for these eigensolutions

This is a solution iff $s = -ke^{-s\tau}$ ← equation for eigenvalues s

$s \in \mathbb{C}$

("expect infinitely many")
(characteristic eqn)

Want to find roots of

$$s + ke^{-s\tau} = 0. \quad \text{given } k, \tau \text{ (both } > 0)$$

Want decaying solutions! $\Rightarrow$ $Re(s) < 0$

$s = a + ib$

$e^{st} = e^{a}\, \overbrace{e^{ibt}}^{\text{oscillating}}$ decays when $a < 0$, $a = Re(s)$

That's why in control theory eigenvalues live in left-half plane.

Want to find condition such that all zeros of $f(s)$ have $Re(s) < 0$, where

$$f(s) = s + ke^{-s\tau}$$ (natural function in this problem)

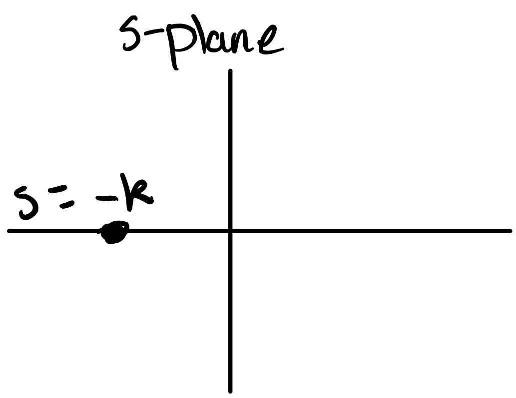
Simple Case

$\tau = 0$ (no delay)

$$f(s) = s + k = 0$$

So, $f = 0$ when

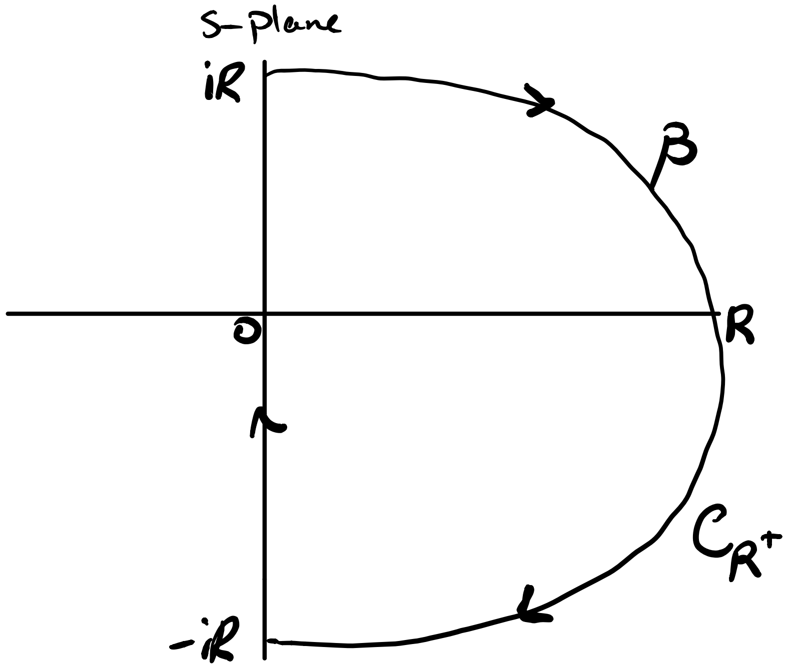
$$s = -k$$ (only one, not infinite amount. As $\tau \to 0$ infinitely many roots go out to infinity)



This is stable (solutions decay to $x = 0$)

By continuity, might expect stability to hold if τ is close to 0.
How much time delay can we tolerate? $\underbrace{\tau < \frac{\pi}{2}\left(\frac{1}{k}\right)}_{\text{prove later}}$ (*)

Want conditions s.t. $\nexists$ zeros in right half plane



$R \to \infty$

Show no zeros in right half-plane

Want to show $f(\beta) = \gamma$ has winding number 0 about origin when $\circledast$ holds