Given $\Omega, \mathbb{P}$, and $X: \Omega \to \mathbb{R}$ a rv. Say X is a **continuous rv** when there exists a function $f_X(x)$ – called the probability density function (pdf) of X – such that "any" $V \in \mathbb{R}$,

$$\mathbb{P}(\{X \in V\}) = \int_V f_X(x)\,dx$$

$f_X(x)$ has to be reasonable enough for integrals to make sense

- $f_X(x) \geq 0 \;\forall x$ (need this to ensure $\mathbb{P}(\{x \in V\}) \geq 0 \;\forall V \subset \mathbb{R}$)
- $\lim_{R\to\infty} \int_{-R}^{+R} f_X(x)\,dx = 1 \to \int_{-\infty}^{\infty} f_X(x)\,dx = \mathbb{P}(\{X \in (-\infty,\infty)\}) = 1$
- Given $x \in \mathbb{R}$, $f_X(x)$ is **NOT** $\mathbb{P}$(some event) – in particular,

$$f_X(x) \neq \mathbb{P}(\{X = x\})$$

Turns out $\mathbb{P}(\{X = x\}) = \mathbf{0} \;\forall x \in \mathbb{R}$ when X is a continuous random variable

Expected Value

The expected value of a continuous rv X w/ pdf $f_X(x)$:

$$\mathbb{E}[X] = \int_{-\infty}^{+\infty} x f_X(x)\,dx$$

Caution: NOT always defined – integral might fail to exist

Expected Value Rule

Given X w/ pdf $f_X(x)$ and $Y = g(X)$, we have

$$\mathbb{E}[Y] = \int_{-\infty}^{+\infty} g(x) f_X(x)\,dx$$

enables $\mathbb{E}[Y]$ computation w/out finding $f_Y(y)$ or $p_Y(y)$

Variance

Variance of continuous rv:

$$Var(X) = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right]$$

By expected value rule, we have

$$Var(X) = \int_{-\infty}^{+\infty} (x - \mathbb{E}[X])^2 f_X(x)\,dx$$

Also, as before,

$$Var(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

Next, define – for **ANY** rv X (discrete OR continuous) – the cumulative distribution function (cdf) by

$$F_X(x) = \mathbb{P}(\{X \leq x\}) \;\forall x \in \mathbb{R}$$

Observation: If X is a continuous rv w/ pdf $f_X(x)$, then since

$$\mathbb{P}(\{X \leq x\}) = \int_{-\infty}^{x} f_X(t)\,dt$$

we have

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt \quad \text{and} \quad f_X(x) = \frac{d}{dx} F_X(x)$$

Discrete version: If X is a discrete rv w/ pmf $p_X(x)$ we have

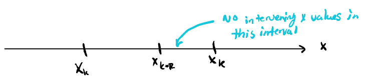
$$F_X(x) = \sum_{\{x_k : x_k \leq x\}} p_X(x_k)$$

set of all possible X-values that don't exceed x

Can invert this formula to get $p_X(x)$ in terms of $F_X(x)$:

$$p_X(x_k) = F_X(x_k) - F_X(x_{k-1})$$

where x_{k-1} is the "**NEXT** largest value" of X below x_k.



General Properties of CDFs

① $\lim_{x\to 0} F_X(x) = 0$ and $\lim_{x\to\infty} F_X(x) = 1$

② When X is a continuous rv, $F_X(x)$ is continuous in x and differentiable "almost everywhere" (corners in $F_X(x)$ correspond to jumps in $f_X(x)$)

③ X is a discrete iff $F_X(x)$ is a piecewise constant.

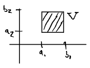
④ $F_X(x)$ is **monotonically** increasing in x.

$$x_1 \leq x_2 \Rightarrow F_X(x_1) \leq F_X(x_2)$$

Say X, Y rvs defined in same $\Omega, \mathbb{P}$ are jointly continuous w/ joint pdf $f_{X,Y}(x,y)$ when

$$\mathbb{P}(\{(X,Y) \in V\}) = \iint_V f_{X,Y}(x,y)\,dx\,dy \quad \forall V \subset \mathbb{R}^2$$

Special case of a V: $[a_1,b_1] \times [a_2,b_2]$



Then

$$\mathbb{P}(\{(X,Y) \in V\}) = \int_{a_1}^{b_1} dx \int_{a_2}^{b_2} dy \left(f_{X,Y}(x,y)\right)$$

Again, have marginals

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dy \; ; \quad f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$

An official way to get this: Get $F_X(x)$ first then take $\frac{d}{dx} F_X(x)$

$$F_X(x) = \mathbb{P}(\{X \leq x\}) = \mathbb{P}(\{(X,Y) \in (-\infty,x] \times (-\infty,\infty)\})$$

$$= \int_{-\infty}^{x} dt \int_{-\infty}^{+\infty} dy \left(f_{X,Y}(t,y)\right)$$

then

$$\frac{d}{dx} F_X(x) = \int_{-\infty}^{+\infty} dy \left(f_{X,Y}(x,y)\right)$$

Could also derive marginal formulas as follows:

$\forall V \subset \mathbb{R}$, $\mathbb{P}(\{X \in V\}) = \mathbb{P}(\{(X,Y) \in V \times (-\infty,\infty)\})$

$$= \int_V dx \int_{-\infty}^{+\infty} dy \left(f_{X,Y}(x,y)\right) = \int_V \left(\left(\int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dy\right)\right) dx$$

Must be $f_X(x)$ integrate over V to get $\mathbb{P}(\{X \in V\})$

Other stuff

- $\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \left(f_{X,Y}(x,y)\right) = 1$
- Joint CDF: $F_{X,Y}(x,y) = \mathbb{P}(\{X \leq x\} \cap \{Y \leq y\}) = \int_{-\infty}^{x} ds \int_{-\infty}^{t} dt \left(f_{X,Y}(s,t)\right)$
- $f_{X,Y}(x,y) = \frac{\partial}{\partial x}\frac{\partial}{\partial y} F_{X,Y}(x,y)$

Conditional Stuff For Continuos Random Variables

Given a continuous rv X on $\Omega, \mathbb{P}$ and some event $A \subset \Omega$, the conditional pdf of X given A "defined" as follows:

For any $V \subset \mathbb{R}$, we have

$$\mathbb{P}(\{X \in V\} | A) = \int_V f_{X|A}(x)\,dx$$

In general, no decent formula for $f_{X|A}(x)$ in terms of $f_X(x)$.
One way to compute it:

- First get conditional cdf of x given A
  $F_{X|A} = \mathbb{P}(\{X \leq x\} | A)$
- Then take d/dx to get $f_X(x)$

However, if A is an event of the form $\{X \in W\}$, and $\mathbb{P}(A) > 0$, we have

$$f_{X|A}(x) = \begin{cases} \dfrac{f_X(x)}{\mathbb{P}(\{X \in W\})} & \text{, when } X \in W \\ 0 & \text{, otherwise} \end{cases}$$

How does this arise?

$$\mathbb{P}(\{X \in V\} | A) = \frac{\mathbb{P}(\{X \in (V \cap W)\})}{\mathbb{P}(\{X \in W\})} = \frac{\int_{V \cap W} f_X(x)\,dx}{\mathbb{P}(\{X \in W\})}$$

Total Probability Theorem in context of $f_{X|A}$:

If X is a continuous rv and $A_1, \ldots, A_n$ are events of positive probability that partition Ω, then

$$f_X(x) = \sum_{k=1}^{n} f_{X|A_k}(x)\,\mathbb{P}(A_k)$$

To see this: go via cdfs.

$$F_{X|A_k} = \frac{\mathbb{P}(\{X \leq x\} \cap A_k)}{\mathbb{P}(A_k)}$$

$$\frac{d}{dx} F_{X|A_k} = f_{X|A_k}(x)$$

By Total Probability Theorem,

$$F_X(x) = \mathbb{P}(\{X \leq x\}) = \sum_{k=1}^{n} F_{X|A_k}\mathbb{P}(A_k) \xrightarrow{\frac{d}{dx}} \sum_{k=1}^{n} f_{X|A_k}\mathbb{P}(A_k) = f_X(x)$$

Comment: this holds when A_k aren't of the special form $\{X \in W_k\}$!

Bottom line: conditional pdf of X given $Y = y$ is

$$f_{X|Y}(x,y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

What you integrate over for any $x \in V$ to get $\mathbb{P}(\{X \in V\} | \{Y = y\})$

Expected value rule for joints.

$$\mathbb{E}(g(X,Y)) = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy\; g(x,y) f_{X,Y}(x,y)$$

As for conditional pmfs in discrete-world, can use conditional pdfs to compute joints, marginals, etc, in situations most naturally expressed in conditional terms.

e.g.

$$f_{X,Y}(x,y) = f_{X|Y}(x|y) f_Y(y)$$

Integrate over x or y to get

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X|Y}(x|y)\,dy \quad \text{OR} \quad f_Y(y) = \int_{-\infty}^{+\infty} f_{Y|X}(y|x) f_X(x)\,dx$$

Conditional Expected Values

Given X, A

$$\mathbb{E}(X|A) = \int_{-\infty}^{+\infty} x f_{X|A}(x)\,dx$$

Given X, Y

$$\mathbb{E}(X|\{Y=y\}) = \int_{-\infty}^{+\infty} x f_{X|Y}(x|y)\,dx \quad \forall y$$

Expected Value Rule

$$\mathbb{E}(g(X)|A) = \int_{-\infty}^{+\infty} g(x) f_{X|A}(x)\,dx$$

$$\mathbb{E}(g(X)|Y=y) = \int_{-\infty}^{+\infty} g(x) f_{X|Y}(x|y)\,dx \quad \forall y$$

Recall the Total Probability-type resuls

- If events $A_1, A_2, \ldots, A_n$ have >0 probability and partition Ω then
  - $f_X(x) = \sum_{k=1}^{n} f_{X|A_k}(x)\,\mathbb{P}(A_k)$
  - $f_X(x) = \int_{-\infty}^{+\infty} f_{X|Y}(x|y) f_Y(y)\,dy$

From these follows

Total Expectation Theorems

$$\mathbb{E}(X) = \sum_{k=1}^{n} \mathbb{E}(X|A_k)\,\mathbb{P}(A_k)$$

$$\mathbb{E}(X) = \int_{-\infty}^{+\infty} \mathbb{E}(X|Y=y) f_Y(y)\,dy$$

For **ANY** pair X, Y (continuous, discrete, or "one and one")

$$X, Y \text{ independent} \Leftrightarrow F_{X,Y}(x,y) = F_X(x) F_Y(y) \quad \forall x, y$$

requires proof

$\Downarrow$ implies (take $\partial/\partial x, \partial/\partial y$)

For X, Y **BOTH** continuous w/ densities $f_X(x), f_Y(y)$; $f_{X,Y}(x,y)$:

$$X, Y \text{ independent} \Leftrightarrow f_{X,Y}(x,y) = f_X(x) f_Y(y) \quad \forall x, y$$

Comment: When X, Y independent, we have

- $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$
- $\mathbb{E}(g(X)h(Y)) = \mathbb{E}(g(X))\mathbb{E}(h(Y))$
- $Var(X+Y) = Var(X) + Var(Y)$

$$f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x) f_X(x)}{f_Y(y)} = \frac{f_{Y|X}(y|x) f_X(x)}{\int_{-\infty}^{+\infty} f_{Y|X}(y|x) f_X(x)\,dx}$$

Continuous Baye's Rule

$X \sim \text{Uniform}[a,b]$

$$f_X(x) = \begin{cases} \frac{1}{b-a} & , x \in (a,b) \\ 0 & , \text{else} \end{cases}$$

$$\mathbb{E}(X) = \frac{b+a}{2}, \quad Var(X) = \frac{(b-a)^2}{12}$$

$$F_X(x) = \begin{cases} 0 & , x < a \\ \frac{x-a}{b-a} & , a \leq x < b \\ 1 & , \text{else} \end{cases}$$

$X \sim \text{Exponential}(\lambda)$

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & , \lambda \geq 0 \\ 0 & , \lambda < 0 \end{cases}$$

$$\mathbb{E}(X) = 1/\lambda \; ; \quad Var(X) = 1/\lambda^2$$

$$F_X(x) = \begin{cases} 1 - e^{-\lambda x} & , x \geq 0 \\ 0 & , \text{else} \end{cases}$$

$X \sim \text{Gaussian}(\mu, \sigma^2)$

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\mathbb{E}(X) = \mu, \quad Var(X) = \sigma^2$$

$$F_X(x) = \int_{-\infty}^{x} e^{-\frac{(t-\mu)^2}{2\sigma^2}}$$

If $Z \sim \text{Gaussian}(0,1)$
$X = \sigma Z + \mu$
If $Y = aX + b$, then
$Y \sim \text{Gaussian}(a\mu + b, a^2\sigma^2)$

That's just the wave