

① MMSE Estimation

X is a continuous random variable with PDF given by

$$f_X(x) = \begin{cases} 1/x, & 1 \leq x \leq e \\ 0, & \text{elsewhere} \end{cases}$$

Given $X=x$, Y has the following conditional pdf:

$$f_{Y|X=x}(y) = \frac{x}{2} e^{-x|y|}, \quad -\infty < y < \infty$$

Find the MMSE estimate of X given Y .

Want

$$\mathbb{E}[X|Y].$$

$$f_{Y|X=x}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

$$f_{X,Y}(x,y) = f_{Y|X=x}(y|x) f_X(x)$$

$$= \begin{cases} \frac{1}{2} e^{-x|y|}, & 1 \leq x \leq e \\ 0, & -\infty < y < \infty \end{cases}$$

Want

$$f_{X|Y=y}(x|y) \rightarrow \int_{-\infty}^{+\infty} x f_{X|Y=y}(x|y) dx$$

$$\begin{aligned}
 f_Y(y) &= \int_{x=1}^e \frac{1}{2} e^{-|x|y} dx \\
 &= -\frac{1}{2|y|} e^{-|x|y} \Big|_1^e \\
 &= \frac{1}{2|y|} (e^{-|y|} - e^{-ey})
 \end{aligned}$$

$$f_{X|Y=y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{\frac{1}{2} e^{-|x|y}}{\frac{1}{2|y|} (e^{-|y|} - e^{-ey})} = \begin{cases} \frac{|y| e^{-|x|y}}{e^{-|y|} - e^{-ey}}, & 1 \leq x \leq e \\ 0, & \text{else} \end{cases}$$

$$\begin{aligned}
 E[X|Y=y] &= \int_{x=1}^e \frac{|y|}{e^{-|y|} - e^{-ey}} \times e^{-|x|y} dx \\
 &= \frac{|y|}{e^{-|y|} - e^{-ey}} \left(-\frac{x}{|y|} e^{-|x|y} - \frac{1}{|y|^2} e^{-|x|y} \Big|_{x=1}^e \right)
 \end{aligned}$$

$u \quad dv$
 $x \quad e^{-|x|y}$
 $-1 \quad -\frac{1}{|y|} e^{-|x|y}$
 $0 \quad \frac{1}{|y|^2} e^{-|x|y}$

$$\begin{aligned}
 &\frac{|y|}{e^{-|y|} - e^{-ey}} \left(\frac{-e e^{-ey} + e^{-|y|}}{|y|} - \frac{1}{|y|^2} (e^{-ey} - e^{-|y|}) \right) \\
 &= \boxed{\frac{e^{-|y|} - e^{(1-e)|y|}}{e^{-|y|} - e^{-ey}} + \frac{1}{|y|}}
 \end{aligned}$$

② Poisson Processes

Let $\{N_1(t)\}_{t \geq 0}$ and $\{N_2(t)\}_{t \geq 0}$ be two independent Poisson processes.

Determine whether each of the following statements is true or false.

(a) $\{N_1(t)\}_{t \geq 0}$ is NOT stationary, but it is wide-sense stationary.

A WSS Poisson process \Rightarrow Stationarity. FALSE

(b) $\{N_1(t)\}_{t \geq 0}$ has independent and stationary increments.

TRUE

(c) $\{N_1(t)\}_{t \geq 0}$ is a Markov Process

TRUE

(d) If it is known $E[N_1(1)] = \alpha$, then $\forall t > 0$,

$$E[N_1(t)] = \text{Var}(N_1(t)) = \alpha t$$

TRUE

(e) Let $X(t) = N_1(t) + N_2(t)$. Then $\{X(t)\}_{t \geq 0}$ is a Poisson Process.

TRUE

$$\begin{aligned} N_1(t) - N_1(s) + N_2(t) - N_2(s) \\ \lambda_1(t-s) + \lambda_2(t-s) \\ \sim (\lambda_1 + \lambda_2)(t-s) \end{aligned}$$

(f) Let $Y(t) = N_1(t) - N_2(t)$. Then $\{Y(t)\}_{t \geq 0}$ is a Poisson Process.

$$\begin{aligned} Y(t) - Y(s) &= N_1(t) - N_2(t) - (N_1(s) - N_2(s)) \\ &= \underbrace{N_1(t) - N_1(s)}_{\sim \lambda_1(t-s)} - N_2(t) + N_2(s) \\ &\sim \lambda_1(t-s) - \lambda_2(t+s) \end{aligned}$$

FALSE

(g) Let $Z(t) = N_1(t+3) - N_1(t)$. Then $\{Z(t)\}_{t \geq 0}$ is wide-sense stationary.

$$\begin{aligned} \mathbb{E}[Z(t)] &= \mathbb{E}[N_1(t+3) - N_1(t)] \\ &= \lambda \cdot 3 \rightarrow \text{constant} \end{aligned}$$

$$R_Z(t_1, t_2) \stackrel{\Delta}{=} \mathbb{E}[Z(t_1)Z(t_2)]$$

$$= \mathbb{E}[(N_1(t_1+3) - N_1(t_1))(N_1(t_2+3) - N_1(t_2))]$$

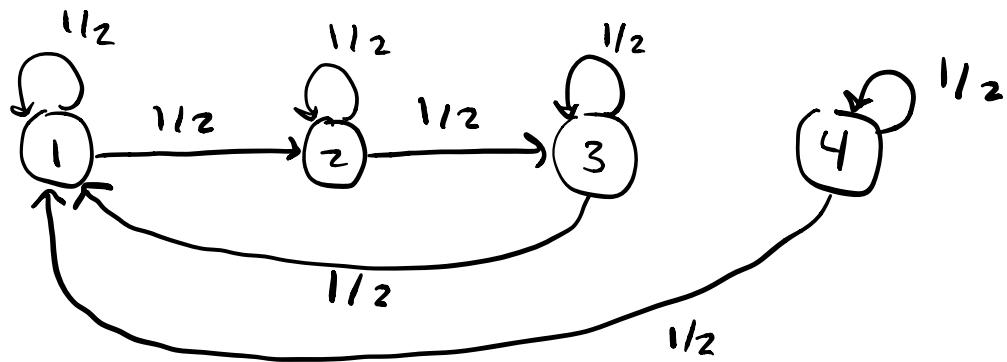
TRUE

③ Markov Chain: Basic Concepts

Consider a discrete-time Markov chain with four states $\{1, 2, 3, 4\}$ and the following one-step transition matrix

$$P = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 2 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 3 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 4 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

(a) Draw the state transition diagram. Identify all communication classes. Find the period of each class. Is the chain irreducible?



$$1 \leftrightarrow 2, \quad 1 \leftrightarrow 3, \quad 2 \leftrightarrow 3 \quad 4 \leftrightarrow 4$$

Communication Classes

$$\{1, 2, 3\}, \{4\} \leftarrow \text{Chain NOT irreducible}$$

Period of each class = 1

(b) How many stationary distributions does the chain have?

Find all stationary distributions

Just one stationary distribution

$$\bar{\pi} \bar{P} = \bar{\pi}$$

$$[\pi_1, \pi_2, \pi_3, \pi_4] \begin{bmatrix} 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ 1/2 & 0 & 0 & 0 \end{bmatrix} = [\pi_1, \pi_2, \pi_3, \pi_4]$$

$$\frac{1}{2}\pi_1 + \frac{1}{2}\pi_3 + \frac{1}{2}\pi_4 = \pi_1 \rightarrow \pi_1 = \pi_3 + \pi_4$$

$$\frac{1}{2}\pi_1 + \frac{1}{2}\pi_2 = \pi_2 \rightarrow \pi_2 = \pi_1$$

$$\frac{1}{2}\pi_2 + \frac{1}{2}\pi_3 = \pi_3 \rightarrow \pi_3 = \pi_2$$

$$\frac{1}{2}\pi_4 = \pi_4 \rightarrow \pi_4 = 0$$

Normalize

$$\sum_{i=1}^4 \pi_i = 1$$

$$\pi_1 + \pi_1 + \pi_1 + 0 = 1$$

$$\pi_1 = 1/3$$

$$\bar{\pi} = [1/3 \ 1/3 \ 1/3 \ 0]$$

(c) Determine the long-run fraction of time the chain spends in State 1 given that it starts in state 4.

State 4 transient \rightarrow eventually moves out to $\{1, 2, 3\}$

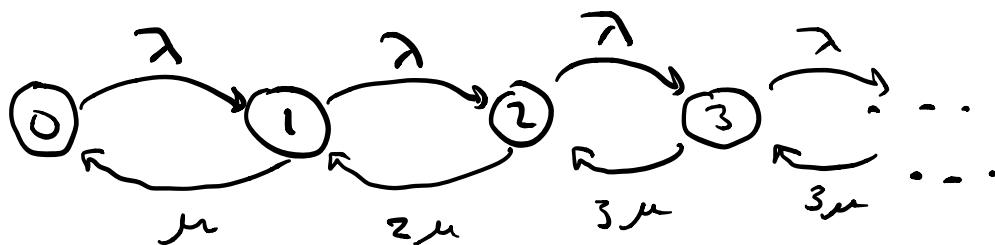
$\Rightarrow \pi_1 = 1/3$ time spent in ① long-run

④ Continuous-Time Markov Chain: M/M/s queuing model

Consider the following M/M/s queuing model.

- There are a total $s=3$ servers.
- Customers arrive according to a Poisson process with rate λ
→ arriving customers wait in the queue and go to the first free server in order of arrival
- Time required to service a customer by any server is exponentially distributed with rate μ .

(a) Construct a continuous-time Markov chain for this queuing model with the number of customers waiting in the queue as a state.



That is,

$$q_{i,i+1} = \lambda \quad \forall i \geq 0$$

$$q_{i,i-1} = \begin{cases} \mu, & i=1 \\ 2\mu, & i=2 \\ 3\mu, & i=3 \end{cases}$$

$\mathcal{X} = \{0, 1, 2, \dots\}$ is our state space.

(b) Identify the condition under which the chain has a stationary distribution. Compute the stationary distribution.

For existence of stationary distribution, arrival rate must be less than service rate for large i .

That is,

$$\begin{aligned}\lambda &< 3\mu \\ \Rightarrow \frac{\lambda}{3\mu} &< 1\end{aligned}$$

$$Q = \begin{bmatrix} -\lambda & \lambda & & & \\ \mu & -(\lambda+\mu) & \lambda & & \\ & 2\mu & -(\lambda+2\mu) & \lambda & \\ & & 3\mu & -(\lambda+3\mu) & \lambda \\ & & & \ddots & \ddots & \ddots \end{bmatrix}; \quad \pi Q = 0$$

$$\pi_0 \lambda - (\lambda + \mu) \pi_1 + \mu \pi_2 = 0 \rightarrow \pi_0 \lambda = \pi_1 \mu$$

$$\pi_1 \lambda - (\lambda + 2\mu) \pi_2 + 2\mu \pi_3 = 0 \rightarrow -\pi_1 \lambda + 2\mu \pi_2 = 0 \rightarrow$$

$$\pi_1 \lambda - (\lambda + 2\mu) \pi_2 + \pi_2 3\mu \rightarrow \pi_1 \lambda = \pi_2 (2\mu)$$

$$2\mu \pi_2 - \lambda \pi_2 = \pi_2 3\mu$$

$$\pi_2 \lambda = \pi_3 (3\mu)$$

⋮

$$\pi_{k+2} \lambda = \pi_{k+1} (3\mu)$$

Thus,

$$\pi_i = \begin{cases} \frac{\lambda}{\mu} \pi_0, & i=1 \\ \frac{\lambda^2}{2\mu^2} \pi_0, & i=2 \\ \frac{9}{2} \left(\frac{\lambda}{3\mu}\right)^i \pi_0, & i>2 \end{cases}$$

Normalizing gives

$$\pi_0 \left(1 + \frac{\lambda}{\mu} + \frac{\lambda^2}{2\mu^2} + \frac{3}{2} \sum_{i=2}^{\infty} \left(\frac{\lambda}{3\mu}\right)^i \right) = 1$$

Since $\lambda/3\mu < 1$ required for existence of $\bar{\pi}$,

$$\pi_0 \left(1 + \frac{\lambda}{\mu} + \frac{\lambda^2}{2\mu^2} + \frac{3}{2} \left(\frac{1}{1 - \frac{\lambda}{3\mu}} - 1 - \frac{\lambda}{3\mu} - \frac{\lambda^2}{9\mu^2} \right) \right) = 1$$

$$\pi_0 = \left(1 + \frac{\lambda}{\mu} + \frac{1}{2} \left(\frac{\lambda}{\mu}\right)^2 + \frac{3}{2} \frac{(\lambda/3\mu)^3}{1 - \lambda/3\mu} \right)^{-1}$$

(c) What is the percentage of time at least one server is idle?

Idle server if < 3 customers.

$$\rightarrow \pi_0 + \pi_1 + \pi_2$$

(d) What is the average queue length in terms of the stationary distribution?

$$\sum_{i=1}^{\infty} i \pi_i$$

⑤ Spectral Analysis and Linear Filtering of Random Processes

Let $\{X_n\}_{n=-\infty}^{+\infty}$ be a wide-sense stationary random process with zero mean and autocorrelation

$$R_X(k) = \begin{cases} 1/2, & \text{if } |k|=1 \\ 1, & \text{if } k=0 \\ 0, & \text{o/w} \end{cases}$$

Suppose that we sample this process to form two new processes.

The process $\{Y_n\}$ consists of even-numbered samples.

That is,

$$Y_n = X_{2n}, \quad -\infty < n < \infty$$

The process $\{Z_n\}$ consists of odd-numbered samples.

That is,

$$Z_n = X_{2n+1}, \quad -\infty < n < \infty$$

(a) Compute the autocorrelation of $\{Y_n\}$. Is this process wide-sense stationary? If so, compute its power spectral density.

$$\begin{aligned} R_Y(n, s) &\stackrel{\Delta}{=} \text{IE}[Y_n Y_s] \\ &= \text{IE}[X_{2n} X_{2s}] = \begin{cases} 1, & n=s \\ 0, & \text{o/w} \end{cases} \end{aligned}$$

$$\mathbb{E}[Y_n] = \mathbb{E}[X_{2n}] = 0$$

$$\begin{aligned}
R_Y(n, n+k) &\triangleq \mathbb{E}[Y_n Y_{n+k}] \\
&= \mathbb{E}[X_{2n} X_{2n+2k}] \\
&= R_X(2n, 2n+2k) \\
&= R_X(2k) \quad (\text{X WSS}) \therefore Y \text{ WSS}
\end{aligned}$$

$$S_Y(f) = \mathcal{F}\{R_X(2k)\} = \mathcal{F}\{\delta(k)\} = 1$$

(b) Compute the autocorrelation of $\{Z_n\}$. Is this process wide-sense stationary? If so, compute its power spectral density.

$$\begin{aligned}
R_Z(n, s) &\triangleq \mathbb{E}[Z_n Z_s] \\
&= \mathbb{E}[X_{2n+1} X_{2s+1}] \\
&= \begin{cases} 1, & n=s \\ 0, & \text{o/w} \end{cases} \quad \underbrace{R_X(2s-2n)}_{\in \mathbb{R}}
\end{aligned}$$

$$\mathbb{E}[Z_n] = \mathbb{E}[X_{2n+1}] = 0$$

$$\begin{aligned}
R_Z(n, n+k) &\triangleq \mathbb{E}[Z_n, Z_{n+k}] \\
&= \mathbb{E}[X_{2n+1} X_{2n+2k+1}] \\
&= R_X(2k) = \delta(k) \rightarrow \sum_{k=0}^{\infty} \delta(k) = 1 \quad \text{WSS}
\end{aligned}$$

$$S_Z(f) = 1$$

(c) Are $\{Y_n\}$ and $\{Z_n\}$ jointly wide-sense stationary?

If so, compute their cross power spectral density.

- Know Y_n, Z_n both WSS from (a), (b).

Need

$$R_{Y,Z}(n, n+k) = R_{Y,Z}(k)$$

$$R_{Y,Z}(n, n+k) \triangleq \mathbb{E}[Y_n Z_{n+k}]$$

$$= \mathbb{E}[X_{2n} X_{2n+2k+1}]$$

$$= R_x(2n+2k+1 - 2n) \quad k=0 \rightarrow R_x(1)$$

$$= R_x(2k+1) \quad k=-1 \rightarrow R_x(-1)$$

$$= \begin{cases} 1/2, & k=0 \\ 1/2, & k=-1 \\ 0, & \text{o/w} \end{cases} = \frac{1}{2} \delta(k) + \frac{1}{2} \delta(k+1)$$

$$S_{Y,Z}(f) = \widetilde{\mathcal{F}}\{R_x(2k+1)\} = \frac{1}{2} + \frac{1}{2} e^{+j2\pi f}$$

(d) If we pass $\{Y_n\}$ through a discrete-time low-pass filter with

$$H(f) = \begin{cases} 1 & , -\frac{1}{4} < f < \frac{1}{4} \\ 0 & , \text{else} \end{cases}$$

compute the average power of the output process.

$$\text{Output } S_o(f) = |H(f)|^2 S_Y(f) = 1, -\frac{1}{4} < f < \frac{1}{4}$$

$$R_o(0) = \int_{-1/2}^{+1/2} S_o(f) df = \int_{-1/4}^{+1/4} 1 df = \frac{1}{2}$$