Math 4310

Homework 8

Due 11/6/19

Name: ______________________

Collaborators: ______________________

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Please print out these pages. I encourage you to work with your classmates on this homework. Please list your collaborators on this cover sheet. (Your grade will not be affected.) Even if you work in a group, you should write up your solutions yourself! You should include all computational details, and proofs should be carefully written with full details. As always, please write neatly and legibly.

Please follow the instructions for the "extended glossary" on separate paper (LaTeX it if you can!) Hand in your final draft, including full explanations and write your glossary in complete, mathematically and grammatically correct sentences. Your answers will be assessed for style and accuracy.

Please **staple** *this cover sheet, your exercise solutions, and your glossary together, in that order, and hand in your homework in class.*

GRADES

Exercises ______________________ / 50

Extended Glossary

Component	Correct?	Well-written?
Definition	/6	/6
Example	/4	/4
Non-example	/4	/4
Theorem	/5	/5
Proof	/6	/6
Total	/25	/25

Exercises.

1. Suppose that $T \in \mathcal{L}(V)$ is invertible, where $\dim V = n$ is a vector space over a field $\mathbb{F}$, and that the distinct eigenvalues of T are $\lambda_1, \ldots, \lambda_m$, and also suppose that the eigenspace $E_{\lambda_i}(T)$ has dimension d_i, for each i. Find the eigenvalues of T^{-1}, and the dimension of its eigenspaces at each eigenvalue.

2. Find $A, B \in \mathbb{R}^{4\times 4}$ such that each A and B have $-1, 3, 10$ as eigenvalues, and they have no other eigenvalues, and such that A and B are not similar. (As always, justify your reasoning!)

3. Consider the stochastic matrix
$$A = \begin{pmatrix} 0.6 & 0.1 & 0.1 \\ 0.1 & 0.9 & 0.2 \\ 0.3 & 0 & 0.7 \end{pmatrix}.$$
   (a) Find the Gershgorin disks for A.
   (b) Find the eigenvalues for A.
   (c) If A is the transition matrix for a 3 state Markov chain, and the initial probability vector (at time step 0) is
$$P = \begin{pmatrix} 0.3 \\ 0.3 \\ 0.4 \end{pmatrix},$$
   Find the proportions in each state after 2 time steps. Find the eventual proportions in each state, (i.e. the limiting proportions, as the time step goes to infinity), by computing the fixed probability vector.

4. Let V be a vector space of dimension n over a field $\mathbb{F}$, and throughout this entire problem, let $S, T \in \mathcal{L}(V)$ be diagonalizable linear operators.

   (a) Show the following: if there is a basis $\mathcal{B}$ of V such that both $[S]_{\mathcal{B}}$ and $[T]_{\mathcal{B}}$ are diagonal matrices, then S and T commute: $ST = TS$.

   (b) Suppose that $U \subset V$ is T-invariant, and that $v_1, \ldots, v_k$ are eigenvectors with respect to different eigenvalues. Show that if $v_1 + \ldots + v_k \in U$, then each $v_i \in U$, for $i = 1..k$.

   (c) If $U \subset V$ is T-invariant, then show that the induced map $T|_U : U \longrightarrow U$ is also diagonalizable.

   (d) Show the converse of part (a): If $ST = TU$, then there is a basis $\mathcal{B}$ of V such that both $[S]_{\mathcal{B}}$ and $[T]_{\mathcal{B}}$ are diagonal matrices, (Remember, in all parts of this problem: S and T are assumed to be diagonalizable! Also, as a hint, you might find the earlier parts of the problem could help with later parts).

5. The vector space of polynomials $\mathbb{F}[x]$ has a multiplication on it too (we can multiply 2 polynomials, and get a new polynomial. This makes $\mathbb{F}[x]$ into what is called an algebra). An *ideal* is a subspace $I \subseteq \mathbb{F}[x]$ which satisfies the property that if $h(x) \in \mathbb{F}[x]$, and $f(x) \in I$, then $h(x)f(x) \in I$.

   (a) Fix a polynomial $f(x)$. Show that the set
   $$\langle f(x) \rangle = \{h(x)f(x) \in \mathbb{F}[x] \mid h(x) \in \mathbb{F}[x]\}$$
   is an ideal in $\mathbb{F}[x]$ (called the *ideal generated by* $f(x)$).

   (b) Suppose that the ideal I contains a polynomial $g(x)$ of degree d, but no nonzero polynomial of lower degree. Show that every polynomial in I is divisible by $g(x)$. Use this to show that $I = \langle g(x) \rangle$.

6. If $f(x)$ is a polynomial, and A is an $n \times n$ matrix, we can form the $n \times n$ matrix $f(A)$. For example, if $f(x) = x^2 - 2x + 3$, then $f(A) = A^2 - 2A + 3I_n$.

   (a) Show that if A is an $n \times n$ matrix, then
   $$\mathrm{ann}(A) := \{f(x) \in \mathbb{F}[x] \mid f(A) = 0_{n \times n}\}$$
   is an ideal in $\mathbb{F}[x]$.

   (b) Find $\mathrm{ann}(A)$, if
   $$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Extended Glossary. There is no extended glossary this week.