Recall

V a vector space over $\mathbb{C}$, $\dim V = n < \infty$

$$E_\lambda(T) = \{\vec{v} \in V \mid (T-\lambda I)v = 0\}$$
$$= \ker(T-\lambda I)$$

Same when T is diagonalizable

$$G_\lambda(T) = \ker(T-\lambda I)^n$$

$\dim E_\lambda(T)$ is called the "geometric multiplicity" of λ in T.

$\dim G_\lambda(T)$ is called the "algebraic multiplicity" of λ in T.

Definition: The characteristic polynomial of T is

$$q(x) = (x-\lambda_1)^{a_1}(x-\lambda_2)^{a_2}\cdots(x-\lambda_m)^{a_m}$$

where $\lambda_1, \dots, \lambda_m$ are distinct eigenvalues of T and

$$a_i = \dim G_{\lambda_i}(T)$$

Last time:

$$V = G_{\lambda_1}(T) \oplus \cdots \oplus G_{\lambda_m}(T)$$

So

$$\deg q(x) = a_1 + a_2 + \cdots + a_m = \dim V = n$$

Want to prove this now:

$$q(x) = \det(xI_n - A)$$

where

$$A = [T]_\beta \quad \text{for some basis } \beta$$

<u>Recall</u>

$\dim V = n$

N $n \times n$ matrix

N is <u>nilpotent</u> if $N^r = 0$ for some r

r s.t. $N^r = 0$, $N^{r-1} \neq 0$ is called the index of n

Proved: N nilpotent $\Rightarrow N^n = 0$

Example

Suppose N is nilpotent and $\exists\, v \in V = \mathbb{C}^n$ s.t.

$$\alpha = (\vec{v}, N\vec{v}, N^2\vec{v}, \dots, N^{n-1}\vec{v})$$

is a basis of V, and

$$\beta = (N^{n-1}\vec{v}, N^{n-2}\vec{v}, \dots, N\vec{v}, \vec{v})$$

be another, basis.

Find $[N]_\alpha$, $[N]_\beta$

Soln

(a)

$$\begin{array}{c|c}
 & \begin{matrix} v & Nv & N^2v & \cdots & N^{n-1}v & \end{matrix} \\
\begin{matrix} v \\ Nv \\ N^2v \\ \vdots \\ \vdots \\ N^{n-1}v \end{matrix} &
\begin{bmatrix}
0 & 0 & & 0 & 0 \\
1 & 0 & & 0 & 0 \\
0 & 1 & \ddots & 0 & 0 \\
\vdots & 0 & \ddots & \vdots & \vdots \\
\vdots & \vdots & \ddots & 1 & \vdots \\
0 & 0 & \cdots & 0 & 0
\end{bmatrix}
\end{array}$$

(β)

$$\begin{array}{c|c}
 & \begin{matrix} N^{n-1}v & N^{n-2}v & \cdots & v \end{matrix} \\
\begin{matrix} N^{n-1}v \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ v \end{matrix} &
\begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & & \vdots \\
\vdots & \vdots & & \ddots & 0 \\
\vdots & \vdots & & & \ddots \\
0 & 0 & 0 & \cdots & 1 \\
0 & 0 & 0 & & 0
\end{bmatrix}
\end{array}$$

(transpose of α!)

e.g $n=4$

$$\alpha \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad \beta \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

definition: Let $\lambda \in \mathbb{C}$, $n \in \mathbb{Z}_{\geq 1}$.

Define

$$J(\lambda, n) = \begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix} \quad (n \times n)$$

$J(\lambda, n)$ is called a Jordan Block.

$$J(\lambda, 4) = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$$

Note: if $A = J(\lambda, n)$, then λ is the only eigenvalue of A, and

$$V = \mathbb{C}^n = \ker(A - \lambda I)^n$$

so $N = A - \lambda I$ is <u>nilpotent</u>

(4×4) example

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Definition: Let $T \in \mathcal{L}(V)$. A basis β of V is called a Jordan basis of T if

$$[T]_\beta = \begin{bmatrix} J(\lambda_1, n_1) & & & \\ & J(\lambda_2, n_2) & & \\ & & \ddots & \\ & & & J(\lambda_N, n_N) \end{bmatrix}$$

$(n \times n)$

called Jordan canonical form of T. It is unique.

$$= J(\lambda_1, n_1) \oplus \cdots \oplus J(\lambda_N, n_N)$$

Theorem: If $\dim V = n < \infty$, V a v.s over $\mathbb{C}$, and $T \in \mathcal{L}(V)$.
Then $\exists$ Jordan basis of T.

Corollary: Every $A \in \mathbb{C}^{n \times n}$ is similar to a direct sum of Jordan blocks.

$A = \begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_r \end{bmatrix}$ A_i $n_i \times n_i$, A $n \times n$, $n = n_1 + \cdots + n_r$

Remark: If $A = A_1 \oplus \cdots \oplus A_r$ and π is a permutation of $1, \ldots, r$ then $B = A_{\pi(1)} \oplus A_{\pi(2)} \oplus \cdots \oplus A_{\pi(r)}$ is similar to A.

Example: Suppose A is $n \times n$, has only one eigenvalue λ. For small n, find the possible Jordan Canonical form's of A.

Sol'n

① $n = 2$, $A_{2\times 2}$

$$A \underset{\text{similar}}{\sim} \left(\begin{array}{c|c} \lambda & \\ \hline & \lambda \end{array}\right) \quad \text{or} \quad \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

	$J(\lambda,1) \oplus J(\lambda,1)$	$J(\lambda,2)$
$\dim E_\lambda(A)$	2	1
$\dim G_\lambda(A)$	2	2

② $n = 3$, $A_{3\times 3}$

$$A \sim \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \text{ or } \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} \text{ or } \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

	$J(\lambda,1) \oplus J(\lambda,1) \oplus J(\lambda,1)$	$J(\lambda,3)$	$J(\lambda,2) \oplus J(\lambda,1)$
$\dim E_\lambda(A)$	3	1	2
$\dim G_\lambda(A)$	3	3	3

Definition: $(k_1, \ldots, k_r)$ is called a partition of $n \in \mathbb{Z}_{>0}$ if

ⓐ $k_1 + \cdots + k_r = n$

ⓑ $k_1 \geq k_2 \geq \cdots \geq k_r > 0$

Definition: If $\underline{k} = (k_1, \dots, k_r)$ is a partition of n,

let $J(\lambda, \underline{k}) = J(\lambda, k_1) \oplus \cdots \oplus J(\lambda, k_r)$

$n = 4$

$$A \sim \left(\begin{array}{c|c|c|c} \lambda & 0 & 0 & 0 \\ \hline 0 & \lambda & 0 & 0 \\ \hline 0 & 0 & \lambda & 0 \\ \hline 0 & 0 & 0 & \lambda \end{array}\right), \quad \left(\begin{array}{cc|cc} \lambda & 1 & 0 & 0 \\ \hline 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{array}\right)$$

$J(\lambda, 1111)$ $\qquad$ $J(\lambda, 21)$

$$\left(\begin{array}{cc|cc} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ \hline 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{array}\right), \quad \left(\begin{array}{ccc|c} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 0 \\ \hline 0 & 0 & 0 & \lambda \end{array}\right)$$

$J(\lambda, 22)$ $\qquad$ $J(\lambda, 31)$

$$\begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix} \; J(\lambda, 4)$$

$\dim E_\lambda(A) = 4, 3, 2, 2, 1$ (in order listed

$\dim G_\lambda(A) = \underline{4}$ (always n)