10) Simplify

$$\left[\frac{2+i}{6i-(1-2i)}\right]^2$$

$$= \left[\frac{2+i}{8i-1}\right]^2 = \left[\frac{2+i}{8i-1}\cdot\frac{-8i-1}{-8i-1}\right] = \left[\frac{6-17i}{65}\right]^2$$

$$= \frac{(6-17i)^2}{65^2} = \frac{-253}{4225} - i\,\frac{204}{4225}$$

21)

$$(1-i)z_1 + 3z_2 = 2-3i$$

$$iz_1 + (1+2i)z_2 = 1$$

$$z_1 = \frac{1-(1+2i)z_2}{i} = \frac{1-z_2-2iz_2}{i}$$

$$(1-i)\left[\frac{1-(1+2i)z_2}{i}\right] + 3z_2 = 2-3i$$

$$(1-i) - (1-i)(1+2i)z_2 + 3iz_2 = 3 \quad +2i$$

$$(1-i) - [1+\cancel{2i}^{\,i}\cancel{-i}+2]z_2 + 3iz_2 = 3+2i$$

$$1-i-[3+i]z_2 + 3iz_2 = 3+2i$$

$$-3z_2 - iz_2 + 3iz_2 = 2+3i$$

$$2iz_2 - 3z_2 = 2+3i$$

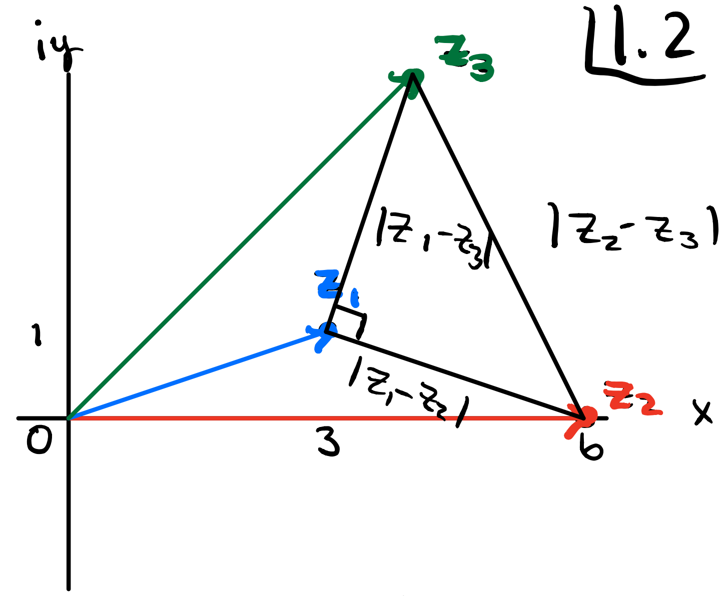
$$z_2[2i-3] = 2+3i$$

$$z_2 = \frac{2+3i}{2i-3} \cdot \frac{-2i-3}{-2i-3} \rightarrow \boxed{z_2 = -i}$$

$$iz_1 + (1+2i)(-i) = 1$$

$$z_1 = \frac{-1-i}{i} = \boxed{1+i}$$

6) $z_1 = 3+i$
$z_2 = 6$
$z_3 = 4+4i$




$$|z_1 - z_2| = \sqrt{3^2+1^2} = \sqrt{10}$$

evidently a right triangle

$$|z_1 - z_3| = \sqrt{3^2+1^2} = \sqrt{10}$$

$$|z_2 - z_3| = \sqrt{2^2+4^2} = \sqrt{20}$$

Edges satisfy Pythagorean Theorem!

$$|z_1 - z_2|^2 + |z_1 - z_2|^2 = |z_2 - z_3|^2$$

7) Describte the set of points

$$|z-1| + |z+1| = 7$$

The equality here describes a set of complex points $z$ whose distance from $(-1,0)$ and $(1,0)$ is constant! This is the definition of foci on an ellipse.

Therefore the set of points described is an ellipse in the complex plane with its foci at $(-1,0)$, $(1,0)$.

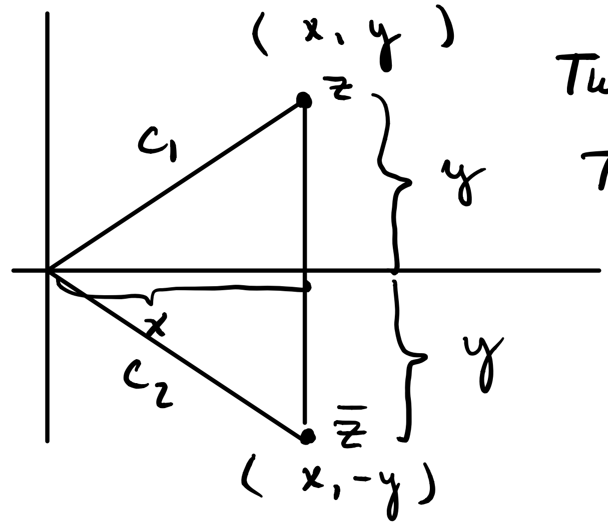
8) Show

$$|z-1| = |\bar{z}-1|$$

Algebraic

$$\sqrt{(x-1)^2+1^2} = \sqrt{(x-1)^2+(-1)^2} \quad \checkmark$$

Geometrically




Two equal Triangles so $c_1 = c_2$

16) Prove that if

$$|z| = 1, \ (z \neq 1)$$

then

$$\text{Re}\left(\frac{1}{1-z}\right) = \frac{1}{2}$$

$$z = x+iy \neq 1$$

$$|z| = \sqrt{x^2+y^2} = 1 \Rightarrow y^2 = 1-x^2$$

$$\text{Re}\left(\frac{1}{1-z}\right) = \text{Re}\left(\frac{1}{1-(x+iy)}\right) = \text{Re}\left(\frac{(1-x)+iy}{(1-x)^2+y^2}\right)$$

$$= \frac{(1-x)}{2(1-x)} = \frac{1}{2}$$

7h) Find the argument of $\dfrac{-\sqrt{7}(1+i)}{\sqrt{3}+i}$

Let

$$z = \frac{-\sqrt{7}(1+i)}{\sqrt{3}+i} \cdot \frac{\sqrt{3}-i}{\sqrt{3}-i} = -\frac{(\sqrt{21}+\sqrt{7})}{4} + i\,\frac{(\sqrt{7}-\sqrt{21})}{4}$$

$$\arg(z) = \arctan\left(\frac{(\sqrt{7}-\sqrt{21})}{-(\sqrt{21}+\sqrt{7})}\right) = \frac{13\pi}{12} + 2\pi k,\ k \in \mathbb{Z}$$

$$|z| = \sqrt{\left(-\frac{(\sqrt{21}+\sqrt{7})}{4}\right)^2 + \left(\frac{(\sqrt{7}-\sqrt{21})}{4}\right)^2} = \frac{\sqrt{14}}{2}$$

Our answer in polar form is thus

$$\frac{\sqrt{14}}{2}\, e^{i\frac{13\pi}{12}}$$

11) Use $(1+i)(5-i)^4$ to derive

$$\frac{\pi}{4} = 4\arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right)$$

$$\arg(z^n) = \arg(z) + \arg(z) + \cdots + \arg(z) = n \cdot \arg(z)$$
$n \in \mathbb{Z}$

$$(1+i)(5-i)^4 = 956-4i$$

$$\arg\left[(1+i)(5-i)^4\right] = \arg(956-4i)$$

$$\Rightarrow \arg(1+i) + 4\arg(5-i) = \arg(956-4i)$$

$$\frac{\pi}{4} + 4\arctan\left(-\frac{1}{5}\right) = \arctan\left(-\frac{1}{239}\right)$$

$$\arctan(-u) = -\arctan(u)$$

$$\frac{\pi}{4} = 4\arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right)$$

5a) Find $(-16)^{1/4}$

$z = -16$

$|z| = 16$

$\arg(z) = \pi + 2\pi k, \ k \in \mathbb{Z}$

$z = 16e^{i\pi + 2\pi k}$

$z^{1/4} = (16)^{1/4} e^{i\frac{\pi}{4} + \frac{\pi}{2}k}$ , $k = 0, 1, 2, 3$

11) Solve $(z+1)^5 = z^5$

$\frac{(z+1)^5}{z^5} = 1$

$\frac{z+1}{z} = 1^{1/5} = e^{i\left(\frac{2\pi k}{5}\right)}$ , $k \neq 5n, \ n \in \mathbb{Z}$

$z\left(e^{i\frac{2\pi k}{5}} - 1\right) = 1$

$z = \frac{1}{e^{i\frac{2\pi k}{5}} - 1}$ , $k \neq 5n, \ n \in \mathbb{Z}$