ECE 4110 Homework 8

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Due 5pm on November 21

1 Reading Material

- Period of a state/class (Chapter 11.3.2).
- Transience and recurrence (Chapter 11.3.2).
- Existence and uniqueness of stationary distribution (Chapter 11.3.3).

2 Assignment

1. **Period, Transience and Recurrence, and Occupancy Rate** (40 points)

   You have identified in HW7 the communication classes for each of the Markov chains with transition probability matrices given below. Now answer the following questions.

   (a) Find the period of each class.
   (b) Which states are transient and which are positive recurrent? (Note that null recurrent states are only possible in infinite-state Markov chains.)
   (c) Permute the state labels and rewrite the transition matrices in their canonical form.
   (d) How many stationary distributions does the chain have? Compute all the stationary distributions.
   (e) For each state i, what is the long-run fraction of time that the chain spends in state i given that it starts in state i?

   Now

$$\mathbf{P}_1 = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 0 & 0 \end{pmatrix} \quad \mathbf{P}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \mathbf{P}_3 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

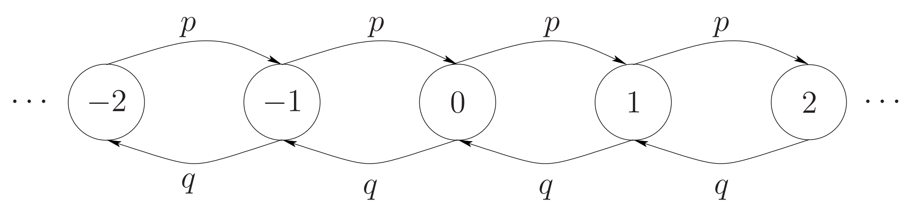
$$\mathbf{P}_4 = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \mathbf{P}_5 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix} \quad \mathbf{P}_6 == \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

2. **Simple 1-dimensional Random Walk** (10 points)

$$X_0 = 0$$
$$X_n = \sum_{i=1}^{n} W_i$$

where $W_1, W_2, \ldots$ are independent random variables with $\Pr(W_i = 1) = 1 - \Pr(W_i = -1) = p$. Show that every state is recurrent when $p = \frac{1}{2}$ and transient when $p \neq \frac{1}{2}$.

(*Hint: Use the recurrence criteria that state i is recurrent if and only if $\sum_{n=1}^{\infty} p_{i,i}^{(n)} = \infty$. Check this for state 0 and use Stirling's formula: $n! \sim \sqrt{2\pi n}(n/e)^n$ as $n \to \infty$.*)



3. **A Markov chain model for Web search** (10 points)
When someone enters "Cornell" into the Spanish search engine El Goog, our webpage comes up either first or second in the listing of webpages. When it comes up first, it is clicked with probability $1/2$. When it comes up second, it is clicked with probabilit $1/4$.

When a user clicks on our webpage, it becomes the first webpage listed for the *next* user to search for "Cornell." When a user does not click on our webpage, our webpage is listed second for the next user.

Let $\{X_n\}_{n=1}^{\infty}$ denote the position of our webpage as seen by the nth user. Note that X_n can only take the values 1 and 2.

(a) Draw the state transition diagram for this Markov chain and write out the one-step transition matrix. Is the chain irreducible?

(b) What is the long-run fraction of users that see our webpage listed first?

(c) Suppose that the process is stationary. Given that a user clicked on our webpage, what is the probability that we were listed first for that user?

Rami Pellumbi

HW8 ECE4110

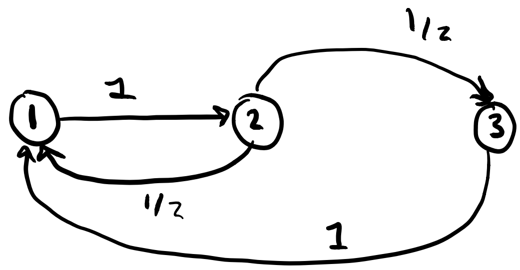
11/21/19 5:00PM

① Period, Transience, Recurrance, and Occupancy Rate

(a) Period of Each Class

$\mathbb{P}_1$

$$\mathbb{P}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \end{pmatrix}$$



Communication class: $\{1,2,3\} \to$ irreducible

Therefore the class period is

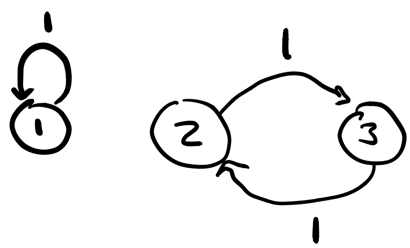
$$d(1) = d(2) = d(3) = \begin{cases} \gcd(N_1) & \text{if } N_i \neq \phi \\ 1 & \text{o/w} \end{cases}$$

$$N_1 = \{ n \geq 1 \mid p_{11}^{(n)} > 0 \} = \phi$$

Thus the class period is 1, and each state is aperiodic.

$\mathbb{P}_2$

$$\mathbb{P}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$



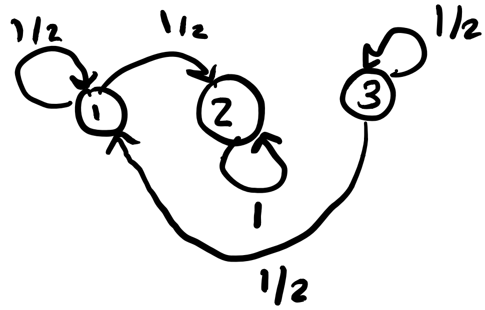
Communication classes: $\{1\}, \{2,3\}$

$$N_1 = \{1,2,3,\ldots\} \Rightarrow d(1) = 1$$

$$N_2 = N_3 = \{2,4,6,\ldots\} \Rightarrow d(2) = d(3) = 2$$

$\mathbb{P}_3$

$$\mathbb{P}_3 = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{pmatrix}$$



Communication Classes

$\{1\}, \{2\}, \{3\}$

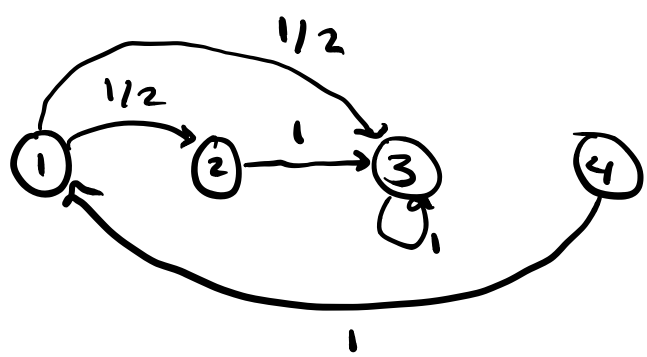
$d(1) = 1$

$d(2) = 1$ since $\{1\} \subset \mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3$

$d(3) = 1$

$\mathbb{P}_4$

$$\mathbb{P}_4 = \begin{pmatrix} 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$



Communication Classes: $\{1\}, \{2\}, \{3\}, \{4\}$

$d(1) = 1$

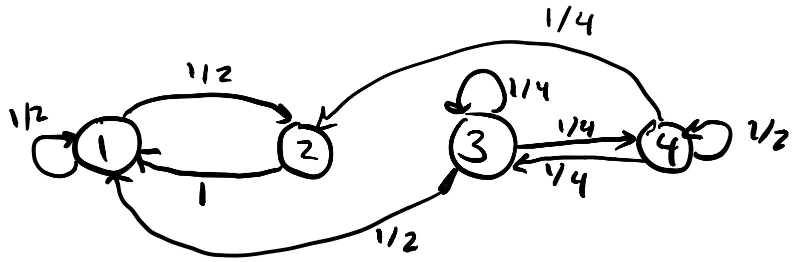
$d(2) = 1$ since $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_4 = \phi$

$d(3) = 1$ $\quad \mathcal{N}_3 = \{1, 2, 3, \ldots\}$

$d(4) = 1$

$\mathbb{P}_5$

$$\mathbb{P}_5 = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1/4 & 1/4 \\ 0 & 1/4 & 1/4 & 1/2 \end{pmatrix}$$



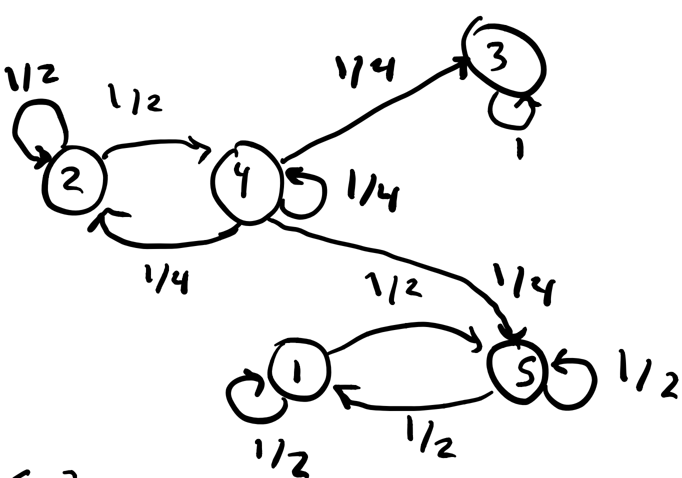
Communication classes: $\{1,2\}$, $\{3,4\}$

$$\mathcal{N}_1 = \{1,2,3,\dots\} \rightarrow d(1) = d(2) = 1$$
$$\mathcal{N}_2 = \{2,3,4,5,6,\dots\}$$

$$\mathcal{N}_3 = \{1,2,3,\dots\} \rightarrow d(3) = d(4) = 1$$

$\mathbb{P}_6$

$$\mathbb{P}_6 = \begin{pmatrix} 1/2 & 0 & 0 & 0 & 1/2 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$



Communication Classes: $\{1,5\}$, $\{3\}$
$\{2,4\}$

$d(1) = d(5) = 1$
$d(2) = d(4) = 1$

$d(3) = 1$

(b) Transient or Recurrent

$\mathbb{P}_1$

The states of a finite state irreducible Markov chain are all recurrent. Show this mathematically.

$$f_i \triangleq \sum_{n=1}^{\infty} f_i^{(n)} = \sum_{n=1}^{\infty} Pr[X_1 \neq i, \ldots, X_{n-1} \neq i, X_n = i \mid X_0 = i]$$

Note:

$$Pr[X_1 = 1 \mid X_0 = 1] = 0$$
$$Pr[X_2 = 1, X_1 \neq 1 \mid X_0 = 1] = 1/2$$
$$Pr[X_3 = 1, X_2 \neq 1, X_1 \neq 1 \mid X_0 = 1] = 0$$
$$Pr[X_4 = 1, X_3 \neq 1, X_2 \neq 1, X_1 \neq 1 \mid X_0 = 1] = 1/2$$

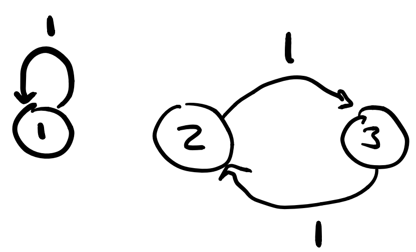
Can only return for the first time on 2^{nd} or 4^{th} transition

$\Rightarrow$ rest are 0.

Thus

$$\sum_{n=1}^{\infty} f_i^{(n)} = 0 + \frac{1}{2} + 0 + \frac{1}{2} + 0 + 0 + \cdots + 0 + \cdots$$
$$= 1$$
$$\Rightarrow \text{recurrent}$$

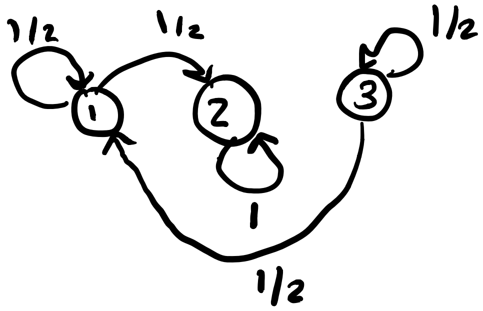
$\mathbb{P}_2$



Clearly, 1 is recurrent, as well as 2, 3 since

$$\sum_{n=1}^{\infty} f_i^{(n)} = 1 \quad \text{for } i = 1, 2, 3$$

$\mathbb{P}_3$

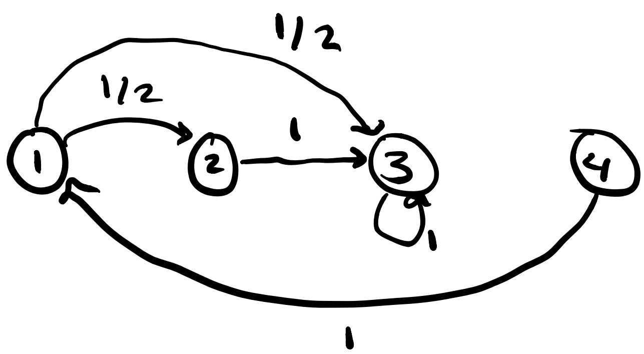


$$\sum_{n=1}^{\infty} f_1^{(n)} = \frac{1}{2} \Rightarrow \text{①} \quad \text{transient}$$

$$\sum_{n=1}^{\infty} f_2^{(n)} = 1 \Rightarrow \text{②} \quad \text{recurrent}$$

$$\sum_{n=1}^{\infty} f_3^{(n)} = \frac{1}{2} \Rightarrow \text{③} \quad \text{transient}$$

$\mathbb{P}_4$

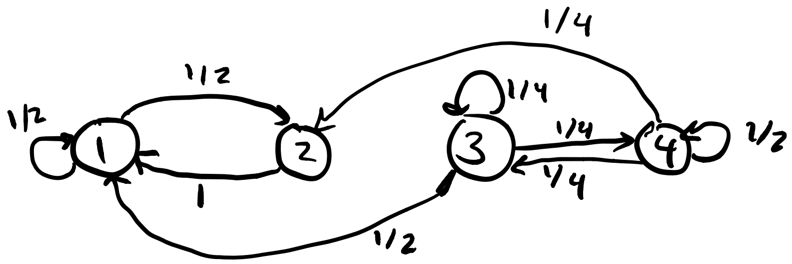


$$\sum_{n=1}^{\infty} f_1^{(n)} = 0 \Rightarrow \text{①} \quad \text{transient}$$

easy to see ②, ④ also transient.

$$\sum_{n=1}^{\infty} f_3 = 1 \Rightarrow \text{③} \quad \text{recurrent}$$

P5



$$\sum_{n=1}^{\infty} f_1^{(n)} = \frac{1}{2} + \frac{1}{2} = 1 \Rightarrow \text{(1) recurrent}$$

$$\sum_{n=1}^{\infty} f_2^{(n)} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = 1 \Rightarrow \text{(2) recurrent}$$

$$\sum_{n=1}^{\infty} f_3^{(n)} = \frac{1}{4} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \cdots$$

$$= \frac{1}{4} + \sum_{n=4}^{\infty} \frac{1}{2^n} = \frac{1}{4} + 2 - \left(1 + \frac{1}{2} + \frac{1}{4}\right) = \frac{1}{2}$$

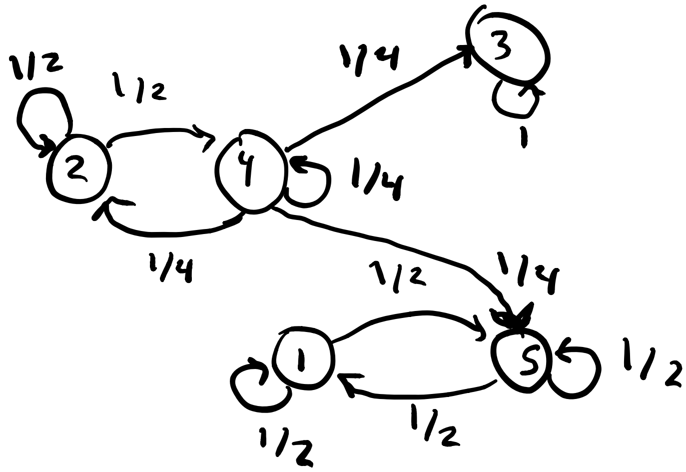
$$\Rightarrow \text{(3) transient}$$

$$\sum_{n=1}^{\infty} f_4^{(n)} = \frac{1}{2} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \cdots$$

$$= \frac{1}{2} + \sum_{n=2}^{\infty} \frac{1}{4^n} = \frac{1}{2} + \frac{4}{3} - \left(1 + \frac{1}{4}\right)$$

$$= \frac{4}{3} - \frac{3}{4} < 1 \Rightarrow \text{(4) transient}$$

P6



$$\sum_{n=1}^{\infty} f_1^{(n)} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = 1 \Rightarrow \text{1 recurrent}$$

Similar reasoning leads to 5 recurrent

$$\sum_{n=1}^{\infty} f_3^{(n)} = 1 \Rightarrow \text{3 recurrent}$$

We note that 2, 4 have a positive probability of moving into a recurrent class and thus are **transient**

(c) Permute the state labels and rewrite the transition matrices in their canonical form

$\mathbb{P}_1$

$$\mathbb{P}_{\mathbb{P}_1} = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \end{bmatrix}$$

since just one communication class

$\mathbb{P}_2$

$$\mathbb{P}_{P_2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} \mathbb{P}_1 & 0 & 0 \\ 0 & & \\ 0 & \mathbb{P}_2 & \end{bmatrix}$$

No transient classes

$\mathbb{P}_3$

$$\mathbb{P}_{\mathbb{P}_3} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

$$\begin{bmatrix} \mathbb{P}_{c_2} & 0 \\ \mathbb{R}_{c_1} & \mathbb{Q} \end{bmatrix}$$

$$\mathbb{P}_{c_2} = [1]$$

$$\mathbb{R}_{c_1} = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$$

$$\mathbb{Q} = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1/2 \end{bmatrix}$$

$\mathbb{P}_4$

$$\mathbb{P}_{\mathbb{P}_4} = \begin{array}{c} \\ 3 \\ 1 \\ 2 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 3 & 1 & 2 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

$\mathbb{P}_5$

$\mathbb{P}$

$$\mathbb{P}_{\mathbb{P}_5} = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1/4 & 1/4 \\ 0 & 1/4 & 1/4 & 1/2 \end{bmatrix}$$ $\mathbb{Q}$

$\mathbb{R}$

Same as original

$\mathbb{P}_6$

$\{1,5\}$, $\{3\}$ recurrent

$\{2,4\}$ transient

$\mathbb{P}_1$

$$\mathbb{P}_{\mathbb{P}_6} = \begin{array}{c} \\ 1 \\ 5 \\ 3 \\ 2 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 5 & 3 & 2 & 4 \end{array} \\ \begin{bmatrix} 1/2 & 1/2 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 1/4 & 1/4 & 1/4 & 1/4 \end{bmatrix} \end{array}$$ $\mathbb{P}_2$ $\mathbb{Q}$

$\mathbb{R}_1$ $\mathbb{R}_2$

(d) How many stationary distributions does the chain have? Compute all stationary distributions.

For each computation, $\vec{\pi}^T = \vec{v}^T$ s.t. $\sum$ elements of $v = 1$ and $\vec{v}^T \in \ker(\mathbb{P}^T - I)$.

$\mathbb{P}_1$

$$\begin{array}{cc} \mathbb{P}^T - I & \vec{v}^T \\ \left[\begin{array}{ccc|c} -1 & 1/2 & 1 & a \\ 1 & -1 & 0 & b \\ 0 & 1/2 & -1 & c \end{array}\right] \end{array} = \begin{array}{c} \vec{0} \\ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{array}$$

$$-a + \frac{b}{2} + c = 0$$
$$a - b = 0$$
$$\frac{b}{2} - c = 0$$

$a = b$

$b = 2c$

$\vec{v} = [2c \quad 2c \quad c]$

Need $2c + 2c + c = 1$

$c = 1/5$

$\vec{\pi} = [2/5 \quad 2/5 \quad 1/5]$

$\mathbb{P}_2$

$$\begin{array}{cc} \mathbb{P}^T - I & \vec{v}^T \\ \left[\begin{array}{ccc|c} 0 & 0 & 0 & a \\ 0 & -1 & 1 & b \\ 0 & 1 & -1 & c \end{array}\right] \end{array} = \begin{array}{c} \vec{0} \\ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{array}$$

$$-b + c = 0$$
$$b - c = 0$$

$a = 0$

$b = c$

OR

$a =$ anything

$b = c = 0$

$\vec{v}_2 = [a \quad 0 \quad 0]$

$\vec{v}_1 = [0 \quad c \quad c]$

$2c = 1$

$c = 1/2$

$\vec{\pi}_1 = [0 \quad 1/2 \quad 1/2] \rightarrow \infty$ stationary distributions

$\vec{\pi}_2 = [1 \quad 0 \quad 0]$

$\mathbb{P}_3$

$$\begin{array}{ccc} \mathbb{P}^T - I & \vec{v}^T & \vec{0} \\ \left[\begin{array}{ccc|c} -1/2 & 0 & 1/2 & a \\ 1/2 & 0 & 0 & b \\ 0 & 0 & -1/2 & c \end{array}\right] & = & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{array}$$

$-\frac{a}{2} + \frac{c}{2} = 0$

$a = c$

$\frac{1}{2} a = 0$

$-\frac{1}{2} c = 0$

$b \to$ anything

$\vec{v} = [0 \quad b \quad 0] \Rightarrow$ one stationary distribution

$\vec{\pi} = [0 \quad 1 \quad 0]$

$\mathbb{P}_4$

$$\begin{array}{ccc} \mathbb{P}^T - I & \vec{v}^T & \vec{0} \\ \left[\begin{array}{cccc|c} -1 & 0 & 0 & 1 & a \\ 1/2 & -1 & 0 & 0 & b \\ 1/2 & 1 & 0 & 0 & c \\ 0 & 0 & 0 & -1 & d \end{array}\right] & = & \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{array}$$

$-d = 0$

$-a + d = 0 \to a = 0$

$\frac{a}{2} + \frac{b}{2} = 0 \to b = 0$

$c \to$ anything

$\vec{v} = [0 \quad 0 \quad c \quad 0]$

$\vec{\pi} = [0 \quad 0 \quad 1 \quad 0] \Rightarrow$ only one stationary distribution

P_5

$$\begin{array}{ccc} P^T - I & \vec{v}^T & \vec{0} \\ \left[\begin{array}{cccc|c} -\frac{1}{2} & 1 & \frac{1}{2} & 0 & a \\ \frac{1}{2} & -1 & 0 & \frac{1}{4} & b \\ 0 & 0 & -\frac{3}{4} & \frac{1}{4} & c \\ 0 & 0 & \frac{1}{4} & -\frac{1}{2} & d \end{array}\right] & = & \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{array}$$

$$-\frac{a}{2} + b + \frac{c}{2} = 0$$

$$\frac{a}{2} - b + \frac{d}{4} = 0$$

$$-\frac{3}{4}c + \frac{d}{4} = 0$$

$$\frac{c}{4} - \frac{d}{2} = 0$$

$$a + b + c + d = 1$$

$$\vec{v} = [\,2b \quad b \quad 0 \quad 0\,]$$

$$\vec{\pi} = [\,\tfrac{2}{3} \quad \tfrac{1}{3} \quad 0 \quad 0\,]$$

One stationary distribution

P_6

$$\begin{array}{ccc} P^T - I & \vec{v}^T & \vec{0} \\ \left[\begin{array}{ccccc|c} -1/2 & 0 & 0 & 0 & 1/2 & a \\ 0 & -1/2 & 0 & 1/4 & 0 & b \\ 0 & 0 & 0 & 1/4 & 0 & c \\ 0 & 1/2 & 0 & -3/4 & 0 & d \\ 1/2 & 0 & 0 & 1/4 & -1/2 & e \end{array}\right] & = & \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{array}$$

$$a = e$$

$$\frac{b}{2} = \frac{d}{4} \rightarrow b = 0$$

$$\frac{d}{4} = 0 \rightarrow d = 0$$

$$\frac{b}{2} - \frac{3d}{4} = 0$$

$$\frac{a}{2} = \frac{e}{2} \rightarrow a = e$$

$c =$ anything

$$\vec{v}_1 = [\,a \quad 0 \quad 0 \quad 0 \quad a\,], \quad \vec{v}_2 = [\,0 \quad 0 \quad c \quad 0 \quad 0\,]$$

$$\vec{\pi}_1 = [\,1/2 \quad 0 \quad 0 \quad 0 \quad 1/2\,]$$

OR

$$\vec{\pi}_2 = [\,0 \quad 0 \quad 1 \quad 0 \quad 0\,]$$

$\rightarrow \infty$ stationary distributions

(e) For each state i, what is the long-run fraction of time that the chain spends in state i given that it starts in state i.

Occupancy rate for state i is

$$r_i = \lim_{n\to\infty} \frac{\sum_{m=1}^{n} \mathbb{1}_{[X_m = i \mid X_0 = i]}}{n}$$

If state i transient, $r_i = 0$

If state i null-recurrent, $r_i = 0$

If state i positive-recurrent, $r_i = \frac{1}{\mathbb{E}[T_i]} = \pi_i > 0$

where

$$T_i \triangleq \min\{n \geq 1 : X_n = i \mid X_0 = i\}$$

$\mathbb{P}_1$ All positive recurrent, so

State ① : $r_1 = \pi_1 = 2/5$

② : $r_2 = \pi_2 = 2/5$

③ : $r_3 = \pi_3 = 1/5$

$\mathbb{P}_2$ State 1 : 0 or 1

2 : 1/2 0 depending on initial

3 : 1/2 0 distribution

i.e.

$$\boxed{\begin{array}{l} 1 : p \\ 2 : (1-p)/2 \quad 3 : (1-p)/2 \end{array}}$$

, p being the amount which start in state 1

$\mathbb{P}_3$ 1, 3 transient $\Rightarrow r_i = 0$

2 $r_2 = 1 = \pi_2$

$\mathbb{P}_4$ 1, 2, 4 : transient $\rightarrow r_i = 0$

3 : $r_3 = 1$

$\mathbb{P}_5$ 3, 4 : transient $\Rightarrow r_i = 0$

2 : $r_2 = 2/3$

4 : $r_4 = 1/3$

$\mathbb{P}_6$ 1 : $r_i = 1/2$

5 : $r_i = 1/2$

2, 3, 4 : $r_i = 0$

i.e

OR

1, 2, 4, 5 : $r_i = 0$

3 : $r_i = 1$

3 : p

1, 5 : $(1-p)/2$

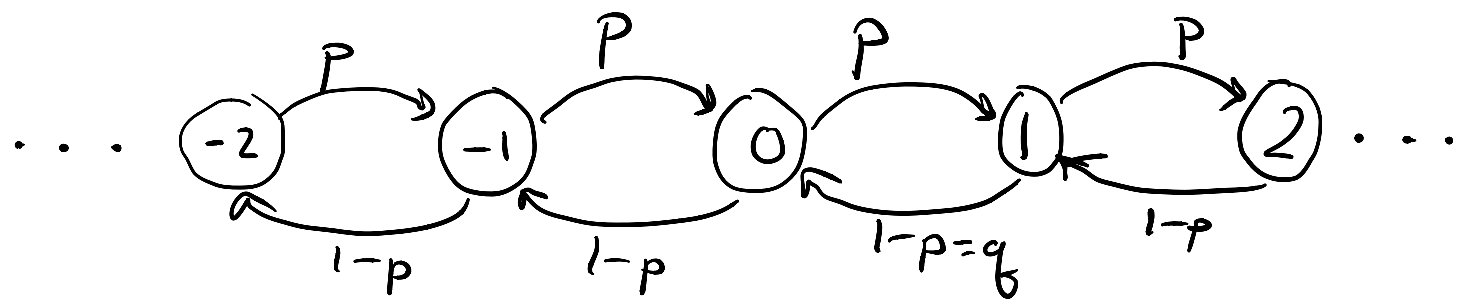
p being amount initially in 3

② Simple 1-dimensional Random Walk

$$X_0 = 0$$
$$X_n = \sum_{i=1}^{n} W_i$$

where $W_1, W_2, \dots$ are independent random variable with

$$\Pr(W_i = 1) = 1 - \Pr(W_i = -1) = p$$



Communication class: $\{\cdots -2, -1, 0, 1, 2, \cdots\}$

i.e irreducible NON-finite state Markov chain

Show that

(i) Every state is recurrent when $p = 1/2$

(ii) Every state is transient when $p \neq 1/2$

(i) Let $p = \frac{1}{2}$. Then $q = \frac{1}{2}$ and $pq = \frac{1}{4}$

Thus

$$p_{0,0}^{(n)} = \binom{2n}{n} p^n q^n = \binom{2n}{n}\left(\frac{1}{4}\right)^n$$

Why $\binom{2n}{n}$? It is the amount of steps

Take

$$\sum_{n=1}^{\infty} P_{0,0}^{(n)} = \sum_{n=1}^{\infty} \binom{2n}{n}\left(\frac{1}{4}\right)^n$$

By Sterlings formula,

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \quad \text{as} \quad n \to \infty$$

this becomes

$$\sum_{n=1}^{\infty} \frac{(2n)!}{n!\,n!}\left(\frac{1}{4}\right)^n \sim \sum_{n=1}^{\infty} \frac{\sqrt{2\pi(2n)}\,\left(\frac{2n}{e}\right)^{2n}}{\left(\sqrt{2\pi n}\right)^2\left(\frac{n}{e}\right)^{2n}} \cdot \left(\frac{1}{4}\right)^n$$

$$= \sum_{n=1}^{\infty} \frac{1}{\sqrt{\pi n}} \cdot \frac{4^n}{4^n} = \frac{1}{\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \longrightarrow \infty$$

Thus state 0 is recurrent. The same analysis applies to every state.

(ii) Let $p > 1/2$. Then $q < 1/2$, and $pq < 1$ $\quad (p, q \in (0,1))$

Similar to above,

$$P_{0,0}^{(n)} = \binom{2n}{n} p^n q^n$$

$$= \frac{(2n)!}{n!\,n!} p^n q^n$$

Then

$$\sum_{n=1}^{\infty} \frac{(2n)!}{n!\,n!} p^n q^n \sim \sum_{n=1}^{\infty} \frac{1}{\sqrt{\pi n}}$$

Let $\varepsilon \in (0, 1/2)$

Let $p = \frac{1}{2} + \varepsilon$

$q = \frac{1}{2} - \varepsilon$

$$(pq)^n = \left(\frac{1}{4} - \varepsilon^2\right)^n$$

Thus have

$$\sum_{n=1}^{\infty} \frac{4^n}{\sqrt{\pi n}} \left(\frac{1}{4} - \varepsilon^2\right)^n$$

$\varepsilon \in (0, 1/2)$
$\Rightarrow \varepsilon = \frac{1}{b}$ for some $b > 2 \in \mathbb{R}$

$$= \sum_{n=1}^{\infty} \frac{4^n}{\sqrt{\pi n}} \left(\frac{1}{4} - \frac{1}{b^2}\right)^n$$

$$= \sum_{n=1}^{\infty} \frac{4^n}{\sqrt{\pi n}} \left(\frac{b^2-4}{4b^2}\right)^n$$

Note: $\underline{b > 2}$

$$= \frac{1}{\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{(b^2-4)^n}{\sqrt{n}\, b^{2n}} \underline{< \infty}$$

Could also see this by noting $\left(\frac{1}{4} - \varepsilon^2\right) < \frac{1}{4}$
so

$$4\left(\frac{1}{4} - \varepsilon^2\right) < 1$$

and then note

$$\frac{4^n\left(\frac{1}{4} - \varepsilon^2\right)^n}{\sqrt{n}} \leq 4^n\left(\frac{1}{4} - \varepsilon^2\right)^n \quad \text{for } n \geq 1$$

thus since

$$\sum_{n=1}^{\infty} 4^n\left(\frac{1}{4} - \varepsilon^2\right)^n = \frac{1}{1 - 4\left(\frac{1}{4} - \varepsilon^2\right)} < \infty$$

$$\sum_{n=1}^{\infty} \frac{4^n\left(\frac{1}{4} - \varepsilon^2\right)^n}{\sqrt{n}} < \infty$$

Thus 0 is a transient state. The same analysis applies to every state. Switching p, q yields same answer.

Q.E.D

③ A Markov Chain Model for Web Search

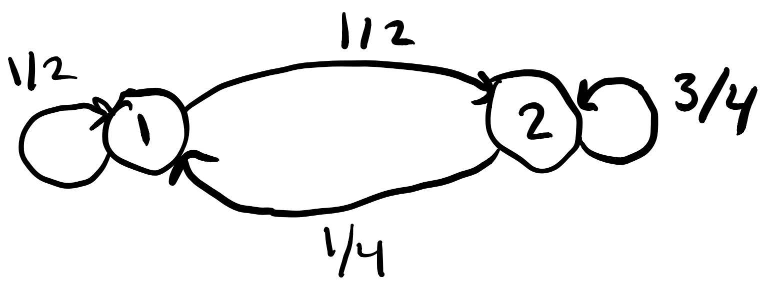
When someone enters "Cornell" into the Spanish search engine El Goog, our webpage comes up either first or second in the listing of webpages. When it comes up first, it is clicked with probability $1/2$. When it comes up second, it is clicked with probabilit $1/4$.

When a user clicks on our webpage, it becomes the first webpage listed for the *next* user to search for "Cornell." When a user does not click on our webpage, our webpage is listed second for the next user.

Let $\{X_n\}_{n=1}^{\infty}$ denote the position of our webpage as seen by the nth user. Note that X_n can only take the values 1 and 2.

(a) Draw the state transition matrix for this Markov chain and write out one-step transition matrix.

Is the chain irreducible?



$$\mathbb{P} = \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix}$$

Communication class: $\{1,2\}$ $\therefore$ irreducible

(b) What is the long-run fraction of users that see our webpage listed first.

$$\bar{\pi} = \bar{\pi} \mathbb{P}$$

$$\mathbb{P}^T \bar{\pi}^T = \bar{\pi}^T \quad \text{w/ eigenvector } \pi^T$$

$\lambda = 1$ eigenvalue

$$\bar{\pi}^T \in \ker(\mathbb{P}^T - I)$$

$$\underbrace{\begin{bmatrix} -1/2 & 1/4 \\ 1/2 & -1/4 \end{bmatrix}}_{\mathbb{P}^T - I} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{\vec{v}^T} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\vec{0}}$$

$$-\frac{a}{2} + \frac{b}{4} = 0$$

$$\frac{a}{2} - \frac{b}{4} = 0$$

$$a - \frac{b}{2} = 0$$

$$a = \frac{b}{2}$$

$$\vec{v}^T = \begin{bmatrix} b/2 \\ b \end{bmatrix}$$

Normalize $\vec{v}^T$

$$\frac{b}{2} + b = 1$$

$$b = 2/3$$

$$\bar{\pi}^T = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$$

$$\Rightarrow \bar{\pi} = \begin{bmatrix} 1/3 & 2/3 \end{bmatrix} = \begin{bmatrix} \pi_1 & \pi_2 \end{bmatrix}$$

Long-run fraction of users that see our webpage first is $\pi_1 = 1/3$.

(c) Process is stationary.

Given a user clicks on our webpage, what is the probability that we were listed first for that user?

Let

A = event we were listed first

C = event user clicks on our webpage

Then

A^c = event we were listed second

Want

$$\Pr(A \mid C) = \frac{\Pr(C \mid A)\Pr(A)}{\Pr(C)}$$

where

$$\begin{aligned}\Pr(C) &= \Pr(A \cap C) + \Pr(A^c \cap C) \\ &= \Pr(C \mid A)\Pr(A) + \Pr(C \mid A^c)\Pr(A^c) \\ &= \frac{1}{2}\cdot\frac{1}{3} + \frac{1}{4}\cdot\frac{2}{3}\end{aligned}$$

Thus

$$\Pr(A \mid C) = \frac{\frac{1}{2}\cdot\frac{1}{3}}{\frac{1}{2}\cdot\frac{1}{3} + \frac{1}{4}\cdot\frac{2}{3}} = \frac{1}{2}$$