n-Step Transition Matrix

n-Step Transition Matrix $\mathbf{P}^{(n)}$

The n-step transition matrix is given by $\mathbf{P}^{(n)} = [p_{i,j}^{(n)}]$ where

$$p_{i,j}^{(n)} \triangleq \Pr\{X_n = j | X_0 = i\}$$

Chapman-Kolmogorov Equations:

$$p_{i,j}^{(m+n)} = \sum_{k \in \mathcal{X}} p_{i,k}^{(m)} p_{k,j}^{(n)}$$

or in matrix form

$$\mathbf{P}^{(m+n)} = \mathbf{P}^{(m)} \mathbf{P}^{(n)}.$$

$$\boxed{\mathbf{P}^{(n)} = \mathbf{P}\mathbf{P}^{(n-1)} = \mathbf{P}\mathbf{P}\mathbf{P}^{(n-2)} = \ldots = \mathbf{P}^n}$$

State Probabilities:

Let $\mathbf{p}(n) = [p_1(n), p_2(n), \ldots]$ be the PMF of the state at time n.

$$\mathbf{p}(n) = \mathbf{p}(0)\mathbf{P}^{(n)} = \mathbf{p}(0)\mathbf{P}^n$$

Stationary Distribution of Markov Chain

Stationary Distribution:

Let $\pi = \{\pi_i, i \in \mathcal{X}\}$ be a probability distribution (a row vector). It is called a stationary distribution of a Markov chain with transition matrix $\mathbf{P}$ if

$$\pi = \pi\mathbf{P}$$

Computing Stationary Distribution:

$$\begin{cases} \pi & = \pi\mathbf{P} \\ \sum_i \pi_i & = 1 \end{cases}$$

Example:

The stationary distribution of the chain for Coin A and Coin B is $\pi = [2/5, 3/5]$.

Stationary Markov Chain:

Let π be a stationary distribution of a Markov chain $\{X_n\}_{n=0}^{\infty}$. If X_0 is distributed according to π, then $\{X_n\}_{n=0}^{\infty}$ is a stationary process.

Intepretations of Stationary Distribution

Long-term occupancy rate:

For a Markov chain with $\mathbf{p}(0)$ given by a stationary distribution π, we have $\Pr(X_n = i) = \pi_i$ for all n. Thus, π_i can be interpreted as the long-term fraction of time the chain spends in state i. For the example of Coin A and Coin B, the stationary distribution is $\pi = [\frac{2}{5}\ \frac{3}{5}]$, which can be interpreted as seeing a "head" 40% of the time and a "tail" 60% of the time.

Equilibrium and balance equations:

- Treat state probabilities as water: one gallon of water is distributed to all state according to $p_i(n)$, which is the amount of water at state i at time n;
- At each time n, each state i gives $p_{i,j}$ fraction of its water to state j;
- The total amount of water giving out by state i at time n is
$$\sum_{j \neq i} p_i(n) p_{i,j} = p_i(n) \sum_{j \neq i} p_{i,j} = p_i(n)(1 - p_{i,i})$$
- The total amount of water received by state i at time n is
$$\sum_{k \neq i} p_k(n) p_{k,i}$$
- When the Markov chain starts at a stationary distribution $\mathbf{p}(0) = \pi$, we have
$$\pi = \pi \mathbf{P} \quad \Longleftrightarrow \quad \pi_i = \sum_{k \in \mathcal{X}} \pi_k p_{k,i} \quad \Longleftrightarrow \quad \underbrace{\pi_i(1 - p_{i,i})}_{\text{water in}} = \underbrace{\sum_{k \neq i} \pi_k p_{k,i}}_{\text{water out}}$$
That is, the amount of water flowing out of state i equals to that flowing into state i. Thus the total amount of water at state i remains π at all times, and the chain is in equilibrium.
- More generally, the net flow through any closed loop must be zero for the chain to be in equilibrium. Based on this, you can create your own set of balance equations by drawing closed loops to solve for the stationary distribution.

Communication Class and Irreducibility

Accessible and Communicating:

- If, for some $n \geq 0$, $p_{i,j}^{(n)} > 0$, we say that j is accessible from i and write $i \rightarrow j$.
- If $i \rightarrow j$ and $j \rightarrow i$, we say that i and j communicate and write $i \leftrightarrow j$.
- "Communication" is an equivalence relation, *i.e.*,
  - □ *reflexive*: $i \leftrightarrow i$
  - □ *symmetric*: $i \leftrightarrow j$ iff $j \leftrightarrow i$
  - □ *transitive*: if $i \leftrightarrow j$ and $j \leftrightarrow k$, then $i \leftrightarrow k$.

Decomposition of State Space into Communication Classes:

The state space $\mathcal{X}$ can be decomposed into disjoint exhaustive communication classes. First put state 1 and all states communicating with 1 in a class C_1. Then pick a state i in $\mathcal{X} \backslash C_1$. Put i and all states communicating with i into another class C_2. Continue this process until all states have been assigned.

Irreducibility:

A Markov chain is irreducible if the state space consists of only one communication class, *i.e.*, for any $i, j \in \mathcal{X}$, we have $i \leftrightarrow j$.

Example:

The chains for Coin A and Coin B and for fish Wanda are irreducible.