

Integrals Involving Multiple Valued Functions

1)

$$I = \int_0^{\infty} \frac{x^{\alpha-1}}{1+x} dx, \quad 0 < \alpha < 1$$

How to define $z^{\alpha-1}$, $0 < \alpha < 1$

Color portrait
Shows branch
cut on positive
real axis

• Above $x=0$,
"just"

$$z^{\alpha-1} = (re^{i\theta})^{\alpha-1} = r^{\alpha-1} e^{i(\alpha-1)\theta}; \quad r > 0, \theta = 0$$

$$= r^{\alpha-1}$$

• Below $x=0$,
"just"

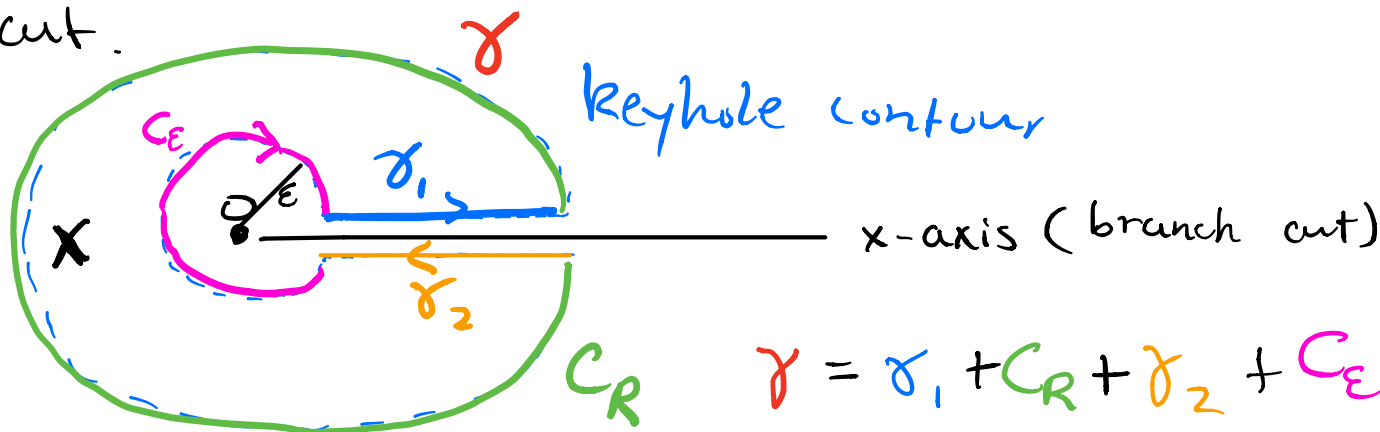
$$z^{\alpha-1} = (re^{i\theta})^{\alpha-1} = r^{\alpha-1} e^{i(\alpha-1)\theta}; \quad r > 0, \theta = 2\pi$$

$$= r^{\alpha-1} e^{i\alpha 2\pi} e^{-i2\pi}$$

$$= r^{\alpha-1} e^{i\alpha 2\pi}$$

When $\alpha = 1/2$ we have an argument change of $e^{i\pi}$

Need a contour that doesn't lie on our branch cut.



$$\int_{\gamma} f(z) dz = 2\pi i \operatorname{Res}(f(z); -1)$$

$$= 2\pi i \lim_{z \rightarrow -1} (z^{\alpha-1})$$

$$= 2\pi i (-1)^{\alpha-1}$$

$$= 2\pi i (e^{i\pi})^{\alpha-1}$$

$$= 2\pi i e^{i\pi\alpha} e^{i\pi}$$

$$= 2\pi i (-e^{i\pi\alpha})$$

$$e^{-i\pi} = \cos(\pi) = \cos(\pi)$$

$$\int_{\gamma} f(z) dz = \int_{\gamma_1} f(z) dz + \int_{C_R} f(z) dz + \int_{\gamma_2} f(z) dz + \int_{C_\epsilon} f(z) dz$$

$$\int_{\gamma_1} f(z) dz = \lim_{\substack{\epsilon \rightarrow 0 \\ R \rightarrow \infty}} \int_{\epsilon}^R \frac{x^{\alpha-1}}{x+1} dx = I \quad \left. \vphantom{\int_{\epsilon}^R} \right\} \gamma_1 \text{ is "just above" case}$$

$$\int_{\gamma_2} f(z) dz = \lim_{\substack{\epsilon \rightarrow 0 \\ R \rightarrow \infty}} \int_R^{\epsilon} \frac{x^{\alpha-1} e^{2\pi i(\alpha)}}{x+1} = -I e^{i2\pi\alpha} \quad \left. \vphantom{\int_R^{\epsilon}} \right\} \gamma_2 \text{ is "just below" case}$$

We expect the integral on C_R to be negligible.

technically less than but equal as $\epsilon \rightarrow 0$

$$\left| \int_{C_R} f(z) dz \right| \leq 2\pi R \max_{z \in C_R} \left| \frac{z^{\alpha-1}}{z+1} \right|$$

via restriction on α this is about

$$\max_{z \in C_R} \left| \frac{z^\alpha}{z^2} \right| \text{ which } \rightarrow 0 \text{ as } R \rightarrow \infty$$

$$\left| \int_{C_\epsilon} f(z) dz \right| \leq \max_{z \in C_\epsilon} \left| \frac{z^{\alpha-1}}{z+1} \right| 2\pi \epsilon$$

$$\leq \max_{z \in C_\epsilon} \left| \frac{z^{\alpha-1}}{1} \right| 2\pi \epsilon$$

$$\leq \frac{\epsilon^\alpha}{\epsilon} 2\pi \epsilon \leq \epsilon^\alpha 2\pi \rightarrow 0 \text{ as } \epsilon \rightarrow 0!$$

So,

$$\int_{\gamma} f(z) dz = \int_{\gamma_1} f(z) dz + \int_{C_R} f(z) dz + \int_{\gamma_2} f(z) dz + \int_{C_\epsilon} f(z) dz$$

$\nearrow I$ $\nearrow 0$ $\nearrow -e^{i2\pi\alpha} I$ $\nearrow 0$

$$2\pi i \operatorname{Res}(f(z); -1) = I(1 - e^{i2\pi\alpha})$$

$$I = \frac{2\pi i (-e^{i\pi\alpha})}{1 - e^{i2\pi\alpha}} = \int_0^\infty \frac{x^{\alpha-1}}{1+x} dx$$

I is real! We had a real integral, so it should be real!

Massage it to get

$$\frac{2\pi i}{-e^{-i\pi\alpha} + e^{i\pi\alpha}} = \frac{\pi}{\sin(\pi\alpha)}$$

So,

$$I = \int_0^{\infty} \frac{x^{\alpha-1}}{1+x} dx = \frac{\pi}{\sin(\pi\alpha)}$$

2)

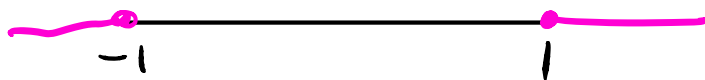
$$I = \int_{-1}^1 \frac{\sqrt{1-x^2}}{\sqrt{1+x^2}} dx$$

$$= \int_{-1}^1 \frac{\sqrt{1-x} \sqrt{1+x}}{1+x^2} dx$$

Want your branch cut on interval of integration.
Right now,

$$f(z) = \frac{\sqrt{1-z} \sqrt{1+z}}{1+z^2}$$

Branch cuts



the cuts don't lie on our interval of integration,

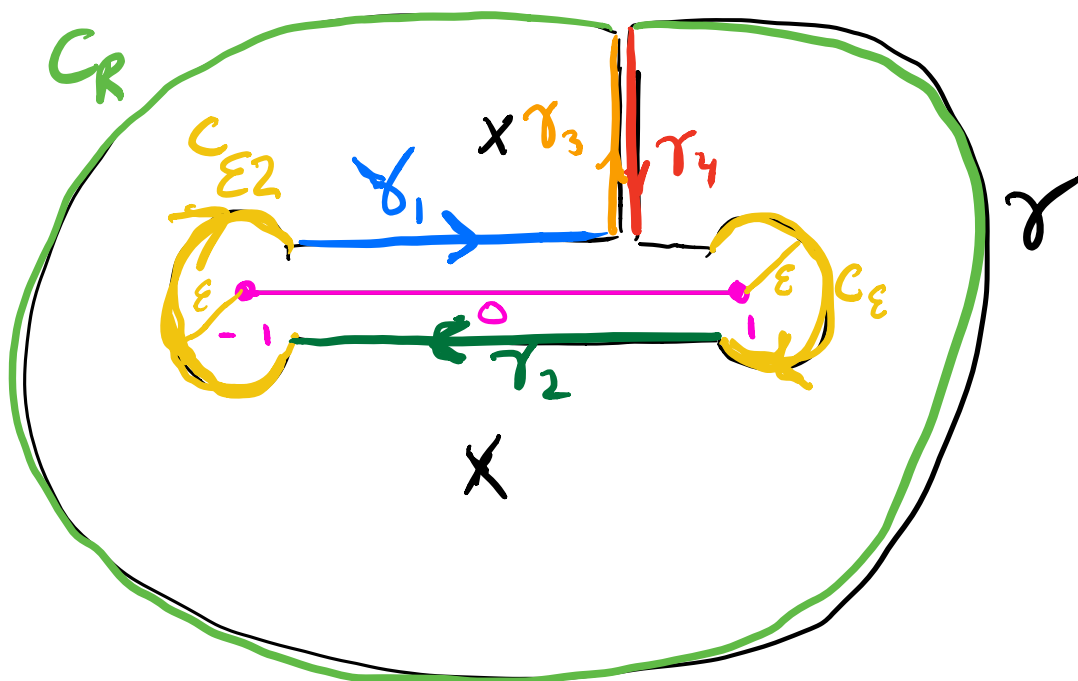
Rewrite the function so it does.

$$f(z) = i \frac{\sqrt{1-z} \sqrt{-1-z}}{1+z^2}$$

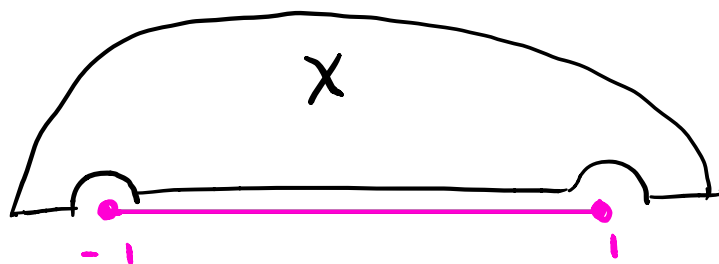
Now



What's our contour? Need to encapsulate poles as well as avoid branch cut.



OR



A lot simpler

γ_1	γ_2	$C_{\epsilon 2}$	C_{ϵ}	C_R	$\underbrace{\gamma_3 + \gamma_4}_{0}$
I	$-e^{i\pi} I$	$\rightarrow 0$	$\rightarrow 0$	2π	

-1

$$I(1 - e^{i\pi}) = 2\pi$$

$$I = \pi$$