

What Have we Done?

Fields

$$\mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}, \quad \mathbb{F}_p = \mathbb{Z}_p \text{ (finite fields)}$$

Vector Spaces over \mathbb{F}

$$-\mathbb{F}^n$$

$$-\mathbb{F}[x]$$

$$-\text{Fun}(X, \mathbb{F})$$

$$-\mathbb{F}^{m \times n}$$

$$-C^\infty(\mathbb{R})$$

$$-\text{Fun}(X, V)$$

↑ "operations essentially done in \mathbb{F} "

Subspaces

$$U \subseteq V, \quad U_1, U_2 \subseteq V \text{ subspaces}$$

$$U_1 + U_2 \subseteq V$$

$$U_1 \oplus U_2 = V, \quad U_1 \cap U_2 = \{0\}$$

Span, LI, LD, dim

- key result, $\dim V$ finite AND v_1, \dots, v_n spans V and w_1, \dots, w_m is LI $\Rightarrow m \leq n$
- any LI set v_1, \dots, v_r of V ($\dim V < \infty$) can be expanded to a basis $(v_1, \dots, v_r, \dots, v_m)$
- any spanning set (v_1, \dots, v_s) has a subset which is a basis

bases $\mathcal{B} = (\vec{v}_1, \dots, \vec{v}_n)$ basis \iff any $\vec{v} \in V$ has a unique map

$$\vec{v} = a_1 \vec{v}_1 + \dots + a_n \vec{v}_n, \quad a_i \in \mathbb{F}$$

write $[\vec{v}]_{\mathcal{B}} = \begin{pmatrix} \end{pmatrix}$

$$- U_1, U_2 \subseteq V, \dim V < \infty$$

$$\dim(U_1 + U_2) + \dim(U_1 \cap U_2) = \dim U_1 + \dim U_2$$

Linear Transforms $T: V \rightarrow W$

def

$$\mathcal{L}(V, W) = (\dim V)(\dim W)$$

$$\mathcal{L}(V) = (\dim V)^2$$

Kernel, Image, Injective, Surjective

$$T: V \rightarrow W$$

$$\dim(V) = \dim \ker(T) + \dim \operatorname{Im}(T)$$

$$\operatorname{rank} T = \dim \operatorname{Im} T$$

$$L_A: \mathbb{F}^n \rightarrow \mathbb{F}^m \quad \text{if } A \in \mathbb{F}^{m \times n}$$

$$x \mapsto Ax$$

$$\ker L_A = \ker A$$

$$\operatorname{Im} L_A = \operatorname{Im} A \quad \operatorname{rank} L_A = \operatorname{rank} A$$

Matrix of T

$$[T]_{\beta \leftarrow \alpha}$$

$$\text{If } T \in \mathcal{L}(V), \quad [T]_{\beta} = [T]_{\beta \leftarrow \beta}$$

Isomorphisms

$$T: V \rightarrow W$$

$$T \text{ isomorphism} \iff T \text{ injective AND } T \text{ surjective}$$

$$\iff \ker T = 0, \quad \text{im } T = W$$

$$\iff \exists T^{-1}: W \rightarrow V \quad \text{s.t.} \quad (T^{-1})T = \text{id}_V \\ T(T^{-1}) = \text{id}_W$$

Suppose $\dim V = \dim W < \infty$

$$V \cong W \iff \dim V = \dim W$$

$$\exists T: V \rightarrow W \text{ isom.}$$

$$T: V \rightarrow W$$

$$(\dim V = \dim W < \infty)$$

$$T \text{ isom.}$$

$$\iff \ker T = 0$$

$$\iff \text{im } T = W$$

$$\iff [T]_{\beta \leftarrow \alpha} \quad n \times n \text{ matrix} \\ \text{has an inverse}$$

Gaussian Elimination

- do it
- row operations are reversible
- $A \rightsquigarrow \text{ref}(A) = B$
 - get $\ker A = \ker B$ $\dim = \# \text{cols} - \# \text{pivots}$
 - $\dim \text{im } A = \dim \text{im } B$
 - $\text{rank } A = \# \text{pivots}$
 - get basis of $\text{im } A$

$$\mathbb{F} = \mathbb{F}_2$$

$$\dim(\mathbb{F}^2, \mathbb{F}) = 4$$

$\uparrow \quad \quad \uparrow$
 $\# \text{elem.} \times \dim$