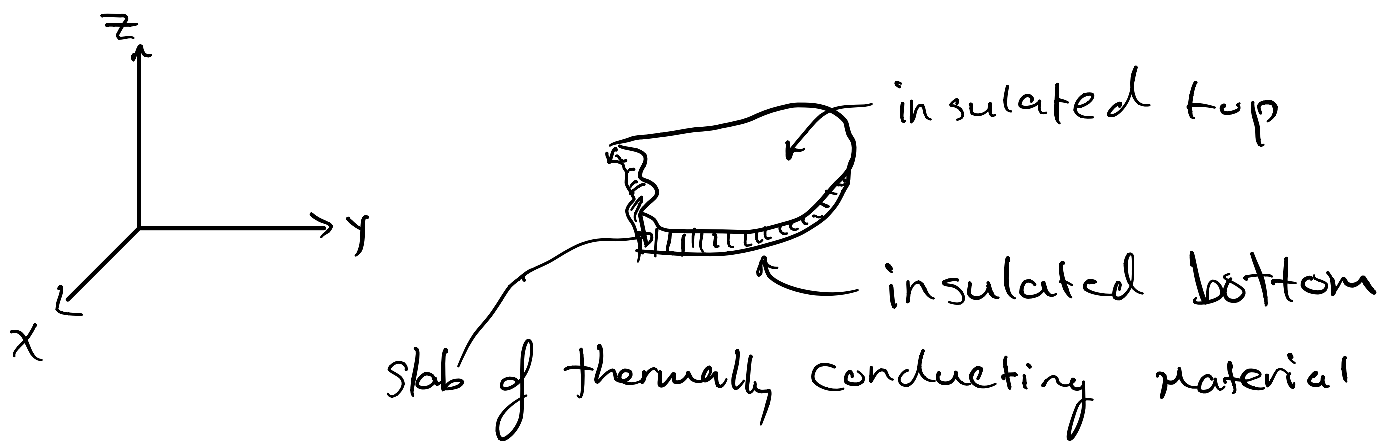
Recall if $\phi(x,y)$ is harmonic then

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

Important since $\nabla^2 \phi$ arises in

- heat/diffusion eqn: $\partial\phi/\partial t = k\nabla^2\phi$
- Black-Scholes eqn:
- wave eqn: $\dfrac{\partial^2 \delta}{\partial t^2} = c^2 \nabla^2 \phi$

Example: Visualize harmonic functions as steady state distributors of temperature.

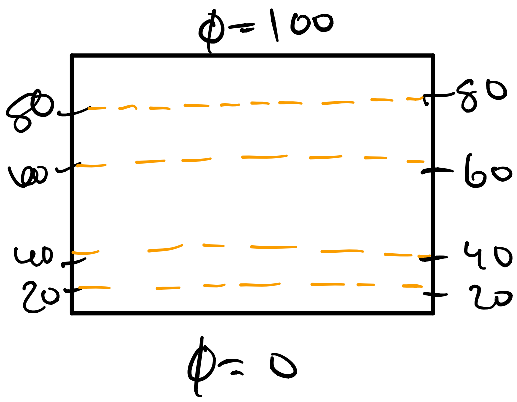


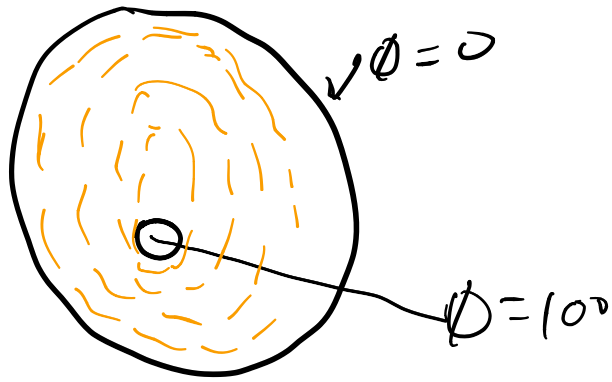
because of insulator, imagine heat flows only laterally.

Next, apply sources & sinks at the boundaries of the 2D region.

Problem: Given $T(x,y)$ of region, find temperature on interior such that $\nabla^2 T = 0$

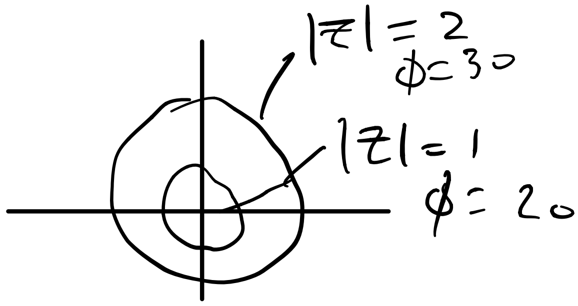
Ex: Sketch the isotherms for regions with boundary conditions





How do we calculate this precisely?

Example: "Washers"



By circular symmetry,
Seek $\phi = \phi(r)$ only, not dependent on θ.
Guess:

$$\phi = A \ln r + B \quad \leftarrow \text{For Washer Probs}$$

Idea: Real and Imaginary part of an analytic $f(z)$ <u>automatically</u> satisfy $\nabla^2 u = 0$ or $\nabla^2 v = 0$

Recall $\log(z) = \ln(r) + i\theta$.

On $r = 1$, $\phi = 20 = A \underbrace{\ln(1)}_{0} + B \Rightarrow B = 20$

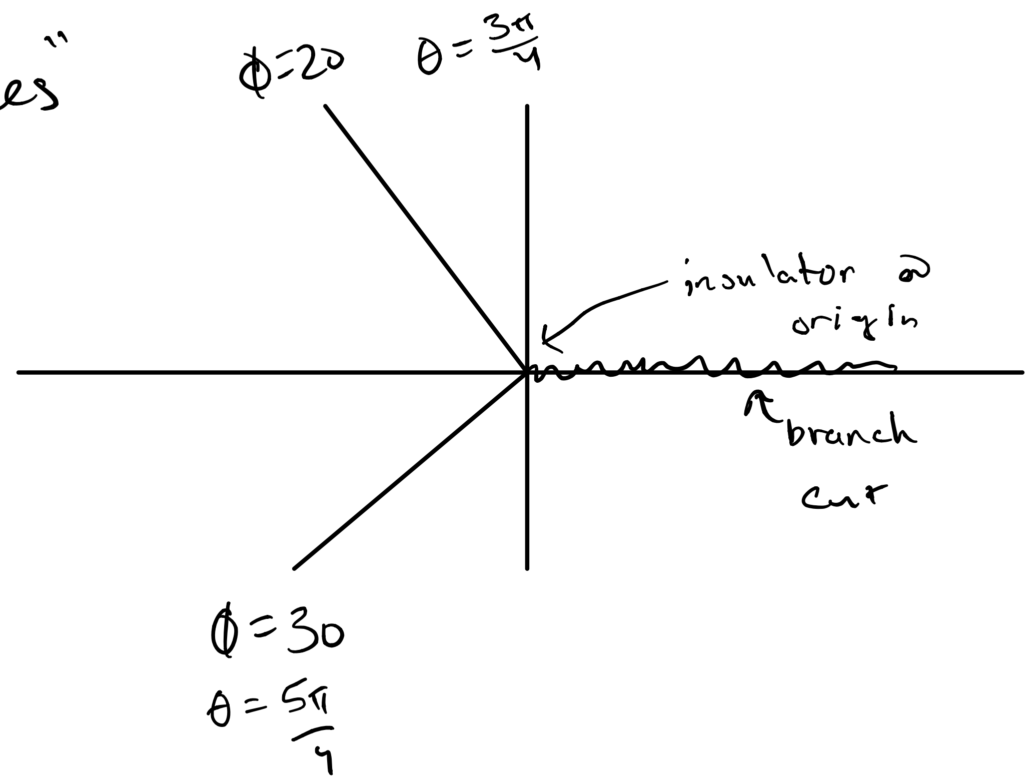
On $r = 2$, $\phi = 30 = A \ln(2) + 20$

$\Rightarrow A = 10/\ln(2)$

$\therefore$

$$\phi(z) = \frac{10}{\ln(2)} \ln(|z|) + 20$$

Example: "Weedges"



$\phi = A\theta + B$ (use when ϕ is constant on radial lines)

$$20 = A\frac{3\pi}{4} + B$$
$$30 = A\frac{5\pi}{4} + B$$

$$-10 = -A\frac{\pi}{2} \qquad B = 5$$

$$A = \frac{20}{\pi} \qquad \Rightarrow \phi = \frac{20}{\pi}\theta + 5$$