Periodicity

Period of State:

Let $\mathcal{N}_i \triangleq \{n \geq 1 : p_{i,i}^{(n)} > 0\}$. The period $d(i)$ of state i is defined as

$$d(i) \triangleq \begin{cases} \text{gcd}\{\mathcal{N}_i\}, & \text{if } \mathcal{N}_i \neq \emptyset \\ 1 & \text{otherwise} \end{cases}$$

If $d(i) = 1$, state i is said to be aperiodic. If $d(i) > 1$, state i is periodic with period $d(i)$.

Remark: If $p_{i,i}^{(n)} > 0$, then n is an integer multiple of $d(i)$, and $d(i)$ is the largest integer with this property. Returns to state i are only possible via paths whose lengths are multiples of $d(i)$.

Period of a Class

All states in a communication class have the same period, also called the class period.

Sufficient Conditions for Aperiodicity

Either of the following is a sufficient condition for an irreducible Markov chain to be aperiodic

- □ $\exists i \in \mathcal{X}, \quad s.t. \quad p_{i,i} > 0$ (*self loop*)
- □ $\exists n > 0, \quad s.t. \quad \mathbf{P}^n > 0$ (*common path length for all state pairs*).

Example:

The chain for Coin A and Coin B is aperiodic. The chain for A Fish Called Wanda has a period of 2.

Transience and Recurrence

Probability of Return:

- ☐ $f_i^{(n)} \stackrel{\Delta}{=} \Pr\{X_1 \neq i, \cdots, X_{n-1} \neq i, X_n = i \mid X_0 = i\}$ is the probability of returning, for the first time, to state i after n steps.
- ☐ $f_i \stackrel{\Delta}{=} \sum_{n=1}^{\infty} f_i^{(n)}$ is the probability of ever returning to state i.

Recurrence and Transience:

- ☐ State i is recurrent if $f_i = 1$.
- ☐ State i is transient if $f_i < 1$.

The Number of Returns:

$$N_i \stackrel{\Delta}{=} \sum_{n=1}^{\infty} 1_{[X_n = i | X_0 = i]}$$

is the number of times the chain returns to state i. We have

$$\begin{cases} \Pr\{N_i = \infty\} = 1, & \text{if } i \text{ recurrent} \\ \mathbb{E}[N_i] = \frac{f_i}{1-f_i} < \infty, & \text{if } i \text{ transient} \end{cases},$$

i.e., if i is recurrent, the chain returns to i infinitely often. Otherwise, the chain visits i only a finite number of times, and the expected number of visits to i is $\frac{f_i}{1-f_i}$.

Criteria for Recurrence:

- ☐ *i is recurrent iff $f_i = 1$.*
- ☐ *i is recurrent iff $\sum_{n=1}^{\infty} p_{i,i}^{(n)} = \infty$.*
- ☐ *If $i \leftrightarrow j$, then i is recurrent iff j is recurrent.*
- ☐ *The states of a finite-state, irreducible Markov chain are all recurrent.*

Positive Recurrence and Null Recurrence

Return Time:

Let $\{X_n\}_{n=0}^{\infty}$ be a Markov chain. Define, for $i \in \mathcal{X}$,

$$T_i \triangleq \min\{n \geq 1 : X_n = i \mid X_0 = i\}.$$

Remark: T_i tells us how long it takes for a chain, started at state i, returns to i for the first time. It is a random variable with PMF given by $[f_i^{(1)}, f_i^{(2)}, \cdots]$.

Positive Recurrence and Null Recurrence:

A recurrent state i is positive recurrent if $\mathbb{E}[T_i] < \infty$ and null recurrent otherwise.

Example: For the simple random walk on $\mathbb{Z}$ with $p = \frac{1}{2}$, it can be shown that $f_i^{(2n)} \sim \frac{C}{n^{3/2}}$. While $f_i = 1$, the expected return time

$$\mathbb{E}[T_i] = \sum_{n=1}^{\infty} 2n f_i^{(2n)} \sim \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \infty$$

Thus, every state is null recurrent.

Occupancy Rate:

Define the occupancy rate r_i of state i (i.e., the long-run fraction of time the chain spent in state i) as

$$r_i = \lim_{n\to\infty} \frac{\sum_{m=1}^{n} 1_{[X_m=i|X_0=i]}}{n}$$

- ☐ If i is transient, the expected number of visits to i is finite, thus $r_i = 0$.
- ☐ If i is null recurrent, the expected number of visits to i is infinite but grows sublinearly with n, thus $r_i = 0$.
- ☐ If i is positive recurrent, $r_i = \frac{1}{\mathbb{E}[T_i]} > 0$.

Recurrent Class

Recurrent Class:

A communication class is a positive (null) recurrent class if one of its members is positive (null) recurrent.

Positive recurrence, null recurrence, and transience are class properties, i.e., if one state in a communication class has the property, all states in this class have the property.

Closed Class:

A communication class $\mathcal{C}$ is closed if for all $i \in \mathcal{C},\ j \notin \mathcal{C}$, we have $p_{i,j} = 0$.

Remark: You can go into a closed set, but you can not go out once you are in.

Every recurrent class is closed.

Proof: Prove by contradiction. Assume there exists $i \in \mathcal{C},\ j \notin \mathcal{C}$ with $p_{i,j} > 0$. Since $\mathcal{C}$ is recurrent, the chain must be able to come back to $\mathcal{C}$ once leaving $\mathcal{C}$ by transiting from i to j. In other words, there must exist $k \in \mathcal{C}$ with $p_{j,k} > 0$. Since $\mathcal{C}$ is a communication class, $p_{j,k} > 0$ implies $j \to i$. Together with $p_{i,j} > 0$, we conclude that j communicates with i. This contradicts with $j \notin \mathcal{C}$.

Canonical Decomposition

Canonical Decomposition:

The state space $\mathcal{X}$ of a Markov chain can be decomposed as

$$\mathcal{X} = \mathcal{T} \cup \mathcal{C}_1 \cup \mathcal{C}_2 \cup \cdots ,$$

where $\mathcal{T}$ consists of transient states ($\mathcal{T}$ is not necessarily one communicating class), $\{\mathcal{C}_i\}$ are closed, disjoint communication classes of recurrent states. If we relabel the states so that the states in each class have consecutive labels with states in $\mathcal{C}_1$ having the smallest indexes, then the transition matrix $\mathbf{P}$ can be rewritten as

$$\mathbf{P} = \begin{pmatrix} \mathbf{P}_1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \mathbf{P}_2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \mathbf{P}_3 & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & & 0 \\ \mathbf{R}_1 & \mathbf{R}_2 & \mathbf{R}_3 & & \cdots & \mathbf{Q} \end{pmatrix}$$

where $\mathbf{P}_i$ is a square stochastic matrix that governs the transitions within $\mathcal{C}_i$. Transitions from states in $\mathcal{T}$ to states in $\mathcal{C}_i$ are governed by $\mathbf{R}_i$. Transitions among states in $\mathcal{T}$ are governed by $\mathbf{Q}$.

Existence of Stationary Distribution

The number of stationary distributions for a discrete-time Markov chain takes three possible values: 0, 1, or ∞:

- If all states are transient or null recurrent, then the chain has no stationary distribution.
- If the chain has a single recurrent class and the recurrent class is positive recurrent, then the chain has a unique stationary distribution $\{\pi_i\}$ given by

$$\pi_i = \begin{cases} \frac{1}{\mathbb{E}[T_i]} & \text{if } i \text{ is positive recurrent} \\ 0 & \text{if } i \text{ is transient} \end{cases}$$

- If the chain has multiple positive recurrent classes, then it has an infinite number of stationary distributions. Specifically, let the states be ordered according to the canonical form with the positive recurrent classes be the first K recurrent classes. Let $\pi^{(k)}$ be the stationary distribution for the kth positive recurrent class. Then for any $0 \leq \alpha_1, \ldots, \alpha_K \leq 1$ with $\sum_{k=1}^{K} \alpha_k = 1$,

$$\left(\alpha_1 \pi^{(1)},\ \alpha_2 \pi^{(2)},\ \ldots,\ \alpha_K \pi^{(K)},\ 0,\ \ldots,\ 0\right)$$

  is a stationary distribution.