

Last Time

$$\beta_1 = \left\{ \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{\vec{u}_1}, \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\vec{u}_2} \right\}$$

$$\beta_2 = \left\{ \underbrace{\begin{pmatrix} -1 \\ 1 \end{pmatrix}}_{\vec{w}_1}, \underbrace{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{\vec{w}_2} \right\}$$

Wrote β_2 in terms of β_1 .

$$\begin{aligned} \vec{w}_1 &= [\vec{u}_1 \ \vec{u}_2] \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\ \vec{w}_2 &= [\vec{u}_1 \ \vec{u}_2] \begin{bmatrix} 2 \\ -1 \end{bmatrix} \end{aligned} \quad \longrightarrow \quad B = \begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix}$$

Then

$$\begin{aligned} \vec{x} = \begin{pmatrix} 10 \\ 20 \end{pmatrix} &= [\vec{w}_1 \ \vec{w}_2] \begin{bmatrix} 0 \\ 10 \end{bmatrix} \\ &= [\vec{u}_1 \ \vec{u}_2] \begin{bmatrix} 20 \\ -10 \end{bmatrix} \end{aligned}$$

$$\text{Point: } \underbrace{B \begin{bmatrix} 0 \\ 10 \end{bmatrix}}_{\substack{\text{coords for } \vec{x} \\ \text{w.r.t. } \beta_2}} = \underbrace{\begin{bmatrix} 20 \\ -10 \end{bmatrix}}_{\substack{\text{coords for } \vec{x} \\ \text{w.r.t. } \beta_1}} \quad \left. \vphantom{\begin{bmatrix} 20 \\ -10 \end{bmatrix}} \right\} \text{ This works } \forall \vec{x}$$

Note this is NOT specific to 2 dim's.

$$\begin{aligned} \text{Give } \forall \text{ } \xi \text{ bases } \beta_1 = \{\vec{u}_1, \dots, \vec{u}_d\} &\xrightarrow{\text{expand}} w_i = [u_1, \dots, u_d] \begin{bmatrix} \beta_{i0} \\ \vdots \\ \beta_{id} \end{bmatrix} \\ \beta_2 = \{\vec{w}_1, \dots, \vec{w}_d\} & \quad \nearrow \\ &\text{Forms } B = (B)_{ij} \text{ which is } d \times d \end{aligned}$$

Confusion from Last Time

$$T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ 3y \end{pmatrix}$$

$$[T]_{\beta_1} = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}, \quad [T]_{\beta_2} = \begin{bmatrix} 7 & 2 \\ 1 & 8 \end{bmatrix}$$

Question: Does

$$[T]_{\beta_1} = B [T]_{\beta_2} B^{-1}$$

OR $B^{-1} [T]_{\beta_2} B$

Answer: $B [T]_{\beta_2} B^{-1}$

LHS inputs \vec{x} in β_1 coords. Outputs β_1 coord of $T\vec{x}$.

RHS input \vec{x} in β_1 coords, B^{-1} outputs β_2 coord of \vec{x} .

$[T]_{\beta_2}$ then outputs β_2 coord of $T\vec{x}$. B outputs β_1 coords of $T\vec{x}$.

Theorem (In General)

$$[T]_{\beta_1} = B [T]_{\beta_2} B^{-1}$$

① what do you need to prove this?

Check B does what we claim.

② What have we suppressed?

The fact B must be invertible!

You can only expand vecs of β_1 in terms of β_2 AND vice-versa

Definition

V, W are isomorphic vector spaces if there exists a bijective linear transform $T: V \rightarrow W$

Fact

V, W are isomorphic $\iff \dim V = \dim W$

\hookrightarrow finite dimensional

\leftarrow this is an isomorphism invariant

If $T: V \rightarrow W$ an isomorphism then $T^{-1}: W \rightarrow V$ is a LT

Example

Let $S, T: V \rightarrow V$ be LTs, $\dim V < \infty$

Then $ST = I \iff TS = I$

Let's Break This - Work w/ $\dim V = \infty$

$V = \{ \text{All infinitely differentiable function } f: \mathbb{R} \rightarrow \mathbb{R} \}$

Define $T: V \rightarrow V$ by $T(f) = \int f(x) dx = F(x) \leftarrow \text{insist } F(0) = 0$

$$S(f) = \frac{d}{dx} f$$

So

$$ST(f) = f$$

$$TS(\text{constant}) = 0_{\text{function}}$$

Laplace Transforms

$$F(s) = \int_0^{\infty} f(t) e^{-st} dt \quad \leftarrow \text{This is a Linear Transformation}$$

Solve

$$f'' - 10f' + 9f = 5t \quad \text{subject to } f(0) = -1 \\ f'(0) = 2$$

Laplace turns differentiation into polynomials.

Get

$$(s^2 - 10s + 9)F(s) + s - 12 = \frac{5}{s^2}$$

Solve for $F(s)$, use inversion to get $f(t)$