ECE 4110 Homework 6

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Due by 5pm on October 31

1 Reading Material

- Gaussian process (Chapter 9.5.1).
- Wiener process and Brownian motion (Chapter 9.5.2 and 9.6.2).
- Markov process: definition and one-step transition matrix (Chapter 11.1 and 11.2).

2 Assignment

1. **Gaussian Process**

   Let $X \sim \mathcal{N}(0,1)$ and $Y \sim \mathcal{N}(0,1)$ be independent Gaussian random variables. Define the random process $\{Z(t)\}_{t=-\infty}^{\infty}$ as

   $$Z(t) = X\cos(2\pi f t) + Y\sin(2\pi f t),$$

   where $f > 0$.

   (a) Is $\{Z(t)\}$ a Gaussian process? Explain.

   (b) Compute the mean and autocorrelation of $\{Z(t)\}$.

   (c) Is $\{Z(t)\}$ wide-sense stationary? Is it stationary?

2. **MMSE Prediction for a Gaussian Process**

   Let $X(t)$ be a stationary Gaussian process with

   $$\mu_X(t) \triangleq \mathbb{E}[X(t)] = 0, \quad R_X(\tau) \triangleq \mathbb{E}[X(t)X(t+\tau)] = 5\cos(\frac{\pi\tau}{2})3^{-|\tau|}$$

   (a) Find the covaraince matrix of $[X(2), X(3), X(4)]^T$.

   (b) Find $\mathbb{E}[X(4)|X(2)]$, the MMSE prediction of $X(4)$ given $X(2)$.

   (c) Find $\mathbb{E}[X(4)|X(2), X(3)]$, the MMSE prediction of $X(4)$ given both $X(2)$ and $X(3)$.

3. **Gaussian Moving Average Process**

   Let $\{X_n\}_{n=-\infty}^{\infty}$ be a sequence of i.i.d. Gaussian random variables with zero mean and unit variance. Define

   $$Y_n = \frac{1}{3}(X_n + X_{n-1} + X_{n-2})$$

   as the moving average of three consecutive X_n's.

(a) Is $\{Y_n\}_{n=-\infty}^{\infty}$ a Gaussian random process? Explain.

(b) Determine the mean function and the autocorrelation function of $\{Y_n\}_{n=-\infty}^{\infty}$.

(c) Is $\{Y_n\}_{n=-\infty}^{\infty}$ (strictly) stationary? Explain.

(d) Does $\{Y_n\}_{n=-\infty}^{\infty}$ have independent increments? Explain.

4. **The Brownian Bridge**

Let $\{X(t)\}_{t\geq 0}$ be a Brownian motion with parameter σ^2. Define the following random process $\{Y(t)\}_{0\leq t\leq 1}$ called a *Brownian bridge* as follows:

$$Y(t) = X(t) - tX(1), \quad 0 \leq t \leq 1.$$

(a) Draw several sample paths of the brownian bridge. Are the sample paths continuous?

(b) Is the Brownian bridge a Gaussian process? Explain.

(c) Compute the mean function and the autocorrelation function of $\{Y(t)\}_{0\leq t\leq 1}$.

(d) Is the Brownian bridge (strictly) stationary? Explain.

(e) Recall that a Brownian motion has independent increments. Does the Brownian bridge have independent increments? Explain.

5. **Markov Chains**

A die is rolled repeatedly. Which of the following sequences $\{X_n\}_{n=0}^{\infty}$ are Markov chains? Prove your statement. For those that are, give the transition matrices.

(a) X_n is the largest number shown up to the nth roll.

(b) X_n is the number of 6's up to the nth roll.

(c) X_n is the number of rolls since the most recent 6.

(d) X_n is the number of rolls until the next 6 (when you see a 6, you reset your count to 0).

6. **A Markov Process with Dependent Increments**

Consider the following first-order Gaussian autoregression process:

$$\begin{aligned} X_0 &= 0 \\ X_{n+1} &= \alpha X_n + W_{n+1} \end{aligned}$$

where $|\alpha| \leq 1$, $\{W_n\}$ are i.i.d. Gaussian with zero mean.

(a) Is $\{X_n\}_{n\geq 0}$ a Gaussian process? Justify your answer.

(b) Show that $\{X_n\}_{n\geq 0}$ is a Markov process (a random process that is Gaussian and Markovian is called a Gaussian-Markov process).

(c) Does $\{X_n\}_{n\geq 0}$ have independent increments? Justify your answer.

① Gaussian Process

Let $X \sim N(0,1)$, $Y \sim N(0,1)$ be independent Gaussian random variables.

Define random process $\{Z(t)\}_{t=-\infty}^{+\infty}$ as

$$Z(t) = X\cos(2\pi f t) + Y\sin(2\pi f t)$$

where $f > 0$.

(a) Is $Z(t)$ a Gaussian process?

Sample $Z(t)$ at 2 times $t_m < t_n$, m,n arbitrary.

Then

$$Z(t_m) = X\cos(2\pi f t_m) + Y\sin(2\pi f t_m)$$

and

$$Z(t_n) = X\cos(2\pi f t_n) + Y\sin(2\pi f t_n)$$

We note that $\cos(2\pi f t)$, $\sin(2\pi f t)$ are just constants in $\mathbb{R}$ $\forall\, t \in \mathbb{R}$ (fixed f).

Since X, Y independent and Gaussian any linear combination of them is Gaussian.

Thus $Z(t_m), Z(t_n)$ Gaussian.

To show $Z(t)$ a Gaussian random process, need to show that $\forall\, n$ and $t_1, \dots, t_n$ the random variables $Z(t_1), \dots, Z(t_n)$ are jointly Gaussian.

Let $\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad , a, b, \dots, n \in \mathbb{R}$

$$\begin{aligned} Z_1 &= aZ(t_1) + bZ(t_2) + \cdots + nZ(t_n) \\ &= X\left[a\cos(2\pi f t_1) + b\cos(2\pi f t_2) + \cdots + n\cos(2\pi f t_n)\right] \\ &\quad + Y\left[a\sin(2\pi f t_1) + b\sin(2\pi f t_2) + \cdots + n\sin(2\pi f t_n)\right] \end{aligned}$$

$\forall n$, this is just a linear combination of X, Y which are two independent Gaussians. Thus Z_1 is jointly Gaussian.

Thus $\{Z(t)\}_{t=-\infty}^{+\infty}$ a Gaussian Random Process.

(b) Mean and Autocorrelation Function of $\{Z(t)\}$?

$\mu_x(t)$

$$\begin{aligned} \mathbb{E}[Z(t)] &= \mathbb{E}\left[X\cos(2\pi f t) + Y\sin(2\pi f t)\right] \\ &= \cos(2\pi f t)\mathbb{E}[X] + \sin(2\pi f t)\mathbb{E}[Y] \\ &= 0 + 0 \quad\quad \forall t \in \mathbb{R} \\ &= 0 \end{aligned}$$

$R_Z(t_1, t_2)$

$$R_Z(t_1, t_2) \triangleq \mathbb{E}[Z(t_1)Z(t_2)]$$

$$\begin{aligned} = \mathbb{E}\Big[& X^2\cos(2\pi f t_1)\cos(2\pi f t_2) \\ & + XY\cos(2\pi f t_1)\sin(2\pi f t_2) \\ & + XY\cos(2\pi f t_2)\sin(2\pi f t_1) \\ & + Y^2\sin(2\pi f t_1)\sin(2\pi f t_2)\Big] \end{aligned}$$

$$= \mathbb{E}[\ X^2\cos(2\pi f t_1)\cos(2\pi f t_2)]$$
$$+ \cancel{\mathbb{E}[\ XY\cos(2\pi f t_1)\sin(2\pi f t_2)]} \rightarrow 0, \text{ X,Y uncorrelated due to J.G. + independence}$$
$$+ \cancel{\mathbb{E}[\ XY\cos(2\pi f t_2)\sin(2\pi f t_1)]} \rightarrow 0$$
$$+ \mathbb{E}[\ Y^2\sin(2\pi f t_1)\sin(2\pi f t_2)]$$

$\text{Var}(X)+\mathbb{E}[X]^2$

$$= \cos(2\pi f t_1)\cos(2\pi f t_2)\,\mathbb{E}[X^2] + \sin(2\pi f t_1)\sin(2\pi f t_2)\,\mathbb{E}[Y^2]$$
$$= \cos(2\pi f t_1)\cos(2\pi f t_2) + \sin(2\pi f t_1)\sin(2\pi f t_2)$$
$$= \cos(2\pi f t_1 - 2\pi f t_2)$$
$$= \cos(2\pi f (t_1 - t_2))$$
$$= \cos(2\pi f (t_2 - t_1)) \quad \text{since cos even}$$

(c) Is $\{Z(t)\}$ Wide-Sense Stationary? Stationary?

WSS

Note $\mathbb{E}[Z(t)] = 0 \quad \forall t$

and $R_X(t_1, t_2) = \cos(2\pi f \tau)$, $\tau = t_2 - t_1$

depends only on τ.

Thus $\{Z(t)\}$ is wide sense stationary.

Stationary?

Since $\{Z(t)\}_{t=-\infty}^{+\infty}$ a Gaussian random process and wide-sense stationary, it is strictly stationary.

② MMSE Prediction for a Gaussian Process

Let $X(t)$ be a stationary Gaussian process with

$$\mu_X(t) \triangleq \mathbb{E}[X(t)] = 0 \qquad R_X(\tau) \triangleq \mathbb{E}[X(t)X(t+\tau)] = 5\cos\left(\frac{\pi\tau}{2}\right)3^{-|\tau|}$$

(a) Find the Covariance Matrix of $[X(2) \quad X(3) \quad X(4)]^T$

$$\mathbb{K} = \{R_X(t_i, t_j) - \mu_X(t_i)\mu_X(t_j)\}_{n\times n} = \{C_X(t_i, t_j)\}_{n \times n}$$

$X(2)$, $X(3)$, $X(4) \Rightarrow$ Sample at time 2, 3, 4. $\mu_X(t) = 0 \ \forall t$.

$$\mathbb{K} = \begin{bmatrix} \mathbb{E}[X(2)X(2)] & \mathbb{E}[X(2)X(3)] & \mathbb{E}[X(2)X(4)] \\ E[X(3)X(2)] & \mathbb{E}[X(3)X(3)] & \mathbb{E}[X(3)X(4)] \\ E[X(4)X(2)] & \mathbb{E}[X(4)X(3)] & \mathbb{E}[X(4)X(4)] \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 & -5/9 \\ 0 & 5 & 0 \\ -5/9 & 0 & 5 \end{bmatrix}$$

(b) Find $\mathbb{E}[X(4) \mid X(2)]$, the MMSE Prediction of $X(4)$ given $X(2)$.

Since $X(4)$, $X(2)$ jointly Gaussian,

$$\mathbb{E}[X(4) \mid X(2)] = \mathbb{E}[X(4)] + \frac{\mathrm{Cov}(X(4), X(2))}{\mathrm{Var}(X(2))}(X(2) - \mathbb{E}[X(2)])$$

$$= 0 + -\frac{5}{9} \cdot \frac{1}{5}(X(2) - 0) = -\frac{1}{9}X(2)$$

This estimator results in following MSE:

$$\mathrm{Var}\left(X(4) \mid X(1)\right) = \mathrm{Var}\left(X(4)\right) - \frac{\mathrm{Cov}^2\left(X(4), X(1)\right)}{\mathrm{Var}\left(X(1)\right)}$$

$$= 5 - \frac{25}{81} \cdot \frac{1}{5} = 5 - \frac{5}{81}$$

(c) Find $\mathbb{E}[X(4) \mid X(2), X(3)]$, the MMSE prediction of $X(4)$ given both $X(2), X(3)$.

Note: $\begin{bmatrix} X(2) \\ X(3) \end{bmatrix}$ a Gaussian random vector w/

since $X(2), X(3)$ zero mean

$$\mu = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathbb{K} = \begin{bmatrix} \mathbb{E}[X(2)X(2)] & \mathbb{E}[X(2)X(3)] \\ \mathbb{E}[X(3)X(2)] & \mathbb{E}[X(3)X(3)] \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

Let

$$\mathbf{X} = [X(4)] \sim \mathcal{N}([0], [5])$$

$$\mathbf{Y} = \begin{bmatrix} X(2) \\ X(3) \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}\right)$$

Then

$$\mathbb{E}[\mathbf{X} \mid \mathbf{Y}] = \underbrace{\mathbb{E}[\mathbf{X}]}_{1\times 1} + \underbrace{\mathrm{Cov}(\mathbf{X}, \mathbf{Y})}_{1 \times 2} \underbrace{\mathrm{Cov}^{-1}(\mathbf{Y})}_{2\times 2} \underbrace{(\mathbf{Y} - \mathbb{E}[\mathbf{Y}])}_{2 \times 1}$$

$$= 0 + \begin{bmatrix} -5/9 & 0 \end{bmatrix} \begin{bmatrix} 1/5 & 0 \\ 0 & 1/5 \end{bmatrix} \begin{bmatrix} X(2) \\ X(3) \end{bmatrix}$$

$$= 0 + \begin{bmatrix} -1/9 & 0 \end{bmatrix} \begin{bmatrix} X(2) \\ X(3) \end{bmatrix}$$

$$= 0 + \frac{-1}{9} X(2) + 0$$

$$= \frac{-1}{9} X(2).$$

Makes sense! $X(4)$ uncorrelated w/ $X(3)$ AND $X(3)$ uncorrelated w/ $X(2)$.

③ Gaussian Moving Average

Let $\{X_n\}_{n=-\infty}^{+\infty}$ be a sequence of iid Gaussian random variables w/ zero mean and unit variance.

Define

$$Y_n = \frac{1}{3}(X_n + X_{n-1} + X_{n-2})$$

as the moving average of X_n's.

(a) Is $\{Y_n\}_{n=-\infty}^{+\infty}$ a Gaussian Random Process?

For arbitrary m, n w/ $m < n$

$$Y_m = \frac{1}{3}(X_m + X_{m-1} + X_{m-2}) \sim \text{Gaussian}$$

$$Y_n = \frac{1}{3}(X_n + X_{n-1} + X_{n-2}) \sim \text{Gaussian}$$

and

$$aY_m + bY_n \qquad a, b \in \mathbb{R}$$

$$= \frac{a}{3}(X_m + X_{m-1} + X_{m-2}) + \frac{b}{3}(X_n + X_{n-1} + X_{n-2}) \sim \text{Gaussian}$$

since X_i iid $\forall\, i$ (overlap case is also included above).

(b) Determine the mean function and the autocorrelation function of $\{Y_n\}_{n=-\infty}^{+\infty}$

$$\mathbb{E}[Y_n] = \frac{1}{3}\mathbb{E}[X_n + X_{n-1} + X_{n-2}]$$

$$= \frac{1}{3}\left(\mathbb{E}[X_n] + \mathbb{E}[X_{n-1}] + \mathbb{E}[X_{n-2}]\right)$$

$$= \frac{1}{3}(0 + 0 + 0)$$

$$= 0$$

Assume $m<n$

$$R_Y(m,n) \hat{=} \mathbb{E}[Y_m Y_n]$$

$$= \frac{1}{9}\mathbb{E}\left[(X_n + X_{n-1} + X_{n-2})(X_m + X_{m-1} + X_{m-2})\right]$$

4 cases.

① $m < n-2$

then $X_i \perp\!\!\!\perp X_j \ \forall i \neq j$ and $\mathbb{E}[Y_m Y_n] = 0$

② $m = n-2$

$$R_Y(m,n) = \frac{1}{9}\mathbb{E}\left[(X_n + X_{n-1} + X_{n-2})(X_{n-2} + X_{n-3} + X_{n-4})\right]$$

$$= \frac{1}{9}\mathbb{E}[X_{n-2}^2] = \frac{1}{9}$$

③ $m = n-1$

$$R_Y(m,n) = \frac{1}{9}\mathbb{E}\left[(X_n + X_{n-1} + X_{n-2})(X_{n-1} + X_{n-2} + X_{n-3})\right]$$

$$= \frac{1}{9}\left(\mathbb{E}[X_{n-1}^2] + \mathbb{E}[X_{n-2}^2]\right) = \frac{2}{9}$$

④ $m = n$

$$R_Y(m,n) = \frac{1}{9}\left(\mathbb{E}[X_n^2] + \mathbb{E}[X_{n-1}^2] + \mathbb{E}[X_{n-2}^2]\right) = \frac{3}{9} = \frac{1}{3}$$

Thus

$$R_Y(m,n) = \begin{cases} 0, & |n-m| > 2 \\ \frac{1}{9}, & |n-m| = 2 \\ \frac{2}{9}, & |n-m| = 1 \\ \frac{1}{3}, & |n-m| = 0 \end{cases}$$

(c) Is $\{Y_n\}_{n=-\infty}^{+\infty}$ (strictly) Stationary?

$\mathbb{E}[Y_n]$ constant and $R_Y(m,n)$ depends only on $|m-n|$.

Thus $\{Y_n\}_{n=-\infty}^{+\infty}$ WSS $\Rightarrow$ Strictly Stationary.

(d) Does $\{Y_n\}_{n=-\infty}^{+\infty}$ have independent increments?

No! For all s and for all $n_1, n_2, \ldots, n_s$

$Y_{n_2} - Y_{n_1}, \ldots, Y_{n_s} - Y_{n_{s-1}}$ not ALWAYS mutually independent.

Easy to see this. Take $2 < 3 < 5 < 8$

$$Y_8 - Y_5 = X_8 + X_7 + X_6 - X_5 - X_4 - X_3$$
$$Y_3 - Y_2 = X_3$$

both increments have X_3 in them.

(4) The Brownian Bridge

Let $\{X(t)\}_{t\geq 0}$ be a Brownian motion with parameter σ^2.

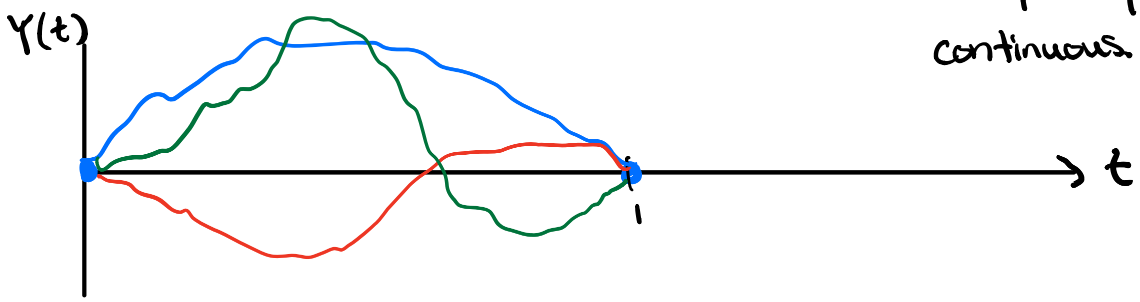
Define the following random process $\{Y(t)\}_{0\leq t\leq 1}$ called a Brownian bridge as follows:

$$Y(t) = X(t) - tX(1), \quad 0 \leq t \leq 1.$$

(a) Draw several sample paths of the brownian bridge.

Are the sample paths continuous?

Sample paths ARE continuous.



(b) Is the Brownian bridge a Gaussian process? Explain.

Take $0 \leq t_1 < t_2 \leq 1$

Then $\quad Y(t_1) + Y(t_2) = X(t_1) + X(t_2) - X(1)(t_1 + t_2)$

Since X a Gaussian process then any linear combination of X is Gaussian and thus $Y(t_1) + Y(t_2)$ is Gaussian.

Extend to general n: $0 \leq t_1 < t_2 < \cdots < t_n \leq 1$

$$Y(t_1) + Y(t_2) + \cdots + Y(t_n) = X(t_1) + X(t_2) + \cdots + X(t_n) - X(1)(t_1 + t_2 + \cdots + t_n)$$

which is also Gaussian - and introduction of a scalar does not change this.

Thus $\{Y\}_{0\leq t\leq 1}$ is a Gaussian process.

(c) Mean function and autocorrelation function of $\{Y(t)\}_{0\le t\le 1}$.

$$\mu_Y(t) \triangleq \mathbb{E}[Y(t)] = \mathbb{E}[X(t) - tX(1)]$$
$$= \mathbb{E}[X(t)] - \mathbb{E}[tX(1)]$$
$$= \mathbb{E}[X(t)] - t\,\mathbb{E}[X(1)]$$
$$= 0 - t\cdot 0$$
$$= 0$$

Take $t_1 < t_2$

$$C_Y(t_1,t_2) \triangleq \mathbb{E}\Big[\big(Y(t_1) - \mathbb{E}[Y(t_1)]\big)\big(Y(t_2) - \mathbb{E}[Y(t_2)]\big)\Big]$$

$R_Y(t_1,t_2)$"

$$= \mathbb{E}[Y(t_1)Y(t_2)]$$
$$= \mathbb{E}[(X(t_1) - t_1X(1))(X(t_2) - t_2X(1))]$$
$$= \mathbb{E}[X(t_1)X(t_2) - X(1)[X(t_1)t_2 + X(t_2)t_1] + t_1t_2X^2(1)]$$
$$= \mathbb{E}[X(t_1)X(t_2)] - t_2\mathbb{E}[X(1)X(t_1)] - t_1\mathbb{E}[X(1)X(t_2)] + t_1t_2\,\mathbb{E}[X^2(1)]$$
$$= \sigma^2 t_1 - t_2\sigma^2 t_1 - t_1\sigma^2 t_2 + t_1t_2\sigma^2$$
$$= \sigma^2 t_1 - t_1t_2\sigma^2$$

In general,

$$R_Y(t_1,t_2) = \min(t_1,t_2)\,\sigma^2 - t_1t_2\sigma^2$$

(d) Is the Brownian bridge strictly stationary?

$\forall\, t\in[0,1]$, $Y(t)\sim\mathcal{N}(0, \sigma^2(t-t^2))$

It's variance changes as a function of t and thus its statistical characteristics change with time $\Rightarrow$ **NOT** strictly stationary.

(e) Does the Brownian bridge have independent increments?

Take n samples of $Y(t)$ at $t_1 < t_2 < \cdots < t_n$.

Need to show

$$Y(t_2)-Y(t_1) \perp\!\!\!\perp Y(t_3)-Y(t_2) \perp\!\!\!\perp \cdots \perp\!\!\!\perp Y(t_n)-Y(t_{n-1})$$

$$\begin{aligned} Y(t_2)-Y(t_1) &= X(t_2)-t_2X(1)-X(t_1)+t_1X(1) \\ &= X(t_2)-X(t_1)-X(1)[t_1-t_2] \end{aligned}$$

$$Y(t_3)-Y(t_1) = X(t_3)-X(t_2)-X(1)[t_2-t_3]$$

$Y(t_2)-Y(t_1)$ clearly **NOT** independent of $Y(t_3)-Y(t_2)$.

Thus the Brownian bridge does **NOT** have independent increments.

⑤ Markov Chains

A die is rolled repeatedly. Which of the following sequences $\{X_n\}_{n=0}^{\infty}$ are Markov chains? Prove your statement. For those that are, give transition matrices.

(a) X_n is the largest number shown up to the n^{th} roll.

Let $X_1 = x_1$.

Then

$$X_2 = \begin{cases} x_2 & \text{if } x_2 > x_1 \\ x_1 & \text{if } x_2 \leq x_1 \end{cases}$$

and

$$X_3 = \begin{cases} x_3 & \text{if } x_3 > x_2 \\ x_2 & \text{if } x_3 \leq x_2 \end{cases}$$

and ...

$$X_n = \begin{cases} x_n & \text{if } x_n > x_{n-1} \\ x_{n-1} & \text{if } x_n \leq x_{n-1} \end{cases}$$

Thus each X_n is dependent only on the value taken at X_{n-1} and this is a Markov process.

Could also note that

$$X_{n-1} = \max(x_1, \ldots, x_{n-1})$$

and

$$X_n = \max(\max(x_1, \ldots x_{n-1}), x_n)$$
$$= \max(x_{n-1}, x_n)$$

$\Pr[X_n = x_n \setminus X_{n-1} = x_{n-1}] = (\# \text{values} > x_{n-1})\frac{1}{6}$

$\Pr[X_n = x_n \mid X_{n-1} = x_{n-1}, \ldots, X_1 = x_1] = (\# \text{values} > x_{n-1})\frac{1}{6}$

The transition matrix is as follows

$$\mathbb{P} = \begin{bmatrix} 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 0 & 2/3 & 1/6 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 1/2 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 0 & 2/3 & 1/6 & 1/6 \\ 0 & 0 & 0 & 0 & 5/6 & 1/6 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) X_n is the number of 6's up to the n^{th} roll.

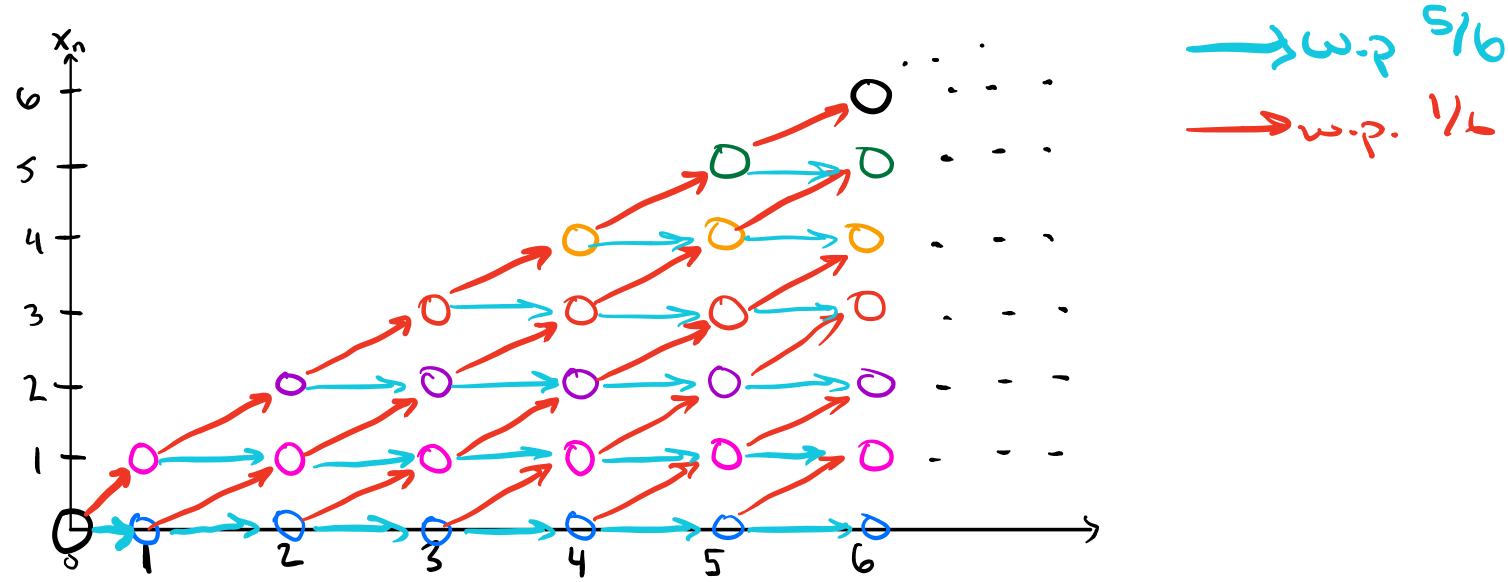
$$X_n = \begin{cases} x_{n-1} & \text{, if 6 is NOT rolled} \\ x_{n-1}+1 & \text{, if 6 is rolled} \end{cases}$$

intuitively a Markov process since X_n depends only on the realization of X_{n-1}.

The transition matrix is as follows

$$\mathbb{P} = \begin{bmatrix} 5/6 & 1/6 & 0 & \cdots & & & \\ 0 & 5/6 & 1/6 & 0 & \cdots & & \\ \vdots & \vdots & 5/6 & 1/6 & 0 & \cdots & \\ \vdots & \vdots & \vdots & 5/6 & 1/6 & 0 & \cdots \\ 0 & \cdots & \cdots & \cdots & 5/6 & \ddots & \end{bmatrix}$$

To better see this is a Markov process, observe the following diagram.



Regardless of state you're in, the next state depends only on where you are and not how you got there as observed in diagram.

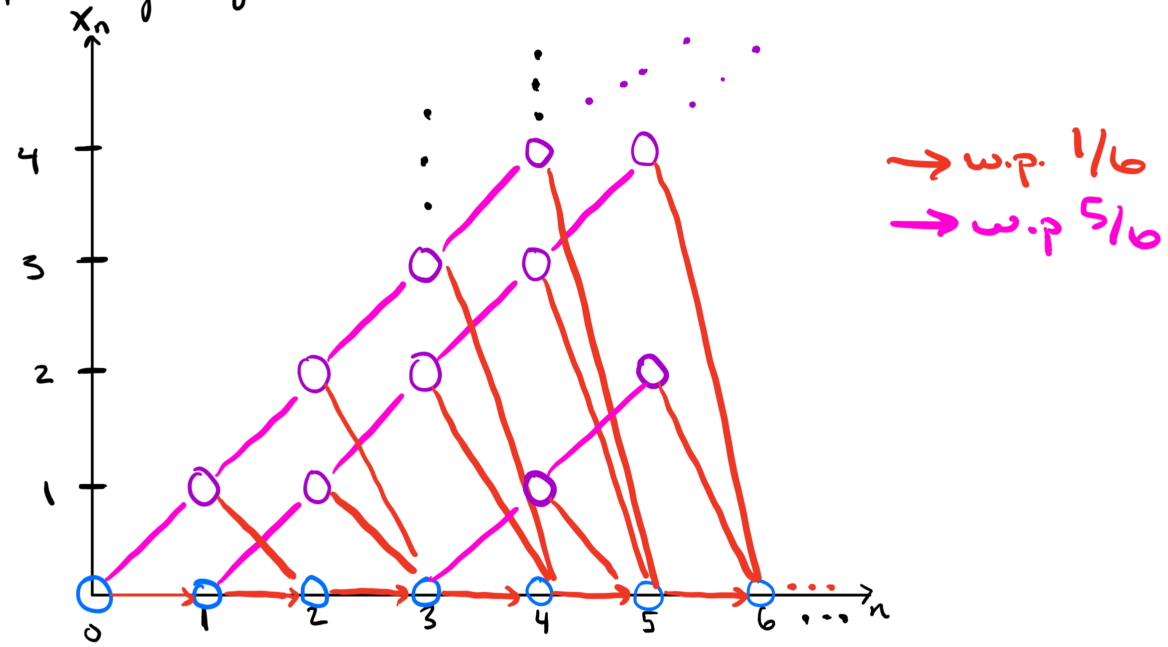
(c) X_n is the number of rolls since the most recent 6.

$$X_n = \begin{cases} 0 & \text{w.p. } 1/6 \\ x_{n-1}+1 & \text{w.p. } 5/6 \end{cases}$$

Each X_n depends only on the realization of X_{n-1} and thus this is intuitively a Markov process.

$$\mathbb{P} = \begin{bmatrix} 1/6 & 5/6 & 0 & \cdots & & & \\ 1/6 & 0 & 5/6 & \cdots & & & \\ \vdots & 0 & 0 & 5/6 & \cdots & & \\ \vdots & \vdots & \vdots & & \ddots & & \\ 1/6 & \cdots & & & \cdots & 5/6 & 0 \ \cdots \\ 1/6 & \cdots & & & & & \cdots \\ \vdots & \cdots & & & & & \cdots \end{bmatrix}$$

To better see this is a Markov process, observe the following diagram.



Regardless of state you're in, the next state depends only on where you are and not how you got there as observed in diagram.

(d) X_n is the number of rolls until the next 6.

We note, that X_n as defined is a geometric random variable with parameter $p = 1/6$.

$$\text{i.e. } X_n \sim (1-p)^n p$$

and X_n iid $\forall n$.

Thus $X_n \mid X_{n-1}, \ldots, X_1 \sim \text{Geom}(p)$

$X_n \mid X_{n-1} \sim \text{Geom}(p)$.

$$\sum_{n=1}^{\infty} \left(\frac{5}{6}\right)^n = \frac{1}{1-5/6} = 6$$

So this is a Markov process.

every row the same

$$\mathbb{P} = \begin{bmatrix} \left(\frac{5}{6}\right)\left(\frac{1}{6}\right) & \left(\frac{5}{6}\right)^2\left(\frac{1}{6}\right) & \left(\frac{5}{6}\right)^3\left(\frac{1}{6}\right) & \cdots & \left(\frac{5}{6}\right)^n\left(\frac{1}{6}\right) & \cdots \\ 1 & 0 & 6 & \cdots & 0 & \cdots \\ 0 & 1 & 0 & \cdots & & \\ \vdots & \vdots & \vdots & \ddots & & \ddots \end{bmatrix}$$

$P_{3,2} = 1$

⑥ A Markov Process with Dependent Increments

Consider the following first-order Gaussian autoregression process:

$$X_0 = 0$$

$$X_{n+1} = \alpha X_n + W_{n+1}$$

where $|\alpha| \leq 1$, $\{W_n\}$ are iid Gaussian with zero mean.
(var σ^2 assumed)

(a) Is $\{X_n\}_{n\geq 0}$ a Gaussian process?

Need $\forall n$ and $0 \leq n_1 < n_2 < \cdots < n_n$ that $X_{n_1}, \ldots, X_{n_n}$ are jointly Gaussian.

First, note that

$$X_1 = \alpha X_0 + W_1 = W_1$$
$$X_2 = \alpha X_1 + W_2 = \alpha W_1 + W_2$$
$$X_3 = \alpha X_2 + W_3 = \alpha^2 W_1 + \alpha W_2 + W_3$$
$$\vdots$$
$$X_n = \alpha X_{n-1} + W_n = \alpha^n W_1 + \alpha^{n-1} W_2 + \cdots + W_n$$

Take

$$a_1 X_1 + a_2 X_2 + \cdots + a_n X_n \quad , \quad a_i \in \mathbb{R} \ \forall i$$

this is equivalent to

$$a_1 W_1 + a_2(\alpha W_1 + W_2) + \cdots + a_n(\alpha^n W_1 + \alpha^{n-1} W_2 + \cdots + W_n)$$
$$= W_1(1 + a_1 + \alpha + \cdots + \alpha^n) + W_2(a_2 + \alpha + \cdots + \alpha^{n-1}) + \cdots + a_n W_n$$

Since W_i iid Gaussian $\forall i$, this is a linear combination of independent Gaussians $\Rightarrow$ sum is Gaussian $\forall n \Rightarrow$ J.G.

Thus $\{X_n\}_{n\geq 0}$ is a Gaussian process.

(b) Show that $\{X_n\}_{n \geq 0}$ is a Markov process.

Need to show

$$Pr[X_n = x_n \mid X_{n-1} = x_{n-1}, \dots, X_0 = x_0] = Pr[X_n = x_n \mid X_{n-1} = x_{n-1}]$$

LHS

$$Pr[X_n = x_n \mid X_{n-1} = x_{n-1}, \dots, X_1 = x_1]$$

Realize that

$$X_1 = x_1 \xrightarrow{\text{Given } X_0 = x_0} W_1 = x_1$$

$$X_2 = x_2 \Rightarrow \alpha X_1 + W_2 = x_2 \xrightarrow{\text{Given } X_1 = x_1, X_0 = 0} x_2 = \alpha x_1 + W_2$$

$$\vdots$$

$$X_n = x_n \Rightarrow \alpha X_{n-1} + W_n = x_n$$

i.e $\alpha^{n-1} W_1 + \alpha^{n-2} W_2 + \dots + W_n = x_n$

Given $X_{n-1} = x_n, \dots, X_0 = 0$

$$\longrightarrow x_n = \alpha^{n-1} x_1 + \alpha^{n-2}(x_2 - \alpha x_1) + \alpha^{n-3}(x_3 - \alpha x_2) + \dots + \alpha(x_{n-1} - \alpha x_{n-2}) + W_n$$

i.e $\quad X_n \mid X_{n-1} = x_n, \dots, X_0 = x_0 = \alpha x_{n-1} + W_n \sim \mathcal{N}(\alpha x_{n-1}, \sigma^2)$

RHS

$$X_n \mid X_{n-1} \rightarrow \alpha x_{n-1} + W_n \sim \mathcal{N}(\alpha x_{n-1}, \sigma^2)$$

Since $X_n \mid X_{n-1}$ and $X_n \mid X_{n-1}, \dots, X_0$ have the same distribution, the probability they take value x_n is the same.

Thus LHS = RHS and $\{X_n\}_{n \geq 0}$ a Markov Process.

(c) Does $\{X_n\}_{n \geq 0}$ have independent increments?

Need $\forall m,\ 0 < n_1 < n_2 < \cdots < n_m$

$$X_{n_1} - X_0 \perp\!\!\!\perp X_{n_2} - X_{n_1} \perp\!\!\!\perp \cdots \perp\!\!\!\perp X_{n_m} - X_{n_{m-1}}$$

Take increments of size 1.

Observe that

$$\begin{aligned} X_3 - X_2 &= \alpha^2 W_1 - \alpha W_1 + \alpha W_2 - W_2 + W_3 \\ &= W_1(\alpha^2 - \alpha) + W_2(\alpha - 1) + W_3 \end{aligned}$$

$$\begin{aligned} X_2 - X_1 &= \alpha W_1 + W_2 - W_1 \\ &= W_1(\alpha - 1) + W_2 \end{aligned}$$

both increments have W_1 dependence.

Thus $\{X_n\}_{n \geq 0}$ does NOT have independent increments.