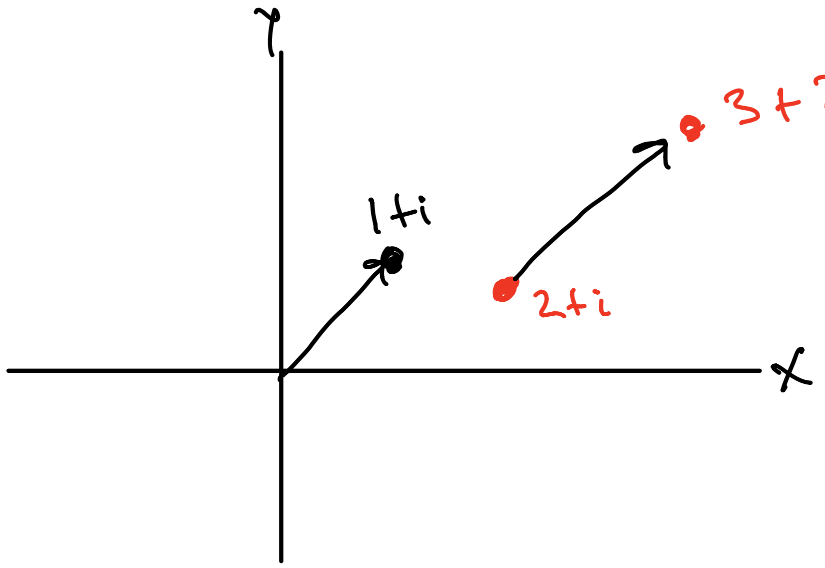
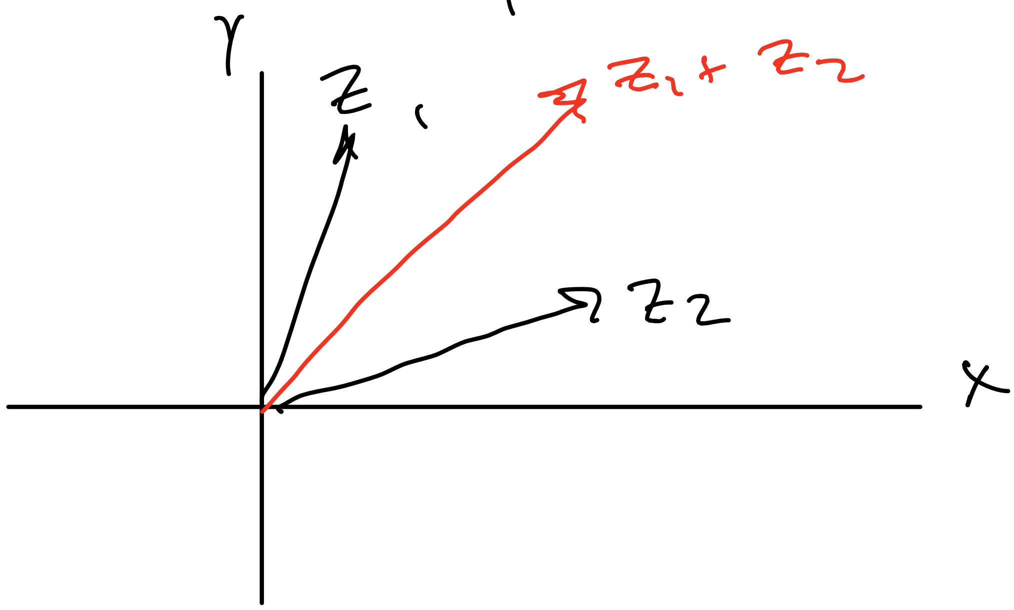


Note how these two vectors have the same magnitude $|z|$.

A vector ∥ x-axis is purely real

A vector ∥ y-axis is purely imaginary



vector addition

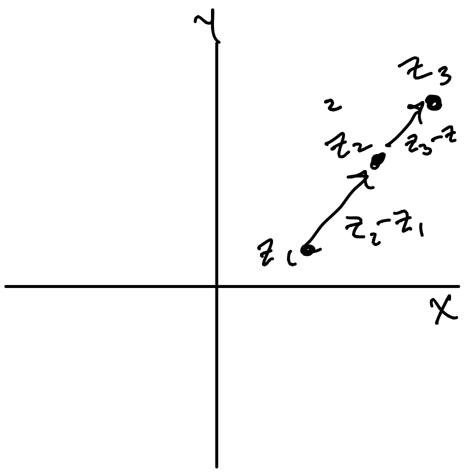
Triangle Inequality: For any two complex numbers z_1 and z_2, we have

$$|z_1+z_2| \leq |z_1| + |z_2|$$

Example 1: Prove 3 distinct points z_1, z_2, z_3 lie on the same straight line iff

$$z_3 - z_2 = c(z_2 - z_1)$$

for some $c \in \mathbb{R}$.

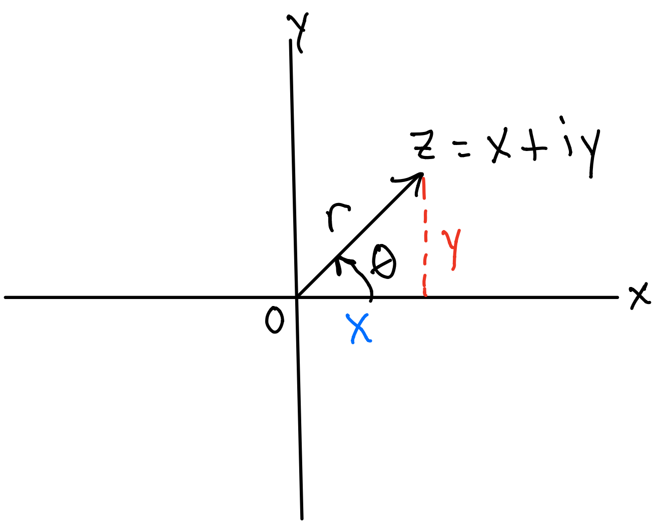


Two vectors are parallel iff one is a (real) scalar multiple of the other. The condition that the points z_1, z_2, z_3 be colinear is equivalent to the statement that the vector $z_3 - z_2$ is parallel to $z_2 - z$. Using our characterization of parallelism the conclusion follows immediately.

Polar Form

r: distance from origin to z.

θ: angle of inclination of the vector z.
measured positively ccw from real-axis.



$x = r\cos(\theta)$

$y = r\sin(\theta)$

$r = \sqrt{x^2 + y^2} = |z|$

$\theta = \tan^{-1}(y/x)$

Note however that our expression for θ is <u>invalid</u> for points z in Quadrants 2/3.
We can adjust for incorrectness by adding/subtracting π radians when appropriate.
More formally, we can use

$$\cos(\theta) = \frac{x}{|z|}, \quad \sin(\theta) = \frac{y}{|z|}$$

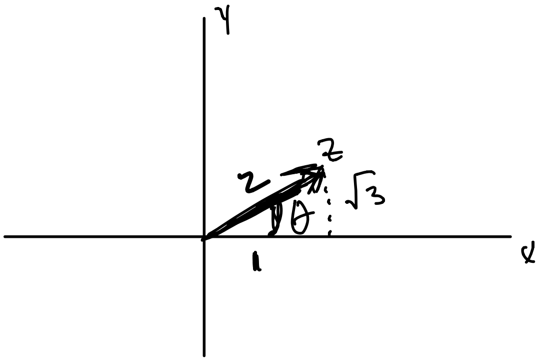
If θ is an identification of z then so is any integer multiple of 2π.
The value shall be denoted

$$\arg(z) \rightarrow \text{phase}$$

Thus if θ_0 qualifies as a value for $\arg(z)$, then so do

$$\theta_0 \pm 2\pi n, \quad n \in \mathbb{Z}$$

Example 2: Find $\arg(1+\sqrt{3}i)$ and write $(1+\sqrt{3}i)$ in polar form.



$$|z| = \sqrt{1+3} = 2$$

$$\theta = \tan^{-1}(\sqrt{3}) = 60^\circ$$

$$\arg(1+\sqrt{3}i) = \frac{\pi}{3} + 2\pi n, \; n \in \mathbb{Z}$$

$$x = 2\cos(\pi/3)$$

$$y = 2\sin(\pi/3);$$

Polar form is therefore $2\left(\cos \pi/3 + i \sin \pi/3\right)$

oftenly written as $2 \operatorname{cis}(\pi/3)$

Polar form lends itself useful during multiplication.

Let

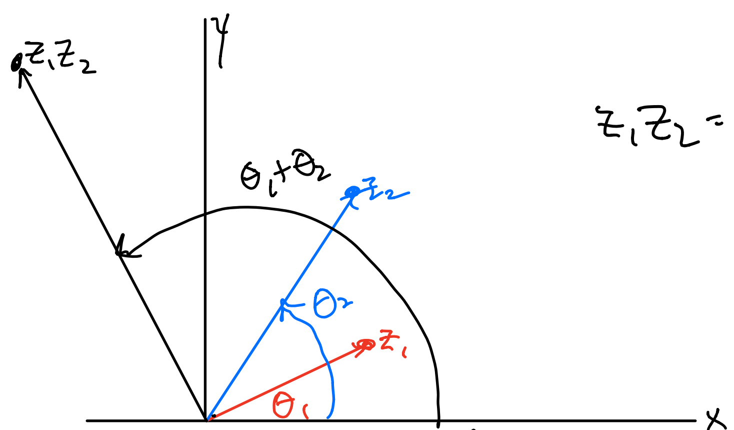
$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$$

$$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

Then we compute

$$z_1 z_2 = r_1 r_2 \left[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\right]$$

and so



Geometric interpretation

$$z_1 z_2 = r_1 r_2\left[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right]$$

The abbreviated version is written

$$z_1 z_2 = (r_1 r_2)\,\text{cis}(\theta_1 + \theta_2)$$

and we see that

The modulus of the product is the product of the moduli:

$$|z_1 z_2| = |z_1||z_2|$$

The argument of the product is the sum of arguments:

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$$

The rules for division then apply as the inverse of multiplication.

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\,\text{cis}(\theta_1 - \theta_2)$$

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

Example 3: Write $\left(\frac{1+i}{\sqrt{3}-i}\right)$ in polar form

$$1+i = |1+i|\,\text{cis}\left(\arg(1+i)\right) = \sqrt{2}\,\text{cis}\left(\pi/4\right)$$

$$\sqrt{3}-i = |\sqrt{3}-i|\,\text{cis}\left(\arg(\sqrt{3}-i)\right) = 2\,\text{cis}\left(-\pi/6\right)$$

$$\frac{1+i}{\sqrt{3}-i} = \frac{\sqrt{2}}{2}\,\text{cis}\left[\frac{\pi}{4} - \frac{-\pi}{6}\right] = \frac{\sqrt{2}}{2}\,\text{cis}\,\frac{5\pi}{12}$$

Example 4: Prove that the line l through points z_1 and z_2 is perpendicular to the line L through points z_3 and z_4 iff

$$\operatorname{Arg} \frac{z_1 - z_2}{z_3 - z_4} = \pm \frac{\pi}{2}$$

The lines l and L are perpendicular iff the vectors $z_1 - z_2$ and $z_3 - z_4$ are perpendicular. Since

$$\arg \frac{z_1 - z_2}{z_3 - z_4} = \arg(z_1 - z_2) - \arg(z_3 - z_4)$$

gives the angle from $z_3 - z_4$ to $z_1 - z_2$, orthogonality holds precisely when this angle is equal to $+/ \pi/2$.

Recall that geometrically, the vector $\bar{z}$ is the reflection in the real axis of the vector z.

Hence we see that the argument of the conjugate of a complex number is the negative of the argument of the number. That is,

$$\arg \bar{z} = -\arg z$$

We also have

$$\arg \frac{1}{z} = -\arg(z)$$

Thus $\bar{z}$ and z^{-1} have the same argument and represent parallel vectors.