

## Markov Chains (part 2)

Stochastic Matrices  $A_{n \times n}$

$$\lim A^m = L$$

If  $A$  diagonalizable, then this limit exists

$$\Leftrightarrow \text{all complex eigenvalues } \lambda \text{ satisfy } |\lambda| < 1 \text{ or } \lambda = 1$$

Note:

$$\lim_{m \rightarrow \infty} A A^m = L = A L$$

So, all columns of  $L$  are eigenvectors w/  $\lambda = 1$ .

Note: if  $\vec{u} = (1, \dots, 1)$

$$\text{then } \lim_{m \rightarrow \infty} (\vec{u} A^m) = \lim_{m \rightarrow \infty} \vec{u} = \vec{u}$$

$$= \vec{u} \lim_{m \rightarrow \infty} A^m$$

$$= \vec{u} L \Rightarrow L \text{ is a stochastic matrix}$$

Gershgorin Disks

$$G_i = \{z \in \mathbb{C} \mid |z - A_{ii}| \leq r_i\}$$

where

$$r_i = \rho_i(A) - |A_{ii}|$$

Theorem: If  $A \in \mathbb{C}^{n \times n}$ , if  $\lambda = \text{eigenvalues of } A$ , then  $\lambda \in \mathbb{C}_i$  for some  $i$ .

Note: if  $z, w \in \mathbb{C}$ ,

$$|z + w| \leq |z| + |w|$$

$$|z - w| \leq |z| + |w|$$

Proof: Suppose  $A\hat{v} = \lambda\hat{v}$ ,  $\hat{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \neq \vec{0}$

So

$$\star \sum_{j=1}^n A_{ij} v_j = \lambda v_i \quad \text{for } i=1, \dots, n$$

Suppose

$$v_k = \text{entry s.t. } |v_k| \geq |v_1|, \dots, |v_n| > 0$$

Show  $\lambda \in \mathbb{C}_k$  use  $i=k$  in  $\star$

$$\sum_{j=1}^n A_{kj} v_j = \lambda v_k$$

$$|\lambda v_k - A_{kk} v_k| = \left| \sum_{j \neq k} A_{kj} v_j \right| \leq \sum_{j \neq k} |A_{kj}| |v_j|$$

$$\leq \sum_{j \neq k} |A_{kj}| |v_k| = r_k |v_k|$$

$$\text{So } |\lambda - A_{kk}| |v_k| \leq r_k |v_k| \rightarrow |\lambda - A_{kk}| \leq r_k$$

Corollary: If  $A$  is stochastic, then any eigenvalue  $\lambda$  lies in  $|\lambda| \leq 1$

Proof: Know  $0 \leq A_{ii} \leq 1$

Know  $\lambda \in C_i = \{z \in \mathbb{C} \mid |z - A_{ii}| \leq r_i\}$

for some  $i = 1, \dots, n$ ,  $r_i = 1 - A_{ii}$

so  $C_i = \{z : |z| \leq 1\}$

Fact (requires proof)

$$\dim E_\lambda(A) = 1$$

$AL = L \Rightarrow$  each column  
of  $L$  identical.