

① MMSE Estimation

$X \sim \text{Unif}[0,2]$. Suppose $Y = Xu$, where $X \perp\!\!\!\perp u$, and $u \sim \text{Unif}[0,1]$.

(a) Find the best LMMSE estimator of X given Y .

$$\hat{X}_{\text{LMMSE}} = \frac{\text{Cov}(X, Y)}{\text{Var}(Y)} (Y - \mathbb{E}[Y]) + \mathbb{E}[X]$$

$$\mathbb{E}[Y] = \mathbb{E}[xu] = \mathbb{E}[x]\mathbb{E}[u] \text{ since } X \perp\!\!\!\perp u$$

$$\mathbb{E}[x] = \frac{2+0}{2} = 1, \quad \mathbb{E}[u] = \frac{1+0}{2} = \frac{1}{2}$$

$$\mathbb{E}[Y] = 1/2$$

$$\text{Var}(Y) = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2$$

$$= \frac{16}{36} - \left(\frac{1}{2}\right)^2$$

$$= \frac{16}{36} - \frac{9}{36}$$

$$= 7/36$$

$$\mathbb{E}[Y^2] = \mathbb{E}[x^2]\mathbb{E}[u^2]$$

$$= (\text{Var}(x) + (\mathbb{E}[x])^2)(\text{Var}(u) + (\mathbb{E}[u])^2)$$

$$= \left(\frac{4}{12} + 1\right)\left(\frac{1}{12} + \frac{1}{4}\right)$$

$$= \frac{16}{12} \cdot \frac{4}{12} = \frac{16}{36}$$

$$\text{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$$

$$\begin{aligned}
&= \mathbb{E}[X^2U] - \mathbb{E}[X]\mathbb{E}[Y] \\
&= \mathbb{E}[X^2]\mathbb{E}[U] - \mathbb{E}[X]\mathbb{E}[Y] \\
&= \frac{4}{3} \cdot \frac{1}{2} - \frac{1}{2} \cdot 1 \\
&= \frac{1}{6}
\end{aligned}$$

Thus

$$\begin{aligned}
\hat{X}_{\text{mmse}} &= \frac{1/6}{7/136} (Y - 1/2) + 1 \\
&= \frac{6}{7}Y - \frac{6}{14} + 1 \\
&= \frac{6}{7}Y + \frac{4}{7}
\end{aligned}$$

(b) Find the best MMSE estimator of X given Y .

$$\hat{X}_{\text{mmse}} = \mathbb{E}[X|Y]$$

$$Y|X=x = xU \sim [0, x]$$

$$f_{Y|X=x}(y|x) = \begin{cases} 1/x, & 0 \leq y \leq x \leq 2 \\ 0, & \text{o/w} \end{cases}$$

Need

$$f_{X|Y=y}(x|y) = \frac{f_{Y|x}(y|x)f_x(x)}{f_Y(y)}$$

$$f_{X|Y}(x,y) = f_{Y|x}(y|x)f_x(x) = \begin{cases} \frac{1}{2x}, & 0 \leq y \leq x \leq 2 \\ 0, & \text{o/w} \end{cases}$$

$$f_Y(y) = \int_{x=y}^2 \frac{1}{2x} dx = \frac{1}{2} \ln|x| \Big|_{x=y}^2 = \frac{1}{2} (\ln|2| - \ln|y|)$$

$$f_Y(y) = \begin{cases} \frac{1}{2}(\ln|2| - \ln|y|), & 0 \leq y \leq 2 \\ 0, & \text{o/w} \end{cases}$$

$$f_{X|Y}(x|y) = \frac{\frac{1}{2x}}{\frac{1}{2}(\ln|2| - \ln|y|)} = \frac{1}{x(\ln|2| - \ln|y|)}, 0 \leq y \leq x \leq 2$$

$$\mathbb{E}[X|Y=y] = \int_y^2 x f_{X|Y}(x|y) dx = \int_y^2 \frac{x}{\ln|2| - \ln|y|} \cdot \frac{1}{x} dx = \frac{2-y}{\ln|2| - \ln|y|}$$

$$\mathbb{E}[X|Y] = \frac{2-Y}{\ln 2 - \ln Y}$$

② Estimating a Cubic

Let X and Y be zero-mean jointly Gaussian random variables with covariance matrix

$$\mathbf{K} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$$

where $|\rho| < 1$

Note:

① For $W \sim N(0, \sigma^2)$

$$\mathbb{E}[W^k] = \begin{cases} 0 & , k \text{ odd} \\ \sigma^k (k-1)! ! & , k \text{ even} \end{cases}$$

where $n! ! = n(n-2)(n-4) \cdots 3 \cdot 1$ for odd n

② For a general Gaussian random variable $Z \sim N(\mu, \sigma^2)$, we have

$$\mathbb{E}[Z^3] = \mu^3 + 3\mu\sigma^2$$

(a) Find $\mathbb{E}[X^3 | Y]$, the best estimator of X^3 given Y , and compute the resulting MSE.

$$X|Y \sim N(\mathbb{E}[X|Y], \text{Var}(X|Y))$$

$$\sim N(\rho Y, 1 - \rho^2)$$

$$\text{IE}[x|Y] = \text{IE}[x] + \frac{\text{Cov}(x, Y)}{\text{Var}(Y)} (Y - \text{IE}[Y]) = pY$$

$$\text{Var}(x|Y) = \text{Var}(x)(1 - p^2_{xy}) = 1 - p^2$$

Using hint ②,

$$\begin{aligned}\text{IE}[x^3|Y] &= \mu^3 + 3\mu\sigma^2 = (\text{IE}[x|Y])^3 + 3(\text{IE}[x|Y])(1-p^2)^2 \\ &= p^3 Y^3 + 3pY(1-p^2)\end{aligned}$$

$$\hat{x}_{\text{MMSE}}^3 = \text{IE}[x^3|Y] = p^3 Y^3 + 3p(1-p^2)Y$$

$$\begin{aligned}\text{MSE} &= \text{IE}[(x^3)^2] - \text{IE}[(\hat{x}_{\text{MMSE}}^3)^2] \\ &= \text{IE}[x^6] - \text{IE}[(\text{IE}[x^3|Y])^2] \\ &= \text{IE}[x^6] - p^6 Y^6 - 9p^2(1-p^2)^2 \text{IE}[Y^2] - 6p^4(1-p^2) \text{IE}[Y^4] \\ &= 15(1-p^6) - 9p^2(1-p^2)^2 - 18p^4(1-p^2)\end{aligned}$$

(b) Find the best linear MMSE estimator of x^3 given Y and compute resulting MSE.

$$\hat{x}_{\text{LMMSE}}^3 = \frac{\text{Cov}(x^3, Y)}{\text{Var}(Y)} (Y - \text{IE}[Y]) + \text{IE}[x^3]$$

$$\begin{aligned}
\text{Cov}(X^3, Y) &= \mathbb{E}[X^3 Y] \\
&= \mathbb{E}_Y[\mathbb{E}[X^3 Y | Y]] \\
&= \mathbb{E}_Y[Y \mathbb{E}[X^3 | Y]] \\
&= \mathbb{E}_Y[Y(p^3 Y^3 + 3p(1-p^2)Y)] \\
&= \mathbb{E}_Y[p^3 Y^4 + 3p(1-p^2)Y^2] \\
&= p^3 \mathbb{E}_Y[Y^4] + 3p(1-p^2) \mathbb{E}[Y^2] \\
&= 3p
\end{aligned}$$

$$\mathbb{E}[X^3] = 0$$

$$X_{\text{mean}}^{13} = 0 + \frac{3p}{1} (Y - \mathbb{E}[Y]) = 3pY$$

$$MSE = \text{Var}(X^3) - \frac{\text{Cov}^2(X^3, Y)}{\text{Var}(Y)}$$

$$\begin{aligned}
\text{Var}(X^3) &= \mathbb{E}[X^6] - (\mathbb{E}[X^3])^2 \\
&= 1(5)(3)(1) - 0 \\
&= 15
\end{aligned}$$

$$MSE = 15 - 9p^2$$

③ Screening Test

A screening test is 98% effective in detecting a certain disease when a person has the disease. However, the test yields a false positive rate of 1% of the healthy persons tested.

If 0.1% of the population has the disease, what is the probability that a person who tests positive actually has the disease.

Let

$A \rightarrow$ event test detects disease

$B \rightarrow$ event person has disease

Given

$$\Pr(A|B) = 0.98$$

$$\Pr(A^c|B) = 0.02$$

$$\Pr(B) = 0.001$$

$$\Pr(B^c) = 0.999$$

False positive occurs when test detects a disease but person does NOT have it.

$$\Pr(A|B^c) = 0.01 \rightarrow \Pr(A^c|B^c) = 0.99$$

Want

$$\Pr(B|A)$$

Via Bayes Rule

$$\Pr(B|A) = \frac{\Pr(A|B) \Pr(B)}{\Pr(A)} = \frac{\Pr(A|B) \Pr(B)}{\Pr(A|B)\Pr(B) + \Pr(A|B^c)\Pr(B^c)}$$
$$= \frac{(0.98)(0.001)}{(0.98)(0.001) + (0.01)(0.999)}$$
$$\approx 0.09$$