Exercises.

Solution to Question 1. Assume v is an eigenvector of T, $Tv = \lambda v$ for some $\lambda \in \mathbb{F}$. Because T is invertible, so we may apply T^{-1} on both sides, and

$$T^{-1}v = \lambda^{-1}v.$$

Therefore, λ^{-1} is an eigenvalue for T^{-1}. Because T is invertible, so $\lambda \neq 0$ and

$$\ker(T - \lambda I) = \ker T^{-1}(T - \lambda I) = \ker \lambda(T^{-1} - \lambda^{-1}I) = \ker(T^{-1} - \lambda^{-1}I).$$

We know $E_\lambda(T) = \ker(T - \lambda I)$ and $E_{\lambda^{-1}}(T^{-1}) = \ker(T^{-1} - \lambda^{-1}I)$, so

$$E_\lambda(T) = E_{\lambda^{-1}}(T^{-1}).$$

In particular,

$$\dim E_\lambda(T) = \dim E_{\lambda^{-1}}(T^{-1}).$$

Solution to Question 2. Let

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix}, \ B = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix}.$$

Because $\dim E_3(A) = 2$ and $\dim E_3(B) = 1$, so A is not similar to B.

Solution to Question 3.

(a)

$$\begin{aligned} R_1 &= 0.1 + 0.1 = 0.2, \\ R_2 &= 0.1 + 0.2 = 0.3, \\ R_3 &= 0.3 + 0 = 0.3. \end{aligned}$$

So the Gershgorin disks are $D(0.6, 0.2)$, $D(0.9, 0.3)$ and $D(0.7, 0.3)$.

(b) Because

$$\det(\lambda I - A) = (\lambda - 1)(\lambda - 0.7)(\lambda - 0.5),$$

so the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 0.7$ and $\lambda_3 = 0.5$.

(c) Because

$$A^2 = \begin{bmatrix} 0.4 & 0.15 & 0.15 \\ 0.21 & 0.82 & 0.33 \\ 0.39 & 0.03 & 0.52 \end{bmatrix},$$

so the proportions in each state after 2 time steps are

$$A^2 P = \begin{bmatrix} 0.4 & 0.15 & 0.15 \\ 0.21 & 0.82 & 0.33 \\ 0.39 & 0.03 & 0.52 \end{bmatrix} \begin{pmatrix} 0.3 \\ 0.3 \\ 0.4 \end{pmatrix} = \begin{pmatrix} 0.225 \\ 0.441 \\ 0.334 \end{pmatrix}.$$

Because

$$\lim_{n\to\infty} A^n = Q \lim_{n\to\infty} \Lambda^n Q^{-1} = \begin{bmatrix} 1/5 & 1/5 & 1/5 \\ 3/5 & 3/5 & 3/5 \\ 1/5 & 1/5 & 1/5 \end{bmatrix},$$

so the eventual proportions are

$$\lim_{n\to\infty} A^n P = \begin{pmatrix} 1/5 \\ 3/5 \\ 1/5 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0.3 \\ 0.3 \\ 0.4 \end{pmatrix} = \begin{pmatrix} 1/5 \\ 3/5 \\ 1/5 \end{pmatrix}.$$

Solution to Question 4.

(a) Because
$$[ST]_{\mathcal{B}} = [S]_{\mathcal{B}}[T]_{\mathcal{B}} = [T]_{\mathcal{B}}[S]_{\mathcal{B}} = [TS]_{\mathcal{B}},$$
so $ST = TS$.

(b) Because U is T-invariant, so if $u \in U$, then $T^r u \in U$ for any r. Therefore
$$v_1 + v_2 + \cdots + v_k \in U$$
implies
$$u_r := T^r(v_1 + v_2 + \cdots + v_k) = T^r v_1 + T^r v_2 + \cdots + T^r v_k = \lambda_1^r v_1 + \lambda_2^r v_2 + \cdots + \lambda_k^r v_k \in U,$$
for $r = 0, 1, \ldots, k-1$. Because
$$(u_0, \ldots, u_{k-1}) = (v_1, \ldots, v_k)A$$
and
$$A = \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \ldots & \lambda_1^{k-1} \\ 1 & \lambda_2 & \lambda_2^2 & \ldots & \lambda_2^{k-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \lambda_k & \lambda_k^2 & \ldots & \lambda_k^{k-1} \end{bmatrix}$$
is a $k \times k$ Vandermonde matrix, which is invertible when $\lambda_1, \ldots, \lambda_k$ are all distinct, so
$$\operatorname{span}\{v_1, \ldots, v_k\} = \operatorname{span}\{u_0, \ldots, u_{k-1}\} \subseteq U.$$
In particular, all v_i's are in U.

(c) Assume $u_1, \ldots, u_r$ span U. Each u_i can be written in terms of
$$u_i = v_{i1} + v_{i2} + \cdots + v_{ik},$$
where $v_{i1}, \ldots, v_{ik}$ are eigenvectors of T with distinct eigenvalue. By part (b), this means every v_{ik} is in U, so if we ran i from 1 to r, then all such v_{ik}'s span U, and there is a subset of them forms a basis $\mathcal{A}$ for U. The elements in $\mathcal{A}$ are eigenvectors of T, therefore $[T|_U]_{\mathcal{A}}$ is a diagonal matrix.

(d) Assume v is an eigenvector of S. Then
$$STv = TSv = T(\lambda v)\lambda Tv.$$
This means Tv is also an eigenvector of S with the same eigenvalue. Therefore $U_\lambda := E_\lambda(S)$ is an T-invariant subspace for each eigenvalue λ of S.
By part (c), $T|_{U_\lambda}$ is diagonalizable for each U_λ, therefore we may take a basis $\mathcal{B}_\lambda$ such that $[T|_{U_\lambda}]_{\mathcal{B}_\lambda}$ is diagonal. Note that $[S|_{U_\lambda}]_{\mathcal{B}_\lambda}$ is also diagonal, because $U_\lambda = E_\lambda(S)$.
Because $V = \oplus_{\lambda \in \Lambda} U_\lambda$, so
$$\mathcal{B} := \cup_{\lambda \in \Lambda} \mathcal{B}_\lambda$$
is a basis for V such that both $[T]_{\mathcal{B}}$ and $[S]_{\mathcal{B}}$ are diagonal.

Solution to Question 5.

(a) If $g(x) \in \langle f(x) \rangle$, then $g(x) = a(x)f(x)$ for some $a(x) \in \mathbb{F}[x]$. For any $h(x) \in \mathbb{F}[x]$, $h(x)g(x) = h(x)a(x)f(x) \in \langle f(x) \rangle$. So $\langle f(x) \rangle$ is an ideal in $\mathbb{F}[x]$.

(b) If there is a polynomial $f(x) \in I$ that is not divisible by $g(x)$, then

$$f(x) = p(x)g(x) + r(x)$$

for some $r(x) \in \mathbb{F}[x]$ such that $\deg r < \deg g$. Because both $f(x)$ and $p(x)g(x)$ are in I, so $r(x) \in I$. But I contains no polynomial of lower degree, so this is a contradiction.
Hence, every polynomial in I is divisible by $g(x)$, which implies $I \subseteq \langle g(x) \rangle$. And $g(x) \in I \implies \langle g(x) \rangle \subset I$, so $I = \langle g(x) \rangle$.

Solution to Question 6.

(a) If $f(x) \in \operatorname{ann}(A)$, then for any $h(x) \in \mathbb{F}[x]$, $h(A)f(A) = h(A) \cdot 0_{n\times n} = 0_{n\times n}$.
So $h(x)f(x) \in \operatorname{ann}(A)$ and thus $\operatorname{ann}(A)$ is an ideal.

(b) Claim: $I = \langle x^2 \rangle$.
Reason: We may check that $x^2 \in \operatorname{ann}(A)$. Because A is not diagonalizable, so there is no degree 1 polynomial $p(x)$ such that $p(A) = 0$. By Problem 5 part (b), this shows that $I = \langle x^2 \rangle$.