

①  $X, Y$  uniformly distributed in  $[0,1]$  and  $[x,1]$  respectively.

- (a) MMSE estimator of  $Y$  using a constant and resulting MSE.
- (b) Linear MMSE estimator of  $X$  given  $Y$  and resulting MSE.
- (c) MMSE estimator of  $X$  given  $Y$ .

$$(a) X \sim U[0,1], Y \sim U[x,1]$$

$$f_X(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & \text{o/w} \end{cases} \quad f_{Y|X=x}(y|x) = \begin{cases} \frac{1}{1-x}, & 0 \leq y \leq 1 \\ 0, & \text{o/w} \end{cases}$$

MMSE estimator of  $y$  using a constant will be  $\mathbb{E}[Y]$   
and the resulting MSE will be  $\text{Var}(Y)$

$$\begin{aligned} \mathbb{E}[Y] &= \mathbb{E}_x [\mathbb{E}_{Y|X}[Y|X]] \\ &\quad \text{f}_{Y|X} = \frac{1}{1-x} \quad 0 \leq x \leq y \leq 1 \\ \mathbb{E}_{Y|X}[Y|X] &= \int_y y f_{Y|X}(y|x) dy \\ &= \int_{y=x}^1 y \left(\frac{1}{1-x}\right) dy \\ &= \frac{\frac{1}{2} - \frac{x^2}{2}}{1-x} = \frac{\frac{1}{2}}{1-x} \frac{1-x^2}{1-x} = \frac{1+x}{2} \end{aligned}$$

$$\mathbb{E}_x[Y] = \int_x \mathbb{E}_{Y|X}[Y|X] f_X(x) dx = \int_{x=0}^1 \left(\frac{1+x}{2}\right) dx = \left[\frac{1}{2} + \frac{x^2}{4}\right]_0^1 = \frac{3}{4}$$

$$\text{So } \hat{Y}_{\text{MSE}} = 5/12$$

The resulting MSE is  $\text{Var}(Y)$

$$\text{Var}(Y) = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2$$

$$\begin{aligned} \mathbb{E}[Y^2] &= \mathbb{E}_x[\mathbb{E}[Y^2|x]] \\ &= \int_{x=0}^1 \left( \int_{y=x}^1 y^2 \left(\frac{1}{1-x}\right) dy \right) f_x(x) dx \\ &= \int_{x=0}^1 \frac{1}{3} \left(\frac{1-x^3}{1-x}\right) dx \\ &= \int_{x=0}^1 \frac{1}{3} (1+x+x^2) dx \\ &= \left. \frac{1}{3}x + \frac{x^2}{6} + \frac{x^3}{9} \right|_0^1 = \frac{1}{3} + \frac{1}{6} + \frac{1}{9} = \frac{11}{18} \end{aligned}$$

thus

$$\text{Var}(Y) = \frac{11}{18} - \left(\frac{3}{4}\right)^2 = \frac{7}{144}$$

(b) LMSE Estimator of  $X$  given  $Y$

$$\begin{aligned} \hat{X}_{\text{LMSE}} &= a^* Y + b^* \\ &= \frac{\text{Cov}(X,Y)}{\text{Var}(Y)} (Y - \mathbb{E}[Y]) + \mathbb{E}[X] \end{aligned}$$

$$\mathbb{E}[X] = \int_{x=0}^1 x f_x(x) dx = \frac{1}{2}$$

$$\mathbb{E}[Y] = 3/4, \quad \text{Var}(Y) = 7/144$$

$$\text{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$$

$$f_{X,Y}(x,y) = f_{Y|X}(y|x)f_X(x)$$

$$= \frac{1}{1-x} \quad 0 \leq x \leq y \leq 1$$

$$\mathbb{E}[XY] = \iint_{x,y} xy f_{X,Y}(x,y) dy dx$$

$$= \int_{x=0}^1 \int_{y=x}^1 \frac{x}{1-x} y dy dx$$

$$= \frac{1}{2} \int_{x=0}^1 \frac{1-x^2}{1-x} x dx = \frac{1}{2} \int_{x=0}^1 x + x^2 dx$$

$$= \frac{1}{2} \left( \frac{5}{2} + \frac{1}{3} \right) = \frac{5}{12}$$

$$\text{Cov}(X, Y) = \frac{5}{12} - \frac{3}{4} \left( \frac{1}{2} \right) = \frac{5}{12} - \frac{3}{8} = \frac{1}{24}$$

$$\hat{X}_{\text{Linnse}} = \frac{\text{Cov}(X, Y)}{\text{Var}(Y)} (Y - \mathbb{E}[Y]) + \mathbb{E}[X]$$

$$\frac{1}{24} \cdot \frac{144}{7} = \frac{6}{7} = \frac{12}{144} (Y - \frac{3}{4}) + \frac{1}{2}$$

$$= \frac{6}{7} (Y - \frac{3}{4}) + \frac{1}{2}$$

$$= \frac{6}{7} Y - \frac{18}{28} + \frac{14}{28} = \frac{6}{7} Y - \frac{1}{4}$$

Resulting MSE is

$$\text{Var}(X) - \frac{(\text{Cov}(X, Y))^2}{\text{Var}(Y)}$$

$$\text{Var}(X) = \frac{1}{12} = \frac{1}{12} - \frac{(0.12)^2}{1/144}$$

(c) MLE estimator of  $X$  given  $Y$ .

$$\hat{X}_{\text{MLE}} = \mathbb{E}[X|Y] = \int_{-\infty}^{\infty} f_{X|Y}(x|y) x \, dx$$

$$f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x) f_X(x)}{f_Y(y)}$$

$$f_Y(y) = \int_{x=0}^y f_{X|Y}(x|y) \, dx = \int_{x=0}^1 \frac{1}{1-x} \, dx = -\ln(1-x) \Big|_0^y = -\ln(1-y)$$

$$f_{X|Y}(x|y) = \frac{1}{1-x} \cdot \frac{-1}{\ln(1-y)} \quad 0 \leq x \leq y \leq 1$$

$$\mathbb{E}[X|Y] = -\frac{1}{\ln(1-y)} \int_{x=0}^y \frac{x}{1-x} \, dx$$

$$\begin{aligned} u &= 1-x \\ du &= -dx \\ x &= 1-u \end{aligned}$$

$$= +\frac{1}{\ln(1-y)} \int_{u=1}^{u=\ln y} \frac{1-u}{u} \, du = \frac{1}{\ln(1-y)} \left( \ln|u| - u \Big|_1^{\ln y} \right)$$

$$= \frac{\ln|1-y|}{\ln(1-y)} - \frac{1-y}{\ln(1-y)} - \left( 0 - \frac{1}{\ln(1-y)} \right) = 1 + \frac{y}{\ln(1-y)}$$

② Let  $X, Y, Z$  be jointly Gaussian random variables with

$$\mathbb{E}[X] = \mathbb{E}[Y] = \mathbb{E}[Z] = 0$$

$$\text{Var}(X) = \text{Var}(Y) = \text{Var}(Z) = 2$$

Let  $P_{XY}, P_{XZ}, P_{YZ}$  denote the correlation coefficients.

(a) MMSE estimator of  $X^2$  given  $Z$

(b) MMSE estimator of  $X^4$  given  $Z$

(a)

$$\mathbb{E}[X^2|Z] = \text{Var}(X|Z) + (\mathbb{E}[X|Z])^2$$

Since  $X, Z$  jointly Gaussian,

$$\mathbb{E}[X|Z] = \hat{X}_{\text{mmse}} = \frac{\text{Cov}(X, Z)}{\text{Var}(Z)} (Z - \mathbb{E}[Z]) + \mathbb{E}[Y]$$

$$= \frac{\rho_{XZ} \sigma_X \sigma_Z}{\sigma_Z^2} (Z)$$

$$= Z \rho_{XZ} f(Z)$$

$$\text{Var}(X|Z) = \text{Var}(X)(1 - \rho_{XZ}^2)$$

$$= 2 - 2\rho_{XZ}^2 \quad \text{constant}$$

$$\mathbb{E}[X^2|Z] = 2 - 2\rho_{XZ}^2 + Z^2 \rho_{XZ}^2$$

$$= 2 + \rho_{XZ}^2 (Z^2 - 2)$$

$$(b) \mathbb{E}[XY|Z]$$

$$2XY = (X+Y)^2 - X^2 - Y^2$$

$$\mathbb{E}[(X+Y)^2 - X^2 - Y^2 | Z] = \mathbb{E}[(X+Y)^2 | Z] - \mathbb{E}[X^2 | Z] - \mathbb{E}[Y^2 | Z]$$

$$\mathbb{E}[X^2 | Z] = 2 + \rho_{xz}^2 (Z^2 - 2)$$

Similarly  $\mathbb{E}[Y^2 | Z] = 2 + \rho_{yz}^2 (Z^2 - 2)$

Realize  $X+Y$  is gaussian since  $X, Y$  jointly Gaussian!

$$\mathbb{E}[(X+Y)^2 | Z] = \text{Var}_r((X+Y) | Z) + (\mathbb{E}[X+Y | Z])^2$$

$$\begin{aligned}\text{Cov}(X+Y, Z) &= \text{Cov}(X, Z) + \text{Cov}(Y, Z) \\ &= \sigma_X \sigma_Z \rho_{xz} + \sigma_Y \sigma_Z \rho_{yz} \\ &= 2\rho_{xz} + 2\rho_{yz}\end{aligned}$$

$$\begin{aligned}\mathbb{E}[X+Y | Z] &= \mathbb{E}[X | Z] + \mathbb{E}[Y | Z] \\ &= Z(\rho_{xz} + \rho_{yz})\end{aligned}$$

$$\begin{aligned}\text{Var}(X+Y) &= \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y) \\ &= 2 + 2 + 2\sigma_X \sigma_Y \rho_{xy} \\ &= 4 + 4\rho_{xy}\end{aligned}$$

$$\begin{aligned}\text{Var}(X+Y|Z) &= \text{Var}(X+Y) - \frac{\text{Cov}^2(X+Y, Z)}{\text{Var}(Z)} \\ &= 4 + 4\rho_{XY} - \frac{(2\rho_{XZ} + 2\rho_{YZ})^2}{2} \\ &= 4 + 4\rho_{XY} - 2(\rho_{XZ} + \rho_{YZ})^2\end{aligned}$$

Thus

$$\begin{aligned}\mathbb{E}[(X+Y)^2|Z] &= 4 + 4\rho_{XY} - 2(\rho_{XZ} + \rho_{YZ})^2 + Z^2(\rho_{XZ} + \rho_{YZ})^2 \\ &= 4 + 4\rho_{XY} + (Z^2 - 2)(\rho_{XZ} + \rho_{YZ})^2\end{aligned}$$

**FINALLY,**

$$\begin{aligned}2\mathbb{E}[XY|Z] &= 4 + 4\rho_{XY} + (Z^2 - 2)(\rho_{XZ} + \rho_{YZ})^2 - (2 + \rho_{XZ}^2(Z^2 - 2)) - (2 + \rho_{YZ}^2(Z^2 - 2)) \\ &= 4\rho_{XY} + (Z^2 - 2)[-\rho_{YZ}^2 - \rho_{XZ}^2 + (\rho_{XZ} + \rho_{YZ})^2]\end{aligned}$$

AND

$$\mathbb{E}[XY|Z] = 2\rho_{XY} + (Z^2 - 2)\rho_{XZ}\rho_{YZ}$$

HOLY CRAP...

③ A total of 11 people, including you, are invited to a party. The times at which people (including you) arrive at the party are independent and uniformly distributed in (0,1).

(a) Find expected amount of people who arrive before you.

(b) Find the variance of the number of people who arrive before you.

(a)

10 people left not including me.

Let  $T$  be the time I arrive. Given  $T = t$

Each person arriving before me is a Bernoulli event, with probability  $t$ .

Have 10 independent Bernoulli events  $\Rightarrow$  Binomial distribution!

Let

$$N \sim B(10, T)$$

Thus

$$\mathbb{E}[N | T = t] = 10t$$

$$\mathbb{E}[N] = \mathbb{E}_T[\mathbb{E}_{N|T}[N|T]]$$

$$= \mathbb{E}_T[10T]$$

$$= 10 \mathbb{E}_T[T] = 5$$

$$f_T(t) = \begin{cases} 1 & 0 \leq t \leq 1 \\ 0 & \text{o/w} \end{cases}, \quad \mathbb{E}[T] = \frac{1}{2}$$

$$\mathbb{E}[T^2] = \int_0^1 t^2 dt = \frac{1}{3}$$

(b)  $\text{Var}(N)$ ?

$$\text{Var}(N|T=t) = nt(1-t)$$

$$\text{Var}(N|T) = 10T(1-T)$$

$$\begin{aligned}\text{Var}(N) &= \mathbb{E}[\text{Var}(N|T)] + \text{Var}(\mathbb{E}[N|T]) \\ &= \mathbb{E}[10T(1-T)] + \text{Var}(10T) \\ &= 10\mathbb{E}[T] - 10\mathbb{E}[T^2] + 100\text{Var}(T) \\ &= 10(\frac{1}{3}) - 10(\frac{1}{3}) + 100(\frac{1}{12}) \\ &= 5 - \frac{10}{3} + \frac{100}{12} \\ &= \frac{5}{3} + \frac{25}{3} = \frac{30}{3} = 10\end{aligned}$$

$$\text{Var}(N) = 10$$