To find a suitable definition for e^z when $z = x + iy$, we want to preserve the basic identities satisfied by the real function e^x.

We therefore postulate that the multiplicative property persists.

$$e^{z_1} e^{z_2} = e^{z_1 + z_2} \qquad (1)$$

This enables the decomposition

$$e^z = e^{x+iy} = e^x e^{iy} \qquad (2)$$

← How do we deal with e^{iy}?

Next we propose that the differentiation law be preserved:

$$\frac{d}{dz}(e^z) = e^z \qquad (3)$$

Consider

$$\frac{d}{d(iy)} e^{(iy)} = e^{iy}$$

Or, equivalently, by the chain rule

$$\frac{d}{dy} e^{iy} = i e^{iy} \qquad (4)$$

If we differentiate again,

$$\frac{d^2}{dy^2} e^{iy} = \frac{d}{dy} i e^{iy} = i^2 e^{iy}$$

iy

$= -e$

In other words, the function $g(y) := e^{iy}$ satisfies the differential equation

$$\frac{d^2}{dy^2} g = -g \qquad (5)$$

Now observe that any function of the form

$$A\cos(y) + B\sin(y) \qquad (A, B \to \text{constants})$$

From the theory of differential equations it is known that every solution of Eq. 5 must have this form.

Hence we can write

$$g(y) = A\cos(y) + B\sin(y) \qquad (6)$$

To evaluate A and B we use the conditions that

$$g(0) = e^{i0} = e^0 = 1 = A\cos(0) + B\sin(0)$$

and

$$\frac{d}{dy} g(0) = i g(0) = i = -A\sin(0) + B\cos(0)$$

Thus $A = 1$ and $B = i$, leading us to the identitification

$$e^{iy} = \cos(y) + i\sin(y) \qquad (7)$$

Eq. 7 is known as Euler's equation.

Definition 5: If $z = x + iy$, then e^z is defined to be the complex number

$$e^z := e^x(\cos y + i\sin y) \qquad (8)$$

It can then be directly verified that e^z, as defined above, satisfied the associated division rule with multiplicative identity (1).

$$\frac{e^{z_1}}{e^{z_2}} = e^{z_1 - z_2} \qquad (9)$$

Example 1: Show that Euler's equation is consistent with the usual Taylor series expansions

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots,$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots,$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots,$$

For now we ignore the questions of convergence, etc, and simply substitute $x = iy$ into the exponential series.

$$e^{iy} = 1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \frac{(iy)^5}{5!} + \cdots$$

$$= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \cdots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \cdots\right)$$

$$= \cos(y) + i\sin(y)$$ WOAH!

Euler's Eqn 7 allows us to write the polar form of a complex number as

$$z = r\,\mathrm{cis}(\theta) = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

Thus we can drop the stupid 'cis' artiface and use

$$z = re^{i\theta} = |z|\,e^{i\arg(z)} \qquad (10)$$

In particular, notice the following identities

$$e^{i0} = e^{2\pi i} = e^{-2\pi i} = e^{4\pi i} = e^{-4\pi i} = \cdots = 1$$

$$e^{(\pi/2)i} = i, \quad e^{(-\pi/2)i} = -i, \quad e^{\pi i} = -1$$ ← Amazing

Observe that $|e^{i \arg z}| = 1$ and that Euler's equation leads to the following representations of the customary trigonometric functions:

$$\cos(\theta) = \mathrm{Re}(e^{i\theta}) = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad (11)$$

$$\sin(\theta) = \mathrm{Im}(e^{i\theta}) = \frac{e^{i\theta} - e^{-i\theta}}{2i} \qquad (12)$$

The rules derived in $\boxed{1.3}$ for multiplying/dividing complex numbers in polar form now find very natural expressions:

$$z_1 z_2 = (r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = (r_1 r_2) e^{i(\theta_1 + \theta_2)} \qquad (13)$$

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \left(\frac{r_1}{r_2}\right) e^{i(\theta_1 - \theta_2)} \qquad (14)$$

and complex conjugation of $z = re^{i\theta}$ is accomplished by changing the sign of i in the exponent:

$$\bar{z} = re^{-i\theta} \qquad (15)$$

Example 2: Compute (a) $\frac{1+i}{\sqrt{3}-i}$ (b) $(1+i)^{24}$

(a) $1+i = \sqrt{2}e^{i\pi/4}$ $\sqrt{3}-i = 2e^{-i\pi/6}$

$$\frac{1+i}{\sqrt{3}-i} = \frac{\sqrt{2}}{2}e^{i\frac{5\pi}{12}}$$

(b)

$$(1+i)^{24} = \left(\sqrt{2}e^{i\pi/4}\right)^{24} = \left(\sqrt{2}\right)^{24} e^{i\frac{24\pi}{4}}$$

$$= 2^{12}e^{i6\pi} = 2^{12}$$

Example 3:

Prove De Moivre's formula:

(16)

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta), \quad n = 1, 2, 3, \ldots$$

$$(e^{i\theta})^n = \underbrace{e^{i\theta}e^{i\theta}\cdots e^{i\theta}}_{n \text{ times}} = e^{i\theta+i\theta+\cdots+i\theta} = e^{in\theta}$$

$$e^{in\theta} = \cos(n\theta) + i\sin(n\theta)$$

Example 4: Express $\cos(3\theta)$ in terms of $\cos(\theta)$, $\sin(\theta)$

$$\cos(3\theta) = \mathrm{Re}\left[\cos(3\theta) + i\sin(3\theta)\right] = \mathrm{Re}(\cos\theta + i\sin\theta)^3 \qquad (17)$$

Thus,

$$\cos(3\theta) = \mathrm{Re}\left[\cos^3\theta + 3\cos^2\theta(i\sin\theta) + 3\cos(\theta)(-\sin^2\theta) - i\sin^3(\theta)\right] \qquad (18)$$

$$= \cos^3\theta - 3\cos\theta\sin^2\theta$$

Example 5: Compute the integral

$$\int_0^{2\pi} \cos^4\theta \, d\theta$$

Note that

$$\cos^4\theta = \left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right)^4 = \frac{1}{2^4}\left(e^{i\theta} + e^{-i\theta}\right)^4$$

and expanding via binomial formula gives

$$\cos^4(\theta) = \frac{1}{16}\left(e^{4i\theta} + 4e^{i3\theta}e^{-i\theta} + 6e^{i2\theta}e^{-2i\theta} + 4e^{i\theta}e^{-3i\theta} + e^{-4i\theta}\right)$$

$$= \frac{1}{16}\left(e^{4i\theta} + 4e^{i2\theta} + 6 + 4e^{-2i\theta} + e^{-4i\theta}\right)$$

$$= \frac{1}{16}\left(6 + 8\cos(2\theta) + 2\cos(4\theta)\right)$$

Thus,

$$\int_0^{2\pi} \cos^4\theta \, d\theta = \int_0^{2\pi} \frac{1}{16}\left(6 + 8\cos(2\theta) + 2\cos(4\theta)\right) d\theta$$

$$= \frac{1}{16}\left[6\theta + 4\sin(2\theta) + \frac{1}{2}\sin(4\theta) \Big|_0^{2\pi} \right]$$

$$\boxed{= \frac{3}{4}\pi}$$