Recall

Have rv X w/ CDF $F_X(x) \triangleq \Pr[X \leq x]$

Discrete RV

$$p_X(x) = \Pr(X = x)$$

Continuous RV

$$f_X(x) = \frac{F_X(x+\Delta) - F_X(x)}{(x+\Delta) - x}$$

$$\lim_{\Delta \to 0} = \frac{F_X(x+\Delta) - F_X(x)}{\Delta}$$

$$= \frac{d}{dx} F_X(x)$$

Have rv X, Y w/ CDF $F_{X,Y}(x,y) \triangleq \Pr(\{X \leq x\} \cap \{Y \leq y\})$

Discrete RV

$$p_{X,Y}(x,y) = \Pr(X = x \cap Y = y)$$

Continuous RV

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \, \partial y}$$

Gives us the definition of conditional pmf, pdf

$$p_{X|Y}(x|y) = \frac{\Pr(X = x \cap Y = y)}{\Pr(Y = y)} = \frac{p_{X,Y}(x,y)}{p_Y(y)}$$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

Conditional Expectation

$$\mathbb{E}[X|Y=y] = \int_{-\infty}^{+\infty} x \, f_{X|Y}(x|y) \, dx$$

$\mathbb{E}[X|Y]$ is a random variable which takes on value $\mathbb{E}[X|Y=y]$ w/ density $f_Y(y)$.

Since $\mathbb{E}_{X|Y}[X|Y]$ is r.v. can take its expectation.

$$\mathbb{E}_Y\left[\mathbb{E}_{X|Y}[X|Y]\right] = \mathbb{E}[X]$$

$$\frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$\int_{-\infty}^{+\infty} y \int_{-\infty}^{+\infty} x f_{X|Y}(x|y)\, dx\, dy$$

$$\int_{-\infty}^{+\infty} y f_{X,Y}(y)\, dy = f_X(x)$$

$$\int_{-\infty}^{+\infty} x f_X(x)\, dx = \mathbb{E}[X]$$

Example: Coins

Coin A - fair

Coin B - two heads

Random Experiment

① First flip A

(i) If H, flip A again

(ii) If T, flip B

Define X: outcome of first flip i.e $\begin{cases} 1 & H \\ 0 & T \end{cases}$

Y: outcome of second flip

Joint PMF:

$p_{X,Y}(1,1) = Pr(1,1) = Pr[X=1]\, Pr[Y=1 \mid X=1] = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

$p_{X,Y}(1,0) = Pr(1,0) = Pr[X=1]\, Pr[Y=0 \mid X=1] = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

$p_{X,Y}(0,1) = Pr(0,1) = Pr[X=0]\, Pr[Y=1 \mid X=0] = \frac{1}{2} \cdot 1 = \frac{1}{2}$

$p_{X,Y}(0,0) = Pr(0,0) = Pr[X=0]\, Pr[Y=0 \mid X=0] = \frac{1}{2} \cdot 0 = 0$

Marginal PMF:

$$p_X(1) = 1/2$$
$$p_X(0) = 1/2$$

OR

By Def

$x=1$: $\sum_y p_{X,Y}(x,y) = p_{X,Y}(1,0) + p_{X,Y}(1,1)$
$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

$x=0$: $\sum_y p_{X,Y}(x,y) = p_{X,Y}(0,0) + p_{X,Y}(0,1)$
$= 0 + \frac{1}{2} = 1/2$

$$p_Y(1) = 3/4 = p_{X,Y}(0,1) + p_{X,Y}(1,1) = \sum_x p_{X,Y}(x,y) \text{ for } y=1$$
$$p_Y(0) = 1/4 = p_{X,Y}(0,0) + p_{X,Y}(0,1) = \sum_x p_{X,Y}(x,y) \text{ for } y=0$$

Conditional PMF $\quad p_{Y|X}(y|x)$

$$p_{Y|X}(y=1|X=1) = \frac{1}{2} = \frac{p_{X,Y}(1,1)}{p_X(1)}$$

$$p_{Y|X}(y=0|X=1) = 1 - p_{Y|X}(y=1|X=1) = 1-\frac{1}{2} = \frac{1}{2} = \frac{p_{X,Y}(1,0)}{p_X(1)}$$

$$p_{Y|X}(y=1|X=0) = 1 = \frac{p_{X,Y}(0,1)}{p_X(0)}$$
$$p_{Y|X}(y=0|X=0) = 1-1 = 0 = \frac{p_{X,Y}(0,0)}{p_X(0)}$$

Conditional Expectation

$$\mathbb{E}_{Y|X}[Y|X=1] = 1\cdot\frac{1}{2} + 0\cdot\frac{1}{2} = 1\cdot \Pr[Y=1|X=1] + 0\cdot \Pr[Y=0|X=1] = \frac{1}{2}$$

$$\mathbb{E}_{Y|X}[Y|X=0] = 1\cdot \Pr[Y=1|X=0] + 1\cdot \Pr[Y=0|X=0] = 1$$

$$\mathbb{E}_{Y|X}[Y|X] = X\cdot\frac{1}{2} + (1-X)\cdot 1 = \begin{cases} \frac{1}{2} & \text{w.p. } \frac{1}{2} \\ 1 & \text{w.p. } \frac{1}{2} \end{cases}$$
$$= 1 - \frac{X}{2}$$

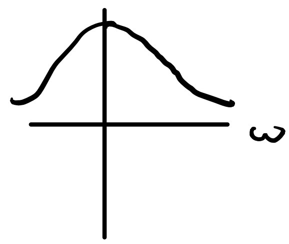
$$\mathbb{E}_X[\mathbb{E}_{Y|X}[Y|X]] = \tfrac{1}{2}\cdot\tfrac{1}{2} + 1\cdot\tfrac{1}{2} = \tfrac{3}{4} = \mathbb{E}[Y]\,!$$

MY WAY

$$\hookrightarrow = 1 - \tfrac{1}{2}\mathbb{E}[X] = 1 - \tfrac{1}{4} = 3/4 = \mathbb{E}[Y]\,!$$

Example - Gaussian Noise

$Y = X + W$ ← Gaussian Noise $f_W(\omega) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{\omega^2}{2\sigma^2}}$



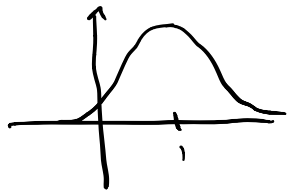
↑ signal in $\{+1, -1\}$

$\Pr[X = 1] = p$

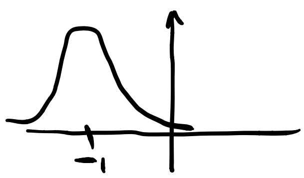
$\Pr[X = 0] = 1 - p$

i.e $p_X(x) = \begin{cases} p & , x = 1 \\ 1-p & , x = 0 \end{cases}$

$f_{Y|X=1}(y|x=1) = 1 + W$ (just shifts the mean!)



$f_{Y|X=0}(y|x=-1) = -1 + W$



$$f_Y(y) = \Pr(X=1)\, f_{Y|X}(y|x=1) + \Pr(X=-1)\, f_{Y|X}(y|x=-1)$$

$$\mathbb{E}[Y|X] = \begin{cases} 1 & \text{w.p } p \\ -1 & \text{w.p } 1-p \end{cases}$$

Want $\max\limits_{\hat{x} \in \{+1,-1\}} \Pr[X = \hat{x} \mid Y = y]$

$$\Pr[X = 1 \mid Y = y] \overset{?}{\gtrless} \Pr[X = -1 \mid Y = y]$$

$$\Pr[X = 1 \mid Y = y] \quad \text{vs.} \quad \Pr[X = -1 \mid Y = y]$$

$$\frac{\Pr[X=1]\, f_{Y|X}(y \mid x=1)}{f_Y(y)} \qquad \frac{\Pr[X=-1]\, f_{Y|X}(y \mid x=-1)}{f_Y(y)}$$

$$p \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(y-1)^2}{2\sigma^2}} \qquad (1-p)\frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(y+1)^2}{2\sigma^2}}$$

Derive This

$$y \underset{\hat{x}=-1}{\overset{\hat{x}=1}{\gtrless}} \tau^* = \frac{\sigma^2}{2}\log\left(\frac{p}{1-p}\right)$$

Correlation

Correlation between X and Y

$$\mathbb{E}[XY] = \int_{y=-\infty}^{y=+\infty}\int_{x=-\infty}^{x=+\infty} xy\, f_{X,Y}(x,y)\, dx\, dy$$

Covariance between X and Y

$$\mathrm{Cov}(X,Y) = \mathbb{E}\left[(X - \mathbb{E}[X])(Y - \mathbb{E}[X])\right]$$

$$= \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y]$$