ECE 3100 **HOMEWORK ASSIGNMENT VII** **Spring 2019**

1. Frodo tosses a coin repeatedly and independently. The probability that the coin comes up heads on any toss is p. Every time Frodo tosses a head, Sam pays him \$1. Let X be the total amount of money Frodo has received after n tosses.

(a) Find $\mathbb{E}(X)$.
(b) Find $\mathrm{Var}(X)$.

2. You roll a pair of fair six-sided dice. Let X be the total of the two numbers facing up. After rolling the dice, you flip a fair coin repeatedly and independently until X heads come up. Let Y be the index of the flip after which you stop.

(a) Find $\mathbb{E}(Y \mid X = x)$ for each possible value x of X.
(b) Find a formula for $\mathbb{E}(Y)$.

3. For the random variables X and Y defined in Problem 3 on Homework VI, find

(a) the conditional pmf $p_{X|Y}(x \mid y)$ for $y = 1$ and $y = 2$ and all values of x.
(b) $\mathbb{E}(X \mid Y = y)$ for $y = 1$ and $y = 2$.
(c) $\mathbb{E}(X)$ from your answer to (b) — note that you solved already for $\mathbb{E}(X)$ on Homework VI.

4. A discrete random variable takes on seven possible values $x_1, x_2, \ldots, x_7$ and has pmf

$$p_X(x) = \begin{cases} 1/4 & \text{when } x \in \{x_1, x_2\} \\ 1/8 & \text{when } x \in \{x_3, x_4, x_5\} \\ 1/16 & \text{when } x \in \{x_6, x_7\}\ . \end{cases}$$

(a) Consider a naive Yes/No questioning scheme where you order the outcomes from most to least probable and ask in order of decreasing probability
  - "Is $X = x_1$?"
  - "Is $X = x_2$?"
  - ...
  - "Is $X = x_6$?"

  Such a scheme always allows you to determine the value of X. Define the discrete random variable L as the number of quesitons you need to ask. L depends on X since, for example, you'll need to ask only one question if $X = x_1$ but three questions when $X = x_3$. Find the pmf of L and also $\mathbb{E}(L)$.

(b) Find the entropy of the random variable X using the defining formula

$$H(X) = -\sum_{k=1}^{7} p_X(x_k) \log_2 (p_X(x_k))\ .$$

  Recall that $H(X)$ is a lower bound for $\mathbb{E}(L)$ in any successful Yes/No questioning scheme for determining the value of X.

(c) Devise a Yes/No questioning scheme for which $\mathbb{E}(L) = H(X)$. Describe the rationale behind your scheme and display it as a binary tree.

5. Let X be the random variable you get by rolling three fair six-sided dice simultaneously and summing up the numbers facing up.

(a) Find the entropy $H(X)$.

(b) Devise a Yes/No questioning scheme, exhibited as a binary tree, for determining the value of X using the Huffman approach we discussed in class. Compare $\mathbb{E}(L)$ for your scheme to $H(X)$. Recall that in class nobody managed to detrmine X with fewer than five questions.

Rami Pellumbi

HW7 ECE3100

① Frodo tosses a coin repeatedly & independently.

$\mathbb{P}(\{\text{coin comes up heads on any toss}\}) = p$

Every time Frodo tosses a head, Sam pays \$1.

Let

X = total amount of money after n tosses.

(a) $\mathbb{E}(X)$? Let's think.

$$X = \begin{cases} \$1 & \text{if } n=1 \text{ AND tosses head, no tails} \\ \$1 & \text{if } n=2 \text{ AND tosses one head, one tail} \\ \vdots \\ \$1 & \text{if } n=n \text{ AND tosses one head, } (n-1) \text{ tails} \end{cases}$$

$$X = \begin{cases} \$2 & \text{if } n=2 \text{ AND tosses two heads, no tails} \\ \$2 & \text{if } n=3 \text{ AND tosses two heads, one tail} \\ \vdots \\ \$2 & \text{if } n=n \text{ AND tosses two heads, } (n-2) \text{ tails} \end{cases}$$

$$X = \begin{cases} \$3 & \text{if } n=3 \text{ AND tosses 3 heads, no tails} \\ \$3 & \text{if } n=4 \text{ AND tosses three heads, one tail} \\ \vdots \\ \$3 & \text{if } n=n \text{ AND tosses three heads, } (n-3) \text{ tails} \end{cases}$$

total amount

Note: multiple ways to toss those heads

$X = \$M$ if you toss M heads in n tosses $\Rightarrow (n-M)$ tails

$$P_X(x) = \begin{cases} p^M (1-p)^{n-M}, & x = M \\ p^{M-1}(1-p)^{n-(M-1)}, & x = M-1 \\ \vdots \\ p^0 (1-p)^{n-0}, & x = 0 \end{cases}$$

in my notation, $M = n$

multiplied by $\binom{n}{l}$, l is how many ways to flip H

i.e

$$P_X(x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad 0 \le x \le M, \quad x \in \mathbb{Z}$$

So,

$$\mathbb{E}(X) = \sum_{x \in X} x p_X(x) = \sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x}$$

Define

$$Z_x = \begin{cases} 1, & x \text{ is } H \\ 0, & x \text{ is } T \end{cases}$$

Note: $X = Z_1 + \cdots + Z_n$

$$\mathbb{E}(Z_x) = \mathbb{E}(Z_1) + \cdots + \mathbb{E}(Z_n)$$
$$= p + \cdots + p$$
$$= np = \mathbb{E}(X)$$

(b) $\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$

$\mathrm{Var}(Z_x) = p(1-p) \quad \forall x$

$\mathrm{Var}(X) = np(1-p)$ by independence

(2) Roll two six-sided die.

Let

$$X = \text{sum of two rolls.}$$

Flip a fair coin repeatedly + independently until X heads come up.

Let

$$Y = \text{index of the flip after which you stop}$$

(a) $\mathbb{E}(Y \mid X=x)\ \forall\, x \in \mathcal{X}$?

$\mathcal{X} = \{x \in \mathbb{N} : 2 \le x \le 12\}$

$\mathcal{Y} = \{y \in \mathbb{N} : 2 \le y \le n\}$; n is the last flip needed such that you flipped X heads total. i.e $n \geq 12$

Let the random variable Z_1 be index of first head. For $i \geq 2$, let Z_i be the number of flips it takes to see the i^{th} head after seeing $(i-1)^{th}$ head.

Thus, Z_i is a Geometric rv with parameter p.

$$\mathbb{E}(Y \mid X=x) = \mathbb{E}\left(\sum_{i=1}^{x} Z_i\right) = \sum_{i=1}^{x} \mathbb{E}(Z_i) = \frac{x}{p}$$

(b) $\mathbb{E}(Y)$?

$$\mathbb{E}(Y) = \sum_x \mathbb{E}(Y \mid X)\, \mathbb{P}(X=x)$$

fancy way to do pmf of rv describing sum of 2 rolls

$$P_X(x) = \begin{cases} \dfrac{6-|x-7|}{36}, & 2 \le x \le 12 \\ 0, & \text{else} \end{cases}$$

Thus

$$\mathbb{E}(Y) = \sum_{x=2}^{12} \frac{x}{p} \cdot \frac{6-|x-7|}{36} = \frac{252}{36p}$$

③ X, Y have joint pmf

$$P_{X,Y} = \begin{cases} cxy^2, & x \in \{1,3,4\},\ y \in \{1,2\} \\ 0, & \text{otherwise} \end{cases}$$

(a) $P_{X|Y}(x|y)$ for $y=1$, $y=2$ and all values of x?

$$P_{X|Y}(1|1) = \frac{1/40}{8/40} = 1/8$$

$P_{X|Y}(1|2) = 1/8$

$P_{X|Y}(3|1) = 3/8$

$P_{X|Y}(3|2) = 3/8$

$P_{X|Y}(4|1) = P_{X|Y}(4|2) = 4/8$

$$P_{X|Y}(x|y) = \begin{cases} \frac{1}{8}, & x=1, \forall y \\ 3/8, & x=3, \forall y \\ \frac{1}{2}, & x=4, \forall y \end{cases}$$

(b) $\mathbb{E}(X|Y=y)$ for $y=1$, $y=2$?

$$\mathbb{E}(X|Y=1) = \sum_{x \in X} \mathbb{P}(X=x|Y=1)\,x$$

$$= 1\left(\tfrac{1}{8}\right) + 3\left(\tfrac{3}{8}\right) + 4\left(\tfrac{4}{8}\right) = 13/4$$

$$\mathbb{E}(X|Y=2) = \sum_{x \in X} \mathbb{P}(X=x|Y=2)\,x = 1(1/8) + 3(3/8) + 4(4/8)$$
$$= 13/4$$

(c) $\mathbb{E}(X)$ from answer to (b)?

$$\mathbb{E}(X) = \mathbb{E}(X \mid Y=1)\,\mathbb{P}(Y=1) + \mathbb{E}(X \mid Y=2)\,\mathbb{P}(Y=2)$$

$$= \frac{13}{4}\left(\frac{1}{5}\right) + \frac{13}{4}\left(\frac{4}{5}\right)$$

$$\boxed{= \frac{13}{4}}$$

(4)

$$p_X(x) = \begin{cases} 1/4, & x \in \{x_1, x_2\} \\ 1/8, & x \in \{x_3, x_4, x_5\} \\ 1/16, & x \in \{x_6, x_7\} \end{cases}$$

$$p_L(l) = \begin{cases} 1/4, & l = 1, 2 \\ 1/8, & l = 3, 4, 5 \\ 1/16, & l = 6, 7 \end{cases}$$

$$\mathbb{E}(L) = \sum_{l \in L} l\, p_L(l) = (1+2)\frac{1}{4} + (3+4+5)\frac{1}{8} + (6+7)\frac{1}{16}$$

$$= 48/16 = 3$$

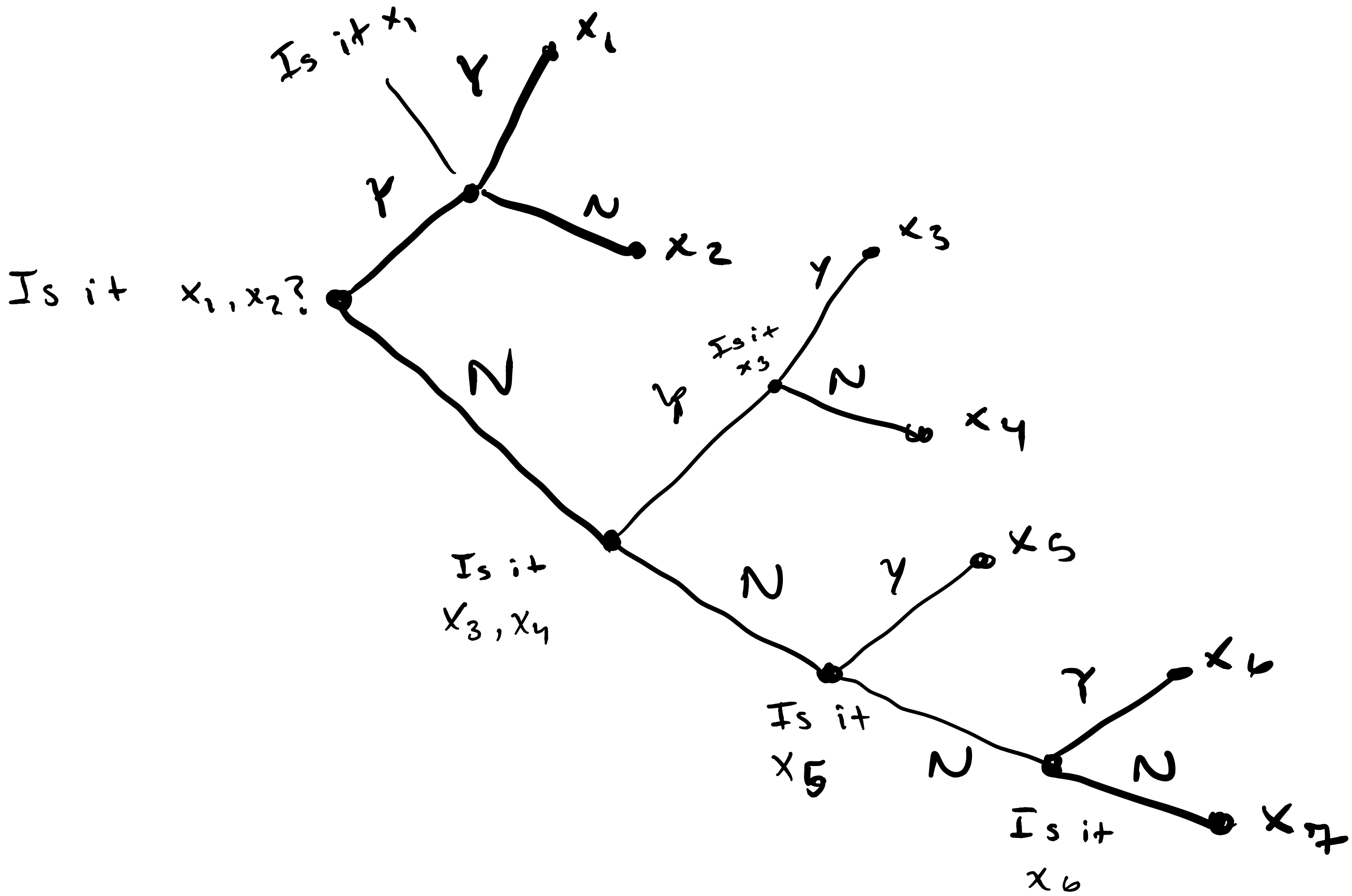
(b) $$H(X) = -\sum_{i=1}^{7} p_X(x_i)\log_2(p_X(x_i))$$

$$= -\left(\frac{1}{4}\cdot\left(\log_2\left(\tfrac{1}{4}\right) + \log_2\left(\tfrac{1}{4}\right)\right) + \frac{1}{8}\left(3\log_2\left(\tfrac{1}{8}\right)\right) + \frac{1}{16}\left(2\cdot\log_2\left(\tfrac{1}{16}\right)\right)\right)$$

distribute negative

$$= \frac{1}{4}\cdot\left(2\log_2(4)\right) + \frac{1}{8}\left(3\cdot\log_2(8)\right) + \frac{1}{16}\left(2\cdot\log_2(16)\right) = 2.63$$

(c)


Is it x_1, x_2?
Is it x_1
Y
Y
N
x_1
x_2
N
Is it x_3, x_4
Y
Is it x_3
Y
N
x_3
x_4
N
Is it x_5
Y
x_5
N
Is it x_6
Y
N
x_6
x_7

5 X = rv from rolling 3 six-sided dice simultaneosly and summing up the numbers facing up.

(a) Entropy H(X)?

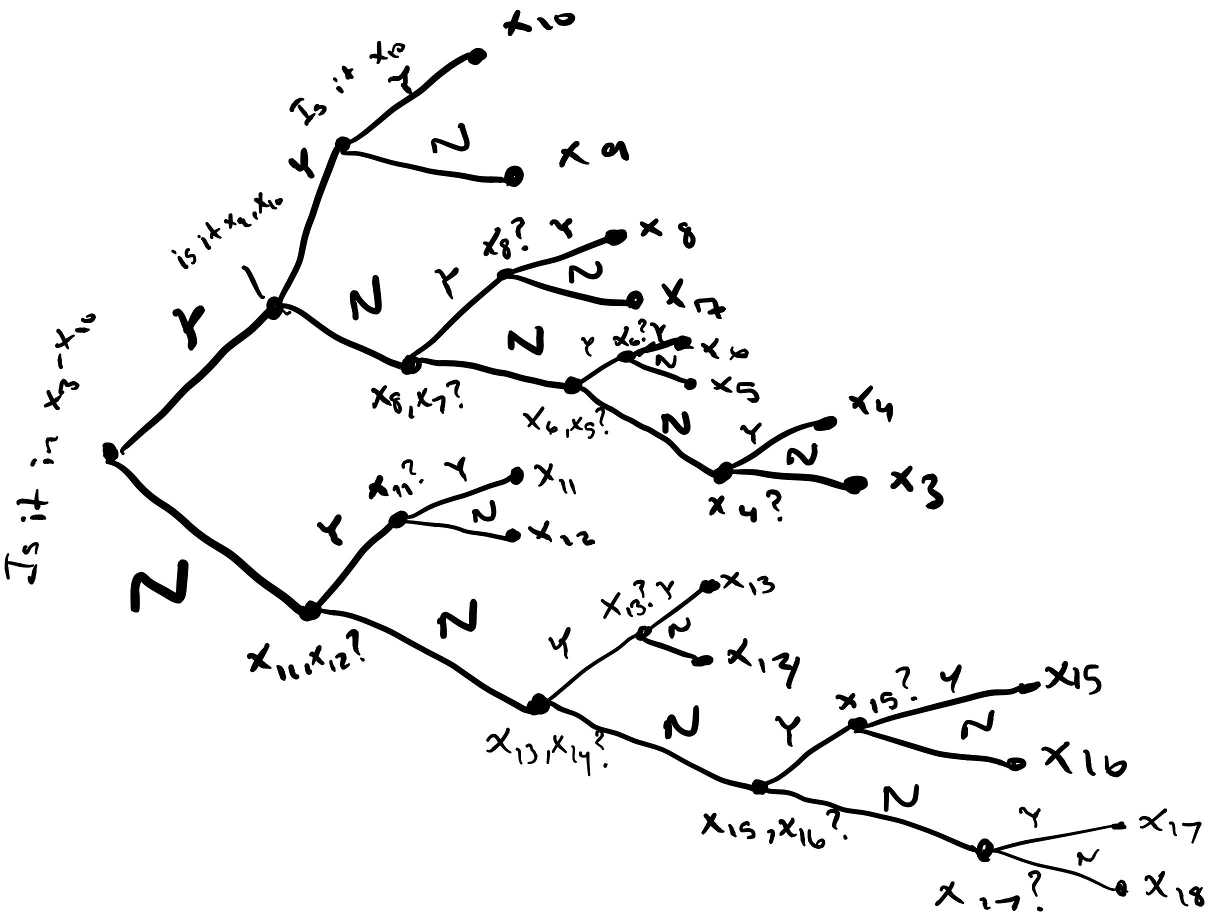
$$H(x) = -\sum_{n=3}^{18} p_x(x_n)\log_2(p_x(x_n)) \quad ?$$

First need $p_x(x)$:

$$P_x(x) = \begin{cases} 1/216 & X=3 \\ 3/216 & X=4 \\ 6/216 & X=5 \\ 10/216 & X=6 \\ 15/216 & X=7 \\ 21/216 & X=8 \\ 25/216 & X=9 \\ 27/216 & X=10 \\ 27/216 & X=11 \\ 25/216 & X=12 \\ 21/216 & X=13 \\ 15/216 & X=14 \\ 10/216 & X=15 \\ 6/216 & X=16 \\ 3/216 & X=17 \\ 1/216 & X=18 \end{cases}$$

$$H(x) = -\sum_{n=3}^{18} p_X(x_n)\log_2(p_X(x_n))$$

$$\simeq 3.6$$



$$\mathbb{E}(L) = \sum_{l \in L} l\, p_L(l)$$

$$= 2\left(\frac{1}{216}\left(27\cdot 3 + 25\cdot 3 + 21\cdot 4 + 15\cdot 4 + 10\cdot 5 + 6\cdot 5 + 3\cdot 5 + 1\cdot 5\right)\right)$$

$$\simeq 3.7$$

$\mathbb{E}(L) > H(L)$ which is expected due to structure of the tree.