Math 4310

Homework 7

Due 10/30/19)

Name: ______________________

Collaborators: ______________________

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Please print out these pages. I encourage you to work with your classmates on this homework. Please list your collaborators on this cover sheet. (Your grade will not be affected.) Even if you work in a group, you should write up your solutions yourself! You should include all computational details, and proofs should be carefully written with full details. As always, please write neatly and legibly.

Please follow the instructions for the "extended glossary" on separate paper (LaTeX it if you can!) Hand in your final draft, including full explanations and write your glossary in complete, mathematically and grammatically correct sentences. Your answers will be assessed for style and accuracy.

Please **staple** *this cover sheet, your exercise solutions, and your glossary together, in that order, and hand in your homework in class.*

GRADES

Exercises ______________________ / 50

Extended Glossary

Component	Correct?	Well-written?
Definition	/6	/6
Example	/4	/4
Non-example	/4	/4
Theorem	/5	/5
Proof	/6	/6
Total	/25	/25

Exercises.

1. Consider the matrix $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ over a field $\mathbb{F}$.
   (a) Find the eigenvalues and eigenvectors of A.
   (b) Show that A is not diagonalizable (similar to a diagonal matrix).

2. Consider the matrix $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.
   (a) Find the eigenvalues and eigenvectors of A over $\mathbb{C}$.
   (b) Show that A is diagonalizable over $\mathbb{C}$, but not over $\mathbb{R}$.

3. Suppose that $\lambda = a + bi$ is a complex number, with $b \neq 0$. Let $\overline{\lambda} = a - bi$ denote the complex conjugate of λ. Consider the polynomial $f(z) = (z - \lambda)(z - \overline{\lambda})$.
   (a) Show that $f(z)$ is a quadratic polynomial, with real coefficients.
   (b) Show that $f(z)$ is irreducible, as a polynomial in $\mathbb{R}[z]$, that is, it cannot be written as a product of 2 polynomials each of strictly smaller degree.
   (c) Show that if $f(z) \in \mathbb{R}[z]$ is a polynomial with real coefficients, then it has a factorization of the form

$$f(z) = a(z - c_1) \dots (z - c_r)(z^2 + p_1 z + q_1) \dots (z^2 + p_s z + q_s),$$

   for some real numbers a, c_i, p_i, q_i, where each polynomial appearing is irreducible. Show that this is unique up to reordering the factors (but note, some factors might appear more than once).

4. Suppose that the $n \times n$ matrix A has an eigenvector v with eigenvalue λ. Suppose that $f \in \mathbb{F}[x]$ is a polynomial. Show that v is also an eigenvector of $f(A)$. Find its corresponding eigenvalue.

5. Let $D \in \mathcal{L}(C^\infty(\mathbb{R}))$ be the differentiation linear operator, that is $D(y(t)) = y'(t)$. In this problem, you may use the fact that if $p(x) \in \mathbb{R}[x]$ has degree d, then $\dim \ker p(D) = d$.
   (a) Show that $(D - rI)(f(t)e^{rt}) = f'(t)e^{rt}$.
   (b) Find a basis for $\ker(D - 2I)^3$ (hint: try functions of the form $y(t) = f(t)e^{rt}$).
   (c) Find a basis for $\ker((D - I)(D - 2I)^2)$.
   (d) The solution set of the functions $y(t)$ which satisfy the differential equation $y''' + y'' - y' + y = 0$ is a vector space. Find a basis for this vector space.

6. Given an $n \times n$ matrix A (over $\mathbb{R}$ or $\mathbb{C}$), we can define another $n \times n$ matrix by the formula

$$e^A = I + A + \frac{A^2}{2} + \frac{A^3}{3!} + \ldots + \frac{A^n}{n!} + \ldots.$$

   It turns out that this infinite sum of matrices converge for all matrices A (i.e. for this problem, you do not need to worry about convergence issues).
   (a) Compute e^A for the diagonal matrix with diagonal entries $\lambda_1, \ldots, \lambda_n$.
   (b) Compute e^A if $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.
   (c) Compute e^A if $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.
   (d) If $B = Q^{-1}AQ$, show that $e^B = Q^{-1}e^AQ$.

Extended Glossary. There is no extended glossary this week.