

# SVD

A  $m \times n$  matrix over  $\mathbb{R}$ , rank  $r$

$A^T A$   $n \times n$  symmetric, rank  $r$ .

Eigenvalues  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r \geq \lambda_{r+1} = 0 = \dots = \lambda_n$

Singular values of  $A$ :

①  $\sigma_1 = \sqrt{\lambda_1} \geq \sigma_2 = \sqrt{\lambda_2} \geq \dots \geq \sigma_r = \sqrt{\lambda_r}$

② Let  $\vec{v}_1, \dots, \vec{v}_n$  be an orthonormal basis of  $\mathbb{R}^n$  consisting of eigenvectors of  $A^T A$ .

So

$$A^T A \vec{v}_i = \lambda_i \vec{v}_i, \quad i = 1, \dots, n$$

③ Let

$$\vec{u}_i = \frac{1}{\sigma_i} A \vec{v}_i, \quad i = 1, \dots, r \quad \checkmark \text{ NOTE}$$

(note  $\vec{u}_i \in \mathbb{R}^m$ )

These are orthonormal, and extend to orthonormal basis  $\vec{u}_1, \dots, \vec{u}_m$  of  $\mathbb{R}^m$ .

Let

$$U = (\vec{u}_1 \ \vec{u}_2 \ \dots \ \vec{u}_m) \quad m \times m \text{ orthogonal matrix}$$

$$V = (\vec{v}_1 \ \vec{v}_2 \ \dots \ \vec{v}_n) \quad n \times n \text{ orthogonal matrix}$$

$$\Sigma = \begin{pmatrix} \sigma_1 & & & & & \\ & \sigma_2 & & & & \\ & & \ddots & & & \\ & & & \sigma_r & & \\ & 0 & & & 0 & \dots & 0 \end{pmatrix} \quad m \times n \text{ matrix}$$

SVD of  $A$ :

$$A = U \Sigma V^T \quad (V^T = V^{-1})$$

since  $V$  orthogonal

Proof

check

$$AV = U\Sigma$$

$$[A\vec{v}_1 \ A\vec{v}_2 \ \dots \ A\vec{v}_n] = [\sigma_1\vec{u}_1 \ \dots \ \sigma_r\vec{u}_r \ 0 \ \dots \ 0] \checkmark$$

Recap

An SVD of  $A$  is a factorization

$$A = U \Sigma V^T$$

such that

①  $U, V$  are orthogonal matrices

②  $\Sigma = \begin{pmatrix} \sigma_1 & & & & \\ & \sigma_2 & & & \\ & & \ddots & & \\ & & & \sigma_r & \\ & 0 & & & 0 & \dots & 0 \end{pmatrix}, \sigma_1, \sigma_2, \dots, \sigma_r > 0$

Example:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \text{rank} = 2$$

find SVD.

$$A^T A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

eigenvalues of  $A^T A$ ,

$$0 \quad \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$1 \quad \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$3 \quad \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

So,

$$\sigma_1 = \sqrt{3}$$

$$\sigma_2 = 1$$

and

$$\vec{v}_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad \vec{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \vec{v}_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$\vec{u}_1 = \frac{1}{\sqrt{3}} A \vec{v}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \vec{u}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

So,

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \end{pmatrix}$$

Proposition: Every  $A \in \mathbb{R}^{m \times n}$  has an SVD (Prove Yourself)

Note:

$$A = (\vec{u}_1 \dots \vec{u}_r \quad \vec{u}_{r+1} \dots \vec{u}_m) \begin{pmatrix} \sigma_1 & \dots & 0 \\ & \ddots & \\ 0 & & \sigma_r & \dots & 0 \\ & & & \ddots & \\ 0 & & & & 0 \end{pmatrix} \begin{pmatrix} \vec{v}_1^T \\ \vec{v}_2^T \\ \vdots \\ \vec{v}_r^T \\ \vec{v}_{r+1}^T \\ \vdots \\ \vec{v}_n^T \end{pmatrix}$$

$$= (\sigma_1 \vec{u}_1 \dots \sigma_r \vec{u}_r \quad 0 \dots 0) \vec{v}^T$$

$$= \sigma_1 \vec{u}_1 \vec{v}_1^T + \sigma_2 \vec{u}_2 \vec{v}_2^T + \dots + \sigma_r \vec{u}_r \vec{v}_r^T$$

"slim, or compact, pretty fine" SVD

$$= (\vec{u}_1 + \dots + \vec{u}_r) \begin{pmatrix} \sigma_1 & \dots & 0 \\ & \ddots & \\ 0 & & \sigma_r \end{pmatrix} \begin{pmatrix} \vec{v}_1^T \\ \vdots \\ \vec{v}_r^T \end{pmatrix}$$

## Slim version of SVD

$$A = U \Sigma V^T$$

but  $U$  is  $m \times r$ ,  $U^T U = I_r$  ( $r = \text{rank } A$ )  
 $V$  is  $n \times r$ ,  $V^T V = I_r$

$$\Sigma = \begin{pmatrix} \sigma_1 & \dots & \sigma_r \\ & & 0 \end{pmatrix}, \sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > 0$$

### NOTE

if  $A = uv^T$ ,  $m \times n$

$$u = m \times 1$$

$$v = n \times 1$$

$$\text{rank } A = 1$$

### Example

$A$   $m \times n$ ,  $1000 \times 1000$

each  $A_{ij} \in [0, 1]$

$A$  is a gray image

often  $\text{rank } A \approx 1000$

$$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_{1000}$$

but MANY very close to 0

Suppose all but 50 very small

$$A = \sigma_1 u_1 v_1^T + \dots + \sigma_{50} u_{50} v_{50}^T$$

Storage for  $A$ : 1000 000 numbers

But w/ ignoring small values,  $50 + 50000 + 50000$  Pretty Good