# The theory of analytic functions.

So far we have viewed a complex function of a complex variable, $f(z)$, as an arbitrary mapping from the $xy$-plane to the $uv$-plane.
Notice there is something special about the pair

$$u_1(x,y) = x^2 - y^2, \qquad v_1(x,y) = 2xy,$$

as opposed to,

$$u_2(x,y) = x^2 - y^2, \qquad v_2(x,y) = 3xy:$$

namely, the complex function

$$u_1 + iv_1$$

treats

$$z = x+iy$$

as a single "unit", because it equals

$$x^2 - y^2 + i2xy = (x+iy)^2$$

and thus it respects the complex structure of

$$z = x+iy.$$

However, the formulation $u_2 + iv_2$ requires us to break apart the real + imaginary parts of $z$.

- In (real) calculus we don't deal with functions that look at a number $3+4\sqrt{2}$ and square the 3 but cube the 4.
- The interesting calculus functions treat the number as an individual module.
- We seek to classify the complex functions that behave this same way with regard to their complex argument.

Thus we want to admit functions such as

$$z = x+iy \qquad \text{(admissable)}$$

$$z^2 = x^2-y^2+i2xy \qquad \text{(admissable)}$$

$$z^3 = x^3-3xy^2+i(3x^2y-y^3) \qquad \text{(admissable)}$$

$$\frac{1}{z} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2} \qquad \text{(admissable)}$$

and their basic arithmetic combinations (sums, products, quotients, powers, and roots) but ban functions such as

$$\operatorname{Re} z = x \qquad \text{(inadmissable)}$$

$$\operatorname{Im} z = y \qquad \text{(inadmissable)}$$

$$x^2-y^2+i3xy \qquad \text{(inadmissable)}$$

Notice we must ban the conjugate function $\bar{z}$, because if we admit it we open the gate to

$$x = \frac{(z+\bar{z})}{2}, \qquad y = \frac{(z-\bar{z})}{2}$$

$$z = x - iy \qquad \text{(inadmissable)}$$

Similarly, admitting $|z|$ would be a mistake as well, since $\bar{z} = |z|^2/z$

$$|z| \qquad \text{(inadmissable)}$$

One could criticize our "inadmissible" classifications of

$$u_2 + iv_2 = x^2 - y^2 + i\,3xy$$

because we have not yet proved that it could be written in terms of $z$ alone.

The following computation is instructive: we set

$$x = \left(\frac{z+\bar{z}}{2}\right), \qquad y = \left(\frac{z-\bar{z}}{2}\right) \qquad (1)$$

in $u_2 + iv_2$ and obtain,

$$u_2 + iv_2 = x^2 - y^2 + i\,3xy$$

$$= \frac{(z+\bar{z})^2}{4} + i\,\frac{3(z+\bar{z})}{2}\,\frac{(z-\bar{z})}{2i}$$

$$= \frac{5}{4}z^2 - \frac{1}{4}\bar{z}^2$$

Now we see that if we admit $u_2 + iv_2$, we would have to admit $\bar{z}^2$ and its undesirable root $\bar{z}$.

**Example 1** Express the following in terms of $z, \bar{z}$:

$$f_1(z) = \frac{x-1-iy}{(x-1)^2+y^2} \qquad f_2(z) = x^2+y^2+3x+1+i3y$$

$$f_1(z) = \frac{\frac{z+\bar{z}}{2} - 1 - i\left(\frac{z-\bar{z}}{2i}\right)}{\left(\frac{z+\bar{z}}{2} - 1\right)^2 + \left(\frac{z-\bar{z}}{2i}\right)^2}$$

$$= \frac{\bar{z}-1}{z\bar{z} - z - \bar{z} + 1} = \frac{1}{z-1}$$

$$f_2(z) = \frac{(z+\bar{z})^2}{4} + \frac{(z-\bar{z})^2}{4i^2} + 3\,\frac{(z+\bar{z})}{2} + 1 + i3\left(\frac{z-\bar{z}}{2i}\right)$$

$$= z\bar{z} + 3z + 1$$

The process of disqualifying functions with $\bar{z}$ in their formulas doesn't lead to a workable criterion.

The criterion we are seeking (the test that will distinguish admissable functions from others) can be expressed simply in terms of differentiability.

**Definition 4.** Let $f$ be a complex-valued function defined in a neighborhood of $z_0$. Then the **derivative** of $f$ at $z_0$ is given by

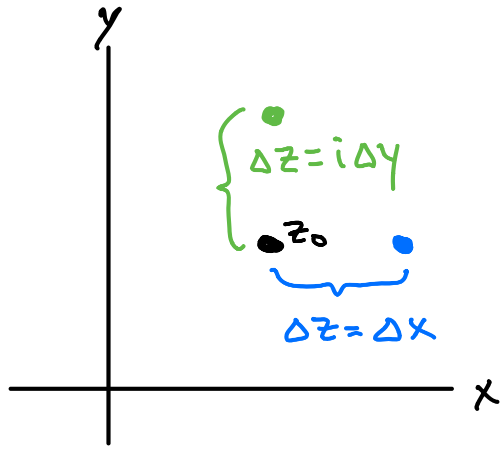
$$\frac{df}{dz}(z_0) \equiv f'(z_0) :=\lim_{z \to z_0} \frac{f(z+z_0) - f(z_0)}{\Delta z}$$

provided this limit exists. (Such an $f$ is said to be differentiable at $z_0$.

The catch here is that $\Delta z$ is a complex number, so it can approach zero in many different ways; but the difference quotient must tend to a unique limit $f'(z_0)$ independent of the manner in which $\Delta z \to 0$.

Let's see why this notion disqualifies $\bar{z}$.

**Example 2:** Show that $f(z) = \bar{z}$ is nowhere differentiable.




← Horizontal + Vertical approach to zero of $\Delta z$.

The difference quotient for this function takes the form:

$$\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{\overline{(z_0 + \Delta z)} - \bar{z}_0}{\Delta z} = \frac{\overline{\Delta z}}{\Delta z}$$

If $\Delta z \to 0$ through real values, then $\Delta z = \Delta x$ and $\overline{\Delta z} = \Delta z$, so the difference quotient is 1.

If $\Delta z \to 0$ from above, then $\Delta z = i\Delta y$ and $\overline{\Delta z} = -\Delta z$, so the quotient is $-1$.

Consequently, there's no way of assigning a unique value to the derivative of $\bar{z}$ at any point.

Hence $\bar{z}$ is NOT differentiable.

**Example 3:** Show that for any positive integer $n$,

$$\frac{d}{dz} z^n = n z^{n-1} \qquad (f(z) = z^n) \qquad (4)$$

$$\frac{d}{dz} z^n = \lim_{\Delta z \to 0} \frac{(z+\Delta z)^n - z^n}{\Delta z}$$

Binomial Formula

$$= \lim_{\Delta z \to 0} \frac{n z^{n-1} \Delta z + \frac{n(n-1)}{2} z^{n-2} (\Delta z)^2 + \cdots + (\Delta z)^n}{\Delta z}$$

$$= n z^{n-1}$$

Note the similarity to the real-variable case

**Theorem 3:** If $f$ and $g$ are differentiable at $z$, then

$$(f \pm g)'(z) = f'(z) \pm g'(z) \qquad (5)$$

$$(cf)'(z) = c f'(z) \qquad (6)$$

$$(fg)'(z) = f(z) g'(z) + f'(z) g(z) \qquad (7)$$

$$(f/g)'(z) = \frac{g(z) f'(z) - f'(z) g(z)}{g(z)^2}, \quad g(z) \neq 0 \qquad (8)$$

If $g$ is differentiable at $z$ and $f$ is differentiable at $g(z)$, then the chain rule holds

$$\frac{d}{dz} f(g(z)) = f'(g(z)) g'(z) \qquad (9)$$

As in the real variable case, differentiability implies continuity.
It follows that any polynomial in $z$,

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

is differentiable in the whole plane and its derivative is given by

$$P'(z) = n a_n z^{n-1} + (n-1) a_{n-1} z^{n-2} + \cdots + a_1$$

**Example 4:** Compute the derivative of

$$f(z) = \left(\frac{z^2-1}{z^2+1}\right)^{100}$$

Unless $z = \pm i$,

$$f'(z) = 100\left(\frac{z^2-1}{z^2+1}\right)^{99} \frac{(z^2+1)2z - (z^2-1)2z}{(z^2+1)^2}$$

$$= 400z \frac{(z^2-1)^{99}}{(z^2+1)^{101}}$$

**Definition 5.** A complex-valued function $f(z)$ is said to be **analytic** on an open set $G$ if it has a derivative at every point $G$.

We **EMPHASIZE** that analyticity is a property defined over open sets, while differentiability could conceivably hold at one point only.

We use "$f(z)$ is analytic at the point $z_0$" to mean that $f(z)$ is analytic in some neighbourhood of $z_0$.

A point where $f$ is NOT analytic but which is the limit of points where $f$ is analytic is known as a singularity.

Thus we can say a rational function $z$ is analytic at every point for which for which its denominator is nonzero, and the zeros of the denominators are singularities.

If $f(z)$ is analytic on the whole complex plane, it is said to be entire.