Recall

**DTMC** $\{X_n\}_{n \geq 0}$

**Definition:** $\Pr[X_{n+1} = j \mid X_n = i, \ldots, X_0 = i]$
$= \Pr[X_{n+1} = j \mid X_n = i]$

**Homogeneity:** $\Pr[X_{n+1} = j \mid X_n = i] \equiv p_{i,j}$

(one-step) transition matrix $\{p_{i,j}\}_{i,j \in \mathcal{X}} = \mathbb{P}$ ← encodes all information about this process!

(n-step) transition matrix $\mathbb{P}^{(n)} = \left\{ p_{i,j}^{(n)} = \Pr[X_n = j \mid X_0 = i] \right\} = \mathbb{P}^n$

**State Probability:** $\vec{p}(n) \triangleq \{\Pr[X_n = i]\}_{i \in \mathcal{X}} \in \mathbb{R}^{1 \times n}$

$$= \vec{p}(0)\, \mathbb{P}^n$$

**Stationary Distribution**

$$\begin{cases} \pi \mathbb{P}^n = \pi & \forall n \\ \sum_{i \in \mathcal{X}} \pi_i = 1 \end{cases}$$

Now,

# Continuous Time Markov Chains

Process denoted by $\{X_t\}_{t \geq 0}$

**Definition:** $\forall\, t_0 < t_1 < \cdots < t_{n+1}$

$$\Pr[X_{t_{n+1}} = j \mid X_{t_n} = i, X_{t_{n-1}} = i_{n-1}, \ldots, X_{t_0} = i_0]$$

$$= \Pr[X_{t_{n+1}} = j \mid X_{t_n} = i]$$

**Homogeneity:**

$$\Pr[X_{t_{n+1}} = j \mid X_{t_n} = i] \equiv p_{i,j}(t_{n+1} - t_n)$$

as a function of the difference

(better seen by

$$\Pr[X_{t+s} = j \mid X_s = i] = p_{i,j}(t)$$

**Transition Matrix**

$$\mathbb{P}(t) = \{p_{i,j}(t)\}_{i,j \in \mathcal{X}}$$

Chapman-Komogorov Equations

$$\mathbb{P}(t+s) = \mathbb{P}(t)\,\mathbb{P}(s) = \left\{ P_{i,j}(t+s) \right\}_{i,j \in \mathcal{X}}$$

State probability

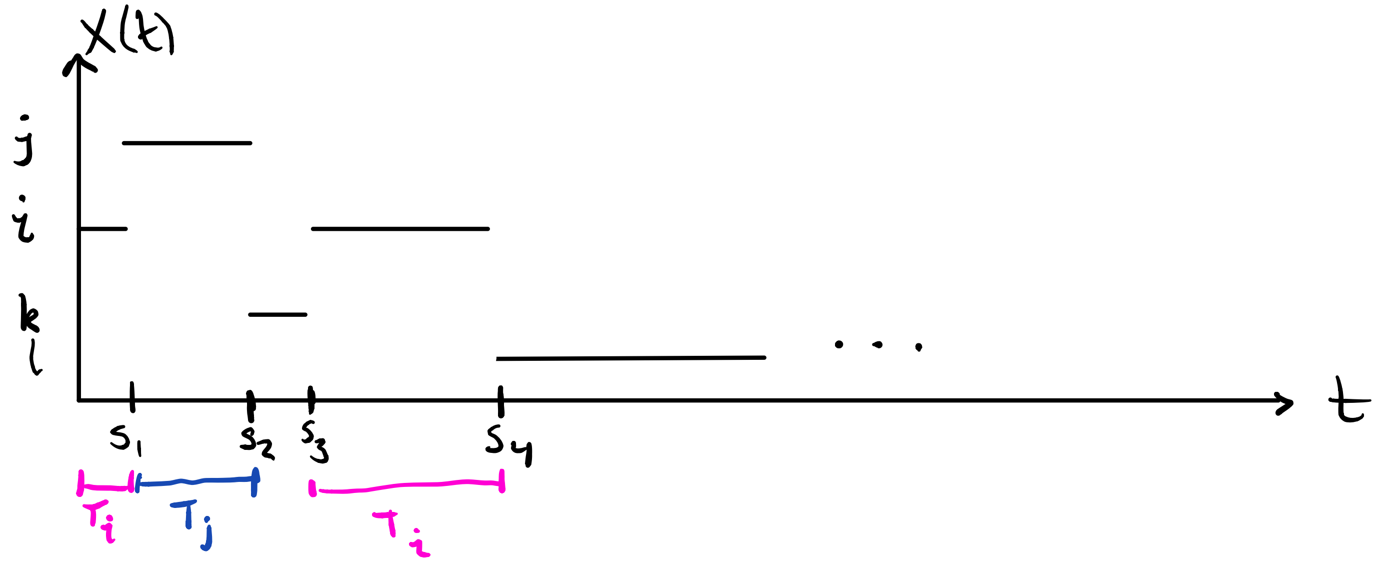
$$\vec{p}(0) \triangleq \left[ \Pr[X_0 = i] \right]_{i \in \mathcal{X}}$$

$$\vec{p}(t) = \vec{p}(0)\,\mathbb{P}(t)$$

Stationary Distribution

$$\begin{cases} \vec{\pi}\,\mathbb{P}(t) = \vec{\pi} \quad \forall\, t \\ \sum_{i \in \mathcal{X}} \pi_i = 1 \end{cases}$$

# Characterization of CTMC




holding time in a particular state

$\{T_i\}$ iid $\forall \, i \in \mathcal{X}$ (every state in state space)

Embedded DTMC: $\{p_{i,j}\}$ $p_{i,i} = 0$

$$\Pr[T_i > t+s \mid T_i > s]$$

$$= \Pr[X_{s+\tau} = i, \ \forall \, 0 < \tau \leq t \mid X_\mu = i \ \forall \, 0 \leq \mu \leq s]$$

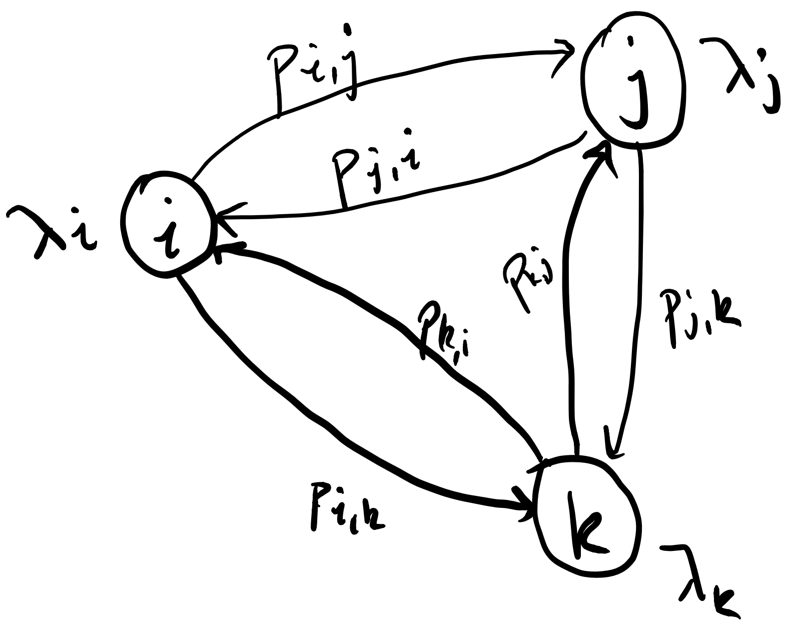
$$= \Pr[X_{s+\tau} = i, \ \forall \, 0 < \tau \leq t \mid X_s = i] \quad \text{(Markov Property)}$$

$$= \Pr[X_\tau = i, \ \forall \, 0 < \tau \leq t \mid X_0 = i] \quad \text{(Homogeneity)}$$

$$= \Pr[T_i > t]$$

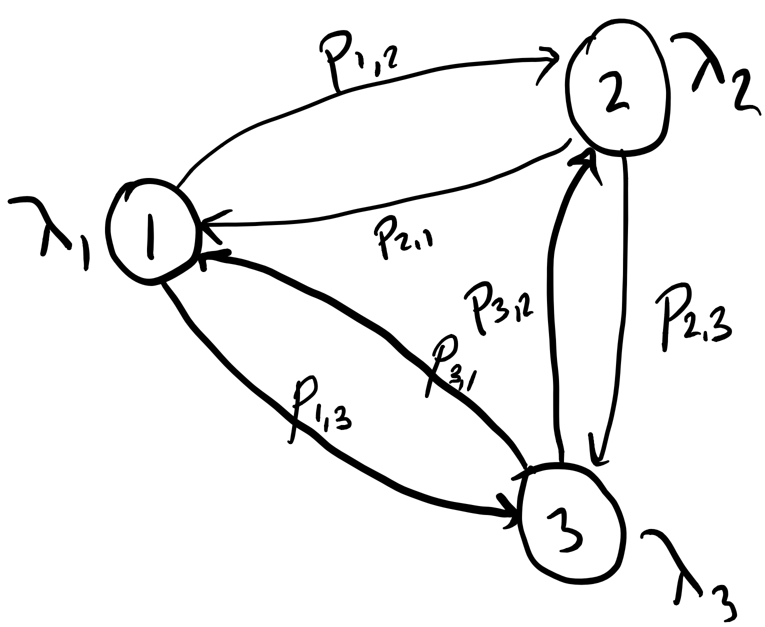
Memoryless Property! $\Rightarrow$ Exponential Distribution

$\therefore$ $T_i \sim \exp(\lambda_i)$ iid random variables




$$\mathbb{P} = \begin{bmatrix} 0 & \cdot & (p_{j,i}) \\ & \cdot & \\ (p_{i,j}) & & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 & \cdots & \lambda_m \end{bmatrix}$$

Equivalent Characterization



In state 1, set two alarms

independent $\longrightarrow$

$$T_{1,2} \sim \exp(p_{1,2}\lambda_1)$$
$$T_{1,3} \sim \exp(p_{1,3}\lambda_1)$$

These 2 alarms tell you when you leave and where you are going

$$T_1 = \min(T_{1,2}, T_{1,3})$$

$$p_{1,2} + p_{1,3} = 1$$

$$\sim \exp(p_{1,2}\lambda_1 + p_{1,3}\lambda_1)$$
$$\sim \exp(\lambda_1)$$

Let $k = \operatorname{argmin}\{T_{1,j}\}_{j \neq 1 \in \mathcal{X}}$

Then

$$\Pr[k=j] = \frac{p_{1,j}}{p_{1,2}+p_{1,3}} = p_{1,j}$$