

# ECE 4110 Homework 4

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*Due by 5pm on October 3*

## 1 Reading Material

- Jointly Gaussian random variables (Chapter 6.4).
- MMSE and LMMSE estimators (Chapter 6.5.3).

## 2 Assignment

### 1. Conditional Variance

You have seen in Problem 2 of Homework 3 the *conditional variance* of  $X$  given  $Y$ . It is defined as

$$\text{Var}(X|Y) \stackrel{\Delta}{=} \mathbb{E}((X - \mathbb{E}(X|Y))^2|Y).$$

- (a) Show that the expectation of the conditional variance is the mean square error of the MMSE estimator of  $X$  given  $Y$ :

$$\mathbb{E}[\text{Var}(X|Y)] = \mathbb{E}[(X - \mathbb{E}[X|Y])^2].$$

- (b) Show that the variance of  $X$  can be decomposed into the variance of the MMSE estimator plus the mean square error of that estimator:

$$\text{Var}(X) = \text{Var}(\mathbb{E}[X|Y]) + \mathbb{E}[\text{Var}(X|Y)].$$

*Hint: You have already shown this in Problem 2 of Homework 3. This time, try to prove it by using the orthogonality principle. First write*

$$\text{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X|Y] + \mathbb{E}[X|Y] - \mathbb{E}[X])^2]$$

*and then expand the square (and then use the orthogonality principle).*

- (c) Based on (b), show that the MMSE of estimating  $X$  based on observations of  $Y$  is smaller than or equal to that of estimating  $X$  without any observations (i.e., estimating  $X$  using a constant). When do these two types of estimators offer the same MMSE?

### 2. MMSE and LMMSE Estimator

Let  $X$  and  $Y$  be jointly uniformly distributed over the triangular region in the  $x$ - $y$  plane with corners  $(0, 0)$ ,  $(0, 1)$ , and  $(1, 2)$ .

- (a) What is the MMSE estimator of  $X$  using a constant? What is the resulting MSE?
- (b) What is the LMMSE estimator of  $X$  given  $Y$ ? What is the resulting MSE? Compare it with the MSE in (a).
- (c) What is the best MMSE estimator of  $X$  given  $Y$ ? What is the resulting MSE? Compare it with the MSE in (a) and (b).

### 3. Marginally Gaussian but not Jointly Gaussian

Consider independent Gaussian random variables  $X, Y \sim \mathcal{N}(0, 1)$ . Let  $Z \in \{\pm 1\}$  be a binary random variable with  $\Pr(Z = 1) = \frac{1}{2}$  and independent of  $X$  and  $Y$ .

- (a) Show that  $W = ZX + Y$  is also Gaussian.  
*(Hint: first show that  $ZX$  is Gaussian by writing out its PDF through conditioning on  $Z$ , then conclude  $W$  is Gaussian based on the independence of  $ZX$  and  $Y$ .)*
- (b) Show that  $X$  and  $W$  are not jointly Gaussian by finding the joint PDF of  $X$  and  $W$ .  
*(Hint: note that conditioning on  $Z = z$ , the random vector  $[X, W]^T$  is a linear transform of the random vector  $[X, Y]^T$ .)*

### 4. MMSE Estimator for Jointly Gaussian Random Variables

Let  $\mathbf{X} = [X_1, X_2, X_3]^T$  be a Gaussian random vector with

$$\mathbb{E}(\mathbf{X}) = \begin{pmatrix} 1 \\ 4 \\ 6 \end{pmatrix} \quad \mathbf{K} = \text{Cov}(\mathbf{X}) = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

- (a) Compute the MMSE estimate of  $X_1$  given  $X_2$  and the resulting MSE.
- (b) Compute the MMSE estimate of  $X_1$  given  $X_3$  and the resulting MSE.
- (c) Compute the MMSE estimate of  $X_1$  given both  $X_2$  and  $X_3$  and the resulting MSE.
- (d) Note that  $X_1$  and  $X_3$  are uncorrelated, and hence independent. Yet the MMSE estimate of  $X_1$  given  $X_2$  and  $X_3$  as obtained in (c) is a function of both  $X_2$  and  $X_3$ . Why is that?

# ① Conditional Variance

- (a) Show that the expectation of the conditional variance is the mean square error of the MMSE estimator of  $X$  given  $Y$ :

$$\text{LHS} \quad \text{RHS} \\ \mathbb{E}[\text{Var}(X|Y)] = \mathbb{E}[(X - \mathbb{E}[X|Y])^2].$$

Know

$$\text{Var}(X|Y) = \mathbb{E}[(X - \mathbb{E}[X|Y])^2 | Y]$$

Thus the LHS is

$$\mathbb{E}[\text{Var}(X|Y)] = \mathbb{E}\left[\mathbb{E}[(X - \mathbb{E}[X|Y])^2 | Y]\right]$$

The rhs is

$$\mathbb{E}_Y[(X - \mathbb{E}[X|Y])^2] = \mathbb{E}_Y\left[\mathbb{E}_{x|y}[(X - \mathbb{E}[X|Y])^2 | Y]\right]$$

by law of iterated expectations.

Thus,

$$\mathbb{E}[\text{Var}(X|Y)] = \mathbb{E}[(X - \mathbb{E}[X|Y])^2 | Y]$$

- (b) Show that the variance of  $X$  can be decomposed into the variance of the MMSE estimator plus the mean square error of that estimator:

$$\text{Var}(X) = \text{Var}(\mathbb{E}[X|Y]) + \mathbb{E}[\text{Var}(X|Y)].$$

Could also use

$$\text{Var}(x) = \mathbb{E}[(x - \mathbb{E}[x|y] + \mathbb{E}[x|y] - \mathbb{E}[x])^2]$$

$$\text{Let } X = X - \mathbb{E}[X|Y] + \mathbb{E}[X|Y]$$

and orthogonality principle!

Then

$$\text{Var}(x) = \text{Var}(X - \mathbb{E}[X|Y] + \mathbb{E}[X|Y])$$

Since  $\mathbb{E}[X|Y]$  is our BEST estimator  $\hat{x}^*$  which minimizes our error  $w = X - \hat{x}$  over ALL possible functions  $g(Y)$ , the orthogonality principle states

$$w \perp g(Y) \text{ for ALL possible functions } g(Y).$$

### Visualization

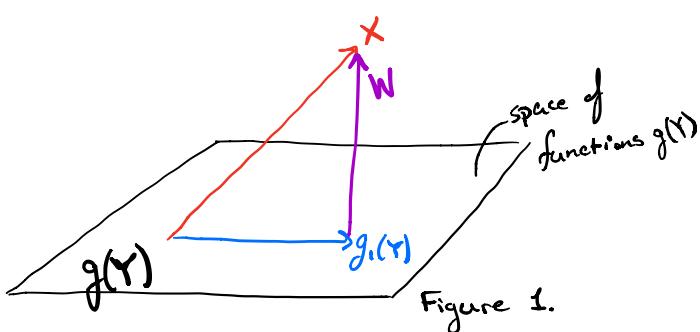


Figure 1.

$$w = x - \hat{x}^* = x - \mathbb{E}[X|Y]$$

$$\text{The } x - \mathbb{E}[X|Y] \perp \mathbb{E}[X|Y]$$

$$\Rightarrow \mathbb{E}[(x - \mathbb{E}[X|Y]) \mathbb{E}[X|Y]] = 0$$

$\Rightarrow$  uncorrelatedness

If  $X, Y$  two uncorrelated random variables,

$$\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y)$$

Since  $w$  is uncorrelated with  $\mathbb{E}[X|Y]$ ,

$$\text{Var}(w + \mathbb{E}[X|Y]) = \text{Var}(w) + \text{Var}(\mathbb{E}[X|Y])$$

Thus

$$\text{Var}(x) = \text{Var}(X - \mathbb{E}[X|Y]) + \text{Var}(\mathbb{E}[X|Y])$$

By Law of Iterated Expectations,

$$\begin{aligned}\text{Var}(X - \mathbb{E}[X|Y]) &= \mathbb{E} \left[ \mathbb{E} \left[ (X - \mathbb{E}[X|Y])^2 | Y \right] \right] \\ &= \mathbb{E}[\text{Var}(X|Y)] \quad \text{Shown in part (a)}$$

Thus

$$\text{Var}(X) = \mathbb{E}[\text{Var}(X|Y)] + \text{Var}(\mathbb{E}[X|Y])$$

- (c) Based on (b), show that the MMSE of estimating  $X$  based on observations of  $Y$  is smaller than or equal to that of estimating  $X$  without any observations (i.e., estimating  $X$  using a constant). When do these two types of estimators offer the same MMSE?

MMSE using NO observations of  $Y$  yields  $\text{Var}(X)$   
MMSE using observations of  $Y$  yields  $\mathbb{E}_Y[\text{Var}(X|Y)]$

} From Lecture Notes, Time crunch prevented rederivation

From part b

$$\text{Var}(X) = \mathbb{E}[\text{Var}(X|Y)] + \text{Var}(\mathbb{E}[X|Y])$$

Rearrange to get

$$\mathbb{E}[\text{Var}(X|Y)] = \text{Var}(X) - \text{Var}(\mathbb{E}[X|Y])$$

✓ From HW3,  
 $\text{Var}(\mathbb{E}[X|Y]) \geq 0$

Thus

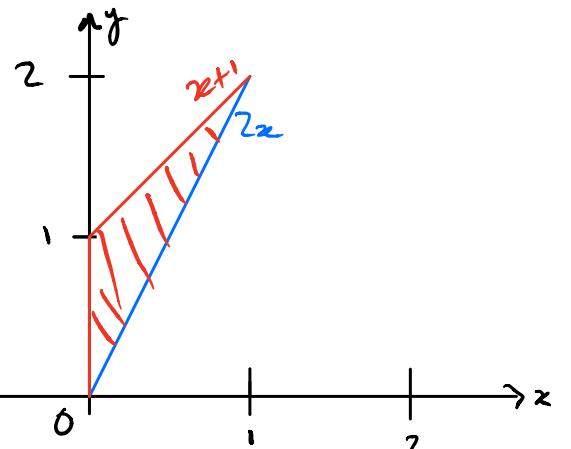
$$\mathbb{E}[\text{Var}(X|Y)] \leq \text{Var}(X)$$

also proved in HW3

With equality when  $X$  is independent of  $Y$ .

$$\begin{aligned}X, Y \text{ independent} \Rightarrow \mathbb{E}[X|Y] &= \mathbb{E}[X] && \text{constant} \\ \text{Var}(\mathbb{E}[X|Y]) &= \text{Var}(\mathbb{E}[X]) = 0\end{aligned}$$

(2)  $X, Y$  jointly uniform over



(a) MMSE estimator of  $X$  using a constant.

Know that best estimator  $\hat{X}$  is  $E(X)$

$$f_{X,Y}(x,y) = \frac{1}{\text{area}} \cdot \begin{matrix} 0 \leq x \leq 1 \\ 2x \leq y \leq x+1 \\ x \leq \frac{y}{2} \leq \frac{x+1}{2} \end{matrix}$$

$$\text{area} = \int_{x=0}^1 x+1 \, dx - \int_{x=0}^1 2x \, dx = \frac{3}{2} - 1 = \frac{1}{2}$$

$$f_{X,Y}(x,y) = 2, \quad \begin{matrix} 0 \leq x \leq 1 \\ 2x \leq y \leq x+1 \end{matrix}$$

$$f_X(x) = \int_{y=2x}^{y=x+1} 2 \, dy = 2(x+1) - 2(2x) = 2(1-x), \quad 0 \leq x \leq 1$$

$$E[X] = \int_{x=0}^1 x(2(1-x)) \, dx = 2 \int_{x=0}^1 x - x^2 \, dx = 2\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{3}$$

The MSE using  $E[X]$  is  $\text{Var}(X)$

$$E[X^2] = 2 \int_{x=0}^1 x^2(1-x) \, dx = 2\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{1}{6}$$

$$\text{Var}(X) = E[X^2] - (E[X])^2$$

$$= \frac{1}{6} - \frac{1}{9} = \frac{1}{18}$$

(b) LMSE estimator of  $X$  given  $Y$ ?

$$\hat{X}_{\text{LMSE}} = aY + b$$

where

$$a = \frac{\text{Cov}(X, Y)}{\text{Var}(Y)}, \quad b = \mathbb{E}[X] - a\mathbb{E}[Y]$$

So

$$\hat{X}_{\text{LMSE}} = \mathbb{E}[X] + \frac{\text{Cov}(X, Y)}{\text{Var}(Y)} (Y - \mathbb{E}[Y])$$

$$\boxed{\mathbb{E}[X] = \frac{1}{3}}$$

$\mathbb{E}[Y]?$

$$f_{Y|X=x}(y|x=x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{2}{2(1-x)} = \frac{1}{1-x}, \quad 2x \leq y \leq x+1$$

$$\mathbb{E}[Y|X=x] = \int_{y=2x}^{y=x+1} \frac{1}{1-x} y \, dy = \frac{1}{1-x} \frac{1}{2} y^2 \Big|_{y=2x}^{y=x+1} = \frac{1}{2(1-x)} [(x+1)^2 - (2x)^2]$$

$$\mathbb{E}[Y|X] = \frac{1}{2(1-x)} [2x+1 - 3x^2]$$

$$\boxed{\mathbb{E}[Y] = \mathbb{E}_x[\mathbb{E}[Y|X]] = \int_{x=0}^1 \mathbb{E}[Y|X=x] f_X(x) dx = \int_{x=0}^1 2x+1 - 3x^2 dx = 1}$$

$\text{Var}(Y)? \quad \text{Var}(Y) = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2$

$$\mathbb{E}[Y^2] = \mathbb{E}_x[\mathbb{E}[Y^2|X]]$$

$$\int_{y=2x}^{y=x+1} y^2 f_{Y|X=x}(y|x) dy - \mathbb{E}[Y^2|X=x] = \int_{y=2x}^{y=x+1} \frac{1}{1-x} y^2 dy = \frac{1}{3(1-x)} [(x+1)^3 - (2x)^3]$$

$$\mathbb{E}[Y^2|X=2] = \frac{x^3 + 3x^2 + 3x + 1 - 8x^3}{3(1-x)}$$

$$= \frac{3x^2 + 3x + 1 - 7x^3}{3(1-x)}$$

$$\mathbb{E}[Y^2|X] = \frac{3X^2 + 3X + 1 - 7X^3}{3(1-X)}$$

$$\begin{aligned} \mathbb{E}[Y^2] &= \mathbb{E}_x[\mathbb{E}[Y^2|X]] = \int_{x=0}^{x=1} \frac{2(1-x)}{3(1-x)} (3x^2 + 3x + 1 - 7x^3) dx \\ &\quad \int_x \mathbb{E}[Y^2|X] f_X(x) dx \\ &= \frac{2}{3} \left[ 1 + \frac{3}{2} + 1 - \frac{7}{4} \right] = \frac{7}{6} \end{aligned}$$

$$\boxed{\text{Var}(Y) = \frac{7}{6} - 1 = 1/6}$$

$\text{Cov}(X, Y)$ ?

$$\mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$$

$$\begin{aligned} \mathbb{E}[XY] &= \iint_{y=x}^{y=1} xy f_{X,Y}(x,y) dx dy = \int_{y=0}^1 \int_{x=0}^{y/2} 2xy dx dy + \int_{y=0}^1 \int_{x=y-1}^{y/2} 2xy dx dy \end{aligned}$$

$$= \int_{y=0}^1 \frac{y^3}{2} dy + \int_{y=0}^1 \left[ \left(\frac{y}{2}\right)^2 - (y-1)^2 \right] y dy$$

$$= 5/12$$

$$\text{Cov}(X, Y) = \frac{5}{12} - 1\left(\frac{1}{3}\right) = \frac{1}{12}$$

Thus

$$\hat{X}_{\text{LMMSE}} = \frac{1}{3} + \frac{1/12}{1/6} (Y - 1)$$

$$MSE_{\text{MMSE}} = \text{Var}(X) - \frac{\text{Cov}^2(X, Y)}{\text{Var}(Y)} = \frac{1}{18} - \frac{\frac{1}{144}}{\frac{1}{16}} = \frac{1}{18} - \frac{1}{24} = \frac{1}{72}$$

This value is LESS THAN previous value of  $\frac{1}{18}$ !

(c) MMSE estimator of  $X$  given  $Y$ ?

This is  $E[X|Y]$ .

$$f_Y(y) = \begin{cases} \int_0^{y/2} f_{X,Y}(x,y) dx = y, & 0 \leq y \leq 1 \\ \int_{y-1}^{y/2} f_{X,Y}(x,y) dx = 2-y, & 1 \leq y \leq 2 \\ 0, & \text{o/w} \end{cases}$$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} \frac{2}{y}, & 0 \leq y \leq 1, 0 \leq x \leq y/2 \\ \frac{2}{2-y}, & 1 \leq y \leq 2, y-1 \leq x \leq y/2 \end{cases}$$

$$E[X|Y] = \begin{cases} \int_{x=0}^{y/2} f_{X|Y}(x|y) \cdot x dx = \frac{Y}{4}, & 0 \leq Y \leq 1 \\ \int_{x=y-1}^{y/2} f_{X|Y}(x|y) \cdot x dx = \frac{3Y-2}{4}, & 1 \leq Y \leq 2 \end{cases}$$

MSE is

$$E[\text{Var}(X|Y)] = \int_y \text{Var}(X|Y=y) f_Y(y) dy = \frac{1}{96}$$

$$\text{Var}(X|Y=y) = E[X^2|Y=y] - (E[X|Y=y])^2$$

③  $X, Y \sim N(0,1)$ ,  $X$  independent of  $Y$

$$Z \text{ is bin} \pm 1, \text{ w/p } P_Z(z) = \begin{cases} 1, & z \geq 0 \\ -1, & z < 0 \end{cases}$$

Let  $V = ZX$

$$\begin{aligned} F_V(v) &= \Pr(V \leq v) = \Pr(ZX \leq v) = \Pr(Z=1) \Pr(X \leq v) + \Pr(Z=-1) \Pr(-X \leq v) \\ &= F_X(v) \end{aligned}$$

Note the symmetry of the Gaussian used!

$$\Pr(X \leq v) = \Pr(X \geq v) = \Pr(-X \leq v)$$

Thus

$$V = ZX \sim N(0,1)$$

Since  $X, Y, Z$  jointly independent,  $V$  is independent of  $Y$ .

Therefore  $V$  and  $Y$  jointly Gaussian.

Thus

$$W = V + Y \sim N(0,2)$$

(b) Show that  $X$  and  $W$  are not jointly Gaussian by finding the joint PDF of  $X$  and  $W$ .

(Hint: note that conditioning on  $Z = z$ , the random vector  $[X, W]^T$  is a linear transform of the random vector  $[X, Y]^T$ .)

By definition we have

$$\begin{pmatrix} X \\ W \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ Z & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$$

Here,  $(\begin{smallmatrix} x \\ y \end{smallmatrix})$  is a Gaussian random vector since  $x, y$  jointly Gaussian.

Conditioning on  $Z=z$ ,  $(\begin{smallmatrix} x \\ w \end{smallmatrix})$  is a linear transform of  $(\begin{smallmatrix} x \\ y \end{smallmatrix})$  and thus a Gaussian random vector with zero mean and

$$K_z = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}^T = \begin{pmatrix} 1 & z \\ z & 1+z^2 \end{pmatrix}$$

We thus have

$$\begin{aligned} f_{x,w}(x,w) &= \Pr(z=1) f_{x,w|z=1}(x,w|z=1) + \Pr(z=-1) f_{x,w|z=-1}(x,w|z=-1) \\ &= \frac{1}{2} \cdot \frac{1}{\sqrt{2\pi} |K|} e^{-\frac{1}{2} (\begin{smallmatrix} x \\ w \end{smallmatrix}) K_z^{-1} (\begin{smallmatrix} x \\ w \end{smallmatrix})^T} + \frac{1}{2} \cdot \frac{1}{\sqrt{2\pi} |K|} e^{-\frac{1}{2} (\begin{smallmatrix} x \\ w \end{smallmatrix}) K_{z=-1}^{-1} (\begin{smallmatrix} x \\ w \end{smallmatrix})^T} \\ &= \frac{e^{-\frac{-2xz+w^2}{2}}}{4\pi} (e^{2zw} + e^{-2zw}) \end{aligned}$$

  
NOT Gaussian!

(4)

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}, \quad \mathbb{E}[\mathbf{X}] = \begin{bmatrix} \mathbb{E}[X_1] \\ \mathbb{E}[X_2] \\ \mathbb{E}[X_3] \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix}$$

$$\mathbf{K} = \text{Cov}(\mathbf{X}) = \begin{bmatrix} \text{Cov}(X_1, X_1) & \text{Cov}(X_1, X_2) & \text{Cov}(X_1, X_3) \\ \text{Cov}(X_2, X_1) & \text{Cov}(X_2, X_2) & \text{Cov}(X_2, X_3) \\ \text{Cov}(X_3, X_1) & \text{Cov}(X_3, X_2) & \text{Cov}(X_3, X_3) \end{bmatrix} = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

(a) MNSE estimate of  $X_1$  given  $X_2$  and resulting MSE.

$$\hat{X}_{\text{muse}} = \mathbb{E}[X_1 | X_2]$$

Since  $X_1, X_2$  jointly Gaussian (any subset of a Gaussian random vector is a Gaussian random vector)

$$\hat{X}_{\text{muse}} = \hat{X}_{2\text{muse}} = \mathbb{E}[X_1] + \frac{\text{Cov}(X_1, X_2)}{\text{Var}(X_2)} (X_2 - \mathbb{E}[X_2])$$

$$= 1 + \frac{1}{2} (X_2 - 4)$$

$$= \boxed{\frac{X_2}{2} - 1}$$

$$\text{MSE} = \text{Var}(X_1) - \frac{\text{Cov}^2(X_1, X_2)}{\text{Var}(X_2)}$$

$$= 3 - \frac{1}{2} = \boxed{\frac{5}{2}}$$

Expected a constant!

(b) MMSE estimate of  $X_1$  given  $X_3$  and resulting MSE.

$$\hat{X}_{\text{MMSE}} = \mathbb{E}[X_1 | X_3]$$

Since  $X_1, X_3$  jointly Gaussian

$$\hat{X}_{\text{MMSE}} = \hat{X}_{2\text{MMSE}} = \mathbb{E}[X_1] + \frac{\text{Cov}(X_1, X_3)}{\text{Var}(X_3)} (X_2 - \mathbb{E}[X_3])$$

$$= 1 + 0$$

$$= 1$$

$$\text{MSE} = \text{Var}(X_1) - \frac{\text{Cov}^2(X_1, X_3)}{\text{Var}(X_3)}$$

$$= 3 - 0 = 3$$

(c) MMSE estimate of  $X_1$  given  $X_2, X_3$ ?

$$\hat{X}_{\text{MMSE}} = \mathbb{E}[X_1 | X_2, X_3]$$

Gaussian random vectors

Let  $\mathbf{Z}_1 = \begin{bmatrix} X_2 \\ X_3 \end{bmatrix} \subset \mathbf{X}$ ,  $\mathbf{Y} = [X_1] \subset \mathbf{X}$

Then

$$\hat{X}_{\text{MMSE}} = \mathbb{E}[\mathbf{Y} | \mathbf{Z}_1] \quad \begin{array}{l} \text{One Gaussian vector} \\ \text{Conditioned on Another} \end{array}$$

$$= \mathbb{E}[\mathbf{Y}] + \text{Cov}(\mathbf{Y}, \mathbf{Z}_1) \text{Cov}^{-1}(\mathbf{Z}_1) (\mathbf{Z}_1 - \mathbb{E}[\mathbf{Z}_1])$$

from lecture notes

$$\text{Cov}(\mathbf{Z}) = \begin{bmatrix} \text{Var}(X_2) & \text{Cov}(X_2, X_3) \\ \text{Cov}(X_3, X_2) & \text{Var}(X_3) \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\text{Cov}^{-1}(\mathbf{Z}) = \frac{1}{2-1} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \quad \begin{array}{l} \text{Then } A \text{ a } 2 \times 2 \text{ w/ } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ A^{-1} = \frac{1}{|A|} \begin{bmatrix} d & -c \\ -b & a \end{bmatrix} \end{array}$$

$$\mathbb{E}[\mathbf{Z}] = \begin{bmatrix} \mathbb{E}[X_2] \\ \mathbb{E}[X_3] \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

$$\mathbb{E}[\mathbf{Y}] = [\mathbb{E}[X_1]] = [1]$$

$$\text{Cov}(\mathbf{Y}, \mathbf{Z}) = \begin{bmatrix} \text{Cov}(X_1, X_2) & \text{Cov}(X_1, X_3) \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

So

$$\hat{X}_{\text{muse}} = [1] + [1 \ 0] \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \left( \begin{bmatrix} X_2 \\ X_3 \end{bmatrix} - \begin{bmatrix} 4 \\ 6 \end{bmatrix} \right)$$

$$= [1] + [1 \ -1] \begin{bmatrix} (X_2 - 4) \\ (X_3 - 6) \end{bmatrix}$$

$$= [1 + (X_2 - 4) - (X_3 - 6)]$$

$$= [1 + X_2 - 4 - X_3 + 6]$$

$$= [X_2 - X_3 + 3]_{1 \times 1}$$

- (d) The muse of  $X_1$  given  $X_2, X_3$  is a function of  $X_2$  AND  $X_3$  (even though  $X_1$  independent of  $X_3$ ) because  $X_2$  is NOT independent of  $X_3$ . <sup>i.e.</sup>  $X_2$  occurring has dependence on  $X_3$ . So,  $X_1 | X_2, X_3$  is dependent on BOTH.