

ECE 5630 - Homework Assignment 1

January 31st, 2020

Due to: Thursday, February 13th, 2020 (at the beginning of the lecture)

Instructions: Submission in pairs is allowed. Prove and explain every step in your answers.

1) **Discrete probability spaces:** Let Ω be a countable set.

a) Show that its power set 2^Ω is a σ -algebra.

b) Let $p : \Omega \rightarrow [0, 1]$ be a probability mass function (PMF), i.e., satisfying $\sum_{\omega \in \Omega} p(\omega) = 1$. Define a function $\mathbb{P}_p : 2^\Omega \rightarrow [0, 1]$ by $\mathbb{P}_p(A) \triangleq \sum_{\omega \in A} p(\omega)$, for $A \in 2^\Omega$. Show that $(\Omega, 2^\Omega, \mathbb{P}_p)$ is a probability space.

2) **σ -algebra:** Let Ω be an arbitrary set. All σ -algebras below are collections of subsets of Ω .

a) Consider a sequence of σ -algebras $\{\mathcal{F}_n\}_{n=1}^\infty$ such that $\mathcal{F}_n \subset \mathcal{F}_{n+1}$, for all $n \in \mathbb{N}$. Is $\mathcal{G} := \bigcup_{n=1}^\infty \mathcal{F}_n$ a σ -algebra? Either a proof (by showing it satisfies the required properties) or provide a counterexample.

b) Let $\{\mathcal{F}_i\}_{i \in I}$ be an arbitrary (possibly uncountable) collection of σ -algebras. Show that $\mathcal{H} := \bigcap_{i \in I} \mathcal{F}_i$ is a σ -algebra.

c) Let \mathcal{F}_1 and \mathcal{F}_2 be arbitrary σ -algebras. Define their Cartesian product $\mathcal{F}_1 \times \mathcal{F}_2 := \{A_1 \times A_2 \mid A_1 \in \mathcal{F}_1, A_2 \in \mathcal{F}_2\}$, where $A_1 \times A_2$, the Cartesian product of two sets A_1 and A_2 , is defined as $A_1 \times A_2 := \{(a_1, a_2) \mid a_1 \in A_1, a_2 \in A_2\}$. Either prove that $\mathcal{F}_1 \times \mathcal{F}_2$ is a σ -algebra or provide a counterexample.

3) **Properties of probability measures:** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Prove the following properties of \mathbb{P} :

a) Law of complement probability: $\mathbb{P}(A) = 1 - \mathbb{P}(A^c)$, $\forall A \in \mathcal{F}$, where $A^c = \Omega \setminus A$ is the complement of A .

b) Monotonicity: If $A, B \in \mathcal{F}$ with $A \subseteq B$, then $\mathbb{P}(A) \leq \mathbb{P}(B)$

c) Union bound: For any $\{A_n\}_{n=1}^\infty \subseteq \mathcal{F}$, we have $\mathbb{P}(\bigcup_{n=1}^\infty A_n) \leq \sum_{n=1}^\infty \mathbb{P}(A_n)$

d) Continuity of probability: Let $\{A_n\}_{n=1}^\infty \subseteq \mathcal{F}$ be a sequence of event increasing to $A \in \mathcal{F}$, i.e., $A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots$ and $\bigcup_{n=1}^\infty A_n = A$. Similarly, let $\{B_n\}_{n=1}^\infty \subseteq \mathcal{F}$ be a sequence of event decreasing to $B \in \mathcal{F}$, i.e., $B_1 \supseteq B_2 \supseteq B_3 \supseteq \dots$ and $\bigcap_{n=1}^\infty B_n = B$. Prove that:

i) $\lim_{n \rightarrow \infty} \mathbb{P}(A_n) = \mathbb{P}(A)$. Deduce that for any $\{A'_n\}_{n=1}^\infty \subseteq \mathcal{F}$, we have $\lim_{m \rightarrow \infty} \mathbb{P}(\bigcup_{n=1}^m A'_n) = \mathbb{P}(\bigcup_{n=1}^\infty A'_n)$.

ii) $\lim_{n \rightarrow \infty} \mathbb{P}(B_n) = \mathbb{P}(B)$. Deduce that for any $\{B'_n\}_{n=1}^\infty \subseteq \mathcal{F}$, we have $\lim_{m \rightarrow \infty} \mathbb{P}(\bigcap_{n=1}^m B'_n) = \mathbb{P}(\bigcap_{n=1}^\infty B'_n)$.

e) Law of Total Probability: Let $\{A_n\}_{n=1}^\infty \subseteq \mathcal{F}$ be a partition of Ω (i.e., (i) $A_n \cap A_m = \emptyset$, $\forall n \neq m$; and (ii) $\bigcup_{n=1}^\infty A_n = \Omega$) and $B \in \mathcal{F}$. Then $\mathbb{P}(B) = \sum_{n=1}^\infty \mathbb{P}(A_n)\mathbb{P}(B|A_n)$. Is your argument valid when $\mathbb{P}(A_{n'}) = 0$ for some $n' \in \mathbb{N}$?

4) **Measurability of indicators:** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. For $A \in \mathcal{F}$, define the function $\mathbb{1}_A : \Omega \rightarrow \mathbb{R}$ by

$$\mathbb{1}_A(\omega) \triangleq \begin{cases} 1, & \omega \in A \\ 0, & \omega \notin A \end{cases}.$$

Prove that $\mathbb{1}_A$ is a random variable over $(\Omega, \mathcal{F}, \mathbb{P})$.

5) **Induced probability measure vs. probability law:** Let $X : \Omega \rightarrow \mathbb{R}^d$ be a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ with probability law \mathbb{P}_X , i.e., $\mathbb{P}_X(B) := \mathbb{P}(X^{-1}(B))$, for $B \in \mathcal{B}(\mathbb{R}^d)$; \mathbb{P}_X is a probability measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$.

- a) Assume X is a discrete random variable, i.e., $\text{supp}(\mathbb{P}_X)$ is countable, and let $p_X : \text{supp}(\mathbb{P}_X) \rightarrow [0, 1]$ be its PMF, defined by $p_X(x) := \mathbb{P}_X(\{x\})$, $x \in \text{supp}(\mathbb{P}_X)$. Prove that the probability measure \mathbb{P}_{p_X} that is induced by the PMF p_X (see Question 1) coincides with the law of X , \mathbb{P}_X . In other words, show that $\mathbb{P}_{p_X}(B) = \mathbb{P}_X(B)$, for all $B \in \mathcal{B}(\mathbb{R}^d)$, where $\mathbb{P}_{p_X}(B) := \sum_{x \in \text{supp}(\mathbb{P}_X) \cap B} p_X(x)$, for $B \in \mathcal{B}(\mathbb{R}^d)$, is the trivial extension of \mathbb{P}_{p_X} from $\text{supp}(\mathbb{P}_X)$ to the entire Borel σ -algebra on \mathbb{R}^d .

Hint: Reduce your analysis from the continuous space $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ to the discrete space $(\text{supp}(\mathbb{P}_X), 2^{\text{supp}(\mathbb{P}_X)})$. Why is it suffices to consider only subsets of the form $B \cap \text{supp}(\mathbb{P}_X)$, where $B \in \mathcal{B}(\mathbb{R}^d)$?

- b) Assume X is continuous with probability density function (PDF) $f_X : \mathbb{R}^d \rightarrow \mathbb{R}_{\geq 0}$, i.e., f_X is the derivative of the cumulative distribution function (CDF) F_X of X . Recall that any PDF f (a nonnegative integrable function with $\int_{\mathbb{R}^d} f(x) dx = 1$) induces a measure \mathbb{P}_f on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ by defining $\mathbb{P}_f(B) := \int_B f(x) dx$, for all $B \in \mathcal{B}(\mathbb{R}^d)$. Prove that \mathbb{P}_{f_X} coincides with \mathbb{P}_X on the generating set of the Borel σ -algebra, i.e.,

$$\mathbb{P}_{f_X}((-\infty, a_1] \times \cdots \times (-\infty, a_d]) = \mathbb{P}_X((-\infty, a_1] \times \cdots \times (-\infty, a_d]), \quad \forall a_1, \dots, a_d \in \mathbb{R}.$$

Comment: In your proof, you may assume $d = 1$.

6) **Generation of samples with arbitrary distribution:** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $X : \Omega \rightarrow \mathbb{R}$ be random variable on it. Denote the probability law of X by \mathbb{P}_X , and let $F_X : \mathbb{R} \rightarrow [0, 1]$ be the CDF, i.e., $F_X(x) = \mathbb{P}_X((-\infty, x])$, for all $x \in \mathbb{R}$.

- a) Show that $Y := F_X(X)$ is a random variable.
- b) Prove that Y is uniformly distributed on $[0, 1]$. It suffices to show that F_Y , the CDF of Y , is given by $F_Y(y) = y$, for all $y \in [0, 1]$, and then explain how the CDF argument extends to the entire probability law \mathbb{P}_Y .
- c) Let $U \sim \text{Unif}[0, 1]$ be a uniformly distributed random variable. Propose a measurable function $T : [0, 1] \rightarrow \mathbb{R}$, such that $T(U)$ has distribution \mathbb{P}_X (or, equivalently, CDF F_X). Prove your answer, but you are not required to establish measurability of T .

Hint: The function T depends on F_X .

7) **Transition kernels:** Let $\Omega = \{0, 1\}$ and consider the following construction of a transition kernel $\kappa(\cdot|\cdot)$ from $(\Omega, 2^\Omega)$ to itself. For $\omega \in \{0, 1\}$, let $\text{Ber}(\alpha_\omega)$ be a Bernoulli distribution with parameter $\alpha \in [0, 1]$, i.e., as defined by the PMF $p_\omega(1) = 1 - p_\omega(0) = \alpha_\omega$. Set $\kappa(\cdot|\omega) = \text{Ber}(\alpha_\omega)$, for $\omega \in \Omega$. This construction of $\kappa(\cdot|\cdot)$ is called the *binary channel with flip parameters* $(\alpha_1, \alpha_2) \in [0, 1]^2$. Prove the following claims:

- i) $\kappa(\cdot|\cdot)$ is a transition kernel (verify the conditions in the definition).
- ii) Let $P_X = \text{Ber}(0.5)$ and define $P_Y(B) := \mathbb{E}[\kappa(B|X)]$, for $B \in 2^\Omega$. Show that whenever $\alpha_1 = \alpha_2$, we have $P_Y = \text{Ber}(0.5)$.
- iii) Give a counterexample to the symmetry of P_Y when $\alpha_1 \neq \alpha_2$.