

# ECE 4110 Homework 5

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*Due by 5pm on October 24*

## 1 Reading Material

- Characterization of random processes (Chapter 9.1-9.2.2).
- Independent increments and Stationary increments (Chapter 9.3).
- Stationarity and wide-sense stationarity (Chapter 9.6-9.6.2).

## 2 Assignment

### 1. Biased Random Walk

Let

$$S_n = \sum_{i=1}^n Y_i$$

where  $\{Y_i\}_{i \geq 1}$  are i.i.d. with PMF given by  $\Pr(Y_i = 1) = 1 - \Pr(Y_i = -1) = p$ . For  $p \neq \frac{1}{2}$ , it is referred to as a *biased* random walk.

- (a) Find the mean function, autocorrelation function, and autocovariance function of  $\{S_n\}_{n \geq 1}$ .
- (b) Is  $\{S_n\}_{n \geq 1}$  a stationary process? Is  $\{S_n\}_{n \geq 1}$  wide-sense stationary? Do your answers change for a balanced random walk with  $p = \frac{1}{2}$ ? Justify your answer.
- (c) Does  $\{S_n\}_{n \geq 1}$  have independent increments? Does  $\{S_n\}_{n \geq 1}$  have stationary increments? Justify your answer.

### 2. A Process with an Abrupt Change at a Random Point

Let  $X_1$  and  $X_2$  be two random variables with zero mean, variance  $\sigma^2$ , and covariance satisfying  $|\text{Cov}(X_1, X_2)| < \sigma^2$ . The random variable  $U$  is uniformly distributed over  $[0, 1]$  and independent of  $X_1$  and  $X_2$ . Define the following random process  $\{X(t), 0 \leq t \leq 1\}$ :

$$X(t) = \begin{cases} X_1 & \text{if } 0 \leq t < U \\ X_2 & \text{if } U \leq t \leq 1 \end{cases}.$$

- (a) Obtain  $\mu_X(t) \triangleq \mathbb{E}[X(t)]$  and  $\text{Var}(X(t))$  as functions of  $t \in [0, 1]$ .
- (b) Assume that  $\tau \geq 0$ . Calculate  $\mathbb{E}[X(t)X(t + \tau)]$  (assume that both  $t$  and  $t + \tau$  belong to  $[0, 1]$ ).

- (c) Assume that  $\tau \leq 0$ . Calculate  $\mathbb{E}[X(t)X(t + \tau)]$  and combine the result with above into a single compact formula for  $R_X(t, t + \tau) = \mathbb{E}[X(t)X(t + \tau)]$  valid for all  $t$  and  $t + \tau$  that belong to  $[0, 1]$ .
- (d) Is  $\{X(t), 0 \leq t \leq 1\}$  wide-sense stationary?

### 3. Random Arrival Times of a Poisson Process

Let  $\{N(t)\}_{t \geq 0}$  be a Poisson process. Given that  $N(\tau) = 1$  (i.e., a single arrival in the interval of  $[0, \tau]$ ). Show that the arrival time  $T_1$  of this given arrival is uniformly distributed in  $[0, \tau]$ .

*Hint: Show that  $\Pr(T_1 \leq x | N(\tau) = 1) = \frac{x}{\tau}$  for all  $0 \leq x \leq \tau$  by noting that the event  $T_1 \leq x$  is equivalent to  $N(x) = 1$ .*

### 4. Poisson Process Probabilities

Consider a Poisson process with rate  $\lambda > 0$ .

- (a) Find the probability that there is exactly one arrival in each of the three intervals  $[0, 1]$ ,  $[1, 2]$ ,  $[2, 3]$ .
- (b) Find the probability that there are two arrivals in the interval  $[0, 2]$  and two arrivals in the interval  $[1, 3]$ .
- Hint: write this event as a union of events defined by arrivals in non-overlapping intervals.*
- (c) Find the probability that there are two arrivals in the interval  $[1, 2]$ , given that there are two arrivals in the interval  $[0, 2]$  and two arrivals in the interval  $[1, 3]$ .

### 5. Merging and Thinning of Poisson Processes

Poisson processes have the following two important properties:

- *Merging independent Poisson processes:* Let  $N_1(t), \dots, N_k(t)$  be independent Poisson processes with rates  $\lambda_1, \dots, \lambda_k$ , respectively. Then the sum (i.e., by counting arrivals in all  $k$  processes together)

$$N(t) = \sum_{i=1}^k N_i(t)$$

is a Poisson process with rate  $\lambda = \sum_{i=1}^k \lambda_i$ .

- *Thinning a Poisson process:* Let  $N(t)$  be a Poisson process with rate  $\lambda$ . For each arrival in this process, label it to be of type  $i$  ( $i = 1, \dots, k$ ) with probability  $p_i$  where  $\sum_{i=1}^k p_i = 1$ . This labeling is independent across arrivals and independent of the arrival times. Then the type  $i$  arrivals form a Poisson process with rate  $p_i \lambda$  for all  $i = 1, \dots, k$ , and these  $k$  Poisson processes are independent.

Use the above properties to solve the follow problem.

In the little town of Acahti, there is a popular eatery called College Town Doughnuts (CTD). Customers arrive at CTD according to a Poisson process with rate  $\lambda$  (per hour). Each customer is of one of three types: the first type wants only a coffee, the second type wants only a doughnut, and the third type wants both a coffee and a doughnut. Each customer is of type  $i$  with probability  $p_i$  ( $i = 1, 2, 3$ ), independent of other customers and the arrival times.

- (a) What is the probability that the first customer to arrive wants a coffee?
- (b) What is the expected time between doughnut sales?
- (c) Are coffee and doughnut sales independent? Justify your answer.

- (d) Given that no coffee is sold during the first hour, what is the expected number of doughnuts sold during the first two hours?

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10/24/19 rp534  
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# ① Biased Random Walk

Let

$$S_n = \sum_{i=1}^n Y_i$$

where  $\{Y_i\}_{i \geq 1}$  are i.i.d. w/ pmf

$$\Pr(Y_i = 1) = 1 - \Pr(Y_i = -1) = p$$

For  $p \neq \frac{1}{2}$  we refer to this as a biased random walk.

(a) Find the mean function, auto correlation, and autocovariance function of  $\{S_n\}_{n \geq 1}$

**MEAN Function**

$$\mu_S(n) = n(2p-1)$$

$$\begin{aligned} \mathbb{E}[Y_i] &= 1 \cdot p + (-1) \cdot (1-p) \\ &= 2p-1 \end{aligned}$$

Since biased over  $\{-1, +1\}$   
earlier if  $\{0, 1\}$

**AUTOCORRELATION Function**

Assume  $m < n$ ,  $m, n \in \mathbb{N}$ .

$$\begin{aligned} R_S(m, n) &= \mathbb{E}[S_m S_n] \\ &= \mathbb{E}[S_m (S_n - S_m + S_m)] \\ &= \mathbb{E}[S_m^2] + \mathbb{E}[S_m (S_n - S_m)] \\ &= \text{Var}(S_m) + \mathbb{E}^2[S_m] + \mathbb{E}[S_m] \mathbb{E}[S_n - S_m] \\ &= m \text{Var}(Y_i) + m^2 (2p-1)^2 + m(2p-1)(n-m)(n-m)(2p-1) \\ &= m - m(2p-1)^2 + m^2(2p-1)^2 + m(n-m)(2p-1)^2 \\ &= m + (mn-m)(2p-1)^2 \end{aligned}$$

Generalize to

$$R_S(m, n) = \min(m, n) + (mn - \min(m, n))(2p-1)^2$$

## AUTOCOVARIANCE Function

$$\begin{aligned}
 C_s(m, n) &\stackrel{\Delta}{=} \mathbb{E}[(S_m - \mu_s(m))(S_n - \mu_s(n))] \\
 &= R_s(m, n) - \mu_s(m)\mu_s(n) \\
 &= \min(m, n) + (mn - \min(m, n))(2p-1)^2 - mn(2p-1)^2 \\
 &= \min(m, n) - \min(m, n)(2p-1)^2 \\
 &= \min(m, n)[1 - (2p-1)^2] \\
 &= \min(m, n)4p(1-p)
 \end{aligned}$$

(b) Is  $\{S_n\}_{n \geq 1}$  a stationary process? Is it wide sense stationary?  
Do these answers change for a balanced random walk ( $p = 1/2$ )?

**STRICTLY STATIONARY?** NO      Take  $k$  values of  $n$   
    WLOG

The joint pmf of  $S_{n_1}, S_{n_2}, \dots, S_{n_k}$  ( $n_1 < n_2 < \dots < n_k$ ) is

$$\begin{aligned}
 &\Pr[S_{n_1} = s_1, S_{n_2} = s_2, \dots, S_{n_k} = s_k] \\
 &= \Pr[S_{n_1} = s_1] \Pr[S_{n_2} = s_2 | S_{n_1} = s_1] \Pr[S_{n_3} = s_3 | S_{n_1} = s_1, S_{n_2} = s_2] \dots \Pr[S_{n_k} = s_k | S_{n_1} = s_1, S_{n_2} = s_2, \dots, S_{n_{k-1}} = s_{k-1}] \\
 &= \Pr[S_{n_1} = s_1] \Pr[S_{n_2} - S_{n_1} = s_2 - s_1] \Pr[S_{n_3} - S_{n_2} = s_3 - s_2] \dots \Pr[S_{n_k} - S_{n_{k-1}} = s_k - s_{k-1}] \\
 &= \binom{n_1}{s_1} \binom{n_2 - n_1}{s_2 - s_1} \dots \binom{n_k - n_{k-1}}{s_k - s_{k-1}} p^{s_k} (1-p)^{n_k - s_k}
 \end{aligned}$$

From here it is easy to observe that 'shifting' the pmf does NOT yield the same answer!

$$\begin{aligned}
 &\Pr[S_{n_1} = s_1 + \Delta, S_{n_2} = s_2 + \Delta, \dots, S_{n_k} = s_k + \Delta] \\
 &= \binom{n_1 + \Delta}{s_1} \binom{n_2 - n_1}{s_2 - s_1} \dots \binom{n_k - n_{k-1}}{s_k - s_{k-1}} p^{s_k + \Delta} (1-p)^{n_k + \Delta - s_k} \\
 &\neq \binom{n_1}{s_1} \binom{n_2 - n_1}{s_2 - s_1} \dots \binom{n_k - n_{k-1}}{s_k - s_{k-1}} p^{s_k} (1-p)^{n_k - s_k}
 \end{aligned}$$

Thus NOT  
strictly stationary

## WIDE SENSE STATIONARY? NO

$\mathbb{E}[S_n]_{n \geq 1} \neq \mu$  ( $\mu$  constant)!  $R_x(m, n)$  doesn't depend on  $m - n$

It is an increasing function in  $n$ .

In fact,  $\mathbb{E}[S_n]_{n \geq 1} = np$

Thus  $\{S_n\}_{n \geq 1}$  is **NOT** wide sense stationary.

## WHAT IF $p = \frac{1}{2}$ ?

It is easy to see the choice of  $p$  does NOT impact the proof for strictly stationary above. For WSS however, the expectation is zero which is a constant **BUT**  $R_x(m, n)$  doesn't depend only on  $m - n$ . So Not WSS.

(c) Does  $\{S_n\}_{n \geq 1}$  have independent increments?

Does  $\{S_n\}_{n \geq 1}$  have stationary increments?

## Independent Increments? YES!

Take  $n > m$ .  $S_n - S_m = n - m$  Bernoulli( $p$ ) trials.

Since each  $Y_i$  i.i.d then  $\forall n > m \quad S_n - S_m \sim \text{Binomial}(n - m, p)$   
and they are independent. Thus every increment is independent.

## STATIONARY Increments? YES!

$\forall k > 0$

$S_{n+k} - S_n \sim \text{Bin}(n+k-n, p) \Rightarrow \text{Bin}(k, p)$  depends only on  $k$  not  $n$  thus all increments are stationary.

## ② A Process With an Abrupt Change at a RANDOM Point

Let  $X_1, X_2$  be two random variables with mean 0, variance  $\sigma^2$ , and covariance satisfying  $|\text{Cov}(X_1, X_2)| < \sigma^2$ .

The random variable  $U$  is uniformly distributed on  $[0, 1]$  and is independent of  $X_1, X_2$ .

Define the random process  $\{X(t)\}_{0 \leq t \leq 1}$  by

$$X(t) = \begin{cases} X_1 & 0 \leq t < U \\ X_2 & U \leq t \leq 1 \end{cases}$$

(a) Obtain  $\mu_x(t) \triangleq \mathbb{E}[X(t)]$  and  $\text{Var}(X(t))$  as functions of  $t \in [0, 1]$ .

$\mu_x(t)$

$$\begin{aligned} \mu_x(t) &= \Pr(U \leq t) \mathbb{E}[X_2] + \Pr(U > t) \mathbb{E}[X_1] \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

$\text{Var}(X(t))$

$$\begin{aligned} \text{Var}(X(t)) &= \mathbb{E}[X^2(t)] - \mu_x^2(t) \\ &= \Pr(U \leq t) \mathbb{E}[X_2^2] + \Pr(U > t) \mathbb{E}[X_1^2] \\ &= \frac{t}{1} \sigma^2 + \frac{(1-t)}{1} \sigma^2 \\ &= \sigma^2 \end{aligned}$$

(b) Assume  $\mathcal{T} \neq 0$ .

Calculate  $\mathbb{E}[X(t)X(t+\tau)]$  - assuming  $t, t+\tau \in [0, 1]$

$$\begin{aligned}\mathbb{E}[X(t)X(t+\tau)] &= \Pr(0 \leq u \leq t)\mathbb{E}[X_1^2] \\ &\quad + \Pr(t < u \leq t+\tau)\text{Cov}(X_1, X_2) \\ &\quad + \Pr(t+\tau < u \leq 1)\mathbb{E}[X_2^2] \\ &= t\sigma^2 + (t+\tau-t)(\text{Cov}(X_1, X_2) + (1-(t+\tau))\sigma^2) \\ &= t\sigma^2 + \sigma^2 - t\sigma^2 - \tau\sigma^2 + \tau(\text{Cov}(X_1, X_2)) \\ &= \sigma^2(1-\tau) + \tau(\text{Cov}(X_1, X_2))\end{aligned}$$

(c) Assume  $\mathcal{T} = 0$ .

Calculate  $\mathbb{E}[X(t)X(t+\tau)]$

$$\begin{aligned}\mathbb{E}[X(t)X(t+\tau)] &= \Pr(t < u \leq 1)\mathbb{E}[X_1^2] \\ &\quad + \Pr(t+\tau < u \leq t)\text{Cov}(X_1, X_2) \\ &\quad + \Pr(0 \leq u \leq t+\tau)\mathbb{E}[X_2^2] \\ &= (1-t)\sigma^2 + (t-(t+\tau))\text{Cov}(X_1, X_2) + (t+\tau)\sigma^2 \\ &= \sigma^2 - t\sigma^2 - \tau(\text{Cov}(X_1, X_2)) + t\sigma^2 + \tau\sigma^2 \\ &= (1+\tau)\sigma^2 - \tau(\text{Cov}(X_1, X_2))\end{aligned}$$

So, for general  $\tau$ , have

$$E[X(t)X(t+\tau)] = \sigma^2(1 - |\tau|) + |\tau| \text{Cov}(x_1, x_2)$$

(d)  $\{X(t), 0 \leq t \leq 1\}$  is WSS.

mean is zero for all  $t \in [0, 1]$ .

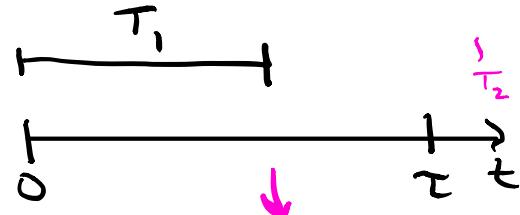
$R_X(t, t+\tau)$  depends only on  $\tau$ .

### ③ Random Arrival Times of a Poisson Process

Let  $\{N(t)\}_{t \geq 0}$  be a Poisson process.

Given  $N(\tau) = 1$ , show that the arrival time  $T_1$  of this given arrival is uniformly distributed in  $[0, \tau]$ .

$N(\tau) \rightarrow \# \text{arrivals in interval } [0, \tau]$ .



$$\Pr(T_1 \leq x \mid N(\tau) = 1) = \Pr(N(x) = 1 \mid N(\tau) = 1)$$

$$= \frac{\Pr(N(\tau) = 1 \mid N(x) = 1) \Pr(N(x) = 1)}{\Pr(N(\tau) = 1)}$$

$\Pr(\text{second arrival after } \tau)$

$$= \frac{e^{-\lambda(\tau-x)} \lambda x e^{-\lambda x}}{\lambda \tau e^{-\lambda \tau}} = \frac{x}{\tau} \text{ for } x \in [0, \tau]$$

$\Pr(T_2 > \tau)$

Note that  $x > \tau$  we always get one by definition.

Thus

$$\Pr(T_1 \leq x \mid N(\tau) = 1) = \begin{cases} \frac{x}{\tau}, & 0 \leq x \leq \tau \\ 0, & x < 0 \\ 1, & x > \tau \end{cases}$$

i.e. for every  $x$  we have shown we satisfy the uniform CDF when conditioning on  $N(\tau) = 1$ , thus  $T_1$  is uniform

## ④ Poisson Process Probabilities

Consider a Poisson process w/ rate  $\lambda$ .

- (a) Find the probability that there is exactly one arrival in each of the three intervals  $[0,1]$ ,  $[1,2]$ ,  $[2,3]$ .

non-overlapping increments.

$$\Pr(N(1) - N(0) = 1) = \lambda e^{-\lambda}$$

$$\Pr(N(2) - N(1) = 1) = \lambda e^{-\lambda} e^{-(2-\lambda)} = \lambda e^{-\lambda}$$

$$\Pr(N(3) - N(2) = 1) = \lambda e^{-\lambda}$$

$$\text{Thus } \Pr[N(3) - N(2) = 1, N(2) - N(1) = 1, N(1) - N(0) = 1]$$

$$= (\lambda e^{-\lambda})^3 = \lambda^3 e^{-3\lambda}$$

- (b) Find the probability that there are two arrivals in the interval  $[0,2]$  and two arrivals in the interval  $[1,3]$ .

$$\begin{aligned}
 &= \Pr[N(2) - N(1) = 2] \cdot \Pr[N(1) - N(0) = 0] \cdot \Pr[N(3) - N(2) = 0] \\
 &\quad + \Pr[N(1) - N(0) = 2] \cdot \Pr[N(3) - N(2) = 2] \cdot \Pr[N(2) - N(1) = 0] \\
 &\quad + \Pr[N(1) - N(0) = 1] \cdot \Pr[N(2) - N(1) = 1] \cdot \Pr[N(3) - N(2) = 1]
 \end{aligned}$$

Both in middle  
 in ends  
 one in each

$$= \frac{\lambda^2 e^{-\lambda}}{2} (e^{-\lambda})^2 + \left(\frac{\lambda^2 e^{-\lambda}}{2}\right)^2 e^{-\lambda} + (\lambda e^{-\lambda})^3$$

$$= e^{-3\lambda} \lambda^2 \left(\frac{\lambda^2}{2} + \lambda + \frac{1}{2}\right)$$

(c) Find the probability that there are two arrivals in the interval  $[1,2]$ , given that there are two arrivals in the interval  $[0,2]$  and two arrivals in the interval  $[1,3]$ .

$$\frac{\lambda^2}{2} + \frac{\lambda}{2} + \frac{1}{4}$$

$$\Pr [N(2) - N(1) = 2 \mid N(2) - N(0) = 2, N(3) - N(1) = 2]$$

$$\Pr [N(2) - N(0) = 2, N(3) - N(1) = 2] = e^{-3\lambda} \lambda^2 \left( \frac{\lambda^2}{4} + \lambda + \frac{1}{2} \right)$$

$$\Pr [N(2) - N(0) = 0, N(3) - N(1) = 0, N(2) - N(1) = 2]$$

$$\Pr [N(2) - N(0) = 2, N(3) - N(1) = 2]$$

$$= \frac{\frac{1}{2} e^{-3\lambda} \lambda^2}{e^{-3\lambda} \lambda^2 \left( \frac{\lambda^2}{4} + \lambda + \frac{1}{2} \right)} = \frac{1}{\frac{\lambda^2}{2} + 2\lambda + 1}$$

$$= \frac{2}{\lambda^2 + 4\lambda + 2}$$

## ⑤ Merging and Thinning of Poisson Process

### Merging Independent Poisson Processes

Let  $N_1(t), \dots, N_k(t)$  be independent Poisson processes w/ rates  $\lambda_1, \dots, \lambda_k$  respectively.

Then

$$N(t) = \sum_{i=1}^k N_i(t)$$

is a Poisson process with rate parameter

$$\lambda = \sum_{i=1}^k \lambda_i$$

### Thinning a Poisson Processes

Let  $N(t)$  be a Poisson process w/ rate parameter  $\lambda$ .

For each arrival in this process, label it to be type  $i$  ( $i=1, \dots, k$ ) with probability  $p_i$ , where  $\sum_{i=1}^k p_i = 1$ .

This labeling is independent across arrivals and independent of arrival times.

Then the type  $i$  arrivals form a Poisson process with rate  $p_i \lambda$   $\forall i=1, \dots, k$ , and these  $k$  Poisson processes are independent.

In the little town of Acahti, there is a popular eatery called College Town Doughnuts (CTD). Customers arrive at CTD according to a Poisson process with rate  $\lambda$  (per hour). Each customer is of one of three types: the first type wants only a coffee, the second type wants only a doughnut, and the third type wants both a coffee and a doughnut. Each customer is of type  $i$  with probability  $p_i$  ( $i = 1, 2, 3$ ), independent of other customers and the arrival times.

(a) Probability first customer to arrive wants a coffee?

People who want coffee arrive at rates  $p_1\lambda$  AND  $p_3\lambda$

So the probability the first arrival wants coffee is contingent it is customer of type 1 or 3. This is  $p_1 + p_3$ .

(b) Expected times between doughnut sales

Each interarrival of a person who wants a doughnut is exponential  $(\lambda(p_2 + p_3))$

via merging of Poisson processes where people buy doughnuts.

Thus the expected time between doughnut sales is

$$\frac{1}{\lambda(p_2 + p_3)}$$

(c) Are coffee and doughnut sales independent?

No!

To see this observe that  $p_1\lambda_1$  and  $p_2\lambda_2$  are not independent of  $p_3\lambda_3$ .

i.e. the customer who wants both items prevents independence of arrived customers wanting just one thing.

Also note, the expectation's don't factor.

i.e

$$\text{Coffee} \sim \text{Pois}(\lambda(p_1 + p_3))$$

$$\text{Doughnut} \sim \text{Pois}(\lambda(p_2 + p_3))$$

$$\text{Coffee} + \text{Doughnut} \sim \text{Pois}(\lambda(p_1 + p_2 + p_3))$$

$$\begin{aligned}\mathbb{E}[\text{coffee} + \text{Doughnut}] &= \lambda(p_1 + p_2 + p_3) \\ &\neq \mathbb{E}[\text{coffee}] \mathbb{E}[\text{Doughnut}] \\ &= \lambda^2(p_1 + p_3)(p_2 + p_3)\end{aligned}$$

(d) Given no coffee sold in first hour, what is the expected amount of doughnuts sold in first two hours?

So in first hour only customers of type 2 showed up (if any).

After, 2 or 3 showed up (if any).

$$\mathbb{E}[\text{Pois}(\lambda p_2)] + \mathbb{E}[\text{Pois}(\lambda(p_2 + p_3))]$$

$$= \lambda p_2 + \lambda p_2 + \lambda p_3$$

$$= 2\lambda p_2 + \lambda p_3$$

$$= \lambda(2p_2 + p_3)$$