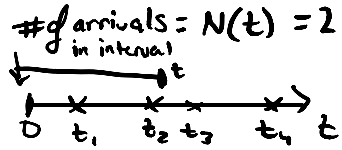
Definitions are bi-directional. $(\Leftrightarrow)$

Properties are one-directional $(\Rightarrow)$

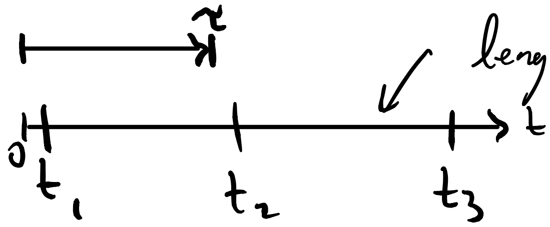
Facts about Poisson Process



$\{N(t)\}_{t \geq 0}$ w/ rate λ. $\lambda \to$ arrival rate / unit time

① Process has independent increments

$$N(t_2) - N(t_1) \perp\!\!\!\perp N(t_3) - N(t_2)$$



length doesn't matter on interval, they just need NOT overlap

② # arrivals in interval $[t_1, t_2] = N(t_2) - N(t_1)$

where

$$N(t_2) - N(t_1) \sim \text{Pois}(\lambda(t_2 - t_1))$$

③ The interarrival times $\{T_i\}_{i=1}^{\infty}$ are iid

where

$$T_i \sim \exp(\lambda)$$

④ Given $N(\tau) = k$, these k arrivals are iid $U[0, \tau)$

i.e $t_i \sim U[0, k]$

⑤ $\mu_N(t) = \mathbb{E}[N(t)] = \lambda t$ i.e arrival rate* time

$Var(N(t)) = \lambda t$ (Poisson property)

$R_N(s,t) \triangleq \mathbb{E}[N(s)N(t)]$ assume $s<t$ (if $s=t$ have var.)

$= \mathbb{E}[N(s)(N(s)+N(t)-N(s))]$

$= \mathbb{E}[(N(s))^2] + \mathbb{E}[\underline{N(s)}\,\underline{(N(t)-N(s))}]$ independ. increments

$= \lambda s + (\lambda s)^2 + \mathbb{E}[N(s)]\,\mathbb{E}[N(t)-N(s)]$ → $\mathbb{E}[N(t)] - \mathbb{E}[N(s)]$

$= \lambda s + (\lambda s)^2 + \lambda s(\lambda(t-s))$

$C_N(s,t) = \lambda \min(s,t)$ ← applies $\underline{\forall s,t}$ (note above we assumed $s<t$)

⑥ Merging

If

$$N_1(t) \perp\!\!\!\perp N_2(t) \perp\!\!\!\perp \cdots \perp\!\!\!\perp N_m(t)$$

w/ arrival λ_1 λ_2 λ_m

then

$$N_1(t) + \cdots + N_m(t) \sim Pois(\lambda_1 + \cdots \lambda_m)$$

Note if $T_1 \perp\!\!\!\perp \cdots \perp\!\!\!\perp T_k$, $T_i \sim exp(\lambda_i)$

then $T_{min} = \min(T_1, \ldots, T_k) \sim exp(\lambda_1 + \cdots + \lambda_k)$

⑦ Thinning (see Notes)

Example: Store opens at 8AM every day. i.e 8AM is "t=0"

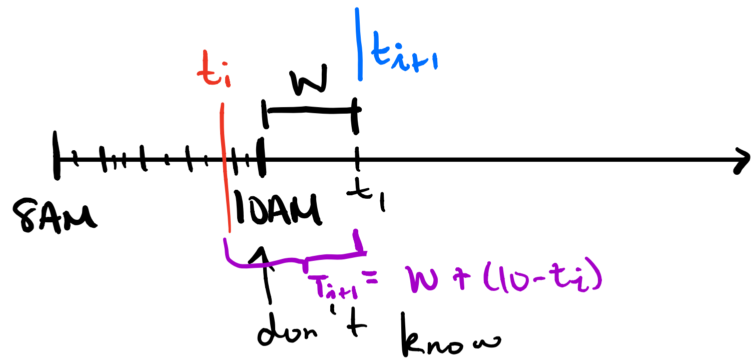
Customers arrivals: Poisson $(\lambda = \frac{\text{arrivals}}{\text{hr}})$

One day, owner opens up late - at 10AM.

① What is the probability owner missed a customer?

② Expected waiting time to see first customer?

①



$1 - Pr[N(2) = 0]$

$1 - Pr[T_1 > 2]$

$1 - e^{-5(2)}$

②

if customer arrived at 10

Condition on t_i = time last customer arrived before 10 AM

$\underline{\forall i, t_i}$

$$Pr[W > w \mid N(2) = i, \text{ the } i^{th} \text{ arrival is at } t_i]$$

$$\Rightarrow Pr[T_{i+1} > 10 - t_i + w \mid T_{i+1} > 10 - t_i]$$

$$= Pr[T_{i+1} > w]$$

$$= e^{-\lambda w}$$

Gaussian Random Processes

Definition:

$X(t)$ is Gaussian if $\forall n$, and $\forall \{t_i\}_{i=1}^{n}$

r.v.s

$X(t_1), \ldots, X(t_n)$ are JOINTLY Gaussian

"Completely specified by first two moments"

Properties

① completely specified by its mean function $\mu_X(t)$ and autocorrelation function $R_X(t)$/ autocovariance function $C_X(t)$.

Example

"READ RIGHT VALUES!"

$$f_{X(t_1),X(t_2)}(x_1, x_2) \sim \mathcal{N}\left(\begin{bmatrix} \mu_X(t_1) \\ \mu_X(t_2) \end{bmatrix}, \mathbb{K} = \begin{bmatrix} C_X(t_1) & C_X(t_1,t_2) \\ C_X(t_2,t_1) & C_X(t_2) \end{bmatrix} \right)$$

② WSS $\Rightarrow$ S