

Recap

MSE Estimator

Have rv X which correlates w/ Y .

Have estimation $\hat{X} = g(Y)$

Error $W = X - \hat{X}$

Define MSE as $\mathbb{E}[W^2]$

① For \hat{X} a constant,

$$\hat{X}_{\text{mse}}^* = \mathbb{E}[X]$$

$$\text{MSE} = \text{Var}(X)$$

② For \hat{X} a function of Y

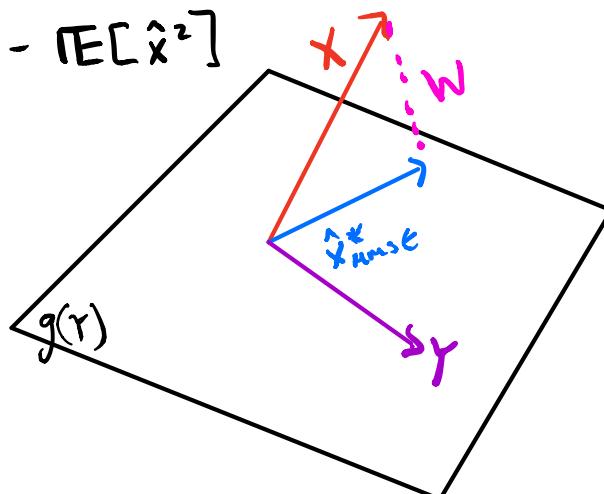
$$\hat{X}_{\text{mse}}^* = \mathbb{E}[X|Y]$$

$$\text{MSE} = \mathbb{E}_Y [\text{Var}(X|Y)]$$

Showed that

$$\text{MSE} = \mathbb{E}[X^2] - \mathbb{E}[\hat{X}^2]$$

$$W \perp g(Y)$$



Linear Minimum Mean Squared Error

Now our estimator $g(Y)$ takes the form

$$g(Y) = aY + b$$

The MSE becomes

$$\begin{aligned} \text{MSE} &= \mathbb{E}[(X - g(Y))^2] \\ &= \mathbb{E}[(X - aY - b)^2] \end{aligned}$$

Want

$$\{a^*, b^*\} = \underset{a,b}{\operatorname{argmin}} \mathbb{E}[(X - aY - b)^2]$$

Use orthogonality principle

$$W \perp g(Y) \Rightarrow X - a^*Y - b^* \perp aY + b \quad \cancel{\perp a, b}$$

Consider the following cases

$$\textcircled{1} \quad a=0, b=1$$

$$\begin{aligned} X - a^*Y - b^* + 1 &\Rightarrow \mathbb{E}[X - a^*Y - b^*] = 0 \\ \mathbb{E}[X] - \mathbb{E}[a^*Y] - \mathbb{E}[b^*] &= 0 \quad \begin{matrix} \textcolor{red}{X \perp Y \Leftrightarrow \mathbb{E}[XY]} \\ \textcolor{pink}{W \rightarrow \mathbb{E}[W=0]} \\ \textcolor{blue}{\Rightarrow \text{UNBIASED Estimator}} \end{matrix} \\ \rightarrow b^* &= \mathbb{E}[X] - a^* \mathbb{E}[Y] \end{aligned}$$

$$\textcircled{2} \quad a=1, b=0$$

$$X - a^*Y - b^* \perp Y \Rightarrow \mathbb{E}[(X - a^*Y - b^*)Y] = 0$$

$$\begin{aligned} &\mathbb{E}[(X - a^*Y - b^*)Y] \\ &= \mathbb{E}[(X - a^*Y - \mathbb{E}[X] - a^*\mathbb{E}[Y])Y] \\ &= \mathbb{E}[(X - \mathbb{E}[X] - a^*(Y - \mathbb{E}[Y]))Y] \\ &\xrightarrow{\text{First term } W} \Rightarrow \text{Cov}((X - \mathbb{E}[X] - a^*(Y - \mathbb{E}[Y])), Y) = 0 \end{aligned}$$

Note $\text{Cov}(wY) = w\mathbb{E}[wY]$
here

$$\text{Cov}(X, Y) - a^* \text{Cov}(Y, Y) = 0$$

$$a^* = \text{Cov}(X, Y) / \text{Var}(Y) \leftarrow \text{Cov}(Y, Y)$$

Aside

$$\text{Cov}(X, aY + bZ) = a \text{Cov}(X, Y) + b \text{Cov}(X, Z)$$

$$\text{Cov}(X - \text{IE}[X], Y - \text{IE}[Y]) = \text{Cov}(X, Y)$$

So

$$\begin{aligned} X_{\text{LME}}^* &= a^*Y + b^* \\ &= \frac{\text{Cov}(X, Y)}{\text{Var}(Y)} Y + \text{IE}[X] - \frac{\text{Cov}(X, Y)}{\text{Var}(Y)} \text{IE}[Y] \\ &= \frac{\text{Cov}(X, Y)}{\text{Var}(Y)} (Y - \text{IE}[Y]) + \text{IE}[X] \end{aligned}$$

and the MSE is

$$\begin{aligned} \text{MSE} &= \text{IE}[W^2] = \text{Cov}(W, W) \\ &= \text{Cov}((X - \text{IE}[X] - a^*(Y - \text{IE}[Y])), (X - \text{IE}[X] - a^*(Y - \text{IE}[Y]))) \\ &= \text{Cov}(X, X) + a^* \text{Cov}(Y, Y) - 2a^* \text{Cov}(X, Y) \\ &= \text{Var}(X) + \left(\frac{\text{Cov}(X, Y)}{\text{Var}(Y)} \right)^2 \text{Var}(Y) - 2 \left(\frac{\text{Cov}(X, Y)}{\text{Var}(Y)} \right) \text{Cov}(X, Y) \\ &= \text{Var}(X) - \frac{(\text{Cov}(X, Y))^2}{\text{Var}(Y)} \end{aligned}$$

Example

$$X \sim U(-1, 1) \quad Y = X^2$$

$$f_X(x) = \begin{cases} \frac{1}{2}, & -1 \leq x \leq 1 \\ 0, & \text{o/w} \end{cases}$$

MSE X to estimate Y

$$\hat{Y}_{\text{MSE}} = \mathbb{E}[Y|X] = X^2$$

$$\text{MSE} = \mathbb{E}((Y - \hat{Y})^2) = 0$$

$$\hat{Y}_{\text{LMSE}} = \frac{\text{cov}(X, Y)}{\text{var}(X)} (X - \mathbb{E}(X)) + \mathbb{E}(Y)$$

$$= \mathbb{E}(Y)$$

$$\text{MSE} = \mathbb{E}[(Y - \hat{Y})^2] = \mathbb{E}[(Y - \mathbb{E}[Y])^2] = \text{var}(Y)$$



Shows LMSE isn't always preferred w/o linear relations

Example - Find LMSE

$$f_{X,Y}(x,y) = \begin{cases} x+y, & 0 \leq x, y \leq 1 \\ 0, & \text{o/w} \end{cases}$$

$$\hat{X} = g(Y) = \frac{\text{Cov}(X,Y)}{\text{Var}(Y)} (Y - \text{IE}[Y]) + \text{IE}[X]$$

Need to compute 4 terms

$$\text{IE}[X] = \iint_{y=0, x=0}^1 x(x+y) dx dy = \int_{y=0}^1 \left[\frac{x^3}{3} + \frac{x^2}{2} y \right]_{x=0}^1 dy = \left[\frac{1}{3}y + \frac{1}{4}y^2 \right]_{y=0}^1 = \frac{7}{12}$$

$$\text{IE}[Y] = \iint_{y=0, x=0}^1 y(x+y) dx dy = \frac{7}{12}$$

$$\text{IE}[XY] = \iint_{x=0, y=0}^1 (xy)(x+y) dy dx = \frac{1}{3}$$

$$\text{Cov}(X,Y) = \text{IE}[XY] - \text{IE}[X]\text{IE}[Y] = \frac{1}{3} - \left(\frac{7}{12}\right)^2 = -\frac{1}{144}$$

$$\begin{aligned} \text{Var}(Y) &= \text{IE}[Y^2] - (\text{IE}[Y])^2 \\ &\quad \left\{ \begin{aligned} \text{IE}[Y^2] &= \int_{x=0}^1 \int_{y=0}^1 y^2(x+y) dy dx = \frac{5}{12} \\ &\quad = \frac{5}{12} - \left(\frac{7}{12}\right)^2 = \frac{11}{144} \end{aligned} \right. \end{aligned}$$

Thus

$$\hat{X}_{\text{LMSE}}^* = -\frac{1}{11} \left(Y - \frac{7}{12}\right) + \frac{7}{12}$$

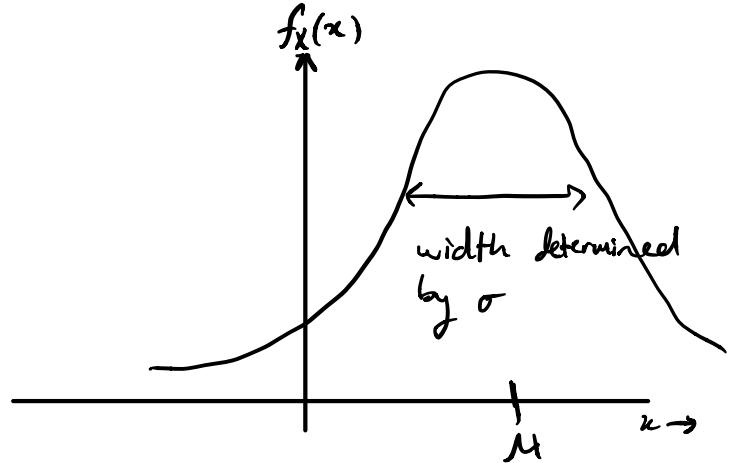
Jointly Gaussian Random Variables

If

$$X \sim N(\mu, \sigma^2)$$

then

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$



Definition: X, Y are jointly Gaussian if $aX + bY$ is Gaussian $\forall a, b \in \mathbb{R}$.

Definition: Joint pdf of such a distribution is

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{|K|}} \exp\left(-\frac{1}{2} \begin{pmatrix} x - \mu_x \\ y - \mu_y \end{pmatrix}^T K^{-1} \begin{pmatrix} x - \mu_x \\ y - \mu_y \end{pmatrix}\right)$$

Note] K is our covariance matrix.

Here,

$$K = \begin{pmatrix} \text{Cov}(X,X) & \text{Cov}(X,Y) \\ \text{Cov}(Y,X) & \text{Cov}(Y,Y) \end{pmatrix}$$

$$= \mathbb{E} \left[\begin{pmatrix} X - \mu_x \\ Y - \mu_y \end{pmatrix} \begin{pmatrix} X - \mu_x & Y - \mu_y \end{pmatrix}^T \right]$$

Implications

① Jointly Gaussian \Rightarrow Marginally Gaussian

X, Y

✗

X, Y

Example

$$X \sim N(0,1)$$

$$Y = \begin{cases} X & \text{if } |X| > 1 \\ -X & \text{if } |X| \leq 1 \end{cases}$$

$$\Pr(A) = \Pr(A \cap B) \cup \Pr(A \cap B^c)$$

$$F_Y(y) = \Pr(Y \leq y) = \Pr(Y \leq y, |X| > 1) + \Pr(Y \leq y, |X| \leq 1)$$

$$= \Pr(X \leq y \cap (X > 1 \text{ or } X < -1)) + \Pr(X \geq y, -1 \leq X \leq 1)$$

$$= \Pr(X \leq y, X > 1) + \Pr(X \leq y, X < -1) + \Pr(X > -y, -1 \leq X \leq 1)$$

$$= \begin{cases} \Pr(1 \leq X \leq y) + \Pr(X < -1) + \Pr(-1 \leq X \leq 1) = \Pr(X \leq y), & y > 1 \\ 0 + \Pr(X < -1) + \Pr(-y \leq X \leq 1) = \Pr(-1 \leq X \leq y), & -1 < y < 1 \\ 0 + \Pr(X \leq y) + 0 = \Pr(X \leq y), & y \leq -1 \end{cases}$$

Thus

$$F_Y(y) = F_X(y) \Rightarrow X, Y \text{ marginally Gaussian}$$

But X, Y NOT jointly Gaussian.

Assume towards a contradiction $X+Y = 1/2$ (uses definition 1)

then

$$X+Y = \begin{cases} 2X, & |X| > 1 \\ 0, & |X| \leq 1 \end{cases} \Rightarrow |X| > 1 \Rightarrow X = \frac{1}{4} \quad \text{Contradiction!}$$