1. This is a variant of the last problem on the previous homework assignment. The input X to a binary communication channel is a continuous random variable distributed uniformly on the interval $[-c, c]$, Here, $c > 0$ is given. The channel output is

$$Y = X + N ,$$

where N is Gaussian with zero mean and variance 1.

(a) Find the marginal pdf $f_Y(y)$. Please provide a closed-form solution for your answer in terms of the standard normal cdf Φ.
(b) Plot $f_Y(y)$ on the interval $-10 \le y \le 10$ for $c = 0.1$ and $c = 5$.

2. Two jointly continuous random variables X and Y have joint pdf

$$f_{X,Y}(x,y) = \begin{cases} 12xy(1-x) & \text{when } x \in [0,1] \text{ and } y \in [0,1] \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find $\mathbb{E}(X)$
(b) Find $\mathbb{E}(Y)$.
(c) Find $\text{Var}(X)$.
(d) Find $\text{Var}(Y)$.
(e) Are X and Y independent?

3. While an excellent archer, Legolas falls short as a dart thrower. When he throws a dart at a circular dart board of radius R, it lands at a point (X, Y) distributed uniformly over the board, where the center of the board lies at $X = Y = 0$. That is, the dart's landing coordinates X and Y have a uniform joint pdf

$$f_{X,Y}(x,y) = \begin{cases} c & \text{when } x^2 + y^2 \le R^2 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the constant c.
(b) Find the marginal pdfs $f_X(x)$ and $f_Y(y)$.
(c) Let D be the distance the dart lands from the center of the board. Find $\mathbb{P}(\{D \le d\})$ as a function of d for $d \in (0, R]$.
(d) Find $\mathbb{E}(D)$.

4. Let $X_1, X_2, \ldots, X_n$ be independent random variables with respective cdfs $F_{X_k}(x)$ for $1 \le k \le n$.

(a) Find the cdf of the random variable $Y = \max\{X_1, \ldots, X_n\}$ in therms of the F_{X_k}'s.
(b) Find the cdf of the random variable $Z = \min\{X_1, \ldots, X_n\}$ in therms of the F_{X_k}'s.

5. Computers are good at generating values of a random variable X distributed uniformly on the interval $[0, 1]$, and we can exploit that capability to simulate other random variables. Suppose, for example, that $G(y)$ takes values in the interval $[0, 1]$ and is strictly increasing in y on some interval $a < y < b$ with $G(a) = 0$ and $G(b) = 1$ — $a = -\infty$ and/or $b = \infty$ are allowed. Define a random variable Y as follows: $Y = h(X)$, where for each $x \in [0, 1]$ $h(x)$ is the unique value of y such $G(y) = x$.

(a) Show that the cdf $F_Y(y)$ is $G(y)$. Explain how to use the computer to generate a sequence of numbers that looks like a sequence of independent draws of the random variable Y.
(b) Give the details of implementing the procedure when the random variable you want to simulate is exponential with rate parameter λ.
(c) How might you use the computer's uniform-X-generation capability to simulate a discrete integer-valued random variable? For starters, consider a Bernoulli random variable with parameter p.

6. Let X_1, X_2, X_3, and X_4 be independent continuous random variables with a common pdf $f(x)$ and cdf $F(x)$. Define Y_1, Y_2, Y_3, and Y_4 as

$$\begin{aligned} Y_1 &= \text{largest of } X_1, \ldots, X_4 \\ Y_2 &= \text{second largest of } X_1, \ldots, X_4 \\ Y_3 &= \text{third largest of } X_1, \ldots, X_4 \\ Y_4 &= \text{smallest of } X_1, \ldots, X_4 \ . \end{aligned}$$

(a) Find $f_{Y_k}(y)$, the marginal pdf of Y_k, for $1 \le k \le 4$
(b) Find $\mathbb{E}\,(Y_2)$.

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-rp534-

HWX 4/22/19

① $X \sim \text{Uniform}[-c, +c]$, $c>0$ given.

$N \sim \text{Gaussian}(\mu=0, \sigma^2=1)$

$$Y = X + N$$

$$Y = x + N \sim \text{Gaussian}(x, 1)$$

(a) $f_Y(y)$?

For each $x \in X$ we can find $f_{Y|X}(y|x)$.

It is

$$f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-x)^2}{2}}$$

Thus by Total Probability Theorem,

$$f_Y(y) = \int_{-\infty}^{+\infty} f_{Y|X}(y|x) f_X(x) \, dx$$

$$= \int_{x=-c}^{x=+c} \frac{1}{2c} \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-x)^2}{2}} \, dx \qquad \begin{matrix} t = y - x \\ dt = -dx \end{matrix}$$

$$= -\int_{t=y+c}^{t=y-c} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \, dt \cdot \frac{1}{2c}$$

$$= \int_{y-c}^{y+c} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \, dt \cdot \frac{1}{2}$$

$$= \frac{\Phi(y+c) - \phi(y-c)}{2c}$$

(b) Plots

(2)

$$f_{X,Y}(x,y) = \begin{cases} 12xy(1-x) & , (x,y) \in [0,1]\times[0,1] \\ 0 & , \text{else} \end{cases}$$

(a) $\mathbb{E}(X)$?

$$\mathbb{E}(X) = \int_{-\infty}^{+\infty} x f_X(x)\,dx = \int_{x=-\infty}^{x=+\infty} x \left(\int_{y=-\infty}^{y=+\infty} f_{X,Y}(x,y)\,dy \right) dx$$

$$= \int_{y=0}^{y=1} 12xy - 12x^2y\,dy = 12x\left(\tfrac{1}{2}y^2\right) - 12x^2\left(\tfrac{1}{2}y^2\right)\Big|_{y=0}^{y=1} = 6x - 6x^2 = 6x(1-x),\ x \in [0,1]$$

$$= \int_{x=0}^{x=1} x \cdot 6x(1-x)\,dx = \int_{x=0}^{x=1} 6x^2 - 6x^3\,dx = 6\left(\tfrac{1}{3}x^3\right) - 6\left(\tfrac{1}{4}x^4\right)\Big|_{x=0}^{x=1}$$

$$\boxed{\mathbb{E}(X) = \frac{6}{3} - \frac{6}{4} = \frac{1}{2}}$$

(b) $\mathbb{E}(Y)$?

$$\mathbb{E}(Y) = \int_{y=-\infty}^{y=+\infty} y f_Y(y)\,dy = \int_{y=-\infty}^{y=+\infty} \left(\int_{x=-\infty}^{x=+\infty} f_{X,Y}(x,y)\,dx \right) dy$$

$$= \int_{x=0}^{x=1} 12xy - 12x^2y\,dx = 12y\left(\tfrac{1}{2}x^2\right) - 12y\left(\tfrac{1}{3}x^3\right)\Big|_{x=0}^{x=1} = \frac{12y}{2} - \frac{12y}{3} = 2y,\ y \in [0,1]$$

$$\mathbb{E}(Y) = \int_{y=0}^{y=1} y \cdot 2y\,dy = \frac{2}{3}y^3\Big|_{y=0}^{y=1} = \frac{2}{3}$$

(c) $Var(X)$?

$$\mathbb{E}(X^2) = \int_{x=-\infty}^{x=+\infty} x^2 f_X(x)\, dx = \int_{x=0}^{x=1} x^2 \left(6x(1-x)\right) dx = \int_{x=0}^{x=1} 6x^3 - 6x^4 \, dx$$

$$= \frac{6}{4}x^4 - \frac{6}{5}x^5 \Big|_{x=0}^{x=1} = \frac{6}{4} - \frac{6}{5} = \frac{6}{20} = \frac{3}{10}$$

$$Var(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \frac{6}{20} - \frac{1}{4} = \frac{6}{20} - \frac{5}{20} = \boxed{\frac{1}{20}}$$

(d) $Var(Y)$?

$$\mathbb{E}(Y^2) = \int_{y=-\infty}^{y=+\infty} y^2 f_Y(y)\, dy = \int_{y=0}^{y=1} y^2 (2y)\, dy = \frac{2}{4} y^4 \Big|_{y=0}^{y=1} = \frac{1}{2}$$

$$Var(Y) = \mathbb{E}(Y^2) - (\mathbb{E}(Y))^2 = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{2} - \frac{4}{9} = \frac{9}{18} - \frac{8}{18} = \boxed{\frac{1}{18}}$$

(e) X, Y independent $\iff f_{X,Y}(x,y) = f_X(x) f_Y(y)$

$$12xy(1-x) \stackrel{?}{=} 6x(1-x)\, 2y \quad , \quad x \in [0,1],\ y \in [0,1]$$

$$12xy(1-x) = 12xy(1-x)$$

$\therefore$ X, Y independent

③ Throw a dart at circular dart board of radius R. It lands at a point (X, Y) distributed uniformally over the board. Center ⇒ (X, Y) = (0, 0). That is, the dart's landing coordinates X and Y have a uniform joint pdf

$$f_{X,Y}(x,y) = \begin{cases} c, & x^2+y^2 \le R^2 \\ 0, & \text{else} \end{cases}$$

$x^2+y^2 \le R^2$

$x^2 \le R^2 - y^2$ | $y^2 \le R^2 - x^2$

$x \le +\sqrt{R^2-y^2}$ | $y \le +\sqrt{R^2-x^2}$

$x \ge -\sqrt{R^2-y^2}$ | $y \ge -\sqrt{R^2-x^2}$

(a) Find constant c.

$$\int_{y=-\infty}^{+\infty}\int_{x=-\infty}^{+\infty} f_{X,Y}(x,y)\,dx\,dy = 1$$

$r \in [0, R]$

$\theta \in [0, 2\pi]$

$$= \iint_{\{(x,y):\, x^2+y^2 \le R^2\}} c\,r\,dr\,d\theta \Rightarrow c \cdot \pi R^2 = 1$$

$0 \le \theta = \tan^{-1}\left(\frac{y}{x}\right) \le 2\pi$

$0 < r = \pm\sqrt{x^2+y^2} \le R$

$$c = \frac{1}{\pi R^2}$$

(b) $f_X(x)$, $f_Y(y)$?

$$f_X(x) = \int_{y=-\infty}^{y=+\infty} f_{X,Y}(x,y)\,dy$$

$$f_Y(y) = \int_{x=-\infty}^{x=+\infty} f_{X,Y}(x,y)\,dx$$

$$= \int_{y=-\sqrt{R^2-x^2}}^{y=+\sqrt{R^2-x^2}} \frac{1}{2\pi R}\,dy = \boxed{\frac{2\sqrt{R^2-x^2}}{\pi R^2}}$$

$$= \int_{x=-\sqrt{R^2-y^2}}^{x=\sqrt{R^2-y^2}} \frac{1}{2\pi R}\,dx = \boxed{\frac{2\sqrt{R^2-y^2}}{\pi R^2}}$$

(c) Let D be the distance the dart lands from the center of the board. $\mathbb{P}(\{D \le d\})$, as a function of d, for $d \in (0, R]$

$$\mathbb{P}(\{D \le d\}) = \int_0^{2\pi}\int_0^{d} c\,r\,dr\,d\theta = \frac{2\pi}{\pi R^2} \cdot \frac{1}{2} d^2 = \boxed{\frac{d^2}{R^2}}$$

(d) $$\mathbb{E}(d) = \int_{-\infty}^{+\infty} d \cdot f_D(d)\,dd = \int_0^R d \cdot \frac{2d}{R^2}\,dd$$

$$= \frac{2}{R^2} \cdot \frac{1}{3} d^3 \Big|_0^R = \frac{2}{3}\frac{R^3}{R^2} \quad \boxed{\frac{2R}{3}}$$

④ $X_1, X_2, \ldots, X_n$ independent random variables with respective cumulative distribution functions $F_{X_k}(x)$ for $1 \le k \le n$.

(a) cdf of rv $Y = \max\{X_1, \ldots, X_n\}$ in terms of $F_{X_k}(x)$'s. i.e $F_Y(y)$?

$Y = g(X)$

$X_1, \ldots, X_n$ independent $\iff F_{X_1}(x) F_{X_2}(x) \cdots F_{X_n}(x) = F_{X_1, X_2, \ldots, X_n}(x)$

$$F_Y(y) = \mathbb{P}(\{Y \le y\}) = \mathbb{P}(\{\max\{X_1, \ldots, X_n\} \le y\}) = \mathbb{P}(\{X_1 \le y\} \cap \cdots \cap \{X_n \le y\})$$
$$= \mathbb{P}(\{X_1 \le y\}) \cdots \mathbb{P}(\{X_n \le y\})$$
$$\boxed{= \prod_{k=1}^{n} F_{X_k}(y)}$$

(b) cdf of rv $Z = \min\{X_1, \ldots, X_n\}$ in terms of $F_{X_k}(x)$'s.

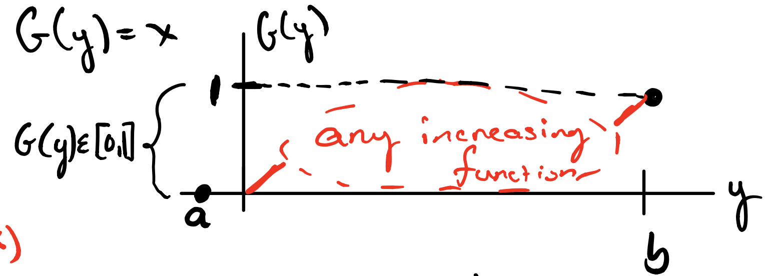
$$\mathbb{P}(\{Z > z\}) = \mathbb{P}(\{X_1 > z\}) \cdots \mathbb{P}(\{X_n > z\})$$
$$= (1 - F_{X_1}(z)) \cdots (1 - F_{X_n}(z))$$
$$= \prod_{i=1}^{n} (1 - F_{X_i}(z))$$

⑤ $X \sim \text{Uniform}[0,1]$ generated by computer

$G(y) \in [0,1]$, and is strictly increasing in y on some interval $a < y < b$ with $G(a) = 0$ and $G(b) = 1$ ($a = -\infty$ and/or $b = \infty$ allowed)

Define rv Y as follows:

$Y = h(X)$, where for each $x \in [0,1]$ $h(x)$ is the unique value of y such that $G(y) = x$



(a) Show cdf $F_Y(y)$ is $G(y)$

$Y = h(x)$. Computer takes an $x \in [0,1]$ (say x^*). For this $x^* \in X$, have $y^* \in Y$ such that **THIS** y^* plugged into $G(y)$ (i.e $G(y^*)$) **equals x^*.**

So $\mathbb{P}(\{Y \le y\}) = \mathbb{P}(\{h(x) \le y\}) = \mathbb{P}(\{G(h(x)) \le G(y)\}) = \mathbb{P}(X \le G(y))$
$= G(y)$

So computer generates Y by mapping each X onto Y.

(b) $X \sim \text{Exponential}(\lambda)$?

Sample enough points to have enough data to simulate exponential

(c) Use computer to generate Bernoulli(p)

```
if (!Bernoulli_p())
    generate_Bernoulli_p();
```

⑥ Let X_1, X_2, X_3, X_4 be independent continuous random variables with a common pdf $f_X(x)$ and cdf $F(x)$.

Define Y_1, Y_2, Y_3, Y_4 as:

$$Y_1 = \text{largest of } X_1, X_2, X_3, X_4$$
$$Y_2 = \text{second largest of } X_1, X_2, X_3, X_4$$
$$Y_3 = \text{third largest of } X_1, X_2, X_3, X_4$$
$$Y_4 = \text{Smallest of } X_1, X_2, X_3, X_4$$

(a) $f_{Y_k}(y)$ for $1 \le k \le 4$?

$Y_1 = \max\{X_1, X_2, X_3, X_4\}$

$$F_{Y_1} = \mathbb{P}(\{Y_1 \le y\})$$

$$\begin{aligned}
F_{Y_1} &= \mathbb{P}(\{\max\{X_1, X_2, X_3, X_4\} \le y\}) \\
&= \mathbb{P}(\{X_1 \le y\} \cap \cdots \cap \{X_4 \le y\}) \\
&= \mathbb{P}(\{X_1 \le y\}) \cdots \mathbb{P}(\{X_4 \le y\}) \\
&= F(y) \cdot F(y) \cdot F(y) \cdot F(y) \\
&= (F(y))^4
\end{aligned}$$

Thus

$$\boxed{\begin{aligned}
f_{Y_1} &= \frac{d}{dy} F_{Y_1} = \frac{d}{dy} (F(y))^4 \\
&= 4[F(y)]^3 \cdot \frac{d}{dy} F(y) \\
&= 4(F(y))^3 f(y)
\end{aligned}}$$

$$Y_2 = \max\left(\underbrace{\{X_1, X_2, X_3, X_4\} - \{Y_1\}}\right)$$

the set which consist of X_1, X_2, X_3, X_4 MINUS result of Y_1 i.e set has 3 elements

$$Y_3 = \max(\{X_1, X_2, X_3, X_4\} - \{Y_1, Y_2\})$$

$$Y_4 = \min(\{X_1, X_2, X_3, X_4\})$$

Y_4 also simple.

$$f_{Y_4} = \frac{d}{dy} F_{Y_4} = \frac{d}{dy}\left(1 - (1 - F(y))^4\right)$$

$$\boxed{= +4(1-F(y))^3 f(y)}$$

Y_2, Y_3 not so simple.

(b) $\mathbb{E}(Y_2) = \int_{-\infty}^{+\infty} y_2 \, f_{Y_2}(y) \, dy$