

If the function

$$f(z) = u(x, y) + i v(x, y)$$

is differentiable at $z_0 = x_0 + iy_0$, then the limit

$$f'(z_0) = \lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

can be computed by allowing $\Delta z = \Delta x + i \Delta y$ to approach zero from any convenient direction in the complex plane.

If it approaches horizontally, then $\Delta z = \Delta x$, and we obtain

$$\begin{aligned} f'(z_0) &= \lim_{\Delta x \rightarrow 0} \frac{u(x_0 + \Delta x, y_0) + i v(x_0 + \Delta x, y_0) - u(x_0, y_0) - i v(x_0, y_0)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \left[\frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} \right] + i \lim_{\Delta x \rightarrow 0} \left[\frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x} \right] \end{aligned}$$

We deduce that

$$f'(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i \frac{\partial v}{\partial x}(x_0, y_0) \quad (1)$$

On the other hand, if $\Delta z = i \Delta y$ and approaches zero vertically

$$f'(z_0) = \lim_{\Delta y \rightarrow 0} \left[\frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{i \Delta y} \right] + i \lim_{\Delta y \rightarrow 0} \left[\frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{i \Delta y} \right]$$

Hence,

$$f'(z_0) = -i \frac{\partial u}{\partial y}(x_0, y_0) + \frac{\partial v}{\partial y}(x_0, y_0) \quad (2)$$

But $(1) = (2)$ so,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad (3)$$

must hold at $z_0 = x_0 + iy_0$. Eqn (3) are called the **Cauchy-Riemann equations**.

We've thus established

Theorem 4: A necessary condition for a function $f(z) = u(x,y) + iv(x,y)$ to be differentiable at a point z_0 is that the Cauchy-Riemann equations hold at z_0 .

Consequently, if f is analytic in an open set G , then the Cauchy-Riemann equations must hold at every point G .

A tip to remember Cauchy-Riemann equations.

Horizontal = Vertical

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial (iy)} \rightarrow \frac{\partial(u+iv)}{\partial x} = \frac{1}{i} \frac{\partial(u+iv)}{\partial y}$$

and equate real and imaginary parts

Example 1: Show that the function

$$f(z) = (x^2+y^2) + i(y^2-x^2)$$

is NOT analytic at any point.

$$u(x,y) = x^2+y$$

$$v(x,y) = y^2-x$$

$$\frac{\partial u}{\partial x} = 2x$$

$$\frac{\partial v}{\partial y} = 2y$$

$$\frac{\partial u}{\partial y} = 1$$

$$\frac{\partial v}{\partial x} = -1$$

Hence the Cauchy-Riemann equations are simultaneously satisfied only on the line $x=y$ and therefore in no open disk. Thus by Theorem 4 the function $f(z)$ is nowhere analytic.

Being **mathematically precise**, we point out that the Cauchy-Riemann equations alone are **NOT** sufficient to ensure differentiability.

You also need the additional hypothesis of continuity of the first partial derivatives of u and v .

The complete story is the following theorem.

Theorem 5. Let $f(z) = u(x, y) + i v(x, y)$ be defined in some open set G containing the point z_0 . If the first partial derivatives of u and v exist in G , are continuous at z_0 , and satisfy the Cauchy-Riemann equations at z_0 , then f is differentiable at z_0 .

Consequently, if the first partial derivatives are continuous and satisfy the Cauchy-Riemann equations at all points of G , then f is analytic in G .

PROOF

The difference quotient for f at z_0 can be written as

$$\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

$$= \frac{[u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)] + i[v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)]}{\Delta x + i \Delta y} \quad (4)$$

Where $z_0 = x_0 + iy_0$ and $\Delta z = \Delta x + i \Delta y$.

* The above expressions are well defined if $|\Delta z|$ is so small that the closed disk with center z_0 and radius $|\Delta z|$ lies entirely in G .

Let us rewrite the difference

$$u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)$$

as

$$[u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0 + \Delta y)] + [u(x_0, y_0 + \Delta y) - u(x_0, y_0)] \quad (5)$$

Because the partial derivatives exist in G , the mean-value theorem says that there is a number x^* between x_0 and $x_0 + \Delta x$ such that

???

$$u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0 + \Delta y) = \Delta x \frac{\partial u}{\partial x}(x^*, y_0 + \Delta y)$$

Furthermore, since the partial derivatives are continuous at (x_0, y_0) , we can write

$$\frac{\partial u}{\partial x}(x^*, y_0 + \Delta y) = \frac{\partial u}{\partial x}(x_0, y_0) + \epsilon,$$

where

$$\begin{aligned} \epsilon &\rightarrow 0 & \text{as } x^* \rightarrow x_0 \\ && \Delta y \rightarrow 0 \end{aligned} \left. \begin{array}{l} \text{in particular} \\ \text{as } \Delta z \rightarrow 0 \end{array} \right\}$$

Thus,

$$u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0 + \Delta y) = \Delta x \left[\frac{\partial u}{\partial x}(x_0, y_0) + \epsilon \right]$$

(Next three proofs were done solo)

Similarly, because the partial derivatives exist in G , the mean-value theorem says that there is a number y^* between y_0 and $y_0 + \Delta y$ such that

$$u(x_0, y_0 + \Delta y) - u(x_0, y_0) = \Delta y \left[\frac{\partial u}{\partial y}(x_0, y^*) \right]$$

Furthermore, since the partial derivatives are continuous at (x_0, y_0) , we can write

$$\frac{\partial u}{\partial x}(x_0, y^*) = \left[\frac{\partial u}{\partial y}(x_0, y_0) + \epsilon_2 \right]$$

where

$$\epsilon_2 \rightarrow 0 \quad \text{as} \quad \Delta z \rightarrow 0$$

Thus,

$$u(x_0, y_0 + \Delta y) - u(x_0, y_0) = \Delta y \left[\frac{\partial u}{\partial y}(x_0, y_0) + \epsilon_2 \right]$$

In a similar fashion, let us rewrite the difference
 $i[\nabla(x_0 + \Delta x, y_0 + \Delta y) - \nabla(x_0, y_0)]$

as

$$i[\nabla(x_0 + \Delta x, y_0 + \Delta y) - \nabla(x_0, y_0 + \Delta y)] + [\nabla(x_0, y_0 + \Delta y) - \nabla(x_0, y_0)] \quad (6)$$

Because the partial derivatives exist in G, the mean-value theorem says that there is a number x^* between x_0 and $x_0 + \Delta x$ such that

$$i[\nabla(x_0 + \Delta x, y_0 + \Delta y) - \nabla(x_0, y_0 + \Delta y)] = i \Delta x \frac{\partial \nabla}{\partial x}(x^*, y_0 + \Delta y)$$

Furthermore, since the partial derivatives are continuous at (x_0, y_0) , we can write

$$\frac{\partial \nabla}{\partial x}(x^*, y_0 + \Delta y) = \frac{\partial \nabla}{\partial x}(x_0, y_0) + \epsilon_3$$

where

$$\epsilon_3 \rightarrow 0 \quad \text{as} \quad \begin{cases} x^* \rightarrow x_0 \\ \Delta y \rightarrow 0 \end{cases} \quad \begin{cases} \text{in particular} \\ \text{as } \Delta z \rightarrow 0 \end{cases}$$

Thus,

$$i \left[v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0 + \Delta y) \right] = i \Delta x \left[\frac{\partial v}{\partial x} (x_0, y_0) + \epsilon_3 \right]$$

Similarly, because the partial derivatives exist in G, the mean-value theorem says that there is a number y^* between y_0 and $y_0 + \Delta y$ such that

$$i \left[v(x_0, y_0 + \Delta y) - v(x_0, y_0) \right] = i \Delta y \left[\frac{\partial v}{\partial y} (x_0, y^*) \right]$$

Furthermore, since the partial derivatives are continuous at (x_0, y_0) , we can write

$$\frac{\partial v}{\partial x} (x_0, y^*) = \left[\frac{\partial v}{\partial x} (x_0, y_0) + \epsilon_4 \right]$$

where

$$\epsilon_4 \rightarrow 0 \quad \text{as} \quad \Delta z \rightarrow 0$$

Thus,

$$i \left[v(x_0, y_0 + \Delta y) - v(x_0, y_0) \right] = i \Delta y \left[\frac{\partial v}{\partial y} (x_0, y_0) + \epsilon_4 \right]$$

Solo work ends here

We ultimately have

$$\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{\Delta x \left[\frac{\partial u}{\partial x} + \epsilon_1 + i \frac{\partial v}{\partial x} + i \epsilon_3 \right] + \Delta y \left[\frac{\partial u}{\partial y} + \epsilon_2 + i \frac{\partial v}{\partial y} + i \epsilon_4 \right]}{\Delta x + i \Delta y}$$

where each partial derivative is evaluated at (x_0, y_0) and where each $\varepsilon_i \rightarrow 0$ as $\Delta z \rightarrow 0$.

Now we use the Cauchy-Riemann equations to express the difference quotient as

$$\frac{\Delta x \left[\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right] + i \Delta y \left[\frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y} \right]}{\Delta x + i \Delta y} + \frac{\lambda}{\Delta x + i \Delta y} \quad (7)$$

where

$$\lambda := \Delta x (\varepsilon_1 + i \varepsilon_3) + \Delta y (\varepsilon_2 + i \varepsilon_4)$$

Since

$$\begin{aligned} \left| \frac{\lambda}{\Delta x + i \Delta y} \right| &\leq \left| \frac{\Delta x}{\Delta x + i \Delta y} \right| |\varepsilon_1 + i \varepsilon_3| + \left| \frac{\Delta y}{\Delta x + i \Delta y} \right| |\varepsilon_2 + i \varepsilon_4| \\ &\leq |\varepsilon_1 + i \varepsilon_3| + |\varepsilon_2 + i \varepsilon_4| \end{aligned}$$

We see that the last term in (7) approaches zero as $\Delta z \rightarrow 0$, and so

$$\lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{\partial u}{\partial x}(x_0, y_0) + i \frac{\partial v}{\partial x}(x_0, y_0)$$

i.e.

$f'(z_0)$ exists!

Example 2: Prove that the function $f(z) = e^z$ is entire and find its derivative.

$$f(z) = e^z = e^x e^{iy} = e^x [\cos y + i \sin y]$$

$$u(x, y) = e^x \cos y \quad v(x, y) = e^x \sin y$$

$$\frac{\partial u}{\partial x} = e^x \cos y \quad \frac{\partial v}{\partial y} = e^x \cos y$$

$$\frac{\partial u}{\partial y} = -e^x \sin y \quad \frac{\partial v}{\partial x} = e^x \sin y$$

Since the first partial derivatives are continuous and satisfy the Cauchy-Riemann equations at every point in the plane.

Hence $f(z)$ is entire. From (1) we see that

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = e^x [\cos y + i \sin y]$$

Not surprisingly,

$$f'(z) = f(z)$$

Theorem 6. If $f(z)$ is analytic in a domain D and if $f'(z)=0$ everywhere in D , then $f(z)$ is constant in D .

Before the proof, observe that the **connectedness** property of the domain is essential. Indeed, if $f(z)$ is defined by

$$f(z) = \begin{cases} 0 & \text{if } |z| < 1 \\ 1 & \text{if } |z| > 2 \end{cases}$$

then f is analytic and $f'(z)=0$ on its domain of definition **WHICH IS NOT ITS DOMAIN.**

Proof of Theorem 6

Since $f'(z)=0$ in D , we see from Eqn (1) + Eqn (2) that all the first partial derivatives of u and v vanish in D ; that is

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$$

Thus by **Theorem 1** (in 1.6) we have

$u=v=\text{constant}$ in D . Consequently, $f=u+iv$ is also constant in D .

A consequence of theorem 6 is the fact that if f and g are two functions analytic in a domain D , then $f = g + \text{constant}$ in D . (Prob 7)

Using theorem 6 and the Cauchy-Riemann equations, it can be shown that an analytic function $f(z)$ must be constant when any of the following conditions hold in a domain D :

$\operatorname{Re} f(z)$ is constant;

$\operatorname{Im} f(z)$ is constant;

$|f(z)|$ is constant

(8)

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