Recall

- Conditional pmfs, conditional expectation.
- Total expectation = avg of conditional expectations
  - ("Law of Total Expectation")

Example - Envelopes (from last time)

Have 2 envelopes. One has twice the amount of money as the other.

i.e

one has m dollars

one has $2m$ dollars

Call the envelopes X and Y.

let Z = total amount of money between the two

Say you find

$$X = m$$

Get to keep m if you want. Should you?

Figure out $\mathbb{E}(Y|X=m)$ using following NAIVE approach:

$$\boxed{\mathbb{P}(\{Y=2m\}|\{X=m\}) = \mathbb{P}\{Y=\tfrac{m}{2}|\{X=m\}) = 1/2}$$

This line of reasoning is incorrect

This leads you to believe you should switch.

Thus

$$\mathbb{E}(Y|X=m) = 2m\left(\tfrac{1}{2}\right) + \tfrac{m}{2}\left(\tfrac{1}{2}\right) = \tfrac{5}{4}m > m$$

The reasoning is specious because it is based on a woefully incomplete probability model.

How's it incomplete?

- Are X, Y integers? Continuous valued?
  - Eg if integers, if X is 19 you should definitely switch

- What's $p_Z(z)$? Any complete model would enable you to find this...

Let us make it more complete.

- Dealer picks z "at random" – a multiple of 3 dollars
- Divides into two piles. $z/3$; $2z/3$.
- Flip a fair coin to decide how to allot piles to 2 envelopes, X, Y.

Say you open X and find m dollars.

$$Z = \begin{cases} 3m, & \text{if } X \text{ is smaller envelope} \\ \frac{3}{2}m, & \text{if } X \text{ is larger envelope} \end{cases}$$

So you ALWAYS switch

If m is odd, can't have $Z = \frac{3}{2}m$ – but say m is even.

Fact: It's $\boxed{\text{impossible}}$ to have

$$p_{Z|X}(3m|m) = p_{Z|X}\left(\tfrac{3}{2}m \,\middle|\, m\right) = \tfrac{1}{2}$$

where we stopped in lecture 12

start of Lec 13

If m is even, Y contains either $2m$ or $\frac{m}{2}$.

In terms of Z, $Z = 3m$ or $\frac{3}{2}m$.

$$p_{Z|X}(3m|m) = \frac{\mathbb{P}(X=m \mid Z=3m)\,\mathbb{P}(Z=3m)}{\mathbb{P}(X=m)}$$

$$p_{Z|X}\left(\tfrac{3}{2}m \,\middle|\, m\right) = \frac{\mathbb{P}(X=m \mid Z=\frac{3}{2}m)\,\mathbb{P}(Z=\frac{3}{2}m)}{\mathbb{P}(X=m)}$$

If

$$p_{Z|X}(3m|m) = p_{Z|X}\left(\tfrac{3}{2}m \,\middle|\, m\right) = \frac{1}{2}$$

then must have

$$\mathbb{P}(Z=3m) = \mathbb{P}\left(Z=\tfrac{3}{2}m\right) \;;\; m=2n,\ n\in\mathbb{N}\setminus\{0\}$$

Thus need

$$\begin{array}{ll} m=2 & \mathbb{P}(Z=6) = \mathbb{P}(Z=2) \\ m=4 & \mathbb{P}(Z=12) = \mathbb{P}(Z=6) \\ m=8 & \mathbb{P}(Z=24) = \mathbb{P}(Z=12) \end{array}$$

i.e

$$\mathbb{P}(Z=2) = \mathbb{P}(Z=6) = \mathbb{P}(Z=12) = \mathbb{P}(Z=24) = \cdots$$

infinite chain of equalities

So if Z is chosen w/ positive probability or all positive integer multiples of 3, can't have this chain of equalities

Moral: If you flesh out the probability model - by putting a pmf on Z - find that specious reasoning fails

i.e $\mathbb{P}(Z=3m \mid X=m) \neq \mathbb{P}(Z=\tfrac{3}{2}m \mid X=m) \neq \tfrac{1}{2} \;\forall$ even m

Example - Conditional Expectation

A tricky way to get $\mathbb{E}(X)$ and $\mathrm{Var}(X)$ when X is geometric.

i.e.

$$p_X(k) = p(1-p)^{k-1}, \quad k > 0$$

Figured out $\mathbb{E}(X) = 1/p$ already "by math"

Here's another way.

Let

$$A = \{X = 1\}$$

$$\Rightarrow A^c = \{X > 1\}$$

Observe that

$$X = 1 + (X - 1)$$

Use linearity of $\mathbb{E}$ to get

"Law of Total Expectation"

$$\mathbb{E}(X) = 1 + \mathbb{E}(X-1)$$

$$= 1 + \underbrace{\mathbb{E}(X-1 \mid A)}_{0}\underbrace{\mathbb{P}(A)}_{p} + \underbrace{\mathbb{E}(X-1 \mid A^c)}_{\mathbb{E}(X)}\underbrace{\mathbb{P}(A^c)}_{1-p}$$

Observe that $X-1 \mid A^c$ is distributed the same as $X \mid 0$ i.e it's geometric! Thus $\mathbb{E}(X-1 \mid A^c) = \mathbb{E}(X)$

So,

$$\mathbb{E}(X) - \mathbb{E}(X)(1-p) = 1$$

$$\mathbb{E}(X)[1 - (1-p)] = 1$$

$$\mathbb{E}(X) = \frac{1}{p}$$

How about $Var(X)$? "By Math" is hard.
Let's use the conditional set up.

Know,

$$Var(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$$

Need $\mathbb{E}(X^2)$.

Let

$$X = 1 + (X-1)$$

Then

$$X^2 = 1 + 2(X-1) + (X-1)^2$$

$$\begin{aligned}
\mathbb{E}(X^2) &= \mathbb{E}(1) + \mathbb{E}(2(X-1)) + \mathbb{E}((X-1)^2) \\
&= 1 + 2\,\mathbb{E}(X-1) + \mathbb{E}((X-1)^2) \\
&= 1 + 2\,\mathbb{E}((X-1)|A)\mathbb{P}(A) + 2\,\mathbb{E}(X-1|A^c)\mathbb{P}(A^c) \\
&\quad + \mathbb{E}((X-1)^2|A)\mathbb{P}(A) + \mathbb{E}((X-1)^2|A^c)\mathbb{P}(A^c)
\end{aligned}$$

Neater,

$$\mathbb{E}(X^2) = 1 + \cancel{\underbrace{2\,\mathbb{E}((X-1)|A)}_{0}\mathbb{P}(A)} + 2\underbrace{\mathbb{E}(X-1|A^c)}_{1/p}\underbrace{\mathbb{P}(A^c)}_{(1-p)} + \cancel{\underbrace{\mathbb{E}((X-1)^2|A)}_{0}}\mathbb{P}(A) + \underbrace{\mathbb{E}((X-1)^2|A^c)}_{\mathbb{E}(X^2)}\underbrace{\mathbb{P}(A^c)}_{(1-p)}$$

Again since $p_{X-1|A^c}(k)$ same as $p_X(k)$, we have

So,

$$\mathbb{E}(X^2) = 1 + 2 \cdot \frac{1-p}{p} + \mathbb{E}(X^2)(1-p)$$

$$\mathbb{E}(X^2)[1-(1-p)] = 1 + 2 \cdot \frac{1-p}{p}$$

$$Var(X) = \frac{1}{p} + 2\frac{1-p}{p^2} - \left(\frac{1}{p}\right)^2 = \frac{1-p}{p^2}$$

Independence

Given Ω, $\mathbb{P}$ and a discrete rv X defined on Ω.

Given an event A, say X is independent of A when every event $\{X=x\}$ is independent of A (event-wise).

i.e

$$\mathbb{P}(\{X=x\}\cap A) = p_X(x)\,\mathbb{P}(A)$$

Note: when $\mathbb{P}(A)>0$, same as $p_{X|A}(x) = p_X(x) \quad \forall\, x$

Say two rvs X and Y are independent when

equivalent statements

- X is independent of every event $\{Y=y\}$
- Y is independent of every event $\{X=x\}$
- $p_{X,Y}(x,y) = p_X(x)\,p_Y(y) \quad \forall\, x\in X,\ y\in Y$

Extends to multiple rvs $X_1,\dots,X_n$ - Say they're independent when

$$\underbrace{p_{X_1,X_2,\dots X_n}(x_1,\dots,x_n)}_{\text{Joint}} = \underbrace{p_{X_1}(x_1)\,p_{X_2}(x_2)\cdots p_{X_n}(x_n)}_{\text{Product of Marginals}} \quad \forall\ x_1,\dots,x_n$$

Important Fact: X, Y independent $\Rightarrow \mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$

To see this,

$$\mathbb{E}(XY) = \sum_{x\in X, y\in Y} xy\, p_{X,Y}(x,y) = \sum_{x\in X, y\in Y} xy\,(p_X(x)p_Y(y)) = \sum_{x\in X} x\Big(\sum_{y\in Y} y\,p_Y(y)\Big)p_X(x)$$

$$= \mathbb{E}(X)\,\mathbb{E}(Y)$$

Extension of this: X, Y independent $\Rightarrow \mathbb{E}(g(X)h(Y)) = \mathbb{E}(g(X))\,\mathbb{E}(h(Y))$

Another Important Fact: X, Y independent. Then

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$$