

(Continued from Lec 1)

### 3. Covariance

Given 2 rvs  $X, Y$ ,

$$\text{cov}(X, Y) = \mathbb{E}[(X - \mu_X)(Y - \mu_Y)] = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (x - \mu_X)(y - \mu_Y) f_{X,Y}(x,y) dx dy$$

Covariance measures dependence of  $X$  and  $Y$ .

Definition:  $X, Y$  uncorrelated if  $\text{cov}(X, Y) = 0$

### Independence

Two rvs  $X, Y$  are independent if

$$f_{X,Y}(x,y) = f_X(x) f_Y(y)$$

i.e. joint density = product of densities

If  $X, Y$  independent rvs, then

$$(i) F_{X,Y}(x,y) = F_X(x) F_Y(y)$$

$$(ii) f_{X|Y}(x|y) = f_X(x)$$

Remark: In data science it is typically assumed that noise + data are independent at each time

$X, Y$  indept  $\Rightarrow X, Y$  uncorrelated

# Random Variables

1. Uniform RV  $X \sim U(a, b)$

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0, & \text{otherwise} \end{cases}$$

$$\mathbb{E}[X] = \frac{b+a}{2}, \quad \text{var}(X) = \frac{1}{12}(b-a)^2$$

2. Gaussian RV,  $X \sim N(\mu, \sigma^2)$

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

$$F_X(\mu) = 1/2, \quad F_X(-\infty) = 0, \quad F_X(\infty) = 1$$

$$\mathbb{E}[X] = \mu, \quad \text{Var}(X) = \sigma^2$$

Result 1: Gaussian input to linear system yields Gaussian Output

$$\text{If } Y = aX + b, \quad X \sim N(\mu, \sigma^2) \rightarrow Y \sim N(a\mu + b, a^2\sigma^2)$$

To show  $Y$  is Gaussian,

$$\begin{aligned}F_Y(y) &= \Pr(Y \leq y) = \Pr(aX + b \leq y) \\&= \Pr(X \leq \frac{y-b}{a}), a > 0 \\&= F_X\left(\frac{y-b}{a}\right)\end{aligned}$$

## Prospect Theory for Human Decision Making

An expected utility maximizer chooses a decision which maximizes expected utility: Given two choices  $a=1, a=2$ , with utility  $U_1(x)$  and  $U_2(x)$ , then

$$IE[U_1(x)] = \sum_x U_1(x_i) p_1(i)$$

$$IE[U_2(x)] = \sum_x U_2(x_i) p_2(i)$$

An expected utility maximizer chooses:

$$a=1 \quad \text{if } IE[U_1(x)] > IE[U_2(x)]$$

$$a=2 \quad \text{if } IE[U_2(x)] > IE[U_1(x)]$$

Example: Suppose there are two choices

1. Either: I give you 10 dollars. So

$$\mathbb{E}[u_1(x)] = 10$$

2. I flip a fair coin. Heads you get \$100.  
Tails: \$0

$$\mathbb{E}[u_2(x)] = 100 \cdot \frac{1}{2} + 0 \cdot \frac{1}{2} = 50$$

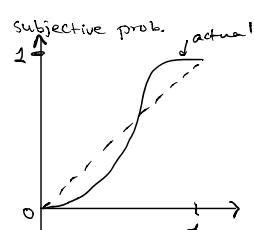
A (rational) utility maximizer chooses option

$$\arg \max_a \mathbb{E}[u_a(x)] = 2$$

BUT MOST HUMANS WOULD RATHER CHOOSE  
OPTION ONE

Humans are not expected utility maximizers!

Humans are risk averse decision makers



# Behavioral Economics Model

1. People have strong preference for certainty. They choose actions that yield lower expected utility with higher certainty.

- I'd rather have \$10 for certain than \$50 w/ uncertainty
- But risk-seeking for loss
  - I'd rather bet 50-50 on losing \$100 or nothing than certainly losing \$20.

2. People give losses more weight than gains. Loss averse.

3. Relative positioning: People are more interested in relative gain/loss than absolute values

- If you get a 10% raise and neighbor gets a 10% raise, you won't feel better off
- If you get a 10% raise and neighbor gets NO raise, you feel amazing

4. People over-react to low probability events and under-react to high probability events

How do we model human decision making?

## Prospect Theory

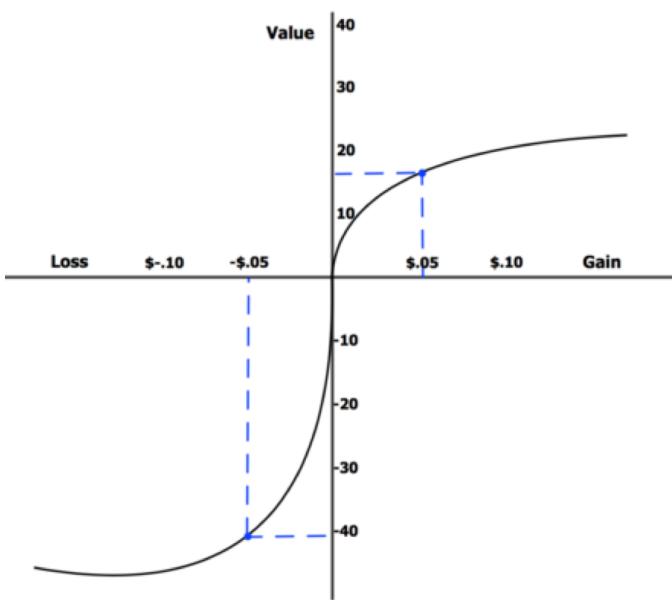
Utility for choice a is:

$$V_a = \sum_i v_a(x) \pi(p_a(i))$$

where  $\pi(\cdot)$  is the subjective probability.

$\pi(\cdot)$  models the fact that people over-react to small probability events and under-react to large probability events.

$v_a$  is the psychological value function for decision a.



- (i) S-shaped curve,  
**ASYMMETRICAL**
- (ii) Convex for losses, concave  
for gains.  
i.e. losses HURT more than  
gains feel GOOD  
→ Need \$150 to compensate  
\$100 loss

Compare prospect theory with expected utility maximizer for action  $a$

$$IE[U_a(x)] = \sum_x U_a(x_i) p_a(i)$$

**Example:** Consider the following choices:

- A. \$240 w/ certainty
- B. \$1000 w.p.  $\frac{1}{4}$ , \$0 w.p.  $\frac{3}{4}$
- C. -\$700 w.p. 1
- D. -\$1000 w.p.  $\frac{3}{4}$ , \$0 w.p.  $\frac{1}{4}$

Classical expected utilities are

$$IE[U_A] = 240$$

$$IE[U_B] = \frac{1}{4}(1000) + \frac{3}{4}(0) = 250$$

$$IE[U_C] = -700$$

$$IE[U_D] = -750$$

↙ from plot above  
like a

Prospect values are

$$v(1000) = 3.5$$

$$v(240) = 1.5$$

$$v(-700) = -6$$

$$v(-1000) = -6.5$$

So, using prospect theory)

Correctly Models!

$$IE[V_A] = 1.5$$

Humans choose guaranteed amount!

$$IE[V_B] = 3.5 \cdot \frac{1}{4} = 0.875$$

$$IE[V_C] = -6$$

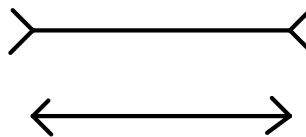
$$IE[V_D] = -6.5 \cdot \frac{3}{4} = -4.875$$

Humans would rather choose the risky loss D than the certain loss C

# Framing Effect

Cognitive bias where people decide on options based on whether they have positive or negative connotations.

i.e.



Visual framing

## Effect of Framing

Suppose U.S. is preparing for outbreak of a disease expected to kill 600 people.

Consider two alternative programs to combat the disease:

- Program A: 200 ppl saved (72% favor)
- Program B: 600 ppl saved w.p.  $\frac{1}{3}$ , 0 w.p.  $\frac{2}{3}$  (28%)

Same cover story, different framing

- Program A: 400 ppl die (22% favor)
- Program B: 0 ppl die w.p.  $\frac{1}{3}$ , 600 w.p.  $\frac{2}{3}$  (78%)

Prospect theory explains these results!

Psychology has a profound effect on decision making.