Recall

Independence of discrete rvs.

↳ $X_1, \dots, X_n$ independent means $p_{X_1,\dots,X_n}(x_1,\dots,x_n)$ is the product $p_{X_1}(x_1)p_{X_2}(x_2)\cdots p_{X_n}(x_n)$

Showed if X_1, X_2 independent, then $\mathbb{E}(X_1X_2) = \mathbb{E}(X_1)\mathbb{E}(X_2)$.

Another useful fact: If X, Y independent, then

$$Var(X+Y) = Var(X) + Var(Y)$$

Proof of fact:

$$Var(X+Y) = \mathbb{E}((X+Y)^2) - (\mathbb{E}(X+Y))^2$$

$$= \mathbb{E}(X^2 + 2XY + Y^2) - (\mathbb{E}(X) + \mathbb{E}(Y))^2$$

$$= \mathbb{E}(X^2) + 2\mathbb{E}(X)\mathbb{E}(Y) + \mathbb{E}(Y^2) - (\mathbb{E}(X))^2 - 2\mathbb{E}(X)\mathbb{E}(Y) - (\mathbb{E}(Y))^2$$

$$= \mathbb{E}(X^2) - (\mathbb{E}(X))^2 + \mathbb{E}(Y^2) - (\mathbb{E}(Y))^2$$

$$= Var(X) + Var(Y)$$

This generalizes to n independent rvs

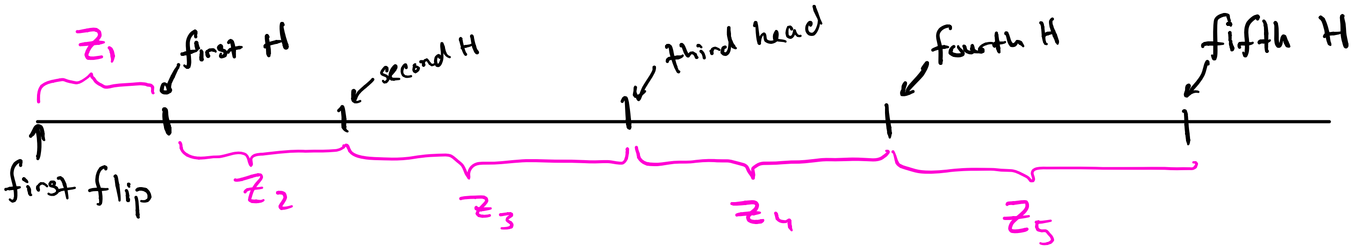
– i.e $Var(X_1 + \cdots + X_n) = Var(X_1) + \cdots + Var(X_n)$

Example - where this property helps

Have a coin, $\mathbb{P}(\{H\}) = p$. Flip it; let X = index of 5^{th} head.

Find $\mathbb{E}(X)$, $Var(X)$.

One approach: use a "stretchy timeline"



All these marked times - $Y_1, Y_2, Y_3, Y_4, Y_5 = X$ - are random.

write X as

$$X = \underbrace{Z_1}_{Y_1} + \underbrace{Z_2}_{Y_2 - Y_1} + \underbrace{Z_3}_{Y_3 - Y_2} + \underbrace{Z_4}_{Y_4 - Y_3} + \underbrace{Z_5}_{Y_5 - Y_4}$$

All Z's are geometric rvs w/ parameter p - **and** they're independent.

(Aside: computing pmf of X would be UGLY)

$$\mathbb{E}(Z_k) = \frac{1}{p}, \quad Var(Z_k) = \frac{1-p}{p^2} \quad \text{for } k \in \{1,2,3,4,5\}$$

$$\mathbb{E}(X) = \sum_{k=1}^{5} Z_k = \frac{5}{p}, \quad Var(X) = \sum_{k=1}^{5} Var(Z_k) = \frac{5(1-p)}{p^2}$$

Example - Binomial r.v

Let's find $\mathrm{Var}(X)$ when X is $\boxed{\text{binomial}(n,p)}$

$$p_X(k) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k} & ; \; 0 \le k \le n \\ 0 & ; \; \text{else} \end{cases}$$

Saw $\mathbb{E}(X) = np$ - saw by writin $X = Z_1 + \cdots + Z_n$ where

$$Z_m = \begin{cases} 1 & \text{if } H \text{ on flip } m \\ 0 & \text{if } T \text{ on flip } m \end{cases} \quad ; \quad \mathbb{E}(Z_m) = p$$

independent Bernoulli p rvs

Meanwhile, $\mathrm{Var}(Z_k) = p(1-p)$ - by independence,

$$\mathrm{Var}(X) = \sum_{m=1}^{n} \mathrm{Var}(Z_m) = np(1-p)$$

This is **WAY EASIER** than

$$\mathrm{Var}(X) = \sum_{k=0}^{n} (k-np)^2 \binom{n}{k} p^k (1-p)^{n-k}$$

One last independence-related item: estimating stats by sample means.
Have a sequence $X_1, X_2, \ldots, X_n$ of independent rvs (can think of them as Bernoulli p but don't have to).

For every $n > 0$, let

$$S_n = \frac{1}{n}\sum_{m=1}^{n} X_m$$

Note: S is a rv

If all the X_m have the same $\mathbb{E}$ (let's use the case when the X_m are Bernoulli p), we have

by independence

assumes X is Bernoulli p

$$\mathbb{E}(S_n) = \frac{1}{n}\sum_{m=1}^{n}\mathbb{E}(X_m) = \frac{1}{n}(np) = p \quad \forall n$$

by independence

$$\mathrm{Var}(S_n) = \sum_{m=1}^{n}\mathrm{Var}\left(\frac{1}{n}X_m\right) = \frac{1}{n^2}\sum_{m=1}^{n}\mathrm{Var}(X_m)$$

$$= \frac{1}{n^2}\cdot n(p(1-p)) = \frac{p(1-p)}{n}$$

Summary:

- $\mathbb{E}(S_n)$ = common $\mathbb{E}$ of all X_m's, $\forall n > 0$
- $\mathrm{Var}(S_n) \to 0$ as $n \to \infty$

- This helps at estimating an unknown p for a p-coin by flipping
- This is an elementary instance of a <u>law of of large numbers</u>