11.7

(5)

Describe the projections on the Riemann Sphere of the following sets in the complex plane.

a. Right-half plane $\{z : Re(z) > 0\}$

$$z = x + iy, \quad x > 0$$

The north pole of the Riemann Sphere is

$$(x_1, x_2, x_3) \to (0, 0, 1)$$

A point in the complex plane is then

$$(x_1, x_2, x_3) \to (x, y, 0)$$

the line through $(0, 0, 1)$ and $(x, y, 0)$ is given by

$$x_1 = tx \ (x>0), \quad x_2 = ty, \quad x_3 = 1 - t$$

where

$$-\infty < t < \infty$$

The line cuts the splere when t satisfies

$$1 = x_1^2 + x_2^2 + x_3^2 = t^2x^2 + t^2y^2 + (1-t)^2$$

whose roots are $t = 0$ (North pole) and

$$t = \frac{2}{x^2+y^2+1} = \frac{2}{|z|^2+1}$$

So,

$$X_1 = \frac{2\,\mathrm{Re}(z)}{|z|^2+1}, \quad X_2 = \frac{2\,\mathrm{Im}(z)}{|z|^2+1}, \quad X_3 = \frac{|z|^2-1}{|z|^2+1}$$

$$X_1 = \frac{2x}{|z|^2+1}, \quad X_2 = \frac{2y}{|z|^2+1}, \quad X_3 = \frac{|z|^2-1}{|z|^2+1}$$

where $x>0$!

So, $\{z : \mathrm{Re}(z)>0\}$ maps onto the right hemisphere the Reimann Sphere.

b. the disk $\{z : |z| < 1/2\}$

$$z = \frac{1}{2}e^{i\theta} = \frac{1}{2}\cos(\theta) + \frac{i}{2}\sin(\theta)$$

$$X_1 = \frac{2\,\mathrm{Re}(z)}{|z|^2+1}, \quad X_2 = \frac{2\,\mathrm{Im}(z)}{|z|^2+1}, \quad X_3 = \frac{|z|^2-1}{|z|^2+1}$$

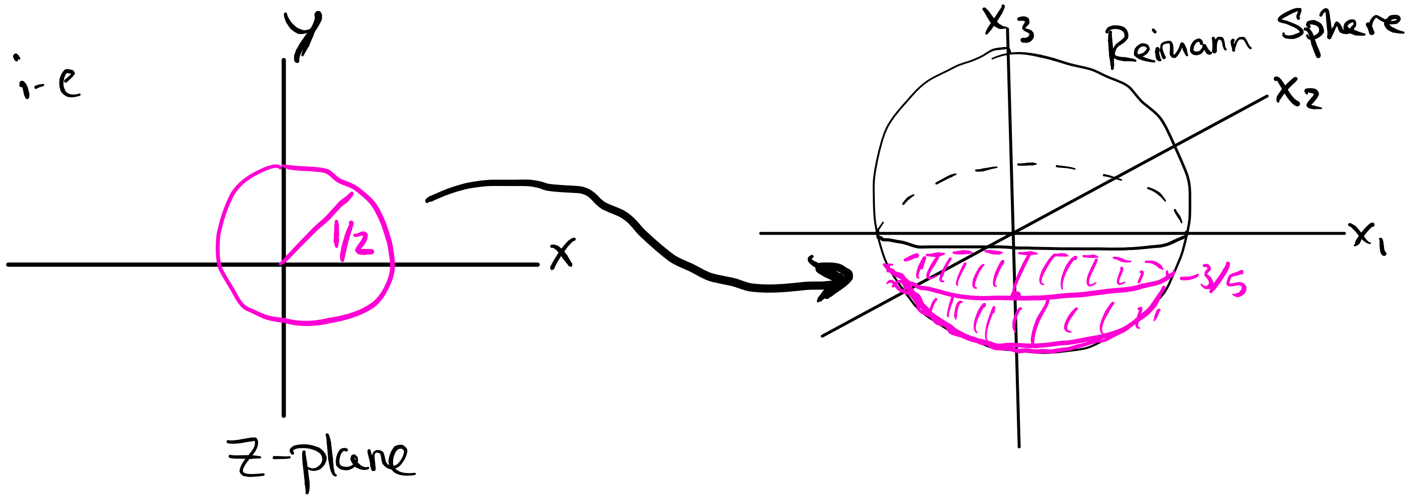
subbing in for $|z|$,

$$X_1 < \frac{\cos(\theta)}{\frac{1}{4}+1}, \quad X_2 < \frac{\sin(\theta)}{\frac{1}{4}+1}, \quad X_3 < \frac{\frac{1}{4}-1}{\frac{1}{4}+1}$$

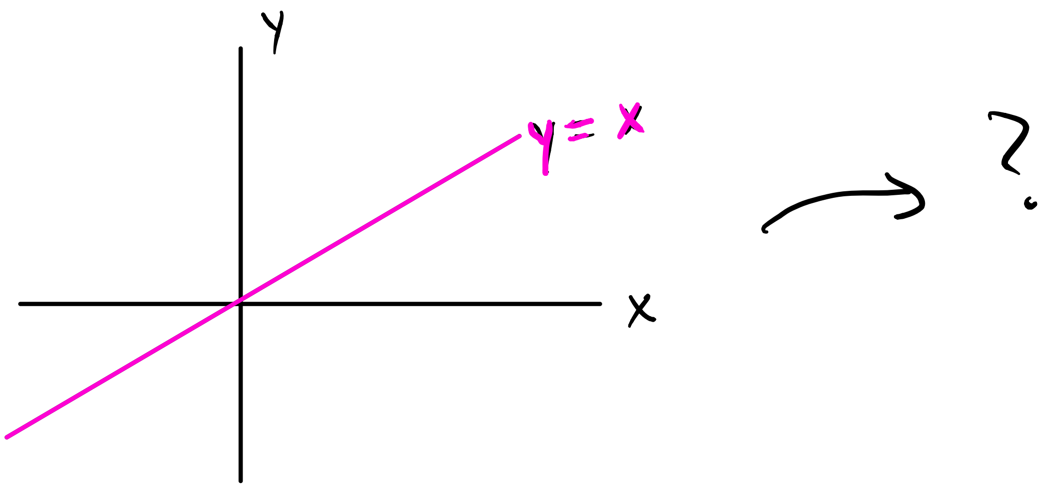
$$\Rightarrow X_1 < \frac{4}{5}\cos(\theta), \quad X_2 < \frac{4}{5}\sin(\theta), \quad X_3 < \frac{-3}{5}$$

So, the disk $\{z : |z| < 1/2\}$ maps onto the "bowl" $x_3 < -3/5$ on the Riemann sphere

i.e



e. The line $y = x$ (including point at infinity).



$$z = x + iy = x + ix = y + iy$$

$$x_1 = \frac{2\operatorname{Re}(z)}{|z|^2+1}, \quad x_2 = \frac{2\operatorname{Im}(z)}{|z|^2+1}, \quad x_3 = \frac{|z|^2-1}{|z|^2+1}$$

$\boxed{x_1 = x_2}$ why? because $y = x$ so $\operatorname{Re}(z) = \operatorname{Im}(z)$

$$|z| = \sqrt{x^2+y^2} = \sqrt{2y^2} = \sqrt{2x^2} = \sqrt{2}\,y = \sqrt{2}\,x$$

$$|z|^2 = 2y^2 = 2x^2$$

$$x_3 = \frac{2y^2-1}{2y^2+1} = \frac{2x^2-1}{2x^2+1}$$

as $y = x \to \infty$ similarly for $-\infty$

$x_3 \to 1$ (via L'hopitals rule)

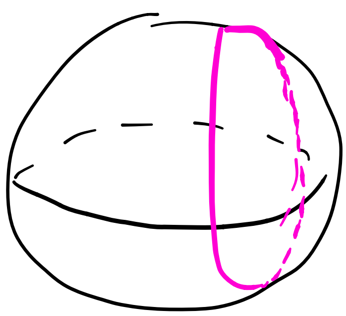
for $y = x = 0$

$x_3 = -1$

So, $x_1 = x_2$, $-1 \leq x_3 \leq 1$ } only x_3 changes as we traverse the line

What does this correspond to?

A circle on the sphere!



(14) Show

a. The mapping
$$w = e^{i\phi} z$$
corresponds to a rotation of the Riemann sphere about the x_3-axis through an angle ϕ.

Let $Z = (x_1, x_2, x_3)$ denote the stereographic projection of the point z
Let $W = (\hat{x}_1, \hat{x}_2, \hat{x}_3)$ denote the gstereographic projection of $e^{i\phi} z$

$$x_1 = \frac{2\,\mathrm{Re}(z)}{|z|^2+1}, \quad x_2 = \frac{2\,\mathrm{Im}(z)}{|z|^2+1}, \quad x_3 = \frac{|z|^2-1}{|z|^2+1}$$

$$\hat{x}_1 = \frac{2\,\mathrm{Re}(e^{i\phi}z)}{|e^{i\phi}z|^2+1}, \quad \hat{x}_2 = \frac{2\,\mathrm{Im}(e^{i\phi}z)}{|e^{i\phi}z|^2+1}, \quad \hat{x}_3 = \frac{|e^{i\phi}z|^2-1}{|e^{i\phi}z|^2+1}$$

Let $z = re^{i\theta}$
$$\mathrm{Re}(e^{i\phi}z) = r\cos(\theta+\phi)$$
$$\mathrm{Im}(e^{i\phi}z) = r\sin(\theta+\phi)$$
$$|e^{i\phi}z|^2 = |z|^2 = r^2$$

$$x_1 = \frac{2r\cos\theta}{r^2+1}, \quad x_2 = \frac{2r\sin\theta}{r^2+1}, \quad x_3 = \frac{r^2-1}{r^2+1}$$

$$\hat{x}_1 = \frac{2r\cos(\theta+\phi)}{r^2+1}, \quad \hat{x}_2 = \frac{2r\sin(\theta+\phi)}{r^2+1}, \quad \hat{x}_3 = \frac{r^2-1}{r^2+1}$$

x_3 is preserved whereas $\hat{x}_1, \hat{x}_2$ rotate x_1, x_2 respectively by ϕ! So W is the stated rotation of Z.

b. The mapping $w = -1/z$ corresponds to a $180°$ rotation of the Riemann sphere about the x_2-axis (imaginary axis).

Let $Z = (x_1, x_2, x_3)$ denote the stereographic projection of the point z

Let $W = (\hat{x}_1, \hat{x}_2, \hat{x}_3)$ denote the gstereoraphic projection $-1/z$

$$x_1 = \frac{2\,\mathrm{Re}(z)}{|z|^2+1}, \quad x_2 = \frac{2\,\mathrm{Im}(z)}{|z|^2+1}, \quad x_3 = \frac{|z|^2-1}{|z|^2+1}$$

$$\hat{x}_1 = \frac{2\,\mathrm{Re}(-1/z)}{\left|\frac{-1}{z}\right|^2+1}, \quad \hat{x}_2 = \frac{2\,\mathrm{Im}(-1/z)}{\left|\frac{-1}{z}\right|^2+1}, \quad \hat{x}_3 = \frac{\left|\frac{-1}{z}\right|^2-1}{\left|\frac{-1}{z}\right|^2+1}$$

$$\frac{-1}{z} = \frac{-1}{x+iy} = \frac{-(x-iy)}{x^2+y^2} = \frac{-x}{x^2+y^2} + i\frac{y}{x^2+y^2}$$

Using $\mathrm{Re}\left(\frac{-1}{z}\right) = -\mathrm{Re}(z)/|z|^2$

$\mathrm{Im}(-1/z) = \mathrm{Im}(z)/|z|^2$

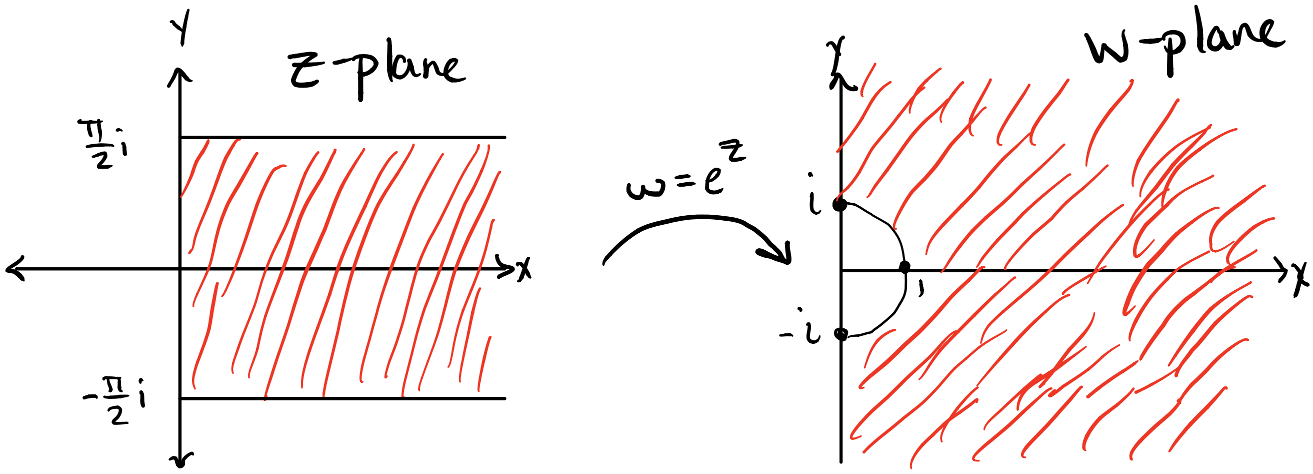
We get after simplification that

$$\hat{x}_1 = \frac{-2\,\mathrm{Re}(z)}{1+|z|^2}, \quad \hat{x}_2 = \frac{+2\,\mathrm{Im}(z)}{1+|z|^2}, \quad \hat{x}_3 = \frac{1-|z|^2}{1+|z|^2}$$

$$\hat{x}_1 = -x_1, \quad \hat{x}_2 = x_2, \quad \hat{x}_3 = -x_3$$

A rotation about x_2-axis preserves x_2 and negates x_1, x_3; so indeed W is the stated rotation of Z.

① Show that $w=e^z$ maps the half strip $x>0$, $-\frac{\pi}{2} < y < \frac{\pi}{2}$ onto the portion of the right-half w-plane that lies outside the unit circle. ①



What harmonic function $\Psi(w)$ does the w-plane "inherit" via this mapping, from the harmonic function $\phi(z) = x+y$? ②

What harmonic function $\phi(z)$ is inherited from $\Psi(w) = u+v$? ③

1 $w = e^z = e^{x+iy} = e^x(\cos(y) + i\sin(y))$

for $x > 0, \; e^x > 0$

for lower y-bound ($y = -\pi/2$)

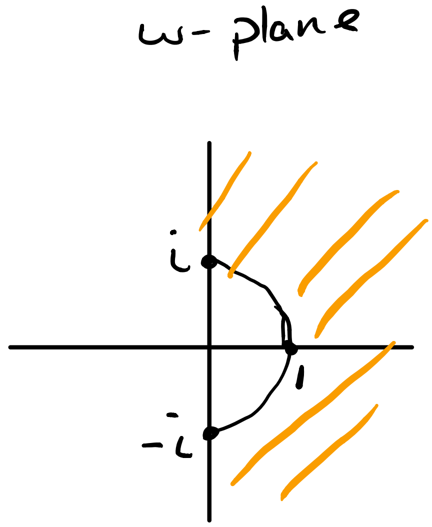
$e^{-i\frac{\pi}{2}} = -i$

for upper y-bound ($y = +\pi/2$)

$e^{i\frac{\pi}{2}} = i$

for $y = 0$

$e^{i0} = 1$



So for $z = x+iy$, $x > 0$, $-\frac{\pi}{2} < y < \frac{\pi}{2}$

The transformation

$$w = e^z$$

thus is the stated mapping.

(2) For $\phi(z) = x+y$

the inherited harmonic function $\Psi(w)$ is

$\Rightarrow$ $w = u+iv = e^{z} = e^{x}\cos(y) + ie^{x}\sin(y)$ $\Leftarrow$

$u = e^{x}\cos(y)$, $v = e^{x}\sin(y)$

$\Psi(w) = u + v$

$\Psi(w) = e^{x}(\cos y + \sin y)$

(3) For $\Psi(w) = u+v$ the inherited function $\phi(z)$ is

$\Rightarrow$ $w = e^{z} = u+iv$

$z = \text{Log}(w) = \text{Log}(u+iv)$ $\Leftarrow$

$z = x+iy = \text{Log}(u+iv) = \text{Log}|w| + i\theta$

$\phi(z) = (x+y) = \text{Log}|w| + \theta$

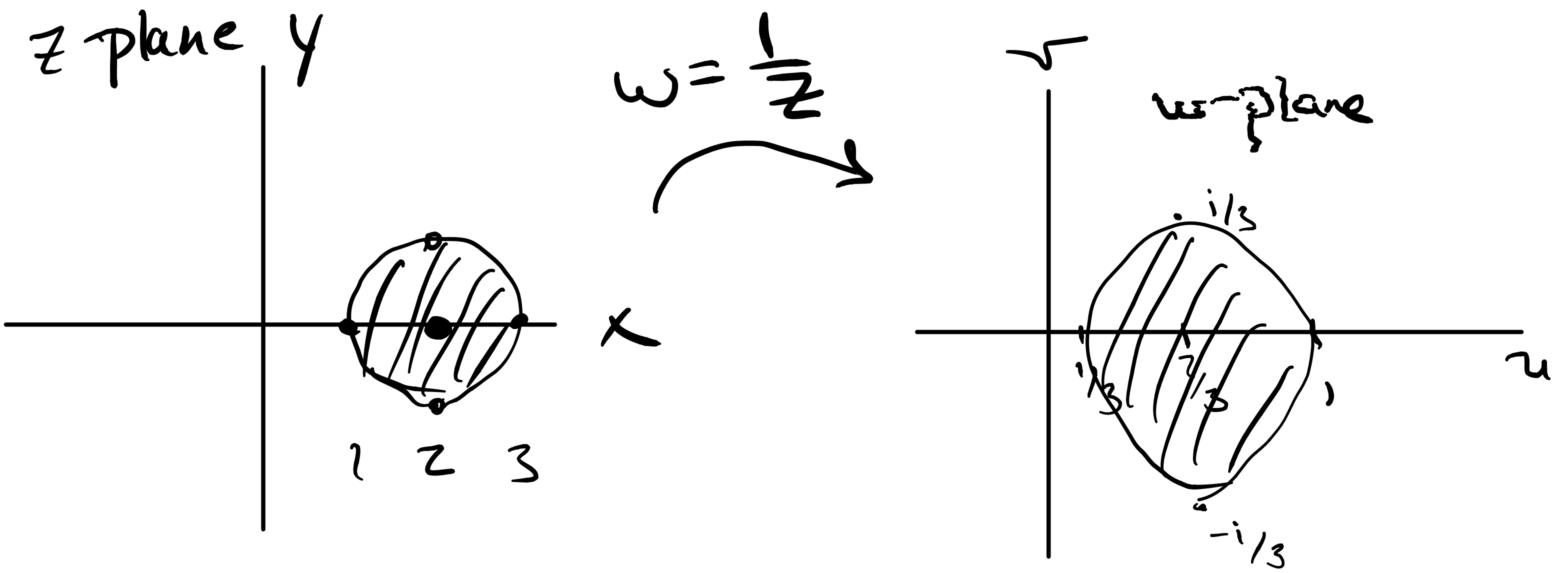
Arg(w)

Image of the circle

3e) $|z-2| = 1$

and its interior unde

$$w = \frac{1}{z}$$



Intuition:

- Circle Maps to another circle in the w-plane

Actual: (Next Page)

$$w = \frac{1}{z} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$

$(2,0) \xrightarrow{\omega} \left(\frac{1}{2}, 0\right)$

$(3,0) \xrightarrow{\omega} \left(\frac{1}{3}, 0\right)$

$(1,0) \xrightarrow{\omega} (1, 0)$

$(2,1) \xrightarrow{\omega} \left(2/3, -1/3\right)$

$(2,-1) \xrightarrow{\omega} \left(2/3, 1/3\right)$

So

$$|z-2| = 1 \xrightarrow{\omega} \left|z - \frac{2}{3}\right| = \frac{1}{3}$$

5) Möbius transform mapping

$|z|<1$ onto the right-half plane and taking $z=-i$ to the origin.

$$w = f(z) = \frac{ad-bc}{(cz+d)^2}$$

- For $w = f(-i)$ to equal zero must have a $(z+i)$ in numerator.

- To map whole right plane must approach infinity at a bound, either 0 or 1.

so $\frac{1}{z}$ or $\frac{1}{z-1}$

So $\cancel{\frac{z+i}{z}}$ or $\frac{z+i}{z-1}$

can't be this or we'd map to a line!

So,

$$w = f(z) = \frac{z+i}{z-1}$$

$$z = i \overset{w}{\Longrightarrow} 0 \quad \checkmark$$

$$z = 0 \overset{w}{\Longrightarrow} -i \quad \times$$ Need it to map to $+i$!

So we rotate our answer by π.

$$w = f(z) = e^{i\pi}\left(\frac{z+1}{z-1}\right)$$

$$\boxed{= \frac{z+i}{1-z}}$$

7a) $0, 1, \infty \longrightarrow 0, i, \infty$

$0 \to 0$

$1 \to i$

$\infty \to \infty$

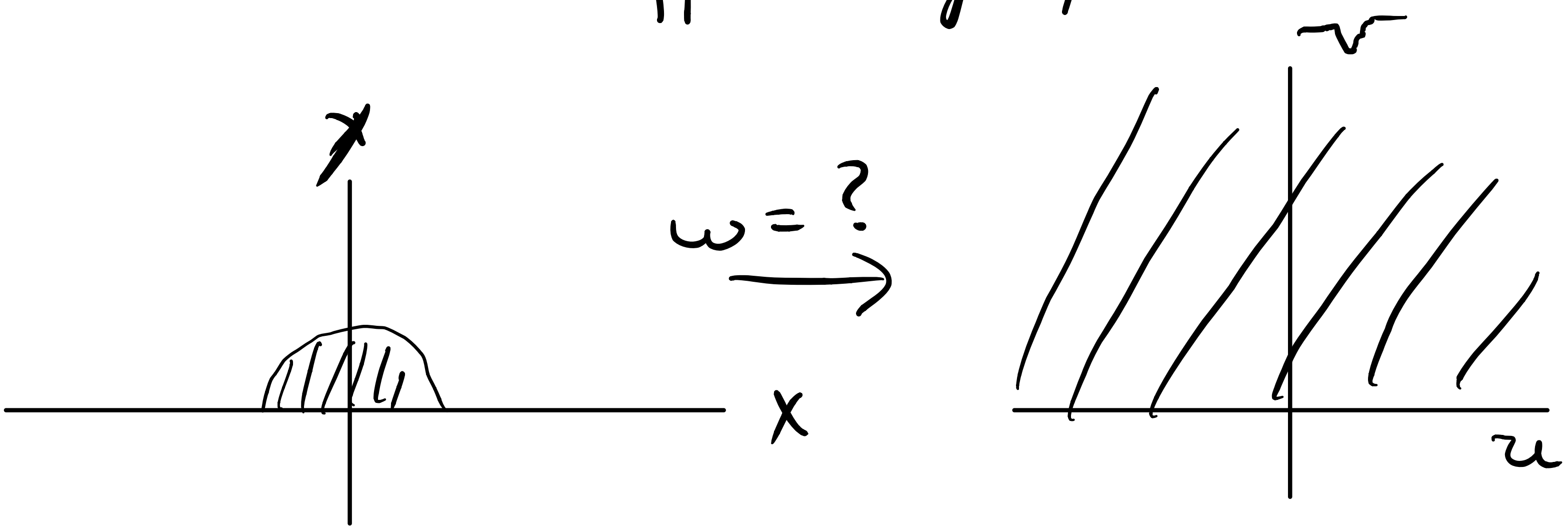
$w = z$ covers $0, \infty$ condition

$w = iz$ covers $1 \to i$ condition

$\boxed{w = iz}$

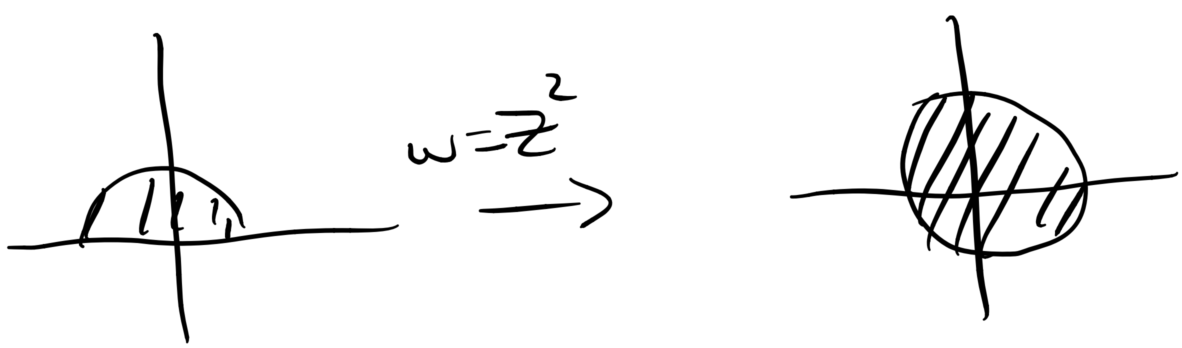
10) $|z|<1$, $\operatorname{Im} z > 0$

onto upper half plane



So z^2 maps $|z|<1$, $\operatorname{Im} z > 0$

to $|z|<1$



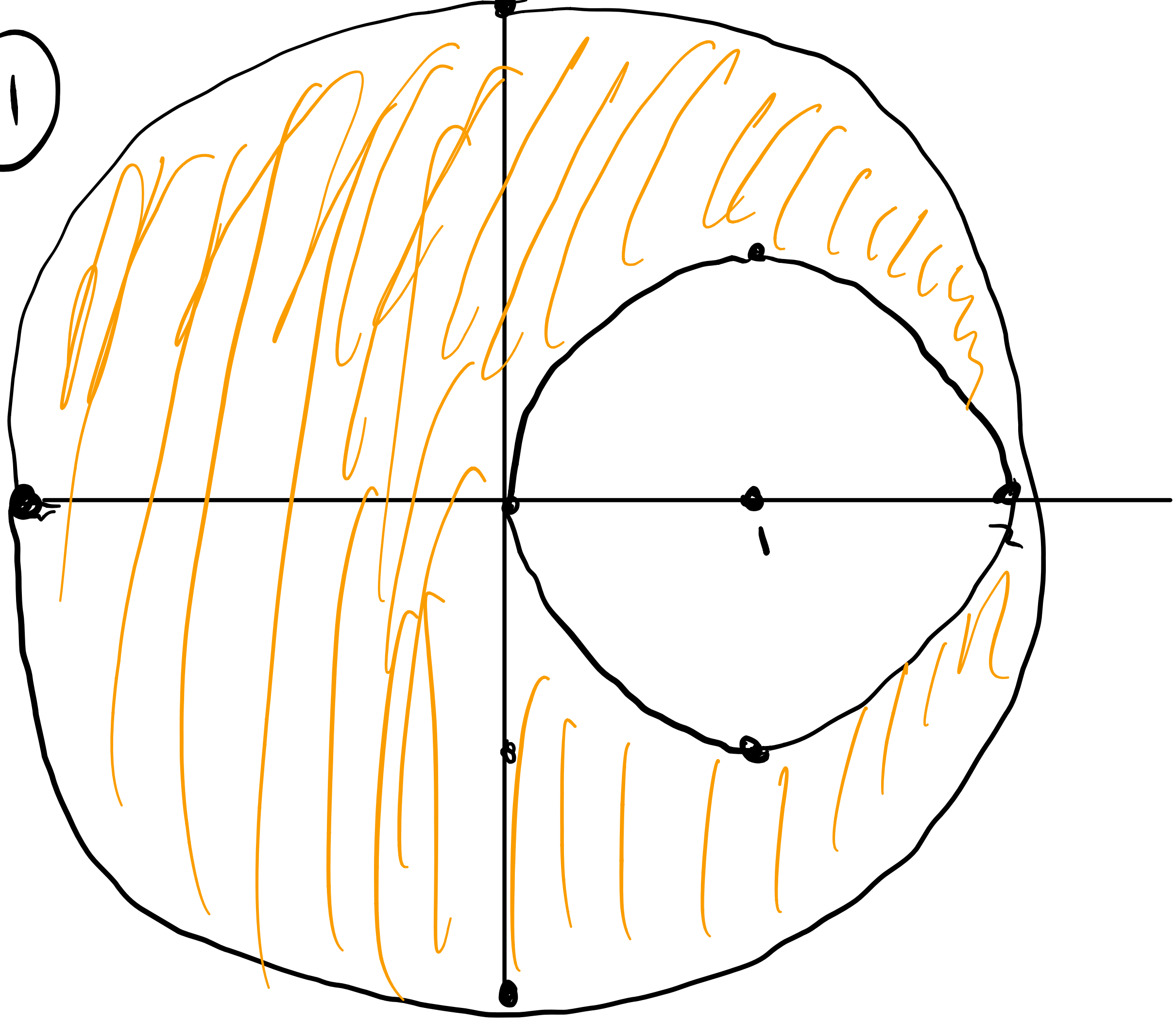
need to map to x axis so must have a pole to map to infinity

$$\frac{1}{z+i} \quad \text{or} \quad \frac{1}{z-1} \quad \text{or} \quad \frac{1}{z+1} \quad \text{or} \quad \frac{1}{z-i}$$

$w = \dfrac{1+z}{1-z}$ maps the open upper semidisk $D = \{|z|<1, \operatorname{Im}(z)>0\}$ into the open first quadrant.

So w^2 is the desired mapping

(11)



$w = f(z)$

$w_1 = \frac{1}{z-2}$ maps $2 \to \infty$

$0 \to -\frac{1}{2}$

$1+i \to -\frac{1}{2} - \frac{i}{2}$

$1 \to -1$

$1-i \to -\frac{1}{2} + i/2$