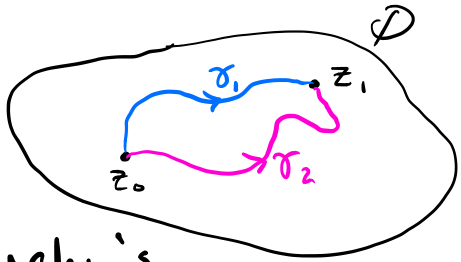
Consequences of Cauchy's Theorem

① Path independence is now obvious.

- Suppose $f(z)$ is analytic in a simply connected domain $\mathcal{D}$, and γ_1 + γ_2 are two curves that have the same endpoints.

Then,



$$\int_{(\gamma_1-\gamma_2)} f(z)\,dz = 0$$ by Cauchy's Theorem ... $(\gamma_1-\gamma_2)$ is a closed curve)

Implication?

$$\int_{(\gamma_1-\gamma_2)} f(z)\,dz = \int_{\gamma_1} f(z)\,dz - \int_{\gamma_2} f(z)\,dz = 0$$

$$\Rightarrow \int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz$$

Note: This is different from earlier path independence result!

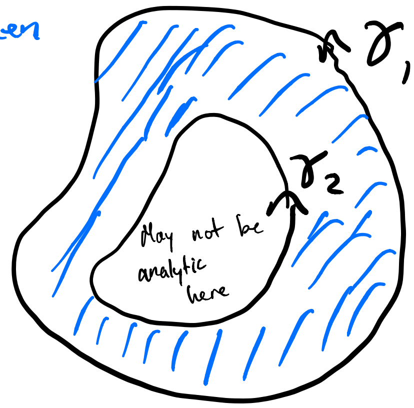
- Earlier result assumed f had an antiderivative
- New resut assumes f has a derivative (analytic)

(2) Even if f is <u>NOT</u> analytic everywhere, as long as f is analytic **ON AND BETWEEN** closed contours γ_1 and γ_2 ← assumes curves are similarly oriented

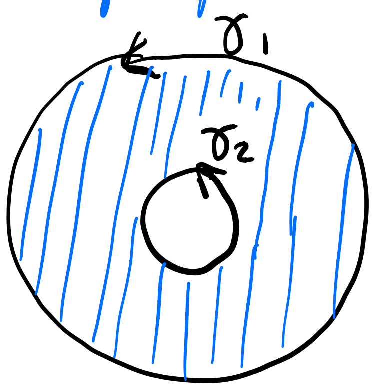
then

$$\oint_{\gamma_1} f(z)\,dz = \oint_{\gamma_2} f(z)\,dz$$

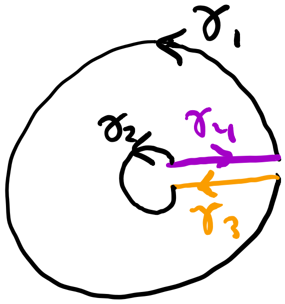
– analytic on + between γ_1, γ_2



Proof: Consider the region (region of interest)



Deform the contour like so



γ_3, γ_4 lie on same line, drawn like so for clarity.

$\gamma_3 = -\gamma_4$

traverse it opposite of shown orientation

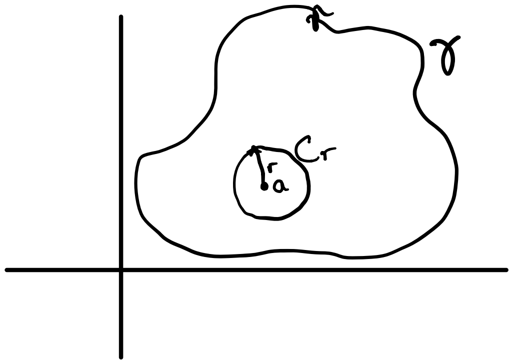
So

$$\int_{\gamma_1} f(z)\,dz - \int_{\gamma_2} f(z)\,dz + \cancelto{0}{\int_{\gamma_3} f(z)\,dz + \int_{\gamma_4} f(z)\,dz} = 0$$

$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz$$

Example:

Find $\oint_\gamma \frac{dz}{z-a}$ where γ is any contour which encircles a once positively



$\frac{1}{z-a}$ is analytic everywhere EXCEPT $z=a$.

By last result, we can deform contour to a small circle C_r around a.

$$\int_\gamma \frac{dz}{z-a} = \int_{C_r} \frac{dz}{z-a} \qquad \begin{aligned} z &= re^{it} + a \qquad 0 \leq t \leq 2\pi \\ dz &= rie^{it}\,dt \end{aligned}$$

$$\Rightarrow \int_0^{2\pi} \frac{rie^{it}}{re^{it}+a-a}\,dt = 2\pi i \quad \Big\} \text{ as we saw earlier!}$$

CAUCHY'S INTEGRAL FORMULA

Let

- γ be any positively oriented closed contour.
- f be analytic in a simply connected domain containing γ
- a be any point inside γ

THEN

$$\boxed{f(a) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z-a}\,dz}$$

AMAZING

Cauchy's Integral formula says that we can determine the values of $f(z)$ everywhere inside γ, just by knowing the values of f everywhere on the boundary γ.

Proof: Deform γ to C_r.

Then

$$\oint_\gamma \frac{f(z)}{z-a}\,dz = \oint_{C_r} \frac{f(z)}{z-a}\,dz$$

On C_r, $f(z) \simeq f(a)$ if r is very small.

Precisely,

$$f(z) = f(a) + \varepsilon(z) \quad \left.\right\} \begin{array}{l} \text{where } \varepsilon(z) \to 0 \\ \text{as } z \to a \\ \text{by continuity of } f \end{array}$$

$$\oint_\gamma \frac{f(z)}{z-a}\,dz = \oint_{C_r} \frac{f(a)+\varepsilon(z)}{z-a}\,dz = \oint_{C_r} \frac{f(a)}{z-a}\,dz + \oint_{C_r} \frac{\varepsilon(z)\,dz}{z-a}$$

$$= f(a)\oint_{C_r} \frac{dz}{z-a} + \oint_{C_r} \frac{\varepsilon(z)}{z-a}\,dz$$

ML estimate

$|z-a| = r$ on C_r

$$\left| \oint_{C_r} \frac{\varepsilon(z)}{z-a}\,dz \right| \le \underbrace{\max_{z\in C_r} \left| \frac{\varepsilon(z)}{z-a} \right|}_{M} \cdot \underbrace{2\pi r}_{L}$$

$$\left| \oint_{C_r} \frac{\varepsilon(z)}{z-a}\, dz \right| \le \max_{z \in C_r} |\varepsilon(z)| \cdot \frac{2\pi \cancel{r}}{\cancel{r}}$$

As $r \to 0$

$z \to a \implies \varepsilon(z) \to 0$

by continuity assumption

$$\therefore \quad \left| \oint_{C_r} \frac{\varepsilon(z)}{z-a}\, dz \right| = 0$$

Must be equal to 0; two things independent of r imply its exactly 0.

In conclusion,

$$\oint_{\gamma} \frac{f(z)}{z-a}\, dz = f(a) \cdot 2\pi i + 0$$

$$\Rightarrow \quad f(a) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{(z-a)}\, dz$$