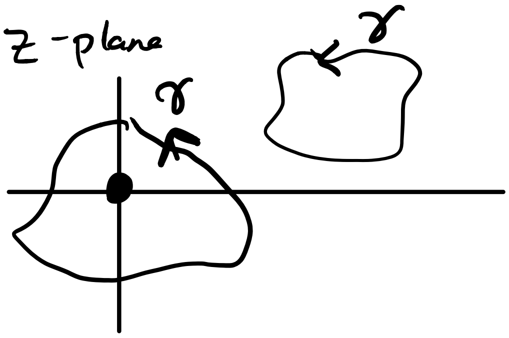
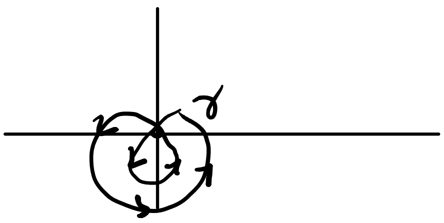
Winding Numbers and Rouché's Theorem

Recall

$$\frac{1}{2\pi i}\oint_\gamma \frac{1}{w}\,dw = \begin{cases} 1, & \gamma \text{ encloses origin} \\ 0, & \gamma \text{ does NOT enclose origin} \end{cases}$$



But say we looped over the origin more then once.



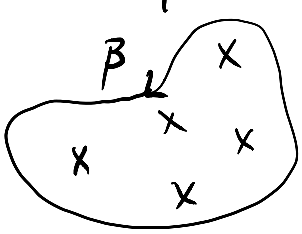
Then

$$\frac{1}{2\pi i}\oint_\gamma \frac{1}{w}\,dw = \begin{cases} 2, & \gamma \text{ encloses origin} \\ 0, & \gamma \text{ does NOT enclose origin} \end{cases}$$

We can therefore define the winding number

$$w(\gamma;\ \underset{\text{singularity}}{0}) = \frac{1}{2\pi i}\oint_\gamma \frac{1}{w}\,dw$$

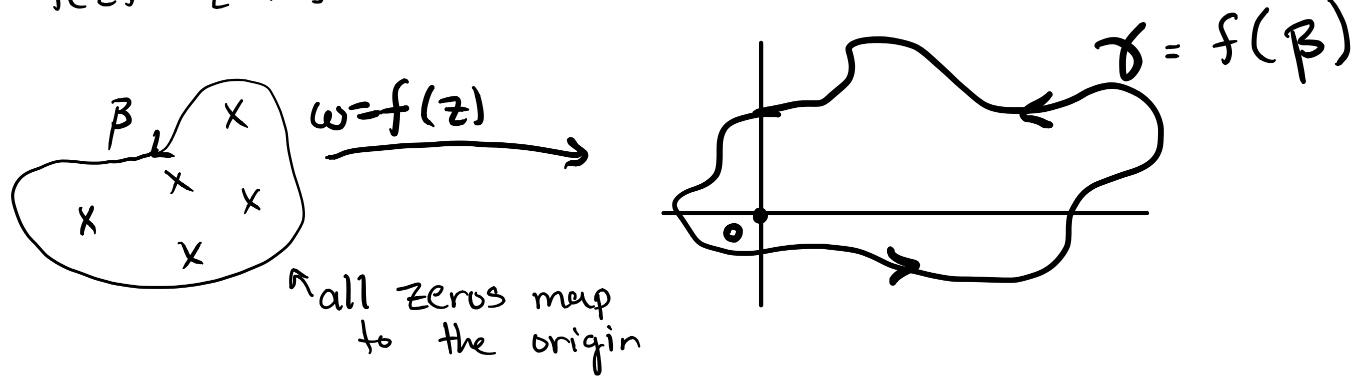
which is the amount of times γ encircles our singularity.
This is a device for counting zeros of an analytic function.
Say $f(z)$ is analytic in and inside β
where β is



Assumption, $f(z) \neq 0$ on β.

Map β onto another region using f.

For $f(z) = z^2 + 3$



Note: Simple $\beta \Rightarrow$ simple γ (0 is not on γ)

$$W(\gamma; 0) = \frac{1}{2\pi i} \oint_\gamma \frac{1}{\omega}\, d\omega$$

$f(z) = 0$ when $z = \pm\sqrt{3}\, i$

$\omega = f(z)$

$d\omega = f'(z)dz$

$\rightarrow \dfrac{1}{2\pi i} \oint_\beta \dfrac{1}{f(z)} f'(z)\, dz$ = # zeros inside β, count multiplicities

Suppose f is analytic inside and on β, and $\neq 0$ on β.

Then

$$\#\text{Zeros of } f \text{ inside } \beta, \text{ count multiplicities} = \frac{1}{2\pi i} \oint_\beta \frac{f'(z)}{f(z)}\, dz$$

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Suppose $a_1, a_2, \ldots, a_m$ are zeros of f inside β wit $n_1, n_2, \ldots, n_m$ multiplicities.

Then

$$f(z) = (z-a_1)^{n_1} h(z)$$

where

$h(z)$ does **NOT** have a root of a_1.

Continuing,

$$f(z) = (z-a_1)^{n_1}(z-a_2)^{n_2} \cdots (z-a_m)^{n_m} h(z)$$

$\Rightarrow h(z)$ doesn't have any zeros inside or on β

Aside:

If $f = f_1 f_2$, $f' = f_1' f_2 + f_1 f_2'$

$$\frac{f'}{f} = \frac{f_1' f_2 + f_1 f_2'}{f_1 f_2} = \frac{f_1'}{f_1} + \frac{f_2'}{f_2}$$

So

$$\frac{f'(z)}{f(z)} = \frac{n_1 (z-a_1)^{n_1 - 1}}{(z-a_1)^{n_1}} + \frac{n_2 (z-a_2)^{n_2-1}}{(z-a_2)^{n_2}} + \cdots \frac{n_m}{(z-a_m)} + \frac{h'(z)}{h(z)}$$

$$= \frac{n_1}{(z-a_1)} + \frac{n_2}{(z-a_2)} + \cdots + \frac{n_m}{(z-a_m)} + \frac{h'(z)}{h(z)}$$

$\frac{h'}{h}$ is analytic in + on β

So,

$$\frac{1}{2\pi i}\left[\oint_{\beta} \frac{n_1}{(z-a_1)} + \frac{n_2}{(z-a_2)} + \cdots + \frac{n_m}{(z-a_m)} + \frac{h'(z)}{h(z)}\, dz\right]$$

$$= \frac{1}{2\pi i}\left[2\pi i n_1 + 2\pi i n_2 + \cdots + 2\pi i n_m + 0\right]$$

$$= n_1 + n_2 + \cdots + n_m$$

Back to example, $f(z) = z^2+3$; count zeros inside circle of radius R at 0

$$\frac{1}{2\pi i}\oint_{|z|=R} \frac{2z}{z^2+3}\, dz$$

if $R > \sqrt{3}$

$$= \frac{1}{2\pi i}\left[\operatorname{Res}\left\{\frac{f'}{f}, \sqrt{3}i\right\} + \operatorname{Res}\left\{\frac{f'}{f}, -\sqrt{3}i\right\}\right]$$

$$= \frac{1}{2\pi i}\left[2\pi i + 2\pi i\right] = 4$$

OR

$$\frac{1}{2\pi i}\int_0^{2\pi} \frac{2Re^{it}}{(Re^{it})^2+3}\, Rie^{it}\, dt$$

In practice! Don't always know root location.

n??. Trapezoid??

Rouche's Theorem

Baby version: Given two closed curves γ_1, γ_2. Then provided that $|\gamma_2(t)| \leq |\gamma_1(t)|$ for all t, and γ_1, γ_2 do not intersect 0,

$$\text{Then } W(\gamma_1 ; 0) = W(\gamma_1 + \gamma_2 ; 0)$$

Let $f(z)$, $g(z)$ be analytic functions on and inside a simple closed curve β, and $|f(z)| > |g(z)|$.
Then

$$\# \text{ zeros of } f = \# \text{ of zeros of } f+g$$

$$\frac{1}{2\pi i} \int_\beta \frac{f'}{f} dz = W(f(\beta) ; 0)$$

$$= \frac{1}{2\pi i} \int_\beta \frac{(f+g)'}{f+g} dz = W(f(\beta) + g(\beta) ; 0)$$

Example: Determine number of zeros of $z^4 - 2z^3 + 9z^2 + z - 1$ inside $|z| = 2$

Pick

$$f(z) = 9z^2$$
$$g(z) = z^4 - 2z^3 + z - 1$$

$$|f(z)| = 36$$
$$|g(z)| \leq 16 + 16 + 2 + 1 = 35$$

$\Rightarrow$ 2 zeros inside circle