7) $\displaystyle\oint_{|z|=1} e^{1/z} \sin\left(\frac{1}{z}\right) dz$

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{z^2}\frac{1}{2!} + \frac{1}{z^3}\frac{1}{3!} + \cdots + \frac{1}{z^n}\frac{1}{n!}$$

$$= \sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n \frac{1}{n!} = \sum_{n=0}^{\infty} \frac{z^{-n}}{n!} \quad \Big\}\ \text{Laurent Series of } e^{1/z}$$

$$\sin\left(\frac{1}{z}\right) = \frac{1}{z} - \frac{\left(\frac{1}{z}\right)^3}{3!} + \frac{\left(\frac{1}{z}\right)^5}{5!} - \frac{\left(\frac{1}{z}\right)^7}{7!} + \cdots -$$

$$= \sum_{n=0}^{+\infty}\left(\frac{1}{z}\right)^{2n+1}\left(\frac{1}{(2n+1)!}\right)(-1)^n = \sum_{n=0}^{+\infty} \frac{z^{-(2n+1)}}{(2n+1)!}(-1)^n \quad \Big\}\ \text{Laurent series of } \sin(1/z)$$

So $e^{1/z}\sin\left(\frac{1}{z}\right)$

$$= \left(\sum_{n=0}^{+\infty} \frac{z^{-n}}{n!}\right)\left(\sum_{n=0}^{+\infty} \frac{z^{-(2n+1)}(-1)^n}{(2n+1)!}\right)$$

$$= \sum_{n=0}^{+\infty} \frac{(z^{-n})(z^{-2n})(z^{-1})}{(n!)[(2n+1)!]}(-1)^n = \sum_{n=0}^{+\infty} \frac{(z^{-3n-1})}{n![(2n+1)!]}(-1)^n$$

$$\Rightarrow \oint_{|z|=1} \sum_{n=0}^{+\infty} \frac{z^{-(3n+1)}}{n![(2n+1)!]}(-1)^n \, dz \quad \Big\}\ \text{What our integral becomes}$$

Interchanging $\Sigma$ and $\oint$,

**Note:**
$$\oint_{\gamma} (z-a)^n \, dz \quad a \in \mathbb{C}$$
$$= \begin{cases} 0, & \text{else} \\ 2\pi i, & n \neq -1 \end{cases}$$

$$\sum_{n=0}^{+\infty} \frac{(-1)^n}{n![(2n+1)!]} \oint_{|z|=1} \frac{1}{(z)^{3n+1}} \, dz$$

It's easy to see we only are for $\frac{1}{z}$ term.

$$\Rightarrow 3n+1 = 1$$
$$n = 0$$

So our residue term is when $n=0$!

$$\frac{(-1)^0}{0!(1!)} \oint_{|z|=1} \frac{1}{z} \, dz = \boxed{2\pi i}$$

6.2

3) $$\int_0^{\pi} \frac{d\theta}{(3+2\cos\theta)^2} = \frac{3\pi\sqrt{5}}{25}$$

$$z = e^{i\theta}$$
$$dz = ie^{i\theta}\,d\theta \Rightarrow d\theta = \frac{1}{i}\frac{dz}{z}$$
$$\cos\theta = \frac{z + 1/z}{2}$$

Note: $\int_0^{\pi} \frac{d\theta}{3+2\cos\theta} = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{3+2\cos\theta}$

So our integral becomes

$$\frac{1}{2}\int_0^{2\pi} \frac{dz/zi}{\left[3+2\left(\frac{z+1/z}{2}\right)\right]^2} = \frac{1}{2i}\int_0^{2\pi} \frac{z}{(3z+z^2+1)^2}\,dz$$

Side work

denominator

$$\frac{(z^2+3z+1)^2}{z^{\cancel{2}}}\ \cancel{z}$$

The roots of

$$z^2+3z+1$$

given by the quadratic formula are

$$z_{1,2} = \frac{-3 \pm \sqrt{9-4}}{2} \Rightarrow z_1 = \frac{-3}{2} + \frac{\sqrt{5}}{2}\ ;\ z_2 = \frac{-3}{2} - \frac{\sqrt{5}}{2}$$

Therefore our integral becomes

$$\frac{1}{2i}\int_{|z|=1} \frac{dz}{[(z-z_1)(z-z_2)]^2}$$

Only $z_1$ lies in our contour. Therefore $z_1$ is a pole of order 2.

$$\Rightarrow \operatorname{Res}\left(f(z); z_1\right) = \lim_{z \to z_1} \frac{1}{(m-1)!}\left(\frac{d}{dz}\right)^{m-1} (z-z_1)^m f(z)$$

$$= \lim_{z \to z_1} \frac{d}{dz}\left[(z-z_1)^2 f(z)\right]$$

$$z_1 - z_2 = \sqrt{5}$$

$$= \lim_{z \to z_1} \frac{d}{dz}\left[\frac{z}{(z-z_2)^2}\right]$$

$$= \lim_{z \to z_1}\left[\frac{(z-z_2)^2 - 2z(z-z_2)}{(z-z_2)^4}\right]$$

$$= \lim_{z \to z_1}\left[\frac{(z-z_2) - 2z}{(z-z_2)^3}\right]$$

$$= \frac{\sqrt{5} + 3 - \sqrt{5}}{(\sqrt{5})^3} = \frac{3}{5\sqrt{5}} = \frac{3\sqrt{5}}{25}$$

So $$\frac{1}{2i}\oint_{|z|=1} \frac{dz}{(z-z_1)^2(z-z_2)^2} = \frac{2\pi i}{2i}\, 3\frac{\sqrt{5}}{25} = \boxed{\frac{3\pi\sqrt{5}}{25}}$$

9) $$\int_0^{2\pi} (\cos\theta)^{2n}\, d\theta = \frac{\pi\,(2n)!}{2^{2n-1}\,(n!)^2} \qquad n = 1, 2, \ldots$$

Wowza, let's begin

$$\cos\theta = \frac{z + 1/z}{2} \qquad z = e^{i\theta}$$

$$d\theta = \frac{dz}{iz}$$

$$\int_{|z|=1} \left(\frac{z + \frac{1}{z}}{2}\right)^{2n} \frac{dz}{iz} \qquad z + \frac{1}{z} = \frac{z^2+z}{z}$$

$$(2z)^{2n}\, z = 2^{2n}\, z^{(2n+1)}$$

$$\frac{1}{i} \int_{|z|=1} \left(\frac{z^2+z}{2z}\right)^{2n} \frac{dz}{z}$$

$$\frac{1}{2^{2n}\, i} \int_{|z|=1} \frac{(z^2+z)^{2n}}{z^{2n+1}}\, dz$$

Our answers beginning to take shape!

$$\frac{2\pi i}{2^{2n}\, i} \sum \text{Res}(\text{Singularities inside } |z|=1)$$

$$= \frac{\pi}{2^{2n-1}} \sum \text{Res}(\text{Singularities inside } |z|=1)$$

$\dfrac{(z^2+z)^{2n}}{z^{2n+1}}$ ~~is~~ has a pole of order $(2n+1)$ at $z=0$.

So $\text{Res}\left(\dfrac{(z^2+z)^{2n}}{z^{2n+1}} ; 0\right) = \lim_{z\to 0} \dfrac{1}{(2n)!}\left(\dfrac{d}{dz}\right)^{2n} (z-z_0)^{2n+1} f(z)$

$$= \lim_{z\to 0} \frac{1}{(2n)!} \frac{d^{2n}}{dz^{2n}}\left[(z^2+z)^{2n}\right] = \frac{1}{2n!}\left(\frac{(2n)!}{n!}\right)^2 = \frac{(2n)!}{(n!)^2}$$

Not obvious but must be true

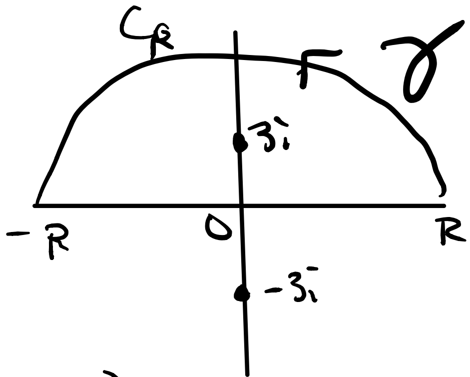
Our answer is therefore

$$\boxed{\frac{\pi}{2^{2n-1}} \frac{(2n)!}{(n!)^2}, \quad n = 1, 2, 3 \ldots}$$

26.3

2) $$\int_{-\infty}^{+\infty} \frac{x^2}{(x^2+9)^2}\,dx = \frac{\pi}{6}$$

Have $$\lim_{R\to\infty} \int_{-R}^{+R} \frac{x^2}{(x^2+9)^2}\,dx$$




$$\int_\gamma \frac{z^2}{(z^2+9)^2}\,dz = \lim_{R\to\infty}\int_{-R}^{+R} \frac{x^2}{(x^2+9)^2}\,dx + \int_{C_R} \frac{z^2}{(z^2+9)^2}\,dz$$

$\downarrow$

$= 2\pi i \sum$ Res (isolated singularities inside $\gamma$)

$$= 2\pi i \lim_{z\to 3i} \frac{1}{1!}\left(\frac{d}{dz}\right)^1 \cancel{(z-3i)^2} \frac{z^2}{(z+3i)^2\cancel{(z-3i)^2}}$$

$$= 2\pi i \lim_{z\to 3i} \frac{d}{dz}\left[\frac{z^2}{(z+3i)^2}\right] \rightarrow \frac{2(z+3i)^{\cancel{2}}z - 2\cancel{(z+3i)}z^2}{(z+3i)^3}$$

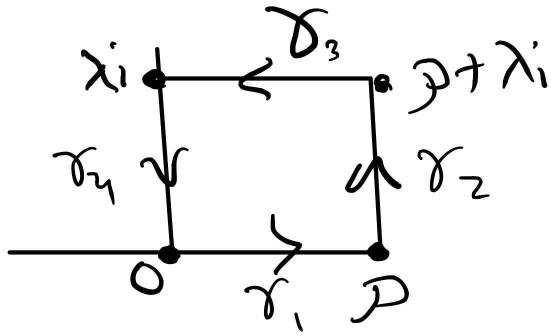
$$= 2\pi i \lim_{z\to 3i}\left[\frac{(z+3i)^2 2z - z^2 2(z+3i)}{(z+3i)^4}\right]$$

$$= 2\pi i\left[\frac{(6i)^3 - 2(3i)^2(6i)}{(6i)^4}\right] = 2\pi i\left[\frac{(6i)^2 - 2(3i)^2}{(6i)^3}\right]$$

$$= 2\pi i\left[\frac{-36+18}{-216i}\right] = 2\pi \cancel{i}\,\frac{-18}{-216\cancel{i}} = \boxed{\frac{\pi}{6}}$$

10a) $$I = \int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$$

integrate $e^{-z^2}$ around $\gamma$, w/ $\lambda > 0$, and let $p \to \infty$ to derive




$$\int_0^\infty e^{-x^2}\cos(2\lambda x)\, dx = \frac{\sqrt{\pi}}{2} e^{-\lambda^2}$$

$e^{-z^2}$ is entire

$$\int_\gamma e^{-z^2} dz = \boxed{\int_{\gamma_1} e^{-z^2} dz} + \boxed{\int_{\gamma_2} e^{-z^2} dz} + \boxed{\int_{\gamma_3} e^{-z^2} dz} + \boxed{\int_{\gamma_4} e^{-z^2} dz} = 0$$

$$\int_{\gamma_1} e^{-z^2} dz = \lim_{p\to\infty} \int_0^p e^{-x^2} dx = I$$

$z = x$

$dz = dx$

$$\int_{\gamma_2} e^{-z^2} dz = \lim_{p\to\infty} \int_0^\lambda e^{-z^2}\, dz$$

$z = p + iy \to dz = i\,dy$

$$= \lim_{p\to\infty} \int_0^{\lambda} e^{-(p+iy)^2}\, i\,dy$$

$$= \lim_{p\to\infty} \int_0^{\lambda} e^{-p^2} e^{y^2} e^{-2ipy}\, i\,dy$$

As $p \to \infty$ $e^{-p^2} \to 0$ and dominates so this integral goes to zero!

$z = x + \lambda i$

$dz = dx$

$$\int_{\gamma_3} e^{-z^2}\,dz = \lim_{p\to\infty} \int_p^0 e^{-z^2}\,dx$$

$$= -\lim_{p\to\infty} \int_0^p e^{-(x+i\lambda)^2}\,dx = -\lim_{p\to\infty} \int_0^p e^{\lambda^2} e^{-x^2} e^{-2ix\lambda}\,dx$$

$\int_{\gamma_4} e^{-z^2}\,dz$, $z = iy$

$dz = i\,dy$

$$= \int_\lambda^0 e^{-(iy)^2}\, i\, dy = -\int_0^\lambda e^{y^2}\, i\, dy$$

We can ignore this integral on $\gamma_4$ since it is entirely imaginary. Our loop sums to zero so the real part must be zero which is what we care about

So,

$$\int_\gamma = \boxed{\int_{\gamma_1} e^{-z^2}dz} + \boxed{\int_{\gamma_2} e^{-z^2}dz} + \boxed{\int_{\gamma_3} e^{-z^2}dz} + \boxed{\int_{\gamma_4} e^{-z^2}dz}$$

only want real part!

$$= \frac{\sqrt{\pi}}{2} + 0 - \lim_{p\to\infty}\int_0^p e^{\lambda^2}e^{-x^2}e^{-2ix\lambda} = 0$$

(sin part $\to 0$ b/c sin odd)

$$\lim_{p\to\infty}\int_0^p e^{\lambda^2}e^{-x^2}\cos(2\lambda x)\,dx = \frac{\sqrt{\pi}}{2}$$

So

$$\int_0^\infty e^{-x^2}\cos(2\lambda x)\,dx = e^{-\lambda^2}\frac{\sqrt{\pi}}{2}\ !$$

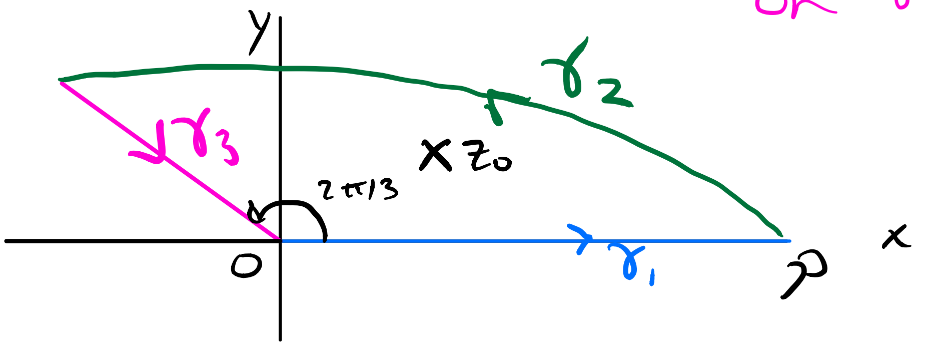
11) Show

$$I = \int_0^\infty \frac{dx}{x^3+1} = \frac{2\pi\sqrt{3}}{9}$$

by integrating $\frac{1}{z^3+1}$ around the boundary of the circular sector

$$S_\rho : \{ z = re^{i\theta} : 0 \le \theta \le 2\pi/3 \ , \ 0 \le r \le \rho \}$$

and letting $\rho \to \infty$




On $\gamma_3$, $z = \overset{\rho}{r} e^{i\left(-\frac{2\pi}{3}\right)}$

$dz = dx$

$z = x$

$$\int_\gamma \frac{1}{z^3+1}\,dz = \int_{\gamma_1} \frac{1}{z^3+1}\,dz + \int_{\gamma_2} \frac{1}{z^3+1}\,dz + \int_{\gamma_3} \frac{1}{1+z^3}\,dz$$

Let $f(z) = \frac{1}{z^3+1}$.

$f$ has poles at $z_k = e^{i\frac{\pi}{3} + \frac{2\pi}{3}k}$, $k = 0, 1, 2$

$z_0$ lies in our contour. $z_0 = e^{i\pi/3}$

$$2\pi i \, \mathrm{Res}(f; z_0) = \int_{\gamma_1} \frac{1}{z^3+1}\,dz + \int_{\gamma_2} \frac{1}{z^3+1}\,dz + \int_{\gamma_3} \frac{1}{1+z^3}\,dz$$

$$\int_{\gamma_1} \frac{1}{z^3+1}\,dz = \lim_{\rho\to\infty} \int_0^{\rho} \frac{1}{x^3+1}\,dx = I$$

$z = x$

$dz = dx$

$$\int_{\gamma_2} \frac{1}{z^3+1}\,dz = \underbrace{\lim_{r\to\rho\to\infty} \int_0^{\frac{2\pi}{3}} \frac{1}{(re^{i\theta})^3+1}\, rie^{i\theta}\,d\theta}_{\text{Clearly this goes to zero!}}$$

$z = re^{i\theta}$

$dz = rie^{i\theta}d\theta$

$$\frac{ire^{i\theta}}{r^3e^{i3\theta}} \longrightarrow 0 \text{ as } r\to\infty$$

$$\int_{\gamma_3} \frac{1}{1+z^3}\,dz = \lim_{\rho\to\infty} \int_{\rho}^{0} \frac{1}{1+r^3\underbrace{e^{i(2\pi)}}_{1}} e^{i\left(\frac{2\pi}{3}\right)}\,dx$$

$z = re^{\left(i\frac{2\pi}{3}\right)}$

$dz = dre^{\left(i\frac{2\pi}{3}\right)}$

So,

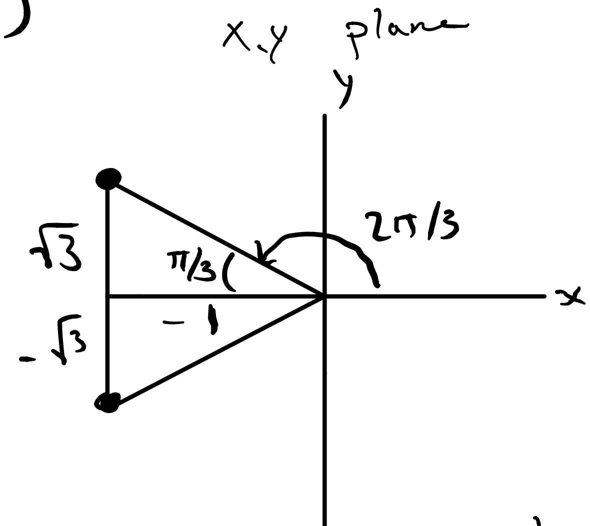
$$2\pi i \,\mathrm{Res}(f; z_0) = \int_{\gamma_1} \frac{1}{z^3+1}\,dz + \int_{\gamma_2} \frac{1}{z^3+1}\,dz + \int_{\gamma_3} \frac{1}{1+z^3}\,dz$$

$$\underbrace{2\pi i \,\mathrm{Res}(f; z_0)}_{} = I - I e^{i2\pi/3}$$

$$\lim_{z \to z_0} \frac{(z - z_0)}{1+z^3} \xrightarrow{\text{L'hopitals}} \lim_{z \to z_0} \frac{1}{3z^2} = \frac{1}{3e^{i2\pi/3}}$$

So

$$\frac{2\pi i}{3e^{i2\pi/3}} = I\left(1 \quad e^{i2\pi/3}\right)$$




$$I = \frac{2\pi i}{3} \cdot \frac{1}{e^{i2\pi/3}\left(1 - e^{i2\pi/3}\right)}$$

$$= \frac{2\pi i}{3} \cdot \frac{1}{e^{i2\pi/3} - e^{i4\pi/3}}$$

$$= \frac{2\pi i}{3} \cdot \frac{1}{\sqrt{3}\, i} = \boxed{\frac{2\pi\sqrt{3}}{9}}$$