

(5)

Describe the projections on the Riemann Sphere of the following sets in the complex plane.

a. Right-half plane $\{z : \operatorname{Re}(z) > 0\}$

$$z = x + iy, \quad x > 0$$

The north pole of the Riemann Sphere is
 $(x_1, x_2, x_3) \rightarrow (0, 0, 1)$

A point in the complex plane is then
 $(x_1, x_2, x_3) \rightarrow (x, y, 0)$

the line through $(0, 0, 1)$ and $(x, y, 0)$ is given by

$$x_1 = tx (x \neq 0), \quad x_2 = ty, \quad x_3 = 1 - t$$

where

-\infty < t < \infty

The line cuts the sphere when t satisfies

$$1 = x_1^2 + x_2^2 + x_3^2 = t^2 x^2 + t^2 y^2 + (1-t)^2$$

whose roots are $t=0$ (North pole) and

$$t = \frac{2}{x^2 + y^2 + 1} = \frac{2}{|z|^2 + 1}$$

So,

$$x_1 = \frac{2 \operatorname{Re}(z)}{|z|^2 + 1}, \quad x_2 = \frac{2 \operatorname{Im}(z)}{|z|^2 + 1}, \quad x_3 = \frac{|z|^2 - 1}{|z|^2 + 1}$$

$$x_1 = \frac{2x}{|z|^2 + 1}, \quad x_2 = \frac{2y}{|z|^2 + 1}, \quad x_3 = \frac{|z|^2 - 1}{|z|^2 + 1}$$

where $x > 0$!

So, $\{z : \operatorname{Re}(z) > 0\}$ maps onto the right hemisphere of the Riemann Sphere.

b. the disk $\{z : |z| < 1/2\}$

$$z = \frac{1}{2} e^{i\theta} = \frac{1}{2} \cos(\theta) + \frac{i}{2} \sin(\theta)$$

$$x_1 = \frac{2 \operatorname{Re}(z)}{|z|^2 + 1}, \quad x_2 = \frac{2 \operatorname{Im}(z)}{|z|^2 + 1}, \quad x_3 = \frac{|z|^2 - 1}{|z|^2 + 1}$$

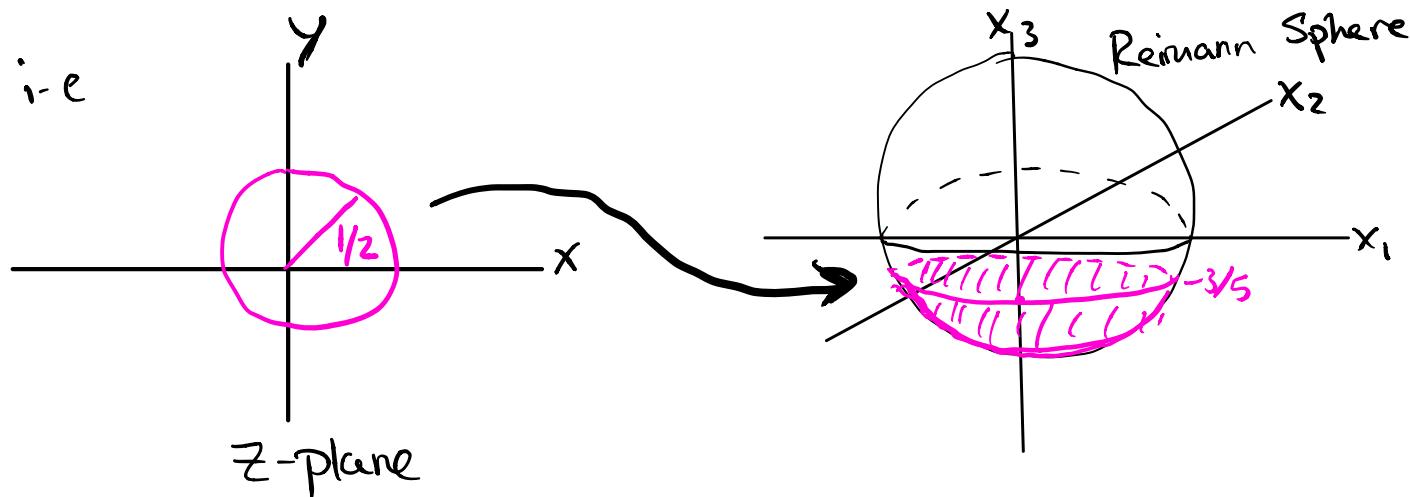
Subbing in for $|z|$,

$$x_1 < \frac{\cos(\theta)}{\frac{1}{4} + 1}, \quad x_2 < \frac{\sin(\theta)}{\frac{1}{4} + 1}, \quad x_3 < \frac{\frac{1}{4} - 1}{\frac{1}{4} + 1}$$

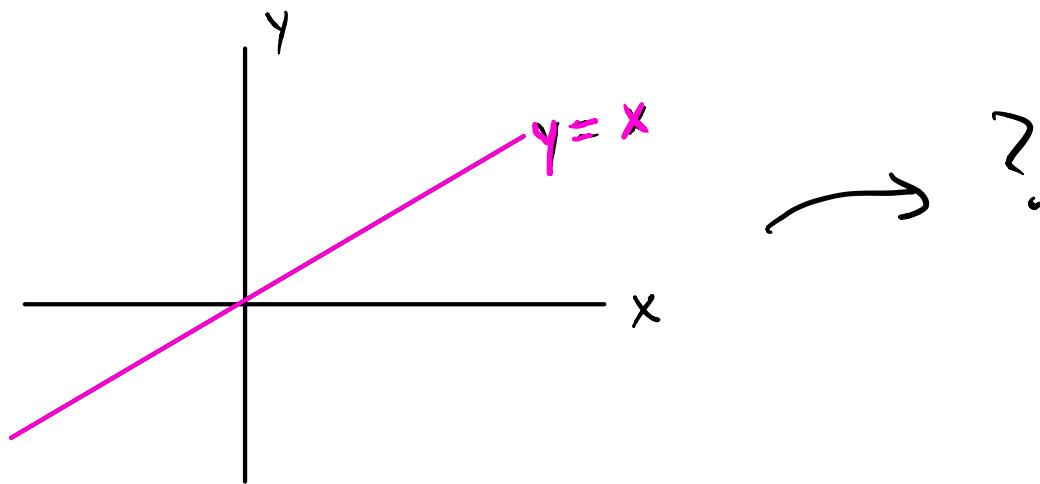
$$\Rightarrow x_1 < \frac{4}{5} \cos(\theta), \quad x_2 < \frac{4}{5} \sin(\theta), \quad x_3 < -\frac{3}{5}$$

So, the disk $\{z : |z| < 1/2\}$ maps onto the "bowl" $x_3 < -3/5$ on the Riemann sphere

i.e



e. The line $y = x$ (including point at infinity).



$$z = x + iy = x + ix = y + iy$$

$$x_1 = \frac{2\operatorname{Re}(z)}{|z|^2 + 1}, \quad x_2 = \frac{2\operatorname{Im}(z)}{|z|^2 + 1}, \quad x_3 = \frac{|z|^2 - 1}{|z|^2 + 1}$$

$x_1 = x_2$ Why? because $y = x$ so
 $\operatorname{Re}(z) = \operatorname{Im}(z)$

$$|z| = \sqrt{x^2 + y^2} = \sqrt{2y^2} = \sqrt{2x^2} = \sqrt{2}y = \sqrt{2}x$$

$$|z|^2 = 2y^2 = 2x^2$$

$$x_3 = \frac{2y^2 - 1}{2y^2 + 1} = \frac{2x^2 - 1}{2x^2 + 1}$$

as $y = x \rightarrow \infty$ similarly for $-\infty$
 $x_3 \rightarrow 1$ (via L'hospital's rule)

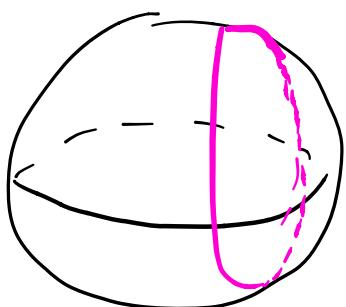
for $y = x = 0$

$$x_3 = -1$$

So, $x_1 = x_2, -1 \leq x_3 \leq 1$ } only x_3 changes as we traverse the line

What does this correspond to?

A circle on the sphere!



(14) Show

a. The mapping

$$w = e^{i\phi} z$$

corresponds to a rotation of the Riemann sphere about the x_3 -axis through an angle ϕ .

Let $Z = (x_1, x_2, x_3)$ denote the stereographic projection of the point z

Let $W = (\hat{x}_1, \hat{x}_2, \hat{x}_3)$ denote the stereographic projection of $e^{i\phi} z$

$$x_1 = \frac{2 \operatorname{Re}(z)}{|z|^2 + 1}, \quad x_2 = \frac{2 \operatorname{Im}(z)}{|z|^2 + 1}, \quad x_3 = \frac{|z|^2 - 1}{|z|^2 + 1}$$

$$\hat{x}_1 = \frac{2 \operatorname{Re}(e^{i\phi} z)}{|e^{i\phi} z|^2 + 1}, \quad \hat{x}_2 = \frac{2 \operatorname{Im}(e^{i\phi} z)}{|e^{i\phi} z|^2 + 1}, \quad \hat{x}_3 = \frac{|e^{i\phi} z|^2 - 1}{|e^{i\phi} z|^2 + 1}$$

Let $z = r e^{i\theta}$

$$\operatorname{Re}(e^{i\phi} z) = r \cos(\theta + \phi)$$

$$\operatorname{Im}(e^{i\phi} z) = r \sin(\theta + \phi)$$

$$|e^{i\phi} z|^2 = |z|^2 = r^2$$

$$x_1 = \frac{2r \cos \theta}{r^2 + 1}, \quad x_2 = \frac{2r \sin \theta}{r^2 + 1}, \quad x_3 = \frac{r^2 - 1}{r^2 + 1}$$

$$\hat{x}_1 = \frac{2r \cos(\theta + \phi)}{r^2 + 1}, \quad \hat{x}_2 = \frac{2r \sin(\theta + \phi)}{r^2 + 1}, \quad \hat{x}_3 = \frac{r^2 - 1}{r^2 + 1}$$

x_3 is preserved whereas \hat{x}_1, \hat{x}_2 rotate x_1, x_2 respectively by ϕ ! So W is the stated rotation of Z .

b. The mapping $w = -1/z$ corresponds to a 180° rotation of the Riemann sphere about the x_2 -axis (imaginary axis).

Let $\tilde{z} = (x_1, x_2, x_3)$ denote the stereographic projection of the point z .
 Let $w = (\hat{x}_1, \hat{x}_2, \hat{x}_3)$ denote the stereographic projection $^{-1}/z$

$$x_1 = \frac{2 \operatorname{Re}(z)}{|z|^2 + 1}, \quad x_2 = \frac{2 \operatorname{Im}(z)}{|z|^2 + 1}, \quad x_3 = \frac{|z|^2 - 1}{|z|^2 + 1}$$

$$\hat{x}_1 = \frac{2 \operatorname{Re}(-1/z)}{\left|\frac{-1}{z}\right|^2 + 1}, \quad \hat{x}_2 = \frac{2 \operatorname{Im}(-1/z)}{\left|\frac{-1}{z}\right|^2 + 1}, \quad \hat{x}_3 = \frac{\left|\frac{-1}{z}\right|^2 - 1}{\left|\frac{-1}{z}\right|^2 + 1}$$

$$\frac{-1}{z} = \frac{-1}{x+iy} = -\frac{(x-iy)}{x^2+y^2} = \frac{-x}{x^2+y^2} + i \frac{y}{x^2+y^2}$$

Using $\operatorname{Re}(-1/z) = -\operatorname{Re}(z)/|z|^2$
 $\operatorname{Im}(-1/z) = \operatorname{Im}(z)/|z|^2$

We get after simplification that

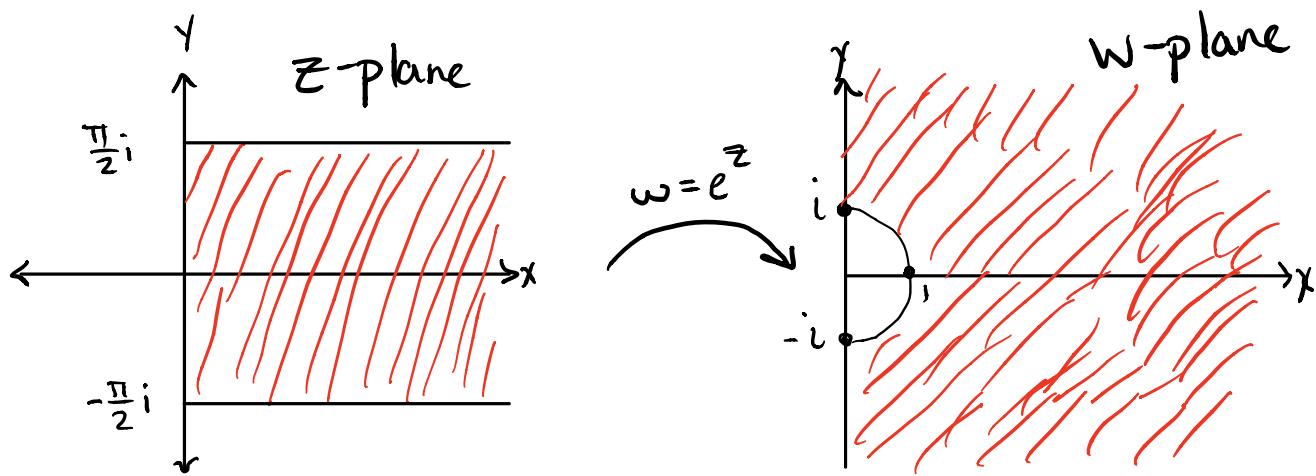
$$\hat{x}_1 = \frac{-2 \operatorname{Re}(z)}{1 + |z|^2}, \quad \hat{x}_2 = \frac{+2 \operatorname{Im}(z)}{1 + |z|^2}, \quad \hat{x}_3 = \frac{1 - |z|^2}{1 + |z|^2}$$

$$\hat{x}_1 = -x_1, \quad \hat{x}_2 = x_2, \quad \hat{x}_3 = -x_3$$

A rotation about x_2 -axis preserves x_2 and negates x_1, x_3 ; so indeed W is the stated rotation of \tilde{z} .

① Show that $w = e^z$ maps the half strip $x > 0, -\frac{\pi}{2} \leq y < \frac{\pi}{2}$ onto the portion of the right-half w -plane that lies outside the unit circle.

(1)



What harmonic function $\Psi(w)$ does the w -plane "inherit" via this mapping, from the harmonic function $\phi(z) = x + y$?

(2)

What harmonic function $\psi(z)$ is inherited from $\Psi(w) = u + v$?

(3)

$$\textcircled{1} \quad w = e^z = e^{x+iy} = e^x (\cos(y) + i \sin(y))$$

for $x > 0, e^x > 0$

for lower y -bound ($y = -\pi/2$)

$$e^{i\frac{\pi}{2}} = -i$$

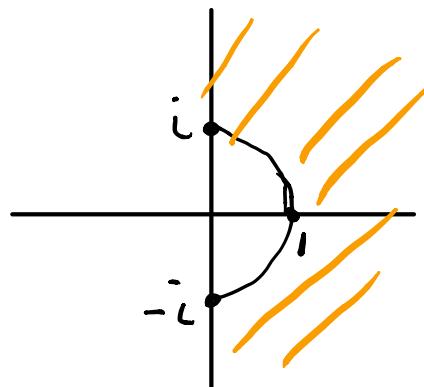
for upper y -bound ($y = +\pi/2$)

$$e^{i\frac{\pi}{2}} = i$$

for $y = 0$

$$e^{i0} = 1$$

w -plane



So for $z = x + iy, x > 0, -\frac{\pi}{2} < y < \frac{\pi}{2}$

The transformation

$$w = e^z$$

thus is the stated mapping.

② For $\phi(z) = x + y$

the inherited harmonic function $\Psi(w)$ is

$$\Rightarrow w = u + iv = e^z = e^x(\cos y + i \sin y) \Leftarrow \\ u = e^x \cos y, v = e^x \sin y$$

$$\Psi(w) = u + v$$

$$\Psi(w) = e^x(\cos y + \sin y)$$

③ For $\Psi(w) = u + v$ the inherited function
 $\phi(z)$ is

$$\Rightarrow w = e^z = u + iv \\ z = \operatorname{Log}(w) = \operatorname{Log}(u + iv) \Leftarrow$$

$$z = x + iy = \operatorname{Log}(u + iv) = \operatorname{Log}|w| + i\theta$$

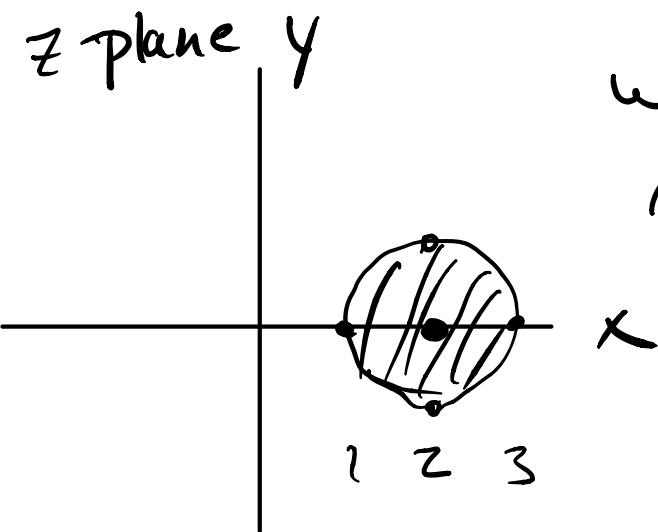
$$\phi(z) = (x + y) = \underbrace{\operatorname{Log}|w|}_{\operatorname{Arg}(w)} + \theta$$

Image of the circle

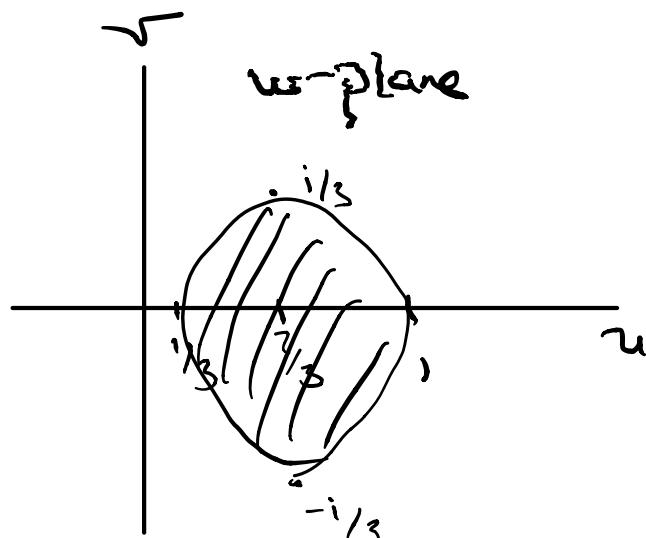
$$3e) |z - 2| = 1$$

and its interior under

$$\omega = \frac{1}{z}$$



$$\omega = \frac{1}{z}$$



Intuition:

- Circle Maps to another circle in the w-plane

Actual: (Next Page)

$$\omega = \frac{1}{z} = \frac{x}{x^2+y^2} - i \frac{y}{x^2+y^2}$$

$$\begin{array}{ll}
 (z, 0) \xrightarrow{\omega} \left(\frac{1}{2}, 0\right) & (z, 1) \xrightarrow{\omega} \left(\frac{2}{3}, -\frac{1}{3}\right) \\
 (3, 0) \xrightarrow{\omega} \left(\frac{1}{3}, 0\right) & (z, -1) \xrightarrow{\omega} \left(\frac{2}{3}, \frac{1}{3}\right) \\
 (1, 0) \xrightarrow{\omega} (1, 0)
 \end{array}$$

so

$$|z - z_1| = 1 \xrightarrow{\omega} |z - \frac{2}{3}| = \frac{1}{3}$$

5) Möbius transform mapping

$|z| < 1$ onto the right-half plane
and taking $z = -i$ to the origin.

$$w = f(z) = \frac{ad - bc}{(cz + d)^2}$$

- For $w = f(-i)$ to equal zero must have a $(z+i)$ in numerator.
- To map whole right plane must approach infinity at a bound, either 0 or 1.

so $\frac{1}{z}$ or $\frac{1}{z-1}$

so ~~$\frac{z+i}{z}$~~ or $\frac{z+i}{z-1}$

Can't be this or we'd map to a line!

So,

$$w = f(z) = \frac{z+i}{z-1}$$

$$z=i \Rightarrow 0 \quad \checkmark$$

$$z=0 \Rightarrow -i \quad \times \quad \text{Need it to map to } +i!$$

So we rotate our answer by π .

$$w = f(z) = e^{i\pi} \left(\frac{z+i}{z-1} \right)$$

$$\boxed{= \frac{z+i}{1-z}}$$

$$7a) 0, 1, \infty \rightarrow 0, i, \infty$$

$$0 \rightarrow 0$$

$$1 \rightarrow i$$

$$\infty \rightarrow \infty$$

$w = z$ covers

$0, \infty$

condition

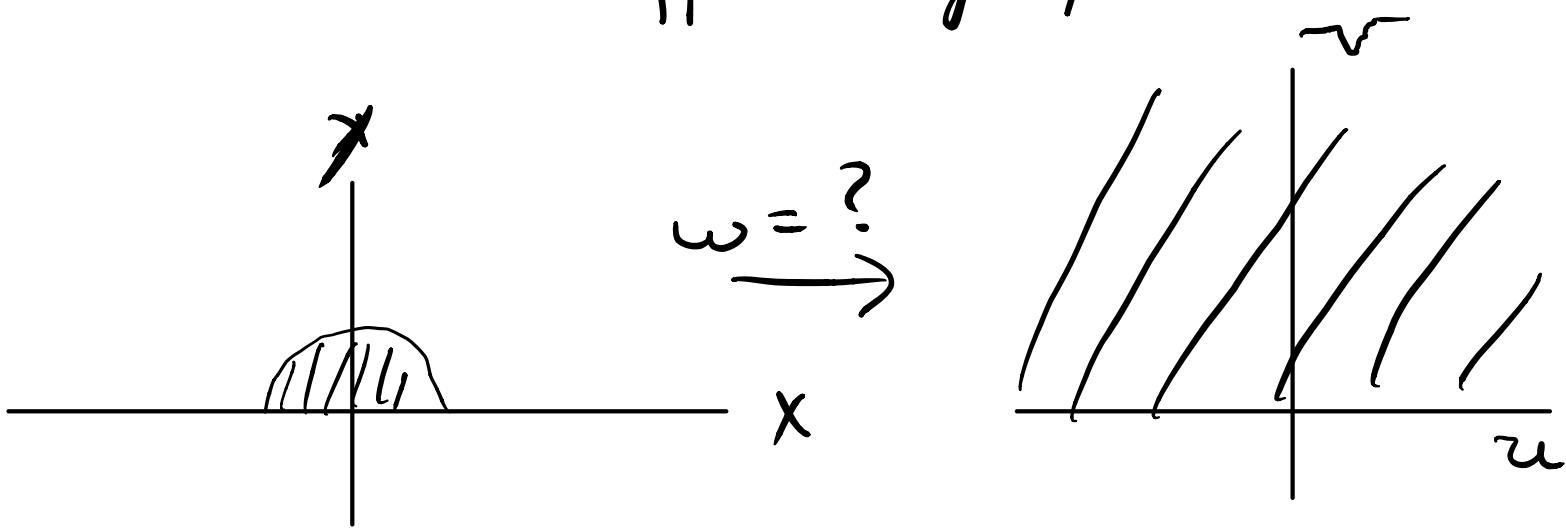
$w = iz$ covers

$1 \rightarrow i$

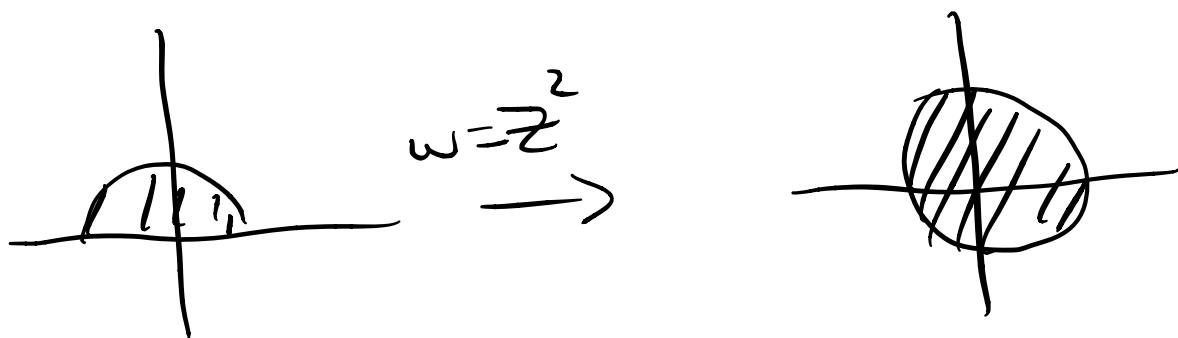
condition

$$w = iz$$

10) $|z| < 1$, $\operatorname{Im} z > 0$
 onto upper half plane



So z^2 maps $|z| < 1$, $\operatorname{Im} z > 0$
 to $|z| < 1$

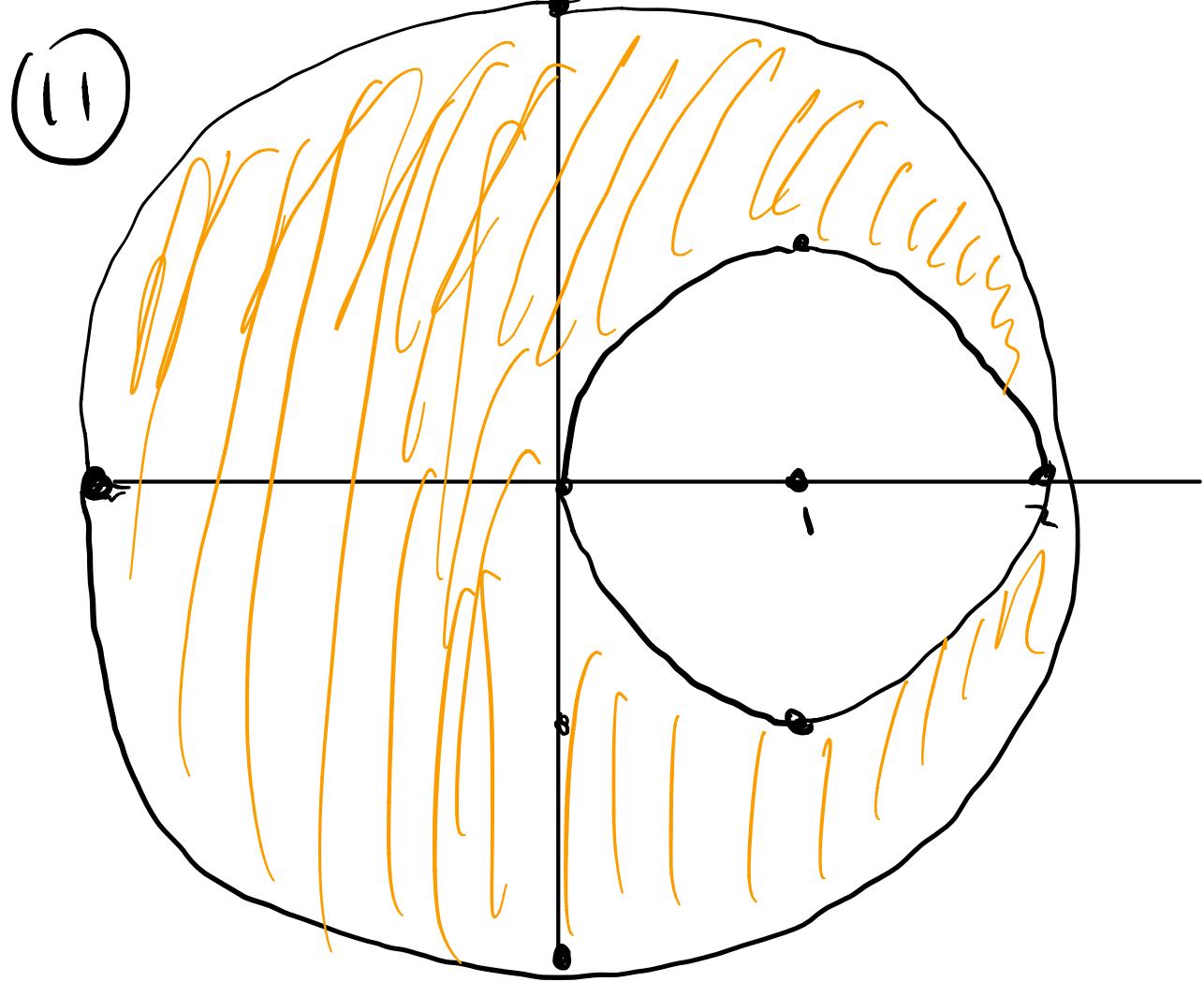


need to map to x-axis so must have
 a pole to map to infinity

$$\frac{1}{z+i} \text{ or } \frac{1}{z-1} \text{ or } \frac{1}{z+i} \text{ or } \frac{1}{z-i}$$

$w = \frac{1+z}{1-z}$ maps the open upper semidisk $D = \{z \mid |z| < 1, \operatorname{Im}(z) > 0\}$ into the open first quadrant.

so w^2 is the desired mapping



$$w = f(z)$$

$$w_1 = \frac{1}{z-2} \text{ maps } 2 \rightarrow \infty$$

$$0 \rightarrow -\frac{1}{2}$$

$$1+i \rightarrow -\frac{1}{2} - \frac{i}{2}$$

$$-1 \rightarrow -1$$

$$1-i \rightarrow -\frac{1}{2} + \frac{i}{2}$$