The meaning of a "neighborhood" in the complex plane.

The set of all all points that satisfy the inequality

$$|z-z_0| < \rho ,$$

where ρ is a positive real number, is called an open disk or circular neighborhood of z_0.

This set consists of all possible points that lie inside the circle of radius ρ about z_0.

In particular, the solution sets of the inequalities

$$|z-2| < 3 \quad , \quad |z+i| < \frac{1}{2} \quad , \quad |z| < 8$$

are circular neighborhoods of the respective points 2, $-i$, and 0.

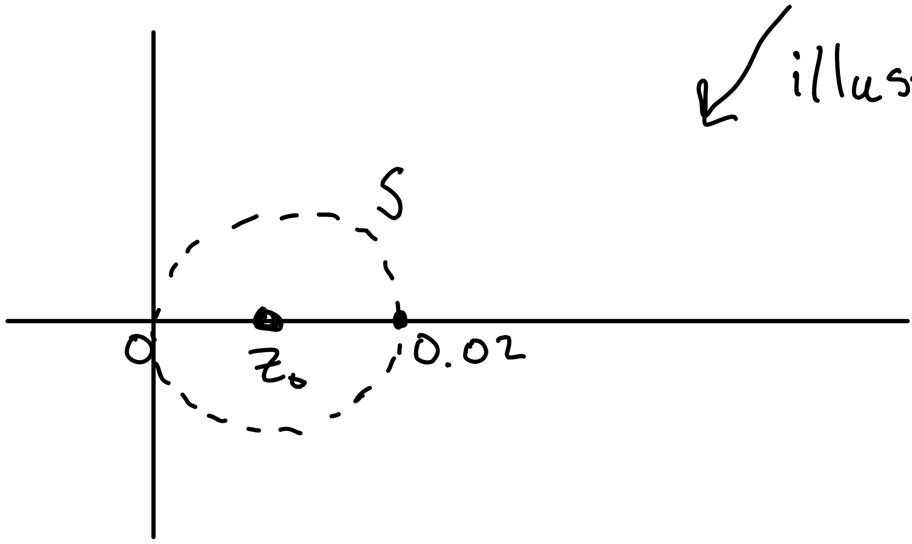
Neighborhood $|z| < 1$ will be frequently referenced. It is the 'open unit disk'

A point z_0 which lies in a set S is called an interior point of S if there is some circular neighborhood of z_0 that is completely contained in S.

i.e

if S is the right half-plane $\operatorname{Re} z > 0$ and $z_0 = 0.01$, then z_0 is an interior point of S because S contains the neighborhood $|z - z_0| < 0.01$

illustrated



If every point of a set S is an interior point of S, we say S is an open set.

* Any open disk is an open set.

The following inequalities describe open sets:

(a) $\rho_1 < |z - z_0| < \rho_2$

(b) $|z - 3| > 2$

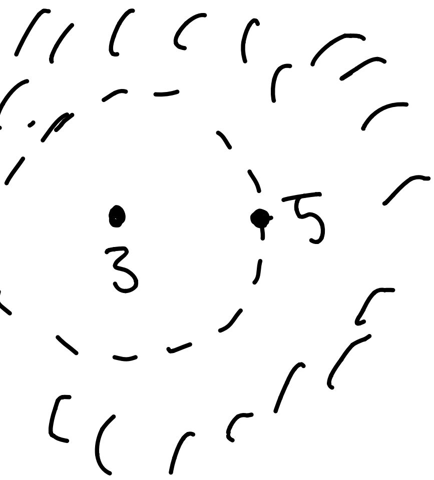
(c) $\operatorname{Im} z > 0$

(d) $1 < \operatorname{Re} z < 2$

see below

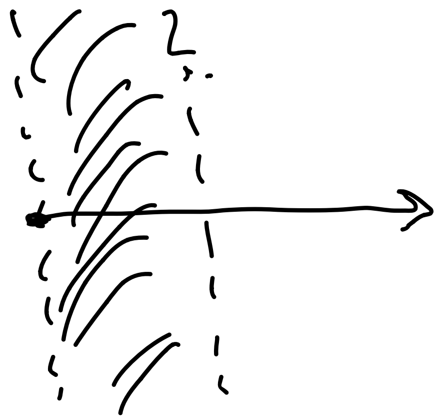
(a)

(b)



(c)

(d)



Note two things:

1) an open interval of the real axis is <u>NOT</u> an open set since it contains no open disk.

2) The solution set T of the inequality $|z-3| \geq 2$ is <u>NOT</u> an open set since no point on the circle $|z-3|=2$ is an interior point of T.

Let $\omega_1, \omega_2, \ldots, \omega_{n+1}$ be $n+1$ points in a plane. For each $k = 1, 2, \ldots, n$, Let l_k denote the line segment joining ω_k to ω_{k+1}.

Then the successive line segments $l_1, l_2, \ldots, l_n$ form a continuous chain known as a polygonal path that joins ω_1 to ω_{n+1}.

An open set S is said to connected if every pair of points z_1, z_2 in S can be joined by a polygonal path that lies entirely in S. [See (a) above].

Roughly speaking, this means that S consists of a "single piece". Each of the above Sets is connected.

We call an open connected set a domain.

Therefore each of the above sets is a domain.

Theorem 1: Suppose $u(x,y)$ is a real-valued function defined in a domain D. If the first partial derivatives of u satisfy

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0 \qquad (1)$$

at all points of D, then $u \equiv \text{constant}$ in D.

Proof: (Also see problem 22, 24)

The assumption $\partial u/\partial x = 0$ implies that u remains constant along any horizontal line segment contained in D; indeed, on such a segment, u is a function of a single variable (namely, x) whose derivative vanishes. Similarly, the assumption $\frac{\partial u}{\partial y} = 0$ means that u is constant along any vertical line segment that lies in D. Putting these facts together we see that u remains unchanged along any polygonal path in D that has all its segments parallel to the coordinate axes.

Any polygonal path in D with segments not parallel to the coordinate axes can be replaced by a chain of small horizontal and vertical segments lying in D. Thus, Theorem 1 follows.

What is crucial for theorem 1 is the connectedness property of domains; in fact, the theorem is no longer true if D is merely assumed to be an open set, because then "piecewise constant" functions would satisfy the hypothesis. (Prob 19)

Example 1:

A real valued function $u(x,y)$ satisfies

$$\frac{\partial u}{\partial x} = 3 \quad \text{and} \quad \frac{\partial u}{\partial y} = 6 \qquad (2)$$

at every point in the open disk $D = \{z : |z| < 1\}$. Show that $u(x,y) = 3x + 6y + c$ for some constant c.

Let $v(x,y) = 3x + 6y$ and consider the function

$$w(x,y) := u(x,y) - v(x,y)$$

From (2) and the definition of $v(x,y)$ we have

$$\frac{\partial w}{\partial x} = 3 - 3 = 0 \quad \text{and} \quad \frac{\partial w}{\partial y} = 6 - 6 = 0$$

at each point of D. Since D is a domain, Theorem 1 asserts that $w(x,y)$ is constant in D, say $w(x,y) = c$.

Thus

$$\begin{aligned} u(x,y) &= v(x,y) + w(x,y) \\ &= v(x,y) + c \\ &= 3x + 6y + c \end{aligned}$$

Continuing with planar sets,

A point z_0 is said to be a boundary point of a set S if every neighborhood of z_0 contains at least one point in S and one point not in S.

The set of all boundary points of S is called the boundary or frontier of S.

Since each point of a domain D is an interior point of D, it follows that a domain cannot contain any of its boundary points.

A set S is said to be closed if it contains all of its boundary points. (See prob (3))

i.e. The set described by the inequality

$$0 < z \leq 1$$

is NOT closed since it does not contain the boundary point 0.

Whereas the set of points z that satisfy the inequality

$$|z - z_0| \leq \rho \quad (\rho > 0)$$

is a closed set, for it contains its boundary $|z-z_0| = \rho$. Therefore, we call this set a closed disk.

A set of points S is said to be bounded if there exists a positive real number R such that $|z| < R$ for every z in S. In other words, S is bounded if it is contained in some neighborhood of the origin.

An unbounded set is one that is NOT bounded. A set that is both closed and bounded is said to be compact.

A region is a domain together with some, none, or all of its boundary points. In particular, every domain is a region.