Solutions of the two-dimensional Laplace equation

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad (1)$$

are among the most important functions in mathematical physics.

Definition 6. A real-valued function $\phi(x,y)$ is said to be **harmonic** in a domain D if all its second-order partial derivatives are continuous in D and if, at each point of D, ϕ satisfies Laplace's equation (1)

The sources of these harmonic functions are the real + imaginary parts of analytic functions.

Theorem 7. If $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D, then each of the functions $u(x,y)$ and $v(x,y)$ is harmonic in D.

Proof

Mixed partial derivatives can be taken in any order, i.e.

$$\frac{\partial}{\partial y}\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\frac{\partial u}{\partial y}$$

(2)

Using the Cauchy-Riemann Equations for the first derivatives we get

$$\frac{\partial^2 v}{\partial y^2} = -\frac{\partial^2 v}{\partial x^2}$$

which is equivalent to Eqn (1). Thus v is harmonic in D. Exact same proof for u.

Conversely, if we're given a function $u(x,y)$ is harmonic in an open disk, then we can find another harmonic function $v(x,y)$ so that $u+iv$ is an analytic function of z in the disk. Such a function v is called a harmonic conjugate of u.

Example 1: Construct an analytic function whose real part is $u(x,y) = x^3 - 3xy^2 + y$

First verify that u is harmonic in the whole plane.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x - 6x = 0 \checkmark$$

Now we must find a mate $v(x,y)$ for u such that the Cauchy-Riemann equations are satisfied. Thus, we need

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 3x^2 - 3y^2 \qquad (3)$$

AND

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 6xy - 1 \qquad (4)$$

If we hold x constant and integrate Eqn (3) with respect to y,

$$v(x,y) = 3x^2y - y^3 + \text{constant}.$$

The constant need only be independent of y. Therefore, we write

$$v(x,y) = 3x^2y - y^3 + \Psi(x)$$

We can find $\Psi(x)$ by plugging into Eqn (4):

$$\frac{\partial v}{\partial x} = 6xy + \Psi'(x) = 6xy - 1 \qquad (5)$$

This yields $\Psi'(x) = -1$ and so

$$\Psi(x) = -x + a$$

where a is some (genuine) constant.

It follows that a harmonic conjugate of $u(x,y)$ is given by

$$v(x,y) = 3x^2y - y^3 - x + a$$

and the analytic function

$$f(z) = x^3 - 3xy^2 + y + i(3x^2y - y^3 - x + a),$$

which we recognize as

$$z^3 - i(z - a)$$

solves the problem.

This procedure will always work for an $u(x,y)$ in a disk. (Prob 20)

The harmonic functions forming the real and imaginary parts of an analytic function $f(z)$ each generate a family of curves in the xy-plane, namely the level curves or isotimic curves

$$u(x,y) = \text{constant} \qquad (6)$$

and

$$v(x,y) = \text{constant} \qquad (7)$$

For the function

$$f(z) = z^2 = x^2 - y^2 + i\,2xy,$$

the level curves

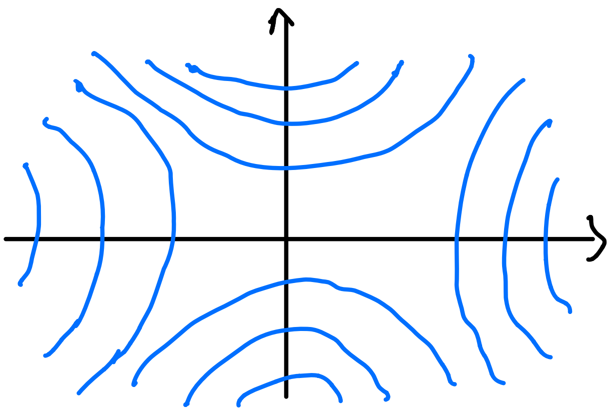
$$u(x,y) = x^2 - y^2 = \text{constant}$$

are hyperbolas asymptotic to $y = \pm x$.

The curves

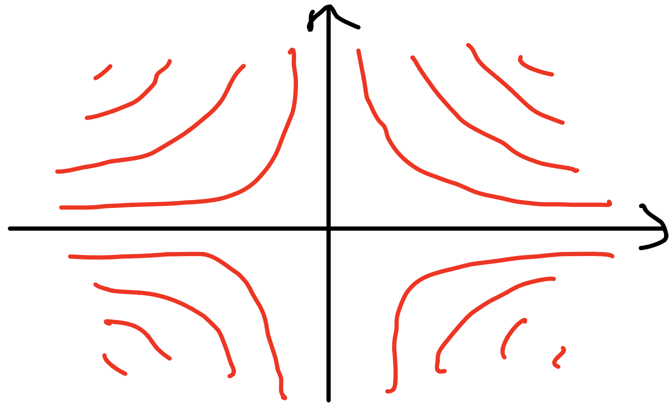
$$v = 2xy = \text{constant}$$

are hyperbolas asymptotic to coordinate axes



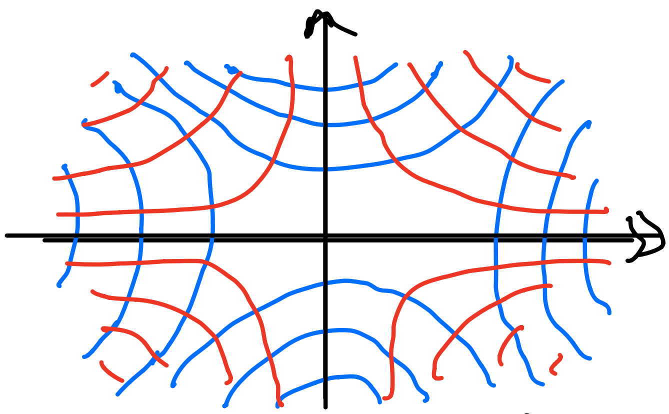
$x^2 - y^2 = \text{const}$

Re $f(z)$



$2xy = \text{const}$

Im $f(z)$



Re $f(z)$ superimposed with Im $f(z)$

Notice when two families of curves are superimposed they appear to intersect at right angles. This is no accident! The level curves of the real and imaginary parts of an analytic function $f(z)$ will (unless $f'(z)=0$ @ point of intersection) always intersect at right angles.

Recall that

$$\left\langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z} \right\rangle$$

is the gradient of u and is normal to the level curves of u. (Same for v)

The scalar product of these gradient vectors is

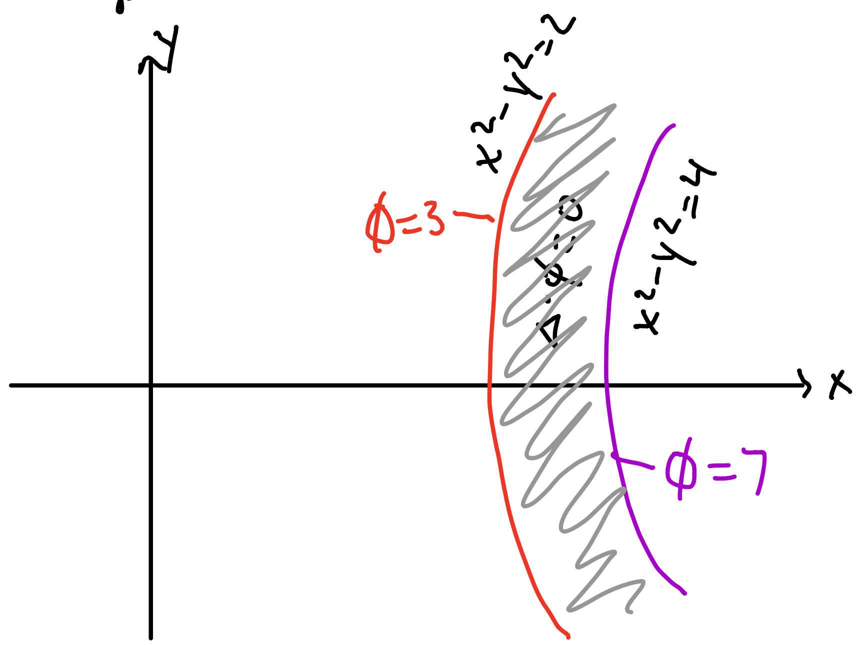
$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}=\frac{\partial v}{\partial y}\frac{\partial v}{\partial x}-\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}=0$$

by the Cauchy-Riemann equations.

Thus if these gradients are nonzero, they are perpendicular, and hence so are the level curves.

Level curves of harmonic functions and their harmonic conjugates intersect at right angles.

Example 2: Find a function $\phi(x,y)$ that is harmonic in the region of the right half-plane between the curves $x^2-y^2=2$ and $x^2-y^2=4$ and takes the value 3 on the left edge and the value 7 on the right edge.



We recognize $x^2-y^2=2$ as $\mathrm{Re}\, z^2$, so the boundary curves are level curves of a known harmonic function.

To meet specified boundary conditions we consider

$$\phi(x,y) = A(x^2-y^2) + B = \mathrm{Re}(Az^2+B), \quad A, B \in \mathbb{R}$$

and adjust A and B accordingly.

When $x^2-y^2=2$, $\phi=3$

$$A(2) + B = 3$$

When $x^2-y^2=4$, $\phi=7$

$$A(4) + B = 7$$

$$\Longrightarrow \quad A = 2, \quad B = -1$$

In conclusion,

$$\phi(x,y) = 2(x^2-y^2) - 1$$