

## Why study complex analysis?

Algebra: need complex numbers to solve

$$P_2(x) = ax^2 + bx + c = 0 \rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Similarly, there are formulas for roots of cubic and quartic equations.

$$P_3(x) = 0, P_4(x) = 0$$

No formula for the roots of the fifth-degree polynomial.

↳ solutions do exist as complex numbers

$P_n(x) = 0$  has  $n$  solutions in complex numbers  $\mathbb{C}$   
"Fundamental Theorem of Algebra" (we'll prove it)

## Exponential & Trig Functions

$$e^{i\theta} = \cos \theta + i \sin \theta \quad \text{"Euler's formula"}$$

$$e^{i\pi} = -1$$

$$e^{i\pi} + 1 = 0$$

Euler's formula is of great use for solving linear differential equations.

## Power Series

Complex numbers give a much deeper understanding - in calculus,

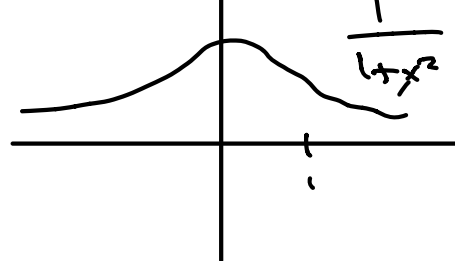
$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 \dots$$

Converges if  $|x| < 1$

diverges if  $|x| > 1$  ← gives silly results

What about  $x=1$ ?

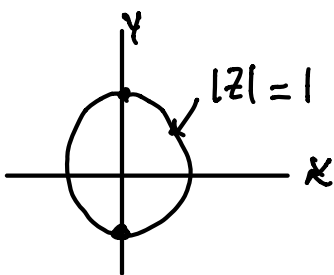
$$\frac{1}{2} = 1 - 1 + 1 - 1 + 1 \dots$$



Something weird happens at  $x = \pm 1$ .

However, in the complex plane bad things happen at  $\pm i$ !

$$\frac{1}{1+z^2} = 1 - z^2 + z^4 - z^6 \dots \text{ valid in circle } |z| < 1$$



Behavior at  $\pm i$  controls behavior on the real axis.

## Definite Integrals

$$\int_0^{\infty} \frac{\sin x}{x} dx, \quad \int_{-\infty}^{+\infty} \frac{1}{1+x^2} dx, \quad \int_0^{\infty} \frac{\cos x}{1+x^2} dx$$

Will be able to do the above via residue theory / contour integrals.

## Fourier & Laplace Transforms

- best to understand w/ complex numbers.

## 2-D potential Theory

- Solving Laplace equation:  $\nabla^2 \phi = 0$

where

$$\nabla^2 \phi = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \phi$$

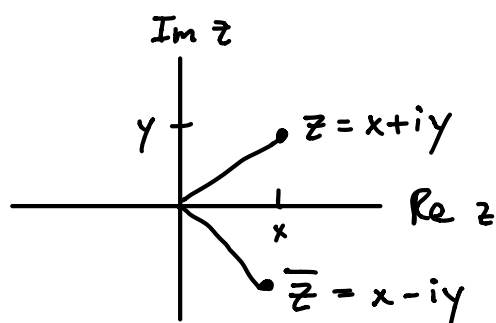
this is what governs airflow over a wing

Now the course begins

Basics of  $i$

class exercise on division

$$\left\{ \begin{aligned} \frac{2+i}{3+4i} \cdot \frac{(3-4i)}{(3-4i)} &= \frac{6-8i+3i+4}{9+16} = \frac{10-5i}{25} \\ &= \frac{10}{25} - \frac{5}{25}i = \frac{2}{5} - \frac{i}{5} \end{aligned} \right.$$



$$y = \text{Im } z$$

$$x = \text{Re } z$$

$$|z| = \text{Magnitude} = \sqrt{x^2 + y^2}$$

Note: Multiplying by  $i$  is a  $90^\circ$  rotation CCW

sanity check

$$z = x + iy$$

$$iz = ix + i^2y = -y + ix$$

Class exercise

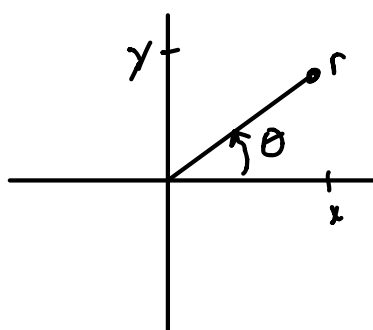
Which points in  $\mathbb{C}$  satisfy

$$|z-1| = \text{Re } z + 1$$

$$\sqrt{(x-1)^2 + y^2} = x + 1$$

$$y^2 = 4x$$

# Polar Form



$r \geq 0$  by convention

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$z = r(\cos \theta + i \sin \theta)$$

$$= r \operatorname{cis} \theta$$

$\theta$ : argument of  $z$

$$\theta = \arg z \quad \text{Ex: } \arg(i) = \pi/2$$

But there's ambiguity, since  $\theta, \theta \pm 2\pi n$ ,  $n \in \mathbb{Z}$  refer to the same  $z$ .

The function  $\arg(z)$  is "multiple-valued" in this sense.  
 ↑ (not really a function)

To make it single valued we would restrict  $\theta$  to lie in  $-\pi < \theta \leq \pi$

After restriction this is now  $\operatorname{Arg}(z)$ .

$\operatorname{Arg}(z)$  is discontinuous w/ this definition.  
 YUCK!

What is  $i^i$  though?

$$e^{i\theta} = \cos \theta + i \sin \theta$$

$$e^{i\pi/2} = 0 + i$$

$$e^{i3\pi/2} = -i$$

$$e^{-i\pi/2} = -i$$

For complete correctness however

$e^{i(\pi/2 + 2\pi k)}$ ,  $k \in \mathbb{Z}$  is the more correct answer.