Recall

$U \subseteq V$ subspace

defined $V/U = \{\vec{v} + U : \vec{v} \in V\}$

and $\pi : V \to V/U$ LT $\ker \pi = U$

$\vec{v} \mapsto \vec{v} + U$ $\operatorname{im} \pi = V/U$ (surjective)

rank-nullity formula $\Rightarrow \dim V/U = \dim V - \dim U$

Another induced map

Suppose we are given $T : V \to W$ LT.

Define a new LT

$$\bar{T} : V/\ker(T) \to W$$

by

$$\vec{v} + \ker(T) \mapsto T(\vec{v})$$

Proposition: In this setting,

① $\bar{T} : V/\ker(T) \to W$ is well-defined and is a LT

② $\ker \bar{T} = \ker T = 0$ $\left(\ker T / \ker T\right)$

③ $\operatorname{im} \bar{T} = \operatorname{im} T$

④ T surjective $(\operatorname{im} T = W)$, then $\bar{T} : V/\ker(T) \to W$ is an isomorphism.

Proof of ②: Get to assume ①

Need $\ker \bar{T} = 0$ NOT COMPLETED

$\bar{T}(\vec{v} + \ker T) = 0_W$

then $\vec{v} + \ker T = \ker T$ $\leftrightarrow$ i.e. $\vec{v} \in \ker T$

Example: $F[x] = V$

$$U = \{x^2 h(x) \mid \text{all } h(x) \in F(x)\} \quad (\langle x^2 \rangle)$$

basis: $U = \text{span}(x^2, x^2, x^4, \ldots)$

Find a vector space isomorphic to $V/U = F[x]/\langle x^2 \rangle$

Solution: $\mathbb{F}[x]/\langle x^2 \rangle \cong \mathbb{F}^2$

$T : \mathbb{F}[x] \to \mathbb{F}^2$ This is a LT

$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots \mapsto \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}$

$\ker T = \langle x^2 \rangle \subseteq \mathbb{F}[x]$

T is surjective and $\therefore \ \overline{T} : \mathbb{F}[X]/\ker(T) \longrightarrow \mathbb{F}^2$ is an isomorphism

Example: Suppose $V = U \oplus W$

i.e $U \leq V$ supspace, W = complement of V

define the projection map

$$P : V \to W$$
$$u + w \mapsto w$$

Check: P a LT

Note: $\ker P = U$ $\quad \therefore \ \overline{P} : V/U \xrightarrow{\sim} W$ is an isomorphism

$\text{im } P = W$

Next Topics: Polynomials

Eigenvalues & Eigenvectors

Diagonilizability

Goal: $T: V \to W$ $\quad \dim V = n < \infty$

Want a basis β s.t. $n \times n$ $[T]_\beta$ is "as nice" as possible.

Polynomials

① Division Algorithm

For integers, if p, s non-negative integers $(s \neq 0)$

then $\exists!$ non-neg integers q, r s.t.

ⓐ $p = qs + r$

ⓑ $\leq r < s$

We want the same for polynomials

i.e $\quad p = x^3 + 1$, $\quad s = x^2 + 1$

$$\begin{array}{r|l} & \;\; x \\ \hline x^2+1 & \;\; x^3 + 1 \\ & -x^3 + x \\ \hline & \quad -x + 1 \end{array}$$

so $\underbrace{x^3 + 1 = x}_{q}(x^2+1) + \underbrace{(-x+1)}_{r}$ and $\deg r < \deg s$

Theorem

Suppose $p, s \in \mathbb{F}[x]$, $s \neq 0$, then $\exists!$ polynomials $q, r \in \mathbb{F}[x]$ such that

ⓐ $p = q \cdot s + r$

ⓑ $\deg r < \deg s$ (or $r = 0$)

Proof: Let $n = \deg p(x)$

$m = \deg s(x)$

if $m > n$, then $q(x) = 0$, $r(x) = p(x)$

So, assume $m \leq n$

Define $T: \mathbb{F}[x]_{\leq n-m} \times \mathbb{F}[x]_{\leq m-1} \longrightarrow \mathbb{F}[x]_{\leq n}$

by

$$(q(x), r(x)) \longmapsto q(x)s(x) + r(x)$$

Note: T is a LT.

ⓐ $\operatorname{Ker} T = \{(\underbrace{q(x)}_{\deg \leq n-m}, \underbrace{r(x)}_{\deg \leq m-1}) \mid q \cdot s + r = 0\}$

$= \{(0,0)\}$

$\Rightarrow \dim \mathbb{F}[x]_{\leq n-m} \times \mathbb{F}[x]_{\leq m-1} = (n-m+1) + (m)$

$= n+1$

$\therefore$ T an isomorphism $\Rightarrow$ surjective