① Show that the mapping

$$w = \frac{(1+z)}{(1-z)}$$

Corresponds to a 90° counterclockwise rotation of the Riemann sphere about the x_2-axis.

Let

$$Z = (x_1, x_2\, x_3)$$

denote the stereographic projection of the point z.

Let

$$W = (\hat{x}_1, \hat{x}_2\, \hat{x}_3)$$

denote the stereographic projection of $w = \frac{1+z}{1-z}$

$$x_1 = \frac{2\,\mathrm{Re}(z)}{|z|^2+1}, \quad x_2 = \frac{2\,\mathrm{Im}(z)}{|z|^2+1}, \quad x_3 = \frac{|z|^2-1}{|z|^2+1}$$

$$\hat{x}_1 = \frac{2\,\mathrm{Re}(w)}{|w|^2+1}, \quad \hat{x}_2 = \frac{2\,\mathrm{Im}(w)}{|w|^2+1}, \quad \hat{x}_3 = \frac{|w|^2-1}{|w|^2+1}$$

A 90° rotation about x_2 means

$$\hat{x}_1 = x_3 \qquad \hat{x}_2 = x_2 \qquad \hat{x}_3 = x_1$$

ALGEBRA

$$z = re^{i\theta}, \quad |z|^2 = r^2$$

$$\frac{1+z}{1-z} = \frac{1+re^{i\theta}}{1-re^{i\theta}} = \frac{(1+re^{i\theta})(1-re^{-i\theta})}{(1-re^{i\theta})(1-re^{-i\theta})}$$

$$= \frac{1-re^{-i\theta}+re^{i\theta}-r^2}{1-re^{-i\theta}-re^{i\theta}+r^2}$$

$$= \frac{1+r(e^{i\theta}-e^{-i\theta})-r^2}{1-r(e^{i\theta}+e^{-i\theta})+r^2}$$

$$= \frac{(1-r^2)+i2r\sin(\theta)}{(1+r^2)-2r\cos(\theta)}$$

$$\text{Re}(w) = \frac{(1-r^2)}{(1+r^2)-2r\cos(\theta)}, \quad \text{Im}(w) = \frac{2r\sin(\theta)}{(1+r^2)-2r\cos(\theta)}$$

$$|w|^2 = w\bar{w}$$

$$= \left[\frac{(1-r^2)+i2r\sin(\theta)}{(1+r^2)-2r\cos(\theta)}\right]\left[\frac{(1-r^2)-i2r\sin(\theta)}{(1+r^2)-2r\cos(\theta)}\right]$$

$$|w|^2 = \frac{(1-r^2)^2 + 4r^2\sin^2(\theta)}{(1+r^2)^2 - 4r(1+r^2)\cos(\theta) + 4r^2\cos^2(\theta)}$$

So W has points (blue denotes denominator)

$$\hat{X}_1 = \frac{\dfrac{2(1-r^2)}{(1+r^2)-2r\cos(\theta)}}{\dfrac{(1-r^2)^2 + 4r^2\sin^2(\theta)}{(1+r^2)^2 - 4r(1+r^2)\cos(\theta) + 4r^2\cos^2(\theta)} + 1}$$

$$= \frac{\dfrac{2(1-r^2)\cdot\left[(1+r^2)^2 - 4r(1+r^2)\cos(\theta) + 4r^2\cos^2(\theta)\right]}{(1+r^2)-2r\cos(\theta)}}{(1-r^2)^2 + 4r^2\sin^2\theta + (1+r^2)^2 - 4r(1+r^2)\cos(\theta) + 4r^2\cos^2\theta}$$

$$= \frac{2(1-r^2)\left[(1+r^2)^2 - 4r(1+r^2)\cos(\theta) + 4r^2\cos^2\theta\right]}{\left[(1+r^2)-2r\cos(\theta)\right]\left[(1-r^2)^2 + 4r^2\sin^2\theta + (1+r^2)^2 - 4r(1+r^2)\cos(\theta) + 4r^2\cos^2\theta\right]}$$

$$= \frac{r^2-1}{r^2+1}$$

$$X_2 = \frac{2r\sin(\theta)}{r^2+1}$$

$$\hat{X}_2 = \frac{\dfrac{4r\sin(\theta)}{(1+r^2)-2r\cos(\theta)}}{\dfrac{(1-r^2)^2+4r^2\sin^2(\theta)}{(1+r^2)^2-4r(1+r^2)\cos(\theta)+4r^2\cos^2(\theta)}+1}$$

$$= \frac{2r\sin(\theta)}{1+r^2}$$