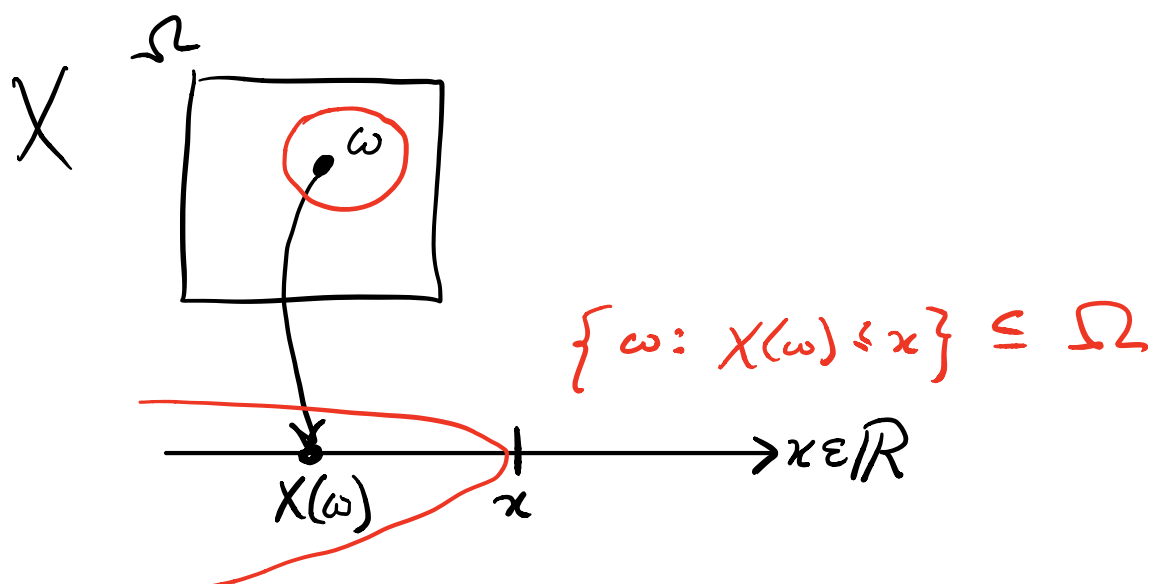


A r.v.  $X: \Omega \rightarrow \mathbb{R}$



Example: Throwing a Die

$$\Omega = \left\{ \begin{matrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 1, 2, 3, 4, 5, 6 \end{matrix} \right\}$$

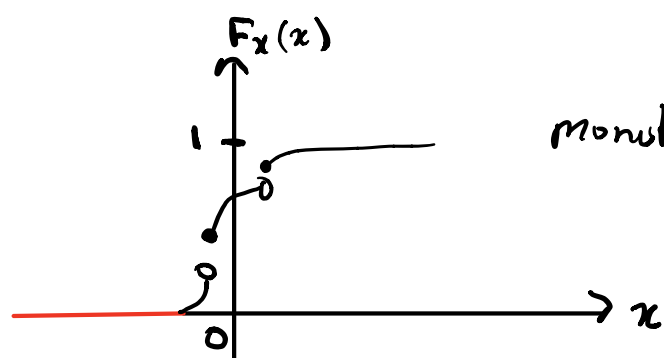
$X$        $\downarrow$     $\downarrow$     $\downarrow$     $\downarrow$     $\downarrow$     $\downarrow$   
               $5$     $5$     $5$     $5$     $5$     $5$   
               $\downarrow$     $\downarrow$     $\downarrow$     $\downarrow$     $\downarrow$     $\downarrow$   
               $-5$     $-5$     $-5$     $-5$     $-5$     $-5$



$$X = \left\{ \begin{matrix} 5 & \Pr(5) = \frac{1}{2} \\ -5 & \Pr(-5) = \frac{1}{2} \end{matrix} \right\}$$

CDF

$$F_X(x) = \Pr[X \leq x] \quad \forall x \in \mathbb{R}$$



$\forall x < y$   
 monotonically increasing;  $F_X(x) \leq F_Y(y)$   
 like die CDF example for equality (last lecture)

## Properties of CDF

$$\textcircled{1} \quad \lim_{x \rightarrow \infty} F_X(x) = 1$$
$$\lim_{x \rightarrow -\infty} F_X(x) = 0$$

$$\textcircled{2} \quad \forall x < y \quad F_X(x) \leq F_X(y)$$

$\textcircled{3}$   $F$  is right continuous

$$\lim_{x \rightarrow x_0^+} F_X(x) = F_X(x_0)$$

$$\textcircled{4} \quad \Pr[x \leq X \leq y]$$
$$= F_X(y) - F_X(x)$$

## PMF for Discrete Random Variables

$$P_X(x) = \Pr[X=x]$$

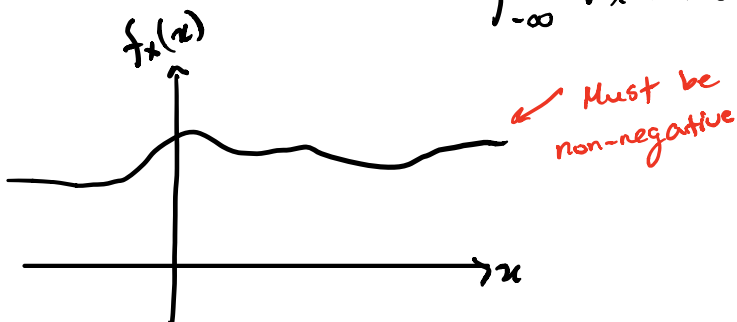
$$F_X(x) = \sum_{u \leq x} P_X(u)$$

## PDF for Continuous Random Variables

$$f_X(x) = \frac{dF_X(x)}{dx} \quad \left. \begin{array}{l} \text{measures how} \\ \text{we accumulate} \end{array} \right\} \text{fast probability}$$

Thus

$$F_X(x) = \int_{-\infty}^x f_X(u) du$$



Properties of PDF

$$① f_X(x) \geq 0 \quad \forall x$$

$$② \int_{-\infty}^{+\infty} f_X(x) = 1 = F_X(\infty)$$

$$③ \Pr[X \leq y] = \int_{-\infty}^y f_X(u) du$$

Expectation

$$E[X] = \begin{cases} \sum_k k \Pr[X=k] & , \quad X \text{ discrete} \\ \int_{-\infty}^{+\infty} x f_X(x) dx & , \quad X \text{ continuous} \end{cases}$$

Properties

LOTUS Rule

If

then

$$Y = g(X)$$

$$E[Y] = \int_{-\infty}^{+\infty} x f_Y(y) dy$$

*(Note: In the original image, a red arrow points from  $f_X(x)$  to  $f_Y(y)$  with the label  $g$ )*

$$E[g(X)] = \int_{-\infty}^{+\infty} g(x) f_X(x) dx$$

# Linearity of Expectation

$$E[\alpha X + \beta Y] = \alpha E[X] + \beta E[Y], \quad \alpha, \beta \in \mathbb{R}$$

## Preservation of Order

If

$$\Pr[X \geq Y] = 1$$

then

$$E[X] \geq E[Y]$$

## Integration by Parts

Simple Case

A discrete, non-negative r.v.  $X = 0, 1, 2, 3, \dots$

$$E[X] = \sum_{i=0}^{\infty} i \Pr(X=i)$$

$$= \sum_{i=0}^{\infty} \underbrace{\Pr(X > i)}_{1 - F_X(i)} \quad \leftarrow \text{Tail Probability}$$

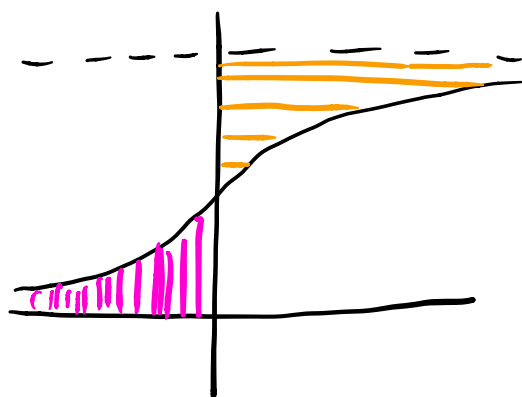
To see this, observe

$$\begin{aligned} \sum_{i=0}^{\infty} i \Pr(X=i) &= 0 \cdot \Pr(X=0) + 1 \cdot \Pr(X=1) + 2 \cdot \Pr(X=2) + \dots \\ \sum_{i=0}^{\infty} \Pr(X > i) &\quad \begin{matrix} i=0 \\ i=1 \\ \vdots \\ \text{etc} \end{matrix} \quad \begin{matrix} \checkmark \\ \checkmark \\ \checkmark \end{matrix} \end{aligned}$$

Thus

$$E[X] = \int_0^{\infty} (1 - F_X(x)) dx = \int_{-\infty}^0 F_X(x) dx$$

To see this, observe



$$E[X] = \int_{-\infty}^{+\infty} x dF_X(x)$$

$$= \int_0^{\infty} x dF_X(x) - \int_{-\infty}^0 (-x) dF_X(x)$$

$$= \int_0^{\infty} (1 - F_X(x)) dx - \int_{-\infty}^0 F_X(x) dx$$

## Variance

How much a r.v varies from its expectation

$$\text{Var}(X) = E[\overbrace{(X - E[X])}^{g(x)}]^2]$$

$$= E[X^2 - 2E[X]X + (E[X])^2]$$

$$= E[X^2] + (E[X])^2$$

$$\text{Var}(X) = \int_{-\infty}^{+\infty} \overbrace{g(x)}^{(x - E[X])^2} f_X(x) dx = \int_{-\infty}^{+\infty} (x - E[X])^2 f_X(x) dx$$

$N^{\text{th}}$  Moment

$$E[X^n] = \int_{-\infty}^{+\infty} x^n f_X(x) dx$$

$N^{\text{th}}$  Central Moment

$$E[(X - E[X])^n]$$

Multiple Random Variables

Joint Distribution of  $(X_1, X_2, \dots, X_n)$

Joint CDF

$$F_{X_1, \dots, X_n}(x_1, \dots, x_n) = \Pr[X_1 \leq x_1, \dots, X_n \leq x_n]$$

Joint PMF

$$P_{X_1, \dots, X_n}(x_1, \dots, x_n) = \Pr[X_1 = x_1, \dots, X_n = x_n]$$

Joint PDF

$$f_{X_1, \dots, X_n}(x_1, \dots, x_n) = \frac{\partial^n F_{X_1, \dots, X_n}(x_1, \dots, x_n)}{\partial x_1 \dots \partial x_n}$$

Marginal CDF  
of  $X_i$

$$F_{X_i} = F_{X_1, \dots, X_n}(x_i, \infty, \infty, \dots, \infty)$$

$$f_{X_i} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} f_{X_1, \dots, X_n}(x_1, x_2, \dots, x_n) dx_2 dx_3 \dots dx_n$$

# Independence

$$F_{X_1, \dots, X_n}(x_1, \dots, x_n) = F_{X_1}(x_1) \cdots F_{X_n}(x_n)$$

$$\Pr[X_1 \leq x_1, \dots, X_n \leq x_n] = \Pr[X_1 \leq x_1] \cdots \Pr[X_n \leq x_n]$$

## Conditioning on Random Variables

Suppose  $X$  and  $Y$  have joint pmf  $p_{X,Y}(x,y)$

The conditional pmf of  $X$  given  $\{Y=y\}$  is

$$p_{X|Y}(x|y) \triangleq \Pr[\overbrace{X=x}^A | \overbrace{Y=y}^B]$$

$$= \frac{\Pr(\{X=x\} \cap \{Y=y\})}{\Pr(\{Y=y\})} = \frac{p_{X,Y}(x,y)}{p_Y(y)}$$

Similarly for continuous r.v.'s

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} \quad \text{if } f_Y(y) \neq 0 \quad \forall y$$

# Conditional Expectation

$$E[X] = \int_{-\infty}^{+\infty} x f_X(x) dx$$

$$E[X|Y=y] = \int_{-\infty}^{+\infty} x f_{X|Y=y}(x|y) dx$$

$E[X|Y]$  is a r.v w/ respect to  $Y$ !  
i.e a function of  $Y$