Recall that a function f is a rule that assigns to each element in a set A one **and only one** element in a set B. If f assigns the value b to the element a in A, we write

$$b = f(a)$$

and call b the image of a under f.

The set A is the domain of definition of f (even if A is not a domain in the sense of chapter 1), and the set of all images $f(a)$ is the range of f.
We sometimes refer to f as a mapping of A into B.

We will concern ourselves with complex-valued functions of a complex variable, so that the domains of definition and the ranges are subsets of the complex numbers.

Say

$$f(z) = \frac{z^2 - 1}{z^2 + 1}$$

then, unless stated otherwise, we take the domain of f to be the set of all z for which the formula is well-defined.

If ω denotes the value of the function f at the point z, we then write

$$\omega = f(z).$$

Just as z decomposes into

$$z = x + iy \quad \text{(real + imaginary part)}$$

the real + imaginary parts of ω are each (real-valued) functions of z or, equivalently, of x and y, and so we customarily write

$$\omega = u(x,y) + i\,v(x,y)$$

with u, v denoting the real + imaginary parts, respectively, of ω.

Thus a complex-valued function of a complex variable is, in essence, a pair of real functions of two variables.

Example 1: Write $\omega = f(z) = z^2 + 2z$ in the form $\omega = u(x,y) + i\,v(x,y)$

Setting $z = x + iy$,

$$\omega = f(z) = (x+iy)^2 + 2(x+iy)$$

$$\omega = x^2 - y^2 + i2xy + 2x + i2y$$

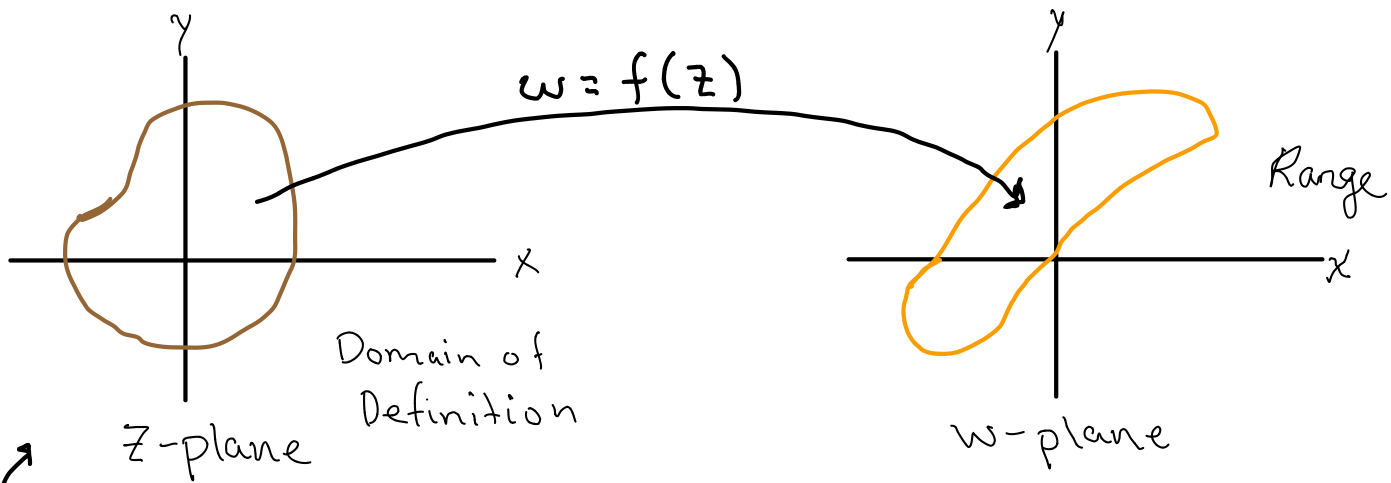
$$\omega = (x^2 - y^2 + 2x) + i(2xy + 2y)$$

Unfortunately, it is generally impossible to draw the graph of a complex function; to display two real functions of two real variables graphically would require four dimensions.

We can visualize some of the properties of a complex function

$$w = f(z)$$

by sketching its domain of definition in the z-plane and its range in the w-plane.



Representation of a complex function

Example 2: Describe the range of the function $f(z) = x^2 + 2i$ defined on the closed unit disk

$$|z| \le 1$$

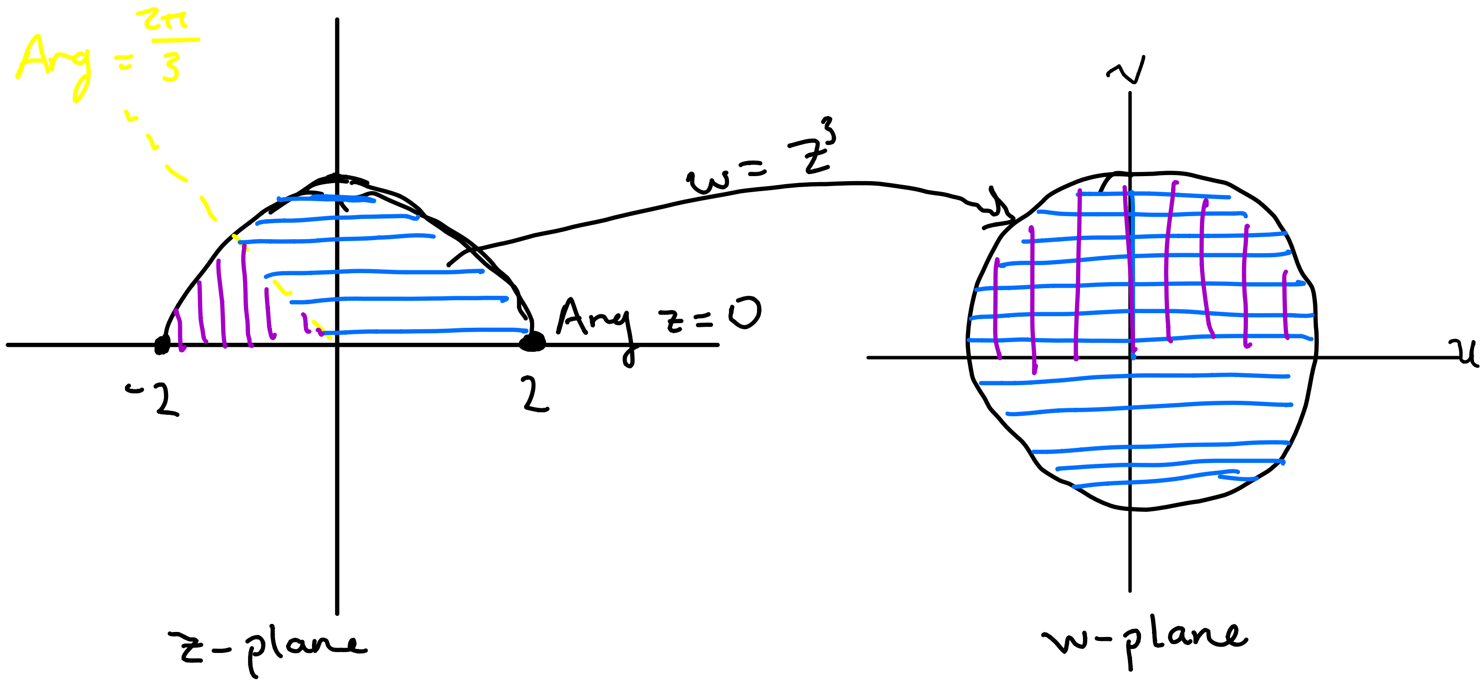
We have

$$u(x,y) = x^2 \qquad v(x,y) = 2$$

Thus as z varies over the closed unit disk, u varies between 0 and 1, and v is constant.

The range is therefore the line segment from
$w = 0 + 2i$ to $w = 1 + 2i$

Example 3: Describe the function $f(z) = z^3$ for z in the semidesk given by $|z| \leq 2$, $\text{Im}\, z \geq 0$ (figure below)



The points z in the sector of the semidisk from $\text{Arg}\, z = 0$ to $\text{Arg}\, z = 2\pi/3$, when cubed, cover the entire disk $|w| \leq 8$. The cubes of the remaining z-points also fall on this disk, overlapping it in the upper half. Depicted above.

The function $f(z) = 1/z$ is called the inversion mapping.
It is an example of a one-to-one function because it maps distinct points to distinct points.
i.e if $z_1 \neq z_2$, $f(z_1) \neq f(z_2)$

Example 4: Show that the inversion mapping $w = 1/z$ corresponds to a rotation of the Riemann sphere by $180°$ about the x_1-axis.

Let $Z = (x_1, x_2, x_3)$ denote the stereographic projection of the point z
Let $W = (\hat{x}_1, \hat{x}_2, \hat{x}_3)$ denote the gstereographic projection $1/z$

$$x_1 = \frac{2\,\mathrm{Re}(z)}{|z|^2+1}, \quad x_2 = \frac{2\,\mathrm{Im}(z)}{|z|^2+1}, \quad x_3 = \frac{|z|^2-1}{|z|^2+1}$$

$$\hat{x}_1 = \frac{2\,\mathrm{Re}(1/z)}{\left|\frac{1}{z}\right|^2+1}, \quad \hat{x}_2 = \frac{2\,\mathrm{Im}(1/z)}{\left|\frac{1}{z}\right|^2+1}, \quad \hat{x}_3 = \frac{\left|\frac{1}{z}\right|^2-1}{\left|\frac{1}{z}\right|^2+1}$$

Using $\mathrm{Re}\left(\frac{1}{z}\right) = \mathrm{Re}(z)/|z|^2$
$\mathrm{Im}(1/z) = -\mathrm{Im}(z)/|z|^2$

We get after simplification that

$$\hat{x}_1 = \frac{2\,\mathrm{Re}(z)}{1+|z|^2}, \quad \hat{x}_2 = \frac{-2\,\mathrm{Im}(z)}{1+|z|^2}, \quad \hat{x}_3 = \frac{1-|z|^2}{1+|z|^2}$$

$$\hat{x}_1 = x_1, \quad \hat{x}_2 = -x_2, \quad \hat{x}_3 = -x_3$$

A rotation about x_1-axis preserves x_1 and negates x_2, x_3; so indeed W is the stated rotation of Z.

A consequence of this example is the fact that an inversion mapping preserves the class of circles and lines (Prob 17)