Exercises.

Solution to Question 1. Let $(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3) = (\begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\1\\-1\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix})$. Then

$$\mathsf{T}(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3) = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) \begin{bmatrix}1 & 1 & 1\\1 & 1 & 0\\1 & 0 & 0\end{bmatrix}$$

and

$$(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3) = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\-1 & -1 & 1\end{bmatrix}.$$

So

$$\begin{aligned}\mathsf{T}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = \mathsf{T}(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3) \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\-1 & -1 & 1\end{bmatrix}^{-1} &= \mathsf{T}(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3) \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\1 & 1 & 1\end{bmatrix}\\ = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) \begin{bmatrix}1 & 1 & 1\\1 & 1 & 0\\1 & 0 & 0\end{bmatrix} \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\1 & 1 & 1\end{bmatrix} &= (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) \begin{bmatrix}2 & 2 & 1\\1 & 1 & 0\\1 & 0 & 0\end{bmatrix}.\end{aligned}$$

Therefore,

$$[\mathsf{T}]_{\mathcal{S}\leftarrow\mathcal{S}} = \begin{bmatrix}2 & 2 & 1\\1 & 1 & 0\\1 & 0 & 0\end{bmatrix}.$$

Similarly,

$$\begin{aligned}\mathsf{T}(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3) &= (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) \begin{bmatrix}1 & 1 & 1\\1 & 1 & 0\\1 & 0 & 0\end{bmatrix}\\ &= (\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3) \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\1 & 1 & 1\end{bmatrix} \begin{bmatrix}1 & 1 & 1\\1 & 1 & 0\\1 & 0 & 0\end{bmatrix}\\ &= (\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3) \begin{bmatrix}1 & 1 & 1\\1 & 1 & 0\\3 & 2 & 1\end{bmatrix}.\end{aligned}$$

Therefore,

$$[\mathsf{T}]_{\mathcal{B}\leftarrow\mathcal{B}} = \begin{bmatrix}1 & 1 & 1\\1 & 1 & 0\\3 & 2 & 1\end{bmatrix}.$$

Solution to Question 2. We may check that for any $c_1, c_2 \in \mathbb{F}$ and $f, g \in V$,

$$T_a(c_1 f + c_2 g) = (c_1 f + c_2 g)(a) = c_1 f(a) + c_2 g(a) = c_1 T_a(f) + c_2 T_a(f).$$

So T_a is a linear transformation.

By definition,

$$\ker(T_a) = \{f \in \mathrm{Fun}(X, V) : f(a) = 0\}.$$

We know T_a is surjective, because for any $v \in V$, there exists $f \in \mathrm{Fun}(X, V)$ such that $f(a) = v$. Hence, $\mathrm{Im}(T_a) = V$.

Now assume that $X = \{x_1, x_2, \ldots, x_n\}$. Let T be a linear transformation defined by

$$\begin{aligned} T : \mathrm{Fun}(X, V) &\to \oplus_{i=1}^{n} V \\ f &\mapsto (f(x_1), f(x_2), \ldots, f(x_n)). \end{aligned}$$

Claim: T is an isomorphism between vector spaces.
Reason: Since T is already a linear transformation, we only need to show that T is a bijection.
First of all, T is injective, because if $f(x) = g(x)$ for all $x \in X$, then $f = g$.
Next, T is surjective: For any $(v_1, v_2, \ldots, v_n) \in \oplus_{i=1}^{n} V$, there exists $f \in \mathrm{Fun}(X, V)$ such that $f(x_i) = v_i$.
Therefore, we have proved that $\mathrm{Fun}(X, V)$ is isomorphic to $\oplus_{i=1}^{n} V$.

Hence,

$$\dim \mathrm{Fun}(X, V) = \dim \oplus_{i=1}^{n} V = n \dim V.$$

Because

$$\dim \mathrm{Im}(T_a) = \dim V,$$

so

$$\dim \ker(T_a) = \dim \mathrm{Fun}(X, V) - \dim \mathrm{Im}(T_a) = (n-1) \dim V.$$

Solution to Question 3.

(a) Because $d_i \circ d_{i+1} = 0$, so $\operatorname{im}(d_{i+1}) \subseteq \ker(d_i)$.

(b) Since $\ker(T) \subseteq V$, f is the inclusion. g is defined by $g(v) = T(v)$.
Therefore, $\operatorname{im}(f) = \ker(T) = \ker(g)$. Also notice that f is injective, i.e. $\ker(f) = 0 = \operatorname{im}(0)$ and that $\operatorname{im}(g) = \operatorname{im}(T) = \ker(0)$. Hence this is an exact chain complex.

(c) If the chain complex
$$0 \to V \xrightarrow{T} W \xrightarrow{S} U \to 0$$
is an exact sequence, then
$$\ker(T) = \operatorname{im}(0) = 0,$$
which means T is an inclusion. So
$$\ker(S) = \operatorname{im}(T) = V,$$
and
$$\operatorname{im}(S) = \ker(0) = U.$$
Hence,
$$\dim W = \dim \ker(S) + \dim \operatorname{im}(S) = \dim V + \dim U.$$

(d) We want to show that if C is an exact sequence, then
$$\sum_{i=0}^{p} (-1)^i \dim V_i = 0. \tag{1}$$
For $p = 1$, the exact sequence is
$$0 \to V_1 \xrightarrow{d_1} V_0 \to 0.$$
So d_1 both injective and surjective, i.e., $V_1 \simeq V_0$. Hence $\dim V_1 = \dim V_0$.
For $p = 2$, by part (c), ,
$$\dim V_2 - \dim V_1 + \dim V_0 = 0.$$
Now we assume for $p \le n-1$, the equation (1) always holds. Now take
$$0 \to V_n \xrightarrow{d_n} V_{n-1} \cdots \to V_1 \xrightarrow{d_1} V_0 \to 0.$$
We may break it into two exact sequences
$$0 \to \ker(d_{n-2}) \to V_{n-2} \xrightarrow{d_{n-2}} V_{n-3} \to \cdots \to V_1 \xrightarrow{d_1} V_0$$
and
$$0 \to V_n \xrightarrow{d_n} V_{n-1} \xrightarrow{d_{n-1}} \operatorname{im}(d_{n-1}) \to 0.$$

By induction hypothesis, we have

$$(-1)^{n-1}\dim\ker(d_{n-2}) + \sum_{i=0}^{n-2}\dim V_i = 0 \tag{2}$$

and

$$(-1)^2\dim V_n + (-1)\dim V_{n-1} + \dim\operatorname{im}(d_{n-1}) = 0. \tag{3}$$

Notice that $\operatorname{im}(d_{n-1}) = \ker(d_{n-2})$, so by adding $(-1)^{n-2}$(3) to (2), we have

$$\sum_{i=0}^{n}(-1)^i\dim V_i = 0.$$

This completes the induction step.

Solution to Question 4.

(a) For any $v \in V$, we want to prove that $v = u + w$ such that $u \in \ker(T)$ and $w \in \ker(T - 1_V)$. In fact, let $u = (1_V - T)v$ and $w = Tv$. Because $T^2 - T = 0$, so

$$Tu = T \circ (1_V - T)v = 0_V v = 0.$$

and

$$(T - 1_V)w = (T - 1_V) \circ Tv = 0.$$

Hence, $u \in \ker(T)$, $w \in \ker(T - 1_V)$ and $v = u + w$. This implies $V = \ker(T) + \ker(T - 1_V)$. To prove that it is a direct sum, notice that if $v \in \ker(T) \cap \ker(T - 1_V)$, then $Tv = 0$ and $(T - 1_V)v = 0$, which implies $v = 0$. So $\ker(T) \cap \ker(T - 1_V) = \{0\}$.

(b) We only need to show that for any $v \in V$, $(T^2 - T)v = 0$.
Because $V = \ker(T) + \ker(T - 1_V)$, so each $v \in V$ can be written as $v = u + w$ with $u \in \ker(T)$ and $w \in \ker(T - 1_V)$. Therefore,

$$(T^2 - T)v = (T^2 - T)u + (T^2 - T)w = 0 + T \circ (T - 1_V)w = 0.$$

(c) Choose $V = \mathbb{R}^2$. Take T given by

$$T(\begin{pmatrix} 1 \\ 0 \end{pmatrix}) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \ T(\begin{pmatrix} 0 \\ 1 \end{pmatrix}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Then

$$T^2(\begin{pmatrix} 1 \\ 0 \end{pmatrix}) = T(\begin{pmatrix} 0 \\ -1 \end{pmatrix}) = -\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and

$$T^2(\begin{pmatrix} 0 \\ 1 \end{pmatrix}) = T(\begin{pmatrix} 1 \\ 0 \end{pmatrix}) = -\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Hence, $T^2 = -1_V$.

(d) If $T \circ T = 0_V$, then $\mathrm{im}(T) \subseteq \ker(T)$. By problem 3,

$$\dim V = \dim\ker(T) + \dim\mathrm{im}(T) \geq 2\dim\mathrm{im}(T) = 2\mathrm{rank}(T).$$

Extended Glossary.

For any vector spaces V and W over field $\mathbb{F}$, we define the **the external direct product** $V \times W$ as the set of ordered pairs (v, w) with $v \in V$ and $w \in W$. We also define $+$ on $V \times W$ by

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2)$$

and the scalar multiplication

$$c(v, w) = (cv, cw) \text{ for any } c \in \mathbb{F},$$

so that $V \times W$ is a vector space.

Now assume that V and W are both finite dimensional vector spaces, and that $(v_1, \dots, v_n)$ is a basis for V, while $(w_1, \dots, w_m)$ is basis for W. Then

$$((u_1, 0), \dots, (u_n, 0), (0, w_1), \dots, (0, w_n))$$

is a basis for $V \times W$. Hence, $\dim V \times W = \dim V + \dim W$.

Theorem 1. *There exists a short exact sequence*

$$0 \to V \xrightarrow{i_V} V \times W \xrightarrow{p_W} W \to 0. \tag{4}$$

Proof. Define i_V by $i_V(v) = (v, 0)$ and p_W by $p_W(v, w) = w$. Then we may check that

- i_V is injective, so $\ker(i_V) = 0$.
- p_W is surjective, so $\operatorname{im}(p_W) = W$.
- $\operatorname{im}(i_V) = \ker(p_W)$. This is because both of them are equal to

$$\{(v, 0) \in V \times W : v \in V\}.$$

Therefore, (4) is a short exact sequence. □

Use the notation as in Theorem 1. Because $V \simeq i_V(V)$ as vector spaces, so we may just consider V as a subspace of $V \times W$. Similarly, W may be considered as a subspace of $V \times W$, too. Then we may talk about $V + W$ and $V \cap W$ as subspaces of $V \times W$. It is easy to check that $V \cap W = (0, 0)$.

Theorem 2. *As subspaces of* $V \times W$,

$$V \times W \simeq V \oplus W.$$

Proof. Define a map

$$\begin{aligned} h : V \times W &\to V + W, \\ (v, w) &\mapsto v + w. \end{aligned}$$

This map is a linear map. Clearly, it is surjective.
To prove it is also injective, notice that as subspaces of $V \times W$, $V \cap W = \{(0, 0)\}$. If $h(v, w) = v + w = 0$, then $v = 0$ and $w = 0$. So h is injective.
Hence, $V \times W \simeq V + W = V \oplus W$. □