4.4

(1)

$$D = \mathbb{C} - \{0, 2i, 4\}$$

Contours continuously deformable to Γ in D.

a) Deformable ✓

b) **NOT** deformable ← passes through 0 ✗

c) Deformable ✓

d) **NOT** deformable ✗

(2) Which are simply connected

a) Simply Connected ✓

b) **NOT** Simply Connected ✗

c) Simply Connected ✓

d) Simply Connected ✓

e) **NOT** Simply Connected ✗

f) Simply Connected ✓

(16) Show that if f is of the form

$$f(z) = \frac{A_k}{z^k} + \frac{A_{k-1}}{z^{k-1}} + \cdots + \frac{A_1}{z} + g(z) \quad (k \geq 1)$$

where g is analytic inside & on the circle $|z|=1$, then

$$\oint_{|z|=1} f(z)\,dz = 2\pi i\, A_1$$

$$\oint_{|z|=1} \frac{A_k}{z_k} + \frac{A_{k-1}}{z_{k-1}} + \cdots + \frac{A_1}{z} + g(z)\, dz \qquad (k \geq 1)$$

0 by analyticity of $g(z)$

$$= \cancelto{0}{\oint_{|z|=1} \frac{A_k}{z_k}\,dz} + \cancelto{0}{\oint_{|z|=1} \frac{A_{k-1}}{z_{k-1}}\,dz} \cancelto{0}{+ \cdots} + \oint_{|z|=1} \frac{A_1}{z}\,dz + \cancelto{0}{\oint_{|z|=1} g(z)\,dz}$$

zero due to result $\oint_{\gamma} (z-a)^n\,dz = 0$ for $n \neq -1$! Here $a=0$

$$= A_1 \oint_{|z|=1} \frac{1}{z}\,dz = \boxed{A_1 2\pi i}$$

(18)

Let

$$I := \oint_{|z|=2} \frac{dz}{z^2(z-1)^3}$$

Given Justification

Proof for $I = 0$

(a) For every $R > 2$, $I = I(R)$ where

$$I(R) := \oint_{|z|=R} \frac{1}{z^2(z-1)^3}\, dz$$

If $R > 2$, we can "reverse" deform the contour, analyzing the function on a circle of radius R. (C_R) Therefore,

$$I = I(R) := \oint_{|z|=R} \frac{1}{z^2(z-1)^3}\, dz$$

(b) $|I(R)| \leq \dfrac{2\pi}{R(R-1)^3}$ for $R > 2$

Using the ML estimate,

$$|I(R)| \leq \max_{z \in C_R} |I(R)| \cdot \text{Length}(I(R))$$

$$|I(R)| \le \underbrace{\frac{1}{R\,(R-1)^3}}_{M} \cdot \underbrace{2\pi R}_{L}$$

$$\le \frac{2\pi}{R^2(R-1)^3} \qquad \underline{\text{For } R > 2}$$

(c) $\lim_{R\to+\infty} I(R) = 0$

$$\lim_{R\to+\infty} I(R) = \lim_{R\to+\infty} \oint_{|z|=R} \frac{1}{z^2(z-1)^3}\,dz$$

This limit is less then the ML estimate above.

$\therefore \approx 0$

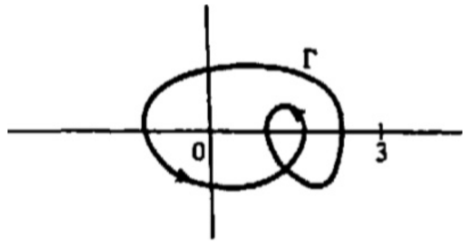
(d) $I = 0$

Varying R doesn't change the result.

$\boxed{4.5}$

7

$$\int_{\Gamma} \frac{\cos(z)}{z^2(z-3)}\,dz$$



$f(z)$ is **NOT** analytic **IN** Γ, but is **ON** Γ.

$z=0$ is an issue

$$\frac{\cos z}{z^2(z-3)} = \frac{\cos z/(z-3)}{z^2}$$

$z_0 = 0,\ m = 2$

$$\Rightarrow \int_{\Gamma} \frac{\cos(z)}{z^2(z-3)}\,dz = \frac{2\pi i f'(0)}{1!}$$

$$= \frac{2\pi i}{1}\left[\frac{-(z-3)\sin(z) - \cos(z)(1)}{(z-3)^2}\right]$$

$$= 2\pi i\left[\frac{0-1}{(-3)^2}\right] = \boxed{\frac{-2\pi i}{9}}$$

(8) If f is analytic in and on $|z-z_0|=r$.
Then

Show This $$f(z_0)=\frac{1}{2\pi}\int_0^{2\pi} f(z_0+re^{i\theta})\,d\theta \quad (1)$$

Prove, more generally that

$$f^{(n)}(z_0)=\frac{n!}{2\pi r^n}\int_0^{2\pi} f(z_0+re^{i\theta})e^{-in\theta}\,d\theta \quad (2)$$

$$f(z_0)=\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-z_0}\,dz$$

Let $z=z_0+re^{i\theta}$ } circle of radius r centered at z_0

$0\le\theta\le 2\pi$

$$f(z_0)=\frac{1}{2\pi i}\int_0^{2\pi}\frac{f(z_0+re^{i\theta})re^{i\theta}i}{re^{i\theta}}\,d\theta$$

$$\boxed{f(z_0)=\frac{1}{2\pi}\int_0^{2\pi} f(z_0+re^{i\theta})\,d\theta}$$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\, dz \quad , n = 1, 2, 3, \ldots$$

$$z = z_0 + re^{i\theta} \quad , \quad 0 \leq \theta \leq 2\pi$$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{(re^{i\theta})^{n+1}} \cdot ire^{i\theta}\, d\theta$$

$$= \frac{n!}{2\pi} \int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{(re^{i\theta})^n (re^{i\theta})} re^{i\theta}\, d\theta$$

$$\boxed{f^{(n)}(\theta) = \frac{n!}{2\pi r^n} \int_0^{2\pi} f(z_0 + re^{i\theta})\, e^{-in\theta}\, d\theta}$$

(9) f is analytic in and on $|z|=1$.
Prove that if

$$|f(z)| \le M \quad \text{for } |z|=1$$

then

$$|f(0)| \le M$$

and

$$|f'(0)| \le M.$$

Estimate

$$|f^{(n)}(0)|$$

$$f(0) = \frac{1}{2\pi i} \int_{|z|=1} \frac{f(z)}{z}\, dz$$

$$|f(0)| \le \left| \frac{1}{2\pi i} \int_{|z|=1} \frac{f(z)}{z}\, dz \right|$$

$$= \frac{1}{2\pi} \max_{z \in |z|=1} \frac{|f(z)|}{\cancel{r}} \cdot 2\pi \cancel{r}$$

$$= \max_{z \in |z|=1} |f(z)| = M$$

$$\boxed{\Rightarrow |f(0)| \le M}$$

$$f'(0) = \frac{1}{2\pi i} \int_{|z|=1} \frac{f(z)}{z^2}\, dz$$

$r=1$!

$$|f'(0)| \le \frac{1}{\cancel{2\pi}} \max_{z \in |z|=1} \frac{|f(z)|}{r^{\cancel{2}}} \cdot \cancel{2\pi r}$$

$$= \max_{z \in |z|=1} |f(z)| = M$$

$$\boxed{|f'(0)| \le M}$$

$$\boxed{\text{Since } r=1 \quad |f^{(n)}(0)| \le \max_{z \in |z|=1} |f(z)| \le M}$$

$\boxed{4.6}$

4) $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ is a polynomial and $\max |p(z)| = M$ for $|z| = 1$, show each a_k is bounded by M: $k = 0, \ldots, n$

For $|z| = 1$,

$$\max |p(z)| = \max |a_0 + a_1 z + \cdots + a_n z^n| \le M$$

$$\max |p(z)| \le \max |a_0| + \max |a_1||z| + \cdots + \max |a_n||z_n|$$

$$\le \max |a_0| + \max |a_1| + \cdots + \max |a_n|$$

$$\le M$$

Therefore each a_k, $k = 0, 1, \ldots, n$ is bounded by M

10) Find all functions

f analytic in $D: |z| < R$ that satisfy

$f(0) = i$ and $|f(z)| \leq 1$ for all z in D.

Theorem 23

If f is analytic in domain D and $|f(z)|$ achieves its maximum value at a point z_0 in D, then f is constant in D.

Since f is constant an $f(0) = i$

$\boxed{f(z) = i}$ by Theorem 23

17) $\max\limits_{|z| \leq 1} |(z-1)(z+\tfrac{1}{2})|$

Let $z = e^{it}$, $0 \leq t \leq 2\pi$

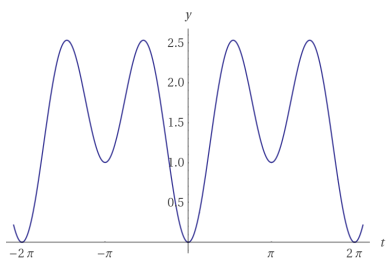
$$|z^2 - \tfrac{z}{2} - \tfrac{1}{2}|^2 = \left(e^{i2t} - \frac{e^{it}}{2} - \frac{1}{2}\right)\left(e^{-i2t} - \frac{e^{-it}}{2} - \frac{1}{2}\right)$$

$$= \cancel{1} - \cancel{\frac{e^{it}}{2}} - \cancel{\frac{e^{i2t}}{2}} - \cancel{\frac{e^{-it}}{2}} + \cancel{\frac{1}{4}} + \frac{e^{it}}{4} - \cancel{\frac{1}{2}e^{-i2t}} + \frac{e^{-it}}{4} + \cancel{\frac{1}{4}}$$

$$\frac{3}{2} - \left[\frac{e^{it}+e^{-it}}{2}\right] - \left[\frac{e^{i2t}+e^{-i2t}}{2}\right] + \left[\frac{e^{it}+e^{-it}}{2}\right]\frac{1}{2}$$

$$= \frac{3}{2} - \cos(t) - \cos(2t) + \frac{\cos(t)}{2}$$

$$= \frac{3}{2} - \frac{\cos(t)}{2} - \cos(2t)$$ PLOT $\rightarrow$

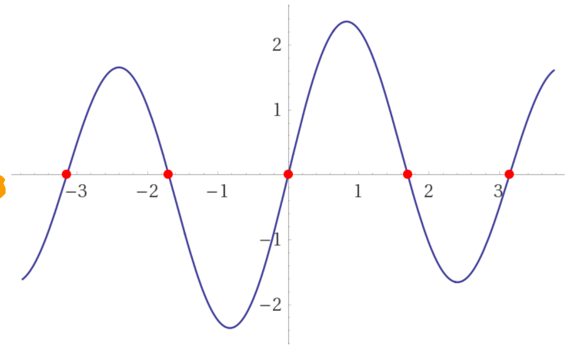


(t from −6.6 to 6.6)

$$\frac{d}{dt}\left[\frac{3}{2} - \frac{\cos(t)}{2} - \cos(2t)\right]$$

$$= \frac{\sin(t)}{2} + 2\sin(2t) = 0$$

Plot



$$\Rightarrow t = \pi n \quad \text{MIN}$$

$$\left.\begin{aligned} t &= 2\pi n - 2\tan^{-1}\left(\frac{3}{\sqrt{7}}\right) \\ t &= 2\pi n + 2\tan^{-1}\left(\frac{3}{\sqrt{7}}\right) \end{aligned}\right\} \text{MAX}$$

Plugging the solutions in yields a max of 2.5

So $\boxed{\sqrt{2.5}}$ is the solution to our problem.

20) Prove

$$\sum_{j=0}^{\infty} z^j = \sum_{j=0}^{\infty} \left(\frac{z}{z_0}\right)^j, \text{ where } z_0 = 1$$

does **NOT** converge uniformally to

$\frac{1}{1-z}$ on open disk $|z|<1$; $\Rightarrow |z| < z_0$

Given $\varepsilon > 0$ we have to show that the remainder after $n+1$ terms will be less than ε for all z in the disk when n is large enough.

A note:

$\sum_{j=0}^{\infty} \left(\frac{z}{z_0}\right)^j$ is uniformally convergent in every closed disk $|z| \leq r$, if $r < |z_0|$

Proof:

$$\left| \frac{(z/z_0)^{n+1}}{1 - z/z_0} \right| \leq \frac{(r/|z_0|)^{n+1}}{1 - r/|z_0|}$$

This can be made arbitrarily small since $r < |z_0|$

For our problem, $z_0 = 1 \Rightarrow |z_0| = 1$

$|z| < 1$ (open disk).

$$\left| \frac{(z/z_0)^{n+1}}{1 - \frac{z}{z_0}} \right| \leq \frac{|z|}{1 - |z|}$$

Can **NOT** be made arbitrarily small and therefore is not less then ε

Therefore we do **NOT** converge uniformally

$\boxed{5.2}$

11b) first 3 nonzero terms in Maclaurin expansion of

$$\frac{e^z}{z-1}$$

$$\left. f(0) + f'(0)z + \frac{f''(0)z^2}{2!} \right\}$$ First 3 nonzero terms in Maclaurin series of $e^z/z-1$

$$f(0) = \frac{1}{-1} = -1$$

$$f'(0) = \frac{(z-1)e^z - e^z}{(z-1)^2} = \frac{e^z(z-2)}{(z-1)^2} = \frac{(-2)}{(-1)^2} = -2$$

$$f''(0) = \frac{(z-1)^2\left[e^z + (z-2)e^z\right] - \left[e^z(z-2)2(z-1)\right]}{(z-1)^4}$$

$$= \frac{(-1)^2\left[1 + -2\right] - \left[4\right]}{(-1)^4} = -5$$

$$\Rightarrow \boxed{-1 - 2z - \frac{5}{2}z^2}$$

13) Find explicit formula for the analytic function $f(z)$ which has Maclaurin series expansion

$$\sum_{k=0}^{\infty} k^2 z^k$$

$$= 0 + z^k + 4z^2 + 9z^3 + 16z^4 + 25z^5 + \cdots$$

First realize

$$\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k$$

Then

$$z \frac{d}{dz}\left(\frac{1}{1-z}\right) = \sum_{k=0}^{\infty} k z^{(k-1)} z = \sum_{k=0}^{\infty} k z^k$$

And finally

$$z \frac{d}{dz}\left[z \frac{d}{dz}\left(\frac{1}{1-z}\right)\right] = \sum_{k=0}^{\infty} k^2 z^{k-1+1} = \sum_{k=0}^{\infty} k^2 z^k$$

$$\frac{d}{dz}\left[\frac{1}{1-z}\right] = +(1-z)^{-2}$$

$$\frac{d}{dz}\left[z(1-z)^{-2}\right] = z \cdot 2(1-z)^{-3} + (1-z)^{-2}$$

$$= \frac{2z}{(1-z)^3} + \frac{1}{(1-z)^2} = \frac{2z+1-z}{(1-z)^3} = \frac{z+1}{(1-z)^3}$$

$$\boxed{\Rightarrow f(z) = z\frac{(z+1)}{(1-z)^3}}$$