Math 4310

Homework 5

Due 10/9/19

Name: ______________________________

Collaborators: ______________________________

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Please print out these pages. I encourage you to work with your classmates on this homework. Please list your collaborators on this cover sheet. (Your grade will not be affected.) Even if you work in a group, you should write up your solutions yourself! You should include all computational details, and proofs should be carefully written with full details. As always, please write neatly and legibly.

Please follow the instructions for the "extended glossary" on separate paper (LaTeX it if you can!) Hand in your final draft, including full explanations and write your glossary in complete, mathematically and grammatically correct sentences. Your answers will be assessed for style and accuracy.

Please **staple** *this cover sheet, your exercise solutions, and your glossary together, in that order, and hand in your homework in class.*

GRADES

Exercises ______________________ / 50

Extended Glossary

Component	Correct?	Well-written?
Definition	/6	/6
Example	/4	/4
Non-example	/4	/4
Theorem	/5	/5
Proof	/6	/6
Total	/25	/25

Exercises.

1. Given a matrix $A = \begin{pmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_n \end{pmatrix} \in \mathcal{M}at_{m,n}(\mathbb{F})$ (where the $\mathbf{a}_i$ are the columns of A), let B be the reduced row echelon form of A. (Note: we have not shown that it is unique! It is, but that doesn't matter for this problem).

   (a) Show that the rank of A is the number of non-zero pivot columns.

   (b) Show that the subset of $\mathbf{a}_1, \dots, \mathbf{a}_n$ consisting of those $\mathbf{a}_i$, where i is a pivot column, is a basis of $\operatorname{im} A$.

   (c) The row span of the matrix A is the span of the rows of A, a subspace of $\mathbb{F}^n$. Find a basis for this subspace, and show that the dimension of the row span of A is the same as the dimension of the column span (image) of A (i.e. the rank of A).

   (d) Suppose that the reduced row echelon form of A is

   $$B = \begin{pmatrix} I_r & C \\ 0 & 0 \end{pmatrix},$$

   where C is an $r \times (n-r)$ matrix. Find a matrix whose columns form a basis for $\ker A$ (justify your answer. Your answer should depend on C of course!)

2. Is the following linear transformation T invertible? If so, find its inverse: $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, given by $T(a, b, c) := (3a - 2c, b, 3a + 4b)$. Justify your answers!

3. Let
$$B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$
Consider the linear transformation $T : \mathbb{R}^{2\times 2} \longrightarrow \mathbb{R}^{2\times 2}$ defined by $T(A) := AB - BA$. Consider the basis $\mathcal{B} = (E_{11}, E_{12}, E_{21}, E_{22})$, of $\mathbb{R}^{2\times 2}$ where E_{ij} is the 2×2 matrix with a 1 in the ith row and jth column, and zeros elsewhere.

   (a) Find the matrix of T with respect to $\mathcal{B}$.
   (b) Find bases and dimensions for the kernel and image of T.

   As always, justify your answers!

4. Suppose that V, W, U are vector spaces, and that $T : V \longrightarrow W$ and $S : W \longrightarrow U$ are linear transformations.

   (a) Prove: if ST is injective, then T is injective.
   (b) Prove: if ST is surjective, then S is surjective.
   (c) Prove: if S is an isomorphism, then $\mathrm{rank}(T) = \mathrm{rank}(ST)$.
   (d) Suppose that S and T are in $\mathcal{L}$. Prove: ST is an isomorphism if and only if S and T are both isomorphisms.

5. Let A be an $n \times n$ matrix over $\mathbb{R}$, let $L_A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ denote the corresponding linear map. Let std denote the standard basis of $\mathbb{R}^n$, and let $\mathcal{B}$ be another basis for $\mathbb{R}^n$. Let $B = [L_A]_{\mathcal{B}}$ be the matrix of L_A in the basis $\mathcal{B}$. In this problem, we try to relate the two matrices A and B. Let $\mathrm{id} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ denote the identity map, and let $S = [\mathrm{id}]_{\mathrm{std}\leftarrow\mathcal{B}}$ be the matrix of id with respect to these bases. It might help to recall that:
$$[ST]_{\mathcal{C}\leftarrow\mathcal{A}} = [S]_{\mathcal{C}\leftarrow\mathcal{B}}[T]_{\mathcal{B}\leftarrow\mathcal{A}}.$$

   (a) What is the matrix $[L_A]_{\mathrm{std}}$?
   (b) Show that $S^{-1} = [\mathrm{id}]_{B\leftarrow\mathrm{std}}$.
   (c) One of the following two statements holds, (1) $B = S^{-1}AS$, or (2) $B = SAS^{-1}$. Which is correct? (Justify your answers!)
   (d) Now we apply this to a specific matrix A:
$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}.$$
   let $\mathcal{B}$ denote the following basis of $\mathbb{R}^3$:
$$\mathcal{B} = \left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right)$$
      i. Find the matrix B of L_A in this basis
      ii. Find the matrices S and S^{-1}.
      iii. Check your answer in (c) above!.

Extended Glossary. There is no extended glossary this week.