Gaussian Process

$X(t)$ Gaussian if $\forall_n \{t_i\}_{i=1}^{n}$

$X(t_1), \ldots, X(t_n)$ are JG

Process fully determined by

$$\mu_X(t), \quad C_X(t_1, t_2)$$

and

$$\text{WSS} \Rightarrow \text{Strictly Stationary}$$

Example: Gaussian Process $X(t)$

$$\mu_X(t) = 3t$$

$$C_X(t_1, t_2) = 9e^{-2|t_2 - t_1|}$$

Recall

$$C_X(t_1, t_2) \triangleq \mathbb{E}\left[[X(t_1) - \mu_X(t_1)][X(t_2) - \mu_X(t_2)]\right]$$

Find ① $\Pr[X(3) < 6]$
② $\Pr[X(1) + X(2) > 2]$

①

$$X(3) \sim N(\mu_X(t=3), C_X(t_1=3, t_2=3)) \Rightarrow N(9, 9)$$

②

$$X(1) + X(2) \sim N(\mu_X(1) + \mu_X(2), \sigma^2 \quad)$$

$$\sim N(7, 9e^{-2})$$

$$P[X(1) + X(2) > 2] = 1 - \Pr[X(1) + X(2) \leq 2]$$

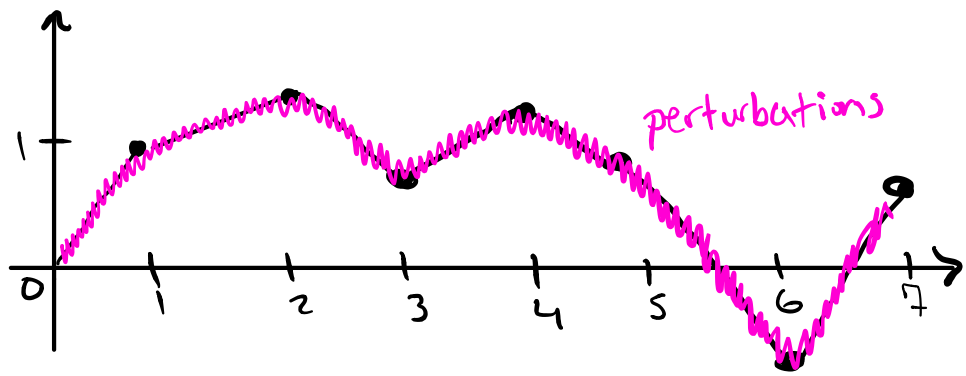
$$\sigma^2 = \mathbb{E}\left[\left(X(1)+X(2) - \mathbb{E}[X(1)+X(2)]\right)^2\right]$$
$$= \mathbb{E}\left[(X(1)-\mathbb{E}[x])^2 + (X(2)-\mathbb{E}[x])^2 + 2(X(1)-\mathbb{E}[X(1)])(X(2)-\mathbb{E}[x])\right]$$
$$= C_x(1,1) + C_x(2,2) + 2C_x(1,2)$$

Can also do MMSE estimates.

Say i.e. $t=5$, $X(t=5)$ based on $X(t=4)$, $X(t=3)$.

Brownian Motion

Discrete-time random walk



$\delta \to 0$

$h \to 0$

$h \sim \sigma\sqrt{\delta} \Rightarrow RW \to BM$

Definition: A Brownian Motion (also called a Wiener Process) with parameter σ^2 is a random process $\{X(t)\}_{t\geq 0}$ such that

1. $X(0) = 0$
2. $\{X(t)\}_{t \geq 0}$ has independent increments
3. $X(t_2) - X(t_1) \sim \mathcal{N}(0, \sigma^2(t_2 - t_1))$
4. Every sample path is continuous

Properties of Brownian Motion

1. $\mu_X(t) \triangleq \mathbb{E}[X(t)] = \mathbb{E}[X(t) - X(0)] = 0$
2. $R_X(t_1, t_2) = C_X(t_1, t_2) \quad t_2 > t_1$

$$= \mathbb{E}[X(t_1)X(t_2)]$$

$$= \mathbb{E}[X(t_1)(X(t_1) + X(t_2) - X(t_1))]$$

$$= \mathbb{E}[X^2(t_1)] + \mathbb{E}[X(t_1)]\,\mathbb{E}[X(t_2) - X(t_1)]$$

just illustrates increments $\longrightarrow$ $= \mathbb{E}[(X(t_1) - X(0))^2] + \mathbb{E}[X(t_1) - X(0)]\,\mathbb{E}[X(t_2) - X(t_1)]$

$$= \sigma^2 t_1$$

In general,

$$C_X(t_1, t_2) = R_X(t_1, t_2) = \min(t_1, t_2)\,\sigma^2$$

3. $X(t) \sim \mathcal{N}(0, \sigma^2 t)$

Also have $X(t_1), X(t_2), \ldots, X(t_n)$ J.G $\forall n$.

i.e

$$\begin{bmatrix} X(t_1) \\ \vdots \\ X(t_n) \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & 0 \cdots & 0 \\ 1 & 1 & 1\; 0 \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} X(t_1) - X(0) \\ X(t_2) - X(t_1) \\ \vdots \\ X(t_n) - X(t_{n-1}) \end{bmatrix}$$

Independence Review

Independent Events: $A \perp\!\!\!\perp B$ if $\Pr[A \cap B] = \Pr[A]\Pr[B]$

or equivalenty: $\Pr[A \mid B] = \Pr[A]$

Independent Random Variables:

$$X \perp\!\!\!\perp Y \text{ if } f_{X,Y}(x,y) = f_X(x)\, f_Y(y)$$

or equivalently,

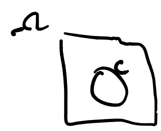
$$f_{X|Y}(x \mid y) = f_X(x)$$

Conditional Independent Events

$A \perp\!\!\!\perp B$ conditioned on C if

$$\Pr[A \cap B \mid C] = \Pr[A \mid C]\Pr[B \mid C]$$

ie C is "new universe".



$C = \Omega'$

or equivalently,

$$\Pr[A \mid B \cap C] = \Pr[A \mid C]$$

Conditional Independent Random Variables:

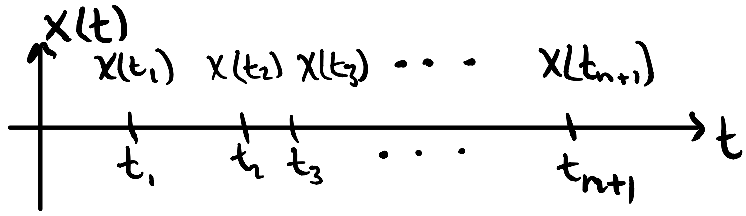
$$X \perp\!\!\!\perp Y \text{ if } f_{X,Y|Z}(x,y \mid z) = f_X(x \mid z)\, f_Y(y \mid z)$$

or equivalently,

$$f_{X|Y,Z}(x \mid y,z) = f_{X|Z}(x \mid z)$$

Markov Process

A random process $\{X(t)\}_{t \geq 0}$ is a Markov process if $\forall n$, and $t_1 < t_2 < \dots < t_{n+1}$



$$\Pr[X(t_{n+1}) = x_{n+1} \mid X(t_n) = x_n, X(t_{n-1}) = x_{n-1}, \dots, X(t_1) = x_1]$$

$$= \Pr[X(t_{n+1}) = x_{n+1} \mid X(t_n) = x_n]$$

i.e. conditioned on where you are now, the future is independent of the past.

We call the value we see the state.

Once you know the current state, future state $\perp\!\!\!\perp$ past states

Example: Random Walk

$$S_n = \sum_{i=1}^{n} X_i \quad , \quad X_i \sim \text{Bernoulli}(p = \tfrac{1}{2})$$

Claim: $\{S_n\}_{n \geq 0}$ is Markov. Note, $S_0 = 0$

LHS = $\Pr[S_{n+1} = s_{n+1} \mid S_n = s_n, \dots, S_1 = s_1]$

$= \Pr[S_{n+1} - S_n = s_{n+1} - s_n \mid S_n = s_n, \ldots, S_1 = s_1]$ increments

$= \Pr[S_{n+1} - S_n = s_{n+1} - s_n \mid S_n = s_n]$

$=$ RHS

Example:

$$Z_n = \begin{cases} 1 & \text{if } S_n \geq 0 \\ -1 & \text{if } S_n < 0 \end{cases}$$

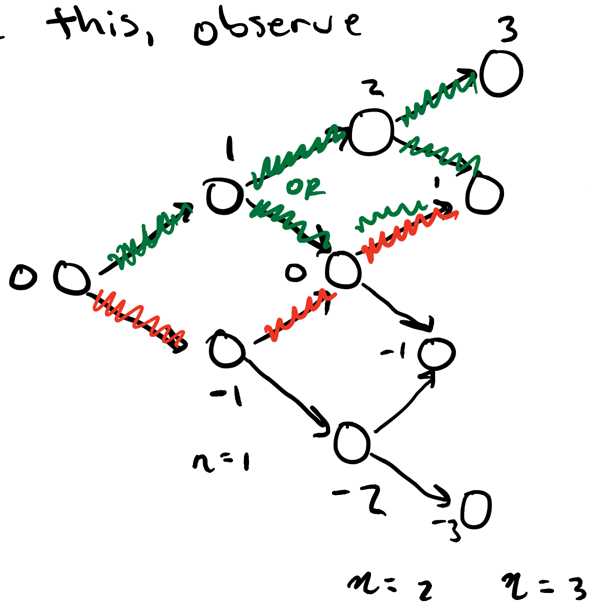
"called Hidden Markov"

Is $\{Z_n\}_{n \geq 0}$ Markov?

To show it is not, it suffices to show one case where it does not hold.

$$\Pr[Z_3 = 1 \mid Z_2 = 1, Z_1 = -1] \neq \Pr[Z_3 = 1 \mid Z_2 = 1, Z_1 = +1]$$

To see this, observe



$\frac{1}{2} \neq \frac{3}{4}$