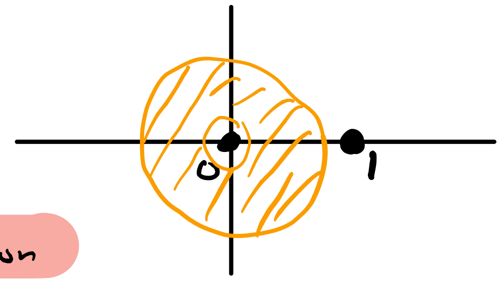
Laurent Series

Ex 1: Consider $f(z) = \dfrac{1}{z^2(1-z)}$. $f(z)$ has isolated singularities at $z=0,\ z=1$



Is it possible to find a series expansion for $f(z)$ valid in $0<|z|<1$? → Such a region is called an **Annulus**

Know That

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots \quad \text{valid for } |z|<1$$

$$\Rightarrow f(z) = \frac{1}{z^2}\left(1 + z + z^2 + z^3 + \cdots\right)$$

$$\boxed{= \frac{1}{z^2} + \frac{1}{z} + 1 + z + z^2 + \cdots}$$

Laurent Series for f in this region

"In different regions there will be different series"

More generally, suppose $f(z)$ analytic in annulus $r<|z-z_0|<R$. Then $f(z)$ has a convergent Laurent series

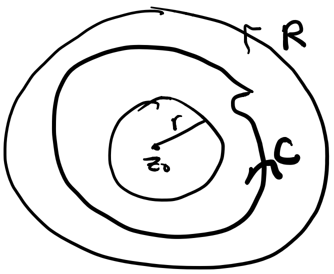
$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$$

note (−) powers!

where

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(w)}{(w-z_0)^{n+1}}$$

and



$C =$ closed contour in annulus that encircles z_0 once counterclockwise

Notes:

1. Series may terminate at either end.
i.e in example 1, the left terminated @ $\frac{1}{z^2}$ and the right did NOT terminate

2. For $n \geq 0$, the a_n would be the usual Taylor coefficients if $f(z)$ were analytic in the whole disk $|z - z_0| < R$
$\Rightarrow$ Laurent generalizes Taylor series and polynomials

3. **SEE PROOF IN BOOK**
- said to look @ it like 10 times, similar to Taylor series proof

4. To caclulate a_n, can do integrals

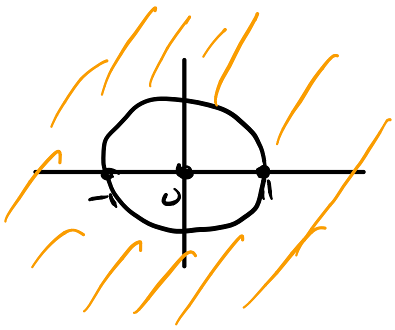
$$\frac{1}{2\pi i} \oint \frac{f(w)}{(w - z_0)^{n+1}}$$ <u>but</u>

easier ways exist

Example 2:

Expand

$$f(z) = \frac{1}{z^2}\left(\frac{1}{1-z}\right) \quad \text{for } |z| > 1$$



Do **NOT** use $\frac{1}{1-z} = 1 + z + z^2 + \cdots$

Valid **ONLY** for $|z| < 1$!

What to do? Use a geometric series somehow. Seek a variable whose magnitude is less then 1.

Note for $z > 1$ $\left|\frac{1}{z}\right| < 1$. Define $u = \frac{1}{z}$. u has the desirable property $|u| < 1$.

Write $\frac{1}{1-z}$ in terms of u

$$\Rightarrow \frac{1}{1-z} = \frac{1}{1-\frac{1}{u}} = \frac{u}{u-1} = -u\left(\frac{1}{1-u}\right)$$

$$= -u\left(1 + u + u^2 + u^3 + \cdots\right) \Big\} \text{ correct because } |u| < 1$$

Hence for $|z| > 1$, we have

$$\frac{1}{1-z} = -\frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots\right)$$

$$= -\frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} - \cdots$$

So

$$f(z) = \frac{1}{z^2}\left(\frac{1}{1-z}\right) = -\frac{1}{z^3} - \frac{1}{z^4} - \frac{1}{z^5} - \frac{1}{z^6} - \cdots$$

for $|z|>1$ — specify region

Example 3:

Not going to do but **LOOK AT IN BOOK**

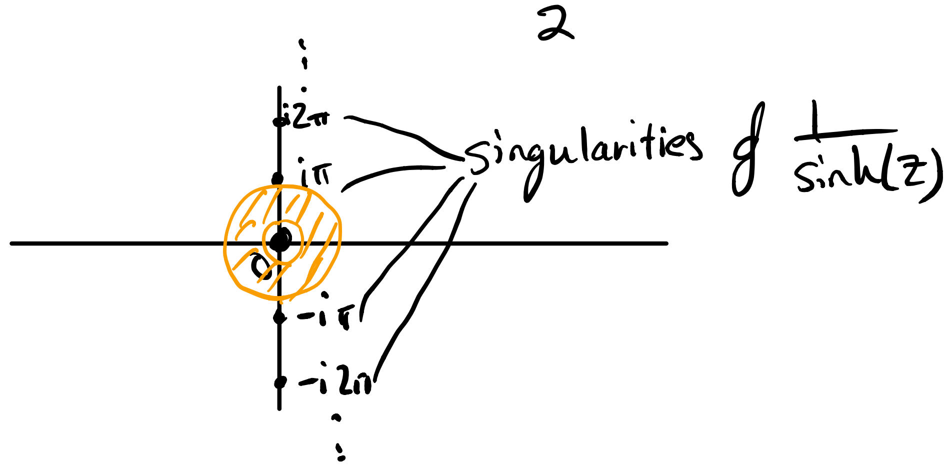
To do w/ partial fractions → **Example 2: Page 274-275**

Example 4 (Long Division)

Laurent series for $f(z) = \dfrac{1}{\sinh(z)}$?

Recall $\sinh(z) = 0$ for $z = ik\pi$, $k \in \mathbb{Z}$

$$\sinh(z) = \frac{e^z - e^{-z}}{2} = -i\sin(iz)$$



Should be able to find a series in $0 < |z| < \pi$

For small z, $\sinh(z) \simeq z$ $\left(\text{since } \begin{array}{l} e^z = 1 + z + \cdots \\ e^{-z} = 1 - z - \cdots \end{array}\right)$

So we expect $\frac{1}{\sinh(z)}$ behaves like $\frac{1}{z}$ + analytic stuff for $z \approx 0$

Saying $\frac{1}{\sinh(z)}$ is about this singular plus moderate stuff like an analytic function

This will be a Maclaurin Series

$$\sinh(z) = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \cdots$$

$$= z + \frac{z^3}{6} + \frac{z^5}{120} + \cdots$$

use long diving to get analytic stuff

Then using long division, $\frac{1}{\sinh(z)}$ becomes

$$\begin{array}{r|l}
 & \frac{1}{z} - \frac{z}{6} + \frac{7z^3}{360} \cdots \\
\hline
z + \frac{z^3}{6} + \frac{z^5}{120} + \cdots & 1 \\
 & -\left(1 + \frac{z^2}{6} + \frac{z^4}{120} + \cdots\right) \\
\hline
 & 0 - \frac{z^2}{6} - \frac{z^4}{120} - \cdots \\
 & -\left(-\frac{z^2}{6} - \frac{z^4}{36} - \cdots\right) \\
\hline
 & 0 + \frac{7z^4}{360} \cdots
\end{array}$$

The long division suggests that

$$\frac{1}{\sinh(z)} = \underbrace{\frac{1}{z} - \frac{z}{6} + \frac{7z^3}{360} + \cdots}_{\text{our Laurent series expansion}} \quad \text{for } 0<|z|<\pi$$

Isolated Singularities

Suppose z_0 is an isolated singularity of $f(z)$ {meaning $f(z)$ is analytic in some punctured disk $0<|z-z_0|<R$ for some $R>0$}

1) Removable

meaning finite / doesn't blow up

$|f(z)|$ bounded as $z \to z_0$

2) pole

$|f(z)| \to \infty$ as $z \to z_0$

3) Essential

$f(z)$ has no limit as $z \to z_0$

(oscillates **WILDLY** as $z \to z_0$)

Examples

1) Removable : $f(z) = \dfrac{\sin z}{z}$ } strictly speaking at $z=0$ we have a singularity but

$$\frac{\sin z}{z} = \frac{z + \frac{z^3}{3!} - \frac{z^5}{5!}}{z} = 1 + \frac{z^2}{3!} - \frac{z^4}{5!} \cdots$$

So as $z \to 0$ $\frac{\sin z}{z} \to 1$. i.e by defining $f(0) = 1$

The singularity has been "removed"

A theoretical point:

At a removable singularity the Laurent series has no negative powers of $z - z_0$

2) POLES

$\frac{1}{z - z_0}$ = simple pole of order 1

$\frac{1}{(z-z_0)^m} \to$ pole of order m, $m > 0$

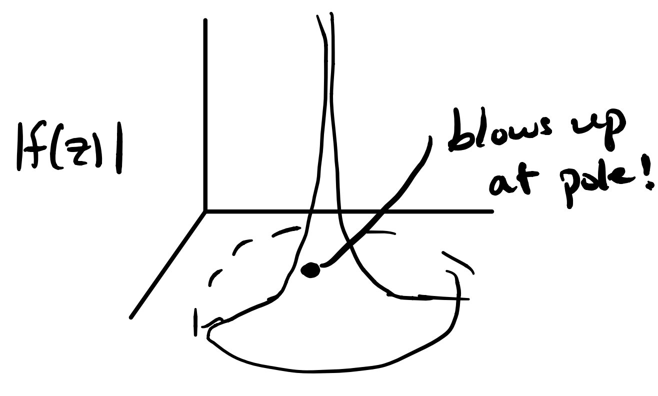
If $f(z)$ has a pole of order m, Laurent series begins w/ $-m$ power!

See book for more detail

$\Leftrightarrow$

$$f(z) = \frac{a_{-m}}{(z-z_0)^m} + \frac{a_{-m+1}}{(z-z_0)^{m+1}} + \cdots + a_0 + \cdots a_1(z - z_0) \cdots$$

Why called a pole. Visualize.



z_0

3) Essential $f(z)$ is neither bounded nor blows up as $z \to z_0$

i.e

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \frac{1}{3!\,z^3} + \cdots \Bigg\} \text{ powers of } z \text{ going to } -\infty$$

$z_0 = 0$ essential

let

$z \to 0$ along real values

$f(z) \to 0$ as $z \to 0$ from negative side

$f(z) \to \infty$ as $z \to 0$ from positive side

let

$z = iy$ along imaginary values

$$f(z) = e^{-\frac{i}{y}} = \cos\left(\frac{1}{y}\right) - i\sin\left(\frac{1}{y}\right)$$

graph, crazy oscillations