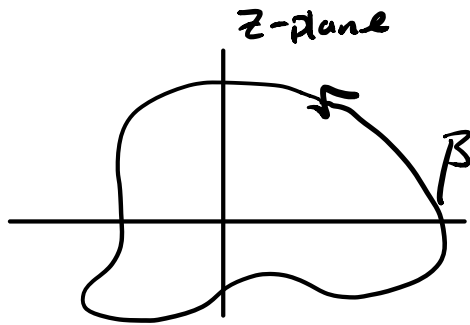


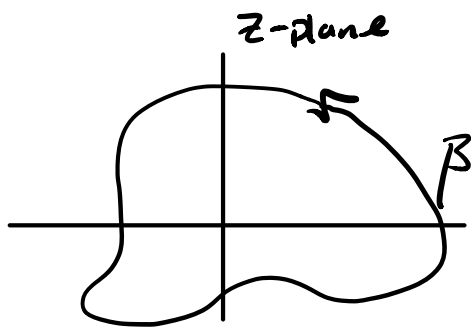
# Argument Principle and Counting Zeros

$f \rightarrow$  analytic function (think of it as mapping)

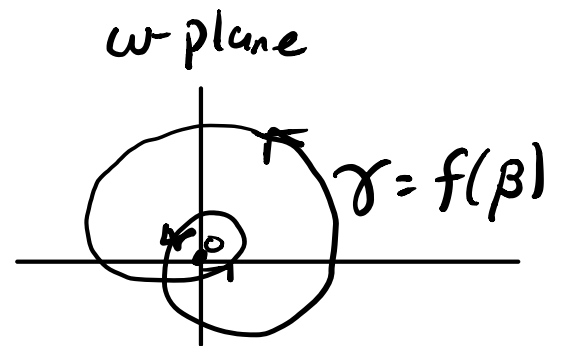
$$w = f(z)$$



Q: Are there any zeros of  $f$  inside  $\beta$ ?



$$w = f(z)$$



$\rightarrow W = 2$   
winding number of  $\gamma$  about 0

So,

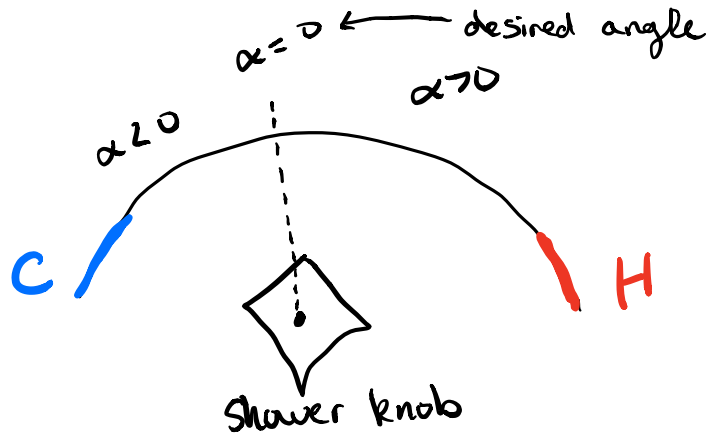
$$\# \text{zeros inside } \beta = W(\gamma; 0)$$

'essentially negative feedback control system'  
 $\swarrow$

## Application of Argument Principle

Control of water temperature in shower, assuming delayed response of plumbing.

Let  $u(t)$  = difference between actual and desired water temp  
 $\uparrow$  "error"



$\alpha$  = angle on knob

$\tau$  = time delay (due to old pipes)

$k_1$  = constant (converts angle to temp.)

$\tau \geq 0$  (zero if instantaneous response)

$$1) u(t) = k_1 \alpha(t - \tau)$$

2) Control Scheme:

If (water too hot)

make colder

else if (water too cold)

make hotter

else

do nothing

← water perfect temp

So, change angle at a rate proportional to error signal.

$$\frac{d\alpha}{dt} = -k_2 u(t)$$

Note:  $k_1, k_2 > 0$

↑  
strength of attempted correction (gain)

Now have a "rough" model

Combine equations to get

$$\frac{d\alpha}{dt} = -k_1 k_2 \alpha(t - \tau)$$

} Let  $k = k_1 k_2 > 0$

$$\frac{d\alpha}{dt} = -k\alpha(t-\tau)$$

} infinite-dimensional  
dynamical systems  
(time-delay differential equations)

Note how our differential equation is linear in  $\alpha$   
( $\alpha$  and its derivatives appear to first power only)

"Exponentials are the eigenfunctions of linear operators"

Guess eigensolution of form

$$\alpha(t) = \alpha_0 e^{st}$$

←  $s$  is an eigenvalue

$$\frac{d\alpha(t)}{dt} = \alpha_0 s e^{st}$$

and

$$\frac{d\alpha}{dt} = -k\alpha(t-\tau) = -k\alpha_0 e^{s(t-\tau)} = -k\alpha_0 e^{st} e^{-s\tau}$$

$\frac{d\alpha}{dt} = -k\alpha(t-\tau)$  reduces to

$$\cancel{\alpha_0 s e^{st}} = -k \cancel{\alpha_0 e^{-s\tau}} \cancel{e^{st}} \quad \text{for these eigensolutions}$$

This is a solution  
iff

$$s = -k e^{-s\tau}$$

$$s \in \mathbb{C}$$

← equation

for eigenvalues  $s$   
("expect infinitely many")  
(characteristic eqn)

Want to find roots of

$$s + k e^{-s\tau} = 0, \quad \text{given } k, \tau \text{ (both } > 0)$$

Want decaying solutions!  $\Rightarrow \operatorname{Re}(s) < 0$

$$s = a + ib \quad \text{oscillating}$$
$$e^{st} = \underbrace{e^a}_{\text{decays when } a < 0} \underbrace{e^{ibt}}_{\text{oscillating}} \quad a = \operatorname{Re}(s)$$

That's why in control theory eigenvalues live in left-half plane.

Want to find condition such that all zeros of  $f(s)$  have  $\operatorname{Re}(s) < 0$ , where

$$f(s) = s + ke^{-s\tau} \quad \left( \begin{array}{l} \text{natural function in} \\ \text{this problem} \end{array} \right)$$

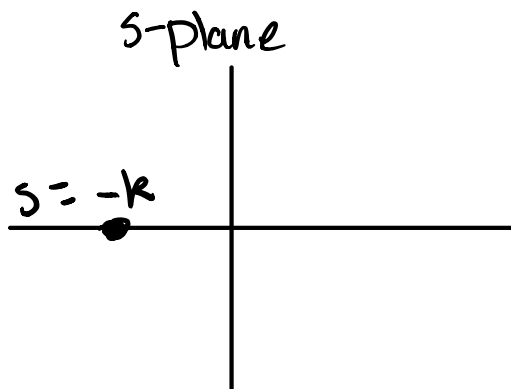
Simple Case

$\tau = 0$  (no delay)

$$f(s) = s + k = 0$$

So,  $f = 0$  when

$$s = -k \quad \left( \begin{array}{l} \text{only one, not infinite amount.} \\ \text{As } \tau \rightarrow 0 \text{ infinitely many roots go} \\ \text{out to infinity} \end{array} \right)$$

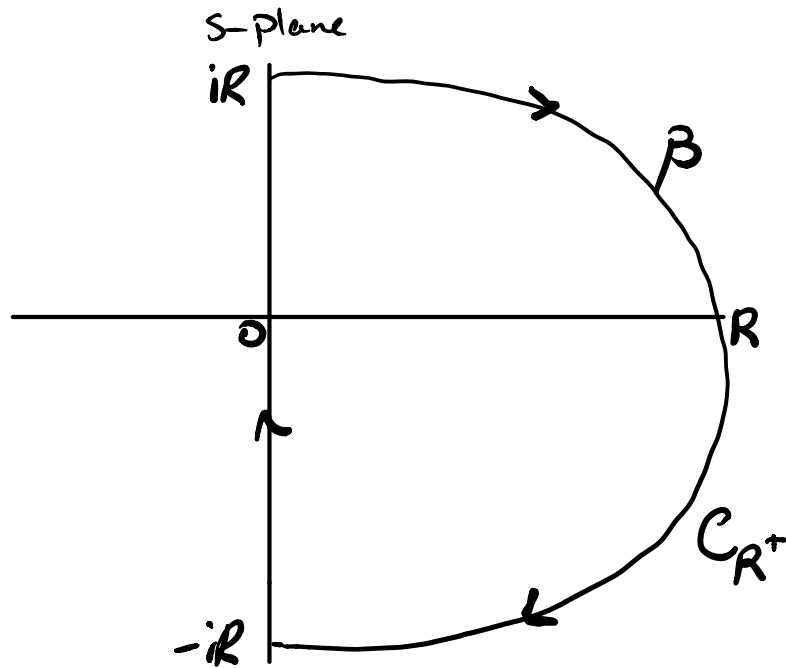


This is stable (solutions decay to  $x = 0$ )

By continuity, might expect stability to hold if  $\tau$  is close to 0.

How much time delay can we tolerate?  $\tau < \frac{\pi}{2} \left( \frac{1}{k} \right) (*)$   
prove later

Want conditions s.t. ~~A~~ zeros in right-half plane



$R \rightarrow \infty$

Show no zeros in  
right half-plane

Want to show  $f(\beta) = \gamma$  has winding number 0 about origin  
when  $\textcircled{*}$  holds