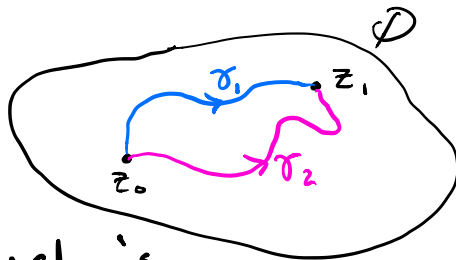


Consequences of Cauchy's Theorem

① Path independence is now obvious.

- Suppose $f(z)$ is analytic in a simply connected domain D , and γ_1, γ_2 are two curves that have the same endpoints.

Then,



$$\int_{(\gamma_1 - \gamma_2)} f(z) dz = 0 \quad \text{by Cauchy's Theorem} \dots (\gamma_1 - \gamma_2) \text{ is a closed curve}$$

Implication?

$$\int_{(\gamma_1 - \gamma_2)} f(z) dz = \int_{\gamma_1} f(z) dz - \int_{\gamma_2} f(z) dz = 0$$

$$\Rightarrow \int_{\gamma_1} f(z) dz = \int_{\gamma_2} f(z) dz$$

Note: This is different from earlier path independence result!

- Earlier result assumed f had an antiderivative
- New result assumes f has a derivative (analytic)

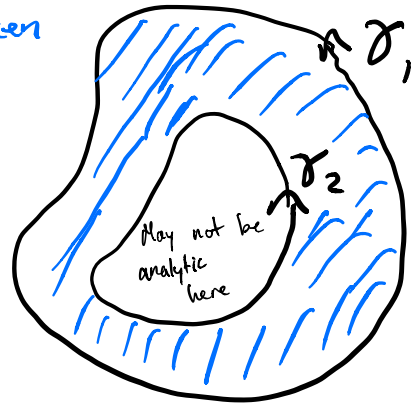
(2) Even if f is NOT analytic everywhere, as long as f is analytic **ON AND BETWEEN** closed contours γ_1 and γ_2

← assumes curves are similarly oriented

then

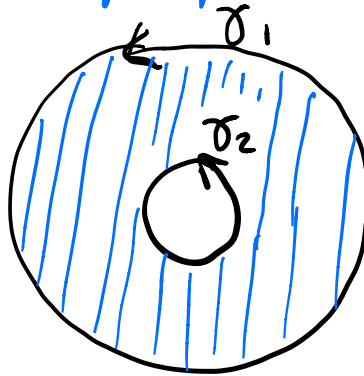
$$\oint_{\gamma_1} f(z) dz = \oint_{\gamma_2} f(z) dz$$

-analytic on + between γ_1, γ_2

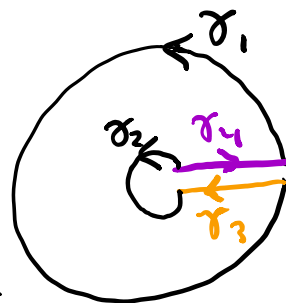


Proof: Consider the region

(region of interest)



Deform the contour like so



γ_3, γ_4 lie on same line, drawn like so for clarity.

$$\gamma_3 = -\gamma_4$$

traverse it opposite of shown orientation

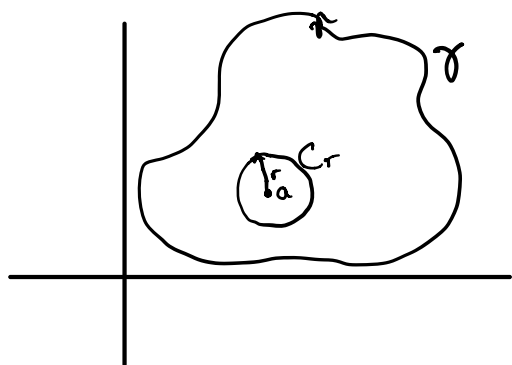
So

$$\int_{\gamma_1} f(z) dz - \int_{\gamma_2} f(z) dz + \int_{\gamma_3} f(z) dz + \int_{\gamma_4} f(z) dz = 0$$

$$\int_{\gamma_1} f(z) dz = \int_{\gamma_2} f(z) dz$$

Example:

Find $\oint_{\gamma} \frac{dz}{z-a}$ where γ is any contour which encircles a once positively



$\frac{1}{z-a}$ is analytic everywhere EXCEPT $z=a$.

By last result, we can deform contour to a small circle C_r around a .

$$\int_{\gamma} \frac{dz}{z-a} = \int_{C_r} \frac{dz}{z-a} \quad \begin{aligned} z &= re^{it} + a & 0 \leq t \leq 2\pi \\ dz &= rie^{it} dt \end{aligned}$$

$$\Rightarrow \int_0^{2\pi} \frac{rie^{it}}{re^{it} + a - a} dt = 2\pi i \quad \left. \vphantom{\int_0^{2\pi}} \right\} \text{as we saw earlier!}$$

CAUCHY'S INTEGRAL FORMULA

Let

- γ be any positively oriented closed contour.
- f be analytic in a simply connected domain containing γ
- a be any point inside γ

THEN

$$f(a) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{z-a} dz$$

AMAZING

Cauchy's Integral formula says that we can determine the values of $f(z)$ everywhere inside γ , just by knowing the values of f everywhere on the boundary γ .

Proof: Deform γ to C_r .

Then

$$\oint_{\gamma} \frac{f(z)}{z-a} dz = \oint_{C_r} \frac{f(z)}{z-a} dz$$

On C_r , $f(z) \approx f(a)$ if r is very small.

Precisely,

$$f(z) = f(a) + \varepsilon(z) \left. \begin{array}{l} \text{where } \varepsilon(z) \rightarrow 0 \\ \text{as } z \rightarrow a \\ \text{by continuity of } f \end{array} \right\}$$


$$\oint_{\gamma} \frac{f(z)}{z-a} dz = \oint_{C_r} \frac{f(a) + \varepsilon(z)}{z-a} dz = \oint_{C_r} \frac{f(a)}{z-a} dz + \oint_{C_r} \frac{\varepsilon(z) dz}{z-a}$$

$$= f(a) \oint_{C_r} \frac{dz}{z-a} + \oint_{C_r} \frac{\varepsilon(z)}{z-a} dz$$

ML estimate

$$\left| \oint_{C_r} \frac{\varepsilon(z)}{z-a} dz \right| \leq \underbrace{\max_{z \in C_r} \left| \frac{\varepsilon(z)}{z-a} \right|}_M \cdot \underbrace{2\pi r}_L$$

$|z-a| = r$ on C_r



$$\left| \oint_{C_r} \frac{\varepsilon(z)}{z-a} dz \right| \leq \max_{z \in C_r} |\varepsilon(z)| \cdot \cancel{2\pi r} \quad \begin{array}{l} \text{As } r \rightarrow 0 \\ z \rightarrow a \Rightarrow \varepsilon(z) \rightarrow 0 \\ \uparrow \\ \text{by continuity assumption} \end{array}$$

$$\therefore \left| \oint_{C_r} \frac{\varepsilon(z)}{z-a} dz \right| = 0 \quad \left. \begin{array}{l} \text{Must be equal to 0; two things} \\ \text{independent of } r \text{ imply it's exactly 0.} \end{array} \right\}$$

In conclusion,

$$\oint_{\gamma} \frac{f(z)}{z-a} dz = f(a) \cdot 2\pi i + 0$$

$$\Rightarrow f(a) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{(z-a)} dz$$