$\text{cis}\,\theta = \cos\theta + i\sin\theta = e^{i\theta}$

Useful Properties

$\text{cis}\,\theta_1\,\text{cis}\,\theta_2 = \text{cis}(\theta_1+\theta_2)$ // Can prove using trig

$= [\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2] + i[\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2]$

$= \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)$

$= \text{cis}(\theta_1+\theta_2)$

More generally,

$$z_1 z_2 = (r_1\,\text{cis}\,\theta_1)(r_2\,\text{cis}\,\theta_2)$$
$$= r_1 r_2\,\text{cis}(\theta_1+\theta_2)$$

If $z_1 = z_2$,

$$z^2 = r^2\,\text{cis}(2\theta)$$

More generally,

$$z^n = r^n\,\text{cis}(n\theta)$$, for any positive integer n

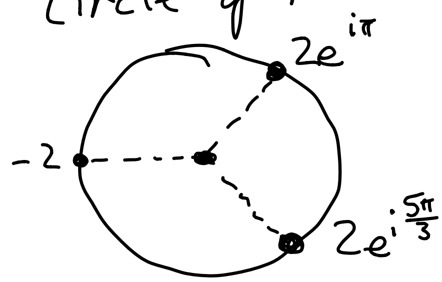
actually $(n \in \mathbb{R})$

De Moivre's Theorem

Example $\sqrt[3]{-8} = \sqrt[3]{8}\, e^{i\frac{2\pi k+\theta}{3}} = 2e^{i\left(\frac{\pi}{3}+\frac{2\pi k}{3}\right)}$, $k = 0, 1, 2$

$z = -8$ 2

circle of radius



$k=0$, $2e^{i\pi/3} = 1+\sqrt{3}\,i$

$k=1$, $2e^{i\pi} = -2$

$k=2$, $2e^{i5\pi/3} = +1-\sqrt{3}\,i$

Example: Trig identities via De Moivre's Theorem

$$(\text{cis}\,\theta)^2 = \cos 2\theta + i\sin 2\theta = \text{cis}\,2\theta$$

$$(\cos\theta + i\sin\theta)^2 = (\cos^2\theta - \sin^2\theta) + i(2\sin\theta\cos\theta)$$

Example: Show

$$\frac{\pi}{4} = 2\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right)$$

Get this by expanding $(3-i)^2(1+i)$!

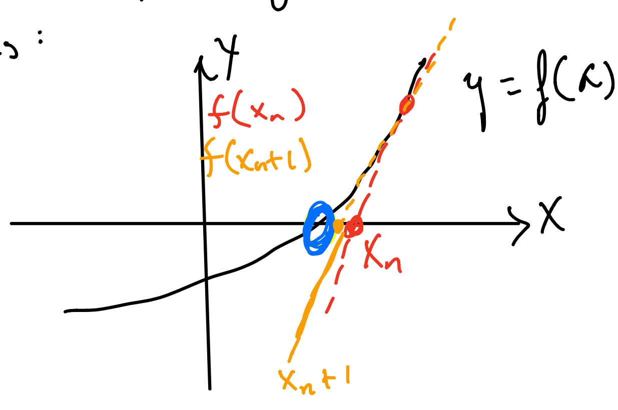
$$\arg\left[(3-i)^2(1+i)\right] = \arg\left[(3-i)(3-i)\right] + \arg((1-i)) = \arg\left[14+2i\right]$$

$$-2\tan^{-1}\left(\frac{1}{3}\right) + \frac{\pi}{4} = \tan^{-1}\left(\frac{1}{7}\right)$$

$$\frac{\pi}{4} = 2\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right)$$

Newton's Method: finding solutions to $f(z) = 0$

Recall from Calculus:



Basically draw tangent lines to the curve until you approach the zero

From picture:

$$\frac{f(x_n) - 0}{x_n - x_{n+1}} = f'(x_n)$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Complex Version:

$$z_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}$$ "iterative mapping"

A solution to $f(z)=0$ gives a fixed point $(z_{n+1} = z_n)$ for Newton iteration.

Example: Find a soln of $z^2 = 3+4i$ in first quadrant
$x \geq 0$, $y \geq 0$

Define $f(z) = z^2 - (3+4i)$
Assuming derivatives stay the same,
$f'(z) = 2z$

$$z_{n+1} = z_n - \frac{z_n^2 - (3+4i)}{2z_n} = \frac{z_n^2 + 3 + 4i}{2z}$$

Start at some reasonable guess, say, $x_0 = 1$, $y_0 = 1$

just a neat way to simplify

$$z_{n+1} = \frac{(x_n^2 - y_n^2 + 3) + i(4 + 2x_n y_n)}{2(x_n + iy_n)}$$

$x_0 = 1, \quad y_0 = 1$

$x_1 = 1.25, \quad y_1 = 0.75$

$x_2 = 1.991667, \quad y_2 = 0.975$ ← 2 correct

$x_3 = 1.99993, \quad y_3 = 1.00014$ ← 4 correct

$x_4 = 1.999999995, \quad y_4 = 0.999999997$ ← 8 correct

$x_5 = 2, \quad y_5 = 1$ ← 16 correct

typically doubles # of correct digits

Number of correct digits asymptotically approaches the right answer

How do we choose initial guess?

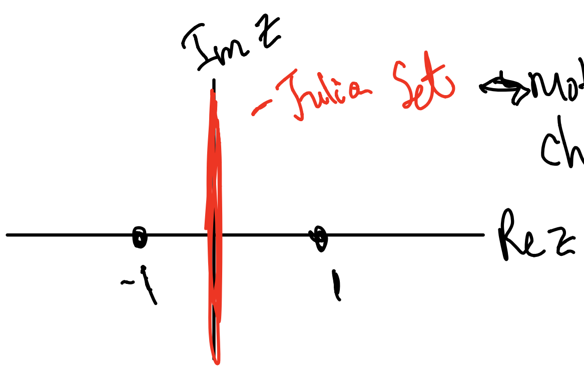
What if $\exists$ 2 or more solutions - which one does it find?

Example

$f(z) = z^2 - 1 = 0 \qquad$ Sol/n, ± 1

Newton: $\quad z_{n+1} = z_n - \dfrac{(z_n^2 - 1)}{2z_n} = \dfrac{z_n^2 + 1}{2z_n}$

If we start on an equidistant point (Im axis) we stay there forever



Motion on this set is chaotic