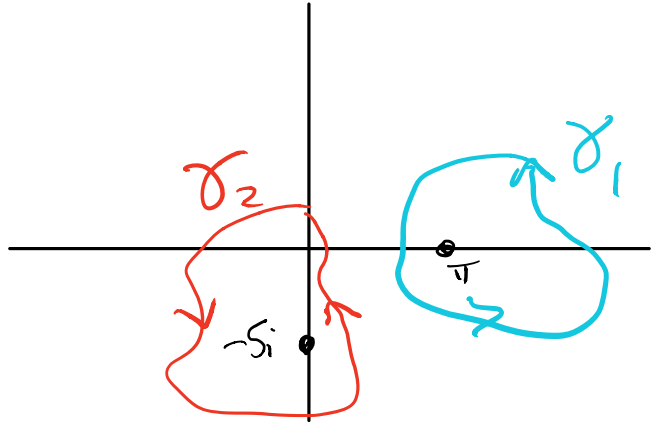


Evaluate

$$\oint \frac{\cos(z)}{(z-\pi)(z+5i)} dz$$

along $\gamma_1 + \gamma_2$



On γ_1 , use

$$2\pi i f(\pi) = \oint_{\gamma_1} \frac{\cos(z)/(z+5i)}{(z-\pi)} dz \quad \left\{ f(z) = \frac{\cos(z)}{(z+5i)} \right.$$

On γ_2 , use

$$2\pi i f(5i) = \oint_{\gamma_2} \frac{\cos(z)/(z-\pi)}{z+5i} dz \quad \left\{ f(z) = \frac{\cos(z)}{(z-\pi)} \right.$$

↑ Need appropriate choices of f and a !
Need f to be analytic **ON AND INSIDE**
 γ to apply Cauchy's Integral Formula

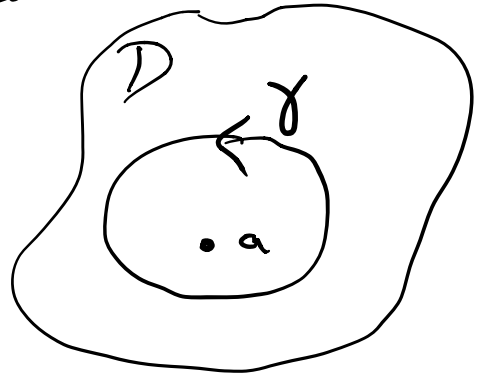
In conclusion,

$$\begin{aligned} \oint_{\gamma_1 + \gamma_2} \frac{\cos z}{(z-\pi)(z+5i)} dz &= \oint_{\gamma_1} \frac{f(z)}{z-\pi} dz + \oint_{\gamma_2} \frac{f(z)}{z+5i} \\ &= 2\pi i \frac{\cos(\pi)}{\pi+5i} + 2\pi i \frac{\cos(5i)}{5i-\pi} \end{aligned}$$

Collarics of Cauchy's Integral Formula

Derivatives: Suppose $f(z)$ is analytic on and inside a closed, positively oriented γ .

Let a be a point inside γ and let f be analytic in D .



$$\text{Then } f(a) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{z-a} dz$$

Regard $f(a)$ as a function of a .
Then

$$f'(a) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{(z-a)^2} dz \quad \left. \vphantom{\frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{(z-a)^2} dz} \right\} \begin{array}{l} \text{Differentiating} \\ \text{Order} \\ \text{the Integral Sign} \end{array}$$

Similarly,

$$f''(a) = \frac{2}{2\pi i} \oint_{\gamma} \frac{f(z)}{(z-a)^3} dz$$

$$\text{In General, } f^{(n)}(a) = \frac{n!}{2\pi i} \oint_{\gamma} \frac{f(z)}{(z-a)^{n+1}} dz$$

AMAZING! ^(analytic) $f(z)$ once differentiable $\Rightarrow f(z)$ infinitely differentiable

Example 1: Let $f(z) = \cos z$

Evaluate $\oint_{\gamma} \frac{f(z)}{(z-i)^3} dz$ at $z=i$

$$f^{(n)}(i) = \frac{n!}{2\pi i} \oint \frac{f(z)}{(z-i)^3} dz \quad \left. \vphantom{\frac{n!}{2\pi i}} \right\} \begin{matrix} 3 = n+1 \\ n=2 \end{matrix}$$

$$\frac{2\pi i}{2!} f''(i) = \oint \frac{f(z)}{(z-i)^3} dz$$

$$\pi i \left[\frac{d^2}{dz^2} [\cos(z)] \right]_{z=i} = -i\pi \cos(i)$$

Bounds on Derivatives: Cauchy's Estimate.

Suppose $f(z)$ analytic ON AND INSIDE circle of radius R about a .

Suppose $M = \max_{z \in C_R} |f(z)|$

Then

$$|f^{(n)}(a)| \leq \frac{n! M}{R^n}$$

Proof:
$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_{C_R} \frac{f(z)}{(z-a)^{n+1}} dz$$

$$\text{ML Estimate} \Rightarrow |f^{(n)}(a)| \leq \underbrace{\max_{z \in C_R} \left| \frac{f(z)}{(z-a)^{n+1}} \right|}_{M} \underbrace{\frac{n!}{2\pi} \cdot 2\pi R}_L$$

$$= \max_{z \in C_R} |f(z)| \frac{n!}{R^{n+1}} R \quad \uparrow \text{Note: } |z-a| = R$$

$$= M \frac{n!}{R^n}$$

$$\text{So, } |f^{(n)}(a)| \leq \frac{M n!}{R^n}$$

Applications of this estimate are seen in Liouville's Theorem. Suppose $f(z)$ is entire and is bounded by some constant: $|f(z)| \leq M \forall z$. Then $f(z)$ has to be constant.

Proof: $|f'(a)| \leq \frac{M}{R} \forall R$ (Since M is independent of R by boundedness)

$$A \leq R \rightarrow \infty, |f'(a)| \leq 0$$

$$\Rightarrow f(a) = C \text{ (constant)}$$

Fundamental Theorem of Algebra

Every non-constant polynomial $P(z)$ has a root in \mathbb{C} .

$$\text{Let } P(z) = C_0 + C_1 z + \dots + C_n z^n$$

Then $P(z) = 0$ for some $z \in \mathbb{C}$

Proof: Suppose $P(z) \neq 0 \quad \forall z$.

$$\text{Consider } f(z) = \frac{1}{P(z)}$$

$f(z)$ is now analytic since $P(z) \neq 0$ anywhere,
i.e.

$f(z)$ is entire, and also bounded

$$\text{As } z \rightarrow \infty \quad f(z) = \frac{1}{P(z)} \approx \frac{1}{C_n z^n}$$

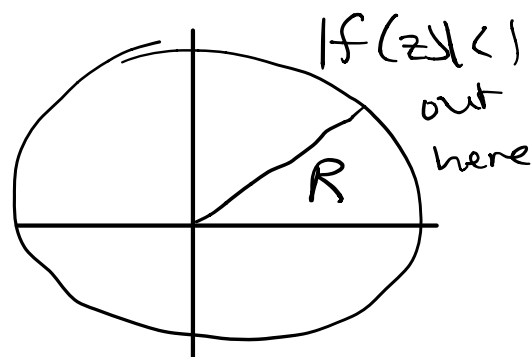
So outside some big circle $|z| = R$

we have

$$|z| = R$$

we have

$$|f(z)| < 1$$



[a continuous function on a compact set achieves its maximum and minimum value]

↗
from real analysis.

So inside circle $|f(z)|$ is bounded by M .

Then $|f(z)| \leq \max(M, 1)$. So f is bounded and entire which implies $f(z)$

So then $P(z)$ must be a constant by Liouville's Theorem!