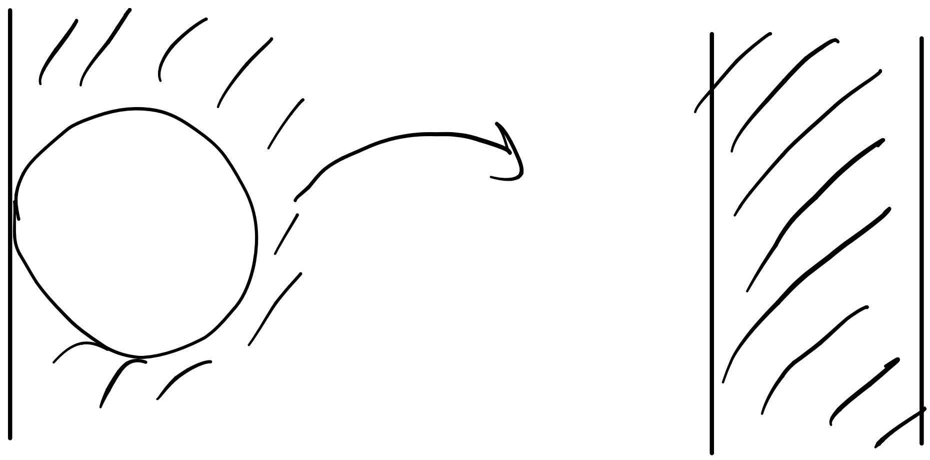
The "inversion" map $w = 1/z$

Useful for problems involving lines and circles as parts of boundaries.

From last time



$\{$ Circles and lines $\} \longrightarrow \{$ circles and lines $\}$

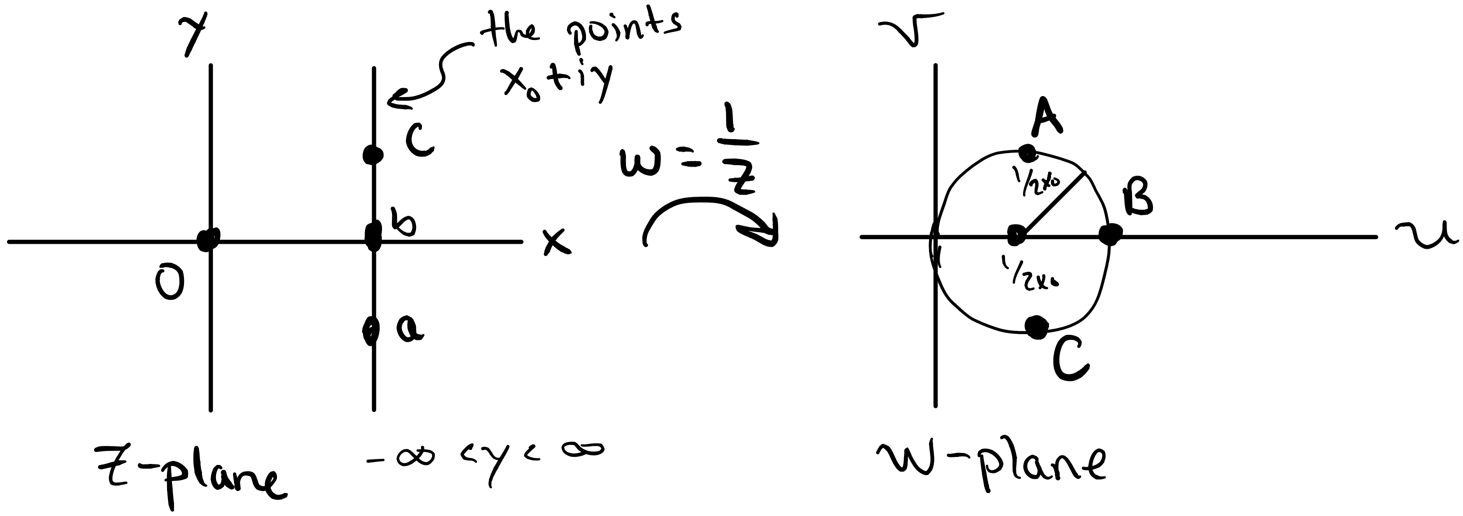
Circles that do contain 0 $\longrightarrow$ lines that do not contain 0

Circles that do NOT contain 0 $\longrightarrow$ circles that do not contain 0

lines that do contain 0 $\longrightarrow$ lines that contain 0

lines that do NOT contain 0 $\longrightarrow$ circles that do contain 0

First: Consider effect of $1/z$ on vertical lines



$$w = \frac{1}{x_0+iy} = \frac{x_0-iy}{x_0^2+y^2}$$

$$u(x,y) = \frac{x_0}{x_0^2+y^2}, \qquad v(x,y) = \frac{-y}{x_0^2+y^2}$$

show this is a circle

$$u^2+v^2 = \frac{(x_0^2+y^2)}{(x_0^2+y^2)^2} = \frac{1}{x_0^2+y^2}$$

Note that: $$\frac{u}{x_0} = \frac{1}{x_0^2+y^2}$$

$$u^2+v^2 = \frac{u}{x_0} \xrightarrow{\text{Rearrange}} v^2+\left(u^2-\frac{u}{x_0}\right) = 0$$

complete the square

$$v^2+\left(u-\frac{1}{2x_0}\right)^2 = \left(\frac{1}{2x_0}\right)^2$$

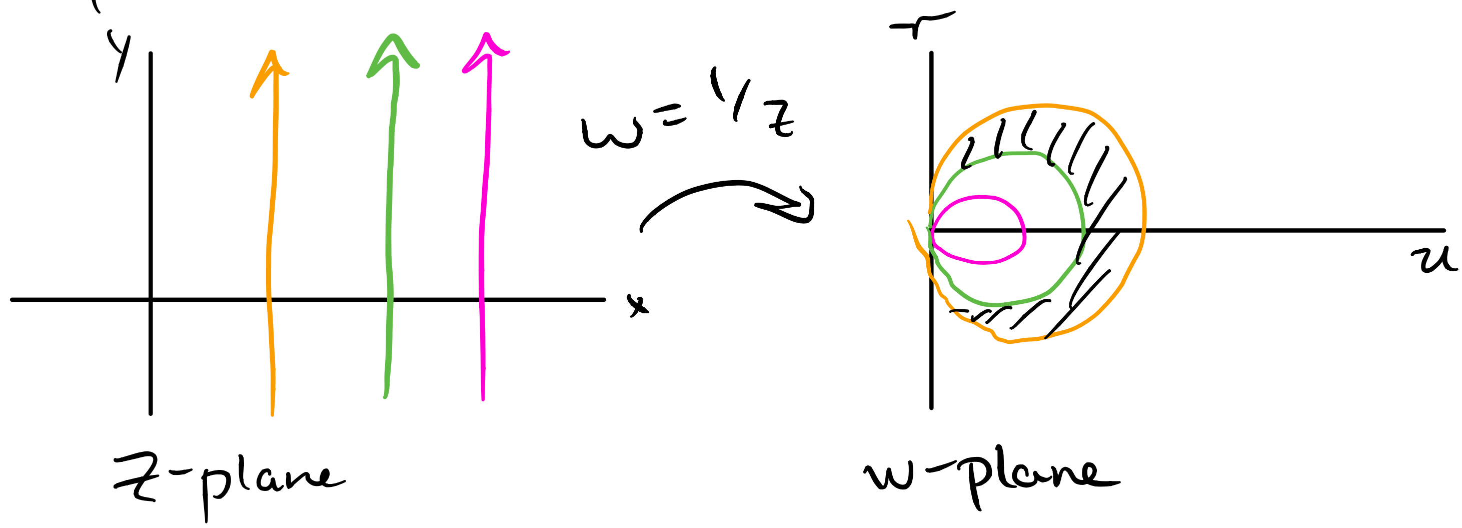
This is a circle of radius

$$\frac{1}{2x_0}$$

at center

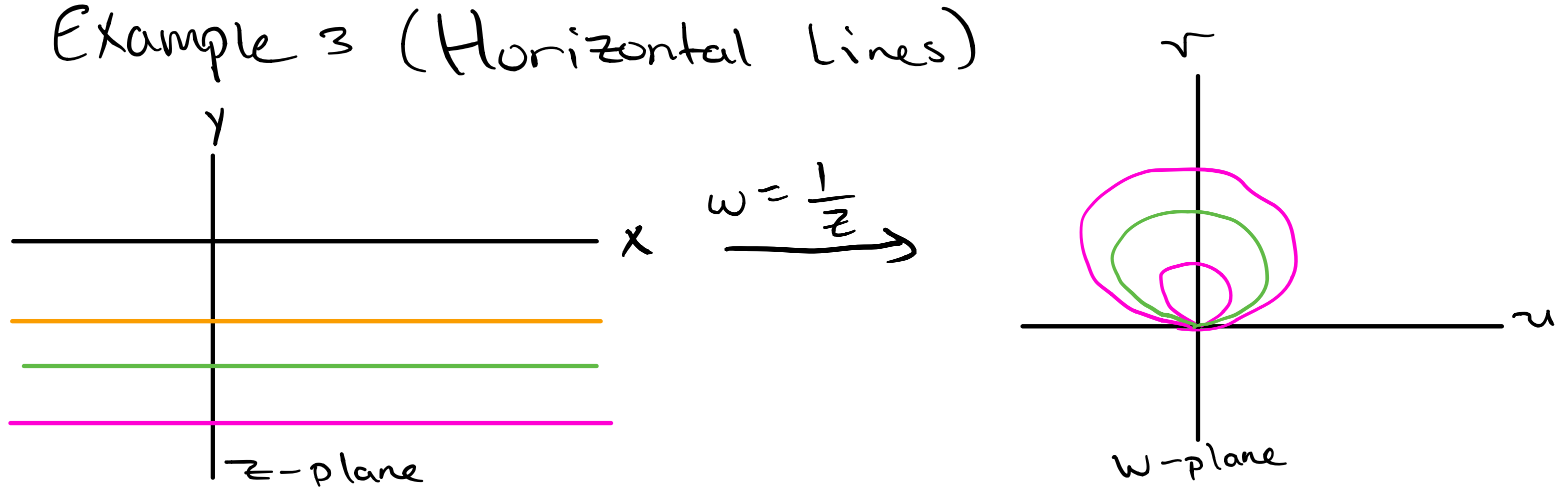
$$u_0 = \frac{1}{2x_0}, \quad v_0 = 0$$

Example 2

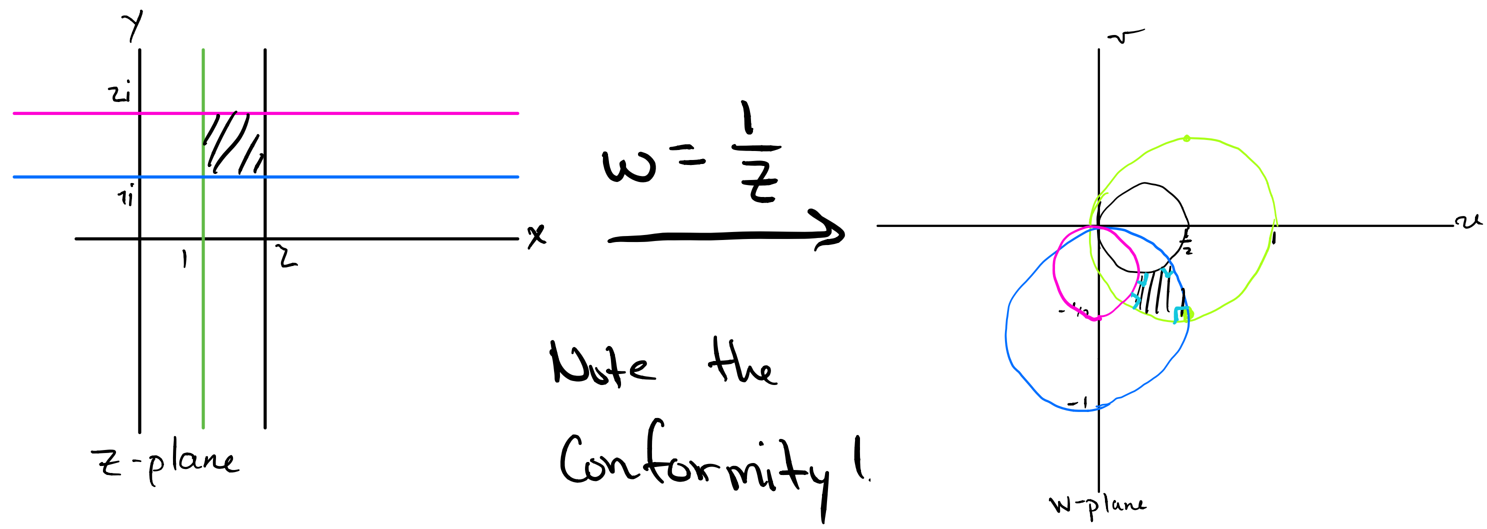


So we can now solve $\nabla^2\phi = 0$ in a crescent (see w-plane above)

Example 3 (Horizontal Lines)



Example 4 (image of a square):



Note the Conformity!

From Handout

Top image = Image of circles at $(x,y) = (1,0)$, mapped into w-plane via $w = 1/z$.

circles

$$(x-1)^2 + y^2 = R^2$$

$$x = \frac{u}{u^2+v^2}$$

$$y = \frac{-v}{u^2+v^2}$$

$$x+iy = z = \frac{1}{w} = \frac{1}{u+iv} = \frac{u-iv}{u^2+v^2}$$

$$x^2 + 1 - 2x + y^2 = R^2$$

$$x^2 + y^2 = \frac{1}{u^2+v^2}$$

$$\frac{1}{u^2+v^2} - 2\left(\frac{u}{u^2+v^2}\right) + 1 = R^2$$

$$\left(u + \frac{1}{R^2-1}\right)^2 + v^2 = \left(\frac{R}{R^2-1}\right)^2$$

Circle, radius $\frac{R}{R^2-1}$

center $u_0 = \frac{1}{1-R^2}$

$v_0 = 0$