Jordan Canonical Form

<u>Goal</u> (working over $\mathbb{C}$)

① $T \in \mathcal{L}(V)$, $\dim V = n < \infty$

find a basis β such that $\underset{\beta}{[T]} \overset{B}{}$ is "awesome"

② Start with matrix $A_{n\times n}$. Find a similar matrix B to A which is "awesome".

Definition: If A is a square matrix, and

$$A = \begin{bmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_m \end{bmatrix}$$

where A_i is a square matrix, we say A is in block diagonal form and we write

$$A = A_1 \oplus A_2 \oplus \cdots \oplus A_m$$

Remarks

(1)

$$A = \left[\begin{array}{cc|c|ccc} 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 3 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{array}\right]$$

(2) If each A_i is 1×1 then A is diagonal.

(3) If

$$A = \begin{array}{c} \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \end{array} \begin{array}{l} m \\ n-m \end{array}$$

Let $T_A = L_A : V \longrightarrow V$, $V = \mathbb{F}^n$

$\beta_1 = (\vec{e}_1, \dots, \vec{e}_m)$ $\quad W_1 = \text{span}(\beta_1)$, $\dim W_1 = m$

$\beta_2 = (\vec{e}_{m+1}, \dots, \vec{e}_n)$ $\quad W_2 = \text{span}(\beta_2)$, $\dim W_2 = n-m$

Know $V = W_1 \oplus W_2$, W_1, W_2 T-invariant.

$$T|_{W_1} : W_1 \longrightarrow W_1$$
$$v \mapsto Av$$

$$[T|_{W_1}] = A_1$$

Similarly for W_2.

(4) If also $A_2 = B_2 \oplus B_3$

then $A = A_1 \oplus B_2 \oplus B_3$

i.e.

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} = \left[\begin{array}{c|cc} A_1 & & \\ \hline & B_2 & \\ & & B_3 \end{array}\right]$$

Basic Method ("Divide and Conquer")

(1) $\dim V = n$

(2) $T: V = W_1 \oplus W_2$

(3) $T: V \to V$

(4) W_1, W_2 T-invariant

We will find bases β_1 of W_1,

β_2 of W_2

$\beta = (\beta_1, \beta_2)$ of V

s.t.

$$[T]_\beta = A_1 \oplus A_2 = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$$

← this holds for ANY bases β_1, β_2 of W_1, W_2

→ we try to choose "nice" ones

Divide & Conquer

Let $T \in \mathcal{L}(V)$, $\dim V = n < \infty$, then

ⓐ $\ker T \subseteq \ker T^2 \subseteq \ker T^3 \subseteq \cdots$

ⓑ if $\ker(T^m) = \ker(T^{m+1})$

then $\ker(T^m) = \ker(T^{m+1}) = \cdots = \ker(T^r) = \cdots \quad r \geq m$

ⓒ $\ker T^n = \ker T^{n+1} = \cdots$

Proof: ⓐ $T\vec{v} = \vec{0}$
$\vec{T}(T\vec{v}) = 0$ $\rightarrow$ $\ker(T^i) \subseteq \ker(T^{i+1})$

ⓑ Let $k \in \mathbb{Z}_+$.

$\ker T^{m+k} = \ker T^{m+k+1}$ ← Want to prove

Know $\ker T^{m+k} \subseteq \ker T^{m+k+1}$ by ⓐ

Suppose $v \in \ker T^{m+k+1}$

Then $T^{m+1}(T^k\vec{v}) = T^{m+1+k}\vec{v} = \vec{0}$

Hence $T^k v \in \ker(T^{m+1}) = \ker(T^m)$.

Thus $T^{m+k}\vec{v} = T^m(T^k\vec{v}) = \vec{0}$

$\Rightarrow v \in \ker(T^{m+k})$

So $\ker T^{m+k+1} \subseteq \ker T^{m+k}$

and

$$\ker T^{m+k} = \ker T^{m+k+1}$$

Ⓒ Suppose $\ker T^n \neq \ker T^{n+1}$

$$\{0\} = \ker T^0 \subset \ker T \subset \ker T^2 \subset \cdots \subset \ker T^n \subset \ker T^{n+1}$$

$$\dim \ker T \geq 1$$

$$\dim \ker T^2 \geq 2$$

$$\vdots$$

$$\dim \ker T^{n+1} \geq n+1$$ $\underline{!\,C}$ $(n+1 > \dim V)$

Proposition: Let $T \in \mathcal{L}(V)$, $\dim V = n$.
Then
$$V = \ker(T^n) \oplus \operatorname{im}(T^n)$$

Proof: Suffices to show $\ker(T^n) \cap \operatorname{im}(T^n) = \{0\}$

Suppose $v \in \ker(T^n) \cap \operatorname{im}(T^n)$
Then
$$T^n v = 0$$
and $\exists\, w \in V$ s.t. $v = T^n u$

$$T^n v = T^{2n} u = 0$$

$\therefore$ $u \in \ker T^{2n} = \ker T^n \Rightarrow v = 0$

Definition

If λ = eigenvalue of T

$$E_\lambda(T) = \ker(T - \lambda I)$$

$$G_\lambda(T) = \ker((T-\lambda I)^n) \qquad n = \dim V$$

$G_\lambda(T)$ is called the generalized λ-eigenspace of T.

Note $E_\lambda(T) \subseteq G_\lambda(T)$

Theorem

If $\lambda_1, \dots, \lambda_m$ are the complex distinct eigenvalues of $T \in \mathcal{L}(V)$, then

(a) each $G_{\lambda_i}(T)$ is T-invariant

(b) $V = G_{\lambda_1}(T) \oplus \cdots \oplus G_{\lambda_n}(T)$

(c) If β_i is a basis of $G_{\lambda_i}(T)$, and

$$B_i = \left[T|_{G_{\lambda_i}(T)} \right]_{\beta_i}$$

then $[T]_\beta = B_1 \oplus \cdots \oplus B_n$

("Proof", OH)

$$V = G_{\lambda_1}(A) \oplus G_{\lambda_2}(A)$$

basis: $(v_1, \dots, v_a)$ $\quad$ $(w_1, \dots, w_b)$

basis of V: $(v_1, \dots, v_a, w_1, \dots, w_b)$

$$[T]_\beta = \begin{array}{c} \\ v_1 \\ \vdots \\ v_a \\ w_1 \\ \vdots \\ w_b \end{array} \begin{array}{c} \begin{array}{cc} v_1 \cdots v_a & w_1 \cdots w_b \end{array} \\ \left[\begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array}\right] \end{array}$$