Last Time

Division algorithm

$$p, s \in \mathbb{F}[x], \quad s \neq 0$$
$$\Rightarrow \exists q, r \in \mathbb{F}[x]$$
s.t. $\quad p = qs + r$
$\quad \deg r < \deg s$

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i.e. NOT identically 0

Proposition: Suppose $p(x) \in \mathbb{F}[x]$, $\lambda \in \mathbb{F}$ assume $p(x) \not\equiv 0$

then $p(\lambda) = 0 \Leftrightarrow p(x) = (x - \lambda) q(x)$

Proof $(\Leftarrow)$ ✓

$(\Rightarrow)$ by division algorithm

(a) $p(x) = (x - \lambda) q(x) + r(x)$

(b) $\deg r < \deg(x - \lambda) = 1$

i.e $r(x) = r \in \mathbb{F}$ is a constant

(c) plug in λ

$$p(\lambda) = 0 \cdot q(\lambda) + r(\lambda) = r \in \mathbb{F} \Rightarrow r = 0$$

$\therefore \; p(x) = (x - \lambda) q(x)$

Corollary: For any field $\mathbb{F}$, if $m = \deg p(x) \geq 0$ then

$$\# \text{zeros of } p(x) = \#\{\lambda \in \mathbb{F} \mid p(\lambda) = 0\} \leq m = \deg p(x)$$

Proof Use induction on $\deg p(x) = m$

if $m = 0$, # zeros $= 0 = m$

Suppose statement is true for $m-1$, show it for m

Consider $p(x)$ deg m

- if no roots, statement is true

let $\lambda \in \mathbb{F}$ be a root

- By prop. $p(x) = (x-\lambda) q(x)$, $q(x)$ has deg $m-1$

$\therefore$

roots of $p(x) = 1 +$ # roots of $q(x) \leq 1 + m - 1 \leq m$

$\therefore$

by induction the corollary holds

Fundamental Theorem of Algebra

Every non-constant polynomial with non-constant coefficients has a zero

Proof / Take Analysis

Corollary: If $p(x) \in \mathbb{C}[x]$, $\exists$ unique factorization $\quad m = \deg p(x)$

$$p(x) = a(x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_m)$$

where $a, \lambda_1, \dots, \lambda_m \in \mathbb{C}$

NEXT

Applying a polynomial to a matrix A $(n \times n)$ over $\mathbb{F}$ or a linear operator $T: V \to V$ $(T \in \mathcal{L}(V))$, V a v.s. over $\mathbb{F}$,

Definition: If $p(x) \in \mathbb{F}[x]$, $p(x) = a_0 + a_1 x + \cdots + a_m x^m$, $a_i \in \mathbb{F}$

then define

$$p(A) = a_0 I_{n \times n} + a_1 A + a_2 A^2 + \cdots + a_m A^m$$

often $id_V = I$

$$p(T) = a_0 id_V + a_1 T + a_2 T^2 + \cdots + a_m T^m$$

Proposition: if $f(x) = p(x) q(x)$, $p, q \in \mathbb{F}[x]$

then $f(T) = p(T) q(T) = q(T) f(T)$

Proof

Suppose $p(x) = \sum_{j=0}^{m} a_j x^j$

$q(x) = \sum_{k=0}^{n} b_k x^k$

$$p(x) q(x) = \sum_{j=0}^{m} a_j x^j \sum_{k=0}^{n} b_k x^k = \sum_{j=0}^{m} \sum_{k=0}^{n} a_j b_k x^{(j+k)}$$

$$p(T) q(T) = \left(\sum_{j=0}^{m} a_j T^j \right) \sum_{k=0}^{n} b_k T^k = \sum_{j=0}^{m} \sum_{k=0}^{n} \left(a_j b_k T^j T^k \right)$$

$f(T) = p(T) q(T)$ $\rightarrow$ other side works to

Example

Consider

$$x''(t) + 3x'(t) + 2x(t) = 0$$

Let

$$V = \{ x(t) \in C^\infty(\mathbb{R}) \mid x'' + 3x' + 2x = 0 \}$$

DE class $\rightsquigarrow$ V is a vector space of dimension 2

Let $D: C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$

$$x(t) \mapsto x'(t)$$

then

$$V = \ker(D^2 + 3D + 2I)$$

and if $p(x) = x^2 + 3x + 2$ Note: $p(x) = (x+2)(x+1)$

then $V = \ker(p(D))$

$$V = \ker((D+2I)(D+I))$$

$$= \ker((D+I)(D+2I))$$

Note: V contains $\ker(D+I)$

i.e if $x' + x = 0$ then $x \in V$

$$x(t) = Ae^{-t}, \quad A \in \mathbb{R}$$

i.e $\operatorname{span}(e^{-t}) \subseteq V$

Similarly, for $\ker(D+2I)$

$$\operatorname{span}(e^{-2t}) \subseteq V$$

$\therefore$ $\operatorname{Span}(e^{-t}, e^{-2t}) \subseteq V$

but these are LI $\Rightarrow$ basis for V.