Exercises.

Solution to Question 1.

(a) It is clear that the projection

$$p : U \to \mathbb{R}^2$$
$$(\alpha_1, \alpha_2, \dots) \mapsto (\alpha_1, \alpha_2)$$

is a surjective linear map.
In fact, $(\alpha_1, \alpha_2, \dots) \in U$ is determined by $(\alpha_1, \alpha_2) \in \mathbb{R}^2$. This means this projection p is also injective. Therefore, $U \simeq \mathbb{R}^2$ and $\dim U = 2$.

(b) Let $\mathbf{e}_i \in \mathbb{R}^\infty$ be the sequence of numbers that the i-th entry is 1 and all other entries are 0.
Let $V := \operatorname{span}\{\mathbf{e}_i \ : \ i \geq 3\}$. We claim $U \oplus V = \mathbb{R}^\infty$:
First of all, we can see $U \cap V = \{0\}$, because

$$V = \{(0, 0, \alpha_3, \alpha_4, \dots)\}.$$

If $\mathbf{a} \in U \cap V$, then the first two entries of $\mathbf{a}$ are 0, hence all entries of it are 0.
Next, any element $\mathbf{b} \in \mathbb{R}^\infty$ can be written as $\mathbf{b} = \mathbf{u} + \mathbf{v}$, with $u \in U$ and $v \in V$. To do this, we simply let $\mathbf{u} = p(\mathbf{b})$ and $\mathbf{v} = \mathbf{b} - \mathbf{u}$, where p is the projection map defined in part (a).

(c) Let $W := \{\mathbf{e}_i : i \text{ is odd}\}$ and $X := \{\mathbf{e}_i : i \text{ is even}\}$. Then $W \oplus X = \mathbb{R}^\infty$ and both $\mathbb{R}^\infty/X$ and $\mathbb{R}^\infty/W$ are infinite dimensional.

Solution to Question 2.

(a) Because $V = W \oplus U_1$, so any $u \in U_2$ can be uniquely written as $u = w + u'$ with $w \in W$ and $u' \in U_1$. Therefore, we may define a map $p : U_2 \to U_1$ by $p(u) = u'$. Similarly, we may define a map $q : U_1 \to U_2$ by the same way. We claim that $p \circ q = \mathrm{id}_{U_1}$. This is because $w = u - u'$ is unique. Now, by HW5, we know $p \circ q$ is an isomorphism if and only if both p and q are isomorphisms.

(b) But U_1 is not necessarily equal to U_2. Example:
Let $V = \mathbb{R}^2$ and $W = \operatorname{span}\{(1, 0)\}$. Then

$$U_1 = \operatorname{span}\{(0, 1)\}$$

and

$$U_2 = \operatorname{span}\{(1, 1)\}$$

are both complements of W in V.

Solution to Question 3. We may check that

- $O(\mathbb{R}) \cap E(\mathbb{R}) = \{0\}$: If $g \in O(\mathbb{R}) \cap E(\mathbb{R})$, then $g(x) = g(-x) = -g(-x)$. So $2g(-x) = 0$, which means $g(x) = 0$.
- $O(\mathbb{R}) + E(\mathbb{R}) = C^\infty(\mathbb{R})$: For any $h \in C^\infty(\mathbb{R})$, let

  $$u(x) := \frac{1}{2}(f(x) + f(-x))$$

  and

  $$v(x) := \frac{1}{2}(f(x) - f(-x)).$$

  Then we obtain $f = u + v$ with $u \in E(\mathbb{R})$ and $v \in O(\mathbb{R})$.

Solution to Question 4. Consider the surjective linear map

$$\begin{aligned} p : U \times V &\to (U/X) \times (V/Y) \\ (u, v) &\mapsto (\bar{u}, \bar{v}). \end{aligned}$$

Because $(\bar{u}, \bar{v}) = (0, 0)$ if and only if $u \in X$ and $v \in Y$, so the kernel of p is $X \times Y$. Hence

$$(U \times V)/(X \times Y) = (U \times V)/\ker p \simeq (U/X) \times (V/Y).$$

Solution to Question 5. We claim that

$$C^\infty(\mathbb{R})/W \simeq \mathbb{R}[x]_{\leq n}.$$

Reason: Consider the linear map

$$\begin{aligned} \phi : C^\infty(\mathbb{R}) &\to \mathbb{R}[x]_{\leq n} \\ f &\mapsto \sum_{i=0}^{n} \frac{d^n f}{dx^n}(0)x^n. \end{aligned}$$

Clearly, ϕ is surjective. The kernel of ϕ is just W.
So there is an isomorphism $\psi : C^\infty(\mathbb{R})/W \to \mathbb{R}[x]_{\leq n}$ induced by ϕ. Then $\psi^{-1}(1), \psi^{-1}(x), \dots, \psi^{-1}(x^n)$ is a basis for $C^\infty(\mathbb{R})/W$.

Solution to Question 6.

(a) Check that

$$\phi_j(x^j) = 1,$$

and that if $i \neq j$,

$$\phi_j(x^i) = 0.$$

(b) In fact,

$$(1, x-3, (x-3)^2, \ldots, (x-3)^m) = (1, x, x^2, \ldots, x^m) \begin{bmatrix} 1 & -3 & (-3)^2 & \ldots & (-3)^m \\ 0 & 1 & 2(-3) & \ldots & m(-3)^{m-1} \\ 0 & 0 & 1 & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{bmatrix}.$$

The matrix on the right is an upper triangle matrix with 1s on the diagonal. So this matrix is invertible. Hence $\mathcal{B} = (1, x-3, (x-3)^2, \ldots, (x-3)^m)$ is a basis for V.

(c) The dual basis $\mathcal{B}^* = (\psi_0, \psi_1, \ldots, \psi_m)$, where

$$\psi_j(p(x)) = \frac{p^{(j)}(3)}{j!}.$$

We may check that

$$\psi_j((x-3)^i) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$