

Thm: Let  $T: V \rightarrow W$  be a linear map.

$$\text{Then } \dim(V) = \dim \underset{\substack{\uparrow \\ \text{Null space}(T)}}{\text{Ker}(T)} + \dim \underset{\substack{\uparrow \\ \text{Range}(T)}}{\text{Im}(T)}$$

Rks.

### Prose Summary of Ideas

① Take bases  $B$  of kernel  $T$  and  $B'$  of  $\text{Im}(T)$ .

But  $B' \in W$ , NOT  $V$ , so take inverse images under  $T$  to get  $B'' \in V$ . Hope  $B \cup B''$  is a basis of  $V$ .

② If  $\dim V = \infty$  or  $\dim W = \infty$  or both?

Q: If  $\dim V = \infty$ , then is one of  $\dim \text{Ker}(T)$ ,  $\dim \text{Im}(T)$   $\infty$  and other  $\emptyset$ ?

A: NOish. one must be  $\infty$ , other could be anything.

Example

$$P \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

$$P: \mathbb{R}^3 \rightarrow \mathbb{R}^3 \text{ (onto } xy\text{-plane)}$$

$\dim=1$   $\text{Ker } P: z\text{-axis}$

$\text{Ker}(I - P) = xy\text{-plane } \dim = 2$

$\dim=2$   $\text{Image } P: xy\text{-plane}$

$\text{Im}(I - P) = z\text{-plane } \dim = 1$

## Conjecture

$$\text{Ker}(P) = \text{Im}(I - P)$$

$$\text{Ker}(I - P) = \text{Im}(P)$$

when  $P$  is a projection i.e.  $P^2 = P$

Proof: Let  $\vec{v} \in \text{Ker}(P)$ .  $P\vec{v} = \vec{0}$

$$\text{Then } (I - P)\vec{v} = \vec{v} - P\vec{v} = \vec{v} \Rightarrow \vec{v} \in \text{Image}(I - P)$$

$$\text{So } \text{Ker}(P) \subseteq \text{Image}(I - P)$$

$$\text{Let } \vec{v} \in \text{Im}(I - P), \text{ thus } \vec{v} = (I - P)\vec{w}$$

$$\text{and } P\vec{v} = P(I - P)\vec{w}$$

$$\vec{0} = P\vec{w} - P^2\vec{w}$$

Use hypothesis  
that  $P = P^2$

$$\longrightarrow = P\vec{w} - P\vec{w} = \vec{0}$$

$$\text{So } \text{Image}(I - P) \subseteq \text{Ker}(P)$$

Thus

$$\text{Ker}(P) = \text{Image}(I - P)$$

Definitions:  $T: V \rightarrow W$  is injective if  $\vec{v} \neq \vec{w} \Rightarrow T\vec{v} \neq T\vec{w}$

surjective if  $\vec{w} \in W \Rightarrow \exists \vec{v} \in V \text{ s.t. } T\vec{v} = \vec{w}$

Q: How does injectivity relate to  $\text{Ker}(T)$ ?

A)  $T$  is injective exactly when  $\dim \text{Ker}(T)$  is 0.

Statement |  $T$  is surjective  $\Leftrightarrow \dim \text{Im}(T) = \dim W$

Fact:  $T: \mathbb{R}^{17} \rightarrow \mathbb{R}^{432}$  can NOT be surjective  
 $17 \neq \dim \ker T + 432$    
~~can NOT be negative~~

Similarly,  $T: \mathbb{R}^{432} \rightarrow \mathbb{R}^{17}$  can NOT be injective.

$$432 = \dim \ker T + \dim \operatorname{Im} T$$

$$\neq 0 + 17$$

Remark:  $\exists$  a continuous function

$$f: [0,1] \rightarrow [0,1]^2 \text{ that is onto}$$

We will use Matrices to describe Linear Transformations.

We can compute

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix} \rightarrow T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$$

$\begin{matrix} 2 \times 2 & 2 \times 1 & \rightarrow & 2 \times 1 \\ \text{matrix} & \text{matrix} & & \text{result} \end{matrix}$

Recall) Characteristic Polynomial of  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  is  $\det \begin{pmatrix} \lambda & -1 \\ -1 & \lambda \end{pmatrix}$   
 $= \lambda^2 - 1 = (\lambda - 1)(\lambda + 1) \Rightarrow$  eigenvalues  $1, -1$   
 $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

We have a basis of  $\mathbb{R}^2$  consisting of eigenvectors of  $T$

Example

$$\begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 8 & -1 \\ 1 & 1 \end{pmatrix}$$

↙ ij entry is row i of matrix 1, column j of matrix 2

This bizarre rule of matrix multiplication is composition of LTs

$$T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x + 2y \\ y \end{pmatrix}$$

$$S \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2u - v \\ u + v \end{pmatrix}$$

$$T \circ S \begin{pmatrix} u \\ v \end{pmatrix} = T \begin{pmatrix} 2u - v \\ u + v \end{pmatrix}$$

$$= \begin{pmatrix} 3(2u - v) + 2(u + v) \\ u + v \end{pmatrix}$$

CARE about  
this

$$= \begin{pmatrix} 8u - v \\ u + v \end{pmatrix}$$

Writing down a matrix  $T: V \rightarrow W$  w.r.t bases  $B_V$  &  $B_W$  will give varying matrices as  $B_V, B_W$  vary but NO new info!