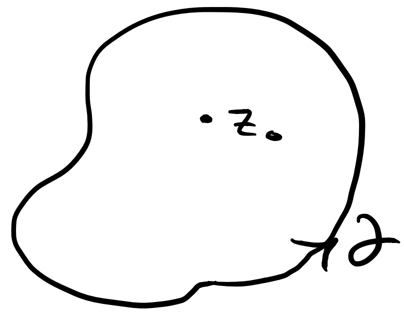
Residue Theory (Contour Integrals)

Suppose $f(z)$ has an isolated singularity at z_0.
Write

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n (z-z_0)^n$$

← Laurent Series about z_0

Have a contour γ around z_0
positively oriented



Consider

$$\oint_\gamma f(z)\, dz$$

can integrate Laurent series term by term

The point :

$$\oint_\gamma f(z)\, dz = \sum_{n=-\infty}^{+\infty} a_n \oint_\gamma (z-z_0)^n\, dz$$

$$\oint_\gamma (z-z_0)^n\, dz = \begin{cases} 0 & , n \neq -1 \\ 2\pi i & , n = -1 \end{cases}$$

And the a_{-1} coefficient is called the residue of z at z_0.

$$a_{-1} = \operatorname{Res}(f(z); z_0)$$

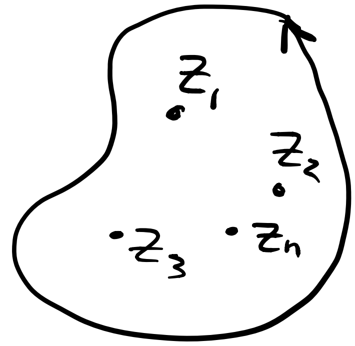
← notation

So calculating the contour boils down to finding the residue.

Suppose we had more than one singularity; say, $z_1, \dots, z_n$ (finite amount)

Then

$$\oint_\gamma f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f(z); z_k)$$



prove by integrating sub contours made by adding ridges

How to Calculate Residues

1) Laurent Series expansion.

$$f(z) = z^3 e^{6/z} = z^3 \underbrace{\left[1 + \frac{6}{z} + \frac{1}{2!}\left(\frac{6}{z}\right)^2 + \frac{1}{3!}\left(\frac{6}{z}\right)^3 + \dots\right]}_{\text{Maclaurin Series for } e^{6/z}}$$

$$= \dots \quad \frac{6^4}{4!} \frac{1}{z} \quad \dots$$

don't care b/c they integrate to zero

The residue term

2) If a simple pole at z_0,

$$\operatorname{Res}(f(z); z_0) = \lim_{z \to z_0} (z - z_0) f(z)$$

Proof:

$$f(z) = \cdots a_{-1} \frac{f(z)}{z - z_0} + a_0 + a_1(z - z_0) \cdots$$

All that's left is the residue!

Example:

$$\text{Res}\left(\frac{e^z}{\sin z}; \pi\right)$$

$\frac{1}{\sin z}$ has a simple pole at $z = \pi$

$$\text{Res} = \lim_{z \to \pi} \frac{(z - \pi) e^z}{\sin z} \overset{\text{L'hospitals!}}{=} \lim_{z \to \pi} \frac{(z-\pi)e^z + e^z}{\cos z}$$

$$= \frac{e^\pi}{-1} = -e^\pi$$

$a_{-1} = -e^\pi$ The residue

3) If a pole of order m, then (slightly more complicated)

$$\text{Res}(f(z); z_0) = \lim_{z \to z_0} \frac{1}{(m-1)!} \left(\frac{d}{dz}\right)^{m-1} (z - z_0)^m f(z)$$

Proof: use Laurent series - Page 310

Using Residues to find Trigonometric Integrals on $[0, 2\pi]$

Example

$$I = \text{Integral} = \int_0^{2\pi} \frac{d\theta}{a + \cos\theta}, \quad \text{where } a > 1$$

technically a function of a

allows integral to be finite

Let $z = e^{i\theta}$ $\longrightarrow$ $dz = e^{i\theta} i \, d\theta$

$$d\theta = -i \frac{dz}{e^{i\theta}} = -i \frac{dz}{z}$$

Then

$$\cos\theta = \frac{z + 1/z}{2}$$

This converts I to an integral around unit circle

$$I = \oint_{|z|=1} \frac{-\frac{i}{z} dz}{a + \frac{z + 1/z}{2}} = \oint_{|z|=1} \frac{-2i}{z^2 + 2az + 1} dz$$

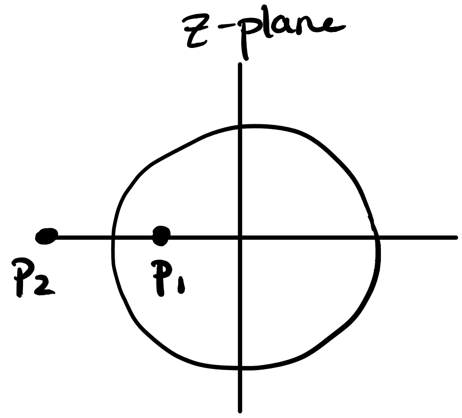
$$= -2i \oint_{|z|=1} \frac{dz}{z^2 + 2az + 1}$$

any isolated singularities inside unit circle?

quadratic has two roots $\Rightarrow$ poles

$$-\frac{2a}{2} \pm \frac{\sqrt{4a^2 - 4}}{2} = -a \pm \sqrt{a^2 - 1}$$

Poles: $P_1 = -a + \sqrt{a^2 - 1}$ $\quad$ $P_2 = -a - \sqrt{a^2 - 1}$



So only residue from p_1 contributes to I.

$$I = 2\pi i \sum_i \text{Res}\,(\text{singularities inside } \gamma)$$

$$= 2\pi i \ \text{Res}\left[\frac{2}{i}\,\frac{1}{z^2+2az+1}\ ;\ p_1\right]$$

$$z^2+2az+1 = (z-p_1)(z-p_2) \qquad = 2\pi i \lim_{z\to p_1} \frac{2}{i}\,\cancel{(z-p_1)}\,\frac{1}{\cancel{(z-p_1)}(z-p_2)}$$

$$p_1 - p_2 = 2\sqrt{a^2-1}$$

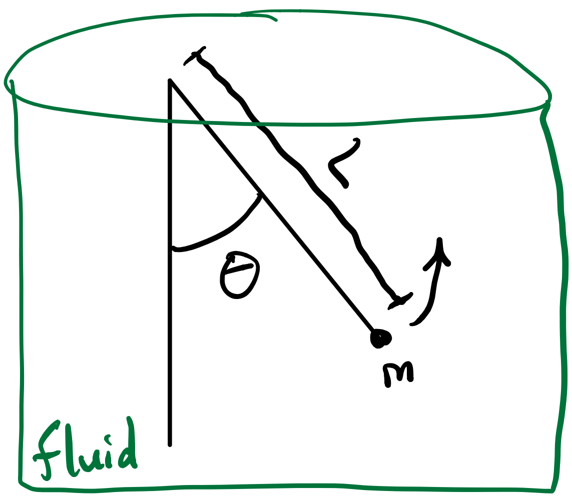
$$= \frac{4\pi}{p_1-p_2} = \frac{2\pi}{\sqrt{a^2-1}}$$

Wowza!

$$I(a) = \int_0^{2\pi} \frac{d\theta}{a+\cos\theta} = \frac{2\pi}{\sqrt{a^2-1}}$$ } This is a neat result

Physics Application

Period of whirling, overdamped pendulum driven by a constant torque.



torque Γ (constant)

the torque will cause this to "whirl" in the sense that it will rotate all around.

overdamped

$$\cancelto{0}{\frac{d}{dt}\left(mL + L\dot{\theta}\right)} + b\theta + mgL\sin(\theta) = \Gamma$$

damping so large, inertia term doesn't apply

$$T = \int dt = \int_0^{2\pi} \frac{d\theta}{a - \sin\theta}, \quad \frac{d\theta}{dt} + \sin\theta = a$$

$$dt = \left[\frac{a - \sin\theta}{d\theta}\right]^{-1}$$