# Recall

Let $V$ = vector space over $\mathbb{F}$.

$U \subseteq V$ is a subset.

$U$ is called a **subspace** if

① $0_V \in U$

② $(U, +, \overset{\text{scalar mult.}}{\cdot}, 0_V)$ is a vector space over $\mathbb{F}$

where $+$, scalar mult. are induced from $V$

Proposition: Let $U \subseteq V$ be a subset.

$U$ is a subspace iff

① $0_V \in U$

② $U$ is closed under $+$

if $x, y \in U$; $x + y \in U$

③ $U$ is closed under $\cdot$

$c \in \mathbb{F}$, $x \in U$, $cx \in U$

Examples

① $V = \mathbb{F}^3$, $U = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \middle| \begin{array}{l} a+b+c=0 \\ a,b,c \in \mathbb{F} \end{array} \right\}$

is $U$ a subspace of $\mathbb{F}^3$?

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0_V \in U$$

$$\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} + \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} a_1+a_2 \\ b_1+b_2 \\ c_1+c_2 \end{pmatrix} \in U$$

similar for scalar multiplication.

(2) $V = \mathbb{F}^3$, $U = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \middle| a+b+c=1 \right\}$

NO! $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \notin U$

(3) $\mathbb{F}^2 \cancel{\subseteq} \mathbb{F}^3$ NOT EVEN A SUBSET

(4) Let $C(\mathbb{R}, \mathbb{R}) \subseteq \text{Fun}(\mathbb{R}, \mathbb{R})$

where $C$ is $\{ f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous} \}$

is $C$ a subspace? YES

Similarly,

$$C^\infty(\mathbb{R}, \mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{R} \mid f \text{ has derivatives of all order} \}$$

(5) Set of all polynomials over $\mathbb{F}$.

A polynomial over $\mathbb{F}$, in variable $x$, is

$$f = a_0 + a_1 x + a_2 x^2 + \cdots + a_d x^d, \quad x_i \in \mathbb{F} \ \forall i$$

Consider two polynomials the same if

$$f = 2 + x + 3x^3 + 5x^{10}$$
$$g = 2 + x + 0 \cdot x^2 + 3x^3 + 0x^4 + 5x^{10} + \cdots 0x^{100}$$

Let $\mathbb{F}[x]$ denote the set of all polynomials over $\mathbb{F}$
(also say $P(\mathbb{F})$, $Pol(\mathbb{F})$)

Make $\mathbb{F}[x]$ into a vector space over $\mathbb{F}$.

① $0_{\mathbb{F}[x]} = 0 + 0x + \cdots = 0$

② $f = a_0 + a_1 x + \cdots + a_n x^n$

$g = b_0 + b_1 x + \cdots + b_n x^n$

Define $f+g = (a_0+b_0) + (a_1+b_1)x + \cdots + (a_n+b_n)x^n$

$cf = ca_0 + ca_1 x + \cdots + ca_n x^n$

Proposition: $\mathbb{F}[x]$ is a vector space over $\mathbb{F}$

Def: if $f = 0$, say $\deg f = -\infty$

if $f = a_0 + a_1 x + \cdots + a_n x^n$ say $\deg f = n$

Let $\mathbb{F}[x]_{\leq d} = \{ f \in \mathbb{F}[x] \mid \deg f \leq d \} \subseteq \mathbb{F}[x]$

Is $\mathbb{F}[x]_{\leq d}$ a subspace? Yes!

# Sums of Subspaces

Def: If $U_1, U_2 \subseteq V$ are supspaces ($V$ = vector space)

Let

$$U_1 + U_2 = \left\{ v + w \;\middle|\; \begin{matrix} v \in U_1 \\ w \in U_2 \end{matrix} \right\}$$

Similarly, define

$$U_1 + U_2 + \cdots + U_n = \{ v_1 + \cdots + v_n \mid v_i \in U_i \ \forall i \} \subseteq V$$

(each $U_i \subseteq V$ is a subspace)

Example $\mathbb{R}^3$

Let $U_1$ = line through origin

$U_2$ = Another line through origin

$$U_1 + U_2 = \begin{cases} U_1 & ; \quad U_1 = U_2 \\ \text{Plane spanned by } U_1, U_2 & ; \quad U_1 \neq U_2 \end{cases}$$

**Theorem:** If $U_1, U_2$ are subspaces of $V$, then

(a) $U_1 + U_2$ is a subspace

(b) $U_1 \cap U_2$ is a subspace

More generally, $U_1 + \cdots + U_n$ is a subspace, $U_1 \cap \cdots \cap U_n$ is a subspace if $\forall i$ $U_i$ is a subspace

Proof (a)

Show $U_1 + U_2$ is a subspace of $V$

Given $U_1, U_2$ are subspaces,

Show

(1) $0_V \in U_1 + U_2$ Since $0_V \in U_1$, $0_V \in U_2$

$$0_V + 0_V = 0_V$$

(2) $U_1 + U_2$ closed under $(+)$

Note: $U_1 \subseteq U_1 + U_2$
$U_2 \subseteq U_1 + U_2$

if $v_1 + v_2 \in U_1 + U_2$ $(v_1 \in U_1, v_2 \in U_2)$

and $w_1 + w_2 \in U_1 + U_2$ $(w_1 \in U_1, w_2 \in U_2)$

then $v_1 + v_2 + w_1 + w_2 \in U_1 + U_2$

Since $(v_1 + w_1) + (v_2 + w_2) \in U_1 + U_2$

(3) Similar for scalar multiplication

## Span

Let $V$ be a vector space over $\mathbb{F}$.

__Definition:__ A linear combination of vectors in $V$ is a vector of the form

$$a_1 v_1 + \cdots + a_m v_m \text{ for } a_1, \ldots, a_m \in \mathbb{F}$$

__Definition__:

$$\text{span}(v_1, \ldots, v_m) = \{a_1 v_1 + \cdots + a_m v_m \mid a_i \in \mathbb{F}\ \forall i\} \subseteq V$$

$$\text{span}(\ ) = \{0_V\} \leftarrow \text{by convention}$$

Question: If $S \subseteq V$ is a subset, define $\text{span}(S) \subseteq V$.

True: $\text{span}(v_1, \ldots, v_m)$ is a subspace if it is the smallest containing $v_1, \ldots, v_m$