Recall

functions of >1 rv - derived pdfs, pmfs, etc

Recipe: ① $F_Z(z)$

② $\frac{d}{dz}(F_Z(z)) = f_Z(z)$

discrete version also holds

X, Y independent; $Z = X+Y \Rightarrow f_Z(z) =$ convolution of $f_X(x)$ w/ $f_Y(y)$

NEXT Family of Topics: covariance, uncorrelatedness, conditional expectation revisited (difficult)

Have $\Omega, \mathbb{P}$

$X: \Omega \to \mathbb{R}$

$Y: \Omega \to \mathbb{R}$

Covariance

Given any X, Y rvs (discrete, continuous, whatever) defined on **same** probability space, the **covariance** of X and Y defined as

$$\text{Cov}(X,Y) = \mathbb{E}\big((X-\mathbb{E}(X))(Y-\mathbb{E}(Y))\big)$$

Note:

$$\text{Cov}(X,X) = \mathbb{E}((X-\mathbb{E}(X))^2) = \text{Var}(X)$$

$$\text{Cov}(X,Y) = \mathbb{E}(XY) - \mathbb{E}(\mathbb{E}(X)Y) - \mathbb{E}(X\,\mathbb{E}(Y)) + \mathbb{E}(X)\mathbb{E}(Y)$$

$$= \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$$

Terminology: When $\text{Cov}(X,Y)=0$; say X and Y are <u>uncorrelated</u>.

Fact: If X,Y independent, then X,Y uncorrelated

If statement; NOT iff

Independent $\Rightarrow \mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y) \Rightarrow \text{Cov}(X,Y) = 0$

Converse is **NOT** true

Example - Uncorrelated but NOT Independent

Consider (X,Y) that take on values

$$(3,0),\ (0,3),\ (-3,0),\ (0,-3)$$

w/ equal probabilities $1/4$.

X,Y uncorrelated $\Rightarrow \mathbb{E}(XY) = \mathbb{E}(0) = 0$ and $\mathbb{E}(X) = \mathbb{E}(Y) = 0$

BUT, NOT independent

also saw this when X,Y independent

Fact: If X,Y uncorrelated, then $\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y)$

To see this, observe

$$\text{Var}(X+Y) = \mathbb{E}\left((X-\mathbb{E}(X)+Y-\mathbb{E}(Y))^2\right)$$

$$= \text{Var}(X) + \underbrace{2\mathbb{E}\left((X-\mathbb{E}(X))(Y-\mathbb{E}(Y))\right)}_{2\,\text{Cov}(X,Y)=0} + \text{Var}(Y)$$

Terminology: Given X,Y, the correlation coefficient of X and Y is

$$\rho = \frac{\text{Cov}(X,Y)}{\sigma_X \sigma_Y}$$

Turns out, $|\rho| \leq 1$ - turns out this is a version of the Cauchy-Schwarz inequality.

Note: $\rho = 0 \overset{\text{iff}}{\Longleftrightarrow} X,Y$ uncorrelated and

$\rho = \pm 1 \Longleftrightarrow X$ and Y are "aligned" in sense $X = \alpha Y$ for some $\alpha \neq 0$

Next,

Conditional Expectation Revisited

Have already seen: for any two X, Y on same probability space $\mathbb{E}(X|Y=y)$.

Get this as follows:

Discrete

① Find $p_{X|Y}(x|y)$

② $\mathbb{E}(X|Y=y) = \sum_x x\, p_{X|Y}(x|y)$

OR

Continuous

① Find $f_{X|Y}(x|y)$

② $\mathbb{E}(X|Y=y) = \int_{-\infty}^{+\infty} x f_{X|Y}(x|y)\, dx$

i.e someone tells you $Y=y$; can then figure out $\mathbb{E}(X|Y=y)$

Consider "automating" the computation – build a box that takes y, spits out $\mathbb{E}(X|Y=y)$

$$y \to \boxed{\quad ? \quad} \longrightarrow \mathbb{E}(X|Y=y)$$

This implements a function $g: \mathbb{R} \to \mathbb{R}$

$$y \to \boxed{\qquad} \longrightarrow g(y) = \mathbb{E}(X|Y=y)$$

Finally, hook up Y at input – out comes $g(Y)$ – just a function of a random variable. $g(Y)$ is also a random variable

We call $g(Y)$ "$\mathbb{E}(X|Y)$" i.e $g(Y) = \mathbb{E}(X|Y)$ here.

Terminology: $\mathbb{E}(X|Y)$ = conditional expectation of X given Y

Question: What is $\mathbb{E}(\mathbb{E}(X|Y))$?

Fact: Law of iterated expectations

$$\mathbb{E}(X) = \mathbb{E}(\mathbb{E}(X|Y))$$

Idea: $\mathbb{E}(X|Y) = g(Y)$ for some function Y.

Thus

$$\mathbb{E}(\mathbb{E}(X|Y)) = \mathbb{E}(g(Y))$$

← use expected value rule to get this

Discrete

$$\mathbb{E}(g(Y)) = \sum_{y \in Y} g(y) p_Y(y)$$

$$= \sum_{y \in Y} \mathbb{E}(X|Y=y) p_Y(y)$$

law of total expectation

$$= \mathbb{E}(X)$$

OR

Continuous

$$\mathbb{E}(g(Y)) = \int_{-\infty}^{+\infty} g(y) f_Y(y)\, dy$$

$$= \int_{-\infty}^{+\infty} \mathbb{E}(X|Y=y) f_Y(y)\, dy$$

$$= \mathbb{E}(X)$$

Example -

Y uniform on $[0,1]$; when $Y=y$, X is Bernoulli(y).

i.e $\quad \mathbb{P}(\{X=1\}) = y \;;\; \mathbb{P}(\{X=0\}) = 1-y$

Get $\mathbb{E}(X|Y=y)$ as follows

- First get $\mathbb{E}(X|Y=y) = y = 1 \times y + 0 \times (1-y) = xy$
- Plug in Y for y. $\mathbb{E}(X|Y) = Y$

Example -

Sum of n independent Bernoulli(y) rvs

Y is uniform $[0,1]$, X binomial(n, y)

Given $Y = y$, $\mathbb{E}(X|Y=y) = ny$; plug in Y for y

$$\mathbb{E}(X|Y) = nY$$

Useful Fact: $\mathbb{E}(h(Y)X|Y) = h(Y)\mathbb{E}(X|Y)$

Idea: $\forall\ y$,

$$\mathbb{E}(h(Y)X|Y=y) = \mathbb{E}(h(Y)X|Y=y) = h(y)\underbrace{\mathbb{E}(X|Y=y)}_{g(y)}$$

Plug in Y for y,

$$h(Y)\underbrace{\mathbb{E}(X|Y)}_{g(Y)}$$

Can view $\mathbb{E}(X|Y)$ as an estimate of X given Y.

Sometimes we write

$$\hat{X} = \mathbb{E}(X|Y)$$

and

$$\tilde{X} = X - \hat{X} = \text{"estimation error"}$$

Estimator-worthy properties of $\mathbb{E}(X|Y) = \hat{X}$:

- $\mathbb{E}(\hat{X}) = \mathbb{E}(X)$ by law of iterated expectations

Finish next time