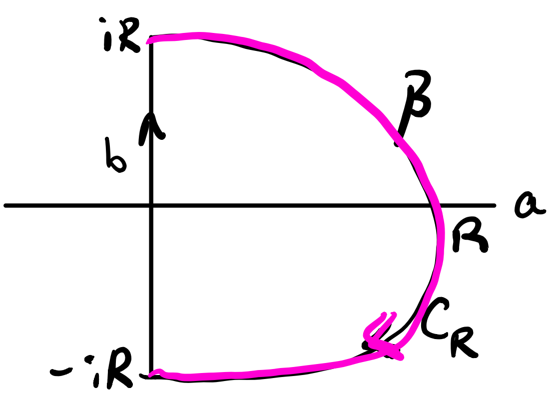
We want to find $s \in \mathbb{C}$ which solves $f(s)=0$, where $f(s)$ is given by

Too Ambitious.

$$f(s) = s + ke^{-\tau s}, \quad \tau > 0 \text{ "delay"}, \quad k > 0 \text{ "gain"}$$

Instead, settle for finding conditions on k, τ s.t. $\mathrm{Re}(s) < 0$ for all zeros of $f(s)$.

stability condition.



s-plane, $s = a + ib$

For stability we want no zeros of $f(s)$ inside β (as $R \to \infty$)

By the argument principle, this holds if $W(\gamma; 0) = 0$ where $\gamma = f(\beta)$

What does γ look like?

First ask what does $f(C_R)$ look like?

On C_R,

$$s = Re^{i\theta}, \quad \pi/2 \le \theta \le -\pi/2$$

$$\Rightarrow f(s) = s + ke^{-\tau s} = Re^{i\theta} + ke^{-\tau Re^{i\theta}}$$

$$= Re^{i\theta} + ke^{-\tau R(\cos\theta + i\sin\theta)}$$

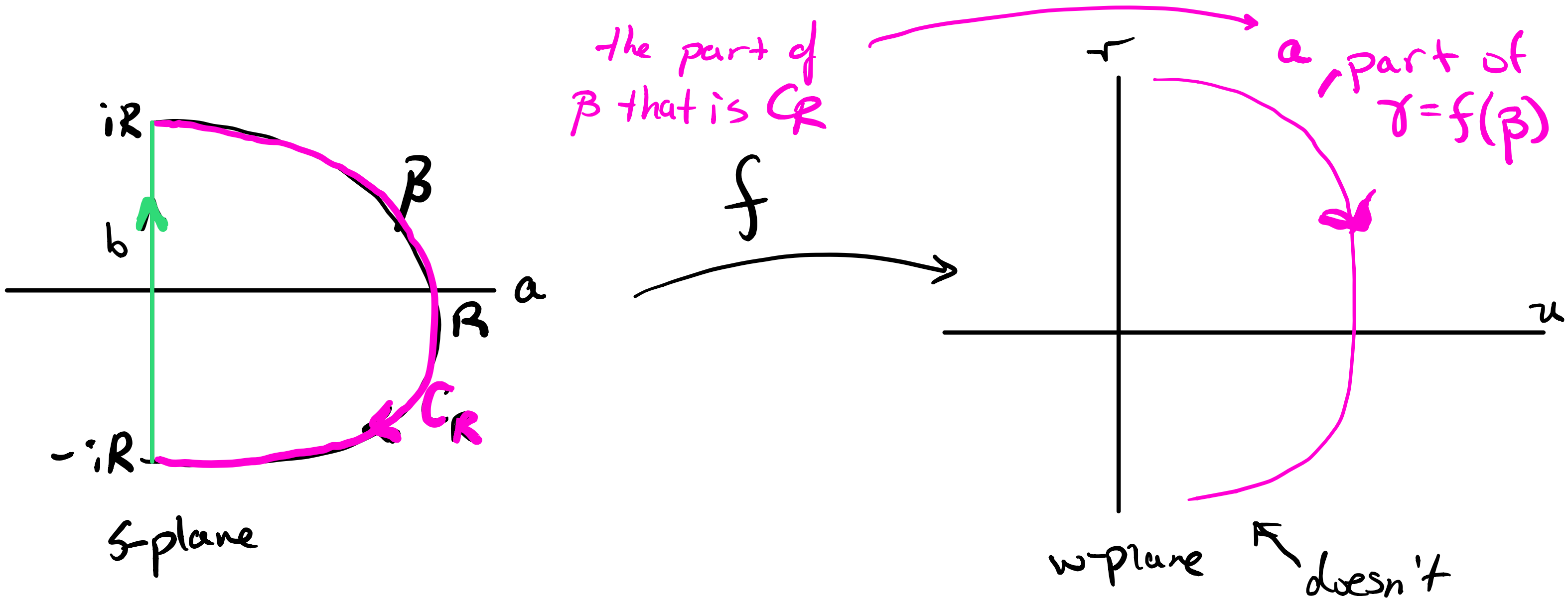
$$= Re^{i\theta} + ke^{-\tau R\cos\theta} e^{-i\tau R\sin(\theta)}$$

We're thinking of arbitrarily large R.

$$Re^{i\theta} + \underbrace{ke^{-\tau R\cos\theta}}_{\to 0 \text{ as } R\to\infty} e^{-i\tau R\sin(\theta)}$$

"oscillates like crazy w/ negligible amplitude"

So the image of the semicircle is another big semicircle.



$f(s) \simeq Re^{i\theta}, \quad R >> 1$

Need to map imaginary axis as well.

* Look at $f(ib)$ as b runs from $-R$ to R *

$$f(ib) = ib + ke^{-ib\tau}$$
$$= \underbrace{k\cos(\tau b)}_{u(b)} + i\underbrace{(b - k\sin(\tau b))}_{v(b)}$$

w plane is u+iv!

What does the curve $(u(b), v(b))$ look like?

"Think of b (the parameter in the parametric equations for a curve) as a kind of 'time'!"

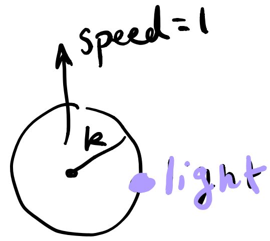
$$u(b) = k\cos(\tau b)$$
$$v(b) = -k\sin(\tau b) + b$$

} circular motion + vertical translation

circular motion: frequency τ, radius k

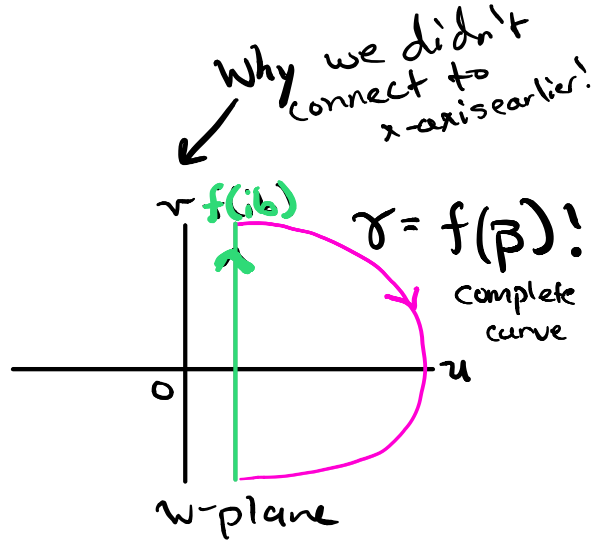
vertical translation: upward "speed" of 1

$f(ib)$ = path traced by moving lightbulb.



If $\tau = 0$ (warm up)

$$u(b) = k$$
$$v(b) = b$$



$W(\gamma;0) = 0$ here, so no zeros of s in righ half plane

$\Rightarrow$ control system stable (as expected, since no delay)

If τ large,

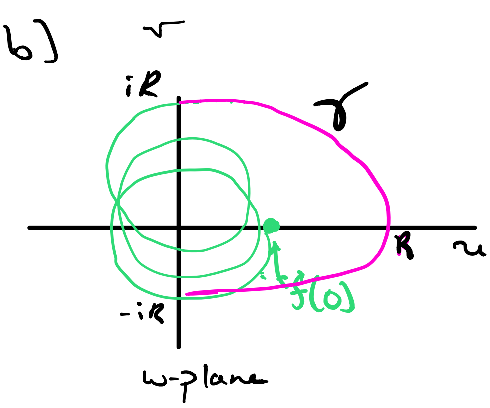
$$u(b) = k\cos(\tau b), \quad -R \leq b \leq R \;;\; R \to \infty$$
$$v(b) = -k\sin(\tau b) + b$$

when $b = 0$, $f(ib) = k = u(b) + iv(b)$

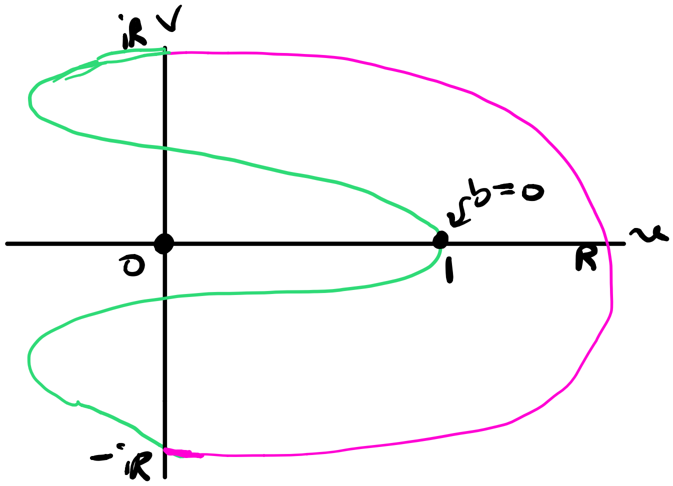
as b increases the center of our circle changes.

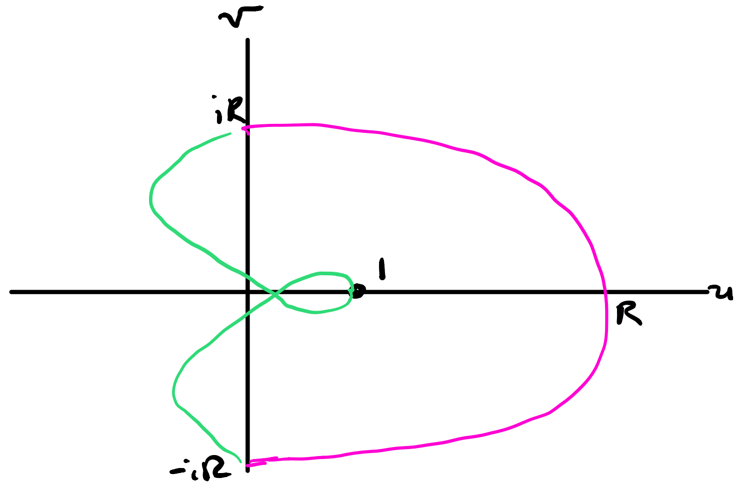
i.e we encircle the origin many times

$\Rightarrow$ wickidly unstable



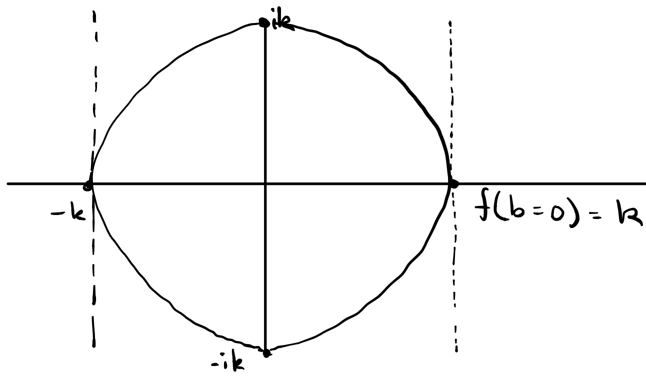
For small $\tau > 0$, (0.1) $(k=1)$ Slightly larger $(\tau = 1.3)$





What's the critical point?

The critical case is when the lowest point due to circular motion is the same as the vertical distance traveled by translation.



$u(b) = k\cos(\tau b)$

$v(b) = -k\sin(\tau b) + b$

$\tau b = \pi/2 \Rightarrow$ lowest pt on circle

$v(b) = b - k$

$b = k = \dfrac{\pi}{2\tau}$

So $\tau_c = \dfrac{\pi}{2k}$