

4.4

①

$$D = \mathbb{C} - \{0, 2i, 4\}$$

Contours continuously deformable to Γ in D .

- a) Deformable ✓
- b) NOT deformable ← passes through 0 ✗
- c) Deformable ✓
- d) NOT deformable ✗

② Which are simply connected

- a) Simply Connected ✓
- b) NOT Simply Connected ✗
- c) Simply Connected ✓
- d) Simply Connected ✓
- e) NOT Simply Connected ✗
- f) Simply Connected ✓

(16) Show that if f is of the form

$$f(z) = \frac{A_k}{z^k} + \frac{A_{k-1}}{z^{k-1}} + \cdots + \frac{A_1}{z} + g(z) \quad (k \geq 1)$$

where g is analytic inside & on the circle $|z|=1$, then

$$\oint_{|z|=1} f(z) dz = 2\pi i A_1,$$

$$\oint_{|z|=1} \frac{A_k}{z^k} + \frac{A_{k-1}}{z^{k-1}} + \cdots + \frac{A_1}{z} + g(z) dz \quad (k \geq 1)$$

$$= \oint_{|z|=1} \frac{A_k}{z^k} dz + \oint_{|z|=1} \frac{A_{k-1}}{z^{k-1}} dz + \cdots + \oint_{|z|=1} \frac{A_1}{z} dz + \oint_{|z|=1} g(z) dz$$

0 by analyticity of $g(z)$

zero due to

result

$$\oint_{\gamma} (z-a)^n dz = 0 \text{ for } n \neq -1 ! \text{ Here } a=0$$

$$= A_1 \oint_{|z|=1} \frac{1}{z} dz = \boxed{A_1 2\pi i}$$

(18)

Let

$$I := \oint_{|z|=2} \frac{dz}{z^2(z-1)^3}$$

Given

Justification

Proof for $I=0$ (a) For every $R > 2$, $I = I(R)$ where

$$I(R) := \oint_{|z|=R} \frac{1}{z^2(z-1)^3} dz$$

If $R > 2$, we can "reverse" deform the contour, analyzing the function on a circle of radius R . (C_R)

Therefore,

$$I = I(R) := \oint_{|z|=R} \frac{1}{z^2(z-1)^3} dz$$

$$(b) |I(R)| \leq \frac{2\pi}{R(R-1)^3} \quad \text{for } R > 2$$

Using the ML estimate,

$$|I(R)| \leq \max_{z \in C_R} |I(z)| \cdot \text{Length}(C_R)$$

$$|I(R)| \leq \underbrace{\frac{1}{R(R-1)^3}}_{M} \cdot \underbrace{2\pi R}_{L}$$

$$\leq \frac{2\pi}{R^2(R-1)^3} \quad \underline{\text{For } R \geq 2}$$

(c) $\lim_{R \rightarrow +\infty} I(R) = 0$

$$\lim_{R \rightarrow +\infty} I(R) = \lim_{R \rightarrow +\infty} \oint_{|z|=R} \frac{1}{z^2(z-1)^3} dz$$

This limit is less than the ML estimate above.
 $\therefore \leq 0$

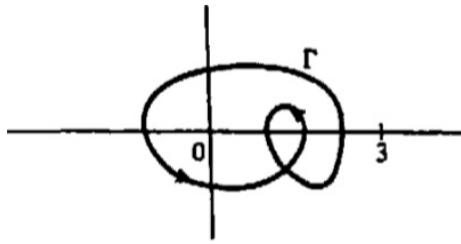
(d) $I=0$

Varying R doesn't change the result.

4.5

(7)

$$\int_{\Gamma} \frac{\cos(z)}{z^2(z-3)} dz$$



$f(z)$ is NOT analytic IN Γ , but is ON Γ .
 $z=0$ is an issue

$$\frac{\cos z}{z^2(z-3)} = \frac{\cos z / (z-3)}{z^2}$$

$$z_0 = 0, m = 2$$

$$\Rightarrow \int_{\Gamma} \frac{\cos(z)}{z^2(z-3)} dz = \frac{2\pi i f'(0)}{1!}$$

$$= \frac{2\pi i}{1} \left[\frac{(z-3) \sin(z) - \cos(z)(1)}{(z-3)^2} \right]$$

$$= 2\pi i \left[\frac{0 - 1}{(-3)^2} \right] = \boxed{\frac{-2\pi i}{9}}$$

⑥ If f is analytic in and on $|z - z_0| = r$.
Then

Show This This

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta \quad ①$$

Prove, more generally that

$$f^{(n)}(z_0) = \frac{n!}{2\pi r^n} \int_0^{2\pi} f(z_0 + re^{i\theta}) e^{-in\theta} d\theta \quad ②$$

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz$$

Let $z = z_0 + re^{i\theta}$ } circle of radius
 r centered at z_0
 $0 \leq \theta \leq 2\pi$

$$f(z_0) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{i\theta}) re^{i\theta}}{re^{i\theta}} ; d\theta$$

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta$$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{\gamma} \frac{f(z)}{(z-z_0)^{n+1}} dz , \quad n=1,2,3,\dots$$

$$z = z_0 + r e^{i\theta}, \quad 0 \leq \theta \leq 2\pi$$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + r e^{i\theta})}{(r e^{i\theta})^{n+1}} i r e^{i\theta} d\theta$$

$$= \frac{n!}{2\pi} \int_0^{2\pi} \frac{f(z_0 + r e^{i\theta})}{(r e^{i\theta})^n (r e^{i\theta})} r e^{i\theta} d\theta$$

$$\boxed{f^{(n)}(\theta) = \frac{n!}{2\pi r^n} \int_0^{2\pi} f(z_0 + r e^{i\theta}) e^{-in\theta} d\theta}$$

(9) f is analytic in and on $|z|=1$.

Prove that if

$$|f(z)| \leq M \quad \text{for } |z|=1$$

then

$$|f(0)| \leq M$$

and

$$|f'(0)| \leq M.$$

Estimate

$$|f^{(n)}(0)|$$

$$f(0) = \frac{1}{2\pi i} \int_{|z|=1} \frac{f(z)}{z} dz$$

$$|z|=1$$

$$|f(0)| \leq \left| \frac{1}{2\pi i} \int_{|z|=1} \frac{f(z)}{z} dz \right|$$

$$= \frac{1}{2\pi} \max_{z \in |z|=1} \frac{|f(z)|}{r} \cdot 2\pi r$$

$$= \max_{z \in |z|=1} |f(z)| = M$$

$$\Rightarrow |f(0)| \leq M$$

$$f'(0) = \frac{1}{2\pi i} \int_{|z|=1} \frac{f(z)}{z^2} dz$$

$$r=1!$$

$$|f'(0)| \leq \frac{1}{2\pi} \max_{z \in |z|=1} \frac{|f(z)|}{r^2} \cdot 2\pi r$$

$$= \max_{z \in |z|=1} |f(z)| = M$$

$$|f'(0)| \leq M$$

$$\text{Since } r=1 \quad |f^{(n)}(0)| \leq \max_{z \in |z|=1} |f(z)| \leq M$$

4.6

4) $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ is a polynomial
and $\max |p(z)| = M$ for $|z| = 1$, show
each a_k is bounded by M : $k = 0, \dots, n$

For $|z| = 1$,

$$\max |p(z)| = \max |a_0 + a_1 z + \cdots + a_n z^n| \leq M$$

$$\begin{aligned} \max |p(z)| &\leq \max |a_0| + \max |a_1||z| + \cdots + \max |a_n||z^n| \\ &\leq \max |a_0| + \max |a_1| + \cdots + \max |a_n| \\ &\leq M \end{aligned}$$

Therefore each a_k , $k = 0, 1, \dots, n$ is bounded
by M

10) Find all functions

f analytic in $D : |z| \in \mathbb{R}$ that satisfy
 $f(0) = i$ and $|f(z)| \leq 1$ for all z in D .

Theorem 23

If f is analytic in domain D and $|f(z)|$ achieves its maximum value at a point z_0 in D , then f is constant in D .

Since f is constant and $f(0) = i$

$f(z) = i$ by Theorem 23

$$17) \max_{|z| \leq 1} |(z-1)(z+1)|$$

Let $z = e^{it}$, $0 \leq t \leq 2\pi$

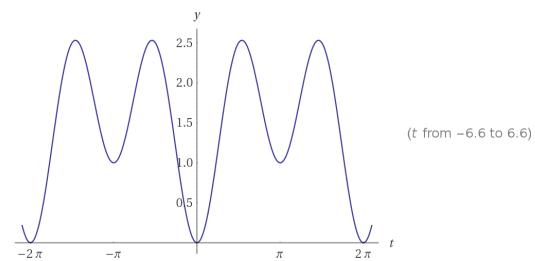
$$\left| z^2 - \frac{z}{2} - \frac{1}{2} \right|^2 = \left(e^{i2t} - \frac{e^{it}}{2} - \frac{1}{2} \right) \left(e^{-i2t} - \frac{e^{-it}}{2} - \frac{1}{2} \right)$$

$$= \cancel{\frac{1}{2} - \frac{e^{it}}{2} - \frac{e^{it}}{2}} - \cancel{\frac{e^{-it}}{2}} + \cancel{\frac{1}{4} + \frac{e^{it}}{4}} - \cancel{\frac{1}{2} e^{-it}} + \cancel{\frac{e^{-it}}{4}} + \cancel{\frac{1}{4}}$$

$$= \frac{3}{2} - \left[\frac{e^{it} + e^{-it}}{2} \right] - \left[\frac{e^{i2t} + e^{-i2t}}{2} \right] + \left[\frac{e^{it} + e^{-it}}{2} \right] \frac{1}{2}$$

$$= \frac{3}{2} - \cos(t) - \cos(2t) + \frac{\cos(t)}{2}$$

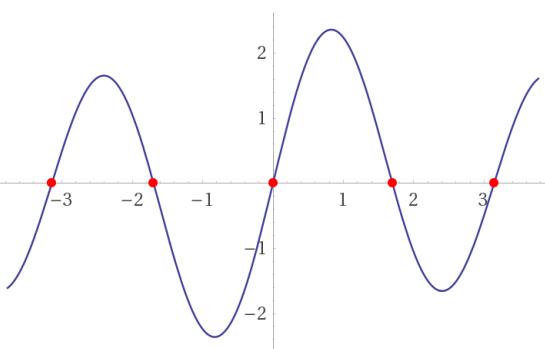
$$= \frac{3}{2} - \underline{\frac{\cos(t)}{2}} - \cos(2t) \quad \text{PLOT} \rightarrow$$



$$\frac{d}{dt} \left[\frac{3}{2} - \frac{\cos(t)}{2} - \cos(2t) \right]$$

$$= \frac{\sin(t)}{2} + 2 \sin(2t) = 0$$

Plot 



$$\Rightarrow t = \pi n \text{ MIN}$$

$$t = 2\pi n - 2\tan^{-1}\left(\frac{3}{\sqrt{7}}\right)$$

$$t = 2\pi n + 2\tan^{-1}\left(\frac{3}{\sqrt{7}}\right)$$

Plugging the solutions in yields a
max of 2.5

so $\boxed{\sqrt{2.5}}$ is the solution to
our problem.

20) Prove

5,1

$$\sum_{j=0}^{\infty} z^j = \sum_{j=0}^{\infty} \left(\frac{z}{z_0}\right)^j, \text{ where } z_0 = 1$$

does NOT converge uniformly to

$$\frac{1}{1-z} \text{ on open disk } |z| < 1; \Rightarrow |z| < z_0$$

Given $\epsilon > 0$ we have to show that the remainder after $n+1$ terms will be

less than ϵ for all z in the disk when n is large enough.

A note:

$\sum_{j=0}^{\infty} \left(\frac{z}{z_0}\right)^j$ is uniformly convergent in every closed disk $|z| \leq r$, if $r < |z_0|$

Proof:

$$\left| \frac{\left(\frac{z}{z_0}\right)^{n+1}}{1 - \frac{z}{z_0}} \right| \leq \frac{\left(\frac{r}{|z_0|}\right)^{n+1}}{1 - \frac{r}{|z_0|}}$$

This can be made arbitrarily small since $r < |z_0|$

For our problem, $z_0 = 1 \Rightarrow |z_0| = 1$

$|z| < 1$ (open disk).

$$\left| \frac{\left(\frac{z}{z_0}\right)^{n+1}}{1 - \frac{z}{z_0}} \right| \leq \frac{|z|}{1 - |z|}$$

Can NOT be made arbitrarily small and therefore is not less than ϵ

Therefore we do NOT converge uniformly

5.2

11b) first 3 nonzero terms in MacLaurin expansion of

$$\frac{e^z}{z-1}$$

$$f(0) + f'(0)z + \frac{f''(0)z^2}{2!} \quad \left. \begin{array}{l} \text{First 3 nonzero terms} \\ \text{in MacLaurin series} \\ \text{of } e^z/z-1 \end{array} \right\}$$

$$f(0) = \frac{1}{-1} = -1$$

$$f'(0) = \frac{(z-1)e^z - e^z}{(z-1)^2} = \frac{e^z(z-2)}{(z-1)^2} = \frac{(-2)}{(-1)^2} = -2$$

$$\begin{aligned} f''(0) &= \frac{(z-1)^2 [e^z + (z-2)e^z] - [e^z(z-2)2(z-1)]}{(z-1)^4} \\ &= \frac{(-1)^2 [1 + -2] - [4]}{(-1)^4} = -5 \end{aligned}$$

$$\Rightarrow \boxed{-1 - 2z - \frac{5}{2}z^2}$$

13) Find explicit formula for the analytic function $f(z)$ which has Maclaurin series expansion

$$\sum_{k=0}^{\infty} k^2 z^k$$

$$= 0 + z^k + 4z^2 + 9z^3 + 16z^4 + 25z^5 + \dots$$

First realize

$$\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k$$

Then

$$z \frac{d}{dz} \left(\frac{1}{1-z} \right) = \sum_{k=0}^{\infty} k z^{k-1} = \sum_{k=0}^{\infty} k z^k$$

And finally

$$z \frac{d}{dz} \left[z \frac{d}{dz} \left(\frac{1}{1-z} \right) \right] = \sum_{k=0}^{\infty} k^2 z^{k-1+1} = \sum_{k=0}^{\infty} k^2 z^k$$

$$\frac{d}{dz} \left[\frac{1}{1-z} \right] = + (1-z)^{-2}$$

$$\begin{aligned} \frac{d}{dz} [z(1-z)^{-2}] &= -z \cdot 2(1-z)^{-3} + (1-z)^{-2} \\ &= \frac{2z}{(1-z)^3} + \frac{1}{(1-z)^2} = \frac{2z+1-z}{(1-z)^3} = \frac{z+1}{(1-z)^3} \end{aligned}$$

$$\boxed{\Rightarrow f(z) = z \frac{(z+1)}{(1-z)^3}}$$