Math 4310 Name: ______________________________

Homework 3 Collaborators: ______________________

Due 9/25/19 ______________________________

Please print out these pages. I encourage you to work with your classmates on this homework. Please list your collaborators on this cover sheet. (Your grade will not be affected.) Even if you work in a group, you should write up your solutions yourself! You should include all computational details, and proofs should be carefully written with full details. As always, please write neatly and legibly.

Please follow the instructions for the "extended glossary" on separate paper (LaTeX it if you can!) Hand in your final draft, including full explanations and write your glossary in complete, mathematically and grammatically correct sentences. Your answers will be assessed for style and accuracy.

Please **staple** *this cover sheet, your exercise solutions, and your glossary together, in that order, and hand in your homework in class.*

GRADES

Exercises ______________________ / 50

Extended Glossary

Component	Correct?	Well-written?
Definition	/6	/6
Example	/4	/4
Non-example	/4	/4
Theorem	/5	/5
Proof	/6	/6
Total	/25	/25

Exercises.

1. Let f_1, f_2 and f_3 be vectors in the vector space $\mathcal{F}un(\mathbb{R}, \mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R}\}$ over the field $\mathbb{R}$.

   A. For three distinct real numbers x_1, x_2 and $x_3 \in \mathbb{R}$, define a 3×3 matrix with $\mathbb{R}$ entries by evaluating the function:

   $$[f_j(x_i)] = \begin{bmatrix} f_1(x_1) & f_2(x_1) & f_3(x_1) \\ f_1(x_2) & f_2(x_2) & f_3(x_2) \\ f_1(x_3) & f_2(x_3) & f_3(x_3) \end{bmatrix}.$$

   Prove that if the columns of the matrix $[f_j(x_i)]$ are linearly independent in $\mathbb{R}^3$, then the functions f_1, f_2 and f_3 are linearly independent in $\mathcal{F}un(\mathbb{R}, \mathbb{R})$.

   B. Show that the functions $f_1(x) = e^{-x}$, $f_2(x) = x$ and $f_3(x) = e^x$ are linearly independent in $\mathcal{F}un(\mathbb{R}, \mathbb{R})$.

   C. Show that the functions $f_1(x) = e^x$, $f_2(x) = \sin(x)$ and $f_3(x) = \cos(x)$ are linearly independent in $\mathcal{F}un(\mathbb{R}, \mathbb{R})$.

   Note: The test given in part A is sufficient to guarantee linear independence, but the functions f_1, f_2 and f_3 may still be linearly independent even when the columns of $[f_j(x_i)]$ are linearly dependent.

2. Prove: If $U \subset V$ is a subspace of a finite dimensional vector space V, then $\dim U \leq \dim V$. Also, if $U \neq V$ as well, then $\dim U < \dim V$.

3. Let V be the vector space $\mathbb{F}[x]_{\leq 4}$ of polynomials of degree at most 4 over the field $\mathbb{F}$. (You do not need to prove that this is a subspace, but you should be able to do so!)
   (a) Find a basis for the subspace $U := \{p(x) \in V \mid p(1) = 0, p(2) = 0\}$.
   (b) Extend this basis to a basis of V.
   (c) Find a subspace W, and a basis for it, such that $V = U \oplus W$.
4. Suppose that V is a vector space of dimension n. Show that:
   (a) If $(v_1, \ldots, v_n)$ spans V, then it is also a basis for V.
   (b) If $(v_1, \ldots, v_n)$ is linear indpendent, then it is also a basis for V.
5. Let V be a finite-dimensional vector space over a field $\mathbb{F}$, and let U, W, X be subspaces.
   (a) If $U \cap W = 0$, prove that $\dim(U + W) = \dim U + \dim W$.
   (b) In general, prove $\dim(U + W) = \dim U + \dim W - \dim(U \cap W)$.
   (c) If you're familiar with the "inclusion-exclusion principle", you might guess that part (b) generalizes to the equality

$$\dim(U+W+X) = \dim(U)+\dim(W)+\dim(X)-\dim(U\cap W)-\dim(U\cap X)-\dim(W\cap X)+\dim(U\cap W\cap X).$$

   Provide a counterexample to show this "equality" can be false!
   (Hint: You can take the field $\mathbb{F}$ to be $\mathbb{R}$ and the vector space V to be $\mathbb{R}^2$).

Extended Glossary. For any set X, a **relation** on X is a rule for deciding for any pair of elements $x, y \in X$ whether or not x stands in a given relation to y. For example, one relation on the set $X = \mathbb{R}$ is "**less than or equal to**". In that example, $5 \leq 10$, but $10 \not\leq 5$, so the order of the two elements does matter! For a general relation, we write $x \sim y$ to mean that x has the given relationship to y.

In your extended glossary this week, please give a definition of an **equivalence relation**. Then give an example of an equivalence relation, an example of a relation that is not an equivalence relation, and state and prove a theorem about equivalence relations. If you want more guidance about some interesting statements to try to prove, ask me, Prof. Ramakrishna, or our T.A. Beihui.

You may work in groups, but please write up your solutions **yourself**. If you do work together, your group should come up with at least two examples, two non-examples, and two theorems. Each one (example/non-example/theorem) should be included in some group member's extended glossary. Your solutions should be written formally, so that we could cut and paste them into a textbook.