5) $$I = \int_{-\infty}^{\infty} \frac{x \sin(3x)}{x^4+4}\, dx$$

There are simple poles at $z_k = \exp\left(\frac{\pi}{4} + \frac{\pi}{2}k\right)$, $k = 0, 1, 2, 3$

It can be noted that

$$\left| \frac{z \sin(3z)}{z^4+4} \right| = \left| \frac{z(e^{i3z} - e^{-i3z})}{2(z^4+4)} \right|$$

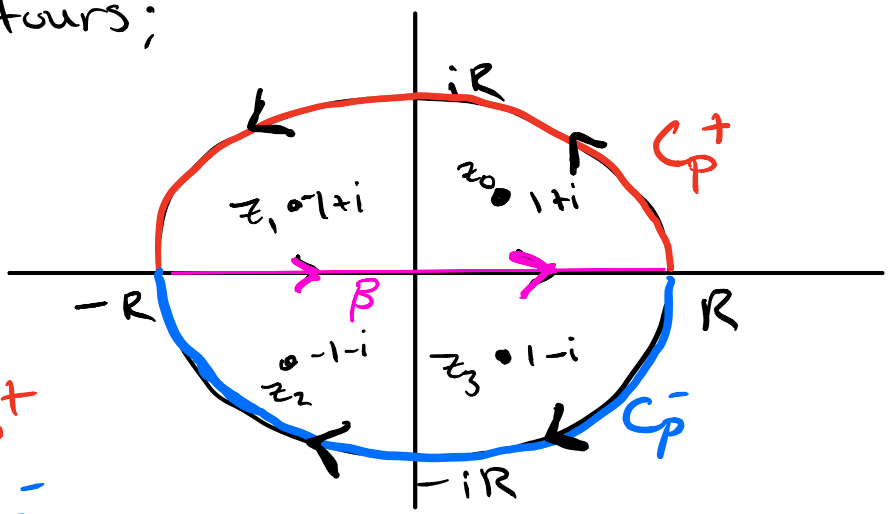
$$= \frac{\left(e^{-3R} + e^{3R}\right)|z|}{2\,|z^4+4|}$$

where $R \to \infty$. This means

- e^{i3z} is bounded in the upper half plane
- e^{-i3z} is bounded in the lower half plane

Thus, I shall use 2 contours;

1) C_p^+
2) C_p^-



where z_0, z_1 lie in C_p^+
and z_2, z_3 lie in C_p^-

where $\gamma_1 = \beta + C_p^+$
$\gamma_2 = \beta + C_p^-$

Let

$$I = \int_{-\infty}^{+\infty} \frac{x\left(e^{i3x} - e^{-i3x}\right)}{2(x^4+4)}\,dx = I_1 + I_2$$

upper half plane (I_1) lower half plane (I_2)

$$= \frac{1}{i}\int_{-\infty}^{+\infty} \frac{xe^{i3x}}{2(x^4+4)}\,dx - \frac{1}{i}\int_{-\infty}^{+\infty} \frac{xe^{-i3x}}{2(x^4+4)}\,dx$$

Now define

$$f_1(z) := \frac{ze^{i3z}}{i2(z^4+4)}$$

which has singularities at

$$z_0 = 1+i \quad , \quad z_1 = -1+i$$

Then note

$$|f_1(z)| = |f_1(x+iy)| = \frac{|e^{i3x}e^{-3y}|\,|z|}{2|z^4+4|}$$

In the upper half plane $y \geq 0$, therefore

$$|f_1(z)| \leq \frac{|z|}{2|z^4+4|}$$

Thus for $R > \sqrt{2}$, the integral over C_p^+ is bounded by

$$\left|\int_{C_p^+} f_1(z)\,dz\right| \leq \pi R \max_{z \in C_p^+} \left|\frac{z}{2|z^4+4|}\right|$$

as $R \to \infty$ essentially $\frac{1}{z^3}$

$$\leq 0$$

So, for $R > \sqrt{2}$

$$\int_{\gamma_1} f_1(z)dz = \lim_{R\to\infty} \int_{-R}^{R} f_1(x)\,dx + \cancel{\int_{C_p^+} f(z)\,dz} \to 0$$

$$2\pi i \sum \text{Res} = \lim_{R\to\infty} \int_{-R}^{R} \frac{x e^{i3x}}{i2(x^4+4)}\,dx = I_1$$

$$\text{Res}(f_1(z); z_1) = \lim_{z\to z_1} \cancel{(z-z_1)} \frac{z e^{i3z}}{2\cancel{(z-z_1)}(z-z_2)(z-z_3)(z-z_4)\,i}$$

$$\frac{z_1 e^{i3z_1}}{2(z_1-z_2)(z_1-z_4)(z_1-z_3)} \qquad e^{i3(1+i)} = e^{i3}e^{-3}$$

$$= \frac{e^{i3}\cancel{(1+i)}}{2e^3\,(2)(2i)\cancel{(2+2i)}2} = \frac{e^{i3}}{e^3 16 i^2}$$

$$\text{Res}(f_1(z); z_2) = \frac{z_2 e^{i3z_2}}{2(z_2-z_3)(z_2-z_1)(z_2-z_4)i}$$

$$= \frac{\cancel{(-1+i)}\,e^{-i3}}{e^3(-2)(-2i)\cancel{(-2+2i)}-2}$$

$$= \frac{-e^{-i3}}{e^3 16 i^2}$$

Result of I_1

$$2\pi i \sum \text{Res} = \frac{\cancel{2}\pi \cancel{i}}{4\cancel{8}\cancel{i}e^3}\left(\underbrace{\frac{e^{i3}-e^{-i3}}{2i}}_{\sin(3)}\right) = \boxed{\frac{\pi\sin(3)}{4e^3}}$$

Now for the I_2! (expect same result.)

Let

$$f_2(z) := -\frac{z}{2i} \frac{e^{-3iz}}{(z^4+4)}$$

$$|f_2(z)| = \frac{|e^{-3ix} e^{3y}|\,|z|}{2\,|z^4+4|}$$

In the lower half plane $y \leq 0$, therefore

$$|f_2(z)| \leq \frac{|z|}{2|z^4+4|}$$

So for $R > \sqrt{2}$

$$\left| \int_{C_-} f_2(z)\,dz \right| \to 0 \quad \text{as } R \to \infty$$

(essentially $1/z^3$)

So

γ_2 is negatively oriented

$$\int_{\gamma_2} f_2(z)\,dz = -\lim_{R\to\infty} \int_{-R}^{+R} \frac{-ze^{-i3z}}{2i(z^4+4)}\,dz + \cancel{\int_{C^-} \frac{-ze^{-i3z}}{2i(z^4+4)}}^{\,\to 0}$$

$$2\pi i \sum \text{Res} = -I_2$$

$$I_2 = -2\pi i \left(\text{Res}(f_2(z); z_2) + \text{Res}(f_2(z); z_3) \right)$$

$$I_2 = \frac{-2\pi i}{e^3 8i} \left[\frac{-e^{i3} + e^{-i3}}{2i} \right] = \frac{\pi \sin(3)}{4e^3}$$

So $I = I_1 + I_2 = \boxed{\dfrac{\pi \sin(3)}{2e^3}}$

12) Given

$$I_G = \int_0^{\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$

Integrate e^{iz^2} around

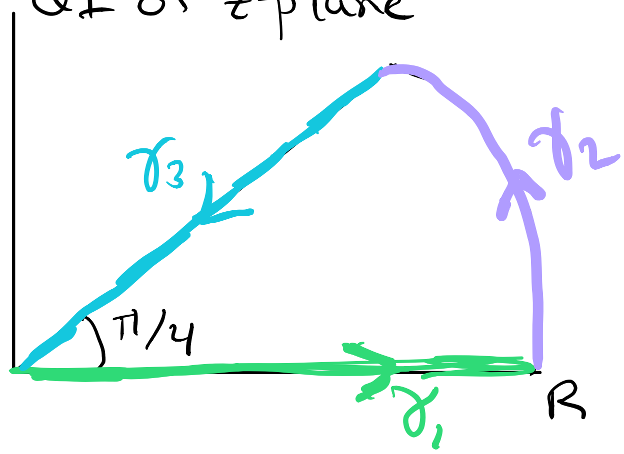
$$S_p : \{ z = re^{i\theta} : 0 \le \theta \le \pi/4 \,,\; 0 \le r \le R \}$$

and let $R \to +\infty$ to prove that

$$I = \int_0^{\infty} e^{ix^2}\,dx = \frac{\sqrt{2\pi}}{4}(1+i) = \frac{\sqrt{\pi}}{2} \cdot \underbrace{\frac{\sqrt{2}}{2}(1+i)}_{\text{multiple of } I_G}$$

let $\gamma = \gamma_1 + \gamma_2 + \gamma_3$ (Shown below)

QI of z-plane



let $f(z) = e^{iz^2}$. Note that $f(z)$ is entire so an integral around γ must sum to zero. $\left(\int_\gamma\right)$

Expect one of curves to be zero, one to be I, and one to be multiple of I_G

$$\int_\gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz + \int_{\gamma_3} f(z)\,dz = 0$$

On γ_1, $z = x$, $dz = dx$

$$\int_{\gamma_1} f(z)\,dz = \lim_{R\to\infty} \int_0^R e^{ix^2}\,dx = I$$

On γ_2,

$$z = Re^{i\theta} \qquad 0 \le \theta \le \pi/4)$$
$$z^2 = R^2 e^{i2\theta}$$

$$f(z) = e^{iz^2}$$

$$\left|\int_{\gamma_2} f(z)\,dz\right| \le \left|\int_0^{\pi/4} e^{i(\rho^2 e^{i2\theta})}\, i\rho e^{i\theta}\, d\theta\right|$$

$$\le \int_0^{\pi/4} \rho\, e^{-\rho^2 \sin(2\theta)}\, d\theta \le 0$$

decays faster than ρ increases as $\rho \to \infty$

On γ_3,

$$z = xe^{i\pi/4}, \quad z^2 = x^2 e^{i\pi/2} = x^2 i$$
$$dz = e^{i\pi/4}\,dx$$

$$\int_{\gamma_3} e^{z^2}\,dz = \lim_{R\to\infty} \int_R^0 e^{i(x^2 i)} e^{i\pi/4}\,dx = -\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i\right) \lim_{R\to\infty} \int_0^R e^{-x^2}\,dx = -\frac{\sqrt{\pi}}{2}\cdot\frac{\sqrt{2}}{2}(1+i)$$

In conclusion,

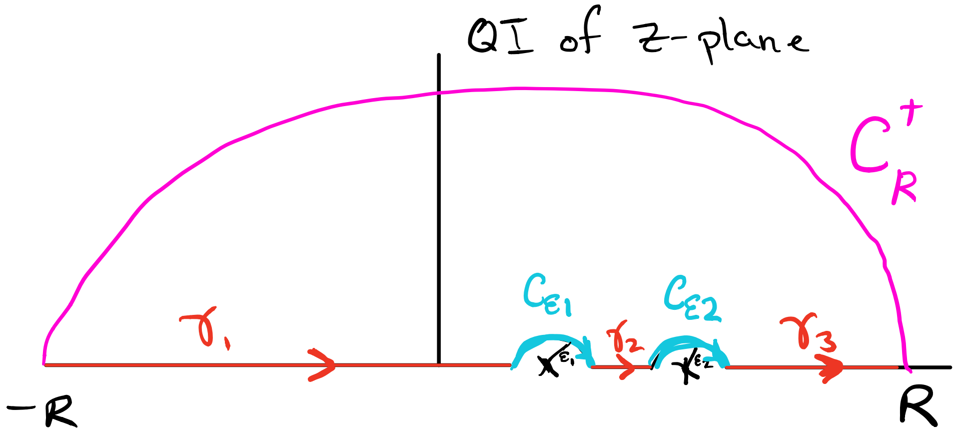
$$\int_\gamma f(z)\,dz = \underbrace{\int_{\gamma_1} f(z)\,dz}_{I} + \cancelto{0}{\int_{\gamma_2} f(z)\,dz} + \underbrace{\int_{\gamma_3} f(z)\,dz}_{-\frac{\sqrt{2\pi}}{4}(1+i)} = 0$$

$$\boxed{I = \lim_{R\to\infty} \int_0^R e^{ix^2}\,dx = \frac{\sqrt{2\pi}}{4}(1+i)}$$

3) $$\lim_{R\to\infty} \int_{-R}^{R} \frac{e^{ix}}{(x-1)(x-2)}\,dx = \pi i\left(e^{2i} - e^{i}\right)$$

$f(x) = \dfrac{e^{ix}}{(x-1)(x-2)}$ has simple poles at $x = 1, 2$

So, our contour becomes



Let $$f(z) := \frac{e^{iz}}{(z-1)(z-2)}$$

So, for $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + C_{\varepsilon_1} + C_{\varepsilon_2} + C_R^+$

we are analytic in our contour

$$\int_\gamma f(z)\,dz = \left(\int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} + \int_{C_{\varepsilon_1}} + \int_{C_{\varepsilon_2}} + \int_{C_R^+}\right) f(z)\,dz = 0$$

By Jordan's lemma,

$$\lim_{R\to\infty} \int_{C_R^+} f(z)\,dz = 0$$

By Lemma 4 (Page 340)

$$\lim_{\varepsilon\to 0^+} \int_{C_{\varepsilon 1}} f(z)dz = i(0-\pi)\,\mathrm{Res}(f(z);1)$$

$$= -i\pi \lim_{z\to 1} \frac{e^{iz}}{(z-2)} = \frac{e^1}{-1}(-i\pi) = \pi e^1$$

$$\lim_{\varepsilon\to 0^+} \int_{C_{\varepsilon 2}} f(z)dz = i(0-\pi)\,\mathrm{Res}(f(z);2)$$

$$= -i\pi \lim_{z\to 2} \frac{e^{iz}}{(z-1)} = -e^{i2}i\pi$$

Then we note

$$\left(\int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3}\right) f(z)\,dz = \lim_{\substack{R\to\infty \\ \varepsilon\to 0}} \left[\left(\int_{-R}^{1-\varepsilon} + \int_{1+\varepsilon}^{2-\varepsilon} + \int_{2+\varepsilon}^{R}\right) f(x)dx\right]$$

where

$$\lim_{\substack{R\to\infty \\ \varepsilon\to 0}} \left[\left(\int_{-R}^{1-\varepsilon} + \int_{1+\varepsilon}^{2-\varepsilon} + \int_{2+\varepsilon}^{R}\right) f(x)dx\right] = I$$

In conclusion,

$$\int_{\gamma} f(z)dz = \Big(\underbrace{\int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3}}_{I} + \underbrace{\int_{C_{\varepsilon 1}}}_{e^1 i\pi} + \underbrace{\int_{C_{\varepsilon 2}}}_{-e^{i2}i\pi} + \underbrace{\int_{C_R^+}}_{0}\Big) f(z)dz = 0$$

$$\Rightarrow \boxed{I = (e^{i2} - e^{1})i\pi}$$

5) $I = \lim_{R\to\infty} \int_0^R \frac{\cos x - 1}{x^2}\,dx = \frac{-\pi}{2}$ } Note: $I = \frac{1}{2} \lim_{R\to\infty} \int_{-R}^{R} \frac{\cos x - 1}{x^2}\,dx$

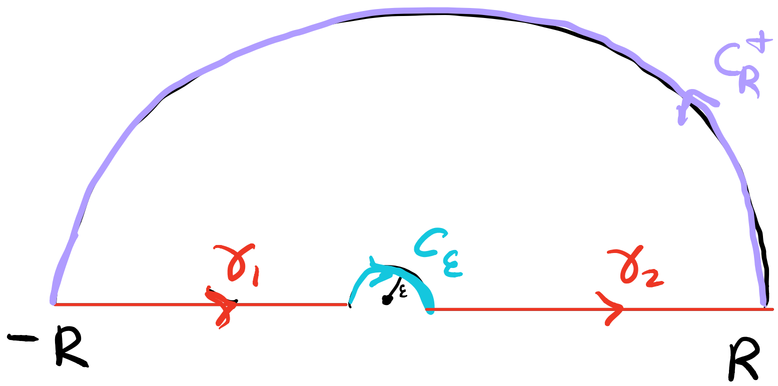
DONT WORK WITH COSINE

let $\cos(x) = e^{ix}$

$$\lim_{R\to\infty} \int_0^R \frac{e^{ix} - 1}{x^2}\,dx$$ } pole of order 2 at $x=0$

• Also note a zero at $x=0$!

$\gamma = \gamma_1 + \gamma_2 + C_\varepsilon + C_R^+$



let

$$f(z) := \frac{e^{iz} - 1}{z^2}$$

$$\int_\gamma f(z)\,dz = \left(\int_{\gamma_1} + \int_{\gamma_2} + \int_{C_\varepsilon} + \int_{C_R^+} \right) f(z)\,dz = 0$$

By Jordan's lemma,

$$\lim_{R\to\infty} \int_{C_R^+} f(z)\,dz = 0$$

By Lemma 4 (Page 340)

$$\lim_{\varepsilon\to 0^+}\int_{C_{\varepsilon 1}} f(z)\,dz = i(0-\pi)\,\mathrm{Res}(f(z);1)$$

$$= -i\pi\,\boxed{\lim_{z\to 0}\frac{d}{dz}\left[\frac{e^{iz}-1}{z^2}\right]}$$

$$= \pi$$

Comes out to i

Then we note

$$\left(\int_{\gamma_1}+\int_{\gamma_2}\right)f(z)\,dz = \lim_{\substack{R\to\infty\\ \varepsilon\to 0}}\left[\left(\int_{-R}^{-\varepsilon}+\int_{\varepsilon}^{R}\right)f(x)\,dx\right] = 2I$$

So,

$$\int_\gamma f(z)\,dz = \Bigg(\overbrace{\int_{\gamma_1}+\int_{\gamma_2}}^{2I}+\overbrace{\int_{C_\varepsilon}}^{+\pi}+\overbrace{\cancel{\int_{C_R^+}}}^{0}\Bigg)f(z)\,dz = 0$$

$$2I = -\pi$$

$$I = \lim_{R\to\infty}\int_0^R \frac{\cos x - 1}{x^2}\,dx = -\frac{\pi}{2}$$

1)

16.6

$$I = \int_0^\infty \frac{\sqrt{x}}{x^2+1}\,dx = \frac{\pi}{\sqrt{2}}$$

Let $z = re^{i\theta}$; Restrict θ ! $0 \le \theta \le 2\pi$

Just "above" x-axis

$$z^{1/2} = r^{1/2} e^{i\theta/2} \quad , \theta = 0 \ , \ r > 0$$
$$= r^{1/2}$$

Just "below" x-axis

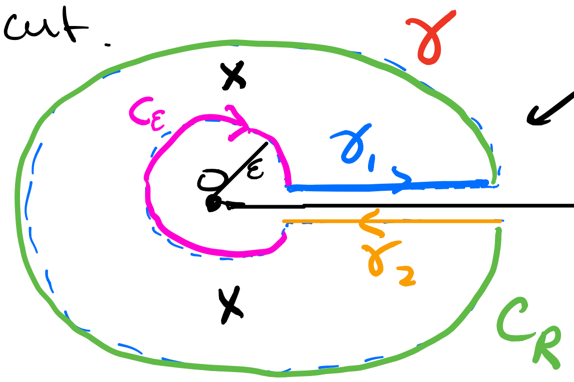
$$z^{1/2} = r^{1/2} e^{i\theta/2} \quad , \ \theta = 2\pi \ , \ r > 0$$
$$= r^{1/2} e^{i\pi} = -r^{1/2}$$

We have an argument change of $e^{i\pi}$ as a result of our branch cut on the positive x-axis.

Let

$$f(z) := \frac{z^{1/2}}{z^2+1}$$

Need a contour that doesn't lie on our branch cut.



keyhole contour (poles inside at $\pm i$)

x-axis (branch cut)

$$\gamma = \gamma_1 + C_R + \gamma_2 + C_\varepsilon$$

$$\int_{\gamma} f(z)\,dz = 2\pi i\left[\operatorname{Res}(f(z);\, i\,) + \operatorname{Res}(f(z);\, -i)\right]$$

$$= 2\pi i\left[\lim_{z\to i}\frac{z^{1/2}}{z+i} + \lim_{z\to -i}\frac{z^{1/2}}{z-i}\right]$$

$$= 2\pi i\left[\frac{(i)^{1/2}}{2i} + \frac{(-i)^{1/2}}{-2i}\right]$$

KEY

$i = e^{i\pi/2}$

$-i = e^{-i\pi/2}$

$$= 2\pi\left[\frac{\left(e^{i\pi/2}\right)^{1/2} + e^{\left(-i\pi/2\right)^{1/2}}}{2}\right] = 2\pi\cos\left(\frac{\pi}{4}\right)$$

$$\int_{\gamma} f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{C_R} f(z)\,dz + \int_{\gamma_2} f(z)\,dz + \int_{C_\varepsilon} f(z)\,dz$$

$$\int_{\gamma_1} f(z)\,dz = \lim_{\substack{\varepsilon\to 0\\ R\to\infty}} \int_{\varepsilon}^{R} \frac{x^{1/2}}{x^2+1}\,dx = I$$

} γ_1 is "just above" case

$$\int_{\gamma_2} f(z)\,dz = \lim_{\substack{\varepsilon\to 0\\ R\to\infty}} \int_{R}^{\varepsilon} \frac{-x^{1/2}}{x^2+1}\,dx = I$$

} γ_2 is "just below" case

$$\int_{C_R} f(z)\,dz = 0, \quad \int_{C_\varepsilon} f(z)\,dz = 0$$

} expect but will verify

$$\left|\int_{C_R} f(z)\,dz = 0\right| \leq \lim_{R\to\infty} 2\pi R \max_{z\in C_R}\left|\frac{z^{1/2}}{z^2+1}\right|$$

$$\Rightarrow \lim_{R\to\infty} \frac{2\pi R\,(R^{1/2})}{R^2+1} = 0$$

$$\left|\int_{C_\varepsilon} f(z)\,dz\right| \leq \max_{z\in C_\varepsilon}\left|\frac{z^{1/2}}{z^2+1}\right| 2\pi\varepsilon$$

$$\leq \max_{z\in C_\varepsilon}\left|\frac{z^{1/2}}{1}\right| 2\pi\varepsilon$$

$$\Rightarrow \lim_{\varepsilon\to 0} \varepsilon^{3/2}\, 2\pi = 0$$

In conclusion,

$$\cancelto{\frac{2\pi}{\sqrt{2}}}{\int_{\gamma} f(z)\,dz} = \cancelto{I}{\int_{\gamma_1} f(z)\,dz} + \cancelto{0}{\int_{C_R} f(z)\,dz} + \cancelto{I}{\int_{\gamma_2} f(z)\,dz} + \cancelto{0}{\int_{C_\varepsilon} f(z)\,dz}$$

$$2\lim_{R\to\infty}\int \frac{x^{1/2}}{x^2+1}\,dx = \frac{2\pi}{\sqrt{2}}$$

$$\boxed{I = \lim_{R\to\infty}\int \frac{x^{1/2}}{x^2+1}\,dx = \frac{\pi}{\sqrt{2}}}$$

4)
$$\int_0^\infty \frac{x^\alpha}{(x^2+1)^2}\,dx = \frac{\pi(1-\alpha)}{4\cos\left(\alpha\frac{\pi}{2}\right)} \qquad -1 < \alpha < 3, \quad \alpha \neq 1$$

Let $z = re^{i\theta}$; Restrict θ ! $-\pi \leq \theta \leq \pi$

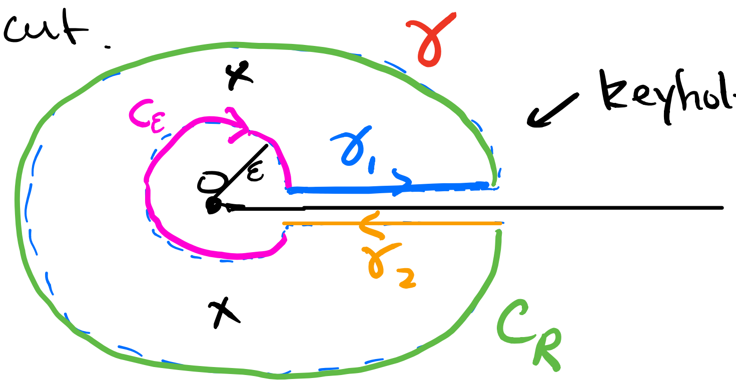
Just "above" x-axis
$$z^\alpha = r^\alpha e^{i\alpha\theta}, \quad \theta = 0, \quad r > 0$$
$$= r^\alpha$$

Just "below" x-axis
$$z^\alpha = r^\alpha e^{i\alpha\theta}, \quad \theta = 2\pi, \quad r > 0$$
$$= r^\alpha e^{i2\pi\alpha}$$

We have an argument change of $e^{i2\pi\alpha}$ as a result of our branch cut on the positive x-axis.

Let
$$f(z) := \frac{z^\alpha}{((z-i)(z+i))^2}$$
which has two poles of order 2 at $x = \pm i$ $(n=2)$

Need a contour that doesn't lie on our branch cut.



keyhole contour (pole inside at -9)

x-axis (branch cut)

$$\gamma = \gamma_1 + C_R + \gamma_2 + C_\varepsilon$$

$$\int_\gamma f(z)\,dz = 2\pi i\left[\mathrm{Res}(f(z); i) + \mathrm{Res}(f(z); -i)\right]$$

$$\lim_{z\to i} \frac{d}{dz}\left[\frac{z^\alpha}{(z+i)^2}\right] = \frac{(z+i)^2 \alpha z^{\alpha-1} - z^\alpha 2(z+i)}{(z+i)^4}$$

$$= \frac{+4\alpha\left(e^{i\frac{\pi\alpha}{2}}\right)i - e^{i\frac{\pi\alpha}{2}}4i}{16} = \frac{-4ie^{i\frac{\pi\alpha}{2}}[1-\alpha]}{16}$$

$$\lim_{z\to -i} \frac{d}{dz}\left[\frac{z^\alpha}{(z-i)^2}\right] = \frac{(z-i)^2 \alpha z^{\alpha-1} - 2z^\alpha(z-i)}{(z-i)^4}$$

$$= \frac{-4\alpha e^{-i\frac{\pi\alpha}{2}} i + 2(+2i)e^{-i\frac{\pi\alpha}{2}}}{16} = \frac{4ie^{-i\frac{\pi\alpha}{2}}[1-\alpha]}{16}$$

$$2\pi i\left[\mathrm{Res}(f(z); i) + \mathrm{Res}(f(z); -i)\right]$$

$$= 2\pi\frac{\left[4e^{i\frac{\pi\alpha}{2}}[1-\alpha] - 4e^{-i\frac{\pi\alpha}{2}}[1-\alpha]\right]}{16}$$

$$= \frac{8\pi(1-\alpha)}{16}\left[4e^{i\frac{\pi}{2}} - 4e^{-\frac{i\pi}{2}}\right]$$

$$\int_{\gamma_1} f(z)dz = \lim_{\substack{\varepsilon\to 0\\ R\to\infty}} \int_{\varepsilon}^{R} \frac{z^\alpha}{(z^2+1)^2}\,dx = I \Bigg\} \begin{array}{l}\gamma_1 \text{ is "just} \\ \text{above" case}\end{array}$$

$$\int_{\gamma_2} f(z)dz = \lim_{\substack{\varepsilon\to 0\\ R\to\infty}} \int_{R}^{\varepsilon} \frac{z^\alpha e^{i2\pi\alpha}}{(z^2+1)^2}\,dx = -Ie^{i2\pi\alpha} \Bigg\} \begin{array}{l}\gamma_2 \text{ is "just} \\ \text{below" case}\end{array}$$

$$\int_{C_R} f(z)dz = 0 \ , \quad \int_{C_\varepsilon} f(z)dz = 0 \ \Bigg\} \begin{array}{l}\text{expect but} \\ \text{will verify}\end{array}$$

$$\left| \int_{C_R} f(z)dz = 0 \right| \leq \lim_{R\to\infty} 2\pi R \max_{z\in C_R} \left| \frac{z^\alpha}{(z^2+1)^2} \right|$$

$$\Rightarrow \lim_{R\to\infty} \frac{2\pi R^{\alpha+1}}{R^4} = 0 \ \Bigg\} \text{ by restrictions on } \alpha$$

$$\left| \int_{C_\varepsilon} f(z)dz \right| \leq \max_{z\in C_\varepsilon} \left| \frac{z^\alpha}{(z^2+1)^2} \right| 2\pi\varepsilon$$

$$\Rightarrow \lim_{\varepsilon\to 0} \frac{(\varepsilon)^{\alpha+1}}{9} 2\pi = 0$$

In conclusion,

$$\int_{\gamma} f(z)dz = \int_{\gamma_1} f(z)dz + \cancelto{0}{\int_{C_R} f(z)dz} + \int_{\gamma_2} f(z)dz + \cancelto{0}{\int_{C_\varepsilon} f(z)dz}$$

$$\int_{\gamma} f(z)\,dz = I(1 - e^{i2\pi})$$

$$I = \frac{2\pi i\left[\mathrm{Res}(f(z); i) + \mathrm{Res}(f(z); -i)\right]}{(1 - e^{i2\pi\alpha})}$$

$$= 8\pi \frac{\left[e^{i\pi\frac{\alpha}{2}} - e^{-i\frac{\pi\alpha}{2}}\right](1-\alpha)}{16(1 - e^{i2\pi\alpha})}$$

Intense Massaging

$\Rightarrow$

$$\boxed{\frac{\pi(1-\alpha)}{4\cos(\alpha\pi/2)}}$$