

# ECE 4110 Homework 9

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*Due 5pm on December 5*

## 1 Reading Material

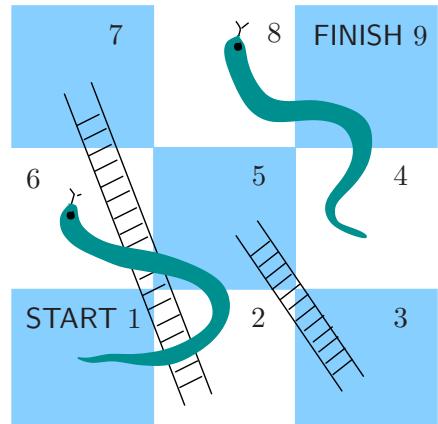
- Absorption time and absorption probability.
- Continuous-time Markov chain (Chapter 11.4).

## 2 Assignment

### 1. Absorption Time and Absorption Probability

A simple game of “snakes and ladders” is played on a board of nine squares. At each turn a player tosses a fair coin and advances one or two places according to whether the coin shows head or tail. If you land at the foot of a ladder, you climb to the top; if you land at the head of a snake, you slide down to the tail.

- Define a Markov chain with the state  $X_n$  being the square that the player is in at time  $n$ . Give the transition matrix.
- How many turns on average does it take to complete the game?
- What is the probability that a player who has reached the middle square will complete the game without slipping back to square 1?



### 2. A Dinner Treat

You are playing a game with your friend John who is majoring in Literature and thinks “Markov Chain” is a new franchise of fancy Russian cuisine.

The game is based on the outcomes of flipping a fair coin in succession. Each of you chooses a pattern consisting of three consecutive coin flip outcomes. Whoever’s pattern appears first will be the winner and will be treated to a lavish dinner. To show your generosity, you tell your friend: “I will let you choose first. You can choose any pattern from the 8 possible ones. I will

then choose a pattern from the remaining 7." Your friend is grateful, not knowing that for any of the eight patterns that he chooses, you can find a pattern that is at least twice better (in terms of the chance to win) than his pattern.

- (a) Show that if John chooses HHH, you can have a winning chance 7 times better than John by choosing THH.
- (b) Show that if John chooses HTT, you can have a winning chance twice better by choosing HHT.
- (c) (Optional) If you do want to play this game with a friend, you might want to fill out the table below. You will see that there is a general rule for finding the best winning pattern against every pattern your friend may choose.

John's Pattern	Your Pattern	Pr[you win]
HHH		
HHT		
HTH		
HTT		
THH		
THT		
TTH		
TTT		

### 3. Properties of Exponential Random Variables

Let  $T_1, \dots, T_n$  be independent random variables with  $T_i$  being exponentially distributed with parameter  $\lambda_i$ . Let

$$T = \min(T_1, \dots, T_n)$$

denote the minimum of these  $n$  random variables and let  $K$  denote the index of the random variable that equals the minimum

$$K = \arg \min(T_1, \dots, T_n).$$

- (a) Show that the probability of  $T_k$  being the minimum is given by

$$\Pr[K = k] = \frac{\lambda_k}{\sum_{i=1}^n \lambda_i}$$

- (b) Show that  $T$  is exponentially distributed with parameter  $\sum_{i=1}^n \lambda_i$ .
- (c) Show that  $T$  and  $K$  are independent.

### 4. Erlang's Formula

At the times of a Poisson process with rate  $\lambda$ , someone in Acahti places a call to the Big City. The phone network can handle  $k$  such calls. If there is spare capacity when a call is placed, the call is connected and lasts an exponentially-distributed amount of time with mean  $1/\mu$ , independent of the other calls. If someone attempts to place a call and  $k$  calls are currently in progress, the person hears a fast busy signal and gives up.

- (a) Argue that the number of connected calls is a Markov chain and draw its transition rate diagram. Is the chain irreducible?

- (b) Find all stationary distributions of the chain.
- (c) When the state is  $k$ , the network cannot handle any additional calls. What is the long-term fraction of time that the chain is in this state?

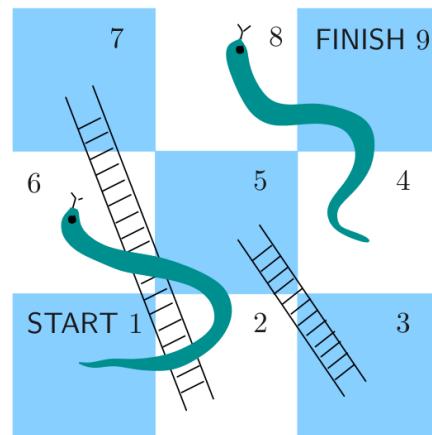
### 5. Inventory Control

The Elppa Store sells chic Orange Watch Series 5 for a tidy sum. Customers enter the store at the times of a Poisson process with rate  $\lambda$ . If an Orange Watch is available when a customer arrives, the customer purchases it. Otherwise, the customer makes a scene and then leaves. When the stock of Orange Watch in the store reaches one, the store orders 4 more, which arrive after an exponentially-distributed amount of time with mean  $1/\mu$ . Suppose that initially the store has 5 Orange Watch in stock.

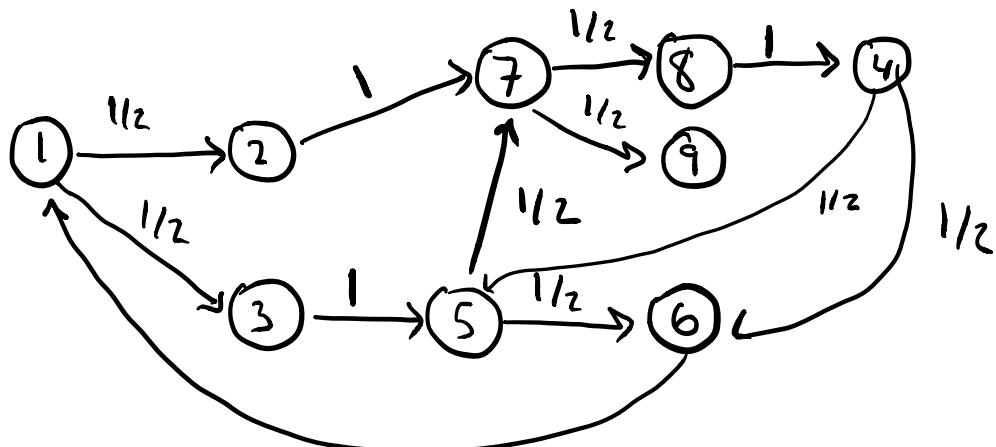
- (a) Argue that the number of Orange Watch in the store is a Markov chain and draw its transition rate diagram. Is the chain irreducible?
- (b) Compute all stationary distributions for the chain.
- (c) What is the long-run fraction of time that the store is out of stock of Orange Watch?

# ① Absorption Time and Absorption Probability

A simple game of "snakes and ladders" is played on a board of nine squares. At each turn a player tosses a fair coin and advances one or two places according to whether the coin shows head or tail. If you land at the foot of a ladder, you climb to the top; if you land at the head of a snake, you slide down to the tail.



(a) Define a Markov chain with the state  $X_n$  being the square that a player is in at time  $n$ . Give the transition matrix.



$$P = \begin{bmatrix} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 6 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 8 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$X = \{1, 4, 5, 7, 9\}$

↑

$(2, 7)$     $(3, 5)$     $(6, 1)$

$(8, 4)$  can be combined to single states

Note ↗

Thus P simplifies to

$$P = \begin{matrix} & \begin{matrix} 1 & 4 & 5 & 7 & 9 \end{matrix} \\ \begin{matrix} 1 \\ 4 \\ 5 \\ 7 \\ 9 \end{matrix} & \left[ \begin{matrix} 0 & 0 & 1/2 & 1/2 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1 \end{matrix} \right] \end{matrix}$$

(b) How many turns on average does it take to complete the game?

Define

$$e_i \triangleq \mathbb{E}[\text{remaining time until absorption} \mid \text{current state is } i]$$

then

$$e_1 = \frac{1}{2}(e_5 + 1) + \frac{1}{2}(e_7 + 1)$$

$$e_4 = \frac{1}{2}(e_1 + 1) + \frac{1}{2}(e_5 + 1)$$

$$e_5 = \frac{1}{2}(e_1 + 1) + \frac{1}{2}(e_7 + 1)$$

$$e_7 = \frac{1}{2}(e_4 + 1) + \frac{1}{2}(e_9 + 1)$$

$$e_9 = 0$$

Solving yields

$$e_1 = 7, e_4 = 8, e_5 = 7, e_7 = 5$$

So, on average it takes  $e_1 = 7$  moves to complete the game.

(c) What is the probability that a player who has reached the middle square will complete the game without slipping back to square 1?

Define

$$g_i = \Pr[\text{Winning the game w/o going back to } 1 \mid \text{started at state } i]$$

then

$$g_1 = \frac{1}{2}g_7 + \frac{1}{2}g_5$$

$$g_4 = \frac{1}{2}g_5$$

$$g_5 = \frac{1}{2}g_7$$

$$g_7 = \frac{1}{2}g_9 + \frac{1}{2}g_4$$

$$g_9 = 1$$

Solving gives

$$g_1 = \frac{3}{7}, g_4 = \frac{1}{7}, g_5 = \frac{2}{7}, g_7 = \frac{4}{7}$$

Middle square is 5, thus answer is  $g_5 = \frac{2}{7}$

## ② A Dinner Treat

You are playing a game with your friend John who is majoring in Literature and thinks "Markov Chain" is a new franchise of fancy Russian cuisine.

The game is based on the outcomes of flipping a fair coin in succession. Each of you chooses a pattern consisting of three consecutive coin flip outcomes. Whoever's pattern appears first will be the winner and will be treated to a lavish dinner. To show your generosity, you tell your friend: "I will let you choose first. You can choose any pattern from the 8 possible ones. I will then choose a pattern from the remaining 7." Your friend is grateful, not knowing that for any of the eight patterns that he chooses, you can find a pattern that is at least twice better (in terms of the chance to win) than his pattern.

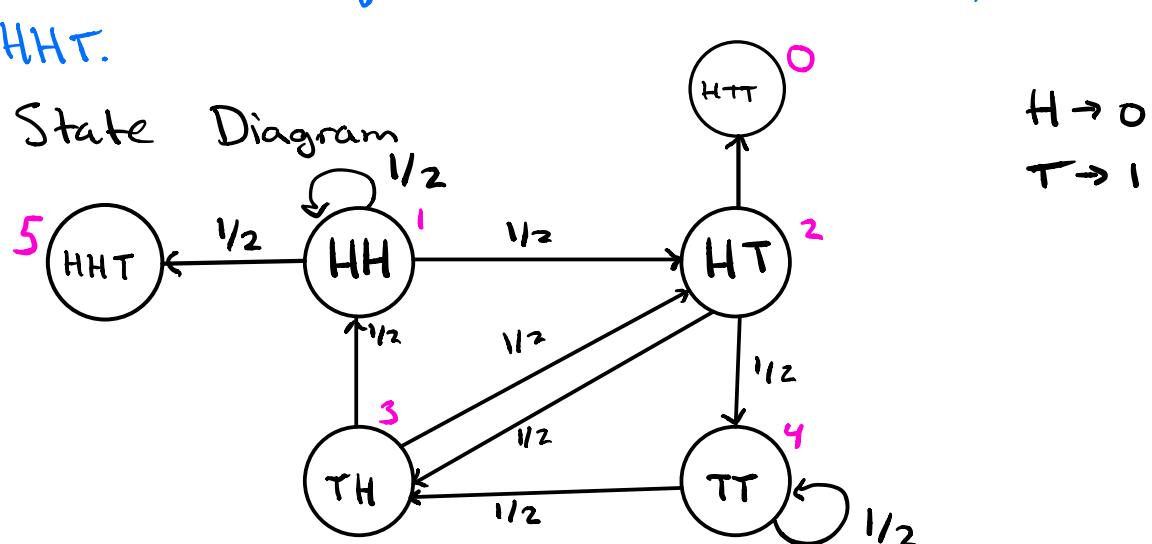
(a) Show that if John chooses HHH, you can have a winning chance of 7 times better than John by choosing THH.

If first flip is T, John never wins. Thus he only wins if first flip is H and following two flips are H.

$$\Pr[\text{HHH}] = 1/8$$

thus I win w/ probability  $1 - 1/8 = 7/8$

(b) Show that if John chooses HTT, you can have a winning chance twice better by choosing HHT.



Define

$$g_i \triangleq \Pr[\text{HTT appears first} \mid \text{current state is } i]$$

$$g_0 = 1$$

$$g_1 = \frac{1}{2}g_5 + \frac{1}{2}g_0 \rightarrow g_1 = 0$$

$$g_2 = \frac{1}{2} + \frac{1}{2}g_3 \rightarrow g_2 = \frac{2}{3}$$

$$g_3 = \frac{1}{2}g_1 + \frac{1}{2}g_2 \rightarrow g_3 = \frac{1}{3}$$

$$g_4 = \frac{1}{2}g_3 + \frac{1}{2}g_4 \rightarrow g_4 = \frac{1}{3}$$

$$g_5 = 0$$

$$\Pr[HH] = \Pr[HT] = \Pr[TH] = \Pr[TT] = \frac{1}{4}$$

$$\Pr[A \text{ wins}] = \frac{1}{4}g_1 + \frac{1}{4}g_2 + \frac{1}{4}g_3 + \frac{1}{4}g_4 = \frac{1}{3}$$

$$\Pr[B \text{ wins}] = 1 - \Pr[A \text{ wins}] = \frac{2}{3}$$

### ③ Properties of Exponential Random Variables

Let  $T_1, \dots, T_n$  be iid  $\exp(\lambda_i)$ . ( $\lambda_i$  corresponds to  $T_i$ )

Let

$$T = \min(T_1, \dots, T_n)$$

denote the minimum of these n random variables and let  $K$  denote the index of the random variable that equals the minimum

$$K = \operatorname{argmin}(T_1, \dots, T_n)$$

(a) Show that the probability of  $T_k$  being the minimum is given by

$$\Pr[K = k] = \frac{\lambda_k}{\sum_{i=1}^n \lambda_i}$$

Conditioning on  $T_k$  and using the total probability theorem yields

$$\begin{aligned} \Pr[K = k] &= \int_{t_k=0}^{\infty} \Pr[T_1 > t, \dots, T_{k-1} > t, T_{k+1} > t, \dots, T_n > t \mid T_k = t] f_{T_k}(t) dt \\ &= \int_{t_k=0}^{\infty} \prod_{i \neq k} \Pr[T_i > t] f_{T_k}(t) dt \\ &= \int_{t_k=0}^{\infty} \prod_{i \neq k} e^{-\lambda_i t} \lambda_k e^{-\lambda_k t} dt \end{aligned}$$

$$= \int_{t_k=0}^{\infty} \lambda_k e^{-t \sum_{i=1}^n \lambda_i} dt$$

$$= \lambda_k \left( \frac{1}{\sum_{i=1}^n \lambda_i} e^{-t \sum_{i=1}^n \lambda_i} \Big|_{t=0}^{\infty} \right)$$

$$= \frac{\lambda_k}{\sum_{i=1}^n \lambda_i}$$

(b) Show that  $T$  is exponentially distributed with parameter  $\sum_{i=1}^n \lambda_i$

$$\Pr[T > t] = \Pr[T_1 > t, \dots, T_n > t]$$

$$= \prod_{i=1}^n \Pr[T_i > t]$$

$$= \prod_{i=1}^n e^{-\lambda_i t}$$

$$= e^{-t \sum_{i=1}^n \lambda_i} \Rightarrow \text{parameter} = \sum_{i=1}^n \lambda_i$$

(L) Show that  $T$  and  $K$  are independent.

Need

$$\Pr[T > t, K = k] = \Pr[T > t] \Pr[K = k]$$

$$\Pr[T > t, K = k] = \int_t^\infty \Pr[T > t, K = k \mid T_k = s] f_{T_k}(s) ds$$

$$= \int_t^\infty \Pr[T_1 > s, \dots, T_{k-1} > s, T_{k+1} > s, \dots, T_n > s] f_{T_k}(s) ds$$

$$= \int_t^\infty \prod_{i \neq k} \Pr[T_i > s] f_{T_k}(s) ds = \int_t^\infty \prod_{i \neq k} e^{-\lambda_i s} \lambda_k e^{-\lambda_k s} ds$$

$$= \lambda_k \int_t^\infty e^{-s \sum_{i \neq k} \lambda_i} e^{-\lambda_k s} ds$$

$$= \lambda_k \cdot \frac{1}{\sum_{i=1}^n \lambda_i} e^{-s \sum_{i=1}^n \lambda_i} \Big|_t^\infty$$

$$= \frac{\lambda_k}{\sum_{i=1}^n \lambda_i} e^{-s \sum_{i=1}^n \lambda_i} = \Pr[K = k] \Pr[T > t]$$

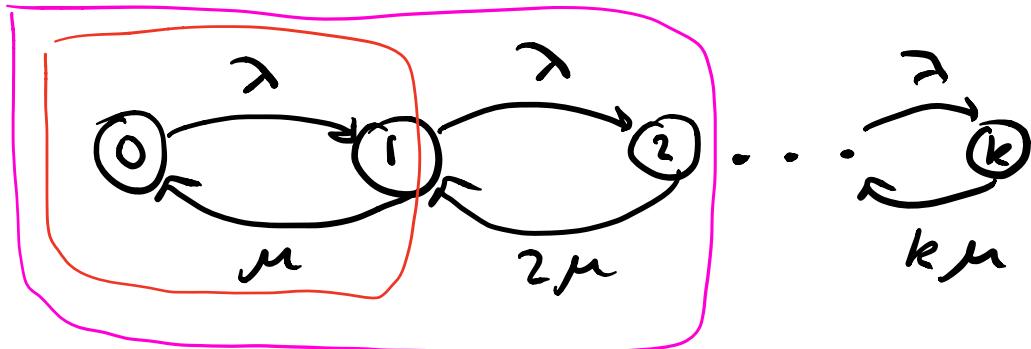
## ④ Erlang's Formula

At the times of a Poisson process with rate  $\lambda$ , someone in Acahti places a call to the Big City. The phone network can handle  $k$  such calls. If there is spare capacity when a call is placed, the call is connected and lasts an exponentially-distributed amount of time with mean  $1/\mu$ , independent of the other calls. If someone attempts to place a call and  $k$  calls are currently in progress, the person hears a fast busy signal and gives up.

- (a) Argue that the number of connected calls is a Markov chain and draw its transition rate diagram.  
Is the chain irreducible?

$$q_{i,i+1} = \lambda, \quad i = 0, 1, \dots, k-1$$

$$q_{i,i-1} = i\mu, \quad i = 1, \dots, k$$



the chain is irreducible.

- (b) Find all stationary distributions of the chain.

$$\begin{cases} \bar{\pi} Q = 0 \\ \sum_{i \in X} \pi_i = 0 \end{cases}$$

$$Q = \begin{bmatrix} -\lambda & \lambda & & & \\ \mu & -(\lambda+\mu) & \lambda & & \\ & 2\mu & -(\lambda+2\mu) & \lambda & \\ & & \ddots & \ddots & \ddots \end{bmatrix}$$

$$\lambda \pi_0 = \mu \pi_1$$

$$\lambda \pi_1 = 2\mu \pi_2 \quad \Rightarrow \quad \pi_i = \frac{1}{i!} \left(\frac{\lambda}{\mu}\right)^i \pi_0$$

$$\lambda \pi_{k-1} = k\mu \pi_k$$

$$\sum_{i=1}^k \pi_i = 1 \rightarrow \pi_0 = \frac{1}{\sum_{i=0}^k \frac{1}{i!} \left(\frac{\lambda}{\mu}\right)^i}$$

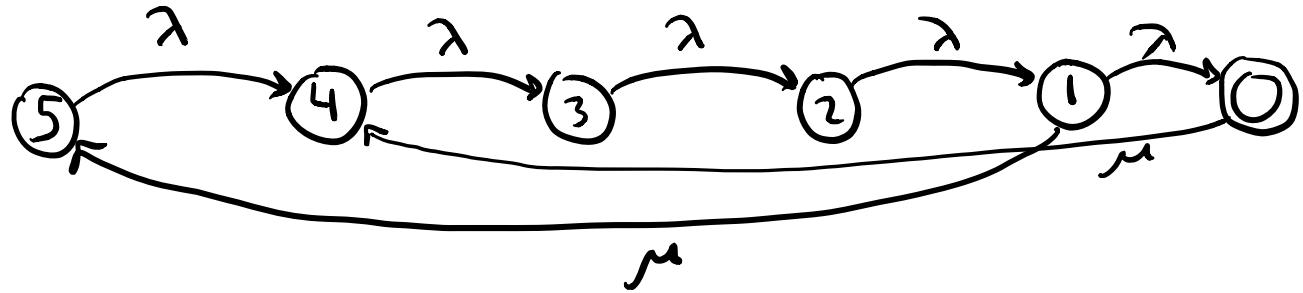
(c) When the state is  $k$ , the network cannot handle any additional calls. What is the long-term fraction of time that the chain is in this state?

$$\pi_k = \frac{1}{k!} \left(\frac{\lambda}{\mu}\right)^k \overline{\sum_{i=0}^k \frac{1}{i!} \left(\frac{\lambda}{\mu}\right)^i}$$

## ⑤ Inventory Control

The Elppa Store sells chic Orange Watch Series 5 for a tidy sum. Customers enter the store at the times of a Poisson process with rate  $\lambda$ . If an Orange Watch is available when a customer arrives, the customer purchases it. Otherwise, the customer makes a scene and then leaves. When the stock of Orange Watch in the store reaches one, the store orders 4 more, which arrive after an exponentially-distributed amount of time with mean  $1/\mu$ . Suppose that initially the store has 5 Orange Watch in stock.

(a) Argue that the number of Orange Watches in the store is a Markov Chain and draw its transition rate diagram. Is the chain irreducible?



$$q_{i,i+1} = \lambda, \quad i = 5, 4, 3, 2, 1$$

$$q_{i,i+4} = \mu, \quad i = 0, 1$$

The chain is irreducible.

$$Q = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ -\mu & 0 & 0 & 0 & \mu & 0 \\ \lambda & -(\lambda + \mu) & 0 & 0 & 0 & \mu \\ 0 & \lambda & -\lambda & 0 & 0 & 0 \\ 0 & 0 & \lambda & -\lambda & 0 & 0 \\ 0 & 0 & 0 & \lambda & -\lambda & 0 \\ 0 & 0 & 0 & 0 & \lambda & -\lambda \end{bmatrix}$$

(b) Compute all stationary distributions of the chain.

$$\pi_1 = \frac{\mu}{\lambda} \pi_0$$

$$\pi_2 = \frac{(\lambda + \mu)}{\lambda} \pi_1 = \frac{(\lambda + \mu)\mu}{\lambda^2} \pi_0$$

$$\pi_3 = \pi_2$$

$$\pi_4 = \pi_3$$

$$\pi_5 = \frac{\mu^2}{\lambda^2} \pi_0$$

$$\pi_0 \left( 1 + \frac{\mu}{\lambda} + 3 \frac{(\lambda + \mu)\mu}{\lambda^2} + \frac{\mu^2}{\lambda^2} \right) = 1$$

$$\pi_0 = \frac{\lambda^2}{(\lambda + 2\mu)^2}$$

(c) What is the long-run fraction of time the store is out of stock?

$$\pi_0 = \left( \frac{\lambda}{\lambda + 2\mu} \right)^2$$