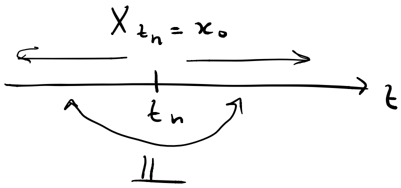
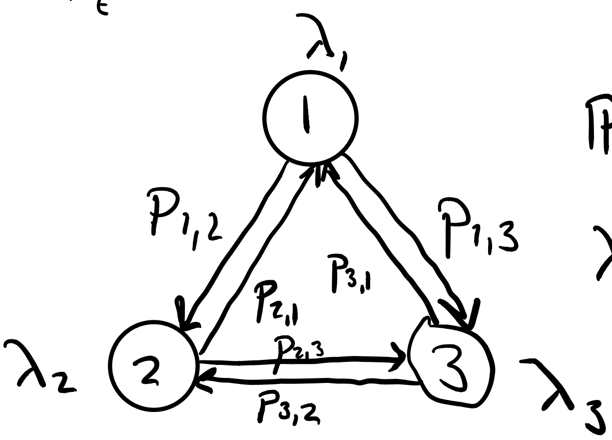
$$\{X(t)\}_{t\geq 0} \to \{T_i\}_{i\in\mathcal{X}} \text{ iid } f_T(t), \quad T_i \sim \exp$$

$$\{X(t)\}_{t\geq 0} \to \{P_{i,j}\}_{i,j\in\mathcal{X}}, \quad P_{i,i} = 0$$

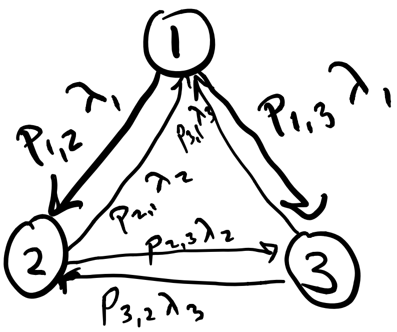




$$\mathbb{P} = [p_{i,j}]_{M\times M}$$

$$\lambda = [\lambda_1, \ldots, \lambda_m]$$

Equivalent characterization



for state 2: $T_{1,2} \sim \exp(p_{1,2}\lambda_1)$

for state 3: $T_{1,3} = \exp(p_{1,3}\lambda_1)$

$$p_{1,2} + p_{1,3} = 1$$

$$T_{1,2} \perp\!\!\!\perp T_{1,3}$$

$$T_1 = \min(T_{1,2}, T_{1,3}) \sim \exp((p_{1,2} + p_{1,3})\lambda_1)$$

$$\sim \exp(\lambda_1)$$

$$\Pr[T_{1,2} < T_{1,3}] = \frac{p_{1,2}\lambda_1}{p_{1,2}\lambda_1 + p_{1,3}\lambda_1} = p_{1,2}$$

Now need a

$$Q = [q_{i,j}]_{i,j \in \mathcal{X}}$$ } transition rate matrix

$$q_{i,j} = p_{i,j} \lambda_i \quad (i \neq j)$$

transition rate from state i to state j.

$$q_{i,i} = -\sum_{i \neq j} q_{i,j}$$

Poisson w/ rate λ

$$\begin{tikzcd} \underset{\lambda}{\textcircled{0}} \arrow[r, "1"] & \underset{\lambda}{\textcircled{1}} \arrow[r] & \underset{\lambda}{\textcircled{2}} \arrow[r] & \underset{\lambda}{\textcircled{3}} & \cdots \end{tikzcd}$$

$$\mathbb{P} = \begin{bmatrix} 0 & 1 & & \ddots & \\ 0 & 0 & 1 & & 0 \\ 0 & 0 & 0 & 1 & \ddots \\ & 0 & & & \ddots \end{bmatrix}$$

$$\mathbb{Q} = \begin{bmatrix} -\lambda & \lambda & & 0 \\ 0 & -\lambda & \lambda & \\ & 0 & -\lambda & \lambda \; \ddots \end{bmatrix}$$

From $\mathbb{Q}$ to state probability at t

$$\mathbb{P}(t) = \{p_{i,j}(t)\}_{i,j \in \mathcal{X}}$$

$$p_{i,j}(t) \triangleq P_r[X(t) = j \mid X(0) = i]$$

want to characterize $\mathbb{P}(t)$ from $\mathbb{Q}$

$$p(t) = p(0)\mathbb{P}(t)$$

From $\mathbb{Q}$ to $\mathbb{P}(t)$

$$\begin{cases} \mathbb{P}'(t) = \mathbb{P}(t)\mathbb{Q} \\ \mathbb{P}(0) = \mathbb{I} \end{cases}$$

$$\mathbb{P}'(t) = \lim_{\delta \to 0} \frac{\mathbb{P}(t+\delta) - \mathbb{P}(t)}{\delta}$$

$$= \lim_{\delta \to 0} \frac{\mathbb{P}(t)\mathbb{P}(\delta) - \mathbb{P}(t)}{\delta}$$

$$= \mathbb{P}(t) \lim_{\delta \to 0} \frac{\mathbb{P}(\delta) - \mathbb{I}}{\delta}$$

$$\lim_{\delta \to 0} \frac{\mathbb{P}(\delta) - \mathbb{I}}{\delta} = \begin{cases} q_{i,j} & , i \neq j \\ q_{i,j} & , i = j \end{cases}$$

$$= \mathbb{Q}$$

$$\mathbb{P}'(t) = \mathbb{P}(t)\mathbb{Q}$$

$$\mathbb{P}(0) = \mathbb{I}$$

$$\rightarrow \mathbb{P}(t) = e^{t\mathbb{Q}} = \sum_{k=0}^{\infty} \frac{(t\mathbb{Q})^k}{k!}$$

Stationary Distribution

DTMC

$$\bar{\pi} = \bar{\pi}\mathbb{P}^n \quad \forall n$$

$$\begin{cases} \bar{\pi} = \bar{\pi}\mathbb{P} \\ \sum_{i \in \mathcal{X}} \bar{\pi}_i = 1 \end{cases}$$

CTMC

$$\bar{\pi} = \bar{\pi}\,\mathbb{P}(t) \;\forall t$$

$$\begin{cases} \bar{\pi} = \bar{\pi}\,? \\ \sum_{i \in \mathcal{X}} \pi_i = 1 \end{cases}$$

use

$$0 = \pi \mathbb{P}'(t)$$
$$0 = \pi \mathbb{P}(t) \mathbb{Q} \quad \forall t$$
$$0 = \pi \mathbb{Q}$$

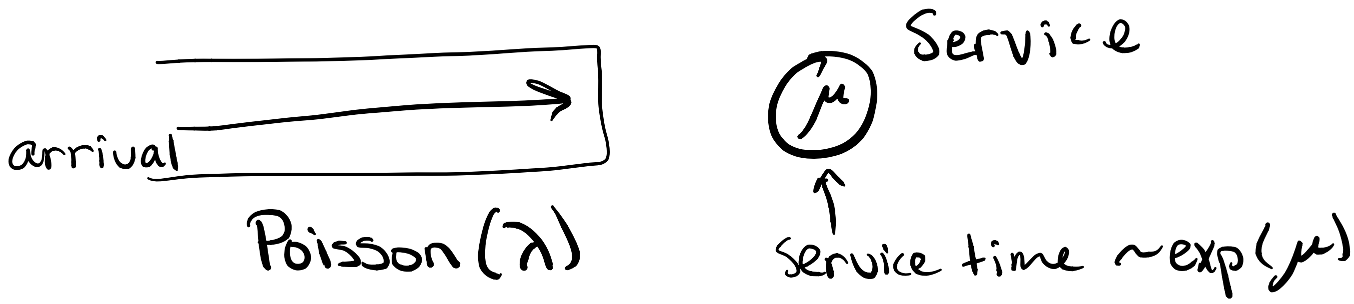
$$\begin{cases} 0 = \pi \mathbb{Q} \\ \sum_{i \in \mathcal{X}} \pi_i = 1 \end{cases}$$

$$\sum_{i \in \mathcal{X}} \pi_i q_{i,j} = 0 \quad \forall j$$

$$\pi_j q_{j,j} + \sum_{k \neq j} \pi_k q_{k,j} = 0$$

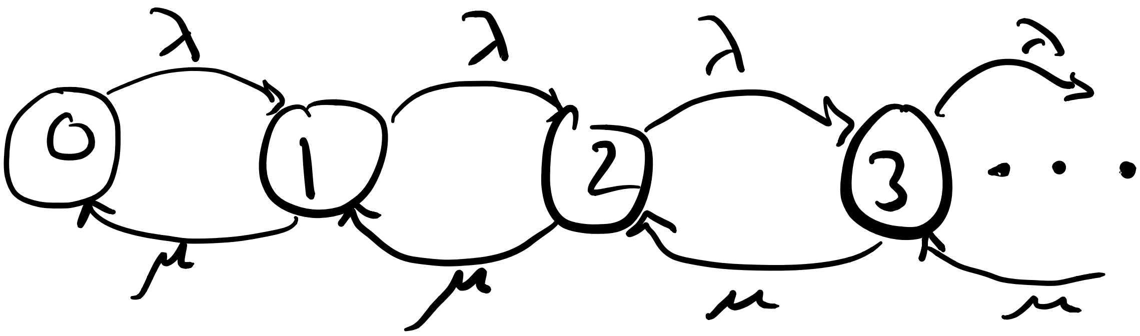
$$\sum \pi_k q_{k,j} = -\pi_j q_{j,j} \quad \forall j$$

Example: M/M/1 Queue



- distribution of the queue length
- stability (will the queue grow unboundedly)

$X(t)$: the # of customers in the system $\triangleq$ queue length



$$Q = \begin{bmatrix} -\lambda & \lambda & & & 0 \\ \mu & -(\mu+\lambda) & \lambda & & \\ & \mu & -(\mu+\lambda) & \lambda & \\ & & \ddots & \ddots & \ddots \\ 0 & & & \ddots & \ddots \end{bmatrix}$$

$$\pi_0 \lambda = \mu \pi_1 \Rightarrow \pi_1 = \frac{\lambda}{\mu} \pi_0$$

$$\pi_1 \lambda = \mu \pi_2 \Rightarrow \pi_2 = \frac{\lambda}{\mu} \pi_1$$

$$\vdots$$

$$\pi_n = \left(\frac{\lambda}{\mu}\right)^n \pi_0$$

$$\forall n > 0$$

Use $\sum_{i=0}^{\infty} \pi_i = 1$

If $\frac{\lambda}{\mu} < 1$ we have an answer,

else no stationary distribution exists

$$\pi_0 = 1 - \frac{\lambda}{\mu} = 1 - \rho \; , \; \rho = \frac{\lambda}{\mu}$$