1. Legolas prepares for each shot during archery practice by drawing an arrow with probability p from a quiver hanging over his left shoulder and with probability $1-p$ from a quiver hanging over his right shoulder. He makes each selection independently of earlier selections. Assume that each quiver contains n arrows to start with, so Legolas begins with $2n$ arrows altogether. Let X be the discrete random variable equal to the number of arrows remaining at the moment when Legolas reaches for an arrow and discovers that the corresponding quiver is empty.

(a) What are the possible values of X?

(b) Find the pmf of X. (Suggestion: For $0 \leq k \leq n$, let L_k be the event that k arrows remain in the right quiver when Legolas first discovers that the left quiver is empty. Define R_k similarly.)

2. Suppose a coin comes up heads with probability p, but you don't know p. You flip the coin n times independently. Let X_k for $1 \leq k \leq n$ be the Bernoulli random variable that equals 1 when flip k comes up heads and 0 when flip k comes up tails.

(a) Suppose you observe a sequence $x_1, x_2, \ldots, x_n$ of outcomes on the n flips — here each x_k is either 0 or 1. As a function of the x_k's, find the value of p that maximizes the probability of the sequence you observed, i.e maximizes

$$\mathbb{P}\left(\{X_1 = x_1\} \cap \{X_2 = x_2\} \cap \cdots \cap \{X_n = x_n\}\right) .$$

Discuss your answer. The maximizing p-value is known as the *maximum likelihood estimate* of p.

(b) You can use MATLAB to generate n independent outcomes of a Bernoulli p random variable using the code

```
for k = 1:n:
 x(k) = binornd(1,p);
end
```

Set $p = 0.5$ and use this code to generate a sequence of 1000 independent outcomes of the corresponding Bernoulli random variable. Using that sequence and the result of part (a), compute for each n the maximum likelihood estimate of p based on observations 1 through n for each $n = 1, 2, 3, \ldots, 1000$ and plot that estimate as a function of n. Discuss the behavior of your estimator. Please include your MATLAB code.

3. Now you perform a sequence of independent experiments with the coin from the previous problem. Number the experiments $m = 1, 2, 3, \ldots$. In experiment m, you flip the coin repeatedly and independently until heads comes up. For each m, define X_m as the geometric random variable that equals k when the first head in experiment m occurs on flip k.

(a) Suppose you observe a sequence $x_1, x_2, \ldots, x_n$ of outcomes of the first n experiments — here each x_m is is a positive integer, namely the value of X_m you observed on experiment m. As a function of the x_m's, find the value of p that maximizes the natural log of the probability of the sequence you observed, i.e maximizes

$$\ln\left(\mathbb{P}\left(\{X_1 = x_1\} \cap \{X_2 = x_2\} \cap \cdots \cap \{X_n = x_n\}\right)\right) .$$

Discuss your answer.

(b) Explain why the value of p you computed in (a) also maximizes

$$\mathbb{P}\left(\{X_1 = x_1\} \cap \{X_2 = x_2\} \cap \cdots \cap \{X_n = x_n\}\right) .$$

4. In this problem you roll a pair of fair six-sided dice. Suppose the dice are labeled A and B.

(a) You repeatedly and independently roll the two dice simultaneously until you get doubles. Let X be the number of rolls it takes. Find the pmf, expected value, and variance of X.

(b) You repeatedly and independently roll the two dice simultaneously until one comes up 3 and the other comes up 4 on the same roll. Let Y be the number of rolls it takes. Find the pmf, expected value, and variance of Y. What is the probability that the last toss of the die labeled A is a 3?

5. This problem pertains to repetition coding in digital communications. A transmitter wants to send a bit through a noisy channel. Suppose the channel flips any transmitted bit with probability $\epsilon > 0$ and doesn't flip the bit with probability $1 - \epsilon$. The transmitter is equally likely to want to send a 0 or a 1 through the channel. The transmitter and receiver agree on the following scheme: the transmitter will send its bit three times in succession, and the receiver will decide what bit the transmitter sent based on majority rule — i.e. if two or three of the received bits are 1's, the receiver decides the transmitter sent a 1; otherwise, the receiver decides the transmitter sent a zero. What is the probability that the receiver decides incorrectly?

6. Let X be a random variable that takes on only nonnegative integer values and suppose $\mathbb{E}(X)$ is finite. Show that

$$\mathbb{E}(X) = \sum_{k=1}^{\infty} \mathbb{P}(\{X \geq k\}) .$$

(Suggestion: after expressing the right-hand side as a double summation, interchange the order of summation.)

① (a)

$$X = \{n, n-1, n-2, \cdots, n-n\}$$ i.e integer k s.t $0 \le k \le n$

So $(n+1)$ total possibilities

(b) Let L_k = (event $k-1$ arrows remain in right quiver when L discovers right is empty)

Note $0 \le k \le n$

R_k = (event k arrows remain in left quiver when L discovers left is empty)

$$\mathbb{P}(L_k) = p^{n+1}(1-p)^{n-k}\binom{2n-k}{n}$$

$$\mathbb{P}(R_k) = (1-p)^{n+1}p^{n-k}\binom{2n-k}{n}$$

$$p_X(k) = \mathbb{P}(L_k) + \mathbb{P}(R_k)$$

$$= \binom{2n-k}{n}\left[p^{n+1}(1-p)^{n-k} + (1-p)^{n+1}p^{n-k}\right]$$

② X_k ; $1 \le k \le n$; Bernoulli p.

$$X_k = \begin{cases} 0, & \text{flip } k \text{ is tails} \\ 1, & \text{flip } k \text{ is heads} \end{cases}$$

(a) Observe $x_1, \dots, x_n$.

Maximize

$$\mathbb{P}(\{X_1 = x_1\} \cap \dots \cap \{X_n = x_n\})$$

Intuitively, this will be the mean of the flips.

i.e

$$\boxed{\frac{1}{n} \sum_{k=1}^{n} x_k}$$

To see this, observe the following.

Let

$$S = \sum_{k=1}^{n} x_k$$

be the amount of heads flipped.

Want to maximiz

$$p^s (1-p)^{n-s}$$

with respect to p.

- If $s=0$, have $(1-p)^n$ which decreases monotonically from $1 \to 0$ for $p=0$ to $p=1$. So maximizing p value is $p=0$.
- If $s=n$, have p^n. So maximizing value of $p=1$.
- If $0 < s < n$,

$$\frac{d}{dp}\left[p^s(1-p)^{n-s}\right] = s(1-p) - (n-s)p = 0 \Rightarrow p = \frac{s}{n}$$

③ $m = 1, 2, 3, \ldots$

X_m = geometric rv that equas k when the firsts head in experiment m occurs on flip k

As a function of X_m's find p that maximizes

$$\ln\left(\mathbb{P}\left(\{X_1 = x_1\} \cap \{X_2 = x_2\} \cap \cdots \cap \{X_n = x_n\}\right)\right)$$

Note: $\mathbb{P}\left(\{X_1 = x_1\} \cap \{X_2 = x_2\} \cap \cdots \cap \{X_n = x_n\}\right)$

$$= \left(p(1-p)^{x_1 - 1}\right)\left(p(1-p)^{x_2-1}\right)\cdots\left(p(1-p)^{x_n-1}\right)$$

$$= p^n (1-p)^{x_1 + x_2 + \cdots + x_n - (1+1+\cdots+1)}$$

$$= p^n (1-p)^{S_n - n} \quad ; \quad S_n := \sum_{m=1}^{n} x_m$$

Let this be $g(p)$.

$$\ln g(p) = n \ln(p) + (S_n - n)\ln(1-p)$$

$$\frac{d}{dp}\left[\ln(g(p))\right] = \frac{n}{p} + \frac{S_n - n}{1-p} = \frac{n(1-p) - (S_n - n)p}{p(1-p)} = 0$$

thus, for $p \in (0, 1]$ the only p is

$$\boxed{p = \frac{n}{S_n}}$$

← to realize this as a max take second derivative

(4) (a) A , B . rolls are independent!
X = # (rolls it takes until doubles)

X is a geometric random variable! $X = \{x \in \mathbb{N} : x > 0\}$

$$p_X(k) = p(1-p)^{k-1} \quad , \quad 1 \le k < \infty$$

p = probability you roll doubles = $1/6$
$1-p$ = probability you don't roll doubles = $5/6$

$$\boxed{p_X(x) = \begin{cases} \frac{1}{6} \cdot \left(\frac{5}{6}\right)^{x-1} & , \quad 1 \le x < \infty \\ 0 & , \quad \text{else} \end{cases}}$$

$$\mathbb{E}(X) = \sum_{x=1}^{\infty} x \cdot \frac{1}{6}\left(\frac{5}{6}\right)^{x-1} = \sum_{x=1}^{\infty} x \cdot p(1-p)^{x-1}$$

$$= p \sum_{x=1}^{\infty} x(1-p)^{x-1}$$

$$= p\left(-\frac{d}{dp} \sum_{x=1}^{\infty} (1-p)^{x-1}\right)$$

$$= \frac{1}{6}\left(-\frac{d}{dp} \cdot \frac{1-p}{p}\right) = p\left(\frac{1}{p^2}\right) = \frac{1}{p}$$

$$\boxed{\mathbb{E}(X) = 1/\tfrac{1}{6} = 6}$$

$$\mathrm{Var}(X) = \frac{1-p}{p^2} = \frac{5/6}{1/36} = \boxed{30}$$

WIKI on Geometric RV

(b) $P_Y(y) = \left(\frac{1}{18}\right)\left(\frac{17}{18}\right)^{y-1}$

$$\mathbb{E}(Y) = \frac{1}{1/18} = 18$$

$$\mathrm{VAR}(Y) = \frac{17/18}{(1/18)^2} = 17 \cdot 18 = 306$$

(5) Let

$$A = \{\text{receiver decides incorrectly}\}$$
$$B = \{\text{Probability transmitter sends a } 0\}$$

Then

$$\mathbb{P}(A) = \mathbb{P}(A|B)\mathbb{P}(B) + \mathbb{P}(A|B^c)\mathbb{P}(B^c)$$

Where

- $\mathbb{P}(B) = \mathbb{P}(B^c) = 1/2$ given
- $\mathbb{P}(A|B)$ is the probability receiver sees either 110, 101, 011, or 111 given that the transmitter sends 000.
    - $\mathbb{P}(110\ R \mid 000\ S) = \varepsilon^2(1-\varepsilon)$
    - $\mathbb{P}(101\ R \mid 000\ S) = \varepsilon^2(1-\varepsilon)$
    - $\mathbb{P}(011\ R \mid 000\ S) = \varepsilon^2(1-\varepsilon)$
    - $\mathbb{P}(111\ R \mid 000\ S) = \varepsilon^3$
- $\mathbb{P}(A|B^c)$ is the same as $\mathbb{P}(A|B)$ with zeros + ones interchanged.

Thus

$$\mathbb{P}(A) = \frac{3\varepsilon^2(1-\varepsilon) + \varepsilon^2}{2} + \frac{3\varepsilon^2(1-\varepsilon) + \varepsilon^2}{2}$$

$$= 3\varepsilon^2(1-\varepsilon) + \varepsilon^3$$

More simple way:

This is simply the probability 2 are incorrect **OR** 3 are incorrect.

So,

$$\binom{3}{2}\varepsilon^2(1-\varepsilon) + \varepsilon^3$$

(6) Show that

$$\mathbb{E}(X) = \sum_{k=1}^{\infty} \mathbb{P}(\{X \geq k\})$$

for X a rv which takes on only nonnegative integer values; $\mathbb{E}(X)$ finite.

$$\sum_{k=1}^{\infty} \mathbb{P}(\{X \geq k\}) = \sum_{k=1}^{\infty} \left(\sum_{n=k}^{\infty} \mathbb{P}(\{X = n\}) \right.$$

$$= \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} \mathbb{P}(\{X = n\}) \right)$$

$$= \sum_{k=1}^{\infty} \mathbb{P}(\{X \geq k\}) = \sum_{n=1}^{\infty} n\,\mathbb{P}(\{X = n\}) = \mathbb{E}(X)$$