**Exercises.**

**Solution to Question 1.**

$$A^T A = \begin{bmatrix} 3 & 0 & 0 & 3 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 3 & 0 & 0 & 3 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 4 & 0 & 2 \\ 0 & 6 & 0 \\ 2 & 0 & 4 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} \sqrt{6} & 0 & 0 & 0 \\ 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \end{bmatrix}, \; V = \begin{bmatrix} 1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1/\sqrt{2} & 0 & 0 & -1/\sqrt{2} \end{bmatrix}, \; U = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix}.$$

**Solution to Question 2.** If $A$ is an $n \times m$ matrix, then so are $\Sigma_1$ and $\Sigma_2$. By definition, $\Sigma_1$ and $\Sigma_2$ have non-zero entries only on the diagonal. Therefore, if they have same diagonal entries, then $\Sigma_1 = \Sigma_2$.
Assume

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0$$

are eigenvalues of $A^T A$, in non-ascending order. So for $i = 1, 2$, the non-zero diagonal entries of $\Sigma_i$ are $\sqrt{\lambda_1}, \sqrt{\lambda_2}, \ldots, \sqrt{\lambda_r}$. Therefore, $\Sigma_1 = \Sigma_2$.

**Solution to Question 3.**

(a) Assume the eigenvalues of $A$ are

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0.$$

Because $A^T A = A^2$, so the eigenvalues of $A^T A$ are

$$\lambda_1^2 \geq \lambda_2^2 \geq \cdots \geq \lambda_r^2.$$

Therefore, the singular values are $\lambda_i = |\lambda_i|$ for all $i = 1, \ldots, r$.

(b) If we have

$$\lambda_1 > \lambda_2 > \cdots > \lambda_r > 0,$$

then this is true.
If not, $U$ and $V$ may differ by an orthonormal matrix $Q$, i.e. $V^T U = Q$.

**Solution to Question 4.**

(a) Assume $A = U\Sigma V^T$, where

$$U = \begin{bmatrix} u_1 & u_2 & \dots & u_n \end{bmatrix}, \; V = \begin{bmatrix} v_1 & v_2 & \dots & u_n \end{bmatrix}.$$

Because $\operatorname{rank}(\Sigma) = \operatorname{rank}(A) = 1$, so

$$\Sigma = \begin{bmatrix} \alpha & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{bmatrix}$$

for some $\alpha > 0$. Let $x = u_1$ and $y = v_1$. Then $A = \alpha x y^T$.

(b) By part (a), $A = \alpha x y^T$. So

$$A = x \begin{bmatrix} \alpha \end{bmatrix} y^T$$

is a slim singular value decomposition for $A$.

(c) If $A = \alpha x y^T$, then $A^T A = \alpha^2 y y^T$. So $y$ is an eigenvector of $A^T A$ with eigenvalue $\alpha^2$. Because $\dim \ker A = n - \operatorname{rank}(A) = n - 1$, so $\dim E_{\lambda=0}(A) = n - 1$. Let

$$E_{\lambda=\alpha^2}(A) = \text{span}\,\{y\}.$$

Then

$$E_{\lambda=0}(A) = E_{\lambda=\alpha^2}(A)^{\perp}.$$

**Solution to Question 5.**

$$Q = I \cdot Q.$$

**Solution to Question 6.**

(a) Let $c_{ij}$ be the $(i, j)$-entry of $A^T B$. Then

$$c_{ii} = \sum_{j=1}^{r} A_{i,j} B_{i,j}.$$

So

$$\operatorname{trace}(A^T B) = \sum_{i=1}^{r} \sum_{j=1}^{r} A_{i,j} B_{i,j}.$$

(b) By part (a),

$$\begin{aligned}
\langle M_i, M_j \rangle &= \operatorname{trace}(M_i^T M_j) \\
&= \operatorname{trace}(v_i u_i^T u_j v_j^T) \\
&= \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}
\end{aligned}$$

So $M_i$ are orthonormal. Note

$$\|A\|^2 = \langle A, A \rangle = \operatorname{trace}(A^T A),$$

and

$$\begin{aligned}
A^T A &= (\sigma_1 M_1 + \sigma_2 M_2 + \cdots + \sigma_r M_r)^T (\sigma_1 M_1 + \sigma_2 M_2 + \cdots + \sigma_r M_r) \\
&= \sum_{i,j=1}^{r} \sigma_i \sigma_j M_i^T M_j \\
&= \sum_{i=1}^{r} \sigma_i^2 M_i^T M_i.
\end{aligned}$$

So

$$\operatorname{trace}(A^T A) == \sum_{i=1}^{r} \sigma_i^2.$$

(c) Because

$$A - A_k = \sigma_{k+1} u_{k+1} v_{k+1}^T + \sigma_{k+2} u_{k+2} v_{k+2}^T + \cdots + \sigma_r u_r v_r^T,$$

so

$$\begin{aligned}
\|A - A_k\|^2 &= \operatorname{trace}\Big( \sum_{i=k+1}^{r} \sigma_i^2 M_i^T M_i \Big) \\
&= \sum_{i=k+1}^{r} \sigma_i^2.
\end{aligned}$$

(d) Because

$$\langle QA, QA\rangle = \text{trace}((QA)^{\mathsf{T}}QA) = \text{trace}(A^{\mathsf{T}}A) = \langle A, A\rangle,$$

so

$$\|QA\| = \|A\|.$$

Similarly,

$$\langle AQ, AQ\rangle = \text{trace}(AQ(AQ)^{\mathsf{T}}) = \text{trace}(A^{\mathsf{T}}A) = \langle A, A\rangle,$$

Therefore, if $A = U\Sigma V^{\mathsf{T}}$ is an SVD of $A$, then

$$\|A\| = \|\Sigma\|.$$

**Part 2.**

**Solution to Question 1.**
Let $J$ be a Jordan canonical form of $A$, then $J^T$ is a Jordan canonical form of $A^T$. We only need to prove $J \sim J^T$. In particular, we only need to prove $J(\lambda, k) \sim J(\lambda, k)^T$ for each Jordan block $J(\lambda, k)$. This is true because

$$P^{-1} J(\lambda, k) P = J(\lambda, k)^T,$$

where

$$P = P^{-1} = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{bmatrix}.$$

**Part 2.**

**Solution to Question 2.**

(a) Because $\dim\ker J(0,k) = 1$ and $\operatorname{rank}(J(0,k)) = k-1$, so

$$\operatorname{rank}(N) = \sum_{i=1}^{r} \operatorname{rank}(J(0,k_i)) = n - r,$$

and

$$\dim\ker N = r.$$

(b) Note that

$$N^k = J(0,k_1)^k \oplus J(0,k_2)^k \oplus \cdots \oplus J(0,k_r)^k.$$

Because $J(0,k_1)^{k_1} = 0$ and $J(0,k_1)^{k_1-1} \neq 0$, so the index of $N$ is $k_1$.

(c) Note that $\lambda$ is the only eigenvalue of $A$, so the algebraic multiplicity is $n$. And the geometric multiplicity of $A$ is the number of Jordan blocks, $r$.

(d) For each Jordan block $J(0,k_i)$,

$$\operatorname{rank}(J(0,k_i)^m) = \max\{0, k_i - m\}.$$

Therefore,

$$\operatorname{rank}(N^m) = \sum_{i=1}^{r} \max\{0, k_i - m\},$$

and

$$\dim\ker(N^m) = n - \operatorname{rank}(N^m) = n - \sum_{i=1}^{r} \max\{0, k_i - m\}.$$

(e) **Claim**: Knowing $\dim\ker N^m$ for all $m$ determine $(k_1, k_2, \dots, k_r)$.

*Proof.* Assume $d_m = \dim\ker N^m$. Then by part (d), $\Delta_m := (d_m - d_{m-1})$ is the number of $J(0,k)$ with $k \geq m$. So

$$\Delta_m - \Delta_{m+1} = 2d_m - d_{m+1} - d_{m-1}$$

is the number of $J(0,k)$ in $N$ with $k = m$. $\square$

**Part 2.**

**Solution to Question 3.** Let

$$N = T - I = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Then

$$N^2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$N^3 = 0.$$

Assume $d_m = \dim \ker N^m$. Then $d_1 = 3$, $d_2 = 5$ and $d_m = 6$ for $m \geq 3$. The number of $J(0, k)$ in $N$ with $k = m$ is

$$n(m) = 2d_m - d_{m-1} - d_{m+1} = \begin{cases} 1, & \text{for } m = 1, \\ 1, & \text{for } m = 2, \\ 1, & \text{for } m = 3, \\ 0, & \text{for } m \geq 4. \end{cases}$$

Therefore, the Jordan canonical form of $T$ is

$$\begin{aligned} J &= J(1,3) \oplus J(1,2) \oplus J(1,1) \\ &= \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$