1. Let X, Y, and Z be discrete random variables on the same probability space. Show that

$$\mathbb{E}(X \mid Z) = \mathbb{E}(\mathbb{E}(X \mid Y, Z) \mid Z) \ .$$

To make things easier, let

$$h(y, z) = \mathbb{E}(X \mid Y = y, Z = z) \ ,$$

so

$$\mathbb{E}(X \mid Y, Z) = h(Y, Z)$$

and, by the expected value rule,

$$\mathbb{E}(h(Y, Z) \mid Z = z) = \sum_y h(y, z) p_{Y|Z}(y \mid z) \ .$$

2. (Problem 4.19 in the book) Suppose a random variable X satisfies $\mathbb{E}(X) = 0$, $\mathbb{E}\left(X^2\right) = 1$, $\mathbb{E}\left(X^3\right) = 0$, and $\mathbb{E}\left(X^4\right) = 3$. Let $Y = a + bx + cX^2$. Find the correlation coefficient $\rho(X, Y)$.

3. (Problem 4.26 in the book) Let X and Y be independent random variables. Use the law of total variance to show that

$$\mathrm{Var}(XY) = (\mathbb{E}(X))^2 + (\mathbb{E}(Y))^2 + \mathrm{var}(X)\mathrm{Var}(Y) \ .$$

4. (Problem 4.27 in the book) Q is a random variable with mean μ and variance σ^2. We toss n times a coin whose probability of heads is Q and for $1 \leq k \leq n$ define X_k as the Bernoulli random variable whose value is 1 when the kth toss is a head. Assume that the X_k are conditionally independent given $Q = q$. Let X be the number of heads among the n tosses.

(a) Use the law of iterated expectation to find $\mathbb{E}(X_k)$ and $\mathbb{E}(X)$.
(b) Find $\mathrm{Cov}(X_k, X_l)$. Are $X_1, \ldots, X_n$ independent?
(c) Use the law of total variance to find $\mathrm{Var}(X)$. Verify your answer using the covariance result of part (b).