Problem to ponder

Consider $A \in \mathbb{F}^{n \times n}$

A^T = transpose A

Q: do A, A^T have the same eigenvalues?

We showed

$$\dim \operatorname{im} A = \dim \operatorname{im} A^T$$
$$\dim \ker A = \dim \ker A^T$$

If λ = eigenvalue of A, $A\vec{v} = \lambda \vec{v}$ for some $\vec{v} \neq 0$.

then

$$\lambda = \text{eigenvalue of } A$$
$$\Leftrightarrow \dim \ker(A - \lambda I) \geq 1$$
$$\Leftrightarrow \dim \ker((A-\lambda I)^T) \geq 1$$
$$\Leftrightarrow \dim \ker(A^T - \lambda I) \geq 1$$
$$\Leftrightarrow \lambda = \text{eigenvalue for } A^T.$$

Last Time: $A \in \mathbb{C}^{n \times n}$, $T \in \mathcal{L}(V)$ (over $\mathbb{C}$).

- $\exists$ eigenvector of A over T
- eigenspace $E_\lambda(A) = \ker(A - \lambda I)$
- Direct sums

$$V = V_1 \oplus \cdots \oplus V_r$$

basis of V is union of bases of $V_1, \ldots, V_r$

and

$$\dim V = \sum_{i=1}^{r} \dim V_i$$

Proposition: Let $T \in \mathcal{L}(V)$, $\dim V < \infty$ (over $\mathbb{F}$).
Suppose $\lambda_1, \dots, \lambda_m$ are the distinct eigenvalues of T then if

$$W = E_{\lambda_1}(T) + \dots + E_{\lambda_m}(T) \subseteq V$$

then

$$W = E_{\lambda_1}(T) \oplus \dots \oplus E_{\lambda_m}(T) \subseteq V$$

Proof: Need to show that if $\vec{v}_i \in E_\lambda(T)$, $i = 1, \dots, m$
then $\vec{v}_i \neq 0 \ \forall i$, then $\vec{v}_1 + \dots + \vec{v}_m = 0$
KNOW $\vec{v}_1, \dots, \vec{v}_m$ is LI since $\vec{v}_i$ is an eigenvector w/ eigenvalue λ_i and all λ_i distinct.

$$\Rightarrow \vec{v}_1 + \dots + \vec{v}_m \neq 0$$

Theorem: Let $T \in \mathcal{L}(V)$, $\dim V = n < \infty$, (over $\mathbb{F}$)
The following are equivalent:

1. T is diagonalizable
2. $[T]_\beta$ is diagonalizable for any basis β
3. V has a basis consisting of eigenvectors (of T)
4. If $\lambda_1, \dots, \lambda_m$ are the distinct eigenvalues (of T), then $V = E_{\lambda_1}(T) \oplus \dots \oplus E_{\lambda_m}(T)$
5. $\dim V = \sum_{i=1}^{m} \dim E_{\lambda_i}(T)$

Proof) ① ⇒ ②

⇒ $[T]_\alpha$ diagonal, for some basis α

but we know $[T]_\beta = Q^{-1}[T]_\alpha Q$ for some

Q $n\times n$ invertible, and thus $[T]_\beta$ diagonalizable.

⇐ "trivial"

① ⇔ ③ "example" from class a few lectures ago

See a GOOD proof

Corollary: If T or A has n distinct eigenvalues, then T is diagonalizable $(\dim V = n)$

Proof: let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of and $\vec{v}_1, \ldots, \vec{v}_n$ be the corresponding eigenvectors.

We know $(\vec{v}_1, \ldots, \vec{v}_n)$ is LI but $\dim V = n$

$\therefore$ this is a basis

One value of diagonalizability?

Let A be $n\times n$ s.t.

$$Q^{-1} A Q = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} = D$$

then

$$A = QDQ^{-1}$$

$$A^2 = QD^2Q^{-1}$$

$$\vdots$$

$$A^N = QD^NQ^{-1}$$

Fibonacci Numbers

Defined by

$$F_1 = 1$$
$$F_2 = 1$$
$$F_n = F_{n-2} + F_{n-1} \quad , n \geq 3$$

i.e

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots,$$

World's Worst Computer Program

$$\begin{aligned} \text{fib } n &= 1 \quad \text{if } n=1 \text{ or } n=2 \\ &= \text{fib}(n-2) + \text{fib}(n-1) \quad \text{if } n \geq 3 \end{aligned}$$

If we have $\begin{pmatrix} F_{n-1} \\ F_n \end{pmatrix}$, we can get $\begin{pmatrix} F_n \\ F_{n+1} \end{pmatrix} = \begin{pmatrix} F_n \\ F_n + F_{n-1} \end{pmatrix}$

i.e given $\begin{pmatrix} a \\ b \end{pmatrix}$, next steb $\begin{pmatrix} b \\ a+b \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$

let

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix},$$

Notice $A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$

also note $A^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$\vdots$$

$$A^n \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} F_n \\ F_{n+1} \end{pmatrix}$$

Eigenvalues of A? is A diagonalizable?

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{pmatrix} b \\ a+b \end{pmatrix} = \begin{pmatrix} \lambda a \\ \lambda b \end{pmatrix}$$

assume $a \neq 0$ since $\begin{pmatrix} a \\ b \end{pmatrix}$ is to be an eigenvector.

$$\lambda a = b$$

$$\lambda b = a + b$$

$$\lambda^2 a = a + \lambda a$$

$$\Rightarrow \lambda^2 - \lambda - 1 = 0$$

$$\lambda_1 = \frac{1 + \sqrt{5}}{2}$$

$$\lambda_2 = \frac{1 - \sqrt{5}}{2}$$

eigenvectors of A: set $a = 1$, $b = \lambda$

So have

$$\vec{v}_1 = \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix}$$

and thus

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = Q^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} Q$$

for $$Q = \begin{array}{c} \begin{matrix} \vec{v}_1 & \vec{v}_2 \end{matrix} \\ \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix} \end{array}$$

So

$$A = Q\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} Q^{-1}$$

$$\begin{pmatrix} F_n \\ F_{n+1} \end{pmatrix} = A^n \begin{pmatrix} 0 \\ 1 \end{pmatrix} = Q\begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} Q^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix} \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

So

$$F_n = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{5}} \left(\lambda_1^n - \lambda_2^n \right)$$

$$F_{n+1} = \frac{1}{\sqrt{5}} \begin{pmatrix} \lambda_1 & \lambda_2 \end{pmatrix} \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{5}} \left(\lambda_1^{n+1} - \lambda_2^{n+1} \right)$$