Useful definitions and facts:

- The cumulative distribution function (cdf) of a continuous random variable X with probability density function (pdf) $f_X(x)$ is
$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt \ .$$
- In the same context, the expected value of X is
$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x f_X(x)dx \ .$$
- In the same context, the variance of X is
$$\mathrm{Var}(X) = \mathbb{E}\left((X - \mathbb{E}(X)^2\right) \ .$$

1. This one is a variation on Problem 1 of Homework VII. Frodo tosses a coin repeatedly and independently. The probability that the coin comes up heads on any toss is p. Every time Frodo tosses a tail followed by a head, Sam pays him \$1. Let X be the total amount of money Frodo has received after 8 tosses. Find $\mathbb{E}(X)$ and $\mathrm{Var}(X)$.

2. A continuous random variable X has pdf
$$f_X(x) = \begin{cases} \frac{1}{\alpha} e^{-(x-\beta)/\alpha} & \text{when } x \geq \beta \\ 0 & \text{otherwise,} \end{cases}$$
where $\alpha > 0$ and β are given real numbers.

(a) Find the cdf $F_X(x)$ of X.
(b) Find $\mathbb{E}(X)$ and $\mathrm{Var}(X)$.
(c) Find the real number m that satisfies $F_X(m) = 1/2$. This number m is known as the median of the random variable X.

3. A random variable X has cdf
$$F_X(x) = \begin{cases} 1 - e^{-(x/\lambda)^\beta} & \text{when } x \geq 0 \\ 0 & \text{otherwise,} \end{cases}$$
where β and λ are given positive real numbers.

(a) Find the pdf $f_X(x)$ of X.
(b) Plot the cdf and pdf of X for $\lambda = 1$ and $\beta = 0.5$, 1.5, and 5.

4. Aragorn and Boromir are bidding on a construction project. The low bidder will win the contract and then pay Gimli's stonemason crew \$100,000 to do the work. Aragorn believes that Boromir's bid is a random variable X distributed uniformly on the interval $[70,000, 140,000]$ dollars. What should Aragorn bid to maximize his expected profit?

5. Let X have exponential pdf
$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{when } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$
with $\lambda = 3$. Let $Y = \lfloor X \rfloor$. Find $\mathbb{E}(Y)$ and $\mathrm{Var}(Y)$.

6. This is a continuous version of Problem 6 from Homework V. Let X be a nonnegative continuous random variable with pdf $f_X(x)$ and cdf $F_X(x)$. Assuming $\mathbb{E}(X)$ exists, show that

$$\mathbb{E}(X) = \int_0^\infty (1 - F_X(x))dx \; .$$

Note that $1 - F_X(x) = \mathbb{P}(\{X > x\})$.

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ECE 3100

HW8 3/29/19

(Late submission Granted)

① $\mathbb{P}(\text{coin is heads}) = p$

8 tosses.

If a head is tossed immediately following a tail get \$1.

So max amount paid is \$4.

X = amount received after 8 tosses $\in \{0,1,2,3,4\}$

$$\mathbb{E}(X) = \cancel{0\left[(1-p)^8 + p^8\right]} \to 0 + 1\left[\binom{8}{1}(1-p)p^7\right] + 2\left[\binom{8}{2}(1-p)^2p^6\right]$$
$$+ 3\left[\binom{8}{3}(1-p)^3(p)^5\right] + 4\left[\binom{8}{4}(1-p)^4p^4\right]$$

$$\mathbb{E}(X^2) = \cancel{0^2\left[(1-p)^8 + p^8\right]} \to 0 + 1^2\left[\binom{8}{1}(1-p)p^7\right] + 2^2\left[\binom{8}{2}(1-p)^2p^6\right]$$
$$+ 3^2\left[\binom{8}{3}(1-p)^3(p)^5\right] + 4^2\left[\binom{8}{4}(1-p)^4p^4\right]$$

$$\mathrm{Var}(X) = 2\left[\binom{8}{2}(1-p)^2p^6\right] + 6\left[\binom{8}{3}(1-p)^3p^5\right] + 12\left[\binom{8}{4}(1-p)^4p^4\right]$$

... It makes sense in my head

Looking back don't need $\binom{n}{k}$ I was trying to make order matter

(2)

$$f_X(x) = \begin{cases} \frac{1}{\alpha} e^{-(x-\beta)/\alpha} & , x \geq \beta \\ 0 & , \text{otherwise} \end{cases}$$

$\alpha > 0$, $\beta \in \mathbb{R}$

(a)

$$F_X(x) = \begin{cases} \int_\beta^x \frac{1}{\alpha} e^{(t-\beta)/\alpha} dt = e^{-(t-\beta)/\alpha} \Big|_\beta^x = 1 - e^{-(x-\beta)/\alpha} & , x \geq \beta \\ 0, & x < \beta \end{cases}$$

(b)

$$\mathbb{E}(X) = \int_\beta^\infty x \cdot \frac{1}{\alpha} e^{-(x-\beta)/\alpha} dx$$

$$= \frac{e^{\beta/\alpha}}{\alpha} \int_\beta^\infty x e^{-x/\alpha} dx$$

u dv

$x \searrow e^{-x/\alpha}$

$-1 \searrow -\alpha e^{-x/\alpha}$

$0 \quad \alpha^2 e^{-x/\alpha}$

$$= \frac{e^{\beta/\alpha}}{\alpha} \left[-\alpha e^{-x/\alpha} - \alpha^2 e^{-x/\alpha} \Big|_\beta^\infty \right] = \beta + \alpha$$

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$$

$$\mathbb{E}(X^2) = \int_\beta^\infty \frac{x^2}{\alpha} e^{-(x-\beta)/\alpha}\, dx$$

u	dv
x^2	$e^{-x/\alpha}$
$-2x$	$-\alpha e^{-x/\alpha}$
2	$\alpha^2 e^{-x/\alpha}$
0	$-\alpha^3 e^{-x/\alpha}$

$$= \frac{e^{\beta/\alpha}}{\cancel{\alpha}} \left[-x^2 \cancel{\alpha} e^{-x/\alpha} - 2x\alpha^{\cancel{2}} e^{-x/\alpha} - 2\alpha^{\cancel{3}\,2} e^{-x/\alpha} \right] \Big|_\beta^\infty$$

Thus

$$\mathrm{Var}(X) = \beta^2 + 2\alpha\beta + 2\alpha^2 - (\alpha+\beta)^2 = \alpha^2$$

(c) Need

$$F_X(m) = 1/2$$

$$1 - e^{\frac{-(m-\beta)}{\alpha}} = \frac{1}{2}$$

$$e^{-(m-\beta)/\alpha} = \frac{1}{2}$$

$$\frac{-(m-\beta)}{\alpha} = \ln\left(\frac{1}{2}\right)$$

$$\frac{m-\beta}{\alpha} = \ln(2)$$

$$\boxed{m = \alpha \ln(2) + \beta}$$

③

$$F_X(x) = \begin{cases} 1 - e^{-\left(\frac{x}{\lambda}\right)^{\beta}} & , \ x \geq 0 \\ 0 & , \ \text{otherwise} \end{cases}$$

$\beta, \lambda \in \mathbb{R}$

(a)

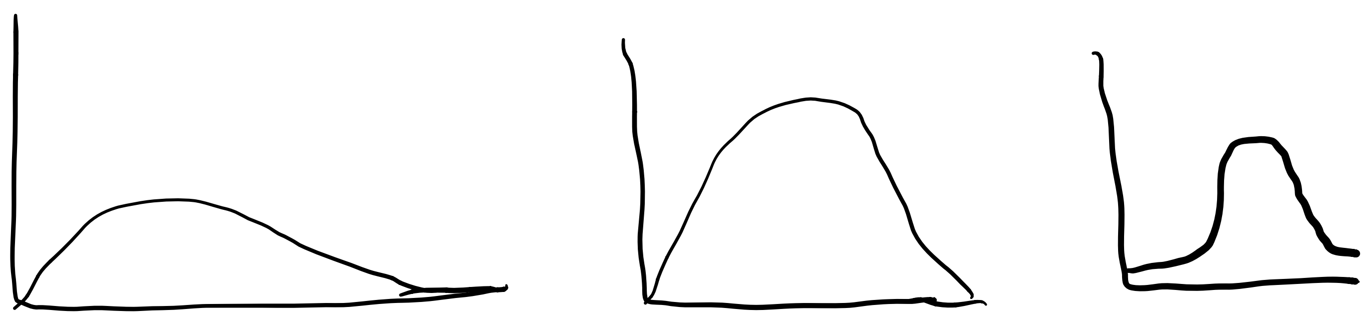
$$f_X(x) = \frac{d}{dx} F_X(x) = \begin{cases} \dfrac{\beta x^{\beta-1}}{\lambda^{\beta}} e^{-(x/\lambda)^{\beta}} & , \ x \geq 0 \\ 0 & , \ \text{otherwise} \end{cases}$$

(b)

$\lambda = 1$

CDF $\quad \beta = \frac{1}{2} \quad \beta = \frac{3}{2} \quad \beta = 5$

PDF



(4) If Boromir bid 70,000, then Aragorn cannot win. Similarly, if Boromir bid over 140,000 then Aragorn always wins.

$X \in [70{,}000,\ 140{,}000]$ ← Boromirs bid

$$\mathbb{P}(\text{Aragorn wins}) = \begin{cases} \dfrac{x - 70{,}000}{70{,}000} & 70{,}000 \le x \le 140{,}000 \\ 0, & \text{otherwise} \end{cases}$$

Maximize profit bid?

middle of interval

$$\mathbb{E}\text{xpected profit} = \frac{x - 70{,}000}{70{,}000}\,(105{,}000 - x)$$

Max @ $x = 87{,}500$ (Wolfram)

profit = $105{,}000 - 87{,}500 = \boxed{17{,}500}$

⑤

$$p_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$\lambda = 3.$

$Y = \lfloor X \rfloor$. $\mathbb{E}(Y)$? $Var(Y)$?
$= g(X)$

$$\mathbb{E}(Y) = \int_{-\infty}^{+\infty} g(x) p_X(x)\, dx = \int_0^{\infty} \lfloor x \rfloor \lambda e^{-\lambda x}$$

Online $\Longrightarrow$ .052
Calculator

$(Y) = .044$

⑥ X a nonnegative continuous rv w/ pdf $f_X(x)$;
cdf $F_X(x)$.

Assuming $\mathbb{E}(X)$ exists, show

$$\mathbb{E}(X) = \int_0^\infty (1 - F_X(x))\,dx$$

$$1 - F_X(x) = \mathbb{P}(\{X > x\}) = \int_x^\infty f_X(t)\,dt$$

Taking integral from $0 \to \infty$ with respect to x we get

$$\int_0^\infty \left(\int_x^\infty f_X(t)\,dt \right) dx$$

Using the indicator function,

$$= \int_{x=0}^{x=\infty} \left[\int_{t=0}^{t=\infty} f_X(t)\ 1\{t \geq x\}\,dt \right] dx$$

$$= \int_{t=0}^{t=\infty} f_X(t) \left[\int_{x=0}^{\infty} 1\{t \geq x\}\,dx \right] dt$$

$$= \int_{t=0}^{t=\infty} f_X(t) \left[\int_{x=0}^{t} dx \right] dt$$

$$= \int_{t=0}^{\infty} t f_X(t) = \int_{t=-\infty}^{+\infty} t f_X(t)\,dt$$