

# ECE 4110 Homework 3

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*Due by 5pm on September 26*

## 1 Key Concepts and Reading Material

- Correlation, independence, and covariance (Chapter 5.6).
- Conditional distribution and conditional expectation (Chapter 3.4, Chapter 5.7).
- MMSE estimator (Chapter 6.5.3).

## 2 Assignment

### 1. Computing Expectations via Conditioning

The number of customers that arrive at a service station during a time  $t$  is a Poisson random variable with parameter  $\beta t$ . The time  $T$  required to service each customer is an exponential random variable with parameter  $\alpha$ .

Let  $N$  denote the number of customers arrived during the service time of a specific customer. Find the mean and variance of  $N$ .

(*Hint: In many cases, it is easier to compute the expectation of  $X$  by first conditioning on another random variable  $Y$  and then taking expectation over  $Y$ . Specifically, for any function  $h(\cdot)$  of  $X$ , we have*

$$\mathbb{E}[(h(X))] = \mathbb{E}\left[\mathbb{E}[h(X)|Y]\right].$$

*Use this to find the mean and variance of  $N$  by conditioning on  $T$ .)*

### 2. Conditioning Reduces Variance

Given random variables  $X$  and  $Y$ , the conditional expectation  $\mathbb{E}(X|Y)$  is a random variable itself with a well defined mean and variance. We have shown that the mean values of  $\mathbb{E}(X|Y)$  and  $X$  are the same, i.e.,

$$\mathbb{E}(\mathbb{E}(X|Y)) = \mathbb{E}(X).$$

Show that the variance of  $\mathbb{E}(X|Y)$  is smaller than or equal to the variance of  $X$ , i.e.,

$$\text{Var}(\mathbb{E}(X|Y)) \leq \text{Var}(X)$$

(*Hint: show that (by working from the right-hand side of the equation)*

$$\text{Var}(X) = \mathbb{E}(\text{Var}(X|Y)) + \text{Var}(\mathbb{E}(X|Y)),$$

where the conditional variance  $\text{Var}(X|Y)$  of  $X$  given  $Y$  is defined as

$$\text{Var}(X|Y) \triangleq \mathbb{E}((X - \mathbb{E}(X|Y))^2|Y).$$

The result thus follows since the conditional variance  $\text{Var}(X|Y)$  is a nonnegative random variable whose expectation must be nonnegative.)

### 3. Sum of Uncorrelated Random Variables

Show that if  $X_1, \dots, X_m$  are pairwise uncorrelated, then

$$\text{Var}\left(\sum_{i=1}^m X_i\right) = \sum_{i=1}^m \text{Var}(X_i).$$

### 4. Independence, Correlation, and MMSE Estimation

Consider two random variables  $X$  and  $Y$  with the following joint PDF.

$$f_{XY}(x, y) = \begin{cases} ce^{-x}e^{-y} & 0 \leq y \leq x < \infty \\ 0 & \text{elsewhere} \end{cases}$$

- (a) Find the value of the constant  $c$  in the joint PDF.
- (b) Calculate  $\Pr(X + Y \leq 1)$ .
- (c) Are  $X$  and  $Y$  independent (the joint PDF appears to factor)? Justify your answer.
- (d) Find the correlation, covariance, and correlation coefficient of  $X$  and  $Y$ .
- (e) Find the MMSE estimate of  $X$  using a constant and the resulting mean squared error.
- (f) Find the MMSE estimate of  $X$  based on  $Y$  (among all functions  $g(Y)$  of  $Y$ ).

Rami Pellumbi - rp534  
ECE 4110 HW3  
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## 1. Computing Expectations via Conditioning

The number of customers that arrive at a service station during a time  $t$  is a Poisson random variable with parameter  $\beta t$ . The time  $T$  required to service each customer is an exponential random variable with parameter  $\alpha$ .

Let  $N$  denote the number of customers arrived during the service time of a specific customer. Find the mean and variance of  $N$ .

Arrival,  $A \sim \text{Pois}(\beta t)$  ( $\lambda = \beta t$ )

Service Time,  $T \sim \exp(\alpha)$

$N \rightarrow$  number of customers arrived during the service time of a specific customer

Want  $E[N], \text{Var}(N)$ .

Know

$$F_T(t) = \begin{cases} 1 - e^{-\alpha t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$F_A(a) = \sum_{i=0}^a \frac{e^{-\beta t} (\beta t)^i}{i!}$$

Via Law of Iterated Expectation,

$$E_N[N] = E_T[E_{N|T}[N|T]]$$

A customer is being serviced.

Want average number of arrivals of new customers during this service time.

So we have

$$\mathbb{E}[N|T] = \mathbb{E}[\text{Pois}(\beta T)] = \beta T$$

$$\mathbb{E}_T [\mathbb{E}_{N|T}[N|T]] = \mathbb{E}_T [\beta T] = \beta \mathbb{E}_T [T] = \frac{\beta}{\alpha}$$

Thus,

$$\mathbb{E}[N] = \frac{\beta}{\alpha}$$

By law of total variance, know

$$\text{Var}(N) = \mathbb{E}[\text{Var}(N|T)] + \text{Var}(\mathbb{E}[N|T])$$

$$\text{Var}(\mathbb{E}[N|T]) = \text{Var}(\beta T) = \beta^2 \text{Var}(T) = \frac{\beta^2}{\alpha^2}$$

$$\text{Var}(N|T) = \text{Var}(\text{Pois}(\beta T)) = \beta T$$

$$\mathbb{E}[\text{Var}(N|T)] = \mathbb{E}[\beta T] = \frac{\beta}{\alpha}$$

Thus

$$\text{Var}(N) = \frac{\beta}{\alpha} + \frac{\beta^2}{\alpha^2}$$

$$= \frac{\beta\alpha + \beta^2}{\alpha^2}$$

## 2. Conditioning Reduces Variance

Given random variables  $X$  and  $Y$ , the conditional expectation  $\mathbb{E}(X|Y)$  is a random variable itself with a well defined mean and variance. We have shown that the mean values of  $\mathbb{E}(X|Y)$  and  $X$  are the same, i.e.,

$$\mathbb{E}(\mathbb{E}(X|Y)) = \mathbb{E}(X).$$

Show that the variance of  $\mathbb{E}(X|Y)$  is smaller than or equal to the variance of  $X$ , i.e.,

$$\text{Var}(\mathbb{E}(X|Y)) \leq \text{Var}(X)$$

We first show

$$\text{Var}(X) = \text{Var}(\mathbb{E}[X|Y]) + \mathbb{E}[\text{Var}(X|Y)]$$

where

$$\text{Var}(X|Y) \triangleq \mathbb{E}[(X - \mathbb{E}[X|Y])^2|Y]$$

$$\begin{aligned}\mathbb{E}[\text{Var}(X|Y)] &= \mathbb{E}[\mathbb{E}[(X - \mathbb{E}[X|Y])^2|Y]] \\ &= \mathbb{E}[\mathbb{E}[X^2 - 2X\mathbb{E}[X|Y] + (\mathbb{E}[X|Y])^2|Y]] \\ &= \mathbb{E}[\mathbb{E}[X^2|Y] - 2(\mathbb{E}[X|Y])^2 + (\mathbb{E}[X|Y])^2] \\ &= \mathbb{E}[X^2] - \mathbb{E}[(\mathbb{E}[X|Y])^2]\end{aligned}$$

$$\begin{aligned}\text{Var}(\mathbb{E}[X|Y]) &= \mathbb{E}[(\mathbb{E}[X|Y])^2] - (\mathbb{E}[\mathbb{E}[X|Y]])^2 \\ &= \mathbb{E}[(\mathbb{E}[X|Y])^2] - (\mathbb{E}[X])^2\end{aligned}$$

It follows that

$$\begin{aligned}\text{Var}(X) &= \text{Var}(\mathbb{E}[X|Y]) + \mathbb{E}[\text{Var}(X|Y)] \\ &= \mathbb{E}[(\mathbb{E}[X|Y])^2] - (\mathbb{E}[X])^2 + \mathbb{E}[X^2] - \mathbb{E}[(\mathbb{E}[X|Y])^2] \\ &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2\end{aligned}$$

Thus, since  $\text{Var}(X|Y)$  a positive rv,  $\mathbb{E}[\text{Var}(X|Y)] > 0$  and

$$\text{Var}(\mathbb{E}[X|Y]) \leq \text{Var}(X)$$

### 3. Sum of Uncorrelated Random Variables

Show that if  $X_1, \dots, X_m$  are pairwise uncorrelated, then

$$\text{Var}\left(\sum_{i=1}^m X_i\right) = \sum_{i=1}^m \text{Var}(X_i).$$

$X_1, X_2$  uncorrelated  $\Rightarrow \text{Cov}(X_1, X_2) = 0 \Rightarrow \text{IE}[X_1 X_2] = \text{IE}[X_1] \text{IE}[X_2]$   
 pairwise uncorrelated means for any  $X_i, X_j$ ,  $1 \leq i, j \leq m, i \neq j$ ,  
 $\text{Cov}(X_i, X_j) = 0$ .

Thus  $\text{IE}[X_i X_j] = \text{IE}[X_j] \text{IE}[X_i] \quad \forall i \neq j, 1 \leq i, j \leq m$

$$\begin{aligned} \text{Var}\left(\sum_{i=1}^m X_i\right) &= \text{Var}(X_1 + X_2 + \dots + X_m) \\ &= \text{IE}\left[\left((X_1 + \dots + X_m) - \text{IE}[X_1 + \dots + X_m]\right)^2\right] \\ &= \text{IE}\left[(X_1 + \dots + X_m)^2\right] - (\text{IE}[X_1 + \dots + X_m])^2 \end{aligned}$$

First Term

$$\begin{aligned} (X_1 + \dots + X_m)^2 &= \sum_{i=1}^m X_i \left( \sum_{j=1}^m X_j \right) \\ &= \sum_{i=j}^m X_i^2 + \sum_{i < j} X_i X_j + \sum_{i > j} X_i X_j \quad \text{The same} \quad i, j \in [1, m] \\ &= \sum_{i=j}^m X_i^2 + 2 \sum_{i \neq j} X_i X_j \end{aligned}$$

Thus

$$\begin{aligned} \text{IE}\left[(X_1 + \dots + X_m)^2\right] &= \text{IE}\left[\sum_{i=j}^m X_i^2\right] + 2 \text{IE}\left(\sum_{i \neq j} X_i X_j\right) \\ &= \sum_{i=j}^m \text{IE}[X_i^2] + 2 \sum_{i \neq j} \text{IE}[X_i X_j] \quad \text{by linearity of expectation} \end{aligned}$$

## Second Term

$$\text{IE}[x_1 + \dots + x_m] = \text{IE}[x_1] + \text{IE}[x_2] + \dots + \text{IE}[x_m] = \sum_{i=1}^m \text{IE}[x_i]$$

$$\begin{aligned} (\text{IE}[x_1 + \dots + x_m])^2 &= \left( \sum_{i=1}^m \text{IE}[x_i] \right)^2 \\ &= \sum_{i=1}^m \text{IE}[x_i] \left( \sum_{j=1}^m \text{IE}[x_j] \right) \\ &= \sum_{i=j} (\text{IE}[x_i])^2 + 2 \sum_{i \neq j} \text{IE}[x_i] \text{IE}[x_j] \end{aligned}$$

$\text{IE}[(x_1 + \dots + x_m)^2] - (\text{IE}[x_1 + \dots + x_m])^2$  is thus

$$\begin{aligned} &\sum_{i=j} \text{IE}[x_i^2] + 2 \sum_{i \neq j} \text{IE}[x_i x_j] - \left( \sum_{i=j} (\text{IE}[x_i])^2 + 2 \sum_{i \neq j} \text{IE}[x_i] \text{IE}[x_j] \right) \\ &= \sum_{i=j} \text{IE}[x_i^2] - (\text{IE}[x_i])^2 + \cancel{2 \sum_{i \neq j} (\text{IE}[x_i x_j] - \text{IE}[x_i] \text{IE}[x_j])}^0 \text{ by pairwise independence} \\ &= \text{IE}[x_1^2] - (\text{IE}[x_1])^2 + \text{IE}[x_2^2] - (\text{IE}[x_2])^2 + \dots + \text{IE}[x_m^2] - (\text{IE}[x_m])^2 \\ &= \text{Var}(x_1) + \text{Var}(x_2) + \dots + \text{Var}(x_m) \\ &= \sum_{i=1}^m \text{Var}(x_i) \end{aligned}$$

Thus,

$$\text{Var}(\sum_{i=1}^m x_i) = \sum_{i=1}^m \text{Var}(x_i)$$

if  $x_1, \dots, x_m$  are pairwise uncorrelated.

#### 4. Independence, Correlation, and MMSE Estimation

Consider two random variables  $X$  and  $Y$  with the following joint PDF.

$$f_{XY}(x, y) = \begin{cases} ce^{-x}e^{-y} & 0 \leq y \leq x < \infty \\ 0 & \text{elsewhere} \end{cases}$$

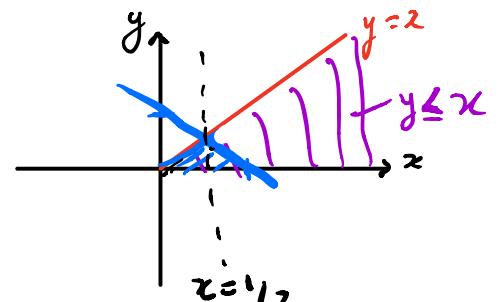
(a) Find the value of the constant  $c$  in the joint PDF.

$$\begin{aligned} & \int_{x=0}^{\infty} \int_{y=0}^{x} ce^{-y} e^{-x} dy dx = 1 \\ &= c \int_{x=0}^{\infty} e^{-x} \left( -e^{-y} \Big|_{y=0}^x \right) dx \\ &= c \int_{x=0}^{\infty} e^{-x} - e^{-2x} dx = 1 \\ &= c \left[ 1 - \frac{1}{2} \right] \rightarrow \boxed{c = 2} \end{aligned}$$

(b) Calculate  $\Pr(X + Y \leq 1)$ .

$$\Pr(X + Y \leq 1) = \Pr(Y \leq 1 - X)$$

Integrate over blue region



$$\int_{x=0}^{1/2} \int_{y=0}^x f_{X,Y}(x,y) dy dx + \int_{x=1/2}^1 \int_{y=0}^{1-x} f_{X,Y}(x,y) dy dx$$

$\xrightarrow{\text{Wolfram}}$

$$\boxed{0.264241}$$

Now we do some precomputations

### Marginal PDFs

$$f_y(y) = \int_{x=y}^{\infty} f_{x,y}(x,y) dx = \int_{x=y}^{\infty} 2e^{-x} e^{-y} dx = 2e^{-2y} \quad \text{for } 0 \leq y < \infty$$

$$f_x(x) = \int_{y=0}^{y=x} f_{x,y}(x,y) dy = \int_{y=0}^{y=x} 2e^{-x} e^{-y} dy = 2e^{-x}(1 - e^{-x}), \quad \text{for } y \leq x < \infty$$

### Conditional PDFs

$$f_{x|y}(x|y) = \frac{f_{x,y}(x,y)}{f_y(y)} = \frac{2e^{-x} e^{-y}}{2e^{-2y}} = e^{-x} e^y$$

$$f_{y|x}(y|x) = \frac{f_{x,y}(x,y)}{f_x(x)} = \frac{2e^{-x} e^{-y}}{2(1 - e^{-x})} = \frac{e^{-x}}{1 - e^{-x}} e^{-y}$$

### Expectations

$$\mathbb{E}[Y] = \int_{y=0}^{\infty} y 2e^{-2y} dy = \frac{1}{4} = \frac{1}{2}$$

$$\mathbb{E}[X|Y=y] = \int_{x=y}^{\infty} x f_{x|y}(x|y) dx = \int_{x=y}^{\infty} x e^{-x} e^y dx = y + 1$$

$$\mathbb{E}[X|Y] = (Y+1)$$

$$\mathbb{E}[X] = \mathbb{E}_Y [\mathbb{E}_{X|Y} [X|Y]] = \mathbb{E}[Y] + 1 = \frac{1}{2} + 1 = \frac{3}{2}$$

Correlation b/t X,Y

$$\text{IE}[XY] = \int_{x=0}^{\infty} \int_{y=0}^{x} xy f_{X,Y}(x,y) dy dx = \int_{x=0}^{\infty} \int_{y=0}^{x} xy (2e^{-x} e^{-y}) dy dx = 1$$

Covariance b/t X,Y

$$\begin{aligned}\text{Cov}(X,Y) &= \text{IE}[XY] - \text{IE}[X]\text{IE}[Y] \\ &= 1 - \frac{3}{4} = \frac{1}{4}\end{aligned}$$

(c) Are  $X$  and  $Y$  independent (the joint PDF appears to factor)? Justify your answer.

X,Y independent if

$$f_{X,Y}(x,y) = f_X(x)f_Y(y)$$

$$f_X(x) = \begin{cases} 2(e^{-x} - e^{-2x}), & y \leq x < \infty \\ 0, & \text{o/w} \end{cases} \quad f_Y(y) = \begin{cases} 2e^{-2y}, & 0 \leq y \leq x \\ 0, & \text{o/w} \end{cases}$$

$$f_X(x)f_Y(y) = 2(e^{-x} - e^{-2x})(e^{-2y}) \neq 2e^{-x}e^{-y}; 0 \leq y \leq x < \infty$$

Thus X and Y are NOT independent

Also easy to see that x has y dependence for bounds.

(d) Find the correlation, covariance, and correlation coefficient of  $X$  and  $Y$ .

$$\text{Correlation} = \mathbb{E}[XY] = 1$$

$$\text{Cov}(X,Y) = 1/16$$

$$P_{X,Y} = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}}$$

$$\text{Var}(Y) = \int_{y=0}^{+\infty} y^2 f_Y(y) dy - \left( \int_{y=0}^{+\infty} y f_Y(y) dy \right)^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$$\text{Var}(X) = \text{Var}(\mathbb{E}[X|Y]) + \mathbb{E}[\text{Var}(X|Y)]$$

$$\text{Var}(X|Y=y) = \int_{x=y}^{\infty} x^2 f_{X|Y=y}(x|y) dx - \left( \int_{x=y}^{\infty} x f_{X|Y=y}(x|y) dx \right)^2$$

$$= y^2 + 2y + 2 - (y+1)^2$$

$$= y^2 + 2y + 2 - (y^2 + 2y + 1)$$

$$= y^2 + 2y + 2 - y^2 - 2y - 1 = 1$$

$$\text{Var}(X|Y) = 1$$

Law of Total Variance

$$\text{Var}(X) = \text{Var}(Y+1) + 1 = \mathbb{E}[(Y+1)^2] - (\mathbb{E}[Y+1])^2 + 1$$

$$= \frac{5}{2} - \left(\frac{9}{4}\right) + 1$$

$$= \frac{10}{4} - \frac{9}{4} + \frac{4}{4}$$

$$= \frac{5}{4}$$

$$\int_{y=0}^{\infty} (y+1)^2 f_Y(y) dy = \int_{y=0}^{\infty} (y+1)^2 2e^{-y} dy = \frac{5}{2}$$

$$P_{X,Y} = \frac{\frac{1}{4}}{\sqrt{\left(\frac{1}{4}\right)\left(\frac{5}{4}\right)}} = \frac{\frac{1}{4}}{\sqrt{\frac{5}{16}}} = \frac{1}{\sqrt{5}} = 0.447$$

(e) MMSE estimate of  $X$  using a constant.

$$\hat{X}_{\text{MMSE}} = a^*$$

where

$$a^* = \underset{a \in \mathbb{R}}{\operatorname{argmin}} \mathbb{E}[(X-a)^2]$$

$$\text{MSE} = \mathbb{E}[(X-a)^2]$$

$$= \mathbb{E}[(X - \mathbb{E}[X] + \mathbb{E}[X] - a)^2]$$

$$= \mathbb{E}[(X - \mathbb{E}[X])^2] + 2 \mathbb{E}[(X - \mathbb{E}[X])(\mathbb{E}[X] - a)] + \mathbb{E}[(\mathbb{E}[X] - a)^2]$$

→ 0

Want to minimize

$$\mathbb{E}[(X - \mathbb{E}[X])^2] + \mathbb{E}[(\mathbb{E}[X] - a)^2]$$

$$\rightarrow \text{Var}(X) + (\mathbb{E}[X] - a)^2$$

$$\text{Take } a = \mathbb{E}[X]$$

So,

$$\hat{X}_{\text{MMSE}}^* = \mathbb{E}[X] = \frac{3}{2}$$

Resulting MSE is

$$\text{MSE} = \text{Var}(X) = \frac{5}{4}$$

(f) MMSE estimate of  $X$  based on  $Y$

Want

$$\hat{X}_{\text{MMSE}} = Y^*$$

where

$$Y^* = \underset{Y \in h(Y)}{\operatorname{argmin}} \mathbb{E}[(X - h(Y))^2]$$

For ANY  $h(Y)$ ,

$$\begin{aligned} \mathbb{E}[(X - h(Y))^2] &= \mathbb{E}\left((X - \mathbb{E}(X|Y)) + (\mathbb{E}(X|Y) - h(Y))^2\right) \\ &= \mathbb{E}[(X - \mathbb{E}(X|Y))^2] + 2 \mathbb{E}[(X - \mathbb{E}(X|Y))(\mathbb{E}(X|Y) - h(Y))] + \mathbb{E}[(\mathbb{E}(X|Y) - h(Y))^2] \end{aligned}$$

So,

$$\mathbb{E}[(X - h(Y))^2] = \mathbb{E}[(X - \mathbb{E}(X|Y))^2] + \mathbb{E}[(\mathbb{E}(X|Y) - h(Y))^2]$$

The choice of  $h(Y)$  which minimizes this is  $h(Y) = \mathbb{E}[X|Y]$

Thus,  $\mathbb{E}(X|Y)$  is the function of  $Y$  that minimizes  $\mathbb{E}((X - h(Y))^2)$  over ALL functions  $h(Y)$

Conclude

$$\boxed{\hat{X}_{\text{MMSE}}^* = \mathbb{E}[X|Y] = (Y+1)}$$