① $X \sim Y \sim \text{Uniform}[0,1]$; X,Y independent

(a) $\text{Cov}(X,Y) = \mathbb{E}((X-E(X))(Y-\mathbb{E}(Y))$

$$= \mathbb{E}((X-\tfrac{1}{2})(Y-\tfrac{1}{2}))$$

$$= \mathbb{E}\left[XY - \tfrac{1}{2}X - \tfrac{1}{2}Y + \tfrac{1}{4}\right]$$

$$= \mathbb{E}[XY] - \tfrac{1}{2}\mathbb{E}(X) - \tfrac{1}{2}E(Y) + \tfrac{1}{4}$$

$$= \mathbb{E}(XY) - \tfrac{1}{4} - \tfrac{1}{4} + \tfrac{1}{4}$$

$$= \mathbb{E}(XY) - \tfrac{1}{4}$$ Note: this is $\mathbb{E}(X)\mathbb{E}(Y)$

$\downarrow$ independence

$$= \mathbb{E}(X)\mathbb{E}(Y) - \tfrac{1}{4}$$

$$= \tfrac{1}{4} - \tfrac{1}{4} = 0$$

Thus

$$\text{Cov}(X,Y) = 0$$

Define event $A = \{X \leq Y\}$

(b) X,Y independent $\Longleftrightarrow$ $f_{X,Y}(x,y) = f_X(x) f_Y(y)$

$$f_{X,Y}(x,y) = \begin{cases} 1, & 0 < x,y < 1 \\ 0, & o/w \end{cases}$$

$$f_{X,Y|A}(x) = \begin{cases} \dfrac{f_{X,Y}(x,y)}{\mathbb{P}(A)} = \dfrac{1}{1/2} & , \quad 0 < x \leq y < 1 \\ 0 & , \quad o/w \end{cases}$$

$$\mathbb{P}(A) = \int_{y=0}^{y=1} \int_{x=0}^{x=y} f_{X,Y}(x,y)\,dx\,dy = 1/2$$

Thus

$$f_{X|A}(x) = \int_{y=x}^{1} f_{X,Y|A}(x,y)\,dy = \int_x^1 2\,dy = 2(1-x)$$

(c) $$\mathbb{E}[X|A] = \int_{-\infty}^{+\infty} x f_{X|A}(x)\,dx = \int_0^1 x[2(1-x)]dx = 1 - \frac{2}{3} = \frac{1}{3}$$

(d) Var(Z)?

$$\begin{aligned} Var(Z) &= \mathbb{E}((XY)^2) - (\mathbb{E}(XY))^2 \\ &= \mathbb{E}(X^2)\mathbb{E}(Y^2) - (\mathbb{E}(X)\mathbb{E}(Y))^2 \\ &= \left(\int_{-\infty}^{+\infty} x^2 f_X(x)\right)\left(\int_{-\infty}^{+\infty} y^2 f_Y(y)\right) - \left(\frac{1}{4}\right)^2 \\ &= \left(\int_0^1 x^2\,dx\right)\left(\int_0^1 y^2\,dy\right) - \left(\frac{1}{4}\right)^2 \\ &= \left(\frac{1}{3}\right)\left(\frac{1}{3}\right) - \frac{1}{16} = \frac{1}{9} - \frac{1}{16} = 7/144 \end{aligned}$$

(e) CDF $F_Z(z)$ and PDF $f_Z(z)$?

$$F_Z(z) = \mathbb{P}(Z \le z) = \mathbb{P}(XY \le z) = \int_y \mathbb{P}(XY \le z \mid Y = y)\, \overset{1}{f_Y(y)}\, dy$$

$$= \int_0^1 \mathbb{P}(X \le z/y)\, dy$$

$$\mathbb{P}\left(X \le \frac{z}{y}\right) = \begin{cases} 0, & \text{o/w} \\ \dfrac{\frac{z}{y} - 0}{1 - 0}, & 0 < \left(\frac{z}{y}\right) < 1 \\ 1, & \left(\frac{z}{y}\right) > 1 \end{cases}$$

So

$$F_Z(z) = \int_0^z dy + \int_z^1 \frac{z}{y}\, dy$$

$$= z + z\ln(1) - z\ln(z)$$

$$= z(1 - \ln(z))$$

0 < z < 1

$$f_Z(z) = \frac{\partial}{\partial z} F_Z(z) = z - \ln(z) \cdot z$$

$$= 1 - \ln(z) + \frac{-z}{z} = \boxed{-\ln(z)}$$

② $X \sim \text{Uniform}[0,4]$

$Y \sim \text{exponential}(\lambda = 2)$ <u>X, Y independent</u>

(a) Mean and variance of $2X - 3Y$

$$\mathbb{E}(2X - 3Y) = 2\mathbb{E}(X) - 3\mathbb{E}(Y)$$
$$= 2(2) - \frac{3}{2} = 4 - \frac{3}{2} = \boxed{\frac{5}{2}}$$

$$\text{Var}(2X - 3Y) \overset{\text{independence}}{=} \text{Var}(2X) + \text{Var}(3Y)$$
$$= 4\text{Var}(X) + 9\text{Var}(Y)$$
$$= 4\,\frac{(4)^2}{12} + \frac{9}{2^2}$$
$$= \frac{64}{12} + \frac{\cancel{9}^{27}}{\cancel{4}_{12}} = \frac{91}{12}$$

(b) $\text{Cov}(X,Y) = 0$ (X, Y independent)

Why?

$$\text{Cov}(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$$

independence $\rightarrow$ $= (\mathbb{E}(X))(\mathbb{E}(Y)) - \mathbb{E}(X)\mathbb{E}(Y) = 0$

(c) $\mathbb{P}(X \le Y)$?

$\mathbb{P}(X \le x) = F_X(x) \rightarrow \mathbb{P}(X \le Y)$

$$\mathbb{P}(X \le Y) = \int_y \mathbb{P}(X \le Y \mid Y = y)\, f_Y(y)\, dy$$

$$= \int_y \mathbb{P}(X \le y)\, f_Y(y)\, dy$$

$$\mathbb{P}(X \le y) = \begin{cases} 0 & ; \quad y < 0 \\ y/4 & ; \quad 0 \le x < y < 4 \\ 1 & ; \quad x \le 4 < y \end{cases}$$

Thus

$$\mathbb{P}(X \le y) = \int_{y=0}^{y=4} \frac{y}{4} \lambda e^{-\lambda y}\, dy + \int_4^\infty \lambda e^{-\lambda y}\, dy$$

$$= -\frac{\cancel{\lambda}}{4}\left[\frac{y}{\cancel{\lambda}} e^{-\lambda y} + \frac{1}{\lambda^2} e^{-\lambda y} \Big|_0^4 \right] - e^{-\lambda y} \Big|_4^\infty$$

$$= -\frac{4}{4} e^{-8} - \frac{1}{8} e^{-8} + \frac{1}{2} \cdot \frac{1}{4} - (0 - e^{-8})$$

$$= -e^{-8} + \frac{1}{8} + e^{-8} - \frac{1}{8} e^{-8}$$

$$\boxed{= \frac{1}{8}(1 - e^{-8})}$$

(d)

$$f_{X,Y|A}(x,y) = \begin{cases} \dfrac{f_{X,Y}(x,y)}{\mathbb{P}(A)} = \dfrac{\frac{1}{4}xe^{-xy}}{c} = \dfrac{e^{-2y}}{2c}, & x \in [0,4] \\ & x \le y \\ 0 & , o/w \end{cases}$$

(e) Define rv W as follows:

- Toss a fair coin independent of Y
- If H; let $W = Y$
- If T: let $W = 2 + Y$

$$\mathbb{P}(\{H\} \mid \{W=3\}) = \frac{\mathbb{P}(\{W=3\} \mid \{H\})\,\mathbb{P}(\{H\})}{\mathbb{P}(\{W=3\})}$$

$$\mathbb{P}(W=3) = \mathbb{P}(W=3 \mid H)\mathbb{P}(H) + \mathbb{P}(W=3 \mid T)\mathbb{P}(T) \Big\} \text{Law of Total Probability}$$

Thus

$$\mathbb{P}(\{H\} \mid \{W=3\}) = \frac{\mathbb{P}(\{W=3\} \mid \{H\})\,\mathbb{P}(\{H\})}{\mathbb{P}(W=3 \mid H)\mathbb{P}(H) + \mathbb{P}(W=3 \mid T)\mathbb{P}(T)}$$

$$\mathbb{P}(H) = \mathbb{P}(T) = 1/2$$

$$\mathbb{P}(W=3 \mid H) = \mathbb{P}(Y=3 \mid H) = \frac{\mathbb{P}(Y=3)\,\mathbb{P}(H)}{\mathbb{P}(H)} = \mathbb{P}(Y=3)$$

$$\mathbb{P}(W=3 \mid T) = \mathbb{P}(Z+Y=3 \mid T) = \frac{\mathbb{P}(Y=1)\,\mathbb{P}(T)}{\mathbb{P}(T)} = \mathbb{P}(Y=1)$$

Thus

$$\mathbb{P}(H \mid W=3) = \frac{f_Y(3)\,^1\!/_2}{f_Y(3)\cdot {}^1\!/_2 + f_Y(1)\cdot {}^1\!/_2}$$

$$= \frac{2e^{-6}\cdot {}^1\!/_2}{2e^{-6}\cdot {}^1\!/_2 + 2e^{-2}\cdot {}^1\!/_2} = \frac{e^{-6}}{e^{-6}+e^{-2}}$$

③ $N \sim \text{geometric}(p = 1/2)$; number of rounds

Each round i, win X_i.

Assume

$$\mathbb{E}(X_i] = \mu > 0 \quad \text{and} \quad Var(X_i) = \sigma^2 \quad \forall i$$

All random variables are independent.

Define

$$Y = \sum_{i=1}^{N} X_i$$

as total amount won.

(a) $\mathbb{E}(Y|N=n) = \sum_{i=1}^{n} \mathbb{E}(X_i) = \mu n \Rightarrow \mathbb{E}(Y) = \mathbb{E}(\mathbb{E}(Y|N)) = \mu N$

(b) $Var(Y|N=n) = \sum_{i=1}^{n} Var(X_i) = n\sigma^2$

By independence

$$Var(Y|N) = N\sigma^2$$

$$Var(N) = \frac{1-p}{p^2} = \frac{\frac{1}{2}}{1/4} = 2$$

$$\mathbb{E}(N) = \frac{1}{p} = 2$$

$$\begin{aligned} Var(Y) &= Var(\mathbb{E}(Y|N)) + \mathbb{E}(Var(Y|N)) \\ &= Var(\mu N) + \mathbb{E}(N\sigma^2) \\ &= \mu^2 Var(N) + \sigma^2 \mathbb{E}(N) \\ &= 2\mu^2 + 2\sigma^2 \end{aligned}$$

(c) $\text{Cov}(Y,N) = \mathbb{E}\Big((Y-\mathbb{E}(Y))(N-\mathbb{E}(N))\Big)$

$$= \mathbb{E}[YN] - \mathbb{E}[Y]\mathbb{E}[N]$$

$$= \mu\mathbb{E}[N^2] - 4\mu$$

$$= \mu[\text{Var}(N)+(\mathbb{E}(N))^2] - 4\mu$$

$$= 6\mu - 4\mu = \boxed{2\mu}$$

$$\mathbb{E}[YN] = \mathbb{E}[\mathbb{E}[YN|N]]$$

$$= \mathbb{E}[N\mathbb{E}[Y|N]]$$

$$\mathbb{E}[N^2\mu]$$

$$= \mu\mathbb{E}[N^2]$$

(d)

$$\rho(Y,N) = \frac{\text{Cov}(Y,N)}{\sigma_Y\sigma_N} = \frac{\cancel{2}\mu}{\sqrt{(2\mu^2+2\sigma^2)(2}} = \frac{\mu}{\sqrt{\mu^2+\sigma^2}}$$

$$\left.\begin{array}{l} ④\ \Theta \sim \text{Uniform}(0, l>0) \\ W \sim \text{Unifom}[0, \pi/2] \end{array}\right\rangle \text{independent}$$

$$X = \Theta \cos W$$

(a) LMSE $\mathbb{E}[\Theta \mid X]$?

$$\mathbb{E}(\theta \mid X = x) = \int_{\theta} \theta \, f_{\theta \mid X = x}(\theta) \, d\theta$$

$$f_{\theta \mid x}(\theta \mid x) = \frac{f_{x \mid \theta}(x \mid \theta) f_{\theta}(\theta)}{f_x(x)} = \frac{f_{x \mid \theta}(x \mid \theta) f_{\theta}(\theta)}{\int_{\theta = -\infty}^{+\infty} f_{x \mid \theta}(x \mid \theta) f_{\theta}(\theta) \, d\theta}$$

$$\begin{aligned} F_{X \mid \theta}(x \mid \theta) &= \mathbb{P}(X \leq x \mid \theta = \theta) \\ &= \mathbb{P}(\theta \cos W \leq x \mid \theta = \theta) \\ &= \mathbb{P}(\cos W \leq x/\theta) \\ &= 1 - \mathbb{P}(W \leq \cos^{-1}(\tfrac{x}{\theta})) \\ &= 1 - F_W(\cos^{-1}(x/\theta)) \\ &= 1 - \frac{2}{\pi} \cos^{-1}(x/\theta) \end{aligned}$$

Thus $$F_{X \mid \theta}(x \mid \theta) = \begin{cases} 0 & , \ o/w \\ 1 - \frac{2}{\pi} \cos^{-1}[x/\theta) & , \ 0 < x \leq \theta < l \\ 1 & , \ x > 0 \end{cases}$$

So

$$f_{X|\Theta}(x|\theta) = \frac{d}{dx} F_{X|\Theta}(x|\theta) = \begin{cases} 0 & , \text{ o/w} \\ \frac{2}{\pi\theta} \cdot \frac{1}{\sqrt{1-\left(\frac{x}{\theta}\right)^2}} & , 0 < x < \theta \end{cases}$$

$$f_{X|\Theta}(x|\theta) = \begin{cases} 0 & , \text{ o/w} \\ \frac{2}{\pi\sqrt{\theta^2 - x^2}} & , 0 < x \leq \theta < l \end{cases}$$

$$f_X(x) = \int_0^l \frac{2}{\pi\sqrt{\theta^2 - x^2}} \cdot \frac{1}{l} \, d\theta$$

$$= \frac{2}{\pi l} \ln\left(\frac{l + \sqrt{l^2 - x^2}}{x}\right) \sqrt{\theta^2 - x^2}$$

So,

$$f_{\Theta|X}(\theta|x) = \begin{cases} \dfrac{\frac{2}{L\pi\sqrt{\theta^2 - x^2}}}{\frac{2}{\pi l} \ln\left(\frac{l + \sqrt{l^2 - x^2}}{x}\right)} & , 0 < x \leq \theta < L \\ 0 & , \text{ o/w} \end{cases}$$

$$= \begin{cases} \dfrac{1}{\sqrt{\theta^2 - x^2} \ln\left(\frac{L + \sqrt{L^2 - x^2}}{x}\right)} & , 0 < x \leq \theta < L \\ 0 & , \text{ o/w} \end{cases}$$

Thus

$$\mathbb{E}[\Theta \mid X = x] = \frac{1}{\ln\left(\frac{L + \sqrt{L^2 - x^2}}{x}\right)} \int_0^L \frac{1}{\sqrt{\theta^2 - x^2}} \, d\theta$$

$$= \frac{1}{\ln\left(\frac{L + \sqrt{L^2 - x^2}}{x}\right)} \sqrt{L^2 - x^2}$$

⑤ X, Y jointly continuous rv.

$$f_{X,Y}(x,y) = \begin{cases} 1, & -\frac{1}{2} < y < \frac{1}{2} \\ & y-\frac{1}{2} < x < y+\frac{1}{2} \\ 0, & \text{else} \end{cases}$$

(a) MMSE estimator of X given observation Y. i.e $\mathbb{E}(X|Y)$

$$\mathbb{E}[X|Y=y] = \int_x x\, f_{X|Y}(x|y)\, dx$$

where

$$f_{X|Y}(x|y) = \begin{cases} \dfrac{f_{X,Y}(x,y)}{f_Y(y)} = \dfrac{1}{\int_{y-\frac{1}{2}}^{y+1/2} f_{X,Y}(x,y)\, dx} = 1, & |y| < 1/2 \\ & y-\frac{1}{2} < x < y+\frac{1}{2} \\ 0 & \text{, else} \end{cases}$$

$$\mathbb{E}[X|Y=y] = \int_{y-1/2}^{y+1/2} x\, dx = \frac{1}{2}x^2 \Big|_{y-1/2}^{y+1/2} = y$$

Thus

$$\mathbb{E}[X|Y] = Y$$

$$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}[Y]$$

(b) $\hat{a} = \dfrac{\mathrm{Cov}(X,Y)}{\mathrm{Var}(Y)}$, $\hat{b} = \mathbb{E}(X) - \hat{a}\,\mathbb{E}(Y)$

$$\mathrm{Cov}(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xy\, f_{X,Y}(x,y)\,dx\,dy - \cancelto{0}{\int_{-\infty}^{+\infty} x f_X(x)\,dx} \int_{-\infty}^{+\infty} x f_Y(y)\,dy$$

$$= \int_{y=-1/2}^{+1/2}\int_{x=y-1/2}^{x=y+1/2} xy\,dx\,dy = \int_{y=-1/2}^{+1/2}\left(\frac{1}{2}x^2\Big|_{y-1/2}^{y+1/2}\right) y\,dy$$

$$= \frac{1}{3}y^3\Big|_{-\frac{1}{2}}^{1/2}$$

$$= \frac{1}{3}\left(\frac{1}{8}+\frac{1}{8}\right) = \frac{1}{12}$$

$$\mathrm{Var}(Y) = \frac{b-a}{12} = 1/12$$

Thus $\hat{a} = \dfrac{1/12}{1/12} = 1$, $\hat{b} = \mathbb{E}[X] - \hat{a}\,\mathbb{E}(Y)$

$= \mathbb{E}[Y] - \mathbb{E}(Y) = 0$!

So $\hat{X}_{opt} = Y$ as well!

(c) QMMSE of X given observation Y

$$\mathbb{E}\left[\left(X - (aY^2 + bY + c)\right)^2\right]$$

Since MMSE is $\mathbb{E}[X|Y] = Y$ is linear there does **NOT** exist a nonlinear estimator w/ smaller mean so

$$\mathbb{E}\left[\left(X - (aY^2 + bY + c)\right)^2\right] = Y$$