Recall

- Central Limit Theorem
  - $X_1, \dots, X_k$ iid; common mean μ, variance σ^2
  - $M_n = \dfrac{X_1 + \cdots + X_n}{n}$
  - Set $Z_n = \dfrac{\sqrt{n}}{\sigma}(M_n - \mu)$

    - Z_n has mean 0, variance 1 $\forall\, n > 0$
    - As $n \to \infty$, $Z_n \to$ Gaussian mean 0, variance 1

      in the sense that

$$F_{Z_n}(z) \longrightarrow \Phi(z) \ \forall\, z \text{ as } n \to \infty$$

Caution: people often use CLT "carelessly" to generate quantatative info

Let's see where care is needed.

Example - Be Careful

Say n large; given $c > 0$, what is $\mathbb{P}(X_1 + \cdots + X_n \geq c)$?

Provided n is large enough so Z_n above are "close to Gaussian", estimate this as follows:

Note

$$X_1 + \cdots + X_n \geq c$$

is the same as

$$M_n \geq c/n$$

which is the same as

$$M_n - \mu \geq c/n - \mu$$

which is the same as

$$\frac{\sqrt{n}}{\sigma}\left(M_n - \mu\right) \geq \frac{\sqrt{n}}{\sigma}\left(\frac{c}{n} - \mu\right)$$

which is the same as

$$Z_n \geq \frac{\sqrt{n}}{\sigma}\left(\frac{c}{n} - \mu\right)$$

Thus

$$\mathbb{P}(X_1 + \cdots + X_n) \geq c \simeq \mathbb{P}\left(Z_n \geq \frac{\sqrt{n}}{\sigma}\left(\frac{c}{n} - \mu\right)\right)$$

$$\simeq 1 - \Phi\left(\frac{\sqrt{n}}{\sigma}\left(\frac{c}{n} - \mu\right)\right)$$

The "caution" thing here: n must be large enough so Gaussian approximation of Z_n holds. Unfortunately, no systematic way – even given complete pmf/pdf info about the X_k's – to determine how large n has to be.

Example – Unknown p-coin

Have a coin; don't know $p = \mathbb{P}(\{H\})$

Want to estimate p by repeatedly flipping

Let $X_k = 1$ if flip k is H

From

$$M_n = \frac{X_1 + \cdots + X_n}{n}$$

by WLLN, converges to p (in probability)

Question: How big does n have to be so

$$\mathbb{P}(|M_n - p| \geq 0.01) \leq 0.05$$

Chebyshev says need $n \sim 50{,}000$

How to use CLT to get a smaller n that will ensure this?

First, note that for large n, M_n distributed roughly symmetrically around p, so

$$\mathbb{P}(|M_n - p| \geq 0.01) \approx 2\,\mathbb{P}(M_n - p \geq 0.01)$$

Next, proceed as in previous example - turn this into a statement about Z_n.

$$M_n - p \geq 0.01$$

is the same as

$$\frac{\sqrt{n}}{\sigma}(M_n - p) \geq 0.01\,\frac{\sqrt{n}}{\sigma}$$

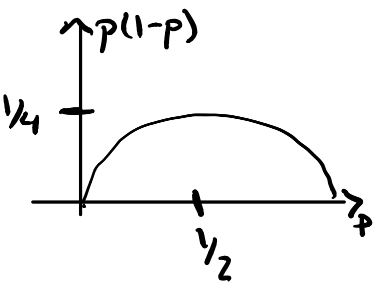
Thus

$$2\,\mathbb{P}(M_n - p \geq 0.01)$$

is equal to

$$2\,\mathbb{P}\left(Z_n \geq 0.01\,\frac{\sqrt{n}}{\sigma}\right)$$

unlike last example, σ unknown! $\sigma = \sqrt{p(1-p)}$; don't know p.



Since you don't know σ, use the fact $p(1-p) \geq 1/4$ to bound σ by $1/2$

This yields a higher, more conservative, estimate of

$$\mathbb{P}\left(Z_n \geq 0.01\,\frac{\sqrt{n}}{\sigma}\right)$$

Bottom line: Find how big n has to be so that

$$2\,\mathbb{P}(Z_n \geq 0.02\sqrt{n}) \leq 0.05 \xrightarrow{\text{from } \Phi\text{-table}} n \simeq 9604$$

Last thing: convergence w/ probability 1 of a sequence $Y_1, Y_2, \dots$ of random variables.

Consider the sequence $\{Y_n : n > 0\}$.
Given some random variable Y,

$$\left\{ \lim_{n\to\infty} Y_n = Y \right\}$$

is an event – need to refer back to Ω, $\mathbb{P}$, etc.

Say $Y_n \to Y$ with probability 1 (w.p.1)

$Y_n \xrightarrow{w.p.1} Y$

$Y_n \xrightarrow{a.s} Y$ a.s → almost surely

when this event has probability 1.

Turns out: Convergence with probability 1 ⇒ convergence in probability

Example – Convergence With Probability 1

Say $W_1, W_2, \dots$ are iid Uniform $[0,1]$ random variables.
let

$$Y_n = \min\{W_1, \dots, W_n\}$$

Turns out

- Y_n is decreasing in n in the sense that $\mathbb{P}(Y_{n+1} \le Y_n) = 1 \;\forall n$
- Y_n is bounded below by 0.

Thus Y_n converges to a limit as $n \to \infty$ ← NOT obvious. See appendix

This is a random variable Y

- Can show that $\mathbb{P}(Y=0)=1$
  - This is because for any $\delta>0$

$$\mathbb{P}(Y \geq \delta) \leq \mathbb{P}(Y_n \geq \delta) = (1-\delta)^n \ \forall n$$

meaning $\mathbb{P}(Y \geq \delta)=0$

Because $\mathbb{P}(Y=0)=1$, we have

$$Y_n \xrightarrow{w.p.1} 0 \quad \text{as} \quad n \to \infty$$

Strong Law of Large Numbers: When X_n iid, common mean μ, common variance σ^2, and $M_n = \frac{X_1 + \cdots + X_n}{n}$, we have

$$M_n \xrightarrow{w.p.1} \mu \quad \text{as} \quad n \to \infty$$

Appendix (NOT done in lecture; added for my own clarity)

Definition: A sequence $\{s_n\}$ of real numbers is said to be

(i) monotonically increasing if $s_n \leq s_{n+1}$ $(n=1,2,3,\dots)$

(ii) monotonically decreasing if $s_n \geq s_{n+1}$ $(n=1,2,3,\dots)$

Theorem: Suppose $\{s_n\}$ is monotonic. Then $\{s_n\}$ converges if and only if it is bounded.

Proof

($\Leftarrow$) Suppose $s_n \leq s_{n+1}$

Let E be the range of $\{s_n\}$.

If $\{s_n\}$ is bounded, let $s = \sup E$.

Then

$$s_n \leq s \quad (n=1,2,3,\dots)$$

For every $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$s - \varepsilon < s_N < s$$

for otherwise $s-\varepsilon$ would be an upper bound of E.

Since $\{s_n\}$ increases,

$$n \geq N \Rightarrow \quad s - \varepsilon < s_n \leq s$$

Thus $\{s_n\}$ converges to s. [analogous proof for decreasing]

($\Rightarrow$) Suppose $p_n \to p$. $\exists N \in \mathbb{N}$ such that $n > N \Rightarrow d(p, p_n) < M$.

Put

$$r = \max(M, d(p_1, p), \dots, d(p_N, p))$$

then $d(p_n, p) \leq r$ for $n = 1, 2, 3, \dots$

Q.E.D