Since last prelim

Dual Vector Spaces

define $f: V \to \mathbb{F}$ as . . .

$f \in V^*$ means f is a LT

- $V^* = \mathrm{Lin}(V, \mathbb{F})$
- $\dim V^* = \dim V$
- given a basis V, dual basis for V^*

Direct Sums

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_r$$

if $V = W_1 + W_2 + \cdots + W_r$

and if $\vec{w}_1 + \vec{w}_2 + \cdots + \vec{w}_r = 0$ then $\vec{w}_i \; \forall i$

basis of $W_1 \oplus \cdots \oplus W_r$:= union of basis of each

and

$$\dim V = \sum_{i=1}^{r} \dim W_i$$

Quotient Vector Spaces

$W \subseteq V$, define V/W

2 Natural Maps

① $T: V \to V/W$

$\ker T = W$

$\operatorname{im} T = V/W$

② if $T: V \to W$ $\qquad \ker \bar{T} = \{0\}$

define $\tilde{T}: V/\ker(T) \to W$ $\qquad \operatorname{im} \bar{T} = \operatorname{im} T$

- $\dim V/W = \dim V - \dim W$

- if $V = W \oplus U_1$, then $V/W \cong U$

if $V = W \oplus U_2$, $U_1 \neq U_2$ often, but

$U_1 \cong U_2 \cong V/W$

Polynomials

division algorithm

roots

fund. Thm of Algebra

ideals. $I \subseteq \mathbb{F}[x]$, $\langle f(x) \rangle$

each ideal I: $\exists$ $f(x)$ s.t. $I = \langle f(x) \rangle$ } From HW

$$D = \frac{d}{dx}$$

$p(D)$, $\ker(p(D))$ solution to DE's

Eigenvalues + Diagonalizable

Similar Matrices

$$B = Q^{-1} A Q$$

$[T]_\alpha$, $[T]_\beta$ similar

and this is how all similar matrices occur.

Diagonalizable

$T : V \to V$ $(\dim V = n)$

$A_{n \times n}$

Eigenvalues, Eigenvectors

- $A\vec{v} = \lambda\vec{v}$

Define

$$E_{\lambda_1}(A), E_{\lambda_2}(A), \ldots, E_{\lambda_n}(A) \quad ; \quad E_{\lambda_i}(A) = \ker(A - \lambda_i I)$$

LI

$v_1, \ldots, v_k$ LI if v_i is an eigenvector w/ λ_i,
λ_i ALL distinct

$E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_r}(T)$
$= E_{\lambda_1} + \cdots + E_{\lambda_r}(T)$ for $i \neq j$ $\lambda_i \neq \lambda_j$

$\Leftrightarrow$ T diagonalizable $\Leftrightarrow$ basis of eigenvectors

Key Important Fact

If $T \in \mathcal{L}(V)$, $\dim V = n < \infty$ over $\mathbb{C}$,
then T has an eigenvalue

If $p(x) \in \mathbb{F}[x]$, considered $P(T), p(A)$.

If $A^T =$ transpose, $(A\ n \times n)$
A^T, A have the same eigenvalues
rank $A^T =$ rank A
$A^{n \times n}$ $\dim \ker(A) = \dim \ker(A^T)$ NOT true if $m \times n$

actually, if A in $\boxed{n\times n}$

$$\dim \ker(A-\lambda I) = \dim \ker(A^T-\lambda I)$$

$$\dim E_\lambda(A) = \dim E_\lambda(A^T)$$

Markov Chains

- Stochastic Matrices
- Regular Stochastic Matrices

$$\lim_{n\to\infty} A^n = L \qquad \left(\lim_{n\to\infty} BA^nC = BLC\right)$$

(exists over $\mathbb{C}$)

if A diagonalizable and all eigenvalues of A satisfy $|\lambda| < 1$ or $\lambda = 1$ then $\lim_{n\to\infty} A^n$ exists

Gerschgorin Disks

A matrix in $\mathbb{C}^{n\times n}$.

λ = eigenvalue $\Rightarrow$ $\lambda \in C_i$ $\quad i = 1, \dots n$

$C_i = \{ |z - A_{ii}| \leq$ sum of all abs values of off diagonal entries in row i $\}$

Main Theorem:

If A is diagonalizable, Stochastic, regular then

ⓐ $\lim_{n\to\infty} A^n$ exists $\quad [\vec{p} \quad \vec{p} \quad \cdots \quad \vec{p}]$ all cols same

ⓑ $\dim E_{\lambda_1}(A) = 1$ and $A\vec{p} = \vec{p}$

ⓒ $\lim_{n\to\infty} A^n \vec{q} = \vec{p}$

Inner Products

def, norm, $\perp$

$W^\perp$ $(\dim W < \infty)$

$W \subseteq V$ if $\dim W < \infty$

$W^\perp$

$W \oplus W^\perp = V$

orthogonal sets, orthonormal sets $\rightsquigarrow$ LI elements

Gram-Schmidt

Cor: If $W \subseteq V$ is finite dim'l

then W has an orthonormal basis

$A^T B =$ matrix of dot products