

Riesz Representation Theorem

Suppose V is finite dimensional and $\varphi \in L(V)$. Then \exists a unique vector $u \in V$ such that

$$\varphi(v) = \langle v, u \rangle \quad \forall v \in V$$

Proof

Let e_1, \dots, e_n be an orthonormal basis for V . Then

$$\begin{aligned} \exists \quad \varphi(v) &= \varphi(\langle v, e_1 \rangle e_1 + \dots + \langle v, e_n \rangle e_n) \\ &= \varphi(e_1) \langle v, e_1 \rangle + \dots + \varphi(e_n) \langle v, e_n \rangle \\ &= \langle v, \overline{\varphi(e_1)} e_1 \rangle + \dots + \langle v, \overline{\varphi(e_n)} e_n \rangle \\ &= \langle v, \underbrace{\overline{\varphi(e_1)} e_1 + \dots + \overline{\varphi(e_n)} e_n}_u \rangle \end{aligned}$$

!

Suppose $u_1, u_2 \in V$ such that

$$\varphi(v) = \langle v, u_1 \rangle = \langle v, u_2 \rangle \quad \forall v \in V.$$

Then

$$0 = \langle v, u_1 - u_2 \rangle \rightarrow u_1 - u_2 = 0 \rightarrow u_1 = u_2$$

Orthogonal Complement, U^\perp

If $U \subseteq V$, then the orthogonal complement of U , denoted U^\perp , is the set of all vectors in V that are orthogonal to every vector in U

$$U^\perp = \{v \in V \mid \langle v, u \rangle = 0 \quad \forall u \in U\}$$

Properties of Orthogonal Complement

(a) If $U \subseteq V$, then $U^\perp \subseteq V$

(b) $\{0\}^\perp = V$

(c) $V^\perp = \{0\}$

(d) If $U \subseteq V$, then $U \cap U^\perp = \{0\}$

(e) If U, W subsets of V and $U \subseteq W$, then $W^\perp \subseteq U^\perp$

Direct Sum of Subspace and Orthogonal Complement

Suppose U is a finite-dimensional subspace of V . Then

$$V = U \oplus U^\perp \Rightarrow \dim U^\perp = \dim V - \dim U$$

Proof

$$v = u + v - u$$

$$u = \langle v, e_1 \rangle e_1 + \dots + \langle v, e_m \rangle e_m$$

Orthogonal Complement of Orthogonal Complement

$U \subseteq V$, $\dim U = m < \infty$

$$U = (U^\perp)^\perp$$

Orthogonal Projection, P_U

Suppose U is a finite-dimensional subspace of V . The orthogonal projection of V onto U is the operator $P_U \in \mathcal{L}(V)$ defined as follows:
For $v \in V$, write $v = u + w$, where $u \in U$ and $w \in U^\perp$. Then $P_U v = u$

Properties of the Orthogonal Projection, P_U

Suppose U is a finite-dimensional subspace of V and $v \in V$. Then

(a) $P_U v \in U$

(b) $P_U v = v \iff v \in U$

(c) $P_U w = 0 \iff w \in U^\perp$

(d) range $P_U = U$

(e) null $P_U = U^\perp$

(f) $v - P_U v \in U^\perp$

(g) $P_U^2 = P_U$

(h) $\|P_U v\| \leq \|v\|$

(i) For every orthonormal basis e_1, \dots, e_m of U ,

$$P_U v = \langle v, e_1 \rangle e_1 + \dots + \langle v, e_m \rangle e_m$$

Proof

(a) Need to show $P_U : V \rightarrow U$ a linear map.

Suppose $v_1, v_2 \in V$.

Write $v_1 = u_1 + w_1, \quad v_2 = u_2 + w_2$

with $u_1, u_2 \in U, \quad w_1, w_2 \in U^\perp$

thus $v_1 + v_2 = (u_1 + u_2) + (w_1 + w_2)$

where $u_1 + u_2 \in U, \quad w_1 + w_2 \in U^\perp$

concluding $P_U(v_1 + v_2) = u_1 + u_2 = P_U v_1 + P_U v_2$

Now let $\lambda \in \mathbb{F}$ so that $\lambda v = \lambda u_1 + \lambda w_1$.

Then $P_U(\lambda v) = \lambda u_1 = \lambda P_U v$ as desired

(b) Suppose $u \in U$. $u = u + 0$, where $u \in U$, $0 \in W^\perp$

then $P_U u = u$

(c) Suppose $w \in U^\perp$. $w = 0 + w$, where $0 \in U$, $w \in U^\perp$.

then $P_U w = 0$

(d) range $P_U = \{P_U v \mid v \in V\}$

every $v \in V$ can be written $v = u + w$, $u \in U$, $w \in W^\perp$

$$\Rightarrow P_U v = u \in U \text{ so that range } P_U \subseteq U$$

$$(b) \Rightarrow U \subseteq \text{range } P_U \rightarrow \text{range } P_U = U$$

(e) null $P_U = \{v \in V \mid P_U v = 0\} \subseteq U^\perp$

$$(c) \Rightarrow U^\perp \subseteq \text{null } P_U \rightarrow \text{null } P_U = U^\perp$$

(f) $v = u + w$, $u \in U$, $w \in U^\perp$

$$v - P_U v = v - u = w \in W^\perp$$

(g) $P_U^2 = P_U$

$$v = u + w, \quad u \in U, \quad w \in U^\perp$$

$$P_U^2 v = P_U P_U v = P_U u = u$$

$$(h) \|P_U v\|^2 = \langle P_U v, P_U v \rangle = \langle u, u \rangle = \|u\|^2$$

$$\|u\|^2 \leq \|u\|^2 + \|w\|^2 = \|v\|^2$$

(i) Every $v \in V$ can be written as a linear combination of orthonormal basis

Minimizing the Distance to a Subspace

Suppose U is a finite-dimensional subspace of V , $v \in V$, $u \in U$.

Then

$$\|v - P_U v\| \leq \|v - u\|$$

Furthermore, the inequality above is an equality iff $u = P_U v$.

Proof

$$\begin{aligned}\|v - P_U v\|^2 &\leq \|v - P_U v\|^2 + \|P_U v - u\|^2 \\ &= \|v - P_U v + P_U v - u\|^2 \\ &= \|v - u\|^2\end{aligned}$$

Adjoint

Suppose $T \in L(V, W)$. The adjoint of T is the function $T^*: W \rightarrow V$ such that

$$\langle T v, w \rangle = \langle v, T^* w \rangle$$

$\forall v \in V, w \in W$.

Example

Define $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ by $(x_1, x_2, x_3) \mapsto (x_2 + 3x_3, 2x_1)$.

Find a formula for $T^*: \mathbb{R}^2 \rightarrow \mathbb{R}^3$.

Fix $(y_1, y_2) \in \mathbb{R}^2$. Then for every $(x_1, x_2, x_3) \in \mathbb{R}^3$ we have

$$\begin{aligned}\langle (x_1, x_2, x_3), T^*(y_1, y_2) \rangle &= \langle T(x_1, x_2, x_3), (y_1, y_2) \rangle \\ &= \langle (x_1, x_2, x_3), (2y_2, y_1, 3y_1) \rangle\end{aligned}$$

$$\text{Thus } T^*(y_1, y_2) = (2y_2, y_1, 3y_1)$$

The Adjoint is a Linear Map

If $T \in \mathcal{L}(V, W)$, then $T^* \in \mathcal{L}(W, V)$.

Proof

Suppose $T \in \mathcal{L}(V, W)$, $w_1, w_2 \in W$, $\lambda_1, \lambda_2 \in \mathbb{R}$.

Then $\forall v \in V$,

$$\begin{aligned}\langle v, T^*(\lambda_1 w_1 + \lambda_2 w_2) \rangle &= \langle Tv, \lambda_1 w_1 + \lambda_2 w_2 \rangle \\&= \langle Tv, \lambda_1 w_1 \rangle + \langle Tv, \lambda_2 w_2 \rangle \\&= \bar{\lambda}_1 \langle Tv, w_1 \rangle + \bar{\lambda}_2 \langle Tv, w_2 \rangle \\&= \bar{\lambda}_1 \langle v, T^* w_1 \rangle + \bar{\lambda}_2 \langle v, T^* w_2 \rangle \\&= \langle v, \bar{\lambda}_1 T^* w_1 + \bar{\lambda}_2 T^* w_2 \rangle\end{aligned}$$

Properties of the Adjoint

(a) $(S+T)^* = S^* + T^*$ $\forall S, T \in \mathcal{L}(V, W)$

(b) $(\lambda T)^* = \bar{\lambda} T^*$ $\forall \lambda \in \mathbb{F}$, $T \in \mathcal{L}(V, W)$

(c) $(T^*)^* = T$ $\forall T \in \mathcal{L}(V, W)$

(d) $I^* = I$, I is the identity operator on V

(e) $(ST)^* = T^* S^*$ $\forall T \in \mathcal{L}(V, W)$ and $S \in \mathcal{L}(W, U)$

(here, U an inner product space over \mathbb{F})

Proof (all similar)

(e) For $T \in \mathcal{L}(V, W)$, $S \in \mathcal{L}(W, U)$; if $v \in V$, $u \in U$

$$\langle v, (ST)^* u \rangle = \langle (ST)v, u \rangle = \underbrace{\langle S[Tv], u \rangle}_{\in U} = \langle Tv, S^* u \rangle = \langle v, T^* S^* u \rangle$$

Null Space and Range of T^*

Suppose $T \in L(V, W)$. Then

(a) $\ker T^* = (\text{range } T)^\perp$

(b) $\text{range } T^* = (\ker T)^\perp$

(c) $\ker T = (\text{range } T^*)^\perp$

(d) $\text{range } T = (\ker T^*)^\perp$

Matrix of T^*

Suppose A is an orthonormal basis for V and β is an orthonormal basis for W . Then if

$$A = [T]_{\beta \leftarrow A}$$
$$[T^*]_{A \leftarrow \beta} = A^T \quad (\text{conj. transpose if over } \mathbb{C})$$

Proof

Suppose $\dim V = n < \infty$, $\dim W = m < \infty$,

so that $A = (v_1, \dots, v_n)$, $\beta = (w_1, \dots, w_m)$; $1 \leq j \leq m$
 $1 \leq k \leq n$

and $A = [T]_{\beta \leftarrow A}$, $B = [T^*]_{\beta \leftarrow A}$

Then

$$T^* w_j = \sum_{r=1}^n B_{r,j} v_r \quad \forall j$$

$$T v_k = \sum_{r=1}^m A_{r,k} w_r \quad \forall k$$

$$\langle v_k, T^* w_j \rangle = \langle T v_k, w_j \rangle = B_{k,j} = A_{j,k} \quad \forall j, k \rightarrow B = A^T$$

Self-Adjoint Operators

An operator $T \in L(V)$ is called self-adjoint if $T = T^*$. In other words, $T \in L(V)$ is self-adjoint if and only if

$$\langle Tv, w \rangle = \langle v, Tw \rangle \quad \forall v, w \in V$$

Eigenvalues of Self-Adjoint Operators are Real

Every eigenvalue of a self-adjoint operator is real.

Proof

$$\begin{aligned} \lambda \|v\|^2 &= \langle \lambda v, v \rangle = \langle Tv, v \rangle = \langle v, Tr \rangle = \langle v, \bar{\lambda} v \rangle = \bar{\lambda} \|v\|^2 \\ \Rightarrow \lambda &= \bar{\lambda} \Rightarrow \lambda \in \mathbb{R} \end{aligned}$$

Self-adjoint Operators Have Eigenvalues

Suppose $V \neq \{0\}$ and $T \in L(V)$ is a self-adjoint operator. Then T has an eigenvalue.

Self-adjoint Operators and Invariant Subspaces

Suppose $T \in L(V)$ is self-adjoint and U is a subspace of V that is invariant under T . Then

- U^\perp is invariant under T
- $T|_{U^\perp}$ is self-adjoint
- $T|_U$ is self-adjoint

Real Spectral Theorem

Suppose $\text{IF} = \mathbb{R}$ and $T \in \mathcal{L}(V)$. Then the following are equivalent:

- (a) T is self-adjoint
- (b) V has an orthonormal basis consisting of eigenvectors of T
- (c) T has a diagonal matrix with respect to some orthonormal basis of V
- (d) (\exists orthogonal matrix Q , diagonal matrix Σ s.t. $[T] = Q\Sigma Q^{-1}$)

Isometry

An operator $S \in \mathcal{L}(V)$ is called an isometry if

$$\|Sv\| = \|v\| \quad \forall v \in V$$

Characterization of Isometries

Suppose $S \in \mathcal{L}(V)$. Then the following are equivalent:

- (a) S is an isometry
- (b) $\langle Su, Sv \rangle = \langle u, v \rangle \quad \forall u, v \in V$
- (c) $S e_1, \dots, S e_n$ is orthonormal for every orthonormal list (e_1, \dots, e_n)
- (d) $\exists (e_1, \dots, e_n)$ of V s.t. $S e_1, \dots, S e_n$ orthonormal
- (e) $S^* S = I$
- (f) $S S^* = I$
- (g) S^* is an isometry
- (h) S is invertible and $S^{-1} = S^*$

Sequence of Increasing Null Spaces

Suppose $T \in L(V)$. Then

$$\{0\} = \ker T^0 \subset \ker T^1 \subset \cdots \subset \ker T^k \subset \ker T^{k+1} \subset \cdots$$

Proof

Suppose $k \in \mathbb{Z}_{\geq 1}$ and $v \in \ker T^k$. Then $T(T^k v) = T^{k+1} v = 0$
 $\rightarrow v \in \ker T^{k+1} \rightarrow \ker T^k \subset \ker T^{k+1}$

Equality in the Sequence of Null Spaces

Suppose $T \in L(V)$. Suppose m is a nonnegative integer such that $\ker T^m = \ker T^{m+1}$. Then

$$\ker T^m = \ker T^{m+1} = \ker T^{m+2} = \cdots$$

Proof

Let $k \in \mathbb{Z}_{\geq 1}$. Want

$$\ker T^{m+k} = \ker T^{m+k+1}$$

Know $\ker T^{m+k} \subset \ker T^{m+k+1}$

Need $\ker T^{m+k+1} \subset \ker T^m$

Suppose $v \in \ker T^{m+k+1}$.

$$T^{m+1}(T^k v) = T^{m+1+k} v = 0$$

$$\rightarrow T^k v \in \ker T^{m+1} = \ker T^m$$

$$\Rightarrow v \in \ker T^{m+k}$$

So

$$\ker T^{m+k+1} \subset \ker T^m \text{ as desired.}$$

Null Spaces Stop Growing

Suppose $T \in \mathcal{L}(V)$. Let $n = \dim V$. Then

$$\ker T^n = \ker T^{n+1} = \ker T^{n+2} = \ker T^{n+3} = \dots$$

Proof

Only need $\ker T^n = \ker T^{n+1}$.

Suppose $\ker T^n \neq \ker T^{n+1}$.

Then $\{0\} = \ker T^0 \subsetneq \ker T \subsetneq \dots \subsetneq \ker T^n \subsetneq \ker T^{n+1}$.

At each strict inclusion, $\dim \ker$ increases by at least one. This give $\dim \ker T^{n+1} > n+1$!

V is the Direct Sum of $\ker T^{\dim V}$ and $\text{im } T^{\dim V}$

Suppose $T \in \mathcal{L}(V)$. Let $n = \dim V$. Then

$$V = \ker T^n \oplus \text{im } T^n.$$

Proof

First show $\ker T^n \cap \text{im } T^n = \{0\}$

Suppose $v \in \ker T^n \cap \text{im } T^n$

then $T^n v = 0$ and $\exists u \in V$ s.t. $T^n v = u$

Applying T^n , $T^{2n} v = T^n u$

$$T^{2n} = T^n \rightarrow T^{2n} v = 0 \rightarrow T^n u = 0 \rightarrow u = 0$$

\Rightarrow direct sum

Generalized Eigenvector

Suppose $T \in L(V)$ and λ an eigenvalue of T . A vector $v \in V$ is called a **generalized eigenvector** of T corresponding to λ if $v \neq 0$ and

$$(T - \lambda I)^j v = 0$$

for some positive integer j .

Generalized Eigenspace

Suppose $T \in L(V)$ and $\lambda \in \mathbb{F}$. The **generalized eigenspace** of T corresponding to λ , denoted $G(\lambda, T)$, is defined to be the set of all generalized eigenvectors of T corresponding to λ , along with the zero vector.

$$\text{Note: } E(\lambda, T) \subset G(\lambda, T)$$

Description of Generalized Eigenspaces

Suppose $T \in L(V)$ and $\lambda \in \mathbb{F}$. Then $G(\lambda, T) = \ker(T - \lambda I)^{\dim V}$

Proof

Suppose $v \in \ker(T - \lambda I)^{\dim V}$. Then $v \in G(\lambda, T) \Rightarrow \ker(T - \lambda I)^{\dim V} \subset G(\lambda, T)$
(Conversely, suppose $v \in G(\lambda, T)$. Then $\exists j \in \mathbb{Z}_+$ s.t.

$$v \in \ker(T - \lambda I)^j \subset \ker(T - \lambda I)^{\dim V}$$

$$\Rightarrow G(\lambda, T) \subset \ker(T - \lambda I)^{\dim V}$$

Nilpotent

An operator is called **nilpotent** if some power of it equals 0.

Nilpotent Operator Raised to Dimension of Domain is 0

Suppose $N \in L(V)$ is nilpotent. Then $N^{\dim V} = 0$.

Proof

Because N is nilpotent, $G(0, N) = V$.

Then $\ker N^{\dim V} = V$

Matrix of a Nilpotent Operator

Suppose N is a nilpotent operator on V . Then there is a basis of V with respect to which the matrix N has the form

$$\begin{pmatrix} 0 & * & & \\ 0 & \ddots & & \\ & & \ddots & 0 \\ & & & 0 \end{pmatrix}$$

The Null Space and Range of $p(T)$ are Invariant Under T

Suppose $T \in L(V)$ and $p \in P(\mathbb{F})$. Then $\ker p(T)$ and range $p(T)$ are invariant under T .

Description of Operators on Complex Vector Spaces

Suppose V is a complex vector space and $T \in \mathcal{L}(V)$. Let $\lambda_1, \dots, \lambda_m$ be the distinct eigenvalues of T . Then

- (a) $V = G(\lambda_1, T) \oplus \cdots \oplus G(\lambda_m, T)$
- (b) each $G(\lambda_j, T)$ is invariant under T
- (c) each $(T - \lambda_j I)|_{G(\lambda_j, T)}$ is nilpotent

A Basis of Generalized Eigenvectors

Suppose V is a complex vector space and $T \in \mathcal{L}(V)$. Then there is a basis of V consisting of generalized eigenvectors of T .

Multiplicity

Suppose $T \in \mathcal{L}(V)$. The multiplicity of an eigenvalue λ of T is defined to be the dimension of the corresponding generalized eigenspace $G(\lambda, T)$.

Sum of Multiplicities Equals $\dim V$

Suppose V is a complex vector space and $T \in \mathcal{L}(V)$. Then the sum of all the eigenvalues of T equals $\dim V$.

Note:

- Geometric Multiplicity = $\dim E(\lambda_i, T)$
- Algebraic Multiplicity = $\dim G(\lambda_i, T)$

Block Diagonal Matrix

A block diagonal matrix is a square matrix of the form

$$\begin{pmatrix} A_1 & & & \\ & \ddots & & 0 \\ & & \ddots & \\ 0 & & & A_m \end{pmatrix}$$

where A_1, \dots, A_m are square matrices lying along the diagonal and all other entries of the matrix equal 0.

Block Diagonal Matrix With Upper Triangular Blocks

Suppose V is a complex vector space and $T \in \mathcal{L}(V)$. Let $\lambda_1, \dots, \lambda_m$ be the distinct eigenvalues of T , with multiplicities d_1, \dots, d_m . Then there is a basis of V with respect to which T has a block diagonal matrix of the form

$$\begin{pmatrix} A_1 & & 0 & \\ & \ddots & & \\ & & \ddots & \\ 0 & & & A_m \end{pmatrix}$$

where each A_i is the $d_i \times d_i$ upper triangular matrix of the form

$$\begin{pmatrix} \lambda_1 & & * & \\ & \ddots & & \\ & & \ddots & \\ & & & \lambda_m \end{pmatrix}$$

Characteristic Polynomial

Suppose V is a complex vector space and $T \in L(V)$. Let $\lambda_1, \dots, \lambda_m$ denote the distinct eigenvalues of T , with multiplicities d_1, \dots, d_m . The polynomial

$$(z - \lambda_1)^{d_1} \cdots (z - \lambda_m)^{d_m}$$

is called the characteristic polynomial of T .

Degrees and Zeros of Characteristic Polynomial

Suppose V is a complex vector space and $T \in L(V)$. Then

- (a) the characteristic polynomial of T has degree $\dim V$
- (b) the zeros of the characteristic polynomial of T are the eigenvalues of T

Proof

Follows from $\sum d_i = \dim V$

Cayley Hamilton Theorem

Suppose V is a complex vector space and $T \in L(V)$. Let g denote the characteristic polynomial of T . Then

$$g(T) = 0$$

Proof

Let $\lambda_1, \dots, \lambda_m$ be the distinct eigenvalues of T with d_1, \dots, d_m algebraic multiplicity each

For each $j \in \{1, \dots, m\}$ we have $(T - \lambda_j I) \Big|_{G(\lambda_j, T)}$
is nilpotent.

Thus

$$(T - \lambda_j I)^{d_j} \Big|_{G(\lambda_j, T)} = 0$$

Every $v \in V$ can be written as

$$v = g_1 + \dots + g_m ; g_j \in G(\lambda_j, T)$$

Suffices to show

$$g(T) \Big|_{G(\lambda_j, T)} = 0 \quad \forall j$$

Fix $j \in \{1, \dots, m\}$. Then

$$g(T) = (T - \lambda_1 I)^{d_1} \cdots (T - \lambda_m I)^{d_m}$$

All operators on left-hand side commute.

Thus can rearrange to

$$g(T) = (T - \lambda_1 I) \cdots (T - \lambda_m I)^{d_m} (T - \lambda_j I)^{d_j}$$

Since

$$(T - \lambda_j I)^{d_j} \Big|_{G(\lambda_j, T)} = 0 \quad \forall j$$

then

$$g(T) \Big|_{G(\lambda_j, T)} = 0 \quad \forall j$$

as desired.

Monic Polynomial

A monic polynomial is a polynomial whose highest-degree coefficient equals 1.

Minimal Polynomial

Suppose $T \in \mathcal{L}(V)$. Then \exists a unique monic polynomial p of smallest degree such that $p(T) = 0$.

Proof

Let $n = \dim V$ (existence)

Then the list

$$I, T, T^2, \dots, T^{n^2} \quad \{n+1 \text{ elements}\}$$

is NOT linearly independent in $\mathcal{L}(V)$ since $\dim \mathcal{L}(V) = n^2$.

Let m be the smallest integer such that

$$I, T, T^2, \dots, T^m$$

is linearly dependent.

Then T^m is a linear combination of I, \dots, T^{m-1} by how we chose m .

Thus $\exists a_0, a_1, \dots, a_{m-1} \in \mathbb{F}$ such that

$$a_0 I + a_1 T + \dots + a_{m-1} T^{m-1} + T^m = 0$$

Define $p \in P(\mathbb{F})$ as

$$a_0 + a_1 z + \dots + a_{m-1} z^{m-1} + z^m.$$

Then $p(T) = 0$ is implied by construction.

(Uniqueness)

Choice of m gives that no monic polynomial $g \in P(\mathbb{F})$ with degree $< m$ satisfies $g(T) = 0$.

Suppose now $g \in P(\mathbb{F})$ has degree m and $g(T) = 0$.

Then

$$(P - g)T = 0$$

and

$$\deg(P - g) < m.$$

Choice of $m \Rightarrow g = P$.

Q.E.D

Minimal Polynomial

Suppose $T \in L(V)$. Then the minimal polynomial of T is the unique monic polynomial p of smallest degree such that $p(T) = 0$.

Note: has at MOST degree $\dim V$

$g(T) = 0 \Rightarrow g$ is a Multiple of Minimal Polynomial

Suppose $T \in L(V)$ and $g \in P(\mathbb{F})$. Then $g(T) = 0$ if and only if g is a polynomial multiple of the minimal polynomial of T .

Proof

Let p denote the minimal polynomial of T .

(\Leftarrow) Suppose g is a polynomial multiple of p .

Then $\exists s \in P(F)$ such that

$$g = ps$$

We have

$$g(T) = p(T)s(T) = 0 \quad s(T) = 0$$

(\Rightarrow) Suppose $g(T) = 0$.

By the division algorithm, $\exists s, r \in P(F)$ such that

$$g = ps + r$$

and $\deg r < \deg p$.

We have

$$g(T) = p(T)s(T) + r(T) = 0$$

Suppose $r(T) \neq 0$. Then dividing $r(T)$ by its highest degree coefficient would produce a monic polynomial that when applied to T equals zero.

This polynomial would have a smaller degree than the minimal polynomial ! C

Thus,

$$r(T) = 0$$

and $g(T)$ is a polynomial multiple of $p(T)$.

Characteristic Polynomial is a Multiple of Minimal Polynomial

Eigenvalues are the Zeros of the Minimal Polynomial
 Let $T \in \mathcal{L}(V)$. Then the zeros of the minimal polynomial
 of T are precisely the eigenvalues of T .

Proof

Let

$$p(z) = a_0 + a_1 z + \dots + a_{m-1} z^{m-1} + z^m$$

be the minimal polynomial of T .

Suppose λ is a zero of p . Then

$$p(z) = (z - \lambda) q_f(z)$$

where q_f is a monic polynomial with coefficients in \mathbb{F}
 and $\deg q_f = \deg p - 1$. Since $p(T) = 0$, we have

$$0 = (T - \lambda I)(q_f(T)v)$$

Then $\exists v \in V$ such that $q_f(T)v \neq 0$, and

$$T(q_f(T)v) = \lambda(q_f(T)v)$$

This gives λ is an eigenvalue of T .

Suppose now $\lambda \in \mathbb{F}$ is an eigenvalue of T .

Then $\exists v \neq 0 \in V$ such that

$$Tv = \lambda v.$$

Applying T to both sides repeatedly gives

$$T^j v = \lambda^j v \quad j = 1, 2, \dots$$

Thus

$$\begin{aligned} 0 = p(T)v &= (a_0 I + a_1 T + \dots + a_{m-1} T^{m-1} + T^m)v \\ &= (a_0 + a_1 \lambda + \dots + a_{m-1} \lambda^{m-1} + \lambda^m)v \\ &= p(\lambda)v \Rightarrow p(\lambda) = 0 \end{aligned}$$

Q.E.D