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Homework 11

Please print out these pages. I encourage you to work with your classmates on this homework. Please list your collaborators on this cover sheet. (Your grade will not be affected.) Even if you work in a group, you should write up your solutions yourself! You should include all computational details, and proofs should be carefully written with full details. As always, please write neatly and legibly.

Please follow the instructions for the "extended glossary" on separate paper (LaTeX it if you can!) Hand in your final draft, including full explanations and write your glossary in complete, mathematically and grammatically correct sentences. Your answers will be assessed for style and accuracy.

Please **staple** *this cover sheet, your exercise solutions, and your glossary together, in that order, and hand in your homework in class.*

GRADES

Exercises ______ / 50

Extended Glossary

Component	Correct?	Well-written?
Definition	/6	/6
Example	/4	/4
Non-example	/4	/4
Theorem	/5	/5
Proof	/6	/6
Total	/25	/25

Exercises.

1. Find a singular value decomposition for the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & -2 & 1 \\ 1 & -1 & 1 & 1 \end{pmatrix}.$$

We first realize

$$\operatorname{rank}(A) = 3.$$

Next, compute $A^T A$

$$A^T A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & -2 & 1 \\ 1 & -1 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 0 & 0 & 3 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 3 & 0 & 0 & 3 \end{pmatrix}$$

and realize

$$\operatorname{rank}(A^T A) = 3.$$

Next, compute the eigenvalues for $A^T A$,

$$\begin{pmatrix} 3 & 0 & 0 & 3 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 3 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

Computing the matrix multiplication yields the following system of equations:

$$3a + 3d = \lambda a,$$

$$2b = \lambda b,$$

$$4c = \lambda c,$$

$$3a + 3d = \lambda d.$$

Immediately we see that $\lambda = 2$ and $\lambda = 4$ are eigenvalues of $A^T A$ from equations (2) and (3).
Adding equation (1) to (4) yields

$$6(a + d) = \lambda(a + d)$$

giving $\lambda = 6$ as an eigenvalue.
Subtracting (1) from (4) yields

$$0 = \lambda(d - a)$$

giving $\lambda = 0$ as an eigenvalue.
To recap, our distinct eigenvalues of $A^T A$ are

$$\lambda_1 = 2, \lambda_2 = 4, \lambda_3 = 6, \lambda_4 = 0.$$

The eigenvectors are found via finding the eigenspace of each corresponding eigenvalue.

$$E(\lambda = 2, A^T A) = \ker(A^T A - 2I) = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$E(\lambda = 4, A^T A) = \ker(A^T A - 4I) = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$$E(\lambda = 6, A^T A) = \ker(A^T A - 6I) = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$E(\lambda = 0, A^T A) = \ker(A^T A) = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \right\}$$

Taking the nonzero vectors in each eigenspace and dividing each element by the norm of the vector it's in gives the orthonormal list

$$(\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}) = \left(\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{-1}{\sqrt{2}} \end{pmatrix} \right).$$

Since we have an orthonormal list of length 4, this list is a basis for $\mathbb{R}^4$. Let

$$V = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}$$

Taking the 3 nonzero distinct eigenvalues of A^TA we compute the corresponding singular values.

$$\sigma_1 = \sqrt{\lambda_1} = \sqrt{2}$$
$$\sigma_2 = \sqrt{\lambda_2} = \sqrt{4} = 2$$
$$\sigma_3 = \sqrt{\lambda_3} = \sqrt{6}$$

This gives

$$\Sigma = \begin{pmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 \\ 0 & 0 & \sigma_3 & 0 \end{pmatrix} = \begin{pmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 \end{pmatrix}$$

For the nonzero eigenvalues of A^TA we compute U using the corresponding eigenvectors.

$$\vec{u_1} = \frac{1}{\sigma_1} A\vec{v_1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & -2 & 1 \\ 1 & -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{-1}{\sqrt{2}} \end{pmatrix}$$

$$\vec{u_2} = \frac{1}{\sigma_2} A\vec{v_2} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & -2 & 1 \\ 1 & -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -1 \\ \frac{1}{2} \end{pmatrix}$$

$$\vec{u_3} = \frac{1}{\sigma_3} A\vec{v_3} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & -2 & 1 \\ 1 & -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$$

The list $(\vec{u_1}, \vec{u_2}, \vec{u_3})$ is an orthonormal list of length three and is thus an orthonormal basis for $\mathbb{R}^3$. Letting

$$U = \begin{pmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{\sqrt{3}} \\ 0 & -1 & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

we see that

$$\begin{bmatrix} A\vec{v}_1 & A\vec{v}_2 & A\vec{v}_3 & A\vec{v}_4 \end{bmatrix} = \begin{bmatrix} \sigma_1\vec{u}_1 & \sigma_2\vec{u}_2 & \sigma_3\vec{u}_3 & 0 \end{bmatrix}$$

so that

$$AV = U\Sigma$$

yielding

$$A = U\Sigma V^{-1}.$$

Since V is an orthogonal matrix, $V^{-1} = V^T$ and we conclude

$$A = U\Sigma V^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{\sqrt{3}} \\ 0 & -1 & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{-1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & -2 & 1 \\ 1 & -1 & 1 & 1 \end{pmatrix}$$

2. Show that if $A = U_1\Sigma_1V_1^T$, and $A = U_2\Sigma_2V_2^T$ are two singular value decompositions of A,then $\Sigma_1 = \Sigma_2$, that is, the singular values of A are uniquely determined by any SVD of A.

Proof. Suppose A is an $m \times n$ matrix. We have

$$A = U_2\Sigma_2V_2^T = U_1\Sigma_1V_1^T$$

and

$$A^TA = V_2\Sigma_2^TU_2^TU_2\Sigma_2V_2^T = V_1\Sigma_1^TU_1^TU_1\Sigma_1V_1^T.$$

Since U is an orthogonal $m \times m$ matrix, $U^TU = I_{m\times m}$. We can therefore simplify the above to

$$A^TA = V_2\Sigma_2^T\Sigma_1V_2^T = V_1\Sigma_1^T\Sigma_1V_1^T.$$

But Σ is exactly the diagonal matrix with the entries being the square root of the eigenvalues of A^TA in decreasing order. Thus $\Sigma^T\Sigma$ is the diagonal matrix with entries of eigenvalues of A^TA in decreasing order. Since the eigenvalues of a square matrix A are unique, it must be that $\Sigma_2^T\Sigma_2 = \Sigma_1^T\Sigma_1$. □

3. Prove that if A is positive definite, then

   (a) the singular values of A are the same as the (nonzero) eigenvalues of A.

Proposition 1. If A is positive definite, $A^T A$ is positive definite.

Proof. Taking $(A\vec{v})^T(A\vec{v})$ for $\vec{v} \neq \vec{0}$ gives

$$(A\vec{v})^T(A\vec{v}) = \|A\vec{v}\|^2 = \vec{v}^T A^T A\vec{v} \geq 0$$

Since A is positive definite it is a full rank matrix, and therefore $A^T A$ is a full rank matrix giving $A^T A$ is invertible and therefore zero is not an eigenvalue allowing us to strengthen the above inequality to

$$\|A\vec{v}\|^2 = \vec{v}^T A^T A\vec{v} > 0.$$

□

Now we begin the proof.

Proof. Suppose $A \in \mathbb{R}^{n\times n}$. Matrix A being positive definite means

$$\langle \vec{v}, \vec{w} \rangle = \vec{v}^T A\vec{w}$$

is an inner product on $\mathbb{R}^n$ for $\vec{v}, \vec{w} \in \mathbb{R}^n$, as well as that A is real and symmetric. It also gives us the fact that A is a full rank matrix. That is, A has no zero eigenvalues.
The singular values of A are the square root of the nonzero eigenvalues of $A^T A$. The n nonzero eigenvalues of A are

$$A\vec{v} = \lambda\vec{v}$$

for $\vec{v} \neq \vec{0} \in \mathbb{R}^n$. Applying A^T to both sides gives

$$A^T A\vec{v} = \lambda\vec{v}$$

$$A^T A\vec{v} = \lambda A^T \lambda\vec{v}$$

$$A^T A\vec{v} = \lambda\lambda A\vec{v}$$

$$A^T A\vec{v} = \lambda^2\vec{v}.$$

That is, the nonzero eigenvalues of $A^T A$ are the square of the eigenvalues of A.
Thus singular values of A are the square root of these squared lambdas, i.e.

$$\sigma_1 = \sqrt{\lambda_1^2} \geq \sigma_2 = \sqrt{\lambda_2^2} \geq ... \geq \sigma_r = \sqrt{\lambda_n^2},$$

giving that the singular values of A are the same as the eigenvalues of A.

□

(b) if $A = U\Sigma V^T$ is a singular value decomposition of A, then $U = V$.

Proof. Suppose that $A \in \mathbb{R}^{n\times n}$ is positive definite and that A has singular value decomposition $A = U\Sigma V^T$.
As shown in part (a),

$$\Sigma_{n\times n} = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}.$$

Right multiplying both sides of the SVD by V gives

$$AV = U\Sigma$$

since V is an orthogonal matrix in the decomposition. Consider, for $1 \leq i \leq n$, the vector by vector multiplication. This yields

$$A\vec{v_i} = \vec{u_i}\lambda_i.$$

But since $\vec{v_i}$ is an eigenvector of A (proved in part A), this becomes

$$A\vec{v_i} = \vec{u_i}\lambda_i = \vec{v_i}\lambda_i$$

giving $U = V$. □

A real symmetric matrix is ***positive definite*** *if for all $\vec{x} \neq 0$, $\vec{x}^T A\vec{x} > 0$*

4. Let $A \in \mathbb{R}^{m\times n}$ be a rank one matrix.

(a) Show that there exists unit vectors $\vec{x} \in \mathbb{R}^m$and $\vec{y} \in \mathbb{R}^n$, and a number $\alpha > 0$ such that $A = \alpha\vec{x}\vec{y}^T$.

Proof. A being a rank 1 matrix means every single column of A is linearly dependent. That is, A can be written as an $m \times 1$ column vector multiplied by a $1 \times n$ vector of scalars. For $\vec{v} \in \mathbb{R}^m \vec{u} \in \mathbb{R}^n$

$$A = \vec{v}\vec{u}^T$$

where $\vec{v}$ is the first column of A and $\vec{u}$ consists of entries u_i - the scalars needed to multiply v to get column i of A.
Normalizing things yields

$$A = \|\vec{v}\|\|\vec{u}\|\frac{\vec{v}}{\|v\|}\frac{\vec{u}^T}{\|u\|}.$$

Letting $\vec{x} = \frac{\vec{v}}{\|v\|}$, $\vec{y}^T = \frac{u^T}{\|u\|}$ and $\alpha = \|\vec{v}\|\|\vec{u}\|$ completes the proof. □

(b) Find a compact singular value decomposition for A.
This is actually the same result as in part (a), but with some things to be noted.
Know A is of the form
$$A = \begin{pmatrix} u_1\vec{v_1} & \dots & u_n\vec{v_1} \end{pmatrix}$$
where $\vec{v}_1 \in \mathbb{R}^m$. Then
$$A = \vec{v_1}\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}^T = \vec{v_1}\vec{u}^T.$$
The singular value is a 1×1 matrix needed to "undo" the normalization of $\vec{u}, \vec{v}$. Thus
$$A = \|u\|\|v\|\frac{\vec{v}}{\|v\|}\frac{u^T}{\|u\|} = \alpha\vec{x}\vec{y}^T$$
is a singular value decomposition for A.

(c) If $A = \alpha\vec{x}\vec{y}^T$ as in (a), find the eigenvalues and eigenvectors of A^TA.
Since A is a rank 1 matrix, A^TA is a rank 1 matrix. Thus
$$\dim\ker(A^TA) = n - 1$$
giving that $\lambda = 0$ is an eigenvalue. The eigenvectors corresponding to $\lambda = 0$ are contained in the kernel of A giving there are $n-1$ linearly independent eigenvectors in the kernel.
The only nonzero eigenvalue arises from squaring the singular value in the singular value decomposition of A. That is,
$$\lambda = \|\vec{u}\|^2\|\vec{v}\|^2$$
is the eigenvalue of A ($\vec{v}, \vec{u}$ as in (a)).
The eigenvector corresponding to this eigenvalue is the $\vec{v} \neq \vec{0} \in \mathbb{R}^n$ such that
$$A^TA\vec{v} = \alpha^2\vec{v}.$$

5. If Q is an orthogonal $n \times n$ matrix, find a singular value decomposition for Q.
If Q is an orthogonal matrix then
$$Q^TQ = I_{n\times n}.$$
The $n \times n$ identity has eigenvalue 1 with multiplicity n. That is,
$$\Sigma = I_{n\times n} = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} = \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{pmatrix} = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}.$$
An orthogonal basis of eigenvectors of Q^TQ is the standard basis, $(\vec{e_1}, ..., \vec{e_n})$. That is
$$V = V^T = I_{n\times n}.$$

Thus

$$Q = U\Sigma V^T = UI_{n\times n}I_{n\times n} = U.$$

That is, for Q and orthogonal $n \times n$ matrix, Q is its own singular value decomposition.

6. Recall that the Frobenius norm of a matrix $A \in V = R^{m\times n}$ is

$$\|A\| = \sqrt{\langle A, A\rangle} = \sqrt{\sum_{i,j} A_{i,j}^2}$$

where

$$\langle A, B\rangle = \sum_{i,j} A_{i,j}B_{i,j}$$

This is sometimes written as $\|A\|_F$.

We also assume that $A = \sigma_1\vec{u}_1\vec{v}_1^T + ... + \sigma_r\vec{u}_r\vec{v}_r^T$ arises from a compact singular value decomposition of A.

For $k \le r$, let

$$A_k := A = \sigma_1\vec{u}_1\vec{v}_1^T + ... + \sigma_k\vec{u}_k\vec{v}_k^T.$$

(a) Show that for $A, B \in \mathbb{R}^{m\times n}$

$$\langle A, B\rangle = trace(A^TB) = trace(BA^T).$$

Recall that the tr(C) of a square matrix C is the sum of its diagonal entries. Also note the key property that $tr(AB) = tr(BA)$ *whenever* A *and* B *are well defined.*

Proof. For $1 \le i \le m, 1 \le j \le n$,

$$\langle A, B\rangle = \sum_{i,j} A_{i,j}B_{i,j}$$

A^TB is an $n \times n$ matrix where the j^{th} diagonal entry is

$$\sum_{i=1}^{m} A_{j,i}^T B_{i,j}.$$

Summing over the n diagonals gives

$$trace(A^TB) = \sum_{j=1}^{n}\sum_{i=1}^{m} A_{i,j}B_{i,j}.$$

Similarly, BA^T is the $m \times m$ matrix where the i^{th} diagonal entry is

$$\sum_{j=1}^{n} B_{i,j}A_{j,i}^T$$

giving

$$\mathtt{trace}(BA^T) = \sum_{j=1}^{n} B_{i,j}A_{i,j}.$$

Summing over the m diagonals gives

$$\mathtt{trace}(BA^T) = \sum_{i=1}^{m}\sum_{j=1}^{n} B_{i,j}A_{i,j}$$

as desired. □

(b) Show that the matrices

$$M_i = \vec{u_i}\vec{v_i}^T$$

are orthonormal, and consequently

$$\|A\|^2 = \sigma_1^2 + ... + \sigma_r^2$$

A fact used in this proof is that the squared norm of an orthonormal linear combination is the sum of squares of the coefficients. That is, if $e_1, ..., e_n$ is an orthonormal list of vectors in a finite inner product space V over $\mathbb{F}$,

$$\|a_1\vec{e_1} + ... + a_n\vec{e_n}\|^2 = |a_1|^2 + ..._|a_n|^2$$

for all $a_1, ..., a_n \in \mathbb{F}$.

Proof. Consider $1 \leq i \leq r$. To show that M_i are orthonormal we need to show that

$$\langle M_i, M_j\rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}.$$

Consider first $i = j$. Then

$$\langle M_i, M_i\rangle = \langle \vec{u_i}\vec{v_i}^T, \vec{u_i}\vec{v_i}^T\rangle = \mathtt{trace}(\vec{v_i}\vec{u_i}^T\vec{u_i}\vec{v_i}^T)$$
$$= \mathtt{trace}(\vec{v_i}\vec{v_i}^T) = 1.$$

Next, consider $i \neq j$, then

$$\langle M_i, M_j\rangle = \langle \vec{u_i}\vec{v_i}^T, \vec{u_j}\vec{v_j}^T\rangle = \mathtt{trace}(\vec{v_i}\vec{u_i}^T\vec{u_j}\vec{v_j}^T) = 0$$

as desired.
Now we know that $(M_1, ..., M_r)$ is an orthonormal list and so using the fact above

$$\langle A, A\rangle = \|A\|^2 = \|\sigma_1 M_1 + ... + \sigma_r M_r\|^2 = \sigma_1^2 + ... + \sigma_r^2$$

as desired.

□

(c) Find $\|A - A_k\|$. First note

$$A - A_k = \sigma_{k+1} M_{k+1} + \ldots + \sigma_r M_r$$

so that

$$\|A - A_k\| = \sqrt{\sigma_{k+1}^2 + \ldots + \sigma_r^2}.$$

(d) Show that if Q is an orthogonal $m \times m$ matrix, then

$$\|QA\| = \|A\|.$$

What is $\|A\| = \left\|U\Sigma V^T\right\|$ for a singular value decomposition of A?

Proof. If Q is an orthogonal $m \times m$ matrix, and $A \in \mathbb{R}^{m\times n}$ then

$$\|QA\|^2 = \langle QA, QA\rangle = trace(A^T Q^T QA) = trace(A^T I_{m\times m} A) = \|A\|^2$$

so that

$$\|QA\|^2 = \|A\|^2 \implies \|QA\| = \|A\|.$$

For $A = U\Sigma V^T$,

$$\|A\| = \left\|U\Sigma V^T\right\|$$

so that

$$\|A\|^2 = \left\|U\Sigma V^T\right\|^2$$

giving

$$\langle A, A\rangle = trace(V\Sigma^T \Sigma V^T) = \lambda_1 + \ldots + \lambda_r$$

where the λ_i are the nonzero eigenvalues of $A^T A$ ordered so that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r$. Thus,

$$\|A\| = \sqrt{\lambda_1 + \ldots + \lambda_r}$$

□

Part 2

1. Prove that if $A \in \mathbb{C}^{n\times n}$, then A and A^T are similar. (Hint: use Jordan canonical form!).

   This was done in a roundabout fashion, but uses a lot of course material to do so I left it as is.

   Proof. First we show that A and A^T have the same eigenvalues (without using similarity). Suppose that $\vec{v} \in \mathbb{C}^n$ is an eigenvector corresponding to eigenvalue λ of A. Let $\vec{w} \in \mathbb{C}^n$. Then by definition of adjoint have

   $$\langle A\vec{v}, \vec{w}\rangle = \langle \vec{v}, A^*\vec{w}\rangle = \langle \vec{v}, \lambda\vec{w}\rangle = \langle \lambda\vec{v}, \vec{w}\rangle$$
   $$\implies \langle \vec{v}, \lambda\vec{w} - A^*\vec{w}\rangle = 0.$$

   Since $\vec{v} \neq 0$ it must be that

   $$\lambda\vec{w} - A^*\vec{w} = \vec{0} \implies A^*\vec{w} = \lambda\vec{w}$$

   giving λ is an eigenvalue of A^*. Not shown is that the eigenspace dimension of each eigenvalue remains the same as a result of the rank-nullity theorem.

   Next, suppose the general case where A has $m \leq n$ distinct eigenvalues $\lambda_1, ..., \lambda_m$. Then A is similar to a direct sum of Jordan blocks:

   $$A \sim J(\lambda_1, \underline{k}^{(1)}) \oplus ... \oplus J(\lambda_m, \underline{k}^{(m)})$$

   where $\underline{k}^{(i)}$ is the partition consisting of $\mathrm{dim}(E_{\lambda_i})$ parts.

   As proved above, A^T has the same eigenvalues as A, and the dimension of the eigenspace of each eigenvalue remains the same. Thus, similarly,

   $$A^T \sim J(\lambda_1, \underline{k}^{(1)}) \oplus ... \oplus J(\lambda_m, \underline{k}^{(m)})$$

   where $\underline{k}^{(i)}$ is the partition consisting of $\mathrm{dim}(E_{\lambda_i})$ parts.

   Since similarity is an equivalence relation, it follows that $A \sim A^T$. □

2. Let
   $$N = J(0; k_1, ...k_r) = J(0, k_1) \oplus ... \oplus J(0, k_r),$$
   where $k_1 \geq ... \geq k_r > 0$ and $J(\lambda, p)$ is the Jordan block with eigenvalue λ of size $p \times p$.

   The fact N is a nilpotent matrix may be used without proof.

   *Recall that a nilpotent matrix is said to be **of index** m if* $N^m = 0$ *but* $N^{m+1} \neq 0$.

   Find, in terms of the partition $(k_1, ..., k_r)$, the following:

   (a) dim ker(N), $\mathrm{rank}(N)$.

   $$\text{dim ker}(N) = \text{dim ker}(J(0, k_1) + ... + J(0, k_r)) = \sum_{i=1}^{r} \text{dim ker}(0, k_i).$$

That is, the dimension of the kernel of N is the sum of the dimension of the kernel of each Jordan block.

By rank-nullity, it follows that

$$\mathrm{rank}(N) = (k_1 + ... + k_r) - \sum_{i=1}^{r} \text{dim ker } J(0, k_i)$$

(b) the index of the nilpotent matrix N.

The index of N is either the rank(N) or the rank(N) + 1, where the rank(N) is as defined above.

(c) the geometric and algebraic multiplicity of λ for A if $A = \lambda I + N$.

Suppose that $n = k_1 + ... + k_r$.

The algebraic multiplicity is the dimension of the generalized eigenspace of λ, and the geometric multiplicity is the dimension of the eigenspace of λ. They are as follows:

$$\text{dim } G_\lambda(A) = \text{dim } \mathrm{ker}((A - \lambda I)^n) = \sum_{i=1}^{r} k_i = n$$

$$\text{dim } E_\lambda(A) = \text{dim } \mathrm{ker}(A - \lambda I) = \text{dim } \mathrm{ker}(N) = \sum_{i=1}^{r} \text{dim } \mathrm{ker}(J(0, k_i))$$

(d) dim ker(N^2), dim ker(N^3). What about dim ker (N^m) in general.

dim ker(N^2) = $k_2 + \text{dim ker } N = k_1 + k_1$

dim ker(N^3) = $k_3 + \text{dim ker } N^2 = k_3 + k_2 + k_1$

dim ker(N^m) = $k_m + \text{dim ker } N^{m-1}$

(e) Does knowing dim ker (N^m) for all m determine $(k_1, ..., k_r)$? Either prove this or give a counter example.

Proof. Suppose that we know dim ker(N^m) for all m. Also suppose that N is an $n \times n$ matrix.

The rank of N is

$$\nabla_1 = n - \text{dim ker}(N)$$

giving that the Jordan Canonical Form of N has ∇_1 ones.

Assume there are r blocks, and let the i^{th} block be denoted by k_i. Then $k_1 + ... + k_r = n$ in a way such that $k_1 \geq ... \geq k_r > 0$.

Assume there are j blocks. We can rewrite the rank in terms of these blocks as

$$\nabla_1 - j = (k_1 - 1) + ... + (k_r - 1) \implies j = \text{dim ker N}.$$

So the partition will consist of j blocks.

The first block will have dimension

$$k_1 = \text{dim ker N},$$

the second will have

$$k_2 = \dim \ker N^2 - \dim \ker N,$$

$$\vdots$$

The last will have

$$k_r = \dim \ker (N^r) - \dim \ker N^{r-1},$$

all subject to constrictions above. That is, once one has value 1, the rest will have value 1.

$\square$

3. Find the Jordan canonical form of the linear transformation $T : \mathbb{C}^6 \to \mathbb{C}^6$ whose matrix in the standard basis of $\mathbb{C}^6$ is

$$A = [T]_{\mathcal{E}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(Hint: the only eigenvalue of this matrix is $\lambda = 1$.)

Begin first by noting that A is full rank and thus $\lambda = 1$ is an eigenvalue with multiplicity 6 and $\exists$ 6 linearly independent eigenvectors of A.

Want to find

$$\dim (E_{\lambda=1}(A))$$

and

$$\dim (G_{\lambda=1}(A)).$$

Calculating

$$E_{\lambda=1}(A) = \left\{ \vec{v} \in \mathbb{C}^n \mid (A - I_{6\times 6})\vec{v} = \vec{0} \right\} = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

gives that

$$\dim (E_{\lambda=1}(A)) = 3.$$

Then we know that

$$\dim (G_{\lambda=1}(A)) = n = 6.$$

The partition we seek is the one consisting of $\dim (E_{\lambda=1})$ "parts". For $n = 6$, three such partitions exist.

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Thus A has only three possible Jordan Canonical forms. The possibilities are

$$J_1 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$J_2 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$J_3 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The partition required is

$$\underline{k} = (3, 2, 1)$$

so that J_2 is our final answer.