Recall

Continuous rvs, pdfs, cdfs, etc.

X w/ pdf $f_X(x)$ - $F_X(x) = \mathbb{P}(\{X \le x\}) = \int_{-\infty}^{x} f_X(t)\,dt$

Note: every random variable has a cdf, even discrete ones.

$$F_X(x) = \mathbb{P}(\{X \le x\})$$

X is Gaussian (normal) w/ mean μ, variance σ^2 when

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$$

CDF of a Gaussian $= \int_{-\infty}^{x} f_X(t)\,dt$

Special Case: $\mu = 0,\ \sigma = 1$ ← Standard normal distribution then get

$$\Phi(x) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\,dt$$

as cdf.

Example- Communication Channel

Want to send one bit through a noisy channel. Encode a 1 as $+C$, a 0 as $-C$ (signal levels).

Thus

$$X = \text{input to channel} = \begin{cases} C & \text{w/ probability } p \\ -C & \text{w/ probability } 1-p \end{cases}$$

(often $p = \frac{1}{2}$ assumed)

Channel output

$$Y = X + N.$$

where N = noise - model as a zero-mean, some variance σ^2

<u>Decoding Rule</u>: Decide $+C$ was sent when $Y \geq 0$, decide $-C$ was sent when $Y < 0$ ← optimal for $p = 1/2$. Else would need to skew towards one result.

What is $\mathbb{P}(\text{error in decoding})$?

It's

$$\mathbb{P}(\text{error} \mid C \text{ sent})\, p + \mathbb{P}(\text{error} \mid -C \text{ sent})(1-p)$$

similar calculation gives $1 - \Phi(\frac{c}{\sigma})$ as well

$$= \mathbb{P}(\{Y \geq 0\} \mid \{X = -C\}) = \mathbb{P}(-C + N \geq 0) = \mathbb{P}(N \geq +C)$$

Idea: get an error when N is positive enough to "push the $-C$ into the 'decode as $+C$' zone"

Get an appropriate value for $\mathbb{P}(\{N \geq c\})$ from Φ-table - N zero mean, $\sigma^2 \Rightarrow N/\sigma$ standard normal, $\mu = 0$, $\sigma = 1$

Hence

$$\mathbb{P}(\{N \geq \sigma\}) = \mathbb{P}(\{\frac{N}{c} \geq \frac{c}{\sigma}\}) = 1 - \Phi(\frac{c}{\sigma})$$

Comment: Turns out that if X and Y are Gaussian defined on same probability space $\Omega, \mathbb{P}$, then any linear combo $Z = c_1 X + c_2 Y$ is **also** Gaussian. ← More on this later

One other CDF-type thing

Example - Experimental Meets Geometric Pt. 2

Consider first T, an exponential(λ) rv

Then

$$f_T(t) = \begin{cases} \lambda e^{-\lambda t} & , t \geq 0 \\ 0 & , t < 0 \end{cases}$$

and

$$F_T(t) = \int_{-\infty}^{\tau} f_T(\tau)\, d\tau = \begin{cases} 1 - e^{-\lambda t} & , t \geq 0 \\ 0 & , t < 0 \end{cases}$$

Next, consider the discrete rv X obtained as follows:

- flip a p-coin in real time
- flip at times $\delta, 2\delta, 3\delta, \ldots$ δ small
- $X =$ the time of the first flip **that is a heads**

k an integer

$\forall k > 0$, $\mathbb{P}(\{X = k\}) = p(1-p)^{k-1}$.

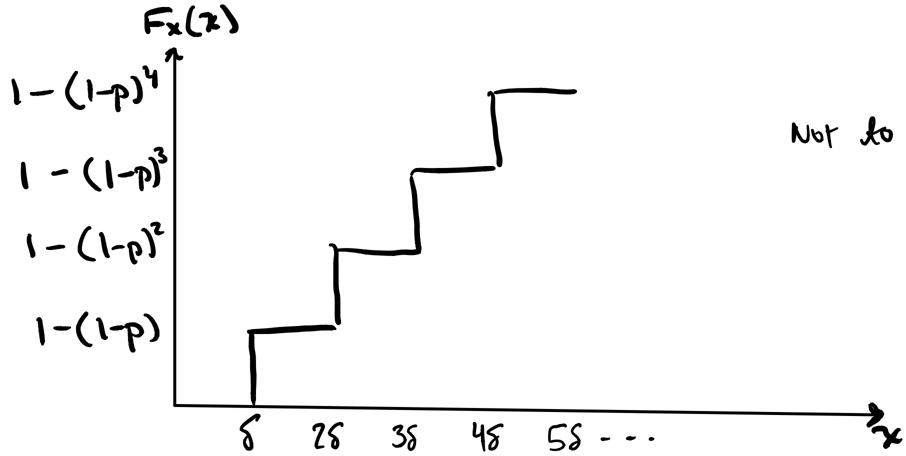
CDF of X?

$$F_X(x) = \mathbb{P}(\{X \leq x\}) = \sum_{\{m : m\delta < x\}} p(1-p)^m$$

Staircase function of x

← $m \in \mathbb{Z}$

Value in the interval $[k\delta, (k+1)\delta)$ is $1 - (1-p)^k$ $\forall k \geq 0$



Not to scale

← Visualization of Staircase function

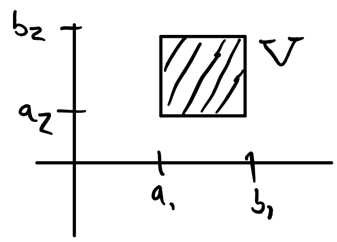
Observation: When $\lambda = -\dfrac{\ln(1-p)}{\delta}$, we have $F_X(k\delta) = F_T(k\delta)\ \forall k \geq 0$

Let δ get smaller, staircase approximation gets better; etc.

Moving along – say X, Y rvs defined in same $\Omega, \mathbb{P}$ are jointly continuous w/ joint pdf $f_{X,Y}(x,y)$ when

$$\mathbb{P}(\{(X,Y) \in V\}) = \iint_V f_{X,Y}(x,y)\,dx\,dy \quad \forall V \subset \mathbb{R}^2$$

Special case of a V: $[a_1, b_1] \times [a_2, b_2]$



Then

$$\mathbb{P}(\{(X,Y) \in V\}) = \int_{a_1}^{b_1} dx \int_{a_2}^{b_2} dy \left(f_{X,Y}(x,y) \right)$$

Again, have marginals

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dy \quad ; \quad f_Y(y) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dx$$

An official way to get this: Get $F_X(x)$ first then take $\frac{d}{dx} F_X(x)$

$$F_X(x) = \mathbb{P}(\{X \leq x\}) = \mathbb{P}(\{(X,Y) \in (-\infty, x] \times (-\infty, +\infty)\})$$

$$= \int_{-\infty}^{x} dt \int_{-\infty}^{+\infty} dy \left(f_{X,Y}(t,y) \right)$$

then

$$\frac{d}{dx} F_X(x) = \int_{-\infty}^{+\infty} dy \left(f_{X,Y}(x,y) \right)$$

Could also derive marginal formulas as follows:

$\forall\, V \subset \mathbb{R}$, $\mathbb{P}(\{X \in V\}) = \mathbb{P}(\{(X,Y) \in V \times (-\infty, \infty)\})$

$$= \int_{V} dx \int_{-\infty}^{+\infty} dy \left(f_{X,Y}(x,y) \right) = \int_{V} \left(\int_{-\infty}^{+\infty} f_{X,Y}(x,y)\, dy \right) dx$$

Must be $f_X(x)$ integrate over V to get $\mathbb{P}(\{X \in V\})$

Other stuff

- $\displaystyle\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \left(f_{X,Y}(x,y) \right) = 1$
- Joint CDF: $F_{X,Y}(x,y) = \mathbb{P}(\{X = x\} \cap \{Y = y\}) = \displaystyle\int_{-\infty}^{x} ds \int_{-\infty}^{t} dt \left(f_{X,Y}(s,t) \right)$
- $f_{X,Y}(x,y) = \dfrac{\partial}{\partial x} \dfrac{\partial}{\partial y} F_{X,Y}(x,y)$
- generalization to >2 rvs pretty straightforward

As for discrete rvs, joint determines the marginals; **NOT** vice-versa.

Example - Joint PDF

One example of a joint PDF: given any set $V_0 \subset \mathbb{R}^2$ w/ nonzero area, say (X,Y) **jointly uniform** on V_0 when

$$f_{X,Y} = \begin{cases} \frac{1}{\text{area}(V_0)}, & (x,y) \in V_0 \\ 0, & \text{otherwise} \end{cases}$$

Next,

Conditional Stuff For Continuos Random Variables

Given a continuous rv X on $\Omega, \mathbb{P}$ and some event $A \subset \Omega$, the conditional pdf of X given A "defined" as follows:

For any $V \subset \mathbb{R}$, we have

$$\mathbb{P}(\{X \in V\} | A) = \int_V f_{X|A}(x)\,dx$$

In general, no decent formula for $f_{X|A}(x)$ in terms of $f_X(x)$.

One way to compute it:

- First get conditional cdf of x given A

$$F_{X|A} = \mathbb{P}(\{X \le x\} | A)$$

- Then take d/dx to get $f_X(x)$

However, if A is an event of the form $\{X \in W\}$, and $\mathbb{P}(A) > 0$, we have

$$f_{X|A}(x) = \begin{cases} \dfrac{f_X(x)}{\mathbb{P}(\{X \in W\})}, & \text{when } x \in W \\ 0, & \text{otherwise} \end{cases}$$

How does this arise?

$$\mathbb{P}(\{X \in V\} | A) = \frac{\mathbb{P}(\{X \in (V \cap W)\})}{\mathbb{P}(\{X \in W\})} = \frac{\int_{V \cap W} f_X(x)\,dx}{\mathbb{P}(\{X \in W\})}$$

Define the <u>indicator function</u> of W via

"chi" $\rightarrow$ $$\chi_W(x) = \begin{cases} 1, & x \in W \\ 0, & x \notin W \end{cases}$$

So,

must be $f_{X|A}(x)$

$$\frac{\int_{V \cap W} f_X(x)\,dx}{\mathbb{P}(\{X \in W\})} = \int_V \left(\frac{f_X(x)\chi_W(x)}{\mathbb{P}(\{X \in W\})} \right) dx = \begin{cases} \dfrac{f_X(x)}{\mathbb{P}(\{X \in W\})}, & x \in W \\ 0, & \text{else} \end{cases}$$