# Conformal Mapping

Useful for solving Laplace's equation $\nabla^2 \phi = 0$ in some weirdly shaped region, subject to same boundary conditions (BC's).

## Strategy?

① Map to a simpler region

② Solve Laplace's equation in simpler region.

- Laplace's equation maps on to itself!

③ Map back using inverse mapping

Applications?

- Lift on an airplane wing
- Fringe field for a capacitor with finite plates

## Preliminaries:

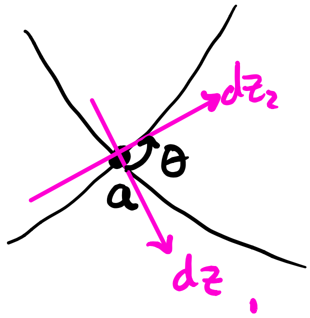
① what's "conformal"?

- Angles are preserved

Suppose $f(z)$ is analytic. Consider mapping $w = f(z)$.

Then $f(z)$ "preserves angles" at points where $f'(z) \neq 0$

Proof: What do we mean by angle?

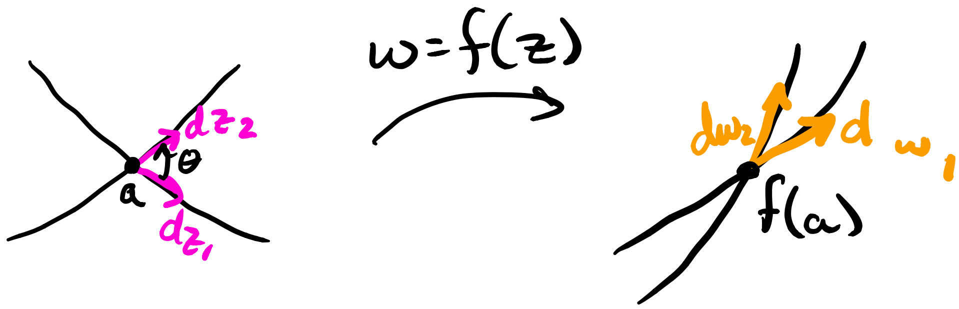



Assume $f'(a) \neq 0$

$dz_1$, $dz_2$ are infinitesimal vectors tangent to curve at point $a$.

$\theta$ is angle between them.

angle




differentiate $w = f(z)$

$$\Rightarrow \quad dw = f'(z)\, dz$$

$$dw = f'(a)\, dz$$

Write

$$f'(a) = re^{i\beta}$$

(b/c $f'(a)$ is a complex number)

where

$r, \beta$ are just constants that depend on $a$.

$$dw = re^{i\beta} dz \longleftarrow$$

just a stretching of $dz$ by $r$ and rotation of $\beta$. So all angles are preserved by mapping!

Visual complex analysis
Tristan Needham

check it out

So locally, in infinitesimal neighborhood of a, $f$ just stretches all $dz$'s by same $r>0$ and rotates them by $\beta$. So angle between them does NOT change.

## ② Invariance of Laplace's Equation

Suppose

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 .$$

Suppose we change variables

$$(x,y) \longrightarrow (u,v)$$

via $w = f(z)$ where $z = x+iy$ and $w = u+iv$, and $f$ is analytic and invertible.

So, $u, v$ satisfy Cauchy-Riemann equations!

$$\left.\begin{aligned} x &= x(u,v) & u &= u(x,y) \\ y &= y(u,v) & v &= v(x,y) \end{aligned}\right\} \text{Keep note of this}$$

$$\phi(x,y) = \phi(x(u,v), y(u,v)) = \tilde{\phi}(u,v)$$

Claim that $\tilde{\phi}$ satisfies

$$\frac{\partial^2 \tilde{\phi}}{\partial u^2} + \frac{\partial^2 \tilde{\phi}}{\partial v^2} = 0$$

Proof:

Write $\phi(x,y)$ as $\mathrm{Re}\{g(z)\}$, where $g(z)$ is analytic — ok b/c $\phi$ is harmonic

Then

$$\tilde{\phi}(w) = \mathrm{Re}\{g(f^{-1}(w))\}$$

analytic

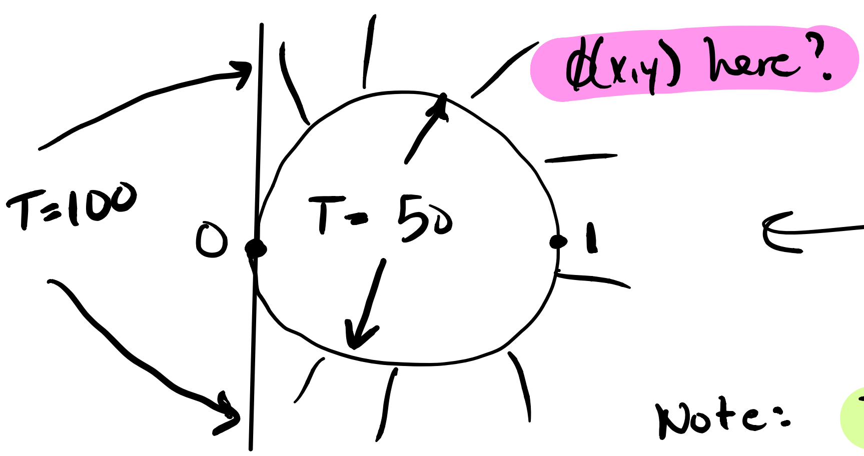
$$\frac{1}{f'(z)} = \frac{d}{dw}\left[f^{-1}(w)\right] \xleftarrow{\text{derivation}} \frac{d}{dw}\, f(f^{-1}(w)) = W$$

$$\hookrightarrow f'(f^{-1}(w))\,\frac{d}{dw}\left[f^{-1}(w)\right] = 1$$

we exclude where $f' = 0$

This is the real part of an analytic function therefore it satisfies Laplace's equation!

Example: Find steady state temperature in the shaded domain, with given BC's.




circle centered at $\frac{1}{2}$, radius $\frac{1}{2}$

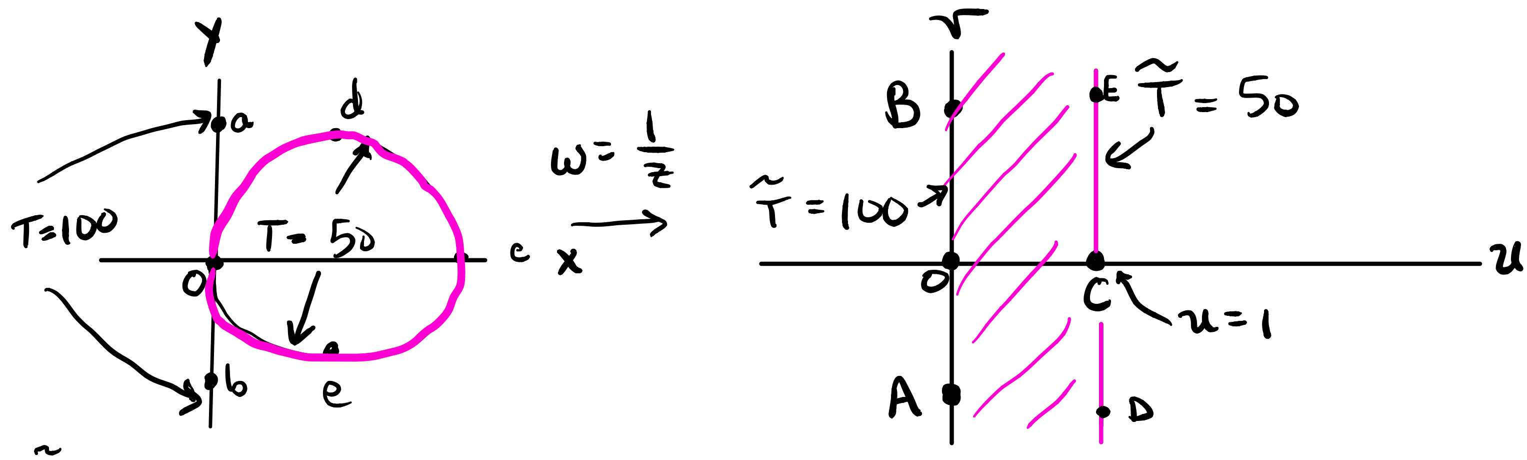
Note: T is bounded as $|z| \to \infty$

$$\nabla^2 T = 0 = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

BC1: $T = 100$ along $x = 0$

BC2: $T = 50$ on $(x - \frac{1}{2})^2 + y^2 = (\frac{1}{2})^2$

Trick? examine offset of $\boxed{w = 1/z}$ "inversion mapping"




$\tilde{T} = 100$ on $u = 0$

$\tilde{T} = 50$ on $u = 1$

$\tilde{T} = 100 - 50u$ satisfies $\nabla^2 \tilde{T} = 0$ and BC's

$$w = \frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$

Want to map back.

$$u(x,y) = \frac{x}{x^2+y^2}, \quad v(x,y) = \frac{-y}{x^2+y^2}$$

$$\boxed{T = 100 - \frac{50x}{x^2+y^2}}$$

# Library of Solutions in Simple Geometries

| $f$ | $\mathrm{Re}(f)$ | $\mathrm{Im}(f)$ | |
|---|---|---|---|
| $w$ | $u$ | $v$ | |
| $w^2$ | $u^2 - v^2$ | $2uv$ | |
| $w^n$ | $r^n \cos(n\theta$ | $r^n \sin(n\theta)$ | |
| $\ln(w)$ | $\log r$ | $\theta$ | $\rightarrow \log\left[\sqrt{u^2+v^2}\right] \quad \tan^{-1}\left(\frac{v}{u}\right)$ |