Last time we showed,

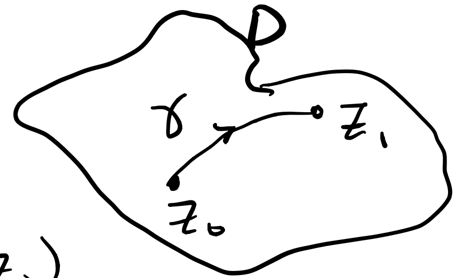
$$\int_{\gamma_1} \bar{z}\, dz \neq \int_{\gamma_2} \bar{z}\, dz$$

Theorem: (antiderivative)

- Assuming
  - f is continuous on an open, connected set D.
  - γ is any contour in this set D
  - $f(z)$ has an antiderivative $F(z)$.

where

$$F'(z) = f(z) \rightarrow F(z) \text{ is single valued in } D.$$



Then
$$\int_{\gamma} f(z)\, dz = F(z_1) - F(z_0)$$

depends on endpoints but Not curve

$\Rightarrow$ path independence if antiderivative exists

Proof: Parametrize γ by t.

let $\gamma = z(t)$, $0 \le t \le 1$, where $z(0) = z_0$ and $z(1) = z_1$

$$\int_{\gamma} f(z)\, dz = \int_{t=0}^{t=1} f(z(t)) \frac{dz}{dt}\, dt = \int_{t=0}^{1} F'(z(t)) \frac{dz}{dt}\, dt$$

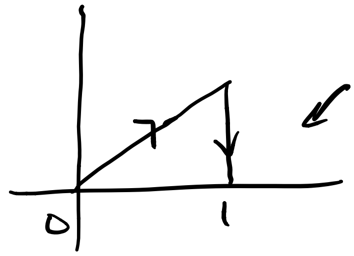
$$= \int_{t=0}^{1} \frac{d}{dt}\Big(F\big(z(t)\big)\Big)\,dt = F(z(1)) - F(z(0))$$

Fundamental THM of Calculus.

$$= F(z_1) - F(z_0)$$

Example:

$\int_\gamma z^2\,dz$ where γ is shown



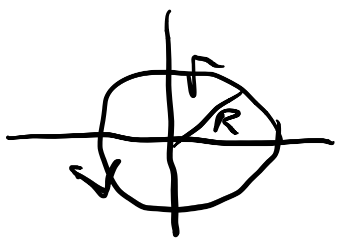
Path DOESNT Matter

Since z^2 has an antiderivative $\frac{z^3}{3}$,

$$\int_0^1 z^2\,dz = \frac{1}{3}$$

Example: Soln ①

$\int_\gamma z\,dz$ where γ = circle of radius R, centered at origin. $\circlearrowleft \times 1$



$x = R\cos(t)$

$y = R\sin(t)$

$z(t) = Re^{it}$

$dz = Re^{it}i\,dt$

$0 \le t \le 2\pi$

$$\int_\gamma z\,dz = \; = \int_0^{2\pi} R^2 i e^{i2t}\,dt = R^2 i\left[e^{i4\pi} - e^0\right]$$

$$= 0$$

Soln (2)

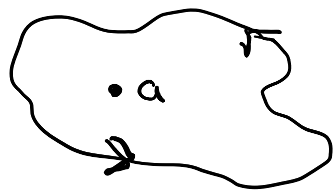
$$z_{2\pi} = z_0$$

$$\int_\gamma z\,dz = \int_{z_0}^{z_{2\pi}} z\,dz = \left.\frac{z^2}{2}\right|_{z_0}^{z_{2\pi}} = 0$$

"work around a closed loop is zero"

More General Result

Let γ = any closed curve that encircles the point $a \in \mathbb{C}$ once CCW.



THEN

$$\int_\gamma (z-a)^n\,dz = 0 \qquad \text{for } n \neq -1$$

Proof: Antiderivative of $(z-a)^n$ is $\frac{1}{n+1}(z-a)^{n+1}$

single valued and analytic in $\underline{\mathbb{C} - \{a\}}$

open connected set

punctured plane

"worry about a b/c we have negative powers"

What goes wrong when $n = -1$?

$\int_\gamma \frac{dz}{z-a}$ } antiderivative would be $\log(z-a)$ BUT that is **NOT** single valued in $\mathbb{C} - \{a\}$

Special Case: $\gamma =$ $\overbrace{\text{positively oriented}}^{\text{ccw}}$ circle of radius R about a.

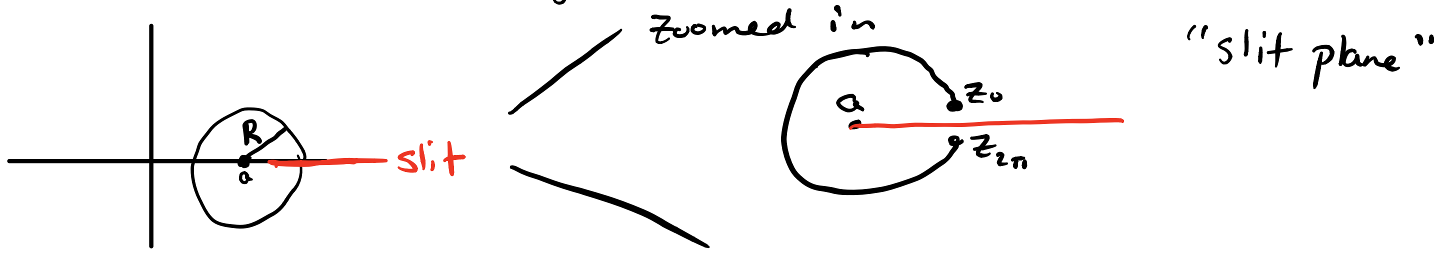
$\int_\gamma \frac{dz}{z-a}$, $z(t) = a + Re^{it} \to z - a = Re^{it}$

$dz = iRe^{it}\, dt$ $\quad 0 \le t \le 2\pi$

$= \int_0^{2\pi} i\, dt = 2\pi i \neq 0$!

TANGENT BEGINS

Say antiderivative is $\log(z-a)$ like real value case.



$\log(z-a)\Big|_{z_0}^{z_{2\pi}} = \ln r + i\theta \Big|_{z_0}^{z_{2\pi}} = 2\pi i$

$\log(z-a)$ is analytic and single valued in slit plane, $\mathbb{C}$ - branch cut.

Note: slit plane doesn't include whole circle.

TANGENT ENDS

Cauchy's Theorem - One of the big theorems

Recall: $\int_\gamma (z-a)^n \, dz = 0$, $\gamma \to$ closed curve, $n \neq -1$

So any polynomial $p(z) = c_0 + c_1 z + c_2 z^2 + \dots + c_n z^n$ also integrates to 0 around any closed curve.

i.e $$\oint_\gamma p(z)\, dz = 0$$

Cauchy's Theorem is a generalization of this:

$\oint_\gamma f(z)\, dz = 0$ if f is analytic in a "simply connected" domain D.
$\gamma \to$ any rectifiable closed curve in D.

"simply connected" - no holes.

Precisely: D is simply connected if every closed loop can be continuously deformed to a point while staying in D