Last Time

Linear Maps

We had a Theorem: $T: V \to W$ a linear map is uniquely determined by where it takes a basis of V.

Example: $T: \mathbb{R}^2 \to \mathbb{R}^2$

$$T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}4\\3\end{pmatrix} \qquad T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}3\\4\end{pmatrix} \Rightarrow T(x,y) = \begin{pmatrix}4x+3y\\3x+4y\end{pmatrix}$$

$$x T\begin{pmatrix}1\\0\end{pmatrix} + y T\begin{pmatrix}0\\1\end{pmatrix}$$

But what about $f\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x^2+3\\y^2+3\end{pmatrix} \to f\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}3\\4\end{pmatrix}$, $f\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}4\\3\end{pmatrix}$ ← NOT a Linear map

Last Time: $\mathcal{L}(V,W)$ ← all linear maps from V to W forms a vector space over $\mathbb{F}$

Form $\mathcal{L}(V,W)$ as it helps us understand linear transformations better.

What's 0 in $\mathcal{L}(V,W)$? $0: V \to W$, $S: V \to W$ $\Rightarrow S + 0 = 0 + S$

$$0(\vec{v}) = 0 \quad \forall \vec{v} \in V$$

Check that if $S, T \in \mathcal{L}(V, W)$, $\lambda \in \mathbb{F}$, such that

$$(S + \lambda T)(v_1 + \alpha \vec{v}_2) \overset{??}{=} (S + \lambda T)(\vec{v}_1) + \alpha (S + \lambda T)(\vec{v}_2)$$

Special Case

$$S, T \in \mathcal{L}(V, W)$$

$$\begin{tikzcd} V \arrow[r, "S"] \arrow[rr, bend right, "T \circ S"'] & V \arrow[r, "T"] & V \end{tikzcd}$$

$$S \circ T \overset{?}{=} T \circ S$$

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[rr, bend right, "S \circ T"'] & V \arrow[r, "S"] & V \end{tikzcd}$$

Example: $P : \mathbb{R}^3 \to \mathbb{R}^3$

a projection

$$P\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

Definition: A Linear Transformation $P : V \to V$ is called a projection if $P^2 = P$

Fact: P a projection $\Rightarrow$ $I - P$ a projection

$$\begin{aligned} (I - P)^2 &= I^2 - 2P + P^2 && \leftarrow P \text{ a projection } \Rightarrow P^2 = P \\ &= I^2 - 2P + P && \leftarrow I^2 = I \\ &= I - P \end{aligned}$$

What is I−P in our example?

"does the perpendicular thing"

projection to z-axis

$$(I-P)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix}$$

$P : \mathbb{R}^3 \to \mathbb{R}^3$ (same P as above)

P takes z-axis to $\vec{0} \in \mathbb{R}^3$

P takes xy-plane to itself

<u>Definition</u>: The null-space (kernel) of $T : V \to W$ is the set

$$\{\vec{v} \in V \mid T(\vec{v}) = \vec{0}\} \subseteq V$$

Squished to 0

NOT proved in class

The image (= range) of T is

$$\{\vec{w} \in W \mid \exists \vec{v} \in V \text{ with } T(\vec{v}) = \vec{w}\} \subseteq W$$

i.e Stuff in W actually hit by T

Stuff not squished to 0

Question: $T : \mathbb{R}^{50} \to W$

what happens to 50 dims of stuff we start with?

Answer: Squished to $\vec{0}$ OR NOT

Theorem: Let $T: V \to W$ be a linear transformation.

Then

$$\dim(V) = \dim \text{Null Space}(T) + \dim \text{Image}(T)$$

Proof: Let $\{\vec{u}_1, \ldots, \vec{u}_r\}$ be a basis of $NS(T)$.

Let $\{\vec{w}_1, \ldots, \vec{w}_k\}$ be a basis of $\text{Image}(T)$.

$\exists\ \vec{v}_1, \ldots, \vec{v}_k \in V$ such that $T(\vec{v}_i) = \vec{w}_i$ ← $\vec{v}_i$ NOT unique

'Hope' $B = \{\vec{v}_1, \ldots, \vec{v}_k, \vec{u}_1, \ldots, \vec{u}_r\}$ is a basis for V

B is independent would mean

$$\sum_{i=1}^{k} \alpha_i \vec{v}_i + \sum_{j=1}^{r} \beta_j \vec{u}_j = \vec{0}$$

Apply T

$$\sum_{i=1}^{k} \alpha_i \vec{w}_i + \vec{0} = \vec{0}$$

Basis $\Rightarrow$ all $\alpha_i = 0$

All $\alpha_i = 0$ and basis $\Rightarrow \beta_j = 0$

Spanning

Let $\vec{v} \in V$. We need to right $\vec{v}$ as a linear combination of B.

$$T(\vec{v}) = \sum_{i=1}^{k} \gamma_i \vec{w}_i = \sum_{i=1}^{k} \gamma_i T(\vec{v}_i) = T\left(\sum_{i=1}^{k} \gamma_i \vec{v}_i\right)$$

$$\Rightarrow T\Big(\underbrace{\vec{v} - \sum_{i=1}^{k} \gamma_i \vec{v}_i}_{\text{in } NS(T)}\Big) = \vec{0}$$

$$\Rightarrow \vec{v} - \sum_{i=1}^{k} \gamma_i \vec{v}_i = \sum_{j=1}^{r} \beta_j u_j$$

$$\Rightarrow \vec{v} = \sum_{i=1}^{k} \gamma_i \vec{v}_i + \sum_{j=1}^{r} \beta_j u_j$$

Q.E.D