Probability Space

$(\Omega, \mathcal{F}, Pr)$ — probability measure on $\mathcal{F}$

- Ω: Sample Space
- $\mathcal{F}$: collection of events I want to assign probabilities to
- level of resolution we observe our random experiment at

Axioms

$\mathcal{F}$

(i) $\Omega \in \mathcal{F}$

(ii) If $A \in \mathcal{F}$, $A^c \in \mathcal{F}$

(iii) If $A_i \in \mathcal{F}$, $\bigcup_i A_i \in \mathcal{F}$ ← countable intersections also there! $A \cap B = (A^c \cup B^c)^c$!

and we argue by countable union to belu to inf

Pr $\quad \mathcal{F} \to [0,1]$

(i) $0 \le Pr(A) \le 1$

(ii) $Pr(\Omega) = 1$ — if one event occurs the other events do not

(iii) If $A_1, A_2, \ldots$, is a sequence of mutually exclusive events then $Pr(\bigcup_i A_i) = \sum_i Pr(A_i)$

Properties that follow

(i) $Pr(A^c) = 1 - Pr(A)$

Proof: $Pr(A \cup A^c) = Pr(A) + Pr(A^c)$

$Pr(\Omega) = 1 = Pr(A) + Pr(A^c)$

$Pr(A^c) = 1 - Pr(A)$

(ii) $Pr(\emptyset) = 0$, $\emptyset = \Omega^c$

(iii) If $A \subseteq B$ then $Pr(A) \le Pr(B)$

Proof: $B = A \cup (A^c \cap B)$

$Pr(B) = Pr(A) + Pr(A^c \cap B) \ge Pr(A)$

(iv) $Pr(A \cup B) = Pr(A) - Pr(B) - Pr(A \cap B)$

Union Bound

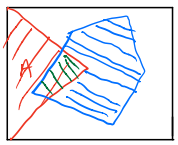
$Pr(\bigcup_{i=1}^n A_i) \le \sum_{i=1}^n Pr(A_i)$

Conditional Probability

If A and B are two events, $Pr(B) \neq 0$, then

$$Pr(A|B) = \frac{Pr(A \cap B)}{Pr(B)}$$

"B becomes the new universe, $B = \Omega'$"



Total Probability Theorem

If $\{E_1, E_2, \ldots, E_k\}$ partition Ω then

$$Pr(A) = \sum_{i=1}^k Pr(A \cap E_i) = \sum_{i=1}^k Pr(A|E_i) Pr(E_i)$$

This makes more sense since you know how event which occurred — generative models

Example of Partition for k=4

No intersections $\bigcup_{i=1}^k E_i = \Omega$

Bayes Rule

Want to find "ground truth" that gave this observation.

$$Pr[E_j|A] = \frac{Pr[A \cap E_j]}{Pr[A]} = \frac{Pr[E_j] Pr(A|E_j)}{\sum_{i=1}^k Pr[E_i] Pr[A|E_i]}$$

prior knowledge, generative model; denominator $= Pr(A)$ by TPT

Independence

Two events, A_1, A_2, are independent if $Pr(A_1 \cap A_2) = Pr(A_1) Pr(A_2)$

$$\Rightarrow Pr[A_1|A_2] = \frac{Pr[A_1 \cap A_2]}{Pr[A_2]} = \frac{Pr(A_1) Pr(A_2)}{Pr(A_2)} = Pr(A_1)$$

Can be extended:

Events $\{A_1, A_2, \ldots, A_n\}$ are independent if

$$Pr(A_1 \cap A_2 \cap \cdots \cap A_k) = Pr(A_1) Pr(A_2) \cdots Pr(A_k) \quad \forall k \in [2,n], k \in \mathbb{Z}$$

$$F_{X_1, \ldots, X_n}(x_1, \ldots, x_n) = F_{X_1}(x_1) \cdots F_{X_n}(x_n)$$

$$Pr[X_1 \le x_1, \ldots, X_n \le x_n] = Pr[X_1 \le x_1] \cdots Pr[X_n \le x_n]$$

Random Variables

Given $(\Omega, \mathcal{F}, Pr)$ a random variable is a function $X: \Omega \to \mathbb{R}$ such that $\forall x \in \mathbb{R}$, $\{\omega \mid X(\omega) \le x\} \in \mathcal{F} \subseteq \Omega$

M^{th} Moment: $\mathbb{E}[X^n] = \int_{-\infty}^{+\infty} x^n f_X(x)\, dx$

M^{th} Central Moment: $\mathbb{E}[(X - \mathbb{E}[X])^n]$

Cumulative Distribution Function

The CDF is defined as

$$F_X(x) = Pr(X \le x) \quad \forall x \in \mathbb{R}$$
$$= Pr(\{\omega \mid X(\omega) \le x\})$$

Properties of CDF

① $\lim_{x \to +\infty} F_X(x) = 1$, $\lim_{x \to -\infty} F_X(x) = 0$

② $\forall x < y \quad F_X(x) \le F_X(y)$

③ F is right continuous: $\lim_{x \to x_0^+} F_X(x) = F_X(x_0)$

④ $Pr[x < X \le y] = F_X(y) - F_X(x)$

PMF for Discrete Random Variables

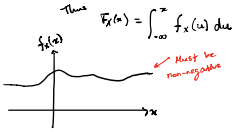
$P_X(x) = Pr[X = x]$

$F_X(x) = \sum_{u \le x} P_X(u)$

PDF for Continuous Random Variables

$f_X(x) = \frac{dF_X(x)}{dx}$ } measures how fast we accumulate probability

Thus $F_X(x) = \int_{-\infty}^{x} f_X(u)\, du$



Properties of PDF

① $f_X(x) \ge 0 \quad \forall x$

② $\int_{-\infty}^{+\infty} f_X(x) = 1 = F_X(\infty)$

③ $Pr[x \le X \le y] = \int_x^y f_X(u)\, du$

Variance

How much a r.v varies from its expectation

$$Var(X) = \mathbb{E}[\underbrace{(X - \mathbb{E}[X])^2}_{g(X)}]$$
$$= \mathbb{E}[X^2 - 2\mathbb{E}[X]X + (\mathbb{E}[X])^2]$$
$$= \mathbb{E}[X^2] + (\mathbb{E}[X])^2$$

$$Var(X) = \int_{-\infty}^{+\infty} g(x) f_X(x)\, dx = \int_{-\infty}^{+\infty} (x - \mathbb{E}[X])^2 f_X(x)\, dx$$

Correlation

Correlation between X and Y

$$\mathbb{E}[XY] = \int_{y=-\infty}^{+\infty} \int_{x=-\infty}^{+\infty} xy\, f_{X,Y}(x,y)\, dx\, dy$$

$$\mathbb{E}[XY] = \sum_{x_i} \sum_{y_j} x_i y_j P_{X,Y}(x_i, y_j)$$

Covariance

$$Cov(X,Y) \triangleq \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])]$$
$$= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$$

If $\mathbb{E}[XY] = 0$, we say X is orthogonal to Y.

If $Cov(X,Y) = 0$, we say X is uncorrelated to Y.

Properties

Independence of r.v's X, Y: $f_{X,Y}(x,y) = f_X(x) f_Y(y) \Rightarrow$ uncorrelatedness

X, Y, Z r.v.'s

$$Cov(X, aY + bZ) = a\,Cov(X,Y) + b\,Cov(X,Z)$$

$$Cov(X - \mathbb{E}[X], Y - \mathbb{E}[Y]) = Cov(X,Y)$$

Expectation

$$\mathbb{E}[X] = \begin{cases} \sum_k k\, Pr[X = k], & X \text{ discrete} \\ \int_{-\infty}^{+\infty} x\, f_X(x)\, dx, & X \text{ continuous} \end{cases}$$

Properties

LOTUS Rule

If $Y = g(X)$ ($f_X(x)$, g)

then $$\mathbb{E}[Y] = \int_{-\infty}^{+\infty} x\, f_Y(y)\, dy$$

$$\mathbb{E}[g(X)] = \int_{-\infty}^{+\infty} g(x) f_X(x)\, dx$$

Linearity of Expectation

$$\mathbb{E}[\alpha X + \beta Y] = \alpha \mathbb{E}[X] + \beta \mathbb{E}[Y], \quad \alpha, \beta \in \mathbb{R}$$

Preservation of Order

If $Pr[X \ge Y] = 1$

then $\mathbb{E}[X] \ge \mathbb{E}[Y]$

Integration by Parts

A discrete, non-negative r.v. $X = 0, 1, 2, 3, \ldots$

$$\mathbb{E}[X] = \sum_{i=0}^{\infty} i\, Pr(X = i)$$
$$= \sum_{i=0}^{\infty} \underbrace{Pr(X > i)}_{1 - F_X(i)}$$ ← Tail Probabilit.

To see this, observe

$\sum_{i=0}^{\infty} i\, Pr(X = i) = 0 \cdot Pr(X=0) + 1 \cdot Pr(X=1) + 2 \cdot Pr(X=2) + \cdots$

$\sum_{i=0}^{\infty} Pr(X > i)$ $\quad i=0$ ✓ $\quad i=1$ ✓ $\quad \vdots$ etc

Thus

$$\mathbb{E}[X] = \int_0^{\infty} (1 - F_X(x))\, dx - \int_{-\infty}^{0} F_X(x)\, dx$$

Conditioning on Random Variables

Suppose X and Y have joint pmf $P_{X,Y}(x,y)$

The conditional pmf of X given $\{Y = y\}$ is

$$P_{X|Y}(x|y) \triangleq Pr[\overset{A}{X = x} \mid \overset{B}{Y = y}]$$
$$= \frac{Pr(\{X = x\} \cap \{Y = y\})}{Pr(\{Y = y\})} = \frac{P_{X,Y}(x,y)}{P_Y(y)}$$

Similarly for continuous r.v.'s

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} \quad \text{if } f_Y(y) \neq 0 \ \forall y$$

Conditional Expectation

$$\mathbb{E}[X|Y=y] = \int_{-\infty}^{+\infty} x\, f_{X|Y}(x|y)\, dx$$

$\mathbb{E}[X|Y]$ is a random variable which takes on value $\mathbb{E}[X|Y=y]$ w/ density $f_Y(y)$.

Since $\mathbb{E}_{X|Y}[X|Y]$ is r.v. can take its expectation.

$$\mathbb{E}_Y[\mathbb{E}_{X|Y}[X|Y]] = \mathbb{E}[X]$$

Cauchy-Schwartz Inequality

$$|\mathbb{E}[XY]| \le \sqrt{|\mathbb{E}[X^2]\mathbb{E}[Y^2]|}$$

Correlation Coefficient

normalization comes from Cauchy-Schwartz inequality

$$\rho_{X,Y} = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}} \quad (|\rho_{X,Y}| \le 1)$$

If $X_1, \dots, X_m$ are pairwise uncorrelated, then

$$\mathrm{Var}(X_1 + X_2 + \cdots + X_m) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2) + \cdots + \mathrm{Var}(X_m)$$

$$\mathrm{Var}\left(\sum_{i=1}^m X_i\right) = \sum_{i=1}^m \mathrm{Var}(X_i)$$

Linear Minimum Mean Squared Error

Now our estimator $g(Y)$ takes the form $g(Y) = aY + b$

The MSE becomes

$$\mathrm{MSE} = \mathbb{E}[(X - g(Y))^2] = \mathbb{E}[(X - aY - b)^2]$$

Want

$$\{a^*, b^*\} = \underset{a,b}{\mathrm{argmin}}\ \mathbb{E}[(X - aY - b)^2]$$

Use orthogonality principle

$$W \perp g(Y) \Rightarrow X - a^*Y - b^* \perp aY + b \quad \forall\, a, b$$

Consider the following cases

① $a = 0, b = 1$ ($X, Y \Leftrightarrow \mathbb{E}[X], \mathbb{E}[Y]$)

$$X - a^*Y - b^* \perp 1 \Rightarrow \mathbb{E}[X - a^*Y - b^*] = 0$$

$$\mathbb{E}[X] - \mathbb{E}[a^*Y] - \mathbb{E}[b^*] = 0 \quad (W \to \mathbb{E}[W] = 0)$$

$$\to b^* = \mathbb{E}[X] - a^*\mathbb{E}[Y]$$

b^* UNBIASED Estimator

② $a = 1, b = 0$

$$X - a^*Y - b^* \perp Y \Rightarrow \mathbb{E}[(X - a^*Y - b^*)Y] = 0$$

$$\mathbb{E}[(X - a^*Y - b^*)Y]$$

$$= \mathbb{E}[(X - a^*Y - \mathbb{E}[X] - a^*\mathbb{E}[Y])Y]$$

First set

$$= \mathbb{E}[(X - \mathbb{E}[X] - a^*(Y - \mathbb{E}[Y]))Y]$$

$$\Rightarrow \mathrm{Cov}\left((X - \mathbb{E}[X] - a^*(Y - \mathbb{E}[Y]),\ Y\right) = 0$$

Note: $\mathrm{Cov}(W, Y) = \mathbb{E}[WY]$ here

$$\mathrm{Cov}(X,Y) - a^*\mathrm{Cov}(X,Y) = 0$$

$$a^* = \mathrm{Cov}(X,Y)/\mathrm{Var}(Y) \leftarrow \mathrm{Cov}(X,Y)$$

Aside

$$\mathrm{Cov}(X, aY + bZ) = a\,\mathrm{Cov}(X,Y) + b\,\mathrm{Cov}(X,Z)$$

$$\mathrm{Cov}(X - \mathbb{E}[X],\ Y - \mathbb{E}[Y]) = \mathrm{Cov}(X,Y)$$

So

$$\hat{X}^*_{\mathrm{LMMSE}} = a^*Y + b^*$$

$$= \frac{\mathrm{Cov}(X,Y)}{\mathrm{Var}(Y)}Y + \mathbb{E}[X] - \frac{\mathrm{Cov}(X,Y)}{\mathrm{Var}(Y)}\mathbb{E}[Y]$$

$$= \frac{\mathrm{Cov}(X,Y)}{\mathrm{Var}(Y)}(Y - \mathbb{E}[Y]) + \mathbb{E}[X]$$

and the MMSE is

$$\mathrm{MSE} = \mathbb{E}[W^2] = \mathrm{Cov}(W,W)$$

$$= \mathrm{Cov}\left((X - \mathbb{E}[X]) - a^*(Y - \mathbb{E}[Y]),\ (X - \mathbb{E}[X]) - a^*(Y - \mathbb{E}[Y])\right)$$

$$= \mathrm{Cov}(X,X) + a^{*2}\mathrm{Cov}(Y,Y) - 2a^*\mathrm{Cov}(X,Y)$$

$$= \mathrm{Var}(X) + \left(\frac{\mathrm{Cov}(X,Y)}{\mathrm{Var}(Y)}\right)^2\mathrm{Var}(Y) - 2\left(\frac{\mathrm{Cov}(X,Y)}{\mathrm{Var}(Y)}\right)\mathrm{Cov}(X,Y)$$

$$= \mathrm{Var}(X) - \frac{(\mathrm{Cov}(X,Y))^2}{\mathrm{Var}(Y)}$$

$$W = X - \hat{X}_{\mathrm{LMMSE}}$$

$$= X - \left(\mathbb{E}[X] + \frac{\mathrm{Cov}(X,Y)}{\mathrm{Var}(Y)}(Y - \mathbb{E}[Y])\right)$$

$$= X - \frac{\mathrm{Cov}(X,Y)}{\mathrm{Var}(Y)}Y - \left(\mathbb{E}[X] + \frac{\mathrm{Cov}(X,Y)}{\mathrm{Var}(Y)}\mathbb{E}[Y]\right)$$

Know W & Y are Jointly Gaussian

$$\mathbb{E}[W] = 0$$

$$\mathbb{E}[WY] = 0 \quad \text{(orthogonality)}$$

So $\mathrm{Cov}(W,Y) = 0 \Rightarrow$ independence by property 2

Need W is error given by best MMSE

$$\mathbb{E}[Wg(Y)] = 0 \quad \forall\, g(Y) \Rightarrow$$

Trivially true since W independent of Y.

$$\mathbb{E}[Wg(Y)] = \mathbb{E}[W]\mathbb{E}[g(Y)] = 0$$

Proof of ④

Hilbert Space

A Hilbert space is a vector space that

① Has an inner product $\langle \cdot, \cdot \rangle$ defined

② Is complete w/ respect to the norm

$\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x} \rangle}$ induced by $\langle \cdot, \cdot \rangle$

→ complete in the sense every Cauchy sequence converges

Hilbert Space of Random Variables

All random variables w/ finite 2nd moments form a Hilbert space w/ inner product (r.v.'s not vectors)

$$\langle X, Y \rangle \triangleq \mathbb{E}[XY]$$

Inner Product ⇒ norm/length ⇒ distance metric

length/norm of X $= \sqrt{\langle X, X \rangle} = \sqrt{\mathbb{E}[X^2]}$

Angle b/t X, Y; θ_{XY}: $\cos(\theta_{XY}) = \frac{\langle X,Y \rangle}{\|X\|\|Y\|} = \frac{\mathbb{E}[XY]}{\sqrt{\mathbb{E}[X^2]\mathbb{E}[Y^2]}}$ (Cauchy-Schwarz ⇒ this ≤ 1)

Projection of X onto Y (a random variable)

$$\Pi_Y(X) = \frac{\langle X, Y \rangle}{\|Y\|^2} Y = \frac{\mathbb{E}[XY]}{\mathbb{E}[Y^2]} Y$$

Orthogonality

$$\langle X, Y \rangle \triangleq \mathbb{E}[XY] = 0$$

$$\Pi_Y(X) = 0$$

Pythagorean (all X_i pairwise orthogonal)

$$\mathbb{E}\left[\left(\sum_{i=1}^m X_i\right)^2\right] = \sum_{i=1}^m \mathbb{E}[X_i^2]$$

Jointly Gaussian Random Variables

If $X \sim N(\mu, \sigma^2)$ then $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{\sigma^2}}$

Note: K is our covariance matrix. Here, $K = \begin{pmatrix} \mathrm{Cov}(X,X) & \mathrm{Cov}(X,Y) \\ \mathrm{Cov}(Y,X) & \mathrm{Cov}(Y,Y) \end{pmatrix}$

$$= \mathbb{E}\left[\begin{pmatrix} X - \mu_X \\ Y - \mu_Y \end{pmatrix}\begin{pmatrix} X - \mu_X \\ Y - \mu_Y \end{pmatrix}^T\right] = \begin{pmatrix} \sigma_X^2 & \mathrm{Cov}(X,Y) \\ \mathrm{Cov}(Y,X) & \sigma_Y^2 \end{pmatrix}$$

Definition: X, Y are jointly Gaussian if $aX + bY$ is Gaussian $\forall\, a, b \in \mathbb{R}$.

Definition: Joint pdf of such a distribution is

$$f_{XY}(x,y) = \frac{1}{2\pi\sqrt{|K|}} \exp\left(-\frac{1}{2}\begin{pmatrix} x - \mu_X \\ y - \mu_Y \end{pmatrix}^T K^{-1}\begin{pmatrix} x - \mu_X \\ y - \mu_Y \end{pmatrix}\right)$$

Properties of Jointly Gaussian Random Variables

① J.G. ⇒ M.G. ⇍

② Uncorrelated JG rv's are independent ← Special!

independence ⇒ uncorrelated ⇍ in general

③ If X, Y jointly Gaussian, then $a_1X + b_1Y + c_1$, $a_2X + b_2Y + c_2$ are jointly Gaussian

④ For JG X, Y

$$\hat{X}_{\mathrm{MMSE}} = \mathbb{E}[X|Y] = \hat{X}_{\mathrm{LMMSE}} = \mathbb{E}[X] + \frac{\mathrm{Cov}(X,Y)}{\mathrm{Var}(Y)}(Y - \mathbb{E}[Y])$$

$$\mathrm{MMSE} = \mathrm{Var}(X)(1 - \rho_{XY}^2) = \mathrm{Var}(X) - \frac{\mathrm{Cov}^2(X,Y)}{\mathrm{Var}(Y)}$$

⑤ For J.G X, Y

$$\mathbb{E}[X|Y=y] = \mathbb{E}[X] + \frac{\mathrm{Cov}(X,y)}{\mathrm{Var}(y)}(y - \mathbb{E}[y])$$

$$\mathbb{E}[X|Y] = \mathbb{E}[X] + \frac{\mathrm{Cov}(X,Y)}{\mathrm{Var}(Y)}(Y - \mathbb{E}[Y])$$

$$\mathrm{Var}(X|Y=y) = \mathrm{Var}(X) - \frac{\mathrm{Cov}^2(X,y)}{\mathrm{Var}(y)}$$

$$\mathrm{Var}(X|Y) = \mathrm{Var}(X) - \frac{\mathrm{Cov}^2(X,Y)}{\mathrm{Var}(Y)}$$

⑥ The conditional distribution of X given $Y = y$ is Gaussian

$$f_{X|Y}(x|y) = \frac{1}{\sqrt{2\pi\mathrm{Var}(X|Y)}} e^{-(\cdots)}$$

Proof! $X = \hat{X}_{\mathrm{MMSE}} + W$

where W is independent of Y Given $Y = y$ (an observation),

- $\hat{X}_{\mathrm{MMSE}}$ is a constant
- W maintains its distribution

Thus $X \sim N(\mu_W + \hat{X}_{\mathrm{MMSE}}, \sigma_W^2)$

Mean Squared Error (MSE)

Have r.v. X which can't be directly observed. Want to estimate $\hat{X}$.

Note: we are in a Hilbert space of random variables

The estimation error $W = X - \hat{X}$

The MSE of $\hat{X}$ is

$$\mathbb{E}[(X - \hat{X})^2] = \mathbb{E}[W^2] = \langle W, W \rangle = \|W\|^2 = (d(X,\hat{X}))^2$$

Minimum Mean Squared Error

$$\hat{X}_{\mathrm{MMSE}} = \underset{\hat{X}}{\mathrm{argmin}}\ \mathbb{E}[(X - \hat{X})^2]$$

MMSE of X using a constant

$$\hat{X}_{\mathrm{MMSE}} = a^*$$

where

$$a^* = \underset{a \in \mathbb{R}}{\mathrm{argmin}}\ \mathbb{E}[(X - a)^2]$$ } it's $\mathbb{E}[X]$!

$$\mathrm{MSE} = \mathbb{E}[(X - a)^2]$$

$$= \mathbb{E}[(X - \mathbb{E}[X] + \mathbb{E}[X] - a)^2]$$

$$= \mathbb{E}[(X - \mathbb{E}[X])^2] + 2\mathbb{E}[(X - \mathbb{E}[X])(\mathbb{E}[X] - a)] + \mathbb{E}[(\mathbb{E}[X] - a)^2]$$ (the middle term → 0)

Want to minimize

$$\mathbb{E}[(X - \mathbb{E}[X])^2] + \mathbb{E}[(\mathbb{E}[X] - a)^2]$$

$$\to \mathrm{Var}(X) + (\mathbb{E}[X] - a)^2$$

Take $a = \mathbb{E}[X]$

MMSE of X given $Y = y$ (similar proof)

$$\hat{X}_{\mathrm{MMSE}} = \mathbb{E}[X|Y]$$

$$\mathrm{MSE}(\hat{X}_{\mathrm{MMSE}}) = \mathbb{E}[\mathrm{Var}(X|Y)]$$

The error

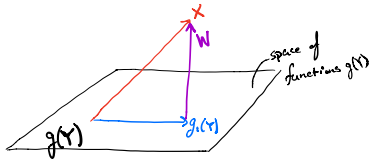
$$W = X - \mathbb{E}[X|Y]$$

has

$$\mathbb{E}[W] = \mathbb{E}[X - \mathbb{E}[X|Y]] = \mathbb{E}[X] - \mathbb{E}[X] = 0$$

we call this type of estimator UNBIASED

VISUALIZATION



$X, Y, \hat{X}$

$W = X - \hat{X}^*$

want $\min_{\hat{X}} \mathbb{E}[W^2]$

- Orthogonality principle:

$$W \perp g(Y)\ \forall\, g$$ } necessary AND sufficient condition

W induced by $\hat{X}^*$ (best estimator)

claim $X - \mathbb{E}[X|Y] \perp g(Y)\ \forall\, g$

Proof $\mathbb{E}[(X - \mathbb{E}[X|Y])\,g(Y)] = 0$ (inner product $\langle X - \mathbb{E}[X|Y], g(Y) \rangle$)

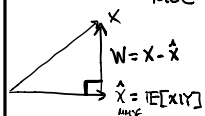
LHS

$$\mathbb{E}_{XY}[Xg(Y)] - \mathbb{E}_Y[(\mathbb{E}_{X|Y}[X|Y])g(Y)]$$

$$\mathbb{E}_Y[\mathbb{E}_{X|Y}[Xg(Y)|Y]] - \mathbb{E}_Y[\mathbb{E}_{X|Y}[X|Y]\,g(Y)]$$

$$\mathbb{E}_Y[g(Y)\mathbb{E}_{X|Y}[X|Y]] - \mathbb{E}_Y[g(Y)\mathbb{E}_{X|Y}[X|Y]] = 0 \checkmark$$

MSE Using Pythagorean Theory



$\|W\|^2 = \langle W, W \rangle$ in Hilbert Space

$$\mathrm{MSE} = \mathbb{E}[W^2] = \mathbb{E}[X^2] - \mathbb{E}[\hat{X}^2]$$

$$= \mathrm{Var}(X) + (\mathbb{E}[X])^2 - (\mathrm{Var}(\hat{X}_{\mathrm{MMSE}}) + (\mathbb{E}[\hat{X}_{\mathrm{MMSE}}])^2)$$

$$\mathbb{E}[X] = \mathbb{E}[\hat{X}] = \mathrm{Var}(X) - \mathrm{Var}(\hat{X}_{\mathrm{MMSE}})$$