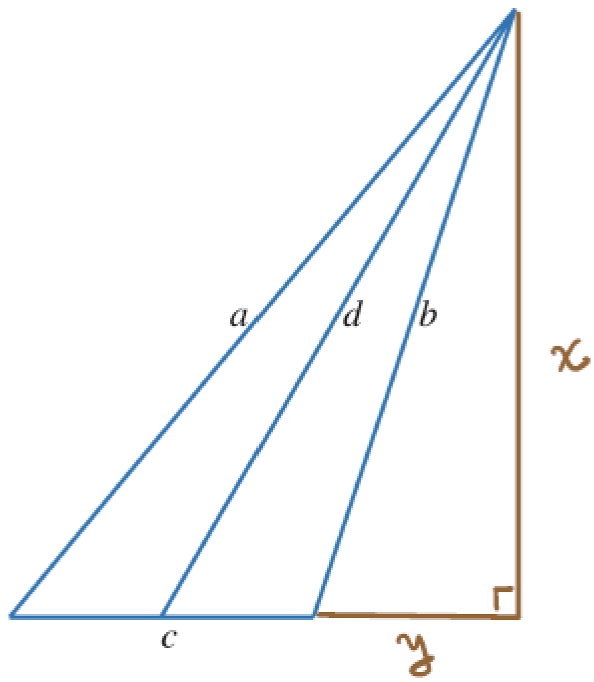
Exercises.



Solution to Question 1.
Let x and y be the length of the two sides as shown in the picture. Then

$$\begin{aligned} b^2 &= x^2 + y^2, \\ a^2 &= x^2 + (c+y)^2, \\ d^2 &= x^2 + (\frac{c}{2} + y)^2. \end{aligned}$$

So

$$\begin{aligned} &a^2 + b^2 - 2d^2 \\ =&y^2 + (c+y)^2 - 2(\frac{c}{2} + y)^2 \\ =&\frac{c^2}{2}. \end{aligned}$$

Solution to Question 2. Take a basis $\mathcal{A} = (1, x, x^2)$ of $\mathbb{R}[x]_{\leq 2}$.

$$\langle 1, 1 \rangle = \int_0^1 1 dx = 1.$$

So $\beta_1 = 1$. Let $b_2 = x - \langle 1, x \rangle \cdot 1 = x - \frac{1}{2}$. Check

$$\langle b_2, b_2 \rangle = \int_0^1 (x - \frac{1}{2})^2 dx = \frac{1}{12}.$$

So $\beta_2 = \sqrt{12} b_2 = \sqrt{12}(x - \frac{1}{2})$.
Let

$$\begin{aligned} b_3 &= x^2 - \langle x^2, \beta_1 \rangle \cdot \beta_1 - \langle x^2, \beta_2 \rangle \cdot \beta_2 \\ &= x^2 - \frac{1}{3} - \frac{\sqrt{12}}{12} \cdot \sqrt{12}(x - \frac{1}{2}) \\ &= x^2 - x + \frac{1}{6}. \end{aligned}$$

Check

$$\begin{aligned} \langle b_3, b_3 \rangle &= \int_0^1 (x^2 - x + \frac{1}{6})^2 dx \\ &= \frac{1}{180}. \end{aligned}$$

So $\beta_3 = \sqrt{180} b_3$. $\mathcal{B} = (\beta_1, \beta_2, \beta_3)$ is an orthogonal basis of $\mathbb{R}[x]_{\leq 2}$.

Solution to Question 3. Let

$$\mathbf{w}' = \frac{\langle \mathbf{w}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}\mathbf{v}$$

be the projection of $\mathbf{w}$ onto the span of $\mathbf{v}$ and let $\mathbf{u} = \mathbf{v} - \mathbf{w}'$. Then

$$\langle \mathbf{w}, \mathbf{w} \rangle = \langle \mathbf{w}', \mathbf{w}' \rangle + \langle \mathbf{u}, \mathbf{u} \rangle.$$

Because $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$, so

$$\begin{aligned} \langle \mathbf{w}, \mathbf{w} \rangle &\geq \langle \mathbf{w}', \mathbf{w}' \rangle \\ &= \frac{\langle \mathbf{w}, \mathbf{v} \rangle^2}{\langle \mathbf{v}, \mathbf{v} \rangle}. \end{aligned}$$

Hence, $\langle \mathbf{w}, \mathbf{w} \rangle \langle \mathbf{v}, \mathbf{v} \rangle \geq \langle \mathbf{w}, \mathbf{v} \rangle^2$.

(a) Let $\mathbf{w} = (\sqrt{a}, \sqrt{b}, \sqrt{c}, \sqrt{d})$ and $\mathbf{v} = (\frac{1}{\sqrt{a}}, \frac{1}{\sqrt{b}}, \frac{1}{\sqrt{c}}, \frac{1}{\sqrt{d}})$. Then

$$\begin{aligned} \langle \mathbf{w}, \mathbf{v} \rangle &= 4, \\ \langle \mathbf{w}, \mathbf{w} \rangle &= a + b + c + d, \\ \langle \mathbf{v}, \mathbf{v} \rangle &= \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}. \end{aligned}$$

By Cauchy-Schwarz formula,

$$16 \leq (a + b + c + d)(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}).$$

(b) Let $\mathbf{w} = (1, \dots, 1)$ and $\mathbf{v} = (a_1, \dots, a_n)$. Then

$$\begin{aligned} \langle \mathbf{w}, \mathbf{v} \rangle &= a_1 + \cdots + a_n, \\ \langle \mathbf{w}, \mathbf{w} \rangle &= n, \\ \langle \mathbf{v}, \mathbf{v} \rangle &= a_1^2 + \cdots + a_n^2. \end{aligned}$$

By Cauchy-Schwarz formula,

$$(a_1 + \cdots + a_n)^2 \leq n(a_1^2 + \cdots + a_n^2).$$

(c) Check that

$$\langle f, g \rangle = \int_0^1 f(x)g(x)dx$$

is an inner product: Symmetric and linearity are clear. We only need to show it is positive definite, i.e. for all non-zero continuous function $f(x)$,

$$\int_0^1 f^2(x)dx > 0.$$

This is because if $f(x_0)^2 > 0$ at a point x_0, then by continuity of f, there exist an open segment I of $[0,1]$ containing x_0 such that $f(x)^2 > 0$ for $x \in I$. Therefore,

$$\int_0^1 f^2(x)dx \geq \int_I f^2(x)dx > 0.$$

So $\langle \cdot, \cdot \rangle$ is an inner product. By Cauchy-Schwarz formula,

$$|\langle f, g \rangle|^2 \leq \langle f, f \rangle \langle g, g \rangle.$$

Solution to Question 4.

(a) We need to check

- $\langle \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{w}, \mathbf{v} \rangle$. This is because

$$\mathbf{v}^\mathsf{T} A \mathbf{w} = (\mathbf{v}^\mathsf{T} A \mathbf{w})^\mathsf{T} = \mathbf{w}^\mathsf{T} A^\mathsf{T} \mathbf{v} = \mathbf{w}^\mathsf{T} A^\mathsf{T} \mathbf{v}.$$

- Linearity in the first argument.

$$\langle c\mathbf{v}, \mathbf{w} \rangle = c\mathbf{v}^\mathsf{T} A \mathbf{w} = c\langle \mathbf{v}, \mathbf{w} \rangle$$
$$\langle \mathbf{v}_1 + \mathbf{v}_2, \mathbf{w} \rangle = (\mathbf{v}_1 + \mathbf{v}_2)^\mathsf{T} A \mathbf{w} = (\mathbf{v}_1^\mathsf{T} + \mathbf{v}_2^\mathsf{T}) A \mathbf{w} = \mathbf{v}_1^\mathsf{T} A \mathbf{w} + \mathbf{v}_2^\mathsf{T} A \mathbf{w} = \langle \mathbf{v}_1, \mathbf{w} \rangle + \langle \mathbf{v}_2, \mathbf{w} \rangle.$$

Now A is positive definite if and only if $\langle \cdot, \cdot \rangle$ is positive definite.

(b) Let $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ be a basis of $\mathbb{R}^n$. Assume A is an $n \times n$ matrix. Denote the (i, j) entry of A by a_{ij}. Let

$$a_{ij} = \langle \mathbf{e}_i, \mathbf{e}_j \rangle.$$

Then

$$\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v}^\mathsf{T} A \mathbf{w}.$$

(c) We may check that for $\mathbf{v} = (1, -1)$,

$$\mathbf{v}^\mathsf{T} A \mathbf{v} = 0.$$

So A is not positive definite. Therefore, by part (a), $\langle \cdot, \cdot \rangle$ is not an inner product.

Solution to Question 5.

(a) Attention: This would only be true if we assume V is finite dimensional. Assume that $\dim W = m$ and $\dim V = m + n$. Let $\mathcal{B} = (\mathbf{w}_1, \ldots, \mathbf{w}_m)$ be a basis of W. We may extend $\mathcal{B}$ to a basis of V

$$\mathcal{A} = (\mathbf{w}_1, \ldots, \mathbf{w}_m, \mathbf{u}_1, \ldots, \mathbf{u}_n).$$

By applying the Gram-Schmidt process to $\mathcal{A}$, we get another basis of V,

$$\mathcal{A}' = (\mathbf{e}_1, \ldots, \mathbf{e}_1 m + n).$$

Now

$$\operatorname{span}(\mathbf{e}_1, \ldots, \mathbf{e}_m) = \operatorname{span}(\mathbf{w}_1, \ldots, \mathbf{w}_m) = W$$

and

$$\operatorname{span}(\mathbf{e}_{m+1}, \ldots, \mathbf{e}_n) = W^{\perp}.$$

So $V = W \oplus W^{\perp}$. Because

(b)

$$\ker A = \{w \in V \mid Aw = 0\},$$

and

$$(\ker A)^{\perp} = \{v \in V \mid v^{\mathsf{T}} w = 0 \text{ for all } w \in \ker A\},$$

so it is clear that $\operatorname{image}(A^{\mathsf{T}}) \subset (\ker A)^{\perp}$.
Assume that $\operatorname{rank}(A) = k$. Then $\operatorname{rank}(A^{\mathsf{T}})$ is also k. So $\dim \operatorname{image}(A^{\mathsf{T}}) = k$.
On the other hand, $\dim \ker A = n - k$, so $\dim(\ker A)^{\perp} = n - (n - k) = k$.
Therefore, $\operatorname{image}(A^{\mathsf{T}}) = (\ker A)^{\perp}$.