12) Prove that if r, θ polar, then

$r^n \cos(n\theta)$ and $r^n \sin(n\theta)$, $n \in \mathbb{Z}$
are harmonic as functions of x, y

Let $z = re^{i\theta}$

$$(re^{i\theta})^n = (r\cos\theta + ir\sin\theta)^n = r^n e^{in\theta}$$

$$= r^n \cos(n\theta) + ir^n \sin(n\theta)$$

Show it is harmonic by showing it's the real part of something analytic.

$$Re(z^n) = r^n \cos(n\theta)$$

$$Im(z^n) = r^n \sin(n\theta)$$

$\therefore$

harmonic

$$r^n = (x^2+y^2)^{n/2}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$r^n \cos(n\theta) = (x^2+y^2)^{n/2} \cos\left(n \tan^{-1}\left(\frac{y}{x}\right)\right)$$

$$r^n \sin(n\theta) = (x^2+y^2)^{n/2} \sin\left(n \tan^{-1}\left(\frac{y}{x}\right)\right)$$

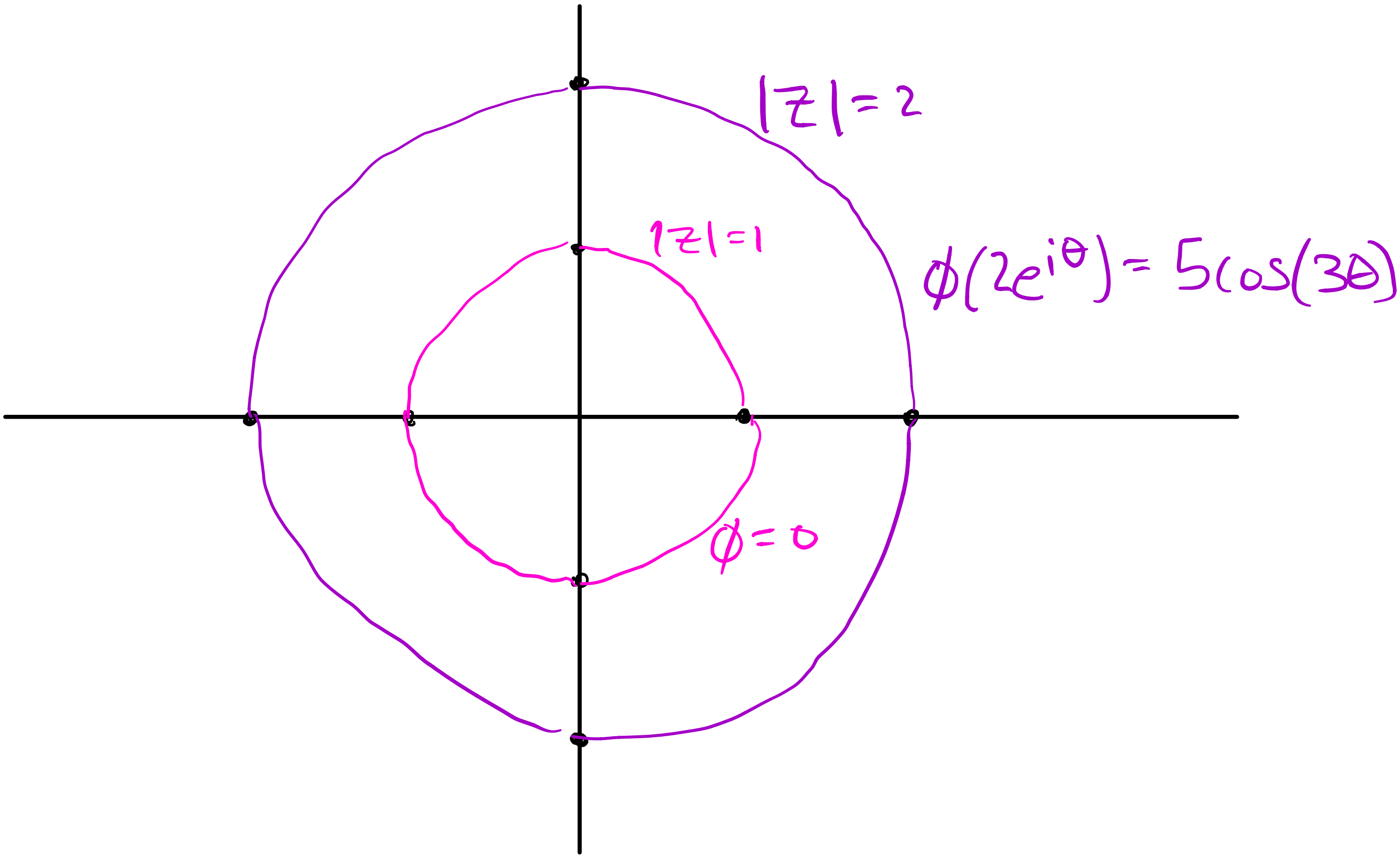
Show it is harmonic by showing it's the real part of something analytic.

$$z^n = r^n e^{in\theta} = r^n \cos(n\theta) + i r^n \sin(n\theta)$$

$$\mathrm{Re}(z^n) = r^n \cos(n\theta) \qquad \therefore \text{ harmonic}$$

$$\mathrm{Im}(z^n) = r^n \sin(n\theta)$$

(15) $\phi(z)$ such that



Hint suggests solution of the form

$$A(z^n + z^{-n})$$

$$z^3 = x^3 - 3xy^2 + y + i(3x^2y - y^2 - x)$$

$$z^3 = r^3\cos(3\theta) + i\,r^3\sin(3\theta) \quad \text{ooh!}$$

Our B.C. on $|z|=2$ look like $\mathrm{Re}|z^3|$

$$z^{-3} = r^{-3}\cos(-3\theta) + i r^{-3}\sin(3\theta)$$

$$\mathrm{Re}(z^{-3}) = r^{-3}\cos(-3\theta)$$

So our solutio is of the form

$$\phi(z) = \text{Re}\left\{A\left(z^3 + z^{-3}\right)\right\}$$

$$= A\left(r^3\cos(3\theta) + r^{-3}\cos(-3\theta)\right)$$

for $r=1$, $\phi = 0$

$$0 = A\left(\cos(3\theta) + \cos(-3\theta)\right) \quad (1)$$

for $r=2$, $\phi = 5\cos(3\theta)$

$$5\cos(3\theta) = A\left(8\cos(3\theta) + \frac{1}{8}\cos(-3\theta)\right) \quad (2)$$

(2) − 8(1)

$$5\cos(3\theta) = A\left(8\cos(3\theta) + \frac{1}{8}\cos(-3\theta)\right) \quad (2)$$

$$- \quad 0 = 8A\left(\cos(3\theta) + \cos(-3\theta)\right) \quad 8(1)$$

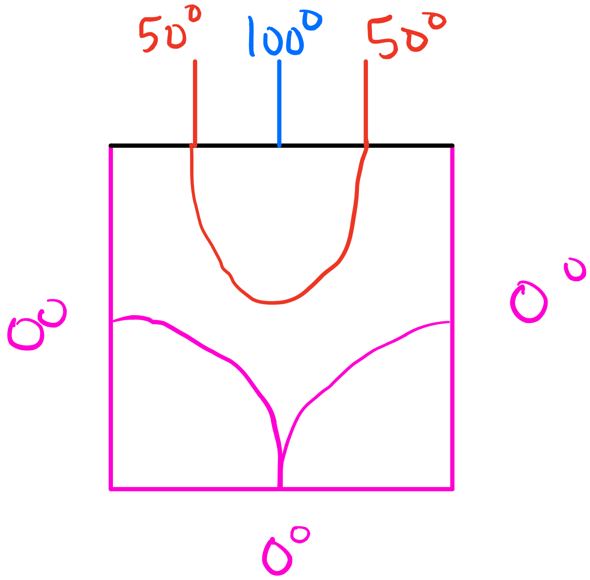
$$5\cos(3\theta) = 0 - A\frac{63}{8}\cos(-3\theta)$$

$$A = \frac{40}{63}\,!$$

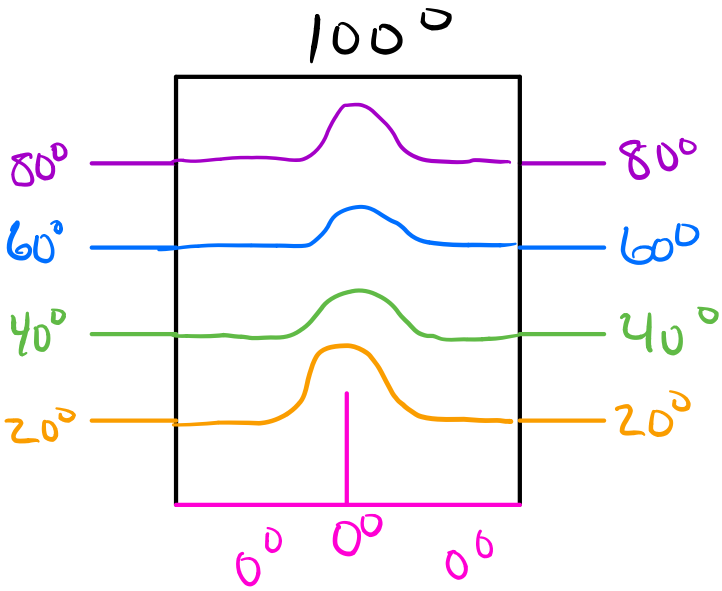
$$\boxed{\phi(z) = \text{Re}\left\{\frac{40}{63}\left(z^3 + z^{-3}\right)\right\}}$$

12.6

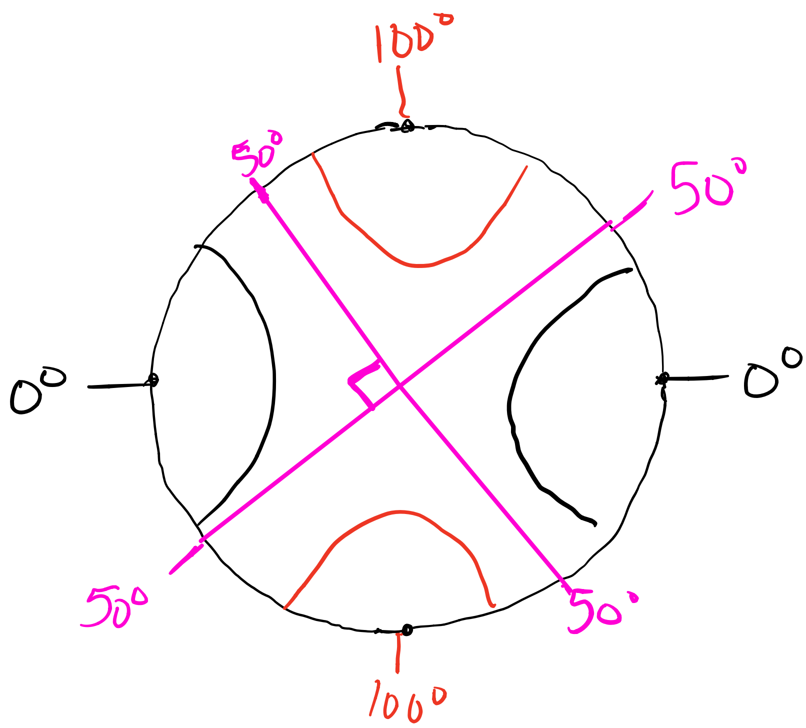
1d)



1e)



③



3.2

5a) $e^{2+i\pi/4} = e^2 e^{i\pi/4}$

$= e^2(\cos(\pi/4) + i\sin(\pi/4))$

$\boxed{= e^2\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)}$

5f) $\underbrace{\cosh\left(i\frac{\pi}{2}\right)}_{\cosh(x) = \frac{e^x + e^{-x}}{2}} = \dfrac{e^{i\pi/2} + e^{-i\pi/2}}{2} = \cos\left(\frac{\pi}{2}\right) = 0$

11) $\mathrm{Re}\left(\dfrac{\cos z}{e^z}\right)$ is harmonic in the whole plane because it is the real part of an analytic function.

13a) $\sin(x+iy) = \sin x \cosh(y) + i\cos x \sinh(y)$

$\sin(x+iy) = \text{Im}\{e^{i(x+iy)}\}$

$= \text{Im}\{e^{ix}e^{i(iy)}\} = \text{Im}\{[\cos(x)+i\sin(x)][\cos(iy)+i\sin(iy)]\}$

$= \cos x \sin(iy) + \sin(x)\cos(iy)$

$\sin(iy) = i\sinh(y)$
$\cos(iy) = \cosh(y)$

$\Rightarrow \sin(x+iy) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$

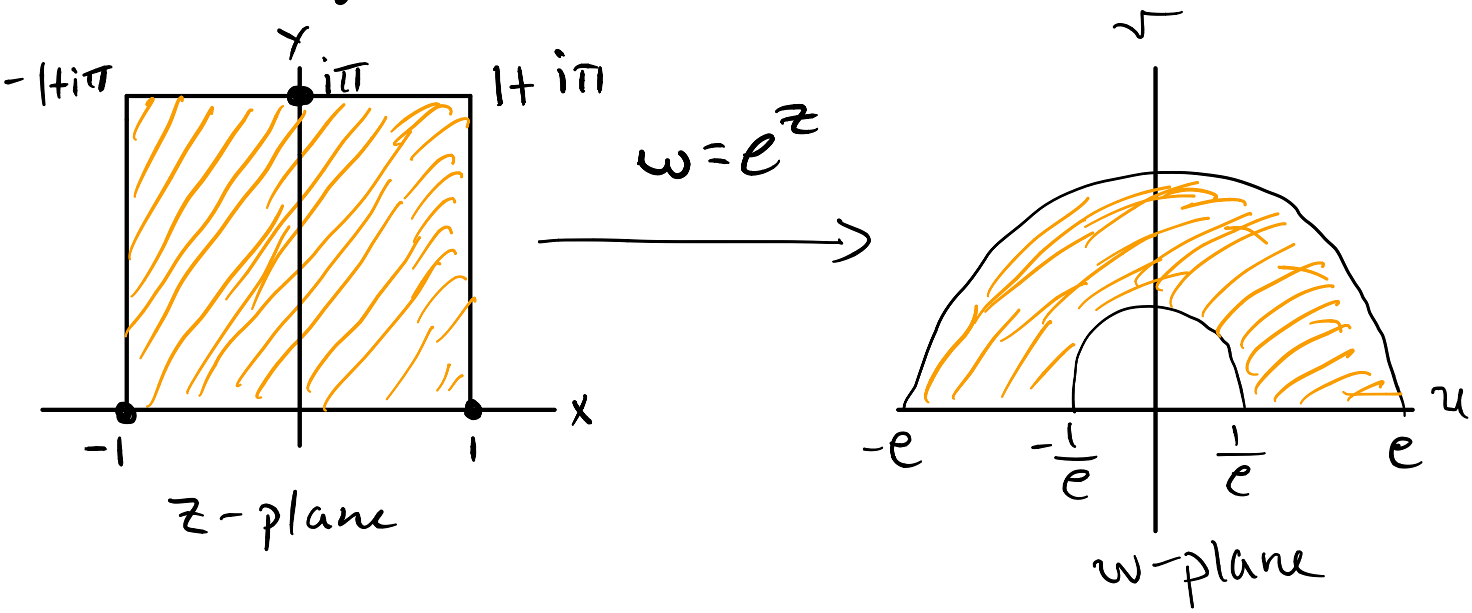
17b) $e^{iz} = 3$

$\log_e(e^{iz}) = \log_e(3)$

$iz = \ln(3) + 2\pi n$

$z = \dfrac{\ln(3)}{i} = \boxed{-i\ln(3) + 2\pi n,\ n \in \mathbb{Z}}$

20) Show $w = e^z$ performs the following mapping



Considering the rectangular region in the z-plane in rectangular coordinates and the semi-annulus in the w-plane in polar coordinates.

The mapping is

$$\{x, y\} \mapsto (r = e^x, \theta = y)$$

It's easy to see that the image of the rectangle is contained in the semi-annulus under the exponential mapping as

$$-1 \leq x \leq 1$$

$$\Rightarrow \quad \frac{1}{e} \leq r \leq e$$

A function is one-to-one and onto if it is invertible.

So we have

$$(x,y) \mapsto (r = e^x, \theta = y) \mapsto (\operatorname{Log} e^x, y)$$

as the identity on the rectangle and

$$(r,\theta) \mapsto (x = \operatorname{Log} r, y = \theta) \mapsto (e^{\operatorname{Log} r}, \theta)$$

is the identity on our semi-annulus.
So the proposed inverse is correct.

3.3

1a) $\log i = \ln(r) + i\theta + 2\pi n,\ n \in \mathbb{Z}$

$r = 1,\ \theta = \frac{\pi}{2}$

$\log(i) = 0 + i\frac{\pi}{2} + 2\pi n$

$\boxed{= i\frac{\pi}{2} + 2\pi n,\quad n \in \mathbb{Z}}$

1c) $\operatorname{Log}(-i) = \ln(r) + i\theta$

$= 0 - i\frac{\pi}{2} = \boxed{-i\frac{\pi}{2}}$

3) If $z_1 = i$ and $z_2 = i - 1$, $\operatorname{Log} \in [-\pi, \pi]$

Then

$\operatorname{Log} z_1 z_2 \neq \operatorname{Log} z_1 + \operatorname{Log} z_2$

$\operatorname{Log}(z_1 z_2) = \operatorname{Log}(i^2 - i) = \operatorname{Log}(-1-i)$

$= \operatorname{Log}|1-i| + i\operatorname{Arg}(-1-i)$

$= \operatorname{Log}(\sqrt{2}) - i\frac{3\pi}{34}$

$\operatorname{Log} z_1 = i\frac{\pi}{2}$

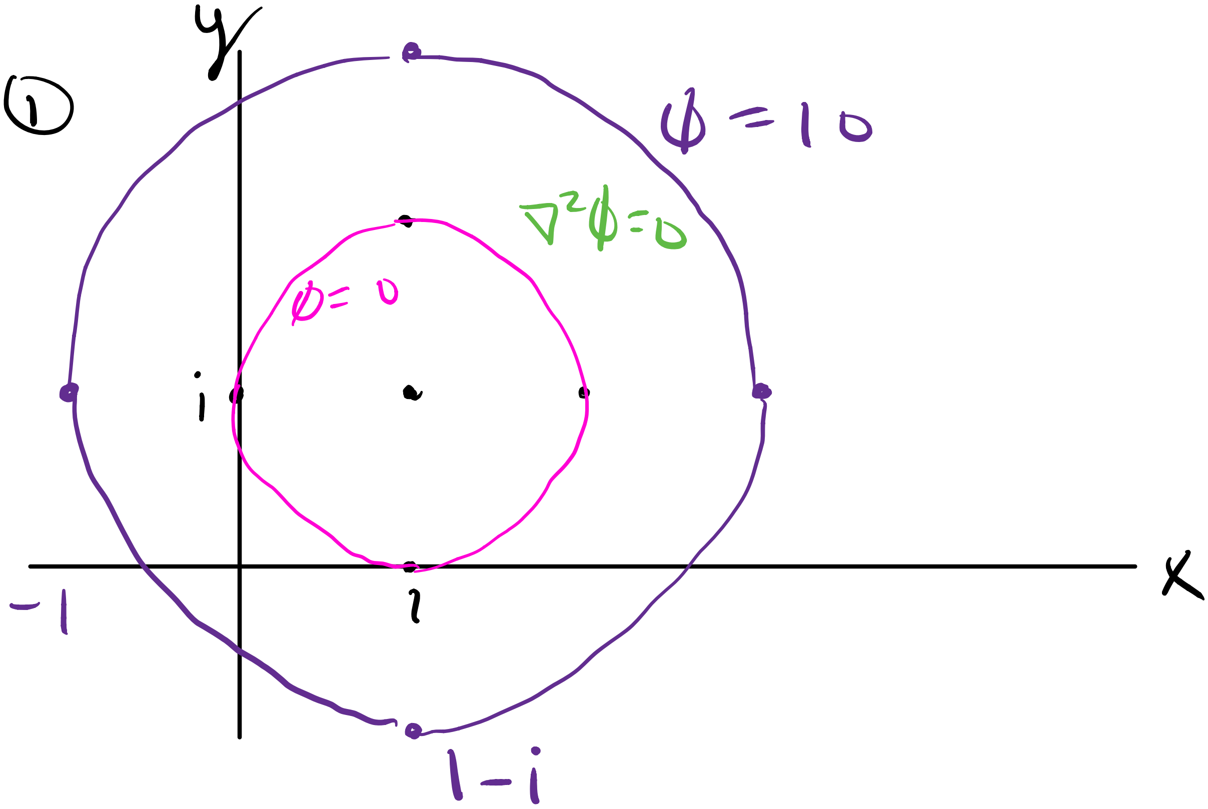
$\operatorname{Log} z_2 = \operatorname{Log}(\sqrt{2}) + i\frac{3\pi}{4}$ $\rightarrow$ $\operatorname{Log} z_1 + \operatorname{Log} z_2$

$= \operatorname{Log}\sqrt{2} + \frac{5\pi}{4}$

$\neq \operatorname{Log}(z_1 z_2)$

3.4

1



Solution of the for washers is generally

$$\phi = A \,\mathrm{Log}(r) + B$$

$$0 = A \,\cancel{\mathrm{Log}(1)}^{\,0} + B \Rightarrow B = 0$$

$$10 = A \,\mathrm{Log}(2) + B$$

$$A = \frac{10}{\mathrm{Log}(2)}$$

So

$$\phi(z) = \frac{10}{\mathrm{Log}(2)} \mathrm{Log}\,|z - z_0|$$

At $(0,0)$

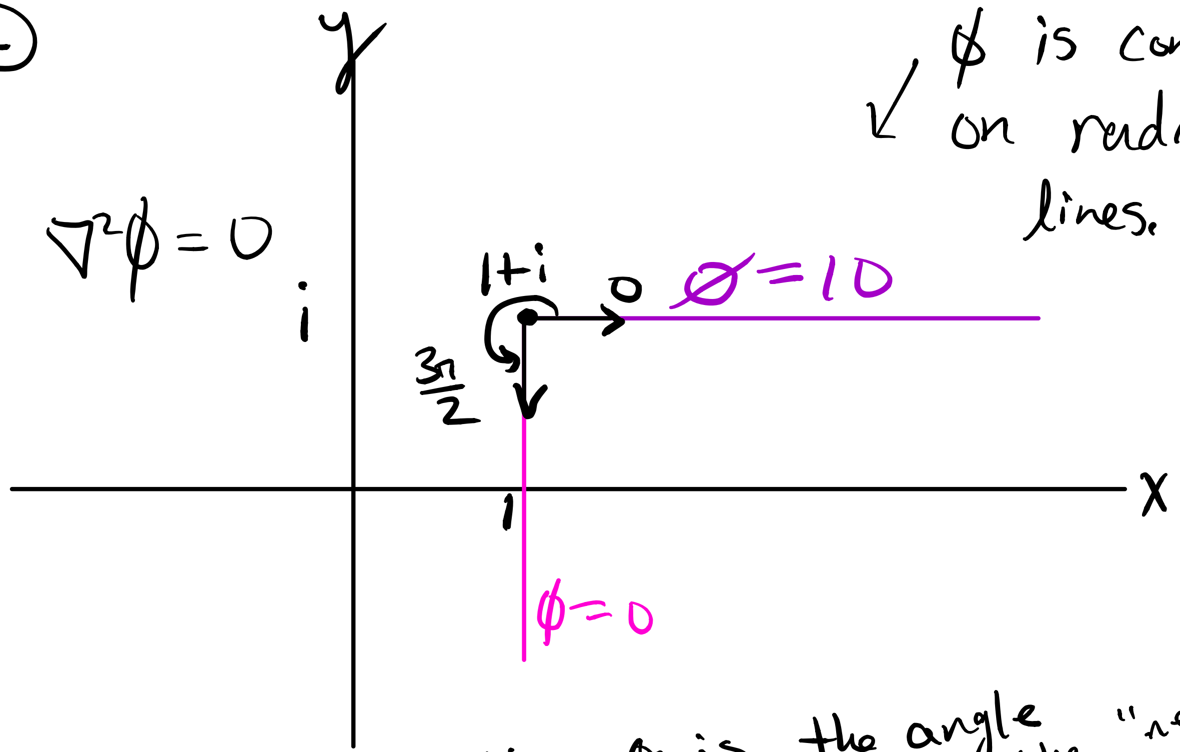
$$\frac{10}{\mathrm{Log}(2)} \mathrm{Log}\,|-i-1|$$

$$= \frac{10\,\mathrm{Log}\sqrt{2}}{\mathrm{Log}\,2} = 5$$

(2)

$\nabla^2 \phi = 0$

$\swarrow$ ϕ is constant on radial lines.



this θ is the angle from 1+i, i.e. 1+i is the "new origin"

$$\phi(\theta) = A\theta + B$$

At 1, $\theta = 0$

At i, $\theta = 3\pi/2$

$$\phi(0) = 10 = B$$

$$\phi\left(\frac{3\pi}{2}\right) = 0 = 10 + A\frac{3\pi}{2}$$

$$A = -\frac{20}{3\pi}$$

$$\phi(\theta) = -\frac{20}{3\pi}\theta + 10$$

$\downarrow$ as a function of z

$$\phi(z) = \frac{-20}{3\pi} \operatorname{Arg}(z - z_0) + 10$$