① MMSE Estimation

$X \sim \text{Unif}[0,2]$. Suppose $Y = XU$, where $X \perp\!\!\!\perp U$, and $U \sim \text{Unif}[0,1]$.

(a) Find the best LMMSE estimator of X given Y.

$$\hat{X}_{LMMSE} = \frac{\text{Cov}(X,Y)}{\text{Var}(Y)}(Y - \mathbb{E}[Y]) + \mathbb{E}[X]$$

$\mathbb{E}[Y] = \mathbb{E}[XU] = \mathbb{E}[X]\mathbb{E}[U]$ since $X \perp\!\!\!\perp U$

$$\mathbb{E}[X] = \frac{2+0}{2} = 1, \quad \mathbb{E}[Y] = \frac{1+0}{2} = \frac{1}{2}$$

$$\mathbb{E}[Y] = 1/2$$

$$\text{Var}(Y) = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2$$

$$= \frac{16}{36} - \left(\frac{1}{2}\right)^2$$

$$= \frac{16}{36} - \frac{9}{36}$$

$$= 7/36$$

$$\mathbb{E}[Y^2] = \mathbb{E}[X^2]\,\mathbb{E}[U^2]$$

$$= (\text{Var}(X) + (\mathbb{E}[X])^2)(\text{Var}(U) + (\mathbb{E}[U])^2)$$

$$= \left(\frac{4}{12} + 1\right)\left(\frac{1}{12} + \frac{1}{4}\right)$$

$$= \frac{16}{12} \cdot \frac{4}{12} = \frac{16}{36}$$

$$\mathrm{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$$

$$= \mathbb{E}[X^2 U] - \mathbb{E}[X]\mathbb{E}[Y]$$

$$= \mathbb{E}[X^2]\mathbb{E}[U] - \mathbb{E}[X]\mathbb{E}[Y]$$

$$= \frac{4}{3} \cdot \frac{1}{2} - \frac{1}{2} \cdot 1$$

$$= 1/6$$

Thus

$$\hat{X}_{LMMSE} = \frac{1/6}{7/36}\left(Y - 1/2\right) + 1$$

$$= \frac{6}{7}Y - \frac{6}{14} + 1$$

$$= \frac{6}{7}Y + \frac{4}{7}$$

(b) Find the best MMSE estimator of X given Y.

$$\hat{X}_{MMSE} = \mathbb{E}[X|Y]$$

$$Y|X=x = xU \sim [0, x]$$

$$f_{Y|X=x}(y|x) = \begin{cases} 1/x, & 0 \le y \le x \le 2 \\ 0, & o/w \end{cases}$$

Need

$$f_{X|Y=y}(x|y) = \frac{f_{Y|X}(y|x) f_X(x)}{f_Y(y)}$$

$$f_{X,Y}(x,y) = f_{Y|X}(y|x) f_X(x) = \begin{cases} \frac{1}{2x} & , \ 0 \le y \le x \le 2 \\ 0 & , \ o/w \end{cases}$$

$$f_Y(y) = \int_{x=y}^{2} \frac{1}{2x} \, dx = \frac{1}{2} \ln|x| \Big|_{x=y}^{2} = \frac{1}{2}(\ln|2| - \ln|y|)$$

$$f_Y(y) = \begin{cases} \frac{1}{2}(\ln|2| - \ln|y|) & , \ 0 \le y \le 2 \\ 0 & , \ o/w \end{cases}$$

$$f_{X|Y}(x|y) = \frac{1/2x}{1/2\,(\ln|2| - \ln|y|)} = \frac{1}{x(\ln|2| - \ln|y|)} \ , \ 0 \le y \le x \le 2$$

$$\mathbb{E}[X|Y=y] = \int_{x=y}^{2} x f_{X|Y}(x|y)\,dx = \int_{y}^{2} \frac{x}{\ln|2| - \ln|y|} \cdot \frac{1}{x} \, dx = \frac{2-y}{\ln|2| - \ln|y|}$$

$$\mathbb{E}[X|Y] = \frac{2-Y}{\ln 2 - \ln Y}$$

② Estimating a Cubic

Let X and Y be zero-mean jointly Gaussian random variables with covariance matrix

$$\mathbb{K} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$$

where $|\rho| < 1$

Note:

① For $W \sim \mathcal{N}(0, \sigma^2)$

$$\mathbb{E}[W^k] = \begin{cases} 0 & , k \text{ odd} \\ \sigma^k (k-1)!! & , k \text{ even} \end{cases}$$

where $n!! = n(n-2)(n-4) \cdots 3 \cdot 1$ for odd n

② For a general Gaussian random variable $Z \sim \mathcal{N}(\mu, \sigma^2)$, we have

$$\mathbb{E}[Z^3] = \mu^3 + 3\mu\sigma^2$$

(a) Find $\mathbb{E}[X^3 | Y]$, the best estimator of X^3 given Y, and compute the resulting MSE.

$$X|Y \sim \mathcal{N}(\mathbb{E}[X|Y], \text{Var}(X|Y))$$

$$\sim \mathcal{N}(\rho Y, 1-\rho^2)$$

$$\mathbb{E}[X|Y] = \mathbb{E}[X] + \frac{\mathrm{Cov}(X,Y)}{\mathrm{Var}(Y)}(Y - \mathbb{E}[Y]) = \rho Y$$

$$\mathrm{Var}(X|Y) = \mathrm{Var}(X)(1-\rho^2_{XY}) = 1-\rho^2$$

Using hint ②,

$$\mathbb{E}[X^3|Y] = \mu^3 + 3\mu\sigma^2 = (\mathbb{E}[X|Y])^3 + 3(\mathbb{E}[X|Y])(1-\rho^2)^2$$
$$= \rho^3 Y^3 + 3\rho Y(1-\rho^2)$$

$$\hat{X}^3_{MMSE} = \mathbb{E}[X^3|Y] = \rho^3 Y^3 + 3\rho(1-\rho^2)\, Y$$

$$\begin{aligned} MSE &= \mathbb{E}[(X^3)^2] - \mathbb{E}[(\hat{X}^3_{MMSE})^2] \\ &= \mathbb{E}[X^6] - \mathbb{E}[(\mathbb{E}[X^3|Y])^2] \\ &= \mathbb{E}[X^6] - \rho^6 Y^6 - 9\rho^2(1-\rho^2)^2\,\mathbb{E}[Y^2] - 6\rho^4(1-\rho^2)\mathbb{E}[Y^4] \\ &= 15(1-\rho^6) - 9\rho^2(1-\rho^2)^2 - 18\rho^4(1-\rho^2) \end{aligned}$$

(b) Find the best linear MMSE estimator of X^3 given Y and compute resulting MSE.

$$\hat{X}^3_{LMMSE} = \frac{\mathrm{Cov}(X^3, Y)}{\mathrm{Var}(Y)}(Y - \mathbb{E}[Y]) + \mathbb{E}[X^3]$$

$$\begin{aligned}
\mathrm{Cov}(X^3, Y) &= \mathbb{E}[X^3 Y] \\
&= \mathbb{E}_Y[\mathbb{E}[X^3 Y \mid Y]] \\
&= \mathbb{E}_Y[Y\,\mathbb{E}[X^3 \mid Y]] \\
&= \mathbb{E}_Y[Y(\rho^3 Y^3 + 3\rho(1-\rho^2)Y)] \\
&= \mathbb{E}_Y[\rho^3 Y^4 + 3\rho(1-\rho^2)Y^2] \\
&= \rho^3\,\mathbb{E}_Y[Y^4] + 3\rho(1-\rho^2)\,\mathbb{E}[Y^2] \\
&= 3\rho
\end{aligned}$$

$$\mathbb{E}[X^3] = 0$$

$$\hat{X}^3_{LMMSE} = 0 + \frac{3\rho}{1}(Y - \mathbb{E}[Y]) = 3\rho Y$$

$$MSE = \mathrm{Var}(X^3) - \frac{\mathrm{Cov}^2(X^3, Y)}{\mathrm{Var}(Y)}$$

$$\begin{aligned}
\mathrm{Var}(X^3) &= \mathbb{E}[X^6] - (\mathbb{E}[X^3])^2 \\
&= 1(5)(3)(1) - 0 \\
&= 15
\end{aligned}$$

$$MSE = 15 - 9\rho^2$$

③ Screening Test

A screening test is 98% effective in detecting a certain disease when a person has the disease. However, the test yields a false positive rate of 1% of the healthy persons tested.
If 0.1% of the population has the disease, what is the probability that a person who tests positive actually has the disease.

Let

$A \rightarrow$ event test detects disease
$B \rightarrow$ event person has disease

Given

$$\Pr(A|B) = 0.98$$
$$\Pr(A^c|B) = 0.02$$
$$\Pr(B) = 0.001$$
$$\Pr(B^c) = 0.999$$

False positive occurs when test detects a disease but person does NOT have it.

$$\Pr(A|B^c) = 0.01 \rightarrow \Pr(A^c|B^c) = 0.99$$

Want

$$\Pr(B|A)$$

Via Bayes Rule

$$Pr(B|A) = \frac{Pr(A|B)\,Pr(B)}{Pr(A)} = \frac{Pr(A|B)\,Pr(B)}{Pr(A|B)\,Pr(B) + Pr(A|B^c)\,Pr(B^c)}$$

$$= \frac{(0.98)(0.001)}{(0.98)(0.001) + (0.01)(0.999)}$$

$$\approx 0.09$$