Recall

Definition: $\{X(t)\}_{t \geq 0}$ is a markov process if $X(t_n) = x_n$ is all that matters for determining the value at $X(t_{n+1})$

i.e.

$\forall n$, $t_1 < t_2 < \cdots < t_n < t_{n+1}$

$$\Pr\left[X(t_{n+1}) = x_{n+1} \mid X(t_n) = x_n, X(t_{n-1}) = x_{n-1}, \ldots, X(t_1) = x_1\right] = \Pr\left[X_{n+1} = x_{n+1} \mid X_n = x_n\right]$$

State: $X(t)$ is called the state at time t.

All the values $X(t)$ can take is called the state-spac $\mathcal{X}$

If $\mathcal{X}$ is discrete, then we call this a <u>Markov chain</u>.

Example: Random Walk, $\mathcal{X} = \mathbb{Z}$

is a discrete state-space $\Rightarrow$ Markov Chain

Example: Brownian motion: $\mathcal{X} = \mathbb{R}$

is a continuous state-space $\Rightarrow$ Markov process

Example: Binomial Counting: $\mathcal{X} = \mathbb{N}$ Disrete state-space Markov chain

Example: Poisson Process : $\mathcal{X} = \mathbb{N}$ continuous state-space Markov chain

Discrete-Time Markov Chain

$\{X_n\}_{n\geq 0}$ is a random process

$k, \{n_i\}_{i=1}^{k}$

$X_{n_1}, \ldots, X_{n_k}$ → need joint pmf to describe it

$$\Pr[X_0 = x_0, X_1 = x_1, \ldots, X_k = x_k]$$

$$= \Pr[X_0 = x_0]\Pr[X_1 = x_1 \mid X_0 = x_0]\Pr[X_2 = x_2 \mid X_1 = x_1, x_0 = 0] \cdots \Pr[X_k = x_k \mid X_{k-1} = x_{k-1}, \ldots, X_0 = x_0]$$

$$= \Pr[X_0 = x_0]\underbrace{\Pr[X_1 = x_1 \mid X_0 = x_0]}_{\text{one-step transition probability}}\Pr[X_2 = x_2 \mid X_1 = x_1] \cdots \Pr[X_k = x_k \mid X_{k-1} = x_{k-1}]$$

i.e. for Markov process only need the marginal distribution of the initial state and the transition probabilities from one state to the next.

~~i.e~~ Need $p_{X_0}(x_0)$ and $\{\Pr[X_{n+1} = j \mid X_n = i]\}_{i,j \in \mathcal{X}}$

Homogenous Markov Chain → Time INVARIANT

$$\Pr[X_{n+1} = j \mid X_n = i] \triangleq p_{i,j} \quad \forall n$$

Transition Matrix:

$\mathcal{X} = \{1, 2, \ldots, M\}$

$0 \leq p_{i,j} \leq 1$

$$\mathbb{P} = \begin{bmatrix} p_{1,1} & p_{1,2} & \cdots & p_{1,M} \\ p_{2,1} & p_{2,2} & \cdots & p_{2,M} \\ \vdots & \vdots & \ddots & \vdots \\ p_{M,1} & p_{M,2} & \cdots & p_{M,M} \end{bmatrix}$$

$\sum_j p_{i,j} = 1 \quad \forall i$

Example:

Coin A and Coin B

A: fair B: biased, $Pr[H] = \frac{1}{4}$

X_0 : flip A $\begin{pmatrix} H = 1 \\ T = 2 \end{pmatrix}$

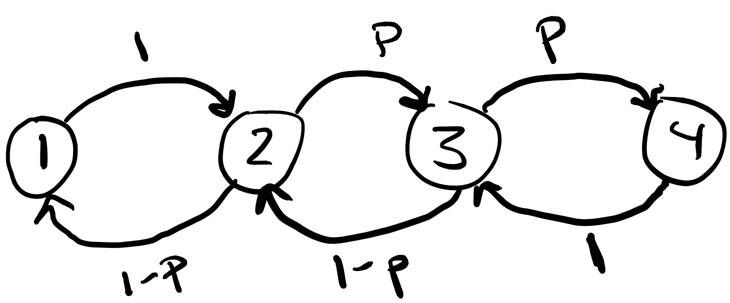
X_{n+1} : outcome of flip B if $X_n = 1$

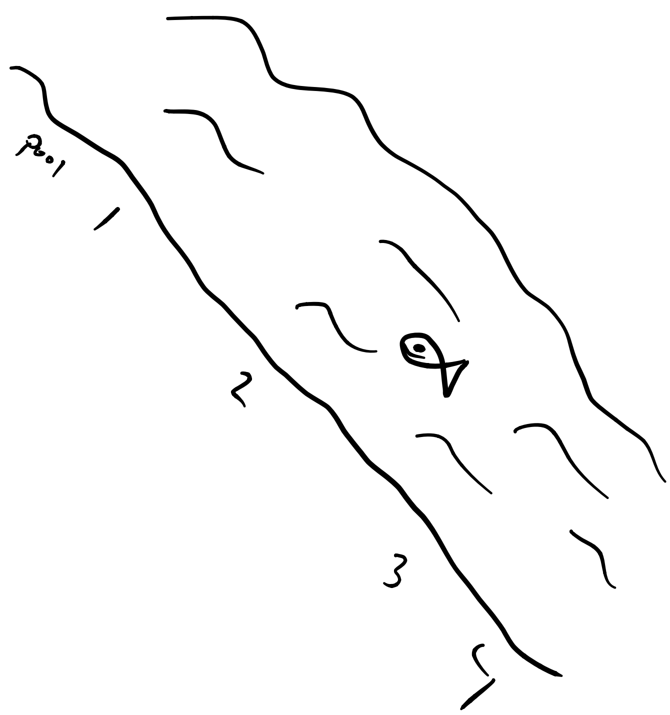
outcome of flip A if $X_n = 2$

$$Pr[X_0 = 1] = Pr[X_0 = 2] = \frac{1}{2}$$

$$\mathbb{P} = \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \end{array}$$

Example: A Fish Called Wanda





$$\mathbb{P} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1-p & 0 & p & 0 \\ 0 & 1-p & 0 & p \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

n-Step Transition Matrix

$$\mathbb{P}^{(n)} = \left\{ p_{i,j} \triangleq \Pr\left[X_n = j \mid X_0 = i\right] \right\}$$

Chapman-Korogorov Equation

$$p^{(m+n)}_{i,j} = \sum_{k \in \mathcal{X}} \underset{i \times k}{p^{(m)}_{i,k}} \; \underset{k \times j}{p^{(n)}_{k,j}}$$

$\swarrow i \times j$

$$\mathbb{P}^{(m+n)} = \mathbb{P}^{(m)} \mathbb{P}^{(n)}$$

Can use this to obtain

$$\begin{aligned} \mathbb{P}^{(n)} &= \mathbb{P}\,\mathbb{P}^{(n-1)} \\ &= \mathbb{P}\,\mathbb{P}\,\mathbb{P}^{(n-2)} \\ &\;\vdots \\ &= \mathbb{P}^{n} \end{aligned}$$

$$\{X_n\}_{n \geq 0} : \quad \vec{p}(0) = \Pr\underbrace{\left[X_0 = 1, \ldots, X_0 = m\right]}_{}, \quad \begin{cases} \text{pmf of } X_0 \\ \mathbb{P} \end{cases}$$

row vector by definition

$$\vec{p}(n) \triangleq \Pr\left[X_n = 1, \ldots, X_n = m\right]$$

$$\begin{aligned} \Pr\left[X_n = j\right] &= \sum_{i=1}^{m} \Pr\left[X_0 = i\right] \Pr\left[X_n = j \mid X_0 = i\right] \\ &= \vec{p}(0)\,\mathbb{P}^{(n)} \end{aligned}$$

State Probability

This is the distribution at time n.

$$\mathbb{p}(n) = \left[Pr[X_n = 1] \quad \cdots \quad Pr[X_n = m] \right]$$ ↙ vector

$$\mathbb{p}(0) = \left[Pr[X_0 = 0] \quad \cdots \quad Pr[X_0 = m] \right]$$

$$\mathbb{p}(1) = \mathbb{p}(0)\,\mathbb{P}$$

$$\vdots$$

$$\mathbb{p}(n) = \mathbb{p}(0)\,\mathbb{P}^n$$

Stationary Distribution

$\pi = \pi \mathbb{P}$, π is stationary