$\log z = \ln z$ (i.e "base e")

Want $\log(z)$ to be the inverse of e^z

Let $z = re^{i\theta}$.

Hope: $\log(z) = \log(r) + i\theta$

Problems arise because $\arg(z)$ is multivalued!
(in the sense that θ, $\theta + 2\pi$, $\theta + 4\pi$ ... all
correspond to the same point z, but give
different values of $\log(z)$).

Example:

$$\log(1) = \log(e^{i2\pi k}) = i2\pi k, \quad k \in \mathbb{Z}$$

$$\log(-2) = \log(2) + i\,2\pi k + i\pi$$

$\log(0)$ is still VERY nasty!

* Not allowed, $\log(r)_{r \to 0} = -\infty$

BUT $i\theta$ could be anything...

Need to make $\log(z)$ into an analytic single
valued function.

Restrict θ to be single valued. $-\pi < \theta \leq \pi$

NOT GOOD ENOUGH - still not analytic.

Such a definition would give us a
discontinuous function.

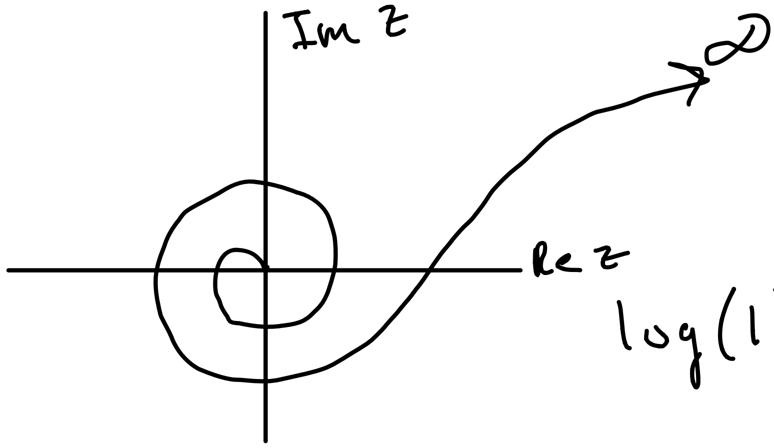
If we want $\log(z)$ to be analytic, we must forbid encirclement of the origin. Otherwise θ jumps by $2\pi i$ when we cross the negative real axis.

We must cut the plane to prevent paths encircling the origin.

Branch point: for a function $f(z)$ there is a point z_0 such that on any path encircling z_0, $f(z)$ is discontinuous at some point on that path.

A branch is a single-valued, continuous function whose values agree with one choice of the original function's multiple values.

Example: Spiral branch cut of $\log(z)$



Look at branch of $\log(z)$ where

$\log(1) = -8\pi i$. What's $\log(4)$?

$$-8\pi i + \underbrace{4\pi i}_{\text{rotating around cut}} + \log(4) = \log(4) - 4\pi i$$

Other multivalued functions

$f(z) = z^{\alpha}$, $\alpha \neq$ integer

$z^{\alpha} = e^{\log z^{\alpha}} = e^{\alpha \log z}$ } log has branch cuts so z^{α} has branch cuts

With $\alpha = 1/2$

$$z^{1/2} = r^{1/2} e^{i(\theta + 2\pi k)/2} = \left[r^{1/2} e^{i\theta/2}\right] e^{i\pi k}$$

two values

Slit plane $\neq$ plane − negative real axis