Not on the exam

- determinants
- normal operators
- adjoint operators
- inner products over $\mathbb{C}$

Since LAST exam:

- $A = QR$
- Spectral Theorem (symmetric matrices)
- SVD
- JCF

Spectral theorem Part

- A $n \times n$ symmetric over $\mathbb{R}$
- eigenvalues are real
- eigenvectors $A\vec{v}_i = \lambda_i \vec{v}_i$, $\lambda_i \neq \lambda_j$
  $\Rightarrow \vec{v}_i \perp \vec{v}_j$

- $\mathbb{R}^n = E_{\lambda_1}(A) \oplus \cdots \oplus E_{\lambda_m}(A)$
- So A is diagonalizable (orthonormally)

So,

$$A = VDV^T$$

$$V^TV = I \quad (V \text{ orthogonal})$$

$D =$ diagonal matrix of eigenvalues

$W \subseteq V$ supspace,

$$V = W \oplus W^{\perp} \text{ if } \dim W < \infty$$

$$(W^{\perp})^{\perp} = W \text{ if } \dim V < \infty$$

SVD

$A \in \mathbb{R}^{m \times n}$

$$\underset{m\times n}{A} = \underset{m\times m}{U}\,\underset{m\times n}{\Sigma}\,\underset{n\times n}{V^T}, \quad U, V \text{ orthogonal}$$

$$\operatorname{rank}(A) = r$$

Σ = diagonal matrix w/ entries $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > \sigma_{r+1} = \cdots = \sigma_n = 0$

$$\sigma_i = \sqrt{\lambda_i}, \quad \lambda_i \text{ eigenvalue of } A^T A$$

COMPACT SVD

$$A = U_1 \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{pmatrix} V_1^T = \sigma_1 u_1 v_1^T + \cdots + \sigma_r u_r v_r^T$$

JCF

A $n \times n$ matrix over $\mathbb{C}$.

① Block Diagonal Matrices

$$A = A_1 \oplus A_2 \oplus \cdots \oplus A_r = \begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_r \end{bmatrix}$$

$$\dim \ker A = \sum_{i=1}^{r} \dim \ker(A_i)$$

$$\operatorname{rank}(A) = \sum_{i=1}^{r} \operatorname{rank}(A_i) = n - \dim\ker(A)$$

Eigenvalues of A vs those of $A_1, \ldots, A_r$

$$E_\lambda(A) = \ker(A - \lambda I) = \ker(A_1 - \lambda \underset{\text{smaller } I}{I}) \oplus \cdots \oplus \ker(A_r - \lambda I)$$

Eigenvectoros of A for λ

② $G_\lambda(A)$'s : $\quad G_\lambda(A) = \ker(A - \lambda I)^n$

key fact

$$V = \mathbb{C}^n = G_{\lambda_1}(A) \oplus \cdots \oplus G_{\lambda_m}(A)$$

if $\lambda_1, \ldots, \lambda_m$ are eigenvalues of A

each is A invariant

So

$$A = A_1 \oplus \cdots \oplus A_m$$

$$A_i = (A - \lambda I)\big|_{G_{\lambda_i}(A)}$$

③ Jordan Blocks

$$J(\lambda, n) = \begin{pmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix} = J(0, n) = N$$

(nilpotent matrix)

know: dim ker $J(0,n) = 1$

index of nilpotence $= n$

Char poly $= q_N(x) = x^n$

min poly $= m_N(x) = x^n$

Theorem: $A \in \mathbb{C}^{n \times n}$ is similar to a direct sum of Jordan blocks

Case 1: A has only eigenvalue λ

$A \sim J(\lambda, k_1) \oplus \cdots \oplus J(\lambda, k_r)$

where

$$k_1 \geq k_2 \geq \cdots \geq k_r \geq 1$$

$$n = k_1 + k_2 + \cdots + k_r \rightarrow \text{partition of } n$$

$$J(\lambda, k_1) \oplus \cdots \oplus J(\lambda, k_r) = J(\lambda, k_1, \dots, k_r)$$

Case 2: General Case: $A \in \mathbb{C}^{n \times n}$, $\lambda_1, \dots, \lambda_m$ eigenvalues

$$A \sim J(\lambda_1, \underline{k}^{(1)}) \oplus J(\lambda_2, \underline{k}^{(2)}) \oplus \cdots \oplus J(\lambda_m, \underline{k}^{(m)})$$

Example: Suppose A is 8×8 with eigenvalues

Always $\leq$

$$\lambda_1 = 7, \quad \dim E_{\lambda_1} = 1, \quad \dim G_{\lambda_1} = 3$$

$$\lambda_2 = 15, \quad \dim E_{\lambda_2} = 1, \quad \dim G_{\lambda_2} = 1$$

$$\lambda_3 = -1, \quad \dim E_{\lambda_3} = 3, \quad \dim G_{\lambda_3} = 4$$

Find all possible JCF's of A

Sol'n

3×3 only has eigenvalue $\lambda = \lambda_1 = 7$

$$A \sim \underset{3\times 3}{A_1} \oplus \overset{1\times 1}{A_2} \oplus \overset{4\times 4}{A_3}$$

where

$$A \sim \underbrace{J(\lambda, \underline{k}^{(1)})}_{A_1} \oplus \underbrace{J(\lambda, \underline{k}^{(2)})}_{A_2} \oplus \underbrace{J(\lambda, \underline{k}^{(3)})}_{A_3}$$

$\underline{k}^{(1)}$ is a partition of 3 $\rightarrow$ 111, 21, 3

$\underline{k}^{(2)}$ is a partition of 1 $\rightarrow$ 1

$\underline{k}^{(3)}$ is a partition of 4 $\rightarrow$ 1111, 211, 22, 31, 4

NOTE: 4, 2, 2, 3, 1 $\}$ $\dim E_{\lambda_3}$

$\underline{\lambda_1}$

$$\begin{bmatrix} \lambda_1 & 1 & \\ & \lambda_1 & 1 \\ & & \lambda_1 \end{bmatrix}, \quad \left[\begin{array}{cc|c} \lambda_1 & 1 & \\ & \lambda_1 & \\ \hline & & \lambda_1 \end{array}\right], \quad \left[\begin{array}{c|c|c} \lambda_1 & & \\ \hline & \lambda_1 & \\ \hline & & \lambda_1 \end{array}\right]$$

$$\underset{\uparrow \text{ Choose this one}}{3} \qquad\qquad 21 \qquad\qquad 111$$

$\underline{\lambda_2}$

No work required

$\underline{\lambda_3}$

Choose partition 211 to satisfy $\dim E_{\lambda_3}$

Thus

$$JCF = \left[\begin{array}{ccc|c|cccc}
\lambda_1 & 1 & & & & & & \\
 & \lambda_1 & 1 & & & & & \\
 & & \lambda_1 & & & & & \\
\hline
 & & & \lambda_2 & & & & \\
\hline
 & & & & \lambda_3 & 1 & & \\
 & & & & & \lambda_3 & & \\
 & & & & & & \lambda_3 & \\
 & & & & & & & \lambda_3
\end{array}\right]$$