Symmetric Matrices

Situation: $V = \mathbb{R}^n$ w/ standard inner product

$$\langle \vec{x}, \vec{y} \rangle = \vec{x}^T \vec{y} = \vec{y}^T \vec{x}$$

$A_{n \times n}$ symmetric $\Leftrightarrow A^T = A$

$$L_A : \mathbb{R}^n \to \mathbb{R}^n$$

"A symmetric $\Leftrightarrow L_A$ is self-adjoint"

Proposition 1: If $\lambda \in \mathbb{C}$ is an eigenvalue of A, then $\lambda \in \mathbb{R}$

<u>Proof</u> Suppose $\vec{v} \in \mathbb{C}^n$ (NOT $\vec{0}$), $\lambda \in \mathbb{C}$, $A\vec{v} = \lambda \vec{v}$

Note: $\lambda \in \mathbb{R} \Leftrightarrow \bar{\lambda} \in \mathbb{R}$

$$A\vec{v} = \lambda \vec{v}$$

complex conj.

$$\bar{A}\bar{\vec{v}} = \bar{\lambda}\bar{\vec{v}}$$

Note: $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$, $\bar{v}^T = (\bar{v}_1 \cdots \bar{v}_n)$

Transpose

$$\bar{v}^T A = \bar{v}^T \bar{\lambda}$$

TRICK

$$\bar{v}^T A v = \bar{v}^T (\lambda v) = \lambda \bar{v}^T v = \lambda \|v\|^2 \quad (>0)$$

$$\bar{\lambda} \bar{v}^T v = \lambda \bar{v}^T v$$

$$\Rightarrow \bar{\lambda} = \lambda, \quad \lambda \neq 0$$

$$\Rightarrow \lambda \in \mathbb{R}$$

Proposition 2: If $\vec{v}_1 \in E_{\lambda_1}(A)$, $\vec{v}_2 \in E_{\lambda_2}(A)$, $\lambda_1 \neq \lambda_2$,
then $\vec{v}_1 \cdot \vec{v}_2 = 0$

ASSUMING $A = A^T$

<u>Proof</u> Know $Av_1 = \lambda_1 v_1$ $\quad \lambda_1 \neq \lambda_2$

$Av_2 = \lambda_2 v_2$ $\quad v_1 \neq 0$

$v_2 \neq 0$

$$v_2^T A v_1 = v_2^T \lambda_1 v_1 = \lambda_1 (v_2^T v_1) = \lambda_1 (v_1 \cdot v_2)$$

$$\lambda_2 v_2^T v_1 = \lambda_2 (v_1 \cdot v_2)$$

$$\lambda_1 \neq \lambda_2 \Rightarrow v_1 \cdot v_2 = 0$$

Proposition 3: Let $W \subseteq \mathbb{R}^n = V$, Suppose W is A-invariant.
Then $W^\perp$ is also A invariant.
$(A = A^T)$

Proof: Let $v \in W^\perp$.

Then $\forall w \in W$, $\langle v, w \rangle = 0$

i.e $\forall w \in W$, $w^T A v = 0$

but $w^T A v = v^T A w = 0$

$\therefore Av \in W^\perp$

Proposition 4: Assume $A = A^T$.

Let $\mathcal{B} = (u_1, \dots, u_m)$ be an orthonormal basis of $V = \mathbb{R}^n$.

Then $[A]_\beta$ is symmetric

Proof: Let $Q = (u_1, \dots, u_n)$ $n \times n$ matrix

Then

$$\underset{\overset{\shortparallel}{B}}{[L_A]_\beta} = [id]_{\beta \leftarrow std} \; A \; [id]_{std \leftarrow \beta}$$

$$B = Q^{-1} A Q$$

$$B^T = Q^T A^T (Q^{-1})^T$$

Now use β orthonormal: $Q^T Q = I$

then $Q^T = Q^{-1}$

$$B^T = Q^{-1} A Q = B$$

Q.E.D

Proposition 5: $A = A^T$. Let $\mathcal{B} = (u_1, \dots, u_r, u_{r+1}, \dots u_n)$ be an orthonormal basis of $V = \mathbb{R}^n$

such that

$$W = \mathrm{Span}(u_1, \dots, u_r)$$
$$W^\perp = \mathrm{span}(u_{r+1}, \dots, u_n)$$

Suppose W is A-invariant,

then

$$[A]_\beta = \begin{array}{c} u_1 \\ \vdots \\ u_r \\ u_{r+1} \\ \vdots \\ u_n \end{array} \overset{\qquad\qquad u_{r+1} \cdots u_n}{\left[\begin{array}{c|c} \beta_1 & 0 \\ \hline 0 & \beta_2 \end{array}\right]}, \quad \begin{array}{l} \beta_1 \text{ symmetric } r \times r \\ \beta_2 \text{ symmetric } (n-r) \times (n-r) \end{array}$$

Proposition 6: If $A = A^T$, and $\lambda_1, \dots, \lambda_m \in \mathbb{R}$ are the distinct eigenvalues of A, then

$$V = E_{\lambda_1}(A) \oplus \cdots \oplus E_{\lambda_m}(A)$$

Proof) Note: A has a real eigenvalue since it has a complex eigenvalue but by prop 1 this is real.

need to show: ① $E_{\lambda_1}(A) + \cdots + E_{\lambda_m}(A) = V$

② $E_{\lambda_1}(A) + \cdots + E_{\lambda_m}(A) = E_{\lambda_1}(A) \oplus \cdots \oplus E_{\lambda_m}(A)$

② Need $v_1 + \cdots + v_m = 0 \Rightarrow v_1 = 0 = \cdots = v_m$

Know $v_i \cdot v_j = 0$ for $i \neq j$

$v_i (v_1 + \cdots + v_m) = v_i = 0 \Rightarrow v_i = 0 \quad \forall i$

① know $V = W \oplus W^{\perp}$

Let β = orthonormal basis

$$Q = (\underbrace{u_1, \dots, u_r}_{W}, \underbrace{u_{r_1}, \dots, u_n}_{W^{\perp}})$$

then

$$Q^T A Q = B = \left[\begin{array}{c|c} B_1 & 0 \\ \hline 0 & B_2 \end{array}\right]$$

β_2 symmetric.

if $r < n_1$, β_2 has eigenvector. $\beta_2 v = \lambda v$

BAD, do on your own