Math 4310

Homework 11

Due 12/6/19

Name: ______________________________

Collaborators: ______________________________

____

Please print out these pages. I encourage you to work with your classmates on this homework. Please list your collaborators on this cover sheet. (Your grade will not be affected.) Even if you work in a group, you should write up your solutions yourself! You should include all computational details, and proofs should be carefully written with full details. As always, please write neatly and legibly.

Please follow the instructions for the "extended glossary" on separate paper (LaTeX it if you can!) Hand in your final draft, including full explanations and write your glossary in complete, mathematically and grammatically correct sentences. Your answers will be assessed for style and accuracy.

Please ***staple*** *this cover sheet, your exercise solutions, and your glossary together, in that order, and hand in your homework in class.*

GRADES

Exercises ______________________ / 50

Extended Glossary

Component	Correct?	Well-written?
Definition	/6	/6
Example	/4	/4
Non-example	/4	/4
Theorem	/5	/5
Proof	/6	/6
Total	/25	/25

Exercises.

1. Find a singular value decomposition for the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & -2 & 1 \\ 1 & -1 & 1 & 1 \end{pmatrix}$$

Do this computation by hand. Then check your work using software (e.g. Matlab, Julia, Macaulay2).

2. Show that if $A = U_1\Sigma_1V_1^T$, and $A = U_2\Sigma_2V_2^T$ are two singular value decompositions of A, then $\Sigma_1 = \Sigma_2$, that is, the singular values of A are uniquely determined by any SVD of A.

3. Prove that if A is positive definite, then

   (a) the singular values of A are the same as the (nonzero) eigenvalues of A.

   (b) if $A = U\Sigma V^T$ is a singular value decomposition of A, then $U = V$.

   Recall that a symmetric real matrix A is called **positive definite** if for all $\mathbf{x} \neq \mathbf{0}$, then $\mathbf{x}^TA\mathbf{x} > 0$.

4. Rank 1 matrices. Let $A \in \mathbb{R}^{m\times n}$ be a rank one matrix.

   (a) Show that there exists unit vectors $\mathbf{x} \in \mathbb{R}^m$ and $\mathbf{y} \in \mathbb{R}^n$, and a number $\alpha > 0$ such that $A = \alpha xy^T$.

   (b) If A has rank 1, find a compact singular value decomposition for A.

(c) If $A = \alpha x y^T$ as in (a), find the eigenvalues and eigenvectors of A.

5. If Q is an orthogonal $n \times n$ matrix, find a singular value decomposition for Q.

6. Recall that the Frobenius norm of a matrix $A \in V = \mathbb{R}^{m \times n}$ is

$$||A|| = \langle A, A \rangle = \sqrt{\sum_{i,j} A_{ij}^2},$$

where

$$\langle A, B \rangle = \sum_{i,j} A_{i,j} B_{i,j}.$$

This is sometimes written as $||A||_F$. We also assume that $A = \sigma_1 u_1 v_1^T + \ldots + \sigma_r u_r v_r^T$ arises from a compact singular value decomposition of A. For $k \leq r$, let $A_k := A = \sigma_1 u_1 v_1^T + \ldots + \sigma_k u_k v_k^T$.

(a) Recall that the trace $\mathrm{tr}(C)$ of a square matrix C is the sum of its diagonal entries. One key property (which you do not need to show) is that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, whenever both of these matrix products are well defined. Show that

$$\langle A, B \rangle = \mathrm{trace}(A^T B) = \mathrm{trace}(B A^T)$$

(b) Show that the matrices $M_i = u_i v_i^T$ are orthonormal, and consequently, $||A||^2 = \sigma_1^2 + \ldots + \sigma_r^2$.

(c) Find $||A - A_k||$.

(d) Show that if Q is an orthogonal $m \times m$ matrix, then $||QA|| = ||A||$. What is $||A|| = ||U \Sigma V^T||$ for an SVD of A?

This is the beginning of the proof of the Eckart-Young theorem, which states that for every $m \times n$ matrix B of rank at most k, then

$$||A - A_k|| \leq ||A - B||,$$

i.e. that A_k is the closest rank k matrix to A. See also https://en.wikipedia.org/wiki/Low-rank_approximation#Proof_of_Eckart%E2%80%93Young%E2%80%93Mirsky_theorem_(for_Frobenius_norm)

Extended Glossary. There is no extended glossary this week.