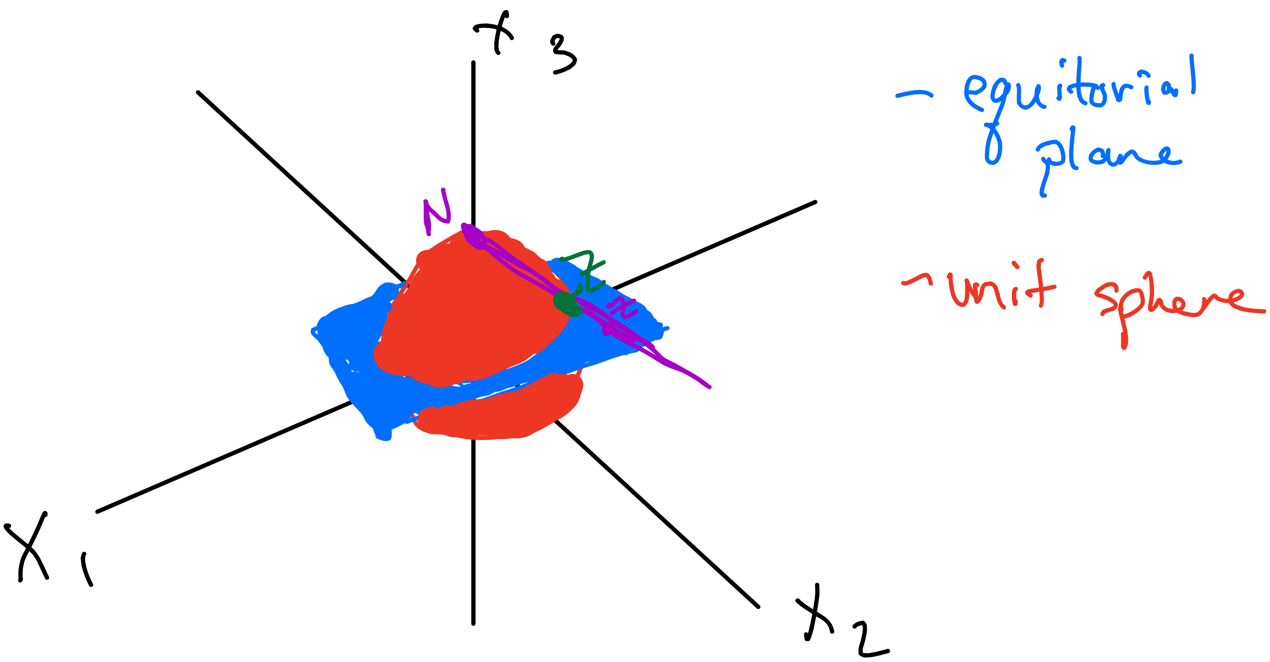
Here we look at stereographic projections - identifying points on the surface of a sphere with points in the complex plane.

Consider the unit sphere in three dimensions (x_1, x_2, x_3) whose equation is given by

$$x_1^2 + x_2^2 + x_3^2 = 1 \quad \text{(depicted below)}$$



Our goal is to associate with each point z in the equitorial plane a unique point Z on the sphere.
For this purpose we construct the line through the North Pole $N = (0, 0, 1)$ of the sphere and the given point z in the $x_1 x_2$ - plane.

This line pierces the spherical surface in exactly one point Z (see above) and we say that the point Z is the stereographic projection of the point z.

If we identify the equitorial plane as the complex plane (or z-plane), the unit sphere is called the Riemann Sphere.

Under stereographic projection, points on the unit circle $|z|=1$ (in z-plane) remain fixed. (that is $z=Z$) forming the equator.

→ Points for which $|z|>1$ projects to northern hemisphere

→ Points for which $|z|<1$ projects to southern hemisphere

→ Origin projects to south pole of Reimann Sphere

Stereographic projection preserves the angles of intersections of these curves: circles centered at the origin intersect lines through the origin at right angles in the z-plane, and longitudes intersect latitudes at right angles on the sphere.

Example 1: Showing formula of projection

Show that if

$$Z = (x_1, x_2, x_3)$$

is the projection on the Riemann sphere of the point

$$z = x + iy$$

in the complex plane, then

$$x_1 = \frac{2\,\mathrm{Re}\, z}{|z|^2+1}, \quad x_2 = \frac{2\,\mathrm{Im}\, z}{|z^2+1}, \quad x_3 = \frac{|z|^2-1}{|z|^2+1} \qquad (1)$$

The line through the north pole $N = (0,0,1)$ and $(x, y, 0)$ is given by the parametric equations

$$x_1 = tx, \quad x_2 = ty, \quad x_3 = 1 - t, \quad -\infty < t < \infty \qquad (2)$$

The line cuts the sphere precisely when t satisfies

$$1 = x_1^2 + x_2^2 + x_3^2 = t^2x^2 + t^2y^2 + (1-t)^2$$

OR

$$1 = t^2(x^2+y^2+1) + 1 - 2t$$

whose roots are

$$t = 0 \text{ (North pole)}$$

and

$$t = \frac{2}{x^2+y^2+1} = \frac{2}{|z|^2+1} \qquad (3)$$

Substituting our root t into Eqn (2) yields

$$x_1 = \frac{2x}{|z|^2+1}, \quad x_2 = \frac{2y}{|z|^2+1}, \quad x_3 = 1 - \frac{2}{|z|^2+1}$$

$$x_3 = \frac{|z|^2-1}{|z|^2+1} \quad \checkmark$$

Conversely, if we start with the point (x_1, x_2, x_3) on the Riemann sphere, we get from Eqn (2) that its corresponding point $x+iy$ in the z-plane is given by

$$x = \frac{x_1}{t}, \quad y = \frac{x_2}{t}, \quad t = 1 - x_3 \qquad (4)$$

and by eliminating t we derive

$$x = \frac{x_1}{1-x_3}, \quad y = \frac{x_2}{1-x_3}$$

Example 2: Show that all lines and circles in the z-plane correspond under stereographic projection to circles on the Riemann sphere.

The general equation for a circle/line in the $z = x + iy$ plane is given by

$$A(x^2 + y^2) + Cx + Dy + E = 0 \qquad (5)$$

Substituting for x, y, we get

$$A\left[\left(\frac{x_1}{1-x_3}\right)^2 + \left(\frac{x_2}{1-x_3}\right)^2\right] + \frac{Cx_1}{1-x_3} + \frac{Dx_2}{1-x_3} + E = 0$$

which simplifies to

$$A(x_1^2 + x_2^2) + Cx_1(1-x_3) + Dx_2(1-x_3) + E(1-x_3)^2 = 0$$

Recalling: $x_1^2 + x_2^2 + x_3^2 = 1$ we get

$$A(1-x_3^2) + Cx_1(1-x_3) + Dx_2(1-x_3) + E(1-x_3)^2 = 0$$

$$\Longrightarrow A(1+x_3) + Cx_1 + Dx_2 + E(1-x_3) = 0$$

OR
$$Cx_1 + Dx_2 + (A-E)x_3 + (A+E) = 0 \qquad (6)$$

BUT Eqn (6) is just the equation of a plane in 3-D space. We've thus shown that the projection of a line or circle in the z-plane must lie in the plane described by (6), as well as on the Riemann sphere.

Argument may be reversed; every circle on the Riemann sphere is the projection of either a line or circle in the z-plane. (Prob 10)

Now to look at the North pole $N = (0,0,1)$. Note in eqn (4) how x_3 <u>cannot</u> equal 1. N does not arise as the projection of any point in the complex plane.

Nonetheless, we give meaning to this exceptional point if we think of points in the complex plane that are very large in modulus. Such points project onto points near the north pole, and as

$$|z| \to +\infty$$

their projections tend toward this pole.

In this context we associate with N the extended complex number "∞", and call

$$\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$$

the extended complex plane.

Example 3: Show that the distance (in 3-space) between the projections Z, W of the points z, w in the complex plane is given by

$$\operatorname{dist}(Z,W) = \frac{2|z-w|}{\left(\sqrt{1+|z|^2}\right)\left(\sqrt{1+|w|^2}\right)}$$

(Probs 6, 7) for geometric approach

$$= \frac{2\operatorname{dist}(z,w)}{\sqrt{1+|z|^2}\ \sqrt{1+|w|^2}} \qquad (7)$$

Let (x_1, x_2, x_3) and $(\hat{x}_1, \hat{x}_2, \hat{x}_3)$ denote the coordinates of Z and W, and set $d := \text{dist}(Z, W)$.

Then,

$$d^2 = (x_1 - \hat{x}_1)^2 + (x_2 - \hat{x}_2)^2 + (x_3 - \hat{x}_3)^2$$

Since

$$x_1^2 + x_2^2 + x_3^2 = 1 \quad \text{and} \quad \hat{x}_1^2 + \hat{x}_2^2 + \hat{x}_3^2 = 1$$

$$d^2 = 2\left[1 - (x_1\hat{x}_1 + x_2\hat{x}_2 + x_3\hat{x}_3)\right] \qquad (8)$$

Using $z = x + iy$, $w = u + iv$

$$1 - (x_1\hat{x}_1 + x_2\hat{x}_2 + x_3\hat{x}_3) =$$

$$1 - \frac{4xu}{(|z|^2+1)(|w|^2+1)} - \frac{4yu}{(|z|^2+1)(|w|^2+1)} - \left(\frac{|z|^2-1}{|z|^2+1}\right)\left(\frac{|w|^2-1}{|w|^2+1}\right)$$

$$= \frac{2|z|^2 + 2|w|^2 - 4xu - 4yu}{(|z|^2+1)(|w|^2+1)}$$

$$= 2\frac{(x-u)^2 + (y-v)^2}{(|z|^2+1)(|w|^2+1)} \quad \checkmark \qquad (9)$$

Note:

$$(x-u)^2 + (y-v)^2 = |z-w|^2$$

The Euclidean distance between the projections Z and W that have just been computed is called the chordal distance between the (original) complex numbers z and w and is denoted by χ, that is,

$$\chi[z,w] := \frac{2|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}} \qquad (10)$$

Like the ordinary Euclidean distance between points in the plane, χ is a metric in the sense that it satisfies the triangle inequality

$$\chi[z_1, z_2] \leq \chi[z_1, w] + \chi[w, z_2] \qquad (11)$$

and the familiar identities

$$\chi[z,z] = 0 \qquad \chi[z,w] = \chi[w,z]$$

for any points in the plane.
The χ-metric is also meaningful for the extended complex plane $\hat{\mathbb{C}}$; we calculate the chordal distance from z to ∞ by manipulating (10)

$$\chi[z,\infty] = \lim_{|w|\to\infty} \frac{2|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}}$$

$$= \lim_{|w|\to\infty} \frac{2\left[\frac{z}{w} - 1\right]}{\sqrt{1+|z|^2}\sqrt{\frac{1}{|w|^2}+1}}$$

evaluate limit $\Rightarrow \chi[z,\infty] = \dfrac{2}{\sqrt{1+|z|^2}}$

(12)

This quantifies our earlier image of neighborhoods in $\hat{\mathbb{C}}$; the spherical cap described by

$$\mathrm{dist}(z,N) \equiv \chi[z,\infty] < \rho$$

for

$$\rho < 2 \quad \text{(diameter of sphere)}$$

is the projection of all points in the plane that lie outside the circle

$$|z| = \sqrt{\left(\frac{4}{\rho}\right)^2 - 1}$$