Two Examples of LLN

1. Monte-Carlo Integration

Compute difficult integrals numerically by stochastic simulation and LLN.

$$\int_{\mathbb{R}} f(x)\,dx = \int_{\mathbb{R}} \frac{f(x)}{\pi(x)}\pi(x)\,dx = \mathbb{E}\left\{\frac{f(x)}{\pi(x)}\right\}, \quad x \sim \pi(x)$$

By LLN, if $x_i \sim \pi(x)$ iid, then Monte-Carlo estimate of integral is

$$\hat{I}_N = \frac{1}{N}\sum_{i=1}^{N} \frac{f(x_i)}{\pi(x_i)} \longrightarrow \int_{\mathbb{R}} f(x)\,dx \quad \text{as } N \to \infty$$

Monte-Carlo estimate is unbiased for any N.

$$\mathbb{E}[\hat{I}_N] = \int_{\mathbb{R}} f(x)\,dx$$

$$\mathrm{Var}(\hat{I}_N) \sim \frac{1}{N}$$

2. Average power in an iid signal (SNR calculations)

Recall Power $P_X = \lim_{T \to \infty} \frac{1}{T} \sum_{n=0}^{T} X_n^2$

Therefore, from LLN, average power of iid signal is

$$P_X = \mathbb{E}[X_n^2]$$

If $X_n \in \{x_1, x_2, ..., x_M\}$ iid w/ pmf $f_X(x_i)$, then

$$P_X = \mathbb{E}[X_n^2] = \sum_{m=1}^{M} x_m^2 f_X(x_m)$$

Central Limit Theorem

Let $X_1, X_2, \dots$ be a sequence of iid random variables with finite variance $Var(X_i) = \sigma^2$.

LLN states

$$\mu_N \triangleq \frac{1}{n} \sum_{i=1}^{n} X_i \to \mu \text{ as } n \to \infty \text{ w.p. } 1$$

The CLT states that for large n, μ_n is a Gaussian rv

$$\mu_n \sim \mathcal{N}(\mu, \frac{\sigma^2}{n})$$

Equivalently,

$$\frac{\sqrt{n}}{\sigma}(\mu_n - \mu) \sim \mathcal{N}(0,1)$$

The central limit theorem is best understood as follows:

Suppose $X_1, X_2, \ldots$ iid w/ $\mathbb{E}[X_i] = 0$, $Var(X_i) = \sigma^2$.

Then

LLN

$$\frac{1}{n}\sum_{i=1}^{n} X_i \to 0$$

CLT

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i \to N(0, \sigma^2)$$

i.e. the sum of independent rvs behaves as a Gaussian!

Remarks

1) X_i can have ANY density - as long as they're iid w/ finite variance. Can be extended to Markov.
2) This is why noise is modeled as Gaussian!

Multivariate Central Limit Theorem

First note that an m-variate Gaussian vector $X \sim N(\mu, \Sigma)$ has pdf

$$\frac{1}{(2\pi)^{m/2}\, |\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(\vec{x}-\vec{\mu})^T \Sigma^{-1} (\vec{x}-\vec{\mu})\right), \quad \vec{x} \in \mathbb{R}^m$$

where

$$\vec{\mu} = \mathbb{E}[\vec{x}], \qquad \Sigma = \mathbb{E}\left[(x-\vec{\mu})^T (x-\vec{\mu})\right]$$

Note symmetric, real $\Rightarrow$ positive definite

Let $X_1, X_2, \ldots$ be a sequence of iid random vectors w/ mean $\mathbb{E}[X_k] = \mu \in \mathbb{R}^m$, and covariance $\Sigma = \mathbb{E}[(X_k - \vec{\mu})^T (X_k - \vec{\mu})]$

Define

$$\mu_n \triangleq \frac{1}{n} \sum_{k=1}^{n} X_k$$

then for large n

$$\sqrt{n}\, \Sigma^{-1/2} (\mu_n - \mu) \sim N(0, I_{m \times m})$$

Example

Consider m dim vector $Y_n = \sqrt{n}\, \Sigma^{-1/2} (\mu_n - \mu)$.

Then

$$T = Y_n' Y_n \sim \chi_m^2 \qquad \text{(for large } n\text{)}$$

Sum of squares of m iid Gaussian rvs has χ_m^2 CDF.

So

$$\Pr(T > t) = 1 - \chi_m^2(t)$$