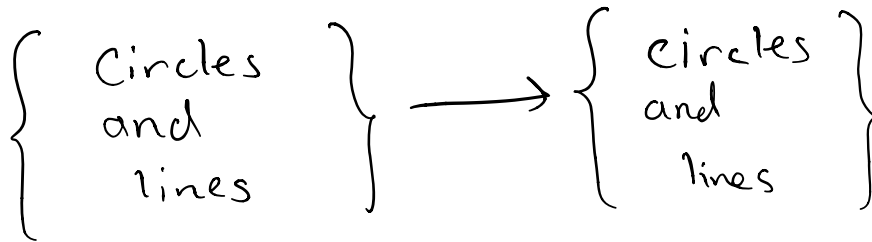
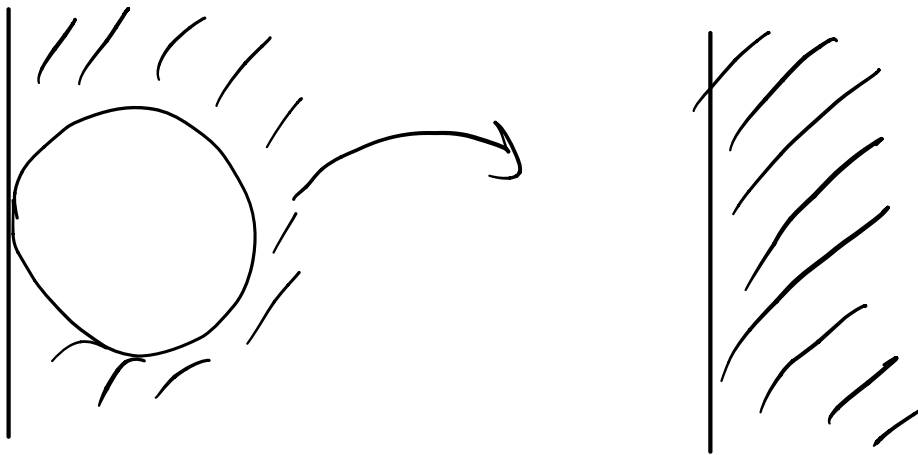


The "inversion" map $w = 1/z$

Useful for problems involving lines and circles as parts of boundaries.

From last time



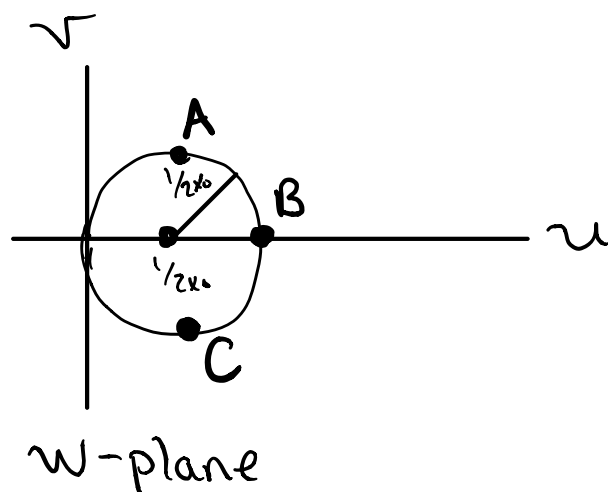
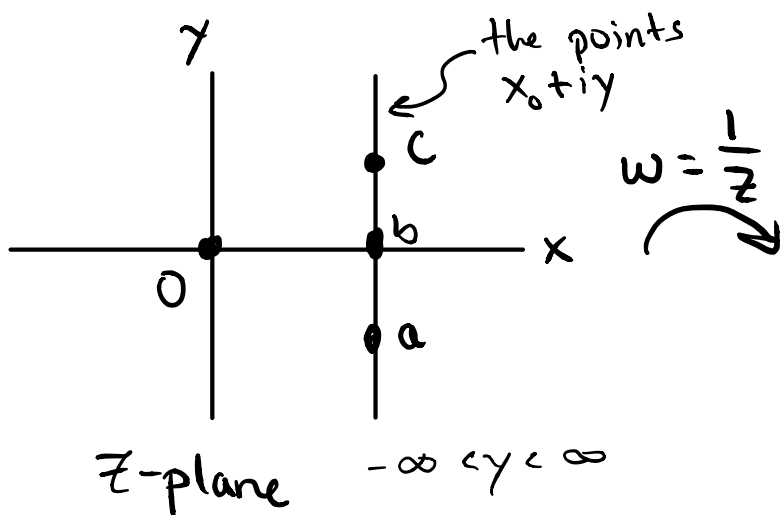
Circles that do contain $0 \longrightarrow$ lines that do not contain 0

Circles that do NOT contain $0 \longrightarrow$ circles that do not contain 0

lines that do contain $0 \longrightarrow$ lines that contain 0

lines that do NOT contain $0 \longrightarrow$ circles that ^{do} contain 0

First: Consider effect of $1/z$ on vertical lines



$$w = \frac{1}{x_0 + iy} = \frac{x_0 - iy}{x_0^2 + y^2}$$

$$u(x, y) = \frac{x_0}{x_0^2 + y^2}$$

$$v(x, y) = \frac{-y}{x_0^2 + y^2}$$

} show this is a circle

$$u^2 + v^2 = \frac{(x_0^2 + y^2)}{(x_0^2 + y^2)^2} = \frac{1}{x_0^2 + y^2}$$

Note that: $\frac{u}{x_0} = \frac{1}{x_0^2 + y^2}$

$$u^2 + v^2 = \frac{u}{x_0} \xrightarrow{\text{Rearrange}} v^2 + \left(u^2 - \frac{u}{x_0}\right) = 0$$

$$\xrightarrow{\text{complete the square}} v^2 + \left(u - \frac{1}{2x_0}\right)^2 = \left(\frac{1}{2x_0}\right)^2$$

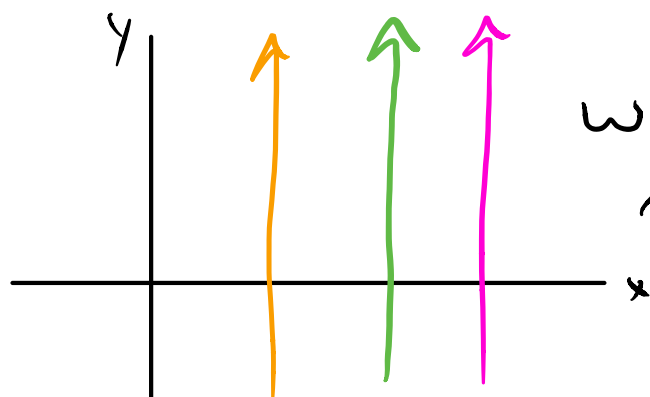
This is a circle of radius

$$\frac{1}{2x_0}$$

at center

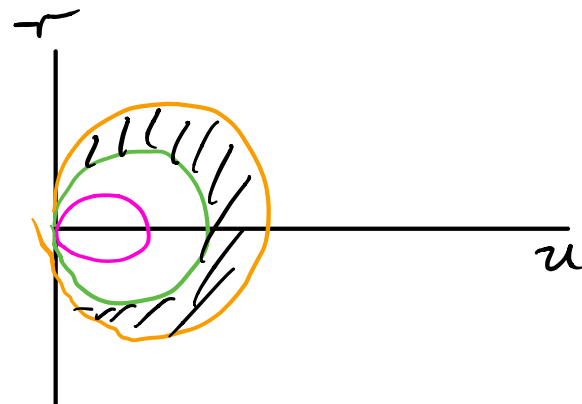
$$u_0 = \frac{1}{2x_0}, \quad v_0 = 0$$

Example 2



z-plane

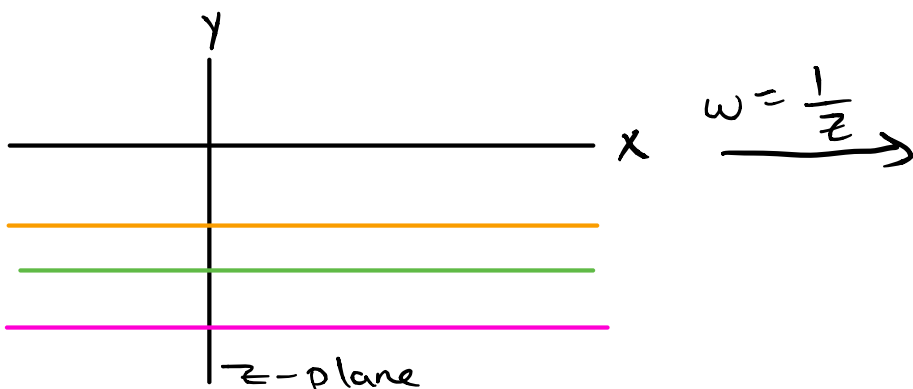
$$w = 1/z$$



w-plane

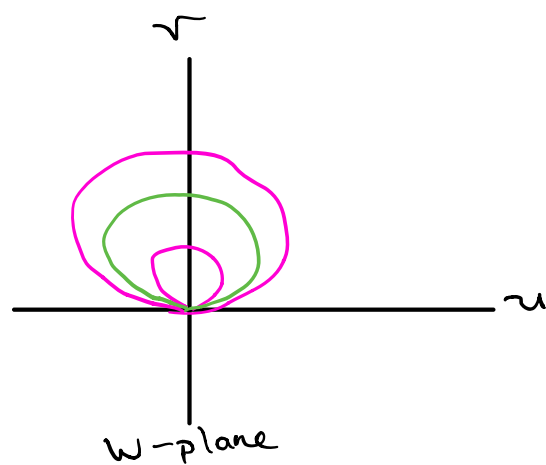
So we can now solve $\nabla^2 \phi = 0$ in a crescent (see w-plane above)

Example 3 (Horizontal Lines)



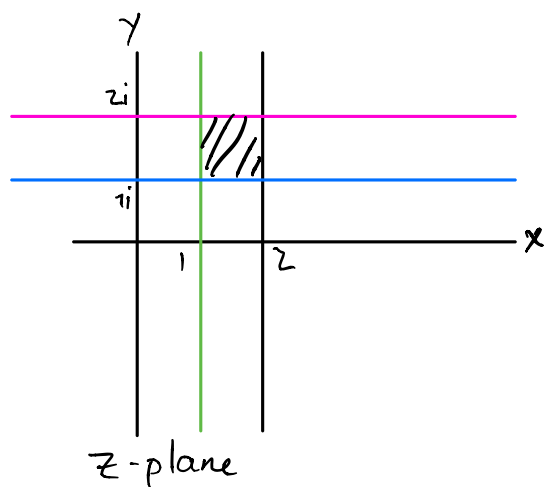
z-plane

$$w = 1/z$$



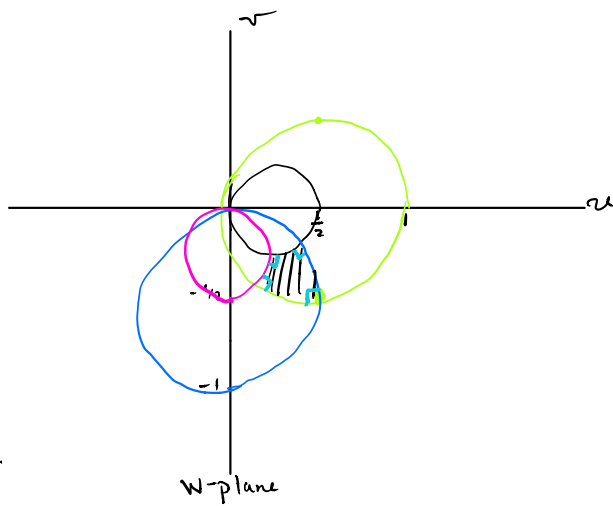
w-plane

Example 4 (image of a square):



$$w = \frac{1}{z}$$

Note the
Conformity!



From Handout

Top image = Image of circles at $(x,y) = (1,0)$,
mapped into w -plane via $w = 1/z$.

Circles

$$x = \frac{u}{u^2 + v^2}$$

$$y = \frac{-v}{u^2 + v^2}$$

$$(x-1)^2 + y^2 = R^2$$

$$x + iy = z = \frac{1}{w} = \frac{1}{u + iv} = \frac{u - iv}{u^2 + v^2}$$

$$x^2 + 1 - 2x + y^2 = R^2$$

$$x^2 + y^2 = \frac{1}{u^2 + v^2}$$

$$\rightarrow \frac{1}{u^2+v^2} - 2\left(\frac{u}{u^2+v^2}\right) + 1 = R^2$$

$$\rightarrow \left(u + \frac{1}{R^2-1}\right)^2 + v^2 = \left(\frac{R}{R^2-1}\right)^2 \left\{ \begin{array}{l} \text{Circle,} \\ \text{radius } \frac{R}{R^2-1} \\ \text{center } u_0 = \frac{1}{1-R^2} \end{array} \right.$$

$v_0 = 0$