

6.1

$$7) \oint_{|z|=1} e^{1/z} \sin\left(\frac{1}{z}\right) dz$$

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{z^2} \frac{1}{2!} + \frac{1}{z^3} \frac{1}{3!} + \dots + \frac{1}{z^n} \frac{1}{n!}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n \frac{1}{n!} = \sum_{n=0}^{\infty} \frac{z^{-n}}{n!} \quad \left. \begin{array}{l} \text{Laurent Series} \\ \text{of } e^{1/z} \end{array} \right\}$$

$$\sin\left(\frac{1}{z}\right) = \frac{1}{z} - \frac{\left(\frac{1}{z}\right)^3}{3!} + \frac{\left(\frac{1}{z}\right)^5}{5!} - \frac{\left(\frac{1}{z}\right)^7}{7!} + \dots -$$

$$= \sum_{n=0}^{+\infty} \left(\frac{1}{z}\right)^{2n+1} \frac{1}{(2n+1)!} (-1)^n = \sum_{n=0}^{+\infty} \frac{z^{-(2n+1)}}{(2n+1)!} (-1)^n \quad \left. \begin{array}{l} \text{Laurent Series} \\ \text{of } \sin\left(\frac{1}{z}\right) \end{array} \right\}$$

$$\text{So } e^{1/z} \sin\left(\frac{1}{z}\right)$$

$$= \left(\sum_{n=0}^{+\infty} \frac{z^{-n}}{n!} \right) \left(\sum_{n=0}^{+\infty} \frac{z^{-(2n+1)} (-1)^n}{(2n+1)!} \right)$$

$$= \sum_{n=0}^{+\infty} \frac{(z^{-n})(z^{-2n})(z^{-1})}{(n!)(2n+1)!} (-1)^n = \sum_{n=0}^{+\infty} \frac{(z^{-3n-1})}{n!(2n+1)!} (-1)^n$$

$$\Rightarrow \oint_{|z|=1} \sum_{n=0}^{+\infty} \frac{z^{-(3n+1)}}{n!(2n+1)!} (-1)^n dz \quad \left. \begin{array}{l} \text{What our} \\ \text{integral becomes} \end{array} \right\}$$

Interchanging \sum and \oint ,

$$\sum_{n=0}^{+\infty} \frac{(-1)^n}{n!(2n+1)!} \oint_{|z|=1} \frac{1}{(z)^{3n+1}} dz$$

Note:

$$\oint_{\gamma} (z-a)^n dz \stackrel{a \in \mathbb{C}}{=} \begin{cases} 0, & \text{else} \\ 2\pi i, & n \neq -1 \end{cases}$$

It's easy to see we only care for $\frac{1}{z}$ term.

$$\Rightarrow 3n+1 = 1 \\ n = 0$$

So our residue term is when $n=0$!

$$\frac{(-1)^0}{0!(1!)} \oint_{|z|=1} \frac{1}{z} dz = \boxed{2\pi i}$$

$$3) \int_0^{\pi} \frac{d\theta}{(3+2\cos\theta)^2} = \frac{3\pi\sqrt{5}}{25}$$

6.2

$$z = e^{i\theta}$$

$$dz = ie^{i\theta} d\theta \Rightarrow d\theta = \frac{1}{i} \frac{dz}{z}$$

$$\cos\theta = \frac{z + 1/z}{2}$$

Note: $\int_0^{\pi} \frac{d\theta}{3+2\cos\theta} = \frac{1}{2} \int_0^{2\pi} \frac{d\theta}{3+2\cos\theta}$

So our integral becomes

$$\frac{1}{2} \int_0^{2\pi} \frac{\frac{dz}{iz}}{\left[3 + 2\left(\frac{z + 1/z}{2}\right)\right]^2} = \frac{1}{2iz} \int_0^{2\pi} \frac{z}{(3z + z^2 + 1)^2} dz$$

The roots of
 $z^2 + 3z + 1$

given by the quadratic formula are

$$z_{1,2} = \frac{-3 \pm \sqrt{9-4}}{2} \Rightarrow z_1 = \frac{-3 + \sqrt{5}}{2}; z_2 = \frac{-3 - \sqrt{5}}{2}$$

Therefore our integral
becomes

$$\frac{1}{2iz} \int \frac{dz}{[(z-z_1)(z-z_2)]^2} \quad |z|=1$$

Side work

$$\frac{(z^2 + 3z + 1)^2}{z^2}$$

denominator

Only z_1 lies in our contour. Therefore z_1 is a pole of order 2.

$$\Rightarrow \text{Res}(f(z); z_1) = \lim_{z \rightarrow z_1} \frac{1}{(m-1)!} \left(\frac{d}{dz} \right)^{m-1} (z - z_1)^m f(z)$$

$$= \lim_{z \rightarrow z_1} \frac{d}{dz} \left[(z - z_1)^2 f(z) \right]$$

$$z_1 - z_2 = \sqrt{5}$$

$$= \lim_{z \rightarrow z_1} \frac{d}{dz} \left[\frac{z}{(z - z_2)^2} \right]$$

$$= \lim_{z \rightarrow z_1} \left[\frac{(z - z_2)^2 - 2z(z - z_2)}{(z - z_2)^4} \right]$$

$$= \lim_{z \rightarrow z_1} \left[\frac{(z - z_2) - 2z}{(z - z_2)^3} \right]$$

$$= \frac{\sqrt{5} + 3 - \sqrt{5}}{(\sqrt{5})^3} = \frac{3}{5\sqrt{5}} = \frac{3\sqrt{5}}{25}$$

$$\text{So } \frac{1}{2i} \oint_{|z|=1} \frac{dz}{(z - z_1)^2 (z - z_2)^2} = \frac{2\pi i}{2i} 3 \frac{\sqrt{5}}{25} = \boxed{\frac{3\pi\sqrt{5}}{25}}$$

$$9) \int_0^{2\pi} (\cos \theta)^{2n} d\theta = \frac{\pi (2n)!}{2^{2n-1} (n!)^2} \quad n=1, 2, \dots$$

Wowza, let's begin

$$\cos \theta = \frac{z + \frac{1}{z}}{2}$$

$$z = e^{i\theta} \quad d\theta = \frac{dz}{iz}$$

$$\int \left(\frac{z + \frac{1}{z}}{2} \right)^{2n} \frac{dz}{iz}$$

$$z + \frac{1}{z} = \frac{z^2 + 1}{z}$$

$$|z|=1$$

$$\frac{1}{i} \int \left(\frac{z^2 + 1}{2z} \right)^{2n} \frac{dz}{z}$$

$$|z|=1$$

$$\frac{1}{2^{2n} i} \int \frac{(z^2 + 1)^{2n}}{z^{2n+1}} dz$$

$$|z|=1$$

Our answers beginning to take shape!

$$\frac{2\pi i}{2^{2n} i} \sum \operatorname{Res}(\text{Singularities inside } |z|=1)$$

$$= \frac{\pi}{2^{2n-1}} \sum \operatorname{Res}(\text{Singularities inside } |z|=1)$$

$\frac{(z^2+z)^{2n}}{z^{2n+1}}$ has a pole of order $(2n+1)$ at $z=0$.

$$\text{So } \operatorname{Res}\left(\frac{(z^2+z)^{2n}}{z^{2n+1}} ; 0\right) = \lim_{z \rightarrow 0} \frac{1}{(2n)!} \left(\frac{d}{dz}\right)^{2n} (z-z_0)^{2n+1} f(z)$$

$$= \lim_{z \rightarrow 0} \frac{1}{(2n)!} \frac{d^{2n}}{dz^{2n}} \left[(z^2+z)^{2n} \right] = \boxed{\frac{1}{2n!} \left(\frac{(2n)!}{n!} \right)^2 = \frac{(2n)!}{(n!)^2}}$$

Our answer is therefore

$$\boxed{\frac{\pi}{2^{2n-1}} \frac{(2n)!}{(n!)^2}, n=1, 2, 3, \dots}$$

Not obvious but
must be true

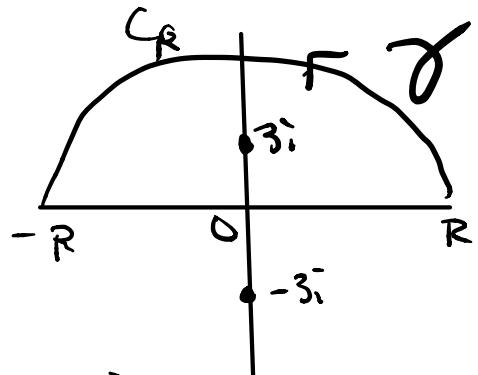
26.3

$$2) \int_{-\infty}^{+\infty} \frac{x^2}{(x^2+9)^2} dx = \frac{\pi}{6}$$

Have

$$\lim_{R \rightarrow \infty} \int_{-R}^{+R} \frac{x^2}{(x^2+9)^2} dx$$

$$\int_{\gamma} \frac{z^2}{(z^2+9)^2} dz = \lim_{R \rightarrow \infty} \int_{-R}^{+R} \frac{x^2}{(x^2+9)^2} dx + \int_{C_R} \frac{z^2}{(z^2+9)^2} dz$$



$$= 2\pi i \sum \text{Res}(\text{isolated singularities inside } \gamma)$$

$$= 2\pi i \lim_{z \rightarrow 3i} \frac{1}{1!} \left(\frac{d}{dz} \right)^1 \cancel{(z-3i)^2} \frac{z^2}{(z+3i)^2 \cancel{(z-3i)^2}}$$

$$= 2\pi i \lim_{z \rightarrow 3i} \frac{d}{dz} \left[\frac{z^2}{(z+3i)^2} \right] \rightarrow \frac{2(z+3i)^{-2} z - 2(z+3i)^{-1} z^2}{(z+3i)^3}$$

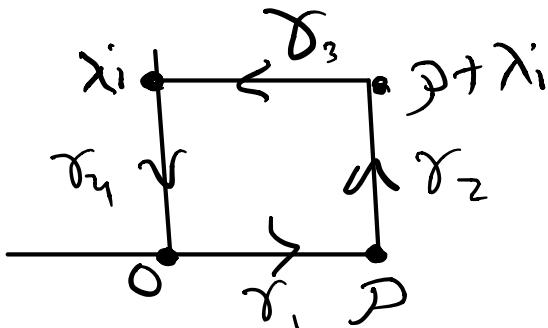
$$= 2\pi i \lim_{z \rightarrow 3i} \left[\frac{(z+3i)^2 2z - z^2 2(z+3i)}{(z+3i)^4} \right]$$

$$= 2\pi i \left[\frac{(6i)^3 - 2(3i)^2(6i)}{(6i)^4} \right] = 2\pi i \left[\frac{(6i)^2 - 2(3i)^2}{(6i)^3} \right]$$

$$= 2\pi i \left[\frac{-36 + 18}{-216i} \right] = 2\pi i \frac{-18}{-216i} = \boxed{\frac{\pi}{6}}$$

$$10a) \quad I = \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

integrate e^{-z^2} around γ ;
 w/ $\lambda > 0$, and let $P \rightarrow \infty$
 to derive



$$\int_0^\infty e^{-x^2} \cos(2\lambda x) dx = \frac{\sqrt{\pi}}{2} e^{-\lambda^2}$$

e^{-z^2} is entire

$$\int_\gamma e^{-z^2} dz = \boxed{\int_{\gamma_1} e^{-z^2} dz} + \boxed{\int_{\gamma_2} e^{-z^2} dz} + \boxed{\int_{\gamma_3} e^{-z^2} dz} + \boxed{\int_{\gamma_4} e^{-z^2} dz} = 0$$

$$\int_{\gamma_1} e^{-z^2} dz = \lim_{P \rightarrow \infty} \int_0^P e^{-x^2} dx = I$$

$$z = x$$

$$dz = dx$$

$$\int_{\gamma_2} e^{-z^2} dz = \lim_{P \rightarrow \infty} \int_0^P e^{-z^2} dz$$

$$z = P + iy \rightarrow dz = idy$$

$$= \lim_{P \rightarrow \infty} \int_0^{\lambda - (P+iy)^2} e^{-(P+iy)^2} i dy$$

$$= \lim_{P \rightarrow \infty} \int_0^{\lambda} e^{-P^2} e^{y^2} e^{-2ipy} i dy$$

As $P \rightarrow \infty$ $e^{-P^2} \rightarrow 0$ and dominates so this integral goes to zero!

$$z = x + \lambda i$$

$$dz = dx$$

$$\int_{\gamma_3} e^{-z^2} dz = \lim_{P \rightarrow \infty} \int_P^0 e^{-z^2} dx$$

$$= -\lim_{P \rightarrow \infty} \int_0^P e^{-(x+i\lambda)^2} dx = -\lim_{P \rightarrow \infty} \int_0^P e^{\lambda^2} e^{-x^2} e^{-2ix\lambda} dx$$

$$\int_{\gamma_4} e^{-z^2} dz, \quad z = iy$$

$$dz = idy$$

$$= \int_{\gamma}^0 e^{-(iy)^2} i dy = - \int_0^{\lambda} e^{y^2} i dy$$

We can ignore this integral on γ since it is entirely imaginary. Our loop sums to zero so the real part must be zero which is what we care about

So,

$$\begin{aligned} \int_{\gamma} &= \boxed{\int_{\gamma_1} e^{-z^2} dz} + \boxed{\int_{\gamma_2} e^{-z^2} dz} + \boxed{\int_{\gamma_3} e^{-z^2} dz} + \boxed{\int_{\gamma_4} e^{-z^2} dz} \\ &= \frac{\sqrt{\pi}}{2} + 0 - \lim_{P \rightarrow \infty} \int_0^P e^{\lambda^2} e^{-x^2} e^{-2ix\lambda} dx = 0 \end{aligned}$$

Only want real part!

(sin part $\rightarrow 0$
b/c sin odd)

$$\lim_{P \rightarrow \infty} \int_0^P e^{\lambda^2} e^{-x^2} \cos(2\lambda x) dx = -\frac{\sqrt{\pi}}{2}$$

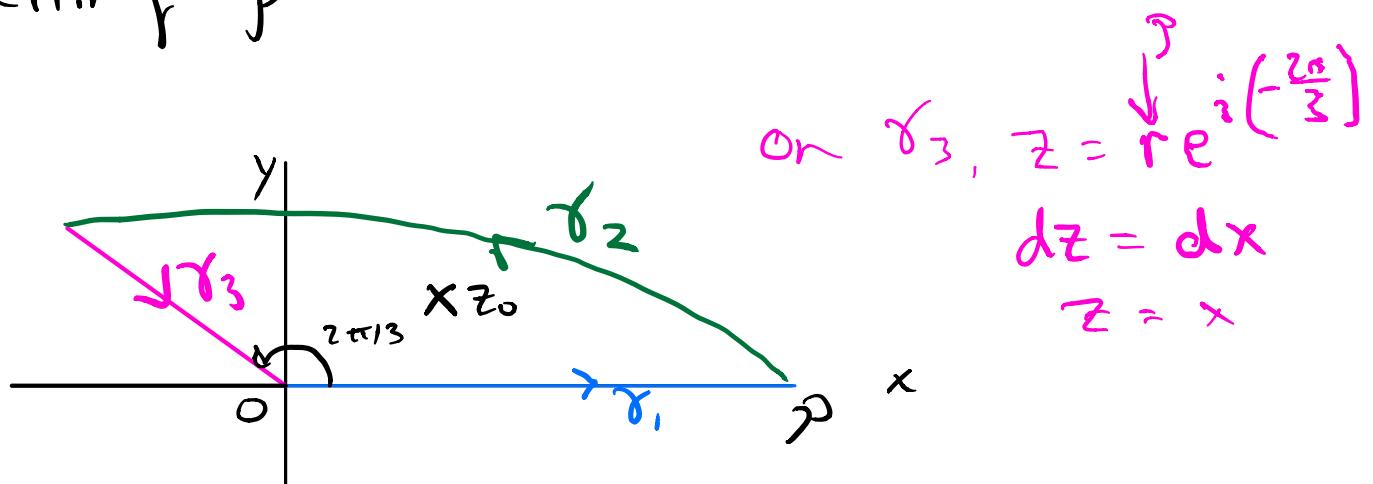
$$\int_0^{\infty} e^{-x^2} \cos(2\lambda x) dx = e^{-\lambda^2} \frac{\sqrt{\pi}}{2} !$$

11) Show

$$I = \int_0^\infty \frac{dx}{x^3 + 1} = \frac{2\pi\sqrt{3}}{9}$$

by integrating $\frac{1}{z^3 + 1}$ around the boundary of the circular sector

$S_p : \{z = re^{i\theta} : 0 \leq \theta \leq 2\pi/3, 0 \leq r \leq p\}$
and letting $p \rightarrow \infty$



$$\int_{\gamma} \frac{1}{z^3 + 1} dz = \int_{\gamma_1} \frac{1}{z^3 + 1} dz + \int_{\gamma_2} \frac{1}{z^3 + 1} dz + \int_{\gamma_3} \frac{1}{z^3 + 1} dz$$

$$\text{Let } f(z) = \frac{1}{z^3 + 1}.$$

f has poles at $z_k = e^{i\frac{\pi}{3} + \frac{2\pi}{3}k}$, $k = 0, 1, 2$

Z_0 lies in our contour. $Z_0 = e^{i\pi/3}$

$$2\pi i \operatorname{Res}(f; z_0) = \int_{\gamma_1} \frac{1}{z^3+1} dz + \int_{\gamma_2} \frac{1}{z^3+1} dz + \int_{\gamma_3} \frac{1}{1+z^3} dz$$

$$\int_{\gamma_1} \frac{1}{z^3+1} dz = \lim_{P \rightarrow \infty} \int_0^P \frac{1}{x^3+1} dx = I$$

$$z = x$$

$$dz = dx$$

$$\int_{\gamma_2} \frac{1}{z^3+1} dz \underset{\sim}{=} \lim_{r \rightarrow P \rightarrow \infty} \int_0^{\frac{2\pi}{3}} \frac{1}{(re^{i\theta})^3+1} rie^{i\theta} d\theta$$

$\underbrace{\hspace{10em}}$

$z = re^{i\theta}$ Clearly this goes to zero !
 $dz = rie^{i\theta} d\theta$

$$\frac{rie^{i\theta}}{r^3 e^{i3\theta}} \rightarrow 0 \text{ as } r \rightarrow \infty$$

$$\int_{\gamma_3} \frac{1}{1+z^3} dz = \lim_{P \rightarrow \infty} \int_P^0 \frac{1}{1+r^3 e^{i(2\pi)}} e^{i(\frac{2\pi}{3})} dx$$

$$z = re^{(i\frac{2\pi}{3})}$$

$$dz = dre^{(i\frac{2\pi}{3})}$$

So,

$$2\pi i \operatorname{Res}(f; z_0) = \int_{\gamma_1} \frac{1}{z^3+1} dz + \int_{\gamma_2} \frac{1}{z^3+1} dz + \int_{\gamma_3} \frac{1}{1+z^3} dz$$

$$2\pi i \operatorname{Res}(f; z_0) = I - I e^{i 2\pi/3}$$

$$\lim_{z \rightarrow z_0} \frac{(z - z_0)}{1 + z^3} \xrightarrow{\text{L'Hopital's}} \lim_{z \rightarrow z_0} \frac{1}{3z^2} = \frac{1}{3e^{i 2\pi/3}}$$

So

$$\frac{2\pi i}{3e^{i 2\pi/3}} = I (1 - e^{i 2\pi/3})$$

$$I = \frac{2\pi i}{3} \cdot \frac{1}{e^{i 2\pi/3} (1 - e^{i 2\pi/3})}$$

$$= \frac{2\pi i}{3} \cdot \frac{1}{e^{i 2\pi/3} - e^{i 4\pi/3}}$$

$$= \frac{2\pi i}{3} \cdot \frac{1}{\sqrt{3}i} =$$

$$\boxed{\frac{2\pi\sqrt{3}}{9}}$$

