POSSIBLY USEFUL INFORMATION

1. A geometric random variable X with parameter p takes on integer values and has pmf

$$p_X(k) = \begin{cases} p(1-p)^{k-1} & \text{when } k > 0 \\ 0 & \text{when } k < 0 \ . \end{cases}$$

$\mathbb{E}(X) = 1/p$ and $\text{Var}(X) = (1-p)/p^2$.

2. An exponential random variable X with rate parameter $\lambda > 0$ takes on real values and has pdf

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{when } x \geq 0 \\ 0 & \text{when } x < 0 \ . \end{cases}$$

$\mathbb{E}(X) = 1/\lambda$ and $\text{Var}(X) = 1/\lambda^2$.

3 A random variable X uniform on $[a, b]$, where a and b are real numbers with $a < b$, has pdf

$$f_X(x) = \begin{cases} 1/(b-a) & \text{when } x \in [a, b] \\ 0 & \text{otherwise.} \end{cases}$$

$\mathbb{E}(X) = (a+b)/2$ and $\text{Var}(X) = (b-a)^2/12$.

4. A Gaussian random variable with mean μ and variance σ^2 has pdf

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2} \quad \text{for all} \quad x \in \mathbb{R} \ .$$

The cdf of the standard normal random variable, i.e. the Gaussian with zero mean and variance 1, is

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt \ .$$

1. (22 points) First suppose X is Gaussian with mean μ and variance σ^2, and $Y = aX + b$, where $a \neq 0$ and b are real numbers.

(a) Find $\mathbb{E}(Y)$ and $\mathrm{Var}(Y)$.
(b) Is Y necessarily Gaussian? A simple yes or no suffices.
(c) Write $\mathbb{P}(\{Y > 17\})$ in terms of Φ defined on the previous page.

Now suppose X is exponential with rate parameter λ, Y is independent of X and is exponential with rate parameter ζ, $W = XY$, and $Z + X + Y$. All these random variables are defied on the same probability space.

d ~~(c)~~ Find $\mathbb{E}(W)$ and $\mathrm{Var}(W)$.
e ~~(d)~~ Find $\mathbb{E}(Z)$ and $\mathrm{Var}(Z)$.
f ~~(e)~~ Find $\mathbb{E}(Z \mid \{Y > 1\})$.

(a) $\mathbb{E}[Y] = \mathbb{E}[aX+b] = a\mathbb{E}[X] + b = a\mu + b$

$\mathrm{Var}(Y) = \mathrm{Var}(aX+b) = a^2 \mathrm{Var}(X) = a^2\sigma^2$

(b) Y is Gaussian$(a\mu+b, a^2\sigma^2)$

(c) $\mathbb{P}(Y>17) = 1 - \mathbb{P}(Y \le 17)$

$= 1 - \mathbb{P}(aX+b \le 17)$

$$= \begin{cases} 1 - \mathbb{P}\left(X \le \frac{17-b}{a}\right), & a>0 \\ 1 - \mathbb{P}\left(X > \frac{17-b}{a}\right), & a<0 \end{cases}$$

$Z \sim N(0,1)$

Then $X = \sigma Z + \mu$

$$= \begin{cases} 1 - \mathbb{P}\left(Z \le \frac{\frac{17-b}{a} - \mu}{\sigma}\right), & a>0 \\ 1 - \mathbb{P}\left(Z > \frac{\frac{17-b}{a} - \mu}{\sigma}\right), & a<0 \end{cases} = \begin{cases} 1 - \Phi\left(\frac{\frac{17-b}{a} - \mu}{\sigma}\right), & a>0 \\ \Phi\left(\frac{\frac{17-b}{a} - \mu}{\sigma}\right), & a<0 \end{cases}$$

$X \sim \text{Exponential}(\lambda)$ $\quad W = XY \quad$ $(X, Y \text{ independent})$

$Y \sim \text{Exponential}(\varphi)$ $\quad Z = X+Y$

(d) $\mathbb{E}(W) = \mathbb{E}(XY) \overset{\text{independence}}{=} \mathbb{E}(X)\mathbb{E}(Y) = \frac{1}{\lambda} \cdot \frac{1}{\varphi}$

$\text{Var}(W) = \text{Var}(XY) = \mathbb{E}[X^2]\,\mathbb{E}[Y^2] - (\mathbb{E}[X]\,\mathbb{E}[Y])^2$

$\mathbb{E}[X^2] = \text{Var}(X) + (\mathbb{E}(X))^2$

$\mathbb{E}[Y^2] = \text{Var}(Y) + (\mathbb{E}(Y))^2$

$$= \left(\frac{2}{\lambda^2}\right)\left(\frac{2}{\varphi^2}\right) - \left(\frac{1}{\lambda^2 \varphi^2}\right)$$

$$= \frac{3}{\lambda^2 \varphi^2}$$

(e) $\mathbb{E}(Z) = \mathbb{E}(X+Y) \overset{\text{Linearity of Expectation}}{=} \mathbb{E}(X) + \mathbb{E}(Y)$

$\text{Var}(Z) = \text{Var}(X+Y) \overset{\text{independence}}{=} \text{Var}(X) + \text{Var}(Y)$

$$= \frac{1}{\lambda^2} + \frac{1}{\varphi^2}$$

(f) $\mathbb{E}(Z \mid \{Y > 1\}) = \mathbb{E}((X+Y) \mid \{Y>1\})$

$\mathbb{P}(Y>1) = 1 - \mathbb{P}(Y \le 1)$
$= 1 - F_Y(1)$
$= 1 - (1 - e^{-\varphi})$
$= e^{-\varphi}$

$f_{Y \mid \{Y>1\}}(y) = \begin{cases} \frac{f_Y(y)}{\mathbb{P}(A)}, & y>1 \\ 0, & o/w \end{cases}$

$= \mathbb{E}(X \mid \{Y>1\}) + \mathbb{E}(Y \mid \{Y>1\})$

$= \mathbb{E}(X) + \mathbb{E}(Y \mid \{Y>1\})$

$= \frac{1}{\lambda} + \mathbb{E}(Y \mid \{Y>1\})$

$$\mathbb{E}(Y \mid \{Y>1\}) = \int_{-\infty}^{+\infty} y\, f_{Y\mid\{Y>1\}}(y)\, dy$$

$$= e^{\varphi} \int_1^{\infty} \varphi e^{-\varphi y}\, y\, dy = e^{\varphi} \int_{t=0}^{t=+\infty} \varphi e^{-(t+1)\varphi}(t+1)\, dt = \mathbb{E}(Y+1)$$
$$= 1 + \frac{1}{\varphi}$$

Thus, $\mathbb{E}(Z \mid \{Y>1\}) = 1 + \frac{1}{\lambda} + \frac{1}{\varphi}$

2. (18 points) X and Y are jointly continuous random variables with joint pdf

$$f_{X,Y}(x,y) = \begin{cases} \frac{7}{13}e^{-7x} & \text{when } 0 \le y \le 13 \text{ and } x \ge 0 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the marginal pdf $f_X(x)$.
(b) Find the marginal pdf $f_Y(y)$.
(c) Are X and Y independent? Why or why not?

(a) $f_X(x) = \begin{cases} \int_{y=-\infty}^{y=+\infty} \frac{7}{13} e^{-7x}\, dy, & x \geq 0 \\ 0, & o/w \end{cases} = \begin{cases} 7e^{-7x}, & x \geq 0 \\ 0, & \text{else} \end{cases}$

(b) $f_Y(y) = \begin{cases} \int_{x=-\infty}^{x=+\infty} \frac{7}{13} e^{-7x}\, dx, & 0 \leq y \leq 13 \\ 0, & o/w \end{cases} = \begin{cases} \frac{1}{13}, & 0 \leq y \leq 13 \\ 0, & o/w \end{cases}$

(c) X, Y independent $\iff f_{X,Y}(x,y) = f_X(x) f_Y(y)$

$f_X(x) f_Y(y) = \begin{cases} \frac{7}{13} e^{-7x}, & 0 \leq y \leq 13,\ x \geq 0 \\ 0, & o/w \end{cases} = f_{X,Y}(x,y)$

Thus independent.

3. (20 points) Q is a discrete random variable distributed uniformly on the set

$$\{1/n, 2/n, 3/n, \ldots, (n-1)/n\},$$

where $n > 2$ is a given integer.

(a) You flip a coin with probability Q of heads repeatedly and independently until heads comes up., Let X be the index of the first heads flip. Find $\mathbb{E}(X)$.

(b) You keep on flipping the same coin until a second heads comes up. Let Y be the index of the second heads flip. Find $\mathbb{E}(Y)$.

(c) Let A be the event $\{Q = 1/n\} \cap \{X = 7\}$. Find

- $p_{Y|A}(3)$
- $p_{Y|A}(13)$
- $\mathbb{E}(Y \mid A)$

$Q \sim$ Discrete uniform ; $n > 2$

X = index of first flip

$p_X(k) =$

$p_Q(k) = \frac{1}{n-1}$

(a) $\mathbb{E}(X) = \mathbb{E}[\mathbb{E}(X|Q)]$

$\mathbb{E}[X|Q=q] = \frac{1}{q}$

$\mathbb{E}[X|Q] = 1/Q$

$\mathbb{E}[\mathbb{E}[X|Q]] = \mathbb{E}[1/Q]$

$\frac{1}{Q} = \{n, \frac{n}{2}, \ldots, \frac{n}{n-1}\}$

also discrete uniform

$\mathbb{E}[\frac{1}{Q}] = \frac{1}{n-1}\left(n + \frac{n}{2} + \cdots + \frac{n}{n-1}\right)$ ← avg on interval

(b) Y = index of second flip. $\mathbb{E}(Y)$?

Let $Y = Z + X$. Z, X distributed identically ; Independent.

So $\mathbb{E}(Y) = 2\,\mathbb{E}(X) = \frac{2}{n-1}\left(n + \frac{n}{2} + \cdots + \frac{n}{n-1}\right)$

(c) Let $A = \{Q = \frac{1}{n}\} \cap \{X = 7\}$

$P_{Y|A}(y) =$ Probability second head occurs at y given first head occurs at flip 7; probability $\frac{1}{n}$ coin

- $P_{Y|A}(3) = 0$; second head must be after first at $x = 7$

- $P_{Y|A}(13)$?

$X = 7$. $y = 13$.

Thus we flip tails 5 times then heads the 6^{th}.

So,

$$P_{Y|A}(13) = \frac{1}{n}\left(1 - \frac{1}{n}\right)^5$$

- $\mathbb{E}(Y|A) = \sum y P_{Y|A}$

$Y|A = Y + 7$

$\mathbb{E}[Y|A] = \mathbb{E}[Y + 7] = 7 + \frac{1}{1/n} = 7 + n$

4. (15 points) Gandalf flips a fair coin. If it comes up heads, he spins a spinner whose outcomes are A, B, C, and D with respective probabilities 1/2, 1/4, 1/8, and 1/8. If it comes up tails, he spins a different spinner whose outcomes are 1, 2, 3, 4, 5, and 6 with respective probabilities 1/2, 1/4, 1/16, 1/16, 1/16, and 1/16. Frodo wants to determine the outcome of Gandalf's experiment by asking Gandalf at most ten Yes-No questions.

(a) Why do we know trivially that, given these rules, Frodo will always be able to determine the outcome? Please don't mention the word entropy.

(b) Frodo wants to be efficient, i.e. to ask as few questions on average as possible. What's a good first question for Frodo to ask? Justify your answer with one sentence.

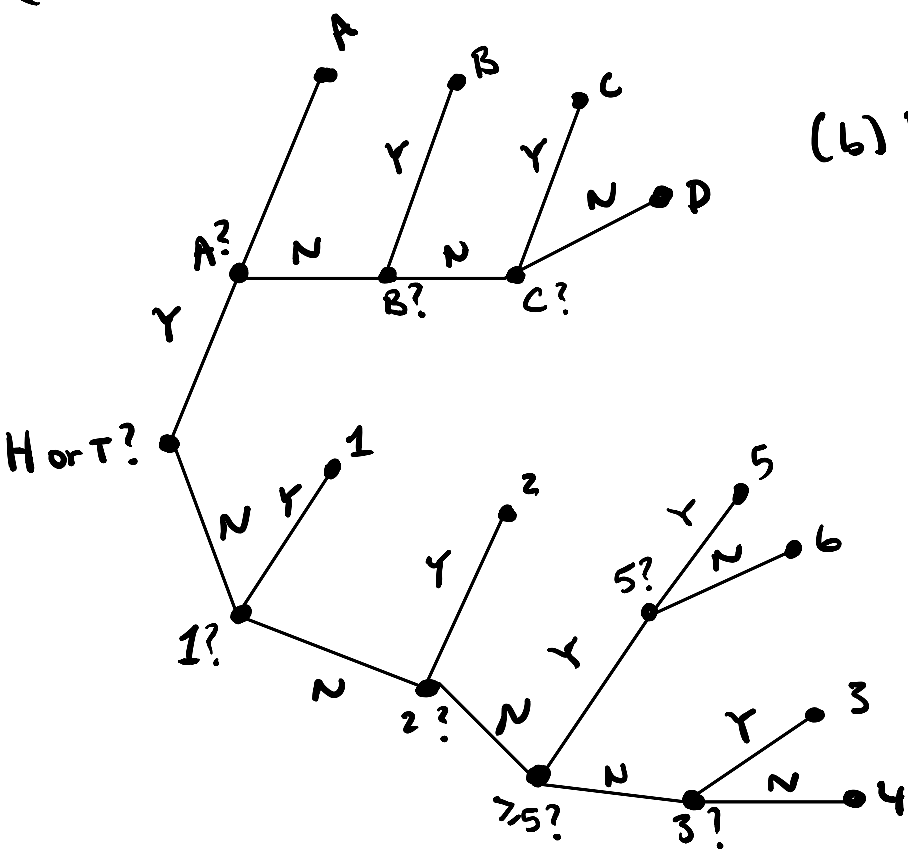
(c) Depicted below is a binary tree intended to represent an optimal Yes-No questioning scheme for Frodo, where optimal means lowest expected number of questions to determine the outcome among all Yes-No questioning schemes. Label the terminal nodes of the tree with the possible outcomes of Gandalf's experiment so the scheme is indeed optimal.

(a) He could ask if it's each one one by one which is 10 questions at most.

(c)

(b) Was the result of the flip H or T?

This eliminates an entire array of values



5. (25 points) Sam's bicycle has a Michelin tire on the front and a Pirelli on the back. The time in months X by which the Michelin will fail (i.e. require replacement) is distributed exponentially with rate parameter $\lambda = 7$, and the time Y by which the Pirelli will fail is distributed exponentially with rate parameter $\zeta = 5$. The tires' behaviors are independent.

(a) Let A be the event that the Michelin (front) tire fails first. Find $\mathbb{P}(A)$.
(b) Let Z be the time at which the first tire failure occurs. Find $f_Z(z)$ and $\mathbb{E}(Z)$.
(c) Find $f_{Y|A}(y)$, where A is defined in (a).

$X \sim$ exponential$(\lambda = 7)$ – front tire
$Y \sim$ exponential$(\zeta = 5)$ – back tire

(a) $\mathbb{P}(\{\text{Front fails first}\}) = \mathbb{P}(A) = \mathbb{P}(\{X < Y\})$

X, Y independent $\Leftrightarrow f_X(x) f_Y(y) = f_{X,Y}(x,y)$

$$f_{X,Y}(x,y) = \begin{cases} \lambda \zeta e^{-\lambda x} e^{-\zeta y} &, x, y > 0 \\ 0 &, o/w \end{cases}$$

$$\mathbb{P}(A) = \int_{y=0}^{y=+\infty} \int_{x=0}^{x=y} f_{X,Y}(x,y)\,dx\,dy = \zeta \int_{y=0}^{+\infty} e^{-\zeta y} dy \Bigg(\underbrace{\int_{x=0}^{x=y} \lambda e^{-\lambda x} dx}_{F_X(y)} \Bigg)$$

$$= \zeta \int_{y=0}^{+\infty} e^{-\zeta y} (1 - e^{-\lambda y})\, dy$$

$$= \frac{\lambda}{\zeta + \lambda} = \frac{7}{12}$$

(b) Z = time at which first failure occurs. $f_Z(z)$? $F_Z(z)$?

$$Z = \min\{X, Y\}$$

$\forall z > 0$,

$$\begin{aligned}
1 - F_Z(z) &= \mathbb{P}(Z > z) \\
&= \mathbb{P}(\{X > z\} \cap \{Y > z\}) \\
&= \mathbb{P}(\{X > z\})\, \mathbb{P}(\{Y > z\}) \\
&= (1 - F_X(z))(1 - F_Y(z) \\
&= (1 - (1 - e^{-\lambda z}))(1 - (1 - e^{-\varphi z})) \\
&= (e^{-\lambda z})(e^{-\varphi z}) \\
&= e^{-(\lambda + \varphi) z} \quad ; \ 0 \text{ otherwise}
\end{aligned}$$

$$F_Z(z) = \begin{cases} 1 - e^{-(\lambda+\varphi) z}, & z > 0 \\ 0, & o/w \end{cases}$$

exponential w/ parameter $\lambda + \varphi$

$$f_Z(z) = \frac{d}{dz} F_Z(z) = \begin{cases} (\lambda + \varphi) e^{-(\lambda+\varphi) z}, & z > 0 \\ 0, & o/w \end{cases}$$

Thus $\mathbb{E}(Z) = \dfrac{1}{\lambda + \varphi}$

(c) $f_{Y|A}(y)$? A as in (a).

$$F_{Y|A}(y) = \frac{\mathbb{P}(\{Y \leq y\} \cap \{X < y\})}{\mathbb{P}(\{X < y\})}$$

$$\mathbb{P}(\{Y \leq y\} \cap \{X < y\}) = \int_{t=0}^{t=y} \varphi e^{-\varphi t} \left(\int_{x=0}^{x=t} \lambda e^{-\lambda x} \, dx \right) dt$$

$$= \int_{t=0}^{y} \varphi e^{-\varphi t} \left(1 - e^{-\lambda t} \right) dt$$

$$= \int_{t=0}^{y} \varphi e^{-\varphi t} - \varphi e^{-(\varphi + \lambda) t} \, dt$$

$$= -e^{-\varphi t} + \frac{\varphi}{\varphi + \lambda} e^{-(\varphi + \lambda) t} \Bigg|_{t=0}^{y}$$

$$= -e^{-\varphi y} + \frac{\varphi}{\varphi + \lambda} e^{-(\varphi + \lambda) y} - \left(-1 + \frac{\varphi}{\varphi + \lambda} \right)$$

$$= 1 - e^{-\varphi y} + \frac{\varphi}{\varphi + \lambda} \left(1 + e^{-(\varphi + \lambda) y} \right)$$

valid for $y > 0$; 0 else

$$\mathbb{P}(\{X < y\}) = \frac{\varphi}{\varphi + \lambda} \rightarrow F_{Y|A}(y) = \frac{1 - e^{-\varphi y}}{\varphi / \varphi + \lambda} + \left(1 + e^{-(\varphi + \lambda) y} \right)$$

$$f_{Y|A}(y) = \frac{d}{dy} F_{Y|A}(y) = (\varphi+\lambda)e^{-\varphi y} - (\varphi+\lambda)e^{-(\varphi+\lambda)y} \quad ; \; y \geq 0$$

else 0

differs from posted answer
so verify this