

Recall

Derived Distributions; given $f_X(x)$, find $f_Y(y)$ when $Y=g(X)$.

Not easy in general - looked at a few easy cases.

Next, suppose X, Y jointly continuous w/ joint pdf $f_{X,Y}(x,y)$

Goal: find $f_Z(z)$ where $Z=g(X,Y)$

- Find $F_Z(z)$

- $f_Z(z) = \frac{d}{dz} F_Z(z)$

Not easy in general, but...

Example - $X \sim Y \sim \text{Uniform}[0,1]$

$X, Y \sim \text{Uniform}[0,1]$; X, Y independent.

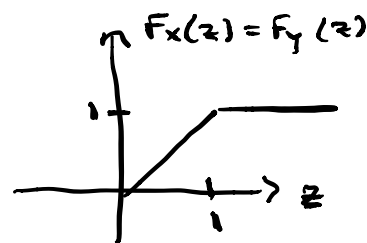
Let $Z = \max\{X, Y\}$

Then

$$F_Z(z) = P(\{Z \leq z\}) = P(\{X \leq z\} \cap \{Y \leq z\}) = P(\{X \leq z\}) P(\{Y \leq z\})$$

So

$$P(\{X \leq z\}) = P(\{Y \leq z\}) = \begin{cases} z & , z \in [0,1] \\ 0 & , \text{else} \end{cases}$$



Hence

$$F_Z(z) = \begin{cases} 0 & , z < 0 \\ z^2 & , 0 \leq z \leq 1 \\ 1 & , 1 < z \end{cases} \Rightarrow f_Z(z) = \begin{cases} 0 & , z < 0 \\ 2z & , z \in [0,1] \\ 0 & , z > 1 \end{cases}$$

Example - $X, Y \sim \text{exponential}(\lambda)$

X, Y independent.

$$Z = g(X, Y) = X - Y$$

want $f_Z(z)$.

$$F_Z(z) = P(\{Z \leq z\}) = P(\{X - Y \leq z\})$$

$$\text{independence} \Rightarrow f_{X,Y}(x,y) = \begin{cases} \lambda^2 e^{-(x+y)} & , x > 0, y > 0 \\ 0 & , \text{else} \end{cases}$$

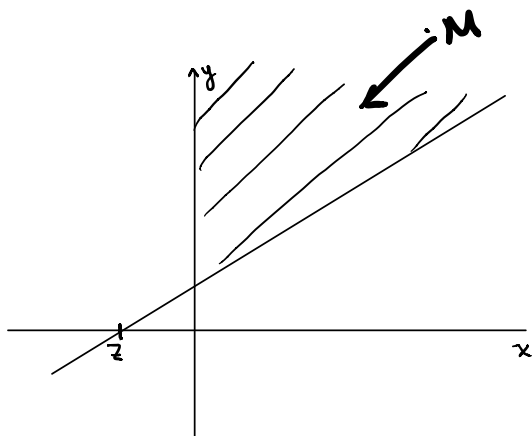
The set where

$$x - y \leq z$$

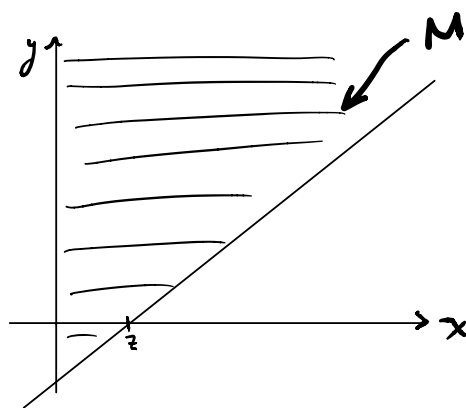
is the same as

$$y \geq x - z, \quad x = \text{anything}$$

when $z < 0$



when $z > 0$



Thus

$$F_Z(z) = \iint_{\substack{\text{shaded} \\ \text{region}}} f_{X,Y}(x,y) dx dy$$

When $z < 0$,

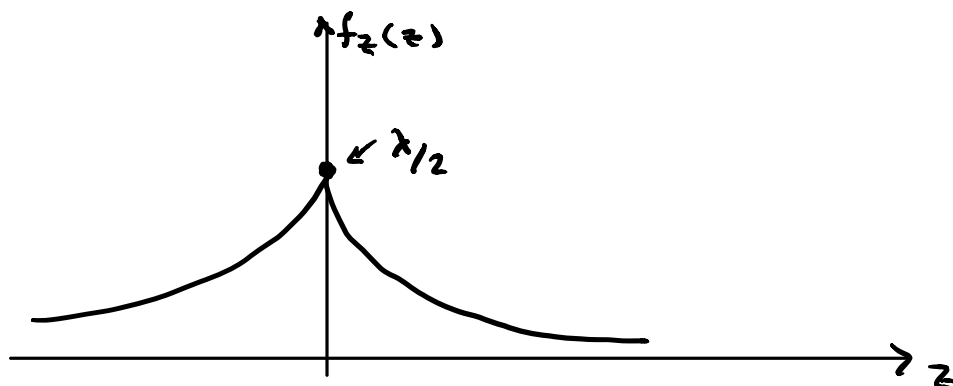
$$F_z(z) = \int_0^{\infty} dx \int_{x-z}^{\infty} dy \left(\lambda^2 e^{-\lambda(x+y)} \right) = \frac{1}{2} e^{\lambda z}$$

Similarly, when $z \geq 0$,

$$F_z(z) = \frac{1}{2} - \frac{1}{2} e^{-\lambda z}$$

In conclusion,

$$f_z(z) = \begin{cases} \frac{\lambda}{2} e^{\lambda z} & , z < 0 \\ \frac{\lambda}{2} e^{-\lambda z} & , z \geq 0 \end{cases} = \frac{\lambda}{2} e^{-\lambda |z|}$$



In both of these examples, X, Y independent - consider X, Y independent; respective pdfs $f_X(x), f_Y(y)$; find $f_z(z)$ when $z = X + Y$.

Instead of following recipe, recall that (saw on problem 1, HW IX) even when X, Y not independent,

$$f_z(z) = \int_{-\infty}^{+\infty} f_{X,Y}(x, z-x) dx$$

Independent $X, Y \Rightarrow f_{X,Y}(x,y) = f_X(x)f_Y(y) \Rightarrow f_{X,Y}(x, z-x) = f_X(x)f_Y(z-x)$

Conclude: when X, Y independent and $Z = X + Y$,

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x)f_Y(z-x) dx \quad \leftarrow \text{always works}$$

Could redo difference of exponentials example using this!

$$Z = X - Y = X + [-Y]; \quad X, Y \text{ independent.}$$

Suppose however X, Y independent $N(\mu_1, \sigma_1^2), N(\mu_2, \sigma_2^2)$ random variables respectively.

Consider $Z = X + Y$.

Know immediately

$$E(Z) = \mu_1 + \mu_2 \quad \text{Var}(Z) = \sigma_1^2 + \sigma_2^2$$

But what is $f_Z(z)$?

$f_Z(z) = \text{Combo of } f_X(x), f_Y(y) = \text{some HIDEOUS integral}$

Miraculously, (tedious to show), Z is also Gaussian!

Thus

$$f_Z(z) = \frac{1}{\sqrt{2\pi(\sigma_1^2 + \sigma_2^2)}} e^{-\frac{(z - (\mu_1 + \mu_2))^2}{2(\sigma_1^2 + \sigma_2^2)}}$$

Similarly, for any α, β not both 0, $Z = \alpha X + \beta Y$ is Gaussian,

mean $\mu = \alpha\mu_1 + \beta\mu_2$ and variance $\sigma^2 = \alpha^2\sigma_1^2 + \beta^2\sigma_2^2$

Mantra: Any nontrivial linear combo of independent Gaussians is also Gaussian.

Turns out there's a discrete version of convolution thing.

Suppose X, Y are integer-valued random variables w/ a joint pmf $p_{X,Y}(m,n)$.

Let $Z = X + Y$.

Turns out

$$p_Z(k) = \sum_{m=-\infty}^{+\infty} p_{X,Y}(m, k-m) \quad \forall k$$

If in addition X, Y independent, so that $p_{X,Y}(x,y) = p_X(x)p_Y(y)$

$$p_Z(k) = \sum_{m=-\infty}^{+\infty} p_X(m) p_Y(k-m) \quad \leftarrow \text{convolution of marginal pmfs}$$