Probability model $(\Omega, \mathcal{F}, Pr)$

$\uparrow$ a collection of subsets of Ω

-sigma field-

Union Bound

$$Pr\left[\bigcup_{i=1}^{n} A_i\right] \leq \sum_{i=1}^{n} Pr(A_i)$$

equality only holds if mutually exclusive

or

$A_i \cap A_j = \emptyset \quad \forall i,j$

Conditional Probability

$$Pr(A|B) = \frac{Pr(A \cap B)}{Pr(B)}$$

$\uparrow$ new sample space

$\leftarrow$ normalization factor

Total Probability Theorem

If $\{E_1, E_2, \dots, E_k\}$ partition Ω

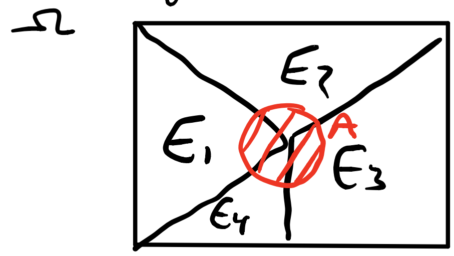
then

$$Pr(A) = \sum_{i=1}^{k} Pr(A \cap E_i)$$

This makes more sense since you know base event which occurred

$$= \underbrace{\sum_{i=1}^{k} Pr(A|E_i) Pr(E_i)}_{\text{generative models}}$$

<u>Example of Partition for k=4</u>



No intersections

$\bigcup_{i=1}^{k} E_i = \Omega$

Bayes Rule

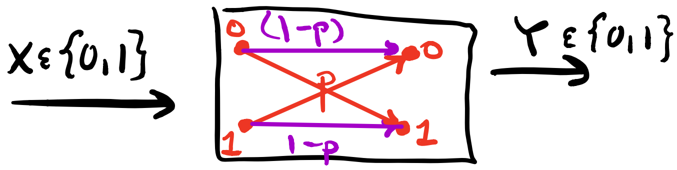
Want to find "ground truth" that gave this observation.

prior knowledge — generative model

$$\Pr[E_j \mid A] = \frac{\Pr[A \cap E_j]}{\Pr[A]} = \frac{\Pr[E_j]\Pr(A \mid E_j)}{\sum_{i=1}^{k} \Pr[E_i]\Pr[A \mid E_i]}$$

← Pr(A) by TPT

Example: Binary Symmetric Channel



$$\Pr(X=0) = \Pr(X=1) = \frac{1}{2}$$

$$\Pr(Y=0 \mid X=0] = 1-p$$

$$\Pr(Y=1 \mid X=0] = p$$

If $Y=0$, what is $\hat{X}_{opt}$?

$\hat{X} = 0$ wrong if $X=1 \rightarrow Y=0$

$\Pr(\text{error}) = \Pr(X=1 \mid Y=0)$

or

$\hat{X} = 1$ wrong if $X=0 \rightarrow Y=1$

$\Pr(\text{error}) = \Pr(X=0 \mid Y=1)$

Want the answer which is LESS likely to be wrong?

$$\Pr(\text{error}) = \frac{\Pr(X=1)\Pr(Y=0 \mid X=1)}{\Pr(X=1)\Pr(Y=0 \mid X=1) + \Pr(X=0)\Pr(Y=1 \mid X=0)} = \frac{\frac{1}{2}p}{\frac{1}{2}p + \frac{1}{2}(1-p)}$$

Independence

Two events, A_1, A_2, are independent if $\Pr(A_1 \cap A_2) = \Pr(A_1)\Pr(A_2)$

$$\Longrightarrow \Pr[A_1 | A_2] = \frac{\Pr[A_1 \cap A_2]}{\Pr[A_2]} = \frac{\Pr(A_1)\Pr(A_2)}{\Pr(A_2)}$$

Can be extended:

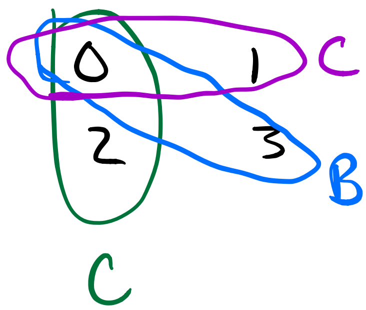
Events $\{A_1, A_2, \ldots, A_n\}$ are independent if

$$\Pr(A_1 \cap A_2 \cap \cdots \cap A_k) = \Pr(A_1)\Pr(A_2)\cdots\Pr(A_k) \quad \forall \begin{smallmatrix} k \in [2,n] \\ k \in \mathbb{Z} \end{smallmatrix}$$

Note: Pairwise independence $\not\Rightarrow$ independence!

Example: $\Omega = \{0,1,2,3\}$; all occur w/ equal probability

$A = \{0,1\}$ $\quad$ $B = \{0,2\}$ $\quad$ $C = \{0,3\}$



$\Pr(AB) = \Pr(\{0\}) = 1/4$

$\Pr(BC) = \Pr(\{0\}) = 1/4$

$\Pr(AC) = \Pr(\{0\}) = 1/4$

$\Pr(ABC) = \Pr(\{0\}) = 1/4$

BUT $\Pr(A)\Pr(B)\Pr(C) = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$!

Random Variables

Given $(\Omega, \mathcal{F}, Pr)$ a random variable is a <u>function</u>

$X: \Omega \to \mathbb{R}$ such that $\forall x \in \mathbb{R}$, $\{\omega \mid X(\omega) \leq x\} \in \mathcal{F} \subseteq \Omega$

Example: Throw two dice: d_1, d_2

$\Omega = \{(1,1), \ldots, (6,6)\}$ w/ each event occuring w/ Pr of $1/36$

Define $X = \max(d_1, d_2) - \min(d_1, d_2)$

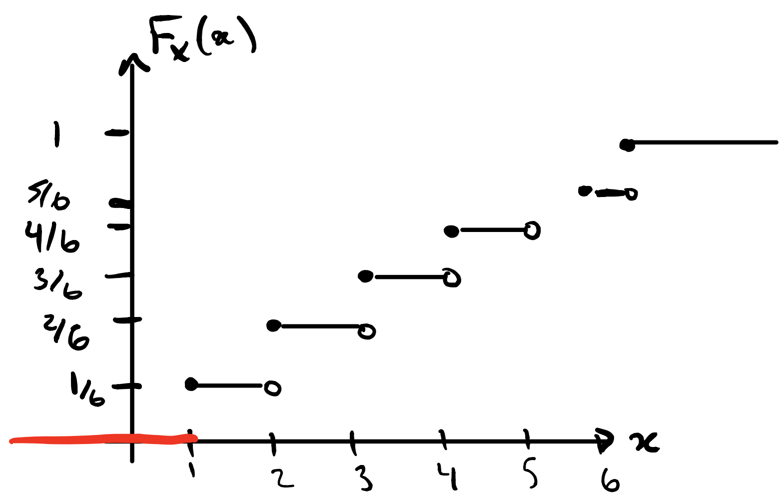
$$Pr(X=3) = Pr(\{\omega \mid X(\omega) = 3\})$$
$$= Pr(\{(1,4), (2,5), (3,6), (4,1), (5,2), (6,3)\})$$
$$= \frac{1}{6} \leftarrow \frac{6 \text{ outcomes}}{36 \text{ total outcomes}}$$

Cumulative Distribution Function

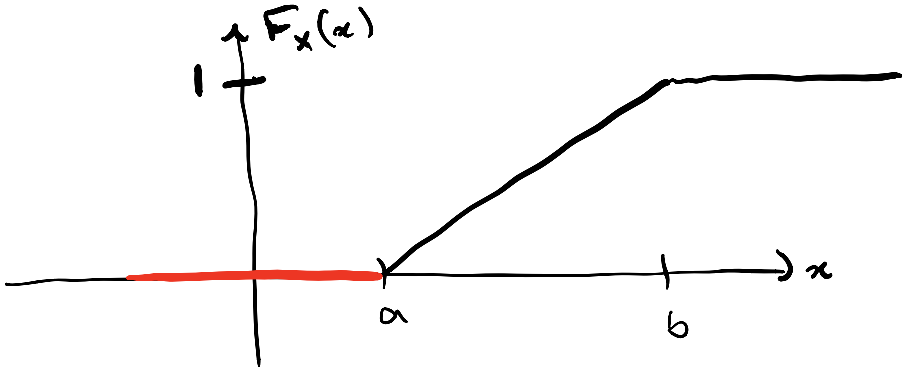
The CDF is defined as

$$F_X(x) = Pr(X \leq x) \quad \forall x \in \mathbb{R}$$
$$= Pr(\{\omega \mid X(\omega) \leq x\})$$

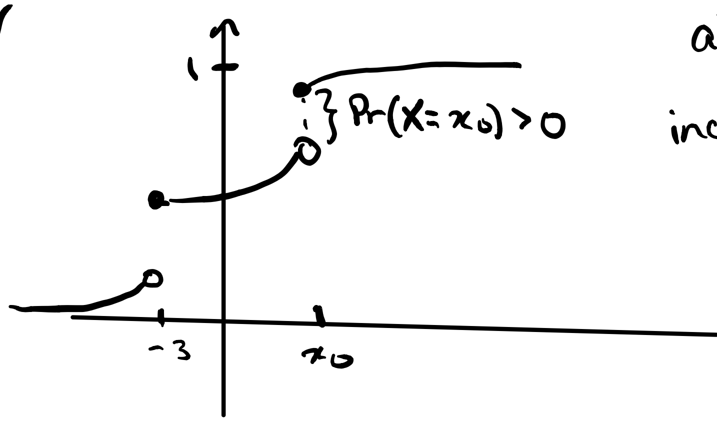
Throw a die



Uniform



Mixed RV



always monotonically increasing

Probability Mass Function

For a discrete r.v.

$$P_X(x) = Pr(X = x)$$

i.e Die