Recall

$1+2+3+4+\cdots$ "=" $-\frac{1}{12}$

$\sum \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$ Basel Problem

Euler realized

Zeta function (2)

$$\sum \frac{1}{n^2} = \frac{\pi^2}{6} = \zeta(2)$$

where

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$$

is the zeta function

How did Euler realize

$$\sum \frac{1}{n^2} = \frac{\pi^2}{6} ?$$

Well, realize that a function with simple zeros at $0, \pi$ has to be

$$f(x) = C(x-0)(x-\pi)$$

So, sin x can be written as

$$\sin x = C(x-0)(x-\pi)(x+\pi)(x-2\pi)(x+2\pi)(x-3\pi)(x+3\pi)\cdots$$

{factor out x} $$= Cx\left(1-\frac{x}{\pi}\right)\left(1+\frac{x}{\pi}\right)\left(1-\frac{x}{2\pi}\right)\left(1+\frac{x}{2\pi}\right)\left(1-\frac{x}{3\pi}\right)\left(1+\frac{x}{3\pi}\right)\cdots$$

So, $C=1$ since $\sin x \simeq x$ for small x

Simplifying,

$$\sin x = x\left(1-\frac{x^2}{\pi^2}\right)\left(1-\left[\frac{x}{2\pi}\right]^2\right)\left(1-\left[\frac{x}{3\pi}\right]^2\right)$$

So,

$$\sin x = x\prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2\pi^2}\right)$$ } Euler's infinite product for sin(x)

Next, match powers of x^3 on both sides!

We know,

$$\sin x = x - \boxed{\frac{x^3}{3!}} + \frac{x^5}{5!} - \cdots = x\prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2\pi^2}\right)$$

$$x - \boxed{x^3\left[\frac{1}{\pi^2}+\frac{1}{2^2\pi^2}+\frac{1}{3^2\pi^2}+\cdots\right]} + (\ \text{stuff}\ )x^5 - \cdots$$

COMPARE $$\frac{x^3}{6} = \frac{x^3}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

So, $\longrightarrow$ $$\boxed{\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2}}$$

Similar could be done for all even Z!

Back to,

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$$

Euler found a product form for this!

Recall how to find prime numbers!

Civ of Aristophones

$$1 \quad 2 \quad 3 \quad \cancel{4} \quad 5 \quad \cancel{6} \quad 7 \quad \cancel{8} \quad \cancel{9} \quad \cancel{10}$$

$$\frac{1}{2^z}\zeta(z) = \frac{1}{2^z}\left(1 + \frac{1}{2^z} + \frac{1}{3^z} + \frac{1}{4^z} + \cdots\right)$$

$$= \frac{1}{2^z} + \frac{1}{4^z} + \frac{1}{6^z} + \frac{1}{8^z}$$

So,

$$\zeta(z) - \frac{1}{2^z}\zeta(z)$$

kills off all even denominators raised to the z-power.

$$1 + \frac{1}{3^z} + \frac{1}{5^z} + \frac{1}{7^z} + \frac{1}{9^z} + \cdots$$

$$= \zeta(z)\left[1 - \frac{1}{2^z}\right]$$

Now take

$$\frac{1}{3^z}\left[\zeta(z)\left[1-\frac{1}{2^z}\right]\right]$$

$$= \frac{1}{3^z} + \frac{1}{9^z} + \frac{1}{15^z} + \frac{1}{21^z} + \cdots$$

So, $\zeta(z)\left[1-\frac{1}{2^z}\right] - \frac{1}{3^z}\left[\zeta(z)\left[1-\frac{1}{2^z}\right]\right]$

$$= \zeta(z)\left[1-\frac{1}{2^z}\right]\left[1-\frac{1}{3^z}\right]$$

$$= 1 + \frac{1}{5^z} + \frac{1}{7^z} + \frac{1}{11^z} + \frac{1}{13^z} + \cdots$$

Got rid of denominators which are multiples of three to the z

$\vdots$ Doing this to infinity

$\vdots$

$$1 = \zeta(z)\prod_{\text{all primes } p}\left(1-\frac{1}{p^z}\right)$$

So, $\zeta(z) = \dfrac{1}{\prod_{\text{all primes } p}\left(1 - \frac{1}{p^z}\right)}$

Prime Obsession
John Derbyshire

i.e $\zeta(z) = \sum_{n=1}^{\infty} \dfrac{1}{n^z} = \dfrac{1}{\prod_{\text{all primes } p}\left(1 - \frac{1}{p^z}\right)}$