

2.1

If) $G(z) = e^z + e^{-z}$

$$\omega = e^x e^{iy} + e^{-x} e^{-iy}$$

$$\omega = e^x [\cos(y) + i \sin(y)] + e^{-x} [\cos(y) - i \sin(y)]$$

$$\omega = e^x \cos(y) + e^{-x} \cos(y) + i [e^x \sin(y) - e^{-x} \sin(y)]$$

$$\omega = u(x,y) + i v(x,y)$$

4) $\omega = f(z) = \frac{1}{z}$ maps

(a) $|z|=r$ onto $|\omega| = \frac{1}{r}$

$$\omega = \frac{1}{z} = \frac{1}{x+iy} = \frac{x}{x^2+y^2} - i \frac{y}{x^2+y^2} \quad \text{Note} \quad |z| = \sqrt{x^2+y^2} = r$$

$$u(x,y) = \frac{x}{x^2+y^2} \quad v(x,y) = \frac{-y}{x^2+y^2}$$

$$|\omega| = \sqrt{\frac{x^2+y^2}{(x^2+y^2)^2}} = \sqrt{\frac{1}{x^2+y^2}} = \frac{1}{r}$$

(b) The ray $\operatorname{Arg} z = \theta_0$, $-\pi < \theta_0 < \pi$, onto
the ray $\operatorname{Arg} \omega = -\theta_0$

$$\operatorname{Arg} z = \theta_0 = \operatorname{Arg} \left(\frac{y}{x} \right)$$

$$\operatorname{Arg} w = \operatorname{Arg} \left(\frac{v(x,y)}{u(x,y)} \right) = \operatorname{Arg} \left(\frac{-y}{x} \right) = -\operatorname{Arg} \left(\frac{y}{x} \right)$$

(c) The circle $|z-1|=1$ onto the vertical line $x=\frac{1}{2}$

$$\omega = \frac{1}{z} = \frac{x}{x^2+y^2} - i \frac{y}{x^2+y^2}$$

$$|z-1| = |x+iy-1| = 1$$

$$(x-1)^2 + y^2 = 1$$

$$x^2 + y^2 - 2x = 0$$

$$x^2 + y^2 = 2x$$

$$\begin{aligned}\Rightarrow \omega &= \frac{x}{2x} - i \frac{y}{2x} \\ &= \frac{1}{2} - i \frac{y}{2x}\end{aligned}$$

Now we must show that $\frac{y}{x}$ spans from $-\infty$ to $+\infty$ as we traverse our circle.

Let

$$x = 1 + \cos \phi$$

$$y = \sin \phi$$

Then

$$\frac{y}{x} = \frac{\sin \phi}{1 + \cos \phi} = \tan\left(\frac{\phi}{2}\right)$$

As we traverse the circle y/x spans from $-\infty$ to $+\infty$ and therefore the circle $|z-1|=1$ is mapped onto the line $x=\frac{1}{2}$.

$$6) \quad \omega = J(z) = \frac{1}{2} \left(z + \frac{1}{z} \right) \quad \left. \begin{array}{l} \text{Joukowki} \\ \text{Mapping} \end{array} \right.$$

(a) Show $J(z) = J(\frac{1}{z})$

$$J(z) = \frac{1}{2} \left(z + \frac{1}{z} \right)$$

$$J\left(\frac{1}{z}\right) = \frac{1}{2} \left(\frac{1}{z} + \frac{1}{\frac{1}{z}} \right) = \frac{1}{2} \left(z + \frac{1}{z} \right) = J(z)$$

(b) \Im maps the unit circle $|z|=1$ onto real interval $[-1,1]$

$$\begin{aligned}\Im(z) &= \frac{1}{2}(z + \frac{1}{z}) \\ w &= \frac{1}{2}(x+iy + \frac{1}{x+iy}) \quad x^2+y^2=1 \\ &= \frac{1}{2} \left(x+iy + \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2} \right) \quad x^2 = 1-y^2 \\ &= \frac{1}{2} \left(x+iy + \frac{x}{1} - i\frac{y}{1} \right) \quad x = \sqrt{1-y^2} \\ &= \frac{1}{2}x + \frac{1}{2}x + \frac{1}{2}iy - \frac{1}{2}iy \\ &= x\end{aligned}$$

By the restriction $|z|=1$, it is automatically known that the bounds on x,y are

$$\begin{aligned}-1 &\leq x \leq 1 \\ -1 &\leq y \leq 1\end{aligned}$$

Since $w=\Im(z)=x$ we therefore have a mapping onto the real axis from -1 to 1 .

(c) \Im maps $|z|=r$ ($r>0, r\neq 1$) onto the ellipse

$$\frac{u^2}{\left[\frac{1}{2}\left(r+\frac{1}{r}\right)\right]^2} + \frac{v^2}{\left[\frac{1}{2}\left(r-\frac{1}{r}\right)\right]^2} = 1, \text{ which has foci at } \pm 1.$$

$x^2+y^2=r^2$

$$w = \Im(z) = \frac{1}{2}(z + \frac{1}{z}), \quad z = x+iy$$

$$w = \frac{1}{2}\left(x+iy + \frac{1}{x+iy}\right) = \frac{1}{2}\left(x+iy + \frac{x-iy}{x^2+y^2}\right) = \frac{1}{2}\left(x + \frac{x}{r^2}\right) + \frac{1}{2}i\left(y - \frac{y}{r^2}\right)$$

$$w = \frac{1}{2} \cdot \frac{x}{r} \left(r + \frac{1}{r}\right) + \frac{1}{2} \cdot \frac{y}{r} \left(r - \frac{1}{r}\right) \quad \left. \begin{array}{l} \frac{x}{r} = \cos\theta, \\ \frac{y}{r} = \sin\theta \end{array} \right\}$$

$$w = \underbrace{\frac{\cos\theta}{2} \left(r + \frac{1}{r}\right)}_{u(r,\theta)} + i \underbrace{\frac{\sin\theta}{2} \left(r - \frac{1}{r}\right)}_{v(r,\theta)}$$

$$u^2 = \frac{\cos^2 \theta}{4} \left(r + \frac{1}{r}\right)^2, \quad v^2 = \frac{\sin^2 \theta}{4} \left(r - \frac{1}{r}\right)^2$$

$$\frac{u^2}{\left[\frac{1}{2} \left(r + \frac{1}{r}\right)\right]^2} = \cos^2 \theta \quad \frac{v^2}{\left(\frac{1}{2} \left(r - \frac{1}{r}\right)\right)^2} = \sin^2 \theta$$

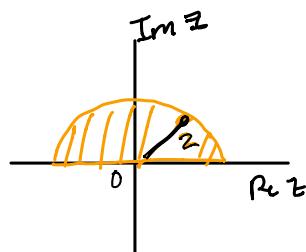
$\cos^2 \theta + \sin^2 \theta = 1 \quad \checkmark \quad \text{Everything works!}$

7a) $F(z) = z + c$, where $c \in \mathbb{C}$, generates a translation mapping

Sketch image of $|z| \leq 2$, $\operatorname{Im} z \geq 0$ under F when $c=3$.

$$F(z) = z + 3$$

$$= (x+3) + iy$$



$$\begin{cases} x^2 + y^2 \leq 4 \\ y \geq 0 \end{cases}$$

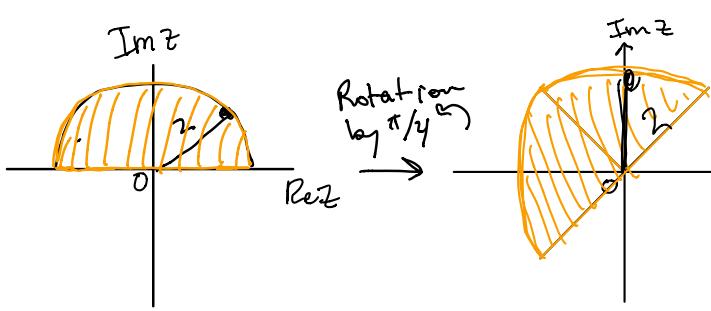
$\operatorname{Im} z$

$0 \quad 1 \quad 2 \quad 3 \quad 4$ $\operatorname{Re} z$

} Every point gets shifted on the real axis by 3

8a) $G(z) = e^{i\phi} z$, $\phi \in \mathbb{R}$, generates a ROTATION MAPPING.

Sketch $|z| \leq 2$, $\operatorname{Im} z \geq 0$ under G when $\phi = \pi/4$



} every point $z = re^{i\theta}$ gets rotated by ϕ

(D) Let $F(z) = z + i$, $G(z) = e^{i\pi/4} z$, and $H(z) = \bar{z}/2$.

Sketch

$$|z| \leq 2, \operatorname{Im} z \geq 0 \text{ under}$$

$G(H(z))$

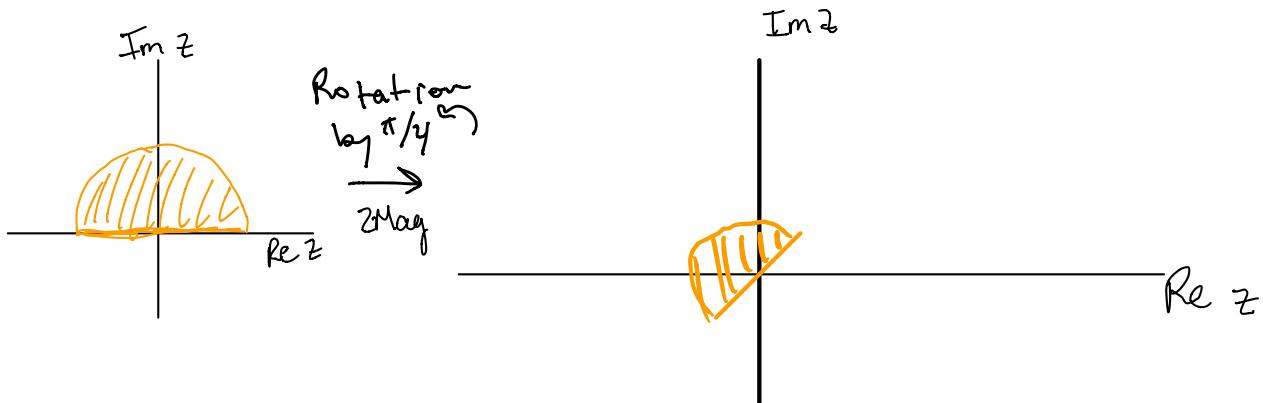
$$G(H(z)) = G(\bar{z}/2) = e^{i\pi/4} \frac{z}{2} \quad \left. \begin{array}{l} \text{rotation by} \\ \pi/4 \text{ and} \\ \frac{1}{2} \cdot \text{Magnitude} \end{array} \right\}$$

$$\left| \frac{z}{2} \right| \leq 2$$

$$\sqrt{\left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2} \leq 2$$

$$\frac{1}{2} \sqrt{x^2 + y^2} \leq 2$$

$$\sqrt{x^2 + y^2} \leq 4$$



2.3

4a) Show $\operatorname{Re} z$ is nowhere differentiable

$$\frac{df}{dz}(\operatorname{Re} z) \equiv f'(\operatorname{Re} z) := \lim_{\Delta z \rightarrow 0} \frac{\operatorname{Re}(z + \Delta z) - \operatorname{Re}(z)}{\Delta z} = \frac{\operatorname{Re}(\Delta z)}{\Delta z}$$

if $\Delta z \rightarrow 0$ through real values, then $\Delta z = \Delta x$ and

$$\frac{\operatorname{Re}(\Delta z)}{\Delta z} = \frac{\Delta x}{\Delta x} = 1$$

However, if $\Delta z \rightarrow 0$ through imaginary values, then $\Delta z = i\Delta y$ and

$$\frac{\operatorname{Re}(\Delta z)}{\Delta z} = 0$$

Consequently, there is no way of assigning a unique value to the derivative of $\operatorname{Re}(z)$ at any point.

9b) Determine the points at which the function is not analytic

$$\frac{iz^3 + 2z}{z^2 + 1}$$

The function is not analytic at $z^2 + 1 = 0 \Rightarrow z = i, -i$ as the denominator is zero at those two points and therefore is not differentiable.

10) $f(z) = |z|^2$. Show f is differentiable at $z=0$ but not at any other point.

$$f'(z) := \lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \rightarrow 0} \frac{|z + \Delta z|^2 - |z|^2}{\Delta z}$$

$$= \lim_{\Delta z \rightarrow 0} \frac{(z + \Delta z)(\overline{z + \Delta z}) - z\bar{z}}{\Delta z} = \lim_{\Delta z \rightarrow 0} \frac{z\overline{z} + \bar{z}\Delta z + \Delta z\bar{z} - z\bar{z}}{\Delta z}$$

$$= \lim_{\Delta z \rightarrow 0} \frac{z\overline{z} + \bar{z}\Delta z + \Delta z\bar{z} - z\bar{z}}{\Delta z}$$

At $z = 0$

$$(0 + 0 + 0) = 0$$

At $z \neq 0$ we would imply the differentiability of \bar{z}

2.4

2) show $h(z) = x^3 + 3xy^2 - 3x + i(y^3 + 3x^2y - 3y)$ is differentiable on the coordinate axes but nowhere analytic.

To be differentiable on the coordinate axes means to be differentiable at each x, y where $xy = 0$.

$$\frac{\partial u}{\partial x} = 3x^2 + 3y^2 - 3$$

$$\frac{\partial v}{\partial x} = 6xy$$

$$\frac{\partial u}{\partial y} = 6xy$$

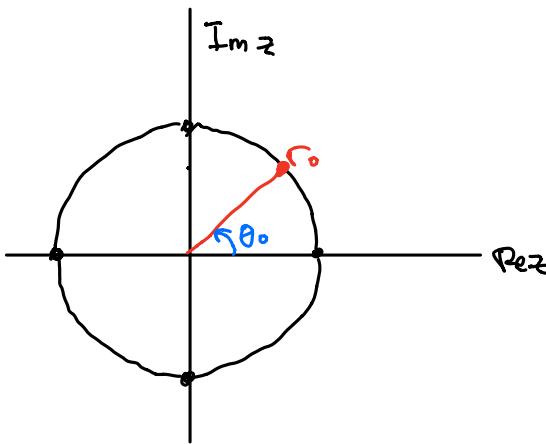
$$\frac{\partial v}{\partial y} = 3y^2 + 3x^2 - 3$$

Since the first partial derivatives are continuous $h(z)$ is differentiable on the coordinate axes.
 $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$ however, and therefore $h(z)$ is nowhere analytic.

6) Show

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}$$

Using the hint,

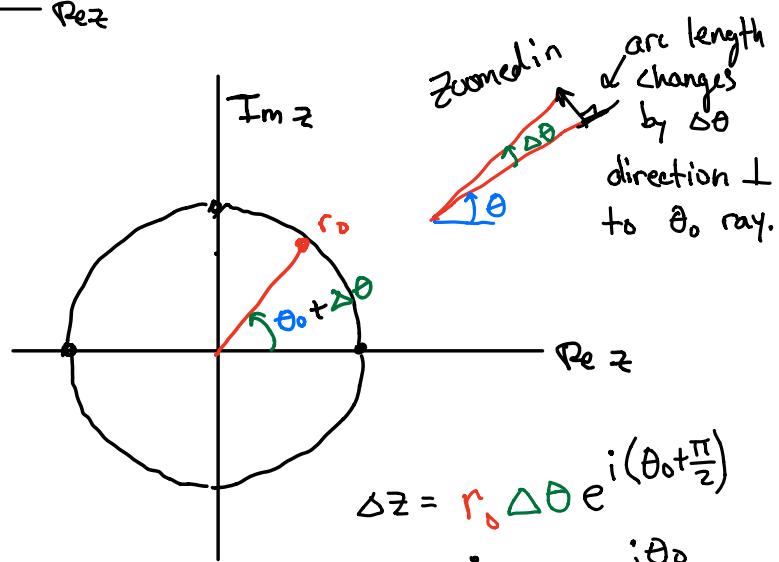


First let's change θ while keeping r constant.

$$f(z) = u(r, \theta) + i r(r, \theta)$$

$$f'(z) = \lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = \frac{\Delta f}{\Delta z}$$

$$f'(z) = \frac{\Delta f}{\Delta z} = \frac{\Delta f}{i r_0 \Delta \theta e^{i \theta_0}} = \frac{\Delta f}{\Delta \theta} \cdot \frac{1}{i r_0 e^{i \theta_0}} = \frac{\partial f}{\partial \theta} \cdot \frac{1}{i r_0 e^{i \theta_0}}$$



$$\Delta z = r_0 \Delta \theta e^{i(\theta_0 + \frac{\pi}{2})}$$

$$\Delta z = i r_0 \Delta \theta e^{i \theta_0}$$

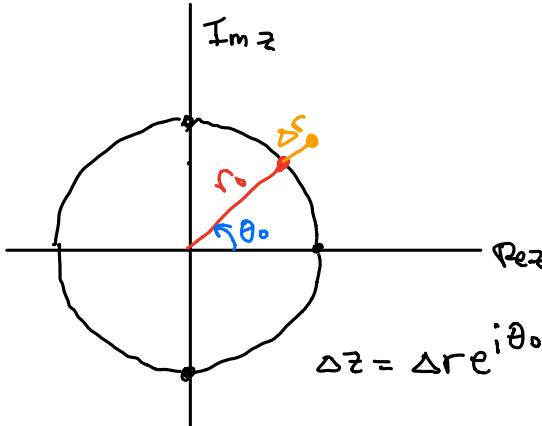
Now let's change r while keeping θ constant.

$$f'(z) = \frac{\Delta f}{\Delta z} = \frac{\Delta f}{\Delta r e^{i \theta_0}} = \frac{\partial f}{\partial r} \cdot \frac{1}{e^{i \theta_0}}$$

Equating our $f'(z)$,

$$f'(z) = \frac{\partial f}{\partial \theta} \cdot \frac{1}{i r_0 e^{i \theta_0}} = \frac{\partial f}{\partial r} \cdot \frac{1}{e^{i \theta_0}}$$

$$\frac{\partial f}{\partial \theta} = i r_0 \frac{\partial f}{\partial r}$$



$$\frac{\partial f}{\partial \theta} = \frac{\partial u}{\partial \theta} + i \frac{\partial v}{\partial \theta}; \quad \frac{\partial f}{\partial r} = \frac{\partial u}{\partial r} + i \frac{\partial v}{\partial r}$$

$$\Rightarrow \frac{\partial u}{\partial \theta} + i \frac{\partial v}{\partial \theta} = i r_0 \left[\frac{\partial u}{\partial r} + i \frac{\partial v}{\partial r} \right] \Rightarrow \frac{\partial u}{\partial \theta} = -r_0 \frac{\partial v}{\partial r}, \quad \frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}$$

10) Show if $f(z) \in \mathbb{R}$ & analytic, then $f(z) = c$

If $f(z)$ is real that means

$$f(z) = u(x,y) + i v(x,y) = u(x,y) + i(0) = u(x,y).$$

If $f(z)$ is analytic,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Since $v(x,y) = 0$,

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$$

Therefore $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$

By Theorem 6, if $f(z)$ is analytic and $f'(z) = 0 \forall z$,
 $f(z)$ must be constant.