Recall: Ω = set of all outcomes of some "uncertain" experiment.

Question: how to pick Ω in a given context?

There is no unique answer

Example

2 different experiments involving 10 coin flips.

① Flip 10 times, you get paid \$ #(Heads)

② Flip 10 times, you get \$1 every flip until you see the first head, then \$2 that flip and others until next head, \$4 ....

Natural to use as Ω

① $\Omega = \{0, 1, 2, \dots, 10\}$

② $\Omega = \{\text{ordered 10-tuples of } H, T\}$ ← 2^{10} elements

Could also use second Ω for case ① - insted of having "You collect \$4" as an <u>outcome</u>, its an $\boxed{\text{event}}$

In this context, an <u>event</u> is <u>ANY</u> subset A of Ω (including $A = \emptyset$, $A = \Omega$)

If we use big Ω for case ①, then

"get paid \$4" = {all 10-tuples containing exactly 4 H's} ← a subset of Ω i.e an event!

Comment: can't go wrong using larger Ω's (more refined) to cover a given situation — modulo over-modeling / irrelevant details.

Given Ω is chosen, a probability law on Ω is a mapping $\mathbb{P}$ that assigns a number to every event (i.e to every subset of Ω) such that:

1) $\mathbb{P}(A) \geq 0$ for every event A
2) $\mathbb{P}(\Omega) = 1$ (normalization)
3) Additivity Rules
   i) If $A \cap B = \emptyset$ (A, B are events) then
   $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$
   ii) If $A_1, A_2, A_3, \dots$ is a countable sequence of mutually disjoint events (i.e $A_i \cap A_j = \emptyset \; \forall i,j$), then $\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(A_n)$

$\bigcup_{n=1}^{\infty} A_n \leftarrow$ union of all of them

So, given an event $A \subset \Omega$, $\mathbb{P}(A)$ is a model for "the likelihood that the outcome of the uncertain experiment is in A"

"Event A occurs" means "outcome of experiment is in A"

Can derive numerous properties of $\mathbb{P}$ from the rules!

$$\mathbb{P}(\emptyset) = 0$$

because

$$1 = \mathbb{P}(\Omega) = \mathbb{P}(\underbrace{\Omega \cup \emptyset}_{\text{disjoint}}) = \mathbb{P}(\Omega) + \mathbb{P}(\emptyset) = 1$$

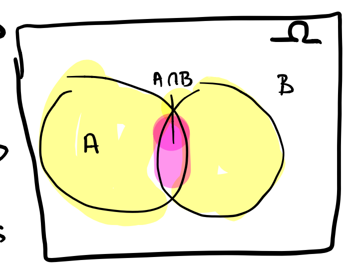
So

$$\mathbb{P}(\emptyset) = 0$$

Another useful one: for any events A, B

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

Can prove via venn diagrams



A consequence of this is

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cup B)$$

holds if $A \cap B = \emptyset$, but can also hold other times

$$\mathbb{P}(A \cup B) \leq \mathbb{P}(A) + \mathbb{P}(B) \quad \text{w/ equality iff } \mathbb{P}(A \cap B) = 0$$

When Ω is <u>finite</u> - say $\Omega = \{s_1, s_2, \dots, s_n\}$ - then rules imply

$$1 = \mathbb{P}(\Omega) = \mathbb{P}(\{s_1\} \cup \{s_2\} \cup \dots \cup \{s_n\}) = \sum_{k=1}^{n} \mathbb{P}(\{s_k\})$$

singleton sets

by disjoint additivity rule.

and similarly for any event $A \subset \Omega$

$$\mathbb{P}(A) = \sum_{s \in A} \mathbb{P}(\{s\})$$

Consequences:

(1) The probability law $\mathbb{P}$ is determined completely by its values on the "singleton events" $\{s_1\}, \{s_2\}, \ldots, \{s_n\}$

(2) If all outcomes $s_1, \ldots, s_n$ are "equally likely", then since $\mathbb{P}(\{s_k\})$ same for all k, and since $\sum_k \mathbb{P}(\{s_k\}) = \frac{1}{n}$, $1 \le k \le n$

and in fact, for any event $A \subset \Omega$ in this case,

$$\mathbb{P}(A) = \frac{\#\text{elements in } A}{n}$$

<u>Terminology:</u> For finite Ω, then $\mathbb{P}(\{s\}) = \frac{1}{n}$, $n = \text{size}(\Omega)$ for all $s \in \Omega$ call $\mathbb{P}$ a (discrete) <u>uniform probability law</u>

Observe that when Ω is infinite, there is no way to have $\mathbb{P}(\{s\})$ the same for all $s \in \Omega$.

↳ If $\mathbb{P}(\{s\}) = \alpha \geq 0$, then can't have $\alpha = 0$ ('cause $\mathbb{P}(\Omega) = 1$) but can't have $\alpha > 0$ 'cause sum of more than $\lceil \frac{1}{\alpha} \rceil$ probs > 1 which is NOT allowed, since any finite event is contained in Ω, hence $\mathbb{P}(\text{it}) \le 1$ needed

Implicitly using $\mathbb{P}(A) \le \mathbb{P}(B)$ when $A \subset B$ – another easy to derive $\mathbb{P}$-rule

Simple Examples:

Three flips of a coin. $\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

A couple of events.

A_1 = Two Heads (exactly) $= \{HHT, HTH, THH\}$

A_2 = Second flip is an H $= \{HHH, HHT, THH, THT\}$

A good probability law: uniform — $\mathbb{P}$(every outcome) $= 1/8$ — So

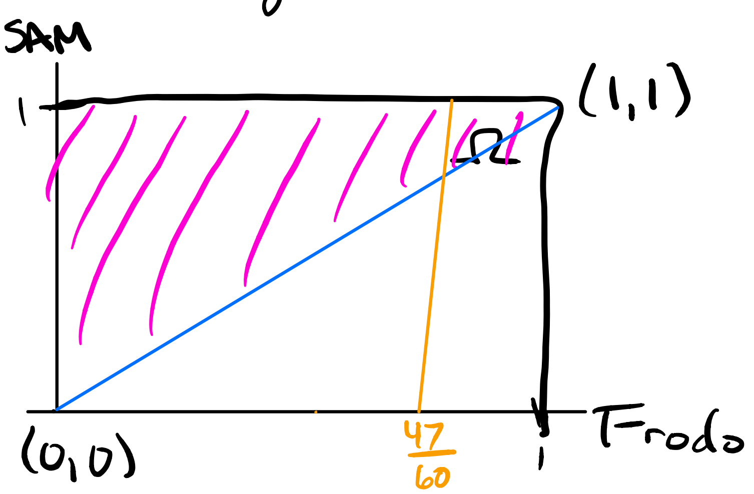
$\mathbb{P}(A_1) = 3/8$, $\mathbb{P}(A_2) = 1/2$

A continuous example:

Frodo and Sam went to meet up; arrive at some specified location between 0 and 1 hour.

- any pair of arrival times is "equally likely"

What's a good Ω? Claim $\Omega = [0,1] \times [0,1]$ = "unit square"



Frodo arrives first event

$\{(x,y) \in \Omega,\ x \le y\}$

Arrive simultaneously event

$\{(x,y) \in \Omega,\ x = y\}$

Frodo arrives at 47 minutes into the hour event

Good $\mathbb{P}$ for this: $\mathbb{P}(A) = \text{Area}(A)$ for any $A \subset \Omega$