1. X and Y are exponential with respective rates λ and ζ. Show that $Z = \min\{X, Y\}$ is exponential with rate $\lambda + \zeta$.

2. In class we saw that $\mathbb{E}(X \mid Y)$ is the minimum mean-squared error estimate (MMSE estimate) of X given Y in the sense that it minimizes the mean-squared error $\mathbb{E}\left((X - h(Y))^2\right)$ over all functions $h(Y)$. Consider now the following problem: find $\hat{a}$ and $\hat{b}$ to minimize the quantity

$$\mathbb{E}\left((X - aY - b)^2\right)$$

over all a and b. We call $\hat{a}Y + \hat{b}Y$ the linear minimum mean-squared error estimate (LMMSE estimate) of X given Y. In general, the LMMSE estimate won't do as well as the MMSE estimate in estimating X given Y, but in practice finding the LMMSE estimate is easier than finding the MMSE estimate.

(a) Show that for any a and b we have

$$\begin{aligned}\mathbb{E}\left((X - aY - b)^2\right) &= \operatorname{Var}(X - aY - b) + (\mathbb{E}(X) - a\mathbb{E}(Y) - b)^2 \\ &= \operatorname{Var}(X - aY) + (\mathbb{E}(X) - a\mathbb{E}(Y) - b)^2 \ .\end{aligned}$$

The first term in the variance of the estimation error and the second term is the square of what a statistician would call the bias of the estimate.

(b) Show that the optimal values $\hat{a}$ and $\hat{b}$ are

$$\hat{a} = \frac{\operatorname{Cov}(X, Y)}{\operatorname{Var}(Y)}$$

and

$$\hat{b} = \mathbb{E}(X) - \hat{a}\mathbb{E}(Y) = \mathbb{E}(X) - \frac{\operatorname{Cov}(X, Y)}{\operatorname{Var}(Y)}\mathbb{E}(Y) \ .$$

Thus in this setting we can simultaneously minimize the variance of the estimate and zero out the the bias of the estimate, thereby minimizing the mean-squared error of the estimate over all linear estimators of X given Y. A statistician would say that we encounter no "bias-variance tradeoff" here. That's not the case in other contexts such as machine learning, where similar but more complicated mean-square estimation and prediction roblems arise.

3. (Problem 4.28 in the book) The zero-mean bivariate Gaussian pdf takes the form

$$f_{X,Y}(x, y) = ce^{-q(x,y)} \ ,$$

where $q(x, y)$ is a quadratic function of x and y, specifically

$$q(x, y) = \frac{1}{2(1 - \rho)^2}\left(\frac{x^2}{\sigma_x^2} - 2\rho\frac{xy}{\sigma_x\sigma_y} + \frac{y^2}{\sigma_y^2}\right) \ ,$$

where σ_x and σ_y are positive constants, ρ is a constant satisfying $-1 < \rho < 1$, and $c > 0$ is a normalizing constant.

(a) Re-write $q(x, y)$ in the form $(\alpha x - \beta y)^2 + \gamma y^2$ by completing the square.

(b) Show that X and Y are zero-mean Gaussian random variables with respective variances σ_x and σ_y.

(c) Find the normalizing constant c.

(d) Show that $f_{X|Y}(x \mid y)$ is Gaussian. Find its mean $\mathbb{E}(X \mid Y = y)$ and variance $\operatorname{Var}(X \mid Y = y)$.

(e) Show that the correlation coefficient of X and Y is ρ.

(f) Show that X and Y are independent if and only if they are uncorrelated (we know the "only if" part already — the new thing is the "if").