Quotient Vector Spaces

Let V be a vector space over $\mathbb{F}$

Let $U \subseteq V$ be a subspace

goal: define + understand V/U

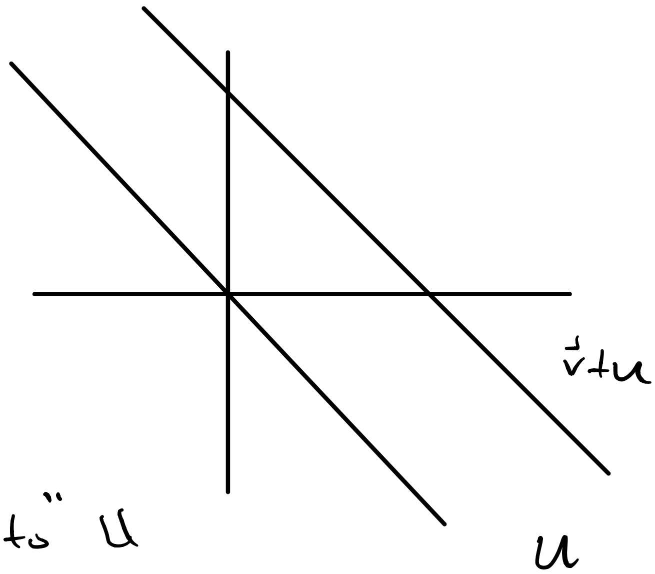
<u>Def</u>: If $\vec{v} \in V$, define $\vec{v} + U = \{\vec{v} + \vec{u} \mid \vec{u} \in U\}$

Example: $V = \mathbb{R}^2$

$U = \text{line} = \left\{ \begin{pmatrix} x \\ -2x \end{pmatrix} \in \mathbb{R}^2 \mid x \in \mathbb{R} \right\}$

$= \operatorname{span}\left(\begin{pmatrix} 1 \\ -2 \end{pmatrix} \right)$

if $\vec{v} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, what is $\vec{v} + U$?



Notation: We say $\vec{v} + U$ is "parallel to" U

(it is a parallel translate of U)

- $\vec{v} + U$ is called an <u>affine subset</u> of V

<u>Def</u>: The quotient space V/U is the set of all affine subsets of V parallel to U:

$$V/U = \{\vec{v} + U \mid \vec{v} \in V\}$$

In example above, $\mathbb{R}^2/U$ = set of parallel lines which are a parallel translate of U

Next Step: Make V/U into a vector space (over $\mathbb{F}$).

- Addition "should be" $(\vec{v}_1 + U) + (\vec{v}_2 + U) = (\vec{v}_1 + \vec{v}_2) + U$

$\star$

- Scalar Multiplication $c(\vec{v} + U) = c\vec{v} + U$

Lemma: Suppose $U \in V$ is a supspace; $\vec{v}, \vec{w} \in V$

Then the following are equivalent

① $\vec{v} + U = \vec{w} + U$

② $(\vec{v} + U) \cap (\vec{w} + U) \neq \phi$

③ $\vec{v} - \vec{w} \in U$

Proof

① ⇒ ②

$(\vec{v}+U) \cap (\vec{w}+U) \overset{\text{by hypothesis}}{=} \vec{v} + U \neq \phi$ since $\vec{v} \in \vec{v} + U$

② ⇒ ③

by hypothesis, suppose $\exists\ \vec{u}_1, \vec{u}_2 \in U$ s.t. $\vec{v} + \vec{u}_1 = \vec{w} + \vec{u}_2$

then $\vec{v} - \vec{w} = \vec{u}_2 - \vec{u}_1 \in U$

③ ⇒ ① by hypothesis, suppose $\exists\ \vec{u} \in U$ s.t. $\vec{v} - \vec{w} = \vec{u}$

show $\vec{v} + U = \vec{w} + U$

Choose $\vec{u}_1 \in U$. Show $\vec{v} + \vec{u}_1 \in \vec{w} + U$

i.e show $\exists\ \vec{u}_2$ s.t. $\vec{v} + \vec{u}_1 = \vec{w} + \vec{u}_2$. Choose $\vec{u}_2 = \vec{u} + \vec{v}_1$

So the $\{\vec{v} + U\}$ are all disjoint

Lemma: the operations ($\star$) are well-defined

i.e. if $\vec{v}_1 + U = \vec{v}_1' + U$

$\vec{v}_2 + U = \vec{v}_2' + U$

then (a) $(\vec{v}_1 + \vec{v}_2) + U = (v_1' + v_2') + U$

(b) $(c\vec{v}_1) + U = (cv_1') + U$

Proof (a) holds iff $(\vec{v}_1 + \vec{v}_2) - (\vec{v}_1' + \vec{v}_2') \in U$

Since $\vec{v}_1 + U = \vec{v}_1' + U$

$\vec{v}_2 + U = \vec{v}_2' + U$

then $\vec{v}_1 - \vec{v}_1' \in U$, $\vec{v}_2 - \vec{v}_2' \in U \Rightarrow$

Theorem With these operations, V/U is a vector space over $\mathbb{F}$

Proof

zero elem. $\vec{0}_{V/U} = \vec{0} + U = U$!

add. inv. $-(\vec{v} + U) = (-\vec{v}) + U$

Others "easy" to check

Question: What is $\dim V/U$?

(1) $\mathbb{R}^2 / \underbrace{\text{span}\begin{pmatrix} 1 \\ -2 \end{pmatrix}}_{U} \rightarrow \dim = 1$

Spanning set: $\begin{pmatrix} 1 \\ 0 \end{pmatrix} + U = \vec{f}_1 \in \mathbb{R}^2/U$

$\begin{pmatrix} 0 \\ 1 \end{pmatrix} + U = \vec{f}_2 \in \mathbb{R}^2/U$

$\vec{0}_{\mathbb{R}^2/U} = \vec{f}_1 - 2\vec{f}_2 = \begin{pmatrix}0\\0\end{pmatrix} + U = U \Rightarrow$ Linear Dependence

Remove $\vec{f}_2$, $\{\vec{f}_1\}$ now spans $\mathbb{R}^2/U$

② $\mathbb{R}^3 / \text{span}\begin{pmatrix}1\\1\\1\end{pmatrix}$, dim = 2

Quotient Map

Define (for $U \subseteq V$ a subspace)

$$\pi : V \to V/U$$
$$\vec{v} \longmapsto \vec{v} + U$$

NOTE

$\ker \pi = U$

$\text{im}\, \pi = V/U$ (π surj.)

$\dim V/U = \dim V - \dim U$

① Is π a LT? YES

Proof: if $\vec{v}, \vec{w} \in V$

then $\pi(\vec{v} + \vec{w}) = \pi(\vec{v}) + \pi(\vec{w})$

equality holds by definition
$$\begin{cases} \pi(\vec{v} + \vec{w}) = (\vec{v} + \vec{w}) + U \\ \pi(\vec{v}) + \pi(\vec{w}) = (\vec{v} + U) + (\vec{w} + U) \end{cases}$$

if $c \in \mathbb{F}$, $\vec{v} \in V$

$\pi(c\vec{v}) = c\pi(\vec{v})$