ECE 4110 Homework 1

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Due by 5pm on September 12

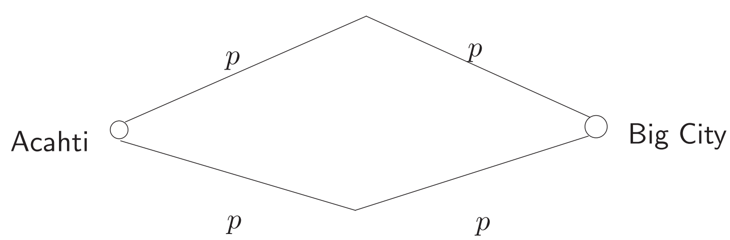
1 Key Concepts and Reading Material

- Probability space and basic properties (Chapter 2.1-2.2).
- Conditional probability, total probability theorem and Bayes' rule (Chapter 2.4).
- Independence of events (Chapter 2.5)

2 Assignment

1. The union bound is useful because it does not require that the events A_i be independent or mutually exclusive. Suppose that a hard drive contains N bytes, each of which is uniformly distributed over the values $\{0, \ldots, 255\}$. Use the union bound to upper bound the probability of the event that somewhere on the hard drive there are k consecutive bytes that are all equal to zero. Evaluate the bound numerically for $k = 4$ and $N = 1024$.

2. The little town of Acahti connects its telephone network to the Big City using two routes, called the upper route and lower route. Both routes consist of two links in series.



Each link fails with probability p, independently of the others. We say Acathi is connected to the Big City if at least one of the two routes has no failed links.

(a) What is the probability that Acahti is connected to the Big City?

(b) A phone company worker checks the link on the lower route that is closest to Acahti and sees that it is working. What now is the probability that Acahti is connected to the Big City?

(c) The worker from (b) then learns, in addition, that Acahti is indeed connected to the Big City. What is the probability that the lower route has no failed links?

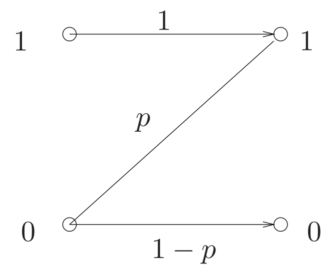
(d) Suppose Acahti demands "five nines" reliability of its connection. How small must p be so that the probability in (a) exceeds .99999?

3. Let A and B be two events. We consider the probability $A \cap B$ conditioned on (i) A occurred; (ii) $A \cup B$ occurred. (Which probability is bigger?) We illustrate this with a concrete case. There are 2 jokers (one black and white, the other colored) in a deck of 54 cards. John picks 4 cards.

   (a) What is the probability that he has both jokers if he tells you that he has the colored joker?

   (b) What is the probability that he has both jokers if he tells you that he has a joker?

4. Consider the asymmetric binary "Z" channel specified in the figure below.



   Suppose that the transmitter transmits $X = 0$ with probability q and $X = 1$ with probability $1 - q$.

   (a) What is the probability that the receiver will receive $Y = 0$?

   (b) Suppose that when the receiver receives $Y = 0$, it detects that 0 was transmitted, and when it receives $Y = 1$, it detects that 1 was transmitted. What is the probability that the receiver will make a mistake?

5. Suppose we toss two fair dice. Let E_1 denote the event that the sum of the dice is six and F denote the event that the first die equals four.

   (a) Are E_1 and F independent?

   (b) Let E_2 be the event that the sum of the dice equals seven. Is E_2 independent of F?

   (c) Give intuitive explanations for your answers to the above.

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ECE 4110 HW1

9/12/19 5:00PM

① Hard drive contains N bytes, each uniformally distributed over the values $\{0,1,\ldots,255\}$.

Use the Union bound to upper bound the probability of the event that somewhere on the hard drive there are k consecutive bytes that are equal to 0.

$\Pr(\bigcup_i A_i) \leq \sum_{i=1}^{n} \Pr(A_i)$

Let b_i be one of N bytes, denote

$$\Omega = \{b_i \mid 1 \leq i \leq N\}$$

Ω is the set of all bytes

For one j value in $1 \leq j \leq N-k+1$, denote the set of k-consecutive bytes as

$$\phi_j = \{b_s \mid j \leq s \leq j+k\}$$

ϕ_j is a set of k consecutive bytes. $\exists$ N-k ϕ_j sets

There are N-k+1 possible j-values so there are N-k+1 possible ϕ_j sets ($\cancel{\text{Note}}$ $\phi_1 \cup \phi_2 \cup \cdots \cup \phi_{N-k} = \Omega$)

The probability k-consecutive bytes equals zero is thus the probability every element of ϕ_j equals 0 for some j.

Denote X_j as the event all elements of ϕ_j equal 0

by independence

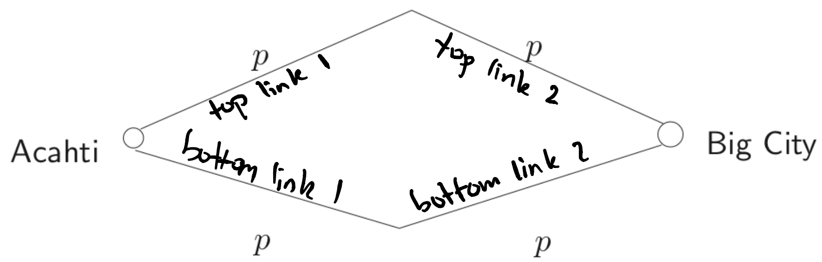
$$\Pr(X_j) = \Pr(b_j = 0 \cap b_{j+1} = 0 \cap \cdots \cap b_{j+k} = 0) = \left(\frac{1}{256}\right)^k \quad \forall j$$

For k-consecutive bits to be zero somewhere on the hard drive, need X_j to occur for at least one j.

$$\Pr\left(\bigcup_{i=1}^{N-k+1} X_i\right) \leq \sum_{i=1}^{N-k+1} \Pr(X_i) = (N-k+1)\left(\frac{1}{256}\right)^k$$

$\boxed{\text{For } N = 1024,\ k = 4\ ;\ \text{get } 2.38\text{e-}7}$

② The little town of Acahti connects its telephone network to the Big City using two routes, called the upper route and lower route. Both routes consist of two links in series.



Each link fails with probability p, independently of the others. We say Acathi is connected to the Big City if at least one of the two routes has no failed links.

(a) $\Pr(\{\text{Acahti is connected to the big city}\})$?

Let $A = \{\text{upper route works}\}$
$B = \{\text{bottom route works}\}$

Want

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$
$$= (1-p)(1-p) + (1-p)(1-p) + (1-p)^4$$
$$= 2(1-p)^2 + (1-p)^4$$

(b) Let $C = \{\text{left link of lower route works}\})$

$$\Pr(A \cup B \mid C) = \frac{\Pr((A \cup B) \cap C)}{\Pr(C)} = \frac{\Pr((A \cap C) \cup (B \cap C))}{\Pr(C)}$$

$$= \frac{\Pr((A \cap C) \cup B)}{\Pr(C)} = \frac{\Pr(A \cap C) + \Pr(B) - \Pr(A \cap B \cap C)}{\Pr(C)}$$

$$= \frac{(1-p)^3 + (1-p)^2 + (1-p)^4}{(1-p)^2} = 1 + (1-p) + (1-p)^2$$

(c) $\Pr(B \mid (A \cup B) \cap C) = \dfrac{\Pr(B \cap [(A \cup B) \cap C])}{\Pr((A \cup B) \cap C))}$

$$
\begin{aligned}
\Pr(B \cap [(A \cup B) \cap C]) &= \Pr(B \cap (A \cup B) \cap B \cap C) \\
&= \Pr(B \cap A \cup B \cap B \cap B \cap C) \\
&= \Pr(A \cap B \cup B \cap B \cap C) \\
&= \Pr(B \cup A \cap B) \\
&= \Pr(B) + \Pr(A \cap B) - \Pr(A \cap B \cap B) \\
&= \Pr(B) = (1-p)^2
\end{aligned}
$$

$\Pr(A \cup B \cap C) = (1-p)^3 + (1-p)^2 + (1-p)^4$

Thus,

$$\frac{\Pr(B \cap [(A \cup B) \cap C])}{\Pr((A \cup B) \cap C))} = \frac{(1-p)^2}{(1-p)^3 + (1-p)^2 + (1-p)^4}$$

(d) $2(1-p)^2 + (1-p)^4 > 0.99999$

$1-p > 0.996837723$

$p < 0.0015823910$

③

Let A and B be two events. We consider the probability $A \cap B$ conditioned on (i) A occurred; (ii) $A \cup B$ occurred. (Which probability is bigger?) We illustrate this with a concrete case. There are 2 jokers (one black and white, the other colored) in a deck of 54 cards. John picks 4 cards.

(a) Probability he has both jokers if he tells you he has the colored joker?

→ John has 4 cards

→ one is colored joker

→ odds one of the three remaining is black/white joker?

Let $A = \{\text{has colored joker}\}$

$B = \{\text{has black/white joker}\}$

$$Pr(A \cap B \mid A) = \frac{Pr(A \cap B \cap A)}{Pr(A)} = \frac{Pr(A \cap B)}{Pr(A)} = \frac{\binom{52}{2} / \binom{54}{4}}{\binom{53}{3} / \binom{54}{4}} = \frac{3}{53}$$

(b) Probability he has both jokers if he tells you he has a joker?

→ John has 4 cards

→ one is **a** joker

→ odds one of the three remaining is <u>the other</u> joker?

Let $A = \{\text{has colored joker}\}$

$B = \{\text{has black/white joker}\}$

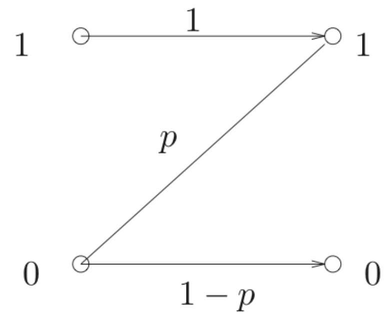
$$Pr(A \cap B \mid A \cup B) = \frac{Pr(A \cap B \cap (A \cup B))}{Pr(A \cup B)} = \frac{Pr((A \cap B \cap A) \cup Pr(A \cap B \cap B))}{Pr(A \cup B)}$$

$$= \frac{Pr(A \cap B \cup A \cap B)}{Pr(A \cup B)} = \frac{Pr(A \cap B)}{Pr(A \cup B)} = \frac{\binom{52}{2} / \binom{54}{4}}{2\binom{53}{3} / \binom{54}{4} - \binom{52}{2} / \binom{54}{4}} = \frac{3}{103}$$

$$Pr(A \cup B) = Pr(A) + Pr(B) = Pr(A \cap B)$$

Thus $Pr(A \cap B \mid A) \geq Pr(A \cap B \mid A \cup B)$ in general

(4) Consider the asymmetric binary "Z" channel specified in the figure below.



$p \in (0,1)$

Suppose that the transmitter transmits $X = 0$ with probability q and $X = 1$ with probability $1 - q$.

$q \in (0,1)$

(a) $Pr(\{\text{Receiver receives } Y = 0\})$?

Transmitter must transmit 0 **AND** it must receive 0

Thus,

$$Pr(\{\text{Receiver receives } Y=0\}) = \underbrace{(1-p)}_{Pr(Y=0|X=0)}\underbrace{q}_{Pr(X=0)} + \underbrace{0}_{Pr(Y=0|X=1)\,Pr(X=1)}$$

$Pr(Y=0|X=1) = 0$, $Pr(X=1) = (1-q)$

(b) Suppose that when the receiver receives $Y = 0$, it detects that 0 was transmitted, and when it receives $Y = 1$, it detects that 1 was transmitted. What is the probability that the receiver will make a mistake?

$Pr(\{\text{receiver makes mistake}\})$?

0 is transmitted with probability q

1 is transmitted with probability $1-q$

If 0 is transmitted $\begin{cases} \text{receive } 0 \text{ w/ probability } (1-p) \\ \text{receive } 1 \text{ w/ probability } p \end{cases}$

If 1 is transmitted $\begin{cases} \text{receive } 0 \text{ w/ probability } 0 \\ \text{receive } 1 \text{ w/ probability } 1 \end{cases}$

Thus, a mistake is made **ONLY** if 0 is transmitted and 1 is detected.

$$Pr(\{\text{receiver makes mistake}\}) = qp \qquad \underbrace{Pr(X=0)}_{q}\,\underbrace{Pr(Y=1|X=0)}_{p}$$

(5) Suppose we toss two fair dice. Let E_1 denote the event that the sum of the dice is six and F denote the event that the first die equals four.

(a) Are E_1 and F independent?

(first roll, second roll)

$$E_1 = \{(1,5),(2,4),(3,3),\boxed{(4,2)},(5,1)\}$$
$$F = \{(4,1),\boxed{(4,2)},(4,3),(4,4),(4,5),(4,6)\}$$

E_1 and F are independent if and only if

$$Pr(E_1 \cap F) = Pr(E_1)Pr(F)$$

amount of tuples in E_1

$Pr(E_1) = 5 \cdot \frac{1}{36}$ possible tuples $\rightarrow Pr(E_1)Pr(F) = \frac{30}{36^2}$

amount of tuples in F

$Pr(F) = 6 \cdot 1/36$ possible tuples

$Pr(E_1 \cap F) = \frac{1}{36} \rightarrow Pr(E_1 \cap F) \neq Pr(E_1)Pr(F)$

Thus E_1 is not independent of F

(b) Let E_2 = event that sum of two dies is 7.

Are E_2 and F independent?

(first die, second die)

$$E_2 = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$$
$$F = \{(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)\}$$

$Pr(E_2 \cap F) = \frac{1}{36} = \left(Pr(E_2) = \frac{6}{36}\right)\left(Pr(F) = \frac{6}{36}\right)$ ∴ Independent!

(c) In part (a) you could roll a 6 on the first roll which hinders the ability to roll a 6 b/t two die. So the event depends on the outcome of the first die.

In part (b), no matter the outcome of the first die a 7 can be rolled. So the events are independent.