2) $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$,
where $a_n \neq 0$.
For a sufficiently large value of R,

$$\oint_{|z|=R} \frac{P'(z)}{P(z)}\, dz = 2n\pi i$$

Consider $P(z)$ as a perturbation of $f(z) = a_n z^n$
which has n zeros at the origin.

$$P(z) - f(z) = a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

is a polynomial of degree $(n-1) < n$!

So on $|z| = R$,

$$|f(z)| = |a_n| R^n$$

Therefore if we choose $R > 1$

$$\frac{|a_{n-1}|}{|a_n|} + \cdots + \frac{|a_1|}{|a_n|} + \frac{|a_0|}{|a_n|} < R$$

$|P(z) - f(z)| < |f(z)|$ is valid on $|z| = R$,

so it must be that $P(z)$ has n zeros!

So by argument principle, $\int_{|z|=R} \frac{P'(z)}{P(z)}\, dz = 2\pi i (n)$!

5) If $f(z)$ is analytic inside and on a simple closed contour C and is one-to-one on C, then $f(z)$ is one-to-one inside C.

If f is nonconstant and analytic in a domain C, then it's range

$$f(C) := \{w \mid w = f(z) \text{ for some } z \text{ in } C\}$$

is an open set.

So that means the open set must be one-to-one if the boundary is one-to-one.

7) $f(z) = z^3 + 9z + 27$ has no roots on $|z| < 2$

Let $h(z) = z^3 + 27 \rightarrow \|h(z)\| \leq |z^3| + 27$

$$|z^3 + 27| < |z^3 + 9z + 27|$$

$$z^3 + 27 = 0$$

$$z^3 = -27$$

$$z = \sqrt[3]{27}\, e^{i\left(\frac{\pi + 2\pi k}{3}\right)} \qquad k = 0, 1, 2$$

$$\left.\begin{aligned} z_0 &= \sqrt[3]{27}\, e^{i\pi/3} \\ z_1 &= \sqrt[3]{27}\, e^{i\pi} \\ z_2 &= \sqrt[3]{27}\, e^{i5\pi/3} \end{aligned}\right\} \text{all zeros of } h(z) \text{ lie outside } |z| = 2$$

So, zeros of $f(z)$ lie outside $|z| = 2$

(10) $z = 2 - e^{-z}$ has exactly one REAL root in the right half plane.

Proof:

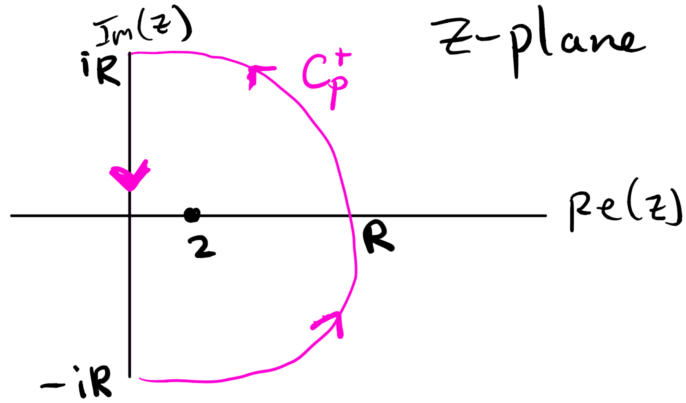
Let

$$f(z) = 2 - z - e^{-z}$$

The roots are found via setting $f(z)$ to 0.

$$f(z) = 2 - z - e^{-z} = 0$$

Define C_p^+ as shown in the z-plane here. where $R \to \infty$



Regard the function

$$f(z) = 2 - z - e^{-z}$$

as a perturbation of

$$g(z) = 2 - z$$

which has exactly one zero in the right half plane. (at $z = 2$)

Then for z on C_p^+, we have

$$\begin{aligned}|h(z)| &= |f(z)-g(z)| \\ &= |-e^{-z}| \\ &= e^{-\operatorname{Re}(z)} \leq e^{-0} = 1\end{aligned}$$

while $g(z)$ is bounded from below on C_p^+ by

$$|g(z)| = |2-z| \geq \begin{cases} 2 & , \text{ for } z = iy \\ |z|-2 = R-2 & , \text{ for } |z| = R \end{cases}$$

Thus for $R > 3$

$$|h(z)| < |g(z)| \quad \forall\, z \in C_p^+$$

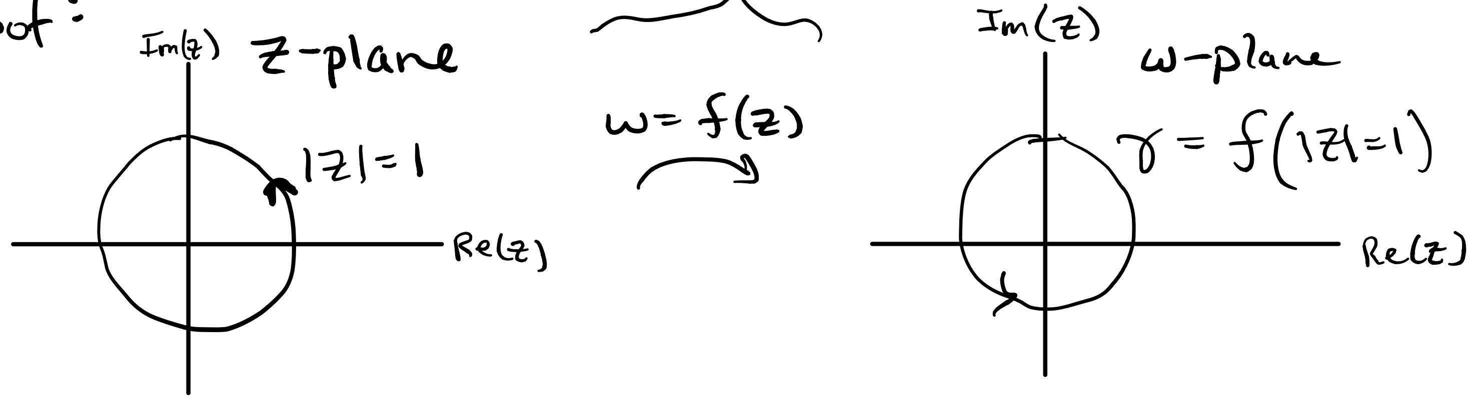
This implies that $f(z)$ also has precisely one (simple) zero inside C_p^+, and hence (letting $R \to \infty$) in the right half-plane. It must be real since if it had an imaginary component we'd also have to include its complex conjugate.

12a) $f(z)$ analytic on $|z| \leq 1$ and satisfies
$|f(z)| < 1$ for $|z| = 1$.

Prove that $f(z) = z$ has exactly one root (counting multiplicity) in $|z| < 1$.

a visual for me, can ignore

Proof:



$f(z) - z = 0$ what we want zeros of

$$\frac{1}{2\pi i} \oint \frac{1}{w}\, dw = \frac{1}{2\pi i} \oint \frac{f'(z)}{f(z) - z}\, dz$$

$$= \frac{1}{2\pi i} \oint_{|z| \leq 1} \frac{f'(z)}{f(z) - z}\, dz$$

$$= \frac{2\pi i}{2\pi i} = 1$$

21) $F(z) = 1 + P(z)$

has its zeros in the left half plane if the feedback control system is stable.

Considering γ_r, shown below,

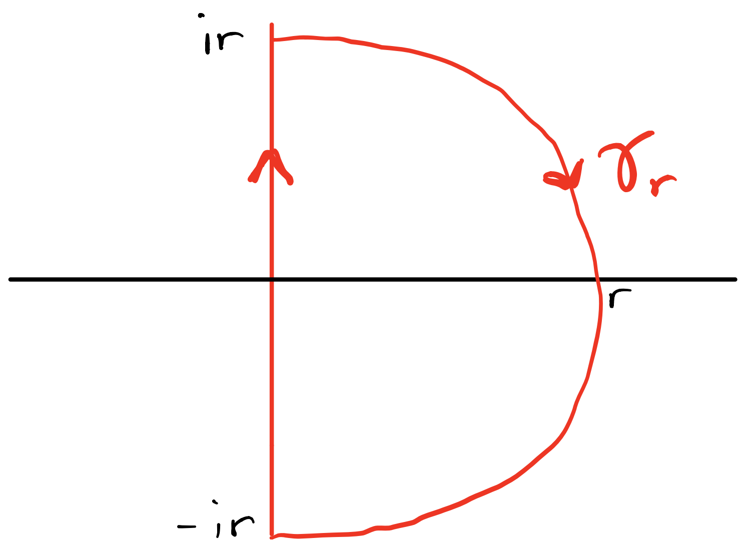


image contour

let m = number of times that $P(\gamma_r)$ encircles $w_0 = -1$ in a counterclockwise direction. (net)
Then let n = number of poles of $P(z)$ with positive real parts.

If $m = n$, for sufficiently large r, then all zeros of $F(z)$ lie in the left half plane; implying system stability.

Proof:

Note how $F(z)=0$ only when $P(z)=-1$.

So for $m = P(\gamma_r)$ encircles -1 (net #)
$n =$ poles of $P(z)$ w/ $Re(z)>0$

We now apply the argument principle

$$\frac{1}{2\pi i}\int_{\gamma_r} \frac{F'(z)}{F(z)}\,dz = \frac{1}{2\pi i}\int_{\gamma_r} \frac{P'(z)}{1+P(z)}\,dz = -N_0(F) + N_p(F)$$

- $F(z)$ has poles when $P(z)$ has poles.
- $F(z)$ has n poles with $Re(z)>0$
- $F(z)$ has a zero at $P(z)=-1$

For $m=n$ it must be that the poles of $P(z)$ equal the amount of times the image contour encircles -1.