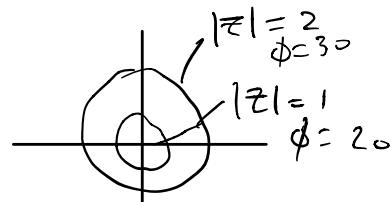


Example: "Washers"



By circular symmetry,

seek  $\phi = \phi(r)$  only, not dependent on  $\theta$ .

Guess:

$$\phi = A \ln r + B \quad \leftarrow \text{For Washer Probs}$$

Idea: Real and Imaginary part of an analytic  $f(z)$   
automatically satisfy  $\nabla^2 u = 0$  or  $\nabla^2 v = 0$

Recall  $\log(z) = \ln(r) + i\theta$ .

$$\text{On } r=1, \quad \phi = 20 = A \ln(1) + B \Rightarrow B = 20$$

$$\text{On } r=2, \quad \phi = 30 = A \ln(2) + 20 \\ \Rightarrow A = 10/\ln(2)$$

$\therefore$

$$\phi(z) = \frac{10}{\ln(2)} \ln(|z|) + 20$$

①

$$\phi(z) = A \ln(r) + B$$

$$\text{On } r=1, \quad 0 = A \ln(1) + B \Rightarrow B = 0$$

$$\text{On } r=2, \quad 10 = A \ln(2)$$

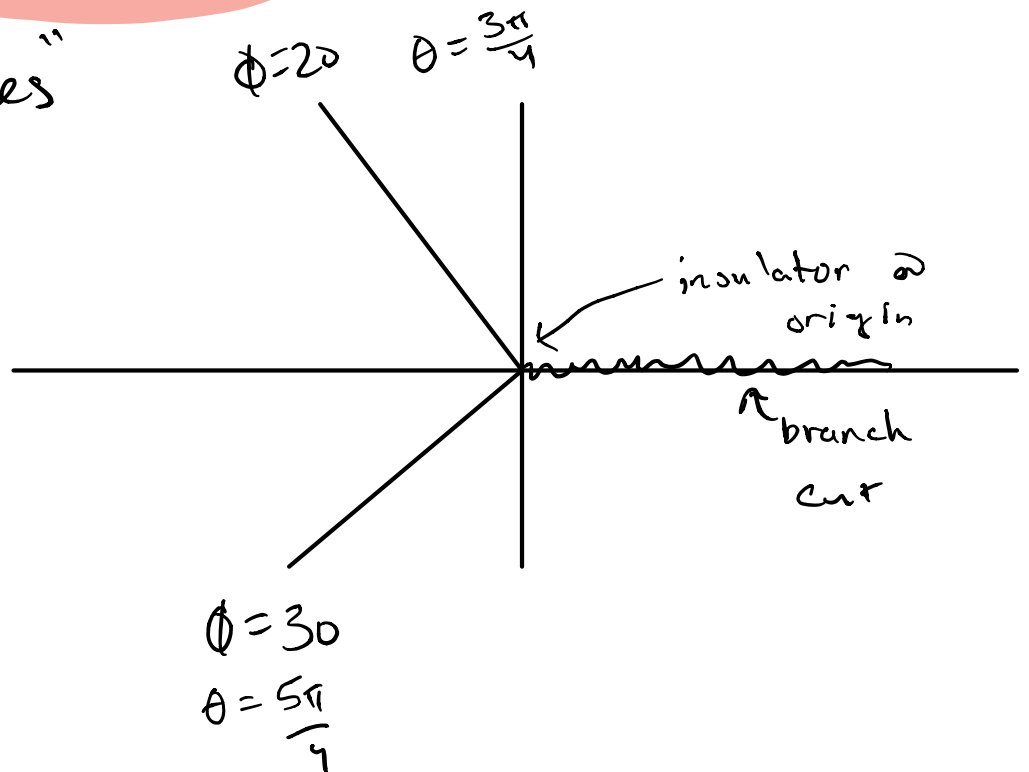
$$A = \frac{10}{\ln(2)}$$

$$\phi(z) = \frac{10}{\ln(2)} \ln(|z - z_0|)$$

At  $z=0$ ,

$$\begin{aligned}\phi(0) &= \frac{10}{\ln(2)} \cdot \ln(10-i-1) \\ &= \frac{10 \ln(\sqrt{2})}{\ln(2)}\end{aligned}$$

Example: "Wedges"



$\phi = A\theta + B$  (use when  $\phi$  is constant on radial lines)

$$20 = A \frac{3\pi}{4} + B$$

$$30 = A \frac{5\pi}{4} + B$$

$$-10 = -A \frac{\pi}{2}$$

$$B = 5$$

$$A = \frac{20}{\pi}$$

$$\Rightarrow \phi = \frac{20}{\pi} \theta + 5$$

$$\textcircled{2} \quad \phi(z) = A\theta + B$$

At  $1, i$ , angles  
 $0, \frac{3\pi}{2}$

$$0 = A \frac{3\pi}{2} + B$$

$$10 = A(0) + B \Rightarrow B = 10$$

$$A = -\frac{20}{3\pi}$$

$$\phi(z) = -\frac{20}{3\pi} \theta + 10$$

$$\text{At } z = 0$$

$$\phi(0) = -\frac{20}{3\pi} \text{Arg}(0-1-i) + 10$$

$$= \frac{-20}{3\pi} \cdot \frac{5\pi}{4} + 10$$

$$= -\frac{20}{12} + 10$$

③ Solution has lines of slope 3

i.e

$$x + y = c$$

So

$$\phi(x, y) = A(x + y) + B$$

$$10 = A(3) + B$$

$$10 = 3A - 20$$

$$-50 = A(-3) + B$$

$$A = 10$$

$$-40 = 2B$$

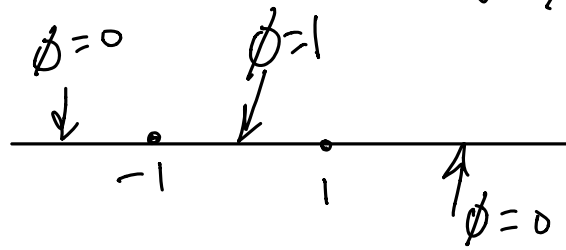
$$B = -20$$

$$\Rightarrow \phi(x, y) = 10(x + y) - 20$$

$$\phi(1, 1) = 10(2) - 20 = 0$$

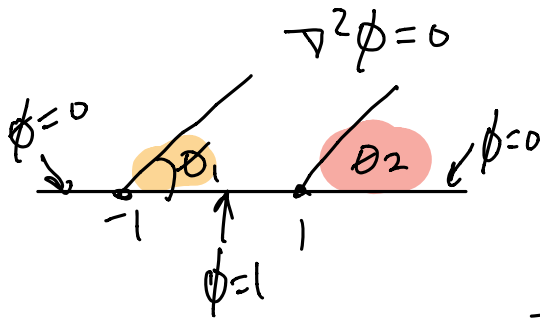
Ex 3: "Wall"

$$\nabla^2 \phi = 0 \text{ here}$$



Sol'n

Define two angles



Seek soln of

$$\phi = A_1 \theta_1 + A_2 \theta_2 + B$$

$$= A_1 \text{Arg}(z+1) + A_2 \text{arg}(z-1) + B$$

$$-\pi < \theta_1 \leq \pi$$

$$-\pi < \theta_2 \leq \pi$$

Impose B.C.

for

$$z = x > 1, \text{ have } \theta_1 = \theta_2 = 0$$

so

$$\phi = A_1(0) + A_2(0) + B$$

$$\Rightarrow B = 0$$

For

$$-1 < x < 1 : \theta_1 = 0, \theta_2 = \pi$$

so

$$\phi = A_1(0) + A_2\pi = 1$$

$$A_2 = \frac{1}{\pi}$$

For  $z = x < -1$  :  $\theta_1 = \theta_2 = \pi$

$$\phi = 0 = A_1\theta_1 + \frac{1}{\pi}\theta_2$$

$$0 = A_1\pi + \frac{1}{\pi}\pi$$

$$A_1 = -\frac{1}{\pi}$$

Therefore,

$$\begin{aligned}\phi &= \frac{1}{\pi} \left[ \text{Arg}(z-1) - \text{Arg}(z+1) \right] \\ &= \frac{1}{\pi} \left[ \theta_2 - \theta_1 \right]\end{aligned}$$

④ "Wall"

Soln of the form

$$\phi(z) = A_1 \text{Arg}|z+1| + A_2 \text{Arg}|z-2| + A_3$$

$$\begin{aligned}\phi(z > 2) &= A_1(0) + A_2(0) + A_3 = 0 \\ &\Rightarrow A_3 = 0\end{aligned}$$

$$\begin{aligned}\phi(z < -1) &= A_1\pi + A_2\pi = 0 \\ A_1 &= -A_2\end{aligned}$$

$$\phi(-1^2 z < 2) = \pi = A_1(0) + A_2 \pi$$

$$\Rightarrow A_2 = 1$$

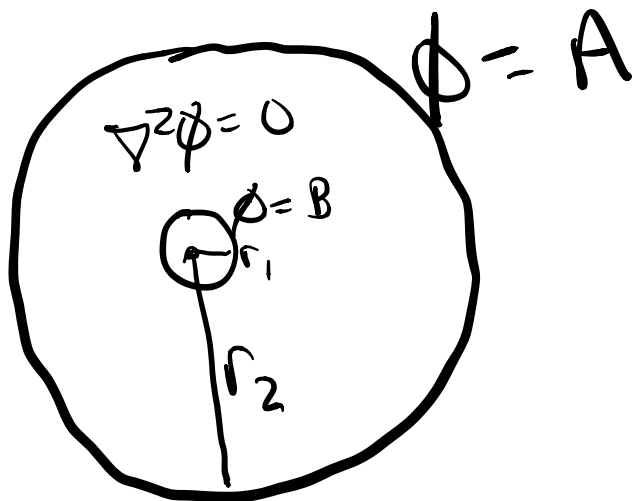
$$A_1 = -1$$

$$\Rightarrow \phi(z) = \text{Arg}|z+1| - \text{Arg}|z-2|$$

$$\phi(2,3) = \text{Arg}|3+3i| - \text{Arg}|0+3i|$$

$$= \frac{\pi}{4} - \frac{\pi}{2} = -\frac{\pi}{2}$$

⑤ General Soln To



$$\phi(z) = C_1 \ln(r) + C_2$$

$$\phi(r_1) = C_1 \ln(r_1) + C_2 = B$$

$$\phi(r_2) = C_1 \ln(r_2) + C_2 = A$$

$$C_1 \ln(r_1) - C_1 \ln(r_2) = B - A$$

$$C_1 = \frac{B-A}{\ln(r_1) - \ln(r_2)}$$

$$C_2 = B - \frac{(B-A)\ln(r_1)}{\ln(r_1) - \ln(r_2)}$$

$$= \frac{B\ln(r_1) - B\ln(r_2) - B\ln(r_1) + A\ln(r_1)}{\ln(r_1) - \ln(r_2)}$$

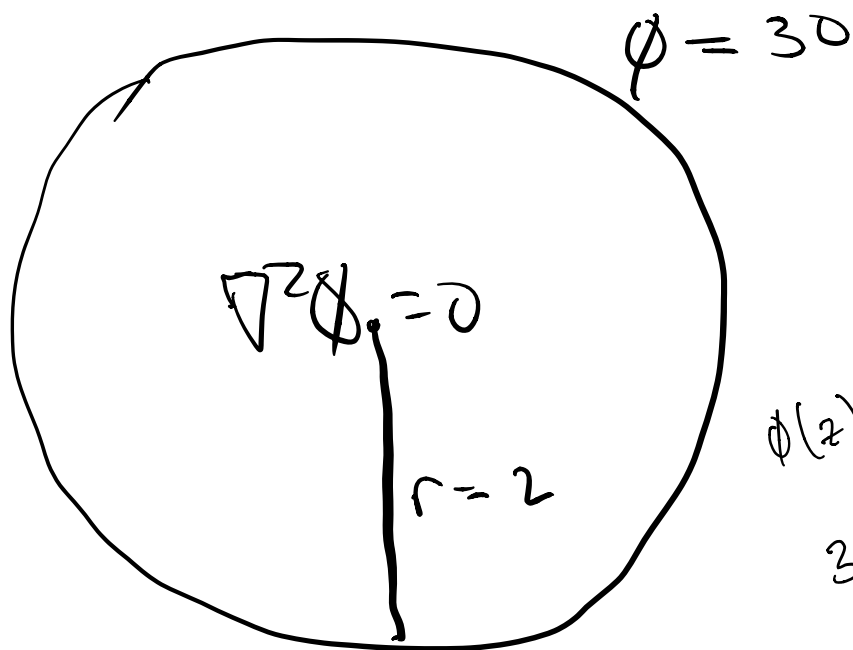
$$= \frac{A\ln(r_1) - B\ln(r_2)}{\ln(r_1) - \ln(r_2)}$$

$$\phi(z) = \frac{B-A}{\ln(r_1) - \ln(r_2)} \ln(|z|) + \frac{A\ln(r_1) - B\ln(r_2)}{\ln(r_1) - \ln(r_2)}$$

$$\lim_{r_1 \rightarrow 0} \phi(z) = \frac{-A}{\ln(r_1) - \ln(r_2)} \ln(|z|) + \frac{A\ln(r_1)}{\ln(r_1) - \ln(r_2)}$$



⑥  $\phi(x, y)$  for



$$\begin{aligned}\phi(z) &= A \ln r + B \\ 30 &= A \ln(2) + B \\ B &= 30 - A \ln(2) \\ \phi(z) &= A(\ln(z) - \ln(2)) + 30\end{aligned}$$

From above, let  $A=30$ ,  $r \rightarrow 0$

$$\phi(z) = \ln(|z|) - \frac{+30}{\ln(2)} = \frac{30}{\ln 2} \ln |z|$$

⑦ Prove that

$(\arg z)(\log |z|)$  is harmonic

$(\arg z)(\log |z|)$  is the imaginary part of

$\frac{(\log z)^2}{2}$  which is an analytic function

therefore it is harmonic!

$$\frac{(\log z)^2}{2} = \frac{(\ln r)^2}{2} + \frac{2i\theta \ln r}{2} - \frac{\theta^2}{2}$$

$$\operatorname{Im}\left(\frac{(\log z)^2}{2}\right) = \theta \ln r = (\arg z)(\log |z|) !$$