Let $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$.

We can then obtain

$$z^2 = r^2 e^{i2\theta}$$

Since $z^3 = z z^2$,

$$z^3 = r^3 e^{i3\theta}$$

Continuing in this manner we arrive at the formula for the n^{th} power of z:

$$z^n = r^n e^{in\theta} = r^n\left(\cos(n\theta) + i\sin(n\theta)\right) \qquad (1)$$

Eqn (1) is an appealing formula for raising a complex number to a positive integer power. It is easy to see that the identity is also valid for negative integers n. The question arises whether the formula will work for $n = 1/m$, so that $\zeta = z^{1/m}$ is an m^{th} root of z satisfying

$$\zeta^m = z \qquad (2)$$

Certainly if we define

$$\zeta = \sqrt[m]{r}\, e^{i\theta/m} \qquad (3)$$

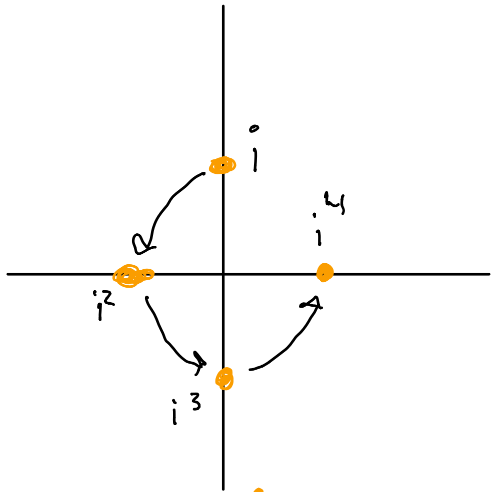
(where $\sqrt[m]{r}$ denotes the customary, positive, m^{th} root, we compute a complex number ξ satisfying Eq (2) [as is easily seen by applying Eq (1)].

It's more complicated than this.
1 for example has two square roots: 1, -1 :
And each of these in turn has two square roots - generating four fourth roots of 1, namely
1, -1, i, -i.

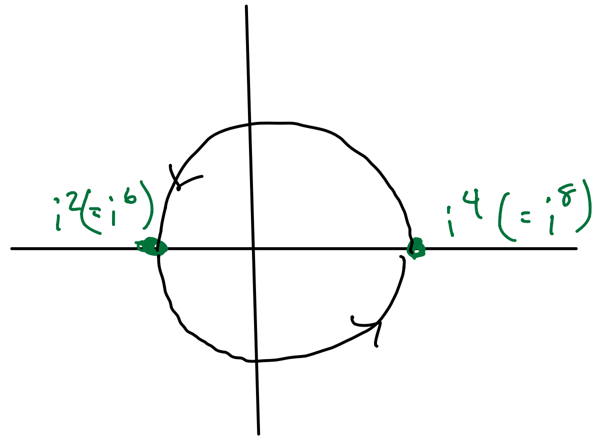
To see how the additional roots fit into the scheme of things, lets work out polar descriptions of the equation $\xi^4 = 1$ for each of these numbers:

$$1^4 = (1e^0)^4 = 1^4 e^{i0} = 1$$
$$i^4 = (1e^{i\pi/2})^4 = 1^4 e^{i2\pi} = 1$$
$$(-1)^4 = (1e^{i\pi})^4 = 1^4 e^{i4\pi} = 1$$
$$(-i)^4 = (1e^{i3\pi/2})^4 = 1^4 e^{i6\pi} = 1$$

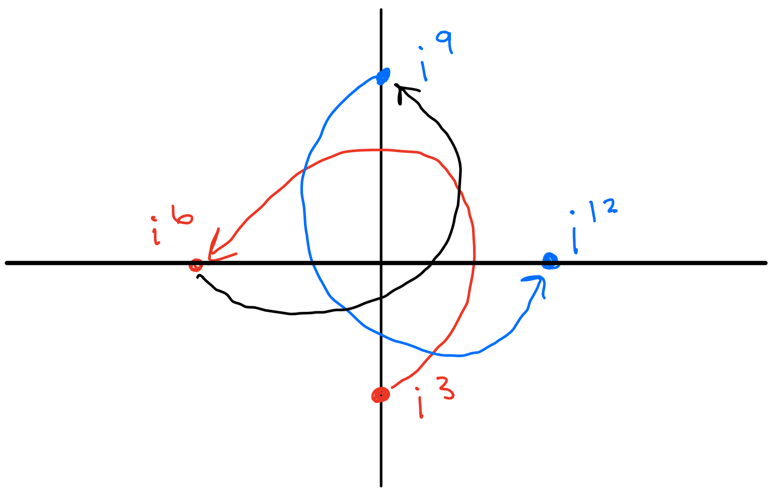
The powers of these roots are explored in the Argand diagram, (below)



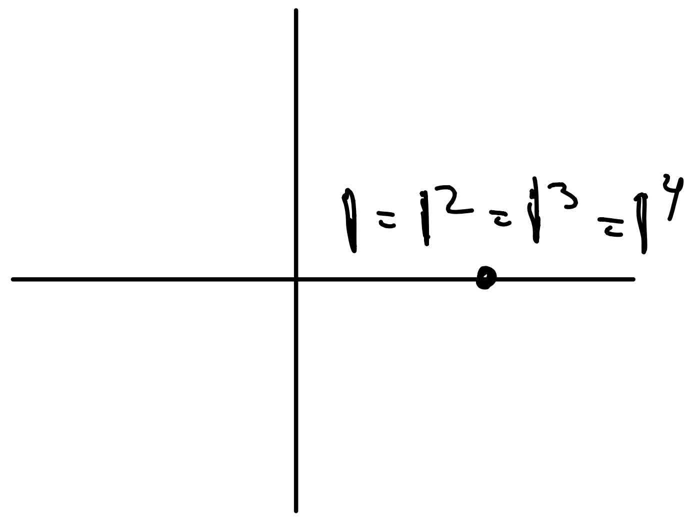
Powers of i



Powers of $i^2 = -1$



Powers of $i^3 = -i$



Powers of $i^4 = 1$

The multiplicity of roots is tied to the ambiguity in representing 1, in polar form as e^{i0}, $e^{i2\pi}$, $e^{i4\pi}$, etc. Thus to compute all the nth roots of a number z, we must apply formula (3) to **EVERY** polar representation of z.

For the cube roots of unity, for example, we would compute as follows:

Polar Representation of 1	Application of (3)
$\vdots$	$\vdots$
$1 = e^{-i6\pi}$	$1^{1/3} = e^{-i6\pi/3} = 1$
$1 = e^{-i4\pi}$	$1^{1/3} = e^{-i4\pi/3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$
$1 = e^{-i2\pi}$	$1^{1/3} = e^{-i2\pi/3} = -\frac{1}{2} - i\sqrt{3}/2$
$1 = e^{i0}$	$1^{1/3} = e^{i0/3} = 1$
$1 = e^{i2\pi}$	$1^{1/3} = e^{i2\pi/3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$
$1 = e^{i4\pi}$	$1^{1/3} = e^{i4\pi/3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$
$1 = e^{i6\pi}$	$1^{1/3} = e^{i6\pi/3} = 1$
$\vdots$	$\vdots$

Obviously the roots recur in sets of three, since $e^{i2\pi m_1/3} = e^{i2\pi m_2/3}$ whenever $m_1 - m_2 = 3$.

Generalizing, we can see that there are exactly m distinct m^{th} roots of unity, denoted by $1^{1/m}$, and they are given by

$$1^{1/m} = e^{i\frac{2\pi k}{m}} = \cos\left(\frac{2\pi k}{m}\right) + i\sin\left(\frac{2\pi k}{m}\right), \quad k = 0, 1, 2, \dots, m-1 \tag{4}$$

The arguments of these roots are $\frac{2\pi}{m}$ radians apart, and the roots themselves form vertices of a regular polygon.

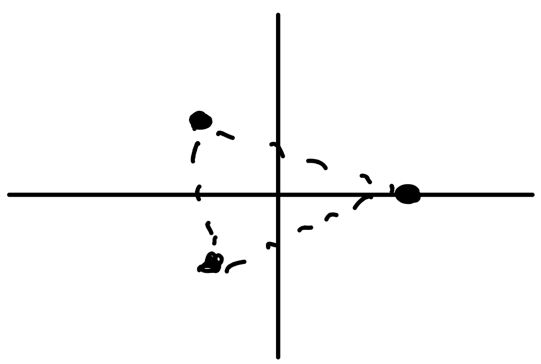
Taking $k=1$ in Eq (4), we obtain the root

$$\omega_m := e^{i2\pi/m} = \cos\left(\frac{2\pi}{m}\right) + i \sin\left(\frac{2\pi}{m}\right)$$

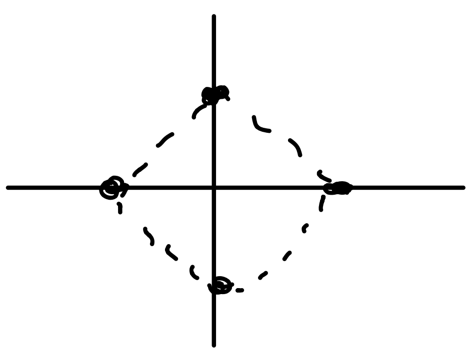
and it is easy to see that the complete set of roots can be displayed as

$$1, \omega_m, \omega_m^2, \ldots, \omega_m^{m-1}$$

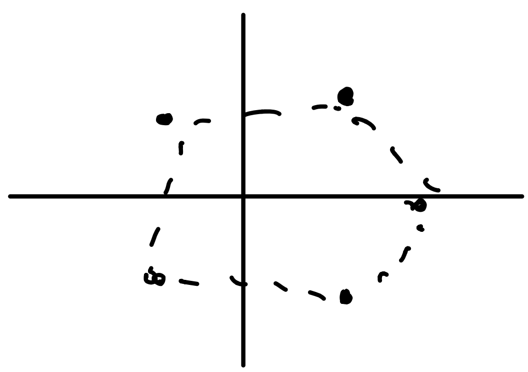
Aside (Polygons formed by roots of unity)



(a) $1^{1/3}$



(b) $1^{1/4}$



(poorly drawn)

Example 1: Prove that

$$1 + \omega_m + \omega_m^2 + \cdots + \omega_m^{m-1} = 0 \qquad (5)$$

From a physical point of view, since, by symmetry, the center of mass $(1+\omega_m+\omega_m^2+\cdots+\omega_m^{m-1})/m$ of the system of m unit masses located at m^{th} roots of unity must be at the origin.

Algebraically,

$$(\omega_m - 1)\left(1+\omega_m+\omega_m^2+\cdots+\omega_m^{m-1}\right) = \omega_m^m - 1 = 0$$

Since $\omega_m \neq 1$, Eqn (5) follows.

To obtain the m^{th} roots of an arbitrary (nonzero) complex number $z = re^{i\theta}$, we generalize the idea displayed by Eqn (4) and, reasoning similarly, conclude that m distinct m^{th} roots of z are given by

$$z^{1/m} = \sqrt[m]{|z|}\, e^{i(\theta + 2\pi k)/m} \quad (k = 0, 1, 2, \ldots, m-1) \qquad (6)$$

Example 2: Find all cube roots of $\sqrt{2} + i\sqrt{2}$

The polar form is $\sqrt{2} + i\sqrt{2} = 2e^{i\pi/4}$

Putting $|z| = 2$, $\theta = \pi/4$, $m = 3$ into Eqn (6), we obtain

$$\left(\sqrt{2} + i\sqrt{2}\right)^{1/3} = \sqrt[3]{2}\, e^{i\left(\frac{\pi}{12} + \frac{2k\pi}{3}\right)}, \quad k = 0, 1, 2$$

Therefore the three cube roots of $\sqrt{2} + i\sqrt{2}$ are

$$\sqrt[3]{2}\, \cos\left(\frac{\pi}{12} + i \sin\frac{\pi}{12}\right)$$

$$\sqrt[3]{2}\left(\cos\frac{3\pi}{4} + i \sin\frac{3\pi}{4}\right)$$

$$\sqrt[3]{2}\left(\cos\frac{17\pi}{12} + i \sin\frac{17\pi}{12}\right)$$

Example 3:

Let a, b, c be complex constants with $a \neq 0$. Prove the solutions of the equation

$$az^2 + bz + c = 0 \qquad (7)$$

are given by the (usual) quadratic formula

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad (8)$$

where $\sqrt{b^2 - 4ac}$ denotes one of the values of $(b^2 - 4ac)^{1/2}$

Multiplying (7) by $4a$,

$$4a^2z^2 + 4abz + 4ac = 0$$

$$4a^2z^2 + 4abz + b^2 = b^2 - 4ac$$

$$(2az + b)^2 = b^2 - 4ac$$

$$2az + b = (b^2 - 4ac)^{1/2} = \pm\sqrt{b^2 - 4ac}$$

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

✓