Real Integrals from $-\infty$ to $+\infty$

How to calculate $\int_{-\infty}^{+\infty} f(x)\,dx$, given $f(x)$.

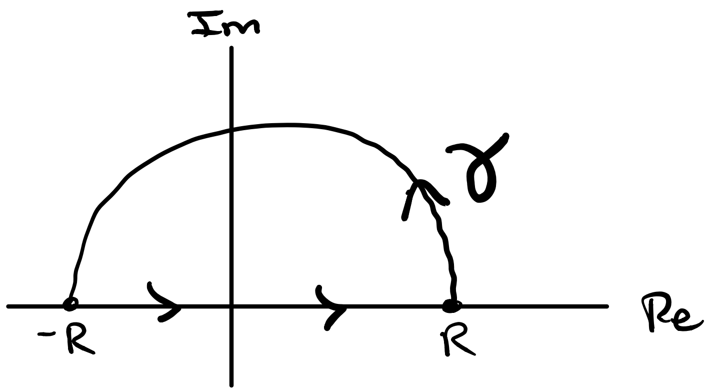
Cauchy principle value $\Rightarrow \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx$

Example 1)

$$I = \int_{-\infty}^{+\infty} \frac{dx}{1+x^4}$$

1. Choose a clever contour.

→ Consider a large semicircle as part of contour (common trick)



Put a semicircle in complex plane connecting points

← what we're interested in

Now look at $\oint_\gamma \frac{dz}{1+z^4} = \int_{-R}^{R} \frac{dx}{1+x^4} + \int_{\text{semi circle}} \frac{dz}{1+z^4}$

Strategy: calculate w/ residues

This is what we want as $R \to \infty$

goes to zero as $R \to \infty$

2. Show why $\displaystyle\int_{\text{semi circle}} \frac{dz}{1+z^4}$ goes to zero as $R \to \infty$

Use ML estimate:

$L = \pi R$ } on semicircle

$$M = \max_{z \in \text{semi circle}} \left| \frac{1}{1+z^4} \right|$$

Sloppy Way (Real Life): $\left| \dfrac{1}{1+z^4} \right| \simeq \left| \dfrac{1}{R^4} \right|$ for $R \gg 1$

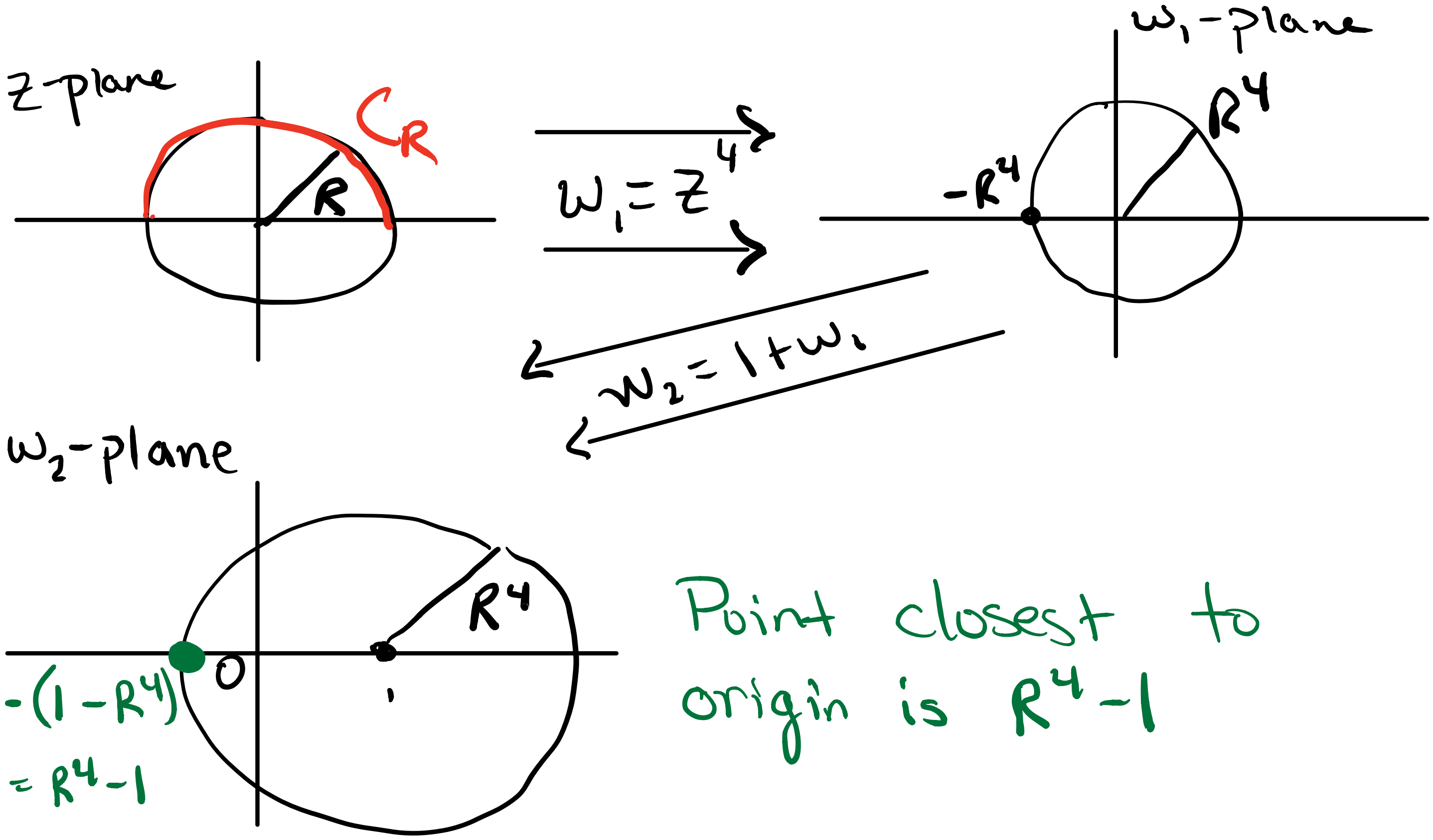
So expect $\displaystyle\left| \int_{\text{semi circle}} \frac{1}{1+z^4} dz \right| \simeq \frac{\pi R}{R^4} = \frac{\pi}{R^3} \to 0$ as $R \to \infty$

Careful Way (Papers, Tests):

$$\left| \int_{\text{semi-circle}} \frac{1}{1+z^4} \right| \leq \frac{\pi R}{R^4 - 1} \quad \forall\ R > 1$$

Let C_R denote semicircle

$$\max_{z \in C_R} \left| \frac{1}{1+z^4} \right| = \frac{1}{\min_{z \in C_R} |1+z^4|}$$



Point closest to origin is $R^4 - 1$

So, $\displaystyle \min_{z \in C_R} |1 + z^4| = R^4 - 1$

And

$$\left| \int_{C_R} \frac{dz}{1+z^4} \right| \leq \frac{\pi R}{R^4 - 1}$$

Hence, $\oint \frac{dz}{1+z^4} = 2\pi i \sum_{\substack{\text{poles inside} \\ \gamma}} \text{Res}\left(\frac{1}{1+z^4}\right)$ as $R \to \infty$

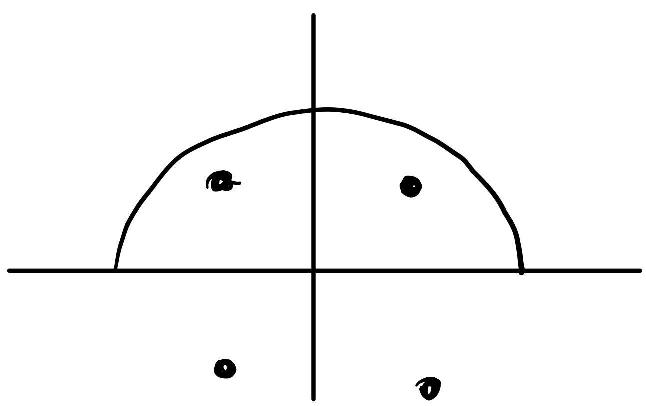
$$= \lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{1+x^4}$$

3. Find Residues

Poles occur when $z^4 = -1$

$$z^4 = e^{i\pi + i2\pi k}$$

$$z = e^{i\pi/4 + \pi/2 ik}, \quad k = 0, 1, 2, 3$$



only 2 of them lie in the contour ($k=0$, $k=1$)

$$\text{Res}\left(f(z); e^{i\pi/4}\right) = \lim_{z \to e^{i\pi/4}} \left[\left(z - e^{i\pi/4}\right) \frac{1}{z^4+1} \right]$$

Use L'hopitals

$$\lim_{z \to e^{i\pi/4}} \frac{1}{4z^3} = \frac{1}{4e^{i3\pi/4}}$$

Similarly,

$$\text{Res}\left(f(z);\ e^{i\frac{3\pi}{4}}\right) = \frac{1}{4e^{i9\pi/4}} = \frac{e^{-i\pi/4}}{4}$$

Now,

$$2\pi i \sum \text{Residues}$$

$$= \frac{2\pi i}{4}\left(e^{-i\frac{3\pi}{4}} + e^{-i\pi/4}\right)$$

$$= \frac{2\pi i}{4}\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i + -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)$$

$$= \frac{2\pi i}{4}\left(-\sqrt{2}\,i\right) = \frac{2\sqrt{2}\,\pi}{4} = \boxed{\frac{\sqrt{2}}{2}\pi}$$

SO,

$$\lim_{R \to \infty} \int_{-R}^{R} \frac{dx}{1+x^4} = \frac{\sqrt{2}}{2}\pi$$

SO MUCH WORK

SO SIMPLE

Example 2

$$I = \lim_{R \to \infty} \int_{-R}^{+R} \frac{\cos(ax)}{1+x^2} \quad , \quad a > 0$$

Can't use the same contour & strategy. $\cos(az)$ is exponentially large on semi-circle as $R \to \infty$.

For instance,

$$\cos(aiR) = \cosh(aR) = e^{aR}/2 \text{ as } R \to \infty$$

Don't wanna deal with cosine!

$$I = \int_{-R}^{R} \frac{\cos(ax)}{1+x^2} dx = \int_{-R}^{R} \frac{e^{iax}}{1+x^2} dx$$

$$= \int_{-R}^{R} \frac{\cos(ax)}{1+x^2} dx + i \underbrace{\int_{-R}^{R} \frac{\sin(ax)}{1+x^2} dx}_{\text{goes to } 0 \text{! odd function over symmetric interval.}}$$

We're led to consider

$$\oint_\gamma \frac{e^{iaz}}{1+z^2}\,dz$$

The point? Unlike $\cos(az)$, e^{iaz} is bounded **ON** semicircle C_R.

$a, y > 0$

$$|e^{iaz}| = |e^{ia(x+iy)}| = \underbrace{|e^{iax}|}_{1}\,\underbrace{|e^{-ay}|}_{\le 1}$$

So,

$$\left|\oint_{C_R} \frac{e^{iaz}}{1+z^2}\right| \le \pi R \cdot \max_{z \in C_R} \left|\frac{e^{iaz}}{1+z^2}\right|$$

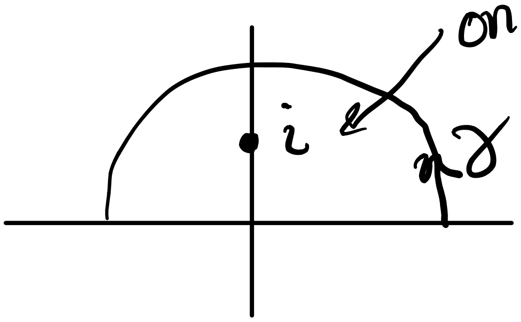
$$\le \frac{\pi R}{R^2-1}$$

So $\quad ML \le \dfrac{\pi R}{R^2-1} \to 0 \quad$ as $\quad R \to \infty$

* If $a<0$, use lower semicircle! *

Thus

$$\oint_\gamma \frac{e^{iaz}}{1+z^2}\, dz = 2\pi i \sum \text{Residues}$$



only pole inside γ at $z=i$

$$\text{Res}(f;i) = \lim_{z\to i} (z-i)\frac{e^{iaz}}{z^2+1} = \frac{e^{-a}}{2i}$$

$$= \frac{\cancel{2}\pi \cancel{i} e^{-a}}{\cancel{2i}} = \pi e^{-a}$$

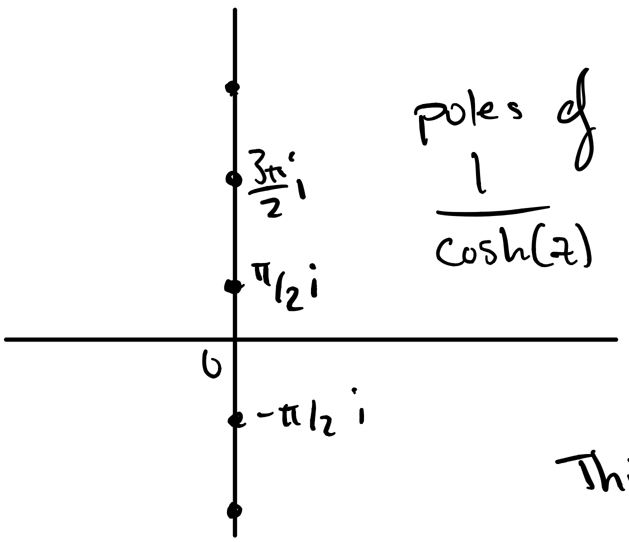
So, $$\lim_{R\to\infty} \int_{-R}^{+R} \frac{\cos(ax)}{1+x^2}\, dx = \pi e^{-a}, \quad a>0$$

Example 3.

$$I = \int_{-\infty}^{+\infty} \frac{dx}{\cosh(x)} \quad \Big\} \text{ use a rectangular contour}$$

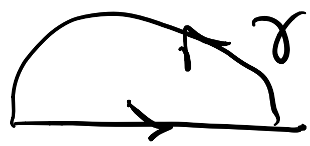
$\cosh(z) = 0$ at $z = i\frac{\pi}{2} + i\pi k, \ k \in \mathbb{Z}$

z-Plane



poles of $\frac{1}{\cosh(z)}$

If we use the usual contour



we get an infinite series when we do sum of residues.

This is OK **BUT** there's an easier way.