Wealth distribution of simple exchange models coupled with extremal dynamics

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H I G H L I G H T S

- We study the wealth distribution simulating two well known asset exchange models.
- We select Yard sale model (YS) and the theft and fraud model (TF).
- Effect of Punctuated Equilibrium (PE) was introduced in YS and TF Models.
- PE makes results from YS and TF Models more realistic avoiding "Economy collapses".
- Wealth distribution for YS Model decays as an asymptotic power law.

A R T I C L E   I N F O

A B S T R A C T

Punctuated Equilibrium (PE) states that after long periods of evolutionary quiescence, species evolution can take place in short time intervals, where sudden differentiation makes new species emerge and some species extinct. In this paper, we introduce and study the effect of punctuated equilibrium on two different asset exchange models: the yard sale model (YS, winner gets a random fraction of a poorer player's wealth) and the theft and fraud model (TF, winner gets a random fraction of the loser's wealth). The resulting wealth distribution is characterized using the Gini index. In order to do this, we consider PE as a perturbation with probability ρ of being applied. We compare the resulting values of the Gini index at different increasing values of ρ in both models. We found that in the case of the TF model, the Gini index reduces as the perturbation ρ increases, not showing dependence with the agents number. While for YS we observe a phase transition which happens around $\rho_c = 0.79$. For perturbations $\rho < \rho_c$ the Gini index reaches the value of one as time increases (an extreme wealth condensation state), whereas for perturbations greater than or equal to ρ_c the Gini index becomes different to one, avoiding the system reaches this extreme state. We show that both simple exchange models coupled with PE dynamics

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give more realistic results. In particular for YS, we observe a power low decay of wealth distribution.

1. Introduction

In the seminal paper of Bak and Sneppen, a Self-Organized Critical (SOC) model [1] is introduced to describe the ecological co-evolution of interacting species. The success of the model in reproducing the punctuated equilibrium behavior proposed by Gould [2], and already observed in the fossil records [3], has attracted many authors to study the model, and variations of it, through several approaches ranging from simulation [4,5] to the renormalization group [6,7]. Bak and Sneppen's model has found interesting applications in economic studies [8,9], bacterial evolution [10] and even optimization problems [11,12].

The Theory of Punctuated Equilibrium emerged as an opposition to Phyletic gradualism, which is a theory of speciation that states evolution occurs uniformly and by the steady and gradual transformation of whole lineages, so that no clear line of demarcation exists between an ancestral species and a descendant one. Punctuated equilibrium, on the contrary, states that evolutionary change takes place in short periods of time tied to speciation and extinction events, separated by large time periods of evolutionary quiescence, called stasis. Evidence for these ideas has been found in the fossil record of bryozoans [13]. This record shows that the first individuals appeared about 140 million years ago, remain unchanged for its first 40 million years (stasis). After that, an explosion of diversification is observed, followed by another period of stasis. Other well known events, observed in the fossil record and explained by Punctuated Equilibrium are the extinction of dinosaurs, about 50 million years ago, or the huge number and sudden emergence of new species during the Cambrian period, in the Paleozoic Era, about 500 million years ago, called the Cambrian Explosion.

On the other hand, the study of wealth and income distributions in society, is a very important and fundamental area of research for practical and theoretical reasons to social scientists, economists, econophysicists, sociologist, philosophers, etc. and also concerns to politicians, government administrators, international bankers, and surely to national security agencies from many countries and of course to every common citizen. Although questions on the origin and causes of inequality are very old; attempts to answer them have been not very successful, even if many ideas have been proposed to understand and solve the problem. Between these ideas, we can mention the following: difference in religious ethics, lack of a qualified workforce, dependence on external technology, low level of internal savings, non-equilibrium between exports and imports, low cognitive and schooling skills of population, level of corruption and quality of democracy, capital's rate of return exceeding rate of output and income, and many more [14–23].

Even more, large scale social and ideological experiments, intended and implemented by force, to solve the inequality problem by centralization of economy, have failed spectacularly with a terrible prior and posterior cost in human suffering and lives, human rights violations, famines, waste of economic resources, political and economical instability, "hot" and "cold" wars, immigration waves, etc.

The important fact is that currently the extreme economic inequality problem seems is not any more only restricted to the beforehand called "Third world countries", but is also becoming a big concern and serious problem in developed economies, where the social and wealth gap between the low-medium income segments of population and the richer one, has been recently increasing fast and systematically [1] (for an extensive and polemic discussion on this topic see Ref. [23]).

The first empirical studies to understand wealth distribution were made by Pareto [24], who proposed that the wealth and income distributions obey a universal power law. Subsequent studies have shown that this is not the case for the whole range of population wealth values. Mandelbrot [25] proposed that the Pareto conjecture only holds at the higher values of wealth and income. The initial part (low wealth or income) of the distribution has been identified with the Gibbs distribution [26,27], while the middle part, according to Gibrat [28], takes the form of a log-normal distribution.

Recently, due to great advances in Complexity Sciences and computing power new ways to model and understand social and economic systems have emerged. Between the most important and well known applications of this computational methods we can mention the use of multi-agent based models to investigate the problem of wealth distribution [27,29–33]

In this work, by using a multi-agent computer methodology, we explore the effect of introducing the extremal dynamics of the already mentioned Bak–Sneppen model, on the wealth distribution produced by two very simple economic exchange models and study their corresponding Gini indices.

In particular, we focus our attention in two well known toy-models of economic interactions that have been used extensively due to their simplicity, such as the so-called "Yard-Sale" (YS) and "Theft and Fraud" (TF) models [34]. Although these two models have the advantage of their simple rules for analysis and simulation, they are over simplified, toy model versions of a real economy and they do not produce realistic wealth distributions. For this reason, several authors have made some refinements to introduce and model more realistic situations, such as the introduction of savings [35], changing the

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[1] Although some economists and policy designers do not make any distinction between inequality and poverty, they are different issues. A society or country can be quite equal with a high number of very poor people or vice versa.

probability of winning according to the relative wealth of the traders [36], allowing the agents to go into debt [29] and by the introduction of altruistic behavior [37,38]. Interesting and deep textbook discussions of multi-agents exchange models in the context of the present work are [39,40].

2. Model description

We can treat an economy in its simplest form as an interchange of wealth between pairs of economic entities (people, companies, countries, etc.), named our "agents" at successive instants of time. Every time two agents interact, wealth flows from one to the other according to some rule. In the case of the YS model, the winner takes a random fraction of wealth from the poorer player, while in the TF case, the winner takes a random fraction of the loser's wealth. There is not production or consumption of wealth in these models, no taxes, savings, etc. Under these circumstances, the YS model produces a collapse in the economy: all the wealth ends in the hands of a single agent, a phenomenon known as extreme wealth condensation. On the other hand, the TF model does not collapse, but leads to a wealth distribution given by the Gibbs distribution [41,27,42]. In particular, the introduction of punctuated equilibrium in these two models has not been studied in depth, and therefore, in this paper we investigate the effect that punctuated equilibrium behavior has on the dynamics of the models and the changes that it can produce on the distribution of the wealth.

As discussed in Ref. [34], after a sufficient number of interactions take place in the YS model, a single agent ends up with (almost) all the money in the system. On the other hand, the TF model produces a distribution with the majority of the agents ending with a wealth close to the average, and no agent becomes extremely rich. Note that in the TF case, a very poor agent that interacts with a rich one, can become rich if he wins the bet and the fraction is enough high, a situation that is not expected to happen in the real world, unless we are dealing with illegal activities, hence the name of the model.

These two outcomes from the YS and TF models actually do not reflect what really happens in a modern economy, where the wealth distribution takes an exponential distribution form for the poor and medium class sectors of the population, and a Pareto distribution for the richer individuals [43,44].

2.1. Simulation implementation

Our simulation runs in the following way: N agents are arranged on a one dimensional lattice with periodic boundary conditions. This lattice is not important in any trading activity between agents, however it will be very important and necessary to keep track of every agent's first neighbors, in order to apply EP rules to introduce wealth mutations in a subsequent step of our simulation as explained below. Initially a certain amount of money M, is distributed equally to all agents in such a way that $\Sigma_{i=1}^{N} m_i = M$. The system is closed, meaning that the total amount of money in the system, M, is always constant (i.e. no production) and the number of agents N remains unchanged (neither dead, birth or migration of agents is allowed). At each time step, two agents are randomly chosen and they interact according to the YS or the TF model. However, with probability ρ, the interaction will be ruled by punctuated equilibrium (PE) dynamics instead of YS or TF. Then, another pair of agents are chosen and the process is repeated K times, which constitutes one Monte Carlo step (MCS). In our simulations and to obtain wealth distributions showed in next section a typical number for $K = 10^6$ and every agent starts the simulation owning 100 monetary units.

Punctuated equilibrium (PE) is introduced in both models in the following way: locate the poorest agent, let us say agent k and assign new values of wealth to agents $k - 1$, k and $k + 1$ at random, but taking care that their combined wealth does not change. Extinction of agents is not allowed in order to maintain the overall number of agents N constant.

In the case of the YS or TF rules, two agents i and j are randomly chosen at time t. The winner, which is also chosen at random, takes an amount T of money from the loser. The traders' wealth w_i and w_j at time $t + 1$, assuming that agent i is the winner and agent j is the loser, will be

$$w_i(t + 1) = w_i(t) + T, \tag{1}$$
$$w_j(t + 1) = w_j(t) - T, \tag{2}$$

where w_j is the wealth of the poorest agent for the YS model and w_j is the wealth of the loser for the TF model.

The amount T of the wealth that changes hands in the bet is defined as

$$T = \alpha MIN(w_i(t), w_j(t)), \tag{3}$$

for the YS model and

$$T = \alpha w_j(t), \tag{4}$$

for the TF model, assuming that agent j is the loser, where the parameter α is a random number from a uniform distribution in the interval [0, 1].

The inequality in the final wealth distribution of the system can be quantified using the Gini index [45], defined by

$$G = \frac{\Sigma_{i=1}^{N} \Sigma_{j=1}^{N} |x_i - x_j|}{2N^2 \mu}, \tag{5}$$

Table 1
Wealth CDF power law fit parameters for values of $\rho = 0.78$ and 0.79. NDF denotes the number of degrees of freedom.

ρ	c	α	χ^2/NDF	Selected fit
0.78	0.029 ± 0.062	0.332 ± 0.062	32.49/998	No
0.79	2.550 ± 0.024	0.729 ± 0.002	4.956/998	Yes

where μ is the average wealth, and x_i and x_j represent the wealth of agents i and j respectively. A perfect distribution of wealth, where everybody has the same amount of money will give a value $G = 0$. The other extreme, where one individual owns all the money has a Gini value of 1.

3. Simulations results

3.1. Gini index analysis

We first consider the Gini index in the YS model as a function of time, for several values of the PE "perturbation" ρ. The results are shown in Fig. 1, where one can see that for small enough values of ρ, the final result is the same as with the "pure" YS model, that is, $G(t)$ reaches the value of 1 as time increases (the economy collapses in a state where a single agent has all the money). However, there is a critical value of ρ_c for which the system does not collapse and G takes an asymptotic value less than 1. As the system size increases, the perturbation necessary to get the system out of collapse increases, as can be seen in Fig. 2, where the asymptotic value of G is shown as function of ρ for several system sizes N. Same figure shows the results for the TF model. In this model the effect of the perturbation only decreases the asymptotic value of G.

3.2. Wealth distribution

Fig. 3 shows in a log scale for the YS model the wealth cumulative distribution function (CDF), that gives the probability P that an agent chosen at random will have a wealth greater than w. Here, $N = 2048$ agents and were simulated the values for the perturbation $\rho = 0.78, 0.79, 0.80, 0.84, 0.90, 0.99$.

From Fig. 3, we can see that, as the perturbation ρ increases, the probability of finding richer agents decreases, giving us a "fairer" distribution of wealth, compared to lower perturbations where only higher values of wealth are found.

We also observe in same figure that for the values of $\rho = 0.78$ and 0.79 it is possible to fit very well these curves with a power law model with the form $F(w) = cw^{-\alpha}$. Parameters c and α of these fits are displayed in Table 1

Again, from Fig. 3 and value of χ^2/NDF in Table 1, we can see that the best power law fit corresponds to $\rho_c = 0.79$, with an exponent $\alpha = 0.729$. This confirms our observation of a phase transition around $\rho_c = 0.79$ described by the analysis of the behavior of the Gini index as a function of time for different ρ values and displayed in Fig. 1.

For $\rho = 0.80$ the corresponding wealth CDF decays asymptotically as a power law with an exponent $\alpha_3 = 1.898 \pm 0.002$. The fit was performed in the region $w > 14764$ on the biggest 647 observations χ^2/NDF obtained is 0.1685/645. This exponent has a value enough close to that observed in real wealth cumulative distribution data that is approximately 2 [41]. Interestingly, the more realistic wealth distribution obtained does not correspond to the value of ρ_c (a pure power law distribution), but to a value close to it, $\rho = 0.80$ which is a like real asymptotic power law distribution with a Pareto exponent close to, but slightly lower than 2.

3.3. System size effect

After being established that the maximum wealth depends on the perturbation, we now proceed to investigate the effect of the system size on our simulations. This is shown in Fig. 4. We divided the Gini index from Fig. 2 by the system size N and then normalized it to 100. We can see that its behavior is independent of N.

For the YS model, in Fig. 5 we show that the critical Gini index G_c, does not depend on the number of agents N involved in our simulations. G_c fluctuates around a mean value of $\langle G_c \rangle = 0.6162 \pm 0251$. The mean value $\langle G_c \rangle$ was obtained by fitting a constant horizontal straight line to our data of G_c values for a different number N of agents. Same numerical value results from averaging directly our G_c values until the fourth position after the decimal point (10^{-4}). This is a good result because if G_c were dependent on N, the wealth distribution for the YS model should change on the number of agents, something that should not happen in a good simulation

Continuing with the YS model, in the upper inset of Fig. 5 we plot the critical ρ_c as a function of $\log(N)$. We can see as was pointed out above, at the beginning of this section, that the critical perturbation increases very slowly with the system size. In fact, since we have defined ρ as the probability of switching on or not PE in the interactions between agents, its numerical value cannot be larger than one. For the extreme case of "an infinite system size" we should have $\rho_c \rightarrow 1$, case where nothing can be done to avoid that the system collapses increasing the strength of the perturbation ρ. Of course this extreme case does not represent any real economic system, since that although they can be constituted of a very big number of agents, their number is always finite.
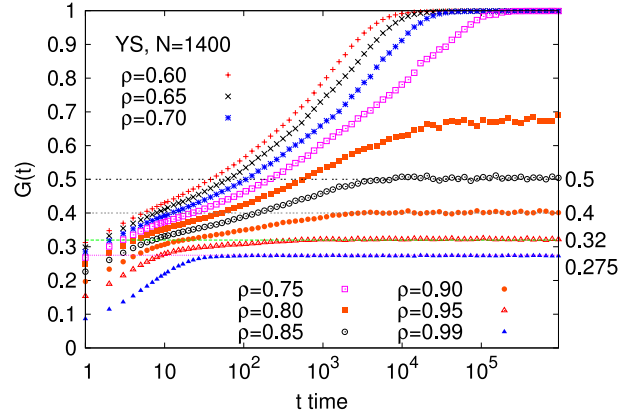
Fig. 1. Gini index $G(t)$ as a function of time for the YS Model for $N = 1400$ agents. We set up the strength of the perturbation ρ to the values $\rho = 0.6, 0.65, 0.7, 0.75, 0.8, 0.85, 0.9, 0.95, 0.99$. In this figure the upper curve corresponds to $\rho = 0.6$, and the bottom one corresponds to $\rho = 0.99$. We observe that as ρ increases, the curves tend to reach the $G = 1$ region more slowly. For $\rho \geq 0.8$ the curves avoid completely that region.
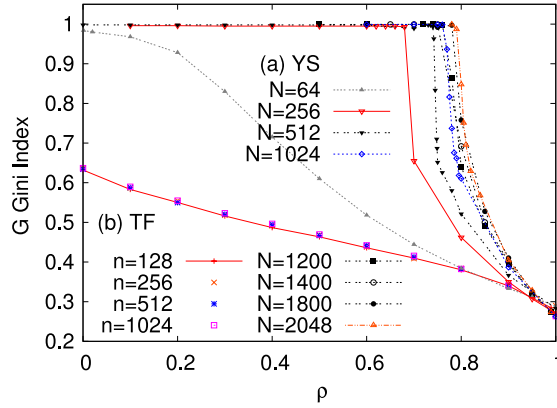

Fig. 2. Gini index $G(p)$ as a function of perturbation for our two models: (a) YS model: $G(\rho)$ curves plotted for a different increasing number of agents N indicated in the figure. (b) TF model: In this case, we use lower case n to indicate the number of agents. $G(\rho)$ curves were plotted for $n = 128, 256, 512, 1024$ values.
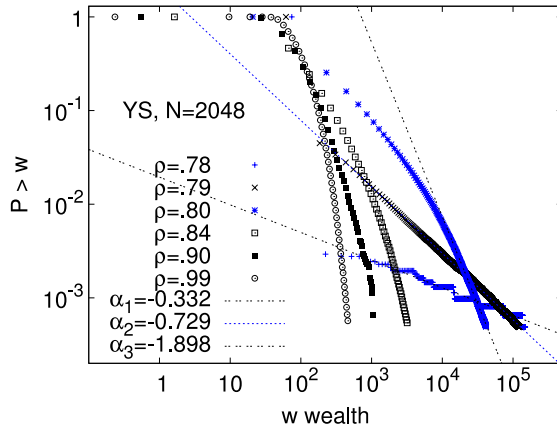

Fig. 3. Wealth Cumulative Distribution Function (CDF) in a log–log scale for the YS model. i.e. the probability P that an agent chosen at random will have a wealth greater than or equal to w. We show simulations with $N = 2048$ agents and different increasing values of $\rho \geq \rho_c$. We observe that as ρ increases, wealth distribution becomes fairer. Power law fits for curves determined by $\rho = 0.78, 0.79$ and 0.80 are displayed. Corresponding exponents are $\alpha_1 = 0.332$, $\alpha_2 = 0.729$ and $\alpha_3 = 0.80$. From second fit, we select $\rho_c = 0.79$. See Table 1. Curve for $\rho_3 = 0.80$ decays asymptotically as a power law fit, and it is the closer to the observed in real data.
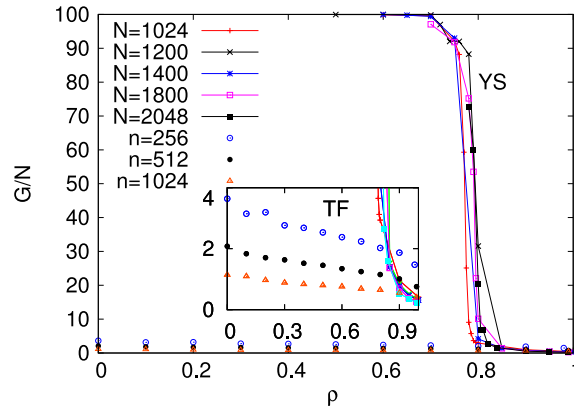
Fig. 4. Gini index divided by the agents number *N* and scaled to 100 as a function of the perturbation for our two models: (a) YS model: figure shows this scaled Gini index for five different number of agents *N* with values indicated in the figure. (b) TF model: Scaled Gini index for three different number of agents *n* = 256, 512, 1024 in the inset. Again we have used lower case *n* to indicate the number of agents for TF case.
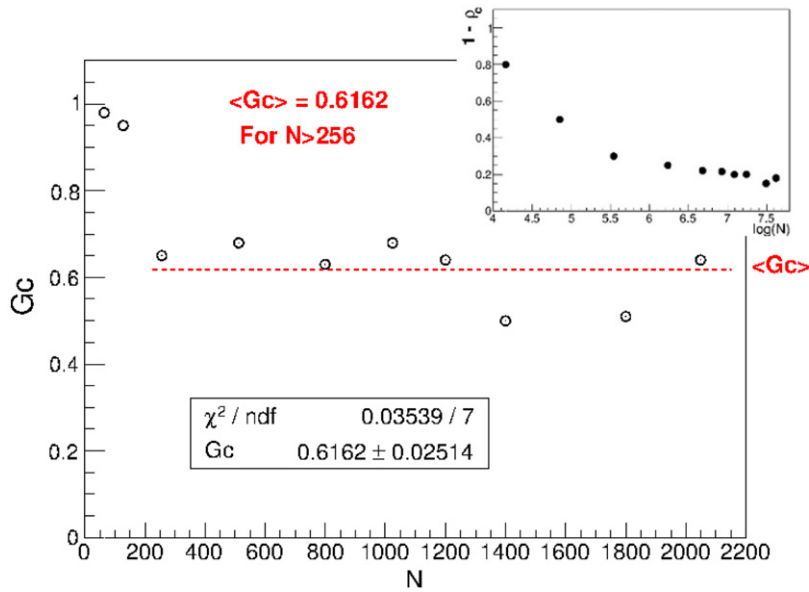

Fig. 5. YS model. Gini index threshold G_c as a function of the number of agents *N*. We can see that G_c seems to become independent of *N*. For *N* > 256 it fluctuates around its mean value $\langle G_c \rangle$. Red broken line was obtained by a χ^2 fit procedure applied to the 8 last data points. We also show in the inset the plot $(1 - \rho_c)$ vs $\log(N)$. It can be observed that $(1 - \rho_c) \to 1$ very slowly and asymptotically.

4. Conclusions

In this work, we analyze the TF and YS models under the influence of the punctuated equilibrium dynamics, which is introduced as a perturbation ρ determined by the probability of applying or not PE in agents exchange of money. Although TF model is weakly affected by the introduction of PE, for the case of YS and for perturbations $\rho > 0.8$, the asymptotic Gini index becomes different to one, meaning that the perturbation avoids the collapse in the economy where a single agent takes all money, re-allocating it between agents in a way that a lower inequality in the distribution of wealth is observed. Even more, a phase transition around a critical value $\rho_c = 0.79$ is observed. For this critical value of ρ, the corresponding wealth distribution displays a power law decay with an exponent $\alpha = 0.729 \pm 0.002$. For a value of $\rho = 0.80$ the wealth distribution decays asymptotically as a power law with an exponent $\alpha_3 = 1.898 \pm 0.002$, consistent with the values observed in real data.

The resulting wealth distributions can be tuned to different values of the Gini index, adjusting the perturbation. A corollary from this, that is possible to extend to real economics is that a fairer wealth distribution is only attainable by means of a mechanism of wealth re-distribution, as by example taxes in real life. The implementation of different re-distribution mechanisms or determining what is the optimum proportion of individual income to taxing or re-distribute are problems that also could be explored through agents modeling methodology.

Even if the evolutionary economic approach presented here seem very naive and far from representing real economic phenomena, it has precisely the virtue that preserving the simplicity of YS and TF models, produces results more realistic than the obtained by the original models without PE dynamics included, such as a finite wealth distribution decaying as a power law. This is important for the agents model theory and implementation goal of constructing a minimum agents model for real economic and social systems. Also, and in our opinion, our results are important for the most ambitious and long term end of constructing a real microeconomics theory, in the sense of statistical physics, if possible [46]. We believe that an initial and main way of attacking successfully this difficult problem is through the intensive use of multi-agent simulations techniques, followed by a formalization of results and techniques emerged from this approach. For a nice discussion on these issues in the context of diverse agents models and SOC, see Ref. [47, chapter 5].

Besides, other studies have been made where the simple YS and TF models are extended including effects of savings, of course taxes and other mechanisms in order to make them more realistic. We believe that our approach using the extremal dynamics of the Bak–Sneppen model which has the effect of re-allocating wealth between agents, is simpler and yields similar results, a feature that can be used to investigate particular economic phenomena with a simpler model. This can benefit both simulation and analytic studies. Any way, the study of many-body real world systems, through the study of computational models is not an easy task, considering that sometimes we do not understand in deep those models. In our case, it would be very interesting to attack the same problem presented here using a more formal approach as a Mean Field Model. This is matter of a future work.

Acknowledgments

References

[1] P. Bak, K. Sneppen, Punctuated equilibrium and criticality in a simple model of evolution, Phys. Rev. Lett. 71 (1993) 4083–4086. http://dx.doi.org/10.1103/PhysRevLett.71.4083.
[2] S.J. Gould, N. Eldredge, Punctuated equilibria; the tempo and mode of evolution reconsidered, Paleobiology 3 (2) (1977) 115–151.
[3] D. Raup, Biological extinction in earth history, Science 231 (4745) (1986) 1528–1533. http://dx.doi.org/10.1126/science.11542058.
[4] P. Grassberger, The Bak–Sneppen model for punctuated evolution, Phys. Lett. A 200 (3–4) (1995) 277–282. http://dx.doi.org/10.1016/0375-9601(95)00179-7.
[5] P. de los Rios, M. Marsili, M. Vendruscolo, High-dimensional Bak–Sneppen model, Phys. Rev. Lett. 80 (1998) 5746–5749. http://dx.doi.org/10.1103/PhysRevLett.80.5746.
[6] M. Marsili, Renormalization group approach to the self-organization of a simple model of biological evolution, Europhys. Lett. 28 (6) (1994) 385. http://dx.doi.org/10.1209/0295-5075/28/6/002.
[7] B. Mikeska, Monte Carlo renormalization-group approach to the Bak–Sneppen model, Phys. Rev. E 55 (1997) 3708–3711. http://dx.doi.org/10.1103/PhysRevE.55.3708.
[8] M. Ausloos, P. Clippe, A. Pekalski, Evolution of economic entities under heterogeneous political/environmental conditions within a Bak–Sneppen-like dynamics, Physica A 332 (2004) 394–402. http://dx.doi.org/10.1016/j.physa.2003.10.004.
[9] T. Yamano, Regulation effects on market with Bak–Sneppen model in high dimensions, Internat. J. Modern Phys. C 12 (09) (2001) 1329–1333. http://dx.doi.org/10.1142/S0129183101002620.
[10] I. Bose, I. Chaudhuri, Bacterial evolution and the Bak–Sneppen model, Internat. J. Modern Phys. C 12 (05) (2001) 675–683. http://dx.doi.org/10.1142/S0129183101001857.
[11] S. Boettcher, A. Percus, Nature's way of optimizing, Artificial Intelligence 119 (1–2) (2000) 275–286. http://dx.doi.org/10.1016/S0004-3702(00)00007-2.
[12] S. Boettcher, A.G. Percus, Optimization with extremal dynamics, Phys. Rev. Lett. 86 (2001) 5211–5214. http://dx.doi.org/10.1103/PhysRevLett.86.5211.
[13] A. Cheetham, J.B.C. Jackson, L.–A.C. Hayek, Quantitative genetics of bryozoan phenotypic evolution II. analysis of selection and random change in fossil species using reconstructed genetic parameters, Evolution 48 (1994) 360–375.
[14] M. Weber, The Protestant Ethic and the Spirit of Capitalism, Roxbury Publishing Company, 1905.
[15] G. Ranis, J. Fei, A theory of economic development, Am. Econ. Rev. 51 (1961) 533–565.
[16] W. Lewis, Economic development with unlimited supplies of labour, Manch. Sch. 22 (1954) 139–191. http://dx.doi.org/10.1111/j.1467-9957.1954.tb00021.x.
[17] H. Chenery, M. Bruno, Development alternatives in an open economy, Econ. J. 72 (1962) 79–103.
[18] L. Taylor, Structuralist Macroeconomics: Applicable Methods for The Third World, New York Basic Books, Inc., 1983.
[19] E. Hanushek, L. Woessmann, Schooling, Cognitive Skills, and The Latin American Growth Puzzle, Working Paper 15066, National Bureau of Economic Research, 2009, URL http://www.nber.org/papers/w15066.
[20] A.L. Hillman, Corruption and public finance: an imf perspective, J. Polit. Econ. 20 (2004) 1067–1077.
[21] H. Li, L. Colin-Xu, H. Zou, Corruption, income distribution, and growth, Econ. Polit. 12 (2000) 155–182.
[22] D. Acemoglu, S. Naidu, P. Restrepo, J. Robinson, Democracy, Redistribution and Inequality, Working Paper No. 19746, National Bureau of Economic Research, 2013, URL http://www.nber.org/papers/w19746.
[23] T. Piketty, Capital in the Twenty-First Century, Belknap Press, 2014.
[24] V. Pareto, Cours d'Économie Politique, Vol. I, 430 pp., Lausanne, F. Rouge, 1896 and 1897.
[25] B.B. Mandelbrot, The Pareto-Levy law and the distribution of income, Internat. Econom. Rev. 1 (2) (1960) 79–106. http://dx.doi.org/10.2307/2525289.
[26] B.K. Chakrabarti, S. Marjit, Self-organisation in game of life and economics, Indian J. Phys. 69 (1995) 681–698.
[27] A. Chakraborti, B. Chakrabarti, Statistical mechanics of money: how saving propensity affects its distribution, Eur. Phys. J. B 17 (1) (2000) 167–170. http://dx.doi.org/10.1007/s100510070173.
[28] R. Gibrat, Les inégalités économiques: applications: aux inégalités des richesses, á la concentracion des enterprises, aux populations des villes, aux statistiques des familles, etc. d'une loi nouvelle, la loi de l'effet proportionnel, Librairie du Recueil Sirey, 1931.
[29] A. Dragulescu, V. Yakovenko, Statistical mechanics of money, Eur. Phys. J. B 17 (4) (2000) 723–729. http://dx.doi.org/10.1007/s100510070114.
[30] J.-P. Bouchaud, M. Mézard, Wealth condensation in a simple model of economy, Physica A 282 (3) (2000) 536–545. http://dx.doi.org/10.1016/S0378-4371(00)00205-3.

[31] A. Chatterjee, B. Chakrabarti, S. Manna, Pareto law in a kinetic model of market with random saving propensity, Physica A 335 (1) (2004) 155–163. http://dx.doi.org/10.1016/j.physa.2003.11.014.
[32] E. Scalas, U. Garibaldi, S. Donadio, Statistical equilibrium in simple exchange games I, Eur. Phys. J. B 53 (2) (2006) 267–272. http://dx.doi.org/10.1140/epjb/e2007-00345-6.
[33] E. Scalas, T. Radivojevic, U. Garibaldi, Wealth distribution and the Lorenz curve: A finitary approach, J. Econom. Interact. Coord. (2014) http://dx.doi.org/10.1007/s11403-014-0136-2.
[34] B. Hayes, Follow the money, Am. Sci. 90 (5) (2002) 400–405.
[35] A. Chakraborti, Distributions of money in model markets of economy, Internat. J. Modern Phys. C 13 (10) (2002) 1315–1321. http://dx.doi.org/10.1142/S0129183102003905.
[36] S. Sinha, Stochastic maps, wealth distribution in random asset exchange models and the marginal utility of relative wealth, Phys. Scr. 2003 (T106) (2003) 59–64. http://dx.doi.org/10.1238/Physica.Topical.106a00059.
[37] R. Trigaux, The wealth repartition law in an altruistic society, Physica A 348 (2005) 453–464. http://dx.doi.org/10.1016/j.physa.2004.09.028.
[38] M. Rodríguez-Achach, R. Huerta-Quintanilla, The distribution of wealth in the presence of altruism in simple economic models, Physica A 361 (1) (2006) 309–318. http://dx.doi.org/10.1016/j.physa.2005.07.001.
[39] U. Garibaldi, E. Scalas, Finitary Probabilistic Methods in Econophysics, Cambridge University Press, 2010.
[40] B. Chakrabarti, A. Chakraborti, S. Chakravarty, A. Chatterjee, Econophysics of Income and Wealth Distributions, Cambridge University Press, 2013.
[41] A. Dragulescu, V.M. Yakovenko, Exponential and power-law probability distributions of wealth and income in the United Kingdom and the United States, Physica A 299 (1–2) (2001) 213–221. http://dx.doi.org/10.1016/S0378-4371(01)00298-9.
[42] S. Ispolatov, P. Krapivsky, S. Redner, Wealth distributions in asset exchange models, Eur. Phys. J. B 2 (2) (1998) 267–276. http://dx.doi.org/10.1007/s100510050249[2].
[43] J. González-Estévez, M. Cosenza, R. López-Ruiz, J. Sánchez, Pareto and Boltzmann–Gibbs behaviors in a deterministic multi-agent system, Physica A 387 (18) (2008) 4637–4642. http://dx.doi.org/10.1016/j.physa.2008.03.013.
[44] I. Wright, The social architecture of capitalism, Physica A 346 (3–4) (2005) 589–620. http://dx.doi.org/10.1016/j.physa.2004.08.006.
[45] Z.M. Berrebi, J. Silber, Income inequality indices and deprivation: A generalization, Q. J. Econ. 100 (3) (1985) 807–810. http://dx.doi.org/10.2307/1884380.
[46] E. Scalas, A proposal, Talk ESHIA/WEHIA 2010, Alessandria, Italy 23–25 June, 2010.
[47] H. Jeldtoft, Self-Organized Criticality. Emergent Complex Behavior in Physical and Biological Systems, Cambridge University Press, 1998.
[48] R. Brun, F. Rademakers, Root: an object oriented data analysis framework, in: Proceedings AIHENP96 Workshop, Lausanne, September, in: Nucl. Instrum. Methods A, 33, 1996.

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[2] To our knowledge, Ref. [42], introduces simple asset exchange models for the first time in the Physics literature.