A logical description of strategizing in social network games

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Abstract

We propose a modal logic for reasoning about strategies in social network games ([12]). In these games, players are connected by a social network graph, and payoffs for players are determined by choices of players in their neighbourhood. We consider improvement dynamics of such games and the formulas of the logic are intended to capture bisimulation classes of improvement graphs. The logic is structured in two layers: local formulas which specify neighbourhood dependent strategization, and global formulas which describe improvement edges and paths. Notions like Nash equilibrium and (weak) finite improvement property are easily defined in the logic. We show that the logic is decidable and that the valid formulas admit a complete axiomatization.

Keywords: Logic of strategies, Social network games, Threshold reasoning, Graphical games, Decidability

1 Background

There has been extensive work on the logical foundations of game theory in the last couple of decades. [13] presents an excellent survey of the logical issues in reasoning about games. Asserting the existence of equilibria and exploring the underlying rationality assumptions forms the crux of many of these studies ([6], [16], [1]). On the other hand, much of game theory studies the *existence* of strategies and the logical approach has led to studying compositional structure in strategies ([10], [15], [2]).

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A fundamental assumption of non-cooperative game theory is that players strategize individually and independent of each other. This was referred to as the *Great Simplification* by von Neumann in 1928 and indeed, this is what leads to the abstraction of normal form games. However, such a flat structure on the set of players is not always realistic. In the case of social networks like Facebook and Twitter, individuals are influenced by their friends, and often seek to influence their friends, in the choices they make. The 'payoff' in such interactive behaviour is often social, in the sense that matching one's friends' choices may indeed be the desired outcome. Such *majority games* (and their dual, so-called *minority games*) are also extensively discussed in the literature. 'Facebook logic' ([11]) and its counterparts discuss such relationships and their impact on decision making.

A specific class of such games on social networks was studied by Apt and Simon ([12]), which is the starting point of departure for our paper. While they study the complexity of computing equilibria in such games, we take their *improvement dynamics* and seek a logical description. The central question we take up is this: how do we abstract away from the details of utilities and preferences, and get to the core of strategization by players in such games?

A natural way for such abstraction is to consider game equivalences and seek logical descriptions of equivalence classes. When outcomes are determined locally, by neighbourhoods in the social network, this induces further structure in the improvement graphs, which leads to interesting bisimulation classes. This naturally leads us to a modal logic as a tool for strategization structure.

The logic we present has two crucial points of departure from other modal descriptions of games: it describes improvement dynamics rather than player preferences; and it describes *threshold reasoning* in strategic choice. The latter is only intended as an instance of local strategization, and other similar logical means would be equally suitable.

We go on to present a complete axiomatization of the logic and show that the satisfiability problem is decidable. These technical results suggest that we have an interesting logical formalism at hand. However, only by specifying different classes of network games and reasoning about their improvement dynamics, can we develop this theory further.

2 Social network games

Fix $[n] = \{1, \ldots, n\}$ a fixed, finite set of players (or agents), and we talk of n-player games. Let Σ denote a set of *choices* or *strategies* available to a player. We only consider games in which all players have the same set of choices. This is for convenience of presentation. [3] We let i, j, etc index players and a, b, etc range over Σ. A *strategy profile* is an element of Σ^n, and we let σ, σ' etc to range over strategy profiles with $\sigma[i]$ denoting the i^{th} element of σ, which is interpreted as the choice of player i.

[3] The material here could be developed with specific choice sets for each player, but this clutters up the formalism without adding significant insight.

Let Ω be a set of *outcomes* (or *payoffs*) and a utility function (or *payoff* function) is a map: $\pi : \Sigma^n \to \Omega^n$. It is assumed that Ω is a partially ordered set, with the order $\preceq$.[4]

Definition 2.1 A game is a tuple $H = (n, \Sigma, \Omega, \pi)$, fixing the number of players, the strategy sets of players, the outcomes and the payoff map.

These are called strategic form (or normal form) games, extensively studied for nearly a century. In games on social networks we assume a graph structure on the set of players, and the edge relation is interpreted to be a form of friendship relation that governs behaviour in a suitable way ([12]).One reason for studying such *graphical games* is that the payoff function above is large, being exponential in n and hence hard to present. In social networks, though the number of players may be large, each player interacts with only a few players. It is often possible to assume that each player, on average, interacts with at most $log\ n$ many players where n is the number of players in the game. In this case, we can consider games where the payoff is determined only by player *neighbourhoods* and the payoff function is then, roughly, of size $\Sigma^{log\ n}$.

Definition 2.2 A **social network game** is a tuple $G = (n, \Sigma, \Omega, E, \pi)$, where $E \subseteq ([n] \times [n])$ is the edge relation of the social network graph, $\pi = (\pi_1, \ldots, \pi_n)$ is the payoff function, one for each player, where $\pi_i : \Sigma^{|N_i|} \to \Omega$, $N_i = \{i\} \cup \{j \mid (j, i) \in E\}$, is the *neighbourhood* of player i.

Clearly π induces a function from $\Sigma^n \to \Omega^n$ which, by abuse of notation, we again denote by π.

Typically, a social network graph is assumed to be simple and undirected: that is, the edge relation is symmetric and has no self-loops. When the edge relation represents a form of friendship, it is surely bi-directional. But we prefer to retain the more general form of directed graphs. For instance the edge from i to j may represent i linking to j on the web, in which case, there is no reason to assume a link in the reverse direction. Self-loops do not matter since we have included every player in its neighbourhood by definition.

How does a player *strategize* in such a game? This clearly depends on the connectivity in the social network graph. For instance, when $E = [n] \times [n]$ it is just the same as reasoning in normal form games.

Consider the edge relation $E_1 = \{(1, 2), (1, 3), (2, 3)\}$. Here $N_1 = \{1\}$, $N_2 = \{1, 2\}$ and $N_3 = \{1, 2, 3\}$. In this game, player 1 makes choices independent of others since her payoff depends only on what she chooses. Player 2 provides his best response to 1's choices, and player 3 provides a best response to every combination of choices of players 1 and 2.

2.1 Modelling examples

Threshold based reasoning is common in games on social networks ([8]), and we present two examples of such modelling.

[4] Again, for convenience, we assume a uniform ordering on outcomes rather than one order $\preceq_i$ for each player i.

The first is that of a **Majority Game**, used in modelling social phenomena such as voting: ([3], [7]). $\Sigma = \{0, 1\}$. Let $\sigma = (a_1, \ldots, a_n)$ be any profile. The payoff for player i at σ is 1 if $\frac{|\{j \in N(i) | a_j = a_i, j \neq i\}|}{|N(i)|} > \frac{1}{2}$, and is 0 otherwise. The trivial equilibria for this game are all players choosing 0, or all choosing 1.

A non-trivial vote will be the following:

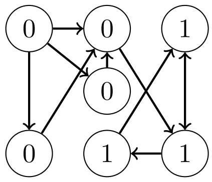

Fig. 1. Non trivial Nash Equilibrium in the Majority Game

For another example, consider the **"Best Shot" Public Goods game** ([8]). Again, $\Sigma = \{0, 1\}$. In this game, there is an option of taking an altruistic action for the public good, or refraining from it. Doing good carries a fixed uniform cost $c \in (0, 1)$. Of course, if some neighbour takes action, it is much better and one can enjoy the result doing nothing. Alas, if everyone thinks so, nobody benefits. This is specified by the payoff function as follows. Again let $\sigma = (a_1, \ldots, a_n)$:

$$\pi_i(\sigma) = \begin{cases} 0 & \text{if } a_i = 0, a_j = 0 \text{ for all } j \in N(i) \\ 1 & \text{if } a_i = 0, a_j = 1 \text{ for some } j \in N(i) \\ 1 - c & \text{if } a_i = 1 \end{cases}$$

Here is a Nash equilibrium for this game:

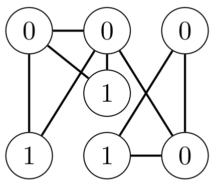

Fig. 2. Nash Equilibrium in the Public Goods Game

2.2 Dynamics

A simple way to study such reasoning is given by the improvement graph dynamics defined as follows.

Definition 2.3 The **improvement graph** I_G, associated with the game G, is the graph $I_G = (\Sigma^n, IE_G)$, where $IE_G \subseteq (\Sigma^n \times [n] \times \Sigma^n)$ is the player labelled edge relation given by $(\sigma, i, \sigma') \in IE_G$ iff $\pi(\sigma)[i] \prec \pi(\sigma')[i]$ and for all $j \neq i$, $\sigma[j] = \sigma'[j]$.

We have an i-labelled edge from a strategy profile to another, if player i can unilaterally deviate from the former to the latter to get an improved payoff.

Note that at a profile, there can be different i-improvement edges leading to different profiles (with perhaps incomparable outcomes). A path in I_G is an improvement path.

Note that the improvement graph can have cycles. For instance, consider the two-player game of matching pennies: both players call heads or tails, the first player wins when the results match, and the second wins when there is a mismatch. We then have the cycle $(H, H) \rightarrow_2 (H, T) \rightarrow_1 (T, T) \rightarrow_2 (T, H) \rightarrow_1 (H, H)$.

In any particular game n is fixed as well as the size of Σ and hence I_G is a finite directed graph, though a large one (its size being exponential in n). It contains a good deal of interesting information about the game G. For instance consider a sink node of I_G, which has no out-going edge: it is easy to see that a strategy profile is a sink node if and only if it constitutes a *Nash equilibrium*, from which no player has any incentive to deviate.

I_G includes structural information from the social network graph as well. For instance, we have the following proposition.

Proposition 2.4 *Let G be a social network game and players i, j such that $N_i \cap N_j = \emptyset$. Then, for all $\sigma_0, \sigma_1, \sigma_2 \in \Sigma^n$ we have:*

- *If $\sigma_0 \rightarrow_i \sigma_1$ and $\sigma_0 \rightarrow_j \sigma_2$, then there exists $\sigma_3 \in \Sigma^n$ such that $\sigma_1 \rightarrow_j \sigma_3$ and $\sigma_2 \rightarrow_i \sigma_3$.*
- *If $\sigma_0 \rightarrow_i \sigma_1$ and $\sigma_1 \rightarrow_j \sigma_2$, then there exists $\sigma_3 \in \Sigma^n$ such that $\sigma_0 \rightarrow_j \sigma_3$ and $\sigma_3 \rightarrow_i \sigma_2$.*

To see this, fix G as above and consider profiles $\sigma_0, \sigma_1, \sigma_2 \in \Sigma^n$ such that $\sigma_0 \rightarrow_i \sigma_1$ and $\sigma_0 \rightarrow_j \sigma_2$. Firstly note that for all $k \in [n]$, $k \neq i$ and $k \neq j$, $\sigma_1[k] = \sigma_2[k]$. Define σ_3 by: $\sigma_3[i] = \sigma_1[i]$, $\sigma_3[j] = \sigma_2[j]$, and for all $k \in [n]$, $k \neq i$ and $k \neq j$, $\sigma_3[k] = \sigma_0[k]$. Note that $\pi(\sigma_0)[i] = \pi(\sigma_2)[i] \prec \pi(\sigma_3)[i]$, and $\pi(\sigma_0)[j] = \pi(\sigma_1)[j] \prec \pi(\sigma_3)[j]$, as required since $N_i \cap N_j = \emptyset$ and π_k is dependent only on N_k, for all players k.

For the other statement, define σ_3 by: $\sigma_3[j] = \sigma_2[j]$ and $\forall k \in [n],\ k \neq j$, $\sigma_3[k] = \sigma_2[k]$. Since $N_i \cap N_j = \emptyset$, the strategy choices of players restricted to the local neighbourhood of j for σ_3 match with that of the strategy choices of players in N_j in σ_2 and similarly the strategy choices made by players in N_i for σ_3 match with that of players in N_i for σ_0. So we have, $\pi(\sigma_0)[j] = \pi(\sigma_1)[j] \prec \pi(\sigma_2)[j] = \pi(\sigma_3)[j]$. Hence, you have, $\sigma_0 \rightarrow_j \sigma_3$. And, $\pi(\sigma_0)[i] = \pi(\sigma_3)[i] \prec \pi(\sigma_1)[i] = \pi(\sigma_2)[i]$. And therefore, $\sigma_3 \rightarrow_i \sigma_2$ as well.

The proposition refers to 'squares' in the improvement graph. In general if we have k players with pairwise disjoint sets of neighbourhoods, we have k-hypercubes embedded in I_G. This may be interpreted as concurrent strategization by players in the game. Such structure has been extensively studied in asynchronous transition systems in concurrency theory ([9]).

In the analysis of games, we are typically interested in questions like whether the game has a Nash equilibrium, whether every improvement path is finite, whether an equilibrium profile is reachable from every profile, etc. Algorithmically all these questions are efficiently solvable, but then the input, namely the improvement graph, is itself large.

Note that the improvement graph dynamics induces a natural **game equivalence**: we can consider two games to be equivalent if they have a similar (but not necessarily isomorphic) improvement structure.

Definition 2.5 Let G, G' be two n player games with strategy sets Σ and Σ', a relation $R \subseteq (\Sigma^n, \Sigma'^n)$ is an improvement bisimulation if whenever $(\sigma, \sigma') \in R$, for all $i \in [n]$,

- whenever $\sigma \rightarrow_i \sigma_1$ in I_G, there exists profile σ_1' in game G' such that $\sigma' \rightarrow_i \sigma_1'$ in $I_{G'}$ and $(\sigma_1, \sigma_1') \in R$.
- whenever $\sigma' \rightarrow_i \sigma_1'$ in $I_{G'}$, there exists profile σ_1 in game G such that $\sigma \rightarrow_i \sigma_1$ in I_G and $(\sigma_1, \sigma_1') \in R$.

The relation is on games in general rather than on social network games. Clearly, the improvement bisimulation relation is an equivalence relation. We say that G and G' are bisimilar if there exists a nonempty bisimulation on their improvement graphs. When we reduce strategic form games by this equivalence, we abstract from specific outcomes as well as player strategies but preserve the *player strategisation structure*. Note that outcomes and orderings on them have entirely disappeared in the bisimulation classes, only the information that some improvement in outcome is possible (or not) is retained. This is a semantic characterization, and we need structural constraints to capture the semantic conditions. What we would like to do is to study player rationale to provide logical structure to the strategization. In the context of social network games, we use threshold reasoning over player neighbourhoods as a way of specifying this rationale.

3 The logic

When we seek a logical description of improvement dynamics, it is natural to consider first order logics and their extensions with least fixed-point operators, since equilibrium computation typically proceeds by finding such fixed points. In earlier work ([4]) we have carried out such an exercise. However, in light of the discussion above, we are not interested in the improvement graphs themselves but in bisimulation classes, and hence define a modal logic for this study. Player strategisation involves reasoning about strategies played by other players in their neighbourhood. Thus mutual strategization by players becomes relevant, and the logic we define below includes precisely such *local* reasoning by players as well as *global* improvement dynamics.

Syntax The formulas of the logic are presented in two layers: **local player formulas** and **global outcome formulas**. The logic is parameterised by n, the number of players, and the strategy set Σ.

The syntax of local formulas is given by:

$$\alpha \in L_i ::= a \in \Sigma \mid e_j \mid \neg\alpha \mid \alpha \vee \alpha' \mid N_{rel\ r}\ \alpha$$

where $rel \in \{\geq, \leq, <, >\}$, $i \in [n]$ and r is a rational number, $0 \leq r \leq 1$.

The atomic formula a asserts that player i chooses a, and the atomic formula e_j asserts that there is a directed edge from j to i, that player i is dependent on j. $N_{rel\ r}\alpha$ considers the size of the neighbourhood choosing α: for instance, $N_{\leq r}\alpha$ asserts that at most an r-fraction of players in the neighbourhood of i support α.

Fix P a finite set of atomic propositions denoting *conditions on outcomes*. These are qualitative outcomes, used to denote levels of satisfaction. We will characterize outcomes by sets of propositions, which can be equivalently thought of as boolean formulas on P.

The syntax of global formulas is given by:

$$\phi \in \Phi ::= p@i,\ p \in P \mid \alpha@i, \alpha \in L_i \mid \neg\phi \mid \phi \vee \phi' \mid \langle i \rangle \phi \mid \Diamond^* \phi$$

The global formulas constitute a standard propositional modal logic of transitive closure built over local formulas. Note that the atomic formulas $p@i$ and $\alpha@i$ are of different sort: the former refers to outcomes and the latter to strategies. The other boolean connectives $\wedge$, $\supset$ and $\equiv$, for conjunction, implication and equivalence are defined in the standard manner, for both local and global formulas. The dual formulas are: $[i]\phi = \neg\langle i \rangle \neg\phi$ and $\Box^*\alpha = \neg\Diamond^*\neg\phi$ We use the abbreviation $\bigcirc\phi = \bigvee_{i \in [n]} \langle i \rangle \phi$ and $\bigodot\phi = \neg\bigcirc\neg\phi$. (We use $\top$ and $\bot$ to refer to the propositional constants 'True' and 'False' which are coded by $p@i \vee \neg p@i$ and $p@i \wedge \neg p@i$, for a fixed propositional symbol p.)

Semantics The formulas are interpreted over strategy profiles of social network games. A model is a social network game $M = (n, \Sigma, 2^P, E, \pi)$ where $\Omega = 2^P$ is the set of outcomes. The ordering can be seen as an ordering on boolean formulas on P.

The semantics is given by assertions of the form $M, \sigma \models \phi$, read as ϕ is true of the strategy profile σ in model M. This in turn depends on the satisfaction relation for local formulas. For $i \in [n]$ and $\alpha \in L_i$, we define i-local satisfaction relations:

- $M, \sigma \models_i a$ if $\sigma[i] = a$.
- $M, \sigma \models_i e_j$ if $(j, i) \in E$.
- $M, \sigma \models_i \neg\alpha$ if $M, \sigma \not\models_i \alpha$.
- $M, \sigma \models_i \alpha \vee \beta$ if $M, \sigma \models_i \alpha$ or $M, \sigma \models_i \beta$.
- $M, \sigma \models_i N_{rel\ r}\ \alpha$ if $\frac{|\{j | M, \sigma \models_j \alpha\}|}{|N_i|}\ rel\ r$.

The semantics of global formulas can then be defined as follows. Below, let $\rightarrow^* = (\cup_i \rightarrow_i)^*$, the reflexive transitive closure of the union of the improvement edge relations.

- $M, \sigma \models p@i$ if $p \in \pi_i(\sigma)$.
- $M, \sigma \models \alpha@i$ if $M, \sigma \models_i \alpha$.
- $M, \sigma \models \neg\phi$ if $M, \sigma \not\models \phi$.

- $M, \sigma \models \phi \vee \psi$ if $M, \sigma \models \phi$ or $M, \sigma \models \psi$.
- $M, \sigma \models \langle i \rangle \phi$ if there exists σ' such that $\sigma \rightarrow_i \sigma'$ and $M, \sigma' \models \phi$.
- $M, \sigma \models \Diamond^* \phi$ if there exists σ' such that $\sigma \rightarrow^* \sigma'$ and $M, \sigma' \models \phi$.

We say that ϕ is satisfiable if there exists a social network game model M and a profile σ such that $M, \sigma \models \phi$. We say that ϕ is valid if $\neg\phi$ is not satisfiable.

It is easy to see that Nash Equilibrium is given by the simple formula: $NE = \bigwedge_{i \in [n]} [i]\bot$. To assert that there is a path from the current profile to a Nash Equilibrium, we write: $\Diamond^*(NE)$. To assert the "Weak Finite Improvement Property", that a Nash Equilibrium profile is reachable from every profile, we write: $\Box^*\Diamond^*(NE)$.

The strategy specification for the majority game is simple. Let the payoff set be given by: $P = \{p_0, p_1\}$ with $\{p_0\} \preceq \{p_1\}$. The formula $(N_{>\frac{1}{2}}(e_j \wedge 1))@i \equiv p_1@i$ defines the payoff map.

For the Public Goods Game, let $P = \{p_0, p_c, p_1\}$ with $\{p_0\} \preceq \{p_c\} \preceq \{p_1\}$. The payoff map is specified by:

$$(\bigwedge_i 0@i) \supset (\bigwedge_i p_0@i) \wedge \bigwedge_i (1@i \supset p_c@i) \wedge \bigwedge_i ((0 \wedge N_{>0}1)@i \supset p_1@i)$$

4 Axiomatization and decidability

We now present an axiomatization of the valid formulas. We have one axiom system Ax_i for each player i in the system, and in addition a global axiom system AX to reason about improvement dynamics. In some sense, this helps to isolate how much global reasoning is required.

Below, we say *rel r entails rel′ r′* when $r \leq r'$ and either $rel = rel' = \leq$ or $rel = <$ and $rel' = \leq$, or $r \geq r'$ and either $rel = rel' = \geq$ or $rel = >$ and $rel' = \geq$. Further we say rel' is the *complement* of rel if one of them is $\geq$ and the other is $<$, or one is $\leq$ and the other $>$.

We use the notation $\vdash_i \alpha$ to mean that the formula $\alpha \in L_i$ is a theorem of system Ax_i. Similarly, $\vdash \phi$ means that ϕ is a theorem of the global system.

Ax_i, The axiom schemes for agent i

($A0_i$) All the substitutional instances of propositional tautologies
($A1_i$) $N_{rel\ r}(\alpha \supset \beta) \supset (N_{rel\ r}\alpha \supset N_{rel\ r}\beta)$
($A2_i$) $\alpha \supset N_{>0}\alpha$
($A3_i$) $N_{rel\ r}\alpha \equiv \neg N_{rel'\ r}\alpha$, $\quad rel'$ complement rel
($A4_i$) $N_{rel\ r}\alpha \supset N_{rel' r'}\alpha$, $\quad rel\ r$ entails $rel'\ r'$

Inference rules

$$(MP_i)\ \frac{\alpha, \quad \alpha \supset \beta}{\beta} \qquad (NG_i)\ \frac{\alpha}{N_{\geq 1}\alpha}$$

The axioms of the local system are quite standard. The Kripke axiom applies to every instance of the $N_{rel\ r}$ modality, and the remaining axioms express properties of inequalities. The rule (NG_i) reflects the fact that properties which are invariant in the system hold throughout the neighbourhood.

In the global axiom system. we have Kripke axioms for $[i]$ modalities and for transitive closure, and an induction rule. We need a "transfer" rule to infer $\alpha@i$ globally when we infer α locally. The remaining axioms relate to social network games: specifying the fact that formulas are asserted at strategy profiles, corresponding to one choice for each player, that payoffs for a player are determined by the player's neighbourhood, and so on.

Global axiom schemes AX

$(B0)$ $(\neg\alpha)@i \equiv \neg\alpha@i$

$(B1)$ $(\alpha \vee \beta)@i \equiv (\alpha@i \vee \beta@i)$

$(B2)$ $[i](\phi_1 \supset \phi_2) \supset ([i]\phi_1 \supset [i]\phi_2)$

$(B3)$ $\Diamond^*\phi \equiv (\phi \vee \bigcirc\Diamond^*\phi)$

$(B4)$ $\alpha@j \equiv [i]\alpha@j, \quad j \neq i$

$(B5)$ $(p@j \equiv [i]p@j) \wedge (\neg p@j \equiv [i]\neg p@j) \quad i \notin N_j$

$(B6)$ $(\bigwedge_{j\in N_i} a_j@j) \supset$
$$((p@i \supset \Box^*(\bigwedge_{j\in N_i} a_j@j \supset p@i)) \wedge (\neg p@i \supset \Box^*(\bigwedge_{j\in N_i} a_j@j \supset \neg p@i))$$

$(B7)$ $e_j@i \equiv \bigodot e_j@i$

$(B8)$ $\Box^* \bigwedge_{i\in[n]} (\bigvee_{a\in\Sigma} (a@i \wedge \bigwedge_{b\neq a} \neg b@i))$

$(B9)$ $(N_{rel\ r}\alpha)@i \supset \bigvee_{J,K\subseteq N_i} (\bigwedge_{j\in J} \alpha@j \wedge \bigwedge_{k\in K} \neg\alpha@j)\ K = N_i - J, \dfrac{|J|}{|J \cup K|}\ rel\ r$

Inference rules

$$(MP)\ \frac{\phi, \quad \phi \supset \psi}{\psi} \qquad (GG)\ \frac{\vdash_i \alpha}{\alpha@i} \qquad (G_i)\ \frac{\phi}{[i]\phi}$$

$$(Conc)\ \frac{\gamma_1 \vee \ldots \vee \gamma_\ell \quad (N_i \cap N_j = \emptyset)}{[\langle i\rangle\phi \wedge \langle j\rangle\psi] \supset \bigvee_{1\leq k\leq \ell} [\langle i\rangle(\phi \wedge \langle j\rangle\gamma_k) \wedge \langle j\rangle(\psi \wedge \langle i\rangle\gamma_k)]}$$

$$(Ind)\ \frac{\psi \supset (\phi \wedge \bigodot\psi)}{\psi \supset \Box^*\phi}$$

The global axioms (B4) and (B5) assert that an i-improvement does not affect the strategies of other players, and hence the payoffs to players that do not have i in their neighbourhoods are unaffected. (B6) asserts that the payoff for a player i is determined only by the strategies of players in the neighbourhood of i. (B7) is a sanity check, that the social network graph is unaltered by improvement dynamics. (B8) asserts that the formulas are asserted over strategy profiles, with every player making a definite choice. (B9) asserts the correctness of neighbourhood threshold formulas.

The rule (Conc) asserts that players can concurrently improve if their neighbourhoods are disjoint, asserting the existence of a square in the improvement graph. This rule typifies the pattern of reasoning in a "true concurrency" based logic.

Proposition 4.1 *Every theorem of AX is valid.*

The soundness of the axioms mostly follow by the semantic definitions. (B4) follows from the fact that when $\sigma \to_i \sigma'$, $\sigma[j] = \sigma'[j]$. (B5) and (B6) follow from the definition of π_i. (B7) is valid since the social network does not vary with profiles. (B8) follows from the definition of strategy profiles. (B9) follows from the semantics of $N_{rel\ r}$ modality.

Among the rules, only the soundness of rule (Conc) is interesting. Assume the validity of the disjunction in the premise, and let $M, \sigma \models \langle i \rangle \phi \wedge \langle j \rangle \psi$. Let $\sigma \to_i \sigma_1$ and $\sigma \to_j \sigma_2$. By Proposition 2.4, there exists a profile σ_3 such that $\sigma_1 \to_j \sigma_3$ and $\sigma_2 \to_i \sigma_3$. Clearly, for some k, $M, \sigma_3 \models \gamma_k$. Hence $M, \sigma_1 \models \phi \wedge \langle j \rangle \gamma_k$, and $M, \sigma_2 \models \psi \wedge \langle i \rangle \gamma_k$. Thus, $M, \sigma \models \langle i \rangle (\phi \wedge \langle j \rangle \gamma_k) \wedge \langle j \rangle (\psi \wedge \langle i \rangle \gamma_k)$, as required.

Theorem 4.2 *AX provides a complete axiomatization of the valid formulas. Satisfiability of a formula ϕ can be decided in nondeterministic exponential time ($2^{O(m \cdot n)}$, where m is the length of ϕ and n is the number of players).*

Proof.

Call a formula ϕ *consistent* if $\not\vdash \neg\phi$. Call $\alpha \in L_i$ *i-consistent* if $\not\vdash_i \neg\alpha$. A finite set of formulas A is consistent if the conjunction of all formulas in A, denoted $\hat{A}$, is consistent. When we have a finite family S of sets of formulas, we write $\tilde{S}$ to denote the disjunction of all formulas $\hat{A}$, where $A \in S$.

For completeness, it suffices to prove that every consistent formula is satisfiable. In fact we show that every consistent formula ϕ is satisfiable in a model of size $2^{O(m \cdot n)}$ where m is the length of ϕ and n is the number of players. From this and soundness, we see a bounded model property: that every satisfiable formula is satisfiable in a model of size exponentially bounded in the size of the formula. This property at once gives a nondeterministic exponential time decision procedure for the logic as well.

Fix a given consistent formula ϕ_0. We confine our attention only to the subformulas of ϕ_0, and maximal consistent sets of subformulas. Towards this, for any i-local formula $\alpha \in L_i$, let $SF_i(\alpha)$ denote the set of subformulas of α. We assume it to be negation closed and to contain α. $|SF_i(\alpha)| = O(|\alpha|)$. Similarly, for any global formula ϕ, define $SF(\phi)$ to be the set of subformulas of ϕ, which is again negation closed and contains ϕ; further, if $\Diamond^*\psi \in SF(\phi)$ then so also $\bigcirc\Diamond^*\psi \in SF(\phi)$. Again, $|SF(\phi)| = O(|\phi|)$.

Let $R \subseteq SF(\phi_0)$. We call R an **atom** if it is a maximal consistent subset (MCS) of $SF(\phi_0)$. Note that, by rule (GG), for any atom R, $A_i = \{\alpha \mid \alpha@i \in R\}$ is i-consistent. Let $(A_1, \ldots, A_n)$ be the tuple of 'local atoms' in R. Let AT denote the set of all atoms. Define $\to \subseteq (AT \times [n] \times AT)$ by: $R_1 \to_i R_2$ iff $\{\phi \mid [i]\phi \in R_1\} \subseteq R_2$. Note that when $R_1 \to_i R_2$ and $\Box^*\phi \in R_1$, $\{\phi, \Box^*\phi\} \subseteq R_2$. Let $G_0 = (AT, \to)$.

Since ϕ_0 is consistent, there exists an MCS $R_0 \in AT$ such that $\phi_0 \in R_0$. Let G_1 be the induced subgraph of G_0 by restricting to atoms reachable from A_0, denoted $(AT_1, \rightarrow)$. We have the following observations on G_1.

- Every R in AT_1 induces a profile σ_R over $[n]$.
- For every R, R' in AT_1, and $i \in [n]$, if $\{a_j@j \in R \mid j \in N_i\} = \{a_j@j \in R' \mid j \in N_i\}$ then $p@i \in R$ iff $p@i \in R'$.
- For every R in AT_1, if A_i is the i-local atom of R, and $N_{rel\ r}\alpha \in A_i$, then $|\{j \mid e_j \in A_i, \alpha \in A_j\}|\ rel\ r \cdot |N_i|$.
- For every R in AT_1, if $\langle i\rangle\phi \in R$ then there exists $R' \in AT_1$ such that $R \rightarrow_i R'$, $\phi \in R'$ and the boolean outcome formula in R' is higher in the preference ordering than the one in R.
- For every R in AT_1, if $\Diamond^*\phi \in R$, then there exists an atom R' in AT_1 reachable from R such that $\phi \in R'$.

Axioms (B8) and (B6) ensure the first two conditions, the third uses the local axiom systems. The (Conc) rule ensures the fourth condition that when $\langle i\rangle\psi \in R$, we can indeed "compute" the maximal consistent set R' such that $R \rightarrow_i R'$. The last condition requires an argument such as the one used for propositional dynamic logic.

Define the game $G_{\phi_0} = (n, \Sigma, \Omega, E, \pi)$ by: $\Omega = 2^{P_0}$ where $P_0 = \{p \mid p@i \in SF(\phi_0), i \in [n]\}$; $E = \{(j,i) \mid \mathrm{e}_j \in A_i \text{ of } R_0\}$; $\pi_i(a_{j_1}, \ldots, a_{j_k}) = \{p@i \in R \in AT_1 \mid a_{j_1}@j_1, \ldots, a_{j_k}@j_k \in R\}$, where $k = |N_i|$ and $\{\top@i\}$ if no such atom R exists, where $\top$ stands for "True" Note that for every profile that occurs in G_1, the payoff map is non-trivial. It is well-defined, by the second condition above. We have a model $M_{\phi_0} = (n, \Sigma, 2^{P_0}, E, \pi)$.

Then we show that for every subformula ϕ and every global maximal consistent set $R \in AT_1$, $\phi \in R$ iff $M_{\phi_0}, \sigma_R \models \phi$. This is proved by induction on the structure of ϕ. The axiom system ensures that neighbourhood specifications are consistent across players and the (Conc) rule ensures that when $\langle i\rangle\psi \in R$, we can indeed "compute" the maximal consistent set R' such that $R \rightarrow_i R'$.

Since there exists a maximal consistent set $R_0 \in AT_1$ such that the given formula $\phi_0 \in A_0$, we now have: $M_0, \sigma_{R_0} \models \phi_0$ and we are done.

$\square$

Thus we have completeness of axiomatization as well as decidability of satisfiability. While we have presented a non-deterministic exponential time decision procedure, we believe that it can be improved to deterministic exponential time: the main idea is to construct the entire atom graph but avoid guessing a good subgraph, but instead delete nodes and edges until what remains is a good subgraph.

Theorem 4.3 *The satisfiability problem is DEXPTIME-hard.*

We will use a version of the propositional dynamic logic whose satisfiability problem is already DEXPTIME-hard due to (citing paper). All that is remaining for us to prove is show satisfiability of the PDL formula is equivalent to the satisfiability of the translated global outcome formulas, Φ . To go about this, we translate the syntax of formulas from the PDL to the global outcome formulas of the strategy logic and vice versa.

PDL Syntax and Semantics Let P be a set of propositions and A be a finite set of actions. Then the following is the syntax for a version of PDL,

$$\alpha \in \Phi_{\mathbf{PDL}} ::= p, \quad p \in P \mid \neg\alpha \mid \alpha \vee \alpha' \mid \langle a \rangle \alpha, \ a \in A \mid \Diamond^* \alpha$$

A model for PDL formulas will be $M_{PDL} = (W, \to, \mathbf{val})$ where W is a finite set of worlds. $\to = \cup_{\{a \in A\}} \to_a$ where for any $a \in A$, $\to_a \subseteq W \times W$ and $\mathbf{val} : W \mapsto 2^P$.

For the direction of showing satisfiability from PDL to **global outcome formulas**.

$\Psi : \Phi_{\mathbf{PDL}} \mapsto \Phi$ is defined inductively, such that only for $\Psi(p) = \vee_{\{a \in A\}} p@a$, $\Psi(\neg\alpha) = \neg\Psi(\alpha)$, $\Psi(\alpha \vee \alpha') = \Psi(\alpha) \vee \Psi(\alpha')$, $\Psi(\langle a \rangle \alpha) = \langle a \rangle \Psi(\alpha)$. Since for converting a $\Phi_{\mathbf{PDL}}$ formula into a Φ formula under the Ψ translation involves changing the leaves of the corresponding formula tree we can safely conclude the translation is polynomial in size of the original $\Phi_{\mathbf{PDL}}$ formula with a size overhead in the order of $O(A)$. And the procedure is also polynomial in time with respect to the size of the original $\Phi_{\mathbf{PDL}}$ formula.

Suppose $\alpha \in \Phi_{\mathbf{PDL}}$ is satisfiable in a model $M_{PDL} = (W, \to, \mathbf{val})$. For the translated global outcome formula $\Psi(\alpha)$, the social network game model where it will be satisfiable will be $M = (|A|, Z, 2^P, \emptyset, \pi = \mathbf{val})$ where Z is any finite set such that $|Z| \leq \frac{\ln(|W|)}{|A|}$. Now, we don't exactly need the entirety of he social network game model. α is satisfiable means, there exists $w \in W$ such that, $M_{PDL}, w \models \alpha$. We would need to work with only the reachability set of w, $\mathbf{Reach}(w) = \{v \in W \mid w \to^* v\}$. Next details yet to work out

For the direction of showing satisfiability from **global outcome formulas** to PDL.

We observe that the Improvement Graph of any game can be thought of as a labelled transition system which are the models of the propositional dynamic logic. Given a social network model , $M = (n, \Sigma, 2^P, E, \pi)$ and a strategy profile σ, we compute the reachability set of σ, $\mathbf{Reach}(\sigma) = \{\sigma' \in \Sigma^n \mid \sigma \to^* \sigma'\}$. $A = [n]$ We translate the formulas similarly as above. $\chi : \Phi \mapsto \Phi_{\mathbf{PDL}}$ Next details yet to work out

The model checking problem asks, given a social network game $M = (n, \Sigma, 2^P, E, \pi)$, and a formula ϕ whether there is some profile σ such that $M, \sigma \models \phi$. This can be solved in time $2^{O(n)} \cdot |E| \cdot |\phi|$, by explicitly constructing the strategy space and then running a standard labelling algorithm. This is exponential in the number of players, which is unavoidable since computing Nash equilibrium in social network games is known to be NP-hard ([12]).

5 Conclusion

We have presented a logic to reason about strategization in social network games. The logic is presented in two layers: local formulas talk about how a player strategizes based on threshold assumptions of strategies of neighbours. Global formulas assert strategy improvement by players and their reachability. We hope that the axiom system demonstrates that such local and global reasoning is sufficiently interesting.

Since we have treated outcomes as propositions, they come with a natural partial order: that of *implication*. However, $\neg p@i \wedge \langle i \rangle p@i$ implicitly asserts that p is a preferred outcome over $\neg p$ for player i. A natural question is to ask what utility functions and preference orderings can be expressed in such a logic (or enrichments thereof), and this leads in interesting directions.

The game equivalence presented here implicitly suggests an algebraic structure in the space of strategy profiles. But this is over strategic form games, which is quite different from that studied in [5], [14] over extensive form games. Moreover, though we have assumed a social relationship between players, the framework is non-cooperative, where players act individually. However social networks encompass both selfish and coalitional behaviour, and it would be interesting to study coalitional powers in social network games.

A very interesting question relates to mixed strategies. Note that the logic remains similar, and improvement graph dynamics can be studied. However, the strategy space is no longer discrete, and the transitive closure operator (based on finite paths) does not have much purchase. The basic modality is much more complicated and convergence issues are challenging.

While we have presented a Hilbert style axiom system, for reasoning about games it would be better to work with *sequents* of the form $\Gamma \vdash \phi$ where Γ is a theory, with formulas describing the game, thus constraining the space of profiles and ϕ is logically entailed. Developing such a proof theory seems to offer interesting challenges.

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