Implicit quantification for modal reasoning in large games

Ramit Das[1], Anantha Padmanabha[2], and R Ramanujam[3]

[1] The Institute of Mathematical Sciences and
Homi Bhabha National Institute Chennai, India
ramitd@imsc.res.in
[2] DI ENS, ENS, CNRS, PSL University & Inria, Paris, France
anantha.padmanabha@inria.fr
[3] Azim Premji University University, Bengaluru, India (Visiting)
jam@imsc.res.in

Abstract. Reasoning about equilibria in normal form games involves the study of players' incentives to deviate unilaterally from any profile. In the case of large anonymous games, the pattern of reasoning is different. Payoffs are determined by strategy distributions rather than strategy profiles. In such a game each player would strategise based on expectations of what fraction of the population makes some choice, rather than respond to individual choices by other players. A player may not even know how many players there are in the game.

Logicising such strategisation involves many challenges as the set of players is potentially unbounded. This suggests a logic of quantification over player variables and modalities for player deviation, but such a logic is easily seen to be undecidable. Instead, we propose a propositional modal logic using player types as names and *implicit quantification* over players. With modalities for player deviation and transitive closure, the logic can be used to specify game equilibrium and interesting patterns of reasoning in large games. We show that the logic is decidable and present a complete axiomatisation of the valid formulas.

Keywords: Large games, modal logic, strategy distributions, improvement dynamics, decidability, implicit quantification

1 Background

Game theory provides elegant models of strategic interaction between players. Solution concepts predict stable play by rational players. When players make choices, reason about how others make choices, consider their own response, and further strategise how others would respond to their own choices, mutual intersubjectivity plays an essential role in strategisation. Common knowledge of, or stable common belief in, players' rationality offers an epistemic justification for equilibrium notions. Central to such reasoning is the modelling of outcomes, assigning an outcome to each player for every strategy profile.

Many social situations have a large number of players involved in interactive and strategic decision making, but the outcome does not depend on who chooses what, but *how many players* make a particular choice. For instance, consider a city (such as Singapore) deciding to make the entire city Wi-Fi enabled. How is it decided that a facility be provided as infrastructure? Typically such analysis involves determining when usage crosses a threshold. But then understanding why usage of one facility increases vastly, rather than another, despite the presence of several alternatives, is tricky. But this is what strategy selection is all about. In such analyses, we are really interested in what fraction of the population chooses each option.

Similar situations occur in the management of the Internet. Policies for bandwidth allocation are not static. They are dynamic, based on studying both volumes of traffic and type of traffic. The popularity of an application like Twitter or YouTube dramatically changes traffic, calling for changes in Internet policies. Predicting such future requirements is tricky, but much wanted by the engineers. Herd mentality and imitation are common in such situations.

These are the situations studied under the rubric of *large games* or *anonymous games*, so-called since player identity does not matter for deciding the outcomes in such games. Payoffs are associated not with strategy profiles, but with choice distributions. Suppose that there are k strategies used by a population of n players. Then the outcome is specified as a map $\mu : \Pi_k(n) \rightarrow P^k$, where $\Pi_k(n)$ is a set of distributions: k-tuples that sum up to n, and P is the set of payoffs. Thus every player playing the j^{th} strategy gets the payoff given by the j^{th} component specified by μ for a given distribution.

The advantage of working with large games is that problems of mutual intersubjectivity are eliminated, as it is virtually impossible for each player to reason about the behaviour of every other player in such games, since a player may not even know how many players are in the game, let alone how they are likely to play. As a result, the strategy spaces are much "smoother", though discontinuities in payoff functions can be a serious issue.

Large anonymous games have been extensively studied by game theorists. See, for instance, [Blo99,Blo00,DP07] and the references therein. What is of interest in such games is that pure strategy equilibria exist for many subclasses, and the fixed-point computation is carried out in a discrete space ([BFH09]).

1.1 Contributions of this paper

This paper is situated in this arena of formal reasoning about games. We are interested in reasoning about how players reason individually when the outcome is collective. Players have common knowledge of how choice distributions determine payoffs and reason about the effect of player deviations. They reason not about the actual number of players that choose an option, but about what fraction of players make that choice. We wish to study paths to equilibrium in such a context.

We study *improvement dynamics* ([SA15]) in large games. This is a graph structure on strategy spaces, where nodes are profiles, and player labeled edges represent a deviation by a player to get an improved outcome. Nash equilibrium is simply a sink node in this graph. In the case of large games, it is appropriate to consider improvement not by single players but by several players at the same time, shifting choice distributions significantly.

The notion of *player types* is central to game theory: it specifies the beliefs that a player has on other players' behaviour, and how the player would act contingently. Logically speaking, they specify player responses to postulated strategy distributions. A typical player type may describe a profile where at least half the players choose a and at most a quarter choose b, from which an improvement path may lead to a profile where less than a quarter choose a and more than two-thirds choose b. So called *herd behaviour* and *runaway phenomena* ([Ban92]) can be thus reasoned about in a logic with such player types. We can then iterate over behaviours set by initial player types to give more complicated player behaviours. We also wish to capture the notion of paths to the solution concept of Nash Equilibrium. Such properties require us to work with a logic that includes closure operators like the transitive closure.

Since the number of players is finite but unbounded, it is natural to expect a first-order logic with quantification over player variables. As it turns out, such a logic is undecidable and has no recursive axiomatisation, hence we study a propositional fragment of such a logic, with constant symbols denoting players, and *implicit quantification* over players. Distribution constraints can still force a lower bound on the number of players in the games. We show that the logic is decidable and present a complete axiomatisation of the valid formulas. We present a bisimulation characterisation of the logic.

The principal technical challenge in such inference is that while a formula is interpreted in a choice profile (an n-tuple where n is the number of players), the formula cannot specify the number n. Thus for a decision procedure or for a completeness proof, we need to *compute* the number of players from the given formula and work out the dynamics. Moreover, when we have implicit quantification, we may know that *some* players deviate without knowing which ones, and then ensuring consistency between profiles is challenging.

Such analysis has an interesting consequence. Logical player types induce an *equivalence on players* in games, and thus analysis of a large game can be reduced to one "between" player types. We also get a *dual use* for the logic on large games: the dynamic aspect of the logic can be used to study the equilibrium dynamics of a given large game; dually, the logic can be used as a compact *specification mechanism* for large games which could in general be quite unwieldy to specify explicitly.

1.2 Related work

The logical foundations of game theory have been studied for some years now, and modal characterisation of equilibria presented by several researchers ([Bon01], [HHMW03], [vB12]). The seminal work of Parikh ([Par85]) presented propositional game logic, a way of reasoning about game composition. Strategic reasoning in extensive form games has been studied in depth in the context of logics of strategic ability such as Alternating Temporal Logics and other process based logics ([AHK02], [HJW05], [RS08]. [Gho08], [GR11]).

However, in general, these studies typically fix n and study n-player games, whereas our principal point of departure in this paper is in reasoning about games with *unboundedly many players*. Moreover these studies associate payoffs with strategy profiles rather than with distributions as we do here. In our earlier work, dynamics in large games modelling imitation behaviour ([PR13]) and dynamic game forms ([PR14]) have been studied.

How choices diffuse across networks ([CH15], [KKT15], [DR21]) is a study similar in spirit to ours but distributions again make the principal difference. Majority dynamics is implicit in our reasoning, and it has been extensively studied as a class of games ([MNT13]).

In [GU08] the authors consider pebble games over a large number of players. The motivation is to understand the expressive power of automata with infinite runs and related logical fragments. In our setting, we do not have the notion of history, our games are one shot for all players. Another work is [JHW15], which formalises the reasoning in multi-agent systems but uses linear temporal logic as specification. Its model is closely related to ours, but we work over one-shot games instead of taking up game dynamics for iterated game play.

Large games arise naturally in *evolutionary game theory* ([Wei97]), in which long range behaviour of a population is determined by distributions of player types among the population. Such games bear a close relationship to the class of *population protocols* ([AAER07]) in distributed computing, where again interactions do not depend on the identity of the interacting processes. In the logic we present, we do use player identities to describe strategisation by players, but the strategies themselves depend only on distributions.

1.3 Outline of the paper

We present the preliminaries in the next section. We define large games and associate improvement graphs with them. In Section 3, we present the logics to reason about these games. We present two such logics, *MSO-STRAT*, a monadic second-order logic and *MIQ-STRAT*, which is a propositional modal logic with implicit quantification.

In Section 4 we discuss the decidability of these logics. The majority of this section focuses on the undecidability of the *finite satisfiability* problem for *MSO-STRAT*, by reduction from the *finite tiling problem*. In Section 5, we present an axiom system for the implicitly quantified propositional modal logic *MIQ-STRAT* which is shown to be complete, yielding a decision procedure for satisfiability. We present a bisimulation characterization for *MIQ-STRAT* in Section 6 and we end the paper with a discussion on unaddressed issues.

2 Large games

Let N denote a finite set of players. We will often consider N to be the set $\{1, 2, \ldots, |N|\}$, and refer to players i, j etc in this set. In general, we will want to consider subsets of players, and use u, v etc. to denote these subsets.

To talk about outcomes of games, rather than work with quantitative payoffs, we use a propositional language to describe abstract conditions on outcomes. Following the line of work in [HW05],[BLSLZ06], [ÅHHW13], and a closer line of work in [PR11], we assign sets of propositions as payoffs for players. We fix $\mathcal{P}$, a countable set of atomic propositions. $\mathcal{P}$ consists of qualitative descriptions of outcomes. Every game involves only a finite set $P \subseteq \mathcal{P}$ of these propositions. In general an outcome for a player is a boolean formula over $\mathcal{P}$. Let $\mathbf{B}(\mathcal{P})$ denote the set of boolean formulas over $\mathcal{P}$.

Fix a finite set of actions (or choices, or strategies) which we denote by $\mathbf{\Gamma}$. An action distribution for n players is a $|\mathbf{\Gamma}|$ tuple of integers $\mathbf{y} = (y_1, \ldots, y_{|\mathbf{\Gamma}|})$ such that $y_j \geq 0$ and $\Sigma_{j=1}^{k} y_j = n, \quad 1 \leq j \leq |\mathbf{\Gamma}|$. That is, the jth component of $\mathbf{y}$ gives the number of players who play action a_j. Let $\mathbf{Y}[n]$ denote the set of all action distributions for n players.

Often we will be interested in *normalised distributions*: if $\mathbf{y} = (y_1, \ldots, y_{|\mathbf{\Gamma}|})$ is an action distribution for n players, then its normalisation $\mathbf{y}_{norm} = (\frac{y_1}{n}, \ldots, \frac{y_{|\mathbf{\Gamma}|}}{n})$. These ratios are independent of the number of players, and can be used to compare distribution profiles across games.

We have an outcome function $\omega : \mathbf{Y}[n] \rightarrow (2^P)^{|\mathbf{\Gamma}|}$ which gives the valuation of the outcome propositions P according to the action distribution of the players: the idea is that every player choosing an action a gets the same outcome.

Thus formally, a game $\mathcal{G}$ over action set $\mathbf{\Gamma}$ is a tuple $\mathcal{G} = (N, P, \omega, \preceq)$, where P is a finite subset of $\mathcal{P}$, ω is an outcome function, and $\preceq$ is a pre-order on 2^P, specifying the common preferences of players. Since every element of 2^P can be viewed as a boolean formula over P, we will consider $(2^P, \preceq)$ as a pre-ordering on boolean formulas. $\prec$ denotes the corresponding strict preference relation.

It is to be noted that we have assumed all players to have a common set of strategy choices and preferences over outcomes; this is for convenience of technical treatment and presentation. The results in the paper can be extended to individual choice sets for players and individual preference relations, but this leads not only to a more laboured presentation, but also underlying philosophical issues about individual preferences being common knowledge which is hard to justify in large anonymous games.

A configuration of the game is given by $\mathbf{a} \in \mathbf{\Gamma}^N$ an *action profile*, which specifies a choice of action for each player. This induces a distribution and hence, implicitly, an outcome for each player. Let Σ denote the set of strategy profiles Γ^N. Note that ω induces a map $\hat{\omega} : \Sigma \rightarrow (2^P)^N$.

In general we use σ, σ' etc to denote profiles, and $\sigma[i]$ denoting the i^{th} element of σ, which is interpreted as the choice of player i. We write $\sigma \prec_i \sigma'$ to denote that $\hat{\omega}(\sigma)[i] \prec_i \hat{\omega}(\sigma')[i]$.

When players act individually, rationality assumptions dictate that, at any profile, if a player can get a better outcome by unilaterally altering the choice, the player would do so. A simple way to study such reasoning is given by the improvement graph dynamics defined as follows.

Definition 1. *A* **player improvement graph** $G_{\mathcal{G}}$*, associated with the game* $\mathcal{G}$*, is the graph* $G_{\mathcal{G}} = (\Sigma, E_{\mathcal{G}})$*, where* $\Sigma = \Gamma^N$*, and* $E_{\mathcal{G}} \subseteq (\Sigma \times N \times \Sigma)$ *is the player labelled edge relation given by:* $(\sigma, i, \sigma') \in E_{\mathcal{G}}$ *iff* $\sigma \prec_i \sigma'$*, and* $\sigma[j] = \sigma'[j]$*, for all* $j \neq i$*.*

We have an i-labelled edge from a strategy profile to another, if player i can unilaterally deviate from the former to the latter to get an improved payoff. Note that at a profile, there can be different i-improvement edges leading to different profiles (with perhaps incomparable outcomes). A path in $G_{\mathcal{G}}$ is an improvement path.

However, in large games, unless the outcome map has major discontinuities, deviation by a single player may cause no change in outcome at all. We may end up having a player improvement graph with very few edges. Moreover consider a swap game where player i deviates from choice a to choice b iff another player j deviates from b to a. This would result in the same distribution and hence may not result in improvement for either player, and hence 'should' not be considered meaningful deviations. We therefore consider deviation by *teams* of players. Note that they need not be acting together intentionally; it is sufficient that they all deviate in different directions, but improving the outcome for all of them.

Definition 2. *An* **improvement graph** $I_{\mathcal{G}}$*, associated with the game* $\mathcal{G}$*, is the graph* $I_{\mathcal{G}} = (\Sigma, E_{\mathcal{G}})$*, where* $\Sigma = \Gamma^N$*, and* $E_{\mathcal{G}} \subseteq (\Sigma \times 2^N \times \Sigma)$ *is the labelled edge relation given by:* $(\sigma, u, \sigma') \in E_{\mathcal{G}}$ *iff for all* $i \in u$*,* $\sigma \prec_i \sigma'$*, and for all* $j \notin u$*,* $\sigma[j] = \sigma'[j]$*.*

We have a u-labelled edge from a strategy profile to another, if every player i in u can unilaterally deviate from the former to the latter to get an improved payoff. Note that at a profile, there can be different u-improvement edges leading to different profiles (with perhaps incomparable outcomes). A path in $I_{\mathcal{G}}$ is an improvement path. When the set u is a singleton $\{i\}$, we will denote the edge as $\rightarrow_i$.

We note again that the players need not form coalitions. This is akin to city traffic, where several cars take a relatively free road, and this changes the traffic pattern, causing an outcome change for other cars as well. Another instance is the network effect of social media platforms. The migration of a population from one social network to another (as in Orkut to Facebook, say), does not require coordination for the collective network effect to take place.

Note that the improvement graph can have cycles. For instance, consider the two-player game of matching pennies: both players call heads or tails, the first player wins when the results match, and the second wins when there is a mismatch. We then have the cycle $(H, H) \rightarrow_2 (H, T) \rightarrow_1 (T, T) \rightarrow_2 (T, H) \rightarrow_1 (H, H)$.

In any particular game, N is fixed, as well as the size of $\mathbf{\Gamma}$, and hence $I_{\mathcal{G}}$ is a finite directed graph, though its size is exponential in the number of players.

It contains interesting information about the game $\mathcal{G}$. For instance consider a *sink* node of $I_{\mathcal{G}}$, which has no out-going edge: it is easy to see that a strategy profile is a sink node if and only if it constitutes a *Nash equilibrium*, from which no player has any incentive to deviate.

There is an equivalent graph structure to the improvement graph that we can work with. Fix a game $\mathcal{G}$ and let σ be a strategy profile. σ has an alternative presentation: $(A_1, \ldots, A_{|\mathbf{\Gamma}|})$, where for each j, the sets A_j constitute a $|\mathbf{\Gamma}|$-way partition of N. Further for every improvement edge $\sigma \rightarrow_u \sigma'$ we can define a corresponding one: $(A_1, \ldots, A_{|\mathbf{\Gamma}|}) \rightarrow_u (B_1, \ldots, B_{|\mathbf{\Gamma}|})$. These two graphs are isomorphic, and we can use either representation, as convenient.

In the analysis of games, we are typically interested in questions like whether the game has a Nash equilibrium, whether every improvement path is finite, whether an equilibrium profile is reachable from every profile, etc. Algorithmically all these questions are efficiently solvable in the size of the improvement graph, which is (unfortunately) exponential in the number of players.

When we consider algorithmic study of large games, an important source of difficulty is the *game specification* itself. For an n-player game, the outcome function is a map from all possible choice distributions over n to the outcome set. This is clearly difficult since we need to consider the number of $|\mathbf{\Gamma}|$-way partitions of n, and this can be very large. In practice, we would like a partial specification mechanism for the outcome function given as a table of distribution vectors with outcomes. This is sufficient to generate the improvement graph. Moreover, player preferences need to be given only over the distribution vectors present in the table. Such specifications may be considered *partial game specifications*. This is of interest since the logical formalism we study can provide a language for such specifications.

Note that the improvement graph dynamics induces a natural **game equivalence**: we can consider two games to be equivalent if they have a similar (not necessarily isomorphic) improvement structure which can be characterized by an appropriate notion of bisimulation. Since we study a propositional modal logic below whose formulas are bisimulation invariant, we can characterize games upto this equivalence.

3 Logics for large games

How do we express strategisation by players in large games in logic? Towards this, we first observe that the best response of a player is not to what other players choose, but to how many players make a particular choice. Therefore deviation by a player is also based on the expectation of how many other players would deviate as well. Such reasoning can be formalised by assertions of the form “If half or more of the players play a then I should respond with b”. Since a player need not know the actual number of players in the game, it makes sense to work with ratios rather than actual numbers. Moreover, by recursion over such formulas, we can build player types. If the above assertion defines a player type t, we can have another type t' which asserts: "If at least one third of the players are of type t, then play a", and so on.

A more tricky question is how we are to refer to players at all in the syntax of the logic. Note that every game comes with a set of players, but player identities are immaterial in a large game. We therefore want only a way of abstractly specifying player types rather than work with actual player identities. The logical device of using *variables* and *quantification* over them is the obvious solution: we would like to assert: $\forall x.(\alpha(x) \implies p@x)$ where ϕ is some type specification, expressing that every player of type α gets an outcome p.

3.1 Player types as formulas

The main characteristic of strategisation in large games is player response to potential choice distributions. Specifically, $\sharp\ a\ rel\ r$ is a *distribution constraint*: it specifies a constraint on the set of players of choosing a. For instance $\sharp\ a\ \leq\ r$ asserts that at most an r-fraction of players choose a.

Distribution Constraint Formulas The syntax of distribution constraint formulas is given by:

$$\delta ::= \sharp\ a\ rel\ r,\ a \in \mathbf{\Gamma} \mid \neg\delta \mid \delta \vee \delta'$$

where $a \in \mathbf{\Gamma}$, $rel \in \{\geq, \leq, <, >\}$, r is a rational number, $0 \leq r \leq 1$.

It is easy to see that a distribution constraint formula specifies a finite set of distribution constraints of the form $\sharp\ a\ rel\ r$.

Fix a game $\mathcal{G}$ and a strategy profile σ in it. Distribution constraint formulas are interpreted on strategy profiles:

- $\sigma \models_d \sharp\ a\ rel\ r$ if $\frac{|\{j|\sigma[j]=a\}|}{|N|}\ rel\ r$.
- $\sigma \models_d \neg\delta$ if $\sigma \not\models_d \delta$.
- $\sigma \models_d \delta \vee \delta'$ if $\sigma \models_d \delta$ or $\sigma \models_d \delta'$.

We call $(rel'\ r)$ *the complement* of $(rel\ r)$ when rel' is the relational complement of rel. For instance, $\geq$ is the complement of $<$ and $>$ is the complement of $\leq$. We say $(rel\ r)$ *entails* $(rel'\ r')$ if whenever $\sharp\ a\ rel\ r$ holds, then $\sharp\ a\ rel'\ r'$ also holds. For instance this holds if rel, rel' are $\leq$ and $r \leq r'$. It also holds if $r \geq r'$ and either both rel, rel' are $\geq$ or rel is $>$ and rel' is $\geq$. We use the abbreviation $\sharp\ a = r$ to denote $\sharp\ a \leq r \wedge \sharp\ a \geq r$.

Let $k = |\mathbf{\Gamma}|$ and $distr(r_1, \ldots, r_k)$ denotes the formula $\bigwedge_{\ell=1}^{k} \sharp a_\ell = r_\ell$, where $\Sigma_\ell\ r_\ell = 1$. Note that this specifies a normalised distribution, independent of the number of players.

We say that a distribution constraint formula δ is satisfiable if there exists a profile σ such that $\sigma \models_d \delta$. Note that some distribution constraint formulas are clearly unsatisfiable. For instance, consider the following, where $\mathbf{\Gamma} = \{a, b\}$: $\sharp\ a < 0$, $\sharp\ a > 1$, $\sharp a > \frac{1}{2} \wedge \sharp b > \frac{1}{2}$, and $\sharp a < \frac{1}{2} \wedge \sharp b < \frac{1}{2}$. A satisfiable instance is : $\sharp a \leq \frac{1}{2} \wedge \sharp b \leq \frac{1}{2}$.

When we have a satisfiable distribution formula like $\sharp a = \frac{3}{7} \wedge \sharp b = \frac{2}{5} \wedge \sharp c > 0$, the number of players, which needs to be an integer, has to be at least 35. One possible distribution that satisfies the constraint is $(15, 14, 6)$, but clearly so also is $(60, 56, 24)$. It is in fact an easy combinatorial exercise to decide when a set of distribution constraints is satisfiable.

Proposition 1. *Let J be a finite set of distribution constraint formulas. If J is satisfiable, then there is a least number n and a set of normalised $\mathbf{\Gamma}$-distributions over $[n]$ each of which satisfies J.*

Proof. The idea is to set up a system of linear inequalities for J and solve it to check if J is satisfiable. Let $\hat{J}$ be the conjunction of formulas of J. Without loss of generality we can assume that $\hat{J}$ is in negation normal form ($\neg$ appears only in front of atomic formulas). Further, we can use propositional validites to obtain a formula in disjunctive normal form J' which is equivalent to $\hat{J}$ where J' is of the form $\vee(\bigwedge(\neg)\sharp\ a\ rel\ r)$. Clearly, $\hat{J}$ is satisfiable iff one of the disjuncts of J' is satisfiable. So we only need to check if one the these disjunction is satisfiable. Let $\phi = \delta_1 \wedge \delta_2 \wedge \ldots \wedge \delta_m$ be such a disjunct.

First if δ_i is of the form $\neg(\sharp\ a \leq r)$ we can equivalently replace it with $\sharp\ a > r$. Similarly the formulas $\neg(\sharp\ a < r)$, $\neg(\sharp\ a > r)$, $\neg(\sharp\ a \geq r)$ can be replaced with $\sharp\ a \geq r$, $\sharp\ a \leq r$ and $\sharp\ a < r$ respectively. Thus we can assume that every δ_i is negation free.

Now, for every $a \in \mathbf{\Gamma}$ consider a variable x_a. Set up the linear inequalities as follows. For each δ_i:

- if δ_i is of the form $\sharp\ a \ \leq r_i$ then have an equation $x_a - r_i \leq 0$
- if δ_i is of the form $\sharp\ a \ < r_i$ then have an equation $x_a - r_i < 0$
- if δ_i is of the form $\sharp\ a \ \geq r_i$ then have an equation $r_i - x_a \leq 0$
- if δ_i is of the form $\sharp\ a \ > r_i$ then have an equation $r_i - x_a < 0$

Let I be the system of equation obtained. We also add the constraints $-x_a \geq 0$ for every $a \in \mathbf{\Gamma}$ and $\sum\limits_{a \in \mathbf{\Gamma}} x_a = 1$. Now we claim that I has a solution iff there is a least number n and a finite set of $\mathbf{\Gamma}$-distributions over $[n]$ each of which satisfies ϕ.

Clearly if such a solution exists for ϕ, then for each x_a we can assign the value $\sharp\ a$ and that satisfies the system of inequalities I. Conversely, if I has a solution, since every coefficient and constant in the system is a rational number, we can obtain a solution (for instance via Gaussian elimination) where every x_i is a rational number. Let $x_a = \frac{p_a}{q_a}$ for every $a \in \mathbf{\Gamma}$. Now define n to be the least common multiple of $\{q_a \mid a \in \mathbf{\Gamma}\}$. It can be verified that the distribution where each $\sharp\ a$ is assigned to $n \cdot \frac{p_a}{q_a}$ satisfies ϕ.

If we use Gaussian elimination, it is also clear that n is at most exponential in r_i that are mentioned in J. Thus we have a polynomial time algorithm that computes n if it exists.

This exercise has a pleasant consequence. Assume that all the rationals in formulas are given in binary. If m is the maximum length of any integer in the distribution constraints, and there are ℓ-many integers occurring, then the least number n referred to above needs only ℓm bits to specify. (Note that the set $\mathbf{\Gamma}$ is fixed for the language, and hence $|\mathbf{\Gamma}|$ is treated as a constant.) Note that though n is described using ℓm bits, the number itself is of the order of $2^{\ell m}$. Thus a distribution constraint formula δ can 'force' at most $2^{|\delta|^2}$-many players.

Player Type Formulas For player types, we will use formulas of the form $\delta \to a$ which asserts a type of player who responds with a when the distribution constraint δ holds. In addition, player types also specify their preferences over outcomes.

As an example, $\sharp\ a\ > \frac{2}{3} \to b$, asserts a type of player who responds with b if more than two-thirds of the players choose a. Moreover we can have type assertions of the form $\sharp\ a\ > \frac{2}{3} \to b$ or even better, $(\sharp\ a\ > \frac{2}{3} \wedge \sharp\ c\ > \frac{1}{10}) \to b$. Note that the types can construct complex patterns of hypothetical responses.

The syntax of player types is given by:

$$\alpha ::= \delta \to a \mid \alpha \vee \alpha' \mid \alpha \wedge \alpha'$$

where δ is a distribution constraint formula and $a \in \mathbf{\Gamma}$.

We have only positive formulas for player types, but note that there is a hidden form of negation: for instance, the formula $\delta \to a \wedge \delta \to b$ is a contradiction when $a \neq b$. On the other hand, $\delta \to a \vee \delta \to b$ denotes nondeterministic choice. Since we are principally interested in specifying choices for different distributions positive formulas suffice for us.

Type formulas are interpreted at profiles, for individual players. As above, fix a game. Let σ be a profile in game $\mathcal{G}$ and i a player.

- $\sigma, i \models_t \delta \to a$ if $(\sigma \models_d \delta)$ then $\sigma[i] = a$.
- $\sigma, i \models_t \alpha \vee \alpha'$ if $\sigma, i \models_t \alpha$ or $\sigma, i \models_t \alpha'$.
- $\sigma, i \models_t \alpha \wedge \alpha'$ if $\sigma, i \models_t \alpha$ and $\sigma, i \models_t \alpha'$.

Though player type formulas need to be interpreted at profiles, the notion of player type depends only on the game $\mathcal{G}$ and obviously does not change at different profiles. We will need to consider this when we reason about games as a whole.

It is reasonable to consider players' preferences on outcomes to also be a part of type specifications as well, but rather than mix these with strategic choices, we will include preference specifications separately.

Let $\mathbf{\Gamma}' \subseteq \mathbf{\Gamma}$ and δ be a distribution constraint formula. Then we denote the formula $\bigvee_{a \in \mathbf{\Gamma}'} (\delta \to a)$ as $\delta \to \bigvee_{a \in \mathbf{\Gamma}'} (a)$. The intuitive meaning of $\sigma, i \models_t \delta \to \bigvee_{a \in \mathbf{\Gamma}'} (a)$ is that if $\sigma \models_d \delta$ then player i plays some action $a \in \mathbf{\Gamma}'$. For technical convenience we will assume that $\delta \to \bigvee_{a \in \mathbf{\Gamma}'} (a)$ are also atomic formulas.

It is easy to see that the Proposition 1 can be employed to prove a similar proposition on type formulas as well.

Proposition 2. *Let J be a finite set of type formulas. If J is satisfiable, then there is a least number n and a profile σ over n players that satisfies J.*

Proof. The proof follows along the same lines as Proposition 1. Let $\hat{J}$ be the conjunction of the type formulas in J and we obtain J' equivalent to J in DNF form $\vee(\bigwedge \beta)$, where each β is of the form $\delta \to a$. Clearly, $\hat{J}$ is satisfiable iff one of the disjuncts of J' is satisfiable. Let $\phi = (\alpha_1 \wedge \alpha_2 \ldots \wedge \alpha_m)$ be such a disjunct where each α_i is of the form $\delta_i \to a_i$. Let $K = \{\delta_1, \ldots, \delta_m\}$, and $A = \{a_1, \ldots, a_m\}$.

Consider the distribution constraint formula $\hat{K}$. Now suppose that $\hat{K}$ is a validity. So for every n and every profile σ over n players, we have $\sigma \models_d \delta$ for every $\delta \in K$. Fix any player k. If $\sigma, k \models_t \phi$ then for all $a_i \in A$, $\sigma[i] = a_i$. This is possible only if $a_1 = \ldots = a_m$. In particular, this holds when we consider $n = 1$.

On the other hand, if $\neg\hat{K}$ is satisfiable (then by Proposition 1) there is a least n and a profile σ over n players where $\neg\hat{K}$ is satisfiable. Note that for every $\delta \in K$ we have $\sigma \not\models_d \delta$. Hence for any arbitrary player k and α_i in ϕ we have $\sigma, k \models_t \alpha_i$ since $\sigma \not\models_d \delta_i$.

Note that the bound on n is again exponential in the size of the type specification. If m is the maximum length of any integer in the distribution constraints occurring in the type formulas, and there are ℓ-many integers occurring, then the least number n referred to above needs only ℓm bits to specify. Note that the set $\mathbf{\Gamma}$ is fixed for the language, and hence $|\mathbf{\Gamma}|$ is treated as a constant so replacing negated formulas with a disjunction over $\mathbf{\Gamma}$ does not blow up the formula size. Thus a type formula can 'force' at most $2^{|\delta|^2}$-many player types. Thus, the type analysis in a given game with unboundedly many players can be reduced to one in which the number of players is bounded (exponentially) by the size of the type specification.

3.2 A logic of quantification

What we have discussed so far are assertions that are evaluated at strategy profiles. But we also want to reason about outcomes, player improvement and paths to equilibrium. However, improvement involves deviation by sets of players, which motivates us to consider *set variables* and quantification over them as well.

Recall that $\mathcal{P}$ is a countable set of outcome propositions fixed for the class of games being studied, and that $\mathbf{B}(\mathcal{P})$ denotes the set of boolean formulas over outcomes.

Thus we are led to a two-level syntax of formulas. We have already considered type formulas at one level. Fix a set of player variables $V = \{x_0, x_1, \ldots\}$ with x, y etc ranging over V and a set of team variables $TV = \{X_0, X_1, \ldots\}$ with X, Y etc ranging over TV. We call this logic the *Monadic Second Order Logic of Strategisation in Large Games*, *MSO-STRAT*, for short. (Here the term 'monadic' refers to set quantification.)

To signal that player types are independent of strategy profiles (though they do need to be evaluated at profiles), we use the syntax $[\alpha@x]$ where x is a player variable and α is a type formula. We will also include player preferences, with formulas of the form $\gamma_1 \preceq \gamma_2$ where $\gamma_1, \gamma_2 \in \mathbf{B}(\mathcal{P})$. These are also independent of strategy profiles, but their evaluation makes it clear. The profile-specific atomic formulas are δ (to refer to the choice distribution at that profile), $a@x$ (to denote player's choice) and $p@x$ (to denote player's outcome).

The syntax of strategisation formulas is given by:

$$\begin{aligned}\phi \in \Phi ::= \gamma_1 \prec \gamma_2 \mid [\alpha@x] \mid \delta \mid p@x \mid a@x \mid x \in X \mid x = y \\ \mid \neg\phi \mid \phi \vee \phi' \mid \langle X\rangle\phi \mid \Box^*\phi \mid \;\mid \exists x.\phi \mid \exists X.\phi\end{aligned}$$

where α is a type formula, $\gamma_1, \gamma_2 \in \mathbf{B}(\mathcal{P})$, $p \in \mathcal{P}, x \in V$ and $X \in TV$.

We do not have constants in the syntax but adding player names and interpreting them in a given model appropriately can be added on standard lines.

The formulas are evaluated at a strategy profile of a given game. $\alpha@x$ asserts that player x is of type α and $p@x$ denotes x gets payoff p. $\langle X\rangle\phi$ asserts that there is an improvement edge from current profile to another where the team of players X switch their strategy so that ϕ holds. $\Diamond^*\phi$ denotes that there is a profile reachable from current profile where ϕ holds. All dual operators are defined in the standard way.

Since we have equality, we can easily specify that a team X is a singleton (denoted by $Sing(X)$): $\exists x.x \in X \wedge \forall y.(y \in X \implies y = x)$. Hence we can assert $\langle x\rangle\phi$ using the formula $\exists X.Sing(X) \wedge x \in X \wedge \langle X\rangle\phi$.

Note that the formula $[(\delta \to a)@x] \implies \Box^*(\delta \implies a@x)$ is a validity. Similarly, $(\gamma_1 \prec \gamma_2)@x \implies \Box^*((\gamma_1 \prec \gamma_2)@x)$ is also valid.

Given a game $\mathcal{G} = (N, P, \omega, (\preceq_i)_{i\in N})$ we evaluate the formulas at strategy profiles of $\mathcal{G}$. We need assignments for player variables and team variables. So, we have the game model $M = (N, P, \omega, \preceq_M, \delta)$ of $\mathcal{G}$ where $\delta = (\delta_1, \delta_2)$ such that $\delta_1 : V \to N$ assigns every variable to a player, and $\delta_2 : TV \to 2^N$ assigns every set variable to a set of players. We denote $\sigma \to_{\delta_2(X)} \sigma'$ if there is an edge in the improvement graph of $\mathcal{G}$ from σ to σ' labeled by $\delta_2(X)$. We denote $\sigma \to^* \sigma'$ if σ' is reachable from σ; that is, $\to^* = (\cup_{u\in 2^N} \to_u)^*$ is the reflexive transitive closure of the union of the improvement edge relations.

The semantics is described as follows:

- $M, \sigma \models \delta$ if $\sigma \models_d \delta$.
- $M, \sigma \models (\gamma_1 \preceq \gamma_2)$ if $\gamma_1 \preceq_M \gamma_2$.
- $M, \sigma \models [\alpha@x]$ if for all $\sigma' \in \Sigma$, we have: $\sigma', \delta_1(x) \models_t \alpha$.
- $M, \sigma \models p@x$ if $p \in \hat{\omega}_{\delta_1(x)}(\sigma)$.
- $M, \sigma \models a@x$ if $\delta_1(x)(\sigma) = a$.
- $M, \sigma \models x \in X$ if $\delta_1(x) \in \delta_2(X)$.
- $M, \sigma \models x = y$ if $\delta_1(x) = \delta_1(y)$.
- $M, \sigma \models \neg\phi$ if $M, \sigma, \delta \not\models \phi$.
- $M, \sigma \models \phi \vee \psi$ if $M, \sigma, \delta \models \phi$ or $M, \sigma' \models \psi$.
- $M, \sigma \models \langle X\rangle\phi$ if there exists σ' such that $\sigma \to_{\delta_2(X)} \sigma'$ and $M, \sigma' \models \phi$.
- $M, \sigma \models \Diamond^*\phi$ if for some σ' such that $\sigma \to^* \sigma'$, we have: $M, \sigma' \models \phi$.

– $M, \sigma \models \exists x\ \phi$ if there is some $i \in N$ such that $M_{[x \to i]}, \sigma \models \phi$.
– $M, \sigma \models \exists X\ \phi$ if there is some $u \subseteq N$ such that $M_{[X \to u]}, \sigma \models \phi$.

where $M_{[x \to i]} = (N, P, \omega, \preceq_M, \delta_{[x \to i]})$ with $\delta' = (\delta_1', \delta_2)$ such that $\delta_1'(y) = i$ when $y = x$ and $\delta_1'(y) = \delta_1(y)$ otherwise. Similarly, $M_{[X \to u]} = (N, P, \omega, \preceq_M, \delta_{[X \to u]})$ with $\delta' = (\delta_1, \delta_2')$ such that $\delta_2'(Y) = u$ when $Y = X$ and $\delta_2'(Y) = \delta_2(Y)$ otherwise.

The logic is expressive enough to force infinitely many players. For instance, the formula $\Box^* \exists x.(\langle x \rangle \top \wedge \Box^* [x] \bot)$ is satisfiable only in game models with infinitely many players. But since we consider only large games with unbounded but finitely many players, such formulas are *unsatisfiable* in our setting. But we can prove that even over models with only *finitely many players* the satisfiability problem for this logic is *undecidable*. We will do this in section 4.2. Here it suffices to point out that the mix of quantifiers and modalities can make the difference between decidability and undecidability, even if we had only first-order variables.

We therefore consider an alternative *propositional* formalisation.

3.3 Implicit quantification

We note that type formulas implicitly define players for us. So we only need to add player names to the logic so that formulas can be interpreted on strategy profiles. However, these names correspond to player types and hence we cannot form sets of types to refer to actual sets of players, which would be needed for specifying improvement dynamics. For this, we take a cue from **Implicitly Quantified Term-Modal logic** ([PR19a]) and leave the modality to implicitly specify a set of players. This leads us to an interesting logic.

Recall that the logic is parameterised by the strategy set $\mathbf{\Gamma}$. Let C denote a set of constant symbols, intended to be interpreted as player names. We let i, j etc to range over C below.

The syntax of formulas is given by:

$$\phi \in \Phi ::= \gamma_1 \preceq \gamma_2 \mid [\alpha@i] \mid \delta \mid p@i \mid a@i \mid \neg\phi \mid \phi \vee \phi' \mid \langle \exists \rangle \phi \mid \Diamond^* \phi$$

where $a \in \mathbf{\Gamma}, p \in \mathcal{P}, i \in C$, δ is a distribution constraint, α is a type formula and $\gamma_1, \gamma_2 \in \mathbf{B}(\mathcal{P})$.

We call this the *Propositional Modal Logic of Implicit Quantification for Strategisation in Large Games*, *MIQ-STRAT*, for short.

The formulas are evaluated at strategy profiles. The atomic formula δ asserts that current profile satisfies the distribution constraint δ and $a@i$ asserts that player i chooses a. The formula $p@i$ specifies that player i receives outcome p. The modality $\langle \exists \rangle \phi$ asserts that there is a team u that has an improvement edge to a profile in which ϕ holds. Note that by each player in u switching to some different choice, the outcome is affected positively for everyone in u. $\Diamond^* \phi$ says that the outcome ϕ is reachable by a finite path from the current profile.

The dual modalities are also interesting: $[\forall]\phi$ asserts that every deviation by every team results in a profile that satisfies ϕ: in some sense, this means that ϕ is an invariant for any successor. $\Box^*\phi$ asserts that ϕ is an invariant property, true at all reachable states.

Semantics The formulas are interpreted over strategy profiles of large games. A model is a large game $M = (N, P, \omega, \preceq_M, \iota)$ where $\omega : \mathbf{Y}[|N|] \rightarrow (2^P)^{\mathbf{\Gamma}}$ is the outcome function, and $\iota : C \rightarrow N$ is a surjective interpretation of constant symbols. We refer also to the induced outcome function for each $a \in \mathbf{\Gamma}$ by $\hat{\omega}_a : \Sigma \rightarrow 2^P$ where Σ is the set of all strategy profiles.

The semantics is given by assertions of the form $M, \sigma \models \phi$, read as ϕ is true of the strategy profile σ in model M. The semantics of improvement formulas can then be defined as follows.

- $M, \sigma \models \gamma_1 \preceq \gamma_2$ if $\gamma_1 \preceq_M \gamma_2$.
- $M, \sigma \models [\alpha@i]$ if for all $\sigma' \in \Sigma$, we have: $\sigma', \iota(i) \models_t \alpha$.
- $M, \sigma \models \delta$ if $\sigma \models_d \delta$.
- $M, \sigma \models p@i$ if $p \in \hat{\omega}_{\iota(i)}(\sigma)$.
- $M, \sigma \models a@i$ if $\sigma[\iota(i)] = a$.
- $M, \sigma \models \neg\phi$ if $M, \sigma \not\models \phi$.
- $M, \sigma \models \phi \vee \psi$ if $M, \sigma \models \phi$ or $M, \sigma \models \psi$.
- $M, \sigma \models \langle\exists\rangle\phi$ if there exists $u \subseteq N$ and σ' such that $\sigma \rightarrow_u \sigma'$ and $M, \sigma' \models \phi$.
- $M, \sigma \models \Diamond^*\phi$ if for some σ' such that $\sigma \rightarrow^* \sigma'$, we have: $M, \sigma' \models \phi$.

The other boolean connectives $\wedge$, $\implies$ and $\equiv$ are defined in the standard way. The dual modalities are: $[\forall]\phi = \neg\langle\exists\rangle\neg\phi$ and $\Box^*\phi = \neg\Diamond^*\neg\phi$.

Fix a constant symbol i and a propositional symbol p. We use $\top$ to denote the proposition that is true in every profile and $\bot = \neg\top$. So $\top@i$ is a tautology and $\bot@i$ is unsatisfiable.

Note that for the truth of a formula ϕ in a model M at a profile σ, it suffices to restrict ourselves to only profiles reachable from σ. We therefore confine our attention to models which are *pointed* at σ: these are improvement graphs such that the entire graph is the reachability set of σ.

Recall the alternate presentation of strategy profiles discussed earlier: every strategy profile σ has an alternative presentation: $(A_1, \ldots, A_{|\mathbf{\Gamma}|})$, where for each j, the sets A_j constitute an $|\mathbf{\Gamma}|$-way partition of N. It is easy to see that the semantics of formulas could be given equivalently on the tuples $(A_1, \ldots, A_{|\mathbf{\Gamma}|})$.

The $\langle\forall\rangle$ modality In the logic of implicit modal quantification, there is another modality: $\langle\forall\rangle$ with the semantics:
$M, \sigma \models \langle\forall\rangle\phi$ if for every $u \subseteq N$, there exists σ' such that $\sigma \rightarrow_u \sigma'$ and $M, \sigma' \models \phi$

The modality $\langle\forall\rangle\phi$ asserts that *every* team of players has an improvement at that profile, making this a 'maximally non-Pareto' situation. Its dual modality $[\exists]\phi$ asserts that there is a team of players for whom every deviation 'together' leads to a profile that satisfies ϕ.

This is logically very interesting and adds a good deal of expressiveness to the logic. However, this kind of reasoning would be of interest when we consider coalitions and co-operative or co-ordinated behaviour of players. Since we are only considering unilateral deviation by players (albeit concurrently by many), we stick with the simpler logic with only one modality of implicit quantification.

Examples The formula $\nu \stackrel{\text{def}}{=} [\forall]\bot$ is interesting: it specifies that the profile it is asserted in is a Nash equilibrium, since no player can improve on their outcome. $\Diamond^*\nu$ says that an equilibrium is reachable, and $\Box^*\Diamond^*\nu$ asserts that from every reachable profile, there is a path to an equilibrium profile. Note that we cannot express the notion that a player's response is the best response to the remaining players' choices.

Let $\mathbf{\Gamma} = \{a, b, c\}$, where, following Blonski ([Blo00]), they represent stocks. The formula $\sharp a \geq \frac{1}{2} \implies a@i$ specifies a trader who owns stock a only if it is favoured by a majority of the population. Now consider a pair of players who make the same choices only when they are guaranteed to receive the same outcome: the formula $\Box^* \bigwedge_{a\in\mathbf{\Gamma}} (a@i \equiv a@j \implies p@i \equiv p@j)$ specifies this.

Consider a special case of the logic where $\mathcal{P}$, the countable set of propositional symbols includes the set of rational numbers in $[0, 1]$. Consider a game $\mathcal{G} = (N, P, \omega)$ where P includes a finite set of rational payoffs. Now consider the formula:

$$\Box^* \bigwedge_{a\in\mathbf{\Gamma}} (a@i \implies (\sharp\ a = r \implies r@i))$$

It says that player i is of *matching type*: the payoff to player i is exactly the proportion of the population whose choice matches i's.

$$\Box^* \bigwedge_{a\in\mathbf{\Gamma}} (a@i \implies (\sharp\ a = r \implies (1-r)@i))$$

specifies a player whose type is *mismatching*. (Note that $(1 - r)$ is Kalai [Kal05] discusses such generalizations of matching pennies games.

Voting rules ([EFSS17]) often use payoffs that are matching: that is, the satisfaction of a voter V who voted for candidate C is given by the fraction of the votes that C received, specified by the same formula as above.

4 Satisfiability problem

We now consider the satisfiability problem for the logics we have discussed. While the propositional modal logic of implicit quantification for large games is decidable, the monadic second-order logic of strategisation for large games is undecidable, as one may expect.

4.1 *MIQ-STRAT*

We first show that the logic *MIQ-STRAT* has the *bounded players* property.

Lemma 1. *If a formula ϕ of MIQ-STRAT is satisfiable, then it is satisfiable in a game model in which the number of players is bounded by $2^{O(|\phi||\mathbf{\Gamma}|)^2}$.*

Proof. The proof principally follows Proposition 2 which asserts that a type formula can force only $2^{O(m^2)}$ many players where m is the maximum number specified in the distribution constraints. Thus we need to consider strategy profiles of bounded length. On such profiles, a standard filtration argument, along the lines of the one for Propositional Dynamic Logic ([KP81]) shows that every satisfiable formula is satisfiable in a model of (exponentially) bounded size. The details are as follows.

For any distribution constraint formula δ, let $SF_d(\delta)$ be the set of subformulas of δ closed under the condition: if $\neg\sharp\ a\ rel\ r \in SF_d(\delta)$, iff $\sharp\ a\ rel'\ r \in SF_d(\delta)$ where rel' is the complement of rel, where we treat $\neg\neg\delta$ to be the same as δ. Similarly, for any type formula α, let $SF_t(\alpha)$ denote the set of type subformulas of α.

Fix a finite subset $D \subseteq C$. We define $SF_D(\phi)$ to be the least set of formulas containing ϕ and closed under the following conditions:

- For all $a \in \mathbf{\Gamma}$ and $i \in D$, $\top@i, a@i \in SF_D(\phi)$.
- For all $\gamma_1, \gamma_2 \in \mathbf{B}(\mathcal{P})(\phi)$ and $i \in D$, $[\gamma_1 \preceq_i \gamma_2@i] \in SF_D(\phi)$.
- If $p@j \in SF_D(\phi)$ and $i \in D$, $p@i \in SF_D(\phi)$.
- If $\delta \in SF_D(\phi)$,then $SF_d(\delta) \subseteq SF_D(\phi)$.
- If $[\alpha@i] \in SF_D(\phi)$ and $\beta \in SF_t(\alpha)$ then $[\beta@i] \in SF_D(\phi)$.
- If $[\alpha@i] \in SF_D(\phi)$ and $j \in D$ then $[\alpha@j] \in SF_D(\phi)$.
- $\neg\psi \in SF_D(\phi)$ iff $\psi \in SF_D(\phi)$ (where we treat $\neg\neg\phi'$ to be the same as ϕ').
- If $\psi_1 \vee \psi_2 \in SF_D(\phi)$, then $\{\psi_1, \psi_2\} \subseteq SF_D(\phi)$.
- If $\langle\exists\rangle\psi \in SF_D(\phi)$, then $\psi \in SF_D(\phi)$.
- If $\Diamond^*\psi \in SF_D(\phi)$, then $\{\psi, \langle\exists\rangle\Diamond^*\psi\} \subseteq SF_D(\phi)$.

Note that $|SF_D(\phi)| = O(|D| \cdot |\phi|^2)$, where we treat $|\mathbf{\Gamma}|$ as a constant.

Fix a formula ϕ_0 and suppose that $M, \sigma_0 \models \phi_0$ where $M = (N, P, \omega, \preceq, \iota)$. If $N \leq 2^{O(|\phi||\mathbf{\Gamma}|)^2}$ there is nothing to prove; hence we assume that $N > 2^{O(|\phi||\mathbf{\Gamma}|)^2}$.

Without loss of generality, we assume that P_{ϕ_0}, the set of propositions occurring in ϕ_0, is contained in P. Otherwise, we can simply expand P as required. Note that for some finite $D \subseteq C$, ι is a bijection between D and N. We fix D for the rest of the proof, and consider subformulas with respect to D. Let Σ be the set of strategy profiles in model M.

We now define $\sim$ on Σ.

$$\sigma \sim \sigma' \text{ iff } \forall\psi \in SF_D\phi_0, (M, \sigma \models \psi \text{ iff } M, \sigma' \models \psi)$$

$\sim$ is an equivalence relation of index bounded by $2^{O(|\phi_0|)}$. Let $[\sigma]$ denote the equivalence class of σ under $\sim$.

For each equivalence class of σ, consider the set of normalised distributions induced by it. All of them satisfy the same distribution constraints in $SF_D\phi_0$. Fix any representative normalised distribution in each and consider the set Y of normalised distributions, one for each equivalence class. Then there exists a number n_Y such that each normalised distribution in Y induces a distribution over n_Y: this can be got by simply taking the LCM of all the integers that occur as denominators in the normalised distributions. It is easily seen that $n_Y \leq 2^{O((|\phi_0||\mathbf{\Gamma}|)^2)}$. By assumption $|N| > n_Y$. Choose $N' \subseteq N$ such that $|N'| = n_Y$. Let Σ' be the set of strategy profiles over N'. Let $D' \subseteq D$ such that $\iota(D') = N'$. We can now consider $SF_{D'}(\phi_0)$.

This gives us a crucial property: for all $\sigma \in \Sigma$, $\sigma' \in \Sigma'$ if σ' is the restriction of σ to N', then for all $\delta \in SF_{D'}(\phi_0)$, $\sigma \models_d \delta$ iff $\sigma' \models_d \delta$. This is proved by induction on the structure of δ. The base case comes from the construction preserving normalised distributions.

We then show that for all $i \in D'$, $[\alpha@i] \in SF_{D'}(\phi_0)$, for all $\sigma \in \Sigma$, $\sigma' \in \Sigma'$ where σ' is the restriction of σ to N', $\sigma, i \models_t \alpha$ iff $\sigma, i \models_t \alpha$. Again this is by induction on the structure of α. The base case follows from the fact that σ' is the restriction of σ.

Note that whenever $\sigma \Rightarrow_u \sigma'$ in model M such that $i \in u, j \notin u$, and M realises the same i-types j-types, then there exists σ'' such that $\sigma \Rightarrow_v \sigma''$ where $v = (u - \{i\}) \cup j$. Thus whenever there is an edge involving players in $N \setminus N'$ we have a corresponding one in N'.

Now define the model $M' = (N', P, \omega', \preceq, \iota')$ where for all $\sigma' \in \Sigma'$, $\omega'(\sigma') = \hat{\omega}([\sigma])$ where σ' is the restriction of σ to N'. Note that this is well defined, since the normalized distribution for the N' restriction is the same for all profiles in the $\sim$-equivalence class of σ.

Now we can show the following:

Claim: For all $\psi \in SF_{D'}(\phi_0)$, for all $\sigma \in \Sigma$, if σ' is the fixed representative of the $\sim$-equivalence class of σ, and $\sigma'' \in \Sigma'$ is the restriction of σ' to N', then $M, \sigma' \models \psi$ iff $M', \sigma'' \models \psi$.

This is proved by a routine induction on the structure of ψ. The base case of atomic formulas of the form δ and $[\alpha@i]$ follow from observations above. $p@i$ and $a@i$ follow from the fact that ω and σ'' are restrictions. For the induction step, the boolean cases are routine. The case of $\langle\exists\rangle$ follows from the observation above regarding $\Rightarrow_u$ edges. The case of $\Diamond^*$ is by standard induction on path length.

From the claim, if if σ'_0 is the fixed representative of the $\sim$-equivalence class of σ_0, and $\sigma''_0 \in \Sigma'$ is the restriction of σ'_0 to N', then $M', \sigma''_0 \phi_0$. Thus ϕ_0 is satisfiable in a model with a bounded number of players, as required.

Incidentally, we have also shown that every satisfiable formula has a model of bounded size, thus showing that the satisfiability problem is decidable. We will later see another proof of the same result via the completeness theorem of an axiom system.

Theorem 1. *The satisfiability problem for the logic MIQ-STRAT can be decided in nondeterministic double exponential time.*

4.2 *MSO-STRAT*

We can prove that even over *finite models* the satisfiability problem for *MSO-STRAT* is undecidable. In fact, we will need neither set variables nor set quantification, the first order fragment is already strong enough to force undecidability.

Consider *FO-STRAT*, the first-order fragment of *MSO-STRAT* defined as follows:

$$\phi \in \Phi ::= \alpha @ x \mid p @ x \mid \neg\phi \mid \phi \vee \phi' \mid \langle x \rangle \phi \mid \Diamond^* \phi \mid \exists x.\phi$$

where α is a type formula, $p \in \mathcal{P}, x \in V$.

To prove undecidability, we use the following version of tiling problem called **finite tiling problem**: Given a tiling instance $\mathcal{T} = (T, V, H, t_0, t_f)$ where T is a finite set of tiles and $H \subseteq (T \times T)$ and $V \subseteq (T \times T)$ are the horizontal and vertical constraints respectively and $t_0, t_f \in T$, does there exist n and a tiling function $f : [0, \ldots n]^2 \to T$ such that $f(0,0) = t_0$ and $f(n,n) = t_f$ and for all i, j we have $[f(i,j), f(i+1,j)] \in H$ and $[f(i,j), f(i,j+1)] \in V$. Deciding whether a given tiling system has a finite tiling is recursively enumerable but not recursive, as we can reduce the halting problem of Turing machines to the finite tiling problem.

Theorem 2. *Given a tiling instance $\mathcal{T} = (T, H, V, t_0, t_f)$, we can construct a formula ϕ_T of FO-STRAT such that $\mathcal{T}$ has a finite tiling iff ϕ_T is satisfiable.*

Fix a tiling instance $\mathcal{T} = (T, V, H, t_0, t_f)$ where T is a finite set of tiles and $H \subseteq (T \times T)$ and $V \subseteq (T \times T)$, are the horizontal and vertical constraints respectively and $t_0, t_f \in T$, and in particular they are, for all $i, j \in [n]$, $[f(i,j), f(i+1,j)] \in H$ and $[f(i,j), f(i,j+1)] \in V$. The tiling problem asks: does there exist n and a tiling function $f : [0, \ldots, n]^2 \to T$ such that $f(0,0) = t_0$ and $f(n,n) = t_f$, respecting the tiling constraints?

We encode the grid into the corresponding strategy profile which in turn induces a distribution. The idea is to associate the grid point (i,j) with the strategy profiles that induce the distribution $(i, j, 2n-(i+j))$. The grid points - $(0,0)$ and (n,n) are associated with profiles that induce the distribution $(0,0,2n)$ and $(n,n,0)$ respectively.

For the successor edges of the grid points, we need to ensure that for every profile that induces the distribution $(i, j, 2n-(i+j))$ there is an edge to two different strategies which induce the distributions $(i+1, j, 2n-(i+j+1))$ and $(i, j+1, 2n-(i+j+1))$ respectively. The tiling information is encoded as the pay-off obtained by players who choose action c. If $f(i,j) = t$, then the corresponding distribution $(i, j, 2n-(i+j))$ will have $t \in \omega_c(i, j, 2n-(i+j))$. But for the distribution $(n,n,0)$ since there are no players who pick c, we associate the payoff t_f to all the players. That is, for $f(n,n) = t_f$, we have $t_f \in \omega_a(n,n,0)$. We also make sure t_f is maximal pay-off, so that there are no outgoing edges and hence is the end point of the grid.

Fix $\mathbf{\Gamma} = \{a, b, c\}$. For every tile $t \in T$, let t be a corresponding proposition and let p, q be new (singleton) payoffs such that the ordering relation, $\prec$, is defined where for all $t \neq t_f$ we have $q \prec t \prec p$ and $p \prec t_f$. Define the game over $2n$ players $M = ([1, \ldots, 2n], P_T, \omega, \prec)$ where $P_T = \{p, q\} \cup \{t \mid t \in T\}$ and the payoffs for the distributions are given by the following table.

Distribution	Condition	ω_a	ω_b	ω_c
$(i, j, 2n-(i+j))$	$i, j < n$	$\{p\}$	$\{p\}$	$\{f(i,j)\}$
$(i, j, 2n-(i+j))$	$i = n$ and $j < n$	$\{p\}$	$\{p\}$	$\{f(i,j)\}$
$(i, j, 2n-(i+j))$	$i < n$ and $j = n$	$\{p\}$	$\{p\}$	$\{f(i,j)\}$
$(n, n, 0)$		$\{t_f\}$	$\{t_f\}$	$\{\}$
$(i, j, 2n-(i+j))$	$i > n$ or $j > n$	$\{q\}$	$\{q\}$	$\{q\}$

The formulas that **force the grid** are given by the following formulas:

$$\begin{aligned}
\phi_1 &:= \Box^*\forall x.\Big(\big(a@x \vee b@x\big) \implies p@x \wedge \Box^* p@x\Big)\\
\phi_2 &:= \Box^*\Big(\big(\sharp a = \tfrac{1}{2} \vee \sharp b = \tfrac{1}{2}\big) \implies \forall x([x]\bot)\Big)\\
\phi_3 &:= \Box^*\Big(\big(\sharp a < \tfrac{1}{2} \wedge \sharp b < \tfrac{1}{2}\big) \implies\\
&\qquad \forall x\ c@x \implies \big(\neg p@x \wedge \neg t_f@x \wedge \langle x\rangle(a@x \wedge \neg b@x \vee \neg a@x \wedge b@x)\ \big)\Big)
\end{aligned}$$

For every $t \in T$ and $x \in V$ define the formula $only(t,x) := t@x \wedge \bigwedge_{t' \neq t} \neg t'@x$ which states that x gets payoff corresponding to tile t (and no other tile). The formulas that **encode the tiling constraints** are as follows:

$$\begin{aligned}
\psi_0 &:= \sharp a \leq 0 \wedge \sharp b \leq 0 \wedge \forall x\Big(c@x \implies only(t_0, x)\Big)\\
\psi_1 &:= \Box^*\Big(\big(\sharp a = \tfrac{1}{2} \wedge \sharp b = \tfrac{1}{2}\big) \vee (\alpha_H \wedge \alpha_V)\Big)\\
\psi_2 &:= \Diamond^*(\sharp c = 0 \wedge \sharp a = \tfrac{1}{2} \wedge \sharp b = \tfrac{1}{2})\ \wedge\\
&\qquad \Box^*\Big(\ \big(\exists x\langle x\rangle(a@x \wedge \sharp c = 0\big) \implies \bigvee_{(t,t_f)\in H}(\forall z(only(t,z)))\ \wedge\\
&\qquad\qquad \big(\exists x\langle x\rangle(b@x \wedge \sharp c = 0\big) \implies \bigvee_{(t,t_f)\in V}(\forall z(only(t,z)))\Big)\\
\psi_3 &:= \big(\sharp c = 0 \implies \forall x.t_f@x\big)\\
\psi_4 &:= \Box^* \bigvee_{t\in T}\Big(\Box^*\big(\forall z(c@z \implies only(t,z)\big)\Big)
\end{aligned}$$

where,

$$\begin{aligned}
\alpha_H &:= \bigvee_{(t,t')\in H}\Big(\forall x(c@x \implies t@x) \wedge \forall y\Big(\langle y\rangle\big(a@y \wedge \forall z(c@z \implies only(t',z))\big)\Big)\Big)\\
\alpha_V &:= \bigvee_{(t,t')\in V}\Big(\forall x(c@x \implies t@x) \wedge \forall y\Big(\langle y\rangle\big(b@y \wedge \forall z(c@z \implies only(t',z))\big)\Big)\Big)
\end{aligned}$$

Claim. 1. ϕ_1 ensures that for every profile σ the players who have already chosen action a or b will not deviate in any reachable profile thereafter.

Proof. For any σ', and $u \subseteq N$, such that $\sigma \to_u \sigma' \to^* \sigma''$. For any $i \in u$ such that $\sigma[i] = a$ (say, can also be b). To prove, $\sigma'[i] = a$ and $\sigma''[i] = a$.

$M, \sigma \models \phi_1$, for $x = i$, we have $M, \sigma' \models p@i$ and $M, \sigma'' \models p@i$. Thus, $p \in \hat{\omega}_i(\sigma')$ and $p \in \hat{\omega}_i(\sigma'')$. By our definition of ω function, we know, for the distributions, p is the payoff given only for players playing strategies a, b. We show that $\sigma'[i] = a$ and similarly $\sigma''[i] = a$. Suppose $\sigma'[i] = b$, that means there is a change in strategy by player i, and since it is an improvement edge, $\sigma[i] \prec \sigma'[i] \prec \sigma''[i]$, but in all these cases the above relation is violated as the payoff is p, contradiction! Thus, $\sigma'[i] = a$ and $\sigma''[i] = a$ as well.

Claim. 2. If σ induces the distribution $(i, j, 2n-(i+j))$, then the only permitted edges are from σ to σ' where σ' induces the distribution of the form $(i+k, j+l, 2n-(i+j+k+l))$, where $k, l \geq 0$ but simultaneously both k, l are never 0

Proof. For the case when both $k, l = 0$, it means we are referring to the same distribution, say, $\mathbf{y}$. From the table we can see that for the same distribution, $\omega_a(\mathbf{y}) = \omega_b(\mathbf{y})$. So players playing strategies a, b won't shuffle amongst each other. And since players who played c get a lower payoff by our ordering relation $\prec$ the only possibility is them deviating either to a or b. But, that means, we can't have the same distribution from two separate strategy profiles. Thus, there can no case, where two strategy profiles have an improvement such that both $k, l = 0$.

From the above argument, we can see that the payoffs for players who choose strategy a, b remain higher than the ones who choose strategy c, by our definition of the ordering relation $\prec$. The players who choose strategy c will deviate to a profitable strategy choice a or b, and thus, the strategy profiles will induce distributions of the form $(i+k, j+l, 2n-(i+j+k+l))$

Claim. 3. ϕ_2 ensures we do not have any transitions from the right or top border grid points.

Proof. If a strategy profile satisfies $\sharp a = \frac{1}{2}$, then it means n many players have chosen strategy a which means our distribution is of the form (n, y, z), which encodes the grid points of the form (n, y), that is the right border. Such strategy profiles by the formula ϕ_2 also satisfy $\forall x([x]\bot)$ that is none of the players improve from there. Thus, we get no further horizontal and vertical constraints from such strategy profiles.

Similarly. for the top border we would have the strategy profile satisfying $\sharp b = \frac{1}{2}$, which would encode grid points, $(*, n)$. A similar argument as above can ascertain our claim for the top border as well.

Claim. 4. ϕ_3 ensures that all points in the interior of the grid necessarily have transitions to the right and top successor grid points.

Proof. Strategy profiles satisfying $\sharp a < \frac{1}{2} \wedge \sharp b < \frac{1}{2}$, means strategies a, b are chosen by less than n players, which corresponds to grid points (i, j) where $i < n$ and $j < n$. By the antecedent of ϕ_3, we have, $M, \sigma \models \forall x \; c@x \implies \big(\neg p@x \wedge \neg t_f@x \wedge \langle x\rangle(a@x \wedge \neg b@x \vee \neg a@x \wedge b@x)$. Which means that for such strategy profiles, players who choose c will not have the payoffs p and t_f and they are able to make a one step deviation by choosing either strategy a or b. This would mean that from a strategy profile σ (corresponding to the interior of the grid) which corresponds to a distribution say, (x, y, z) will at least have an improvement edge to a strategy profile that would correspond to a distribution of the form $(x+k, y, z-k)$ or $(x, y+k, z-k)$, for $k \geq 1$. That means we have grid connections to the left and right from the interiors at all times.

Claim. 5. If $[f(i,j) = t, f(i+1,j) = t'] \in H$, then there exists strategy profiles σ, σ' corresponding to the grid points (i,j) and $(i+1,j)$, such that $\sigma \rightarrow_i \sigma'$ where, $\sigma'[i] = a$, whenever $M, \sigma \models \alpha_H$.

Proof. $M, \sigma \models \alpha_H$ means for one $(t, t') \in H$ we have, $M, \sigma \models \forall x(c@x \implies t@x)$, which means for the players that play strategy c, those players get a payoff of t. And, $M, \sigma \models \forall y\Big(\langle y\rangle\big(a@y \implies \forall z(c@z \implies only(t', z))\big)\Big)$. That is, for any player i, $M_{[y\rightarrow i]}, \sigma \models \langle y\rangle\big(a@y \implies \forall z(c@z \implies only(t', z))\big)$. Thus, for some σ' which satisfies $\sigma \rightarrow_i \sigma'$, we have, $M_{[y\rightarrow i]}, \sigma' \models a@y \implies \forall z(c@z \implies only(t', z))$ which means $\sigma'[i] = a$ and all players in σ' that play strategy c get a payoff of t'. Our σ' corresponds to the strategy profile associated with grid point $(i+1, j)$ because it gets the tiling t'.

Thus α_H encodes the horizontal constraints when the transition player chooses strategy a.

Claim. 6. If $[f(i,j) = t, f(i,j+1) = t'] \in V$, then there exists strategy profiles σ, σ' corresponding to the grid points (i,j) and $(i,j+1)$, such that $\sigma \rightarrow_i \sigma'$ where, $\sigma'[i] = b$, whenever $M, \sigma \models \alpha_V$.

Proof. Similar proof as above.

Thus α_H encodes the vertical constraints when the transition player chooses strategy b.

Claim. 7. ψ_0 ensures that for distribution $(0, 0, 2n)$ all players, playing c, get t_0 as payoff.

Proof. $M, \sigma \models \psi_0$ means $M, \sigma \models \sharp a \leq 0 \wedge \sharp b \leq 0$. Then this means the distribution profile, $\mathbf{y}$ associated with σ has 0 players who have played strategies a, b. Thus, $\mathbf{y} = (0, 0, 2n)$, which corresponds to the grid point $(0, 0)$. $M, \sigma \models \forall x(c@x \implies only(t_0, x))$, this means, $\omega_c(\mathbf{y}) = t_0$, thus $f(0, 0) = t_0$ as required.

Claim. 8. ψ_1 ensures that either the vertical and horizontal tiling constraints are respected or we are at a top or right border grid point.

Proof. By Claim 2. we know σ that induces a distribution of $(i, j, 2n-(i+j))$ has improvement edges to σ' where σ' induces a distribution of the form $(i+k, j+l, 2n-(i+j+k+l))$ where k, l are non zero integers where both are not 0. And by following the claims on α_H and α_V, pick (k, l) as either $(1, 0)$ or $(0, 1)$ and we can see for any σ save the one which corresponds to the grid point (n, n), we will have $M, \sigma \models \alpha_H \vee \alpha_V$.

Claim. 9. ψ_2 ensures ensures that the grid points $(n,*)$ and $(*,n)$ satisfy the tiling constraints to their respective top and right successors.

Proof. The strategy profiles which will satisfy $\Diamond^*(\sharp c = 0 \wedge \sharp a = \frac{1}{2} \wedge \sharp b = \frac{1}{2})$ means a path to the strategy profile corresponding to the grid point (n,n). And within such strategy profiles there shall exist strategy profiles which satisfy $\exists x\langle x\rangle(a@x \wedge \sharp c = 0) \implies \bigvee_{(t,t_f)\in H} \forall z(only(t,z))$ from whom all paths out of it lead to a strategy profile from where one of the players can choose a strategy a which will also result in no players choosing strategy c which means arriving at the top corner, via a horizontal constraint, and, strategy profiles satisfy $\exists x\langle x\rangle(b@x \wedge \sharp c = 0) \implies \bigvee_{(t,t_f)\in V} \forall z(only(t,z))$ similarly players can choose strategy b while none of the players choose strategy c, which means arriving at the top corner via a vertical constraint.

Claim. 10. ψ_3 ensures that the grid point (n,n) gets the tile t_f. And all players get payoff t_f.

Proof. $M,\sigma \models \sharp c \leq 0$ then, the distribution profile associated with σ has 0 players playing strategy c, which means, $\mathbf{y} = (n,n,0)$. According to the payoff associated to this particular distribution profile we know all the players get payoff t_f. Thus, $f(n,n) = t_f$ where the grid point (n,n) is the one associated with the distribution profile $\mathbf{y}$.

Claim. 11. ψ_4 ensures that only one tile is obtained as payoff at any profile for players choosing c.

Proof. Suppose there are two tiles t_1, t_2 obtained at the grid point for a particular strategy profile σ, then $M,\sigma \models \forall x(c@x \implies only(t_1,x) \wedge only(t_2,x))$ but, $M,\sigma \models \psi_4$. This would lead to the contradiction.

Let ϕ_T be $\phi_1 \wedge \phi_2 \wedge \phi_3 \wedge \psi_0 \wedge \psi_1 \wedge \psi_2 \wedge \psi_3 \wedge \psi_4$.
We can now prove the main claim.

Claim. Given a tiling instance $\mathcal{T} = (T,H,V,t_0,t_f)$, we can construct a formula ϕ_T such that $\mathcal{T}$ has a finite tiling iff ϕ_T is satisfiable.

Proof. Suppose $\mathcal{T}$ has a finite tiling. Then, there is some n and a tiling function $f : [0,\dots n]^2 \to T$.

The corresponding game over $2n$ players, $M = ([1,\dots,2n], P_T, \omega, \prec)$.

It can be verified for σ_0 which is the profile that induces the distribution $(0,0,2n)$ we have $M,\sigma_0 \models \phi_T$.

For the other direction, assume that the formula is satisfiable over some game. Let $M,\sigma_0 \models \phi_T$. Then, ψ_1 ensures that the number of players is even. Let $2n$ be the number of players. Define the function $f : [0\dots n]^2 \to T$ such that for all i,j if $i \neq n$ or $j \neq n$, then $f(i,j) = t$ such that there is a profile

σ reachable from σ_0 such that σ induces the distribution $(i, j, 2n - (i + j))$ and $M, \sigma \models \forall x(c@x \implies only(t, x))$. Finally define $f(n, n) = t_f$.

To see that f is well defined, it is enough to prove that for all (i, j) there is some profile σ reachable from σ_0 that induces the distribution $(i, j, 2n-(i+j))$. This is proved by induction on $i + j$. Base case is when $i + j = 0$ and σ_0 satisfies the condition. For the induction step, since $i < n$ and $j < n$ there is some players who played c shift to a or b, and ϕ_3 ensures that there is a next transition to the profile that induces the required distribution. For $i + j - 1$, by our hypothesis we had f defined over possible grid points $(i-1, j)$ or $(i, j-1)$, let them correspond to strategy profiles σ_1, σ_2 respectively and they are reachable from σ_0. At, σ_1, we have $M, \sigma_1 \models \big(\sharp a < \frac{1}{2} \wedge \sharp b < \frac{1}{2}\big) \implies \big(\forall x\ c@x \implies \big(\neg p@x \ \wedge\ \neg t_f@x \wedge\ \langle x\rangle(a@x\ \wedge \neg b@x \vee \neg a@x \wedge b@x)\ \big)$, that is there exists a player which either shifts to strategy a or strategy b, yielding possible distribution profiles, (i, j) or $(i - 1, j + 1)$. The similar situation will play out for strategy profile σ_2 and in both instances the grid point (i, j) shall be possible. Thus the strategy profile corresponding to grid point (i, j), say σ' is again reachable from σ_0.

Finally to see that f satisfies the tiling constraints, note that we have $f(0, 0) = t_0$ and $f(n, n) = t_f$. Also, ψ_1 ensures that horizontal and vertical constraints for all the grid points are satisfied except (n, n) and ψ_2 ensures that these constraints are satisfied for the grid point $(n, n - 1)$ and $(n - 1, n)$ to their respective top and right successor (n, n).

Corollary 1. *The satisfiability problem for FO-STRAT is recursively enumerable but not recursive; hence the validity problem is not recursively enumerable: that is, the logic is not recursively axiomatizable.*

Note that the coding uses only *three* strategies, which can be thought of as rudimentary player types. Further, the proof shows that even the monodic fragment of *FO-STRAT*, where every modal formula has only one variable free in its scope, is undecidable.

5 Axiomatization

We now consider an inference system for validity in *MIQ-STRAT*. The axiomatization follows that of propositional dynamic logic ([KP81]) with some characteristics of modal logic of implicit quantification, but the key element here are the characteristic formulas that describe choice distributions.

We define some notations before describing the axioms. Recall that the abbreviation $\sharp\ a = r$ denotes $\sharp\ a \leq r \wedge \sharp\ a \geq r$. Let $k = |\mathbf{\Gamma}|$ and $distr(r_1, \ldots, r_k)$ stand for $\bigwedge_{\ell=1}^{k} \sharp a_\ell\ =\ r_\ell$, where $\Sigma_\ell\ r_\ell = 1$. This is a complete distribution, independent of the number of players.

Let $X \subseteq \mathbf{\Gamma}$, and $D_X = \{\sharp\ a_j\ rel_j\ s_j \mid a_j \in X\}$. We say that $distr(r_1, \ldots, r_k)$ is a *completion* of D_X if for all $a_j \in X$, and $\sharp\ a_j\ rel_j\ s_j \in D_X$, $(\sharp\ a_j\ =\ r_j)$ entails

($\sharp\ a_j\ rel_j\ s_j$). The set D_X is said to be *coherent* if there exists a completion of D_X.

For a finite set of propositions $A \subseteq P$ and $i \in C$, let $\nu_A@i$ denote the formula $\bigwedge_{p \in A} p@i \wedge \bigwedge_{q \notin A} \neg q@i$. From Proposition 2 if a type formula α is satisfiable, then it is satisfiable in a game with exponential number of players. Let c be a constant such that for every formula α, if α is satisfiable then α is satisfiable in a game with at most $2^{c|\alpha|^2}$ players.

The axiom system for *MIQ-STRAT* is given as follows:

Axioms		
(A0)	Substitutional instances of tautologies of propositional logic	
(A1)	$[\forall](\phi \implies \psi) \implies ([\forall]\phi \implies [\forall]\psi)$	
(A2)	$\lozenge^*\phi \equiv (\phi \vee \langle\exists\rangle\lozenge^*\phi)$	
(A3)	$\bigvee_{a \in \mathbf{\Gamma}} \sharp a > 0$	
(A4 (a))	$(\gamma_1 \preceq \gamma_2) \implies \Box^*(\gamma_1 \preceq \gamma_2)$	
(A4 (b))	$\gamma \preceq \gamma$	
(A4 (c))	$(\gamma_1 \preceq \gamma_2) \wedge (\gamma_2 \preceq \gamma_3) \implies (\gamma_1 \preceq \gamma_3)$	
(A5 (a))	$[\alpha@i] \implies \Box^*[\alpha@i]$	
(A5 (b))	$[(\delta \to a)@i] \equiv (\delta \implies a@i)$	
(A6)	$((distr(r_1, \ldots, r_k) \wedge a@i) \implies p@i) \implies \Box^*((distr(r_1, \ldots, r_k) \wedge a@i) \implies p@i)$	
(A7)	$(a@i \wedge \alpha@i) \implies [\forall]((\neg a@i \wedge \beta@i) \implies (\alpha \prec \beta))$	
(A8)	$[\neg\sharp a\ rel\ r@i] \equiv [\sharp a\ rel'\ r@i]$	where $a \in \mathbf{\Gamma}$ and rel' is the complement of rel
(A9)	$[\sharp a\ rel\ r@i] \implies [\sharp a\ rel'\ r@i]$	where $a \in act$ and $\sharp a\ rel\ r$ entails $\sharp a\ rel'\ r'$

Rules		
(MP)	$\dfrac{\phi, \quad \phi \implies \psi}{\psi}$	
(GEN)	$\dfrac{\phi}{[\forall]\phi}$	
(IND)	$\dfrac{\phi \implies [\forall]\phi}{\Box^*\phi}$	
(NUM)	$\dfrac{\Box^* \bigwedge_{i \in D} \top@i \implies \phi}{\phi}$	where $D \subseteq C$ and $\|D\| = 2^{c\|\phi\|^2}$
(IMPR)	$\dfrac{(\phi \wedge \bigwedge_{i \in u} a_i@i) \implies [\forall](\bigvee_{j \in u} a_j@j \vee \bigvee_{j \notin u} (\neg a_j)@j)}{\phi \implies [\forall]\bot}$	where $u \subseteq_{fin} C$

Axioms (A0), (A1), (A2) and the inference rules (MP), (Gen) and (IND) are standard in any propositional modal logic of transitive closure.

Axiom (A3) is a non-triviality condition, which asserts that some choice is made by some player at any profile.

The group of three (A4) axioms assert that preference formulas form a pre-order; clearly this depends only on the player and is invariant across profiles. Similarly, (A5) axioms specify the semantics of player types and assure that they are indeed only player dependent and independent of strategy profiles. (A6) ensures that outcomes are maps that depend only on choice distributions ie. if the current profile induces the distribution $(r_1, \dots r_k)$ and player i playing a gets payoff γ then in every profile that is reachable and induces the same distribution $(r_1, \dots r_k)$ if player i plays a then she gets payoff γ. This is the characteristic formula for anonymous large games.

(A7) describes improvement dynamics: when player i deviates from strategy a, it is because of some improved outcome. (A8) and (A9) facilitate reasoning with inequalities.

The rule (Num) asserts the essence of Lemma 1 we encountered. To see this, we argue its soundness. Suppose the premise of (Num) is valid but the conclusion is not. Then, $\neg\phi$ is satisfiable. But then it is satisfiable in an n-player game, where $n \leq 2^{c|\phi|^2}$. In that game we interpret D as names of players, and we see that $\Box^* \bigwedge\limits_{i \in D} \top@i \wedge \neg\phi$ is satisfiable, which contradicts our assumption that the premise is valid.

Similarly, the rule (Impr) asserts that whenever a set of players u have an improvement, they deviate from the current profile. This rule, when used along with axiom (A5) ensures that the new outcome for each of the players in u is preferred over the current one.

We argue soundness of rule (Impr) as follows. Suppose that the premise of the rule is valid but the conclusion is not. Then the formula $\phi \wedge \langle\exists\rangle\top$ is satisfied in model M at profile σ. Let u be a set of players such that for some σ', we have: $\sigma \rightarrow_u \sigma'$. Let $a_i = \sigma[i]$. Clearly, the formula $\phi \wedge \bigwedge\limits_{i \in u} a_i@i$ holds at σ. Since this is an improvement edge, we know that for all $i \in u$, $\sigma'[i] \neq a_i$, and for all $j \notin u$, $\sigma'[j] = a_j$. Therefore $\bigwedge\limits_{i \in u} \neg a_i@i \wedge \bigwedge\limits_{j \notin u} a_j@j$ holds at σ'. Therefore, $(\phi \wedge \bigwedge\limits_{i \in u} a_i@i) \wedge \langle\exists\rangle(\bigwedge\limits_{i \in u} \neg a_i@i \wedge \bigwedge\limits_{j \notin u} a_j@j)$ holds at σ, contradicting the validity of the premise.

Theorem 3. *Every consistent formula is satisfiable in a model of size $2^{O(|\phi|^3)}$ with $2^{O(|\phi|^2)}$ players. Therefore the axiom system is complete, and the satisfiabililty problem for the logic can be solved in nondeterministic double exponential time.*

We say $\vdash \phi$ if ϕ is a theorem of the system. We say that α is consistent if $\nvdash \neg\alpha$. A finite set of formulas A is consistent if the conjunction of all formulas in A, denoted $\hat{A}$, is consistent. When we have a finite family R of sets of formulas, we write $\tilde{R}$ to denote the disjunction of all formulas $\hat{A}$, where $A \in R$.

To prove completeness, we show that every consistent formula is satisfiable, and by showing that it is satisfiable in a model of bounded size, we get decidability as well.

Fix a consistent formula ϕ_0. We invoke the rule (Num) to argue that for some set D of constant symbols $\phi_1 = \Box^* \bigwedge_{i\in D} \top@i \wedge \phi_0$ is consistent. Since at most linearly many constant symbols occur in ϕ_0, we may assume that $\{j \in C \mid j$ occurs in $\phi_0\} \subseteq D$. We fix ϕ_1 and D for the rest of the proof. By $P(\phi_1)$ we refer to the set of atomic outcome propositions mentioned in ϕ_1 and $\mathbf{B}(\mathcal{P})(\phi_1)$ denotes the set of boolean formulas over $P(\phi_1)$.

Recall that $SF_d(\delta)$ be the set of subformulas of δ and for any type formula α, $SF_t(\alpha)$ denotes the set of type subformulas of α.

We recall the definition of $SF_D(\phi)$, the least set of formulas containing ϕ and closed under the following conditions:

- For all $a \in \mathbf{\Gamma}$ and $i \in D$, $\top@i, a@i \in SF_D(\phi)$.
- For all $\gamma_1, \gamma_2 \in \mathbf{B}(\mathcal{P})(\phi)$ and $i \in D$, $[\gamma_1 \preceq_i \gamma_2@i] \in SF_D(\phi)$.
- If $p@j \in SF_D(\phi)$ and $i \in D$, $p@i \in SF_D(\phi)$.
- If $\delta \in SF_D(\phi)$,then $SF_d(\delta) \subseteq SF_D(\phi)$.
- If $[\alpha@i] \in SF_D(\phi)$ and $\beta \in SF_t(\alpha)$ then $[\beta@i] \in SF_D(\phi)$.
- $\neg\psi \in SF_D(\phi)$ iff $\psi \in SF_D(\phi)$ (where we treat $\neg\neg\phi'$ to be the same as ϕ').
- If $\psi_1 \vee \psi_2 \in SF_D(\phi)$, then $\{\psi_1, \psi_2\} \subseteq SF_D(\phi)$.
- If $\langle\exists\rangle\psi \in SF_D(\phi)$, then $\psi \in SF_D(\phi)$.
- If $\Diamond^*\psi \in SF_D(\phi)$, then $\{\psi, \langle\exists\rangle\Diamond^*\psi\} \subseteq SF_D(\phi)$.

Note that $|SF_D(\phi)| = O(|D| \cdot |\phi|^2)$, where we treat $|\mathbf{\Gamma}|$ as a constant.

We call $A \subseteq SF_D(\phi_1)$ an atom if it is a maximal consistent subset (MCS) of $SF_D(\phi_1)$: that is, A is consistent, and for any $A \subseteq B \subseteq SF_D(\phi_1)$ such that B is consistent, we have $A = B$. Note that every consistent subset A of $SF_D(\phi_1)$ can be extended to a maximal consistent subset of $SF_D(\phi_1)$. Let AT denote the set of MCS's.

We now prove some "local" properties of MCS's.

L1 Every A in AT induces a profile σ_A over D such that for all $\delta \in SF_D(\phi_1)$, $\delta \in A$ iff $\sigma_A \models \delta$ and for all $a \in \mathbf{\Gamma}$, $i \in D$, $a@i \in A$ iff $\sigma_A \models a@i$.

Proof. Let C_A denote the set of distribution constraint formulas in A, and let ϵ_A denote the set of formulas of the form $a@i$ in A, $a \in \mathbf{\Gamma}$ and $i \in D$. Note that $\hat{A} \wedge \hat{C_A} \wedge \hat{\epsilon_A}$ is consistent. $\hat{C_A}$ is a propositional consistent formula, and since the axiom system is propositionally complete, it is satisfiable. By Proposition 1, there exists $n = O(2^{|C_A|^2})$ and a finite set of distributions E over n satisfying C_A. Note that $n < |D|$ and we can assume the distributions to be over D. For a distribution $\sigma \in E$, let e_σ denote the conjunction of formulas of the form $a@i$ where $\sigma[i] = a$, and let $\tilde{E}$ denote the disjunction $\bigvee_{\sigma\in E} e_\sigma$. Then the propositional formula $\hat{C_A} \implies \tilde{E}$ is consistent (since it is satisfiable), and thus we have that $\hat{A} \wedge \tilde{E} \wedge \hat{\epsilon_A}$ is consistent. Therefore for some $\sigma \in E$, $\hat{A} \wedge e_\sigma \wedge \hat{\epsilon_A}$ is consistent. We denote this profile by σ_A. Clearly, for all $\delta \in SF_D(\phi_1)$, $\delta \in A$ iff $\sigma_A \models \delta$ and for all $a \in \mathbf{\Gamma}$, $i \in D$, $a@i \in A$ iff $\sigma_A \models a@i$.

L2 For every A in AT, there exists a complete distribution formula $distr(r_1, \ldots, r_k)$ such that $\hat{A} \wedge distr(r_1, \ldots, r_k)$ is consistent (where $k = |\mathbf{\Gamma}|$). (Note that $distr(r_1, \ldots, r_k)$ need not be in $SF_D(\phi_1)$.)

Proof. Given any profile σ over D, consider the formula $d_\sigma = \bigwedge_{a \in \mathbf{\Gamma}} \sharp a = r_a$, where $r_a = \frac{\{j \in D \mid \sigma[j]=a\}}{|D|}$. Clearly, the propositional formula $e_\sigma \implies d_\sigma$ is consistent. Since we showed above that $\hat{A} \wedge e_{\sigma_A}$ is consistent, so is $\hat{A} \wedge d_{\sigma_A}$, as required.

L3 For every A in AT and for every type formula $\alpha \in SF_t(\phi_1)$, $[\alpha@i] \in A$ iff $\sigma_A, i \models \alpha$.

Proof. The proof is by a routine induction on the structure of α. The base case follows by Property [L1] above and Axiom (A5)(b).

L4 There exists $A_0 \in AT$ such that $\phi_0 \in A_0$.

Proof. Since ϕ_1 is consistent, there exists an MCS $A_0 \in AT$ such that $\phi_1 \in A_0$. Since ϕ_0 is a conjunct of ϕ_0, $\phi_0 \in A_0$ as well.

For $A \in AT$ and $i \in D$, let $P_i(A) = \{p \mid p@i \in A\}$. For $A, B \in AT$ and for $u \subseteq D$, we write $A \prec_u B$ if for all $i \in u$, $P_i(A) \prec_i P_i(B)$, $\{a@i \in A \mid i \in u\} \cap B = \emptyset$ and $\{a@j \in A \mid j \notin u\} \subseteq B$.

Define $\Rightarrow \subseteq (AT \times D_1 \times AT)$, an edge relation on AT labelled by constant symbols from D_1 by: $A \Rightarrow_u B$ iff $\hat{A} \wedge \langle \exists \rangle \hat{B}$ is consistent, and $A \prec_u B$. Let G denote the graph $(AT, \Rightarrow)$.

As remarked above, there exists an MCS $A_0 \in AT$ such that $\phi_1 \in A_0$. Let G_1 be the induced subgraph of G by restricting to atoms reachable from A_0, denoted AT_1. We have the following observations on G_1.

Define, for all $i \in D$, $\preceq_i = \{(\gamma_1, \gamma_2) \mid [\gamma_1 \prec \gamma_2]@i \in A_0\}$. Though this is defined only with reference to A_0, thanks to Axiom (A4(a)), it applies to all atoms in AT_1, the reachability set of A_0. Moreover, these relations are pre-orders thanks to Axioms (A4(b)) and (A4(c)).

G1 For every A, B in AT_1, if $\sigma_A = \sigma_B$, then for all $i \in D$ and propositions $p \in P(\phi_1)$, $p@i \in A$ iff $p@i \in B$.

Proof. Because of Axiom (A6) we have:

$$\begin{aligned}\hat{A}_0 \implies \Box^*((distr(r_1, \ldots, r_k) \wedge a@i) \implies p@i)\\ \implies (\Box^*((distr(r_1, \ldots, r_k) \wedge a@i) \implies p@i))\end{aligned}$$

A and B are reachable from A_0, so we have similar implications for $\hat{A}$ and $\hat{B}$ as well. Since $\sigma_A = \sigma_B$, $\hat{A} \implies (distr(r_1, \ldots, r_k) \wedge a@i)$ iff $\hat{B} \implies (distr(r_1, \ldots, r_k) \wedge a@i)$. Therefore $\hat{A} \implies p@i$ iff $\hat{B} \implies p@i$, as required.

G2 For every A, B in AT_1 and $u \subseteq D$, if $A \Rightarrow_u B$ and $[\forall]\phi \in A$, then $\phi \in B$.

Proof. Suppose A, B are as above such that $A \Rightarrow_u B$ and $[\forall]\phi \in A$. Suppose that $\phi \notin B$. Then $\neg\phi \in B$. Since $\hat{A} \wedge \langle\exists\rangle\hat{B}$ is consistent, and hence $\hat{A} \wedge \langle\exists\rangle\neg\phi$ is consistent. Therefore, $[\forall]\phi \wedge \langle\exists\rangle\neg\phi$ is consistent, which is a contradiction.

G3 For every A in AT_1, if $\langle\exists\rangle\phi \in A$, then there exists $B \in AT_1$ and $u \subseteq D$ such that $A \Rightarrow_u B$ and $\phi \in B$. (Therefore we also have that for A, B in AT_1, if $\hat{A} \wedge \langle\exists\rangle\hat{B}$ is consistent, then there is an edge from A to B in the graph $G1$.)

Proof. Let $A \in AT_1$, and $\langle\exists\rangle\phi \in A$. Then the formula $\hat{A} \wedge \langle\exists\rangle\phi$ is consistent, and by rule (Impr) and Axiom (A7), for some $u \subseteq D$, the formula

$$(\hat{A} \wedge \bigwedge_{i \in u}(a_i@i \wedge P_i(A)) \wedge \langle\exists\rangle(\bigwedge_{i \in u}(\neg a_i@i \wedge P_i(B) \wedge [P_i(A) \prec P_i(B)@i]))$$

is consistent. Hence there exists $B \in AT$ such that $\hat{A} \wedge \langle\exists\rangle\hat{B}$ is consistent, $\phi \in B$. Moreover for all $i \in u$ if $a_i@i \in A$ then $\neg a_i@i \in B$, thus $\sigma_A[i] \neq \sigma_B[i]$, and $P_i(A) \prec_i P_i(B)$. Further, for all $j \notin u$, $a@j \in A$ iff $a@j \in B$. All this ensures that $A \Rightarrow_u B$ and $\psi \in B$.

G4 For every A, B in AT_1, if $A \Rightarrow^* B$ and $\Box^*\phi \in A$, then $\phi \in B$.

Proof. Suppose A, B are as above such that $A \Rightarrow^* B$.
Let $A = B_0, B_1, \ldots, B_k = B \in AT$ such that for $j : 0 \leq j < k$ and $u_j \subseteq D$, $B_j \Rightarrow_{u_j} B_{j+1}$. Thanks to axiom (A2), if $\Box^*\phi \in B_j$, then $[\forall]\Box^*\phi \in B_j$ as well, and by property [G3] above, $\Box^*\phi \in B_{j+1}$. Proceeding thus, we get $\Box^*\phi \in B_k = B$. Again by axiom (A2), we get that $\phi \in B$.

G5 For every A in AT_1, if $\Diamond^*\phi \in A$, then there exists an atom B in AT_1 reachable from A such that $\phi \in B$.

Proof. Let $R_A = \{B \mid A \rightarrow^* B\}$. Suppose none of the atoms in R_A have ϕ in them. Hence $\neg\phi \in B$ for all $B \in R_A$. Then we have: $\vdash \tilde{R_A} \implies \neg\phi$.
We now make the following claim:

Claim: $\vdash \tilde{R_A} \implies [\forall]\tilde{R_A}$

Assuming the claim, by the rule (IND) we have: $\vdash \tilde{R_A} \implies \Box^*\tilde{R_A}$.

$$\begin{array}{lr}
\vdash \hat{B} \implies \neg\phi & \text{for all } B \in R_A \\
\vdash \tilde{R_A} \implies \neg\phi & \\
\vdash \Box^*(\tilde{R_A} \implies \neg\phi) & \text{by Gen} \\
\vdash \Box^*\tilde{R_A} \implies \Box^*\neg\phi & \text{by B2}
\end{array}$$

$\vdash \tilde{R_A} \implies \Box^*\tilde{R_A}$ by Claim

$$\begin{array}{lr}
\vdash \tilde{R_A} \implies \Box^*\neg\phi & \text{from above} \\
\vdash \hat{A} \implies \tilde{R_A} & A \in R_A \\
\vdash \hat{A} \implies \Box^*\neg\phi &
\end{array}$$

which means we get $\Box^*\neg\phi \in A$ which is the contradiction we need.
Proof of Claim: Suppose not. Then $\tilde{R_A} \wedge \langle\exists\rangle\neg\hat{R_A}$ is consistent. We partition AT and into $R_A = \{X_1, X_2, \ldots, X_m\}$ and $\overline{R_A} = \{Y_1, Y_2, \ldots, Y_n\}$ the complement of R_A. We have $(\hat{X_1} \vee \ldots \vee \hat{X_m}) \wedge \langle\exists\rangle(\hat{Y_1}, \ldots, \hat{Y_n})$ is consistent. Therefore we have, for some $i \in [m],\ j \in [n]$, $\hat{X_i} \wedge \langle\exists\rangle\hat{Y_j}$ is consistent. By the graph property [G3], for some $u \subseteq D$, $X_i \rightarrow_u Y_j$ which means Y_j also belongs to R_A giving us a contradiction, and proving the claim.

We can now define the game $\mathcal{G}_{\phi_1} = (N_1, P_1, (\preceq_i)_{i\in N_1}, \omega_1)$ by: $N_1 = D$, $P_1 = P(\phi_1)$ and $\omega_1(\sigma_A)(a) = \{p@i \mid \{p@i, a@i\} \subseteq A\}$. Recall that $\preceq_i = \{(\gamma_1, \gamma_2) \mid [\gamma_1 \prec \gamma_2]@i \in A_0\}$. The map ω_1 is well defined by the local property [L2] and Axiom (A6). Let σ_A be the profile induced by A, and consider the improvement graph for $\mathcal{G}_1$ given by the reachability set of σ_1. which we denote by Σ_1. Thus we have the model M_1. We define ι to be the identity function on D.

We now show that for all A reachable from A_0 in AT_1, and σ_A induced by A, and for all $\psi \in SF(\phi)$, $\psi \in A$ iff $M_1, \sigma_A \models \psi$.

Proof. This is proved by **induction** on structure of ψ.

Base Case .

i. The base cases follow from the local properties [L1] through [L4] of atoms and the definition of the outcome map ω_1.

Induction Case .

i. The boolean cases are routine.
ii. $\psi := \langle\exists\rangle\phi$ From graph property [G3], if $\langle\exists\rangle\phi \in A$, there exists $B \in AT_1$ and $u \in D$ such that, $A \Rightarrow_u B$ and $\phi \in B$ which by the induction hypothesis means $M_1, \sigma_B \models \phi$. Therefore $M_1, \sigma_A \models \langle\exists\rangle\phi$.
For the other direction : suppose $M_1, \sigma_A \models \langle\exists\rangle\phi$. Then for some $B \in AT_1$ and some $u \subseteq D$, $A \Rightarrow_u B$ and $M_1, \sigma_B \models \phi$. By induction hypothesis, $\phi \in B$. But $\hat{A} \wedge \langle\exists\rangle\hat{B}$ is consistent. Therefore, $\langle\exists\rangle\phi \in A$, as required.
iii. $\psi := \Diamond^*\phi$. If $\Diamond^*\phi \in A$, by graph property [G5], there exists $B \in AT_1$ such that $A \Rightarrow^* B$ and $\phi \in B$. By the induction hypothesis $M_1, \sigma_B \models \phi$. Therefore $M_1, \sigma_A \models \Diamond^*\phi$, as required.
For the other direction : suppose $M_1, \sigma_A \models \Diamond^*\phi$ but $\Diamond^*\phi \notin A$. Since $M_1, \sigma_A \models \Diamond^*\phi$, there exists $B' \in AT$ such that $A \Rightarrow^* B'$ and $M_1, \sigma_{B'} \models \phi$. By induction hypothesis, $\phi \in B'$. On the other hand, since $\Diamond^*\phi \notin A$, then $\Box^*\neg\phi \in A$. Then for all $B \in AT$, if $A \Rightarrow^* B$ then by graph property [G4], $\neg\phi \in B$. In particular, $\neg\phi \in B'$, which contradicts consistency of B'.

Since the given consistent formula $\phi_0 \in A_0$, we have shown that $M_1, A_0 \models \phi_0$. Thus ϕ_0 is satisfiable in a model of bounded size, proving the theorem.

We conjecture that the upper bound can be improved to deterministic double exponential time. We have a lower bound of deterministic exponential time, since we have a logic with two modalities, one of which is a transitive closure of the other.

6 Bisimulation

Bisimulation for *MSO-STRAT* should consider the use of quantifiers over player variables. Since the bisimulation for *MIQ-STRAT* already requires us to consider relations over player names (in addition to relating system states), we discuss only bisimulation for *MIQ-STRAT* here. Bismulation for *MSO-STRAT* can be generalized from that for *MIQ-STRAT* along the lines of the notion for Term Modal Logic ([PR19b], [Pad20]).

Definition 3. *Given two large games $\mathcal{G}$ and $\mathcal{G}'$ with their corresponding models $M = (N, P, \omega, \preceq, \iota)$ and $M' = (N', P, \omega', \preceq, \iota')$ the* improvement bisimulation *is given by $R \subseteq (\Sigma \times \Sigma')$ which is a non-empty set, such that for all $(\sigma, \sigma') \in R$ and for every $i \in C$ the following holds:*

1. $\sigma[\iota(i)] = \sigma'[\iota'(i)]$.
2. $\hat{\omega}_{\iota(i)}(\sigma) = \hat{\omega}'_{\iota'(i)}(\sigma')$.
3. *for every $a \in \mathbf{\Gamma}$ we have $\frac{|\{j|\sigma[j]=a\}|}{|N|} = \frac{|\{k|\sigma'[k]=a\}|}{|N'|}$*
4. (a) *for all $k \subseteq N$ and for all $\sigma \rightarrow_k \sigma_1$ in $I_{\mathcal{G}}$, there exists some $k' \subseteq N'$ and a profile σ'_1 in $I_{\mathcal{G}'}$ such that $\sigma' \rightarrow_{k'} \sigma'_1$ in $I_{\mathcal{G}'}$ and $(\sigma_1, \sigma'_1) \in R$.*
   (b) *for all $k' \subseteq N'$ and for all $\sigma' \rightarrow_{k'} \sigma'_1$ in $I_{\mathcal{G}'}$, there exists some $k \subseteq N$ and a profile σ_1 in $I_{\mathcal{G}}$ such that $\sigma \rightarrow_k \sigma_1$ in $I_{\mathcal{G}}$ and $(\sigma_1, \sigma'_1) \in R$.*
5. (a) *whenever σ_1 is reachable from σ in $I_{\mathcal{G}}$, there exists a profile σ'_1 in $I_{\mathcal{G}'}$ that is reachable from σ' such that $(\sigma_1, \sigma'_1) \in R$*
   (b) *whenever σ'_1 is reachable from σ' in $I_{\mathcal{G}'}$, there exists a profile σ_1 in $I_{\mathcal{G}}$ that is reachable from σ such that $(\sigma_1, \sigma'_1) \in R$*

Intuitively if (σ, σ') are improvement bisimilar profiles, then for every constant $i \in C$ we want the player $\iota(i)$ in σ to 'act like' $\iota'(i)$ in σ'. This is captured in (1) which says both these players play the same action and (2) which says both these players get the same payoff. Condition (3) says that the distribution induced by σ and σ' is the same. Condition $(4, 5)$ are the standard back-forth properties for one step improvement and reachability respectively.

For any two games $\mathcal{G}$ and $\mathcal{G}'$ with their corresponding models $M = (N, P, \omega, \preceq, \iota)$ and $M' = (N', P, \omega', \preceq, \iota')$ for every $\sigma \in I_{\mathcal{G}}$ and $\sigma' \in I_{\mathcal{G}'}$ we say that σ and σ' are improvement bisimilar if there is some improvement bisimulation R over M and M' such that $(\sigma, \sigma') \in R$. And, σ and σ' are *elementarily equivalent* if for every formula ϕ in the logic, *MIQ-STRAT*, $M, \sigma \models \phi$ iff $M', \sigma' \models \phi$. Since the large games are always finite, the improvement bisimilation exactly characterizes elementary equivalence always.

Theorem 4. *Given two large games $M = (N, P, \omega, \iota)$ and $M' = (N', P, \omega', \iota')$ for all $\sigma \in I_{\mathcal{G}}$ and $\sigma' \in I_{\mathcal{G}'}$,*
σ and σ' are improvement bisimilar iff σ and σ' are elementarily equivalent.

Proof. ($\Rightarrow$) Assume that σ and σ' are improvement bisimilar. Let R be the improvement bisimilation relation such that $(\sigma, \sigma') \in R$.

We can first show that for any distribution constraint formula δ, $\pi \models_d \delta$ iff $\pi' \models_d \delta$. The base case follows by condition 3 and the boolean cases are routine. We then show that for any type formula α and $i \in C$, $\pi, \iota(i) \models_t \alpha$ iff $\pi', \iota(i) \models_t \alpha$. For the base case consider a formula of the form $(\delta \to a)$. This follows from the above claim on δ and condition 1.

We now prove that for all $(\pi, \pi') \in R$ and for all formula ϕ we have $M, \pi \models \phi$ iff $M', \pi' \models \phi$. The proof is by induction on the structure of ϕ. (We omit the case $(\gamma_1 \preceq \gamma_2)$ below since the relation is identical for both models.)

- Case δ. $M, \pi \models \delta$ iff $\pi \models_d \delta$ iff, by above, $\pi' \models_d \delta$ iff $M', \pi' \models \delta$.
- Case $[\alpha@i]$. $M, \pi \models [\alpha@i]$ iff $\pi, \iota(i) \models_t \alpha$ iff, by above, $\pi', \iota(i) \models_t \alpha$ iff $M', \pi' \models [\alpha@i]$.
- Case $a@i$: $M, \pi \models a@i$ iff $\pi[\iota(i)] = a$ iff (by condition 1) $\pi'[\iota'(i)] = a$ iff $M', \pi' \models a@i$.
- Case $p@i$: $M, \pi \models p@i$ iff $p \in \hat{\omega}_{\iota(i)}(\pi)$ iff (by condition 2) $p \in \hat{\omega}'_{\iota'(i)}(\pi')$ iff $M, \pi' \models p@i$.
- $\neg\phi$ and $\phi \vee \phi'$ are standard.
- Case $\langle\exists\rangle\phi$: If $M, \pi \models \langle\exists\rangle\phi$, then there is some $k \subseteq N$ and some $\pi \to_k \pi_1$ such that $M, \pi_1 \models \phi$ and by condition (4.a) there is some $k' \subseteq N'$ and some $\pi' \to_{k'} \pi'_1$ such that $(\pi_1, \pi'_1) \in R$. By induction $M', \pi'_1 \models \phi$ and hence $M', \pi' \models \langle\exists\rangle\phi$.
  Conversely, if $M', \pi' \models \langle\exists\rangle\phi$, then it can be similarly argued using condition (4.b) that $M, \pi \models \langle\exists\rangle\phi$.
- Case $\Diamond^*\phi$: if $M', \pi' \models \Diamond^*\phi$, then there is some π'_1 reachable from π such that $M', \pi'_1 \models \phi$ and by condition (5.b) there is some π_1 reachable from π such that $(\pi_1, \pi'_1) \in R$ and hence $M, \pi_1 \models \phi$ which implies that $M, \pi \models \Diamond^*\phi$.
  Similarly if $M, \pi \models \Diamond^*\phi$, then it can be argued using condition (5.a) that $M', \pi' \models \Diamond^*\phi$.

($\Leftarrow$) Assume that σ and σ' are elementarily equivalent.
Define $R = \{(\pi, \pi') \mid \pi \text{ and } \pi' \text{ are elementarily equivalent}\}$. Clearly $(\sigma, \sigma') \in R$. Now we verify that R satisfies all conditions of improvement bisimulation for every $(\pi, \pi') \in R$. Pick some arbitrary $(\pi, \pi') \in R$.

1. If first condition is violated, then there is some $i \in C$ such that $\pi[\iota(i)] = a \neq \pi'[\iota'(i)]$. Then we have $M, \pi \models a@i$ and $M', \pi' \not\models a@i$ which is a contradiction.

2. If second condition is violated, then there is some $i \in C$ and $p \in P$ such that $p \in \hat{\omega}_{\iota(i)}(\sigma)$ and $p \notin \hat{\omega}'_{\iota'(i)}(\sigma')$ (the other case is symmetric). Then we have $M, \pi \models p@i$ and $M', \pi' \not\models p@i$ which is a contradiction.

3. If third condition is violated, then there is some $a \in \mathbf{\Gamma}$ such that $\frac{|\{j \mid \sigma[j]=a\}|}{|N|} = r \neq \frac{|\{k \mid \sigma'[k]=a\}|}{|N'|}$
   Then we have $M, \pi \models (\sharp\ a \ = \ r)$ and $M', \pi' \not\models (\sharp\ a \ = \ r)$ which is a contradiction.

4. If Condition (4.a) is violated, then there is some $k \subseteq N$ and some $\pi \rightarrow_k \pi_1$ such that for all $k' \subseteq N'$ and $\pi' \rightarrow_{k'} \pi_1'$ we have $(\pi_1, \pi_1') \notin R$.
   Let $S = \{\pi_1' \mid \text{for some } k' \subseteq N' \text{ we have } \pi' \rightarrow_{k'} \pi_1\}$. Since $I_{\mathcal{G}'}$ is finite, S is finite.
   Now if $S = \emptyset$ then $M, \pi \models \langle\exists\rangle\top$ and $M', \pi' \models \neg\langle\exists\rangle\top$ which is a contradiction. So, let $S = \{\pi_1', \ldots \pi_s'\}$ and for every $l \leq s$ we have α_l such that $M, \pi_1 \models \alpha_l$ and $M, \pi_l' \models \neg\alpha_l$. Thus we have $M, \pi \models \langle\exists\rangle\big(\bigwedge_{l=1}^{s} \alpha_l\big)$ and $M', \pi' \models \neg\langle\exists\rangle\big(\bigwedge_{l=1}^{s} \alpha_l\big)$ which is a contradiction.
   Similarly it can be argued that violation of Condition (4.b) leads to a contradiction.
5. If Condition (5.b) is violated, then there is some π_1' reachable from π' such that for all π_1 reachable from π we have $(\pi_1, \pi_1') \notin R$.
   Let $S = \{\pi_1 \mid \pi_1 \text{ is reachable from } \pi\}$. Since $I_{\mathcal{G}}$ is finite, S is finite.
   Now if $S = \emptyset$, then $M, \pi \models \Box^*\bot$ and $M', \pi' \models \Diamond^*\top$ which is a contradiction. So, let $S = \{\pi_1, \ldots \pi_t\}$ and for every $l \leq t$ we have β_l such that $M, \pi_l \models \beta_l$ and $M', \pi_1' \models \neg\beta_l$. Thus we have $M, \pi \models \Box^*\big(\bigvee_{l=1}^{t} \beta_l\big)$ and $M', \pi' \models \Diamond^*\big(\bigwedge_{l=1}^{t} \neg\beta_l\big)$ which is a contradiction.
   Similarly it can be argued that violation of Condition (5.a) leads to a contradiction.

Note that since improvement graphs are finite, elementary equivalence implies bisimilarity as well.

7 Discussion

We have considered reasoning about strategisation in large games, where a player responds not to what other specific players choose, but to what fraction of the population choose a specific strategy. Since this involves reasoning about games with unboundedly many players, we presented logics with no player identities but player variables or names for player types. A monadic second-order logic with player variables and team variables is natural for such reasoning, but such a logic becomes undecidable and non-axiomatisable, so we consider a propositional modal logic of implicit quantification which we show to be decidable and for which we present a complete axiom system. The use of implicit quantification in complex distribution constraints characterise the modal logic. We also present a bisimulation characterisation for elementary equivalence.

Note that the (Num) rule and the (Impr) rule in the presented system are infinitary (but recursive). While it may be possible to avoid the (Num) rule by a more careful model construction, it seems difficult to do so in the case of (Impr). We conjecture that there is indeed no finite axiomatisation of even this propositional modal logic.

The *presentation* of large games poses many challenges, since specifying total functions on distributions can be unwieldy. The use of logic to specify player types and thereby computing the distributions for which we need outcomes to be specified opens up new possibilities in the algorithmic analysis of large games. Further, player equivalence on games (based on types) offers an interesting possibility of algebraic analysis.

While we have studied improvement dynamics for teams of players acting individually, it is of great interest to study *coalitions* and coordination. The use of implicit quantification is natural in such contexts.

We have not discussed the model checking algorithm for these logics, which can be given along standard lines. However, for these logics, it may be more interesting to model check *invariant* properties that hold for *every* strategy profile, or *realisability*, of properties that hold at *some* strategy profile.

The logic we have presented here is preliminary and needs more sophistication to be useful. For one thing, players do not respond to actual distributions but to *expectations* on distributions. Other important questions relate to modal characterization of subclasses of large games such as majority games and minority games, or type matching and type mismatching games.

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