A logical description of priority separable games

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Abstract. When we reason about strategic games, implicitly we need to reason about arbitrary strategy profiles and how players can improve from each profile. This structure is exponential in the number of players. Hence it is natural to look for subclasses of succinct games for which we can reason directly by interpreting formulas on the (succinct) game description rather than on the associated improvement structure. Priority separable games are one of such subclasses : payoffs are specified for pairwise interactions, and from these, payoffs are computed for strategy profiles. We show that equilibria in such games can be described in Monadic Least Fixed Point Logic (MLFP). We then extend the description to games over arbitrarily many players, but using the monadic least fixed point extension of existential second order logic.

1 Introduction

Finite strategic games are a well-studied formalism used to analyse strategic behaviour of rational agents. A strategic game is specified by a finite set of players along with a finite set of strategies and a payoff function for each player. Players choose strategies simultaneously and for each player, the corresponding payoff function specifies the utility for the player given the profile of choices. In terms of structural and computational analysis of games, a major drawback of strategic games is that the representation is not compact. The two main parameters in the representation are the number of players and the strategies available for each player. An explicit representation of the payoff functions is exponential in the number of players. Identifying subclasses of strategic game with compact representation, is thus an important first step towards analysing the structural and computational properties of the game model.

There are two possible approaches which are commonly adopted to achieve concise representation in games. First is to retain quantitative payoffs and impose restrictions on the payoff functions. The second is to use an appropriate logical formalism to describe payoffs in a qualitative manner.

In the classical approach with quantitative payoffs, it is possible to achieve compact representation with a careful analysis of the underlying dependency structure in the payoff functions. Such an approach, which restricts the dependency of payoff functions to a "small" number of other agents is adopted in *graphical games* [26]. Another approach is to explicitly impose restrictions on the payoff functions. A simple constraint is to insist that the payoff functions are *pairwise separable*. This results in the well-studied class of games with a compact representation, called *polymatrix games* [25, 11, 10, 35]. In the specific context of coalition formation games, the restriction to pairwise separable payoffs results in the well studied game model called *additively separable hedonic games* [23, 5].

The second approach is to view payoffs as qualitative outcomes described using some logical formalism. Boolean games [19] is a well studied model adopting this approach. In Boolean games, each player controls a disjoint subset of atomic propositions and the payoffs are represented using Boolean formulas over the union of these propositions. Though originally defined as a qualitative version of two player zero-sum games, the model has been extended to reason about multi-player games [20, 9]. Epistemic Boolean games, where payoffs are specified as epistemic formulas, were studied in [2, 21].

In this paper we propose a subclass of strategic games that combine both approaches. In our model, the payoffs are qualitative (but not necessarily Boolean). The payoff functions are restricted to be pairwise separable. This results in a game model which is concise while at the same time being able to represent genuine multi-player games with non-zero sum objective. We show that these priority separable games need not always have a pure Nash equilibrium. Then an immediate question is whether there is an efficient procedure to check if a Nash equilibrium exists in this class of games. We show that checking for the existence of a Nash equilibrium is NP-complete. Nash's Theorem states that the mixed extension of every finite strategic game has a Nash equilibrium (in mixed strategies). It is known that computing such a mixed strategy Nash equilibrium is PPAD-hard [15]. In this paper, we restrict our study to pure strategies and pure Nash equilibrium.

We express the existence of a pure Nash equilibria in priority separable games in a logical language: the *monadic least fixed-point logic* ([33]). This is an extension of first order logic with monadic least fixed-point operators. In this, we follow the spirit of descriptive complexity [24], where extensions of first order logics describe complexity classes. The major question of interest in such investigation is to ask what logical resources are needed to describe the property (which is Nash equilibrium in this case). We would like minimal use of the logical resources needed. Since equilibrium computation involves iterative exploration of the strategy space, by considering every possible player improvement and player response to it, a least fixed-point operator is natural to use in this context. However, we show that it suffices to use a *monadic* least fixed-point operator, where the operator is applied only on sets (rather than on arbitrary relations).

A natural question that arises in the logical study is games over unboundedly many players. In game theory, we typically specify every game with the number of players playing in it. This seems reasonable for logical descriptions as well: we can ask, how many players are *forced* by the structure present in the reasoning? A formula can have different game models, with different numbers of players. With the addition of binary second order variables to the logic (and monadic least fixed-point operators), we describe Nash equilibrium in the logic. There are complexity theoretic implications in this but we do not take it up for study in this paper.

Various logical formalisms have been used in the literature to reason about games and strategies. Action indexed modal logics have often been used to analyse finite extensive form games where the game representation is interpreted as models of the logical language [8, 6, 7]. A dynamic logic framework can then be used to describe games and strategies in a compositional manner [30, 18, 31] and encode existence of equilibrium strategies [20]. The work in [13] employs modal logics similarly to study large games. Alternating temporal logic (ATL) [3] and its variants [22, 36, 12] constitute a popular framework to reason about strategic ability in games. Strategy Logic ([12]) is of specific interest in its explicit use of strategy quantifiers, and hence the existence of solution concepts like Nash equilibrium can be expressed in it. In the context of two-player turn-based sequential games on graphs Strategy Logic provides a logical mechanism for description of a variety of equilibria. [32] provides another logical treatment of explicit reasoning about structured strategies.

Our approach here is different from these, both in the structure of the games studied and in the use of logic for checking properties as in descriptive complexity theory. The models of the logic are strategy spaces: every node is a strategy profile, and edges denote deviations by players. Iteration of such deviations until no more deviation is possible suggests the use of fixed-point operators. These are one-shot strategic form games, as distinct from models of logics like ATL, Strategy Logic and the logic of structured strategies in [32] where the games are turn based and of infinite duration. Moreover, while Strategy Logic discusses two-player games, we talk of multi-player games which are determined by pairwise interactions among players. The work presented here follows [14] but the emphasis there was on improvement graph dynamics, whereas we reason directly with game descriptions here.

2 Our model

2.1 Background

Let $N = \{1, \ldots, n\}$ be the set of players. A *strategic game* is defined as $\mathcal{G} = (N, \{S_i\}_{i \in N}, \{p_i\}_{i \in N})$ consists of the set of players N, and for each player i, a set S_i of strategies along with a payoff function $p_i : S_1 \times \cdots \times S_n \to \mathbb{R}$. A strategy profile is a tuple of strategies, $s = (s_1, \ldots, s_n)$ where for all players i, $s_i \in S_i$. Given a strategy profile s and a player i, let $s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$.

Thus $s = (s_i, s_{-i})$. Let $S = S_1 \times \ldots \times S_n$ denote the set of all strategy profiles and $S_{-i} = S_1 \times \ldots S_{i-1} \times S_{i+1} \times \ldots \times S_n$.

We say that the strategy $s_i \in S_i$ of player i is a *best response* to $s_{-i} \in S_{-i}$ if for all $s'_i \in S_i$, $(s'_i, s_{-i}) \preceq_i s$. A strategy profile s is a *Nash equilibrium* if for all $i \in N$, s_i is a best response to s_{-i}. Existence of Nash equilibrium and computation of an equilibrium profile (when it exists) are important questions in the context of strategic form games. While strategic games are well-studied as a model for games, it has the drawback that the representation is not concise. An explicit representation of the payoff functions is exponential in the number of players. To analyse the computational properties of games, it is important to identify subclasses of strategic form games which have a compact representation. *Polymatrix games* [25] form such a subclass, where the payoff functions are restricted to be pairwise separable. Formally, a *polymatrix game* is a strategic game $\mathcal{G} = (N, \{S_i\}_{i \in N}, \{p_i\}_{i \in N})$ where for all players i and for all $j \neq i$, there exists a partial payoff function $p_{i,j}$ such that for any strategy profile s, $p_i(s) = \Sigma_{j \neq i}\, p_{i,j}(s_i, s_j)$. It can be observed that polymatix games have compact representation, polynomial in $|N|$ and $\max_{i \in N} |S_i|$.

It is often useful to explicitly specify the dependency of the pairwise separable payoff functions in terms of a neighbourhood graph. Let $G = (N, E)$ be a directed graph (without self loops) over the set of players N and for each $i \in N$, let $R(i) = \{\ j \mid (j, i) \in E\ \}$ be the neighbourhood of i in G. For the players not in the neighbourhood of a certain player, say i, we define the partial payoff function values on those instances as 0. That is, for all strategy profiles s, for all $i \in N$, whenever $j \notin R(i)$ then $p_{i,j}(s_i, s_j) = 0$.

2.2 Priority separable games

Since we are interested in the logical study of games, we define a qualitative subclass of polymatrix games called *priority separable games* as follows.

Let N be a finite set of players and for each $i \in N$, let S_i be a finite set of strategies for player i. Let Ω be a finite set of outcomes and for all $i \in N$, let $\ll_i \subseteq \Omega \times \Omega$ be a strict total ordering over the outcome set.

We explicitly model the dependency on payoffs using a graphical structure. Let $G = (N, E)$ be a directed graph (without self loops) and $R(i)$ be the neighbourhood of i in G as defined earlier. We associate a linear priority ordering within the neighbourhood for each node $i \in N$ and denote this by the relation $\triangleright_i \subseteq R(i) \times R(i)$

For $i, j \in N$, let $p_{i,j} : S_i \times S_j \to \Omega$ be a partial payoff function. Given a strategy profile s, the payoff for player $i \in N$ is then defined as the tuple $p_i(s) = (p_{i,j}(s_i, s_j))_{j \in R(i)}$.

A *priority separable game* is defined as the tuple

$$\mathcal{G} = (G, (S_i)_{i \in N}, \Omega, (\ll_i)_{i \in N}, (\triangleright_i)_{i \in N}, (p_{i,j})_{i,j \in N}).$$

Note that in a priority separable game $\mathcal{G}$, the number of payoff entries that need to be specified in $\mathcal{G}$ is bounded by $2 \cdot \max_{i \in N} |S_i|^2 \cdot |N|^2$. Thus $\mathcal{G}$ has a compact representation that is polynomial in both $|N|$ and $\max_{i \in N} |S_i|$.

Given a strategy profile s, let $p_i^*(s)$ denote the reordering of the tuple $p_i(s)$ in decreasing order of the priority of neighbours of i. That is, if $R(i) = \{i_1, \dots, i_k\}$ and $i_1 \rhd_i i_2 \rhd_i \cdots \rhd_i i_k$, then for $j \in \{1, \dots, k\}$, $(p_i^*(s))_j = p_{i,i_j}(s_i, s_{i_j})$. In order to analyse the strategic aspect of the game, we need to define how players compare between strategy profiles. For $i \in N$, we define the relation $\preceq_i \subseteq S \times S$ as follows: $s \preceq_i s'$ if $p_i^*(s) \preceq^{lex} p_i^*(s')$ where $\preceq^{lex}$ denotes the lexicographic ordering.

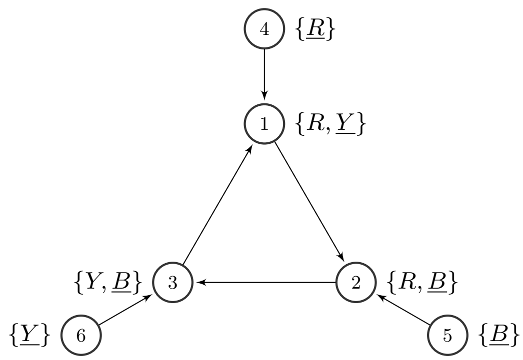

Fig. 1. A priority separable game

Example 1. Consider the game where $N = \{1, \dots, 6\}$ and the graph G is as given in figure 1. For $i \in N$ the set of strategies S_i is specified in Figure 1 as a label next to each node in G. Let $\Omega = \{0, 1\}$ with $0 \ll_i 1$ for all $i \in N$. For $i, j \in N$, let $p_{i,j} = 1$ if $s_i = s_j$ and $p_{i,j} = 0$ if $s_i \neq s_j$. Let $3 \rhd_1 4$, $1 \rhd_2 5$ and $2 \rhd_3 6$. For $j \in \{4, 5, 6\}$, $|S_j| = 1$ and $R(j) = \emptyset$. Consider the strategy profile $s = (Y, B, B, R, B, Y)$ which is denoted with an underline in Figure 1. Note that in s player 1 is not playing its best response and has a profitable deviation to R. Therefore s is not a Nash Equilibrium.

Some classes of priority separable games. Two player zero-sum games form a well studied subclass of strategic games that has compact representation and good computational properties. For instance, a (mixed) Nash equilibrium in two player zero-sum games can be computed in polynomial time. It can be observed that every two player game is a priority separable game. The restriction to separable payoff functions extends the underlying idea of two player interaction to multi-player games while retaining the attractive property of having a concise representation. Priority separable games form a subclass of polymatrix games with qualitative payoffs, making it an ideal model for logical analysis.

Note that it is not the case that all polymatrix games can be translated into a priority separable game which preserves the set of Nash equilibria. This is illustrated in the example given below.

Example 2. Consider the polymatrix game $\mathcal{H}$ defined as follows. The player set $N = \{1, 2, 3, 4\}$ and the graph G is as given in Figure 2. For $i \in N$, S_i is specified

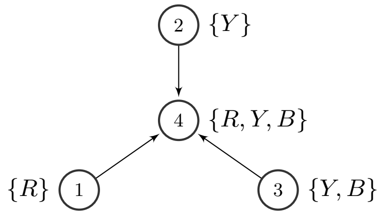

Fig. 2. A priority separable game

in Figure 2 as a label next to each node in G. Note that for all $i \in \{1, 2, 3\}$, $R(i) = \emptyset$. Consider the partial payoff functions given below.

- $p_{4,1}(s_4, s_1) = 3$ if $s_1 = s_4$ and 0 otherwise.
- $p_{4,2}(s_4, s_2) = 2$ if $s_2 = s_4$ and 0 otherwise.
- $p_{4,3}(s_4, s_3) = 2$ if $s_3 = s_4$ and 0 otherwise.

Now consider the priority separable game, $\mathcal{G}$, defined over the same set of players N, the same graph G and the same partial payoff functions $p_{i,j}$ where $\Omega = \{2, 3\}$. For all $i \in N$, let $2 \ll_i 3$. Let $\rhd_4$ be chosen arbitrarily. Let $NE(\mathcal{H})$ and $NE(\mathcal{G})$ denote the set of Nash equlibria in $\mathcal{H}$ and $\mathcal{G}$ respectively.

Below we list all possible orderings for $\rhd_4$ and argue that in each case $NE(\mathcal{H}) \neq NE(\mathcal{G})$.

- $1 \rhd_4 2 \rhd_4 3$: The strategy profile $(R, Y, Y, R) \in NE(\mathcal{G})$ but $(R, Y, Y, R) \notin NE(\mathcal{H})$.
- $2 \rhd_4 1 \rhd_4 3$ or $2 \rhd_4 3 \rhd_4 1$: The strategy profile $(R, Y, B, Y) \in NE(\mathcal{G})$ but $(R, Y, B, Y) \notin NE(\mathcal{H})$.
- $3 \rhd_4 2 \rhd_4 1$ or $3 \rhd_4 1 \rhd_4 1$: The strategy profile $(R, Y, B, B) \in NE(\mathcal{G})$ but $(R, Y, B, B) \notin NE(\mathcal{H})$.

There are various interesting classes of polymatrix games which can be viewed as priority separable games. For instance, consider a polymatrix game over a graph G where for every $i \in N$, we can define an ordering $>_i \subseteq R(i) \times R(i)$ such that for all $s \in S$ and for all $j \in R(i)$, $p_{i,j}(s) > \Sigma_{k \in R(i): j >_i k}\, p_{i,k}(s)$. Such a game can be converted into a priority separable game which is strategically equivalent by defining $\rhd_i$ as $>_i$ for all $i \in N$ and taking $\ll_i$ to be the natural ordering over numbers. Note that priority separable games allows $\ll_i$ to be different for each player $i \in N$.

A resource allocation model. The restriction to priority separable payoffs also arises naturally in other domains like resource allocation. A well-studied model for allocation of indivisible items is the Shapley-Scarf housing market [34] defined as follows. Let $N = \{1, \ldots, n\}$ be a finite set of agents and $A = \{a_1, \ldots, a_n\}$ be a finite set of indivisible items where $|N| = |A|$. An allocation is a bijection $\pi : N \to A$. In the most commonly studied setting, each agent i has a preference ordering over its allocation $\pi(i)$ and is independent of the allocation of the other agents. However, in many practical instances, agents preferences could depend on externalities like the allocation of other agents. This is particularly relevant in the housing market where an agent's preference for a house could depend on identity of other agents in its immediate neighbourhood. This is also a natural criterion in the allocation of office space where individuals might prefer to be located close to their group members. Priority separable externalities can capture many of these situations.

If agents are allowed to exchange items with each other, stability of allocation is a very natural solution concept to study. Core stable outcomes are defined as allocations in which no group of agents have an incentive to exchange their items as part of an internal redistribution within the coalition. For the housing market without externalities, a simple and efficient procedure often termed as Gale's Top Trading Cycle, can compute a stable allocation that is core stable [34]. In the presence of pairwise separable externalities with quantitative payoffs, the computational properties of the model are studied in [17, 28].

Existence of Nash equilibrium A natural question is whether priority separable games always have a pure Nash equilibrium. Below we show that the class of priority separable games need not always have a pure Nash equilibrium using an example which is similar to the one given in [4] for polymatrix games.

Example 3. Consider the game given in example 1 along with the neighbourhood graph given in figure 1. For players $i \in \{4, 5, 6\}$, $R_i = \emptyset$ and since $|S_i| = 1$, for all $s \in S$, s_i is a best response to s_{-i}. Thus in each strategy profile s only the choices made by players $1, 2$ and 3 are relevant. Below we enumerate all such strategy profiles and underline a strategy which is not a best response for each strategy profile. It then follows that this game does not have a Nash equilibrium. $(R, R, \underline{B})$, $(\underline{R}, R, G)$, $(R, \underline{B}, B)$, $(R, \underline{B}, G)$, $(\underline{G}, R, B)$, $(G, \underline{R}, G)$, $(\underline{G}, B, B)$, $(G, B, \underline{G})$.

3 Computing Nash equilibria in priority separable games

Given that priority separable games need not always have a Nash equilibrium, an immediate question is whether there is an efficient procedure to check if a Nash equilibrium exists in this class of games. We show that checking for the existence of a Nash equilibrium is NP-complete. While the upper bound is straightforward, to show NP-hardness we give a reduction from 3-SAT using an argument similar to the one in [4].

Theorem 1. *Given a priority separable game $\mathcal{G}$, deciding if $\mathcal{G}$ has a Nash equilibrium is NP-complete.*

Proof. Given a priority separable game $\mathcal{G}$ and a strategy profile s, we can verify if s is a Nash equilibrium in $\mathcal{G}$ in time polynomial in $|N|$ and $\max_{i \in N} |S_i|$. It follows that the problem is in NP. We show NP-hardness by giving a reduction from 3-SAT.

Consider an instance of 3-SAT given by the formula $\varphi = (a_1 \vee b_1 \vee c_1) \wedge (a_2 \vee b_2 \vee c_2) \wedge \ldots \wedge (a_k \vee b_k \vee c_k)$ with k clauses and m propositional variables $x_1, \ldots, x_m$. For $j \in \{1, \ldots, k\}$, a_j, b_j and c_j are literals of the form x_ℓ or $\neg x_\ell$ for some $\ell \in \{1, \ldots, m\}$. We construct in poly-time a priority separable game $\mathcal{G}_\varphi$ with the neighbourhood graph structure $G = (N, E)$ such that $\mathcal{G}_\varphi$ has a Nash equilibrium iff φ is satisfiable.

For every propositional variable x_ℓ where $\ell \in \{1, \ldots, m\}$, we add a player X_ℓ in $\mathcal{G}_\varphi$ with $S_{X_\ell} = \{\top, \bot\}$. With each clause $a_j \vee b_j \vee c_j$ for $j \in \{1, \ldots, k\}$, we associate 9 players whose neighbourhood is specified by the graph given in figure 3. The strategy set for each such node (or player) in the graph is specified as a label next to the node. We use x, y, z as variables where $x, y, z \in \{\top, \bot\}$ whose values are specified as part of the reduction. We denote this graph by $F_j(x, y, z)$ indicating that x, y and z are parameters.

For a literal d, let $\lambda(d) = \top$ if d is a positive literal and $\lambda(d) = \bot$ if d is a negative literal. For each clause with literals a_j, b_j and c_j, which is of the form x_ℓ or $\neg x_\ell$, we add to $\mathcal{G}_\varphi$ the subgraph $F_j(\lambda(a_j), \lambda(b_j), \lambda(c_j))$ and an edge from X_ℓ to the node A_j, B_j or C_j. Let $\Omega = \{0, 1\}$ with $0 \ll_i 1$ for all $i \in N$. For all $i, i' \in N$, we define $p_{i,i'} = 1$ if $s_i = s_{i'}$ and $p_{i,i'} = 0$ if $s_i \neq s_{i'}$. For each subgraph $F_j(x, y, z)$ corresponding to the clause $(a_j \vee b_j \vee c_j)$ and nodes A_j, B_j and C_j let $X_{\ell[A_j]}, X_{\ell[B_j]}, X_{\ell[C_j]}$ denote the nodes such that $(X_{\ell[A_j]}, A_j) \in E$, $(X_{\ell[B_j]}, B_j) \in E$ and $(X_{\ell[C_j]}, C_j) \in E$ respectively for $\ell[A_j], \ell[B_j], \ell[C_j] \in \{1, \ldots, m\}$. We specify the priority ordering for all players i with $|R(i)| > 1$ as follows. For each subgraph $F_j(x, y, z)$ we have,

- $X_{\ell[A_j]} \rhd_{A_j} 6_j \rhd_{A_j} 7_j \rhd_{A_j} C_j$.
- $X_{\ell[B_j]} \rhd_{B_j} 4_j \rhd_{B_j} 8_j \rhd_{B_j} A_j$.
- $X_{\ell[C_j]} \rhd_{C_j} 5_j \rhd_{C_j} 9_j \rhd_{C_j} B_j$.

The crucial observation used in the reduction is the following. Consider the subgraph H_j induced by the nodes in $F_j(x, y, z)$ for $j \in \{1, \ldots, k\}$ along the nodes $X_{\ell[A_j]}, X_{\ell[B_j]}, X_{\ell[C_j]}$. Consider the priority separable game $\mathcal{G}(H_j)$ induced by nodes in H_j and the neighbourhood structure specified by H_j. Observe that a strategy profile t in $\mathcal{G}(H_j)$ is a Nash equilibrium iff at least one of the following conditions hold: $t_{A_j} = t_{X_{\ell[A_j]}}$ or $t_{B_j} = t_{X_{\ell[B_j]}}$ or $t_{B_j} = t_{X_{\ell[B_j]}}$. Using this observation, we can argue that $\mathcal{G}_\varphi$ has a Nash equilibrium iff φ is satisfiable.

Suppose s is a Nash equilibrium in $\mathcal{G}_\varphi$. Consider the valuation function $v_s : \{x_1, \ldots, x_m\} \to \{\top, \bot\}$ defined as follows: $x_\ell = s_{X_\ell}$. From the observation above, it follows that for every $F_j(x, y, z)$ for $j \in \{1, \ldots, k\}$ at least one of the following conditions hold: $s_{A_j} = s_{X_{\ell[A_j]}}$ or $s_{B_j} = s_{X_{\ell[B_j]}}$ or $s_{B_j} = s_{X_{\ell[B_j]}}$. Assume without loss of generality that $s_{A_j} = s_{X_{\ell[A_j]}}$. By the definition of $\mathcal{G}_\varphi$, we have $S_{A_j} \cap S_{X_{\ell[A_j]}} = \{\lambda(a_j)\}$. By the definition of v_s we have $v_s(x_{\ell[A_j]}) = \lambda(a_j)$. This implies that $v_s \models a_j$ and therefore $v_s \models a_j \vee b_j \vee c_j$. Since this holds for all clauses, it follows that $v_s \models \varphi$.

Conversely, suppose φ is satisfiable and let $v \models \varphi$ for some valuation $v : \{x_1, \ldots, x_m\} \to \{\top, \bot\}$. Consider the partially defined strategy profile s^v where $s^v_{X_\ell} = v(x_\ell)$ for all $\ell \in \{1, \ldots, m\}$. Since $v \models \varphi$, for all clauses $a_j \vee b_j \vee c_j$, for

$j \in \{1, \ldots, k\}$ we have $v \models a_j$ or $v \models b_j$ or $v \models c_j$. Without loss of generality suppose $v \models a_j$. By definition of $\mathcal{G}_\varphi$ we have $S_{A_j} \cap S_{X_{\ell[A_j]}} = \{\lambda(a_j)\}$. Therefore, the unique best response for node A_j in the game $\mathcal{G}_\varphi$ is the strategy $\lambda(a_j)$. This holds for all clauses and therefore, it is possible to extend s^v to a strategy profile which is a Nash equilibrium in $\mathcal{G}_\varphi$.

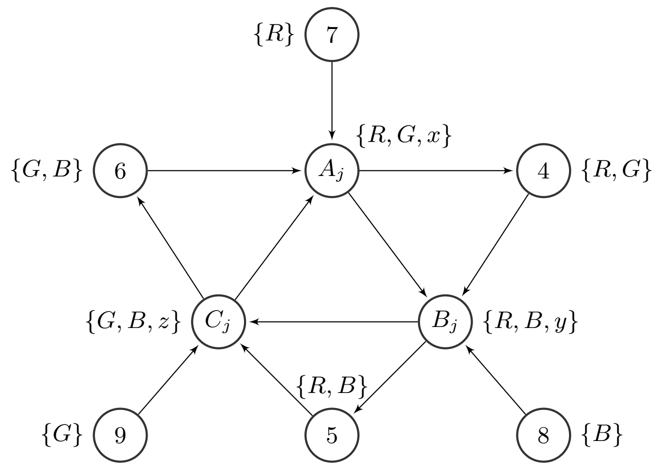

Fig. 3. Gadget $F_j(x, y, z)$

4 Monadic least fixed point logic

We now present Monadic least fixed point logic (*MLFP*) [33], the logical language we will use to reason about separable games. As fist-order logic cannot express properties like the transitive closure of a relation, its extension with the least fixed point operator, termed $FO(LFP)$ is used to describe path properties on graphs. *MLFP* is a monadic restriction of $FO(LFP)$ where the fixed-point operator can be applied only to unary relation variables. The model checking problem for *MLFP* over finite relational structures can be solved in time polynomial in the size of the model, for a fixed formula (in the sense of data complexity ([16])). The logic is also expressive enough to describe various interesting properties of games on finite graphs, as it can describe the transitive closure of a binary relation.

4.1 Syntax

Let $V = \{x_0, x_1, \ldots\}$ be a countable set of first-order variables, and $SV = \{S_0, S_1, \ldots\}$ be a countable set of second-order variables. These sets are disjoint.

A first-order vocabulary $\sigma = \{R_0^{k_0}, R_1^{k_1}, \ldots\}$ is a countable set of relation symbols R_i of arity $k_i > 0$.

The set of all *MLFP* formulas is defined inductively as follows:

$$\alpha \in \Phi_{MLFP} := R(x_1, \ldots, x_k) \mid x = y \mid S(x) \mid \neg\alpha \mid \alpha \wedge \alpha \mid \exists x \alpha \mid [\mathbf{lfp}_{S,x}\ \alpha](u)$$

where R is of arity k, $x_1, \ldots, x_k, x \in V$, $S \in SV$, u does not occur free in α, and all occurrences of S in α are *positive*.

The notion of free occurrence of a variable in a formula is standard. We say that an occurrence of a relation symbol or a second order variable in a formula is *positive* if it occurs under the scope of an even number of negations. (Otherwise we call the occurrence negative.) In the formula, $\phi(R) \stackrel{\text{def}}{=} \neg\forall x \forall y \neg(R(y) \wedge \neg R(x))$ the first occurrence of R is positive and the second is negative. This restriction is needed to provide semantics, where positivity ensures monotonicity of the associated operator and hence guaranteeing existence of a least fixed-point.

Note that in a formula $[\mathbf{lfp}_{S,x}\ \alpha](u)$ there can be free variables other than x (these are often called parameters).

4.2 Semantics

A σ-structure is a pair $\mathcal{D} = (A, \iota)$ where the non-empty domain of the structure is A, ι is an interpretation such that a relation symbol R of arity k is interpreted as a k-ary relation over A. A model is a tuple $M = (A, \iota, \rho_1, \rho_2)$ where (A, ι) is a σ-structure, and ρ_1, ρ_2 are interpretations of first order and second order variables respectively. For a first order variable x, $\rho_1(x) \in A$ and for a second order variable S, $\rho_1(S) \subseteq A$.

The notion that a formula α holds in a model M is defined inductively as follows:

- $M \models R(x_1, \ldots, x_k)$ iff the tuple $(\rho_1(x_1), \ldots, \rho_k(x_k)) \in \iota(R)$.
- $M \models x = y$ iff $\rho_1(x) = \rho_1(y)$.
- $M \models S(x)$ iff $\rho_1(x) \in \rho_2(S)$.
- $M \models \neg\alpha$ iff $M \not\models \alpha$.
- $M \models \alpha \wedge \beta$ iff $M \models \alpha$ and $M \models \beta$.
- $M \models \exists x.\alpha$ iff for some $a \in A$, $M[x \to a] \models \alpha$.
- $M \models [\mathbf{lfp}_{S,x}\alpha](u)$ iff $\rho_1(u) \in \mathrm{lfp}(f_\alpha)$ (where the map f_α is defined below).

Above, $M[x \to a]$ is the model variant $(A, \iota, \rho_1', \rho_2)$ where $\rho_1'(y) = \rho_1(y)$ for $y \neq x$ and $\rho_1'(x) = a$. Similarly, define $M[x \to a, S \to B]$ is the model variant $(A, \iota, \rho_1', \rho_2')$ where $\rho_1'(y) = \rho_1(y)$ for $y \neq x$ and $\rho_1'(x) = a$, and $\rho_2'(S') = \rho_2(S')$ for $S' \neq S$ and $\rho_2'(S) = B$.

For any formula with x and S occurring free in β, $f_\beta : \wp(A) \mapsto \wp(A)$ is defined by: $f_\beta(B) = \{a \in A \mid M[x \to a, S \to B] \models \beta\}$. The map f_β is an operator on the powerset of elements on the structure ordered by inclusion. The positivity restriction ensures that the operator is monotone and hence has a least fixed-point due to the Knaster-Tarski Theorem [27].

5 Expressing Nash equilibria in MLFP

We now consider priority separable games as models for the logic. This requires some interpreted relations, and accordingly in the syntax we use special atomic formulas. We will use many-sorted domains, with elements of different types: players, strategies, outcomes etc. Types will be specified by formulas: $P(x)$ would denote that the domain element denoted by x is a player, whereas $O(x)$ would similarly denote x as an outcome, and $T(x)$ denotes x as a strategy and so on.

Towards this, we fix $n > 0$ and consider only models from n-player games. Let $N = \{1, \ldots, n\}$. The *game vocabulary* is a tuple

$$\sigma_{\mathcal{G}} = (P^1, T^1, O^1, E^2, u^3, (S_i^1)_{i\in N}, \mathcal{S}^{n+2}, \mathcal{O}^{n+2}, \ll^3, \triangleright^2),$$

where P is intended to denote players. T for strategies, O for outcomes, E for edge relation on players, S_i for strategies of player i, u is intended to give the outcomes for pairwise interactions of players - thus it represents the pairwise utility relation, $\mathcal{S}$ for relating players and their strategies, $\mathcal{O}$ for relating players and their outcomes, $\ll$ for the preference on outcomes of players, and $\triangleright$ for the priority relation on players. The superscripts represent the arities of the relational symbols. In addition, we identify a subset of first order and second order *player variables* V_P and SV_P, respectively. We use p, q, p', q' etc to distinguish player variables and Q, Q' etc to designate player set variables.

Correspondingly, $\sigma_{\mathcal{G}}$ structures are interpreted as pairwise separable games. We abuse notation and use the same symbols for the interpreted relations as well. Thus $P \subseteq N$, $T, O \subseteq D$ (the domain that consisted of S_i and Ω) such that $T \cap O = \emptyset$, $E \subseteq (N \times N)$, $u \subseteq (N \times N \times O)$, and for each $i \in N$, $S_i \subseteq T$, $\mathcal{O} \subseteq (N \times O)$, $\ll \subseteq (N \times O \times O)$ and $\triangleright \subseteq (N \times N \times N)$. For variable assignments, we ensure that player variables map to N and player set variables map to subsets of N. However we use x, y etc to denote variables of all types.

We use the abbreviation $\boldsymbol{x}$ to denote the tuple of variables $(x_1, \ldots, x_n)$. The abbrevation $St(\boldsymbol{x})$ will be used for $\bigwedge_{j\in N} S_j(x_j)$, to denote strategy profiles. $Out(\boldsymbol{y})$ for $\bigwedge_{j\in N} O(y_j)$ for tuples of outcomes. We use $\mathcal{S}(\boldsymbol{x}, p, h)$ to denote that the player p chooses strategy h in profile $\boldsymbol{x}$, and $\mathcal{O}(\boldsymbol{a}, p, a)$ to denote that the player p gets outcome a in the vector of outcomes $\boldsymbol{a}$.

Our goal is to describe Nash equilibrium. A profile $\boldsymbol{x}$ is a Nash equilibrium if there is no other profile $\boldsymbol{y}$ such that $\boldsymbol{y}$ is an improvement over $\boldsymbol{x}$ for some player $i \in N$. Thus the formula we are looking for is $Imp(p, \boldsymbol{x}, \boldsymbol{y})$ to denote such an improvement for player p.

There are several steps involved in specifying a player i improvement, some routine, some tricky. For instance we need to specify that for other players, the strategy choices remain the same, which is routine. In each profile, from the pairwise interactions and player priorities we need to compute the outcomes for all players, which is tricky. Indeed this is where we use the least fixed-point operator. We now build the improvement formula, step-by-step.

We first describe formulas relating to the graph structure on players, their priorities and preferences over outcomes.

– p is the player in the neighbourhood of q with the highest priority:
first$(p,q) = E(p,q) \wedge (\forall q'(E(q,q') \implies \triangleright(p,q',q))$
– In the priority preference for player p, player q comes immediately before q':

$$\begin{aligned}\text{nxt}(p,q,q') = E(q,p) \wedge E(q',p) \wedge \triangleright(p,q,q') \wedge \forall p'((E(p',p) \wedge \triangleright(p,q,p') \wedge \\ \triangleright(p,p,q')) \implies (q = p' \vee p' = q'))\end{aligned}$$

The following formulas relate strategy profiles and outcomes.

– Player 1-step: this relates two strategy profiles that differ only in the strategy of player p:

$$\begin{aligned}\text{1-step}(p,\boldsymbol{x},\boldsymbol{y}) = St(\boldsymbol{x}) \wedge St(\boldsymbol{y}) \wedge (\mathcal{S}(\boldsymbol{x},p,g) \wedge \mathcal{S}(\boldsymbol{y},p,h)) \implies g \neq h \\ \wedge \forall q(q \neq p \wedge \mathcal{S}(\boldsymbol{x},q,g) \wedge \mathcal{S}(\boldsymbol{y},q,h)) \implies g = h\end{aligned}$$

– The outcome vector $\boldsymbol{y}$ is consistent with the pairwise utility relation u and the tuple of strategies $\boldsymbol{x}$:
$\text{Con}(\boldsymbol{x},\boldsymbol{y}) = St(\boldsymbol{x}) \wedge Out(\boldsymbol{y}) \wedge \bigwedge_{i \in N} \bigwedge_{j \neq i} u(x_i,x_j,y_j)$
– Fix $i,j \in N$. We want to specify that $p_{i,j}(\boldsymbol{x}) \ll_i p_{i,j}(\boldsymbol{y})$:

$$\begin{aligned}&\psi_1(p,q,\boldsymbol{x},\boldsymbol{a},\boldsymbol{y},\boldsymbol{b}) = \\ &\qquad Con(\boldsymbol{x},\boldsymbol{a}) \wedge Con(\boldsymbol{y},\boldsymbol{b}) \wedge (\mathcal{O}(\boldsymbol{a},q,a) \wedge \mathcal{O}(\boldsymbol{b},q,b)) \implies \ll(p,a,b)\end{aligned}$$

– For a player p the outcomes are equal in two outcome tuples:

$$\psi_2(p,\boldsymbol{a},\boldsymbol{b}) = Out(\boldsymbol{a}) \wedge Out(\boldsymbol{b}) \wedge (\mathcal{O}(\boldsymbol{a},p,a) \wedge \mathcal{O}(\boldsymbol{b},p,b)) \implies a = b$$

Given two outcome vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ we can now compare the outcome for player p using these formulas. Following this we can then write the formula $\text{Dev}(p,\boldsymbol{x},\boldsymbol{a},\boldsymbol{y},\boldsymbol{b})$ that states that $\boldsymbol{y}$ with outcome $\boldsymbol{b}$ is an improvement for player p from $\boldsymbol{x}$ with outcome $\boldsymbol{a}$ by a 1-step deviation.

$$\begin{aligned}&\text{Dev}(p,\boldsymbol{x},\boldsymbol{a},\boldsymbol{y},\boldsymbol{b}) = Con(\boldsymbol{x},\boldsymbol{a}) \wedge Con(\boldsymbol{y},\boldsymbol{b}) \wedge \\ &\quad \text{1-step}(p,\boldsymbol{x},\boldsymbol{y}) \wedge \exists v.\big([\mathbf{lfp}_{M,w}\alpha](p,v,\boldsymbol{x},\boldsymbol{a},\boldsymbol{y},\boldsymbol{b}) \wedge \psi_1(p,v,\boldsymbol{x},\boldsymbol{a},\boldsymbol{y},\boldsymbol{b}))\big) \\ &\quad \alpha(M,p,w,\boldsymbol{x},\boldsymbol{a},\boldsymbol{y},\boldsymbol{b}) = \text{first}(p,w) \vee \exists v\big[M(v) \wedge \text{nxt}(p,w,v) \wedge \psi_2(v,\boldsymbol{a},\boldsymbol{b})\big]\end{aligned}$$

The least fixed point formula helps in checking whether for player i has a lexicographic better payoff from strategy profile $\boldsymbol{x}$ to $\boldsymbol{y}$ by checking against $p_i^*(\boldsymbol{x}) \preceq^{lex} p_i^*(\boldsymbol{y})$. The formula α does the iteration for the lexicographic checking. As long as the partial payoffs are equal (which is why ψ_2 features there), it keeps accumulating the players from the neighbourhood set of p. The **lfp** computation on the operator due to the formula α, scans across the payoffs $p_i^*(\boldsymbol{x})$ and $p_i^*(\boldsymbol{y})$. When there is a mismatch in the payoffs the iteration halts or it accumulates all the vertices, which would mean that the payoffs are same.

Finally, in the Dev formula we have $\exists v \psi_1$ which checks if there is a player in the neighbourhood of i for whom there is a lexicographically greater outcome, and along with the other conditions set we are able to express that $\boldsymbol{x} \preceq_p \boldsymbol{y}$.

Now, the existence of Nash equilibrium can be characterised using the following formula:

– $\mathrm{Imp}(p, \boldsymbol{x}, \boldsymbol{y}) = (\ \exists \boldsymbol{b} \exists \boldsymbol{a} \wedge\ \mathrm{Dev}(p, \boldsymbol{x}, \boldsymbol{a}, \boldsymbol{y}, \boldsymbol{b}))$.
– $\mathrm{NE}(\boldsymbol{x}) = \forall p \forall \boldsymbol{y} \neg \mathrm{Imp}(p, \boldsymbol{x}, \boldsymbol{y})$.
– $\mathcal{G} \models \exists \boldsymbol{x}\ \mathrm{NE}(\boldsymbol{x})$

6 Discussion

In this paper we have studied Nash equilibria in priority separable games and their description in monadic least fixed point logic. Proceeding further, we would like to delineate bounds on the use of logical resources for game theoretic reasoning. For instance, one natural question is the characterization of equilibrium dynamics definable with at most one second order (fixed-point) variable. Moreover, delineating the precise complexity of the logic over games with unboundedly many players requires further work. In this context, it would be especially interesting to explore the framework of parameterized verification ([1]).

Strategy Logic ([12]) is a natural logical framework for description of Nash equilibria. It would be interesting to explore reasoning in such logics over subclasses of polymatrix games, especially in terms of the impact on the model checking problem ([29]).

An important direction is the study of infinite strategy spaces. Clearly the model checking algorithm needs a finite presentation of the input but this is possible and it is then interesting to explore convergence of fixed-point iterations.

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