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EN2570 Digital Signal Processing

Project Report

FIR Filter Design Project

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1 Abstract

Digital filters play a major role in a vast variety of fields. Thus, it is important to understand the fundamental concepts of filter design. In this project, the objective is to get hands on experience building a Finite Impulse Response(FIR) Filter.
In the scope of this project, an FIR filter will be designed using the windowing method in conjunction with the Kaiser window to meet the specifications given.

Table 1: Filter Specifications **(170ABC == 170221)**

Parameter	**Symbol**	**Relationship**	**Value**
Maximum passband ripple	$\tilde{A}_p$	$0.03 + (0.01 \times A)$ dB	**0.05 dB**
Minimum stopband attenuation	$\tilde{A}_a$	45+B dB	**47 dB**
Lower passband edge	ω_{p1}	$(C \times 100) + 400$ rad/s	**500 rad/s**
Upper passband edge	ω_{p2}	$(C \times 100) + 950$ rad/s	**1050 rad/s**
Lower stopband edge	ω_{a1}	$(C \times 100) + 500$ rad/s	**600 rad/s**
Upper stopband edge	ω_{a2}	$(C \times 100) + 800$ rad/s	**900 rad/s**
Sampling frequency	ω_s	$2[(C \times 100) + 1300]$ rad/s	**2800 rad/s**

2 Introduction

2.1 Window Method

Windowing method is one of the simplest ways in which a Finite Impulse Response(FIR) filter can be approximated. Equation (1) represents the **ideal desired frequency response.** It is clear that this is non-causal as well as infinite. Therefore, to get an FIR approximation, we need to truncate this ideal desired frequency response. One way to obtain such truncation is known as *windowing method.*

$$H_d\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} h_d[n]\ e^{-j\omega n} \tag{1}$$

We can obtain a causal FIR filter if we truncate $h_d[n]$ as described in the equation below.

$$h[n] \ = \ \begin{cases} h_d[n], & 0 \leqslant n \leqslant M \\ 0, & otherwise. \end{cases}$$

Equation (2), shows the *general* way how the ideal frequency response is truncated in the windowing method according to an **arbitrary window function w[n]**.

$$h[n] \ = \ h_d[n]\ w[n] \tag{2}$$

There are various window types/functions introduced over the years. The four window types listed below have only 1 adjustable parameter:*window size.*

- Hamming
- von Hann
- Blackman
- Barlett

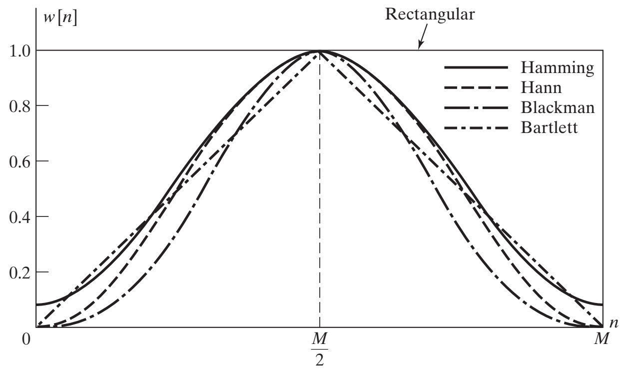


Figure 1: Commonly used windows $w[n]$ [1]

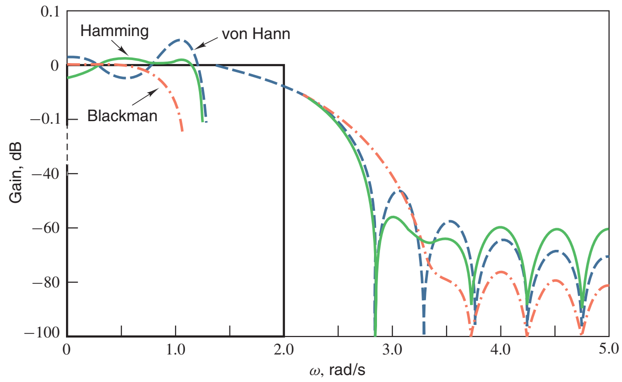


Figure 2: Amplitude response of common window types $w[n]$ [2]

The main problem with most of these windows is that, we cannot control the *peak approximation error (*$20 \log \delta$ *dB)* and the *transition width (*Δw*, Figure 3)* simultaneously. **Kaiser window method**, provides much convenience to overcome these issues. Compared to other window types, the kaiser window method has two parameters: *window length & β*

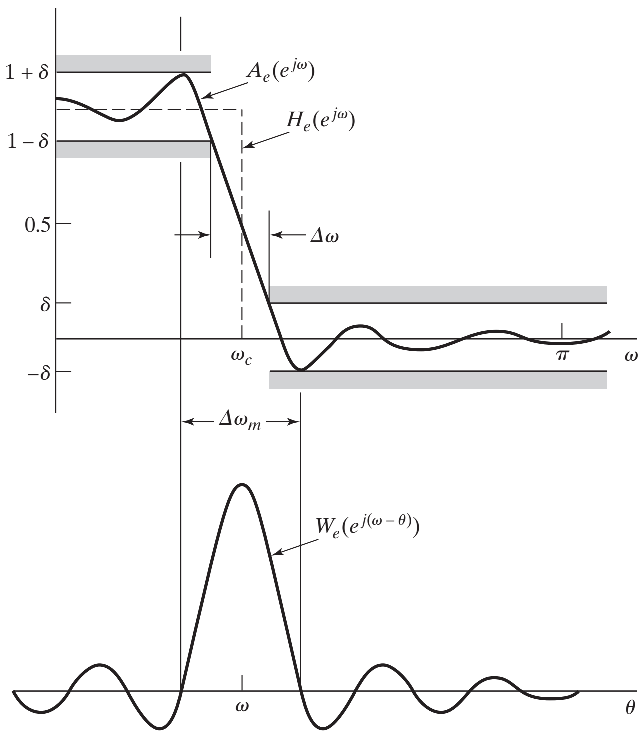


Figure 3: Approximation Errors [1]

2.2 Kaiser Window Method

Our intention when designing a FIR filter is to make the approximation very much closer to an ideal filter. However, the main two causes to approximation errors, *Main-lobe width & Side-lobe area* are very well linked to each other, which makes it hard to get an optimal filter design in a straightforward manner. Kaiser, was able to derive a near-optimal window design which can be formed using the zero$^{\text{th}}$ order Bessel Function of the first kind.

The Kaiser window is defined as,

$$w_K(nT) = \begin{cases} I_0(\beta)/I_0(\alpha) & for\ |n| \leq \frac{N-1}{2} \\ 0 & otherwise \end{cases}$$

Note: [1]

where α is an independent parameter and

$$\beta = \alpha\sqrt{1 - \left(\frac{2n}{N-1}\right)^2} \qquad I_0(x) = 1 + \sum_{k=1}^{\infty}\left[\frac{1}{k!}\left(\frac{x}{2}\right)^k\right]^2$$

As mentioned previously, in this windowing method there are two parameters to be configured.

1. Window size
2. β Parameter

As the name suggests, the Window size means the length (n) of the window. The β parameter however is not straightforward. β controls the side-lobe levels and the main-lobe width.[3] **when β increases, the side-lobe levels decreases. However, at the same time, the main lobe becomes wider, which is an adverse effect.**

[1]**It should be noted that N represents the Order of the filter. Therefore the length is N+1**

3 Kaiser Window - Design Steps

3.1 Deriving the Ideal Impulse Response

3.1.1 Calculating the Realistic Specifications

As this is a bandstop filter, there will be two transition widths. Therefore, we need to select the narrowest transition width.

$$B_{t1} = \omega_{a1} - \omega_{p1} = \mathbf{100\ rad/s}$$

$$B_{t2} = \omega_{p2} - \omega_{a2} = \mathbf{150\ rad/s}$$

$$\star\ \mathbf{B_t} \ = \ min(B_{t1}, B_{t2})$$

Thus, ω_{c1} & ω_{c2} becomes, $\omega_{c1} = \ \omega_{p1} + \frac{\mathbf{B_t}}{2}$, $\omega_{c2} = \ \omega_{p2} - \frac{\mathbf{B_t}}{2}$.

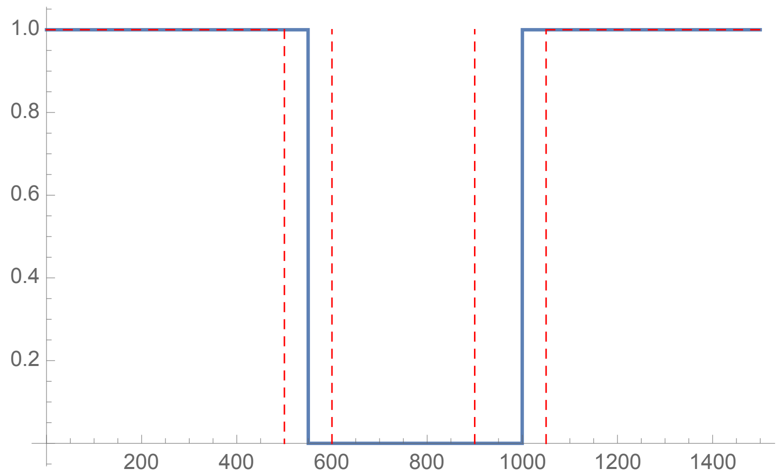


Figure 4: $H(e^{j\omega T})$ - Red dotted lines represent $\omega_{p1}, \omega_{a1}, \omega_{a2}, \omega_{p2}$

3.1.2 Ideal Impulse Response

The frequency response of an ideal bandstop filter with cutoff frequencies ω_{c1} and ω_{c2} is given by

$$H(e^{j\omega T}) = \begin{cases} 1 & for\ 0 \leq |\omega| \leq \omega_{c1} \\ 0 & for\ \omega_{c1} \leq |\omega| \leq \omega_{c2} \\ 1 & for\ \omega_{c2} \leq |\omega| \leq \frac{\omega_s}{2} \end{cases}$$

Using the inverse fourier transform we get,

$$h(nT) = \frac{1}{\omega_s} \int_{-\omega_s/2}^{\omega_s/2} H(e^{j\omega T}) e^{j\omega nT} d\omega$$

$$h(nT) = \frac{1}{\omega_s} \left[\int_{-\omega_s/2}^{-\omega_{c2}} e^{j\omega nT} d\omega + \int_{-\omega_{c1}}^{0} e^{j\omega nT} d\omega + \int_{0}^{\omega_{c1}} e^{j\omega nT} d\omega + \int_{\omega_{c2}}^{\omega_s/2} e^{j\omega nT} d\omega \right]$$

So the ideal impulse response $h(nT)$,

$$h(nT) = \begin{cases} 1 + \frac{2}{\omega_s}(\omega_{c1} - \omega_{c2}) & for\ n = 0 \\ \frac{1}{n\pi}(sin(\frac{\omega_s}{2}nT) + sin(\omega_{c1}nT) - sin(\omega_{c2}nT) & otherwise \end{cases}$$

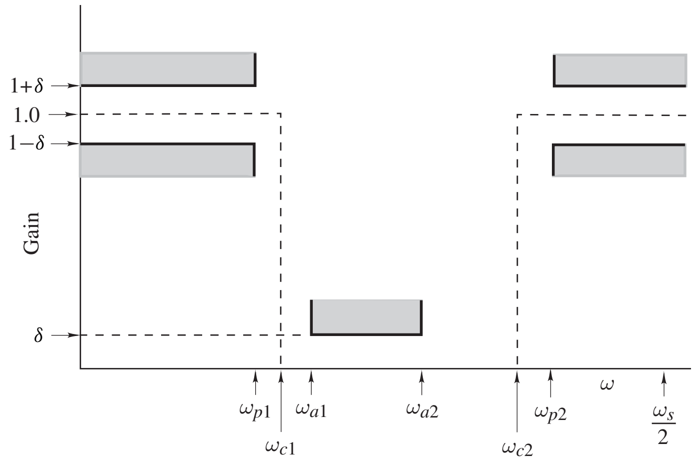


Figure 5: Intended Bandstop Filter Characteristics [2]

In summary we have the derived parameters,

Table 2: Derived Filter Specifications

Specification	Symbol	Derivation	Value	Units
Lower transition width	B_{t1}	$\omega_{a1} - \omega_{p1}$	100	rads^{-1}
Upper transition width	B_{t2}	$\omega_{p2} - \omega_{a2}$	150	rads^{-1}
Critical transition width	B_t	$min(B_{t1}, B_{t2})$	100	rads^{-1}
Lower cutoff frequency	ω_{c1}	$\omega_{p1} + \frac{B_t}{2}$	550	rads^{-1}
Upper cutoff frequency	ω_{c2}	$\omega_{p2} - \frac{B_t}{2}$	1000	rads^{-1}
Sampling period	T	$\frac{2\pi}{\omega_s}$	0.002244	s

3.2 Kaiser Parameters

3.2.1 Deriving δ

The passband amplitude response oscillates between (1 - σ) and (1 + σ). The stopband amplitude response oscillates between 0 and σ.
Therefore we are constrained under, **Maximum passband ripple $\tilde{\mathbf{A}}_{\mathbf{p}}$** and **Minimum stopband attenuation $\tilde{\mathbf{A}}_{\mathbf{a}}$**
Since these constraints are given in decibels, we need to establish a relationship between σ, A_p & A_a

$$A_a = -20log|\delta| \;\; (stopband\ attenuation) \;\Rightarrow\;\; \tilde{\delta_a} = 10^{-0.05\tilde{A_a}}$$

$$A_p = 20log\frac{1+\delta}{1-\delta} \;\; (passband\ ripple) \;\;\Rightarrow\; \tilde{\delta_p} = \frac{10^{0.05\tilde{A_p}} - 1}{10^{0.05\tilde{A_p}} + 1}$$

Because it might not be possible to achieve the Maximum passband ripple $\tilde{\mathbf{A}}_{\mathbf{p}}$ and Minimum stopband attenuation $\tilde{\mathbf{A}}_{\mathbf{a}}$ in exact value with the same δ, we build a filter that might oversatisfy one or both specifications. So we pick δ such that $A_p \leqslant \tilde{A_p}$ and $A_a \geqslant \tilde{A_a}$ conditions satisfy. Therefore,

$$\delta = min\left(\tilde{\delta_p}, \tilde{\delta_a}\right)$$

Minimum stopband attenuation $\mathbf{\tilde{A_a} = 47dB} \Rightarrow \tilde{\delta_a} = 0.00446684$
Maximum passband ripple $\mathbf{\tilde{A_p} = 0.05dB} \Rightarrow \tilde{\delta_p} = 0.00287822$

$$\therefore \delta = \mathbf{0.00287822}$$

3.2.2 Calculating Actual Stopband Loss A_a & Passband Ripple A_p

Now that we've found δ we can find the actual stopband loss/attenuation,

$$A_a = -20log|\delta| = -20log|0.00287822| = 50.8175$$

$$A_p = 20log|\frac{1 + 0.00287822}{1 - 0.00287822}|0 = 0.05$$

3.2.3 Choosing Parameter α

$$\alpha = \begin{cases} 0 & for\ A_a \leq 21dB \\ 0.5842(A_a - 21)^{0.4} + 0.07886(A_a - 21) & for\ 21 < A_a \leq 50dB \\ 0.1102(A_a - 8.7) & for\ A_a > 50dB \end{cases}$$

Since $A_a > 50dB$, $\alpha = 0.1102(A_a - 8.7) = 0.1102(50.8175 - 8.7) = 4.64135$

3.2.4 Choosing Parameter D & Window size N

$$D = \begin{cases} 0.9222 & for\ A_a \leq 21dB \\ \frac{A_a - 7.95}{14.36} & for\ A_a > 21dB \end{cases}$$

Since $A_a > 21dB$, $D = \frac{A_a - 7.95}{14.36} = \frac{50.8175 - 7.95}{14.36} = 2.9852$

N is chosen such that it is the smallest odd integer value satisfying the inequality

$$N \geq \frac{\omega_s D}{B_t} + 1 = \frac{2800rad/s * 2.9852}{100rad/s} + 1 = 84.5856$$

Thus, N = 85.

Table 3: Kaiser Parameters Summary

Parameter	Value	Units
δ	0.00287822	-
A_a	50.8175	dB
A_p	0.05	dB
α	4.64135	-
$I_0(\alpha)$	19.8032	-
D	2.9852	-
N	85	-

4 Results

4.1 Forming the window sequence $w_k(nT)$

$$w_K(nT) = \begin{cases} I_0(\beta)/I_0(\alpha) & for\ |x| \leq \frac{N}{2} \\ 0 & otherwise \end{cases}$$

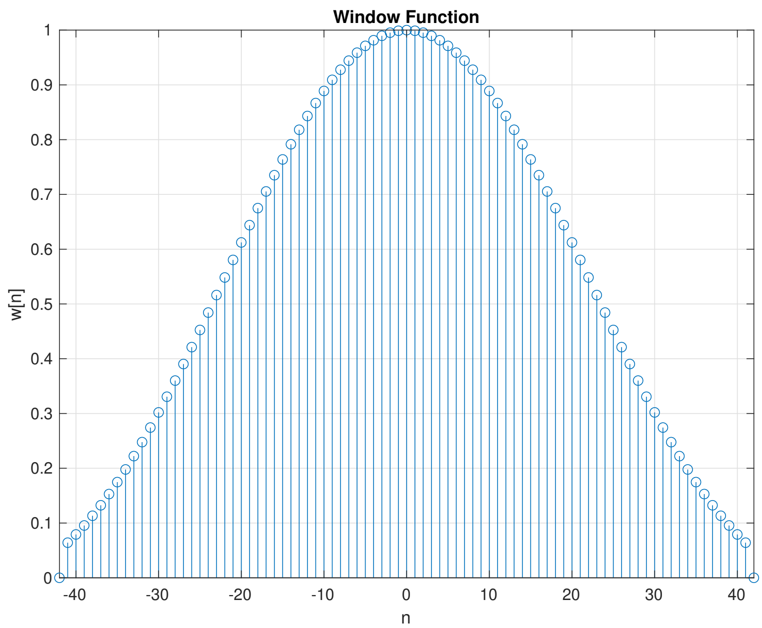


Figure 6: Constructed Kaiser Window

4.2 Impulse Response of Bandstop Filter

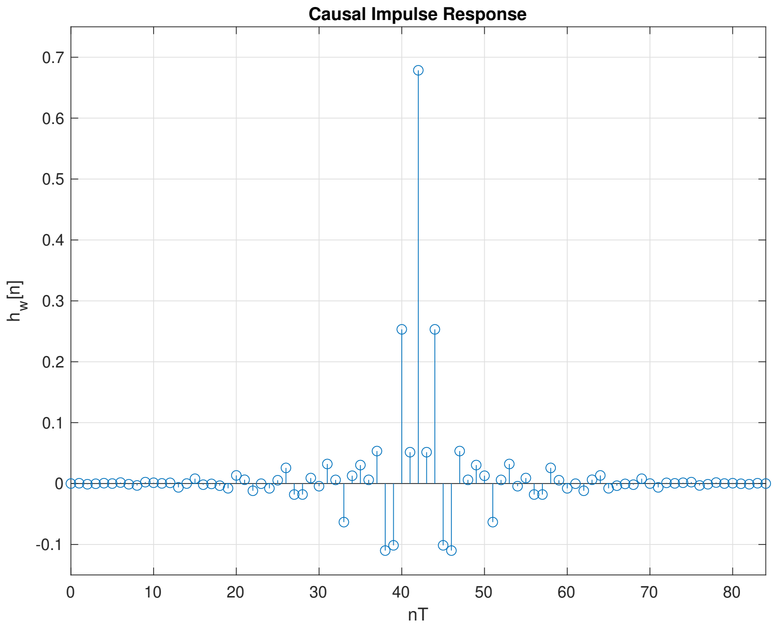


Figure 7: Causal Impulse Response

4.3 Magnitude Response of the Digital Filter

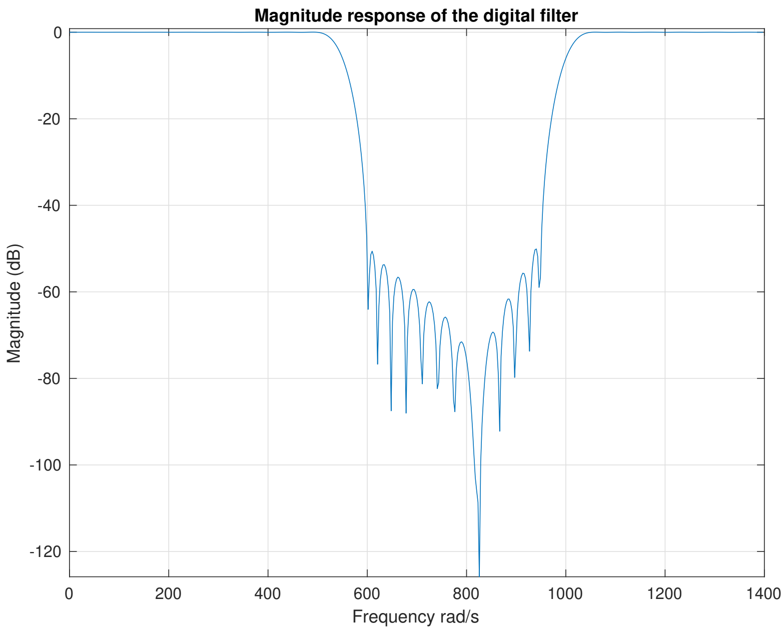


Figure 8: Magnitude Response of the BandStop Filter

4.4 Magnitude Responses of the Passbands

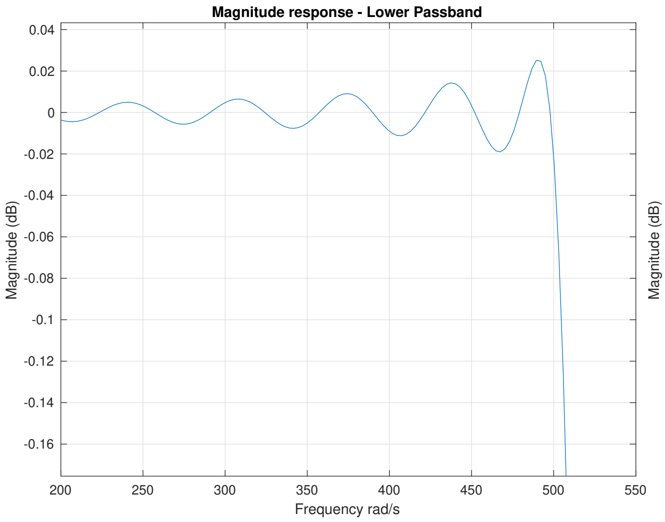


Figure 9: Upper passband (Zoomed)

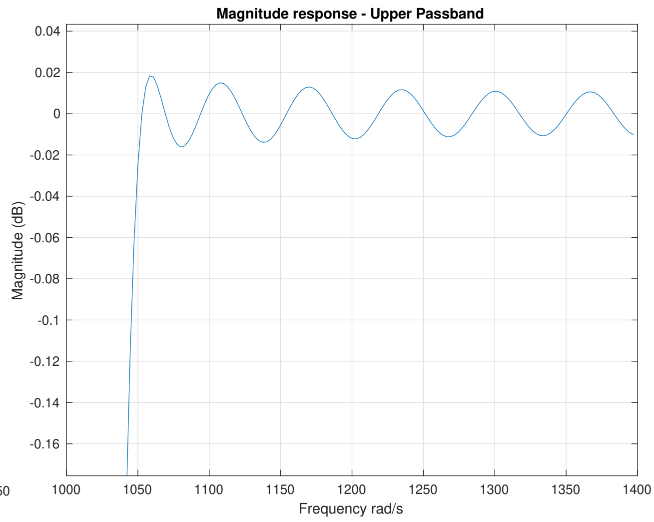


Figure 10: Lower passband (Zoomed)

4.5 Input Signal

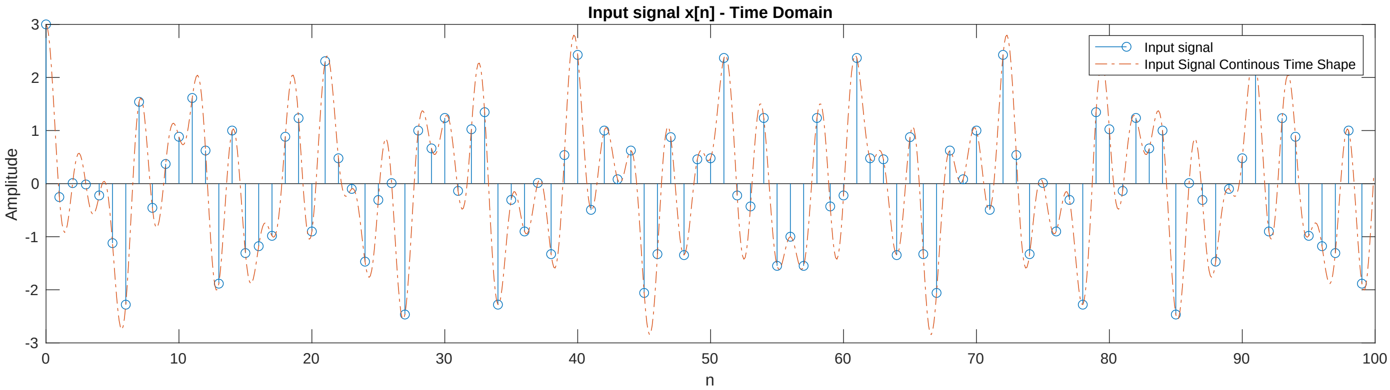
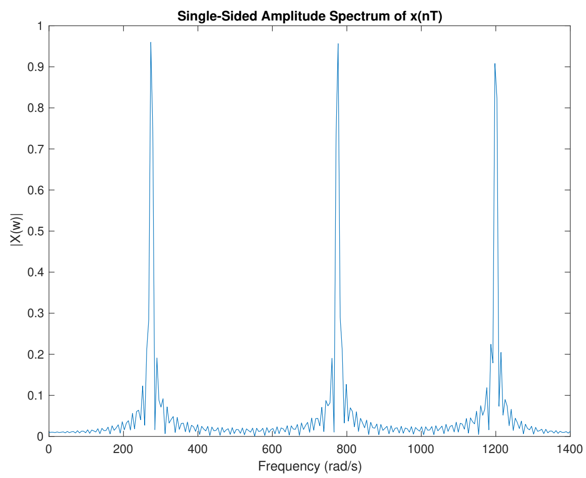


Figure 11: $x(nT) \ = \sum_{i=1}^{3} cos[\Omega_i nT]$



4.6 Filtered Signal

$$Input\ Signal:\quad x(nT) \ = \sum_{i=1}^{3} cos[\Omega_i nT]$$

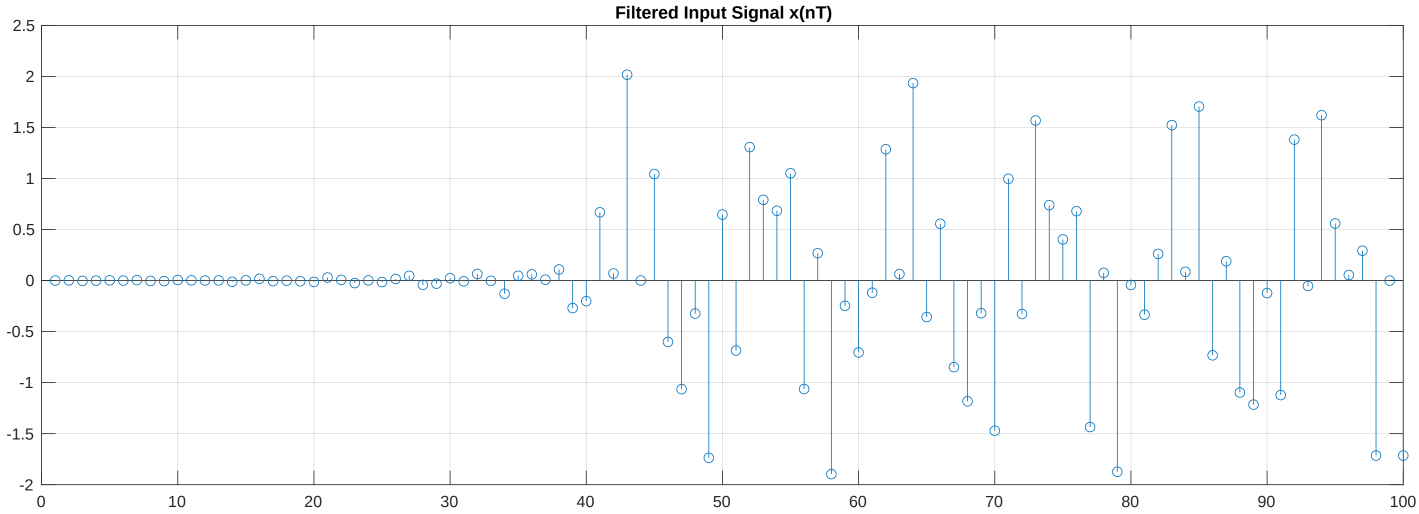


Figure 12: $x(nT) \ * \ h_w(nT)$

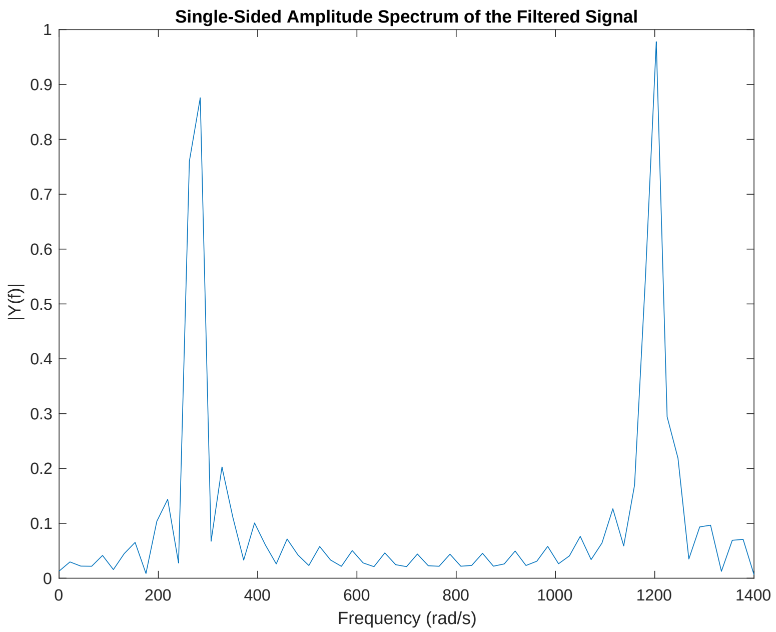


Figure 13: $\mathscr{F}\ (x(nT) \ * \ h_w(nT))$

4.7 Expected Outputs

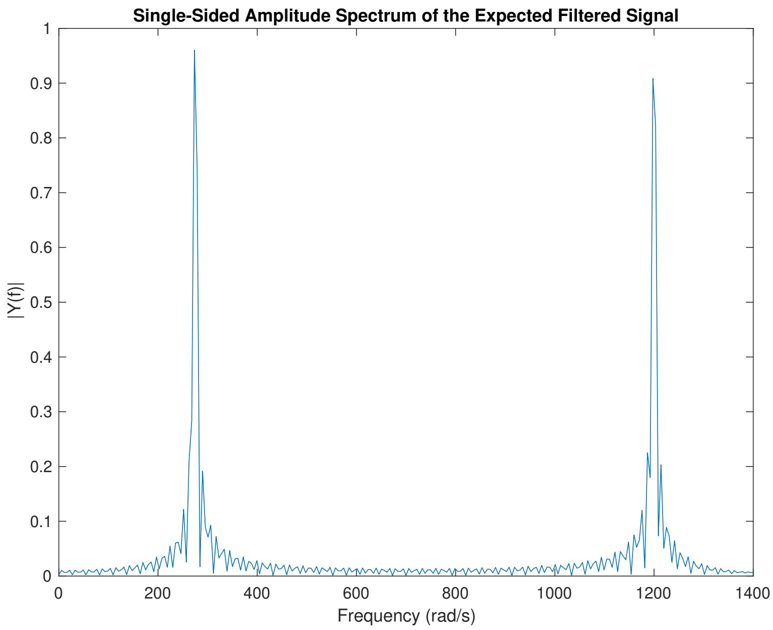


Figure 14: $\mathscr{F}\ (cos[\Omega_1 nT]\ +\ cos[\Omega_3 nT])$

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[1] Alan V. Oppenheim and Ronald W. Schafer. *Discrete-time signal processing.* Pearson, 2010.

[2] Andreas Antoniou. *Digital signal processing.* McGraw-Hill, 2006.

[3] Center for Computer Research in Music and Stanford University Acoustics (CCRMA). - kaiser window beta parameter.