NUMERICAL METHODS FOR SOLVING WAVE EQUATION

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INTRODUCTION TO WAVE EQUATION

A second order wave equation: $$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$

A second order 1-D wave equation: $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

A first order 1-D wave equation also known as 1-D linear convection equation:

$$\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0$$

SOLUTION OF CONVECTION EQUATION

Consider the pure initial value problem,

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

Initial condition: $\mathrm{u(x,0) = f(x)}, \quad -\infty < x < \infty$

Exact solution: $\mathrm{u(x\,,\,t) = f(x\;\text{-}\;c\;t)}$

EULER EXPLICIT METHOD

Converting PDE into FDE (FTCS):

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + C\left(\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}\right) = 0$$

or, $$u_j^{n+1} = u_j^n - \frac{1}{2}\left(\frac{C\Delta t}{\Delta x}\right)\left(u_{j+1}^n - u_{j-1}^\eta\right)$$

Putting $\frac{C\Delta t}{\Delta x} = r$ we get,

Or, $$\boxed{u_j^{n+1} = u_j^n - \frac{1}{2}r\left(u_{j+1}^n - u_{j-1}^\eta\right)} \qquad (1)$$

VON NEUMANN STABILITY ANALYSIS OF EULER EXPLICIT METHOD

Using Von Neumann stability analysis, we can put in (1)

$$u_j^n = G^n e^{ikj\Delta x}$$

Where G is growth factor, k is wave number and $i = \sqrt{-1}$

A method is stable if $|G| \leq 1$

Equation (1) reduces to,

$$G^{n+1} e^{ikj\Delta x} = G^n e^{ikj\Delta x} - \frac{r}{2}\left(G^n e^{ik(j+1)\Delta x} - G^n e^{ik(j-1)\Delta x}\right)$$

Or, $G = 1 - \frac{r}{2}\left(e^{ik\Delta x} - e^{-ik\Delta x}\right)$

VON NEUMANN STABILITY ANALYSIS OF EULER EXPLICIT METHOD(CONT.)

Using the relation $e^{i\theta} - e^{-i\theta} = 2i \sin\theta$

we get, $G = 1 - ir\sin(k\Delta x)$

So,

$$(|G|)^2 = GG^* = \{1 - ir\sin(k\Delta x)\}\{1 + ir\sin(k\Delta x)\}$$

$$= 1 + r^2 \sin^2(k\Delta x)$$

So, $G \geq 1$

Thus Euler Explicit Method for convection equation is unconditionally unstable.

UPSTREAM (FIRST ORDER UPWIND OR WINDWARD) DIFFERENCING METHOD

Changing Forward space to Backward space in Euler explicit method:

Forward time backward space (FTBS):

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + c\,\frac{u_j^n - u_{j-1}^n}{\Delta x} = 0$$

Or, $$\boxed{u_j^{n+1} = (1 - r)u_j^n + r u_{j-1}^n} \qquad (2)$$

This method is conditionally stable. Using Von Neumann stability analysis we get

$$0 \le r \le 1$$

LAX METHOD

Euler Explicit Method: $u_j^{n+1} = u_j^n - \frac{1}{2}r\left(u_{j+1}^n - u_{j-1}^n\right)$

Substituting $u_j^n = \left(u_{j+1}^n + u_{j-1}^n\right)/2$ in Euler Explicit equation we get,

$$u_{j+1}^n = \frac{1}{2}\left(u_{j+1}^n + u_{j-1}^n\right) - \frac{r}{2}\left(u_{j+1}^n - u_{j-1}^n\right)$$

or, $$\boxed{u_j^n = \frac{1}{2}\left\{(1-r)u_{j+1}^n + (1+r)u_{j-1}^n\right\}} \qquad (3)$$

VON NEUMANN STABILITY ANALYSIS OF LAX METHOD

Using Von Neumann stability analysis we get,

$$G^{n+1}e^{ikj\Delta x} = \frac{1}{2}\left\{(1-r)G^n e^{ik(j+1)\Delta x} + (1-r)G^n e^{ik(j-1)\Delta}\right\}$$

Or, $$G = \frac{1}{2}\left\{(1-r)e^{ik\Delta x} + (1+r)e^{-ik\Delta x}\right\}$$

Or, $$G = \frac{1}{2}\left\{\left(e^{ik\Delta x} + e^{-ik\Delta x}\right) - r\left(e^{ik\Delta x} - e^{-ik\Delta x}\right)\right\}$$

Or, $$G = \cos(k\Delta x) - ir\sin(k\Delta x)$$

VON NEUMANN STABILITY ANALYSIS OF LAX METHOD (CONT.)

Now, $|G|^2 = \cos^2(k\Delta x) + r^2 \sin^2(k\,\Delta x)$

Or, $|G|^2 = 1 - (1 - r^2)\sin^2(k\Delta x)$

So, for stability, $1 - r^2 \geq 0$ or, $r \leq 1$

So Lax method is stable for $\boxed{r \leq 1 \Rightarrow \frac{C\Delta t}{\Delta x} \leq 1}$

Lax method is conditionally stable

EULER IMPLICIT METHOD

For implicit method Difference (FTCS) equation is :

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + C\left(\frac{u_{j+1}^{n+1} - u_{j-1}^{n+1}}{2\Delta x}\right) = 0$$

Or,

$$\boxed{\frac{r}{2}u_{j+1}^{n+1} + u_j^{n+1} - \frac{r}{2}u_{j-1}^{n+1} = u_j^n} \qquad (4)$$

Here only u_j^n is known. To find the others values we have to solve system of linear equation.

VON NEUMANN STABILITY ANALYSIS OF EULER IMPLICIT METHOD

Using Von Neumann stability analysis we get,

$$\frac{r}{2}G^{n+1}e^{ik(j+1)\Delta x} + G^{n+1}e^{ikj\Delta x} - \frac{r}{2}G^{n+1}e^{ik(j-1)\Delta x} = G^{n}e^{ijk\Delta x}$$

Or, $\frac{r}{2}Ge^{ik\Delta x} + G - \frac{r}{2}Ge^{ik\Delta x} = 1$

Or, $G\left\{1 + \left(e^{ik\Delta x} - e^{-ik\Delta x}\right)\frac{r}{2}\right\} = 1$

Or, $G\{1 + i\, r\sin(k\Delta x)\} = 1$

Or, $G = \frac{1}{1+ir\sin(k\Delta x)}$

So, $|G|^2 = \frac{1}{1+r^2\sin^2(k\Delta x)} \leq 1$

Thus the Euler implicit method is unconditionally stable

LEAP FROG METHOD

Taking central time and central space difference formula we get equation for Leap Frog scheme.

$$\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} + C\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = 0$$

Or, $\boxed{u_j^{n+1} = u_j^{n-1} - r\left(u_{j+1}^n - u_{j-1}^n\right)}$ (5)

Leap Frog method is stable for $|r| \leq 1$ and It is second order T.E scheme.

Leap Frog method is conditionally stable.

CRANK-NICOLSON METHOD

The difference equation for convection equation using Crank-Nicolson method is given by:

$$\frac{u_j^{n+1}-u_j^n}{\Delta t}+c\,\frac{1}{2}\left[\frac{u_{j+1}^{n+1}-u_{j-1}^{n+1}}{2\Delta x}+\frac{u_{j+1}^{n}-u_{j-1}^{n}}{2\Delta x}\right]=0$$

Or,

$$\boxed{\frac{1}{4}r\,u_{j-1}^{n+1}-u_j^{n+1}-\frac{1}{4}r\,u_{j-1}^{n+1}=-u_j^n+\frac{1}{4}r\left(u_{j+1}^{\eta}-u_{j-1}^{n}\right)} \qquad (6)$$

It is a second order accuracy method.

VON NEUMANN STABILITY ANALYSIS CRANK-NICOLSON METHOD

Using Von Neumann stability analysis, we get:

$$\frac{1}{4}r\, G^{n+1}\, e^{ik(j-1)\Delta x} - G^{n+1}\, e^{ikj\Delta x} - \frac{1}{4}r\, G^{n+1}\, e^{ik(j+1)\Delta x} = -G^{n} e^{ikj\Delta x} + \frac{1}{4}r(G^{n} e^{ik(j+1)\Delta x} - G^{n} e^{ik(j-1)\Delta x})$$

Or, $-G\left(1 + \frac{r}{4}(e^{ik\Delta x} - e^{-ik\Delta x})\right) = -(1 - \frac{r}{4}(e^{ik\Delta x} - e^{-ik\Delta x}))$

Or, $G = \dfrac{1 - \frac{ir}{2}\sin\phi}{1 + \frac{ir}{2}\sin\phi}$; where $\phi = k\Delta x$

VON NEUMANN STABILITY ANALYSIS FOR CRANK-NICOLSON METHOD(CONT.)

Or, $G = \dfrac{\left(1-\frac{r^2 \sin^2 \phi}{4}\right)-ir \sin \phi}{1+\frac{r^2 \sin^2 \phi}{4}}$

So, $$\boxed{|G|^2 = \frac{1}{1+\frac{r^2 \sin^2 \phi}{4}} \leq 1}$$

Thus Crank-Nicolson Method is unconditionally stable.

REFERENCE

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- Computational Fluid Dynamics Volume I by Klaus A. Hoffmann, Steve T. Chiang

Thank You