Mean-Payoff Games and the Max-Atom Problem

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Abstract

The max-atom problem asks for the satisfiability of a system of inequality constraints of the type $x \leq \max(y, z) + c$, where x, y and z are integer variables, and c is an integer constant. We observe that this problem is polynomial-time equivalent to solving mean-payoff games, and therefore at least as hard as solving parity games.

1 Introduction

A max-atom is an inequality constraint of the form $x \leq \max(y, z) + c$, where x, y and z are variables ranging over the integers, and c is an integer constant called offset. Motivated by potential applications in hardware verification, Bezem, Nieuwenhuis and Rodríguez-Carbonell [1] introduced the *max-atom problem*: given a system of max-atoms, is there an assignment of integer values to the variables that satisfies all inequalities? The problem contains as a special case the satisfiability of systems of inequalities of the form $x \leq y + c$. This particular case is a well-studied fragment of linear arithmetic called *difference logic* that has several applications.

Unlike in the special case of difference logic, a polynomial-time algorithm for the max-atom problem is not known. The authors of [1] gave strong evidence that the problem is not NP-hard by showing that it belongs to NP $\cap$ co-NP. They also gave evidence that the problem might not be easy by showing that it is polynomial-time equivalent to a 30-year old problem in control theory, for which polynomial-time algorithms are not known either. This gives the max-atom problem an interesting status shared only by a few other problems. Bezem at al. mention a possible connection between the max-atom problem and solving simple stochastic games. This is a well-known path-forming game on graphs, whose complexity is also between P and NP $\cap$ co-NP [3]. The problem of solving parity games is another famous example with similar complexity status that has many applications in logic and automata.

We show in this note that the max-atom problem is polynomial-time equivalent to solving mean-payoff games. This is yet another path-forming game, introduced by Ehrenfeucht and Mycielsky [5], whose complexity is known to lie between parity and simple stochastic games. A *path-forming game* comes specified by a directed graph $G = (V, E)$ in which every vertex has positive outdegree, and by two disjoint sets of vertices V_0 and V_1. The game is played by two players called 0 and 1 and starts at an initial vertex u_0. After $t \geq 0$ moves have been made, if u_t belongs to V_i, with $i \in \{0, 1\}$, player i chooses u_{t+1} in such a way that (u_t, u_{t+1}) belongs to E. On the other hand, if u_t belongs to $V - (V_0 \cup V_1)$, it is *nature* who chooses the next vertex u_{t+1}, uniformly at random among all those for which (u_t, u_{t+1}) belongs to E. When all vertices of V belong to either V_0 or V_1 the game is called deterministic. Otherwise it is called stochastic.

Mean-payoff games are deterministic. The edges are labeled by an integer weight assignment $w : E \to \{-W, \dots, 0, \dots, W\}$. The goal of player 0 is to maximize the long-run average

$$\liminf_{t\to\infty} \frac{1}{t} \sum_{i=1}^{t} w(u_{i-1}, u_i)$$

of the weight of the walk. The goal of player 1 is to minimize it. Parity games are also deterministic. In this case the vertices a labeled by an integer priority assignment $p : V \to \{0, \dots, k-1\}$. The goal of player 0 is to ensure that the largest priority that appears infinitely often in $p(u_0), p(u_1), \dots$ is even. The goal of player 1 is to ensure that it is odd. Finally, in simple stochastic games, every vertex in $V - (V_0 \cup V_1)$ has outdegree two and the goal of the players is to maximize the probability of reaching corresponding target vertices t_0 and t_1. In all three cases, solving the game means determining whether player 0 has a strategy that ensures her a win. In the case of mean-payoff games, a win is a non-negative long-run average. In the case of parity games, a win is an even largest ultimate priority. And in the case of simple stochastic games, a win is getting probability at least $1/2$ of reaching the target vertex t_0.

Jurdziński [6] reduces solving parity games to solving mean-payoff games. Zwick and Paterson [9] reduce solving mean-payoff games to solving simple stochastic games. Thus, mean-payoff games lie in between. The correctness of both these reductions rely on a key property of the games which states that the optimal strategies for the players can be chosen *memoryless*. This means that the choice made by the players at each stage depends only on the current vertex u_t, and not on the path that led to u_t. This property of games, called *memoryless determinacy*, was first established for mean-payoff games by Ehrenfeucht and Mycielsky [5] and has been revisited several times (see, for example, [2]). It is the key step in showing that the complexity of the problems is in NP $\cap$ co-NP. Here we show that the max-atom problem and mean-payoff games are polynomial-time equivalent without relying on memoryless determinacy. In fact, our proof shows that general strategies can be replaced by memoryless ones as a consequence of a simple lemma on the structure of unsatisfiable max-atom systems. This structure lemma was introduced by Bezem et al. to show that their problem is in NP $\cap$ co-NP. Here we offer a refined version of their lemma with an equally simple proof.

2 Problems

MAX ATOM: A max-atom inequality has the form $z_0 \leq \max(z_1, z_2) + c$, where z_0, z_1 and z_2 are variables ranging over the integers $\mathbb{Z}$, and c is an integer constant in $\mathbb{Z}$. The problem MAX ATOM is the following:

> Given a system of max-atom inequalities S, determine if S is satisfiable over $\mathbb{Z}$.

MAX MIN OFFSET OPERATOR: A max offset operator is a function $F : \mathbb{Z}^n \to \mathbb{Z}$ defined by $(x_1, \dots, x_n) \mapsto \max\ \{x_j + c_j : j \in I\}$ for some non-empty $I \subseteq \{1, \dots, n\}$, where c_j is an integer constant in $\mathbb{Z}$ for every $j \in I$. A min offset operator is the same with min replacing max. In both cases, the arity of the operator is n. A system of operators is a function $F : \mathbb{Z}^n \to \mathbb{Z}^n$ defined by

$(x_1, \ldots, x_n) \mapsto (F_1(x_1, \ldots, x_n), \ldots, F_n(x_1, \ldots, x_n))$, where each F_i is a max or min offset operator of the same arity n. The problem MAX MIN OFFSET OPERATOR is this:

Given a system of operators $F : \mathbb{Z}^n \to \mathbb{Z}^n$, determine if $\mathbf{x} \leq F(\mathbf{x})$ is satisfiable over $\mathbb{Z}$.

Max-min offset operators, exactly as defined here, are the objects of study in a subarea of control and decision theory called max-plus algebra (see [8]). The connection between max-min offset operators and mean payoff games, as we will use it later, appears in [9] and has been revisited more recently by Dhingra and Gaubert [4].

MEAN-PAYOFF GAME: A mean-payoff game is a path-forming game specified by three components: a directed graph $G = (V, E)$ in which every vertex has positive outdegree, a partition $V = V_0 \cup V_1$ of the vertices of G, and a weight assignment $w : E \to \{-W, \ldots, 0, \ldots, W\}$ to the edges of G, where W is a positive integer. A strategy for player $i \in \{0, 1\}$ is a mapping $s_i : V^* \times V_i \to V$ such that $(v, s_i(u, v))$ belongs to E for every $u \in V^*$ and every $v \in V_i$. The play determined by the starting vertex u and the strategies s_0 and s_1 is the sequence $u_0, u_1, \ldots$ defined inductively as $u_0 = u$ and $u_{t+1} = s_i(u_0 \ldots u_{t-1}, u_t)$ if $u_t \in V_i$. The outcome of the play, denoted by $\nu(u, s_0, s_1)$, is $\liminf_{t\to\infty} \frac{1}{t} \sum_{i=1}^{t} w(u_{i-1}, u_i)$. The value of the game at vertex u, denoted by $\nu(u)$, is the supremum over all strategies s_0 for player 0 of the infimum over all strategies s_1 for player 1 of $\nu(u, s_0, s_1)$. The problem MEAN-PAYOFF GAME is this:

Given a mean-payoff game, determine if $\nu(u) \geq 0$ for every starting vertex u.

In all problems above, integers are represented in binary notation.

3 Reductions

MAX ATOM $\leq$ MAX MIN OFFSET OPERATOR:
To every system of max-atom inequalities S we associate a system of operators F_S. Let Z denote the set of variables of S, and let E denote the set of inequalities of S. We use $I = Z \cup E$ as an index set for tuples as in $(x_i : i \in I)$. For every z in Z, let E_z denote the set of inequalities $z_0 \leq \max(z_1, z_2) + c$ in E with $z_0 = z$. To every variable z in Z we associate a min operator $F_z : \mathbb{Z}^I \to \mathbb{Z}$ defined by $(x_i : i \in I) \mapsto \min\ \{x_e : e \in E_z\}$. To every e in E of the form $z_0 \leq \max(z_1, z_2) + c$ we associate a max operator $F_e : \mathbb{Z}^I \to \mathbb{Z}$ defined by $(x_i : i \in I) \mapsto \max\ \{x_{z_1} + c, x_{z_2} + c\}$. Finally, $F_S : \mathbb{Z}^I \to \mathbb{Z}^I$ is the system $(F_i : i \in I)$.

MAX MIN OFFSET OPERATOR $\leq$ MEAN-PAYOFF GAME:
To every system of operators F we associate a mean-payoff game $(G_F, V_{F,0}, V_{F,1}, w_F)$. Let $F = (F_1, \ldots, F_n)$ with F_i given by $(x_1, \ldots, x_n) \mapsto M_i\ \{x_j + c_{i,j} : j \in I_i\}$, where M_i is either max or min and the $c_{i,j}$ are integer constants. Let $G_F = (V_F, E_F)$ be the directed graph whose set of vertices V_F is $\{1, \ldots, n\}$, with an edge (i, j) in E_F if j belongs to I_i. The partition of the vertices $V_F = V_{F,0} \cup V_{F,1}$ is defined as follows: if $M_i = \max$, put i in $V_{F,0}$, otherwise put it in $V_{F,1}$. The weight function $w_F : E_F \to \{-W, \ldots, 0, \ldots, W\}$ is defined by $w_F(i, j) = c_{i,j}$ for $j \in I_i$. Here W is the maximum absolute value of all $c_{i,j}$.

MEAN-PAYOFF GAME $\leq$ MAX MIN OFFSET OPERATOR:
To every mean-payoff game (G, V_0, V_1, w) we associate a system of operators $F_{G,V_0,V_1,w}$. Let $G = (V, E)$ and let $n = |V|$. For every $u \in V$, let $F_u : \mathbb{Z}^V \to \mathbb{Z}$ be the operator defined by $(x_i : i \in V) \mapsto M_u \ \{x_v + w(u,v) : (u,v) \in E\}$, where $M_u = \max$ if $u \in V_0$ and $M_u = \min$ otherwise. Finally, let $F_{G,V_0,V_1,w} = (F_i : i \in V)$.

MAX MIN OFFSET OPERATOR $\leq$ MAX ATOM:
To every system of operators F we associate a system of max-atom inequalities S_F. Let $F = (F_1, \ldots, F_n)$ with F_i given as $(x_1, \ldots, x_n) \mapsto M_i \ \{x_j + c_{i,j} : j \in I_i\}$, where M_i is either max or min and the $c_{i,j}$ are integer constants. For every $i \in \{1, \ldots, n\}$, we introduce one variable z_i. For every i for which $M_i = \max$, we want to impose the constraint $z_i \leq \max \ \{z_j + c_{i,j} : j \in I_i\}$. If $I_i = \{j_0\}$, this is achieved by the max-atom inequality $z_i \leq \max(z_{j_0}, z_{j_0}) + c_{i,j_0}$. If $I_i = \{j_0, \ldots, j_h\}$ with $h \geq 1$, we introduce one more variable z and impose the system

$$\begin{array}{rcl} z_i & \leq & \max(z, z_{j_0}) + c_{i,j_0} \\ z & \leq & \max \ \{z_j + c_{i,j} - c_{i,j_0} : j \in I_i - \{j_0\}\}. \end{array}$$

Note the linear recursion on $|I_i|$ underlying this construction. For every i for which $M_i = \min$, we want to impose the constraint $z_i \leq \min \ \{z_j + c_{i,j} : j \in I_i\}$. If $I_i = \{j_0\}$, this is simply $z_i \leq \max(z_{j_0}, z_{j_0}) + c_{i,j_0}$. If $I_i = \{j_0, \ldots, j_h\}$ with $h \geq 1$, we impose the system

$$\begin{array}{rcl} z_i & \leq & \max(z_{j_0}, z_{j_0}) + c_{i,j_0} \\ z_i & \leq & \min \ \{z_j + c_{i,j} : j \in I_i - \{j_0\}\}. \end{array}$$

Again, note the linear recursion on $|I_i|$ underlying this construction.

4 Proofs

MAX ATOM $\leq$ MAX MIN OFFSET OPERATOR:

It follows directly from the definition of the reduction that S is satisfiable if and only if $\mathbf{x} \leq F_S(\mathbf{x})$ is satisfiable.

MAX MIN OFFSET OPERATOR $\leq$ MEAN-PAYOFF GAME:

Let $G_F = (V, E)$ be the game graph produced by the reduction. We need to show that $\mathbf{x} \leq F(\mathbf{x})$ is satisfiable if and only if $\nu(u) \geq 0$ for every $u \in V$. The forward implication is proved in the following lemma:

Lemma 1. *If $\mathbf{x} \leq F(\mathbf{x})$ is satisfiable, then $\nu(u) \geq 0$ for every $u \in V$.*

Proof. Let $\mathbf{x} \in \mathbb{Z}^n$ be such that $\mathbf{x} \leq F(\mathbf{x})$. For every i for which F_i is a max operator, let $s(i) \in I_i$ be such that

$$x_{s(i)} + c_{i,s(i)} = \max \ \{x_j + c_{i,j} : j \in I_i\}. \tag{1}$$

By the construction of G_F, we may think of s as a (memoryless) strategy s_0 for player 0. We claim that for every $u \in V$ and every strategy s_1 for player 1, memoryless or not, the outcome of the play determined by u, s_0 and s_1 is non-negative. Fix u and s_1 and let $u_0, u_1, \ldots$ be the play. In order to simplify notation, write y_t for x_{u_t}. We want to prove that the following inequality holds:

$$y_0 \leq y_t + \sum_{i=1}^{t} w(u_{i-1}, u_i). \tag{2}$$

Let E_t denote statement (2). We will prove that E_t holds by induction on t. The base case $t = 0$ is obvious because the sum on the right is vacuous. Assume now that $t \geq 0$ and that E_t holds; we prove E_{t+1}. Suppose first that u_t belongs to V_1. From the facts that $\mathbf{x} \leq F(\mathbf{x})$ and that (u_t, u_{t+1}) is an edge of the game graph we get

$$y_t \leq \min \{x_v + w(u_t, v) : v \in I_{u_t}\} \leq y_{t+1} + w(u_t, u_{t+1}). \tag{3}$$

Combining E_t and (3) we get E_{t+1}. Suppose next that u_t belongs to V_0. From $\mathbf{x} \leq F(\mathbf{x})$ and the choice in (1) we get

$$y_t \leq \max \{x_v + w(u_t, v) : v \in I_{u_t}\} = y_{t+1} + w(u_t, u_{t+1}). \tag{4}$$

Combining E_t and (4) we get E_{t+1}. This completes the proof of (2). Dividing through by t and taking lim inf on both sides we conclude that

$$\liminf_{t\to\infty} \frac{1}{t} y_0 \leq \liminf_{t\to\infty} \left(\frac{1}{t} y_t + \frac{1}{t} \sum_{i=1}^{t} w(u_{i-1}, u_i) \right).$$

As y_0 is a fixed integer, the lim inf on the left is 0. As y_t is bounded from below by $\min_{v \in V} x_v$ and from above by $\max_{v \in V} x_v$, the lim inf on the right is not affected by the vanishing term $\frac{1}{t} y_t$. Therefore, the term on the right is precisely $\nu(u, s_0, s_1)$ which means that the outcome of the play is non-negative. $\square$

Next we prove the backward implication. For this we will need some preliminary facts. Let $F = (F_1, \ldots, F_n)$, with F_i given by $M_i \; \{x_j + c_{i,j} : j \in I_i\}$ where $M_i \in \{\min, \max\}$. We will need the following notation:

1. $\mathrm{MIN} = \{k : M_k \text{ is } \min\}$,
2. $\mathrm{MAX} = \{k : M_k \text{ is } \max\}$,
3. $\mathrm{MIN}^+ = \{k : M_k \text{ is } \min \text{ and } |I_k| \geq 2\}$,
4. $\mathrm{MAX}^+ = \{k : M_k \text{ is } \max \text{ and } |I_k| \geq 2\}$.

For $k \in \{1, \ldots, n\}$ for which $|I_k| \geq 2$ and $\ell \in I_k$, we write $F^{k,\ell}$ for the operator that agrees with F in every component except in F_k where it is defined as $M_k \; \{x_j + c_{k,j} : j \in I_k - \{\ell\}\}$. The proof of the following lemma is basically the same as the proof of Lemma 4 in [1].

Lemma 2. *Let $\mathbf{x} \leq F(\mathbf{x})$ be unsatisfiable and let $k \in \{1, \ldots, n\}$.*

1. *If $k \in \mathrm{MIN}^+$, then $\mathbf{x} \leq F^{k,\ell}(\mathbf{x})$ is unsatisfiable for some $\ell \in I_k$.*
2. *If $k \in \mathrm{MAX}^+$, then $\mathbf{x} \leq F^{k,\ell}(\mathbf{x})$ is unsatisfiable for every $\ell \in I_k$.*

Proof. The case $k \in \mathrm{MAX}^+$ is easier: every solution to $\mathbf{x} \leq F^{k,\ell}(\mathbf{x})$ is also a solution to $\mathbf{x} \leq F(\mathbf{x})$ simply because $\max\ S \leq \max\ T$ whenever $S \subseteq T$. We proceed with the case $k \in \mathrm{MIN}^+$. For every $\ell \in I_k$, let $\mathbf{x}^\ell = (x_1^\ell, \ldots, x_n^\ell)$ be such that $\mathbf{x}^\ell \leq F^{k,\ell}(\mathbf{x}^\ell)$. Since adding a fixed integer to every component of a solution yields another solution, we may assume that $x_k^{\ell_1} = x_k^{\ell_2}$ for every $\ell_1, \ell_2 \in I_k$. Let $\mathbf{x} = (x_1, \ldots, x_n)$ be defined as $x_i = \max\ \{x_i^\ell : \ell \in I_k\}$. In particular $x_k = x_k^\ell$ for every $\ell \in I_k$, and $x_i = x_i^\ell$ for some $\ell \in I_k$ whenever $i \neq k$. We claim that $\mathbf{x}$ is a solution to $\mathbf{x} \leq F(\mathbf{x})$. For $i \neq k$, let $\ell \in I_k$ be such that $x_i = x_i^\ell$ and note that

$$x_i = x_i^\ell \leq M_i\ \{x_j^\ell + c_{i,j} : j \in I_i\} \leq M_i\ \{x_j + c_{i,j} : j \in I_i\}. \tag{5}$$

For $i = k$ we have

$$x_k = x_k^\ell \leq \min\ \{x_j^\ell + c_{k,j} : j \in I_k - \{\ell\}\} \leq \min\ \{x_j + c_{k,j} : j \in I_k - \{\ell\}\} \tag{6}$$

for every $\ell \in I_k$. Because $|I_k| \geq 2$, every $j \in I_k$ belongs to some $I_k - \{\ell\}$, which means that the inequality $x_k \leq \min\ \{x_j + c_{k,j} : j \in I_k\}$ is also true. $\square$

The second fact we need is a characterization of unsatisfiable systems without min operators. Let us also note that when $\mathrm{MIN}^+ = \emptyset$, we may think of the system as not having min operators. Indeed, if $|I_i| = 1$, the inequalities $x_i \leq \min\ \{x_j + c_{i,j} : j \in I_i\}$ and $x_i \leq \max\ \{x_j + c_{i,j} : j \in I_i\}$ are equivalent.

Lemma 3. *Let F be a system without min operators. The following are equivalent:*

1. $\mathbf{x} \leq F(\mathbf{x})$ *is unsatisfiable,*
2. *there exists a vertex u in G_F such that every cycle reachable from u in G_F is negative.*

Proof. The proof that 2. implies 1. is similar to the proof of Lemma 1. We prove the contrapositive. Assume that $\mathbf{x} = (x_1, \ldots, x_n)$ satisfies $\mathbf{x} \leq F(\mathbf{x})$. For every $i \in \{1, \ldots, n\}$, let $s(i) \in I_i$ be such that

$$x_{s(i)} + c_{i,s(i)} = \max\ \{x_j + c_{i,j} : j \in I_i\}. \tag{7}$$

Then, for every vertex u in G_F, the sequence $u_0, u_1, \ldots$ defined by $u_0 = u$ and $u_{t+1} = s(u_t)$ is a walk in G_F that reaches a cycle. By construction we have $x_{u_t} \leq x_{u_{t+1}} + c_{u_t,u_{t+1}}$. Adding up all inequalities for the vertices of the cycle we get that its weight is non-negative.

The proof that 1. implies 2. is by induction on n, the arity of F. The base case is $n = 1$, which means that F consists of a single inequality $x_1 \leq \max\ \{x_1 + c_{1,1}\}$. As F is unsatisfiable, necessarily $c_{1,1} < 0$. But G_F consists of a single vertex with a loop labelled by $c_{1,1}$, so the only cycle reachable from 1 is negative. Suppose now that $n > 1$ and that the claim holds for systems of smaller arities. Let F_n be of the form $\max\ \{x_j + c_{n,j} : j \in I_n\}$. We consider two cases: $n \notin I_n$ and $n \in I_n$.

In case $n \notin I_n$, let H be the system obtained from F by *replacing* every occurrence of x_n by $\max\ \{x_j + c_{n,j} : j \in I_n\}$. Formally, this is done as follows. For every $i \in \{1, \ldots, n-1\}$ such that $n \in I_i$, let $d_{i,j} = \max(c_{i,n} + c_{n,j}, c_{i,j})$ for every $j \in I_i \cap I_n$, and let $d_{i,j} = c_{i,j}$ for every $j \in I_i - I_n$. For every $i \in \{1, \ldots, n-1\}$ such that $n \notin I_i$, let $d_{i,j} = c_{i,j}$ for every $j \in I_i$. Then $H = (H_1, \ldots, H_{n-1})$ where H_i is defined as $\max\ \{x_j + d_{i,j} : j \in I_i\}$. We claim that if H were satisfiable, then F would also be satisfiable. For a proof, take a satisfying assignment for H and extend it to a satisfying assignment for F by setting $x_n = \max\ \{x_j + c_{n,j} : j \in I_n\}$. Since $n \notin I_n$, this is well defined and it satisfies F.

We continue with the proof in the case $n \notin I_n$. It follows from the above that H is unsatisfiable. Its arity is $n-1$. By induction hypothesis, there exists a vertex u in G_H for which all cycles reachable from u in G_H are negative. We claim that the same u works for G_F. For a proof, let $u_0, u_1, \ldots, u_r$ be a path to a cycle in G_F starting at u. In other words, $u_0 = u$, and all u_i are different except $u_r = u_s$ for some $s \leq r$. If $u_i \neq n$ for every $i \in \{1, \ldots, r\}$, then this path to a cycle also appears in G_H with the same or bigger weight. Since the weight of this cycle in G_H is negative, its weight in G_F is also negative which is what we want. If $u_i = n$ for some $i \in \{1, \ldots, r\}$, we distinguish two cases: $u_r = u_s = n$ and $u_r = u_s \neq n$. In case $u_r = u_s = n$, we have $1 \leq s \leq r-1$ because $n \notin I_n$. But then

$$u_0, \ldots, u_{s-1}, u_{s+1}, \ldots, u_{r-1}, u_{s+1} \tag{8}$$

is a path to a cycle in G_H. The weight of the edge (u_{s-1}, u_{s+1}) in G_H is at least as big as the sum of the weights of the edges (u_{s-1}, u_s) and (u_s, u_{s+1}) in G_F, and the weight of the edge (u_{r-1}, u_{s+1}) in G_H is at least as big as the sum of the weights of the edges (u_{r-1}, u_r) and (u_r, u_{s+1}). It follows that the weight of the cycle in G_F is bounded by the weight of the cycle in G_H. This is negative, which is what we want. Finally, in case $u_r = u_s \neq n$, we have $1 \leq i \leq r-1$ and

$$u_0, \ldots, u_{i-1}, u_{i+1}, \ldots, u_r \tag{9}$$

is a path to a cycle in G_H. The weight of the edge (u_{i-1}, u_{i+1}) is at least as big as the sum of the weights of the edges (u_{i-1}, u_i) and (u_i, u_{i+1}) in G_F. Again it follows that the weight of the cycle in G_F is bounded by the weight of the cycle in G_H, which is negative.

Next the case $n \in I_n$. If $c_{n,n} < 0$ and $|I_n| = 1$, we let $u = n$ and we are done: the only cycle reachable from u in G_F is negative. If $c_{n,n} < 0$ and $|I_n| > 1$, we can apply the argument of the previous case to $F^{n,n}$ and get a vertex u in $G_{F^{n,n}}$ such that all cycles reachable from u in $G_{F^{n,n}}$ are negative. But then, since the only difference between G_F and $G_{F^{n,n}}$ is a negative loop on n, all cycles reachable from u in G_F are also negative. If $c_{n,n} \geq 0$, we need to proceed differently. First, let H be the system obtained by *trivializing* every inequality of F in which x_n occurs in the right-hand side. Formally this is done as follows. For every $i \in \{1, \ldots, n-1\}$, if $n \in I_i$ let $J_i = \{i\}$ and $d_{i,i} = 0$, and if $n \notin I_i$ let $J_i = I_i$ and $d_{i,j} = c_{i,j}$ for every $j \in I_i$. Then $H = (H_1, \ldots, H_{n-1})$ where H_i is defined as $\max\{x_j + d_{i,j} : j \in J_i\}$. We claim that if H were satisfiable, then F would also be satisfiable. For a proof, take a satisfying assignment for H and extend it to a satisfying assignment for F by setting $x_n = \max\{x_i - c_{i,n} : i \in C_n\}$, where C_n is the set of $i \in \{1, \ldots, n-1\}$ such that $n \in I_i$.

We continue with the proof in the case $n \in I_n$. It follows from the above that H is unsatisfiable. Its arity is $n-1$. By induction hypothesis, there exists a vertex u in G_H for which all cycles reachable from u in G_H are negative. We claim that the same u works for G_F. To see this, note that no $i \in C_n$ is reachable from u in G_H since otherwise u would reach a non-negative cycle in G_H: the loop with weight 0 on i. But then n is not reachable from u in G_F because every path to n must go through some i in C_n. It follows that all the cycles reachable from u in G_F are already in G_H and therefore they are negative. □

We are ready for the converse to Lemma 1.

Lemma 4. *If $\nu(u) \geq 0$ for every $u \in V$, then $\mathbf{x} \leq F(\mathbf{x})$ is satisfiable.*

Proof. We prove the contrapositive. Suppose that $\mathbf{x} \leq F(\mathbf{x})$ is unsatisfiable. Repeated application of part 1 of Lemma 2 until $\mathrm{MIN}^+ = \emptyset$ gives, for every $i \in \mathrm{MIN}$, a vertex $s(i) \in I_i$ such that the system

$$\begin{array}{rcll} x_i & \leq & \max\{x_j + c_{i,j} : j \in I_i\} & \text{if } i \in \text{MAX} \\ x_i & \leq & x_{s(i)} + c_{i,s(i)} & \text{if } i \in \text{MIN} \end{array} \tag{10}$$

is unsatisfiable. By the construction of G_F, we may think of s as a (memoryless) strategy s_1 for player 1. We will show that there exists a vertex $u \in V$ such that, for every strategy s_0 for player 0, memoryless or not, the outcome of the game determined by u, s_0 and s_1 is negative. To define u, let H be the system in (10). Note that H may be thought as a min-free system. Let u be the vertex given by Lemma 3 applied to H.

Fix now a strategy s_0 for player 0. We will show that $\nu(u, s_0, s_1) < 0$. Let $u_0, u_1, \ldots$ be the play determined by u, s_0 and s_1. Let t be an integer and let $p = u_0 \ldots u_t$ denote the first $t+1$ vertices of the play. This forms a walk in G_H. If this walk is longer than the number of vertices $|V|$ of G_H, some vertex repeats in p. Let i_0 be minimal such that u_{i_0} repeats, and let j_0 be minimal such that $j_0 > i_0$ and $u_{j_0} = u_{i_0}$. Then we define another walk $C(p)$ by contracting the cycle $u_{i_0} \ldots u_{j_0}$ in p. In other words,

$$C(p) = u_0 \ldots u_{i_0-1} u_{j_0} \ldots u_t.$$

If the walk $C(p)$ is longer than the number of vertices, we repeat and get another walk $C(C(p))$, obtained from $C(p)$ by contracting its first cycle. Continuing this way, we produce a sequence of walks $p_0, p_1, \ldots, p_m$, where $p_0 = p$ and $p_{i+1} = C(p_i)$, until $|p_m| \leq |V|$. The lengths satisfy $|p_i| \leq |p_{i+1}| + |V|$ for $i \in \{0, \ldots, m-1\}$. Since $|p_0| \geq t+1$ and $|p_m| \leq |V|$, we get

$$m \geq (t+1)\frac{1}{|V|} - 1. \tag{11}$$

Note also that every p_i starts at u.

Let W_i stand for the weight of p_i. For $i \in \{0, \ldots, m-1\}$, let R_i be the weight of the cycle removed from p_i. Note that $W_i = W_{i+1} + R_i$. Therefore,

$$W_0 = W_m + \sum_{i=0}^{m-1} R_i \leq |V|W + \sum_{i=0}^{m-1} R_i, \tag{12}$$

where the inequality follows from $|p_m| \leq |V|$ and the fact that all weights are bounded by W. Note also that each R_i is negative by the choice of u because it is the weight of a cycle reachable from u in G_H.

We continue with the proof that the outcome of the play is negative. We have:

$$\sum_{i=1}^{t} w(u_{i-1}, u_i) \leq |V|W + \sum_{i=0}^{m-1} R_i \leq |V|W - m \leq |V|W - (t+1)\frac{1}{|V|} + 1, \tag{13}$$

where the first inequality comes from (12), the second inequality comes from the fact that R_i is a negative integer for every $i \in \{0, \ldots, m-1\}$, and the last inequality comes from (11). Dividing through by t and taking $\liminf$ on both sides we get

$$\liminf_{t\to\infty} \frac{1}{t} \sum_{i=1}^{t} w(u_{i-1}, u_i) \leq \liminf_{t\to\infty} \left(\frac{1}{t}|V|W - \left(1 + \frac{1}{t}\right) \frac{1}{|V|} + \frac{1}{t} \right). \tag{14}$$

The $\liminf$ on the right is not affected by the vanishing terms $\frac{1}{t}|V|W$ and $\frac{1}{t}$. On the other hand, the middle term approaches $-\frac{1}{|V|}$ as t grows, which means that the right-hand side is negative. Since the left-hand side is precisely $\nu(u, s_0, s_1)$, the proof is complete. □

MEAN-PAYOFF GAME $\leq$ MAX MIN OFFSET OPERATOR:

The correctness of this reduction follows from the fact that if we start with a game G, apply the reduction to get a system of operators F_G, and then the reduction back into a game G_{F_G}, we end up with the same game G we started with (up to isomorphism). Therefore, $\nu(u) \geq 0$ for every u in G if and only if $\nu(u) \geq 0$ for every u in G_{F_G}, and by the above, if and only if F_G is satisfiable.

MAX MIN OFFSET OPERATOR $\leq$ MAX ATOM:

This follows in a straighforward way by inspection of the reduction.

5 Remarks

It is worth noting that the proof of Lemma 1 shows something stronger than it states. It shows that if $\mathbf{x} \leq F(\mathbf{x})$ is satisfiable, then not only $\nu(u) \geq 0$ for every u, but moreover there exists a single memoryless strategy for player 0 that achieves non-negative value at every vertex. Similarly, the proof of Lemma 4 also shows that if $\mathbf{x} \leq F(\mathbf{x})$ is unsatisfiable, then not only $\nu(u) < 0$ for some u, but moreover there exists a memoryless strategy for player 1 that forces negative value at that vertex, and even $\limsup_{t\to\infty} \frac{1}{t}\sum_{i=1}^{t} w(u_{i-1}, u_i)$ is negative. Following along these lines, it is possible to rederive the memoryless determinacy of mean-payoff games in the form originally stated by Ehrenfeucht and Mycielsky. Conversely, if we used memoryless determinacy as a black-box, our proofs would get even simpler at the expense of not being self-contained. In personal communication, Bezem et al. informed us that, according to one of the referees of [1], the general theory of max-min function would also give alternative proofs.

A different point worth noting is that the version of the decision problem for mean-payoff games considered here is equivalent to several other variants. For example, we might want to determine whether $\nu(u) \geq 0$ for a given starting vertex u instead of whether $\nu(u) \geq 0$ for every starting vertex u. Or whether $\nu(u) \geq \nu$ for a given starting vertex u and a given rational value ν, etc. All these versions are polynomial-time equivalent to MEAN-PAYOFF GAME through standard reductions.

On the other hand, the standard reduction from parity games to mean-payoff games produces an instance of mean-payoff games where the goal is to determine if the values are negative or positive. This is the reduction that assigns weight $(-|V|)^{p(v)}$ to every edge going out of v, where $p(v)$ is the priority assigned to v in the parity game. The corresponding max-atom instance gets exponentially large offsets (polynomially-sized when represented in binary) with some special structure. While we do not see a straightforward way of exploiting this structure to speed up the pseudo-polynomial-time algorithm from [1] for this special case of the max-atom problem, it might be worth turning this around and interpreting the known subexponential algorithms for parity games [7] in the language of the max-atom problem with the hope of generalizing them.

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References

[1] M. Bezem, R. Nieuwenhuis, and E. Rodríguez-Carbonell. The max-atom problem and its relevance. In *Proceedings of the 15th International Conference on Logic for Programming, Artificial Intelligence and Reasoning (LPAR)*, pages 47–61. Springer-Verlag, 2008.

[2] H. Björklund, S. Sandberg, and S. Vorobyov. Memoryless determinacy of parity and mean payoff games: a simple proof. *Theoretical Computer Science*, 310:365–378, 2004.

[3] A. Condon. The complexity of stochastic games. *Information and Computation*, 96(2):203–224, 1992.

[4] V. Dhingra and S. Gaubert. How to solve large scale deterministic games with mean payoff by policy iteration. In *Proceedings of the 1st International Conference on Performance Evaluation Methodologies and Tools*, page 12, 2006.

[5] A. Ehrenfeucht and J. Mycielsky. Positional strategies for mean payoff games. *International Journal of Game Theory*, 8(2):109–113, 1979.

[6] M. Jurdziński. Deciding the winner in partity games is in UP $\cap$ co-UP. *Information Processing Letters*, 68:119–124, 1998.

[7] M. Jurdziński, M. Paterson, and U. Zwick. A deterministic subexponential algorithm for solving parity games. *SIAM Journal on Computing*, 38(4):1519–1532, 2008.

[8] G. J. Olsder. Eigenvalues of dynamic min-max systems. *Journal of Discrete Event Dynamic Systems*, 1:177–207, 1991.

[9] U. Zwick and M. Paterson. The complexity of mean payoff games on graphs. *Theoretical Computer Science*, 158:343–359, 1996.