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TROPICALLY CONVEX CONSTRAINT SATISFACTION

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ABSTRACT. A semilinear relation $S \subseteq \mathbb{Q}^n$ is *max-closed* if it is preserved by taking the componentwise maximum. The constraint satisfaction problem for max-closed semilinear constraints is at least as hard as determining the winner in Mean Payoff Games, a notorious problem of open computational complexity. Mean Payoff Games are known to be in $\mathsf{NP} \cap \mathsf{co\text{-}NP}$, which is not known for max-closed semilinear constraints. Semilinear relations that are max-closed and additionally closed under translations have been called *tropically convex* in the literature. One of our main results is a new duality for open tropically convex relations, which puts the CSP for tropically convex semilinear constraints in general into $\mathsf{NP} \cap \mathsf{co\text{-}NP}$. This extends the corresponding complexity result for scheduling under and-or precedence constraints, or equivalently the max-atoms problem. To this end, we present a characterization of max-closed semilinear relations in terms of syntactically restricted first-order logic, and another characterization in terms of a finite set of relations L that allow primitive positive definitions of all other relations in the class. We also present a subclass of max-closed constraints where the CSP is in P; this class generalizes the class of max-closed constraints over finite domains, and the feasibility problem for max-closed linear inequalities. Finally, we show that the class of max-closed semilinear constraints is *maximal* in the sense that as soon as a single relation that is not max-closed is added to L, the CSP becomes NP-hard.

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1. Introduction

A relation $R \subseteq \mathbb{Q}^n$ is *semilinear* if R has a first-order definition in $(\mathbb{Q}; +, \leqslant, 1)$; equivalently, R is a finite union of finite intersections of (open or closed) linear half spaces; see Ferrante and Rackoff [14]. In this article we study the computational complexity of constraint satisfaction problems with semilinear constraints. Informally, a constraint satisfaction problem (CSP) is the problem of deciding whether a given finite set of constraints has a common solution. It has been a fruitful approach to study the computational complexity of CSPs depending on the type of constraints allowed in the input.

Formally, we fix a set D, a set of relation symbols $\tau = \{R_1, R_2 \dots\}$, and a τ-structure $\Gamma = (D; R_1^\Gamma, R_2^\Gamma \dots)$ where $R_i^\Gamma \subseteq D^{k_i}$ is a relation over D of arity k_i. For finite τ the computational problem CSP(Γ) is defined as follows.

CSP(Γ)
INSTANCE: a finite set of formal variables $x_1 \dots x_n$, and a finite set of expressions of the form $R(x_{i_1} \dots x_{i_k})$ with $R \in \tau$
QUESTION: is there an assignment $x_1^s \dots x_n^s \in D$ such that $(x_{i_1}^s \dots x_{i_k}^s) \in R^\Gamma$ for all constraints of the form $R(x_{i_1}, \dots, x_{i_k})$ in the input?

When the domain of Γ is the set of rational numbers $\mathbb{Q}$, and all relations of Γ are semilinear, we say that Γ is semilinear. We adopt the convention that all semilinear relations have rational coefficients. For relations of this kind and the questions studied here, whether we work with $D = \mathbb{Q}$ or with $D = \mathbb{R}$ does not play any role.

It is possible, and sometimes essential, to also define CSP(Γ) when the signature τ is infinite. However, in this situation it is important to discuss how the symbols from τ are represented in the input of the CSP. In our context, Γ is semilinear, it is natural to assume that the relations are given as quantifier-free formulas in disjunctive normal form where the coefficients are represented in binary (by the mentioned result of Ferrante and Rackoff [14], every relation with a first-order definition over $(\mathbb{Q}; +, \leqslant, 1)$ has a definition of this form). However, there are other natural representations, and we will for infinite languages always discuss how the relations are given.

A famous example of a computational problem that can be formulated as CSP(Γ) for a semilinear structure Γ is the feasibility problem for linear programming, which is simply the CSP for the structure Γ with domain $\mathbb{Q}$ that contains a relation for every linear inequality, that is, all relations of the form $\{(x_1, \dots, x_k) \mid c_1 x_k + \dots + c_k x_k \leqslant c_0\}$ for coefficients $c_0, c_1, \dots, c_k \in \mathbb{Q}$. Here, it is natural to assume that the relations in the input are represented via the inequalities that define them, and that the coefficients are represented in binary. This CSP is well-known to be in P [26]. It follows that the CSP for *any* semilinear Γ (represented, say, in DNF) is in NP, because we can non-deterministically select a disjunct from the representation of each of the given constraints, and then verify in polynomial time whether the obtained set of linear inequalities is satisfiable.

We would like to systematically study the computational complexity of CSP(Γ) for all semilinear structures Γ. This is a very ambitious goal. Several partial results are known [6, 7, 8, 24, 25]. Let us also mention that it is easy to find for every structure Δ with a finite domain a semilinear structure Γ so that CSP(Δ) and CSP(Γ) are the same computational problem (in the sense that they have the same satisfiable instances). But already the complexity classification of CSPs for finite structures is open [13].

Even worse, there are concrete semilinear structures whose CSP has an open computational complexity. An important example of this type is the max-atoms problem [5], which is the CSP for the semilinear structure Γ that contains all ternary relations of the form

$$M_c \overset{\mathit{def}}{=} \{(x_1, x_2, x_3) \mid x_1 + c \leqslant \max(x_2, x_3)\}$$

where $c \in \mathbb{Q}$ is represented in binary. It is an open problem whether the max atoms problem is in P, but it is known to be polynomial-time equivalent to determining the winner in mean payoff games (Möhring, Skutella, and Stork [28]), which is known to be in $\mathsf{NP} \cap \mathsf{co\text{-}NP}$. Note that here the assumption that c is represented in binary is important: when c is represented in unary, or when we drop all but a finite number of relations in Γ, the resulting problem is known to be in P.

An important tool to study the computational complexity of CSP(Γ) is the concept of *primitive positive definability*. A primitive positive formula is a first-order formula of the form $\exists x_1, \ldots, x_n (\psi_1 \wedge \cdots \wedge \psi_m)$ where ψ are atomic formulas (in primitive positive formulas, no disjunction, negation, or universal quantification is allowed). Jeavons, Cohen, and Gyssens [22] showed that the CSP for expansions of Γ by finitely many primitive positive definable relations is polynomial-time reducible to CSP(Γ). Let us mention that difficult problems in real algebraic geometry are about primitive positive definability: the conjecture of Helton and Nie [19] for instance can be phrased as "every convex semialgebraic set has a primitive positive definition over the solution spaces of semidefinite programs".

Primitive positive definability in Γ can be studied using the *polymorphisms* of Γ, which are a multi-variate generalization of the endomorphisms of Γ. We say that $f\colon \Gamma^k \to \Gamma$ is a polymorphism of a τ-structure Γ if

$$\left(f(a_1^1, \ldots, a_1^k), \ldots, f(a_m^1, \ldots, f_m^k)\right) \in R^\Gamma$$

for all $R \in \tau$ and $(a_1^1, \ldots, a_m^1), \ldots, (a_1^k, \ldots, a_m^k) \in R^\Gamma$. For finite structures Γ, a relation R is primitive positive definable in Γ *if and only if* R is preserved by all polymorphisms of Γ. And indeed, the *tractability conjecture* of Bulatov, Jeavons, and Krokhin [10] in a reformulation due to Kozik and Barto [4] states that CSP(Γ) is in P if and only if Γ has a polymorphism f which is *cyclic*, this is, has arity $n \geqslant 2$ and satisfies $\forall x_1, \ldots, x_n\ f(x_1, \ldots, x_n) = f(x_2, \ldots, x_n, x_1)$.

Polymorphisms are also relevant when Γ is infinite. For example, a semilinear relation is convex if and only if it has the cyclic polymorphism $(x, y) \mapsto (x + y)/2$. The CSP for such relations is the feasibility problem for strict and non-strict linear inequalities, and it is well-known that this problem is in P. When Γ has a cyclic polymorphism, and assuming the tractability conjecture, Γ cannot interpret[1] primitively positively any hard finite-domain CSP, which is the standard way of proving that a CSP is NP-hard.

A fundamental cyclic operation is the maximum operation, given by $(x, y) \mapsto \max(x, y)$. The constraints for the max-atoms problem are examples of semilinear relations that are not convex, but that have max as a polymorphism; we also say that they are *max-closed*. When a *finite* structure is max-closed, with respect to some ordering of the domain, then the CSP for this structure is known to be in P [21]. The complexity of the CSP for max-closed *semilinear* constraints, on the other hand, is open. Since this problem is more general than the max atoms problem, it is at least as hard as determining the winner of mean payoff games. But unlike the max-atoms problem, it is not known whether the CSP for max-closed semilinear constraints is in co-NP.

2. Results

We show that the CSP for semilinear max-closed relations that are *translation-invariant*, that is, have the polymorphism $x \mapsto x + c$ for all $c \in \mathbb{Q}$, is in NP $\cap$ co-NP (Section 5). Such relations have been called *tropically convex* in the literature [12].[2] This class is a non trivial extension of the max-atoms problem (for instance it contains relations such as $x \leqslant (y+z)/2$), and it is not covered by the known reduction to mean payoff games [28, 1, 3]. Indeed, it is open whether the CSP for tropically convex semilinear relations can be reduced to mean payoff games (in fact, Zwick and Paterson [32] believe that mean payoff games are "strictly easier" than simple stochastic games, which reduce to our problem via the results presented in Section 4). The containment in NP $\cap$ co-NP can be slightly extended to the CSP for the structure that includes additionally all relations $x = c$ for $c \in \mathbb{Q}$ (represented in binary). It follows from our results (Corollary 3.5) that the class of semilinear tropically convex sets is the smallest class of semilinear sets that has the same polymorphisms as the max atoms constraints $x \leqslant \max(y, z) + c$ with $c \in \mathbb{Q}$.

In our proof, we first present a characterization of max-closed semilinear relations in terms of syntactically restricted first-order logic (Section 3). We show that a semilinear relation is max-closed if and only if it can be defined by a *semilinear Horn formula*, which we define as a finite conjunction of *semilinear Horn clauses*, this is, finite disjunctions of the form

$$\bigvee_{i=1}^{m} \bar{a}_i^\top \bar{x} \succ_i c_i$$

[1]Interpretations in the sense of model theory; we refer to Hodges [20] since we do not need this concept further.

[2]The original definition of tropical convexity is for the dual situation, considering min instead of max.

where

(1) $\bar{a}_1 \dots \bar{a}_m \in \mathbb{Q}^n$ and there is a $k \leqslant n$ such that $\bar{a}_{i,j} \geqslant 0$ for all i and $j \neq k$,
(2) $\bar{x} = (x_1 \dots x_n)$ is a vector of variables,
(3) $\succ_i \in \{\geqslant, >\}$ are strict or non-strict inequalities, and
(4) $c_1 \dots c_m \in \mathbb{Q}$ are coefficients.

Example 2.1. The ternary relation M_c from the max-atoms problem can be defined by the semilinear Horn clause $x_2 - x_1 \geqslant c \vee x_3 - x_1 \geqslant c$.

Example 2.2. A linear inequality $a_1x_1 + \dots + a_nx_n \geqslant c$ is max-closed if and only if at most one of $a_1 \dots a_n$ is negative. The relations defined by such formulas are in general *not* tropically convex; consider for example the relation defined by $-x_1 + x_2 + x_3 \geqslant 0$.

Example 2.3. Conjunctions of implications of the form

$$x_1 \leqslant c_1 \wedge \dots \wedge x_n \leqslant c_n \quad \Rightarrow \quad x_i < c_0 \tag{1}$$

are max-closed since such an implication is equivalent to the semilinear Horn clause

$$(-x_i > -c_0) \vee \bigvee_i x_i > c_i \; .$$

It has been shown by Jeavons and Cooper [21] that over finite ordered domains, a relation is max-closed[3] if and only if it can be defined by finite conjunctions of implications of the form (1). Over infinite domains, this is no longer true, as demonstrated by the relations in Example 2.1 and Example 2.2.

We also show that the classes $\mathcal{C}$ of max-closed semilinear relations and $\mathcal{C}_t$ of tropically convex semilinear relations are *finitely related* in the sense of universal algebra, this is, there exists a finite subset Γ_0 of $\mathcal{C}$ (resp. Γ_t of $\mathcal{C}_t$) that can primitively positively define all other relations in $\mathcal{C}$ (resp. $\mathcal{C}_t$). For instance, we show that all relations in $\mathcal{C}$ have a primitive positive definition in

$$\Gamma_0 = (\mathbb{Q}; <, 1, -1, S_1, S_2, M_0)$$

where

$$S_1 = \{(x,y) \mid 2x \leqslant y\} \qquad S_2 = \{(x,y,z) \mid x \leqslant y + z\}$$
$$M_0 = \{(x,y,z) \mid x \leqslant y \vee x \leqslant z\}$$

Note that any other structure Γ_1 with finite relational signature and this property has a polynomial-time equivalent CSP. We show that the primitive positive formulas can even be computed efficiently from a given semilinear Horn formula ϕ, and have linear size in the representation size of ϕ.

Our proof of the containment in $\mathsf{NP} \cap \mathsf{co\text{-}NP}$ is based on a duality for open tropically convex semilinear sets, which extends a duality that has

[3] Also the results in Jeavons and Cooper [21] have been formulated in the dual situation for min instead of max.

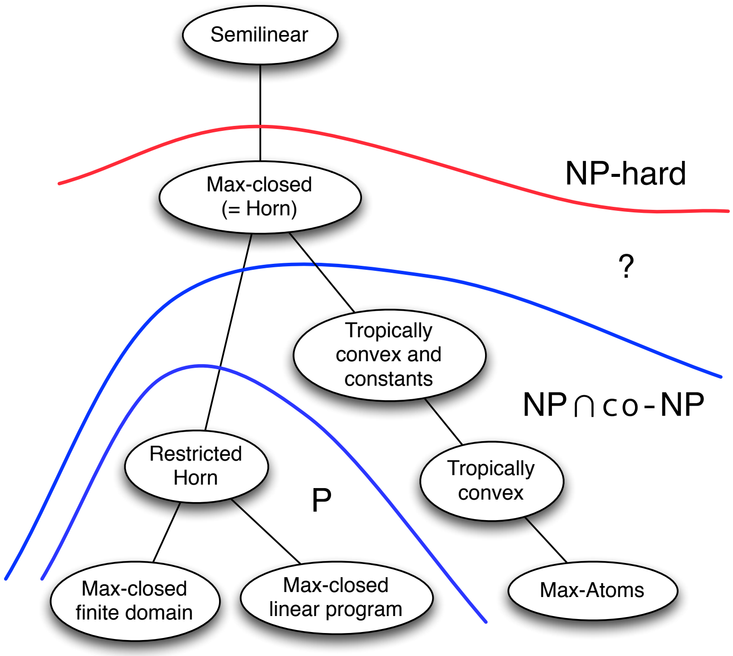

FIGURE 1. An overview of constraint languages relevant for this work (P membership for finite domain max-closed CSPs is in [21], NP ∩ co-NP membership of max-atoms is in [5]).

been observed for the max-atoms problem in [18]. To prove the duality, we translate instances of our problem into a condition on associated mean payoff stochastic games (also called limiting average payoff stochastic games; see [15] for a general reference), and then exploit the symmetry implicit in the definition of such games. The connections between the max-atoms problem, mean payoff deterministic games, and tropical polytopes have been explored extensively in the computer science literature. However, the theory of stochastic games introduces profound changes over the deterministic setting, and employs non-trivial new techniques. Even though this field is active since the 70s, to the best of our knowledge, no application of its results to semilinear feasibility problems has been published yet. Note that to solve mean payoff stochastic games is in NP ∩ co-NP, as a consequence of the existence of optimal positional strategies; see [17] and [27] (for a general reference on the computational complexity of stochastic games, see [2]). However, we cannot use this fact directly, because stochastic games only relate to a subset of tropically convex sets. We conclude our argument combining the duality with semilinear geometry techniques.

Interestingly, at several places in our proofs, we need to replace $\mathbb{Q}$ with appropriate non-Archimedean structures, for instance in the proof of the syntactic characterization of max-closed semi-linear relations, in order to deduce the general case from the easier situation for closed semilinear relations. But also for representing the winning strategies for stochastic games, and for using the duality for closed tropically convex relations to solve the CSP for tropically convex constraints, we work with explicit non-standard models (for another application of non-standard models to CSPs see for instance [8]).

Our next result is the identification of a class of max-closed semilinear relations whose CSP can be solved in polynomial time (Section 6). This class consists of the semilinear relations that can be defined by *restricted semilinear Horn formulas*, which are conjunctions of semilinear Horn clauses satisfying the additional condition that there are $k \leqslant n$ and $l \leqslant m$ such that $(\bar{a}_i)_j \geqslant 0$ for all $i \in \{1, \ldots, m\} \setminus \{l\}$ and $j \in \{1, \ldots, n\} \setminus \{k\}$. An example of a semilinear Horn clause which is not restricted Horn is $x_1 < x_2 \vee x_1 < x_3$.

Finally, we show that the class of max-closed semilinear constraints is *maximal* in the sense that for every relation R that is not max-closed, the problem $\mathrm{CSP}(\Gamma_0, R)$ is NP-hard (Section 7). Figure 1 gives an overview over our results. Of course, all our results apply dually to min-closed semilinear relations as well.

3. A Syntactic Characterization of Max-closure

This section aims to offer a syntactic characterization of semilinear max-closed and semilinear tropically convex relations over $\mathbb{Q}$. In this section, the letter F will denote an ordered field: for technical reasons we need to work in this slightly more general setting.

Definition 3.1. A semilinear Horn clause is called *closed* if all inequalities are non-strict. We say that $X \subset F^n$ is a *basic max-closed set* if it is the graph of a semilinear Horn clause, i.e. there is $k \in \{1 \ldots n\}$ such that X can be written as a finite union

$$X = \bigcup_i \{(x_1 \ldots x_n) \mid a_{i,1}x_1 + \cdots + a_{i,n}x_n \succ_i c_i\}$$

where $\succ_i$ can be either $>$ or $\geqslant$, and $a_{i,j} \geqslant 0$ for all i and all $j \neq k$. We say that X is *basic closed max-closed* if it is the graph of a closed semilinear Horn clause.

Theorem 3.2. (1) Let $X \subset F^n$ be a semilinear set. Then X is max-closed if and only if it is a finite intersection of basic max-closed sets.

(2) Let $X \subset F^n$ be a closed semilinear set. Then X is max-closed if and only if it is a finite intersection of basic closed max-closed sets.

(3) $X \subset \mathbb{Q}^n$ is primitive positive definable in $\Gamma_0 = (\mathbb{Q}; <, 1, -1, S_1, S_2, M_0)$ with

$$S_1 = \{(x, y) \mid 2x \leqslant y\} \qquad S_2 = \{(x, y, z) \mid x \leqslant y + z\}$$
$$M_0 = \{(x,y,z) \mid x \leqslant y \vee x \leqslant z\}$$

if and only if X is semilinear and max-closed.

(4) $X \subset \mathbb{Q}^n$ is primitive positive definable in $\Gamma_0' = (\mathbb{Q}; 1, -1, S_1, S_2, M_0)$ if and only if X is semilinear closed and max-closed.

Definition 3.3. We say that $X \subset F^n$ is a *basic tropically convex set* if there is $k \in \{1 \dots n\}$ such that X can be written as a finite union

$$X = \bigcup_i \{(x_1 \dots x_n) \mid a_{i,1}x_1 + \cdots + a_{i,n}x_n \succ_i c_i\}$$

where $\succ_i$ can be either $>$ or $\geqslant$, and $a_{i,j} \geqslant 0$ for all i and all $j \neq k$, and, moreover

$$\sum_j a_{i,j} = 0$$

for all i.

Theorem 3.4. (1) Let $X \subset F^n$ be a semilinear set. Then X is tropically convex if and only if it is a finite intersection of basic tropically convex sets.

(2) $X \subset \mathbb{Q}^n$ is primitive positive definable in $\Gamma_t = (\mathbb{Q}; <, T_1, T_{-1}, S_3, M_0)$ where

$$T_{\pm 1} = \{(x,y) \mid x \leqslant y \pm 1\} \qquad S_3 = \left\{(x,y,z) \,\middle|\, x \leqslant \frac{y+z}{2}\right\}$$

if and only if X is semilinear and tropically convex.

Corollary 3.5. A semilinear set $X \subset \mathbb{Q}^n$ is tropically convex if and only if it is preserved by every polymorphism that preserves the max-atoms language (i.e. all sets of the form $\{(x,y,z) \mid x \leqslant \max(y,z) + c\}$ for $c \in \mathbb{Q}$).

Proof. Translations and maximum are polymorphisms of max-atoms, so one direction is trivial. For the converse, by Theorem 3.4(2), it suffices to prove that the relations $<$, T_1, T_{-1}, S_3, and M_0 are preserved by all polymorphisms of max-atoms. This is immediate for T_1, T_{-1}, and M_0 since they have primitive positive definitions over max-atoms. The relation $<$, on the other hand, is an monotone union of max-atoms constraints

$$< = \{(x,y) \mid x < y\} = \bigcup_{c \in \mathbb{Q}, c > 0} \{(x,y) \mid x \leqslant y - c\}$$

and S_3 is an intersection of max-atoms constraints

$$S_3 = \left\{(x,y,z) \mid x <= \frac{y+z}{2}\right\} = \bigcap_{c \in \mathbb{Q}} \{(x,y,z) \mid z \leqslant \max(y+c, z-c)\}$$

It's well known that monotone unions and arbitrary intersections of primitive positive definable sets are preserved by polymorphisms [29]. □ □

The following observation is important in view of its implications on the complexity of the constraint satisfaction problems for max-closed and tropically convex sets. The reader will recognize that it follows straightforwardly from the proofs of Theorem 3.2 and Theorem 3.4.

Observation 3.6. Given a max-closed (resp. tropically convex) set X written as a finite intersection of basic max-closed (resp. basic tropically convex) sets with the constants represented in binary, we can compute in polynomial time a primitive positive definition of X in the structure Γ_0 (resp. Γ_t).

In this section, as well as in Section 5, we will make use of a few basic facts about semilinear geometry. Semilinear sets, as defined in the introduction, are finite Boolean combinations of half spaces in $\mathbb{Q}^n$. We already mentioned that this setting can be extended without difficulty to an ordered field F. An example of ordered field that we will need is the field of formal Laurent series $F((\epsilon))$, whose elements are the series of the form

$$\sum_{i=-\infty}^{\infty} a_i \epsilon^i$$

where ϵ is a formal variable, the coefficients a_i are in F, and the set of integers i such that $a_i \neq 0$ is bounded from below. The field $F((\epsilon))$ can be seen as the field of fractions of the ring of formal power series $F[[\epsilon]]$, whose elements are

$$\sum_{i=0}^{\infty} a_i \epsilon^i$$

On both these structures, we give the lexicographic order with $\epsilon^i \ll \epsilon^j$ for $i > j$, the choice of ϵ as the name of the formal variable is indeed meant to be a reminder of this. The open convex semilinear subsets of F^n are precisely the finite intersections of open half spaces (see [11, Corollary 4.9] and also [30]).

Still further, one can consider finite Boolean combinations of half spaces in V^n, where V is an ordered F-vector space. By a half space in V^n here we mean a set defined by $\{\bar{x} \in V^n \mid \bar{a}^\top \bar{x} \succ c\}$, where $\bar{a} \in F^n$ and $c \in V$. From this point of view, one can prove that the semilinear sets are precisely those first order definable in the structure $\Gamma_V = (V; <, +, \alpha\cdot)_{\alpha \in F}$, where $\alpha\cdot$ represents the function mapping $x \in V$ to αx. Now, Γ_V is a so called o-minimal structure, see [31] as a general reference, and, as all o-minimal structures expanding an ordered group, it has definable Skolem functions (definable choice). Namely, given a parametric existential formula $\exists x \phi(x, \bar{p})$ which is true for all $\bar{p} \in V^n$, there is a semilinear function $x = x(\bar{p})$ such that $\phi(x(\bar{p}), \bar{p})$ holds for all $\bar{p} \in V^n$. We will make extensive use of this principle.

We begin with the proof of Theorem 3.2. It is convenient to prove the four points in the order (2)–(1)–(3)–(4). Then we will use Theorem 3.2 to deduce Theorem 3.4.

Claim 3.7. Let $X \subset F^n$ be an open semilinear set, then X is a finite union of convex open semilinear sets.

Proof. A semilinear set $Y \subset F^n$ is said to be *relatively open* if Y is open in the affine subspace of F^n generated by Y. All semilinear sets are finite unions of relatively open convex semilinear sets (e.g. [31, Chapter 1 Corollary 7.8]). It therefore suffices to show that given a relatively open convex semilinear $Y \subset X$, there is an open convex semilinear set Z such that $Y \subset Z \subset X$. Let A be the affine space generated by Y. Then Y can be written in the form

$$Y = \left\{ x \in A \;\middle|\; f_1(x) > 0 \wedge \cdots \wedge f_p(x) > 0 \right\}$$

for some affine $f_1 \dots f_p$. Without loss of generality, we may assume that the projection of A onto the first $d \overset{def}{=} \dim(A)$ coordinates is one-to-one. Consider the function $\delta \colon F^{>0} \to F^{\geqslant 0} \cup \{+\infty\}$ mapping ϵ to the largest $\delta(\epsilon)$ such that for all $x \in Y$ we have

$$f_1(x) \geqslant \epsilon \wedge \dots \wedge f_p(x) \geqslant \epsilon \quad \Rightarrow \quad x + B_{\delta(\epsilon)} \subset X$$

where

$$B_\alpha = \left\{ y \in F^n \;\middle|\; y_1 = \dots = y_d = 0 \wedge y_{d+1} \dots y_n \in (-\alpha, \alpha) \right\}$$

First we claim that $\delta(\epsilon)$ is well defined and positive for all $\epsilon > 0$. Well definedness is immediate. Suppose that for some ϵ we have $\delta(\epsilon) = 0$. This means that for all $\delta' > 0$ we can find an $x'(\delta') \in Y$ such that

$$f_1(x'(\delta')) \geqslant \epsilon \wedge \dots \wedge f_p(x'(\delta')) \geqslant \epsilon \quad \wedge \quad x'(\delta') + B_{\delta'} \not\subseteq X$$

By definable choice we can assume that the function $\delta' \mapsto x'(\delta')$ is semilinear. Consider the limit

$$x'' = \lim_{\delta' \to 0+} x'(\delta')$$

Clearly, for all $\delta'' > 0$

$$f_1(x'') \geqslant \epsilon \wedge \dots \wedge f_p(x'') \geqslant \epsilon \quad \wedge \quad x'' + B_{\delta''} \not\subseteq X$$

contradicting the fact that X is open. Now, $\epsilon \mapsto \delta(\epsilon)$ is first-order definable from semilinear data, hence it is semilinear, and we know that it must be increasing and map positive elements to positive elements. It follows that $\delta(\epsilon) \leqslant a \min(\epsilon, b)$ for some positive a and b. Hence

$$Z = \bigcup_{x \in Y} x + B_{a \min(\epsilon, b)} \subset X$$

It is easy to check that Z is open and convex. □

Definition 3.8. We say that $x \in X \subset F^n$ is *of type* k *in* X, with $k \in \{1 \dots n\}$, if $x - Q_k \overset{def}{=} \{x - y \mid y \in Q_k\} \subset X$ where

$$Q_k = \left\{ y \in (F^{\geqslant 0})^n \;\middle|\; y_k = 0 \right\}$$

Observe that if X is the complement of a max-closed set, then every point of X is of type k in X for at least one k. Figure 2 displays an example of semilinear max-closed set, with points of its complement marked according to their type (notice that max-closed sets need to be neither convex nor connected).

Claim 3.9. Let X be an open semilinear subset of F^n. Then the set X_k of all points of type k in X is an open semilinear set.

Proof. The set X_k is clearly semilinear, hence all we need to prove is that it is open. Pick x on the boundary of X_k, it suffices to prove that $x \notin X_k$. Since x is on the boundary

$$\forall \epsilon > 0 \quad \exists\, \delta(\epsilon) \in F^n \quad |\delta(\epsilon)|_\infty < \epsilon \quad \wedge \quad \delta(\epsilon) + x \notin X_k$$

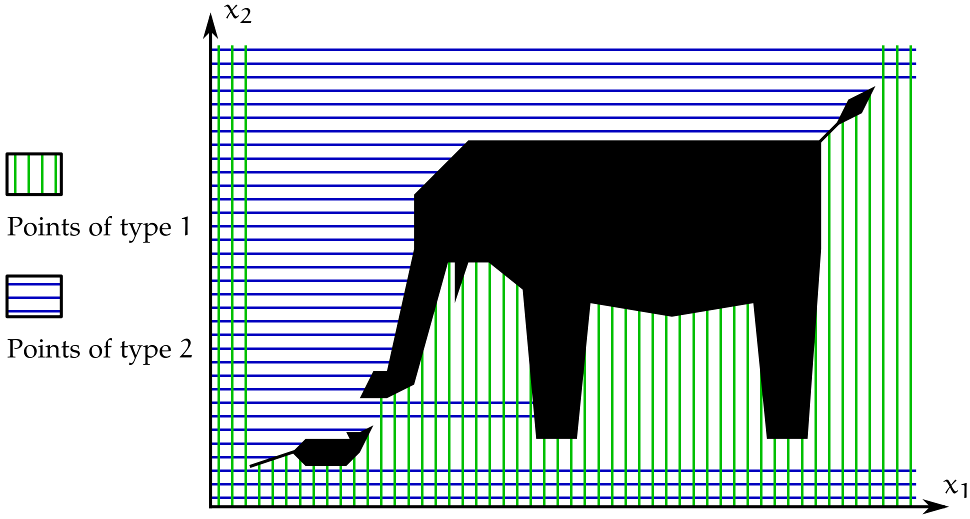

FIGURE 2. A max-closed set in $\mathbb{Q}^2$

where $|\cdot|_\infty$ denotes the maximum norm, and expanding the definition of X_k

$$\forall \epsilon > 0 \quad \exists \delta(\epsilon) \in F^n \quad |\delta(\epsilon)|_\infty < \epsilon \quad \wedge \quad \exists y(\epsilon) \in Q_k \quad \delta(\epsilon) + x - y(\epsilon) \notin X$$

Now, by definable choice, we can assume that the function $\epsilon \mapsto y(\epsilon)$ is semilinear. Hence the limit

$$y_0 = \lim_{\epsilon \to 0+} y(\epsilon)$$

exists. Since Q_k is closed, we have $y_0 \in Q_k$, and, since X is open, we have $x - y_0 \notin X$, therefore $x \notin X_k$. $\square$

Proof of Theorem 3.2 (2). The *if* part is immediate, hence we concentrate on the *only if*. Let $\bar{X}$ be $F^n \setminus X$. Consider the subsets $\bar{X}_1 \dots \bar{X}_n$ of the points in $\bar{X}$ of type $1 \dots n$ respectively. Since X is max-closed, each point of $\bar{X}$ is of type k for some $k \in \{1 \dots n\}$, hence

$$\bar{X} = \bar{X}_1 \cup \dots \cup \bar{X}_n$$

moreover, by Claim 3.9, each of these sets is open and semilinear. Applying Claim 3.7, we can further split each of the sets $\bar{X}_i$ into a finite union

$$\bar{X}_i = \bar{X}_{i,1} \cup \dots \cup \bar{X}_{i,m_i}$$

of convex open semilinear sets. Now, by definition, $X_i = X_i - Q_i$, hence

$$\bar{X}_i = \tilde{X}_{i,1} \cup \dots \cup \tilde{X}_{i,m_i}$$

where $\tilde{X}_{i,j} = \bar{X}_{i,j} - Q_i$. We claim that each of the sets $X_{i,j} = F^n \setminus \tilde{X}_{i,j}$ is definable by a semilinear Horn clause: this is enough to conclude because, by what was said, X is the finite intersection

$$X = \bigcap_{i,j} X_{i,j}$$

It is easy to check that the sets $\tilde{X}_{i,j}$ are open and convex. Therefore each of them is a finite intersection

$$\tilde{X}_{i,j} = \bigcap_k \{(x_1 \dots x_n) \mid a_{i,j,k,1}x_1 + \dots + a_{i,j,k,n}x_n < c_{i,j,k}\}$$

in other words

$$X_{i,j} = \bigcup_k \{(x_1 \dots x_n) \mid a_{i,j,k,1}x_1 + \dots + a_{i,j,k,n}x_n \geqslant c_{i,j,k}\}$$

It remains to check the condition on the coefficients $a_{i,j,k,l}$, namely we claim that $a_{i,j,k,l} \geqslant 0$ for all $l \neq i$. For a contradiction, suppose that $a_{i,j,k,l} < 0$ with $l \neq i$. Then pick a point $(x_1 \dots x_n) \in \tilde{X}_{i,j}$. Then

$$(x_1 \dots x_n) - (0 \dots \overset{l}{N} \dots 0) \notin \tilde{X}_{i,j}$$

for N larger than

$$N' = \frac{a_{i,j,k,1}x_1 + \dots + a_{i,j,k,n}x_n}{-a_{i,j,k,l}}$$

contradicting the construction of $\tilde{X}_{i,j}$. □

Proof of Theorem 3.2—(2)⇒(1). Our strategy is the following. We apply (2) to the field of formal Laurent series $F((\epsilon))$ with coefficients in F. Each semilinear set $X \subset F^n$ has a unique extension X^* to $F((\epsilon))^n$, which is the set defined by the same formula that defines X (clearly which one is chosen is immaterial). Let X denote our max-closed set, which is not necessarily closed. The extension X^* of X is closed if and only if X is, however we will modify X^* slightly to obtain a closed semilinear max-closed set $\tilde{X} \subset F((\epsilon))^n$ such that $\tilde{X} \cap F^n = X$. Our intention is to apply (2) to $\tilde{X}$, and write it as an intersection of basic max-closed sets. This and the relation $\tilde{X} \cap F^n = X$ are not prima facie sufficient to recover a similar expression for X, because the coefficients appearing in the definition of $\tilde{X}$ are, in general, elements of $F((\epsilon))$. Nevertheless we will be able to reduce to the case of coefficients in F by a sequence of formal manipulations.

The first step is the definition of $\tilde{X}$. The only properties that we require are that $\tilde{X}$ must be closed max-closed semilinear and that $\tilde{X} \cap F^n = X$. To this aim, write X as a finite union

$$X = X_1 \cup \dots \cup X_m$$

of relatively open semilinear sets. Call H_i the affine subspace generated by X_i for $i \in \{1 \dots m\}$, so by definition X_i is open in H_i. Now define

$$\tilde{X}_i = \overline{\{p \in H_i^* \mid B_\epsilon(p) \cap H_i^* \subset X_i^*\}}$$

where $B_\epsilon(p)$ denotes the ball of radius ϵ centered at p in the sup distance, and the overline denotes the topological closure. Clearly $X_i \subset \tilde{X}_i \subset X_i^*$, where the first inclusion follows from X_i being relatively open. Hence the following set

$$\tilde{X} = \text{max-closure}\left(\tilde{X}_1 \cup \dots \cup \tilde{X}_k\right)$$

meets our requirements (where max-closure denotes the closure under the maximum operation).

By (2), we can write $\tilde{X}$ as an intersection of basic closed max-closed subsets of $F((\epsilon))^n$. So it suffices to prove that given a basic closed max-closed $B \subset F((\epsilon))^n$, then $B' = B \cap F^n$ is an intersection of basic max-closed subsets of F^n. Let B' be

$$B' = \left\{(x_1 \dots x_n) \in F^n \;\middle|\; a_1x_1 + \cdots + a_nx_n \geqslant c\right\}$$

with $a_1 \dots a_n, c \in F((\epsilon))$ satisfying the condition $a_i \geqslant 0$ for $i \neq k_B$. We will prove that B' is a finite positive Boolean combination of sets of the form

$$\left\{(x_1 \dots x_n) \in F^n \;\middle|\; a'_1x_1 + \cdots + a'_nx_n \succ c'\right\}$$

with $a'_1 \dots a'_n, c' \in F$ all satisfying $a'_i \geqslant 0$ for $i \neq k_B$. From this follows our assertion.

Multiplying the equation

$$a_1x_1 + \cdots + a_nx_n \geqslant c$$

by a suitable power of ϵ we can assume that all the coefficients a_i are in $F[[\epsilon]]$. We will proceed by induction on the number of the coefficients a_i which are not in F. If this number is 0, the only problem is that c may not be in F. Either c has infinite magnitude (i.e., the leading monomial has negative degree), or it is of the form $c = c_0 + \epsilon c_{>0}$ with $c_0 \in F$ and $c_{>0} \in F[[\epsilon]]$. In the first case, B' can only be all of F^n or the empty set. In the second case, B' is

$$B' = \left\{(x_1 \dots x_n) \in F^n \;\middle|\; a_1x_1 + \cdots + a_nx_n \succ c_0\right\}$$

where $\succ$ is $\geqslant$ if $c_{>0} \leqslant 0$, and $\succ$ is $>$ if $c_{>0} > 0$.

Assume that the cardinality of the set $I \stackrel{def}{=} \{i \mid a_i \notin F\}$ is a positive number m. Write $a_i = (a_i)_0 + \epsilon(a_i)_{>0}$ with $(a_i)_0 \in F$. As before, if c has infinite magnitude, then B' is trivial, hence we can also write $c = c_0 + \epsilon c_{>0}$. Then $x \in F^n$ is an element of B' if and only if

$$\begin{gathered} (a_1)_0x_1 + \cdots + (a_n)_0x_n \geqslant c_0 \\ \wedge \\ \big(\; (a_1)_0x_1 + \cdots + (a_n)_0x_n > c_0 \quad \vee \quad (a_1)_{>0}x_1 + \cdots + (a_n)_{>0}x_n \geqslant c_{>0} \;\big) \end{gathered}$$

The first and the second conditions in this expression are immediately of the required form. The third condition is relevant only when

$$(a_1)_0x_1 + \cdots + (a_n)_0x_n = c_0$$

because if $<$ holds then $x \notin B'$ by the first condition, and if $>$ holds then $x \in B'$ by the second condition. We will bring this condition in an equivalent form with less than m coefficients which are not in F.

Let h be such that $|(a_h)_{>0}|$ is maximal, clearly $h \in I$. Dividing

$$(a_1)_{>0}x_1 + \cdots + (a_n)_{>0}x_n \geqslant c_{>0}$$

by $|(a_h)_{>0}|$ we obtain an equivalent condition such that the coefficient of x_h is $\pm 1 \in F$. However, in order to apply the inductive hypothesis, we also need to ensure that all the coefficients except that of x_k are non negative. This can be achieved by adding a suitable multiple of

$$(a_1)_0 x_1 + \cdots + (a_n)_0 x_n = c_0$$

to obtain

$$\left(\frac{(a_1)_{>0}}{|(a_h)_{>0}|} + N \cdot (a_1)_0\right) x_1 + \cdots + \left(\frac{(a_n)_{>0}}{|(a_h)_{>0}|} + N \cdot (a_n)_0\right) x_n \geqslant \frac{c_{>0}}{|(a_h)_{>0}|} + N \cdot c_0$$

in fact, $(a_i)_{>0}$ can be negative only if $(a_i)_0$ is positive. □

Proof of Theorem 3.2 (3)–(4). Clearly the relations in our finite basis are semilinear, closed, and max-closed, hence the *only if* of (3) is immediate. Topological closure is preserved by finite intersections. Therefore, to establish the *only-if* part for (4), we only have to check that projections of semilinear closed sets are closed. For a contradiction, pick a point x_0 in the boundary of $\pi(X)$, then for all $\epsilon > 0$ there is a $x(\epsilon) \in X$ such that $|\pi(x(\epsilon)) - x|_\infty < \epsilon$. As usual, by definable choice, we can assume that the function $\epsilon \mapsto x(\epsilon)$ is semilinear, and taking the limit for $\epsilon \to 0$ we get an $x \in X$ such that $\pi(x) = x_0$.

For the *if* part, first we prove (4). By (2), it suffices to show that basic closed max-closed sets are primitive positive definable using our relations. Consider the set

$$X = \bigcup_i \left\{(x_1 \dots x_n) \mid a_{i,1}x_1 + \cdots + a_{i,n}x_n \geqslant c_i\right\}$$

with $a_{i,j} \geqslant 0$ for all i and all $j \neq k$. We can write equivalently $X = \bigcup_{i=1}^m X_i$ where X_i denotes the set defined by the formula

$$\beta_i x_k \leqslant \sum_j \alpha_{i,j} x_j - c_i$$

and the coefficients β_i, $\alpha_{i,j}$, and c_i are chosen in such a way that they are all integers with $\beta_i > 0$ and $\alpha_{i,j} \geqslant 0$.

First we prove that X_i is primitive positive definable using $\{1, -1, S_1, S_2\}$. To this aim, it suffices to show primitive positive definitions of the sets defined by $2^M x \leqslant y$ and $y \leqslant z_1 + \cdots + z_N$ for any M and N. In fact, intersecting them and projecting along y one obtains the set $2^M x \leqslant z_1 + \cdots + z_N$, and assigning $x_1 \dots x_n$ or ± 1 to the variables x and $z_1 \dots z_N$ it is easy to obtain X_i. The set $2^M x \leqslant y$ is defined inductively on M by

$$2^{M+1} \leqslant y \quad \Leftrightarrow \quad \exists t\ 2x \leqslant t \wedge 2^M t \leqslant y$$

And the set $y \leqslant z_1 + \cdots + z_N$ is defined inductively on N by

$$y \leqslant z_1 + \cdots + z_{N+1} \quad \Leftrightarrow \quad \exists t\ y \leqslant t + z_{N+1} \wedge t \leqslant z_1 + \cdots + z_N$$

Now we show that the union of the sets X_i is primitive positive definable. First we replace the variable x_k on the left hand side in the definition of each of these sets (remember that k is fixed) with a new variable x'_i. Thus we end up with the following sets

$$(x'_i, x_1 \dots x_n) \in X'_i \quad \Leftrightarrow \quad \beta_i x'_i \leqslant \sum_j \alpha_{i,j} x_j - c_i$$

remember that we assumed, without loss of generality, that all coefficients β_i are strictly positive. We can combine the sets X'_i to form a new definition of X

$$(x_1 \dots x_n) \in X \quad \Leftrightarrow \quad \exists x'_1 \dots x'_m \begin{cases} (x'_1, x_1 \dots x_n) \in X'_1 \\ \qquad \wedge \\ (x'_m, x_1 \dots x_n) \in X'_m \\ \qquad \wedge \\ x_k \leqslant x'_1 \vee \dots \vee x_k \leqslant x'_m \end{cases}$$

Now it remains to show a primitive positive definition of the last clause

$$x_k \leqslant x'_1 \vee \dots \vee x_k \leqslant x'_m$$

We can proceed by induction on m observing that it is equivalent to

$$\exists t \ (x_k \leqslant t \vee x_k \leqslant x'_m) \wedge (t \leqslant x'_1 \vee \dots \vee t \leqslant x'_{m-1})$$

The *if* part of (3) is analogous, using (1) in place of (2), and observing that the relations $x < y \vee x \leqslant z$ and $x < y \vee x < z$ are primitive positive definable. $\square$

Proof of Theorem 3.4. We will concentrate on point (1). Point (2) is analogous to Theorem 3.2 (3)–(4). Denote by $\tau_k \colon \mathbb{Q}^n \to \mathbb{Q}^n$ the translation by $k \in \mathbb{Q}$ applied componentwise

$$\tau_k \colon (x_1 \dots x_n) \mapsto (x_1 + k \dots x_n + k)$$

Applying Theorem 3.2, we can write X as an intersection of basic max-closed sets X_i. Let X'_i denote the set

$$X'_i = \bigcap_{k \in \mathbb{Q}} \tau_k(X_i)$$

Since X is translation invariant, for all i, we have $X \subset X'_i \subset X_i$, hence X is the intersection of the sets X'_i. The sets X'_i are clearly translation invariant, therefore we only need to prove that each of them is an intersection of basic tropically convex sets. In other words, given a basic max-closed set B, we want to prove that

$$B' \stackrel{\text{def}}{=} \bigcap_{k \in \mathbb{Q}} \tau_k(B)$$

is a finite intersection of basic tropically convex sets. Let B be

$$B = \bigcup_j \left\{ (x_1 \dots x_n) \mid a_{j,1} x_1 + \dots + a_{j,n} x_n \succ_j c_j \right\}$$

with $a_{j,l} \geqslant 0$ for all $l \neq k_B$. It suffices to write B' as a positive Boolean combination of sets of the form

$$\left\{ (x_1 \dots x_n) \mid a'_{j,1} x_1 + \dots + a'_{j,n} x_n \succ'_j c'_j \right\}$$

with $\sum_l a'_{j,l} = 0$ and $a'_{j,l} \geqslant 0$ for all $l \neq k_B$. By definition we have

$$B' = \bigcap_{k\in\mathbb{Q}} \bigcup_{j} \left\{ (x_1 \dots x_n) \mid a_{j,1}x_1 + \dots + a_{j,n}x_n - ks_j \succ_j c_j \right\}$$

where $s_j = \sum_l a_{j,l}$. We can separate the indices j satisfying $s_j = 0$ writing $B' = B'_0 \cup B'_{\neq 0}$ where

$$B'_0 = \bigcup_{j|s_j=0} \left\{ (x_1 \dots x_n) \mid a_{j,1}x_1 + \dots + a_{j,n}x_n \succ_j c_j \right\}$$

$$B'_{\neq 0} = \bigcap_{k\in\mathbb{Q}} \bigcup_{j|s_j\neq 0} \left\{ (x_1 \dots x_n) \mid a_{j,1}x_1 + \dots + a_{j,n}x_n - ks_j \succ_j c_j \right\}$$

Now, B'_0 is already in the required form. Rearranging the definition of $B'_{\neq 0}$ we get

$$x \in B'_{\neq 0} \Leftrightarrow \mathbb{Q} = \bigcup_{j|s_j\neq 0} \left\{ k \in \mathbb{Q} \mid a_{j,1}x_1 + \dots + a_{j,n}x_n - c_j \succ_j ks_j \right\}$$

On the right hand side, we have a union of left and right (according to the sign of s_j) half lines. This union covers all of $\mathbb{Q}$ if and only if it contains two opposite overlapping half lines. Hence

$$B'_{\neq 0} = \bigcup_{j,l|s_j<0\wedge s_l>0} \left\{ (x_1 \dots x_n) \;\middle|\; \sum_{i=1}^{n} (a_{j,i}s_l - a_{l,i}s_j)x_i \succ_{j,l} c_js_l - c_ls_j \right\}$$

where $\succ_{j,l}$ is $>$ if $\succ_j$ and $\succ_l$ are both $>$, and it is $\geqslant$ otherwise. □

4. A Duality for Max-plus-average Inequalities

Let $\mathcal{O}_n$ be the class of functions mapping $\left(\mathbb{Q} \cup \{+\infty\}\right)^n$ to $\mathbb{Q} \cup \{+\infty\}$ of either of the following forms

$$(x_1 \dots x_n) \mapsto \max(x_{j_1} + k_1 \dots x_{j_m} + k_m)$$
$$(x_1 \dots x_n) \mapsto \min(x_{j_1} + k_1 \dots x_{j_m} + k_m)$$
$$(x_1 \dots x_n) \mapsto \frac{\alpha_1 x_{j_1} + \dots + \alpha_m x_{j_m}}{\alpha_1 + \dots + \alpha_m} + k$$

where $k, k_i \in \mathbb{Q}$ and $\alpha_i \in \mathbb{Q}^{>0}$.

For any given vector $\bar{o} \in \mathcal{O}_n^n$ of n operators in $\mathcal{O}_n$, we consider the following satisfiability problems: the *primal* $P(\bar{o})$ and the *dual* $D(\bar{o})$

$$P(\bar{o})\colon \begin{cases} \bar{x} \in \mathbb{Q}^n \\ \bar{x} < \bar{o}(\bar{x}) \end{cases} \qquad D(\bar{o})\colon \begin{cases} \bar{y} \in \left(\mathbb{Q} \cup \{+\infty\}\right)^n \setminus \{+\infty\}^n \\ \bar{y} \geqslant \bar{o}(\bar{y}) \end{cases}$$

where $<$ and $\geqslant$ are meant to hold component-wise. We intend to prove the following result.

Theorem 4.1. For any $\bar{o} \in \mathcal{O}_n^n$ one and only one of the problems $P(\bar{o})$ and $D(\bar{o})$ is satisfiable.

For the proof of Theorem 4.1, we will make use of zero-sum stochastic games with perfect information, in the flavours known as the discounted and the limiting average payoff. In this context, it is more natural to work over the domain $\mathbb{R}$ of the real numbers instead of $\mathbb{Q}$. Observing that the satisfiability of $P(\bar{o})$ and $D(\bar{o})$ is insensitive to the domain, for the rest of this section, we will work with the real numbers.

The set-up for a stochastic game is a directed graph G with vertex- and edge-labels, all vertices of G must have at least one out-edge. Each vertex of G is either assigned to one of the players, in this case it is labelled with one of the symbols MAX and MIN, or it is a *stochastic vertex*, and in this case it carries the label s. Each edge e of G has a label $\mathrm{po}(e) \in \mathbb{R}$, which represents the payoff earned by MAX (or the penalty incurred by MIN) when that edge is traversed. The out-edges of a stochastic vertex have an additional label $\mathrm{pr}(e)$, which is a rational number representing the probability that each edge is taken when exiting that specific vertex: clearly, the probabilities of the out-edges of each stochastic vertex must sum to 1.

A stochastic game G is played by moving a token along the directed edges of G. First the token is placed on a vertex which we call the *initial position*. Then, repeatedly, the token is moved by MAX when it is on a MAX-vertex, by MIN when it is on a MIN-vertex, and at random when it is on an s-vertex (according to the probabilities assigned to the out-edges of that vertex). A play never ends.

Let $e_1, e_2 \ldots$ be the edges traversed during a play p on G. The *discounted payoff* $\boldsymbol{\nu}_\beta(p)$ of p with discounting factor $\beta \in [0, 1[$ is

$$\boldsymbol{\nu}_\beta(p) \overset{def}{=} (1-\beta) \sum_{i=1}^{\infty} \mathrm{po}(e_i)\beta^{i-1}$$

and the *limiting average payoff* is

$$\boldsymbol{\nu}_1(p) \overset{def}{=} \liminf_{T\to\infty} \frac{1}{T} \sum_{i=1}^{T} \mathrm{po}(e_i)$$

In such formulas we suppress the dependency on G, to ease the notation.

In general, a strategy for player $P \in \{\text{MAX}, \text{MIN}\}$ is a probability distribution on the functions mapping every partial play $v_0 \overset{e_1}{\to} v_1 \ldots \overset{e_T}{\to} v_T$ ending in a P-vertex v_T to one of the out-edges of v_T: such functions are called *pure strategies*. Playing according to a strategy π means to choose one of the pure strategies f at random with the probability distribution π, and then choosing one's moves by calling f. Given two strategies π_{MAX} and π_{MIN} for the two players, one can thus define the value $\boldsymbol{\nu}_\beta(v, \pi_{\text{MAX}}, \pi_{\text{MIN}})$ as the expected $\boldsymbol{\nu}_\beta$ of a play generated by the given pair of strategies from the initial position $v_0 = v$.

For a given game G and $\beta \in [0, 1]$, a strategy π_{MAX} for MAX is optimal if for all vertices v it maximizes the quantity

$$\boldsymbol{\nu}_\beta(v, \pi_{\text{MAX}}, \cdot) \overset{def}{=} \inf_{\pi_{\text{MIN}}} \boldsymbol{\nu}_\beta(v, \pi_{\text{MAX}}, \pi_{\text{MIN}})$$

Conversely a strategy π_{MIN} for MIN is optimal if it minimizes

$$\boldsymbol{v}_\beta(v, \cdot, \pi_{\text{MIN}}) \stackrel{def}{=} \sup_{\pi_{\text{MAX}}} \boldsymbol{v}_\beta(v, \pi_{\text{MAX}}, \pi_{\text{MIN}})$$

Given a stochastic game G and a $\beta \in [0,1]$, there are optimal strategies for each player, moreover, calling π_{MAX} and π_{MIN} two optimal strategies

$$\boldsymbol{v}_\beta(v, \pi_{\text{MAX}}, \cdot) = \boldsymbol{v}_\beta(v, \cdot, \pi_{\text{MIN}})$$

holds for any initial position v. This value does not depend on the pair of optimal strategies chosen, but only on β and the initial position v (and G), hence we may denote it by the notation $\boldsymbol{v}_\beta(v)$. The vector $\boldsymbol{v}_\beta \stackrel{def}{=} (\boldsymbol{v}_\beta(v))_{v \in V(G)}$ is called *value vector* of G with discount factor β.

In the specific case of perfect information stochastic games, one can show that there are optimal strategies which are also pure and *stationary* (also known as *memoryless* or *positional*). In other words there are optimal strategies that prescribe for each possible position of the token (i.e. each MAX- or MIN-vertex) one definite next move. If we further restrict our consideration to the discounted payoff criterion (i.e. $\beta < 1$), then the value vector of a game G admits an explicit description through a condition known as the *limit discount equation* (see [15] Theorem 4.3.2 and also Definition 4.3.12)

$$\boldsymbol{v}_\beta(v) = \begin{cases} \max_{(v,w)\in E(G)} (1-\beta)\,\text{po}(v,w) + \beta \boldsymbol{v}_\beta(w) & \text{if } v \text{ is a MAX-vertex} \\ \min_{(v,w)\in E(G)} (1-\beta)\,\text{po}(v,w) + \beta \boldsymbol{v}_\beta(w) & \text{if } v \text{ is a MIN-vertex} \\ \sum_{(v,w)\in E(G)} \text{pr}(v,w)\big((1-\beta)\,\text{po}(v,w) + \beta \boldsymbol{v}_\beta(w)\big) & \\ & \text{if } v \text{ is a stochastic vertex} \end{cases}$$

In particular we will need the following fact (see [15] Theorem 6.3.7 plus Theorem 6.3.5 and its proof).

Fact 4.2. For any stochastic game G with perfect information

(1) both players possess pure stationary strategies π_{MAX} and π_{MIN} which are optimal for the limiting average payoff and for all discount factors β sufficiently close to 1
(2) calling $\boldsymbol{v}_\beta$ the value vector of G with discount factor β, the value vector for the limiting average payoff can be written as $\boldsymbol{v}_1 = \lim_{\beta\to 1} \boldsymbol{v}_\beta$
(3) the solution $\boldsymbol{v}_\beta$ to the limit discount equation can be written as a power series in $(1-\beta)$—more precisely, consider the field $\mathbb{R}((x))$ of formal Laurent series in x ordered by $0 < x \ll 1$, let $\beta = 1 - x \in \mathbb{R}((x))$, then the limit discount equation for $\boldsymbol{v}_\beta$ has a formal solution in $\mathbb{R}((x))^n$ which, moreover, is convergent in a neighbourhood of 0.

An alternative approach to ours would have been to use the normal form described in [9]. In fact, probably, the duality and the existence of a normal form in the sense of [9] imply each other. However we obtain our result via a different method.

We now map each vector of operators $\bar{o} \in \mathcal{O}_n^n$ to a stochastic game $G_{\bar{o}}$. First we place one vertex in $G_{\bar{o}}$ per component of $\bar{o}$, i.e. formally we fix $V(G_{\bar{o}}) = \{v_1 \dots v_n\}$. Then we stipulate that v_i is of type MAX, MIN, or S according to whether $\bar{o}_i$ is a max, min, or weighted average operator respectively. Finally, for each i, let

$$\bar{o}_i\colon (x_1 \dots x_n) \mapsto \Box_i(x_{j^i_1} + k^i_1 \dots x_{j^i_{m^i}} + k^i_{m^i})$$

where $\Box_i$ can be either of

$$\Box_i\colon (y_1 \dots y_{m^i}) \mapsto \begin{cases} \max(y_1 \dots y_{m^i}) \\ \min(y_1 \dots y_{m^i}) \\ \frac{y_1\alpha^i_1 + \dots + y_{m^i}\alpha^i_{m^i}}{\alpha^i_1 + \dots + \alpha^i_{m^i}} \end{cases}$$

then we introduce an edge $(v_i, v_{j^i_l}) \in E(G_{\bar{o}})$ for each pair of $i \in \{1 \dots n\}$ and $l \in \{1 \dots m_i\}$ with payoff

$$\mathrm{po}(v_i, v_{j^i_l}) = k^i_l$$

and, if $\Box_i$ is a weighted average, with probability

$$\mathrm{pr}(v_i, v_{j^i_l}) = \frac{\alpha^i_l}{\alpha^i_1 + \dots + \alpha^i_{m^i}}$$

Lemma 4.3. Let $\bar{o} \in \mathcal{O}^n_n$ be a vector of operators, and let $\boldsymbol{v}_1$ denote the value vector of the game $G_{\bar{o}}$ with the limiting average payoff. The problem $P(\bar{o})$ is satisfiable if and only if $\boldsymbol{v}_1(v_i) > 0$ for all vertices v_i of $G_{\bar{o}}$.

Proof (if direction). Using Fact 4.2(3), let

$$\boldsymbol{v}_\beta = \sum_{i=0}^{\infty} \bar{a}_i (1-\beta)^i$$

be the value vector of $G_{\bar{o}}$ with discount β. It follows from Fact 4.2(2) that $(\bar{a}_0)_j = \boldsymbol{v}_1(v_j) > 0$ for all $j \in \{1 \dots n\}$. For N denoting a (large) real number, define $\bar{x}_N \in \mathbb{R}^n$ by

$$(\bar{x}_N)_j = N(\bar{a}_0)_j + (\bar{a}_1)_j$$

We claim that, for large enough N, the vector $\bar{x}_N$ satisfies $P(\bar{o})$.

Let us consider the limit discount equation for MAX-, MIN-, and S-vertices separately. We will show that the condition on each vertex v_j implies that the corresponding coordinate $(\bar{x}_N)_j$ satisfies $P(\bar{o})$.

If v_j is of type MAX, then

$$\sum_{i=0}^{\infty} (\bar{a}_i)_j (1-\beta)^i = \max_{l \,|\, (v_j, v_l) \in E(G_{\bar{o}})} \Big((1-\beta)\,\mathrm{po}(v_j, v_l) + \beta \sum_{i=0}^{\infty} (\bar{a}_i)_l (1-\beta)^i \Big)$$

Since series are ordered lexicographically, truncating both sides to the first two terms commutes with the max operation

$$(\bar{a}_0)_j + (\bar{a}_1)_j(1-\beta) = \max_{l \,|\, (v_j, v_l) \in E(G_{\bar{o}})} (\bar{a}_0)_l + \big(\mathrm{po}(v_j, v_l) - (\bar{a}_0)_l + (\bar{a}_1)_l\big)(1-\beta)$$

From which it follows that

$$\begin{aligned}(\bar{a}_0)_j &= \max_{l \mid (v_j, v_l) \in E(G_{\bar{o}})} (\bar{a}_0)_l \\ (\bar{a}_1)_j &= \max_{l \text{ realizing the max above}} \mathrm{po}(v_j, v_l) - (\bar{a}_0)_l + (\bar{a}_1)_l \\ &< \max_{l \text{ realizing the max above}} \mathrm{po}(v_j, v_l) + (\bar{a}_1)_l\end{aligned}$$

Hence, for large values of N

$$N(\bar{a}_0)_j + (\bar{a}_1)_j < \max_{l \mid (v_j, v_l) \in E(G_{\bar{o}})} N(\bar{a}_0)_l + \mathrm{po}(v_j, v_l) + (\bar{a}_1)_l$$

Vertices of type MIN are treated precisely as those of type MAX. Finally, for stochastic vertices, we get

$$(\bar{a}_0)_j + (\bar{a}_1)_j(1-\beta) = \sum_{l \mid (v_j, v_l) \in E(G_{\bar{o}})} (\bar{a}_0)_l + \big(\mathrm{po}(v_j, v_l) - (\bar{a}_0)_l + (\bar{a}_1)_l\big)(1-\beta)$$

whence, multiplying by $(1-\beta)^{-1}$ and evaluating at $1-\beta = 1/N$

$$\begin{aligned}N(\bar{a}_0)_j + (\bar{a}_1)_j &= \sum_{l \mid (v_j, v_l) \in E(G_{\bar{o}})} N(\bar{a}_0)_l + \mathrm{po}(v_j, v_l) - (\bar{a}_0)_l + (\bar{a}_1)_l \\ &< \sum_{l \mid (v_j, v_l) \in E(G_{\bar{o}})} N(\bar{a}_0)_l + \mathrm{po}(v_j, v_l) + (\bar{a}_1)_l\end{aligned}$$

□

Proof (only if direction). Fix a solution $\bar{x}$ of $P(\bar{o})$. We claim that the following strategy π_{MAX} for MAX (which, incidentally, is stationary and pure) satisfies $\boldsymbol{\nu}_1(v_i, \pi_{\text{MAX}}, \cdot) > 0$ for all vertices v_i of $G_{\bar{o}}$. Let v_i be a MAX-vertex, assume that it is MAX's turn and the token rests on v_i, we will say which of the out-edges of v_i the player MAX will elect to move the token along. Let us consider the operator $\bar{o}_i$, which necessarily has the following form

$$\bar{o}_i \colon (x_1 \dots x_n) \mapsto \max(x_{j^i_1} + k^i_1 \dots x_{j^i_{m^i}} + k^i_{m^i})$$

The player MAX then moves the token to a vertex $v_{j^i_l}$ such that $\bar{x}_{j^i_l} + k^i_l$ realizes the maximum (which one he chooses is immaterial).

Now we intend to prove that $\boldsymbol{\nu}_1(v_i, \pi_{\text{MAX}}, \cdot) > 0$. To this aim, because of Fact 4.2(1), it is enough to test our strategy π_{MAX} against all stationary and pure strategies for MIN. Let π_{MIN} be such a strategy. A play generated by the pair of strategies $(\pi_{\text{MAX}}, \pi_{\text{MIN}})$ is a Markov chain on the finite state-space $\{v_1 \dots v_n\}$. The average time spent by the process in each state during plays starting at v_i must converge to a stable distribution μ. The limiting average payoff $\boldsymbol{\nu}_1(v_i, \pi_{\text{MAX}}, \pi_{\text{MIN}})$ can be described using μ by

$$\boldsymbol{\nu}_1(v_i, \pi_{\text{MAX}}, \pi_{\text{MIN}}) = \sum_{a,b \mid (v_a, v_b) \in E(G_{\bar{o}})} \mu(v_a)\, \mathrm{pr}'(v_a, v_b)\, \mathrm{po}(v_a, v_b)$$

where

$$\mathrm{pr}'(\nu_a, \nu_b) = \begin{cases} \mathrm{pr}(\nu_a, \nu_b) & \text{if } \nu_a \text{ is a stochastic vertex} \\ 1 & \text{if } \nu_a \text{ is a MAX-vertex and } \pi_{\text{MAX}}(\cdots \to \nu_a) = \nu_b \\ 1 & \text{if } \nu_a \text{ is a MIN-vertex and } \pi_{\text{MIN}}(\cdots \to \nu_a) = \nu_b \\ 0 & \text{otherwise} \end{cases}$$

We need to prove that the above quantity is strictly positive.

For all $a \in \{1 \dots n\}$ we have

$$\bar{x}_a < \sum_{b \mid (\nu_a, \nu_b) \in E(G_{\bar{o}})} \mathrm{pr}'(\nu_a, \nu_b)(\bar{x}_b + \mathrm{po}(\nu_a, \nu_b))$$

which, for MIN- and S-vertices, is an immediate consequence of $\bar{x}$ being a solution to $P(\bar{o})$, and, when ν_a is a MAX-vertex, it follows for our choice of the strategy π_{MAX}. Splitting the sum on the right hand side

$$\bar{x}_a - \sum_{b \mid (\nu_a, \nu_b) \in E(G_{\bar{o}})} \mathrm{pr}'(\nu_a, \nu_b)\bar{x}_b < \sum_{b \mid (\nu_a, \nu_b) \in E(G_{\bar{o}})} \mathrm{pr}'(\nu_a, \nu_b)\, \mathrm{po}(\nu_a, \nu_b)$$

Then multiplying by $\mu(\nu_a)$ and taking the sum over a

$$\sum_{a=1}^{n} \mu(\nu_a)\bar{x}_a - \sum_{a,b \mid (\nu_a, \nu_b) \in E(G_{\bar{o}})} \mu(\nu_a)\, \mathrm{pr}'(\nu_a, \nu_b)\bar{x}_b < \sum_{a,b \mid (\nu_a, \nu_b) \in E(G_{\bar{o}})} \mu(\nu_a)\, \mathrm{pr}'(\nu_a, \nu_b)\, \mathrm{po}(\nu_a, \nu_b)$$

We need to prove that the left hand side is not negative. By μ being a stable distribution, we have

$$\mu(\nu_b) = \sum_{a \mid (\nu_a, \nu_b) \in E(G_{\bar{o}})} \mu(\nu_a)\, \mathrm{pr}'(\nu_a, \nu_b)$$

hence

$$\begin{aligned} \sum_{a=1}^{n} \mu(\nu_a)\bar{x}_a - & \sum_{a,b \mid (\nu_a, \nu_b) \in E(G_{\bar{o}})} \mu(\nu_a)\, \mathrm{pr}'(\nu_a, \nu_b)\bar{x}_b \\ &= \sum_{a=1}^{n} \mu(\nu_a)\bar{x}_a - \sum_{b=1}^{n} \Big(\sum_{a \mid (\nu_a, \nu_b) \in E(G_{\bar{o}})} \mu(\nu_a)\, \mathrm{pr}'(\nu_a, \nu_b) \Big)\bar{x}_b \\ &= \sum_{a=1}^{n} \mu(\nu_a)\bar{x}_a - \sum_{b=1}^{n} \mu(\nu_b)\bar{x}_b = 0 \end{aligned}$$

□

Lemma 4.4. Let $\bar{o} \in \mathcal{O}_n^n$ be a vector of operators, and let $\boldsymbol{\nu}_1$ denote the value vector of the game $G_{\bar{o}}$ with the limiting average payoff. The problem $D(\bar{o})$ is satisfiable if and only if $\boldsymbol{\nu}_1(\nu_i) \leqslant 0$ for some vertex ν_i of $G_{\bar{o}}$.

The proof of this lemma is similar to the proof of Lemma 4.3 above, we will therefore only point out the relevant differences.

Proof (if direction). As in the proof of Lemma 4.3, using Fact 4.2(3), let

$$\boldsymbol{v}_\beta = \sum_{i=0}^{\infty} \bar{a}_i (1-\beta)^i$$

be the value vector of $G_{\bar{o}}$ with discount β. Let $\boldsymbol{v}_1^{\min} \leqslant 0$ denote the minimum of $\boldsymbol{v}_1(v_j) = (\bar{a}_0)_j$ over all vertices v_j. We define $\bar{y} \in (\mathbb{R} \cup \{+\infty\})^n$ satisfying $D(\bar{o})$ by

$$\bar{y}_j = \begin{cases} (\bar{a}_1)_j & \text{if } (\bar{a}_0)_j = \boldsymbol{v}_1^{\min} \\ +\infty & \text{otherwise} \end{cases}$$

We want to show that the condition imposed by the limit discount equation on each vertex v_j implies that the corresponding coordinate $\bar{y}_j$ satisfies $D(\bar{o})$. As opposed to Lemma 4.3, we can concentrate on those vertices that have minimal value, i.e. $(\bar{a}_0)_j = \boldsymbol{v}_1^{\min}$, because those with larger value give rise to $+\infty$ coordinates for which $D(\bar{o})$ is automatically satisfied. If v_j is of type MIN, then, considering the limit discount equation and truncating the series to the first two terms, we have

$$\boldsymbol{v}_1^{\min} + (\bar{a}_1)_j (1-\beta) = \min_{l \mid (v_j, v_l) \in E(G_{\bar{o}})} (\bar{a}_0)_l + \big(\operatorname{po}(v_j, v_l) - (\bar{a}_0)_l + (\bar{a}_1)_l\big)(1-\beta)$$

The min can be restricted to those values of l for which $(\bar{a}_0)_l = \boldsymbol{v}_1^{\min}$

$$\boldsymbol{v}_1^{\min} + (\bar{a}_1)_j (1-\beta) = \min_{l \mid (v_j, v_l) \in E(G_{\bar{o}}) \wedge (\bar{a}_0)_l = \boldsymbol{v}_1^{\min}} \boldsymbol{v}_1^{\min} + \big(\operatorname{po}(v_j, v_l) - \boldsymbol{v}_1^{\min} + (\bar{a}_1)_l\big)(1-\beta)$$

Subtracting $\boldsymbol{v}_1^{\min}$ from both sides and multiplying by $(1-\beta)^{-1}$

$$\begin{aligned} \bar{y}_j = (\bar{a}_1)_j &= \min_{l \mid (v_j, v_l) \in E(G_{\bar{o}}) \wedge (\bar{a}_0)_l = \boldsymbol{v}_1^{\min}} \operatorname{po}(v_j, v_l) - \boldsymbol{v}_1^{\min} + (\bar{a}_1)_l \\ &\geqslant \min_{l \mid (v_j, v_l) \in E(G_{\bar{o}}) \wedge (\bar{a}_0)_l = \boldsymbol{v}_1^{\min}} \operatorname{po}(v_j, v_l) + (\bar{a}_1)_l \\ &= \min_{l \mid (v_j, v_l) \in E(G_{\bar{o}})} \operatorname{po}(v_j, v_l) + \bar{y}_l \end{aligned}$$

For v_j of type MAX we obtain as before

$$\boldsymbol{v}_1^{\min} + (\bar{a}_1)_j (1-\beta) = \max_{l \mid (v_j, v_l) \in E(G_{\bar{o}})} (\bar{a}_0)_l + \big(\operatorname{po}(v_j, v_l) - (\bar{a}_0)_l + (\bar{a}_1)_l\big)(1-\beta)$$

hence

$$\boldsymbol{v}_1^{\min} = \max_{l \mid (v_j, v_l) \in E(G_{\bar{o}})} (\bar{a}_0)_l$$

and, by the minimality of $\boldsymbol{v}_1^{\min}$, we have that $(\bar{a}_0)_l = \boldsymbol{v}_1^{\min}$ for all l such that $(v_j, v_l) \in E(G_{\bar{o}})$. Therefore

$$\boldsymbol{v}_1^{\min} + (\bar{a}_1)_j (1-\beta) = \max_{l \mid (v_j, v_l) \in E(G_{\bar{o}})} \boldsymbol{v}_1^{\min} + \big(\operatorname{po}(v_j, v_l) - (\bar{a}_0)_l + (\bar{a}_1)_l\big)(1-\beta)$$

and we conclude again subtracting $\boldsymbol{v}_1^{\min}$ and multiplying by $(1-\beta)^{-1}$.

Finally stochastic vertices are dealt with as MAX-vertices observing that

$$\boldsymbol{v}_1^{\min} = \sum_{l \mid (v_j, v_l) \in E(G_{\bar{o}})} \operatorname{pr}(v_j, v_l)(\bar{a}_0)_l$$

implies, again by the minimality of $\boldsymbol{v}_1^{\min}$, that $(\bar{a}_0)_l = \boldsymbol{v}_1^{\min}$ for all l such that $(v_j, v_l) \in E(G_{\bar{o}})$. □

Proof (only if direction). Fix a solution $\bar{y} \in (\mathbb{R} \cup \{+\infty\})^n \setminus \{+\infty\}^n$ of $D(\bar{o})$. We produce a strategy π_{MIN} for MIN that satisfies $\boldsymbol{v}_1(v_i, \pi_{\text{MAX}}, \cdot) \leqslant 0$ for some v_i: specifically $\boldsymbol{v}_1(v_i, \pi_{\text{MAX}}, \cdot) \leqslant 0$ if $\bar{y}_i$ is finite. Assume that it is MIN's turn and the token rests on v_i. If $\bar{y}_i = +\infty$, then it is immaterial which move MIN chooses. Otherwise, let us consider the operator

$$\bar{o}_i \colon (x_1 \dots x_n) \mapsto \min(x_{j_1^i} + k_1^i \dots x_{j_{m^i}^i} + k_{m^i}^i)$$

The player MIN moves to a vertex $v_{j_l^i}$ such that $\bar{x}_{j_l^i} + k_l^i$ realizes the min.

As in the proof of Lemma 4.3, pick a stationary pure strategy π_{MAX} for MAX, and consider the Markov process defined by π_{MIN} and π_{MAX} starting from vertex v_i. If $\bar{y}_i$ is finite, then it is easy to see that, by our choice of π_{MIN}, no vertex $\boldsymbol{v}_j$ with $\bar{y}_j = +\infty$ can be reached by a play starting from $\boldsymbol{v}_i$. Let μ be the stable distribution to which the Markov chain started from v_i converges. The limiting average payoff is

$$\boldsymbol{v}_1(v_i, \pi_{\text{MAX}}, \pi_{\text{MIN}}) = \sum_{a,b \mid (v_a, v_b) \in E(G_{\bar{o}})} \mu(v_a) \operatorname{pr}'(v_a, v_b) \operatorname{po}(v_a, v_b)$$

where

$$\operatorname{pr}'(v_a, v_b) = \begin{cases} \operatorname{pr}(v_a, v_b) & \text{if } v_a \text{ is a stochastic vertex} \\ 1 & \text{if } v_a \text{ is a MAX-vertex and } \pi_{\text{MAX}}(\dots \to v_a) = v_b \\ 1 & \text{if } v_a \text{ is a MIN-vertex and } \pi_{\text{MIN}}(\dots \to v_a) = v_b \\ 0 & \text{otherwise} \end{cases}$$

For $a \in \{1 \dots n\}$ such that $\bar{y}_a$ is finite, we have

$$\bar{y}_a \geqslant \sum_{b \mid (v_a, v_b) \in E(G_{\bar{o}})} \operatorname{pr}'(v_a, v_b)(\bar{y}_b + \operatorname{po}(v_a, v_b))$$

and proceeding as in the proof of Lemma 4.4

$$\begin{aligned} \sum_{a \mid \bar{y}_a \neq +\infty} \mu(v_a)\bar{y}_a - \sum_{a,b \mid (v_a, v_b) \in E(G_{\bar{o}}) \wedge \bar{y}_a \neq +\infty} \mu(v_a) \operatorname{pr}'(v_a, v_b)\bar{y}_b \\ \geqslant \sum_{a,b \mid (v_a, v_b) \in E(G_{\bar{o}}) \wedge \bar{y}_a \neq +\infty} \mu(v_a) \operatorname{pr}'(v_a, v_b) \operatorname{po}(v_a, v_b) \end{aligned}$$

where the sums can be restricted to $\bar{y}_a \neq +\infty$ because the vertices v_a with $\bar{y}_a = +\infty$ are unreachable. We can conclude using the fact that μ is a stable distribution. □

Theorem 4.1 follows immediately from Lemma 4.3 and Lemma 4.4.

5. Complexity of Tropically Convex CSPs

In this section, we will apply our duality to tropically convex constraint satisfaction problems. By Theorem 3.4, we know that the tropically convex relations are precisely those primitive positive definable in the structure $\Gamma_t = (\mathbb{Q}; <, T_1, T_{-1}, S_3, M_0)$ where

$$T_{\pm 1}(x,y) \Leftrightarrow x \leqslant y \pm 1$$
$$S_3(x,y,z) \Leftrightarrow x \leqslant \frac{y+z}{2}$$
$$M_0(x,y,z) \Leftrightarrow x \leqslant \max(y,z)$$

Max-atoms (with constants in binary) is polynomial-time reducible to the CSP of Γ_t, but $\mathrm{CSP}(\Gamma_t)$ is more expressive. We intend to prove the following theorem.

Theorem 5.1. The problem $\mathrm{CSP}(\Gamma_t)$ is in $\mathsf{NP} \cap \text{co-}\mathsf{NP}$.

Instead of the finite constraint language of Γ_t, we could have chosen to work with basic tropically convex sets (in the sense of section 3) encoded with the constants expressed in binary. In view of Observation 3.6 this choice is immaterial. The reader can also check that the same proofs apply to both settings. Also, it is easy to extend Γ_t with relations of the form $x = c$ for rational constants c: the idea is to introduce a new variable z and replace $x = c$ with $x = c + z$ (which is primitive positive definable in Γ_t), then by translation invariance we have a solution if and only if we have a solution with $z = 0$. We begin by proving an analogue of Theorem 4.1 for non-strict inequalities.

Corollary 5.2. For any given vector of operators $\bar{o} \in \mathcal{O}_n^n$ we consider the following satisfiability problems: the *primal* $P'(\bar{o})$ and the *dual* $D'(\bar{o})$

$$P'(\bar{o})\colon \begin{cases} \bar{x} \in \mathbb{Q}^n \\ \bar{x} \leqslant \bar{o}(\bar{x}) \end{cases} \qquad D'(\bar{o})\colon \begin{cases} \bar{y} \in \big(\mathbb{Q} \cup \{+\infty\}\big)^n \setminus \{+\infty\}^n \\ \bar{y} > \bar{o}(\bar{y}) \end{cases}$$

where $\leqslant$ and $>$ are meant to hold component-wise, and we stipulate that $+\infty > +\infty$. Then one and only one of the problems $P'(\bar{o})$ and $D'(\bar{o})$ is satisfiable.

Proof. We claim that the problem $P'(\bar{o})$ is satisfiable if and only if the problem $P_\epsilon(\bar{o})$ defined as follows

$$\bar{x} \in \mathbb{Q}^n$$
$$\bar{x} < \bar{o}(\bar{x}) + \bar{1}\epsilon$$

where $\bar{1}$ denotes the vector $(1, 1 \dots) \in \mathbb{Q}^n$, is satisfiable for all $\epsilon > 0$. The only-if direction is, in fact, immediate. For the if direction, by definable choice, we can choose a solution $\bar{x}(\epsilon)$ of the second problem which is a semilinear function of ϵ. Hence the limit $\bar{x}_0 \stackrel{def}{=} \lim_{\epsilon \to 0^+} \bar{x}(\epsilon)$ exists and it is easy to check that $\bar{x}_0$ satisfies of $P'(\bar{o})$.

Applying Theorem 4.1 we get that $P_\epsilon(\bar{o})$ is satisfiable if and only if the following problem, $D_\epsilon(\bar{o})$

$$\begin{aligned}&\bar{y} \in \left(\mathbb{Q} \cup \{+\infty\}\right)^n \setminus \{+\infty\}^n\\&\bar{y} \geqslant \bar{o}(\bar{y}) + \bar{1}\epsilon\end{aligned}$$

is not satisfiable. In turn, it is immediate that $D_\epsilon(\bar{o})$ is satisfiable for some $\epsilon > 0$ if and only if $D'(\bar{o})$ is satisfiable. □

Before proceeding to the proof of Theorem 5.1 we need one last technical statement.

Definition 5.3. Consider a quantifier free semilinear formula $\phi(t, \bar{x})$ with rational coefficients, where t denotes a variable and $\bar{x}$ denotes a tuple of variables. We say that $\phi(t, \bar{x})$ is *satisfiable in* $0+$ if

$$\exists t_0 > 0 \; \forall t \in \,]0, t_0] \; \exists \bar{x} \; \phi(t, \bar{x})$$

Lemma 5.4. The problem, given ϕ as in Definition 5.3 with coefficients encoded in binary, of deciding whether ϕ is satisfiable in $0+$ is in NP.

Proof. Satisfiability for quantifier free semilinear formulas is in NP by standard linear programming techniques. In our case, consider the two-dimensional ordered vector space over $\mathbb{Q}$

$$V \stackrel{def}{=} \left\{ a + b\boldsymbol{\epsilon} \;\middle|\; a, b \in \mathbb{Q} \right\}$$

where the expression $a + b\boldsymbol{\epsilon}$ is just a formal replacement for (a, b), and the order is lexicographical – in other words, writing $\boldsymbol{\epsilon}$ for $0 + 1\boldsymbol{\epsilon} = (0, 1)$, we have $(0, 0) < (0, 1) \ll (1, 0)$ or $0 < \boldsymbol{\epsilon} \ll 1$. We claim that $\phi(t, \bar{x})$ is satisfiable in $0+$ if and only if $\phi(\boldsymbol{\epsilon}, \bar{x})$ is satisfiable in V. Before proving the claim, we may observe that this is, indeed, sufficient to conclude. In fact

$$\exists \bar{x} \in V^n \;\; \phi(\boldsymbol{\epsilon}, \bar{x})$$

is, by definition, equivalent to

$$\exists \bar{a}, \bar{b} \in \mathbb{Q}^n \;\; \phi(\boldsymbol{\epsilon}, \bar{a} + \bar{b}\boldsymbol{\epsilon})$$

and the subformula $\phi(\boldsymbol{\epsilon}, \bar{a} + \bar{b}\boldsymbol{\epsilon})$ can be easily replaced by a quantifier free semilinear formula over $\mathbb{Q}$, by substituting each basic relation with its component-wise definition.

It remains to prove the claim. At first, we can observe that a basic relation (say $a + b\boldsymbol{\epsilon} < c + d\boldsymbol{\epsilon}$) holds in V, if and only if the corresponding relation ($a + bt < c + dt$) holds for all t sufficiently small. The if part of the claim follows immediately: let $\bar{x} = \bar{a} + \bar{b}\boldsymbol{\epsilon}$ be a satisfying assignment for $\phi(\boldsymbol{\epsilon}, \bar{x})$ in V, then $\phi(t, \bar{a} + \bar{b}t)$ must hold for all sufficiently small positive $t \in \mathbb{Q}$. For the only if part, if $\phi(t, \bar{x})$ is satisfiable for all $t \in \,]0, t_0]$, then, by the definability of Skolem functions, there is a semilinear function

$$\bar{f} \colon \,]0, t_0] \to \mathbb{Q}^n$$

such that $\bar{x} = \bar{f}(t)$ is a satisfying assignment for all $t \in]0, t_0]$. Now, for some positive $t_1 < t_0$, the restriction of f to $]0, t_1]$ must be linear. In other words, there are $\bar{a}$ and $\bar{b}$ in $\mathbb{Q}^n$ such that

$$\forall t \in]0, t_1] \ \phi(t, \bar{a} + \bar{b}t)$$

hence $\bar{x} = \bar{a} + \bar{b}\boldsymbol{\epsilon} \in V$ must satisfy $\phi(\boldsymbol{\epsilon}, \bar{x})$. □

Proof of Theorem 5.1. For the NP part, it suffices to observe that after guessing which of the inputs of each max constraint realizes the maximum, one is left with a linear feasibility problem, which is in P.

For the co-NP part, we will employ the following strategy. First we replace each strict inequality $A < B$ by $\exists \epsilon > 0 \ A \leqslant B - \epsilon$. Hence we can apply Corollary 5.2, so we get that our problem is not satisfiable if and only if for all $\epsilon > 0$ some $D'(\bar{o}_\epsilon)$ is satisfiable. We then conclude by Lemma 5.4.

First let ϕ denote an instance of CSP(Γ_t). Construct a new formula ϕ_ϵ by formally replacing each strict inequality $A < B$ in ϕ by $A \leqslant B - \epsilon$, the new formula ϕ_ϵ is hence a conjunction of non-strict inequalities. Clearly ϕ is satisfiable if and only if there is an $\epsilon > 0$ such that ϕ_ϵ is satisfiable. Now, the formula ϕ_ϵ is almost in the form $\bar{x} \leqslant \bar{o}_\epsilon(\bar{x})$ that we need to apply Corollary 5.2, except that the same variable may appear in the left hand side of more that one constraint. To correct this discrepancy, let Γ'_t be the expansion of Γ_t obtained by addition of relations of the form

$$x_0 \leqslant \min(x_1 \ldots x_n)$$

We rewrite ϕ_ϵ as a new formula ϕ'_ϵ in the language of Γ'_t by replacing each left hand side instance of a variable x_i in ϕ_ϵ with a new variable $x_{i,1}, x_{i,2} \ldots$ and then adding a constraint

$$x_i \leqslant \min(x_{i,1}, x_{i,2} \ldots)$$

for each of the old variables x_i. The formula ϕ'_ϵ is in the form $\bar{x} \leqslant \bar{o}_\epsilon(\bar{x})$, therefore, by Corollary 5.2 it is non-satisfiable if and only if $\bar{y} > \bar{o}_\epsilon(\bar{y})$ has a solution in $\mathbb{Q} \cup \{+\infty\}$ which is not $+\infty$ on every coordinate. Hence we have reduced the non-satisfiability of ϕ to the following formula ψ

$$\forall \epsilon > 0 \quad \exists \bar{y} \in (\mathbb{Q} \cup \{+\infty\})^n \setminus \{+\infty\}^n \quad \bar{y} > \bar{o}_\epsilon(\bar{y})$$

which we intend to prove that can be checked in NP. In fact, if an assignment $\bar{y} = \bar{y}_0$ satisfies $\bar{y} > \bar{o}_\epsilon(\bar{y})$ for $\epsilon = \epsilon_0$, then the same assignment must satisfy $\bar{y} > \bar{o}_\epsilon(\bar{y})$ also for any other $\epsilon > \epsilon_0$. As a consequence ψ is equivalent to the following formula ψ'

$$\exists \epsilon_0 > 0 \quad \forall \epsilon \in]0, \epsilon_0] \quad \exists \bar{y} \in (\mathbb{Q} \cup \{+\infty\})^n \setminus \{+\infty\}^n \quad \bar{y} > \bar{o}_\epsilon(\bar{y})$$

We can therefore apply Lemma 5.4 observing that the domain $\mathbb{Q} \cup \{+\infty\}$ can be coded in $(\mathbb{Q}; +, \leqslant)$ in a quantifier free fashion—for instance by pairs $(a, b) \in \mathbb{Q}^2$ where $(x, 1)$ represents the number x and $(1, 0)$ represents $+\infty$. □

6. A Polynomial-time Tractable Fragment

We present an algorithm that tests satisfiability of a given (quantifier-free) restricted Horn formula Φ. Let V be the set of variables of Φ. Recall that each restricted Horn clause has at most one literal which contains a variable with a negative coefficient. We call this literal the *positive literal* of the clause, and all other literals the *negative literals*.

```
Solve(Φ)
// Input: a restricted Horn formula Φ
Do
    Let Ψ be the clauses in Φ that contain at most one literal.
    If Ψ is unsatisfiable then return unsatisfiable.
    For all negative literals φ in clauses from Φ
        If Ψ ∧ φ is unsatisfiable, then Ψ implies ¬φ:
            remove φ from all clauses in Φ.
Loop until no literal has been removed
Return satisfiable.
```

For testing whether a set of linear inequalities is satisfiable, we use a polynomial-time algorithm for linear program feasibility, such a the ellipsoid method [26]; it is well-known that this method can also be adapted to the situation where some of the inequalities are strict, see e.g. [23]. Since the algorithm always removes false literals, it is clear that if the algorithm returns *unsatisfiable*, then Φ is indeed unsatisfiable. Suppose now that we are in the final step of the algorithm and the procedure returns *satisfiable*. Then for each negative literal φ of Φ the set $\Psi \wedge \varphi$ has a solution $s_\varphi \colon V \to \mathbb{Q}$. Let s be the mapping $s(x) \stackrel{def}{=} \max_{\varphi \text{ negative literal of } \Phi} s_\varphi(x)$. Clearly, s satisfies Ψ since the mappings s_φ satisfy Ψ, and since all literals in Ψ are max-closed. We claim that s satisfies all negative literals $\varphi = (a_1x_1 + \cdots a_nx_n \succ c)$ in Φ, too:

$$\begin{aligned} c &\prec a_1 s_\varphi(x_1) + \cdots + a_n s_\varphi(x_n) && \text{since } s_\varphi \text{ satisfies } \varphi \\ &\leqslant a_1 s(x_1) + \cdots + a_n s(x_n) && \text{since } a_1, \ldots, a_n \text{ are positive.} \end{aligned}$$

Since every clause of Φ either contains a negative literal or it is contained in Ψ, this shows that s satisfies all constraints in Φ.

7. Maximality of Max-closed Semilinear Relations

Proposition 7.1. Let $R \subseteq \mathbb{Q}^n$ an n-ary relation that is *not* max-closed. Then $\mathrm{CSP}(\Gamma_0, R)$ is NP-hard.

Proof. The proposition follows easily from [21, Theorems 6.5–6.6], as it is very short we offer here a direct argument.

Let $s, t \in R$ be such that $m = \max(s, t) \notin R$. Clearly the finite relation $S \stackrel{def}{=} \{s, t, m\}$ is max-closed and semilinear, hence $R \cap S = \{s, t\}$ is primitive positive definable in $(\mathbb{Q}; \Gamma_0, R)$. Now, consider the $3n$-ary relation

$$T(x, y, z) \quad \Leftrightarrow \quad x, y, z \in R \cap S \ \wedge \ \max(x, y, z) = m$$

which is also primitive positive definable in $(\mathbb{Q}; \Gamma_0, R)$. Clearly

$$T = (R \cap S)^3 \setminus \{(s,s,s),(t,t,t)\}$$

and this gives a direct reduction of the NP-complete problem of *positive not-all-equal 3SAT* [16]

$$\text{CSP}\left(\{0,1\};\{0,1\}^3 \setminus \{(0,0,0),(1,1,1)\}\right)$$

to $\text{CSP}(\Gamma_0, R)$. □

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