Unit - 4 [Eigen Value Problem and Matrix Decomposition]

Eigen value problems :

⇒ Power method
⇒ Jacobi rotation method
⇒ Singular value decomposition
⇒ QR decomposition

Power Method :-

Introduction :-

The power method is an iterative technique. The method may not converge very fast. We can accelerate the convergence as well as get Eigenvalues of magnitude intermediate between the largest and smallest by shifting. The power method with its Variations is fine for small matrices. However, if a matrix has two Eigenvalues of equal magnitude, the method fails in the successive normalization factors alternate between two numbers. The duplicated Eigenvalue in this case is the square root of the product of the alternating normalization factors. If we want all the Eigenvalues for a larger matrix, there is a better way.

Definition :- 1 [Dominant Eigenvalue and Dominant Eigenvector]

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of an $n \times n$ matrix A. λ_1 is called the dominant eigenvalue of A if

$$|\lambda_1| > |\lambda_i|, \quad i = 2, \ldots, n$$

The eigenvectors corresponding to λ_1 is called dominant eigen vectors of A.

Note:-

Not every matrix has a dominant eigenvalue.

Ex:1

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$ with eigen values $\lambda_1 = 1$, $\lambda_2 = -1$

$\lambda_1 = 1 = |\lambda_2|$.

$\Rightarrow \lambda_1 \not> |\lambda_2|$ [Ref Def 1.]

$\Rightarrow$ A has no Dominant eigen value.

The power method:-

$\Rightarrow$ power method for approximating eigenvalues is iterative.

$\Rightarrow$ First assume that the matrix A has a dominant eigenvalue with corresponding dominant eigenvectors.

$\Rightarrow$ we choose an initial approximation x_0 of one of the dominant eigenvectors of A.

$\Rightarrow$ This initial approximation must be a nonzero vector in $\mathbb{R}^n$.

Finally, we form the sequence by

(i) $Ax_0 \Rightarrow \lambda . x_1$

(ii) $Ax_1 \Rightarrow \lambda . x_2$

(iii) $Ax_2 \Rightarrow \lambda . x_3$

$\vdots \qquad \vdots$

⇒ continue to iterate until $x_i = x_{i+1}$.

⇒ The coefficient of x_i is Dominant eigenvalue.

⇒ The corresponding matrix x_i is Dominant eigenvector.

problem: 1

Find the numerically largest Eigenvalue of $A = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}$ by power method.

Solution:-

Let $x_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ be an arbitrary initial Eigenvector.

$$Ax_0 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+0+0 \\ 4+0+0 \\ 6+0+0 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix} = 6 \underbrace{\begin{bmatrix} 0.167 \\ 0.667 \\ 1 \end{bmatrix}}_{x_1}$$

$$Ax_1 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.167 \\ 0.667 \\ 1 \end{bmatrix} = \begin{bmatrix} 1(0.167) - 3(0.667) + 2(1) \\ 4(0.167) + 4(0.667) - 1(1) \\ 6(0.167) + 3(0.667) + 5 \end{bmatrix} = \begin{bmatrix} 0.166 \\ 2.336 \\ 8.003 \end{bmatrix}$$

$$= 8.003 \underbrace{\begin{bmatrix} 0.021 \\ 0.292 \\ 1 \end{bmatrix}}_{x_2}$$

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$$Ax_2 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.021 \\ 0.291 \\ 1 \end{bmatrix} = \begin{bmatrix} 1(0.021) - 3(0.291) + 2(1) \\ 4(0.021) + 4(0.291) - 1(1) \\ 6(0.021) + 3(0.291) + 5(1) \end{bmatrix}$$

$$= \begin{bmatrix} 1.145 \\ 0.252 \\ 6.002 \end{bmatrix} = 6.002 \underbrace{\begin{bmatrix} 0.191 \\ 0.042 \\ 1 \end{bmatrix}}_{\to x_3}$$

$$Ax_3 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.191 \\ 0.042 \\ 1 \end{bmatrix} = \begin{bmatrix} 1(0.191) - 3(0.042) + 2(1) \\ 4(0.191) + 4(0.042) - 1(1) \\ 6(0.191) + 3(0.042) + 5(1) \end{bmatrix}$$

$$= \begin{bmatrix} 2.065 \\ -0.068 \\ 6.272 \end{bmatrix} = 6.272 \underbrace{\begin{bmatrix} 0.329 \\ -0.011 \\ 1 \end{bmatrix}}_{\to x_4}$$

$$Ax_4 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.329 \\ -0.011 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.362 \\ 0.272 \\ 6.941 \end{bmatrix} = 6.941 \underbrace{\begin{bmatrix} 0.34 \\ 0.039 \\ 1 \end{bmatrix}}_{\to x_5}$$

$$Ax_5 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.34 \\ 0.039 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.223 \\ 0.516 \\ 7.157 \end{bmatrix} = 7.157 \underbrace{\begin{bmatrix} 0.311 \\ 0.072 \\ 1 \end{bmatrix}}_{\to x_6}$$

$$Ax_6 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.311 \\ 0.072 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.095 \\ 0.532 \\ 7.082 \end{bmatrix} = 7.082 \underbrace{\begin{bmatrix} 0.296 \\ 0.075 \\ 1 \end{bmatrix}}_{\to x_7}$$

$$Ax_7 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.296 \\ 0.075 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.071 \\ 0.484 \\ 7.001 \end{bmatrix} = 7.001 \underbrace{\begin{bmatrix} 0.296 \\ 0.069 \\ 1 \end{bmatrix}}_{\to x_8}$$

$$Ax_8 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.296 \\ 0.069 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.089 \\ 0.46 \\ 6.983 \end{bmatrix} = 6.983 \underbrace{\begin{bmatrix} 0.296 \\ 0.066 \\ 1 \end{bmatrix}}_{\to x_9}$$

$$Ax_9 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.296 \\ 0.066 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.101 \\ 0.46 \\ 6.992 \end{bmatrix} = 6.992 \underbrace{\begin{bmatrix} 0.3 \\ 0.066 \\ 1 \end{bmatrix}}_{\to x_{10}}$$

$$Ax_{10} = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.3 \\ 0.066 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.102 \\ 0.464 \\ 6.998 \end{bmatrix} = 6.998 \underbrace{\begin{bmatrix} 0.3 \\ 0.066 \\ 1 \end{bmatrix}}_{\to x_{11}}$$

$$Ax_{11} = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.3 \\ 0.066 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.102 \\ 0.464 \\ 6.998 \end{bmatrix} = 6.998 \underbrace{\begin{bmatrix} 0.3 \\ 0.066 \\ 1 \end{bmatrix}}_{\to x_{12}}$$

Since $x_{11} = x_{12}$ with $\lambda = 6.998$.

$\Rightarrow$ The Largest Eigenvalue $= 6.998$.

problem : 2

Using power method, find all the Eigenvalues of

$$A = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}.$$

Solution :-

Let $x_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ be an initial eigenvector.

$$Ax_0 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} = 5\underbrace{\begin{bmatrix} 1 \\ 0 \\ 0.2 \end{bmatrix}}_{\to x_1}$$

$$Ax_1 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 5.2 \\ 0 \\ 2 \end{bmatrix} = 5.2\underbrace{\begin{bmatrix} 1 \\ 0 \\ 0.39 \end{bmatrix}}_{\to x_2}$$

$$Ax_2 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.39 \end{bmatrix} = \begin{bmatrix} 5.39 \\ 0 \\ 2.95 \end{bmatrix} = 5.39\underbrace{\begin{bmatrix} 1 \\ 0 \\ 0.55 \end{bmatrix}}_{\to x_3}$$

$$Ax_3 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.55 \end{bmatrix} = \begin{bmatrix} 5.55 \\ 0 \\ 3.75 \end{bmatrix} = 5.55\underbrace{\begin{bmatrix} 1 \\ 0 \\ 0.68 \end{bmatrix}}_{\to x_4}$$

$$Ax_4 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.68 \end{bmatrix} = \begin{bmatrix} 5.68 \\ 0 \\ 4.4 \end{bmatrix} = 5.68\underbrace{\begin{bmatrix} 1 \\ 0 \\ 0.78 \end{bmatrix}}_{\to x_5}$$

$$Ax_5 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.78 \end{bmatrix} = \begin{bmatrix} 5.78 \\ 0 \\ 4.9 \end{bmatrix} = 5.78\underbrace{\begin{bmatrix} 1 \\ 0 \\ 0.85 \end{bmatrix}}_{\to x_6}$$

$$Ax_6 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0.85 \end{bmatrix} = \begin{bmatrix} 5.85 \\ 0 \\ 5.25 \end{bmatrix} = 5.85 \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0.9 \end{bmatrix}}_{\to x_7}$$

$$Ax_7 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0.9 \end{bmatrix} = \begin{bmatrix} 5.9 \\ 0 \\ 5.5 \end{bmatrix} = 5.9 \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0.93 \end{bmatrix}}_{\to x_8}$$

$$A x_8 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.93 \end{bmatrix} = \begin{bmatrix} 5.93 \\ 0 \\ 5.65 \end{bmatrix} = 5.93 \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0.95 \end{bmatrix}}_{\to x_9}$$

$$Ax_9 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.95 \end{bmatrix} = \begin{bmatrix} 5.95 \\ 0 \\ 5.75 \end{bmatrix} = 5.95 \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}_{\to x_{10}}$$

$$\cancel{Ax_{10} = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5.95 \\ 0 \\ 5.75 \end{bmatrix} = 5.95 \Big[}$$

$$Ax_{10} = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 6 \end{bmatrix} = 6 \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}_{\to x_{11}}$$

$$Ax_{11} = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 6 \end{bmatrix} = 6 \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}_{x_{12}}.$$

Since $x_{11} = x_{12}$. with $\lambda_1 = 6$.

$\therefore \lambda_1 = 6$ and Eigenvector $= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

$$B = A - 6I = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix} - \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -8 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

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$$B = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -8 & 0 \\ 1 & 0 & -1 \end{bmatrix} \text{ and take } y_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$By_0 = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -8 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1+0+0 \\ 0+0+0 \\ 1+0+0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = -1 \underbrace{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}_{y_1}$$

[Note:- take the least element outside]

$$By_1 = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -8 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1+0-1 \\ 0+0+0 \\ 1+0+1 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix} = -2 \underbrace{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}_{y_2}$$

Since, $y_1 = y_2$.

$\therefore$ Greater eigenvalue of $B = \boxed{-2 = \Lambda_1}$

$\therefore$ Smallest eigenvalue of $A = \Lambda_1 + \lambda_1 = -2 + 6 = 4$.

(ie) $\boxed{\lambda_2 = 4}$

To find λ_3:-

Trace (A) = 8.

$\lambda_1 + \lambda_2 + \lambda_3 = \text{Trace}(A)$

$6 + 4 + \lambda_3 = 8$

$\boxed{\lambda_3 = -2}$

$\therefore$ All eigenvalues are 6, 4, −2

3. Find the dominant Eigenvalue and the corresponding Eigenvector of $A = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. Find also the least latent root and hence the third Eigenvalue also.

Solution:-

Let $x_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ be an approximate initial eigenvalue.

$$Ax_0 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+0+0 \\ 1+0+0 \\ 0+0+0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1 \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{\to x_1}$$

$$Ax_1 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+6+0 \\ 1+2+0 \\ 0+0+0 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 0 \end{bmatrix} = 7 \underbrace{\begin{bmatrix} 1 \\ 0.429 \\ 0 \end{bmatrix}}_{\to x_2}$$

$$Ax_2 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.429 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+2.574+0 \\ 1+0.858+0 \\ 0+0+0 \end{bmatrix} = \begin{bmatrix} 3.574 \\ 1.858 \\ 0 \end{bmatrix} = 3.574 \underbrace{\begin{bmatrix} 1 \\ 0.5199 \\ 0 \end{bmatrix}}_{\to x_3}$$

$$Ax_3 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.5199 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+3.1194+0 \\ 1+1.0398+0 \\ 0+0+0 \end{bmatrix} = \begin{bmatrix} 4.1194 \\ 2.0398 \\ 0 \end{bmatrix} = 4.1194 \underbrace{\begin{bmatrix} 1 \\ 0.4951 \\ 0 \end{bmatrix}}_{\to x_4}$$

$$Ax_4 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.4951 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+2.9706+0 \\ 1+0.9902+0 \\ 0+0+0 \end{bmatrix} = \begin{bmatrix} 3.9706 \\ 1.9902 \\ 0 \end{bmatrix} = 3.9706 \underbrace{\begin{bmatrix} 1 \\ 0.5012 \\ 0 \end{bmatrix}}_{\to x_5}$$

$$Ax_5 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5012 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+3.0072+0 \\ 1+1.0024+0 \\ 0+0+0 \end{bmatrix} = \begin{bmatrix} 4.3007 \\ 2.0024 \\ 0 \end{bmatrix} = 4.3007 \underbrace{\begin{bmatrix} 1 \\ 0.4656 \\ 0 \end{bmatrix}}_{\to x_6}$$

$$Ax_6 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.4656 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+2.7936+0 \\ 1+0.9312+0 \\ 0+0+0 \end{bmatrix} = \begin{bmatrix} 3.7936 \\ 1.9312 \\ 0 \end{bmatrix} = 3.7936 \underbrace{\begin{bmatrix} 1 \\ 0.5091 \\ 0 \end{bmatrix}}_{\to x_7}$$

$$Ax_7 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5091 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+3.0546+0 \\ 1+1.0186+0 \\ 0+0+0 \end{bmatrix} = \begin{bmatrix} 4.0546 \\ 2.0186 \\ 0 \end{bmatrix} = 4.0546 \underbrace{\begin{bmatrix} 1 \\ 0.4978 \\ 0 \end{bmatrix}}_{\to x_8}$$

$$Ax_8 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.4978 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+2.9868+0 \\ 1+0.9956+0 \\ 0+0+0 \end{bmatrix} = \begin{bmatrix} 3.9868 \\ 1.9956 \\ 0 \end{bmatrix} = 3.9868 \underbrace{\begin{bmatrix} 1 \\ 0.5006 \\ 0 \end{bmatrix}}_{x_9}$$

$$Ax_9 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5006 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+3.0036+0 \\ 1+1.0012+0 \\ 0+0+0 \end{bmatrix} = \begin{bmatrix} 4.0036 \\ 2.0012 \\ 0 \end{bmatrix} = 4.0036 \underbrace{\begin{bmatrix} 1 \\ 0.4999 \\ 0 \end{bmatrix}}_{\to x_{10}}$$

$$Ax_{10} = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.4999 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+2.9994+0 \\ 1+0.9998+0 \\ 0+0+0 \end{bmatrix} = \begin{bmatrix} 3.9994 \\ 1.9998 \\ 0 \end{bmatrix} = 3.9994 \underbrace{\begin{bmatrix} 1 \\ 0.5000 \\ 0 \end{bmatrix}}_{\to x_{11}}$$

$$Ax_{11} = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+3+0 \\ 1+1+0 \\ 0+0+0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} = 4\underbrace{\begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix}}_{\to x_{12}}$$

$$Ax_{12} = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+3+0 \\ 1+1+0 \\ 0+0+0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} = 4\underbrace{\begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix}}_{\to x_{13}}$$

$\Rightarrow x_{12} = x_{13}$.

$\therefore$ Dominant Eigenvalue is $= 4$.

$\Rightarrow \boxed{\lambda_1 = 4}$

Corresponding Eigenvector is $= (1, 0.5, 0)$.

<u>To find</u>: The least Eigenvalue

Let $B = A - 4I = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} -3 & 6 & 1 \\ 1 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

let $y_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ be an initial eigenvector.

$$By_0 = \begin{bmatrix} -3 & 6 & 1 \\ 1 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -3+0+0 \\ 1+0+0 \\ 0+0+0 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} = -3\underbrace{\begin{bmatrix} 1 \\ -0.333 \\ 0 \end{bmatrix}}_{\to y_1}$$

$$By_1 = \begin{bmatrix} -3 & 6 & 1 \\ 1 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -0.333 \\ 0 \end{bmatrix} = \begin{bmatrix} -3-1.998+0 \\ 1+0.666+0 \\ 0+0+0 \end{bmatrix} = \begin{bmatrix} -5 \\ 1.666 \\ 0 \end{bmatrix} = -5\underbrace{\begin{bmatrix} 1 \\ -0.333 \\ 0 \end{bmatrix}}_{\to y_2}$$

$$By_2 = \begin{bmatrix} -3 & 6 & 1 \\ 1 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.333 \\ 0 \end{bmatrix} = \begin{bmatrix} -5 \\ 1.666 \\ 0 \end{bmatrix} = -5 \underbrace{\begin{bmatrix} 1 \\ -0.333 \\ 0 \end{bmatrix}}_{\to y_3}$$

$\Rightarrow y_2 = y_3.$

$\therefore$ The dominant eigenvalue of B is $\boxed{-5 = \Lambda_1}$

To find: the smallest eigenvalue of A.

$$\lambda_2 = \lambda_1 + \Lambda_1 = 4 - 5 = -1$$

$$\Rightarrow \boxed{\lambda_2 = -1}$$

To find λ_3 :-

$$\text{Trace}(A) = 1 + 2 + 3 = 6$$

$$\lambda_1 + \lambda_2 + \lambda_3 = \text{Trace}(A)$$

$$4 - 1 + \lambda_3 = 6$$

$$\boxed{\lambda_3 = 3}$$

$\therefore$ $\boxed{\text{All eigenvalues are } 4, 3, -1}$

4. Find the numerically largest Eigenvalue of

$$A = \begin{pmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{pmatrix}$$ and the corresponding Eigenvector.

Solution :-

Let $x_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ be an arbitrary initial eigenvector.

$$Ax_0 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 25+0+0 \\ 1+0+0 \\ 2+0+0 \end{bmatrix} = \begin{bmatrix} 25 \\ 1 \\ 2 \end{bmatrix} = 25\underbrace{\begin{bmatrix} 1 \\ 0.04 \\ 0.08 \end{bmatrix}}_{\to x_1}$$

$$Ax_1 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 0.04 \\ 0.08 \end{bmatrix} = \begin{bmatrix} 25+0.04+0.16 \\ 1+0.12+0 \\ 2+0-0.32 \end{bmatrix} = \begin{bmatrix} 25.2 \\ 1.12 \\ 1.68 \end{bmatrix} = 25.2\underbrace{\begin{bmatrix} 1 \\ 0.0444 \\ 0.0667 \end{bmatrix}}_{\to x_2}$$

$$Ax_2 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 0.0444 \\ 0.0667 \end{bmatrix} = \begin{bmatrix} 25.1778 \\ 1.1332 \\ 1.7337 \end{bmatrix} = 25.1778\underbrace{\begin{bmatrix} 1 \\ 0.0450 \\ 0.06888 \end{bmatrix}}_{\to x_3}$$

$$Ax_3 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 0.0450 \\ 0.06888 \end{bmatrix} = \begin{bmatrix} 25.1826 \\ 1.135 \\ 1.7248 \end{bmatrix} = 25.1826\underbrace{\begin{bmatrix} 1 \\ 0.0451 \\ 0.0685 \end{bmatrix}}_{\to x_4}$$

$$Ax_4 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 0.0451 \\ 0.0685 \end{bmatrix} = \begin{bmatrix} 25.1821 \\ 1.1353 \\ 1.7260 \end{bmatrix} = 25.1821\underbrace{\begin{bmatrix} 1 \\ 0.0451 \\ 0.0685 \end{bmatrix}}_{\to x_5}$$

we have reached the limit.

$\therefore$ The Dominant eigenvalue is $\boxed{\lambda_1 = 25.1821}$

The corresponding eigenvector is $(1, 0.0451, 0.0685)$.

5. Find the dominant eigenvalue & corresponding eigenvector of the matrix

(a) $A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$ [∴ 3 decimal places]

<u>Solution :-</u>

Let $x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ be an arbitrary initial eigenvector.

$$Ax_0 = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2+0 \\ 1+0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2\underbrace{\begin{bmatrix} 1 \\ 0.5 \end{bmatrix}}_{x_1}$$

$$Ax_1 = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 2-6 \\ 1-2.5 \end{bmatrix} = \begin{bmatrix} -4 \\ -1.5 \end{bmatrix} = -1.5\underbrace{\begin{bmatrix} 2.667 \\ 1 \end{bmatrix}}_{x_2}$$

$$Ax_2 = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 2.667 \\ 1 \end{bmatrix} = \begin{bmatrix} 5.334-12 \\ 2.667-5 \end{bmatrix} = \begin{bmatrix} -6.666 \\ -2.333 \end{bmatrix} = -2.333\underbrace{\begin{bmatrix} 2.857 \\ 1 \end{bmatrix}}_{x_3}$$

$$Ax_3 = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 2.857 \\ 1 \end{bmatrix} = \begin{bmatrix} 5.714-12 \\ 2.857-5 \end{bmatrix} = \begin{bmatrix} -6.286 \\ -2.143 \end{bmatrix} = -2.143\underbrace{\begin{bmatrix} 2.933 \\ 1 \end{bmatrix}}_{x_4}$$

$$Ax_4 = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 2.933 \\ 1 \end{bmatrix} = \begin{bmatrix} 5.866-12 \\ 2.933-5 \end{bmatrix} = \begin{bmatrix} -6.134 \\ -2.067 \end{bmatrix} = -2.067\underbrace{\begin{bmatrix} 2.968 \\ 1 \end{bmatrix}}_{x_5}$$

$$Ax_5 = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 2.968 \\ 1 \end{bmatrix} = \begin{bmatrix} 5.936-12 \\ 2.968-5 \end{bmatrix} = \begin{bmatrix} -6.064 \\ -2.032 \end{bmatrix}$$

$$= -2.032 \underbrace{\begin{bmatrix} 2.984 \\ 1 \end{bmatrix}}_{x_6}$$

$$Ax_6 = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 2.984 \\ 1 \end{bmatrix} = \begin{bmatrix} 5.968-12 \\ 2.984-5 \end{bmatrix} = \begin{bmatrix} -6.032 \\ -2.016 \end{bmatrix}$$

$$= -2.016 \underbrace{\begin{bmatrix} 2.992 \\ 1 \end{bmatrix}}_{x_7}$$

$$Ax_7 = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 2.992 \\ 1 \end{bmatrix} = \begin{bmatrix} 5.984-12 \\ 2.992-5 \end{bmatrix} = \begin{bmatrix} -6.016 \\ -2.008 \end{bmatrix}$$

$$= -2.008 \underbrace{\begin{bmatrix} 2.996 \\ 1 \end{bmatrix}}_{x_8}$$

$$Ax_8 = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 2.996 \\ 1 \end{bmatrix} = \begin{bmatrix} 5.992-12 \\ 2.996-5 \end{bmatrix} = \begin{bmatrix} -6.008 \\ -2.004 \end{bmatrix} = -2.004 \underbrace{\begin{bmatrix} 2.998 \\ 1 \end{bmatrix}}_{x_9}$$

$$Ax_9 = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 2.998 \\ 1 \end{bmatrix} = \begin{bmatrix} 5.996-12 \\ 2.998-5 \end{bmatrix} = \begin{bmatrix} -6.004 \\ -2.002 \end{bmatrix} = -2.002 \underbrace{\begin{bmatrix} 2.999 \\ 1 \end{bmatrix}}_{x_{10}}$$

$$Ax_{10} = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 2.999 \\ 1 \end{bmatrix} = \begin{bmatrix} 5.998-12 \\ 2.999-5 \end{bmatrix} = \begin{bmatrix} -6.002 \\ -2.001 \end{bmatrix} = -2.001 \underbrace{\begin{bmatrix} 3 \\ 1 \end{bmatrix}}_{x_{11}}$$

$$Ax_{11} = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6-12 \\ 3-5 \end{bmatrix} = \begin{bmatrix} -6 \\ -2 \end{bmatrix} = -2 \underbrace{\begin{bmatrix} 3 \\ 1 \end{bmatrix}}_{x_{12}}$$

$\therefore$ The dominant eigenvalues is -2. Eigen vector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

(a) $A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$ [∵ 2 Decimal places]

Solution:-

Let $x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ be an initial Eigenvector.

$$Ax_0 = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2+0 \\ 1+0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2\underbrace{\begin{bmatrix} 1 \\ 0.5 \end{bmatrix}}_{\to x_1}$$

$$Ax_1 = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 2-6 \\ 1-2.5 \end{bmatrix} = \begin{bmatrix} -4 \\ -1.5 \end{bmatrix} = -1.5\underbrace{\begin{bmatrix} 2.67 \\ 1 \end{bmatrix}}_{\to x_2}$$

$$Ax_2 = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 2.67 \\ 1 \end{bmatrix} = \begin{bmatrix} 5.34-12 \\ 2.67-5 \end{bmatrix} = \begin{bmatrix} -6.66 \\ -2.33 \end{bmatrix} = -2.33\underbrace{\begin{bmatrix} 2.86 \\ 1 \end{bmatrix}}_{\to x_3}$$

$$Ax_3 = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 2.86 \\ 1 \end{bmatrix} = \begin{bmatrix} 5.72-12 \\ 2.86-5 \end{bmatrix} = \begin{bmatrix} -6.28 \\ -2.14 \end{bmatrix} = -2.14\underbrace{\begin{bmatrix} 2.93 \\ 1 \end{bmatrix}}_{\to x_4}$$

$$Ax_4 = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 2.93 \\ 1 \end{bmatrix} = \begin{bmatrix} 5.86-12 \\ 2.93-5 \end{bmatrix} = \begin{bmatrix} -6.14 \\ -2.07 \end{bmatrix} = -2.07\underbrace{\begin{bmatrix} 2.97 \\ 1 \end{bmatrix}}_{\to x_5}$$

$$Ax_5 = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 2.97 \\ 1 \end{bmatrix} = \begin{bmatrix} 5.94-12 \\ 2.97-5 \end{bmatrix} = \begin{bmatrix} -6.06 \\ -2.03 \end{bmatrix} = -2.03\underbrace{\begin{bmatrix} 2.99 \\ 1 \end{bmatrix}}_{\to x_6}$$

$$Ax_6 = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 2.99 \\ 1 \end{bmatrix} = \begin{bmatrix} 5.98-12 \\ 2.99-5 \end{bmatrix} = \begin{bmatrix} -6.02 \\ -2.01 \end{bmatrix} = -2.01\underbrace{\begin{bmatrix} 3 \\ 1 \end{bmatrix}}_{\to x_7}$$

$$Ax_7 = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6-12 \\ 3-5 \end{bmatrix} = \begin{bmatrix} -6 \\ -2 \end{bmatrix} = -2\underbrace{\begin{bmatrix} 3 \\ 1 \end{bmatrix}}_{\to x_8}$$

$$Ax_8 = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6-12 \\ 3-5 \end{bmatrix} = \begin{bmatrix} -6 \\ -2 \end{bmatrix} = -2\underbrace{\begin{bmatrix} 3 \\ 1 \end{bmatrix}}_{x_9}.$$

Here $x_8 = x_9$.

$\therefore$ The dominant Eigenvalue is $\boxed{\lambda_1 = -2}$

The corresponding Eigenvector is $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

(b) $A = \begin{bmatrix} 1 & -5 \\ -3 & -1 \end{bmatrix}$

Solution:-

Let $x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ be an initial eigenvector.

$$Ax_0 = \begin{bmatrix} 1 & -5 \\ -3 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+0 \\ -3+0 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix} = 1\underbrace{\begin{bmatrix} 1 \\ -3 \end{bmatrix}}_{x_1}$$

$$Ax_1 = \begin{bmatrix} 1 & -5 \\ -3 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 1+15 \\ -3+3 \end{bmatrix} = \begin{bmatrix} 16 \\ 0 \end{bmatrix} = 16\underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{x_2}$$

$$Ax_2 = \begin{bmatrix} 1 & -5 \\ -3 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+0 \\ -3+0 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix} = 1\underbrace{\begin{bmatrix} 1 \\ -3 \end{bmatrix}}_{x_3}$$

$$Ax_3 = \begin{bmatrix} 1 & -5 \\ -3 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 1+15 \\ -3+3 \end{bmatrix} = \begin{bmatrix} 16 \\ 0 \end{bmatrix} = 16\underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{x_4}.$$

$$x_i \neq x_{i+1}$$

$\Rightarrow$ There is no Dominant Eigen value.

Jacobi Method for Finding Eigen values.

Introduction:-

⇒ We use Jacobi's method to find the eigen values of a real symmetric matrix.

⇒ By this method we can find all eigenvalues and eigenvectors of a symmetric matrix.

⇒ The basic principle involved in this method is to find a sequence of similarity transformations which reduce the given matrix into another whose eigen values can be found easily.

⇒ If a matrix of higher order is given, we have to use only computers to solve the problem.

Jacobi method:-

⇒ Let A be a given real symmetric matrix. Its Eigen values are real and there exists a real orthogonal matrix B such that $B^{-1}AB$ is a diagonal matrix D.

⇒ Jacobi's method consists of diagonalising A by applying a series of orthogonal transformations $B_1, B_2, \dots, B_n$ such that their product B satisfies the equation $D = B^{-1}AB$.

Rotation matrix :-

If $P(x,y)$ is any point in the xy-plane and if OP is rotated (O is the origin) in the clockwise direction through an angle θ, then the new position of $P(x',y')$ is given by

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

(ie) $\begin{pmatrix} x' \\ y' \end{pmatrix} = R \begin{pmatrix} x \\ y \end{pmatrix}$, where $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Hence R is called a Rotation matrix in the xy-plane.
Here R is also an orthogonal matrix since $RR^T = I$.

---

problem: 1 Using Jacobi method, find the Eigenvalues and Eigenvectors of $A = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$.

Solution:-

Let $A = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$

	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
	$0°$	$30°$	$45°$	$60°$	$90°$
$\sin\theta$	0	$1/2$	$1/\sqrt{2}$	$\sqrt{3}/2$	1
$\cos\theta$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$	0
$\tan\theta$	0	$1/\sqrt{3}$	1	$\sqrt{3}$	0

working Rule:

<u>Method :- 1</u> Given matrix A.

(1) Use $\theta = \frac{1}{2}\tan^{-1}\left(\frac{2a_{12}}{a_{11}-a_{22}}\right)$ if $a_{11} \neq a_{22}$.

$\theta = \pi/4$ if $a_{11} = a_{22}$ and $a_{12} > 0$

$\theta = -\pi/4$ if $a_{11} = a_{22}$ and $a_{12} < 0$.

(2) write down $R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ using the value of θ

(3) Get $B = R^T A R$

(4) The diagonal elements of B are the eigen values.

(5) The columns of R are the corresponding eigen vectors

<u>Method : 2</u> Given matrix is A.

(1) $R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ and find $\cot 2\theta = \alpha = \frac{a_{11}-a_{22}}{2a_{12}}$

(2) $\cot\theta = \beta = \alpha \pm \sqrt{\alpha^2+1}$.

(3) Then find $\sin\theta = \frac{1}{\sqrt{1+\beta^2}}$, $\cos\theta = \frac{\beta}{\sqrt{1+\beta^2}}$ if $\beta > 0$

(4) Then calculate the eigen values of A as

$$\boxed{b_{11} = a_{11}\cos^2\theta + a_{12}\sin 2\theta + a_{22}\sin^2\theta}$$

$$b_{22} = a_{11}\sin^2\theta - a_{12}\sin 2\theta + a_{22}\cos^2\theta$$

(ie) $\boxed{b_{22} = a_{11} + a_{22} - b_{11}}$

Method : 2

Example : 1 Using Jacobi method, find eigen values of

(i) $A = \begin{bmatrix} 6 & \sqrt{3} \\ \sqrt{3} & 4 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 0 \end{bmatrix}$

(i) **Solution :-**

Evidently, the given matrix in both problems is symmetric.

Let $R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Now,

$B = R^T A R$

Here, $\cot 2\theta = \alpha = \dfrac{a_{11} - a_{22}}{2a_{12}} = \dfrac{6-4}{2\sqrt{3}} = \dfrac{\cancel{2}}{\cancel{2}\sqrt{3}} = \boxed{\dfrac{1}{\sqrt{3}} = \alpha}$

$\cot\theta = \beta = \alpha \pm \sqrt{1+\alpha^2} = \dfrac{1}{\sqrt{3}} \pm \sqrt{1 + 1/3}$

$= \dfrac{1}{\sqrt{3}} \pm \sqrt{4/3} = \dfrac{1}{\sqrt{3}} \pm \dfrac{2}{\sqrt{3}}$

$= \dfrac{1}{\sqrt{3}} + \dfrac{2}{\sqrt{3}}$ (or) $\dfrac{1}{\sqrt{3}} - \dfrac{2}{\sqrt{3}}$

$= \dfrac{\overset{\sqrt{3}}{\cancel{3}}}{\cancel{\sqrt{3}}}$ (or) $\dfrac{-1}{\sqrt{3}}$

$$\boxed{\cot\theta = \beta = \sqrt{3} \text{ (or) } \frac{-1}{\sqrt{3}}}$$

First take, $\beta = \sqrt{3}$

$$\sin\theta = \frac{1}{\sqrt{1+\beta^2}} = \frac{1}{\sqrt{1+3}} = \frac{1}{\sqrt{4}} = \frac{1}{2}.$$

and

$$\cos\theta = \frac{\beta}{\sqrt{1+\beta^2}} = \frac{\sqrt{3}}{\sqrt{4}} = \frac{\sqrt{3}}{2}$$

(ie) $\boxed{\theta = \pi/6}$

$$b_{11} = a_{11}\cos^2\theta + a_{12}\sin 2\theta + a_{22}\sin^2\theta$$

$$= 6\cdot\left(\sqrt{3}/2\right)^2 + \sqrt{3}\cdot\sin\left(\pi/3\right) + 4\cdot\left(1/2\right)^2$$

$$= \cancel{6}^{3}\cdot 3/\cancel{4}_{2} + \sqrt{3}\cdot\sqrt{3}/2 + \cancel{4}\cdot 1/\cancel{4}$$

$$= 9/2 + 3/2 + 1 = \frac{\cancel{12}^{6}}{\cancel{2}} + 1 = 7.$$

$\boxed{b_{11} = 7}$

$$b_{22} = a_{11} + a_{22} - b_{11} = 6 + 4 - 7 = 3.$$

$\boxed{b_{22} = 3}$

$\therefore$ The eigenvalues are $7, 3$

To find eigenvectors,

$$R = \begin{pmatrix} \cos\pi/6 & -\sin\pi/6 \\ \sin\pi/6 & \cos\pi/6 \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}$$

$\therefore$ The corresponding eigen vectors are,

$$\begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix}, \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix}. \text{ (or) } \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix}$$

Method :- 1

Using Jacobi method, find the Eigen values and Eigenvectors of $A = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$.

Solution :-

Let $A = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$ Here $a_{11} = a_{22} = 4$ and

$a_{21} = a_{12} = 1 > 0$

The rotation matrix is,

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Here,

$$\theta = \frac{1}{2}\tan^{-1}\left[\frac{2a_{12}}{a_{11}-a_{22}}\right]$$

$$= \frac{1}{2}\tan^{-1}\left[\frac{2(1)}{4-4}\right] = \frac{1}{2}\tan^{-1}(\infty) = \frac{1}{2}\cdot\frac{\pi}{2}$$

$$\Rightarrow \boxed{\theta = \pi/4}$$

$\therefore$ Rotation matrix $R = \begin{bmatrix} \cos\pi/4 & -\sin\pi/4 \\ \sin\pi/4 & \cos\pi/4 \end{bmatrix}$

$$= \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$D = R^T A R = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^T \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 4/\sqrt{2} + 1/\sqrt{2} & -4/\sqrt{2} + 1/\sqrt{2} \\ 1/\sqrt{2} + 4/\sqrt{2} & -1/\sqrt{2} + 4/\sqrt{2} \end{bmatrix}$$

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$$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 5/\sqrt{2} & -3/\sqrt{2} \\ 5/\sqrt{2} & 3/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 5/2 + 5/2 & -3/2 + 3/2 \\ -5/2 + 5/\sqrt{2} & 3/2 + 3/2 \end{bmatrix}$$

$$= \begin{bmatrix} \cancel{10}^{5}/\cancel{2} & 0 \\ 0 & \cancel{6}^{3}/\cancel{2} \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}.$$

$$D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}.$$

$\therefore$ The eigenvalues are 5, 3.

The eigenvectors are $\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ and $\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$.

③ using Jacobi method, Find the Eigenvalues and Eigenvectors of $A = \begin{pmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{pmatrix}$.

Solution:-

Here the largest off-diagonal element is $a_{13} = a_{31} = 2$

and $a_{11} = a_{33} = 1$.

Hence take the rotation matrix

$$R_1 = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

To find θ:-

By method: 2

$$\cot 2\theta = \frac{a_{11} - a_{33}}{2a_{13}} = \frac{1-1}{2(2)} = 0.$$

$$\cot 2\theta = 0$$

$$2\theta = \cot^{-1}(0) = \pi/2$$

$$\Rightarrow \boxed{\theta = \pi/4}$$

(or)

By method: 1

$$\theta = \frac{1}{2} \tan^{-1}\left(\frac{2a_{13}}{a_{11} - a_{33}}\right) = \frac{1}{2} \tan^{-1}(\infty)$$

$$= \frac{1}{2} \cdot \frac{\pi}{2}$$

$$\Rightarrow \boxed{\theta = \pi/4}$$

$$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 5/\sqrt{2} & -3/\sqrt{2} \\ 5/\sqrt{2} & 3/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 5/2 + 5/2 & -3/2 + 3/2 \\ -5/2 + 5/\sqrt{2} & 3/2 + 3/2 \end{bmatrix}$$

$$= \begin{bmatrix} \cancel{10}^{5}/\cancel{2} & 0 \\ 0 & \cancel{6}^{3}/\cancel{2} \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}.$$

$$D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}.$$

$\therefore$ The eigenvalues are 5, 3.

The eigenvectors are $\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ and $\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$.

③ using Jacobi method, Find the Eigenvalues and Eigenvectors of $A = \begin{pmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{pmatrix}$.

Solution:-

Here the largest off-diagonal element is $a_{13} = a_{31} = 2$

and $a_{11} = a_{33} = 1$.

Hence take the rotation matrix

$$R_1 = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

To find θ:-

By method : 2

$$\cot 2\theta = \frac{a_{11} - a_{33}}{2a_{13}} = \frac{1-1}{2(2)} = 0.$$

$$\cot 2\theta = 0$$

$$2\theta = \cot^{-1}(0) = \pi/2$$

$$\Rightarrow \boxed{\theta = \pi/4}$$

(or)

By method : 1

$$\theta = \frac{1}{2} \cdot \tan^{-1}\left(\frac{2a_{13}}{a_{11} - a_{33}}\right) = \frac{1}{2} \cdot \tan^{-1}(\infty)$$

$$= \frac{1}{2} \cdot \frac{\pi}{2}$$

$$\Rightarrow \boxed{\theta = \pi/4}$$

(6)

$\therefore$ The rotation matrix is $R_1 = \begin{pmatrix} \cos \pi/4 & 0 & -\sin \pi/4 \\ 0 & 1 & 0 \\ \sin \pi/4 & 0 & \cos \pi/4 \end{pmatrix}$

$$\boxed{R_1 = \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}}$$

To find B_1:-

$B_1 = R_1^T A R_1$

$$= \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}^T \begin{bmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

$$= (1/\sqrt{2})(1/\sqrt{2}) \begin{bmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1+0+2 & 0+2+0 & -1+0+2 \\ \sqrt{2}+0+\sqrt{2} & 0+3\sqrt{2}+0 & -\sqrt{2}+0+\sqrt{2} \\ 2+0+1 & 0+2+0 & -2+0+1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 & 1 \\ 2\sqrt{2} & 3\sqrt{2} & 0 \\ 3 & 2 & -1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 3+0+3 & 2+0+2 & 1+0-1 \\ 0+4+0 & 0+6+0 & 0+0+0 \\ -3+0+3 & -2+0+2 & -1+0-1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 6 & 4 & 0 \\ 4 & 6 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \cancel{2}/\cancel{2}\begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\therefore \boxed{B_1 = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}}$$ ($\because$ Its not a diagonal matrix).

Again anihilate the largest off-diagonal element $a_{12} = a_{21} = 2$ in B_1.

Also, $a_{11} = a_{22} = 3$.

<u>To find θ:-</u>

$$\theta = \frac{1}{2}\cdot\tan^{-1}\left(\frac{2a_{12}}{a_{11}-a_{22}}\right)$$

$$= \frac{1}{2}\cdot\tan^{-1}\left(\frac{4}{0}\right) = \frac{1}{2}\cdot\tan^{-1}(\infty) = \frac{1}{2}\cdot\frac{\pi}{2}.$$

$$\boxed{\theta = \pi/4}$$

$$R_2 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

since $\theta = \pi/4$,

$$R_2 = \begin{bmatrix} \cos \pi/4 & -\sin \pi/4 & 0 \\ \sin \pi/4 & \cos \pi/4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_2 = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now, $B_2 = R_2^T B_1 R_2$

$$= \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}^T \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \left(1/\sqrt{2}\right)\left(1/\sqrt{2}\right) \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3+2+0 & -3+2+0 & 0+0+0 \\ 2+3+0 & -2+3+0 & 0+0+0 \\ 0+0+0 & 0+0+0 & 0+0-\sqrt{2} \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}\begin{bmatrix} 5 & -1 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & -\sqrt{2} \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 5+5+0 & -1+1+0 & 0+0+0 \\ -5+5+0 & 1+1+0 & 0+0+0 \\ 0+0+0 & 0+0+0 & 0+0-(2) \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 10 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

After 2 rotations, A is reduced to the diagonal matrix B_2. Hence the Eigen values of A are 5, 1, −1.

To find : Eigenvectors.

$$R = R_1 R_2 = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)\begin{bmatrix} 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1+0+0 & -1+0+0 & 0+0-\sqrt{2} \\ 0+\sqrt{2}+0 & 0+\sqrt{2}+0 & 0+0+0 \\ 1+0+0 & -1+0+0 & 0+0+\sqrt{2} \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & -1 & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} & 0 \\ 1 & -1 & \sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & -1/2 & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/2 & -1/2 & 1/\sqrt{2} \end{bmatrix}.$$

$\therefore$ The eigen vectors are $\begin{pmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/2 \end{pmatrix}$, $\begin{pmatrix} -1/2 \\ 1/\sqrt{2} \\ -1/2 \end{pmatrix}$ and $\begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$.

(or)

The eigen vectors are $\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$, $\begin{pmatrix} -1 \\ \sqrt{2} \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -\sqrt{2} \\ 0 \\ \sqrt{2} \end{pmatrix}$.

(or)

The eigen vectors are $\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$, $\begin{pmatrix} -1 \\ \sqrt{2} \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$.

(33)

(4) Find the eigen values and eigenvectors of $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$.

Solution :-

Hence, the largest off-diagonal element is $a_{13} = a_{31} = 1$.

Let us annihilate this element.

Take, $R_1 = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$

$$\theta = \frac{1}{2} \cdot \tan^{-1}\left(\frac{2a_{13}}{a_{11} - a_{33}}\right) = \frac{1}{2} \cdot \tan^{-1}\left(\frac{2(1)}{2-2}\right)$$

$$= \frac{1}{2} \cdot \tan^{-1}\left(\frac{2}{0}\right) = \frac{1}{2} \tan^{-1}(\infty) = \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4}.$$

$\therefore \boxed{\theta = \pi/4}$

$$\Rightarrow R_1 = \begin{bmatrix} \cos\pi/4 & 0 & -\sin\pi/4 \\ 0 & 1 & 0 \\ \sin\pi/4 & 0 & \cos\pi/4 \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

To find B_1 :-

$$B_1 = R_1^T A R_1$$

$$= \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ +1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}^T \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

$$= (1/\sqrt{2})(1/\sqrt{2}) \begin{bmatrix} 1 & 0 & +1 \\ 0 & \sqrt{2} & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= 1/2 \begin{bmatrix} 1 & 0 & +1 \\ 0 & \sqrt{2} & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2+0+1 & 0+0+0 & -2+0+1 \\ 0+0+0 & 0+2\sqrt{2}+0 & 0+0+0 \\ 1+0+2 & 0+0+0 & -1+0+2 \end{bmatrix}$$

$$= 1/2 \begin{bmatrix} 1 & 0 & +1 \\ 0 & \sqrt{2} & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & -1 \\ 0 & 2\sqrt{2} & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

$$= 1/2 \begin{bmatrix} 3+0+3 & 0+0+0 & -1+0+1 \\ 0+0+0 & 0+4+0 & 0+0+0 \\ -3+0+3 & 0+0+0 & 1+0+1 \end{bmatrix}$$

$$= 1/2 \begin{bmatrix} 6 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In one rotation we get B as diagonal matrix.

$$a_{12} = a_{23} = 0.$$

$\therefore$ The eigenvalues of A are 3, 2, 1.

The corresponding eigenvectors of A are the columns of R_1

$$\begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \ \& \ \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}.$$

(or)

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \ \& \ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

5. Find eigenvalues and eigenvectors of $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{pmatrix}$.

<u>Solution</u> :-

The element $|a_{23}| = 1$ is to be annihilated.

Take,

$$R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

<u>To find θ</u> :-

$$\theta = \frac{1}{2} \cdot \tan^{-1}\left(\frac{2a_{23}}{a_{22}-a_{33}}\right) = \frac{1}{2} \tan^{-1}\left(\frac{-2}{0}\right)$$

$$= \frac{1}{2} \tan^{-1}(-\infty)$$

$$\boxed{\theta = \pi/4}$$

$$R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$B_1 = R_1^T A R_1$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & +1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 2 & -4 \\ 0 & 2 & 4 \end{bmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

$\therefore$ The eigenvalues are $1, 2, 4$.

The corresponding eigenvectors are columns of R_1,

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ +1/\sqrt{2} \end{pmatrix} \ \& \ \begin{pmatrix} 0 \\ -1/\sqrt{2} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

(or)

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ +1 \end{pmatrix} \ \& \ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

(6) Apply Jacobi process to evaluate the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}$.

Solution :-

Here the largest off-diagonal element is $a_{13} = a_{31} = 1$

and $a_{11} = a_{33} = 5$.

To find : θ

$$\theta = \frac{1}{2} \tan^{-1}\left(\frac{2(1)}{5-5}\right) = \frac{1}{2}\cdot\left(\frac{\pi}{2}\right) = \boxed{\frac{\pi}{4} = \theta}$$

To find : R

$$R = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}$$

The first transformation gives $B = R^T A R$.

$$B = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}$$

$$= \left(1/\sqrt{2}\right)\left(1/\sqrt{2}\right) \begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$= \begin{bmatrix} 6 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

∴ The eigen values of the given matrix are 6, −2, 14.

and the corresponding Eigenvectors are the columns of

$$R = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

∴ eigen vectors are $\begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$; $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ & $\begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$.

Singular Value Decomposition (SVD)

Introduction:-

We considered square matrices only we obtained the LU decomposition of a square matrix. A Similar decomposition is also possible of a rectangular matrix and this is called the Singular Value decomposition (SVD)

The SVD is of great importance in matrix theory because it is useful in finding the generalized inverse of a singular matrix and has several image processing applications.

Moore - penrose inverse.

SVD :-

Let A be an $(m \times n)$ real matrix with $m \geq n$.

Then the matrices $A^T A$ and AA^T are non-negative Symmetric and have identical Eigenvalues, Say λ_n.

We obtain the n orthonormalized Eigenvectors, Say X_n of $A^T A$ Such that

$$A^T A \, X_n = \lambda_n X_n \quad \longrightarrow (1)$$

We assume Y_n to be the n orthonormalized eigenvectors of AA^T,

$$\Rightarrow \quad AA^T Y_n = \lambda_n Y_n \quad \longrightarrow (2)$$

Then A can be decomposed into the form

$$A = UDV^T \longrightarrow (3)$$

where,

$$U^TU = V^TV = VV^T = I_n \longrightarrow (4)$$

and

$$D = \text{diag}\left(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \ldots, \sqrt{\lambda_n}\right) \longrightarrow (5)$$

The decomposition defined by (3) is called the singular value decomposition of A.

<u>Note:-1</u>

⇒ The matrix $V_{n \times n}$ consists of X_n which are the n orthonormalized Eigenvectors of A^TA.

⇒ It follows then that the Eigenvectors of AA^T. (ie) Y_n are given by

$$y_n = \frac{1}{\sqrt{\lambda_n}} A X_n \longrightarrow (6)$$

The matrix D is a diagonal matrix given by

$$D = \begin{bmatrix} \sqrt{\lambda_1} & 0 & 0 & \cdots & 0 \\ 0 & \sqrt{\lambda_2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \longrightarrow (7)$$

where $\sqrt{\lambda_1}, \sqrt{\lambda_2} \dots, \sqrt{\lambda_n}$ are singular values of A and are

such that

$$\sqrt{\lambda_1} > \sqrt{\lambda_2} \geq \dots \geq \sqrt{\lambda_n} \geq 0. \longrightarrow (8)$$

$\Rightarrow$ If the rank of A is $r < n$, then

$$\sqrt{\lambda_{r+1}} = \sqrt{\lambda_{r+2}} = \dots = \sqrt{\lambda_n} = 0 \longrightarrow (9)$$

It can be shown that the SVD of A is unique if the λ_i are distinct and equation (8) is satisfied.

Note: 2 In case, A is a square matrix of order n, then the matrices U, D and V are also square matrices of the same size and the inverse of A can be trivially computed, since

$$A^{-1} = VD^{-1}U^T \longrightarrow (10)$$

and

$$D^{-1} = \text{diag}\left(\frac{1}{\sqrt{\lambda_1}}, \frac{1}{\sqrt{\lambda_2}}, \dots, \frac{1}{\sqrt{\lambda_n}}\right) \rightarrow (11).$$

If any of the λ_i's are zero, then the matrix A is singular.

|||ly, if the λ_i are very small, then the matrix A is very nearly singular.

Hence, the singular-value decomposition of a matrix gives a clear indication whether the matrix is singular (or) very nearly singular.

Singular matrix :-

A square matrix is singular iff its determinant is 0.

problems

1. Obtain the singular value decomposition of $A = \begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix}$

Solution :-

Given,

$$A = \begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix}, \quad A^T = \begin{bmatrix} 4 & 3 \\ 0 & -5 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 0 & -5 \end{bmatrix} = \begin{bmatrix} 16+0 & 12+0 \\ 12+0 & 9+25 \end{bmatrix} = \begin{bmatrix} 16 & 12 \\ 12 & 34 \end{bmatrix}$$

To find the Eigenvalue of AA^T :

$$|AA' - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 16-\lambda & 12 \\ 12 & 34-\lambda \end{vmatrix} = (16-\lambda)(34-\lambda) - 144 = 0$$

$$\lambda^2 - 16\lambda - 34\lambda + 544 - 144 = 0$$

$$\lambda^2 - 50\lambda + 400 = 0$$

$$400$$
$$-40 \quad -10$$

$$(\lambda - 40)(\lambda - 10) = 0$$

$$\Rightarrow \lambda_1 = 40 \quad \& \quad \lambda_2 = 10.$$

Eigenvector for $\lambda_1 = 40$:-

$$(AA^T - \lambda I)X = 0$$

$$\begin{pmatrix} 16-\lambda & 12 \\ 12 & 34-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

put $\lambda = 40$,

$$\begin{pmatrix} 16-40 & 12 \\ 12 & 34-40 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -24 & 12 \\ 12 & -6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -24x_1 + 12x_2 \\ 12x_1 - 6x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow -24x_1 + 12x_2 = 0 \rightarrow (1)$$

and $12x_1 - 6x_2 = 0 \rightarrow (2)$

$$\Rightarrow -2x_1 + x_2 = 0 \rightarrow (3)$$

$$2x_1 - x_2 = 0. \rightarrow (4)$$

Equations ③ and ④ are same

$$\therefore \quad 2x_1 = x_2$$

$$x_1 = \frac{x_2}{2}$$

put $x_2 = 1 \Rightarrow x_1 = 1/2$.

Hence, the corresponding Eigenvector is

$$X_1 = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$$

Normalized form of X_1 is $\begin{bmatrix} 0.5/\sqrt{(0.5)^2+1^2} \\ 1/\sqrt{(0.5)^2+1^2} \end{bmatrix}$

$$= \begin{bmatrix} 0.5/1.1180 \\ 1/1.1180 \end{bmatrix} = \begin{bmatrix} 0.4472 \\ 0.8945 \end{bmatrix}$$

(ie) $u_1 = \begin{bmatrix} 0.4472 \\ 0.8945 \end{bmatrix}$.

|||ly for $\underline{\underline{\lambda_2 = 10}}$:

$$(AA^T - \lambda I)X = 0$$

$$\begin{pmatrix} 16-\lambda & 12 \\ 12 & 34-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

put $\lambda = 10$,

$$\begin{pmatrix} 16-10 & 12 \\ 12 & 34-10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 6 & 12 \\ 12 & 24 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$\Rightarrow$ $6x_1 + 12x_2 = 0 \rightarrow (5)$

$12x_1 + 24x_2 = 0. \rightarrow (6).$

$x_1 + 2x_2 = 0 \rightarrow (7)$

$x_1 + 2x_2 = 0 \rightarrow (8)$

Since Equations (7) & (8) are Same,

$\therefore \quad 2x_2 = -x_1$

put, $x_1 = 1 \Rightarrow x_2 = -1/2$

$$X_2 = \begin{bmatrix} 1 \\ -1/2 \end{bmatrix} = \begin{bmatrix} 1 \\ -0.5 \end{bmatrix}.$$

The normalized form of Eigenvector is

$$\begin{bmatrix} 1/\sqrt{(-0.5)^2+1^2} \\ -0.5/\sqrt{(-0.5)^2+1^2} \end{bmatrix} = \begin{bmatrix} 1/1.1180 \\ -0.5/\sqrt{(1.1180)} \end{bmatrix} = \begin{bmatrix} 0.8945 \\ -0.4472 \end{bmatrix}$$

$$u_2 = \begin{bmatrix} 0.8945 \\ -0.4475 \end{bmatrix}$$

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$$\therefore u_2 = \begin{bmatrix} 0.8945 \\ -0.4472 \end{bmatrix}$$

$$\Rightarrow U = [u_1, u_2] = \begin{bmatrix} 0.4472 & 0.8945 \\ 0.8945 & -0.4472 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} \sqrt{40} & 0 \\ 0 & \sqrt{10} \end{bmatrix} = \begin{bmatrix} 6.325 & 0 \\ 0 & 3.1623 \end{bmatrix}$$

To find V :-

To find the eigenvalues of A^TA :-

$$A^TA = \begin{bmatrix} 4 & 3 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 16+9 & 0-15 \\ 0-15 & 0+25 \end{bmatrix} = \begin{bmatrix} 25 & -15 \\ -15 & 25 \end{bmatrix}.$$

$$|A^TA - \lambda I| = 0$$

$$\begin{vmatrix} 25-\lambda & -15 \\ -15 & 25-\lambda \end{vmatrix} = [25-\lambda]^2 - 15^2 = 0.$$

$$25^2 + \lambda^2 - 50\lambda - 225 = 0$$

$$\lambda^2 - 50\lambda - 225 + 625 = 0$$

$$\lambda^2 - 50\lambda + 400 = 0$$

$$(\lambda-10)(\lambda-40)=0.$$

$$400$$

$$-40-10$$

$$\Rightarrow \lambda_1=10,\ \lambda_2=40.$$

For $\lambda_1=10$:

$$(A^TA-\lambda I)\,X=0$$

$$\begin{pmatrix} 25-\lambda & -15 \\ -15 & 25-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

put $\lambda=10$,

$$\begin{pmatrix} 25-10 & -15 \\ -15 & 25-10 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 15 & -15 \\ -15 & 15 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$15x_1-15x_2=0 \rightarrow (9) \Rightarrow x_1-x_2=0 \rightarrow (11)$

$-15x_1+15x_2=0 \rightarrow (10) \qquad -x_1+x_2=0 \rightarrow (12).$

Equations (11) & (12) are equal.

put $x_1=1 \Rightarrow x_2=1$

$$X_1=\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Normalized form ~~ote~~, is, $\begin{bmatrix} 1/\sqrt{1^2+1^2} \\ 1/\sqrt{1^2+1^2} \end{bmatrix}=\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$$=\begin{bmatrix} 1/1.414 \\ 1/414 \end{bmatrix}=\begin{bmatrix} 0.7072 \\ 0.70772 \end{bmatrix}=V_1$$

$$-x_1 + x_2 = 0$$

$$x_1 = -1 \quad , \quad x_2 = -1$$

$$v_2 = \begin{pmatrix} -0.7072 \\ -0.7072 \end{pmatrix}.$$

$$V = (v_1, v_2) = \begin{bmatrix} 0.7072 & -0.7072 \\ 0.7072 & -0.7072 \end{bmatrix}$$

The Singular value decomposition of A

$$A = U \Sigma V$$

Here, $A = \begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix}$, $U = \begin{bmatrix} 0.4472 & 0.8945 \\ 0.8945 & -0.4472 \end{bmatrix}$

$$\Sigma = \begin{bmatrix} 6.325 & 0 \\ 0 & 3.1623 \end{bmatrix}; \quad V = \begin{bmatrix} 0.7072 & -0.7072 \\ 0.7072 & -0.7072 \end{bmatrix}$$

$$A = \begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 0.4472 & 0.8945 \\ 0.8945 & -0.4472 \end{bmatrix} \begin{bmatrix} 6.325 & 0 \\ 0 & 3.1623 \end{bmatrix} \begin{bmatrix} 0.7072 & -0.7072 \\ 0.7072 & -0.7072 \end{bmatrix}$$

problem : 2

Obtain the singular value decomposition of

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

Solution :-

To find V :-

(i) Find the Eigenvalues of A^TA.

(ii) Find the Eigenvectors of A^TA.

Apply the same method, we get

$$V_1 = (-0.7071, \ 0.7071)$$

$$V_2 = (0.7071, \ 0.7071).$$

$$V = (V_1, V_2) = \begin{bmatrix} -0.7071 & 0.7071 \\ 0.7071 & 0.7071 \end{bmatrix} = \begin{bmatrix} +0.7071 & -0.7071 \\ -0.7071 & -0.70 \end{bmatrix}.$$

$$\Sigma = \begin{bmatrix} \sqrt{10} & 0 \\ 0 & \sqrt{40} \end{bmatrix} = \begin{bmatrix} 3.1623 & 0 \\ 0 & 6.325 \end{bmatrix}.$$

$\therefore$ The singular value decomposition of A

$$A = U \Sigma V^T$$

$$A = \begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix} = \begin{bmatrix} 0.8945 & 0.4472 \\ -0.4472 & 0.8945 \end{bmatrix} \begin{bmatrix} 3.1623 & 0 \\ 0 & 6.325 \end{bmatrix} \begin{bmatrix} +0.7071 & -0.7071 \\ -0.7071 & -0.7071 \end{bmatrix}.$$

problem:- 2

obtain the singular value decomposition of

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

Solution :-

Given, $A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ and

$$A^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

To find U :-

(i) AA^T:

$$AA^T = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+0+1+0 & 0+0+0+0 \\ 0+0+0+0 & 0+1+0+1 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

(ii) Eigen values of AA^T:

$$|AA^T - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (2-\lambda)(2-\lambda) = 0$$

$$\Rightarrow \boxed{\lambda_1 = 2 \text{ \& } \lambda_2 = 2}$$

Eigenvector for $\lambda = 2$:-

$$(AA^T - \lambda I)x = 0.$$

$$\begin{bmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

put $\lambda = 2$,

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$\Rightarrow \quad x_1 = 0 \quad , \quad x_2 = 1$

(or) $\quad x_1 = 1 \quad , \quad x_2 = 0.$

Hence the corresponding Eigenvector $X_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$

$$\text{Normalized form} = \begin{bmatrix} 1/\sqrt{1^2+0^2} \\ 0/\sqrt{1^2+0^2} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

$$\Rightarrow u_1 = [1 \quad 0]$$

|||ly $u_2 = [0, 1].$

$$\therefore U = [u_1, u_2] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

<u>To find V</u> :-

$$A^T A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0 & 0+0 & 1+0 & 0+0 \\ 0+0 & 0+1 & 0+0 & 0+1 \\ 0+0 & 0+1 & 1+0 & 0+0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

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$$A^TA = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$(a_{11})(-1)^{1+1}$ $\quad (a_{12})(-1)^{1+2}$ $\quad (a_{13})(-1)^{1+3}$

$$|A^TA - \lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 0 & 1 & 0 \\ 0 & 1-\lambda & 0 & 1 \\ 1 & 0 & 1-\lambda & 0 \\ 0 & 1 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$= (1-\lambda)(-1)^2 \cdot \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} + \cancel{0 \cdot (-1)^3 \cdot \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix}} \rightarrow 0$$

$$+ 1 \cdot (-1)^4 \cdot \begin{vmatrix} 0 & 1-\lambda & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1-\lambda \end{vmatrix} + \cancel{0} \rightarrow 0$$

$$= (1-\lambda) \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} + \begin{vmatrix} 0 & 1-\lambda & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)\Big[(1-\lambda)\big[(1-\lambda)(1-\lambda) - 0\big] - 0 + 1\big(0 - (1-\lambda)\big)\Big]$$

$$+ 0[\quad] - (1-\lambda)\big[(1-\lambda) + 0\big] + 1(1+0)\Big]$$

$$= (1-\lambda)\left[(1-\lambda)^3 - 0 - (1-\lambda)\right] - (1-\lambda)^2 + 1$$

$$= (1-\lambda)^4 - (1-\lambda)^2 - (1-\lambda)^2 + 1$$

$$\Rightarrow (1-\lambda)^4 - 2(1-\lambda)^2 + 1 = 0.$$

$$\left[(a-b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4.\right.$$

$$1 - 4\lambda + 6\lambda^2 - 4\lambda^3 + \lambda^4 - 2(1+\lambda^2 - 2\lambda) + 1 = 0.$$

$$\lambda^4 - 4\lambda^3 + \underline{6\lambda^2} - \cancel{4\lambda} + \cancel{1} - \cancel{2} - \underline{2\lambda^2} + \cancel{4\lambda} + \cancel{1} = 0$$

$$\lambda^4 - 4\lambda^3 + 4\lambda^2 = 0.$$

$$\lambda^2\left[\lambda^2 - 4\lambda + 4\right] = 0.$$

$$\lambda^2 = 0 \quad \text{(or)} \quad \lambda^2 - 4\lambda + 4 = 0.$$

$$\boxed{\lambda_{1,2} = 0, 0} \qquad \boxed{\lambda_{3,4} = 2, 2}$$

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$-2\ -2$

(i) Eigenvectors for 0, 0, 2, 2:

$v_1 = (0.7071, 0, 0.7071, 0)$

$v_2 = (0, 0.7071, 0, 0.7071)$

$v_3 = (-0.7071, 0, 0.7071, 0)$

$v_4 = (0, -0.7071, 0, 0.7071)$

(i) Eigenvectors for 0, 0, 2, 2 : [

→ (a) Eigenvector for $\lambda_1 = 0$:

$$[A^TA - \lambda I]x = 0$$

$$\begin{bmatrix} 1-\lambda & 0 & 1 & 0 \\ 0 & 1-\lambda & 0 & 1 \\ 0 & 0 & 1-\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

put $\lambda = 0$,

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + x_3 = 0 \longrightarrow (1)$$

$$x_2 + x_4 = 0 \longrightarrow (2)$$

$$x_1 = x_3 = 0$$

$$x_2 = -1, \; x_4 = 1.$$

(ie) $X_1 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$.

Normalized form of $X_1 = \begin{bmatrix} 0/\sqrt{0^2+(-1)^2+0^2+1^2} \\ -1/\sqrt{0^2+(-1)^2+0^2+1^2} \\ 0/\sqrt{0^2+(-1)^2+0^2+1^2} \\ 1/\sqrt{0^2+(-1)^2+0^2+1^2} \end{bmatrix} = \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 \\ -0.7071 \\ 0 \\ 0.7071 \end{bmatrix}$

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(b) Eigenvector for $\lambda_2 = 0$:

$$[A^TA - \lambda I]x = 0$$

(ie) $\begin{bmatrix} 1-\lambda & 0 & 1 & 0 \\ 0 & 1-\lambda & 0 & 1 \\ 1 & 0 & 1-\lambda & 0 \\ 0 & 1 & 0 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

put $\lambda = 0$,

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 + x_3 = 0 \rightarrow$ (3)

$x_2 + x_4 = 0 \rightarrow$ (4)

$x_1 + x_3 = 0 \rightarrow$ (5)

$x_2 + x_4 = 0 \rightarrow$ (6)

let $x_1 = 1$, $x_3 = -1$

and $x_2 = 0$, $x_4 = 0$.

(ie) $X_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$.

Normalized eigenvector of X_2 is $\begin{bmatrix} 0.7071 \\ 0 \\ -0.7071 \\ 0 \end{bmatrix}$.

(c) Eigenvector for $\lambda_3 = 2$:

$$[A^TA - \lambda I]x = 0$$

$$\begin{bmatrix} 1-\lambda & 0 & 1 & 0 \\ 0 & 1-\lambda & 0 & 1 \\ 1 & 0 & 1-\lambda & 0 \\ 0 & 1 & 0 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

put, $\lambda = 2$.

$$\begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$\left.\begin{aligned} -x_1 + x_3 &= 0 \\ -x_2 + x_4 &= 0 \\ x_1 - x_3 &= 0 \\ x_2 - x_4 &= 0. \end{aligned}\right\} \Rightarrow \begin{aligned} x_1 - x_3 &= 0 \\ x_2 - x_4 &= 0 \\ x_1 - x_3 &= 0 \\ x_2 - x_4 &= 0. \end{aligned}$$

$$x_1 = 1 = x_3$$

$$x_2 = 0 = x_4$$

(ie) $X_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

Normalized eigenvector for X_3 is $\begin{bmatrix} 0.7071 \\ 0 \\ 0.7071 \\ 0 \end{bmatrix}$

(d) Eigenvector for $\lambda_4 = 2$:

$$[A^TA - \lambda I]x = 0$$

$$\begin{bmatrix} 1-\lambda & 0 & 1 & 0 \\ 0 & 1-\lambda & 0 & 1 \\ 1 & 0 & 1-\lambda & 0 \\ 0 & 1 & 0 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

put $\lambda = 2$.

$$\begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} -x_1 + x_3 &= 0 \\ -x_2 + x_4 &= 0 \\ x_1 - x_3 &= 0 \\ x_2 - x_4 &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} x_1 - x_3 &= 0 \\ x_2 - x_4 &= 0 \\ x_1 - x_3 &= 0 \\ x_2 - x_4 &= 0. \end{aligned}$$

let $x_1 = x_3 = 0$

and $x_2 = x_4 = +1$

$$X_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Normalized form of an Eigenvector X_4 is $\begin{bmatrix} 0 \\ 0.7071 \\ 0 \\ 0.7071 \end{bmatrix}$.

From x_1, x_2, x_3 & x_4,

$$V = \begin{array}{c} \begin{matrix} & 3 & 2 & 1 \end{matrix} \\ \begin{bmatrix} 0 & -0.7071 & 0 & 0.7071 \\ +0.7071 & 0 & 0.7071 & 0 \\ 0 & +0.7071 & 0 & 0.7071 \\ -0.7071 & 0 & 0.7071 & 0 \end{bmatrix} \end{array}.$$

$\therefore$ The singular value decomposition of A is

$$A = U \Sigma V$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0.7071 & 0 & 0.7071 \\ -0.7071 & 0 & 0.7071 & 0 \\ 0 & -0.7071 & 0 & 0.7071 \\ 0.7071 & 0 & 0.7071 & 0 \end{bmatrix}$$

QR decomposition:

⇒ QR method is the most widdly used general method for obtaining all the Eigenvalues of a matrix.

Orthogonal Matrix:-

A square matrix A is said to be an Orthogonal matrix if $A^T = A^{-1}$ (or) $AA^T = I$.

Upper triangular matrix:-

The upper triangular matrix has all the elements below the main diagonal as zero.

Ex:-

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix} \rightarrow \text{main diagonal.}$$

Invertible matrix:-

A matrix A of dimension $n \times n$ is called invertible iff there exits another matrix B of the same dimension, such that $AB = BA = I$.

B is known as inverse of A.

* Invertible matrix is also known as a non-singular matrix.

Note:-

1. The matrix Q is orthogonal and R is upper triangular. So

$A = QR$ is the required QR-decomposition

2. Every matrix has a QR-decomposition, through R may not always be invertible.

Problem:-1

Find QR decomposition of $A = \begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix}$ by Gauss-Schmidt method.

Solution:-

$$a_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow q_1' = a_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$r_{11} = \|q_1'\| = \sqrt{1^2+1^2+1^2+1^2} = \sqrt{4} = 2.$$

$$q_1 = \frac{q_1'}{\|q_1'\|} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix} \longrightarrow (1)$$

$$a_2 = \begin{bmatrix} -1 \\ 4 \\ 4 \\ -1 \end{bmatrix}$$

$$r_{12} = q_1^T \cdot a_2 = [0.5 \quad 0.5 \quad 0.5 \quad 0.5] \cdot \begin{bmatrix} -1 \\ 4 \\ 4 \\ -1 \end{bmatrix}$$

(e)

$$r_{12} = q_1^T \cdot a_2 = \begin{bmatrix} 0.5 & 0.5 & 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \\ 4 \\ -1 \end{bmatrix}$$

$$= -0.5 + 2 + 2 - 0.5 = 3.$$

$$\boxed{r_{12} = 3}$$

$$q_2' = a_2 - r_{12} \cdot q_1 = \begin{bmatrix} -1 \\ 4 \\ 4 \\ -1 \end{bmatrix} - 3 \cdot \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 4 \\ -1 \end{bmatrix} - \begin{bmatrix} 1.5 \\ 1.5 \\ 1.5 \\ 1.5 \end{bmatrix}$$

$$q_2' = \begin{bmatrix} -2.5 \\ 2.5 \\ 2.5 \\ -2.5 \end{bmatrix}$$

$$r_{22} = \|q_2'\| = \sqrt{(2.5)^2 + (2.5)^2 + (2.5)^2 + (-2.5)^2}$$

$$= \sqrt{6.25 + 6.25 + 6.25 + 6.25}$$

$$= \sqrt{25} = 5.$$

$$\boxed{r_{22} = 5}$$

$$q_2 = \frac{q_2'}{\|q_2'\|} = \begin{bmatrix} -2.5/5 \\ 2.5/5 \\ 2.5/5 \\ -2.5/5 \end{bmatrix} = \begin{bmatrix} -0.5 \\ 0.5 \\ 0.5 \\ -0.5 \end{bmatrix} \rightarrow (2)$$

$$a_3 = \begin{bmatrix} 4 \\ -2 \\ 2 \\ 0 \end{bmatrix}$$

$$r_{23} = q_2^T \cdot a_3 = \begin{bmatrix} -0.5 & 0.5 & 0.5 & -0.5 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \\ 2 \\ 0 \end{bmatrix}$$

$$= -2-1+1+0 = -2$$

$$\boxed{r_{23} = -2}$$

$$r_{13} = q_1^T \cdot a_3 = \begin{bmatrix} 0.5 & 0.5 & 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \\ 2 \\ 0 \end{bmatrix}$$

$$= [2-1+1+0] = 2$$

$$\boxed{r_{13} = 2}$$

$$q_3' = \begin{bmatrix} 2 \\ -2 \\ 2 \\ -2 \end{bmatrix} \quad \text{and} \quad q_3 = \frac{q_3'}{\|q_3'\|} = \begin{bmatrix} 0.5 \\ -0.5 \\ 0.5 \\ -0.5 \end{bmatrix} \rightarrow (3)$$

$$r_3 = \|q_3'\| = \sqrt{2^2+(-2)^2+2^2+(-2)^2} = \sqrt{16} = 4.$$

From (1), (2) & (3)

$$Q = \begin{bmatrix} 0.5 & -0.5 & 0.5 \\ 0.5 & 0.5 & -0.5 \\ 0.5 & 0.5 & 0.5 \\ 0.5 & -0.5 & -0.5 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 2 & 3 & 2 \\ 0 & 5 & -2 \\ 0 & 0 & 4 \end{bmatrix}.$$