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Vector Space

A vector space V over a field F consist of a set on which 2 operations (+ and .) are defined so that for each pair of elements u and v belongs to V, there is a unique element $u+v \in V$ and for each element $a \in F$ and $u \in V$, there is an unique element $au \in V$ such that the following axium holds.

(i) $u+v = v+u \;, \forall u, v \in V$

(ii) $(u+v)+w = u+(v+w) \;, \forall u, v, w \in V$

(iii) $\exists$ (there exists) $0 \in V$ such that $u+0 = u \;, \forall u \in V$

(iv) $\forall u \in V \;, \exists (-u) \in V$ such that $u+(-u) = 0$

(v) $1 . u = u \;, \forall u \in V$

(vi) $(ab)u = a(bu) \;, \forall u \in V$ & $a, b \in F$

(vii) $(a+b)u = au+bu \;, \forall u \in V$ & $a, b \in F$

(viii) $(u+v)a = au+av \;, \forall u, v \in V$ & $a \in F$

1) Show that $F^n = \{(a_1, a_2, \dots, a_n) : \forall a_i \in F\}$ is a vector space over the field F with respect to addition and scalar multiplication defined component wise.

let $u = (a_1, a_2, \dots, a_n)$

$v = (b_1, b_2, \dots, b_n)$

$u+v = (a_1+b_1, a_2+b_2, \dots, a_n+b_n)$

(i) $u+v = v+u \;, \forall u, v \in V$

$u+v = (a_1+b_1, a_2+b_2, \dots, a_n+b_n)$

$= (b_1+a_1, b_2+a_2, \dots, b_n+a_n)$

$= (b_1, b_2, \dots b_n) + (a_1, a_2, \dots a_n)$

$\therefore u+v = v+u$

(ii) $(u+v)+w = u+(v+w) \;, \forall u, v, w \in V$

$(u+v)+w = (a_1+b_1, a_2+b_2, \dots a_n+b_n) + (c_1, c_2, \dots c_n)$

$$=[(a_1+b_1)+c_1,(a_2+b_2)+c_2,\ldots,(a_n+b_n)+c_n]$$

$$=[a_1+(b_1+c_1),a_2+(b_2+c_2),\ldots,a_n+(b_n+c_n)]$$

$$=(a_1,a_2,\ldots,a_n)+(b_1+c_1,b_2+c_2,\ldots,b_n+c_n)$$

$\therefore (u+v)+w=u+(v+w)$

(iii) $\exists\ 0\in V$, such that $u+0=u,\ \forall\ u\in V$

$\exists (0,0,\ldots,0)\in F^n$

$$u+0=(a_1,a_2,\ldots,a_n)+(0,0,\ldots,0)$$
$$=(a_1,a_2,\ldots,a_n)$$

$\therefore u+0=u$

(iv) $\forall\ u\in V,\ \exists (-u)\in V$ such that $u+(-u)=0$

Take $(-u)=(-a_1,-a_2,\ldots,-a_n)$

$$u+(-u)=(a_1,a_2,\ldots,a_n)+(-a_1,-a_2,\ldots,-a_n)$$

$\therefore u+(-u)=0$

(v) $1.u=u,\ \forall u\in V$

$$1.u=1.(a_1,a_2,\ldots,a_n)$$
$$=(a_1,a_2,\ldots,a_n)$$

$\therefore 1.u=u$

(vi) $(ab)u=a(bu),\ \forall\ u\in V\ \&\ a,b\in F$

$$(ab)u=ab(a_1,a_2,\ldots,a_n)$$
$$=(aba_1,aba_2,\ldots,aba_n)$$
$$=a(ba_1,ba_2,\ldots,ba_n)$$

$\therefore (ab)u=a(bu)$

(vii) $(a+b)u=au+bu,\ \forall\ u\in V\ \&\ a,b\in F$

$$(a+b)u=(a+b)(a_1,a_2,\ldots,a_n)$$
$$=[(a+b)a_1,(a+b)a_2,\ldots,(a+b)a_n]$$
$$=(aa_1+ba_1,aa_2+ba_2,\ldots,aa_n+ba_n)$$
$$=a(a_1,a_2,\ldots,a_n)+b(a_1,a_2,\ldots,a_n)$$

$\therefore (a+b)u=au+bu$

(viii) $(u+v)a=au+av,\ \forall\ u,v\in V\ \&\ a\in F$

$$(u+v)a=(a_1+b_1,a_2+b_2,\ldots,a_n+b_n)a$$
$$=(aa_1+ab_1,aa_2+ab_2,\ldots,aa_n+ab_n)$$

$= (aa_1, aa_2, \ldots, aa_n) + (ab_1, ab_2, \ldots, ab_n)$

$\therefore (u+v)a = ua + va$

$\therefore F^n$ is a vector space.

2) Let $V = \{(a_1, a_2), a_1, a_2 \in R\}$ for $(a_1, a_2), (b_1, b_2) \in V$ and $c \in R$ defined $(a_1, a_2) + (b_1, b_2) = (a_1 + 2b_1, a_2 + 3b_2)$
$c(a_1, a_2) = (ca_1, ca_2)$. Is V is a vector space over R with these operation? Justify your answer.

$$(a_1, a_2) + (b_1, b_2) = (a_1 + 2b_1, a_2 + 3b_2)$$

$$(b_1, b_2) + (a_1, a_2) = (b_1 + 2a_1, b_2 + 3a_2)$$

$\therefore (a_1, a_2) + (b_1, b_2) \neq (b_1, b_2) + (a_1, a_2)$

$\therefore$ commutation axiom is not hold.

$\therefore$ V is not a vector space.

3) Determine whether the set of all 2x2 matrix is of the form $\begin{pmatrix} a & a+b \\ a+b & b \end{pmatrix}$, $a, b \in R$ with respect to standard addition and scalar multiplication is a vector space or not. If not, list all the axioms that fail to hold.

Let $A = \begin{pmatrix} a & a+b \\ a+b & b \end{pmatrix}$, $B = \begin{pmatrix} c & c+d \\ c+d & d \end{pmatrix}$

i) $A+B = \begin{pmatrix} a & a+b \\ a+b & b \end{pmatrix} + \begin{pmatrix} c & c+d \\ c+d & d \end{pmatrix}$

$$= \begin{pmatrix} a+c & (a+b)+(c+d) \\ (a+b)+(c+d) & b+d \end{pmatrix}$$

$$= \begin{pmatrix} c+a & (c+d)+(a+b) \\ (c+d)+(a+b) & d+b \end{pmatrix}$$

$$= \begin{pmatrix} c & c+d \\ c+d & d \end{pmatrix} + \begin{pmatrix} a & a+b \\ a+b & b \end{pmatrix}$$

$\therefore A+B = B+A, \forall A, B \in V$

ii) $(A+B)+C = \begin{pmatrix} a+c & (a+b)+(c+d) \\ (a+b)+(c+d) & b+d \end{pmatrix} + \begin{pmatrix} e & e+f \\ e+f & f \end{pmatrix}$

$$= \begin{pmatrix} (a+c)+e & ((a+b)+(c+d))+(e+f) \\ ((a+b)+(c+d))+(e+f) & (b+d)+f \;\; \cancel{(a+c)+e} \end{pmatrix}$$

$$= \begin{pmatrix} a+(c+e) & (a+b)+((c+d)+(e+f)) \\ (a+b)+((c+d)+(e+f)) & b+(d+f) \;\; \cancel{a+(c+e)} \end{pmatrix}$$

$$= \begin{pmatrix} a & a+b \\ a+b & b \end{pmatrix} + \begin{pmatrix} c+e & (c+d)+(e+f) \\ (c+d)+(e+f) & d+f \end{pmatrix}$$

$\therefore (A+B)+C = A+(B+C)$, $\forall A, B, C \in V$

iii) Let $O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$$A+O = \begin{pmatrix} a & a+b \\ a+b & b \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} a & a+b \\ a+b & b \end{pmatrix}$$

$\therefore A+O = A$, $\forall A \in V$

iv) Take $-A = \begin{pmatrix} -a & -(a+b) \\ -(a+b) & -b \end{pmatrix}$

$$\therefore A+(-A) = \begin{pmatrix} a & a+b \\ a+b & b \end{pmatrix} + \begin{pmatrix} -a & -(a+b) \\ -(a+b) & -b \end{pmatrix}$$

$\therefore A+(-A) = 0$; $\forall A \in V$

v) $1 \cdot A = 1 \cdot \begin{pmatrix} a & a+b \\ a+b & b \end{pmatrix}$

$$= \begin{pmatrix} a & a+b \\ a+b & b \end{pmatrix}$$

$\therefore 1 \cdot A = A$, $\forall A \in V$

vi) $(\alpha\beta) A = (\alpha\beta) \begin{pmatrix} a & a+b \\ a+b & b \end{pmatrix}$

$$= \begin{pmatrix} \alpha\beta a & \alpha\beta(a+b) \\ \alpha\beta(a+b) & \alpha\beta b \end{pmatrix}$$

$$= \begin{pmatrix} \alpha(\beta a) & \alpha(\beta(a+b)) \\ \alpha(\beta(a+b)) & \alpha(\beta b) \end{pmatrix}$$

$$= \alpha \begin{pmatrix} \beta a & \beta(a+b) \\ \beta(a+b) & \beta b \end{pmatrix}$$

$\therefore (\alpha\beta)A = \alpha(\beta A), \forall A \in V, \alpha, \beta \in R$

vii) $(\alpha+\beta)A = (\alpha+\beta)\begin{pmatrix} a & a+b \\ a+b & b \end{pmatrix}$

$$= \begin{pmatrix} (\alpha+\beta)a & (\alpha+\beta)(a+b) \\ (\alpha+\beta)(a+b) & (\alpha+\beta)b \end{pmatrix}$$

$$= \begin{pmatrix} \alpha a+\beta a & \alpha(a+b)+\beta(a+b) \\ \alpha(a+b)+\beta(a+b) & \alpha b+\beta b \end{pmatrix}$$

$$= \begin{pmatrix} \alpha a & \alpha(a+b) \\ \alpha(a+b) & \alpha b \end{pmatrix} + \begin{pmatrix} \beta a & \beta(a+b) \\ \beta(a+b) & \beta b \end{pmatrix}$$

$\therefore (\alpha+\beta)A = \alpha A + \beta A, \forall A \in V, \alpha, \beta \in R$

viii) $\alpha(A+B) = \alpha\begin{pmatrix} a+c & (a+b)+(c+d) \\ (a+b)+(c+d) & b+d \end{pmatrix}$

$$= \begin{pmatrix} \alpha(a+c) & \alpha(a+b)+\alpha(c+d) \\ \alpha(a+b)+\alpha(c+d) & \alpha(b+d) \end{pmatrix}$$

$$= \begin{pmatrix} \alpha a+\alpha c & \alpha(a+b)+\alpha(c+d) \\ \alpha(a+b)+\alpha(c+d) & \alpha b+\alpha d \end{pmatrix}$$

$$= \alpha\begin{pmatrix} a & a+b \\ a+b & b \end{pmatrix} + \alpha\begin{pmatrix} c & c+d \\ c+d & d \end{pmatrix}$$

$\therefore \alpha(A+B) = \alpha A + \alpha B, \forall A, B \in V, \alpha \in R$

$\therefore$ It is a vector space.

4) Prove that the set of all polynomials over the field F is a vector space V.

Let $f(x) = a_0 + a_1 x + a_2 x^2 + \dots\dots + a_n x^n$

$g(x) = b_0 + b_1 x + b_2 x^2 + \dots\dots + b_n x^n$

i) $f(x) + g(x) = (a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2 + \dots + (a_n+b_n)x^n$

$= (b_0+a_0) + (b_1+a_1)x + (b_2+a_2)x^2 + \dots + (b_n+a_n)x^n$

$\therefore f(x)+g(x) = g(x)+f(x), \forall f(x), g(x) \in V$

ii) $[f(x)+g(x)] + h(x) = [(a_0+b_0)+(a_1+b_1)x+(a_2+b_2)x^2 + \dots + (a_n+b_n)x^n]$

$+ [c_0 + c_1 x + c_2 x^2 + \dots\dots c_n x^n]$

$= [(a_0+b_0)+c_0] + [(a_1+b_1)+c_1]x + \ldots + [(a_n+b_n)+c_n]x^n$

$= [a_0+(b_0+c_0)] + [a_1+(b_1+c_1)]x + \ldots + [a_n+(b_n+c_n)]x^n$

$\therefore [f(x)+g(x)]+h(x) = f(x)+[g(x)+h(x)], \forall f(x), g(x), h(x) \in V$

iii) Let $o(x) = 0 + 0x + \ldots + 0x^n$

$f(x) + o(x) = (a_0 + a_1 x + \ldots + a_n x^n) + (0 + 0x + \ldots + 0x^n)$

$= (a_0+0) + (a_1+0)x + \ldots + (a_n+0)x^n$

$= a_0 + a_1 x + \ldots a_n x^n$

$\therefore f(x) + o(x) = f(x), \forall f(x) \in V$

iv) Take $-f(x) = -a_0 - a_1 x - a_2 x^2 - \ldots - a_n x^n$

$f(x) + [-f(x)] = (a_0 + a_1 x + \ldots + a_n x^n) + (-a_0 - a_1 x - \ldots - a_n x^n)$

$= (a_0 + (-a_0)) + (a_1 + (-a_1))x + \ldots + (a_n + (-a_n))x^n$

$= 0 + 0x + \ldots + 0x^n$

$\therefore f(x) + [-f(x)] = o(x), \forall f(x) \in V$

v) $1 \cdot f(x) = 1 \cdot (a_0 + a_1 x + \ldots + a_n x^n)$

$= (a_0 + a_1 x + \ldots + a_n x^n)$

$\therefore 1 \cdot f(x) = f(x), \forall f(x) \in V$

vi) $(ab) f(x) = (ab)(a_0 + a_1 x + \ldots + a_n x^n)$

$= aba_0 + aba_1 x + \ldots + aba_n x^n$

$= a(ba_0 + ba_1 x + \ldots + ba_n x^n)$

$\therefore (ab) f(x) = a(bf(x)), \forall f(x) \in V, a, b \in R$

vii) $(a+b) f(x) = (a+b)(a_0 + a_1 x + \ldots + a_n x^n)$

$= (a+b)a_0 + (a+b)a_1 x + \ldots + (a+b)a_n x^n$

$= (aa_0 + aa_1 x + \ldots + aa_n x^n) + (ba_0 + ba_1 x + \ldots + ba_n x^n)$

$= a(a_0 + a_1 x + \ldots + a_n x^n) + b(a_0 + a_1 x + \ldots + a_n x^n)$

$\therefore (a+b) f(x) = af(x) + bf(x), \forall f(x) \in V, a, b \in R$

viii) $a[f(x)+g(x)] = a[(a_0 + a_1 x + \ldots + a_n x^n) + (b_0 + b_1 x + \ldots + b_n x^n)]$

$= a[(a_0+b_0) + (a_1+b_1)x + \ldots + (a_n+b_n)x^n]$

$= a(a_0+b_0) + a(a_1+b_1)x + \dots + a(a_n+b_n)x^n$

$= (aa_0 + aa_1x + \dots + aa_nx^n) + (ab_0 + ab_1x + \dots + ab_nx^n)$

$= a(a_0+a_1x+\dots+a_nx^n) + a(b_0+b_1x+\dots+b_nx^n)$

$\therefore a[f(x)+g(x)] = af(x)+ag(x), \forall f(x), g(x) \in V, a \in R$

$\therefore$ It is a vector space.

5) Let R^+ be the set of all positive real numbers defined addition and scalar multiplication as follows $u+v = uv, \forall u, v \in R^+$. $\alpha u = u^\alpha, \forall u \in R^+, \alpha \in R$, then prove that R^+ is a vector space.

i) $u+v = uv$

$= vu$

$\therefore u+v = v+u, \forall u, v \in R^+$

ii) $(u+v)+w = uv+w$

$= uvw$

$= u(vw)$

$= u(v+w)$

$\therefore (u+v)+w = u+(v+w), \forall u, v, w \in R^+$

iii) Since R^+ is a positive real number.

Hence $0 \notin R^+$

$1+u = 1.u$

$\therefore 1+u = u, \forall u \in R^+$

$\therefore 1$ is the additive identity.

iv) Take $\frac{1}{u}$ is a additive inverse.

$u+\frac{1}{u} = u\cdot\frac{1}{u}$

$\therefore u+\frac{1}{u} = 1, \forall u \in R^+$

v) $1.u = u'$

$\therefore 1.u = u, \forall u \in R^+$

vi) $(\alpha\beta)u = u^{\alpha\beta}$

$= u^{\beta\alpha}$

$= (u^\beta)^\alpha$

$= \alpha(u^\beta)$

$\therefore (\alpha\beta)u = \alpha(\beta u), \forall u \in R^+, \alpha, \beta \in R$

vii) $(\alpha+\beta)u = u^{(\alpha+\beta)}$

$= u^{\alpha}.u^{\beta}$

$= u^{\alpha}+u^{\beta}$

$\therefore (\alpha+\beta)u = \alpha u+\beta u, \forall u, v \in R^{+}, \alpha, \beta \in R$

viii) $\alpha(u+v) = (u+v)^{\alpha}$

$= (uv)^{\alpha}$

$= u^{\alpha}.v^{\alpha}$

$= u^{\alpha}+v^{\alpha}$

$\therefore \alpha(u+v) = \alpha u+\alpha v, \forall u, v \in R^{+}, \alpha \in R$

$\therefore R^{+}$ is a vector space.

Sub-spaces:

A non-empty sub-set W of a vector space V over a field F is called a sub-space of V if W itself is a vector space over F with the same operations as in V.

note:

1) In any vector space V, V and $\{0\}$ are sub-spaces.

2) $\{0\}$ is called a zero subspace of vector space.

Necessary Condition for a subset.

A subset W of a vector space V is a sub-space of V if it satisfies the following axioms:

i) $0 \in W$

ii) $x+y \in W, x, y \in W$

iii) $cx \in W, x \in W, c \in F$

1) Show that a set $W = \{(a_1, a_2, a_3) \in R^3, 2a_1-7a_2+a_3=0\}$ is a subspace of V.

Let $V = R^3$

i) $(0,0,0) \in R^3$ satisfies $2a_1-7a_2+a_3 = 0$ $[\because 0=0]$

$\therefore 0 \in W$

ii) Let $x = (a_1, a_2, a_3) \Rightarrow 2a_1-7a_2+a_3=0$

$y = (b_1, b_2, b_3) \Rightarrow 2b_1-7b_2+b_3=0$

$x+y = (a_1+b_1, a_2+b_2, a_3+b_3)$

$= 2(a_1+b_1)-7(a_2+b_2)+(a_3+b_3)$

$= (2a_1 - 7a_2 + a_3) + (2b_1 - 7b_2 + b_3)$

$= 0 + 0$

$\therefore x+y = 0$

$\therefore x+y \in W$

iii) Let $x = \cancel{(a+b)}\ (a_1, a_2, a_3) \Rightarrow 2a_1 - 7a_2 + a_3 = 0$

$cx = c(a_1, a_2, a_3) = (ca_1, ca_2, ca_3)$

$= 2(ca_1) - 7(ca_2) + (ca_3)$

$= c(2a_1 - 7a_2 + a_3)$

$= c(0)$

$cx = 0$

$\therefore cx \in W$

$\therefore W$ is a sub-space over F.

2) Show that the set $W = \{(a_1, a_2, a_3) \in R^3 ; 5a_1^2 - 3a_2^2 + 6a_3^2 = 0\}$ is not a sub-space of V.

Let $V = R^3$

i) $(0,0,0) \in R^3$ satisfies $5a_1^2 - 3a_2^2 + 6a_3^2 = 0$ $[\because 0 = 0]$

$\therefore 0 \in W$

ii) Let $x = (a_1, a_2, a_3) \Rightarrow 5a_1^2 - 3a_2^2 + 6a_3^2 = 0$

$y = (b_1, b_2, b_3) \Rightarrow 5b_1^2 - 3b_2^2 + 6b_3^2 = 0$

$x+y = (a_1+b_1, a_2+b_2, a_3+b_3)$

$= 5(a_1+b_1)^2 - 3(a_2+b_2)^2 + 6(a_3+b_3)^2$

$= 5(a_1^2+b_1^2+2a_1b_1) - 3(a_2^2+b_2^2+2a_2b_2) + 6(a_3^2+b_3^2+2a_3b_3)$

$= (5a_1^2 - 3a_2^2 + 6a_3^2) + (5b_1^2 - 3b_2^2 + 6b_3^2) + 10a_1b_1 - 6a_2b_2 + 12a_3b_3$

$x+y \neq 0$

$\therefore x+y \notin W$

$\therefore W$ is not a sub-space.

3) Let $V = R^3$, show that $W = \{(a_1, a_2, a_3), a_1 = a_3 + 2\}$ is not a sub-space of V.

$(0,0,0) \in R^3$ does not satisfy $a_1 = a_3 + 2$ $[\because 0 \neq 2]$

$\therefore 0 \notin W$

$\therefore W$ is not a sub-space.

4) Let $V = R^3$, show that $W = \{(a,b,c) ; a^2 + b^2 + c^2 \leq 1\}$ is not a sub-space of V.

i) $(0,0,0) \in R^3$ satisfies $a^2+b^2+c^2 \leq 1$ $[\because 0 \leq 1]$

$\therefore 0 \in W$

ii) Let $x = (1,0,0) \Rightarrow 1^2+0^2+0^2 \leq 1$

$y = (0,1,0) \Rightarrow 0^2+1^2+0^2 \leq 1$

$x+y = (1,1,0) = 1^2+1^2+0^2 \not\leq 1$

$2 \not\leq 1$

$\therefore x+y \notin W$

$\therefore W$ is not a sub-space of V.

5) Show that $W = \{(a_1,a_2,a_3) \in R^3, a_1 = 3a_2, a_3 = -a_2\}$ is not a sub-space of V.

i) $(0,0,0) \in R^3$ satisfies $a_1 = 3a_2, a_3 = -a_2$. $[\because 0=0]$

$\therefore 0 \in W$

ii) Let $x = (a_1, a_2, a_3) \Rightarrow a_1 = 3a_2$ & $a_3 = -a_2$

$y = (b_1, b_2, b_3) \Rightarrow b_1 = 3b_2$ & $b_3 = -b_2$

$x+y = (a_1+b_1, a_2+b_2, a_3+b_3)$

$a_1+b_1 = 3a_2+3b_2 = 3(a_2+b_2)$

$a_3+b_3 = -a_2-b_2 = -(a_2+b_2)$

$\therefore x+y \in W$

iii) Let $x = (a_1,a_2,a_3)$

$cx = c(a_1,a_2,a_3)$

$cx = (ca_1, ca_2, ca_3)$

$ca_1 = c3a_2 = 3(ca_2)$

$ca_3 = c(-a_2) = -(ca_2)$

$\therefore cx \in W$

$\therefore W$ is a sub-space of V.

Linear Combinations and Systems of Linear Equations:

Linear Combination:

Let V be a vector space over a field F. Let $x_1, x_2, \ldots\ldots, x_n \in V$, then an element of the form $a_1x_1 + a_2x_2 + \ldots\ldots + a_nx_n$ where $a_i \in F$ is called a linear combination of $x_1, x_2, \ldots, x_n$.

1) Determine whether $(-2,0,3)$ can be expressed as a linear combination of $(1,3,0)$ and $(2,4,-1)$.

Let $(-2,0,3) = a(1,3,0) + b(2,4,-1)$

$(-2,0,3) = (a,3a,0) + (2b,4b,-b)$

$-2 = a+2b \rightarrow (1)$

$0 = 3a+4b \rightarrow (2)$

$3 = 0-b \rightarrow (3)$

$\Rightarrow b = -3$

$(1) \Rightarrow -2 = a+2(-3)$

$-2 = a-6$

$a = 4$

$(2) \Rightarrow 3a+4b = 0$

$12-12 = 0$

$0 = 0$

$\therefore$ The eq. is consistent

The linear combination can be expressed as,

$-(2,0,3) = 4(1,3,0) - 3(2,4,-1)$

2) Check whether $(1,-2,5)$ is a linear combination of $(1,1,1)$, $(1,2,3)$ and $(2,-1,1)$.

Let $(1,-2,5) = a(1,1,1) + b(1,2,3) + c(2,-1,1)$

$(1,-2,5) = (a,a,a) + (b,2b,3b) + (2c,-c,c)$

$1 = a+b+2c \rightarrow (1)$

$-2 = a+2b-c \rightarrow (2)$

$5 = a+3b+c \rightarrow (3)$

$(1)-(2) \Rightarrow 3 = -\cancel{b}+3c \rightarrow (4)$

$(2)-(3) \Rightarrow -7 = \cancel{-b}-2c \rightarrow (5)$

$\quad (-)\ (+)\ \ (+)\ (+)$

$10 = 5c$

$c = 2$

$(4) \Rightarrow 3 = -b+6$

$b = 3$

$(1) \Rightarrow 1 = a+3+4$

$1 = a+7$

$a = -6$

$(2) \Rightarrow -2 = -6+6-2$

$-2 = -2$

$\therefore$ The eq. is consistent

The linear combination can be expressed as

$(+1,-2,5) = -6(1,1,1) + 3(1,2,3) + 2(2,-1,1)$

3) Check whether $x^3 - 3x + 5$ is a linear combination of $x^3 + 2x^2 - x + 1$ and $x^3 + 3x^2 - 1$.

Let $(x^3 - 3x + 5) = a(x^3 + 2x^2 - x + 1) + b(x^3 + 3x^2 - 1)$

Eq. like coeff.,

$$1 = a + b \rightarrow ①$$
$$-3 = -a \rightarrow ② \Rightarrow a = 3$$
$$5 = a - b \rightarrow ③$$

③ $\Rightarrow 5 = 3 - b$

$b = -2$

① $\Rightarrow 1 = 3 - 2$

$1 = 1$

$\therefore$ The eq. is consistent.

The linear combination can be expressed as

$x^3 - 3x + 5 = 3(x^3 + 2x^2 - x + 1) - 2(x^3 + 3x^2 - 1)$

Linear span:

Let S be the non-empty subset of a vector space V, then the set of all linear combinations of element of S is called linear span of S and it is denoted as $L(S) = a_1x_1 + a_2x_2 + \dots + a_nx_n$.

1) Determine whether the set of vector $(1,1,2)$, $(1,0,1)$, $(2,1,3)$ span r^3.

Let $(1,1,2) = a(1,0,1) + b(2,1,3)$

$(1,1,2) = (a,0,a) + (2b,b,3b)$

$$1 = a + 2b \rightarrow ①$$
$$1 = b \rightarrow ②$$
$$2 = a + 3b \rightarrow ③$$

② $\Rightarrow b = 1$

① $\Rightarrow 1 = a+2$

$a = -1$

③ $\Rightarrow 2 = -1+3$

$2 = 2$

$\therefore$ The eq. is consistent

The linear combination can be expressed as

$(1,1,2) = -1(1,0,1) + 1(2,1,3)$

$\therefore$ The given vectors are in the span of r^3

2) Test whether the indicated vector is in the linear span of S.

i) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$; $S = \left\{ \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right\}$

Let $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} + c\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

$1 = a+c \rightarrow$ ①

$0 = b+c \rightarrow$ ②

$0 = -a \rightarrow$ ③ $\Rightarrow a=0$

$1 = b \rightarrow$ ④ $\Rightarrow b=1$

② $\Rightarrow 0 = 1+c$

$c = -1$

① $\Rightarrow 1 = a+c$

$1 = 0-1$

$1 \neq -1$

$\therefore$ The eq. is inconsistent

$\therefore$ The given vector is not in the span of S.

ii) $(2,-1,1)$; $S = \{(1,0,2), (-1,1,1)\}$ in $R^3(R)$

Let $(2,-1,1) = a(1,0,2) + b(-1,1,1)$

$(2,-1,1) = (a,0,2a) + (-b,b,b)$

$2 = a-b \rightarrow$ ①

$-1 = b \rightarrow$ ② $\Rightarrow b=-1$

$1 = 2a+b \rightarrow$ ③

$① \Rightarrow 2 = a + 1$

$a = 1$

$③ \Rightarrow 1 = 2 - 1$

$1 = 1$

∴ The eq. is consistent

∴ The linear combination can be expressed as

$(2,-1,1) = 1(1,0,2) - 1(-1,1,1)$

∴ The given vectors are in the span of S

3) Show that the vectors $(1,1,0)$, $(1,0,1)$, $(0,1,1)$ generate F^3.

Let $(x_1, x_2, x_3) = a(1,1,0) + b(1,0,1) + c(0,1,1)$

$x_1 = a + b \rightarrow ①$

$x_2 = a + c \rightarrow ②$

$x_3 = b + c \rightarrow ③$

$① - ②$, $x_1 - x_2 = b - c \rightarrow ④$

$② - ③$, $x_2 - x_3 = a - b \rightarrow ⑤$

$① + ⑤$, $x_1 + x_2 - x_3 = 2a$

$$a = \frac{x_1 + x_2 - x_3}{2}$$

$③ + ④$, $x_1 - x_2 + x_3 = 2b$

$$b = \frac{x_1 - x_2 + x_3}{2}$$

$② \Rightarrow a + c = x_2$

$$\frac{x_1 + x_2 - x_3}{2} + c = x_2$$

$$c = x_2 - \left(\frac{x_1 + x_2 - x_3}{2}\right)$$

$$c = \frac{-x_1 + x_2 + x_3}{2}$$

$$\therefore (x_1, x_2, x_3) = \frac{1}{2}(x_1 + x_2 - x_3)(1,1,0) + \frac{1}{2}(x_1 - x_2 + x_3)(1,0,1) + \frac{1}{2}(-x_1 + x_2 + x_3)(0,1,1)$$

4) Show that $\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}, \begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}$ generate $M_{2\times2}(F)$.

Let $\begin{pmatrix}a & b\\ c & d\end{pmatrix} = r\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix} + s\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} + t\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix} + u\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}$

$a = r+s+t \rightarrow (1)$

$b = r+s+u \rightarrow (2)$

$c = r+t+u \rightarrow (3)$

$d = s+t+u \rightarrow (4)$

$a+b+c+d = 3r+3s+3t+3u$

$r+s+t+u = \frac{a+b+c+d}{3} \rightarrow (5)$

$(5) \Rightarrow r+d = \frac{a+b+c+d}{3}$

$r = \frac{a+b+c-2d}{3}$

$(5) \Rightarrow s+c = \frac{a+b+c+d}{3}$

$s = \frac{a+b-2c+d}{3}$

$(5) \Rightarrow t+b = \frac{a+b+c+d}{3}$

$t = \frac{a-2b+c+d}{3}$

$(5) \Rightarrow u+a = \frac{a+b+c+d}{3}$

$u = \frac{-2a+b+c+d}{3}$

$\therefore \begin{pmatrix}a & b\\ c & d\end{pmatrix} = \left[\frac{a+b+c-2d}{3}\right]\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix} + \left[\frac{a+b-2c+d}{3}\right]\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$

$+ \left[\frac{a-2b+c+d}{3}\right]\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix} + \left[\frac{-2a+b+c+d}{3}\right]\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}$

5) Show that $(1,1,1), (0,1,1), (0,1,-1)$ generate R^3.

Let $(x_1, x_2, x_3) = a(1,1,1) + b(0,1,1) + c(0,1,-1)$

$x_1 = a \rightarrow (1)$

$x_2 = a+b+c \rightarrow (2)$

$x_3 = a+b-c \rightarrow (3)$

$① \Rightarrow a = x_1$

$② \Rightarrow x_2 = x_1 + b + c$

$\therefore x_2 - x_1 = b + c \rightarrow ④$

$③ \Rightarrow x_3 = x_1 + b - c$

$x_3 - x_1 = b - c \rightarrow ⑤$

$④ + ⑤, \quad -2x_1 + x_2 + x_3 = 2b$

$$b = \frac{-2x_1 + x_2 + x_3}{2}$$

$$④ \Rightarrow x_2 - x_1 = \frac{-2x_1 + x_2 + x_3}{2} + c$$

$$c = x_2 - x_1 - \left(\frac{-2x_1 + x_2 + x_3}{2}\right)$$

$$c = \frac{x_2 - x_3}{2}$$

$$\therefore (x_1, x_2, x_3) = x_1(1,1,1) + \frac{1}{2}(-2x_1 + x_2 + x_3)(0,1,1) + \frac{1}{2}(x_2 - x_3)(0,1,-1)$$

System of Linear Equations :

1) Solve

$$3x_1 - 7x_2 + 4x_3 = 10$$
$$x_1 - 2x_2 + x_3 = 3$$
$$2x_1 - x_2 - 2x_3 = 6$$

Rearrange the equations,

$$x_1 - 2x_2 + x_3 = 3$$
$$2x_1 - x_2 - 2x_3 = 6$$
$$3x_1 - 7x_2 + 4x_3 = 10$$

$$= \left[\begin{array}{ccc|c} 1 & -2 & 1 & 3 \\ 2 & -1 & -2 & 6 \\ 3 & -7 & 4 & 10 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & 3 \\ 0 & 3 & -4 & 0 \\ 0 & -1 & +1 & 1 \end{array}\right] \quad \begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 3R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & 3 \\ 0 & 3 & -4 & 0 \\ 0 & 0 & -1 & 3 \end{array}\right] \quad R_3 \to 3R_3 + R_2$$

$-x_3 = 3$

$x_3 = -3$

$3x_2 - 4x_3 = 0$

$3x_2 + 12 = 0$

$3x_2 = -12$

$x_2 = -4$

$x_1 - 2x_2 + x_3 = 3$

$x_1 + 8 - 3 = 3$

$x_1 = -2$

$\therefore$ The solution set is $(-2, -4, -3)$

2) solve
$x_1 + 2x_2 - x_3 + x_4 = 5$
$x_1 + 4x_2 - 3x_3 - 3x_4 = 6$
$2x_1 + 3x_2 - x_3 + 4x_4 = 8$

$$= \left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 5 \\ 1 & 4 & -3 & -3 & 6 \\ 2 & 3 & -1 & 4 & 8 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 5 \\ 0 & 2 & -2 & -4 & 1 \\ 0 & -1 & 1 & 2 & -2 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 5 \\ 0 & 2 & -2 & -4 & 1 \\ 0 & 0 & 0 & 0 & -3 \end{array}\right] \quad R_3 \to 2R_3 + R_2$$

$0 \neq -3$ (impossible)

$\therefore$ The given system of eq. has no solution.

3) solve
$2x_1 - 2x_2 - 3x_3 = -2$
$3x_1 - 3x_2 - 2x_3 + 5x_4 = 7$
$x_1 - x_2 - 2x_3 - x_4 = -3$

Rearranging,

$$x_1 - x_2 - 2x_3 - x_4 = -3$$
$$2x_1 - 2x_2 - 3x_3 = -2$$
$$3x_1 - 3x_2 - 2x_3 + 5x_4 = 7$$

$$= \left[\begin{array}{cccc|c} 1 & -1 & -2 & -1 & -3 \\ 2 & -2 & -3 & 0 & -2 \\ 3 & -3 & -2 & 5 & 7 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & -1 & -2 & -1 & -3 \\ 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 4 & 8 & 16 \end{array}\right] \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & -1 & -2 & -1 & -3 \\ 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] R_3 \to R_3 - 4R_2$$

$$x_3 + 2x_4 = 4 \longrightarrow (1)$$
$$x_1 - x_2 - 2x_3 - x_4 = -3 \longrightarrow (2)$$

put $x_4 = t$,

$(1) \Rightarrow x_3 + 2t = 4$

$x_3 = 4 - 2t$

$(2) \Rightarrow x_1 - x_2 - 2(4-2t) - t = -3$

$x_1 - x_2 - 8 + 4t - t = -3$

$x_1 - x_2 = 5 - 3t$

put $x_2 = s$,

$x_1 - s = 5 - 3t$

$x_1 = 5 - 3t + s$

$\therefore$ The sol. set is $\{5 - 3t + s, s, 4 - 2t, t\}$, $s, t \in R$

Part - A

1) Define vector space.

2) Define subspace.

3) state and prove cancellation law for vector addition.

If x, y, z are vectors in vector space V such that $x+z = y+z$, then $x = y$.

Proof: we know that, $\forall z \in V$ there exists $v \in V$ such that $z + v = 0$.

$$\begin{aligned} x &= x + 0 \\ &= x + (z+v) \\ &= (x+z) + v \\ &= (y+z) + v \\ &= y + (z+v) \\ &= y + 0 \\ x &= y \end{aligned}$$

4) Prove that every vector space has an unique additive identity.

suppose 0 and $0'$ are two additive identities of V, ~~and~~ $\forall v \in V$.

$$0 + v = v \quad \& \quad 0' + v = v$$

$$0 + v = v = 0' + v$$

$$0 + v = 0' + v$$

$$0 = 0' \quad (\text{by cancellation law})$$

$\therefore$ The additive identity is unique.

5) Prove that in a vector space V, the additive inverse is unique.

suppose w and w' are additive inverses of v, $\forall v \in V$.

$$w + v = 0 \quad \& \quad w' + v = 0$$

$$\begin{aligned} w &= w + 0 \\ &= w + (w' + v) \\ &= w + (v + w') \\ &= (w + v) + w' \\ &= 0 + w' \\ w &= w' \end{aligned}$$

$\therefore$ The additive inverse is unique.

6) Check $(-2,0,3)$ is a linear combination of $(1,3,0)$ and $(2,4,-1)$.

Let $(-2,0,3) = a(1,3,0) + b(2,4,-1)$

$-2 = a + 2b \rightarrow (1)$
$0 = 3a + 4b \rightarrow (2)$
$3 = -b \rightarrow (3)$

$(3) \Rightarrow b = -3$

$(2) \Rightarrow 0 = 3a - 12$
$3a = 12$
$a = 4$

$(1) \Rightarrow -2 = 4 - 6$
$-2 = -2$

$\therefore$ The eqs. is consistent

$\therefore$ The L.C can be expressed as

$(-2,0,3) = 4(1,3,0) - 3(2,4,-1)$.

7) Check $(3,4,1)$ is a linear combination of $(1,-2,1)$ and $(-2,-1,1)$.

Let $(3,4,1) = a(1,-2,1) + b(-2,-1,1)$

$3 = a - 2b \rightarrow (1)$
$4 = -2a - b \rightarrow (2)$
$1 = a + b \rightarrow (3)$

$(2) + (3)$, $5 = -a$
$a = -5$

$\cancel{(1) \Rightarrow 3 = -5 - 2b}$
$\cancel{2b = -8}$
$\cancel{b = -4}$

$(3) \Rightarrow 1 = -5 + b$
$b = 6$

$(1) \Rightarrow 3 = -5 - 12$
$3 \neq -17$

$\therefore$ The eqs. is inconsistent.

8) Check the set $W = \{(a_1, a_2, a_3) \in R^3, a_1 = a_2 + 2\}$ is subspace or not.

~~Sol~~ $(0,0,0) \in R^3$ does not satisfy $a_1 = a_2 + 2$. $[\because 0 \neq 2]$

$\therefore 0 \notin W$

$\therefore$ The given set is not a sub space.

1) Determine whether the set of vectors $(1,1,2), (1,0,1), (2,1,3)$ span R^3.

Let $(1,1,2) = a(1,0,1) + b(2,1,3)$

$$1 = a + 2b \rightarrow (1)$$
$$1 = b \rightarrow (2)$$
$$2 = a + 3b \rightarrow (3)$$

(2) $\Rightarrow$ $b = 1$

(1) $\Rightarrow$ $a + 2 = 1$

$a = -1$

(3) $\Rightarrow$ $2 = -1 + 3$

$2 = 2$

$\therefore$ The given equations is consistent.

$\therefore$ we can express as a linear combination.

$(1,1,2) = -1(1,0,1) + 1(2,1,3)$

$\therefore$ The given sets are in span R^3.

2) Determine whether the given vectors is span $(2,-1,1)$, $S = \{(1,0,2), (-1,1,1)\}$

Let $(2,-1,1) = a(1,0,2) + b(-1,1,1)$

$$2 = a - b \rightarrow (1)$$
$$-1 = b \rightarrow (2)$$
$$1 = 2a + b \rightarrow (3)$$

(2) $\Rightarrow$ $b = -1$

(1) $\Rightarrow$ $2 = a + 1$

$a = 1$

(3) $\Rightarrow$ $1 = 2 - 1$

$1 = 1$

$\therefore$ The given equations is consistent

$\therefore$ we can express as a linear combination

$(2,-1,1) = 1(1,0,2) - 1(-1,1,1)$

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$\therefore$ The given vectors are in span S.

3) Determine the vectors are in span, $(2,-5,4)$; $S = \{(1,-3,2), (2,-1,1)\}$

Let $(2,-5,4) = a(1,-3,2) + b(2,-1,1)$

$$2 = a + 2b \rightarrow (1)$$
$$-5 = -3a - b \rightarrow (2)$$
$$4 = 2a + b \rightarrow (3)$$

$(2)+(3)$, $-1 = -a$

$a = 1$

$(3) \Rightarrow 4 = 2 + b$

$b = 2$

$(1) \Rightarrow 2 = 1 + 4$

$2 \neq 5$

$\therefore$ The eqs. is inconsistent

$\therefore$ The given vectors are not in span S.

4) Determine whether the vectors are in span, $x^3 - 3x + 5$, $S = \{x^3 + 2x^2 - x + 1, x^3 + 3x^2 - 1\}$

Let $x^3 - 3x + 5 = a(x^3 + 2x^2 - x + 1) + b(x^3 + 3x^2 - 1)$

coeff of $x^3 \Rightarrow 1 = a + b \rightarrow (1)$

coeff of $x \Rightarrow -3 = -a \rightarrow (2)$

coeff of constant $\Rightarrow 5 = a - b \rightarrow (3)$

$(2) \Rightarrow a = 3$

$(3) \Rightarrow 5 = 3 - b$

$b = -2$

$(1) \Rightarrow 1 = 3 - 2$

$1 = 1$

$\therefore$ The given eqs is consistent

$\therefore$ we can express as a linear combination.

$$x^3 - 3x + 5 = 3(x^3 + 2x^2 - x + 1) - 2(x^3 + 3x^2 - 1)$$

$\therefore$ The given vectors are in span S.

Linear Independence and Linear Dependence:

Linearly Independent:

Let V be a vector space over a field F, a finite set of vectors $x_1, x_2, \ldots, x_n$ in V is said to be linearly independent if,

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0 \Rightarrow a_1 = a_2 = \cdots = a_n = 0$$

Linearly Dependent:

Let V be a vector space over a field F, a finite set of vectors $x_1, x_2, \ldots, x_n$ in V is said to be linearly dependent if there exist a scalar $a_1, a_2, \ldots, a_n$ not all zero such that

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$$

1) Determine whether the following set of vectors are linearly independent or linearly dependent in $V_3(R)$

i) $(1,2,1), (2,1,0), (1,-1,2)$

Let $a(1,2,1) + b(2,1,0) + c(1,-1,2) = (0,0,0)$

$$a + 2b + c = 0$$
$$2a + b - c = 0$$
$$a + 2c = 0$$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 1 & -1 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & -2 & 1 & 0 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 9 & 0 \end{array}\right] \quad R_3 \to 3R_3 - 2R_2$$

$$9c = 0$$
$$c = 0$$
$$-3b - 3c = 0$$

$$-3b = 0$$

$$b = 0$$

$$a + 2b + c = 0$$

$$a = 0$$

$$\therefore a = b = c = 0$$

$\therefore$ The given vectors are linearly independent.

ii) $(1,0,0), (0,1,0), (1,1,0)$

Let $a(1,0,0) + b(0,1,0) + c(1,1,0) = (0,0,0)$

$$a + c = 0$$

$$b + c = 0$$

Take $c = t$

$$b + t = 0$$

$$b = -t$$

$$a + t = 0$$

$$a = -t$$

$$\therefore a = -t, \ b = -t, \ c = t$$

$\therefore$ The given vectors are linearly dependent.

iii) $(1,2,3), (2,3,1)$

Let $a(1,2,3) + b(2,3,1) = (0,0,0)$

$$a + 2b = 0 \rightarrow (1)$$

$$2a + 3b = 0 \rightarrow (2)$$

$$3a + b = 0 \rightarrow (3)$$

$$2 \times (1) \Rightarrow 2a + 4b = 0$$

$$(2) \Rightarrow \underset{(-)}{2a} \underset{(-)}{+ 3b} \underset{(-)}{=0}$$

$$b = 0$$

$$(1) \Rightarrow a + 0 = 0$$

$$a = 0$$

$$\therefore a = b = 0$$

$\therefore$ The given vectors are linearly independent.

iv) $(1,4,-2), (-2,1,3), (-4,11,5)$

Let $a(1,4,-2) + b(-2,1,3) + c(-4,11,5) = (0,0,0)$

$$a - 2b - 4c = 0$$
$$4a + b + 11c = 0$$
$$-2a + 3b + 5c = 0$$

$$= \left[\begin{array}{ccc|c} 1 & -2 & -4 & 0 \\ 4 & 1 & 11 & 0 \\ -2 & 3 & 5 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & -4 & 0 \\ 0 & 9 & 27 & 0 \\ 0 & -1 & -3 & 0 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 4R_1 \\ R_3 \to R_3 + 2R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & -4 & 0 \\ 0 & 9 & 27 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_3 \to R_3 + \frac{1}{9}R_2$$

$$9b + 27c = 0$$
$$a - 2b - 4c = 0$$

Take $c = t$

$$9b + 27t = 0$$
$$9b = -27t$$
$$b = -3t$$
$$a + 6t - 4t = 0$$
$$a + 2t = 0$$
$$a = -2t$$

$\therefore a = -2t,\ b = -3t,\ c = t$

$\therefore$ The given vectors are linearly dependent.

v) $\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ -4 & 4 \end{pmatrix}$ in $M_{2\times 2}(R)$

$$a\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} + b\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} + c\begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 2 & 1 \\ -4 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$a - c + 2d = 0$$
$$-b + 2c + d = 0$$
$$-2a + b + c - 4d = 0$$
$$a + b + 4d = 0$$

$$= \left[\begin{array}{cccc|c} 1 & 0 & -1 & 2 & 0 \\ 0 & -1 & 2 & 1 & 0 \\ -2 & 1 & 1 & -4 & 0 \\ 1 & 1 & 0 & 4 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & -1 & 2 & 0 \\ 0 & -1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 0 \end{array}\right] \quad \begin{array}{l} R_3 \to R_3 + 2R_1 \\ R_4 \to R_4 - R_1 \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & -1 & 2 & 0 \\ 0 & -1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 3 & 3 & 0 \end{array}\right] \quad \begin{array}{l} R_3 \to R_3 + R_2 \\ R_4 \to R_4 + R_2 \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & -1 & 2 & 0 \\ 0 & -1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad R_4 \to R_4 - 3R_3$$

$$c + d = 0$$
$$-b + 2c + d = 0$$
$$a - c + 2d = 0$$

Take $d = t$

$$c + t = 0$$
$$c = -t$$
$$-b - 2t + t = 0$$
$$-b - t = 0$$
$$b = -t$$
$$a + t + 2t = 0$$
$$a + 3t = 0$$
$$a = -3t$$

$\therefore a = -3t,\ b = -t,\ c = -t,\ d = t$

$\therefore$ The given vectors are linearly dependent.

Basis and Dimensions:

Basis:

A linearly independent subset S of a vector space V which spans the whole space V is called the basis of the vector space.

1) span $\{\phi\} = \{0\}$ is a basis for 0 vector space

2) In F^n, let $e_1 = (1, 0, \ldots, 0)$ is a standard basis for F^n.

$$e_2 = (0, 1, \ldots, 0)$$
$$\vdots$$
$$e_n = (0, 0, \ldots, 1)$$

3) $M_{m \times n}(F)$, let E_{ij} ~~deno~~ denote the matrix whose only non-zero entry is a '1' in the i^{th} row and j^{th} column is a basis for $M_{m \times n}(F)$.

4) In $P_n(F)$, $= \{1, x, x^2, \ldots, x^n\}$ is a standard polynomial for $P_n(F)$.

Dimension:

Let V be a finite dimensional vector space over a field F, the number of elements in any basis of V(F) is called dimension of V and is denoted by dim V.

1) $\dim \{0\} = 0$

2) $\dim F^n = n$

3) $\dim M_{m \times n}(F) = mn$

4) $\dim P_n(F) = n+1$

note:

Any linearly independent subset of V that contains exactly n vectors is a basis for V.

1) Determine ~~which of~~ whether the following sets are basis: ~~for R^3~~

1) $(1, 0, -1), (2, 5, 1), (0, -4, 3)$ for R^3.

First, we have to check if the given vectors are linearly independent or linearly dependent.

The coefficient matrix,

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 2 & 5 & 1 \\ 0 & -4 & 3 \end{bmatrix}$$

$= 1(15+4) - 1(-8) = 19 + 8$

$= 27 \neq 0$

$\therefore$ The given vectors are linearly independent.

There are 3 vectors and dimension of $R^3 = 3$

$\therefore$ By note theorem, any linearly independent subset of V contains exactly 'n' vector is a basis of V.

$\therefore$ The given set is basis for R^3.

2) $(1, -3, -2), (-3, 1, 3), (-2, -10, -2)$

First, we have to check if the given vectors are linearly independent or linearly dependent.

The coefficient matrix,

$$\Rightarrow \begin{vmatrix} 1 & -3 & -2 \\ -3 & 1 & 3 \\ -2 & -10 & -2 \end{vmatrix}$$

$$= 1(-2+30) + 3(6+6) - 2(30+2)$$

$$= 28 + 36 - 64$$

$$= 64 - 64 = 0$$

$\therefore$ The given vectors are linearly dependent.

$\therefore$ The given set is not a basis for R^3.

3) $\{-1+2x+4x^2, 3-4x-10x^2, -2-5x-6x^2\}$ in $P_2(R)$.

First we have to check the given vectors are linearly independent or linearly dependent.

The coefficient matrix,

$$\Rightarrow \begin{vmatrix} -1 & 2 & 4 \\ 3 & -4 & -10 \\ -2 & -5 & -6 \end{vmatrix}$$

$$= -1(24-50) - 2(-18-20) + 4(-15-8)$$

$$= -1(-26) - 2(-38) + 4(-23)$$

$$= 26 + 76 - 92$$

$$= 10 \neq 0$$

$\therefore$ The given vectors are linearly independent.

There are 3 vectors and dimension of $P_2(R) = 2$

$\therefore$ The given set is basis for $P_2(R)$.

4) Determine the basis and dimension of solutions of space of linear homogeneous system of $x+y-z=0$, $-2x-y+2z=0$, $-x+z=0$

Rearrange the given equations,

$$x+y-z=0$$
$$-x+z=0$$
$$-2x-y-2z=0$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & -1 \\ -1 & 0 & 1 \\ -2 & -1 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 + R_1 \\ R_3 \to R_3 + 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

$\therefore$ dim $= 2$ (no. of non-zero rows in row echelon form)

Basis $= \{(1,1,-1), (0,1,0)\}$.

5) Determine if $(1,0,0)$, $(0,1,0)$ is a basis for R^3.

There are 2 vectors and dimension of $R^3 = 2$.

$\Rightarrow$ The subset of R^3 contains only 2 vectors.

$\therefore$ The given set is not a basis for R^3.

6) The vectors $u_1 = (2,-3,1)$, $u_2 = (1,4,-2)$, $u_3 = (-8,12,-4)$, $u_4 = (1,37,-17)$, $u_5 = (-3,-5,8)$ generate R^3. Find the subset of sets $\{u_1, u_2, u_3, u_4, u_5\}$ is a basis for R^3.

Select the vectors u_1, u_2, u_5.

First, we have to check if the given vectors are linearly independent or linearly dependent.

The coefficient matrix,

$$\Rightarrow \begin{vmatrix} 2 & -3 & 1 \\ 1 & 4 & -2 \\ -3 & -5 & 8 \end{vmatrix}$$

$$= 2(32-10) + 3(8-6) + 1(-5+12)$$

$$= 2(22) + 3(2) + 1(7)$$

$$= 44 + 6 + 7$$

$$= 57 \neq 0$$

∴ The given vectors are linearly independent.

There are 3 vectors and dimension of $R^3 = 3$

∴ The vectors ~~are~~ is basis for R^3.

Basis $= \{(2,-3,1), (1,4,-2), (-3,-5,8)\}$.

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Linear Transformation and Diagonalisation

Linear Transform:

Let V and W be vector spaces over F, we call a function $T: V \to W$ a linear transformation if for all $x, y \in V$ and $c \in F$,

i) $T(x+y) = T(x) + T(y)$

ii) $T(cx) = cT(x)$

Properties:

1) If T is linear, then $T(0) = 0$.

proof:

given : T is linear

$$\therefore T(x+y) = Tx + Ty$$

$$T(cx) = cT(x), \forall x, y \in V, c \in F$$

$$T(0) = T(0x)$$

$$= 0\,T(x)$$

$$T(0) = 0$$

2) T is linear, iff $T(cx+y) = cT(x) + T(y)$.

proof:

given : T is linear

$$\therefore T(x+y) = T(x) + T(y)$$

$$T(cx) = cT(x), \forall x, y \in V, c \in F$$

$$T(cx+y) = T(cx) + T(y)$$

$$T(cx+y) = cT(x) + T(y)$$

converse part:

given : $T(cx+y) = cT(x) + T(y)$

put $c = 1$

$$\therefore T(x+y) = 1.\,T(x) + T(y)$$

$$= T(x) + T(y)$$

put $y = 0$

$$\therefore T(cx+0) = cT(x) + T(0)$$

$\therefore$ T is linear. $T(cx) = cT(x)$

3) If T is linear, then $T(x-y) = T(x) - T(y)$, $\forall x, y \in V$.

proof:

given: T is linear

$T(ax+by) = aT(x) + bT(y)$, $\forall x, y \in V$, $a, b \in F$

put $a = 1$, $b = -1$,

$T(x-y) = 1.T(x) + (-1)T(y)$

$T(x-y) = T(x) - T(y)$

1) Let $T: R^2 \to R^2$ defined by $T(a_1, a_2) = (2a_1 + a_2, a_1)$

Show that T is linear.

Let $x, y \in R^2$ and $c \in F$

where $x = (a_1, a_2)$

$y = (b_1, b_2)$

$\therefore x+y = (a_1+b_1, a_2+b_2)$

$T(x+y) = T(a_1+b_1, a_2+b_2)$

$= [2(a_1+b_1)+(a_2+b_2), (a_1+b_1)]$

$= (2a_1+a_2, a_1) + (2b_1+b_2, b_1)$

$= T(a_1, a_2) + T(b_1, b_2)$

$T(x+y) = T(x) + T(y)$

$T(cx) = T(c(a_1, a_2))$

$= T(ca_1, ca_2)$

$= [(2ca_1 + ca_2), ca_1]$

$= c(2a_1+a_2, a_1)$

$= c\,T(a_1, a_2)$

$T(cx) = c\,T(x)$

$\therefore T$ is linear

2) Let $T: R^2 \to R^2$ defined by $T(a, b) = (2a-3b, a+4b)$

Prove that T is linear.

Let $x, y \in R^2$ and $\alpha \in F$

where $x = (a, b)$

$y = (c, d)$

$\therefore x+y = (a+c, b+d)$

$$T(x+y) = T(a+c, b+d)$$
$$= [2(a+c)-3(b+d), (a+c)+4(b+d)]$$
$$= (2a-3b, a+4b) + (2c-3d, c+4d)$$
$$= T(a,b) + T(c,d)$$
$$T(x+y) = T(x) + T(y)$$
$$T(\alpha x) = T(\alpha(a,b))$$
$$= T(\alpha a, \alpha b)$$
$$= (2\alpha a - 3\alpha b, \alpha a + 4\alpha b)$$
$$= \alpha(2a-3b, a+4b)$$
$$= \alpha T(a,b)$$
$$T(\alpha x) = \alpha T(x)$$

$\therefore$ T is linear

3) Is there a linear transformation $T: R^3 \to R^2$ such that $T(1,0,3) = (1,1)$, $T(-2,0,-3) = (2,1)$?

$$T(-2,0,-6) = T(-2,(1,0,3))$$
$$= -2T(1,0,3)$$
$$= -2(1,1)$$
$$= (-2,-2) \neq (2,1)$$

$\therefore$ T is not linear.

4) Prove that the set of all polynomials of degree $\leq n$ is a linear transformation. The transformation T is defined by,

$$T(a_0 + a_1x + \dots\dots + a_nx^n) = (a_0, a_1, \dots, a_n)$$

Let $f = a_0 + a_1x + \dots\dots + a_nx^n$

$g = b_0 + b_1x + \dots\dots + b_nx^n$

$$f+g = (a_0+b_0) + (a_1+b_1)x + \dots\dots + (a_n+b_n)x^n$$
$$T(f+g) = T[(a_0+b_0) + (a_1+b_1)x + \dots + (a_n+b_n)x^n]$$
$$= (a_0+b_0, a_1+b_1, \dots\dots, a_n+b_n)$$
$$= (a_0, a_1, \dots, a_n) + (b_0, b_1, \dots, b_n)$$
$$T(f+g) = T(f) + T(g)$$
$$T(cf) = T[c(a_0 + a_1x + \dots + a_nx^n)]$$

$= T[ca_0 + ca_1 x + \cdots + ca_n x^n]$

$= (ca_0, ca_1, \ldots, ca_n)$

$= c(a_0, a_1, \ldots, a_n)$

$T(cf) = c\,T(f)$

$\therefore$ T is linear

Matrix Representation of a linear transformation.

1) Obtain the matrix representation of linear transformation $T : V_3(R) \to V_3(R)$ given by $T(a,b,c) = (3a, a-b, 2a+b+c)$ with respect to standard basis $\{e_1, e_2, e_3\}$.

Let the standard basis $\beta = \{e_1, e_2, e_3\}$

where, $e_1 = (1,0,0)$; $e_2 = (0,1,0)$, $e_3 = (0,0,1)$

$T(e_1) = T(1,0,0)$

$= (3(1), 1-0, 2(1)+0+0)$

$= (3,1,2)$

$T(e_2) = T(0,1,0)$

$= (0,-1,1)$

$T(e_3) = T(0,0,1)$

$= (0,0,1)$

$\therefore$ Matrix representation,

$$[T]_\beta = \begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

2) Let $T : V_3(R) \to V_2(R)$ given by, $T(a,b,c) = (a+b, 2c-a)$. Find matrix representation with respect to standard basis.

Let the standard basis $\beta = \{e_1, e_2, e_3\}$.

where, $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$

$T(e_1) = T(1,0,0)$

$= (1,-1)$

$T(e_2) = T(0,1,0)$

$= (1,0)$

$T(e_3) = T(0,0,1)$

$= (0,2)$

$\therefore$ Matrix representation,

$$[T]_\beta = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 2 \end{bmatrix}$$

3) Let $T : V_3 \to V_3$ given by $T(a, b, c) = (3a + c, -2a+b, a+2b+4c)$.

Find the matrix representation with respect to the standard basis;

Let the standard basis $\beta = \{e_1, e_2, e_3\}$

where, $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$

$T(e_1) = T(1, 0, 0) = (3, -2, 1)$

$T(e_2) = T(0, 1, 0) = (0, 1, 2)$

$T(e_3) = T(0, 0, 1) = (1, 0, 4)$

$\therefore$ Matrix representation,

$$[T]_\beta = \begin{bmatrix} 3 & -2 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 4 \end{bmatrix}$$

4) Let $T : P_3(R) \to P_2(R)$ be the linear transformation defined by $T[f(x)] = f'(x)$. Find the matrix representation of standard basis.

Let the standard basis $\beta = \{1, x, x^2, x^3\}$

$T(1) = 0 = 0.1 + 0x + 0x^2$

$T(x) = 1 = 1.1 + 0x + 0x^2$

$T(x^2) = 2x = 0.1 + 2x + 0x^2$

$T(x^3) = 3x^2 = 0.1 + 0x + 3x^2$

$\therefore$ Matrix representation,

$$[T]_\beta = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

5) Find the linear transformation $T : V_3 \to V_3$, determined by the matrix $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$ with respect to the standard basis $\{e_1, e_2, e_3\}$.

$T(e_1) = e_1 + 2e_2 + e_3$

$T(e_2) = 0e_1 + e_2 + e_3$

$T(e_3) = -e_1 + 3e_2 + 4e_3$.

$(a,b,c) = a(1,0,0) + b(0,1,0) + c(0,0,1)$

$T(a,b,c) = aT(1,0,0) + bT(0,1,0) + cT(0,0,1)$

$= aT(e_1) + bT(e_2) + cT(e_3)$

$T(a,b,c) = a(1,2,1) + b(0,1,1) + c(-1,3,4)$

$T(a,b,c) = \{a-c, 2a+b+3c, a+b+4c\}$

This is the required linear transformation.

6) Find the linear transformation $T: V_2 \to V_3$ given by $\begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix}$ with respect to the standard basis.

Let the standard basis $\beta = \{e_1, e_2\}$

where, $e_1 = (1,0)$, $e_2 = (0,1)$

$T(e_1) = (2,1,-1)$

$T(e_2) = (1,1,-1)$

$(a,b) = a(1,0) + b(0,1)$

$T(a,b) = aT(1,0) + bT(0,1)$

$= aT(e_1) + bT(e_2)$

$\therefore T(a,b) = a(2,1,-1) + b(1,1,-1)$

$\therefore T(a,b) = \{2a+b, a+b, -a-b\}$

7) Find the matrix ~~transformation~~ representation with $T: V_2 \to V_2$ given by $T(a,b) = \{-b, a\}$ with respect to the basis $(1,2), (1,-1)$.

$T(1,2) = (-2,1)$.

$T(1,-1) = (1,1)$

$(-2,1) = a(1,2) + b(1,-1)$

$-2 = a+b \to (1)$

$1 = 2a-b \to (2)$

$(1)+(2)$,

$-1 = 3a$

$a = \frac{-1}{3}$

$(1) \Rightarrow -2 = \frac{-1}{3} + b$

$b = -2 + \frac{1}{3} = \frac{-5}{3}$

$\therefore (-2,1) = \frac{-1}{3}(1,2) - \frac{5}{3}(1,-1) \rightarrow (3)$.

$(1,1) = a(1,2) + b(1,-1)$

$1 = a+b \rightarrow (4)$

$1 = 2a - b \rightarrow (5)$

$(4)+(5)$,

$2 = 3a$

$a = \frac{2}{3}$

$(4) \Rightarrow 1 = \frac{2}{3} + b$

$b = 1 - \frac{2}{3} = \frac{1}{3}$

$\therefore (1,1) = \frac{2}{3}(1,2) + \frac{1}{3}(1,-1) \rightarrow (6)$

From (3) & (6),

$$[T] = \begin{bmatrix} \frac{-1}{3} & \frac{-5}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

8) Find the matrix representation $T: V_3 \rightarrow V_3$ given by $T(a,b,c) = \{3a+c, -2a+b, a+2b+4c\}$ with respect to the basis $(1,0,1), (-1,2,1), (2,1,1)$.

$T(1,0,1) = (4,-2,5)$

$T(-1,2,1) = (-2,4,7)$

$T(2,1,1) = (7,-3,8)$

$(4,-2,5) = a(1,0,1) + b(-1,2,1) + c(2,1,1)$

$4 = a - b + 2c$

$-2 = 2b + c$

$5 = a + b + c$

solving these equations,

$a = \frac{27}{4}$

$b = \frac{-1}{4}$

$c = \frac{-3}{2}$

$\therefore (4, -2, 5) = \frac{27}{4}(1,0,1) - \frac{1}{4}(-1,2,1) - \frac{3}{2}(2,1,1)$

$\hookrightarrow$ ①

$(-2, 4, 7) = a(1,0,1) + b(-1,2,1) + c(2,1,1)$

$$-2 = a - b + 2c$$
$$4 = 2b + c$$
$$7 = a + b + c$$

solving these equations,

$$a = \frac{25}{4}$$
$$b = \frac{13}{4}$$
$$c = \frac{-5}{2}$$

$\therefore (-2, 4, 7) = \frac{25}{4}(1,0,1) + \frac{13}{4}(-1,2,1) - \frac{5}{2}(2,1,1)$ $\hookrightarrow$ ②

$(7, -3, 8) = a(1,0,1) + b(-1,2,1) + c(2,1,1)$

$$7 = a - b + 2c$$
$$-3 = 2b + c$$
$$8 = a + b + c$$

solving these equations,

$$a = \frac{21}{2}$$
$$b = \frac{-1}{2}$$
$$c = -2$$

$\therefore (7, -3, 8) = \frac{21}{2}(1,0,1) - \frac{1}{2}(-1,2,1) - 2(2,1,1)$ $\hookrightarrow$ ③

from ①, ②, ③,

$$[T] = \begin{bmatrix} \frac{27}{4} & \frac{-1}{4} & \frac{-3}{2} \\ \frac{25}{4} & \frac{13}{4} & \frac{-5}{2} \\ \frac{21}{2} & \frac{-1}{2} & -2 \end{bmatrix}$$

Eigen values or Eigen vectors:

Eigen ~~vectors~~ values

Let T be a linear operator on a vector space V a non-zero vector $v \in V$ is called an Eigen vector of T if there exists a ~~vector~~ scalar λ $T(v) = \lambda v$

The scalar λ is called Eigen value corresponding to the Eigen vector v.

1) Find Eigen value of $A = \begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix} \in M_{2\times2}(F)$.

$A = \begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix}$

Char. eq.,

$$\lambda^2 - s_1\lambda + s_2 = 0$$

$s_1 = 1 + 1 = 2$ (trace - sum of diagonals)

$s_2 = |A| = 1 - 16 = -15$

$\therefore \lambda^2 - 2\lambda - 15 = 0$

$\lambda = -3, 5$

$\therefore$ Eigen value are $-3, 5$.

2) Let T is a linear operation on $P_2(R)$ defined by $T[f(x)] = f(x) + (x+1)f'(x)$. Find matrix representation with respect to standard basis, find Eigen value and Eigen vectors of T.

Let standard basis $\beta = \{1, x, x^2\}$

$T[1] = 1 + (x+1)(0) = 1 = 1.1 + 0x + 0x^2$

$T[x] = x + (x+1)(1) = 2x + 1 = 1.1 + 2x + 0x^2$

$T[x^2] = x^2 + (x+1)(2x) = 3x^2 + 2x = 0.1 + 2x + 3x^2$

$$[T]_\beta = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 2 & 3 \end{bmatrix}$$

Char. eq.,

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$$

$s_1 = 1 + 2 + 3 = 6$ (trace - sum of diagonals)

$s_2 =$ sum of minors of diagonals

$= 6 + 3 + 2 = 11$

$s_3 = |A| = 1(6) = 6$

$\therefore \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

$\lambda = 1, 2, 3$

$\therefore$ Eigen values are $1, 2, 3$

To find Eigen vectors, solve $(A - \lambda I)X = 0$

$$\begin{bmatrix} 1-\lambda & 0 & 0 \\ 1 & 2-\lambda & 0 \\ 0 & 2 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\lambda = 1$,

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{array}{cccc} 1 & 0 & 1 & 1 \\ 2 & 2 & 0 & 2 \end{array}$$

$$x_1 + x_2 = 0$$

$$2x_2 + 2x_3 = 0$$

$$\frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{2}$$

$$\therefore X_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$\lambda = 2$,

$$\begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x_1 = 0 \Rightarrow x_1 = 0$$

$$2x_2 + x_3 = 0$$

$$2x_2 = -x_3$$

$$\frac{x_2}{-1} = \frac{x_3}{2}$$

$$\therefore X_2 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$$

$\lambda = 3$,

$$\begin{bmatrix} -2 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x_1 = 0$$

$$\Rightarrow x_1 = 0$$

$$2x_2 = 0$$

$$\Rightarrow x_2 = 0$$

$$\therefore X_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ Eigen vectors are $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

Diagonalizable :

A linear operator T on a finite dimensional vector space V is called diagonalizable if, there is an ordered basis β for V such that $[T]_\beta$ is a diagonal matrix.

Theorem :

A linear operator T on a finite dimensional vector space V is diagonalisable if and only if there exists an ordered basis β for V consisting of Eigen vectors of T further more if T is diagonalisable $\beta = \{v_1, v_2, \ldots, v_n\}$ is an ordered basis of eigen vectors of $D = [T]_\beta$ then, T is the diagonal matrix and D_{jj} is the Eigen value corresponding to V_j for $1 \leq j \leq n$.

1) Let $A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & 1 & -4 \\ 2 & 0 & -1 \end{bmatrix} \in M_{n \times n}(F)$

i) Determine all Eigen values of A.

ii) For each eigen value λ of A, find the set of corresponding eigen vectors.

iii) If possible, find a basis for F^n constituting of Eigen vectors of A.

iv) If successful in finding such a basis, determine an invertible matrix Q and diagonal matrix D such that $Q^{-1}AQ = D$.

i) $C.E \Rightarrow \lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$s_1 = 2 + 1 - 1 = 2$

$s_2 = -1 + 0 + 2 = 1$

$\therefore S_3 = 1(-2+2) = 0$

$\therefore \lambda^3 - 2\lambda^2 + \lambda = 0.$

Eigen values : $\lambda = 0, 1, 1$

ii) Eigen vectors : $(A - \lambda I)X = 0$

$$\begin{bmatrix} 2-\lambda & 0 & -1 \\ 4 & 1-\lambda & -4 \\ 2 & 0 & -1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\lambda = 0,$

$$\begin{bmatrix} 2 & 0 & -1 \\ 4 & 1 & -4 \\ 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2x_1 + 0x_2 - x_3 = 0$

$4x_1 + x_2 - 4x_3 = 0$

$2x_1 + 0x_2 - x_3 = 0$

$$\begin{matrix} 0 & & -1 & & 2 & & 0 \\ & \times & & \times & & \times & \\ 1 & & -4 & & 4 & & 1 \end{matrix}$$

$$\frac{x_1}{1} = \frac{x_2}{4} = \frac{x_3}{2}$$

$$\therefore X_1 = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$$

$\lambda = 1,$

$$\begin{bmatrix} 1 & 0 & -1 \\ 4 & 0 & -4 \\ 2 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 - x_3 = 0$

$x_1 = x_3$

$$\frac{x_1}{1} = \frac{x_3}{1}$$

$$\therefore X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Eigen vectors are $\left\{ \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

iii) $$\begin{vmatrix} 1 & 1 & 1 \\ 4 & 1 & 0 \\ 2 & 1 & 1 \end{vmatrix} = -1 \neq 0$$

$\therefore$ The Eigen vectors are linearly independent

$\therefore$ The Eigen vectors $\left\{ \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis.

$$Q = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}, \quad Q^{-1} = \begin{bmatrix} -1 & 0 & 1 \\ 4 & 1 & -4 \\ -2 & -1 & 3 \end{bmatrix}$$

To find D:

$$Q^{-1}AQ = \begin{bmatrix} -1 & 0 & 1 \\ -4 & 1 & -4 \\ -2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ 4 & 1 & -4 \\ 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 4 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = D$$

2) $A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \in M_{2\times 2}(F)$ (same as previous sum)

$C.E \Rightarrow \lambda^2 - s_1\lambda + s_2 = 0$

$s_1 = 2$

$s_2 = -3$

$\therefore \lambda^2 - 2\lambda - 3 = 0$

Eigen values:

$(\lambda+1)(\lambda-3) = 0$

$\lambda = -1, 3$

Eigen vectors:

$(A - \lambda I)X = 0$

$$\begin{bmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\lambda = -1$,

$$\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$2x_1 + x_2 = 0$

$2x_1 = -x_2$

$\frac{x_1}{-1} = \frac{x_2}{2}$

$\therefore X_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$\lambda = 3,$

$$\begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-2x_1 + x_2 = 0$$

$$-2x_1 = -x_2$$

$$2x_1 = x_2$$

$$\frac{x_1}{1} = \frac{x_2}{2}$$

$$\therefore X_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$\therefore$ Eigen vectors are $\left\{ \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$

$$\begin{vmatrix} -1 & 1 \\ 2 & 2 \end{vmatrix} = (-2-2) = -4 \neq 0$$

$\therefore$ The Eigen vectors are linearly independent

$\therefore$ The Eigen vectors $\left\{ \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ is a basis

$$Q = \begin{bmatrix} -1 & 1 \\ 2 & 2 \end{bmatrix}, \quad Q^{-1} = \frac{-1}{4} \begin{bmatrix} 2 & -1 \\ -2 & -1 \end{bmatrix}$$

To find D:

$$Q^{-1}AQ = \frac{-1}{4} \begin{bmatrix} 2 & -1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} = D$$

3) Let L be a linear transformation from R_3 to R_3 whose matrix representation A with respect to the standard basis given below. Find the Eigen values of L and basis of Eigen vectors

$$A = \begin{bmatrix} 1 & 3 & -3 \\ 3 & 1 & -3 \\ -3 & -3 & 1 \end{bmatrix}$$

$$C.E \Rightarrow \lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

$$S_1 = 1 + 1 + 1 = 3$$

$$S_2 = (-8-8-8) = -24$$

$$S_3 = 1(-8) - 3(-6) - 3(-6)$$
$$= -8 + 18 + 18 = 28$$
$$\therefore \lambda^3 - 3\lambda^2 - 24\lambda - 28 = 0$$

Eigen values :

$$\lambda = 7, -2, -2$$

Eigen vectors : $(A - \lambda I)X = 0$

$$\begin{bmatrix} 1-\lambda & 3 & -3 \\ 3 & 1-\lambda & -3 \\ -3 & -3 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\lambda = 7$,

$$\begin{bmatrix} -6 & 3 & -3 \\ 3 & -6 & -3 \\ -3 & -3 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-6x_1 + 3x_2 - 3x_3 = 0$$
$$3x_1 - 6x_2 - 3x_3 = 0$$
$$-3x_1 - 3x_2 - 6x_3 = 0$$

$$-2x_1 + x_2 - x_3 = 0$$
$$x_1 - 2x_2 - x_3 = 0$$

$$\begin{array}{cccccccc} 1 & & -1 & & -2 & & 1 \\ & \times & & \times & & \times & \\ -2 & & -1 & & 1 & & -2 \end{array}$$

$$\frac{x_1}{-3} = \frac{x_2}{-3} = \frac{x_3}{3}$$

$$\therefore X_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

$\lambda = -2$,

$$\begin{bmatrix} 3 & 3 & -3 \\ 3 & 3 & -3 \\ -3 & -3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$3x_1 + 3x_2 - 3x_3 = 0$$
$$x_1 + x_2 - x_3 = 0$$

put $x_1 = 0$,

$$x_2 - x_3 = 0$$
$$x_2 = x_3$$
$$\frac{x_2}{1} = \frac{x_3}{1}$$

$$\therefore X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Let X_3 be orthogonal to x_1 & x_2. $X_3 = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$

$\therefore X_3^T X_1 = 0$

$$[l \; m \; n]\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = 0$$

$$-l - m + n = 0$$

$\therefore X_3^T X_2 = 0$

$$[l \; m \; n]\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = 0$$

$$0l + m + n = 0$$

$$\frac{l}{-2} = \frac{m}{1} = \frac{n}{-1}$$

$$\therefore X_3 = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$$

$\therefore$ Eigen vectors are $\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix} \right\}$.

$$\begin{vmatrix} -1 & 0 & -2 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = -6 \neq 0$$

$\therefore$ The Eigen vectors are linearly independent

$\therefore$ The Eigen vectors are basis.

4) Let $V = P_1(R)$, $T(a+bx) = (6a-6b)+(12a-11b)x$, T is the linear operator on V and $\beta = \{3+4x, 2+3x\}$ is ordered basis. compute $[T]_\beta$ and determine whether the basis consisting of Eigen vectors of T.

$$T(a+bx) = (6a-6b) + (12a-11b)x$$

$$T(3+4x) = -6 - 8x = -2(3+4x) + 0(2+3x)$$

$$T(2+3x) = -6 - 9x = 0(3+4x) - 3(2+3x)$$

$$\therefore [T]_\beta = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}$$

Since the matrix is a diagonal matrix, $\beta = \{3+4x, 2+3x\}$ is a basis contains the Eigen values of T.

5) Let $V = R_2$; $T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 10a - 6b \\ 17a - 10b \end{pmatrix}$ and $\beta = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right\}$ is ordered basis T is a linear operator on V and determine whether β is a basis consisting of Eigen ~~values~~ vectors of T.

$$T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 10a - 6b \\ 17a - 10b \end{pmatrix}$$

$$T\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \end{pmatrix} = -\begin{pmatrix} 2 \\ 3 \end{pmatrix} = 0\begin{pmatrix} 1 \\ 2 \end{pmatrix} - 1\begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$T\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 2 \end{pmatrix} + 0\begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$[T]_\beta = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}$$

Since the matrix is not a diagonal matrix, $\beta = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right\}$ does not contain the Eigen vectors of T.

6) Find the Eigen values of T and an ordered basis β for T such that $[T]_\beta$ is the diagonal matrix and the linear transformation is given by

$T[f(x)] = xf'(x) + f''(x) - f(2)$, $V = P_3(R)$

standard basis $\beta = \{1, x, x^2, x^3\}$.

$T(1) = -1 = -1.1 + 0x + 0x^2 + 0x^3$

$T(x) = x.1 + 0 - 2$

$= x - 2$

$T(x) = -2.1 + 1.x + 0x^2 + 0x^3$

$T(x^2) = x.(2x) + 2 - 4$

$= 2x^2 - 2$

$T(x^2) = -2.1 + 0.x + 2x^2 + 0x^3$

$T(x^3) = x(3x^2) + 6x - 8$

$= 3x^3 + 6x - 8$

$T(x^3) = -8.1 + 6x + 0x^2 + 3x^3$

$f(x) = 1$
$f(2) = 1$

$f(x) = x$
$f(2) = 2$

$$[T] = \begin{bmatrix} -1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -2 & 0 & 2 & 0 \\ -8 & 6 & 0 & 3 \end{bmatrix}$$

$C.E \Rightarrow |A-\lambda I| = 0$

$$\begin{vmatrix} -1-\lambda & 0 & 0 & 0 \\ -2 & 1-\lambda & 0 & 0 \\ -2 & 0 & 2-\lambda & 0 \\ -8 & 6 & 0 & 3-\lambda \end{vmatrix} = 0$$

$$(-1-\lambda)(1-\lambda)(2-\lambda)(3-\lambda) = 0$$

$$\lambda = -1, 1, 2, 3$$

Eigen vectors :

$$(A-\lambda I)X = 0$$

$$\begin{bmatrix} -1-\lambda & 0 & 0 & 0 \\ -2 & 1-\lambda & 0 & 0 \\ -2 & 0 & 2-\lambda & 0 \\ -8 & 6 & 0 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\lambda = -1$,

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ -2 & 0 & 3 & 0 \\ -8 & 6 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x_1 + 2x_2 = 0$$
$$-2x_1 + 3x_3 = 0$$
$$-8x_1 + 6x_2 + 4x_4 = 0$$

$$\Rightarrow -2x_1 = -2x_2$$
$$\frac{x_1}{1} = \frac{x_2}{1}$$

$$-2 + 3x_3 = 0$$
$$3x_3 = 2$$
$$x_3 = \frac{2}{3}$$

$$-8 + 6 + 4x_4 = 0$$
$$4x_4 = 2$$
$$x_4 = \frac{1}{2}$$

$$\therefore X_1 = \begin{bmatrix} 1 \\ 1 \\ \frac{2}{3} \\ \frac{1}{2} \end{bmatrix}$$

$\lambda = 1$,

$$\begin{bmatrix} -2 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ -8 & 6 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x_1 = 0$$
$$\Rightarrow x_1 = 0$$
$$-2x_1 + x_3 = 0$$
$$x_3 = 0$$
$$-8x_1 + 6x_2 + 2x_4 = 0$$
$$6x_2 + 2x_4 = 0$$
$$3x_2 = -x_4$$
$$\frac{x_2}{-1} = \frac{x_4}{3}$$

$$\therefore X_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 3 \end{bmatrix}$$

$\lambda = 2$,

$$\begin{bmatrix} -3 & 0 & 0 & 0 \\ -2 & -1 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ -8 & 6 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-3x_1 = 0$$
$$\Rightarrow x_1 = 0$$
$$-2x_1 - x_2 = 0$$
$$-x_2 = 0$$
$$\Rightarrow x_2 = 0$$
$$-8x_1 + 6x_2 + x_4 = 0$$
$$x_4 = 0$$
$$x_3 = 1 \quad \text{(any least value except 0)}$$

$$\therefore X_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$\lambda = 3$,

$$\begin{bmatrix} -4 & 0 & 0 & 0 \\ -2 & -2 & 0 & 0 \\ -2 & 0 & -1 & 0 \\ -8 & 6 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-4x_1 = 0$$
$$\Rightarrow x_1 = 0$$

$$-2x_1 - 2x_2 = 0$$
$$-2x_2 = 0$$
$$x_2 = 0$$

$$-2x_1 - x_3 = 0$$
$$-x_3 = 0$$
$$x_3 = 0$$

$$\therefore X_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Now

$$\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\begin{array}{cccc} 1 & x & x^2 & x^3 \end{array}$$

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ \frac{2}{3} & 0 & 1 & 0 \\ \frac{1}{2} & 3 & 0 & 1 \end{vmatrix} = 1[-1(1-0)] = 1(-1) = -1 \neq 0$$

$\therefore$ The Eigen vectors are linearly independent

$\therefore$ The Eigen vectors are basis.

$\therefore$ The Eigen vectors $\beta = \{1, 1-x, \frac{2}{3}+x^2, \frac{1}{2}+3x+x^3\}$ is a basis which consisting Eigen vectors of T.

Null space and Ranges:

Null space:

Let V and W be the vector spaces over the field F and $T: V \to W$ be a linear transformation in the null space or kernel $N(T)$ of T is defined by,

$$\ker T = N(T) = \{x \in V : T(x) = 0\}$$

Range:

Let V and W be the vector spaces over the field F and $T: V \to W$ be a linear transformation the range or image $R(T)$ is defined by,

$$R(T) = \{T(x) : x \in V\}$$

1) Let $V: R^3 \to R^2$ be a linear transformation defined by $T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3)$. Find $N(T)$ and $R(T)$.

$$N(T) = \{x = (a_1, a_2, a_3) \in R^3 : T(a_1, a_2, a_3) = 0\}$$
$$= \{x = (a_1, a_2, a_3) \in R^3 : (a_1 - a_2, 2a_3) = 0\}$$
$$= \{x = (a_1, a_2, a_3) \in R^3 : a_1 = a_2, a_3 = 0\}$$
$$N(T) = \{(a_1, a_1, 0), a_1 \in R\}$$
$$R(T) = \{T(a_1, a_2, a_3) : (a_1, a_2, a_3) \in R^3\}$$
$$= \{(a_1 - a_2, 2a_3) : (a_1, a_2, a_3) \in R^3\}$$
$$R(T) = R^2$$

Theorem : 1

Let V and W be the vector spaces and $T: V \to W$ be linear then,

i) $N(T)$ is a sub-space of V

ii) $R(T)$ is a sub-space of W

Proof :

i) By the definition of null space,

$$N(T) = \{x \in V : T(x) = 0\}$$

since $T(0) = 0 \in N(T)$

Let $x, y \in N(T)$, then $T(x) = 0$, $T(y) = 0$

$$T(x+y) = T(x) + T(y)$$
$$= 0 + 0$$
$$= 0$$
$$\therefore x+y \in N(T)$$

Let $x \in N(T)$, then $T(x) = 0$, $c \in F$

$$T(cx) = cT(x) \quad [\because T \text{ is linear}]$$
$$= c(0)$$
$$T(cx) = 0$$
$$\therefore cx \in N(T)$$

$\therefore N(T)$ is a sub-space of V.

ii) By the definition of range,

$$R(T) = \{T(x) : x \in V\}.$$

Since $T(0)=0 \in R(T)$

Let $x, y \in R(T)$, then there exists a vectors $u, v \in V$ such that $T(u)=x$ and $T(v)=y$.

$$T(u+v) = T(u)+T(v)$$
$$= x+y \in R(T)$$

Let $x \in R(T)$ and $c \in F$ then there exists a vector $u \in V$ such that $T(u)=x$

$$T(cu) = cT(u)$$
$$= cx \in R(T)$$

$\therefore R(T)$ is a sub-space of W.

Theorem : 2

Let V and W be vector spaces and let $T: V \to W$ be linear if $\beta = \{v_1, v_2, \ldots, v_n\}$ is a basis for V then, $R(T) = \text{span}(T(\beta)) = \text{span}(T(v_1), T(v_2), \ldots, T(v_n))$

1) Define the linear transformation $T: P_2(R) \to M_{2\times 2}(R)$ by $T(f(x)) = \begin{pmatrix} f(1)-f(2) & 0 \\ 0 & f(0) \end{pmatrix}$. Find the $R(T)$ and dimension $R(T)$.

Let $\beta = (1, x, x^2)$ is a basis for $P_2(R)$.

$T(1) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ $\qquad f(x)=1$, $f(0)=1$, $f(1)=1$, $f(2)=1$

$T(x) = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$ $\qquad f(x)=x$, $f(0)=0$, $f(1)=1$, $f(2)=2$

$T(x^2) = \begin{pmatrix} -3 & 0 \\ 0 & 0 \end{pmatrix}$ $\qquad f(x)=x^2$, $f(0)=0$, $f(1)=1$, $f(2)=4$

Now $R(T) = \text{span}(T(\beta))$

$$= \text{span}(T(1), T(x), T(x^2))$$
$$= \text{span}\left\{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -3 & 0 \\ 0 & 0 \end{pmatrix}\right\}$$
$$= \text{span}\left\{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}\right\}$$

$$a\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$a=0, \quad -b=0 \Rightarrow b=0$$

$\therefore \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$ generate $R(T)$ since $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$ are linearly independent.

$$\therefore \dim(R(T)) = 2$$

Nullity and Rank of T:

Let V and W be the vector spaces and $T: V \to W$ be a linear transformation.

The dimension of $N(T)$ is called nullity of T.

The dimension of $R(T)$ is called rank of T.

Dimension Theorem:

Let V and W be the vector space and $T: V \to W$ be a linear transformation. If D is finite dimensional and then,

$$\dim V = \text{rank } T + \text{nullity } T$$

Proof:

Let $\dim V = n$ and $\dim(N(T)) = k$ and $\{v_1, v_2, \dots, v_k\}$ is a basis for $N(T)$, then $\{v_1, v_2, \dots, v_k\}$ is a linearly independent set in V.

$\therefore \{v_1, v_2, \dots, v_k\}$ can be expanded to a basis $\beta = \{v_1, v_2, \dots, v_n\}$ for V.

Now we claim $S = \{T(v_{k+1}), T(v_{k+2}), \dots, T(v_n)\}$ is a basis for $R(T)$.

Since $\beta = \{v_1, v_2, \dots, v_n\}$ is a basis for V.

$$\therefore R(T) = \text{span}(T(\beta))$$
$$= \text{span}(T(v_1), T(v_2), \dots, T(v_n))$$
$$= \text{span}(T(v_{k+1}), T(v_{k+2}), \dots, T(v_n))$$

$$R(T) = \text{span}(S) \to ①$$

Now, we prove that S is a linearly independent set.

To prove this,

Let $b_{k+1}T(v_{k+1}) + b_{k+2}T(v_{k+2}) + \dots + b_nT(v_n) = 0$

$T(b_{k+1}v_{k+1}) + T(b_{k+2}v_{k+2}) + \dots + T(b_nv_n) = 0$

$T(b_{k+1}v_{k+1} + b_{k+2}v_{k+2} + \dots + b_nv_n) = 0$

$N(T) = \{x \in V : T(x) = 0\}$

$\Rightarrow b_{k+1}v_{k+1} + b_{k+2}v_{k+2} + \dots + b_nv_n \in N(T)$

Since $v_1, v_2, \dots, v_k$ is a basis for $N(T)$

$\therefore b_{k+1}v_{k+1} + b_{k+2}v_{k+2} + \dots + b_nv_n = b_1v_1 + b_2v_2 + \dots + b_kv_k$

$\therefore (-b_1)v_1 + (-b_2)v_2 + \dots + (-b_k)v_k + b_{k+1}v_{k+1} + b_{k+2}v_{k+2} + \dots + b_nv_n = 0$

$\Rightarrow b_1 = b_2 = \dots = b_n = 0$ $\left[\because \{v_1, v_2, \dots, v_n\} \text{ is linearly independent}\right]$

S is linearly independent $\rightarrow$ (2)

from (1) & (2),

S is the basis for $R(T)$.

Now $\dim(R(T)) = n - k$

$\dim(R(T)) = \dim V - \dim(N(T))$

$\dim(R(T)) + \dim(N(T)) = \dim V$

$\therefore \dim V = \text{rank } T + \text{nullity } T$

1) Verify dimension theorem for $T : R_3 \rightarrow R_2$ defined by $T(a_1, a_2, a_3) = \{(a_1 - a_2, 2a_3)\}$.

Here R_3 & R_2 are 2 vector spaces and

$T : R_3 \rightarrow R_2$ is linear

$\therefore \dim(R_3) = 3 \Rightarrow \dim V = 3 \rightarrow$ (1)

Let $\beta = \{(1,0,0), (0,1,0), (0,0,1)\}$

$R(T) = \text{span}(T(\beta))$

$= \text{span}\{T(1,0,0), T(0,1,0), T(0,0,1)\}$

$= \text{span}\{(1,0), (-1,0), (0,2)\}$

$= \text{span}\{(1,0), (0,2)\}$

$\dim(R(T)) = 2$

$\therefore \text{rank } T = 2$

$N(T) = \{T(x) = 0, x \in V\}$

$N(T) = \{T(a_1, a_2, a_3) = 0, (a_1, a_2, a_3) \in R_3\}$

$N(T) = \{(a_1 - a_2, 2a_3) = (0, 0), (a_1, a_2, a_3) \in R_3\}$

$$= \left\{ \begin{matrix} a_1 - a_2 = 0 & , & 2a_3 = 0 & , (a_1, a_2, a_3) \in R_3 \\ a_1 = a_2 & , & a_3 = 0 & \end{matrix} \right\}$$

$$= \{(a_1, a_1, 0), a_1 \in R\}$$

$\dim(N(T)) = 1$

$\therefore$ nullity $T = 1$

$\therefore$ rank T + nullity $T = 2 + 1 = 3 \rightarrow$ ②

from ① & ②,

$\dim V$ = rank T + nullity T.

2) Let $T: R^2 \rightarrow R^3$ defined by $T(a_1, a_2) = (a_1 + a_2, 0, 2a_1 - a_2)$. Verify dimension theorem.

Here R^2 & R^3 are 2 vector spaces and

$T: R^2 \rightarrow R^3$ is linear.

$\therefore \dim(R^2) = 2 \Rightarrow \dim V = 2 \longrightarrow$ ①

Let $\beta = \{(1, 0), (0, 1)\}$

$R(T) = \text{span}(T(\beta))$

$= \text{span}\{T(1, 0), T(0, 1)\}$

$= \text{span}\{(1, 0, 2), (1, 0, -1)\}$

$\dim(R(T)) = 2$

$\therefore$ rank $T = 2$

$N(T) = \{T(x) = 0, x \in V\}$

$N(T) = \{T(a_1, a_2) = 0, (a_1, a_2) \in R^2\}$

$$= \{(a_1 + a_2, 0, 2a_1 - a_2) = (0, 0, 0), (a_1, a_2) \in R^2\}$$

$$= \left\{ \begin{matrix} a_1 + a_2 = 0 & , & 2a_1 - a_2 = 0 & , (a_1, a_2) \\ a_1 = -a_2 & & 2a_1 + a_1 = 0 & \in R^2 \end{matrix} \right\}$$

$3a_1 = 0$

$\Rightarrow a_1 = 0 \Rightarrow a_2 = 0$

$$= \{(0, 0, 0), 0 \in R\}$$

$\dim(N(T)) = 0$

$\therefore$ nullity $T = 0$

$\therefore$ rank T + nullity $T = 2 + 0 = 2 \rightarrow$ ②

from ① & ②,

$$\dim V = \text{rank}\, T + \text{nullity}\, T$$

3) Let $T : R^3 \rightarrow R^2$ be defined by $T(x,y,z) = (2x-y, 3z)$. Verify T is linear or not. Find $N(T)$ and $R(T)$. Hence, verify dimension theorem.

Let $u = (a,b,c)$, $v = (d,e,f)$

$$T(u+v) = T(a+d, b+e, c+f)$$
$$= (2(a+d) - (b+e), 3(c+f))$$
$$= (2a-b, 3c) + (2d-e, 3f)$$
$$= T(a,b,c) + T(d,e,f)$$

$$T(u+v) = T(u) + T(v)$$

Let $u = (a,b,c)$, $\alpha \in F$

$$T(\alpha u) = T(\alpha(a,b,c))$$
$$= T(\alpha a, \alpha b, \alpha c)$$
$$= (2(\alpha a) - \alpha b, 3\alpha c)$$
$$= \alpha(2a-b, 3c)$$
$$= \alpha T(a,b,c)$$

$$T(\alpha u) = \alpha T(u)$$

$\therefore$ T is linear.

Now, R^3 & R^2 are 2 vector spaces and $T : R^3 \rightarrow R^2$ is linear.

$\therefore \dim(R^3) = 3 \Rightarrow \dim V = 3 \rightarrow$ ①

Let $\beta = \{(1,0,0), (0,1,0), (0,0,1)\}$

$$R(T) = \text{span}\,(T(\beta))$$
$$= \text{span}\,\{T(1,0,0), T(0,1,0), T(0,0,1)\}$$
$$= \text{span}\,\{(2,0), (-1,0), (0,3)\}$$
$$= \text{span}\,\{(-1,0), (0,3)\}$$

$$\dim(R(T)) = 2$$

$\therefore$ rank $T = 2$

$$N(T) = \{T(x) = 0, x \in V\}$$
$$N(T) = \{T(x,y,z) = 0, (x,y,z) \in R^3\}$$
$$= \{(2x-y, 3z) = 0, (x,y,z) \in R^3\}$$

$$N(T) = \left\{ \begin{array}{ll} 2x - y = 0, & 3z = 0 \\ 2x = y, & z = 0 \end{array} , (x,y,z) \in R^3 \right\}$$

$\neq \{ (\frac{1}{2}x, 2x \cancel{)}$

$= \{ (\frac{y}{2}, y, 0), y \in R \}$

$\dim(N(T)) = 1$

$\therefore$ nullity $T = 1$

$\therefore$ rank T + nullity $T = 2 + 1 = 3 \longrightarrow (2)$

from (1) & (2),

$\dim V =$ rank T + nullity T

Inner Product Space

Let V be a vector space over a field F at an inner product on V is the function which assigns to reach ordered pair of vectors x, y in V, a scalar in F denoted by $\langle x, y\rangle$ satisfying the following axioms hold.

i) $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$

ii) $\langle cx, y\rangle = c\langle x, y\rangle$

iii) $\langle x, y\rangle = \overline{\langle y, x\rangle}$, where $\overline{\langle y, x\rangle}$ is complex conjugate of $\langle x, y\rangle$.

iv) $\langle x, x\rangle > 0$, $x \neq 0$

note:
Inner product space is also known as unitary space or Euclidean space.

1) Prove that $V_n(R)$ is a real inner product space defined by $\langle x, y\rangle = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$ where $x = (x_1, x_2, \dots, x_n)$, $y = (y_1, y_2, \dots, y_n)$.

This is called standard inner product space of $V_n(R)$

Let $x, y, z \in V_n(R)$ and $c \in R$.

i) $\langle x+y, z\rangle = \langle (x_1+y_1, x_2+y_2, \dots, x_n+y_n), (z_1, z_2, \dots, z_n)\rangle$

$= (x_1+y_1)z_1 + (x_2+y_2)z_2 + \dots + (x_n+y_n)z_n$

$= (x_1 z_1 + x_2 z_2 + \dots + x_n z_n) + (y_1 z_1 + y_2 z_2 + \dots + y_n z_n)$

$\therefore \langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$

ii) $\langle cx, y\rangle = \langle c(x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n)\rangle$

$= \langle (cx_1, cx_2, \dots, cx_n), (y_1, y_2, \dots, y_n)\rangle$

$= cx_1 y_1 + cx_2 y_2 + \dots + cx_n y_n$

$= c(x_1y_1 + x_2y_2 + \dots + x_ny_n)$

$\therefore \langle cx, y\rangle = c\langle x, y\rangle$.

iii) $\langle x, y\rangle = x_1y_1 + x_2y_2 + \dots + x_ny_n$

$= y_1x_1 + y_2x_2 + \dots + y_nx_n$

$\therefore \langle x, y\rangle = \langle y, x\rangle$.

iv) $\langle x, x\rangle = \langle (x_1, x_2, \dots, x_n), (x_1, x_2, \dots, x_n)\rangle$

$= x_1^2 + x_2^2 + \dots + x_n^2 > 0$, $x = (x_1, x_2, \dots, x_n) \neq 0$.

$\therefore V_n(R)$ is an inner product space.

2) Let V be the set of all continuous real value functions defined on the $[0, 1]$, V is a real inner product space with inner product defined by $\langle f, g\rangle = \int_0^1 f(t)g(t)\,dt$.

Let $f, g, h \in V$, $c \in F$ (or) R

i) $\langle f+g, h\rangle = \int_0^1 [f(t)+g(t)]h(t)\,dt$

$= \int_0^1 f(t)h(t)\,dt + \int_0^1 g(t)h(t)\,dt$

$= \langle f, h\rangle + \langle g, h\rangle$

ii) $\langle cf, g\rangle = \int_0^1 c\,f(t)\,g(t)\,dt$

$= c\int_0^1 f(t)g(t)\,dt$

$= c\langle f, g\rangle$

iii) $\langle f, g\rangle = \int_0^1 f(t)\,g(t)\,dt$

$= \int_0^1 g(t)f(t)\,dt$

$= \langle g, f\rangle$

iv) $\langle f, f\rangle = \int_0^1 f(t)f(t)\,dt$

$= \int_0^1 [f(t)]^2\,dt > 0$, $f(t) \neq 0$

$\therefore V$ is a real inner product space.

3) Prove that $V_n(C)$ is a complex inner product space with inner product defined by $\langle x, y\rangle = x_1\overline{y_1} + x_2\overline{y_2} + \dots + x_n\overline{y_n} = \sum_{i=1}^{n} x_i\overline{y_i}$, where

$x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$

Let $x, y, z \in V_n(C)$, $c \in C$

i) $\langle x+y, z\rangle = \sum_{i=1}^{n} (x_i + y_i)\overline{z_i}$

$= \sum_{i=1}^{n} x_i \overline{z_i} + \sum_{i=1}^{n} y_i \overline{z_i}$

$= \langle x, z\rangle + \langle y, z\rangle$

ii) $\langle cx, y\rangle = \sum_{i=1}^{n} c x_i \overline{y_i}$

$= c \sum_{i=1}^{n} x_i \overline{y_i}$

$= c\langle x, y\rangle$

iii) $\langle x, y\rangle = \sum_{i=1}^{n} x_i \overline{y_i}$

$= \sum_{i=1}^{n} \overline{y_i} x_i$

$= \overline{\sum_{i=1}^{n} y_i \overline{x_i}}$

$= \overline{\langle y, x\rangle}$

$z\overline{z} = |z|^2$

iv) $\langle x, x\rangle = \sum_{i=1}^{n} x_i \overline{x_i}$

$= \sum_{i=1}^{n} |x_i|^2 > 0$, $x_i \neq 0$, $i = 1, 2, \dots, n$

$\therefore V_n(C)$ is a complex inner product space.

1) Let $x = (1+i, 4)$, $y = (2-3i, 4+5i)$ in C^2. Find $\langle x, y\rangle$.

$\langle x, y\rangle = \sum_{i=1}^{n} x_i \overline{y_i}$

$\langle x, y\rangle = (1+i)(2+3i) + 4(4-5i)$

$= 2 + 3i + 2i - 3 + 16 - 20i$

$= 15 - 15i$

2) Let V be the vector space of polynomial with inner product given by $\langle f, g\rangle = \int_0^1 f(t) g(t)\, dt$ where $f(t) = (t+2)$, $g(t) = t^2 - 2t - 3$. Find $\langle f, g\rangle$

$\langle f, g\rangle = \int_0^1 (t+2)(t^2 - 2t - 3)\, dt$

$= \int_0^1 (t^3 - 7t - 6)\, dt$

$$\langle f, g\rangle = \left[\frac{t^4}{4} - 7\frac{t^2}{2} - 6t\right]_0^1$$

$$= \frac{1}{4} - \frac{7}{2} - 6 = \frac{1-14-24}{4}$$

$$= \frac{-37}{4}$$

Norm : (length)

Let V be an inner product space for $x \in V$ that norm (or) length of x defined by,

$$\|x\| = \sqrt{\langle x, x\rangle}$$

1) Let $x = (2, 1+i, i)$, $y = (2-i, 2, 1+2i)$ be vectors in C^3. Compute norm x ; norm y , norm $x+y$.
$[\|x\|]$ $[\|y\|]$ $[\|x+y\|]$

$$\|x\|^2 = \langle x, x\rangle$$

$$= \langle [2, (1+i), i], [2, (1+i), i]\rangle$$

$$= 2(2) + (1+i)\overline{(1+i)} + i(\overline{i})$$

$$= 4 + (1+i)(1-i) + i(-i)$$

$$= 4 + 2 + 1$$

$$\|x\|^2 = 7$$

$$\|x\| = \sqrt{7}$$

$$\|y\|^2 = \langle y, y\rangle$$

$$= \langle [(2-i), 2, (1+2i)], [(2-i), 2, (1+2i)]\rangle$$

$$= (2-i)\overline{(2-i)} + 2(2) + (1+2i)\overline{(1+2i)}$$

$$= (2-i)(2+i) + 4 + (1+2i)(1-2i)$$

$$= 5 + 4 + 5$$

$$\|y\|^2 = 14$$

$$\|y\| = \sqrt{14}$$

$$x + y = (4-i, 3+i, 1+3i)$$

$$\|x+y\|^2 = \langle x+y, x+y\rangle$$

$$= \langle [4-i, 3+i, 1+3i], [4-i, 3+i, 1+3i]\rangle$$

$$= (4-i)(4+i) + (3+i)(3-i) + (1+3i)(1-3i)$$

$= 17 + 10 + 10$

$\|x+y\|^2 = 37$

$\|x+y\| = \sqrt{37}$

2) In $C([0,1])$, let $f(t) = t$, $g(t) = e^t$. Find $\|f\|$ and $\|g\|$.

we define $\langle f, g\rangle = \int_0^1 f(t)\, g(t)\, dt$

$\|f\|^2 = \langle f, f\rangle$

$= \int_0^1 f(t) f(t)\, dt$

$= \int_0^t t \cdot t\, dt$

$= \left[\frac{t^3}{3}\right]_0^1$

$\|f\|^2 = \frac{1}{3}$

$\|f\| = \frac{1}{\sqrt{3}}$

$\|g\|^2 = \langle g, g\rangle$

$= \int_0^1 g(t) g(t)\, dt$

$= \int_0^1 e^t \cdot e^t\, dt$

$= \int_0^1 e^{2t}\, dt$

$= \left[\frac{e^{2t}}{2}\right]_0^1$

$\|g\|^2 = \frac{e^2 - 1}{2}$

$\|g\| = \sqrt{\frac{e^2-1}{2}}$

Theorem:

Let V be an inner product space, then for $x, y, z \in V$ and $c \in F$ the following statements are true.

i) $\langle x, y+z\rangle = \langle x, y\rangle + \langle x, z\rangle$

ii) $\langle x, cy\rangle = \bar{c}\langle x, y\rangle$

iii) $\langle x, y\rangle = \langle x, z\rangle \Rightarrow y = z$, $\forall\, x \in V$

Proof:

i) $\langle x, y+z\rangle = \overline{\langle y+z, x\rangle}$

$= \overline{\langle y, x\rangle + \langle z, x\rangle}$

$= \overline{\langle y, x\rangle} + \overline{\langle z, x\rangle}$

$= \langle x, y\rangle + \langle x, z\rangle$

ii) $\langle x, cy\rangle = \overline{\langle cy, x\rangle}$

$= \overline{c\langle y, x\rangle}$

$= \bar{c}\,\overline{\langle y, x\rangle}$

$= \bar{c}\langle x, y\rangle$

iii) given $\langle x, y\rangle = \langle x, z\rangle$

$\langle x, y\rangle - \langle x, z\rangle = 0$

$\langle 0, y-z\rangle = 0$

$\Rightarrow y - z = 0$

$\therefore y = z$

1) Let V be an inner product space over F, then $\forall x \in V$, $c \in F$. Prove that $\|cx\| = |c|\,\|x\|$.

$\|cx\|^2 = \langle cx, cx\rangle$

$= c\langle x, cx\rangle$

$= c\bar{c}\langle x, x\rangle$

$\|cx\|^2 = |c|^2\|x\|^2$

$\Rightarrow \|cx\| = |c|\,\|x\|$

2) Prove that parallelogram law and inner product space V, ie.. show that $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$.

$\|x+y\|^2 = \langle x+y, x+y\rangle$

$= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle$

$\|x+y\|^2 = \|x\|^2 + \langle x, y\rangle + \langle y, x\rangle + \|y\|^2 \rightarrow (1)$

$\|x-y\|^2 = \langle x-y, x-y\rangle$

$= \langle x, x\rangle - \langle x, y\rangle - \langle y, x\rangle + \langle y, y\rangle$

$\|x-y\|^2 = \|x\|^2 - \langle x, y\rangle - \langle y, x\rangle + \|y\|^2 \rightarrow (2)$

① + ②,

$$\Rightarrow \|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

3) Show that in a real inner product space if $\langle x, y\rangle = 0$, then $\|x+y\|^2 = \|x\|^2 + \|y\|^2$ [Pythagorean Theorem]

$$\|x+y\|^2 = \langle x+y, x+y\rangle$$
$$= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle$$
$$= \|x\|^2 + 0 + 0 + \|y\|^2$$

$\langle x, y\rangle = 0$

$\langle y, x\rangle = \overline{\langle x, y\rangle} = \bar{0} = 0$

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2$$

Triangle inequality:

Let V be an inner product space over the field F and for all $x, y \in V$, then $\|x+y\| \leq \|x\| + \|y\|$

$$\|x+y\|^2 = \langle x+y, x+y\rangle$$
$$= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle$$
$$= \|x\|^2 + \langle x,y\rangle + \overline{\langle x,y\rangle} + \|y\|^2$$
$$= \|x\|^2 + 2\,\mathrm{Re}\langle x,y\rangle + \|y\|^2$$
$$\leq \|x\|^2 + 2|\langle x,y\rangle| + \|y\|^2$$
$$\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2$$
$$\leq (\|x\| + \|y\|)^2$$

$z = 1 + i$

$\mathrm{Re}\, z = 1$

$|z| = \sqrt{2}$

$\Rightarrow \mathrm{Re}\, z \leq |z|$

$|\langle x, y\rangle| \leq \|x\| \cdot \|y\|$

$$\therefore \|x+y\| \leq \|x\| + \|y\|$$

Cauchy - Schwarz inequality:

Let V be an inner product space over the field F, $\forall\, x, y \in V$, then $|\langle x,y\rangle| \leq \|x\| . \|y\|$

The inequality is trivially proved when $x = 0$ or $y = 0$.

Hence, let $x \neq 0$, $y \neq 0$.

Consider $z = y - \dfrac{\langle y, x\rangle}{\|x\|^2} x$

$$\langle z, z \rangle \geq 0$$

$$\left\langle y - \frac{\langle y,x\rangle}{\|x\|^2}x \,,\, y - \frac{\langle y,x\rangle}{\|x\|^2}x \right\rangle \geq 0$$

$$\langle y,y\rangle - \frac{\langle y,x\rangle}{\|x\|^2}\langle y,x\rangle - \frac{\langle y,x\rangle}{\|x\|^2}\langle x,y\rangle + \frac{\langle y,x\rangle}{\|x\|^2}\frac{\langle y,x\rangle}{\|x\|^2}\langle x,x\rangle \geq 0$$

$$\|y\|^2 - \cancel{\frac{\langle y,x\rangle\langle y,x\rangle}{\|x\|^2}} - \frac{\langle y,x\rangle\langle x,y\rangle}{\|x\|^2} + \cancel{\frac{\langle y,x\rangle}{\|x\|^2}}\cdot\cancel{\frac{\langle y,x\rangle}{\|x\|^2}}\|x\|^2 \geq 0$$

$$\|y\|^2 - \frac{\langle y,x\rangle}{\|x\|^2}\langle x,y\rangle \geq 0$$

$$\|y\|^2 - \frac{\overline{\langle x,y\rangle}}{\|x\|^2}\langle x,y\rangle \geq 0$$

$$z\bar{z} = |z|^2$$

$$\|y\|^2 - \frac{|\langle x,y\rangle|^2}{\|x\|^2} \geq 0$$

$z = a+ib$

$\bar{z} = a-ib$

$z\bar{z} = a^2+b^2$

$|z| = \sqrt{a^2+b^2}$

$|z|^2 = a^2+b^2$

$$\|x\|^2 \cdot \|y\|^2 - |\langle x,y\rangle|^2 \geq 0$$

$$\|x\|^2\|y\|^2 \geq |\langle x,y\rangle|^2$$

$$|\langle x,y\rangle|^2 \leq \|x\|^2\|y\|^2$$

$$\therefore |\langle x,y\rangle| \leq \|x\|\cdot\|y\|$$

1) In $C([0,1])$, let $f(t) = t$, $g(t) = e^t$. Verify Schwarz and triangular inequality.

Schwarz inequality states that,

$$|\langle f,g\rangle| \leq \|f\|\cdot\|g\|$$

we define $\langle f,g\rangle = \int_0^1 f(t)\,g(t)\,dt$

$$= \int_0^1 te^t\,dt$$

$$= \left[te^t - e^t\right]_0^1$$

$$= [e - e - (0-1)]$$

$$\langle f,g\rangle = 1$$

$$|\langle f,g\rangle| = |1| = 1 \rightarrow (1)$$

$\|f\| = \frac{1}{\sqrt{3}}$, $\|g\| = \frac{1}{\sqrt{2}}\sqrt{e^2-1}$ → refer previous problem

$$\|f\|\cdot\|g\| = \frac{1}{\sqrt{3}}\cdot\frac{1}{\sqrt{2}}\cdot\sqrt{e^2-1}$$

$$= \frac{1}{\sqrt{6}}\cdot\sqrt{e^2-1}$$

$$= 1.032 \rightarrow (2)$$

from (1) & (2),

$$|\langle f,g\rangle| \le \|f\|\cdot\|g\|$$

Triangle inequality states that,

$$\|f+g\| \le \|f\| + \|g\|$$

$$f+g = t+e^t$$

$$\|f+g\|^2 = \langle f+g, f+g\rangle$$

$$= \int_0^1 (f+g)^2\,dt$$

$$= \int_0^1 (t+e^t)^2\,dt$$

$$= \int_0^1 [t^2+e^{2t}+2te^t]\,dt$$

$$= \left[\frac{t^3}{3}+\frac{e^{2t}}{2}+2[te^t-e^t]\right]_0^1$$

$$= \frac{1}{3}+\frac{e^2}{2}+2(e-e)-\left(\frac{1}{2}+2(0-1)\right)$$

$$= \frac{1}{3}+\frac{e^2}{2}-\frac{1}{2}+2$$

$$\|f+g\|^2 = \frac{e^2}{2}+\frac{11}{6} = \frac{3e^2+11}{6}$$

$$\|f+g\| = \sqrt{\frac{3e^2+11}{6}} = 2.3511 \rightarrow (3)$$

$$\|f\| = \frac{1}{\sqrt{3}}, \quad \|g\| = \frac{1}{\sqrt{2}}\sqrt{e^2-1}$$

$$\|f\|+\|g\| = \frac{1}{\sqrt{3}}+\frac{1}{\sqrt{2}}\sqrt{e^2-1}$$

~~0.57 + 1.78~~

$$= 2.3646 \rightarrow (4)$$

from (3) & (4),

$$\|f+g\| \le \|f\|+\|g\|$$

Gram Schmidt orthogonalization process:

Orthogonal:

Let V be an inner product space and let $x, y \in V$. x is said to be orthogonal to y if,

$$\langle x, y \rangle = 0.$$

note:

x is orthogonal to y.

$$\Rightarrow \langle x, y \rangle = 0$$

$$\Rightarrow \overline{\langle y, x \rangle} = 0$$

$$\Rightarrow \langle y, x \rangle = 0$$

$\Rightarrow y$ is orthogonal to x.

Orthogonal set:

Let V be an inner product space, a subset S of V is said to be an orthogonal set if any 2 distinct vectors in S are orthogonal.

Orthonormal set:

A subset S of V is orthonormal set if S is orthogonal and $\|x\| = 1$ for all $\forall x \in S$.

Orthonormal basis:

Let V be an inner product space. A subset S of V is an orthogonal basis for V if it is an ordered basis, ie.. orthonormal.

Gram Schmidt ~~orthogono~~ orthogonalization process:

Every finite dimensional ~~linear~~ inner product space has an orthonormal basis.

Proof:

Let V be an finite dimensional inner product space. Let $\{V_1, V_2, \ldots, V_n\}$ be a basis for V. From this basis, we shall construct an orthonormal basis $\{\omega_1, \omega_2, \ldots, \omega_n\}$

First, we take $\omega_1 = V_1$

Let $\omega_2 = v_2 - \frac{\langle v_2, \omega_1 \rangle}{\|\omega_1\|^2} \omega_1$

we claim, $\omega_2 \neq 0$.

If $\omega_2 = 0$, then v_2 is a scalar multiple of ω_1 and hence of v_1 which is contradiction since v_1, v_2 are linearly independent.

$$\therefore \omega_2 \neq 0$$

Now,

$$\langle \omega_2, \omega_1 \rangle = \langle v_2 - \frac{\langle v_2, \omega_1 \rangle}{\|\omega_1\|^2} \omega_1, \omega_1 \rangle$$

$$= \langle v_2, \omega_1 \rangle - \frac{\langle v_2, \omega_1 \rangle}{\|\omega_1\|^2} \langle \omega_1, \omega_1 \rangle$$

$$= \langle v_2, \omega_1 \rangle - \frac{\langle v_2, \omega_1 \rangle}{\cancel{\|\omega_1\|^2}} \cancel{\|\omega_1\|^2}$$

$$\langle \omega_2, \omega_1 \rangle = 0$$

$\therefore \{\omega_2, \omega_1\}$ is an orthogonal set.

Now, suppose that we have constructed non-zero orthogonal vectors $\omega_1, \omega_2, \dots, \omega_k$ $(k < n)$ then,

$$\omega_{k+1} = v_k + 1 - \sum_{j=1}^{k} \frac{\langle v_{k+1}, \omega_j \rangle}{\|\omega_j\|^2} \omega_j$$

we claim that $\omega_{k+1} \neq 0$.

If $\omega_{k+1} = 0$, then v_{k+1} is a linear combination of $\omega_1, \omega_2, \dots, \omega_k$ and hence of $v_1, v_2, \dots, v_k$ are linearly independent.

$$\therefore \omega_{k+1} \neq 0$$

Now, $1 \leq i \leq k$.

$$\langle \omega_{k+1}, \omega_i \rangle = \langle v_{k+1} - \sum_{j=1}^{k} \frac{\langle v_{k+1}, \omega_j \rangle}{\|\omega_j\|^2} \omega_j, \omega_i \rangle$$

$$= \langle v_{k+1}, \omega_i \rangle - \sum_{j=1}^{k} \frac{\langle v_{k+1}, \omega_j \rangle}{\|\omega_j\|^2} \langle \omega_j, \omega_i \rangle \quad \because \omega_j = \omega_i$$

$$= \langle v_{k+1}, \omega_i \rangle - \frac{\langle v_{k+1}, \omega_i \rangle}{\|\omega_i\|^2} \langle \omega_i, \omega_i \rangle$$

$$= \langle v_{k+1}, \omega_i \rangle - \frac{\langle v_{k+1}, \omega_i \rangle}{\cancel{\|\omega_i\|^2}} \cancel{\|\omega_i\|^2}$$

$$\langle \omega_{k+1}, \omega_i \rangle = 0$$

Thus, continuing in this way, ultimately obtain a non zero orthogonal set $\{w_1, w_2, \ldots, w_n\}$.

This orthogonal set is linearly independent; hence it is a basis.

$\therefore \{w_1, w_2, \ldots, w_n\}$ is an orthonormal set.

— x —

1) Apply Gram Schmidt process to construct an orthonormal basis for V_3 of all $V_3(R)$ with standard inner product for the basis $\{V_1, V_2, V_3\}$ where $V_1 = (1,0,1)$, $V_2 = (1,3,1)$, $V_3 = (3,2,1)$.

Let $w_1 = V_1 = (1,0,1)$

$\therefore \|w_1\|^2 = \langle w_1, w_1 \rangle = 1+0+1 = 2$

$\|w_1\| = \sqrt{2}$

$\langle V_2, w_1 \rangle = 1+0+1 = 2$

$$w_2 = V_2 - \frac{\langle V_2, w_1 \rangle}{\|w_1\|^2} w_1$$

$\langle V_3, w_1 \rangle = 3+0+1 = 4$

$$= (1,3,1) - \frac{2}{2}(1,0,1)$$

$\langle V_3, w_2 \rangle = 0+6+0 = 6$

$$= (1,3,1) - (1,0,1)$$

$$w_2 = (0,3,0)$$

$$\|w_2\|^2 = \langle w_2, w_2 \rangle = 0+9+0 = 9$$

$$\|w_2\| = 3$$

$$w_3 = V_3 - \frac{\langle V_3, w_1 \rangle}{\|w_1\|^2} \cdot w_1 - \frac{\langle V_3, w_2 \rangle}{\|w_2\|^2} \cdot w_2$$

$$= (3,2,1) - \frac{4}{2}(1,0,1) - \frac{6}{9} \cdot (0,3,0)$$

$$= (3,2,1) - 2(1,0,1) - \frac{2}{3}(0,3,0)$$

$$= [(3-2,0), (2-0-2), (1-2+0)]$$

$$w_3 = (1,0,-1)$$

$$\|w_3\|^2 = \langle w_3, w_3 \rangle = 1+0+1 = 2$$

$$\|w_3\| = \sqrt{2}$$

$\therefore$ The orthogonal set is $\{(1,0,1), (0,3,0), (1,0,-1)\}$

$\therefore$ The orthonormal basis is,

$$\left\{ \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), (0,1,0), \left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right) \right\}$$

2) Let V be the set of all polynomials of degree ≤ 2, V is an real inner product space with inner product defined by $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,dx$ starting with the basis $\{1, x, x^2\}$. Obtain an ortho normal basis for V.

Let $v_1 = 1$, $v_2 = x$, $v_3 = x^2$

Let $w_1 = v_1 = 1$

$$\|w_1\|^2 = \langle w_1, w_1\rangle = \int_{-1}^{1} (1)(1)\,dx$$

$$= \int_{-1}^{1} dx = [x]_{-1}^{1}$$

$$= 1 + 1 = 2$$

$$\|w_1\| = \sqrt{2}$$

$$w_2 = v_2 - \frac{\langle v_2, w_1\rangle}{\|w_1\|^2} \cdot w_1$$

$$\langle v_2, w_1\rangle = \int_{-1}^{1} x \cdot 1\,dx = \int_{-1}^{1} x\,dx = 0$$

$$= x - 0$$

$$w_2 = x$$

$$\|w_2\|^2 = \langle w_2, w_2\rangle = \int_{-1}^{1} x \cdot x\,dx$$

$$= 2\int_{0}^{1} x^2\,dx$$

$$= 2\cdot\left[\frac{x^3}{3}\right]_0^1 = \frac{2}{3}$$

$$\|w_2\| = \frac{\sqrt{2}}{\sqrt{3}}$$

$$w_3 = v_3 - \frac{\langle v_3, w_1\rangle}{\|w_1\|^2} \cdot w_1 - \frac{\langle v_3, w_2\rangle}{\|w_2\|^2} \cdot w_2$$

$$\langle v_3, w_1\rangle = \int_{-1}^{1} x^2\,dx = 2\int_{0}^{1} x^2\,dx = \frac{2}{3}$$

$$= x^2 - \frac{\frac{2}{3}}{2} \cdot 1$$

$$\langle v_3, w_2\rangle = \int_{-1}^{1} x^3\,dx = 0$$

$$w_3 = x^2 - \frac{1}{3}$$

$$\|w_3\|^2 = \langle w_3, w_3\rangle = \int_{-1}^{1} \left(x^2 - \frac{1}{3}\right)^2 dx$$

$$= 2\int_{0}^{1} \left[x^4 + \frac{1}{9} - 2x^2\frac{1}{3}\right] dx$$

$$= 2\left[\frac{x^5}{5} + \frac{x}{9} - \frac{2x^3}{3}\cdot\frac{1}{3}\right]_0^1$$

$$= 2\left[\frac{1}{5} + \frac{1}{9} - \frac{2}{9}\right]$$

$$= 2 \times \frac{4}{45} = \frac{8}{45}$$

$$\|w_3\| = \frac{2\sqrt{2}}{3\sqrt{5}}$$

$\therefore$ The orthogonal set is, $\left\{1, x, x^2 - \frac{1}{3}\right\}$

$\therefore$ The orthonormal basis is,

$$\left\{ \frac{1}{\sqrt{2}}, \frac{x}{\sqrt{2}/\sqrt{3}}, \frac{x^2 - \frac{1}{3}}{\frac{2\sqrt{2}}{3\sqrt{5}}} \right\}$$

$$\Rightarrow \left\{ \frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{\sqrt{2}}x, \frac{\sqrt{10}}{4}(3x^2-1) \right\}$$

3) Find an orthogonal basis containing the vector $(1,3,4)$ for $V_3(R)$ with the standard inner product.

Let (x_1, x_2, x_3) be any vector orthogonal to $(1,3,4)$.

$\therefore x_1 + 3x_2 + 4x_3 = 0$. $[1+3-4=0]$

$\therefore$ The solution is $(1,1,-1)$.

$\therefore (1,1,-1)$ is orthogonal to $(1,3,4)$.

Let (y_1, y_2, y_3) be any orthogonal vector to $(1,3,4)$ and $(1,1,-1)$.

$\therefore y_1 + 3y_2 + 4y_3 = 0$

$\therefore y_1 + y_2 - y_3 = 0$

$$\begin{array}{cccc} 3 & 4 & 1 & 3 \\ 1 & -1 & 1 & 1 \end{array}$$

$(-3-4), (4+1), (1-3)$

$$\frac{y_1}{(-3-4)} = \frac{y_2}{(4+1)} = \frac{y_3}{(1-3)}$$

$$\frac{y_1}{-7} = \frac{y_2}{5} = \frac{y_3}{-2}$$

$\therefore (-7, 5, -2)$ is one such vector orthogonal to $(1,3,4), (1,1,-1)$.

$\therefore$ Orthogonal basis is $\{(1,3,4), (1,1,-1), (-7,5,-2)\}$

4) Find an orthogonal basis containing $(1,1,-1)$, $(1,0,1)$ for $V_3(R)$ with standard $\langle \cdot , \cdot \rangle$. inner product

Here, $(1,1,-1)$ is orthogonal to $(1,0,1)$.

Let (y_1, y_2, y_3) be any orthogonal vector to $(1,1,-1)$ and $(1,0,1)$.

$\therefore y_1 + y_2 - y_3 = 0$

$\therefore y_1 + 0y_2 + y_3 = 0$

$$\frac{y_1}{(1-0)} = \frac{y_2}{(-1-1)} = \frac{y_3}{(0-1)}$$

$$\frac{y_1}{1} = \frac{y_2}{-2} = \frac{y_3}{-1}$$

$\therefore (1,-2,-1)$ is one such vector orthogonal to $(1,1,-1)$ and $(1,0,1)$.

$\therefore$ Orthogonal basis is $\{(1,1,-1),(1,0,1),(1,-2,-1)\}$

5) Find the orthonormal basis for the set of values $(1,i,0)$ and $(1-i,2,4-i)$.

Let $v_1 = (1,i,0)$

$v_2 = (1-i,2,4-i)$

$w_1 = v_1 = (1,i,0)$

$\|w_1\|^2 = \langle w_1, w_1\rangle = (1)(1)+(i)(-i)+0$

$= 1+1$

$\|w_1\|^2 = 2$

$\|w_1\| = \sqrt{2}$

$\langle v_2, w_1\rangle = (1-i)(1)+2(-i)+0$

$= 1-i-2i$

$= 1-3i$

$$w_2 = v_2 - \frac{\langle v_2, w_1\rangle}{\|w_1\|^2} w_1$$

$$= (1-i,2,4-i) - \frac{1-3i}{2}(1,i,0)$$

$$= (1-i,2,4-i) - \left(\frac{1-3i}{2}, \frac{i+3}{2}, 0\right)$$

$$w_2 = \left(\frac{1+i}{2}, \frac{1-i}{2}, 4i\right)$$

$$\|w_2\|^2 = \langle w_2, w_2\rangle = \left(\frac{1+i}{2}\right)\left(\frac{1-i}{2}\right) + \left(\frac{1-i}{2}\right)\left(\frac{1+i}{2}\right) + (4i)(-4i)$$

$$= \frac{2}{4} + \frac{2}{4} + 16$$

$\|w_2\|^2 = 17$

$\|w_2\| = \sqrt{17}$

$\therefore$ The orthogonal set is $(1,i,0), \left(\frac{1+i}{2}, \frac{1-i}{2}, 4i\right)$

$\therefore$ The ortho~~gonal~~normal basis is;

$$\left\{\left(\frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}}, 0\right), \left(\frac{1+i}{2\sqrt{7}}, \frac{1-i}{2\sqrt{7}}, \frac{4i}{\sqrt{7}}\right)\right\}$$

6) Apply Gram schmidt process to construct an ~~orthogonal~~ orthonormal basis for $M_{2\times2}(R)$ with

where $\langle A, B\rangle = tr(B^* A)$, $\forall A, B \in M_{2\times2}(R)$

$$v_1 = \begin{pmatrix} 3 & 5 \\ -1 & 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -1 & 9 \\ 5 & -1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 7 & -17 \\ 2 & -6 \end{pmatrix}$$

Let $w_1 = v_1 = \begin{pmatrix} 3 & 5 \\ -1 & 1 \end{pmatrix}$

$$\|w_1\|^2 = \langle w_1, w_1\rangle = tr(w_1^* w_1)$$

$$= tr\left[\begin{pmatrix} 3 & -1 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} 3 & 5 \\ -1 & 1 \end{pmatrix}\right]$$

$$= tr\begin{bmatrix} 10 & 14 \\ 14 & 26 \end{bmatrix}$$

$$\|w_1\|^2 = 10 + 26 = 36$$

$$\|w_1\| = 6$$

$$w_2 = v_2 - \frac{\langle v_2, w_1\rangle}{\|w_1\|^2} w_1$$

$$\langle v_2, w_1\rangle = tr(w_1^* v_2) = tr\left[\begin{pmatrix} 3 & -1 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} -1 & 9 \\ 5 & -1 \end{pmatrix}\right]$$

$$= tr\begin{bmatrix} -8 & 28 \\ 0 & 44 \end{bmatrix}$$

$$\langle v_2, w_1\rangle = -8 + 44 = 36$$

$$w_2 = \begin{pmatrix} -1 & 9 \\ 5 & -1 \end{pmatrix} - \frac{\cancel{36}}{\cancel{36}}\begin{pmatrix} 3 & 5 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -4 & 4 \\ 6 & -2 \end{pmatrix}$$

$$\|w_2\|^2 = \langle w_2, w_2\rangle = tr(w_2^* w_2)$$

$$= tr\left[\begin{pmatrix} -4 & 6 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} -4 & 4 \\ 6 & -2 \end{pmatrix}\right]$$

$$= tr\begin{bmatrix} 52 & -28 \\ -28 & 20 \end{bmatrix}$$

$$\|w_2\|^2 = 52 + 20 = 72$$

$$\|w_2\| = 6\sqrt{2}$$

$$w_3 = v_3 - \frac{\langle v_3, w_1\rangle}{\|w_1\|^2} w_1 - \frac{\langle v_3, w_2\rangle}{\|w_2\|^2} w_2$$

$$\langle v_3, w_1 \rangle = tr(w_1^* v_3) = tr\left[\begin{pmatrix}3 & -1\\5 & 1\end{pmatrix}\begin{pmatrix}7 & -17\\2 & -6\end{pmatrix}\right]$$

$$= tr\begin{bmatrix}19 & -45\\37 & -91\end{bmatrix}$$

$$\langle v_3, w_1 \rangle = 19 - 91 = -72$$

$$\langle v_3, w_2 \rangle = tr(w_2^* v_3) = tr\left[\begin{pmatrix}-4 & 6\\4 & -2\end{pmatrix}\begin{pmatrix}7 & -17\\2 & -6\end{pmatrix}\right]$$

$$= tr\begin{bmatrix}-16 & 32\\24 & -56\end{bmatrix}$$

$$\langle v_3, w_2 \rangle = -16 - 56 = -72$$

$$w_3 = \begin{pmatrix}7 & -17\\2 & -6\end{pmatrix} - \frac{\cancel{-72}^{\,-2}}{\cancel{36}}\begin{pmatrix}3 & 5\\-1 & 1\end{pmatrix} - \frac{\cancel{-72}^{\,-1}}{\cancel{72}}\begin{pmatrix}-4 & 4\\6 & -2\end{pmatrix}$$

$$= \begin{pmatrix}7 & -17\\2 & -6\end{pmatrix} + \begin{pmatrix}6 & 10\\-2 & 2\end{pmatrix} + \begin{pmatrix}-4 & 4\\6 & -2\end{pmatrix}$$

$$w_3 = \begin{pmatrix}9 & -3\\6 & -6\end{pmatrix}$$

$$\|w_3\|^2 = \langle w_3, w_3 \rangle = tr(w_3^* w_3)$$

$$= tr\left[\begin{pmatrix}9 & 6\\-3 & -6\end{pmatrix}\begin{pmatrix}9 & -3\\6 & -6\end{pmatrix}\right]$$

$$= tr\begin{bmatrix}117 & -63\\-63 & 45\end{bmatrix}$$

$$\|w_3\|^2 = 117 + 45 = 162$$

$$\therefore \|w_3\| = 9\sqrt{2}$$

$\therefore$ The orthogonal set is $\begin{pmatrix}3 & 5\\-1 & 1\end{pmatrix}, \begin{pmatrix}-4 & 4\\6 & -2\end{pmatrix}, \begin{pmatrix}9 & -3\\6 & -6\end{pmatrix}$

$\therefore$ The orthonormal basis is,

$$\left\{\frac{1}{6}\begin{pmatrix}3 & 5\\-1 & 1\end{pmatrix}, \frac{1}{6\sqrt{2}}\begin{pmatrix}-4 & 4\\6 & -2\end{pmatrix}, \frac{1}{9\sqrt{2}}\begin{pmatrix}9 & -3\\6 & -6\end{pmatrix}\right\}$$

Least square approximation:

Let $(t_1, y_1), (t_2, y_2), \ldots, (t_n, y_n)$ be a set of observed values of y corresponding to the given time t and let $y = ct + d$ be the line of best fit for this data, then our aim is to determine the constants c and d so that, this line represents the line of best.

To determine c and d, we use the principle of least square.

Let $A = \begin{bmatrix} t_1 & 1 \\ t_2 & 1 \\ \vdots & \vdots \\ t_n & 1 \end{bmatrix}$, $y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$, $x_0 = \begin{pmatrix} c \\ d \end{pmatrix}$

$$\therefore x_0 = (A^* A)^{-1} A^* y$$

Error: $E = \|Ax_0 - y\|^2$

1) Use the least square approximation to find the best fit with both (i) linear function and (ii) quadratic function. Compute the error in both cases for the following data:

$\{(-3,9), (-2,6), (0,2), (1,1)\}$.

i) Linear function:

$$A = \begin{pmatrix} -3 & 1 \\ -2 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad y = \begin{pmatrix} 9 \\ 6 \\ 2 \\ 1 \end{pmatrix}, \quad x_0 = \begin{pmatrix} c \\ d \end{pmatrix}$$

To find c and d:

$$x_0 = (A^* A)^{-1} A^* y$$

$$A^*A = \begin{pmatrix} -3 & -2 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -3 & 1 \\ -2 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 14 & -4 \\ -4 & 4 \end{pmatrix}$$

$$(A^*A)^{-1} = \frac{1}{56-16} \begin{pmatrix} 4 & 4 \\ 4 & 14 \end{pmatrix}$$

$$= \frac{1}{40} \begin{pmatrix} 4 & 4 \\ 4 & 14 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{10} & \frac{1}{10} \\ \frac{1}{10} & \frac{7}{20} \end{pmatrix}$$

$$A^*y = \begin{pmatrix} -3 & -2 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 9 \\ 6 \\ 2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -38 \\ 18 \end{pmatrix}$$

$$\therefore x_0 = (A^*A)^{-1}A^*y$$

$$= \begin{pmatrix} \frac{1}{10} & \frac{1}{10} \\ \frac{1}{10} & \frac{7}{20} \end{pmatrix}\begin{pmatrix} -38 \\ 18 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ \frac{5}{2} \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}$$

$\therefore$ The line for best fit for linear function is,

$$y = ct + d$$

$$\therefore y = -2t + \frac{5}{2}$$

Error: $E = \|Ax_0 - y\|^2$

$$Ax_0 = \begin{pmatrix} -3 & 1 \\ -2 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -2 \\ \frac{5}{2} \end{pmatrix} = \begin{pmatrix} 17/2 \\ 13/2 \\ 5/2 \\ 1/2 \end{pmatrix}$$

$$Ax_0 - y = \begin{pmatrix} 17/2 \\ 13/2 \\ 5/2 \\ 1/2 \end{pmatrix} - \begin{pmatrix} 9 \\ 6 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{pmatrix}$$

$$\|Ax_0 - y\|^2 = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}$$

$$\|Ax_0 - y\|^2 = 1$$

ii) Quadratic function: $y = ct^2 + dt + e$

$$A = \begin{pmatrix} 9 & -3 & 1 \\ 4 & -2 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad y = \begin{pmatrix} 9 \\ 6 \\ 2 \\ 1 \end{pmatrix}, \quad x_0 = \begin{pmatrix} c \\ d \\ e \end{pmatrix}$$

To find c and d:

$$x_0 = (A^*A)^{-1}A^*y$$

$$A^*A = \begin{pmatrix} 9 & 4 & 0 & 1 \\ -3 & -2 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 9 & -3 & 1 \\ 4 & -2 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$A^* A = \begin{pmatrix} 98 & -34 & 14 \\ -34 & 14 & -4 \\ 14 & -4 & 4 \end{pmatrix}$$

$$(A^* A)^{-1} = \begin{pmatrix} \frac{1}{9} & \frac{2}{9} & \frac{-1}{6} \\ \frac{2}{9} & \frac{49}{90} & \frac{-7}{20} \\ \frac{-1}{6} & \frac{-7}{20} & \frac{3}{5} \end{pmatrix}$$

$$A^* y = \begin{pmatrix} 9 & 4 & 0 & 1 \\ -3 & -2 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 9 \\ 6 \\ 2 \\ 1 \end{pmatrix}$$

$$A^* y = \begin{pmatrix} 106 \\ -38 \\ 18 \end{pmatrix}$$

$$x_0 = (A^* A)^{-1} A^* y$$

$$= \begin{pmatrix} 1/9 & 2/9 & -1/6 \\ 2/9 & 49/90 & -7/20 \\ -1/6 & -7/30 & 3/5 \end{pmatrix} \begin{pmatrix} 106 \\ -38 \\ 18 \end{pmatrix}$$

$$x_0 = \begin{pmatrix} 1/3 \\ -4/3 \\ 2 \end{pmatrix} = \begin{pmatrix} c \\ d \\ e \end{pmatrix}$$

$\therefore$ The best fit for this data is,

$$y = \frac{1}{3}t^2 - \frac{4}{3}t + 2$$

Error:

$$E = \| Ax_0 - y \|^2$$

$$Ax_0 = \begin{pmatrix} 9 & -3 & 1 \\ 4 & -2 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1/3 \\ -4/3 \\ 2 \end{pmatrix} = \begin{pmatrix} 9 \\ 6 \\ 2 \\ 1 \end{pmatrix}$$

$$Ax_0 - y = \begin{pmatrix} 9 \\ 6 \\ 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 9 \\ 6 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\| Ax_0 - y \|^2 = 0 + 0 + 0 + 0 = 0$$

2) Let us suppose that data collected are $(1,2)$, $(2,3)$, $(3,5)$, $(4,7)$, then determine the line of best fit and also find the error.

Let $A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{pmatrix}$, $y = \begin{pmatrix} 2 \\ 3 \\ 5 \\ 7 \end{pmatrix}$, $x_0 = \begin{pmatrix} c \\ d \end{pmatrix}$

$$x_0 = (A^*A)^{-1} A^* y$$

$$A^*A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{pmatrix}$$

$1+4+9+16$

$1+2+3+4$

$$A^*A = \begin{pmatrix} 30 & 10 \\ 10 & 4 \end{pmatrix}$$

$\frac{1}{|A|}$ adj A

$$(A^*A)^{-1} = \frac{1}{20} \begin{pmatrix} 4 & -10 \\ -10 & 30 \end{pmatrix}$$

$$(A^*A)^{-1} = \begin{pmatrix} 1/5 & -1/2 \\ -1/2 & 3/2 \end{pmatrix}$$

$$A^*y = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 5 \\ 7 \end{pmatrix} = \begin{pmatrix} 51 \\ 17 \end{pmatrix}$$

$$x_0 = (A^*A)^{-1} A^* y$$

$$= \begin{pmatrix} 1/5 & -1/2 \\ -1/2 & 3/2 \end{pmatrix} \begin{pmatrix} 51 \\ 17 \end{pmatrix}$$

$$x_0 = \begin{pmatrix} \frac{17}{10} \\ 0 \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}$$

$\therefore$ The best fit for this data is,

$$y = ct + d$$

$$\boxed{y = \frac{17}{10} t}$$

Error :

$$E = \| A x_0 - y \|^2$$

$$A x_0 = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} \frac{17}{10} \\ 0 \end{pmatrix} = \begin{pmatrix} 17/10 \\ 34/10 \\ 51/10 \\ 68/10 \end{pmatrix} = \begin{pmatrix} 17/10 \\ 17/5 \\ 51/10 \\ 34/5 \end{pmatrix}$$

$$A x_0 - y = \begin{pmatrix} 17/10 \\ 17/5 \\ 51/10 \\ 34/5 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ 5 \\ 7 \end{pmatrix} = \begin{pmatrix} -3/10 \\ 2/5 \\ 1/10 \\ -1/5 \end{pmatrix}$$

$$\| A x_0 - y \|^2 = \frac{9}{100} + \frac{4}{25} + \frac{1}{100} + \frac{1}{25} = \frac{3}{10}$$

The adjoint of a linear operator

Adjoint operator:

Let V be an inner product space and $T: V \to V$ be a linear operator. Let there exist a unique operator T^* on V such that $\langle T(u), v\rangle = \langle u, T^*(v)\rangle$, $\forall\, u, v \in V$, then T^* is called adjoint operator.

Let V be an inner product space and let T, U be linear operators of V, then prove the following:

1) $(T+U)^* = T^* + U^*$

$$\langle x, (T+U)^*(y)\rangle = \langle (T+U)(x), y\rangle$$
$$= \langle T(x) + U(x), y\rangle$$
$$= \langle T(x), y\rangle + \langle U(x), y\rangle$$
$$= \langle x, T^*(y)\rangle + \langle x, U^*(y)\rangle$$
$$= \langle x, T^*(y) + U^*(y)\rangle$$
$$\langle x, (T+U)^*(y)\rangle = \langle x, (T^*+U^*)y\rangle$$
$$\Rightarrow (T+U)^* = T^* + U^*$$

ii) $(cT)^* = \bar{c}T^*$

$$\langle x, (cT)^*(y)\rangle = \langle (cT)x, y\rangle$$
$$= c\langle T(x), y\rangle$$
$$= c\langle x, T^*y\rangle$$
$$\langle x, (cT)^*(y)\rangle = \langle x, \bar{c}T^*(y)\rangle$$
$$\Rightarrow (cT)^* = \bar{c}T^*$$

iii) $(TU)^* = U^* T^*$

$$\langle x, (TU)^*(y)\rangle = \langle (TU)(x), y\rangle$$
$$= \langle T[U(x)], y\rangle$$
$$= \langle T(x), U^*(y)\rangle$$
$$\langle x, (TU)^*(y)\rangle = \langle x, U^*T^*(y)\rangle$$
$$\Rightarrow (TU)^* = U^*T^*$$

Part - A

1) Let P_2 have the $\langle p, q\rangle = \int_{-1}^{1} p(x)\, q(x)\, dx$.
Find the angle between p and q where

$p = x$ and $q = x^2$ with respect to the inner product on P_2.

$$\langle p, q \rangle = \int_{-1}^{1} p(x)q(x)\,dx$$
$$= \int_{-1}^{1} x \cdot x^2\,dx$$
$$= \int_{-1}^{1} x^3\,dx$$
$$= 0$$

$\therefore$ p and q are orthogonal

$\therefore$ angle $= 90°$

2) Let R^2 have the weighted Euclidean inner product defined as $\langle u, v \rangle = 2u_1v_1 + 3u_2v_2$ and let $u = (1,1)$, $v = (3,2)$, $w = (0,-1)$. Compute the value of $\langle u+v, 3w \rangle$.

$$u + v = (\overset{u_1}{4}, \overset{u_2}{3})$$
$$3w = (\overset{v_1}{0}, \overset{v_2}{-3})$$
$$\langle u+v, 3w \rangle = 2(4)(0) + 3(3)(-3)$$
$$= 0 - 27$$
$$= -27$$

3) Let V be the vector space of polynomials with inner product defined by $\langle f, g \rangle = \int_0^1 f(t)\,g(t)\,dt$ where $f(t) = t+2$, $g(t) = t^2 - 2t - 3$. Find $\langle f, g \rangle$.

$$\langle f, g \rangle = \int_0^1 (t+2)(t^2 - 2t - 3)\,dt$$
$$= \int_0^1 (t^3 - \cancel{2t^2} - 3t + \cancel{2t^2} - 4t - 6)\,dt$$
$$\cancel{\neq \left[\frac{t^4}{4} - \frac{3t^2}{2} - 4t\right]}$$
$$= \int_0^1 (t^3 - 7t - 6)\,dt$$
$$= \left[\frac{t^4}{4} - \frac{7t^2}{2} - 6t\right]_0^1$$
$$= \frac{1}{4} - \frac{7}{2} - 6$$
$$= \frac{-37}{4}$$

4) Define adjoint matrix.

Let $A \in M_{m\times n}(F)$, we ~~adjusted~~ defined the conjugate transpose are adjoint of A denoted by A^* is defined by matrix $(A^*)_{ij} = \overline{A_{ij}}$

5) Let V be the inner product space. Prove that $\langle x, y+z\rangle = \langle x, y\rangle + \langle x, z\rangle$

$$\langle x, y+z\rangle = \overline{\langle y+z, x\rangle}$$
$$= \overline{\langle y,x\rangle + \langle z,x\rangle}$$
$$= \overline{\langle y,x\rangle} + \overline{\langle z,x\rangle}$$
$$\langle x, y+z\rangle = \langle x, y\rangle + \langle x, z\rangle$$

1) Let R^3 have the Euclidean inner product. Use Gram-Schmidt process $\{u_1, u_2, u_3\}$ into an orthogonal basis where $u_1 = (1,1,1)$, $u_2 = (0,1,1)$, $u_3 = (0,0,1)$.

Let $w_1 = u_1 = (1,1,1)$

$$\|w_1\|^2 = \langle w_1, w_1\rangle = 1+1+1 = 3$$
$$\|w_1\| = \sqrt{3}$$

$\langle u_2, w_1\rangle = 0+1+1 = 2$

$$w_2 = u_2 - \frac{\langle u_2, w_1\rangle}{\|w_1\|^2} w_1$$

$\langle u_3, w_1\rangle = 1$

$\langle u_3, w_2\rangle = \frac{1}{3}$

$$= (0,1,1) - \frac{2}{3}(1,1,1)$$
$$= (0,1,1) - \left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right)$$
$$w_2 = \left(\frac{-2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$
$$\|w_2\|^2 = \langle w_2, w_2\rangle = \frac{4}{9} + \frac{1}{9} + \frac{1}{9}$$
$$= \frac{6}{9} = \frac{2}{3}$$
$$\|w_2\| = \sqrt{\frac{2}{3}}$$
$$w_3 = u_3 - \frac{\langle u_3, w_1\rangle}{\|w_1\|^2} w_1 - \frac{\langle u_3, w_2\rangle}{\|w_2\|^2} w_2$$
$$= (0,0,1) - \frac{1}{3}(1,1,1) - \frac{\frac{1}{\cancel{3}}}{\frac{2}{\cancel{3}}}\left(\frac{-2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$
$$= (0,0,1) - \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) - \left(\frac{-1}{3}, \frac{1}{6}, \frac{1}{6}\right)$$
$$w_3 = \left(0, \frac{-1}{2}, \frac{1}{2}\right)$$

$\|w_3\|^2 = \langle w_3, w_3\rangle = 0 + \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

$\|w_3\| = \frac{1}{\sqrt{2}}$

$\therefore$ The orthogonal set is $\left\{(1,1,1), \left(\frac{-2}{3}, \frac{1}{3}, \frac{1}{3}\right), \left(0, \frac{-1}{2}, \frac{1}{2}\right)\right\}$

$\therefore$ The orthonormal basis is,

$$\left\{\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(\frac{\frac{-2}{3}}{\sqrt{\frac{2}{3}}}, \frac{\frac{1}{3}}{\sqrt{\frac{2}{3}}}, \frac{\frac{1}{3}}{\sqrt{\frac{2}{3}}}\right), \left(0, \frac{\frac{-1}{2}}{\frac{1}{\sqrt{2}}}, \frac{\frac{1}{2}}{\frac{1}{\sqrt{2}}}\right)\right\}$$

2) Use Gram - Schmidt process find an orthonormal basis for $(1,0,1)$, $(0,1,1)$, $(1,3,3)$.

Let $w_1 = v_1 = (1,0,1)$

$\|w_1\|^2 = \langle w_1, w_1\rangle = 1 + 0 + 1 = 2$

$\|w_1\| = \sqrt{2}$

$\langle v_2, w_1\rangle = 0 + 0 + 1 = 1$

$\langle v_3, w_1\rangle = 1 + 0 + 3 = 4$

$\langle v_3, w_2\rangle = \frac{-1}{2} + 3 + \frac{3}{2} = 4$

$w_2 = v_2 - \frac{\langle v_2, w_1\rangle}{\|w_1\|^2} w_1$

$= (0,1,1) - \frac{1}{2}(1,0,1)$

$w_2 = \left(\frac{-1}{2}, 1, \frac{1}{2}\right)$

$\|w_2\|^2 = \langle w_2, w_2\rangle = \frac{1}{4} + 1 + \frac{1}{4} = \frac{6}{4} = \frac{3}{2}$

$\|w_2\| = \sqrt{\frac{3}{2}}$

$w_3 = v_3 - \frac{\langle v_3, w_1\rangle}{\|w_1\|^2} w_1 - \frac{\langle v_3, w_2\rangle}{\|w_2\|^2} w_2$

$= (1,3,3) - \frac{4}{2}(1,0,1) - \frac{4}{3/2}\left(\frac{-1}{2}, 1, \frac{1}{2}\right)$

$= (1,3,3) - (2,0,2) - \left(\frac{-4}{3}, \frac{8}{3}, \frac{4}{3}\right)$

$w_3 = \left(\frac{1}{3}, \frac{1}{3}, \frac{-1}{3}\right)$

$\|w_3\|^2 = \langle w_3, w_3\rangle = \frac{1}{9} + \frac{1}{9} + \frac{1}{9} = \frac{3}{9} = \frac{1}{3}$

$\|w_3\| = \frac{1}{\sqrt{3}}$

$\therefore$ The orthogonal set is $\left\{(1,0,1), \left(\frac{-1}{2}, 1, \frac{1}{2}\right), \left(\frac{1}{3}, \frac{1}{3}, \frac{-1}{3}\right)\right\}$

∴ The orthonormal basis is,

$$\left\{\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \left(\frac{\frac{-1}{2}}{\sqrt{\frac{3}{2}}}, \frac{1}{\sqrt{\frac{3}{2}}}, \frac{\frac{1}{2}}{\sqrt{\frac{3}{2}}}\right), \left(\frac{\frac{1}{3}}{\frac{1}{\sqrt{3}}}, \frac{\frac{1}{3}}{\frac{1}{\sqrt{3}}}, \frac{\frac{-1}{3}}{\frac{1}{\sqrt{3}}}\right)\right\}$$

$$\Rightarrow \left\{\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \left(\frac{-1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right), \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{2}}\right)\right\}$$

3) Show that the vector space with inner product defined on the interval $[0, 2\pi]$ with inner product $\langle f, g\rangle = \int_0^{2\pi} f(t)\overline{g(t)}\,dt$ is an inner product space.

i) $\langle f+g, h\rangle = \int_0^{2\pi} [f(t) + g(t)]\overline{h(t)}\,dt$

$= \int_0^{2\pi} f(t)\overline{h(t)}\,dt + \int_0^{2\pi} g(t)\overline{h(t)}\,dt$

$= \langle f, h\rangle + \langle g, h\rangle, \ \forall f, g, h \in V$

ii) $\langle cf, g\rangle = \int_0^{2\pi} cf(t)\overline{g(t)}\,dt$

$= c\int_0^{2\pi} f(t)\overline{g(t)}\,dt$

$= c\langle f, g\rangle$

iii) $\langle f, g\rangle = \int_0^{2\pi} f(t)\overline{g(t)}\,dt$

$= \overline{\int_0^{2\pi} g(t)\overline{f(t)}\,dt}$

$\langle f, g\rangle = \overline{\langle g, f\rangle}$

iv) $\langle f, f\rangle = \int_0^{2\pi} f(t)\overline{f(t)}\,dt$

$= \int_0^{2\pi} |f(t)|^2\,dt > 0$

ie.., $\langle f, f\rangle > 0$

∴ $\langle f, g\rangle$ is an inner product space on V.

4) For the set of data values, used the least square approximation to find the best fit and also find the errors for $\{(1,2), (3,4), (5,7), (7,9), (9,12)\}$.

The best fit for the collected data is,

$$y = ct + d$$

Let $A = \begin{pmatrix} 1 & 1 \\ 3 & 1 \\ 5 & 1 \\ 7 & 1 \\ 9 & 1 \end{pmatrix}$, $y = \begin{pmatrix} 2 \\ 4 \\ 7 \\ 9 \\ 12 \end{pmatrix}$, $x_0 = \begin{pmatrix} c \\ d \end{pmatrix}$

$\therefore$ By least square approximation,

$$x_0 = (A^* A)^{-1} A^* y$$

$$A^* A = \begin{pmatrix} 1 & 3 & 5 & 7 & 9 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 3 & 1 \\ 5 & 1 \\ 7 & 1 \\ 9 & 1 \end{pmatrix}$$

$$A^* A = \begin{pmatrix} 165 & 25 \\ 25 & 5 \end{pmatrix}$$

$$(A^* A)^{-1} = \frac{1}{200} \begin{pmatrix} 5 & -25 \\ -25 & 165 \end{pmatrix} = \begin{pmatrix} \frac{1}{40} & \frac{-1}{8} \\ \frac{-1}{8} & \frac{33}{40} \end{pmatrix}$$

$$A^* y = \begin{pmatrix} 1 & 3 & 5 & 7 & 9 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \\ 7 \\ 9 \\ 12 \end{pmatrix} = \begin{pmatrix} 220 \\ 34 \end{pmatrix}$$

$$\therefore x_0 = (A^* A)^{-1} A^* y$$

$$= \begin{pmatrix} \frac{1}{40} & \frac{-1}{8} \\ \frac{-1}{8} & \frac{33}{40} \end{pmatrix} \begin{pmatrix} 220 \\ 34 \end{pmatrix}$$

$$x_0 = \begin{pmatrix} \frac{5}{4} \\ \frac{11}{20} \end{pmatrix} = \begin{pmatrix} 1.25 \\ 0.55 \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}$$

The best fit for the given data is,

$$y = ct + d = 1.25t + 0.55$$

Error : $E = \|Ax_0 - y\|^2$

$$Ax_0 = \begin{pmatrix} 1 & 1 \\ 3 & 1 \\ 5 & 1 \\ 7 & 1 \\ 9 & 1 \end{pmatrix} \begin{pmatrix} 1.25 \\ 0.55 \end{pmatrix} = \begin{pmatrix} 1.8 \\ 4.3 \\ 6.8 \\ 9.3 \\ 11.8 \end{pmatrix}$$

$$Ax_0 - y = \begin{pmatrix} 1.8 \\ 4.3 \\ 6.8 \\ 9.3 \\ 11.8 \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \\ 7 \\ 9 \\ 12 \end{pmatrix} = \begin{pmatrix} -0.2 \\ 0.3 \\ -0.2 \\ 0.3 \\ -0.2 \end{pmatrix}$$

$$\|Ax_0 - y\|^2 = (-0.2)^2 + (0.3)^2 + (-0.2)^2 + (0.3)^2 + (-0.2)^2$$

$$= 0.3$$

5) Let the vector space P_2 have the inner product $\langle p, q\rangle = \int_0^1 p(x)q(x)\,dx$. Apply Gram-Schmidt process to transform the basis $S = \{u_1, u_2, u_3\} = \{1, x, x^2\}$ into an orthonormal basis.

Let $u_1 = 1$, $u_2 = x$, $u_3 = x^2$

Let $w_1 = u_1 = 1$

$$\|w_1\|^2 = \langle w_1, w_1\rangle = \int_0^1 1.1\,dx = 1$$

$$\therefore \|w_1\| = 1$$

$$w_2 = u_2 - \frac{\langle u_2, w_1\rangle}{\|w_1\|^2} w_1$$

$$= x - \frac{\frac{1}{2}}{1}(1)$$

$$\boxed{w_2 = x - \frac{1}{2}}$$

$$\langle u_2, w_1\rangle = \langle x, 1\rangle = \int_0^1 x\,dx = \left(\frac{x^2}{2}\right)_0^1 = \frac{1}{2}$$

$$\|w_2\|^2 = \langle w_2, w_2\rangle = \int_0^1 \left(x - \frac{1}{2}\right)^2 dx \qquad x^2 \to \frac{x^3}{3}$$

$$= \left[\frac{\left(x-\frac{1}{2}\right)^3}{3}\right]_0^1$$

$$= \frac{1}{24} + \frac{1}{24}$$

$$= \frac{2}{24} = \frac{1}{12}$$

$$\|w_2\| = \frac{1}{\sqrt{12}} = \frac{1}{2\sqrt{3}}$$

$$w_3 = u_3 - \frac{\langle u_3, w_1\rangle}{\|w_1\|^2} w_1 - \frac{\langle u_3, w_2\rangle}{\|w_2\|^2} w_2$$

$$\langle u_3, w_1\rangle = \int_0^1 x^2\,dx = \left(\frac{x^3}{3}\right)_0^1 = \frac{1}{3}$$

$$\langle u_3, w_2\rangle = \int_0^1 x^2\left(x-\frac{1}{2}\right)dx$$

$$= \int_0^1 \left(x^3 - \frac{x^2}{2}\right)dx$$

$$= \left(\frac{x^4}{4} - \frac{x^3}{6}\right)_0^1 = \frac{1}{4} - \frac{1}{6} = \frac{1}{12}$$

$$w_3 = x^2 - \frac{\frac{1}{3}}{1}\,1 - \frac{\cancel{\frac{1}{12}}}{\cancel{\frac{1}{12}}}\left(x-\frac{1}{2}\right)$$

$$= x^2 - \frac{1}{3} - x + \frac{1}{2} \qquad \frac{-1}{3} + \frac{1}{2} = \frac{+1}{6}$$

$$w_3 = x^2 - x + \frac{1}{6}$$

$$\|w_3\|^2 = \langle w_3, w_3\rangle = \int_0^1 \left(x^2 - x + \frac{1}{6}\right)^2 dx.$$

$$= \int_0^1 \left[x^4 + x^2 + \frac{1}{36} + 2(x^2)(-x) + \cancel{2}(-x)\left(\frac{1}{\cancel{6}_3}\right) + \cancel{2}(x^2)\left(\frac{1}{\cancel{6}_3}\right)\right] dx.$$

$$= \int_0^1 \left(x^4 + x^2 + \frac{1}{36} - 2x^3 - \frac{x}{3} + \frac{x^2}{3}\right)dx.$$

$$= \left(\frac{x^5}{5} + \frac{x^3}{3} + \frac{1}{36}x - 2\frac{x^4}{4} - \frac{x^2}{6} + \frac{x^3}{9}\right)_0^1$$

$$= \frac{1}{5} + \frac{1}{3} + \frac{1}{36} - \frac{1}{2} - \frac{1}{6} + \frac{1}{9}$$

$$= \frac{36 + 60 + 5 - 90 - 30 + 20}{180}$$

$$\|w_3\|^2 = \frac{1}{180}.$$

$$\|w_3\| = \frac{1}{\sqrt{180}} = \frac{1}{6\sqrt{5}}$$

$\therefore$ The orthogonal set $\left\{1, \left(x-\frac{1}{2}\right), \left(x^2 - x + \frac{1}{6}\right)\right\}$

The orthonormal basis $\left\{\frac{1}{1}, 2\sqrt{3}\left(x-\frac{1}{2}\right), 6\sqrt{5}\left(x^2 - x + \frac{1}{6}\right)\right\}$.