Broadcasting With Side Information

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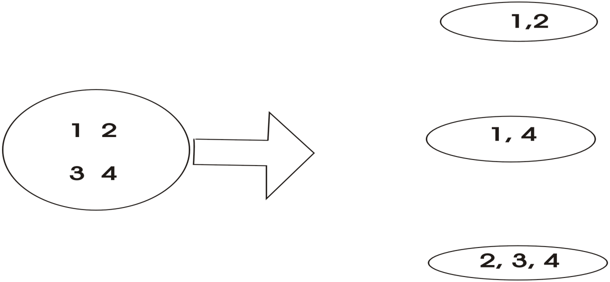
Broadcasting with Side Information

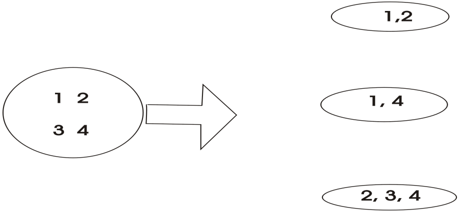
- Central Server
- Broadcast Audience with arbitrary side information



- Aim: Binary linear codes with minimal transmissions
  Example: Code $1+3$, $2+4$

Literature on Index Codes

- Proposed by [Birk and Kol, Trans IT 2006]
- Optimal linear code was characterized as a rank minimization problem by [Bar Yossef et al, FOCS 2006]



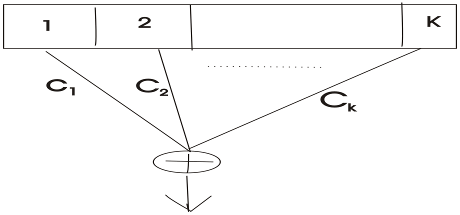
Optimal Code from minimizing the rank of:

$$\begin{bmatrix} x_1 & x_2 & 1 & 0 \\ x_3 & x_4 & 0 & 1 \\ x_5 & 1 & 0 & x_4 \\ x_6 & 0 & 1 & x_7 \\ 1 & x_8 & x_9 & x_{10} \end{bmatrix}$$

Some practical issues

- Intractability of rank minimization
- Collecting ARQ from a large audience
- Losses/Erasures
- Encoding/Decoding Complexity

Random Linear Coding



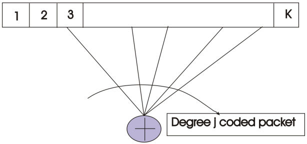
- Coefficients c_i chosen iid non zero with probability 0.5
- Decoding by matrix inversion
- Code remains uniformly random even after subtracting out side information
- Random matrices are close to full rank w.h.p. $\Rightarrow$ asymptotically optimal code

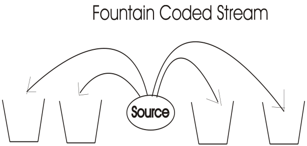
Random Linear Coding

- § Intractability of Computing the Code
- § Collecting ARQ from a large audience
- § Losses/Erasures
- § Encoding/Decoding Complexity

Matrix inversion is too expensive for large block lengths!

Fountain Codes: background





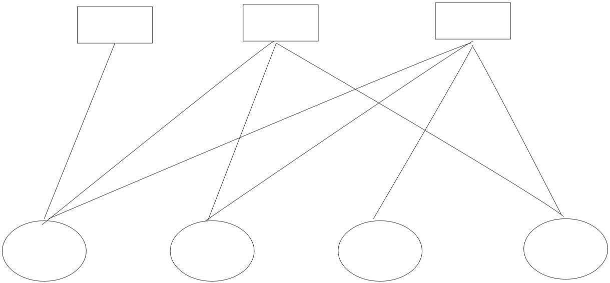
Encoding:

- Degree distribution - object of optimization
- Degree, d - sampled independently for each coded packet (expected value is complexity of coding)
- A uniform random subset of size d ex-or-ed to form coded packet

Decoding:

1. Collect $\phi(k)$ packets (code is good if $\phi(k) \sim k$)
2. Sift for a degree 1 packet and mark it as decoded
3. Subtract the decoded packet off from other coded packets and repeat.

Fountain Codes: Decoding



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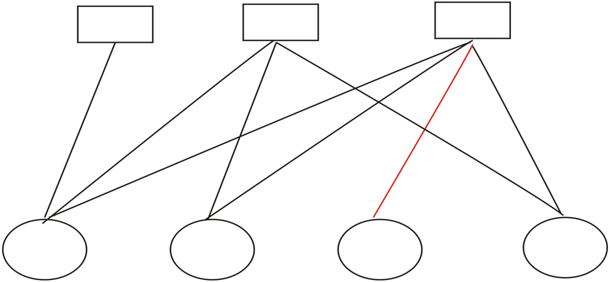

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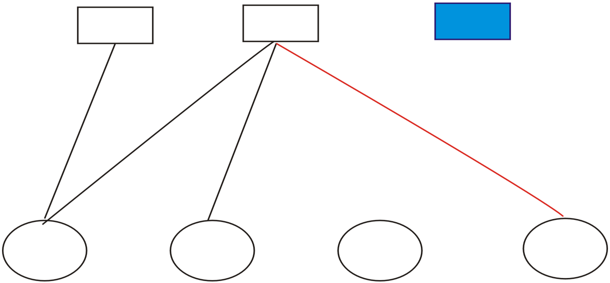

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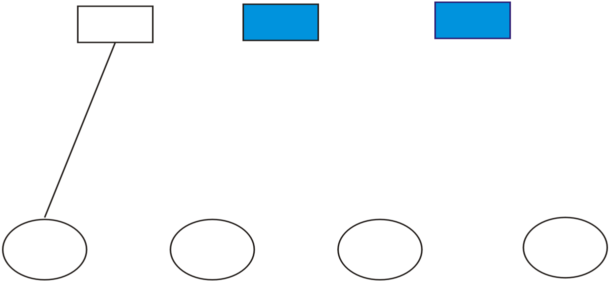

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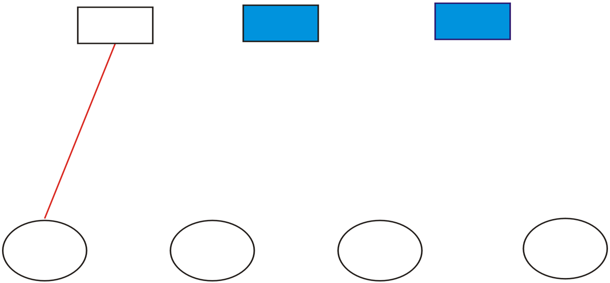

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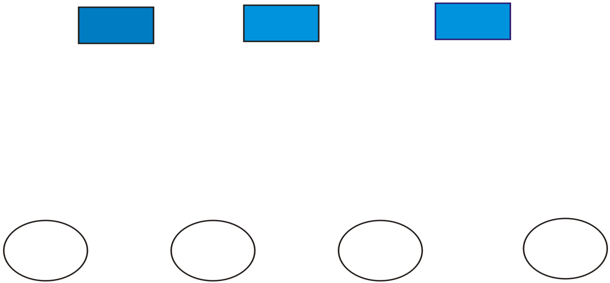

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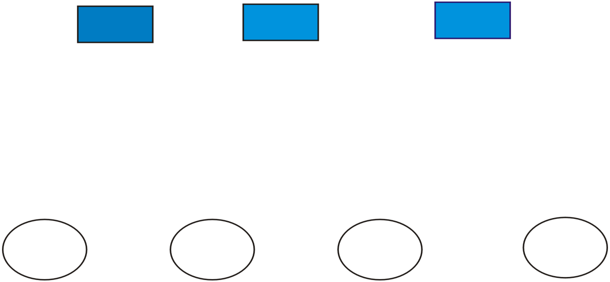
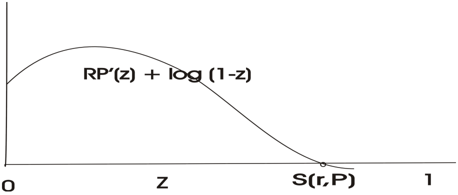


- [Luby, 2006]: There exists degree distribution (the “soliton”) for which decoding recovers all packets whp from asymptotically k coded packets
- Claim via analysis of the asymptotic evolution of degree one packets
- Encoding/Decoding complexities are logarithmic in blocklength (huge improvement over RLC)

Analysis of Iterative Decoding

For degree dist $P(z) = \sum_i p_i z^i$, fraction of recovered packets from rk coded packets given as:

$$s(r,P) = \inf\{z \in [0,1) : rP'(z) + \log(1-z) < 0\} \wedge 1$$



[Darling and Norris, Ann. of Appl. Prob 2006]

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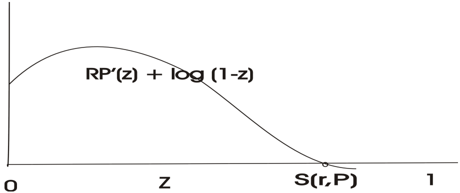

[Darling and Norris, Ann. of Appl. Prob 2006]

§ Example: Just send a random packet at each time. We have the coupon collector setting.

- $P(z) = z$. Then, $s(r, P) < 1$ for any constant r, which is true because its impossible to collect k distinct coupons using rk attempts for any constant r.

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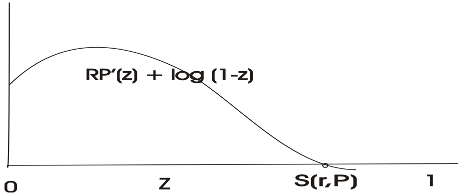


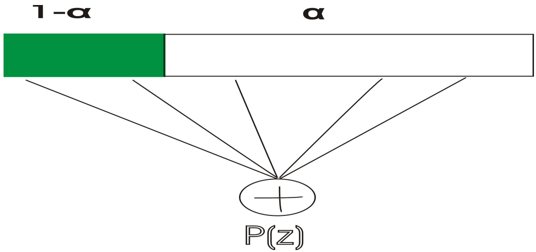
[Darling and Norris, Ann. of Appl. Prob 2006]

§ Example: Soliton distribution used in LT codes is defined by

$$P(z) = \sum_{i \geq 2} \frac{z^i}{i(i-1)}$$

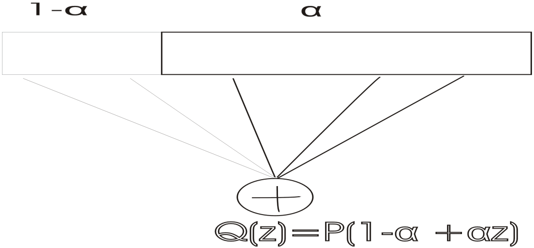
which implies $P'(z) = -\log(1-z)$ and $s(1,P) = 1$.

Decoding with Side Information



- Let $1 - \alpha$ be a fraction that denotes side information
- $P(z)$ is the distribution used for coding.
- $Q(z)$ is the limiting projected distribution after subtracting out the side information.

Decoding with Side Information



- Let $1-\alpha$ be a fraction that denotes side information
- $P(z)$ is the distribution used for coding.
- $Q(z)$ is the limiting projected distribution after subtracting out the side information.

Define:

$$s_\alpha(r,P) = \inf\{z \in [0,1) : r\alpha P'(1-\alpha+\alpha z) + \log(1-z) < 0\} \wedge 1$$

Analysis: Decoding with Side Information

Optimization problem of interest for complete recovery:

$$r_\alpha^* = \min_P r$$
$$\text{subject to : } s_\alpha(r, P) = 1$$

Trivial bounds: $1 \leq r_\alpha^* \leq \frac{1}{\alpha}$

1. *Can't communicate more than what was sent* $\Rightarrow$ need at least $r \geq \frac{\alpha k}{\alpha k}$
2. *Ignore side information* $\Rightarrow$ k packets sufficient, hence $r \leq \frac{k}{\alpha k}$.

Analysis: Decoding with Side Information

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Some Natural Questions:

- Can a soliton based distribution exploit side info?
- Are there capacity achieving distributions for general α?
- Is a constant approximation possible? (i.e. or is r_α^* uniformly bounded as $\alpha \to 0$?)

Analysis: Decoding with Side Information

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Some Natural Questions:

- Can a soliton based distribution exploit side info? *No, as good as throwing away all side information*
- Are there capacity achieving distributions for general α? *No, at least 10% inefficiency unavoidable*
- Is a good approximation possible? (i.e. is r_α^* uniformly bounded as $\alpha \to 0$?) *Yes, by 2*

Performance of LT-Soliton with Side Information

- How does soliton perform? $P(z) = \sum_{i \geq 2} \frac{1}{i(i-1)} z^i$
- Want to keep the following quantity non-negative as $z \rightarrow 1$ for as small an r as possible.

$$
\begin{aligned}
& rQ'(z) + \log(1-z) \\
= r\alpha \sum_{i \geq 1} \frac{(1-\alpha+\alpha z)^i}{i} & + \log(1-z) \\
= r\alpha |\log \alpha| & + (r\alpha - 1)|\log(1-z)|
\end{aligned}
$$

- Thus, $s_\alpha(r, P) = 1 \Rightarrow r > 1/\alpha$
  $\sim$ Soliton as good as no coding for complete recovery.
- Next task: edit P so as to achieve a smaller r.

Modified Soliton for Side Information

- For soliton, we had
  $rQ'(z) + \log(1-z) = r\alpha|\log\alpha| + (r\alpha - 1)|\log(1-z)|$

- Consider the k-shifted soliton: $P(z) = \sum_{i \geq k+1} \frac{k}{i(i-1)} z^i$

$$
\begin{aligned}
& rQ'(z) + \log(1-z) \\
& = r \sum_{i \geq k} \alpha k \frac{(1-\alpha+\alpha z)^i}{i} + \log(1-z) \\
& \geq \dots\dots\dots \\
& \geq r\alpha k \left(|\log\alpha| - \log(1.82k)\right) + (r\alpha k - 1)|\log(1-z)|
\end{aligned}
$$

- Now we only need $r > \frac{1}{\alpha k}$ as long as $k \leq \frac{1}{1.82\alpha}$.
- implies upper bound on r_α^* as $1/(\alpha\lfloor 1/1.82\alpha \rfloor)$

Lower Bounds on r_α^*

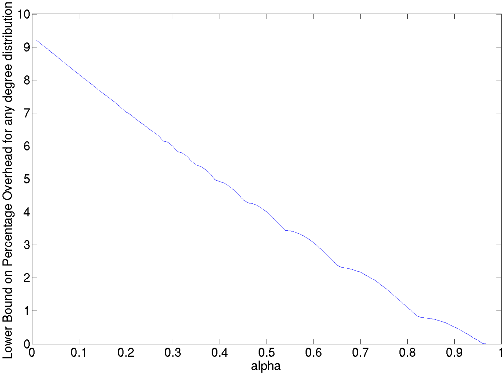
Do capacity achieving distributions exist for $0 < \alpha < 1$?

$$r_\alpha^* = \min_P r$$
$$\text{subject to : } s_\alpha(r, P) = 1$$

- Above optimization can be seen as an LP with infinite variables and constraints
- Can lower bound r_α^* by using feasible solutions to the dual
- Closely intermediate analysis of fountain codes [Sanghavi, ITW 2007]

Lower Bounds on r_α^*

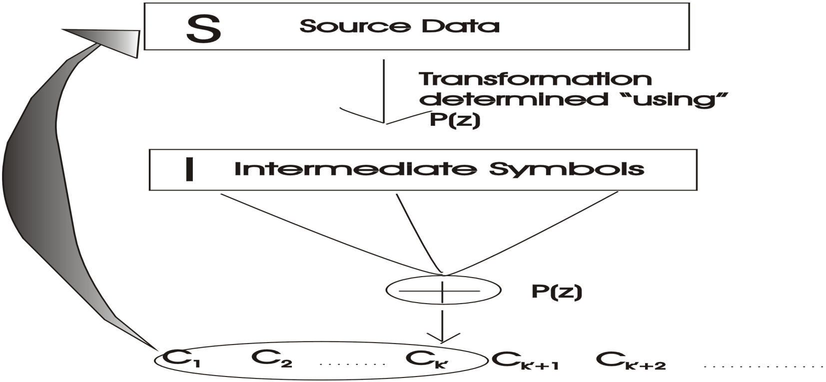
Lower bounds on r_α^* numerically computed via LP duality:



Do capacity achieving distributions exist for $0 < \alpha < 1$?

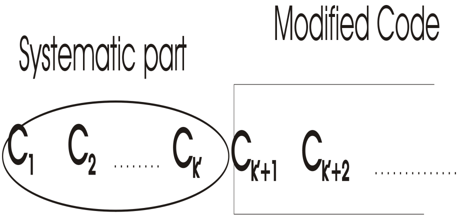
Not without upto a 10% inefficiency

Systematic Codes



- Apply degree distribution to I
- Transformation from S to I defined such that $C_1, \ldots, C_{k'}$ contain S (uncoded).

Systematic Codes



- Transmit code starting only from $c_{k'+1}$
- Think of side information as the non-erased symbols from the systematic part.
- Beats the lower bound for purely degree distribution based codes
- Encoding/Decoding complexity depends on the transformation steps between S and I
- Need to store additional data in the form of I at the encoder
- Don't need an estimate of α

Conclusion

- Systematic Versions necessary to achieve rate optimality in the presence of side information
- Possible to achieve 2-approximation of optimal rate via direct degree distribution coding if we have a good upper bound to α
- Soliton based coding can not exploit side information for complete decoding

Thanks!