Optimal Control of a Broadcasting Server

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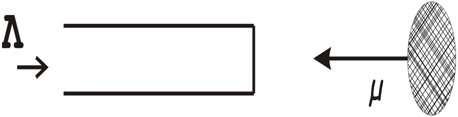
Outline of the Talk

- Introduction
  - § The Basic Broadcast Server Queueing Model
  - § Motivation for studying the Model
- Single Queue Problem
  - § Objective and the convex cost model with broadcast costs
  - § Main Result: Threshold property of the optimal control
  - § Proof Outline
- Two Queues
  - § Cost Model
  - § Switch Curve Property of the Optimal Control
- Conclusion and Further Work

The Basic Broadcast Queueing Model



- Continuous time system
- Poisson Arrivals of rate λ
- Exponential Server of rate μ
- Each service clears the entire queue

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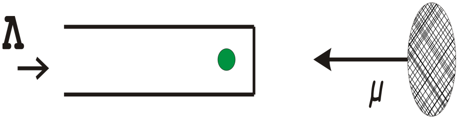

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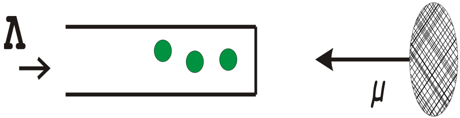

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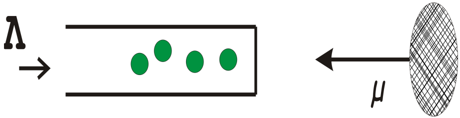

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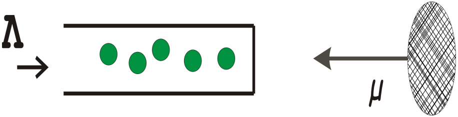

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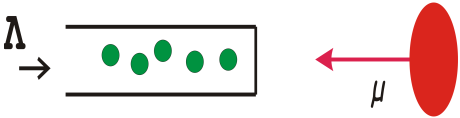

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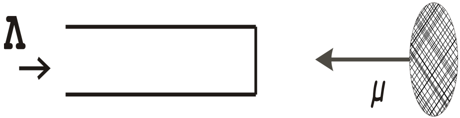

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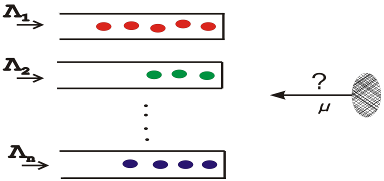
Motivation for studying this model

- § Broadcast Scheduling



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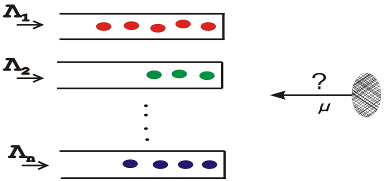

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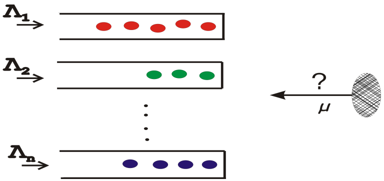
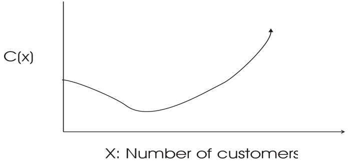


- Batch processing systems with large batch size

- High Interference Scheduling - WPAN
  - n mutually interfering links in close proximity
  - Each link rate without interference is very high
  - Schedule them so as to minimize cost given as some function of the queue sizes

Objective: Single Queue

- $c(x)$ is a cost rate for holding x customers in the system



- c_s is an additional cost per broadcast
- At state x we operate the server at rate $w(x)\mu$ for $0 \leq w(x) \leq 1$
- Describe the optimal control $w(x)$ to minimize:

$$E_x^w \int_0^\infty e^{-\alpha t} c(x_t)\, dt + \sum_{k=1}^{\infty} e^{-\alpha \tau_k} \mathbb{1}\{x_{k-1} \neq 0 \text{ and } x_k = 0\} c_s$$

Cost Models

- Previous work on batch service models shows that $w(x)$ is threshold type for monotone costs, $c(x)$.
  - Deb and Serfozo, *Adv. Appl. Prob '73*
  - Aalto, *Math. Methods of OR, '98, '00*

- Current Work: any convex $c(x)$.

- Practical motivation for convex cost on single queue:
  1. p2p system with strategic cost model abstraction
  2. Heuristics to decompose multiple queue systems to single queue.

Main Result: Single Queue

Theorem

A threshold policy is optimal for discounted infinite horizon cost for convex cost rate $c(x)$ and constant service cost

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- Equivalent to a discrete time problem for minimizing:

$$U^w(x) = E_x^w \sum_{k=0}^{\infty} \beta^k \left(c(x_k) + c_s \mathbb{1}\{x_{k-1} \neq 0, x_k = 0\} \right)$$

Single Queue

- Dynamic programming operator, $\mathcal{T}$ defined as:

$$\mathcal{T}f(x) = c(x) + \beta\{\lambda f(x+1) + \mu \min(f(x), f(0) + c_s)\}$$

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- Optimal Value function:

$$V(x) = \inf_u E_x^u \sum_{k=0}^{\infty} \beta^k (c(x_k) + c_s \mathbb{1}\{x_{k-1} \neq 0, x_k = 0\})$$

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- From Dynamic Programming argument, V satisfies:

$$V = \mathcal{T}V$$

Proof of Threshold Optimality

For a given control $w(x)$, the value function $(U^w(x) = E_x^w \sum_{k=0}^{\infty} c(x_k)\beta^k)$ satisfies a fixed point eqn for:

$$\mathcal{T}^w f(x) = c(x) + \beta(\lambda f(x+1) + \mu(w(x)(f(0)+c_s) + (1-w(x))f(x)))$$

Theorem

Let U_l be the value function for threshold l policy. If $U_l(l-1) \leq U_l(0) + c_s < U_l(l)$, then U_l is quasiconvex

Definition

A function f on Z_+ is quasiconvex (unimin) if $f(x+1) - f(x) \geq 0$ for all $x > y$ whenever $f(y+1) - f(y) > 0$.

Proof of Threshold Optimality

- Suppose we could find an I^* for which:
  1. U_{I^*} is quasiconvex
  2. $U_{I^*}(I^* - 1) \leq U_{I^*}(0) + c_s < U_{I^*}(I^*)$

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- This implies:

$$U_{I^*}(x) \begin{cases} \leq U_{I^*}(0) + c_s & \text{if } x \leq I^* - 1 \\ > U_{I^*}(0) + c_s & \text{if } x \geq I^* \end{cases}$$

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- Then, $f(x) = U_l^*(x)$ is a solution to the fixed point equation for optimal DP operator:

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§ But we only need to look for an l^* for which condition (2) holds since (2) $\Rightarrow$ (1), by theorem.

Proof of Threshold Optimality

Lemma

$l^* = \min\{l : U_l(l) > U_l(0) + c_s\}$ *satisfies (2)*

Proof: Suppose not. $U_{l^*}(l^* - 1) > U_{l^*}(0) + c_s$. Then:

Proof of Threshold Optimality

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$I^* = \min\{I : U_I(I) > U_I(0) + c_s\}$ *satisfies (2)*

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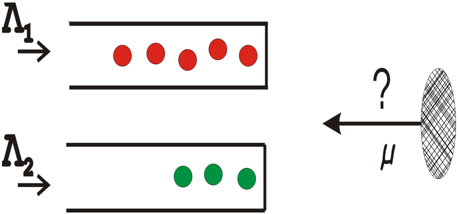
- which would be a contradiction, unless: $U_{I^*-1}(I^* - 1) > U_{I^*-1}(0) + c_s$
- ...which contradicts definition of I^*

Two queues



- Assume cost, $c(x_1, x_2)$ is monotone and has no service costs
- n step value function V_n is recursively given by:

$$V_{n+1}(x_1, x_2) = c(x_1, x_2)$$
$$\beta\{\lambda_1 V_n(x_1 + 1, x_2) + \lambda_2 V_n(x_1, x_2 + 1) +$$
$$\mu \min(V_n(x_1, 0), V_n(0, x_2), V_n(x_1, x_2))\}$$

Two queues: Switch Curve Optimality

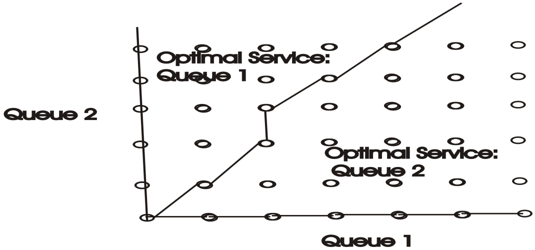
Lemma

V_n is increasing. i.e. if (x_1, x_2) and (y_1, y_2) are such that $x_1 \leq y_1$ and $x_2 \leq y_2$ then $V_n(x_1, x_2) \leq V_n(y_1, y_2)$

The optimal control u_n is:

$$u_n(x_1, x_2) = \begin{cases} 1 & \text{, if } V_n(x_1, 0) \leq V_n(0, x_2) \\ 2 & \text{, otherwise.} \end{cases}$$

Two queues: Switch Curve Optimality



Theorem

The optimal control with n steps to go is given by a switch curve:

$$u_n(x_1, x_2) = \begin{cases} 1 & , \textit{ if } x_2 \geq s_n(x_1) \\ 2 & , \textit{ otherwise.} \end{cases}$$

where

$$s_n(x) = \min\{y : V_n(x, 0) \leq V_n(0, y)\}$$

Further Work: The general problem for $n > 2$ queues

- An index rule is given by n functions $\psi_1, \ldots, \psi_n$ such that the control is given as:

$$u(x_1, \ldots, x_n) = \arg max_{i \in [n]} \{\psi_i(x_i)\}$$

- Can the optimal control be described by index rules?

- Approximate algorithms using index policies
  - Longest queue scheduling corresponds to $\psi_i(x) = x$
  - LWF scheduling, which has been found to be 'competitive' in CS literature corresponds to using an index rule where: $\psi_i(x) = \frac{x}{\sqrt{\lambda_i}}$

Thank you!