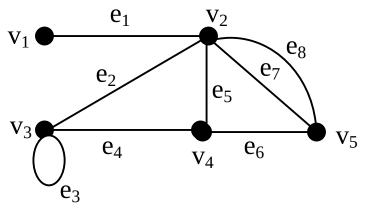
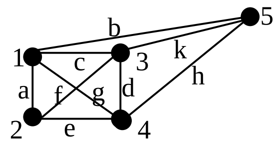
1.1. **What is a graph?**

1.1.2. **Definition**. A **graph** G is a triple (V(G), E(G), ψ_G) consisting of V(G) of **vertices**, a set E(G), disjoint from V(G), of **edges**, and an **incidence** function ψ_G that associates with each edge of G an unordered pair of (not necessarily distinct) vertices of G.
If e is an edge and u and v are vertices such that $\psi_G(e) = \{u, v\}$(or is simply denoted by uv), then e is said to **join** u and v; the vertices u and v are called **endpoints** of e.

Example. $G = (V(G), E(G), \psi_G)$ where $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$, $E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$ and ψ_G is defined by $\psi_G(e_1) = v_1v_2$, $\psi_G(e_2) = v_2v_3$, $\psi_G(e_3) = v_3v_3$, $\psi_G(e_4) = v_3v_4$, $\psi_G(e_5) = v_2v_4$, $\psi_G(e_6) = v_4v_5$, $\psi_G(e_7) = v_2v_5$, $\psi_G(e_8) = v_2v_5$.
$H = (V(H), E(H), \psi_H)$ where $V(H) = \{1, 2, 3, 4, 5\}$, $E(H) = \{a, b, c, d, e, f, g, h, k\}$ and ψ_H is defined by $\psi_H(a) = 12$, $\psi_H(b) = 15$, $\psi_H(c) = 13$, $\psi_H(d) = 34$, $\psi_H(e) = 24$, $\psi_H(f) = 23$, $\psi_H(g) = 14$, $\psi_H(h) = 45$, $\psi_H(k) = 35$. #

Graphs are so named because they can be represented graphically, and it is this graphical representation which helps us understand many of their properties. Each vertex is indicated by a point, and each edge by a line joining the points which represent its ends. In such a drawing it is understood that no line intersects itself or passes through a point representing a vertex which is not an end of the corresponding edge, that is clearly always possible. The diagram itself is then referred to as a graph. Diagram of G and H are shown as follows:

G:


H:


1.1.4. **Definition**. A **loop** is an edge whose endpoints are equal.
Multiple edges are edges having the same pair of endpoints.
In the above diagram, e_3 is a loop, e_7 and e_8 are multiple edges.

A **simple graph** is a graph having no loops or multiple edges, i.e. a simple graph G consists of a **vertex set** V(G), an **edge set** E(G) where E(G) is a set of unordered pairs of vertices or a set of 2-elements subsets of V(G). In the above diagram, H is a simple graph.

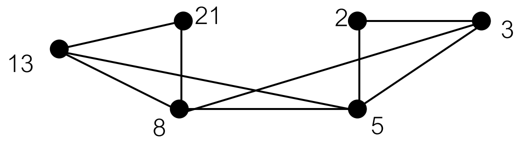
When u and v are endpoints of an edge, they are **adjacent** and are **neighbors**. In the above diagram, v_3 and v_4 are adjacent in G, v_1 and v_3 are not adjacent in G, 1 and 5 are neighbors in H.

A graph is **finite** if its vertex set and edge set are finite. We call a graph with just one vertex **trivial** and all other graphs **nontrivial**.

1.1.6. **Remark**. The **null graph** is the graph whose vertex set and edge set are empty.

We emphasize finite simple graphs with a nonempty set of vertices.

Example. Consider the set S = {2, 3, 5, 8, 13, 21}. There are some pairs of distinct integers belonging to S whose sum or difference(in absolute value) also belongs to S, namely, {2, 3}, {2, 5}, {3, 5}, {3, 8}, {5, 8}, {8, 13}, {8, 21}, and {13, 21}.
There is a more visual way of identifying these pairs, namely, by the graph G of the following figure. In this case, V(G) = {2, 3, 5, 8, 13, 21}and E(G) = {{2, 3}, {2, 5}, {3, 5}, {3, 8}, {5, 8},

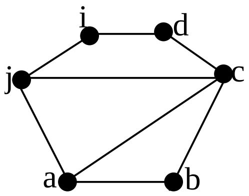


{8, 13}, {8, 21}, {13, 21}}. #

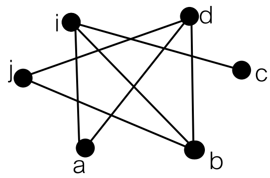
1.1.8. **Definition**. The **complement** $\overline{G}$ of a simple graph G is the simple graph with vertex set V(G) defined by uv $\in$ E($\overline{G}$) if and only if uv $\notin$ E(G).
A **clique** in a graph is a set of pairwise adjacent vertices.
An **independent set** in a graph is a set of pairwise nonadjacent vertices.

Example

G:



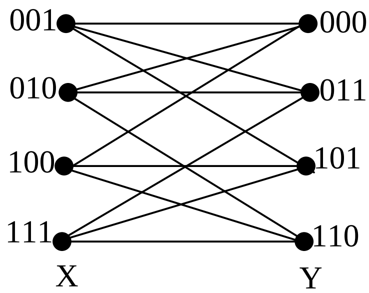
$\overline{G}$:



{a, b, c}is a clique in G, {a, i}is an independent set in G.
{a, b, c}is an independent set in $\overline{G}$, {i, d, b} is a clique in $\overline{G}$. #

1.1.10. **Definition**. A graph G is **bipartite** if V(G) is the union of two disjoint (possibly empty) independent sets called **partite sets** of G.
A graph G is **k-partite** if V(G) is the union of k (possibly empty) independent sets.

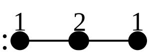
Exercise 1.1.13. Let G be the graph whose vertex set is the set of k-tuples with coordinates in {0,1}, with x adjacent to y when x and y differ in exactly one position.
Determine whether G is bipartite.
Solution. For example, k = 3, V(G) = {000, 001, 010, 011, 100, 101, 110, 111},
E(G) = {{000, 001}, {000, 010}, {000, 100}, {001, 011}, {001, 101}, {010, 011}, {010, 110}, {011, 111}, {100, 101}, {100, 110}{101, 111}, {110, 111}}.
Let X = {001, 010, 100, 111}, Y = {000, 011, 101, 110} be partite sets of G.



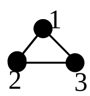
Then, adjacent vertices differ in exactly one position, no edges in X or Y, and G is a bipartite graph.
In general, let X be the set of k-tuples with odd numbers of 1's and
Y be be the set of k-tuples with even numbers of 1's.
Then, adjacent vertices have opposite parity, no edges in X or Y and G is a bipartite graph. #

1.1.12. **Definition**. The **chromatic number** of a graph G, written $\chi(G)$, is the minimum number of colors needed to label the vertices so that adjacent vertices receive different colors.

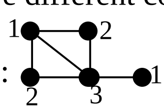
Example. G :



H :

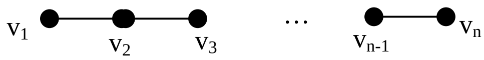


K :

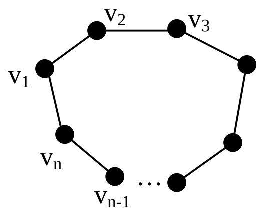


$\chi(G) = 2$, $\chi(H) = 3$, $\chi(K) = 3$. #

1.1.15. **Definition**. A **path** is a simple graph whose vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the list.
More formally, a **path** P_n (a path of n vertices) is a simple graph G with
$V(G) = \{v_1, v_2,\ldots,v_n\}$ and $E(G) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n\}$.

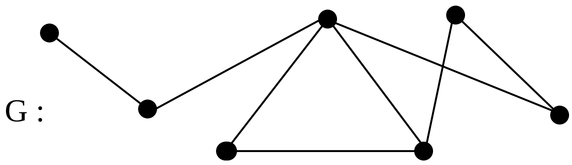


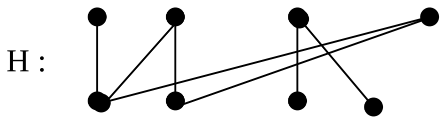
A **cycle** is a graph with an equal number of vertices and edges whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consecutively along the circle.
More formally, a **cycle** C_n (a cycle of n vertices) is a simple graph G with
$V(G) = \{v_1, v_2,\ldots,v_n\}$ and $E(G) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\}$.



1.1.16. **Definition**. A **subgraph** of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ and the assignment of endpoint to edges in H is the same as in G.
We write $H \subseteq G$ and say that "G contains H".
A path in a graph G is a subgraph of G that is a path.
A graph G is **connected** if each pair of vertices in G belongs to a path;
otherwise, G is **disconnected**.

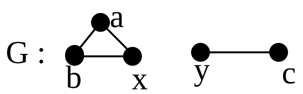
Example.

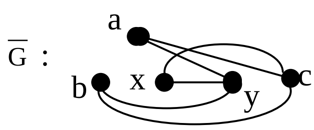




G is connected, while H is disconnected. #

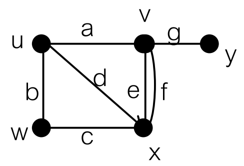
Exercise1.1.10. Prove or disprove: The complement of a simple disconnected graph G must be connected.
Proof. Since G is disconnected, there exist 2 vertices x, y that do not belong to a path.





Thus, $xy \in E(\overline{G})$. Also x and y have no common neighbor in G, otherwise, that would yield a path connecting them. Every vertex not in {x, y} is adjacent in $\overline{G}$ to at least one of {x, y}. Hence every vertex can reach every other vertex in $\overline{G}$ using paths through {x, y}. #

1.1.17. **Definition**. Let G be a loopless graph with vertex set $V(G) = \{v_1, v_2,\ldots,v_n\}$and edge set $E(G) = \{e_1, e_2,\ldots,e_m\}$.
The **adjacency matrix** of G, written A(G), is the n-by-n matrix in which entry a_{ij} is the number of edges in G with endpoints $\{v_i, v_j\}$.
The **incidence matrix** of G, written M(G), is the n-by-m matrix in which entry m_{ij} is 1 if v_i is an endpoint of e_j and otherwise is 0.
If vertex v is an endpoint of edge e, then v and e are incident.
The **degree** of vertex v(in a loopless graph), written d(v) is the number of incident edges

Example. 

$$A(G) = \begin{array}{c} \\ u \\ v \\ w \\ x \\ y \end{array}\!\!\begin{array}{c} \begin{array}{ccccc} u & v & w & x & y \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 2 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}, \quad M(G) = \begin{array}{c} \\ u \\ v \\ w \\ x \\ y \end{array}\!\!\begin{array}{c} \begin{array}{ccccccc} a & b & c & d & e & f & g \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$d(u) = 3$, $d(v) = 4$, $d(w) = 2$, $d(x)$, $d(y) = 1$. #

Exercise1.1.5. Prove or disprove: If every vertex of a simple graph G has degree 2, then G is a cycle
Disproof: Such a graph G can be a disconnected graph with each component a cycle. #

1.1.20. **Definition**. An **isomorphism** from a simple graph G to a simple graph H is a bijection $f : V(G) \to V(H)$ such that $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$.
We say "G is **isomorphic** to H", denoted by $G \cong H$, if there is an isomorphism from G to H.

Remark. $G \cong H \leftrightarrow \overline{G} \cong \overline{H}$.
Proof. ($\to$) Assume that $G \cong H$. Let f be an isomorphism from V(G) to V(H). Then every two adjacent vertices of G are mapped to adjacent vertices of H, also every two nonadjacent vertices of G are mapped to nonadjacent vertices of H.
Since $V(\overline{G}) = V(G)$ and $V(\overline{H}) = V(H)$, the same function $f : V(\overline{G}) \to V(\overline{H})$ also maps adjacent vertices of $\overline{G}$ to adjacent vertices of $\overline{H}$ and nonadjacent vertices of $\overline{G}$ to nonadjacent vertices of $\overline{H}$.
Therefore, $\overline{G} \cong \overline{H}$. #

Let $\mathcal{G}$ be any set of simple graphs. $\cong = \{(G, H) \in \mathcal{G} \times \mathcal{G}$: G is isomorphic to H$\}$ is a relation on $\mathcal{G}$.

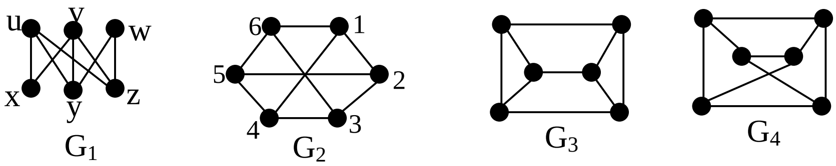
1.1.24. **Proposition**. The isomorphism relation ($\cong$) is an equivalence relation on $\mathcal{G}$.
Proof: *Reflexive property*. The identity permutation on V(G) is an isomorphism from G to itself. Thus $G \cong G$.
Symmetric property. If $f : V(G) \to V(H)$ is an isomorphism from G to H, then f^{-1} is an isomorphism from H to G, because "$uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$" yields $xy \in E(H)$ if and only if $f^{-1}(x)f^{-1}(y) \in E(G)$. Thus $G \cong H$ implies $H \cong G$.
Transitive property. Suppose that $f : V(F) \to V(G)$ is an isomorphism from F to G and $g : V(G) \to V(H)$ is an isomorphism from G to H.
We are given "$uv \in E(F)$ if and only if $f(u)f(v) \in E(G)$" and "$xy \in E(G)$ if and only if $g(x)g(y) \in E(H)$".
Since f is an isomorphism, for every $xy \in E(G)$ we can find $uv \in E(F)$ such that $f(u) = x$ and $f(v) = y$. This yields $uv \in E(F)$ if and only if $g(f(u))g(f(v)) \in E(H)$.
Thus the composition $g \circ f$ is an isomorphism from F to H.
We have prove that $F \cong G$ and $G \cong H$ together imply $F \cong H$. #

1.1.25. **Definition**. An **isomorphic class** of graphs is an equivalence class of graphs under the isomorphism relation.

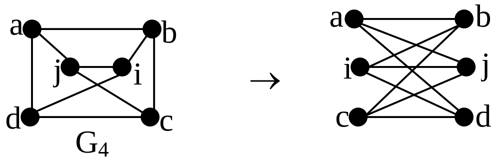
1.1.27. **Definition**. A **complete graph** is a simple graph whose vertices are pairwise adjacent; the unlabeled complete graph with n vertices is denoted K_n..
A **complete bipartite graph**(**biclique**) is a simple bipartite graph such that two vertices are adjacent if and only if they are in different partite sets.
When the sets have sizes r and s, the unlabeled complete bipartite graph is denoted $K_{r,s}$.
So the complete bipartite graph $K_{m,n}$ is a complete graph if and only if $m = n = 1$, i.e. $K_{1,1} \cong K_2$.

1.1.29. **Remark**. When we name a graph without naming its vertices, we often mean its isomorphic class. H is a subgraph of G means that some subgraph of G is isomorphic to H and we say G contains a **copy** of H.

1.1.30. **Example**.

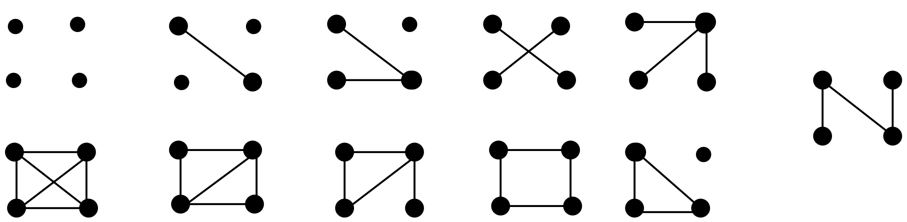


Each graph has 6 vertices and 9 edges and is connected, but these graphs are not pairwise isomorphic.
To prove that $G_1 \cong G_2$, let $f : V(G_1) \to V(G_2)$ defined by $f(u) = 1$, $f(v) = 3$, $f(w) = 5$, $f(x) = 2$, $f(y) = 4$, $f(z) = 6$.
Both G_1 and G_2 are bipartite. they are drawings of $K_{3,3}$ as is G_4.



The graph G_3 contains K_3, so its vertices cannot be partitioned into 2 independent sets.
Thus G_3 is not isomorphic to the others.
Sometimes we can test isomorphism quickly using the complements.
Simple graphs G and H are isomorphic if and only if their complements are isomorphic.
Hence $\overline{G_1}$, $\overline{G_2}$, $\overline{G_4}$ all consist of 2 disjoint 3-cycles and are not connected,
but $\overline{G_3}$ is a 6-cycle and is connected. #

1.1.31. **Example**. When choosing 2 vertices from a set of size n, we can pick one and then the other but don't care about the order, the number of ways is $\binom{n}{2}$.
In a simple graph with n vertices, each vertex pair may form an edge or may not. Making the choice for each pair specifies the graph, so the number of n-vertex simple graphs is $2^{\binom{n}{2}}$.
For example, there are 64 simple graphs on a fixed set of 4 vertices. These graphs form only 11 isomorphism classes.



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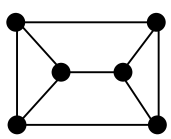
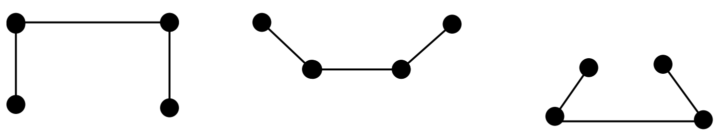
1.1.32. **Definition**. A graph is **self-complementary** if it is isomorphic to its complement.
A **decomposition** of a graph is a list of subgraphs such that each edge appears in exactly one subgraph in the list.

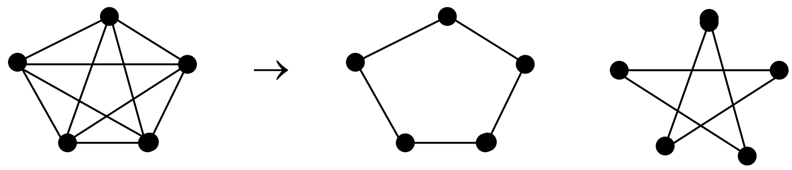
Exercise1.1.6. Determine whether the graph below decomposes into copies of P_4.



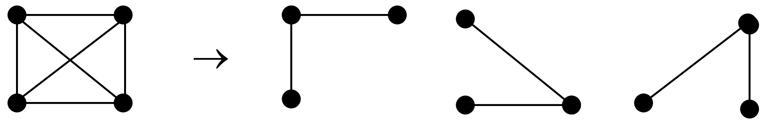


Solution. This graph decomposes into 3 copies of P_4 as shown on the right. #

1.1.33. **Example**. We can decompose K_5 into 5-cycles, and thus the 5-cycle is self-complementary.

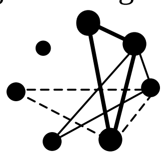


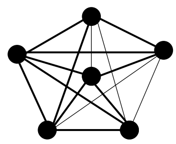
Any n-vertex graph and its complement decompose K_n.
Also $K_{1,n-1}$ and K_{n-1} decompose K_n, even though one of these subgraphs omits a vertex.
Below we show a decomposition of K_4 using 3 copies of P_3.



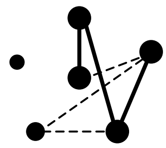
#

1.1.34. **Example**. The question of which complete graphs decompose into copies of K_3 is a fundamental question in the theory of combinatorial designs.
On the left below we suggest a decomposition of K_7 into copies of K_3.
Rotating the triangle through 7 positions uses each edge exactly once.





K_6

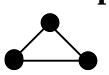


On the right we suggest a decomposition of K_6 into copies of P_4.
Placing one vertex in the center groups the edges into 3 types: the outer 5-cycle, the inner(crossing) 5-cycle on those vertices, and the edges involving the central vertex.
Each 4-vertex path in the decomposition uses one edge of each type; we rotate the picture to get the next path. #

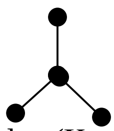
Exercise1.1.7. Prove that a graph with more than 6 vertices of odd degree can not be decomposed into 3 paths.
Proof. Since every vertex of odd degree must be the endpoint of some path in
a decomposition into paths and 3 paths need only 6 endpoints. #

Exercise1.1.36. Prove that if K_n decomposes into triangles, then 6|(n-1) or 6|(n-3).
Proof. A decomposition of K_n into triangles requires the degree of each vertex is even and the number of edges is divisible by 3. To have even degree, n must be odd.
Also $|E(K_n)| = \binom{n}{2} = \frac{n(n-1)}{2}$ is a multiple of 3, so 3|n or 3|(n-1).
If 3|n and n is odd, then 6|(n-3). If 3|(n-1) and n is odd, then 6|(n-1). #

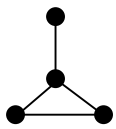
1.1.35. **Example**. The Graph Menagerie.



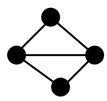
triangle(K_3)



claw($K_{1,3}$)

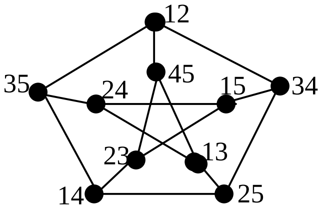


paw($K_{1,3}$ + e)



kite(K_4 – e) #

1.1.36. **Definition**. The **Petersen graph** is the simple graph whose vertices are the 2-element subsets of a 5-element set and whose edges are the pairs of disjoint 2-element subsets.



1.1.37. **Example**. Structure of the Petersen graph.
Using [5] = {1, 2, 3, 4, 5}as our 5-element set, we write the pair {a, b}as ab or ba.
Since 12 and 34 are disjoint, they are adjacent vertices when we form the graph, but 12 and 23 are not. For 2-set ab, there are 3 ways to pick a 2-set from the remaining 3 elements of [5], so every vertex has degree 3.
The Petersen graph consists of 2 disjoint 5-cycles plus edges that pair up vertices on the two 5-cycles.
The disjointness definition tells us that 12, 34, 25, 14, 35 in order are the vertices of a 5-cycle, and similarly this holds for the remaining vertices 13, 24, 15, 23, 45.
Also 24 is adjacent to 35, and 15 is adjacent to 34, and so on. #

1.1.38. **Proposition**. If 2 vertices are nonadjacent in the Petersen graph, then they have exactly 1 common neighbor.
Proof: Nonadjacent vertices are 2-sets sharing 1 element; their union S has size 3. A vertex adjacent to both is a 2-set disjoint from both. Since the 2-sets are chosen from {1, 2, 3, 4, 5}, there is exactly one 2-set disjoint from S. #

1.1.39. **Definition**. The **girth** of a graph with a cycle is the length of its shortest cycle.
A graph with no cycle has infinite girth.

1.1.40. **Corollary**. The Petersen graph has girth 5.
Proof: The graph is simple, so it has no 1-cycle or 2-cycle.
A 3-cycle would require 3 pairwise-disjoint 2-sets, which can't occur among 5 elements.
A 4-cycle in the absence of 3-cycles would require nonadjacent vertices with 2 common neighbors, which Proposition 1.1.38 forbids.
The vertices 12, 34, 25, 14, 35 yields a 5-cycle, so the girth of the Petersen graph is 5. #

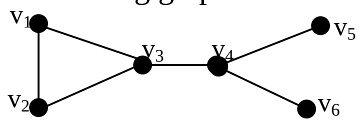
Exercise1.1.26. Let G be a graph with girth 4 in which every vertex has degree k.
Prove that G has at least 2k vertices. Determine all such graphs with exactly 2k vertices.
Proof. Since G has girth 4, thus G is simple and there are at least 4 edges in G, choose $xy \in E(G)$ then x, y has no common neighbors(why?). Thus, the neighborhoods N(x) and N(y) are disjoint sets of size k, G must have at least 2 k vertices.
$K_{k,k}$ is a k-regular graph with girth 4 and has exactly 2k vertices (why?) #

1.1.41. **Definition**. An **automorphism** of a graph G is an isomorphism from G to G. A graph G is **vertex transitive** if for every pair u, v $\in V(G)$ there is an automorphism that maps u to v.

1.1.42. **Example**. Let G be the P_4 with vertex set {1, 2, 3, 4} and edge set {12, 23, 34}.
This graph has 2 automorphisms α_1, α_2 as follows:

$\alpha_1 : V(G) \to V(G)$ defined by $\alpha_1(v) = v$ for every vertex v of G.
$\alpha_2 : V(G) \to V(G)$ defined by $\alpha_2(1) = 4$, $\alpha_2(2) = 3$, $\alpha_2(3) = 2$, $\alpha_2(4) = 1$.
The function $\alpha_3 : V(G) \to V(G)$ defined by $\alpha_3(1) = 2$, $\alpha_2(2) = 1$, $\alpha_2(3) = 3$, $\alpha_2(4) = 4$
is not an automorphism of G, although G is isomorphic to the graph with vertex set {1,2,3,4} and edge set {21,13,34}. #

Example. Consider the following graph G :



There are 4 automorphisms α_1, α_2, α_3, α_4 of G as follows:

$\alpha_1 : V(G) \to V(G)$ defined by $\alpha_1(v) = v$ for every vertex v of G.

$$\alpha_2 : V(G) \to V(G) \text{ defined by } \alpha_2(v) = \begin{cases} v_2 & \text{if } v = v_1 \\ v_1 & \text{if } v = v_2 \\ v & \text{if } v \neq v_1, v_2 \end{cases}$$

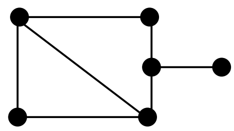
$$\alpha_3 : V(G) \to V(G) \text{ defined by } \alpha_3(v) = \begin{cases} v_6 & \text{if } v = v_5 \\ v_5 & \text{if } v = v_6 \\ v & \text{if } v \neq v_5, v_6 \end{cases}$$

$$\alpha_4 : V(G) \to V(G) \text{ defined by } \alpha_4(v) = \begin{cases} v_2 & \text{if } v = v_1 \\ v_1 & \text{if } v = v_2 \\ v_6 & \text{if } v = v_5 \\ v_5 & \text{if } v = v_6 \\ v & \text{if } v = v_3, v_4 \end{cases} .$$ #

Remark. Since composition of functions is associative, the identity function is an automorphism, the inverse of an automorphism is an automorphism., and the composition of 2 automorphisms is an automorphism, it follows that the set of all automorphisms of a graph G from a group under the operation of composition.
This group is denoted by Aut(G) and is called the automorphism group of G.
For the graph G above, Aut(G) = {α_1, α_2, α_3, α_4}. #

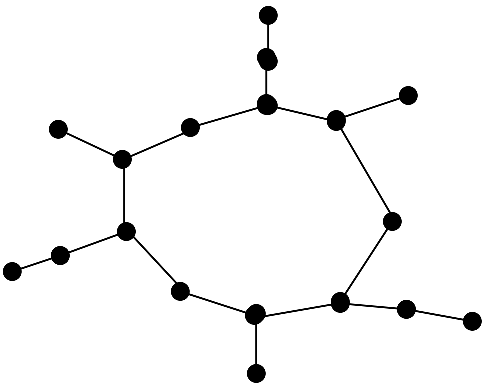
Exercise 1.1.40. Count the automorphisms of P_n, C_n, and K_n.
The number of automorphisms of P_n is 2 since P_n can be left alone or flipped.
The number of automorphisms of C_n is 2n since C_n can be rotated or flipped.
The number of automorphisms of K_n is n! since K_n can be permuted arbitrarily. #

Exercise 1.1.41. Construct a simple graph with 6 vertices that has only one automorphism.



Verify!

Construct a simple graph that has only 3 automorphisms.



Verify! #

Homework 1. 1.1.25, 1.1.34, 1.1.35, 1.1.38, 1.1.41 due on June 18.