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Module 6 Hamilton Graphs

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6.1 Introduction

Much research on Hamilton graphs is driven by a real world optimization problem called the ***traveling salesman problem***. It has resemblance to Chinese Postman Problem, but is known to be a hard open problem.

Traveling Salesman Problem (TSP)

Given n cities and distance between any two cities, a traveling salesman wishes to start from one of these cities, visits each city exactly once and comes back to the starting point. Design an algorithm to find a shortest route.

It can be modeled as a problem in several branches of mathematics. Its graph theoretical model is easy to state.

Graph theoretical model of TSP

Given a weighted graph G on n vertices, design an algorithm to find a cycle with minimum weight containing all the vertices of G.

Its solution carries a one million dollar reward by Clay Institute. The unweighted version of the above problem is equally difficult. It is well-known as Hamilton problem in honor of Sir William Rowan Hamilton who used spanning cycles of graphs to construct non-commutative algebras (1859).

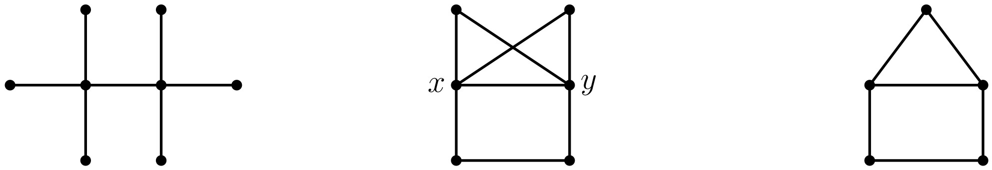
Definitions.

(1) A path (cycle) in a graph containing all its vertices is called a ***Hamilton path*** *(respectively* ***Hamilton cycle****).*

(2) A graph is called a ***Hamilton graph*** *if it contains a Hamilton cycle.*

Remarks.

- Every Hamilton graph is 2-connected (follows by Theorem 2.17).



(a) A graph with no Hamilton path

(b) A graph with a Hamilton path but no Hamilton cycle

(c) A Hamilton graph

Figure 6.1: Examples of Hamilton and non-Hamilton graphs.

- ○ A graph is Hamilton iff its underlying simple graph is Hamilton. **Hence, the study of Hamilton graphs is limited to simple graphs.**
- ○ A graph may contain a Hamilton path but may not contain a Hamilton cycle; see Figure 6.1.

Hamilton problems

1. Find necessary and sufficient conditions for a graph to be Hamilton.
2. Design a polynomial-time algorithm to generate a Hamilton cycle in a given graph G or declare that G has no Hamilton cycle.

In the last chapter, we have seen that the above two problems have good solutions with respect to Eulerian graphs. Despite numerous attempts and impressive contributions by various mathematicians and computer scientists, the Hamilton problems have remained open.

6.2 Necessary conditions and sufficient conditions

As remarked earlier all our graphs are simple.

Theorem 6.1 (Necessary condition). *If G is Hamilton, then*

$$c(G-S) \leq |S|, \text{ for every } S \subseteq V(G),$$

where $c(G-S)$ denotes the number of components in $G-S$.

Proof. (Two way counting technique) Let C be a Hamilton cycle in G and let S be a subset of $V(G)$ with s vertices. Consider the set of edges $F \subseteq E(C)$ with one end in S and another end in $G-S$. Every vertex in G is incident with two edges of C.

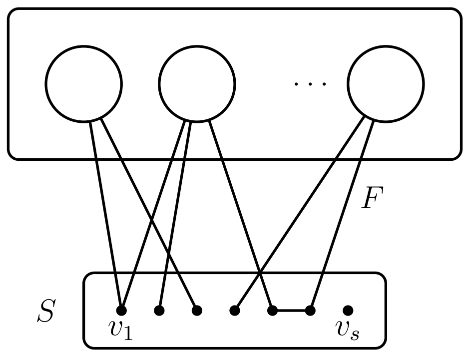


Figure 6.2: Two way counting of $|F|$.

Therefore, every vertex in S is incident with at most two edges of F. Hence,
(1) $|F| \leq 2s = 2|S|$
On the other hand, every component of $G-S$ is incident with at least two edges of F. Hence
(2) $|F| \geq 2c(G-S)$
The inequalities (1) and (2) imply that $c(G-S) \leq |S|$. $\square$

- Using this necessary condition it is easy to show that the graph shown in Figure 6.1b is non-Hamilton; $G - \{x, y\}$ contains 3 components.

Sufficient conditions for the existence of a Hamilton cycle are based on a common intuition that a graph is likely to contain a Hamilton cycle if it contains large number of edges uniformly distributed among the vertices. Many sufficient conditions have been proved by making mathematically precise the term "the large number of edges uniformly distributed among the vertices." We state and prove three such theorems.

Theorem 6.2 (Ore, 1962)**.** *If a graph G on $n(\geq 3)$ is such that*

$$deg(u) + deg(v) \geq n, \text{ for every pair of non-adjacent vertices } u \text{ and } v \tag{Ore}$$

then G is Hamilton.

Proof. (Nash-Williams, 1966). Assume the contrary that G is non-Hamilton though it satisfies (Ore). Let H be a graph obtained from G by successively joining pairs of non-adjacent vertices until addition of any more new edge creates a Hamilton cycle. Then G is a spanning subgraph of H and so $deg_H(v) \geq deg_G(v)$ for every vertex v. Hence, H too satisfies (Ore). Since H is non-Hamilton, there exist two non-adjacent vertices, say u and v. By the construction of H, it follows that $H + (u, v)$ is Hamilton. Hence, there is a Hamilton path $P = (u_1, u_2, \ldots, u_{n-1}, u_n)$ in H connecting u and v, where $u_1 = u$ and $u_n = v$. If u_1 is adjacent to u_k, then u_n is non-adjacent to u_{k-1}.

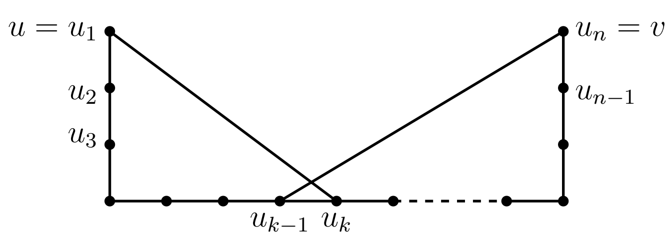


Figure 6.3: Hamilton path P and construction of a Hamilton cycle.

Else,

$$(u_1, u_2, \ldots, u_{k-1}, u_n, u_{n-1}, \ldots, u_k, u_1)$$

is a Hamilton cycle (see Figure 6.3) in H, a contradiction to non-Hamiltonicity of H. Therefore, there are (at least) deg(u) vertices of H which are non-adjacent to v. Hence, $deg(v) \leq (n-1) - deg(u)$, a contradiction to (Ore).

□

Next result due to Dirac (1959) in fact preceded the above theorem.

Corollary (Dirac, 1959)**.** *If a graph G on $n(\geq 3)$ vertices is such that*

$$deg(v) \geq \frac{n}{2}, \text{ for every vertex } v \text{ in } G \qquad \text{(Dirac)}$$

then G is Hamilton.

Proof. An easy consequence of Theorem 6.2. □

It is easily seen that Ore's result is a proper generalization of Dirac's result. That is, we can construct graphs which satisfy (Ore) but do not satisfy (Dirac).

Corollary. *If G is a graph with $n \geq 3$ and $m \geq \dfrac{n^2-3n+6}{2}$, then G is Hamilton.*

Proof. Assume the contrary that G is non-Hamilton. By Theorem 6.2, there exists a pair of non-adjacent vertices u and v such that $deg(u) + deg(v) \leq n-1$. Hence,

$$\begin{aligned} m &\leq |E(G - \{u, v\})| + deg_G(u) + deg_G(v) \\ &\leq \binom{n-2}{2} + n - 1 \\ &= \frac{n^2-3n+4}{2}, \text{a contradiction.} \end{aligned} \qquad \square$$

Corollary. *If a graph G on n vertices is such that*

$$deg(u) + deg(v) \geq n-1, \text{ for every pair of non-adjacent vertices } u \text{ and } v \text{ in } G \quad \text{(P)}$$

then G contains a Hamilton path.

Proof. If $n = 1$, the result is vacuously true. If $n \geq 2$, construct a new graph H by adding a new vertex x and joining x to every vertex of G. Since G satisfies (P), $deg_H(u) + deg_H(v) \geq n + 1 = n(H)$, for every pair of non-adjacent vertices u and v in H. So by Theorem 6.2, H contains a Hamilton cycle C. We can choose C so that x is its initial and terminal vertex. So let $C = (x, v_1, v_2, \ldots, v_n, x)$. Then $(v_1, v_2, \ldots, v_n)$ is a Hamilton path in G. $\square$

Theorem 6.3 (Chvatal, 1972)**.** *Let $n \geq 3$. Suppose a graph G with degree sequence $(d_1 \leq d_2 \leq \cdots \leq d_n)$ satisfies the following condition:*

$$\textit{if there is an integer } k \textit{ such that } 1 \leq k < \frac{n}{2} \textit{ and } d_k \leq k, \textit{ then } d_{n-k} \geq n-k \quad \text{(Chvatal).}$$

Then G is Hamilton.

(Essentially, the condition says that if G contains vertices of small degree, then it contains vertices of large degree too. Before proving the theorem, the reader is encouraged to show that if G satisfies (Chvatal), then G contains no vertex of degree 1; see also Exercise 19.)

Proof. (Proof is similar to the proof of Theorem 6.2 but more deeper.) Assume the contrary that G is non-Hamilton though it satisfies (Chvatal). Let H be a graph obtained from G by successively joining pairs of non-adjacent vertices until addition of any more new edge creates a Hamilton cycle. Then G is a spanning subgraph of H and so $deg_H(v) \geq deg_G(v)$ for every vertex v. Hence, H too satisfies (Chvatal). Henceforth, all our statements are with respect to H;in particular, $(d_1 \leq d_2 \leq \cdots \leq d_n)$ is the degree sequence of H. Since H is non-Hamilton, there exist two non-adjacent vertices, say u and v. Among all such pairs of vertices, we choose two non-adjacent vertices u and v such that

(1) $d(u) + d(v)$ is maximum.

Since, $H + (u, v)$ contains a Hamilton cycle, H contains a Hamilton path $P = (u_1, u_2, \ldots, u_{n-1}, u_n)$ connecting u and v, where $u_1 = u$ and $u_n = v$. If u is adjacent to u_j, then v is non-adjacent to u_{j-1}; else, H contains a Hamilton cycle as in the proof of Theorem 6.2. Hence there are at least $d(u)$ vertices which are non-adjacent to v. So,

(2) $d(u) + d(v) \leq n - 1$.

Without loss of generality, assume that

(3) $d(u) \leq d(v)$; and let $d(u) = k$.

So,

(4) $k < \frac{n}{2}$.

By the maximality of $d(u) + d(v)$, every vertex u_{j-1} that is non-adjacent to v has degree at most $d(u)(= k)$. So, there are k vertices of degree at most k. Since we have arranged the degree sequence in non-decreasing order, it follows that

(5) $d_k \leq k$.

Since, u is adjacent to k vertices, it is non-adjacent to $n - 1 - k$ vertices (other than u). Again by the maximality of $d(u)+d(v)$, each of these $n-1-k$ vertices has degree at most $d(v)(\leq n - 1 - k)$. Moreover, $d(u) \underset{(3)}{\leq} d(v) \underset{(2)}{\leq} n - 1 - k$. Hence, there are at least $n - k$ vertices of degree at most $n - 1 - k$. Since we have arranged the degree sequence in non-decreasing order, it follows that

(6) $d_{n-k} \leq n - 1 - k$.

Thus we have found a k such that $k \underset{(4)}{<} \frac{n}{2}$, $d_k \underset{(5)}{\leq} k$ and $d_{n-k} \underset{(6)}{\leq} n-1-k$. But the existence of such a k is a contradiction to (Chvatal). $\square$

Remark. It can be shown that if G satisfies (Ore), then it satisfies (Chvatal). Moreover, there exist graphs which satisfy (Chvatal) but do not satisfy (Ore). So Chvatal's result is a proper generalization of Ore's result. For example, consider the graph G shown in Figure 6.4. It does not satisfy (Ore). However, its degree sequence $(2,3,3,4,5,5,5,5)$ satisfies (Chvatal). So, Ore's theorem does not guarantee an Hamilton cycle in G but Chvatal's theorem guarantees a Hamilton cycle.

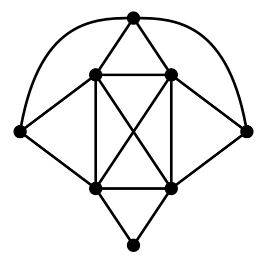


Figure 6.4: A graph G.

For the next theorem, we require a new concept.

Definitions.

- *A set $I \subseteq V(G)$ is called an* ***independent set*** *if no two vertices in I are adjacent.*
- *The parameter $\alpha_0(G) = \max\{|I| : I \text{ is an independent set of } G\}$ is called the* ***independence number*** *of G.*

Clearly, $\alpha_0(P_n) = \lceil \frac{n}{2} \rceil$, $\alpha_0(C_n) = \lfloor \frac{n}{2} \rfloor$, $\alpha_0(K_n) = 1$ and $\alpha_0(K_{m,n}) = \max\{m, n\}$. The independent sets and independence number of a graph will be studied in more detail in Chapter 7.

Theorem 6.4 (Chvatal and Erdös, 1972)**.** *If G is a graph $(n \geq 3)$ with*

$$k_0(G) \geq \alpha_0(G), \text{ where } k_0(G) \text{ is the vertex connectivity of } G, \tag{CE}$$

then G is Hamilton.

Proof. (Contradiction method) Assume the contrary that G is non-Hamilton, though it satisfies (CE). Let C be a longest cycle in G; see Figure 6.5. C does not contain all the vertices, since G is non-Hamilton. Therefore, there exists a component B in $G - V(C)$. Assume that the vertices of C are numbered in a clockwise direction. Let x^+ denote the vertex which succeeds the vertex x on C. Define

$$S = \{x \in V(C) : x \text{ is adjacent to a vertex in } B\}$$

and

$$S^+ = \{x^+ \in V(C) : x \in S\}.$$

Claim 1: If $x \in S$, then $x^+ \notin S$.

On the contrary, let b_1 and b_2 be the neighbors of x and x^+ (respectively) in B. Let $P(b_1, b_2)$ be a path in B. Then the cycle

$$(x, b_1, P(b_1, b_2), b_2, x^+, C(x^+, x))$$

contains more number of vertices than C, a contradiction; see Figure 6.5.

It follows that S is a vertex-cut; so $k_0(G) \leq |S|$.

Claim 2: S^+ is an independent set in G.

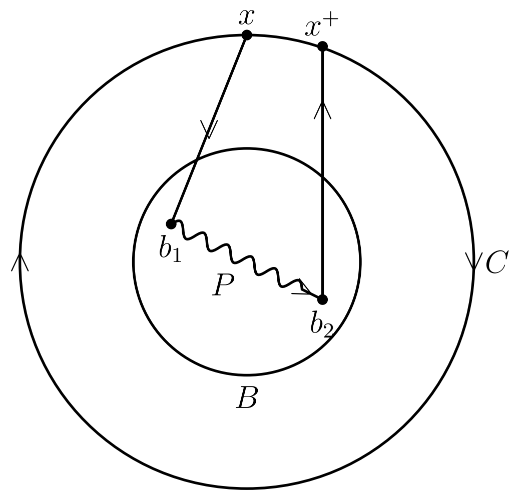


Figure 6.5: Pictorial description for the proof of Claim 1.

On the contrary, suppose that $x^+, y^+ \in S^+$ are adjacent. Let b_1 and b_2 be the vertices of B adjacent to x and y, respectively and let $P(b_1, b_2)$ be a path in B; see Figure 6.6. Then the cycle

$$(x, b_1, P(b_1, b_2), b_2, y, C(y, x^+), y^+, C(y^+, x))$$

contains more number of vertices than C, a contradiction.

Using claims 1 and 2, we conclude that if $b \in B$, then $S^+ \cup \{b\}$ is also an independent set in G. Hence, $k_0(G) \leq |S| = |S^+| < \alpha_0(G)$, a contradiction. So C is a Hamilton cycle. $\square$

Exercises

(All graphs are simple)

1. Draw the following graphs.
   (a) Hamilton and Eulerian.

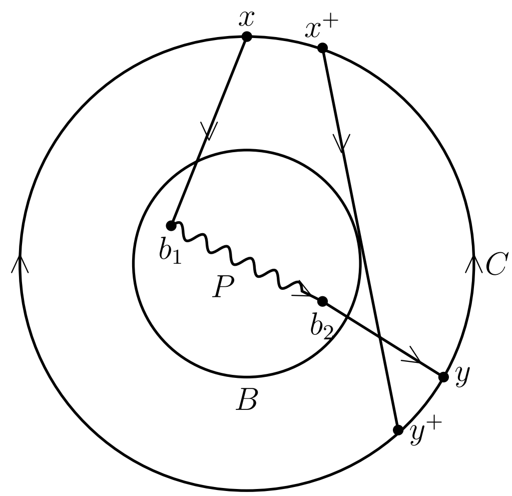


Figure 6.6: Pictorial description for the proof of Claim 2.

   (b) Hamilton and non-Eulerian.
   (c) non-Hamilton and Eulerian.
   (d) non-Hamilton and non-Eulerian.

2. Show that the d-cube Q_d is Hamilton for every $d \geq 2$.

3. (a) Show that the Petersen graph is non-Hamilton.
   (b) Prove or disprove: If P is the Petersen graph, then $P - v$ is Hamilton for every $v \in V(P)$.

4. If G is a non-Hamilton graph such that $G - v$ is Hamilton for every vertex $v \in V(G)$, then show that
   (a) $deg(v) \geq 3$, for every $v \in V(G)$.
   (b) $deg(v) \leq \frac{n-1}{2}$, for every $v \in V(G)$.
   (c) $G - u - v$ is connected for every subset $\{u, v\}$ of $V(G)$.
   (d) $n \geq 10$.
   (e) If $n = 10$, then G is 3-regular.

5. Find 5 edge-disjoint Hamilton cycles in K_{11}.

6. If a graph G has a Hamilton path, then show that

$$c(G - S) \leq |S| + 1, \text{ for every proper subset of } V(G).$$

7. If $deg(u) + deg(v) \geq n + 2$, for every pair of vertices u and v in a graph G, then show that G contains two edge-disjoint Hamilton cycles.

8. Prove that the line graph $L(G)$ of a graph G is Hamilton iff G contains a closed trail W such that every edge of G is incident with a vertex of W.

9. Draw the following graphs :
   (a) A 2-connected non-Hamilton graph on at most 8 vertices.
   (b) A non-Hamilton graph on at most 10 vertices with minimum degree at least 3 which has a Hamilton path.
   (c) A graph on at least 9 vertices such that both G and its complement G^c are connected but both are non-Hamilton.

10. For every k, construct a k-connected simple non-Hamilton graph.

11. Draw a non-Hamilton simple graph with 10 vertices such that for every pair u, v of non-adjacent vertices, we have $deg(u) + deg(v) \geq 9$.

12. Draw the following non-Hamilton graphs.
   (a) $m = \frac{n^2-3n+4}{2}$.
   (b) $deg(u) + deg(v) \geq n - 1$, for every pair of non-adjacent vertices in G.
   (c) $\delta(G) \geq \frac{n-1}{2}$.

13. Let (v_1, v_2, v_3, v_4) be a path P_4 on 4 vertices. Show that the composition graph $P_4(K_n^c, K_n, K_n, K_n^c)$ on $4n$ vertices is a self-complimentary non-Hamilton graph.

14. (a) For what values of p, $K_{1,2,\ldots,p}$ is Hamilton? Justify your answer.
   (b) For what integers n, the graph $K_{2,3,4,n}$ is Hamilton? Justify your answer.
   (c) Show that a complete r-partite graph $K_{n_1,n_2,\ldots,n_r}$ $(n_1 \leq n_2 \leq \cdots \leq n_r)$ is Hamilton iff $n_1 + n_2 + \cdots + n_{r-1} \geq n_r$.
   (d) Show that $K_s + K_{1,t}$ is Hamilton if and only if $t \leq s + 1$.

15. Find necessary and sufficient conditions for the following graphs to be Hamilton.
   (a) $P_s \square P_t$.
   (b) $K_s^c + K_t$.

16. Let G be a graph obtained from K_n $(n \geq 3)$ by deleting any set of at most $n-3$ edges. Using Ore's theorem or otherwise show that G is Hamilton.

17. Show that if a graph satisfies (Ore), then it satisfies (Chvatal).

18. Give an example of a graph which satisfies (Chvatal) but does not satisfy (Ore).

19. If G satisfies (Chvatal) and $d_2 \leq 2$, then show that G contains at least 3 vertices of degree $\geq n - 3$.

20. Which of the following degree sequences of graphs satisfy (Chvatal)(see Theorem 6.3).
    (a) (2,2,2,2,2)
    (b) (2,2,3,3,3,3)
    (c) (3,3,3,3,3,3)
    (d) (2,2,3,4,4,5)

21. Which of the following conditions are satisfied by the graph $K_3 + (K_3^c \cup K_2)$?
    (a) Ore's sufficient condition for a graph to be Hamilton.
    (b) Chvatal's sufficient condition for a graph to be Hamilton.
    (c) Chvatal's necessary condition for a graph to be Hamilton.

22. Let G_1 and G_2 be two simple graphs both on n vertices and satisfying Dirac's condition for the existence of a Hamilton cycle. Let G be a simple graph obtained from G_1 and G_2 by joining edges between G_1 and G_2 such that:
    (a) every vertex in G_1 is joined to at least half the number of vertices in G_2, and
    (b) every vertex in G_2 is joined to at least half the number of vertices in G_1.

    Show that G is Hamilton.

23. Prove or disprove : Graphs shown in Figure 6.7 are Hamilton.

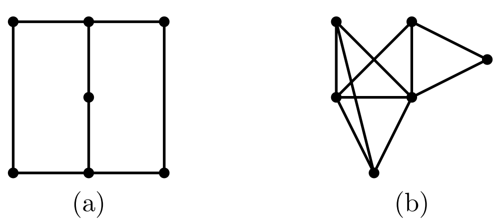


Figure 6.7

The ***closure*** $C(G)$ of a graph G is the graph obtained from G by successively joining pairs of non-adjacent vertices whose degree sum is at least n, until no such pair remains. For example, see Figure 6.8.

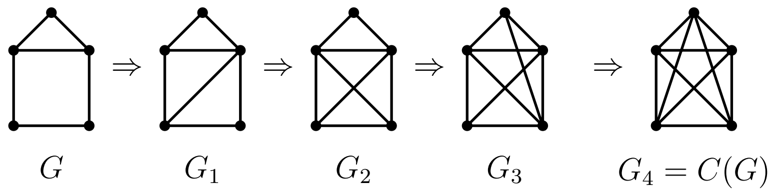


Figure 6.8: Construction of closure of a graph.

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24. (a) Find the closures of the graphs shown in Figure 6.9.

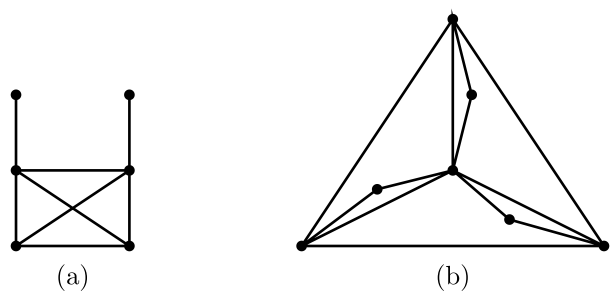

Figure 6.9

(b) Draw an example of a graph G whose closure is neither complete nor G.

25. (a) Let u, v be two non-adjacent vertices in G such that $deg(u) + deg(v) \geq n$. Show that G is Hamilton iff $G + (u, v)$ is Hamilton.

(b) Show that G is Hamilton iff $C(G)$ is Hamilton.

(c) If G satisfies (Chvatal)(Theorem 6.3), then show that $C(G)$ is complete.

A graph G is said to be ***Hamilton-connected***, if any two vertices in G are connected by a Hamilton path.

---

26. (a) Show that $K_5 - e$, where e is an edge, is Hamilton-connected.

(b) Show that $K_4 - e$, where e is an edge, is not Hamilton-connected.

(c) If G is Hamilton-connected, then show that every edge in G belongs to a Hamilton cycle.

(d) Let G be a graph and let w be a vertex of degree 2 adjacent with vertices u and v. Let H be the graph $G - w$ if $(u, v) \in E(G)$; $G - w + (u, v)$, otherwise. Show that G is Hamilton-connected iff H is Hamilton-connected.

(e) If G is Hamilton-connected, then show that $m \geq \lfloor \frac{3n+1}{2} \rfloor$, for $n \geq 4$.

(f) If G is a graph with $m \geq \frac{(n-1)(n-2)}{2} + 3$, then show that G is Hamilton-connected.