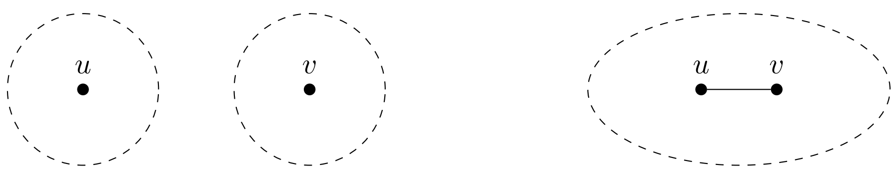
Math 68 - Spring 2014 - Practice problems with solutions

Chapter 1

1.1.10 **If G is simple and disconnected, then $\bar{G}$ is connected.**

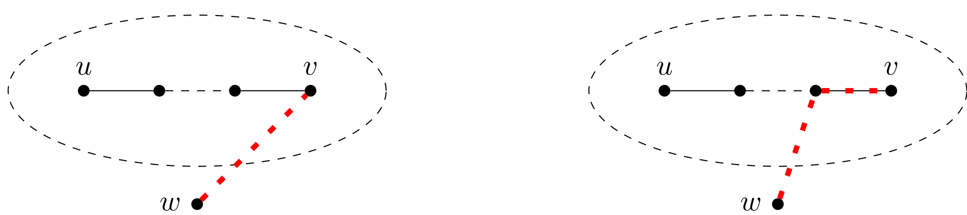
Proof. Consider any two vertices $u, v \in V(\bar{G})$. We'll use the fact that G is disconnected to show that u and v lie on a path in $\bar{G}$. Since this will be true for *any* two vertices in $\bar{G}$, this will imply that $\bar{G}$ is connected.
<u>Case 1: u and v are not connected in G.</u> If two vertices are not connected, then they're certainly not adjacent. So $uv \notin E(G)$, and therefore $uv \in E(\bar{G})$, and so u and v are connected in $\bar{G}$.

G : $\bar{G}$:



<u>Case 2: u and v are connected in G.</u> Since G is not connected, there must be some third vertex w which is not connected to either u or v: *If w is connected to v by a path, then the subgraph which is the union of the two paths w to v and u to v has a path from w to u.*

G : G :

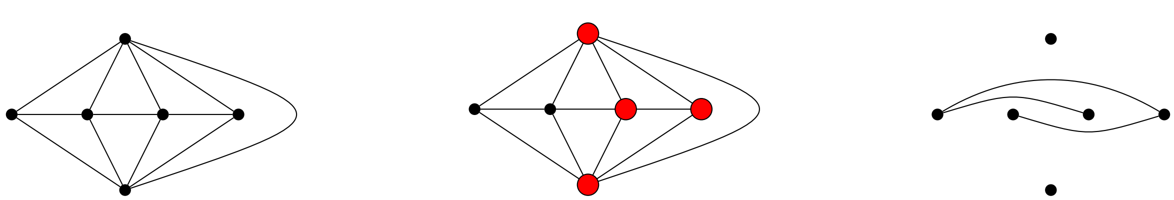


Then w is not adjacent to u or v in G, so uw and uv are both in $E(\bar{G})$. So u, uw, w, wv, v is a path connecting u and v in $\bar{G}$.

□

1.1.11 **Determine the maximum size of a clique and the maximum size of an independent set in the graph G:**

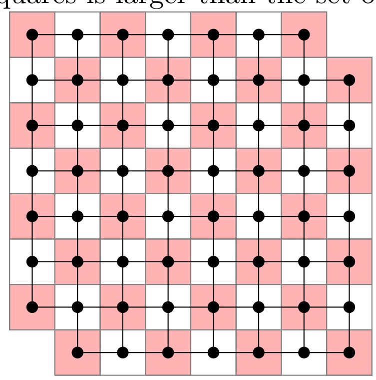
G : Size 4 clique $\bar{G}$:



Proof. In a clique of size n, every vertex has degree $n-1$. In this graph, the maximal degree is 5. However, there are only 2 vertices with degree 5, so there is no clique of size 5. Similarly, there are only four vertices of degree 4, so there is no clique of size 5. There is, however a clique of size 4 (drawn above). An independent set is a clique in $\bar{G}$ (also drawn above). Since $\bar{G}$ is a path, any maximal independent set in G is of size 2. □

1.1.14 **Prove that removing opposite corner squares from an 8-by-8 checkerboard leaves a subboard that cannot be tiled with 1-by-2 rectangles. Using the same argument, make a general statement about all bipartite graphs.**

Proof. Notice that the opposite corners are of the same color. So when you remove those two squares, there are more squares left of one color (say red) than of the other (say white). A tiling of the board is a perfect matching between red squares and white squares, which we can't have if the set of red squares is larger than the set of white squares.



To frame this question in terms of bipartite graphs, we can think about the checkerboard as a map that is two-colorable. Draw vertices for each square and connect two vertices if the squares share a border. Since this adjacency graph is two-colorable it is bipartite (the two partites being the set of white and red squares, respectively). A tiling is a choice of k disjoint edges which, as a set, cover all vertices. In general, this cannot be done if the two partites are of different sizes.

□

1.1.15 **Consider the following four families of graphs:**

$$A = \{\textbf{paths}\}, B = \{\textbf{cycles}\}, C = \{\textbf{complete graphs}\}, D = \{\textbf{bipartite graphs}\}.$$

For each pairs of families, determine all isomorphism classes of graphs belonging to both families.

Answer: Isomorphism classes are just unlabeled graphs. For a fixed number of vertices n the size of the edge sets are as follows:

$$|E(P_n)| = n - 1, \qquad |E(C_n)| = n, \qquad |E(K_n)| = \binom{n}{2} = \tfrac{1}{2}n(n-1).$$

A partite graph with partites of sizes $r \leq s$ has at most r^2 edges.

$\{\textbf{paths}\} \cap \{\textbf{cycles}\} = \emptyset$**:** The edge sets of C_n and P_n are always of different sized (there is no C_1 or C_2 if we restrict to simple graphs).

{paths} ∩ {complete graphs} $= \{P_1 = K_1, P_2 = K_2\}$**:** Counting edges again, $n - 1 = \frac{1}{2}n(n-1)$ has exactly two solutions: $n = 1$ and $n = 2$, so P_1 and P_2 are the only two candidates. Checking each individually, we see that these are, in fact, both complete.

{paths} ∩ {bipartite graphs} = {paths}: By alternating, all paths are two-colorable, and so are bipartite.

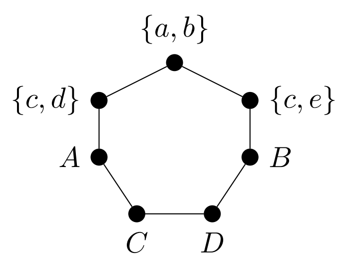
{cycles} ∩ {complete graphs} $= \{C_3 = K_3\}$**:** Back to counting edges, the only solutions to $n = \frac{1}{2}n(n-1)$ are $n = 1$ and $n = 3$. However, there is not simple cycle with one vertex, so C_3 is the only candidate. Upon inspection, $C_3 = K_3$.

{cycles} ∩ {bipartite graphs} $= \{C_{2k}\}_{k \in \mathbb{Z}_{\geq 2}}$**:** A cycle is a type walk from a vertex back to itself. In a bipartite graph, any walk must alternate between the parites, so any walk from a vertex back to itself must be of even length (have an even number of edges). So the even cycles are the only candidates. And, in fact, by alternating colors, we can always 2-color an even cycle, and so every even cycle is bipartite.

{complete graphs} ∩ {bipartite graphs} $= \{K_1, K_2\}$**:** Any bipartite graph with more than two vertices must be missing at least one edge, so is not complete. However, since K_2 is a path, it's bipartite. The single vertex, K_1 is also bipartite with one empty partite.

1.1.25 **Argue that the Petersen graph G has no 7-cycles.**

Proof. Suppose there is some 7-cycle in G. Pick some vertex in G to start at, and for the sake of generality, call it $\{a, b\}$. Its two neighbors cannot have any elements in common, but there are only three numbers left to choose from, so those two neighbors must have an element in common; call it c.

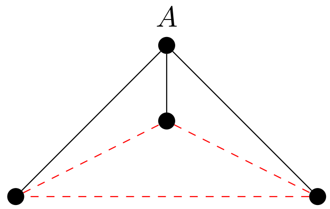


The girth of G is 5. Therefore $\{a, b\}$ must not be adjacent to and of A, B, C, or D. So all of the unknown vertices must have exactly one intersection with $\{a, b\}$. Similarly, C and D must both have exactly one intersection with $\{c, d\}$ and with $\{c, e\}$. So C and D must both contain c, which is a contradiction. □

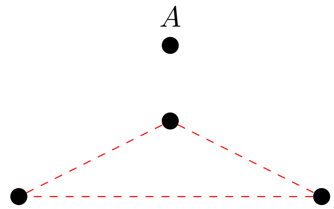
1.1.29 **Prove that every set of six people contains at at least three mutual acquaintances of three mutual strangers.**

Proof. Pick a person A. From the remaining 5 people, A either has at least three people who she knows or three people that she doesn't know. If A knows three people, then either there's some pair of those people who also know each other (forming a clique of size at least three), or no two of them know each other (in which case they form an independent set of size three. If A has three strangers, then either there is some pair of those strangers who also don't know each other (forming an independent set of size three with A), or they all know eachother (forming a clique of size three.

Case 1:



Case 2:



$\square$

1.1.30 **Let G be a simple graph with adjacency matrix A and incidence matrix M. Prove that the degree of v_i is the ith diagonal entry of A^2 and MM^T. What do the entries in position (i, j) of A^2 and MM^T say about G?**

Proof. Since A is symmetric, the ith diagonal entry of A is the dot product square of the vector

$$\mathrm{v}^{(i)} \qquad \text{where } \mathrm{w}_j^{(i)} = \begin{cases} 1 & \text{if } v_i \text{ is adjacent to } v_j, \\ 0 & \text{otherwise.} \end{cases}$$

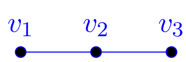
In general, the dot product square of a vector is the sum of the squares of the entries in the vector. Since all of the entries of v are 1 or 0, $\mathrm{v} \cdot \mathrm{v}$ is just the sum of the entries of v, which is the degree of v_i. The ith diagonal entry of MM^T is the dot product square of

$$\mathrm{w}^{(i)} \qquad \text{where } \mathrm{w}_j^{(i)} = \begin{cases} 1 & \text{if } e_j \text{ is incident to } v_i \\ 0 & \text{otherwise.} \end{cases}$$

Again, this is just the sum of the entries in $\mathrm{w}^{(i)}$, which is also the degree of v_i.

In A^2, the (i, j) entry is the dot-product $\mathrm{v}^{(i)} \cdot \mathrm{v}^{(j)}$, which will again be the sum of 1's and 0's: a 1 occurs in the kth summand if v_i and v_j share v_k as a neighbor. So (i, j) entry is the number of neighbors v_i and v_j have in common!

(**Fun fact:** In general, the (i, j) entry of A^k will be the number of length k walks from v_i to v_j. For example, if

$G =$ 

Then

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad A^2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}, \quad A^4 = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 2 \end{pmatrix}, \cdots)$$

Similarly, the (i, j) entry of MM^T is the dot product $\mathrm{w}^{(i)} \cdot \mathrm{w}^{(j)}$, which will again be the sum of 1's and 0's: a 1 occurs in the kth summand if v_i and v_j share e_k as an edge. So (i, j) entry is the number of edges joining v_i to v_j. $\square$

1.2.1 **Determine whether the statements below are true or false.**

(a) **Every disconnected graph has an isolated vertex.**

False. For example, is disconnected and has no isolated vertex.

(b) **A graph is connected if and only if some vertex is connected to all other vertices.**

True. IF the graph is connected, then by definition, any vertex is connected to every other vertex in G. On the other hand, if there is a vertex that is connected to every other vertex in G, then by transitivity (of the connection relation), every other pair of vertices is also connected (it P is a u, v-path and P' is a v, w-path, then P followed by P' is a u, w-walk, which contains a u, w-path).

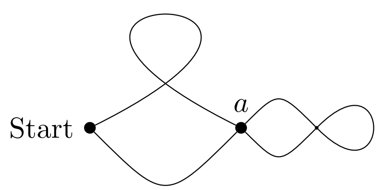
(c) **The edge set of every closed trail can be partitioned into edge sets of cycles.**

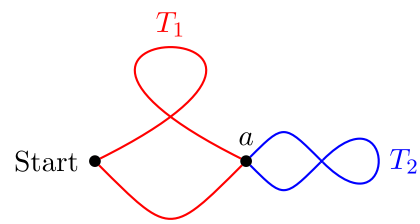
True. This is certainly true for a closed trail which is a single vertex or any trail which is itself a cycle. If a closed trail T is not a cycle, then either (1) the first/last vertex is repeated more that just the twice, or (2) there is some other vertex which is repeated; call this special vertex a. Decompose the trail T into

T_1, the trail which walks to the first occurrence of a and skips everything between then and the second occurence of a and then completes the rest of T, and

T_2, which starts at the first occurrence of a and walks to the second occurrence of a.

Both T_1 and T_2 are closed trails of shorter length than T. Strongly induct on the length of T.





(d) **If a maximal trail in a graph is not closed, then its endpoints have odd degree.**

True. If a trail is not closed, then for each endpoint, the number of edges in the trail which are incident to that endpoint must be odd. If a trail is maximal, then there are no edges incident to the endpoint(s) which are not already in the trail; so the number of edges in the trail which are incident to each endpoint is equal to the degree of that vertex.

1.2.4 **Let G be a loopless graph. For $v \in V(G)$ and $e \in E(G)$, describe the adjacency and incidence matrics of $G - v$ and $G - e$ in terms of the corresponding matrices for G.**

$\underline{G - e}$: Suppose the endpoints of e are u and v. The adjacency matrix is the same as that of G, except in the (u, v) and the (v, u) entries, which are each reduced by exactly 1. The incidence matrix has one fewer columns than $I(G)$ (the one corresponding to e), and is otherwise unchanged.

$\underline{G - v}$: The adjacency matrix is one dimension smaller, and is achieved by deleting the row and column of $A(G)$ corresponding to v. The incidence matrix has one fewer row and degree(v) fewer columns and is achieved by deleting the row corresponding to v and the columns corresponding to all incident edges to v.

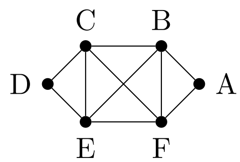
1.2.12 **Convert the proof given in item 1.2.32 to a procedure for finding an Eulerian circuit in a connected even graph.**

Start by trying to draw an Eulerian trail (recording the order of the edges and vertices). If you get stuck in one direction, try to extend the the other end (adding edges and vertices in revers order to the beginning). When you run out of room, you must have a closed trail by Lemma 1.2.31. Since it's closed, think of it as a circuit C, which you can start at any of its vertices.

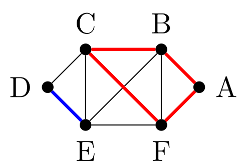
If you missed an edge e, find a path from e to your circuit (which doesn't intersect unnecessarily with C). Then start a new trail which starts at a far endpoint of e, follows the path to C, and then continues around C. If this new trail is not closed, you can close it again by Lemma 1.2.31. Iterate until there are no edges left.

For example:

An even connected graph

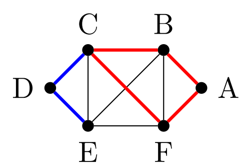


Try 1:



Trail: A–B–C–F–A
Missing edge: DE

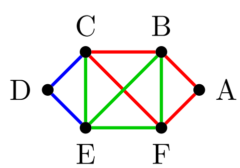
Try 2:



Original Trail: A–B–C–F–A
Path with edge: E–D–C

New Trail: E–D–$\underbrace{\text{C–F–A–B–C}}_{\text{same order as before, but new start point}}$

Closing off try 2:



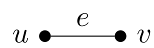
Last Trail: E–D–C–F–A–B–C
Closing it off (in reverse): E–B–F–E–C

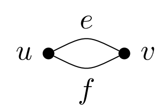
New Trail: C–E–F–B–E–D–C–F–A–B–C

1.2.15 **Let W be a closed walk of length at least 1 that does not contain a cycle. Prove that some edge of W repeats immediately (once in each direction).**

Proof. (By strong induction)
Let $\ell(W)$ denote the length of the walk.
Suppose $\ell(W) = 1$. If W is closed, then it must be a loop (a cycle of length 1), which contradicts the hypothesis. If $\ell(W) = 2$ and W is closed, then W is one of the following:

$W = u, e, v, e, u$ $\qquad\qquad$ $W = u, e, v, f, u$





pro The latter has a cycle, so it can only be the first, in which the edge e immediately repeats.

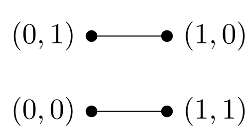
Now assume that any closed walk of length $k < n$ which does not contain a cycle has an edge which immediately repeats. Any closed walk with only one repeated vertex (the first) is itself a cycle, so if $W = v_0, e_1, v_1, \ldots, e_\ell v_\ell$ does not contain a cycle, then it must have another vertex w which is repeated. Now consider the subwalk U of W which starts at the first occurrence of w, continues on W until the second occurrence of w (like T_2 in 1.2.1 (c)). This is a closed walk of length less than $\ell(W)$, and since W does not have a cycle, neither does this subwalk. So, by the induction hypothesis, this subwalk has an edge which immediately repeats. Since consecutive edges of this subwalk also occur consecutively in W, that means that there is an edge of W immediately repeats. □

1.2.18 **Let G be the graph whose vertex set is the set of k-tuples with elements in $\{0, 1\}$. with x adjacent to y if x and y differ in exactly two positions. Determine the number of components of G.**

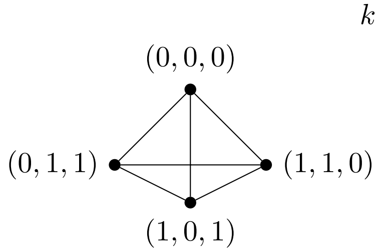
Answer:

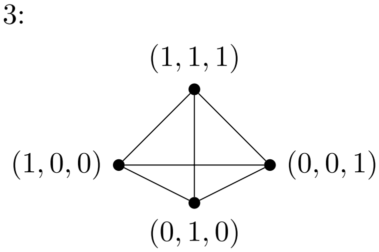
First, some examples:

$k = 2$:



$k = 3$:





In general, there will be **two connected components**:

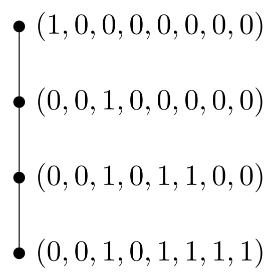
(a) one containing all points with an **odd** number of 1's, and

(b) one containing all points with an **even** number of 1's.

We prove this in two steps:

A. If two points do not share the same parity of 1's, then they're not in the same component. This can be seen by observing that an edge will only connect two vertices if those vertices have the same parity of 1's. Therefore, there is no walk (and therefore no path) in G which contains vertices of both parity's.

B. Any two points with the same parity of 1's are in the same component. We'll apply transitivity and show that every vertex is connected to one of two special vertices. **Even:** you can get to any point with an even number of 1's from the vertex with all 0's by following edges which changes the necessary 0's to 1's two at a time. **Odd:** similarly, you can get to any point with an odd number of 1's from the vertex with 1 in the first place followed by all 0's as follows: if the desired point has a 1 in the first coordinate, then change 0's to 1's as needed in pairs; if it doesn't then switch the 1 to a 0 at the same time as switching a desired 0 to a 1, and then switch 0's to 1's as needed in pairs. For example, (1,0,0,0,0,0,0,0) is connected to (0,0,1,0,1,1,1,1) by the following path:



1.2.38 **Prove that every n-vertex graph with at least n edges contains a cycle.**

Proof. If $n = 1$, and G has one vertex and at least one edge, then it has at least one loop, which is a cycle.

Now suppose $n > 1$ and G has n vertices and at least n edges. Either

(a) **every vertex in G has degree at least 2**, in which case Proposition 1.2.28 implies G contains a cycles, or

(b) G **has a vertex v of degree 1 or 0**. Then $G - v$ has $n - 1$ vertices and at least $n - 1$ edges (we deleted at most 1 edge in removing v). By strong induction, $G - v$ (and therefore G) has a cycle.

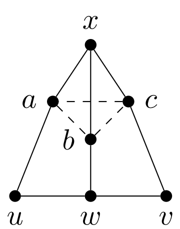
□

1.2.42 **Let G be a connected simple graph not having P_4 or C_4 as an induced graph. Prove that G has a vertex adjacent to all other vertices.**

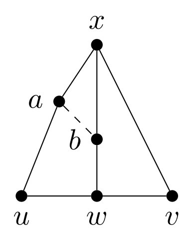
Proof. Since G is connected, any two vertices are connected by a path. Consider two vertices u and v, and pick a minimal u,v-path $P = u, e_1, v_1, \ldots, v_{\ell-1}, e_\ell, v$. Since P is minimal, no two v_i, v_j with $i \neq j \pm 1$ can be neighbors. So $G[V(P)] \cong P_{\ell+1}$ (and so G has P_k as an induced subgraph for all $k \leq \ell$ as well). Thus $\ell \leq 2$, and any pair of vertices in G must either be adjacent, or mutually adjacent to at least one vertex.

Now, either G is a complete graph (in which case every vertex is connected to every other vertex), or there are two non-adjacent vertices u and v. We'll show that any two such vertices have a common neighbor which is adjacent to all other vertices in G.

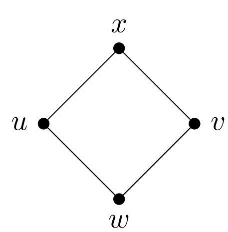
Choose a minimal u,v-path, and call the intermediate vertex w. Pick *any* other vertex x in $V(G)$. Since any minimal path in G connecting two vertices must be of length at most 2, and since G is connected, x is at most 2 away from u, v, and w. If x is not adjacent to w, then G would have one of the following as an induced subgraph (where the dashed edges are unknown):



I. x is not adjacent to u,v, or w.
Contradiction: $G[\{x, a, u, w\}] \cong P_4$



II. x is not adjacent to u or w.
(same as x not adjacent to v or w)
Contradiction: $G[\{x, a, u, w\}] \cong P_4$



III. x is adjacent to u and v, but not to w.
Contradiction: $G[\{u, v, w, x\}] \cong C_4$

Therefore, x (any other vertex in G) is adjacent to w.

□

1.3.9 **In a league with two divisions of 13 teams each, determine whether it is possible to schedule a season with each team playing 9 games against teams within its division and four games against teams in the other division.**

Answer: Formulating this as a graph theory problem, we need a 26-vertex graph which decomposes into a 4-regular bipartite graph with partites X and Y both of order 13, and two 9-regular graphs of order 13 (one on each partite of X and Y). Unfortunately, the latter isn't possible because a graph cannot have an odd number odd-degree vertices!

It is, however, possible to schedule the four games each across the divisions: Call the teams in division X $x_1, x_2, \ldots, x_{13}$, and the teams in division Y $y_1, y_2, \ldots, y_{13}$. For team x_i, schedule games with

$$y_i, \quad y_{i+1 \pmod{13}}, \quad y_{i+2 \pmod{13}}, \quad \text{and} \quad y_{i+3 \pmod{13}}.$$

Then, conversely, y_j plays games with

$$x_j, \quad x_{i-1 \pmod{13}}, \quad x_{i-2 \pmod{13}}, \quad \text{and} \quad x_{i-3 \pmod{13}}.$$

1.3.22 **Let G be a nonbipartite triangle-free simple graph with n vertices and minimum degree k. Let ℓ be the minimum length of an odd cycle in G.**

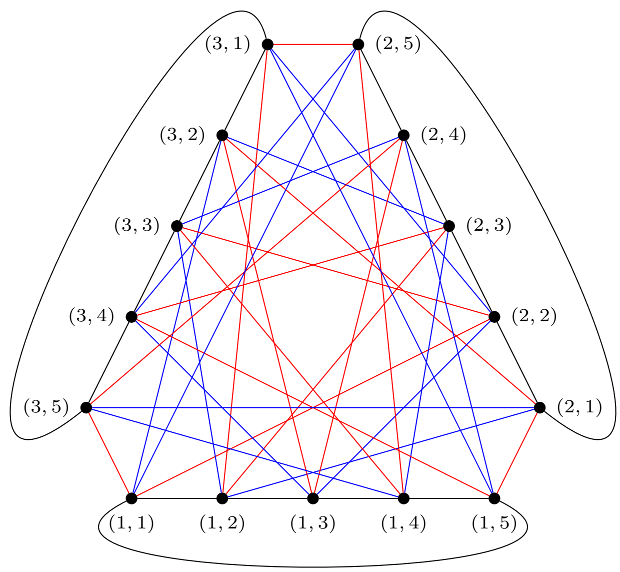
(a) **Let C be a cycle of length ℓ in G. Prove that every vertex not in $V(C)$ has at most two neighbors in $V(C)$.**
Suppose $x \in V(G) - V(C)$ has three neighbors in $V(C)$. Then those neighbors partition C into three pieces. Since ℓ is odd, at least one of those parts has odd length. And since there are no triangles in G, that part has length at most $\ell - 3$. Then by walking from x to one of the endpoints of that part, along that part, and then back to x, you've built an odd cycle of length less than ℓ, a contradiction.

(b) **By counting the edges joining $V(C)$ and $V(G) - V(C)$ in two ways, prove that $n \geq k\ell/2$ (and thus $\ell \leq 2n/k$).**
Since $\delta(G) = k$, the edges coming out of $V(C)$ is at least $k\ell - 2\ell$ (the sum of the degrees minus the edges contributing to C, which were all double-counted). On the other hand, the number of edges coming into $V(C)$ is at most $(|V(G)| - |V(C)|) * 2 = (n - \ell) * 2$. So
$$k\ell - 2\ell \leq (n - \ell) * 2 \implies k\ell \leq 2n \implies k\ell/2 \leq n.$$

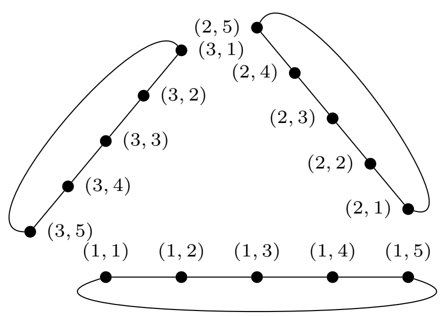
(c) **When k is even, prove that the inequality of part (b) is best possible.**
Let k be even and $\ell > 3$ odd. Then the criteria above are satisfied by the graph G with $V(G) = \{(a, b) \mid 1 \leq a \leq k/2, 1 \leq b \leq \ell\}$ and $E(G) = \{(a, b) - (c, d) \mid b = d \pm 1 \pmod{\ell}\}$ (see the example below). For this graph, $n = (k/2) * \ell$.

Proof. (1) Any cycle which has an even number of edges (a, ℓ)–$(b, 1)$ must be of even length (e.g. a cycle with no such edge will have the property that every second coordinate which appears must appear an even number of times). So the shortest odd cycle is of length ℓ.
(2) Every vertex has degree k since (a,b) will connect to $(c, b + 1 \pmod{\ell})$ and $(c, b - 1 \pmod{\ell})$ for $c = 1, \ldots, k$. .
(3) G is not bipartite since it has an odd cycle, and has no triangles by (1). □

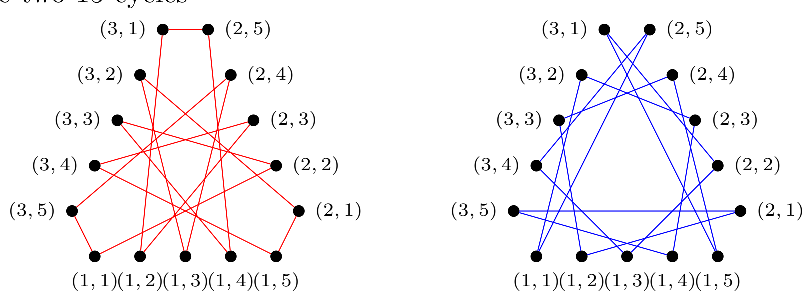
For example, if $k = 6$, $\ell = 5$, then G is



which decomposes into the three 5-cycles



and the two 15-cycles

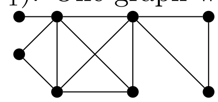


1.3.8 Which of the following are graphic sequences? Provide a construction of a proof of impossibility for each.

(a) $(\mathbf{5,5,4,3,2,2,2,1})$: Using the iteration from Theorem 1.3.31, you get

$$\begin{aligned}(5,5,4,3,2,2,2,1) &\rightarrow (4,3,2,1,1,2,1) = (4,3,2,2,1,1,1)\\ &\rightarrow (2,1,1,0,1,1) = (2,1,1,1,1,0)\end{aligned}$$

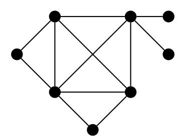
the last of which is graphic ($P_3 + P_2 + P_1$). One graph which has this sequence is



(b) $(\mathbf{5,5,4,4,2,2,1,1})$: Using the iteration from Theorem 1.3.31, you get

$$\begin{aligned}(5,5,4,4,2,2,1,1) &\rightarrow (4,3,3,1,1,1,1)\\ &\rightarrow (2,2,0,0,1,1) = (2,2,1,1,0,0)\end{aligned}$$

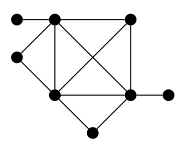
the last of which is graphic ($P_4 + P_1 + P_1$). One graph which has this sequence is



(c) $(\mathbf{5,5,5,3,2,2,1,1})$: Using the iteration from Theorem 1.3.31, you get

$$\begin{aligned}(5,5,5,3,2,2,1,1) &\rightarrow (4,4,2,1,1,1,1) \rightarrow (3,1,0,0,1,1) = (3,1,1,1,0,0)\\ &\rightarrow (0,0,0,0,0)\end{aligned}$$

the last of which is graphic (5 vertices w. no edges). One graph which has this sequence is



(d) $(\mathbf{5,5,5,4,2,1,1,1})$: Using the iteration from Theorem 1.3.31, you get

$$\begin{aligned}(5,5,5,4,2,1,1,1) &\rightarrow (4,4,3,1,0,1,1) = (4,4,3,1,1,1,0)\\ &\rightarrow (3,2,0,0,1,0) = (3,2,1,0,0,0)\end{aligned}$$

which is not graphic since there are not three vertices of non-zero degree to connect the degree vertex to. So (5,5,5,4,2,1,1,1) is not graphic.

1.3.18 For $k \geq 2$, prove that a k-regular bipartite graph has no cut-edge.

Proof. Notice that any component of G is also k-regular and bipartite, so we'll assume, without loss of generality, that G is connected. Suppose there is a cut edge e of G, and consider a H component of $G - e$. Then H is also bipartite, and has $n(H) - 1$ vertices of degree k and one vertex v of degree $k - 1$. But if H has partites X and Y (say $v \in Y$), this means that H simultaneously has $|X| * k$ and $|Y| * k - 1$ edges. So

$$k(|Y| - |X|) = 1$$

which is a contradiction since $k \geq 2$. $\square$

1.3.25 Prove that every cycle of length $2r$ in a hypercube is contained in a subcube of dimension at most r. Can a cycle of length $2r$ be contained in a subcube of dimension less than r?

Proof. Consider the set of bits which are not shared amongst all of the vertices of the cycle C. The cycle sits in the subcube generated by varying only those bits which vary in C. There are at most r of them since the farthest a vertex can be from any other vertex in C half the length of the cycle. So C sits inside some copy of Q_r.

On the other hand,

(0,0,0) — (1,0,0) — (1,1,0) — (1,1,1) — (1,0,1) — (0,0,1) — (0,1,1) — (0,1,0) — (0,0,0)

is a cycle of length $8 = 2 * 4$ in a hypercube of dimension 3. (So yes, but not as an induced subgraph. Also, this is the smallest example since $Q_2 \cong C_4$.)

□

1.3.32 Prove that the number of simple even graphs with vertex set $[n]$ is $2^{\binom{n-1}{2}}$. (Hint: establish a bijection with the set of all simple graphs on $V(G) = [n-1]$.)

Proof. There is a bijection between simple graphs on $[n-1]$ and even simple graphs on $[n]$ given by

$$\begin{aligned} \{\text{ simple graphs on } [n-1]\} &\to \{\text{ simple even graphs on } [n]\} \\ G &\mapsto G + v_n + \{v_n v_i \mid d(v_i) \text{ is odd}\} \end{aligned}$$

(connect the new vertex to every odd vertex in G; since there were an even number of these, not only is the new degree on each odd v_i, but the degree of v_n is also even). The inverse of this map is

$$\begin{aligned} \{\text{ simple even graphs on } [n]\} &\to \{\text{ simple graphs on } [n-1]\} \\ G &\mapsto G - v_n. \end{aligned}$$

(Notice that we're interested in *graphs, not isomorphism classes of graphs*, so both of these maps are one-to-one). Since there are $2^{\binom{n-1}{2}}$ simple graphs on $[n-1]$, there are also $2^{\binom{n-1}{2}}$ simple even graphs on $[n]$

□

1.3.44 Let G be a loopless graph with average vertex degree $a = 2e(G)/n(G)$.

(a) **Prove that $G - x$ has average degree at least a if and only if $d(x) \leq a/2$.**

Proof. Let $n = n(G)$, $k = e(G)$, and $d = d_G(x)$. Since $n(G-x) = n-1$ and $e(G-x) = k-d$, the average degree in $G - e$ is $2(k-d)/(n-1)$. This is greater than or equal to a exactly when

$$n(k-d) \geq (n-1)k \quad \Leftrightarrow \quad nd \leq k \quad \Leftrightarrow \quad d \leq k/n = a/2.$$

□

(b) **Use part (a) to give an algorithmic proof that if $a > 0$, then G has a subgraph with minimum degree greater than $a/2$.**
If G is a regular graph, then we're done. If not, then there is some vertex v of degree less than $a/2$. By (a), $G - v$ has average degree at least $a/2$. Continue deleting vertices of degree less than $a/2$; each time the average degree of the remaining graph G' will go up since we are only deleting vertices of degree $d < \frac{1}{2}a \leq \frac{1}{2}\left(2e(G')/n(G')\right)$. Iterate until there are no more vertices of degree less than $a/2$; we will not run out of vertices to remove since, at every step,

$$n(G') > \Delta(G') \geq 2e(G')/n(G') \geq a/2.$$

(c) **Show that there is no constant c greater than $1/2$ such that G must have a subgraph with minimum degree greater than ca; this proves that the bound in part (b) is best possible.** (Hint: use $K_{1,n-1}$.)

Proof. Consider the star on n vertices. This has average degree $2(n-1)/n < 2$, and so it's not possible to remove a vertex and raise the average degree: by removing vertices, it is only possible to drop the average degree. Moreover, every subgraph of G has $\delta(G) \leq 1$. Since

$$\lim_{n\to\infty} 2\frac{n-1}{n} = 2, \qquad \text{we have } \lim_{n\to\infty} ca = 2c,$$

and so for any $c \geq \frac{1}{2}$, there is some n for which $ca > 1$. □

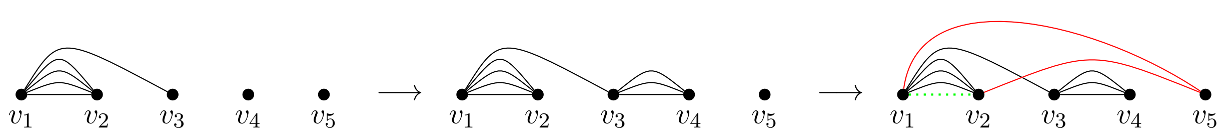
1.3.63 Let $d_1, \ldots, d_n$ be integers such that $d_1 \geq \cdots \geq d_n \geq 0$. Prove that there is a loopless graph (multiple edges allowed) with degree sequence $d_1, \ldots, d_n$ if and only if $\sum_i d_i$ is even and $d_1 \leq d_2 + \cdots + d_n$. (Hakimi [1962])

Proof. If $\sum_i d_i$ is odd, then $d_1, \ldots, d_n$ cannot be a degree sequence. Since there are no loops, there can be at most $D = d_2 + \cdots + d_n$ edges in a graph with degree sequence $d_1, \ldots, d_n$, so $d_1 \leq D$.

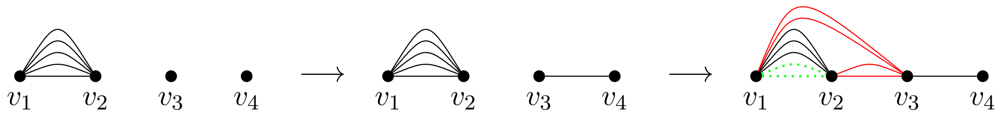
Now assume that $\sum_i d_i$ is even and $d_1 \leq d_2 + \cdots + d_n$. Start with vertices $v_1, \ldots, v_n$. Add d_2 edges incident to v_1 and v_2, then up to d_3 edges to v_3, and so on, until we've added d_1 edges in total. This is possible by our hypothesis.

Now take the last vertex v_j which is not "full" and add edges from v_j to v_{j+1}, and so on until v_j is "full". Iterate this process until there is at most one vertex v_ℓ which is not full (maybe $\ell = j$, but $\ell \neq 2$ since $d_2 \leq d_1$). The number of edges N remaining to be added to v_ℓ is even (since $\sum_i d_i$ is even, and N is $\sum_i d_i$ minus the sum of the degrees in the graph so far). There are at least N edges joining v_1 to v_2 since $N \leq d_\ell \leq d_2$; to avoid loops, remove N edges connecting v_1 and v_2 and add $N/2$ edges connecting v_ℓ to each of v_1 and v_2.

For example, consider the sequence $(5, 4, 4, 3, 2)$, the first few iterations look like



Alternatively, consider $(5, 5, 5, 1)$ (to see why ℓ isn't always n):

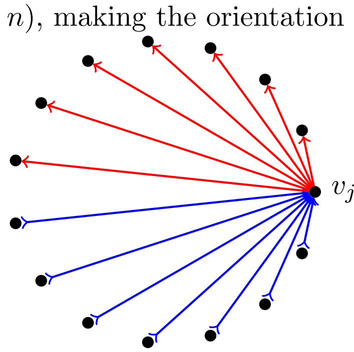


□

1.4.8 Prove that there is an n-vertex tournament with in-degree equal to out-degree at every vertex if and only if n is odd.

Proof. If the in-degree is equal to the out-degree, then, in particular, the underlying graph of the tournament is even. This only happens when n is odd. (The underlying graph is a complete graph, and is regular with degree $n-1$.

Now assume n is odd. One tournament with the same in and out degrees at each vertex is as follows: Label the vertices $v_1, \ldots, v_n$. For each v_i, direct the edges in common with $v_{i+1 \pmod n}, v_{i+2 \pmod n}, \ldots, v_{i+(n-1)/2 \pmod n}$ away (giving it an out-degree of half the available edges). So any vertex v_j points to its $(n-1)/2$ successors and is pointed to by its $(n-1)/2$ predecessors (mod n), making the orientation well-defined.



□

1.4.10 Prove that a digraph is strongly connected if and only if for each partition of the vertex set into non-empty sets S and T, there is an edge from S to T.

Proof. If a digraph is strongly connected, then every pair of vertices u and v has a u, v-path. Now take any two partitions of $V(G)$ into sets S and T, and select vertices $s \in S$ and $t \in T$. Then let P be a s, t-path, and consider the last vertex in P which is not in T. That vertex and its successor in P are joined by an edge from S to T (it has a successor since P's last vertex is in T).

Now assume that for each partition of the vertex set into non-empty sets S and T, there is an edge from S to T. Consider any two vertices u and v; we can construct a u, v-path as follows. Partition $V(G)$ into $S = \{u\}$ and $T = V(G) - u$. Then there is an edge out of u by our assumption. If v is at the tip of one such edge, then we are done. At each step, add vertices which are successors of S, removing them from T (there is at least one, since there is at least one by our hypothesis). As soon as an edge connects to v, we have recovered a u, v-path. □

1.4.14 Let G be an n-vertex digraph with no cycles. Prove that the vertices of G can be ordered as $v_1, \ldots, v_n$ so that if $v_iv_j \in E(G)$, then $i < j$.

Proof. What we are concerned with is the transitivity of order, i.e. a sequence of inequalities produces a new inequality. A sting of inequalities produced by our criteria is equivalent to a walk in G; since G is acyclic, every walk is a u, v-path. So our criteria for the ordering is equivalent to requiring that whenever v_i appears before v_j in any path, i must be less than j. If there is no such order, then there must be some pair of vertices u and v for which there is both a u, v-path and a v, u-path. However, this would produce a cycle (follow the u, v path until it intersects with the v, u-path and then follow the v, u-path back). So there must be some ordering which agrees with all paths. $\square$