Module 4 Special classes of graphs

Contents

There are many classes of graphs which are interesting at an elementary level and also at an advanced level. In this chapter, we introduce three such classes. Before that we define two general concepts.

Definitions.

*(1) A graph theoretic property P is said to be an **hereditary property**, if G has property P, then every subgraph of G has property P.*

*(2) A graph theoretic property P is said to be an **induced hereditary property**, if G has property P, then every **induced subgraph** of G has property P.*

Remarks.

- If G is acyclic, then every subgraph of G is acyclic. Therefore, acyclicity is an hereditary property. It is also induced hereditary.
- If G is connected, then every subgraph of G need not be connected. Therefore, connectedness is not a hereditary property.
- If P is hereditary, then it is induced hereditary. However, the converse is not true. For example, if G is complete, then every subgraph of G need not be complete but every induced subgraph of G is complete.

In the context of induced hereditary properties, the following terminology is useful.

Definitions. *Let H be a graph and $\mathcal{F}$ be a family of graphs.*

(1) A graph G is said to be H-free, if G contains no induced subgraph isomorphic to H.

(2) A graph G is said to be $\mathcal{F}$-free if G contains no induced subgraph isomorphic to any graph in $\mathcal{F}$.

For example, the graph shown in Figure 4.1 is C_4-free but it is not P_4-free. However, it contains a C_4.

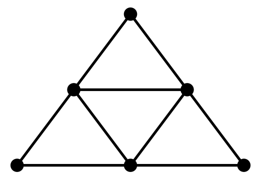


Figure 4.1: A C_4-free graph.

4.1 Bipartite Graphs

Immediate generalization of trees are bipartite graphs.

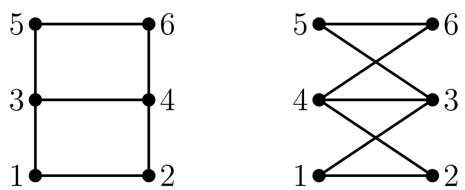
Definition. *A graph G is said to be* ***bipartite****, if $V(G)$ can be partitioned into two parts A and B such that*

(i) no two vertices in A are adjacent, and

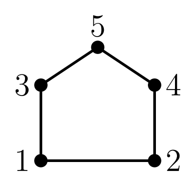
(ii) no two vertices in B are adjacent.

To emphasize, the two parts A and B, we denote a bipartite graph G by $G[A, B]$.

A bipartite graph and a non-bipartite graph are shown in Figure 4.2.



(a) A bipartite graph and its bipartition



(b) A non-bipartite graph

Figure 4.2: Bipartite and non-bipartite graphs.

Clearly,

(i) If G is bipartite and $H \subseteq G$, then H is bipartite. Equivalently, if H is non-bipartite and $H \subseteq G$, then G is non-bipartite. Therefore, "bipartiteness" is a hereditary property.

(ii) Every odd cycle is non-bipartite.

Therefore, if G is bipartite, then it contains no odd cycles. The following characterization of bipartite graphs says that the converse too holds.

Theorem 4.1. *A graph G is a bipartite graph if and only if it contains no odd cycles.*

Proof: (1) G is bipartite $\Rightarrow G$ contains no odd cycles.

Let G be a bipartite graph with bipartition $[A, B]$. Let $C = (v_1, v_2, \ldots, v_k, v_1)$ be a k-cycle in G. The vertices v_i belong alternately to A and B. Let, without loss of generality, $v_1 \in A$. Then, it follows that $v_j \in A$ iff j is odd and $v_j \in B$ iff j is even. Since $v_1 \in A$ and $(v_1, v_k) \in E$, we deduce that $v_k \in B$. So k is even.

(2) G has no odd cycles $\Rightarrow G$ is bipartite.

We give two proofs.

First proof: Clearly, a graph is bipartite iff each of its components is bipartite. So, it is enough if we prove the theorem for connected graphs. Let x be a vertex in G. Define the following sets:

$$A = \{v \in V : dist(x, v) \text{ is even}\} \text{ and } B = \{v \in V : dist(x, v) \text{ is odd}\}.$$

Clearly, $A \cap B = \phi$ and $A \cup B = V(G)$. We claim that no two vertices in A are adjacent. On the contrary suppose that $v_1, v_2 \in A$ are adjacent. So, $dist(x, v_1)$ is even, say $2r$, and $dist(x, v_2)$ is even, say $2s$. Let $P_1(x, v_1)$ and $P_2(x, v_2)$ be the paths of length $2r$ and $2s$ respectively. Let z be the last vertex common between $P_1(x, v_1)$ and $P_2(x, v_2)$. See Figure 4.3.

Since P_1 and P_2 are of minimum length it follows that the subpaths $P_1(x, z)$ and $P_2(x, z)$ have the same length, say k. Then the sequence $P_1(z, v_1), (v_1, v_2)$, $P_2(v_2, z)$ is a cycle of length $(2r-k)+1+(2s-k)$, which is an odd integer. This is a contradiction,

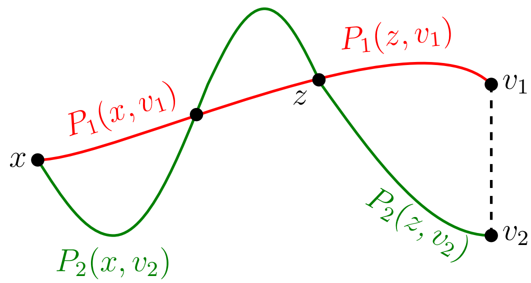


Figure 4.3: $(P(z, v_1), (v_1, v_2), P(v_2, z))$ is a cycle of odd length.

since, G has no odd cycles. Hence, no two vertices in A are adjacent. Similarly, no two vertices in B are adjacent. Hence, $[A, B]$ is a bipartition of G.

Second Proof: By induction on m.

Basic Step: If $m = 0$ or 1, then obviously G is bipartite.

Induction Step: Assume that every graph with $m-1$ edges and with no odd cycles is bipartite. Let G have m edges and let (u, v) be an edge in G. Consider the graph $H = G - (u, v)$. Since G has no odd cycles, H too has no odd cycles. Therefore, by induction hypothesis, H is bipartite. We now make two cases and in each case obtain a bipartition of G.

Case 1: u and v are connected in H.

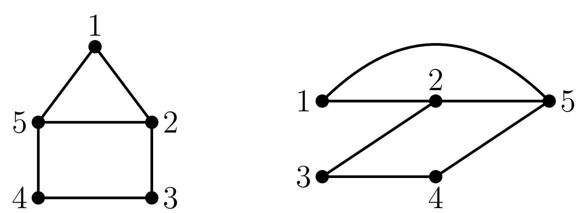
Let $[A, B]$ be a bipartition of H. We claim that u and v are in different sets A and B. On the contrary, suppose that u and v belong to the same set, say A. Since, u and v are connected, there is a path $P(u, v) = (u = u_1, u_2, \ldots, u_{k-1}, u_k = v)$. The vertices u_i belong alternately to A and B and moreover the terminal vertices u and v belong to A; hence the number of vertices in P is odd. Therefore, $P(u, v)$ is a path of even length. But then $(P(u, v), v, u)$ is cycle of odd length in G; a contradiction. So our claim holds; let $u \in A$ and $v \in B$. But then $[A, B]$ is bipartition of $H + (u, v) = G$ too.

Case 2: u and v are not connected in H.

Let D be the component in H which contains u. Since H is bipartite, both D and $H - V(D)$ are bipartite. Let $[A_1, B_1]$ be a bipartition of D and $[A_2, B_2]$ be a bipartition of $H - V(D)$. We can assume that $u \in A_1$ and $v \in B_2$ (otherwise, suitably relabel A_i and B_i). Then $[A_1 \cup A_2, B_1 \cup B_2]$ is a bipartition of G. $\square$

The concept of bipartite graphs can be straightaway generalized to k-partite graphs.

Definition. *A graph G is said to be k-partite, if there exists a partition $(V_1, V_2, \ldots, V_k)$ of $V(G)$ such that no two vertices in V_i $(1 \leq i \leq k)$ are adjacent. It is denoted by $G[V_1, V_2, \ldots, V_k]$.*



(a) G (b) A 3-partition of G.

Figure 4.4: A 3-partite graph G. Note that it is not a bipartite graph.

While bipartite graphs were easily characterized, the characterization of k-partite graphs $(k \geq 3)$ has remained open.

Definition. *A k-partite graph $G[V_1, V_2, \ldots, V_k]$ is said to be a complete k-partite graph if $(x, y) \in G$, whenever $x \in V_i, y \in V_j, i \neq j$. It is denoted by $K_{n_1, n_2, \ldots, n_k}$, if $|V_i| = n_i$, $1 \leq i \leq k$.*

4.2 Line Graphs (Optional)

Given a graph G, we can "derive" many graphs based on G. One such family of graphs called "line graphs" is extensively studied.

Definition. *Let H be a simple graph with n vertices $v_1, v_2, \ldots, v_n$ and m edges $e_1, e_2, \ldots, e_m$. The **line graph** $L(H)$ of H is a simple graph with the vertices $e_1, e_2, \ldots, e_m$ in which e_i and e_j are adjacent iff they are adjacent in H.*

A graph and its line graph are shown below.

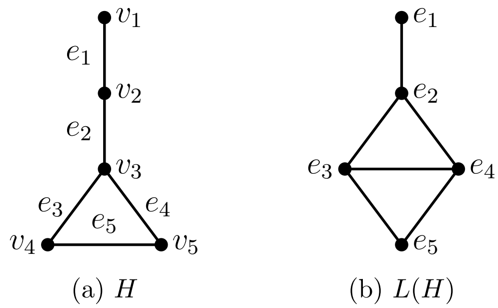


Figure 4.5: A graph and its line graph.

- A graph G is said to be a ***line graph***, if there exists a simple graph H such that $L(H) = G$. The graph shown in Figure 4.5b is a line graph. Two more graphs G_1 and G_2 which are line graphs are shown below. Find simple graphs H_1 and H_2 such that $G_1 = L(H_1)$ and $G_2 = L(H_2)$.

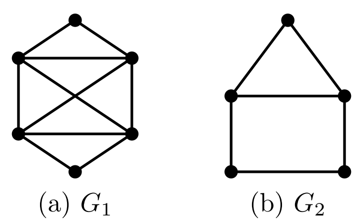
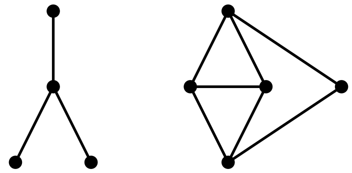


Figure 4.6: Two line graphs.

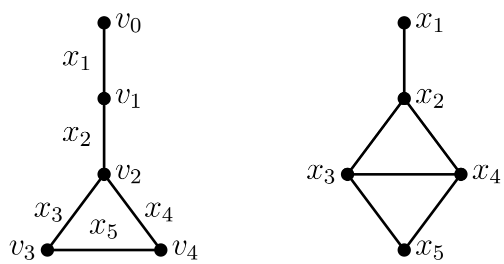
- Two graphs which are not line graphs are shown in Figure 4.7. That is, one can show that there are no graphs H_3 and H_4 such that $G_3 = L(H_3)$ and $G_4 = L(H_4)$, either with case-by-case analysis or by looking at all the graphs on four edges and five edges, respectively. We leave this exercise to the reader.



(a) G_3 (b) G_4

Figure 4.7: Two graphs which are not line graphs.

In view of the existence of graphs which are line graphs and also the existence of graphs which are not line graphs, a natural question arises: Find necessary and sufficient conditions for a graph to be a line graph. Towards this characterization, let us look at the graphs shown in Figure 4.8 more closely.



(a) H (b) G

Figure 4.8: $G = L(H)$

We observe that

$$Q_0 = [\{x_1\}], Q_1 = [\{x_1, x_2\}], Q_2 = [\{x_2, x_3, x_4\}], Q_3 = [\{x_3, x_5\}], Q_4 = [\{x_4, x_5\}]$$

are complete subgraphs of G having the following properties:

(i) Any edge of G belongs to exactly one complete subgraph.

(ii) Any vertex of G belongs to at most two complete subgraphs.

Notice that the vertices of Q_i $(i = 0, 1, 2, 3, 4)$ are the edges incident with the vertex v_i $(i = 0, 1, 2, 3, 4)$ in H. These are the key observations to obtain the first characterization of line graphs. Clearly a graph G is a line graph iff each component of G is a line graph. Moreover, it can be easily verified that every graph on at most 2 vertices is a line graph. So, in the following we assume that G is connected and $n \geq 3$.

Theorem 4.2 (Krausz, 1943)**.** *A connected graph G is a line graph if and only if there exists a family F of complete graphs $\{Q_1, Q_2, \ldots, Q_p\}$ in G such that*

(i) any edge of G belongs to exactly one complete subgraph, and

(ii) any vertex of G belongs to at most two complete subgraphs.

Proof. (1) G is a line graph $\Rightarrow$ G contains a family F of complete subgraphs as described in the theorem.

Since G is a line graph, there exists a simple graph H such that $G = L(H)$.

If v is a vertex of H, then the set of edges $\{x_1, x_2, \ldots, x_d\}$ incident with v in H induces a complete subgraph $Q_v = [\{x_1, x_2, \ldots, x_d\}]$ in G. Our claim is that the family of complete subgraphs $F = \{Q_v \subseteq G : v \in H\}$, satisfies (i) and (ii).

Claim 1: F satisfies (i).

Let $(x, y) \in E(G)$. Then $x = (u, v)$ and $y = (v, w)$ for some vertices u, v, w in H. So, $(x, y) \in E(Q_v)$ in G and no other complete subgraph.

Claim 2: F satisfies (ii).

Let x be a vertex of G; so x is an edge in H. Let $x = (u, v)$, where $u, v \in V(H)$. Then $x \in V(Q_u) \cap V(Q_v)$ and no other complete subgraph in F.

(2) G contains a family $F = \{Q_1, Q_2, \ldots, Q_p\}$ of complete subgraphs as described in the theorem $\Rightarrow$ G is a line graph.

We construct a graph H such that $L(H) = G$. For each Q_i we take a vertex v_i, $i = 1, 2, \ldots, p$. Next, for each vertex x_i of G which belongs to exactly one of the complete subgraphs of F, we take a vertex u_i, $i = 1, 2, \ldots, r$ (say). Then $u_1, u_2, \ldots, u_r$

and $v_1, v_2, \ldots, v_p$ are the vertices of H. Two of these vertices are joined in H iff the corresponding complete graphs share a vertex in G. Then $L(H) = G$. $\square$

We next state two structural characterizations of line graphs. The graph shown in Figure 4.9 is called a ***diamond***.

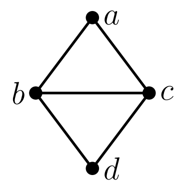


Figure 4.9: A diamond, $K_4 - e$.

A diamond D in G with triangles $[a, b, c]$ and $[b, c, d]$ as shown above is called an odd diamond of G, if

(i) $D \sqsubseteq G$,

(ii) there is a vertex $x \in V(G) - V(D)$ which is adjacent with odd number of vertices of $[a, b, c]$, and

(iii) there is a vertex $y \in V(G) - V(D)$ which is adjacent with odd number of vertices of $[b, c, d]$;

Here the vertices x and y need not be distinct. For example, the three graphs shown in Figure 4.10 contain odd diamonds.

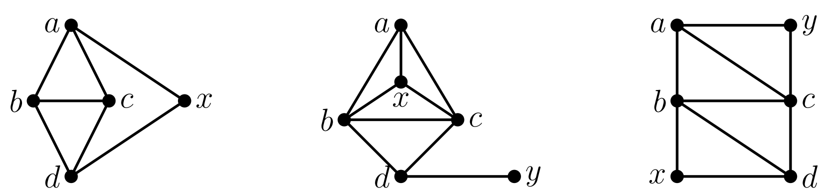
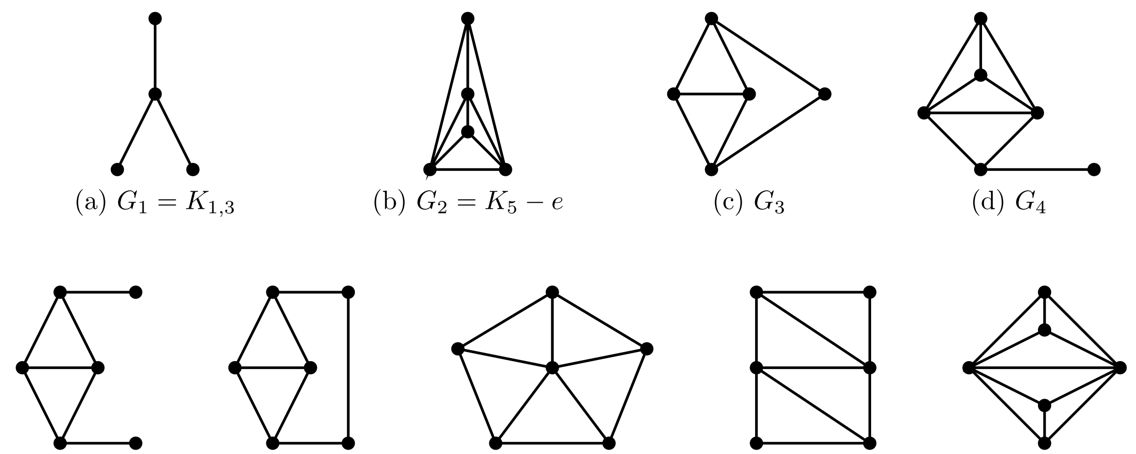


Figure 4.10: Graphs containing odd diamonds (Can you identify them?)

Theorem 4.3 (A.V. Van Rooiz and H.S. Wilf, 1965)**.** *A graph G is a line graph iff it contains no induced $K_{1,3}$ and no odd diamonds.*

Theorem 4.4. *(Beineke, 1968) A graph G is a line graph iff it contains none of the nine graphs shown in Figure 4.11 as an induced subgraph.*



(a) $G_1 = K_{1,3}$ (b) $G_2 = K_5 - e$ (c) G_3 (d) G_4

(e) G_5 (f) G_6 (g) G_7 (h) G_8 (i) G_9

Figure 4.11: Nine forbidden graphs of line graphs

4.3 Chordal Graphs (Optional)

Definitions.

- *Let $C_k (k \geq 4)$ be a cycle in G. An edge of G which joins two non-consecutive vertices of C_k is called a* ***chord****.*
- *A graph G is said to be a* ***chordal graph****, if every cycle $C_k (k \geq 4)$ in G has a chord. A chordal graph is also called a* ***triangulated graph****.*

Remarks.

1. Every induced subgraph of a chordal graph is chordal.
2. A subgraph of a chordal graph need not be a chordal graph.

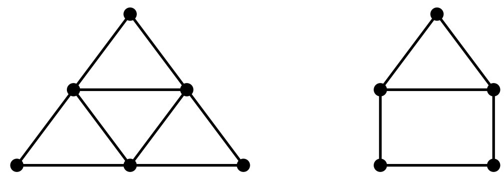


Figure 4.12: A chordal graph and a non-chordal graph

The class of chordal graphs and its subclasses have been a subject of considerable interest because of their structural properties and interesting characterizations. We state and prove two characterizations.

Definitions. *Let a, b be any two non-adjacent vertices in a graph G.*

(i) An ***a-b-separator*** *is a subset S of vertices such that a and b belong to two distinct components $G - S$.*

(ii) A ***minimal a-b-separator*** *S is an a-b-separator such that no proper subset of S is an a-b-separator.*

In the graph shown in Figure 4.13, $\{u, v, x\}$ is a y-z-separator. It is not a minimal y-z-separator, since $\{u, x\}$ is a y-z-separator. In fact, $\{u, x\}$ is a minimal y-z-separator. However, $\{u, v, x\}$ is a minimal y-w-separator. These examples illustrate that the minimality of an a-b-separator depends on the vertices a and b. Next, $\{z\}$ is a minimal w-p-separator and $\{u, x\}$ is also a minimal w-p-separator. These examples illustrate that the minimal separators need not have same number of vertices.

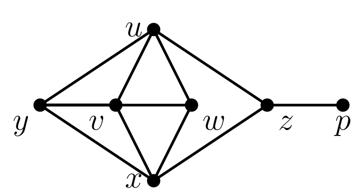


Figure 4.13: Illustration for separators

Remark.

- For any two vertices x, y in G, there exists a x-y-separator if and only if x and y are non-adjacent.

Theorem 4.5 (Dirac, 1961)**.** *A graph is chordal iff every minimal a-b-separator induces a complete subgraph, for every pair of non-adjacent vertices a and b in G.*

Proof. (1) Every minimal a-b-separator, for every pair of non-adjacent a, b in G, induces a complete subgraph in $G \Rightarrow G$ is chordal.

Let $C : (x_1, x_2, \ldots, x_r, x_1)$ be a cycle in G of length ≥ 4; see Figure 4.14.

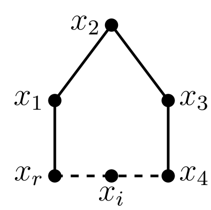


Figure 4.14: $(x_1, x_3) \in E(G)$ or $(x_2, x_i) \in E(G)$.

If (x_1, x_3) is an edge, then it is a chord. Next suppose $(x_1, x_3) \notin E(G)$ and let S be a minimal x_1-x_3-separator. Then S contains x_2 and some x_i $(4 \leq i \leq r)$. Since S induces a complete subgraph we deduce that $(x_2, x_i) \in E(G)$ and it is a chord.

(2) G is chordal $\Rightarrow$ Every minimal a-b-separator, for every pair of non-adjacent a, b in G, induces a complete subgraph in G.

Let S be a minimal a-b-separator in G. If $|S| = 1$, then $[S] = K_1$, and so the assertion holds. If $|S| \geq 2$, let $x, y \in S$. Our goal is to show that $(x, y) \in E(G)$; see Figure 4.15. Let G_a and G_b be the components of $G - S$ containing a and b respectively.

Since S is minimal, the sets of edges $[x, V(G_a)], [y, V(G_a)], [x, V(G_b)]$ and $[y, V(G_b)]$ are all non-empty. Let $(x, a_1, a_2, \ldots, a_r, y)$ be a path in G such that

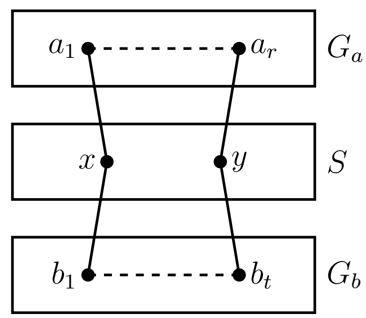


Figure 4.15: Schematic representation of G.

$a_1, a_2, \ldots, a_r \in V(G_a)$ and r is minimum. Let $(x, b_1, b_2, \ldots, b_t, y)$ be a path in G such that $b_1, b_2, \ldots, b_t \in V(G_b)$ and t is minimum.

Consider the cycle $C = (x, a_1, a_2, \ldots, a_r, y, b_t, b_{t-1}, \ldots, b_1, x)$. Since r is minimum, we deduce that

(i) x is non-adjacent to $a_2, a_3, \ldots, a_r$.

(ii) y is non-adjacent to $a_1, a_2, a_3, \ldots, a_{r-1}$.

(iii) a_i, a_j are non-adjacent for every $i, j \in \{1, 2, \ldots, r\}, j \notin \{i+1, i-1\}$; that is, the path $(a_1, \ldots, a_r)$ has no chords.

Similarly, since t is minimum, we deduce that

(i) x is non-adjacent to $b_2, b_3, \ldots, b_t$.

(ii) y is non-adjacent to $b_1, b_2, \ldots, b_{t-1}$.

(iii) b_i, b_j are non-adjacent for every $i, j \in \{1, 2, \ldots, t\}, j \notin \{i+1, i-1\}$; that is, the path $(b_1, \ldots, b_t)$ has no chords.

Since G_a and G_b are components of $G - S$, $(a_i, b_j) \notin E(G)$ for all i, j, $1 \leq i \leq r, 1 \leq j \leq t$. All these non-adjacency properties imply that $(x, y) \in E(G)$, since G is chordal and C is a cycle. Since x and y are arbitrary, it follows that S induces a complete subgraph in G. $\square$

The second characterization of chordal graphs has proved useful in designing algorithms to find various parameters of chordal graphs. Before stating this characterization, we define a new concept and prove a crucial theorem to obtain the characterization.

Definition. *A vertex v of a graph G is called a* ***simplicial vertex****, if its neighborhood $N(v)$ induces a complete subgraph in G.*

In the following graph, x is a simplicial vertex while y and z are non-simplicial vertices.

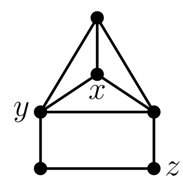


Figure 4.16: A graph with simplicial vertices and non-simplicial vertices.

Theorem 4.6. *Every chordal graph G has a simplicial vertex. Moreover, if G is not complete, then G has two non-adjacent simplicial vertices.*

Proof. We prove the theorem by induction on n. If $n = 1, 2$ or 3, the statement is obvious. So we proceed to the induction step. If G is complete, then every vertex of G is simplicial. Next suppose that G has two non-adjacent vertices say a and b. Let S be a minimal a-b-separator. $[S]$ is complete by the previous theorem. Let G_a, G_b be the components in $G - S$ containing a and b respectively. Let $A = V(G_a)$ and $B = V(G_b)$.

We apply induction hypothesis to the induced subgraph $H = [A \cup S]$ which is chordal. If H is complete, then every vertex of H is a simplicial vertex of H. If H is not complete, then by induction hypothesis, H contains two non-adjacent vertices which are simplicial. One of these vertices is in A, since S is complete. So, in either case A contains a simplicial vertex of H, say x. Since $x \in A$, $N_G(x) = N_H(x)$. So, x

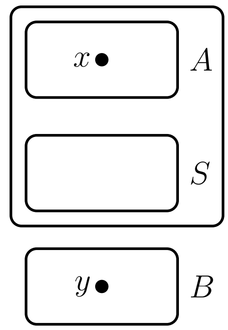


Figure 4.17: Schematic representation.

is also a simplicial vertex of G. Similarly, B contains a simplicial vertex of G, say y. Then x and y are two non-adjacent simplicial vertices of G. □

Definition. *A graph G on n vertices is said to have a* ***perfect elimination ordering (PEO)*** *if its vertices can be ordered $(v_1, v_2, \ldots, v_n)$ such that v_i is a simplicial vertex of the induced subgraph $[\{v_i, v_{i+1}, \ldots, v_n\}]$, $i = 1, 2, \ldots, n-1$.*

In the following graph $(v_1, v_3, v_5, v_2, v_4, v_6)$ is a PEO but $(v_1, v_2, v_3, v_5, v_4, v_6)$ is not a PEO.

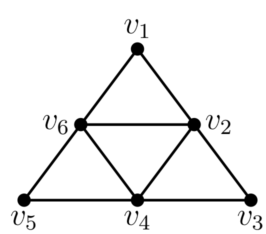


Figure 4.18: A graph with perfect elimination ordering.

Theorem 4.7 (Dirac 1961, Fulkerson and Gross, 1965, Rose 1970)**.** *A graph G is chordal iff it admits a PEO.*

Proof. (1) G is chordal $\Rightarrow$ G admits a PEO.

Since G is chordal, it contains a simplicial vertex, say v_1. Since $G_1 = G - v_1$ is chordal, it contains a simplicial vertex, say v_2. Similarly, $G_2 = G - v_1 - v_2$ contains a simplicial vertex, say v_3. Continuing this process, we obtain an ordering of the vertices $(v_1, v_2, \ldots, v_n)$ which is a PEO of G.

(2) G admits a PEO $\Rightarrow$ G is chordal.

Let $(v_1, v_2, \ldots, v_n)$ be a PEO of G and let C be an arbitrary cycle in G of length ≥ 4. Let v_k be the first vertex in the PEO which belongs to C. Let x and y be the vertices of C adjacent to v_k; see Figure 4.19.

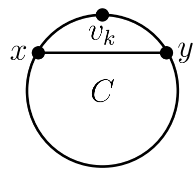


Figure 4.19: C contains the chord (x, y).

By our choice of v_k, it follows that $x, y \in \{v_{k+1}, v_{k+2}, \ldots, v_n\}$. Since v_k is a simplicial vertex, all its neighbors in $\{v_k, v_{k+1}, \ldots, v_n\}$ are mutually adjacent. So, we conclude that x and y are adjacent. Hence, every cycle in G contains a chord. $\square$

Exercises

1. Which of the graphs shown in Figure 4.20 are bipartite? If bipartite, then redraw them with a suitable bipartition.

2. Show that the maximum number of edges in a bipartite graph is $[\frac{n^2}{4}]$. Characterize the bipartite graphs $G[A, B]$ with $[\frac{n^2}{4}]$ edges.

3. If G is a connected bipartite graph, show that the bipartition of $V(G)$ is unique, in the sense that if $[A, B]$ and $[C, D]$ are bipartitions of G, then either (i) $A = C$ and $B = D$ or (ii) $A = D$ and $B = C$.

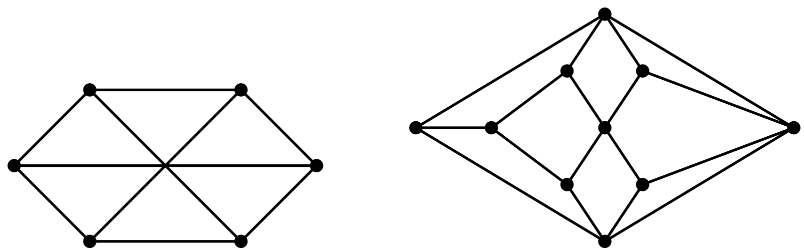


Figure 4.20

4. If G is a connected simple graph such that $G-v$ is bipartite for every $v \in V(G)$, then show that G is either bipartite or G is an odd cycle.

5. If $G[A,B]$ is a k-regular bipartite graph ($k \geq 1$), then show that $|A| = |B|$.

6. If G and H are bipartite, show that $G \Box H$ is bipartite.

7. Show that the hypercube Q_d is bipartite, for every $d \geq 1$.

8. Draw all the simple bipartite graphs G such that G^c is also bipartite.

9. A bipartite simple graph $G[A,B]$ has 22 vertices where $|A| = 12$. Suppose that every vertex in A has degree 3, and every vertex in B has degree 2 or 4. Find the number of vertices of degree 2.

10. Show that a simple graph G is a complete k-partite graph for some $k(\geq 2)$ iff whenever $(u,v) \notin E(G)$ and $(v,w) \notin E(G)$, then $(u,w) \notin E(G)$. (Hint: Consider G^c.)

11. Find the degree sequence and the number of edges of $K_{1,2,\ldots,r}$ and $(K_{1,2,\ldots,r})^c$.

12. Find
    (a) $k_0(K_{1,2,\ldots,r})$ and $k_1(K_{1,2,\ldots,r})$.
    (b) $k_0(K_{p,p,\ldots,p})$ and $k_1(K_{p,p,\ldots,p})$, where p is repeated k times.
    (c) $k_0(K_{a,b,c})$ and $k_1(K_{a,b,c})$, where $1 \leq a \leq b \leq c$.

13. Find the diameter and the girth of $K_{1,2,\ldots,p}$.

14. Suppose that the average degree of a simple graph G is b and x is a vertex of G. Show that $G - x$ has average degree at least b iff $deg_G(x) \leq \frac{b}{2}$. (Average degree of a graph $H := \frac{2m(H)}{n(H)}$.)

15. Let $k \geq 2$. Show that there is no bipartite graph on n vertices with $n-1$ vertices of degree k and one vertex of degree $k-1$.

16. If G is a simple graph on 7 vertices, then show that either $C_3 \subseteq G$ or $C_4 \subseteq G^c$. Construct a simple graph H on 6 vertices such that neither $C_3 \subseteq H$ nor $C_4 \subseteq H^c$.

17. Let $\underline{a} = (a_1 \geq a_2 \geq \cdots \geq a_r)$ and $\underline{b} = (b_1 \geq b_2 \geq \cdots \geq b_s)$ be sequences of non-negative integers such that $\sum_{i=1}^{r} a_i = \sum_{j=1}^{s} b_j$. Show that there exists a simple bipartite graph $G[\{x_1, x_2, \ldots, x_r\}, \{y_1, y_2, \ldots, y_s\}]$ such that $deg(x_i) = a_i$, $1 \leq i \leq r$ and $deg(y_j) = b_j$, $1 \leq j \leq s$ iff

$$\sum_{i=1}^{r} \min\{a_i, k\} \geq \sum_{j=1}^{k} b_j, \text{ for } k = 1, 2, \ldots, s.$$

18. Draw a 3-partite graph on 7 vertices with maximum number of edges.

19. Prove or disprove: If $K_{r,s,t}$ is a regular graph, then $r = s = t$.

20. Draw a spanning tree T of $G = K_{4,4}$ such that $dist_T(u,v) \leq 3 \cdot dist_G(u,v)$, for every pair of vertices u, v.

21. Draw the following spanning trees of $K_{2,3} (= G)$.
    (a) A spanning tree which is a complete binary tree.
    (b) A spanning tree of diameter 3.
    (c) A spanning tree T of G such that $dist_T(x,y) \leq dist_G(x,y) + 1$, for every pair of vertices x, y.

22. If G is a tree with maximum degree 3 and bipartition $[X, Y]$, where $|X| \leq |Y|$, then show that there exist at least $|Y| - |X| + 1$ vertices of degree 1 in G.

23. The following $(0,1)$- matrix represents a bipartite graph G with vertices a, b, c, d in one part and e, f, g, h in another part. Draw G.

	e	f	g	h
a	1	0	1	0
b	0	1	1	0
c	1	0	1	1
d	1	1	1	0

24. Prove or disprove: There is a bipartite graph with bipartition $[X, Y]$, where $X = \{x_1, x_2, x_3, x_4, x_5\}$, $Y = \{y_1, y_2, y_3, y_4, y_5\}$, $deg(x_j) = deg(y_j) = j$ for $j = 1, 2, 4, 5$ and $deg(x_3) = deg(y_3) = 4$.

25. Prove or disprove: If G is a r-regular bipartite simple graph, the G contains a spanning k-regular subgraph H, for every k, $1 \leq k \leq r$.

26. Characterize the trees which are line graphs.

27. Characterize the connected bipartite graphs which are line graphs.

28. Prove or disprove:
    (a) If G is a line graph, then $G - v$ is a line graph, for every $v \in V(G)$.
    (b) If G is such that $G - v$ is a line graph, for every $v \in V(G)$, then G is a line graph.

29. If $e = (u, v) \in E(G)$, then find $deg_{L(G)}(e)$.

30. Characterize the connected graphs G such that $G = L(G)$.

31. If G is p-edge-connected then show the following:
    (a) $L(G)$ is p-vertex-connected.
    (b) $L(G)$ is $(2p - 2)$-edge-connected.

32. For a simple graph $G(n \geq 4)$, show that the following statements are equivalent.
    (a) G is a line graph of a triangle-free graph.
    (b) G is $\{K_{1,3}, K_4 - e\}$-free.

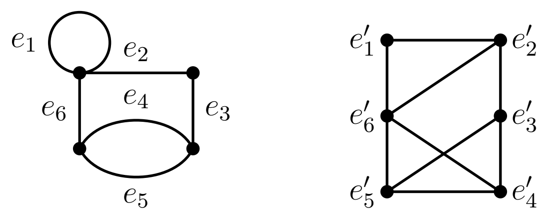
33. For a simple graph $G(n \geq 4)$, show that the following statements are equivalent.
    (a) G is the line graph of a bipartite graph.
    (b) G is $\{K_{1,3}, K_4 - e, C_{2k+1}(k \geq 2)\}$-free.

34. For a simple connected graph G, show that the following statements are equivalent.
    (a) G is the line graph of a tree.
    (b) G is $\{K_{1,3}, K_4 - e, C_k(k \geq 4)\}$-free.

35. Let G be a simple connected graph. Show that $L(G)$ is bipartite iff G is a path or an even cycle.

36. Show that a simple graph G is chordal iff $L(G)$ is chordal.

The line graphs of simple graphs were defined in this chapter. The line graph of any graph (not necessarily simple) can be defined as follows.
Let G be a graph with edges $e_1, e_2, \ldots, e_m$. The line graph $L(G)$ of G has vertices $e'_1, e'_2, \ldots, e'_m$. Two vertices e'_i and e'_j are joined by an edge in $L(G)$ iff e_i and e_j are adjacent in G.
A graph H is said to be a line graph, if there exists a graph G such that $L(G) = H$.



(a) G (b) $L(G)$

Figure 4.21: A graph and its line graph.

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37. Find which of the graphs shown in Figure 4.11 are line graphs of graphs (which are not necessarily simple).

38. Prove or disprove:
    (a) If G is chordal, then G^c is chordal.
    (b) If G is chordal, then $G - v$ is chordal, for every $v \in V(G)$.
    (c) If G is such that $G - v$ is chordal, for every $v \in V(G)$, then G is chordal.

39. Get a PEO of $K_p^c + K_t$.