Module 1 Preliminaries

Contents

1.1 Introduction: Discovery of graphs

We begin the course with a set of problems stated in a quiz format. However, when generalized, these problems lead to deep theorems and interesting applications. So, we urge you to do the following exercises too after solving the quizzes.

- Generalize the problems in various ways.
- Justify your solutions with logical precision.
- Wherever possible design an algorithm to find an optimal solution.
- Code algorithms and run on your desktop. Though coding is not a part of mathematics, this experience will help you to see the solution in a more transparent way.

This whole exercise will enable you to anticipate some of the concepts and theorems before they are stated.

Problem 1: In a college campus, there are seven blocks, Computer Center(C), Library(L), Academic Zone(AC), Administrative Zone(AD), Hospital(H), Guest House(G), Security(S). The problem is to design two LANs satisfying certain conditions:

1. **LAN 1:**
   (i) Two of the blocks are connected to exactly five of the blocks.
   (ii) Two of the blocks are connected to three of the blocks.
   (iii) Three of the blocks are connected to two of the blocks.
2. **LAN 2:**
   (i) Four of the blocks are connected to five of the blocks.
   (ii) Three of the blocks are connected to two of the blocks.

(You can choose the blocks of your choice to satisfy the required conditions. No multiple cables and self loops are permitted.)

Problem 2: There are seven persons. If A calls B, all the information they know is exchanged. Find the minimum number of calls required so that at the end of these many calls, everybody knows the information that everybody else has.

Problem 3: Show that in a party of six persons either (i) there are three persons who know one another or (ii) there are three persons who does not know one another. Can we conclude the same if the party consists of five persons?

Problem 4: A saturated hydrocarbon molecule consists of carbon atoms and hydrogen atoms. Recall from your high school Chemistry that every carbon atom has valency four and every hydrogen atom has valency one. For example, ethane C_2H_6 is such a molecule. If there are n carbon atoms in the molecule, find the number of hydrogen atoms.

Problem 5: In 1996, Nobel prize in Chemistry was awarded to R.F. Curl, R. E. Smalley and H.W. Kroto for their role in the discovery of pure carbon molecules called fullerenes, that is, a fullerene contains carbon atoms and no other atoms. The Kekule structure of a fullerene is a pseudospherical polyhedral shell satisfying the following conditions:

(i) Every corner of the polyhedron is occupied by a carbon atom such that every carbon atom is linked with one other carbon atom with a double bond and with two other carbon atoms with single bonds; notice that this way the valency of every carbon atom is four. (For convenience, the double bonds are replaced by single bonds in creating physical models.)

(ii) Every face of the polyhedron is a pentagonal face or a hexagonal face.

If a fullerene has n carbons atoms, show that $n = 20 + 2k$ for some $k \geq 0$. Thus there is a fullerene with sixty carbon atoms, but no fullerene with sixty nine carbon atoms.

1.2 Graphs

Intuitively, a graph consists of a set of points and a set of lines such that each line joins a pair of points.

• Definitions

- A ***graph*** G is a triple (V, E, I_G), where V, E are sets, and $I_G : E \rightarrow V^{(2)}$ is a function, where $V^{(2)} = \binom{V}{2} \cup \{(v, v) : v \in V\}$. An element $\{u, v\} \in \binom{V}{2}$ is denoted by (u, v) or (v, u). For convenience, it is assumed that $V \cap E = \phi$.
- An element of V is called a ***vertex.***
- An element of E is called an ***edge.***
- I_G is called an ***incidence relation.***

Throughout this course, we assume that V and E are ***finite***, and denote $\lvert V\rvert$ by n and $\lvert E\rvert$ by m. V and E are also denoted by V(G) and E(G) respectively.

An example of a graph:

Let $V = \{v_1, v_2, v_3, v_4, v_5\}$, $E = \{a, b, c, d, e, f, g\}$, and let $I_G : E \rightarrow V^{(2)}$ be defined as follows: $I_G(a) = (v_1, v_2)$, $I_G(b) = (v_2, v_2)$, $I_G(c) = (v_2, v_3)$, $I_G(d) = (v_3, v_4)$, $I_G(e) = (v_3, v_4)$, $I_G(f) = (v_4, v_5)$, $I_G(g) = (v_1, v_5)$. Then (V, E, I_G) is a graph with five vertices and seven edges. It may be observed that I_G is neither one-one nor onto.

• Pictorial representation of a graph

Any graph is represented by a diagram as follows. Each vertex is represented by a point. If $I_G(x) = (u, v)$, then u and v are joined by a line and it is labeled x. This representation is not unique; you can choose to put the points on the plane wherever you like and draw the lines whichever way you like. Two representations of the above graph are shown in Figure 1.1.

Much of the terminology in graph theory is inspired by such a representation.

Definitions. *Let $G(V, E, I_G)$ be a graph.*

- *If $I_G(x) = (u, v)$, then:*
  (i) x is said to **join** *u and v,*
  (ii) x is said to be **incident** *with u and v and vice-versa,*

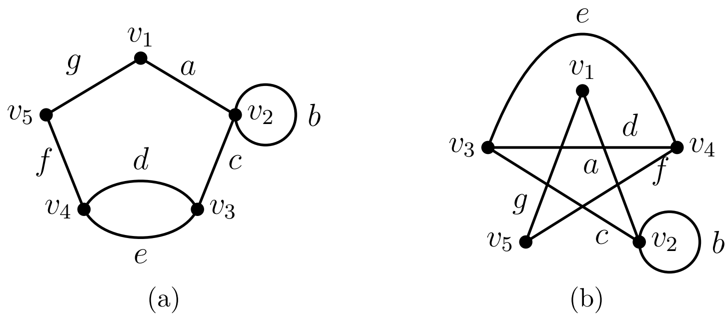


Figure 1.1: Two representations of a graph.

(iii) u and v are said to be **adjacent**,

(iv) u and v are said to be the **end-vertices** *of x,*

(v) x is denoted by $x(u, v)$, to emphasis its end vertices.

- *If there is no x such that $I_G(x) = (u, v)$, then u and v are said to be* ***non-adjacent.***

- *If x and y are edges incident with a vertex v, then they are said to be* ***adjacent edges***.

- *If x and y are edges joining the same pair of vertices, then they are called* ***multiple edges***.

- *If $I_G(x) = (u, u)$, then x is called a* ***loop*** *incident with u.*

- *V is called a* ***simple graph*** *if it has neither multiple edges nor loops. That is, I_G is one-one and $I_G : E \to \binom{V}{2}$.*
  Since I_G is one-one, we can identify any $e \in E$, uniquely with its image $I_G(e)$ $= (u, v) \in \binom{V}{2}$. So, we can alternatively define a simple graph as follows:

- *A* ***simple graph*** *G is a pair (V, E), where V is a non-empty set and $E \subseteq \binom{V}{2}$.*
  For example, consider the graph of Figure 1.1. Here,

  - *v_1 and v_2 are adjacent vertices.*
  - *v_1 and v_3 are non-adjacent vertices.*
  - *a joins v_1 and v_2 and therefore it is incident with v_1 and v_2.*
  - *v_1 and v_2 are the end-vertices of a.*

- *d is not incident with v_1.*
- *b is a loop.*
- *d and e are the multiple edges.*
- *It is not a simple graph.*

A few simple graphs are shown below.

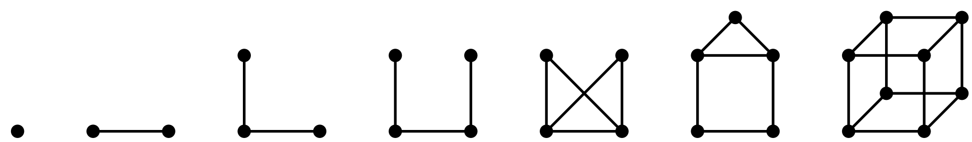


Figure 1.2: Simple graphs

Remark. *While adjacency is a relation among like elements (vertices or edges), incidence is a relation among unlike elements (vertices and edges).*

• Isomorphic graphs

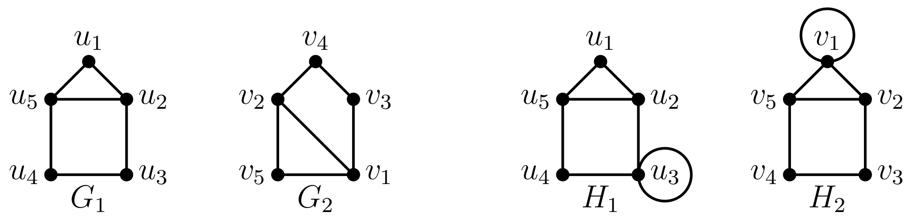
When do we say two graphs are "similar"? The concept of isomorphism is central to all branches of mathematics.

Definitions.

- *Two graphs $G(V, E, I_G)$ and $H(W, F, I_H)$ are said to be* ***isomorphic*** *if there exist two bijections $f : V \to W$ and $g : E \to F$ such that an edge e joins u and v in G if and only if the edge $g(e)$ joins $f(u)$ and $f(v)$ in H.*
- *If G and H are isomorphic, we write $G \simeq H$, and the pair (f, g) is called an* ***isomorphism*** *between G and H.*
- *If $G = H$, then (f, g) is called an* ***automorphism***.

Figure 1.3 shows a pair of isomorphic graphs and a pair of non-isomorphic graphs.

Remarks.



(a) $G_1 \simeq G_2$ (b) $H_1 \not\simeq H_2$

Figure 1.3: Isomorphic and non-isomorphic graphs.

- Two simple graphs $G(V, E)$, and $H(W, F)$ are isomorphic if and only if there exists a bijection $f : V \to W$ such that two vertices u and v are adjacent in G if and only if $f(u)$ and $f(v)$ are adjacent in H.
- Two graphs $G(V, E, I_G)$ and $H(W, F, I_H)$ are isomorphic if and only if there exists a bijection $f : V \to W$ such that two vertices u and v are joined by k edges in G iff $f(u)$ and $f(v)$ are joined by k edges in H.
- Unfortunately, we have no way to "check" whether two given graphs are isomorphic, except by the brute force method. To show that two given graphs are isomorphic, we have to define a bijection f satisfying the properties stated above. To show that two graphs G_1 and G_2 are non-isomorphic, we have to find a property that G_1 has but G_2 does not have. For example, if $|V(G_1)| \neq |V(G_2)|$ or $|E(G_1)| \neq |E(G_2)|$, then $G_1 \not\simeq G_2$. During this course, we will learn many properties which may be used to show the non-existence of isomorphism.

The problem of designing a "good" (technically, polynomial time) algorithm to check whether two given graphs are isomorphic or not, carries a reward of one million dollars. For details, open any search engine and type "Clay Mathematical Institute".

• Subgraphs

Definition. A graph $H(W, F, I_H)$ is called a ***subgraph*** of $G(V, E, I_G)$, if $W \subseteq V$, $F \subseteq E$ and if $e \in F$ joins u and v in H, then e joins u and v in G (note that the converse is not demanded).

We next define various kinds of subgraphs.

Definitions. *Let $H(W, F, I_H)$ be a subgraph of $G(V, E, I_G)$.*

*(i) H is called a **spanning subgraph**, if $W = V$.*

*(ii) H is called an **induced subgraph** if e joins u, v in G, where $u, v \in W$, then e joins u, v in H. H is denoted by $[W]_G$ or $[W]$.*

(iii) If $V_1 \subseteq V$, then $G - V_1$ is the graph $[V - V_1]$. In other words, $G - V_1$ is obtained by deleting every vertex of V_1 and every edge that is incident with a vertex in V_1. In particular, $G - \{v\}$ is denoted by $G - v$.

(iv) If $E_1 \subset E$, then the subgraph $G - E_1$ is obtained by deleting all the edges of E_1 from G. Note that, it is a spanning subgraph of G. If $E_1 = \{e\}$, then $G - \{e\}$ is denoted by $G - e$.

If H is a subgraph of G, we write $H \subseteq G$. If H is an induced subgraph of G, we write $H \sqsubseteq G$.

These various subgraphs are illustrated in Figure 1.4.

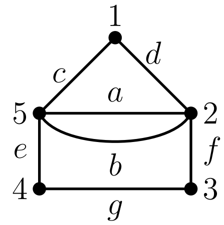
Remarks.

(i) If H is a spanning induced subgraph of G, then $H = G$.

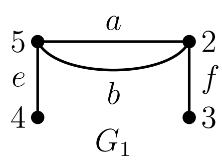
(ii) If H is an induced subgraph of G, then $H = G - V_1$, for some $V_1 \subseteq V$.

• Matrix representations of graphs

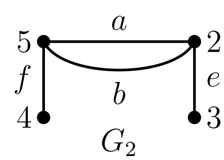
Clearly, we cannot represent a graph pictorially in a computer and hope to compute the parameters associated with graphs. More precisely, pictorial representations do not serve as data structures of graphs. We define below two convenient representations which can serve as data structures.



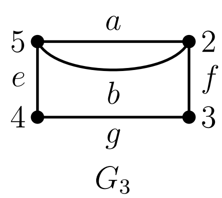
(a) G



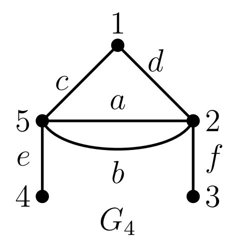
(b) G_1 is a subgraph of G. It is neither an induced subgraph nor a spanning subgraph of G.



(c) G_2 is not a subgraph of G. However, it is isomorphic with a subgraph of G, namely G_1.



(d) G_3 is an induced subgraph of G but it is not a spanning subgraph. In fact, $G_3 = G - 1$.



(e) G_4 is spanning subgraph but not an induced subgraph. In fact, $G_4 = G - g$.

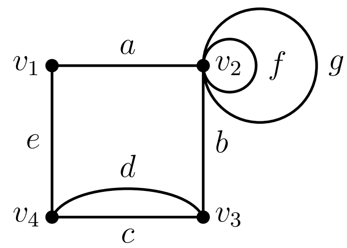
Figure 1.4: A graph G and its various subgraphs.

Definitions. Let G be a graph with vertices $v_1, v_2, \ldots, v_n$ and edges $e_1, e_2, \ldots, e_m$.

(1) The ***adjacency matrix*** $A(G) = [a_{ij}]$ of G, is the $n \times n$ matrix, where

$$a_{ij} = \begin{cases} \text{the number of edges joining } v_i \text{ and } v_j, & \text{if } i \neq j. \\ 2 \times (\text{the number of loops incident with } v_i), & \text{if } i = j. \end{cases}$$

Remarks.



(a) G

	v_1	v_2	v_3	v_4
v_1	0	1	0	1
v_2	1	4	1	0
v_3	0	1	0	2
v_4	1	0	2	0

(b) $A(G)$

	a	b	c	d	e	f	g
v_1	1	0	0	0	1	0	0
v_2	1	1	0	0	0	2	2
v_3	0	1	1	1	0	0	0
v_4	0	0	1	1	1	0	0

(c) $B(G)$

Figure 1.5: A graph G with its adjacency matrix and incidence matrix.

- The adjacency matrix is a symmetric matrix.

- If G is simple, then every entry in $A(G)$ is zero or one and every diagonal entry is zero.

- Any matrix M in which every entry is zero or one is called a ***zero-one*** matrix or a ***binary*** matrix. So, $A(G)$ is a binary matrix if G is simple.

(2) The ***incidence matrix*** $B(G) = [b_{ij}]$ of G, is the $n \times m$ matrix, where

$$b_{ij} = \begin{cases} 0, & \text{if } v_i \text{ is not incident with } e_j. \\ 1, & \text{if } v_i \text{ is incident with } e_j \text{ and } e_j \text{ is not a loop.} \\ 2, & \text{if } v_i \text{ is incident with } e_j \text{ and } e_j \text{ is a loop.} \end{cases}$$

Remark. If G is simple, then B(G) is a zero-one matrix.

• Degree of a vertex

The ***degree*** of a vertex v in a graph G is the number of edges incident with v, with loops counted twice. It is denoted by $deg_G(v)$, $deg(v)$ or simply $d(v)$. The degree of a vertex is also called the ***valency***.

The following equations can be easily observed.

- ∘ $\sum_{j=1}^{n} a_{ij} = deg(v_i)$, for every $i, 1 \leq i \leq n$.
- ∘ $\sum_{i=1}^{n} a_{ij} = deg(v_j)$, for every $j, 1 \leq j \leq n$.
- ∘ $\sum_{i=1}^{n} b_{ij} = 2$, for every $j, 1 \leq j \leq m$.
- ∘ $\sum_{j=1}^{m} b_{ij} = deg(v_i)$, for every $i, 1 \leq i \leq n$.

Theorem 1.1. *In any graph G, $\sum_{v \in V(G)} d(v) = 2m$.*

Proof. Every edge contributes 2 to the left hand side sum.

Alternatively, we can also use the incidence matrix $[b_{ij}]$ to prove the theorem. Let $V(G) = \{v_1, v_2, ..., v_n\}$. Then

$$\sum_{i=1}^{n} deg(v_i) = \sum_{i=1}^{n}\sum_{j=1}^{m} b_{ij} = \sum_{j=1}^{m}\left(\sum_{i=1}^{n} b_{ij}\right) = \sum_{j=1}^{m} 2 = 2m. \qquad \square$$

Corollary. *In any graph G, the number of vertices of odd degree is even.*

Proof. Exercise. □

Definitions.

- ∘ *If u and v are adjacent vertices in G, then u and v are said to be **neighbors**. The set of all neighbors of a vertex x in G is denoted by $N_G(x)$ or $N(x)$. Clearly, if G is simple, then $|N_G(x)| = deg_G(x)$.*

- *If $A, B \subseteq V(G)$ are disjoint non-empty subsets, then*

  $$[A, B] := \{e \in E(G) : e \textit{ has one end-vertex in } A \textit{ and other end-vertex in } B\}.$$

  If $A = \{a\}$, then $[\{a\}, B]$ is denoted by $[a, B]$.

The following two-way counting argument is often useful in estimating $|[A, B]|$. In this technique, we first look at the edges going out of A and then we look at the edges going out of B; notice that these two sets are equal.

Theorem 1.2. *If $A \subset V(G)$ is a proper subset, then*

$$|[A, V - A]| = \sum_{x \in A} \left(d_G(x) - d_{[A]}(x)\right) = \sum_{y \in V - A} \left(d_G(y) - d_{[V-A]}(y)\right).$$

Proof. (Use of two way counting.) We have

(1) $[A, V - A] = \cup_{x \in A}[x, V - A]$,

(2) $[A, V - A] = \cup_{y \in V - A}[A, y]$,

(3) for any $x \in A$, $d_G(x) = |[x, A - x]| + |[x, V - A]| = d_{[A]}(x) + |[x, V - A]|$, and

(4) for any $y \in V - A$, $d_G(y) = |[y, V - A - y]| + |[y, A]| = d_{[V-A]}(y) + |[A, y]|$.

Therefore,

$$\begin{aligned} |[A, V - A]| &= | \cup_{x \in A} [x, V - A]|, \text{(by (1))} \\ &= \sum_{x \in A} |[x, V - A]| \\ &= \sum_{x \in A} (d_G(x) - d_{[A]}(x)), \text{(by (3))}. \end{aligned}$$

Similarly, (2) and (4) yield

$$|[A, V - A]| = \sum_{y \in V - A} (d_G(y) - d_{[V-A]}(y)).$$

□

Definitions.

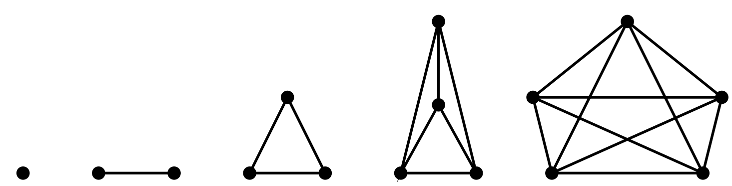
- *A vertex with degree* 0 *is called an* ***isolated vertex****.*
- *A vertex of degree* 1 *is called a* ***pendant vertex****.*
- *The* ***minimum degree*** *of* G*, denoted by* $\delta(G)$*, is the minimum degree among all the vertices of* G*.*
- *The* ***maximum degree*** *of* G*, denoted by* $\Delta(G)$*, is the maximum degree among all the vertices of* G*.*
  Clearly, if G *is a simple graph and* $v \in V(G)$*, then*

$$0 \leq \delta(G) \leq deg(v) \leq \Delta(G) \leq n-1.$$

- *If* $deg(v) = k$*, for every vertex* v *of* G*, then* G *is called a* k***-regular graph****. It is called a* **regular graph***, if it is* k*-regular for some* k*.*

• Special graphs

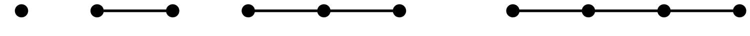
- A simple graph in which any two vertices are adjacent is called a ***complete graph***. A complete graph on n vertices is denoted by K_n. See Figure 1.6.



K_1 K_2 K_3 K_4 K_5

Figure 1.6: Complete graphs.

- A simple graph on n vertices $v_1, v_2, \ldots, v_n$ and $n-1$ edges (v_1, v_2), (v_2, v_3), $\ldots$, (v_{n-1}, v_n) is called a ***path***. It is denoted by $(v_1, v_2, \ldots, v_n)$ or $P(v_1, v_n)$ or P_n. See Figure 1.7



P_1 P_2 P_3 P_4

Figure 1.7: Paths.

- A graph on n distinct vertices $v_1, v_2, \ldots, v_n$ and n edges (v_1, v_2), (v_2, v_3), $\ldots$, (v_{n-1}, v_n), (v_n, v_1) is called a ***cycle.*** It is denoted by $(v_1, v_2, \ldots, v_n, v_1)$ or $C(v_1, v_1)$ or C_n. See Figure 1.8.

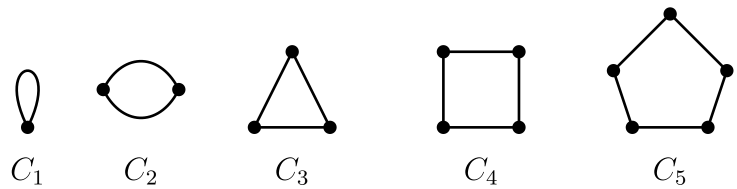


Figure 1.8: Cycles.

- A special graph on ten vertices frequently appears in graph theory. It is called the ***Petersen graph***, after its discoverer J. Petersen (1891). It is shown in Figure 1.9.

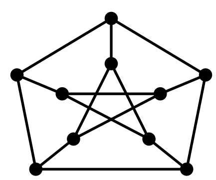


Figure 1.9: Petersen graph.

◦ A class of regular polyhedra, which have appeared in ancient mathematics are shown in Figure 1.10. These are more popularly called as ***Platonic solids***.

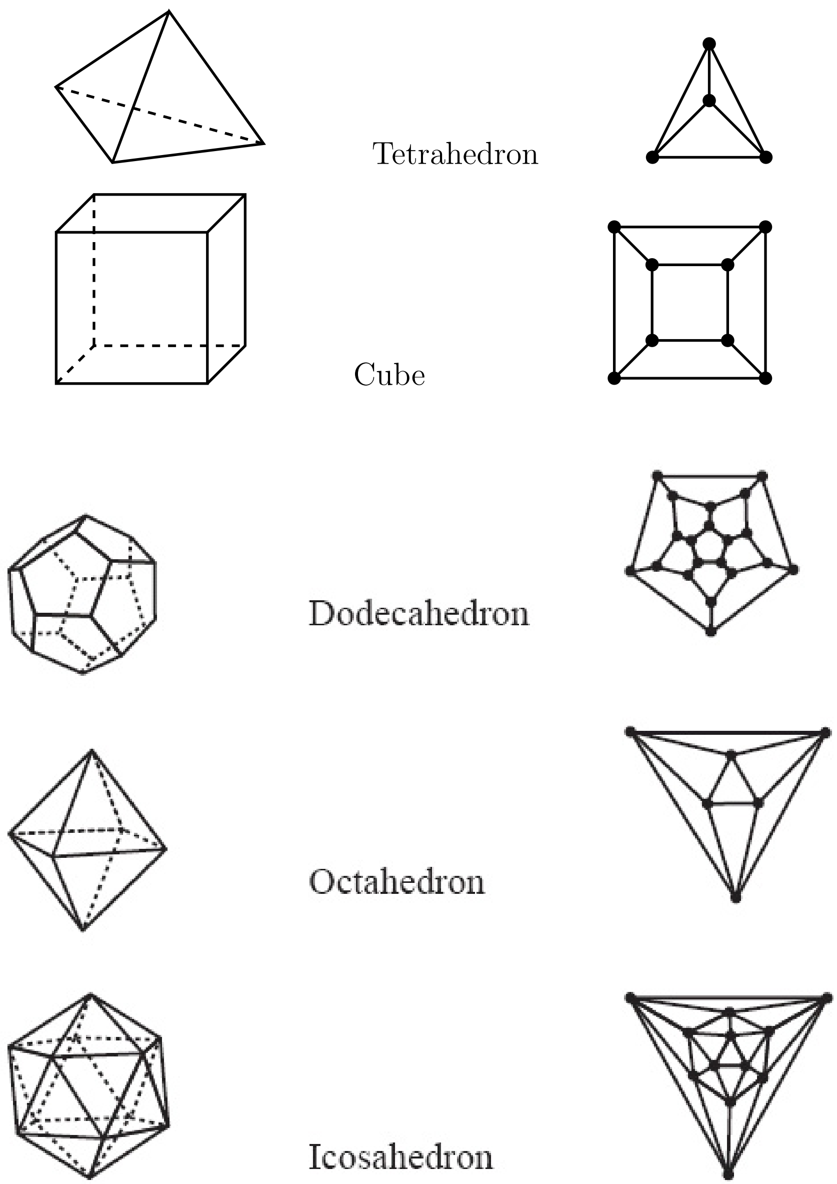


Figure 1.10: Platonic solids and their graphs.

• Complement

- The ***complement*** G^c of a simple graph G has vertex set $V(G)$ and two vertices u, v are adjacent in G^c if and only if they are non-adjacent in G.

- A simple graph G is called a ***self-complementary graph*** if $G \simeq G^c$.

(a) C_4 and its complement   (b) Two self-complementary graphs

Figure 1.11: Complement and self-complementary graphs.

• Larger graphs from smaller graphs

We often require an infinite class of graphs with a given property P rather than a single graph. There are several techniques to construct new graphs by "combining" two or more old graphs. These new graphs preserve some of the properties of old graphs. In this subsection we describe a few such techniques.

Let G_1 and G_2 be vertex disjoint graphs with $|V(G_1)| = n_1$, $|E(G_1)| = m_1$, $|V(G_2)| = n_2$ and $|E(G_2)| = m_2$.

Union

The ***union*** of G_1 and G_2 is the graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. It is denoted by $G_1 \cup G_2$. So, $G_1 \cup G_2$ has $n_1 + n_2$ vertices and $m_1 + m_2$ edges.

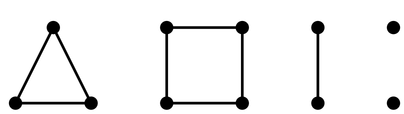
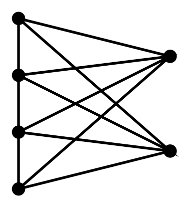


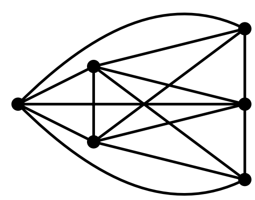
Figure 1.12: Union of graphs: $K_3 \cup C_4 \cup K_2 \cup 2K_1$.

Sum

The ***sum*** or ***join*** of G_1 and G_2 is the graph obtained from $G_1 \cup G_2$ by joining every vertex of G_1 with every vertex of G_2. It is denoted by $G_1 + G_2$. So, $G_1 + G_2$ has $n_1 + n_2$ vertices and $m_1 + m_2 + n_1 n_2$ edges.



(a) $P_4 + K_2^c$.



(b) $K_1 + K_2 + P_3 = K_3 + P_3$.

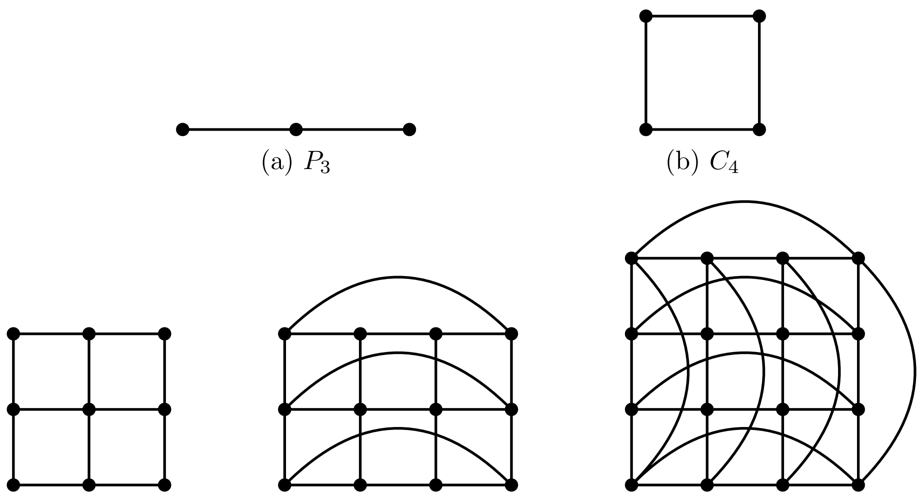
Figure 1.13: Sum of graphs.

The above two definitions can be straightaway extended to the union and the sum of k vertex disjoint graphs $G_1, G_2, \ldots, G_k$. If every $G_i, i = 1, 2, \ldots, k$ is isomorphic with a graph G, then $G_1 \cup G_2 \cup \cdots \cup G_k$ is denoted by kG.

Cartesian Product

The ***Cartesian product*** of simple graphs G_1 and G_2 is the simple graph with vertex set $V(G_1) \times V(G_2)$ in which any two vertices (u_1, v_1) and (u_2, v_2) are adjacent if and only if (i) $u_1 = u_2$, and $(v_1, v_2) \in E(G_2)$ or (ii) $(u_1, u_2) \in E(G_1)$ and $v_1 = v_2$. It is denoted by $G_1 \Box G_2$ or $G_1 \times G_2$. So, $G_1 \Box G_2$ has $n_1 \cdot n_2$ vertices and $n_1 \cdot m_2 + m_1 \cdot n_2$ edges.

The Cartesian product $G_1 \Box G_2 \Box \cdots \Box G_k$ of k simple graphs $G_1, G_2, \ldots, G_k$ has vertex set $V(G_1) \times V(G_2) \times \cdots \times V(G_k)$. Two vertices $(u_1, u_2, \ldots, u_k)$ and $(v_1, v_2, \ldots, v_k)$ are adjacent iff for exactly one i, $1 \leq i \leq k$, $u_i \neq v_i$ and $(u_i, v_i) \in E(G_i)$.



(a) P_3 (b) C_4

(c) $P_3 \Box P_3$ (d) $P_3 \Box C_4$ (e) $C_4 \Box C_4$

Figure 1.14: Cartesian products.

A prime example of the Cartesian product of graphs is the hypercube of dimension d defined by $Q_d = K_2 \Box K_2 \Box \cdots \Box K_2$ (product d times).

Composition

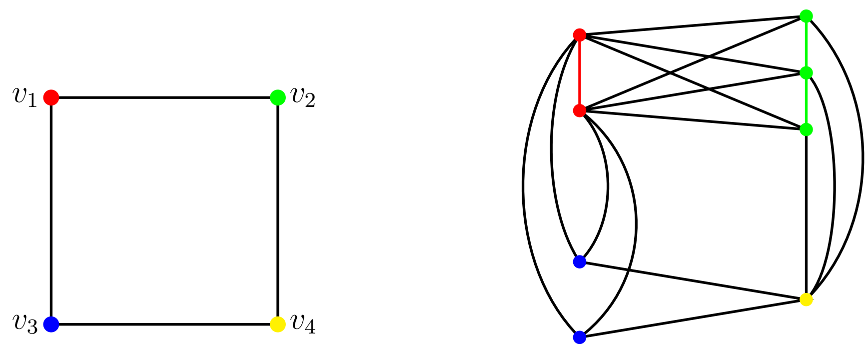
Definition. *Let G have n vertices $v_1, v_2, \ldots, v_n$, and $H_1, H_2, \ldots, H_n$ be any n vertex disjoint graphs. Then the **composition** $G(H_1, H_2, \ldots, H_n)$ of G with $H_1, H_2, \ldots, H_n$ is the graph obtained as follows:*

(i) Replace each vertex v_i of G by H_i, $i = 1, 2, \ldots, n$. Thus $V(G(H_1, H_2, \ldots, H_n)) = \bigcup_{i=1}^{n} V(H_i)$.

(ii) If v_i and v_j are adjacent in G, then join every vertex of H_i with every vertex of H_j.

(iii) If v_i and v_j are non-adjacent in G, then there is no edge between H_i and H_j.

See Figure 1.15 for an example.



(a) C_4. (b) $C_4(K_2, P_3, K_2^c, K_1)$

Figure 1.15: Composition of graphs.

1.3 Graphic sequences

In this section, we build a graph theoretical model of the LAN problems stated in section 1 and solve them. We recall these problems.

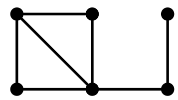
In a college campus, there are seven blocks, Computer Center (C), Library (L), Academic Zone (AC), Administrative Zone (AD), Hospital (H), Guest House (G), Security (S). The problem is to design two LANs satisfying certain conditions:

1. **LAN 1:**
   (i) Two of the blocks are connected to exactly five of the blocks.
   (ii) Two of the blocks are connected to three of the blocks.
   (iii) Three of the blocks are connected to two of the blocks.

2. **LAN 2:**
   (i) Four of the blocks are connected to five of the blocks.
   (ii) Three of the blocks are connected to two of the blocks.

With these problems as motivation, we define the concept of a graphic sequence.

Definition. If G is a graph on n vertices $v_1, v_2, \ldots, v_n$ with degrees $d_1, d_2, \ldots, d_n$ respectively, then the n-tuple $(d_1, d_2, \ldots, d_n)$ is called the ***degree sequence*** of G.



$S : (4, 3, 2, 2, 2, 1)$

Figure 1.16: A graph and its degree sequence.

So every graph G gives rise to a sequence of integers $(d_1, d_2, \ldots, d_n)$. Conversely, we can ask the question: Given a sequence S of integers $(d_1, d_2 \ldots, d_n)$, does there exist a graph G with S as its degree sequence? By Theorem 1.1, one necessary condition for the existence of G is that $\sum_{i=1}^{n} d_i$ is even. It is easy to show that it is also a sufficient condition. However, the question is more difficult if we ask for the existence of a ***simple graph*** G with degree sequence S. Towards this end, we define the following concept.

Definition. *A sequence of non-negative integers $S = (d_1, d_2, \ldots, d_n)$ is said to be **graphic**, if there exists a **simple graph** G with n vertices $v_1, v_2, \ldots, v_n$ such that $deg(v_i) = d_i$, for $i = 1, 2, \ldots, n$. When such a G exists, it is called a **realization** of S.*

• Graph theoretic model of the LAN problem

Problem: Find necessary and sufficient conditions for a sequence $S_n = (d_1, d_2, \ldots, d_n)$ of non-negative integers to be graphic.

This problem leads to the following three problems (and many more).

- Design algorithms to construct a realization of S, if S is graphic.
- When S is graphic, how many non-isomorphic realizations of S are there?
- Given a graph theoretic property P and a sequence of integers $S = (d_1, d_2, \ldots, d_n)$, find necessary and sufficient conditions for the existence of a graph G having the property P and degree sequence $(d_1, d_2, \ldots, d_n)$.

While first two problems have been solved, the third problem is open for many properties P.

We prove two theorems which characterize graphic sequences.

• Havel-Hakimi criterion

Theorem 1.3 (Havel 1955, Hakimi 1962)**.** *A sequence*

$$S : (d_1 \geq d_2 \geq \cdots \geq d_n)$$

of non-negative integers is graphic if and only if the reduced sequence

$$S' : (*, d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n)$$

is graphic.

(Here, S' is obtained from S by deleting d_1 and subtracting 1 from the next d_1 terms. We also assume, without loss of generality, that $d_1 \leq n-1$.)

Proof.

(1): S' is graphic $\Rightarrow S$ is graphic.

Since S' is graphic, there exists a simple graph G' on $n-1$ vertices $v_2, v_3, \ldots,$ v_n with degrees $d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n$, respectively. Add a new vertex v_1 to G' and join it to $v_2, v_3, \ldots, v_{d_1+1}$. The resultant graph is a realization of S.

(2): S is graphic $\Rightarrow$ S$'$ is graphic.

Since S is graphic, there exists a simple graph G on n vertices $v_1, v_2, \ldots, v_n$ with degrees $d_1, d_2, \ldots, d_n$ respectively. If v_1 is adjacent with $v_2, v_3, \ldots, v_{d_1+1}$ then $G - v_1$ is a realization of S'. Else, v_1 is non-adjacent to some vertex v_i, where 2

$\leq i \leq d_1 + 1$. Therefore, v_1 is adjacent to some vertex v_j, where $d_1 + 2 \leq j \leq n$. Since $j > i$, we conclude that $deg(v_i) = d_i \geq d_j = deg(v_j)$. However, v_j is adjacent to v_1 but v_i is not adjacent to v_1. So, there is some v_p such that v_i is adjacent to v_p but v_j is not adjacent to v_p. Hence, G contains the subgraph shown in Figure 1.17.

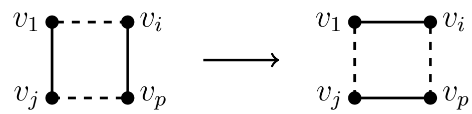


Figure 1.17: Application of 2-switch.

We delete the edges (v_1, v_j), (v_i, v_p) and add the edges (v_1, v_i), (v_j, v_p) (and retain all other edges of G). The resultant graph H is simple and $deg_G(v_k) = deg_H(v_k)$ for every $k, 1 \leq k \leq n$. So, H is also a realization of S in which v_1 is adjacent with ***one more vertex*** in $\{v_2, v_3, \ldots, v_{d_1+1}\}$ than G does. If v_1 is not adjacent to some vertex in $\{v_2, v_3, \ldots, v_{d_1+1}\}$ in H, then we can continue the above procedure to eventually get a realization G^* of S such that v_1 is adjacent to all the vertices in $\{v_2, v_3, \ldots, v_{d_1+1}\}$. Then $G^* - v_1$ is a realization of S. $\square$

• Realization of a graphic sequence

The proof of Havel-Hakimi theorem contains enough information to construct a simple graph with degree sequence $(d_1, d_2, \ldots, d_n)$, if $(d_1, d_2, \ldots, d_n)$ is graphic, else we can use the theorem to declare that $(d_1, d_2, \ldots, d_n)$ is not graphic. We illustrate these remarks by taking the examples of LAN 1 and LAN 2 problems.

LAN 1

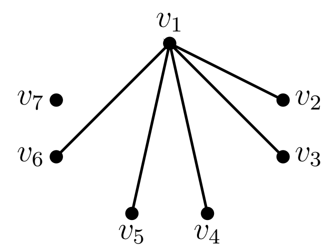
Input: $(5, 5, 3, 3, 2, 2, 2)$.

Output: A simple graph G on seven vertices $v_1, v_2, \ldots, v_7$ with degree sequence $(5, 5, 3, 3, 2, 2, 2)$ if the input is graphic, else declaration that the input is not graphic.

Iteration 1:

Input:$(5_1, 5_2, 3_3, 3_4, 2_5, 2_6, 2_7)$. Here, i_j indicates that the degree of v_j will be i in G, at the end of the algorithm, if the input is graphic.

Output:$(*, 4_2, 2_3, 2_4, 1_5, 1_6, 2_7)$



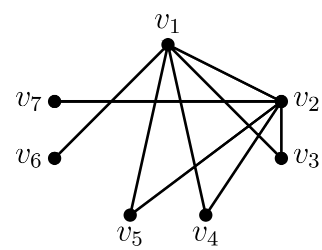
In the figure, we have shown a graph by drawing the vertices $v_1, v_2, \ldots, v_7$ and joining v_1 with v_2, v_3, v_4, v_5, v_6, since we have subtracted 1 from d_2, d_3, d_4, d_5, d_6.

We rearrange this sequence in decreasing order which will be the input for the second iteration.

Iteration 2:

Input: $(*, 4_2, 2_3, 2_4, 2_7, 1_5, 1_6)$

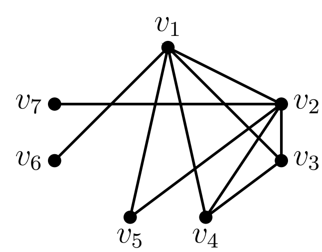
Output: $(*, *, 1_3, 1_4, 1_7, 0_5, 1_6)$



Iteration 3:

Input: $(*, *, 1_3, 1_4, 1_7, 1_6, 0_5)$

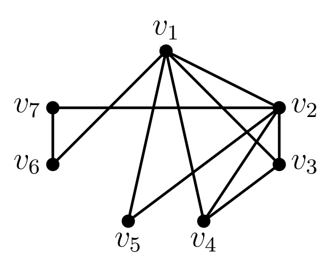
Output: $(*, *, *, 0_4, 1_7, 1_6, 0_5)$



Iteration 4:

Input: $(*, *, *, 1_7, 1_6, 0_4, 0_5)$

Output: $(*, *, *, *, 0_6, 0_4, 0_5)$

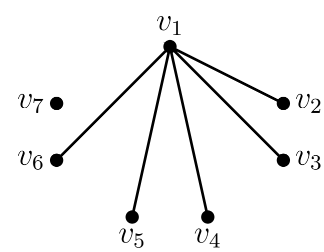


The output in the fourth iteration is obviously graphic and so we stop the algorithm. Also, we have realized a graph G with degree sequence (5, 5, 3, 3, 2, 2, 2) shown above.

LAN 2:

Input: (5, 5, 5, 5, 2, 2, 2)

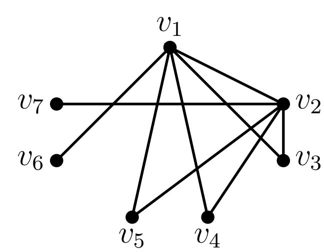
Output: A graph G with degree sequence (5, 5, 5, 5, 2, 2, 2) or a declaration that the input is not graphic.



Iteration 1:

Input: $(5_1, 5_2, 5_3, 5_4, 2_5, 2_6, 2_7)$

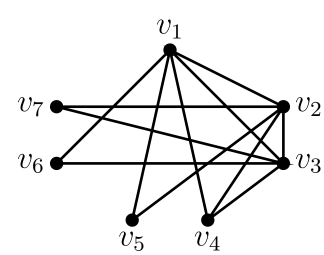
Output: $(*, 4_2, 4_3, 4_4, 1_5, 1_6, 2_7)$



Iteration 2:

Input: $(*, 4_2, 4_3, 4_4, 2_7, 1_5, 1_6)$

Output: $(*, *, 3_3, 3_4, 1_7, 0_5, 1_6)$



Iteration 3:

Input: $(*, *, 3_3, 3_4, 1_7, 1_6, 0_5)$

Output: $(*, *, *, 2_4, 0_7, 0_6, 0_5)$

We stop the algorithm after the third iteration, since $(2, 0, 0, 0)$ is obviously not graphic. Using Havel-Hakimi Theorem, we declare that the given input is not graphic and hence conclude that the construction of LAN 2 is not possible.

Remarks.

- The realization of a graphic sequence constructed as above is not necessarily unique, since we rearrange the sequence of integers during the iterations. In fact, one may realize two non-isomorphic simple graphs with degree sequence S. See exercise 29.

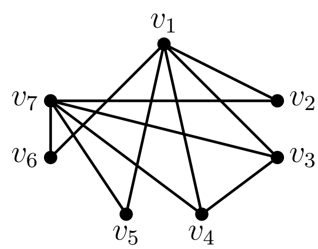


Figure 1.18: Another graph realization of the sequence (5, 5, 3, 3, 2, 2, 2).

- There exist simple graphs which cannot be constructed by the above algorithm. For example, the graph shown in Figure 1.18 cannot be constructed using the Havel-Hakimi algorithm.

• Erdös-Gallai criterion

The next theorem gives an alternative characterization of graphic sequences.

Theorem 1.4 (Erdös and Gallai, 1960)**.** *A sequence* $S : (d_1 \geq d_2 \geq \cdots \geq d_n)$ *of non-negative integers is graphic if and only if the following hold:*

(EG1) $\sum_{i=1}^{n} d_i$ *is even,*

(EG2) $\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{j=k+1}^{n} \min\{k, d_j\}$, *for every* $k = 1, 2, \ldots, n$.

Proof.

(1) S is graphic $\Rightarrow$ S satisfies (EG1) and (EG2).

Since S is graphic, there exists a simple graph G on n vertices $v_1, v_2, \ldots, v_n$ such that $deg_G(v_i) = d_i$, for $i = 1, 2, \ldots, n$. Therefore, $\sum_{i=1}^{n} d_i = \sum_{i=1}^{n} deg_G(v_i)$ which is an even integer by Theorem 1.1.

Next, we show that (EG2) holds. Let k be an integer such that $1 \leq k \leq n$. Let $A = \{v_1, v_2, \ldots, v_k\}$ and $B = \{v_{k+1}, v_{k+2}, \ldots, v_n\}$. We estimate the maximum value of l.h.s sum $M = \sum_{i=1}^{k} d_i = \sum_{i=1}^{k} deg_G(v_i)$.

(i) Any $v_i \in A$ is adjacent to at most $k-1$ vertices in A. Hence, it contributes at most $k-1$ to M.

(ii) Any $v_j \in B$ is adjacent to all the k vertices in A or d_j vertices in A, whichever is minimum. So it contributes $\min\{k, d_j\}$ to M. Hence, $\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{j=k+1}^{n} \min\{k, d_j\}$.

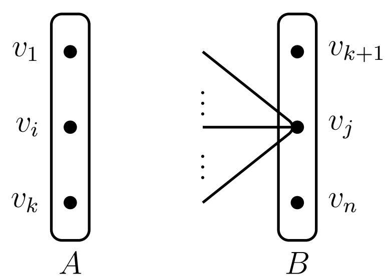


Figure 1.19: $\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{j=k+1}^{n} min\{k, d_j\}$.

(2) (EG1) and (EG2) $\Rightarrow$ S is graphic. We omit the proof. $\square$

We illustrate Erdös-Gallai Theorem by taking LAN 1 and LAN 2 problems.

LAN 1: Here we have to check whether the sequence (5, 5, 3, 3, 2, 2, 2) is graphic.

k	l.h.s	r.h.s		l.h.s $\leq$ r.h.s?
	$d_1+\cdots+d_k$	$k(k-1)$	$\min\{k, d_{k+1}\}+\cdots+\min\{k, d_7\}$	
1	5	0	$1+1+1+1+1+1=6$	✓
2	10	2	$2+2+2+2+2=10$	✓
3	13	6	$3+2+2+2=9$	✓
4	16	12	$2+2+2=6$	✓
5	18	20	$2+2=4$	✓
6	20	30	2	✓
7	22	42	0	✓

Using Erdös-Gallai Theorem we conclude that (5, 5, 3, 3, 2, 2, 2) is graphic.

LAN 2: Here, we have to check whether (5, 5, 5, 5, 2, 2, 2) is graphic.

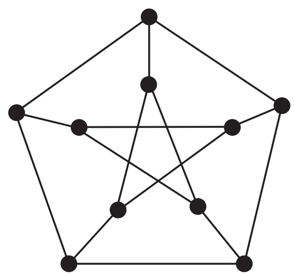
k	l.h.s	r.h.s		l.h.s $\leq$ r.h.s ?
	$d_1+\cdots+d_k$	$k(k-1)$	$\min\{k, d_{k+1}\}+\cdots+\min\{k, d_7\}$	
1	5	0	$1+1+1+1+1+1=6$	✓
2	10	2	$2+2+2+2+2=10$	✓
3	15	6	$3+2+2+2=9$	✓
4	20	12	$2+2+2=6$	×

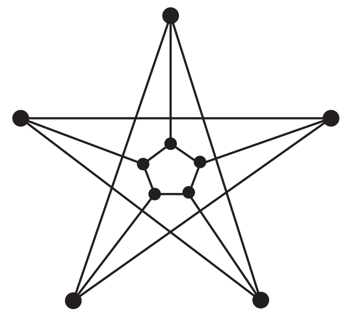
We stop checking, since (EG2) is not satisfied when $k=4$. We conclude that (5, 5, 5, 5, 2, 2, 2) is not graphic.

Remark. Notice that we did not construct a realization of $(5,5,3,3,2,2,2)$. We only verified that the sequence is graphic. So, LAN 1 cannot be solved using Erdös-Gallai Theorem. However, the theorem has good theoretical implications.

Exercises

1. Define an isomorphism between the following two graphs:





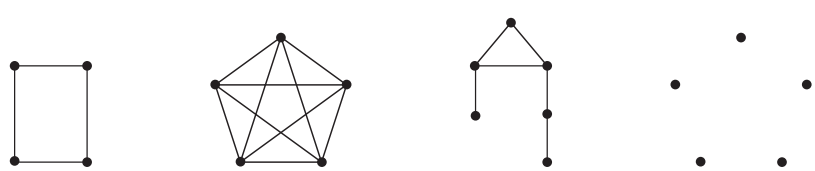
2. (a) If G and H are isomorphic graphs, then show that (i) $|V(G)| = |V(H)|$, (ii) $|E(G)| = |E(H)|$, (iii) if (f, g) is an isomorphism between G and H, then show that $deg_G(v) = deg_H(f(v))$, for every $v \in V(G)$ and (iv) if $f(u) = v$, then $f|_{N(u)}$ is an isomorphism between $[N_G(u)]$ and $[N_H(v)]$.

   (b) Give examples of two non-isomorphic graphs with the same degree sequence.

3. Let $u, v \in V(G)$. If there exists an automorphism (f, g) of G, such that $f(u) = v$, then show that $G - u \simeq G - v$.

4. Draw all the non-isomorphic simple graphs on n vertices for $n = 1, 2, 3, 4$.

5. Show that the set of all automorphisms of a simple graph G form a permutation group under the usual binary operation of functions. Describe the automorphism groups of the following graphs:



G_1 G_2 G_3 G_4

6. If G is simple, then show that G and G^c have the same automorphism group.

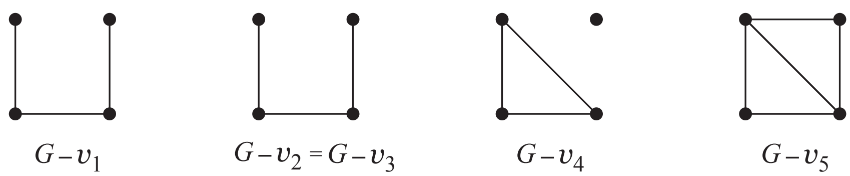
7. If H is a subgraph of a simple graph G, does it follow that H^c is a subgraph of G^c.

8. A simple graph G on 6 vertices $v_1, v_2, v_3, v_4, v_5, v_6$ has (i) 7 edges in $G - v_1$ and $G - v_2$, (ii) 6 edges in $G - v_3$ and $G - v_4$, (iii) 5 edges in $G - v_5$ and $G - v_6$. Find the number of edges in G.

9. Draw all the simple graphs G on 6 vertices such that $G - u \simeq G - v$, for every pair u, v of vertices.

10. A simple graph G on 5 vertices v_1, v_2, v_3, v_4, v_5 is such that (i) $G - v_1 \simeq G - v_2 \simeq G - v_3 \simeq K_2 \cup K_2$, and (ii) $G - v_4 \simeq G - v_5 \simeq K_3 \cup K_1$. Draw G.

11. Let G be a simple graph on v_1, v_2, v_3, v_4, v_5. The graphs $G - v_i, i = 1, 2, 3, 4, 5$ are shown below. Find G.



12. Let G be a simple graph on vertices $v_1, v_2, \ldots, v_n$. Let $G - v_i$ have m_i edges for $i = 1, 2, \ldots, n$. Show the following:
    (i) $m = \frac{1}{n-2} \sum_{i=1}^{n} m_i$,
    (ii) $deg(v_i) = \left(\frac{1}{n-2} \sum_{j=1}^{n} m_j \right) - m_i, i = 1, 2, \ldots, n.$

13. Give an example of a simple graph on 9 vertices and 20 edges which contains no K_3 as a subgraph.

14. Give an example of a graph G on 8 vertices such that neither G contains a K_3 nor G^c contains K_4.

15. (a) Draw a simple graph on 7 vertices with maximum number of edges which contains no complete subgraph on 4 vertices.
    (b) Draw a simple graph on n vertices with maximum number of edges which contains no complete subgraph on p, $2 \le p \le n$ vertices.

16. The diagonal entries of the square of the adjacency matrix of a simple graph G are (3, 3, 2, 1, 1). Draw G and its incidence matrix.

17. The adjacency matrix $A(G_1)$ and the incidence matrices $B(G_2)$ and $B(G_3)$ of three graphs G_1, G_2 and G_3 on 5 vertices and 4 edges are shown below. Verify:

(a) Whether G_1 is isomorphic with G_2.

(b) Whether G_2 is isomorphic with G_3.

Justify your answers.

$$\underset{A(G_1)}{\begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}} \qquad \underset{B(G_2)}{\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}} \qquad \underset{B(G_3)}{\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}}$$

18. Find the adjacency matrix $A(G)$ and draw the simple graph G whose incidence matrix B is such that

$$BB^T = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 \\ 1 & 1 & 4 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 & 3 \end{pmatrix}$$

where B^T is the transpose of B.

19. Draw (i) one self-complementary graph on 4 vertices, (ii) two non-isomorphic self-complementary graphs on 5 vertices, and (iii) two non-isomorphic self-complementary graph on 8 vertices.

20. If G is self-complementary then show that $|V(G)| = 4k$ or $4k+1$, for some k.

21. For each integer n of the form $4k$ or $4k+1$, $k \geq 1$, construct a self-complimentary graph on n vertices using a recursive technique.

22. If G and H are self complimentary graphs, then show that the composition graph $G(H_1, H_2, \ldots, H_n)$ is self complimentary, where $H_i = H$, $i = 1, 2, \ldots, n$.

23. The graph d-cube, denoted by Q_d, is defined as follows:
$V(Q_d) = \{(x_1, x_2, \ldots, x_d) : x_i = 0 \text{ or } 1, 1 \leq i \leq d\}$. Any two vertices $(x_1, x_2, \ldots, x_d)$ and $(y_1, y_2, \ldots, y_d)$ are adjacent in Q_d if and only if $x_i \neq y_i$, for exactly one i, $(1 \leq i \leq d)$. Draw Q_1, Q_2, Q_3. Find (i) $|V(Q_d)|$, (ii) $|E(Q_d)|$, (iii) $deg(x)$, $x = (x_1, x_2, \ldots, x_d) \in Q_d$.

24. Show that $Q_d = K_2 \Box K_2 \Box \cdots \Box K_2$ (Cartesian product d times).

25. Find the minimum integer $k(\geq 1)$ such that there is a simple graph with degree sequence $2^k 4^k 7^k$, where d^k denotes that d is repeated k times.

26. A simple graph G has degree sequence $(d_1, d_2, \ldots, d_n)$. What is the degree sequence of G^c.

27. (a) Verify which of the following sequences are graphic, using (I) Havel-Hakimi Theorem, and using (II) Erdös-Gallai Theorem.
    (i) (5, 5, 5, 2, 2, 2, 1)
    (ii) (4, 4, 4, 4, 2, 2, 0)
    (iii) (7, 6, 5, 4, 4, 3, 2, 1)
    (iv) (5, 5, 4, 4, 2, 2)
    (v) (5, 5, 3, 3, 2, 2)
    (b) Whenever a sequence S is graphic, construct a simple graph with S as degree sequence using Havel-Hakimi Theorem.

28. Given an example of a simple graph that cannot be realized by using the algorithm following Havel-Hakimi criterion.

29. Show that there are only two non-isomorphic realizations of the degree sequence (5, 5, 3, 3, 2, 2, 2).

30. Let $S = (d_1 \geq d_2 \geq \cdots \geq d_n)$ be a sequence of integers. If p $(1 \leq p \leq n)$ is the smallest integer such that $d_p \leq p - 1$, then show that S is graphic iff

    (i) $\sum_i d_i$ is even, and
    (ii) $\sum_{i=1}^{k} \leq k(k-1) + \sum_{j=p}^{n} \min\{k, d_j\}$, for every $k = 1, 2, \ldots, p - 1$.

31. Show that in any group of two or more persons, there are always two persons with exactly same number of friends.

32. Show that any sequence $(d_1, d_2, \ldots, d_n)$ of non-negative integers is a degree sequence of some graph (not necessarily simple) if and only if $\sum_{i=1}^{n} d_i$ is even.

33. Show that any sequence $(d_1, d_2, \ldots, d_n)$ of non-negative integers where $d_1 \geq d_2 \cdots \geq d_n$ is a degree sequence of some loopless graph (it can have multiple edges) if and only if (i) $\sum_{i=1}^{n} d_i$ is even, and (ii) $d_1 \leq d_2 + d_3 + \cdots + d_n$.

34. Give an example of a graphic sequence $(d_1, d_2, \ldots, d_n)$ such that the application of Havel-Hakimi algorithm yields two non-isomorphic graphs. (Choose n as small as possible.)

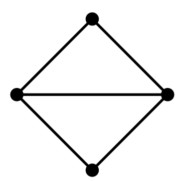
35. Give an example of a simple graph (on as few vertices as you can) which cannot be constructed by Havel-Hakimi algorithm.

36. Let $(4, 4, \ldots, 4, 3, 3, \ldots, 3)$ be sequence of n integers where 4 is repeated k times and 3 is repeated $n - k$ times. Find all the values of k and n for which the sequence is graphic.

37. Let $n \geq 1$ be an integer. Does there exist a simple graph with degree sequence $(n, n, n-1, n-1, \ldots, 3, 3, 2, 2, 1, 1)$? Justify your answer.

38. Show that a regular sequence $(d, d, \ldots, d)$ of length n is graphic if and only if (i) $d \leq n - 1$, (ii) $d \cdot n$ is even.

39. (a) Let G be the graph shown below.



    Construct a 3- regular simple graph H such that $G \sqsubseteq H$.

    (b) Show that any simple graph G has a $\Delta(G)$-regular simple supergraph H such that G is an induced subgraph of H. (If $G \subseteq H$, then H is called a supergraph of G).