Module 10 Planar Graphs

Contents

Planar graphs are a major link between graph theory and geometry/-topology. There are three easily identifiable milestones in planar graph theory.

(1) A formula of Euler that $V - E + F = 2$, for any convex polyhedron with V vertices/corners, E edges and F faces.

(2) A deep characterization of planar graphs due to Kuratowski.

(3) The 4-color-theorem of Appel, Haken and Koch.

Colorings of planar graphs made their first appearance in a problem of map-coloring. Recent applications of planar graphs in the design of chips and VLSI have further boosted the current research on planar graphs.

10.1 Basic concepts

In this chapter, we will be guided more by intuition rather than precise definitions. For example the terms, plane, open region, closed region, boundary, interior, exterior of a region in the Euclidean space are not defined. These can be found in any undergraduate text on calculus.

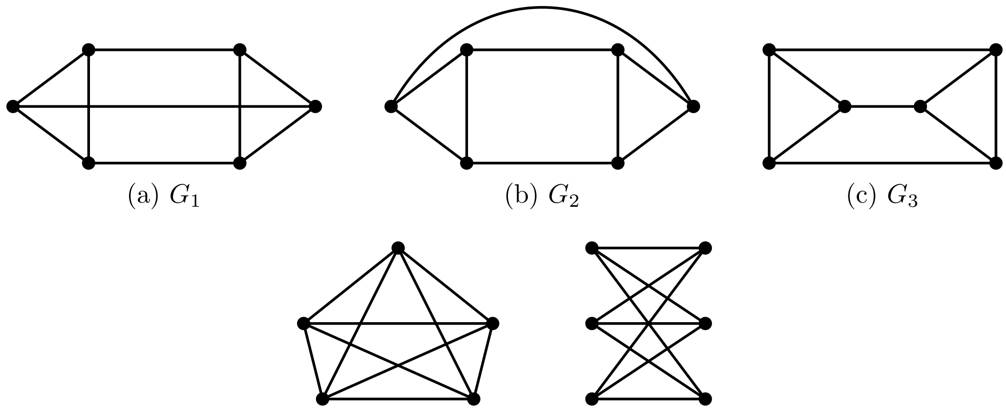
Definition. *A graph G is said to be* ***planar*** *or* ***embeddable in the plane*** *if it can be drawn in the plane so that no two edges intersect except (possibly) at their end vertices; otherwise it is said to be a* ***nonplanar graph****.*

A planar graph embedded in the plane is called a ***plane graph***.

Figure 10.1 shows a planar graph G_1, two plane graphs G_2, $G_3(\simeq G_1)$ and two nonplanar graphs G_4, G_5 (why are they nonplanar?). We emphasize that G_1 is not a plane graph.

Since there are planar graphs and nonplanar graphs, the following three problems arise:

1. Find necessary and sufficient conditions for a graph to be planar.
2. How to test a given graph for planarity?



(a) G_1 (b) G_2 (c) G_3 (d) G_4 (e) G_5

Figure 10.1: Planar graph, plane graphs and nonplanar graphs.

3. Design a (polynomial time) algorithm to draw a given planar graph as a plane graph.

The first problem was solved by Kuratowski in 1930. His characterization uses the hereditary nature of planar graphs. A graph theoretic property P is said to be ***hereditary*** if a graph has property P then all its subgraphs too have property P. Clearly, acyclicity, bipartiteness and planarity are hereditary properties. Kuratowski's characterization has lead to the design of many "good" (:= polynomial time) algorithms to check whether a given graph is planar, and if it is planar to draw it as a plane graph.

A mature reader would have realized that in drawing a planar graph as a plane graph, one would require the knowledge of the famous Jordan Curve Theorem (in the x-y plane).

Given any two points a and b in the plane, any non-self-intersecting continuous curve from a to b is called a ***Jordan curve*** and it is denoted by $J[a, b]$. If $a = b$, then J is called a ***closed Jordan curve***.

Theorem 10.1 (Jordan Curve Theorem)**.**

(i) Any closed Jordan curve J partitions the plane into 3 parts namely, interior of J (int$J$), exterior of J (ext$J$) and J.

(ii) If J is a closed Jordan curve, $s \in intJ$ and $t \in extJ$, then any Jordan curve $J'[s,t]$ contains a point of J (that is, J' intersects J).

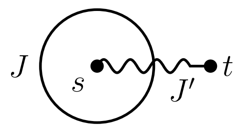


Figure 10.2: Illustration for Jordan Curve Theorem.

If G is a plane graph, then any path in G is identified with a Jordan curve. Similarly, any cycle is identified with a closed Jordan curve. In particular, an edge $e(u,v)$ of G is a Jordan curve from u to v.

Definition. *Let G be a plane graph.*

- *G partitions the plane into several regions. These regions are called the **faces** of G. The set of all faces of G is denoted by $F(G)$, and the number of faces by $r(G)$; so $|F(G)| = r(G)$.*
- *Except one face, every other face is a bounded region. The exceptional face is called the **exterior face** and other faces are called **interior faces** of G. The exterior face is unbounded and interior faces are bounded (:= area is finite).*
- *The **boundary** of a face f is the set of all edges of G which are incident with f. It is denoted by $b_G(f)$ or $b(f)$.*

Importantly, the boundary of a face f need not be a cycle; it can be a walk; see Figure 10.3.

Definition. *The **degree of a face** f in a plane graph G is the number of edges in the boundary of f with cut-edges counted twice (the reason why we count twice will be clear soon). The degree of f is denoted by $deg_G(f)$ or $deg(f)$ or $d(f)$.*

The plane graph of Figure 10.3 contains 4 faces.

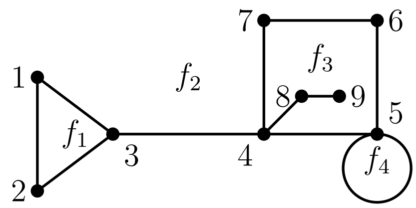


Figure 10.3: A plane graph.

$b(f_1) = \{(1,2),(2,3),(1,3)\}$; $d(f_1) = 3$; $b(f_1)$ is a cycle.

$b(f_2) = \{(1,2),(2,3),(1,3),(3,4),(4,5),(5,5),(5,6),(6,7),(7,4)\}$; $d(f_2) = 10$. Note that (3,4) is counted twice and that $b(f_2)$ does not form a cycle.

$b(f_3) = \{(4,5),(5,6),(6,7),(7,4),(4,8),(8,9)\}$; $d(f_3) = 8$. Note that (4,8), (8,9) are counted twice.

$b(f_4) = \{(5,5)\}$; $d(f_4) = 1$.

Remarks (Consequences of Jordan Curve Theorem)**.**

- A cyclic edge belongs to two faces.
- A cut-edge belongs to only one face.
- A plane graph G is acyclic if and only if $r(G) = 1$.

Theorem 10.2. *If G is a plane graph, then*

$$\sum_{f \in F(G)} deg(f) = 2m.$$

Proof. Every cyclic edge contributes 2 to the left hand side sum since it belongs to 2 faces of G (by Jordan Curve Theorem). Every cut-edge also contributes 2, since it is counted twice although it belongs to only one face. $\square$

Many results on plane graphs become apparent if we look at their "duals".

Definition (Dual of a plane graph)**.** *Let G be a plane graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$, $E(G) = \{e_1, e_2, \ldots, e_m\}$ and $F(G) = \{f_1, f_2, \ldots, f_r\}$. The dual (general) graph G^* of G has vertices $f_1^*, f_2^*, \ldots, f_r^*$ ($f_i \leftrightarrow f_i^*$) and edges $e_1^*, e_2^*, \ldots, e_m^*$ ($e_i \leftrightarrow e_i^*$), where an edge e_j^* joins the vertices f_s^* and f_t^* if and only if the edge e_j is common to the faces f_s and f_t in G.*

The following figure illustrates a drawing of G^*.

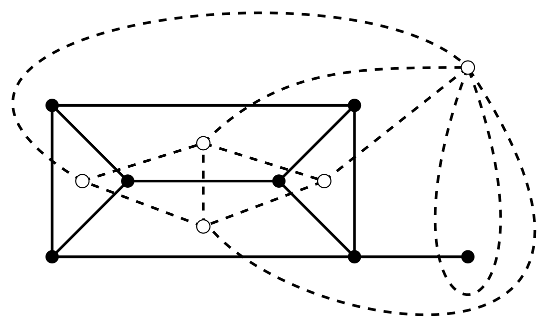


Figure 10.4: A graph G and its dual.

In this drawing, a point is chosen in the interior of every region and two such points f_1^*, f_2^* are joined by a line e^* crossing the edge e only, where e is an edge common to the faces f_1, f_2.

Though the following statements are intuitively obvious from the definition of dual and its drawing, the proofs are tedious and hence they are omitted.

(i) G^* is plane; in fact the above drawing gives a plane embedding of G^*.

(ii) $(G^*)^* = G$.

(iii) e is a cut-edge in G if and only if e^* is a loop in G^*.

Remark. We have defined the dual of a plane graph and not of a planar graph.

10.2 Euler's formula and its consequences

As with many Euler's formulae (like $e^{i\pi} = -1$) his formula for plane graphs too relates three basic parameters and it is both attractive and has many applications.

Theorem 10.3 (Euler, 1758)**.** *For any connected plane graph G,*

$$n - m + r = 2.$$

In polyhedral geometry, the formula is stated using symbols: $V - E + F = 2$.

Proof. (Induction on r) If $r = 1$, then G has no cycles and so it is a tree. Hence $m = n - 1$ and the result holds. Next assume that $r \geq 2$ and that the result holds for all connected plane graphs with $r - 1$ faces. Let G be a plane graph with r faces. Since, $r \geq 2$, G contains a cycle. Let e be a cyclic edge. The graph $G-e$ is a connected plane graph with $n(G - e) = n(G)$, $m(G - e) = m(G) - 1$ and $r(G - e) = r(G) - 1$. The first two equations are obvious and the last equation follows, since e is common to two faces in G and these two faces merge to become one face in $G-e$. By induction hypothesis,

$$n(G - e) - m(G - e) + r(G - e) = 2.$$

So,

$$n(G) - (m(G) - 1) + (r(G) - 1) = 2$$

that is,

$$n(G) - m(G) + r(G) = 2.$$

□

Remark. Euler's formula does not hold for disconnected graphs. Draw a disconnected simple plane graph for which Euler's formula fails.

Corollary. *If G is a connected plane simple graph such that $deg(f) \geq k$, for every face f, then*

$$m \leq \frac{k(n-2)}{k-2}.$$

Proof. Using Theorem 10.2, we find that

$$2m = \sum_{f \in F(G)} deg(f) \geq k \cdot r.$$

Substituting $2m \geq kr$ in Euler's formula, we obtain the result. $\square$

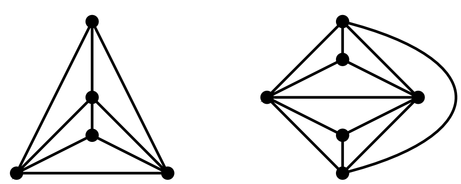
The above result holds for disconnected graphs too.

Corollary. *If G is a plane simple graph such that $deg(f) \geq k$, for every face f, then*

$$m \leq \frac{k(n-2)}{k-2}.$$

$\square$

Definition. A simple planar graph G is said to be a maximal planar graph if $G + e$ is nonplanar, for every $e \in G^c$. Two maximal plane graphs are shown below.



(a) G_1 (b) G_2

Figure 10.5: Maximal plane graphs.

Remarks.

- By definition, every maximal planar graph is simple.

- Every simple planar graph is a spanning subgraph of a maximal planar graph. (Go on adding edges until addition of any edge results in a nonplanar graph.)
- Every face of a maximal plane graph is a triangle. Hence a maximal plane graph is called a ***triangulation of the sphere***.
- Every maximal planar graph ($n \geq 3$) is 2-vertex-connected.

Corollary. *If G is a planar simple graph on at least 3 vertices, then $m \leq 3n - 6$.*

Proof. It is enough if we prove the result for maximal plane graphs; see the remarks above. So, assume that G is a maximal plane graph on at least three vertices. For every face f in G, $deg(f) \geq 3$. So, by putting $k = 3$ in the previous corollary, we find that

$$m \leq \frac{3(n-2)}{3-2} = 3n - 6.$$

□

Corollary. *K_5 is nonplanar.*

Proof. On the contrary, if K_5 is a plane graph, then using the above corollary, we get $10 = m(K_5) \leq 9$, a contradiction. □

Corollary. *$K_{3,3}$ is nonplanar.*

Proof. Assume the contrary that $K_{3,3}$ is a plane graph. Since, it is a bipartite graph, $deg(f) \geq 4$, for every face f. Therefore by putting $k = 4$, in the first corollary to Euler's theorem, we get

$$9 = m(K_{3,3}) \leq \frac{4(4)}{2} = 8,$$

a contradiction. □

Corollary. *If G is a plane simple graph, then $\delta(G) \leq 5$.*

***Proof*.** On the contrary, if $\delta(G) \geq 6$, then $2m = \sum_{v \in V(G)} deg(v) \geq 6n$ which is a contradiction. $\square$

Corollary. *If G is a connected plane simple graph with $\delta(G) \geq 3$, then it contains a face of degree ≤ 5.*

Proof. Assume the contrary that every face f in G has degree ≥ 6. Then

(i) $2m = \sum_{f \in F(G)} deg(f) \geq 6r$.
Since, $\delta(G) \geq 3$, we also have

(ii) $2m = \sum_{v \in V(G)} deg(v) \geq 3n$.

Using these two inequalities in Euler's formula, we get

$$2 = n - m + r \leq \frac{2m}{3} - m + \frac{m}{3} = 0,$$

a contradiction. $\square$

10.3 Polyhedrons and planar graphs (Optional)

Polyhedrons have been fascinating objects since ancient times. In the following, all the polyhedrons are convex.

That the graph of a polyhedron is planar can be practically demonstrated as follows:

Take a polyhedron (P) made of elastic sheet. Color the nodes and the sides of P. Thereupon make a hole in one of the faces of P. Stretch P so that P becomes a plane sheet with the circle of the hole as the boundary. Get a color print of the nodes and the sides of P by pressing it on a paper. The color print is precisely the graph of the polyhedron and it is plane.

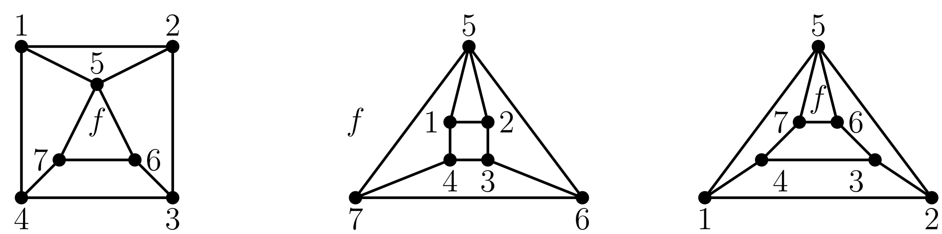
The above operation can be described in precise mathematical formulation known as ***stereographic projection***.

The stereographic projection of a plane graph implies a few crucial facts.

(1) If G is a plane graph and f is any face in G then G can be embedded in the plane so that f is the exterior face.

(2) If G is a plane graph and v is any vertex, then G can be embedded in the plane such that v lies on the exterior face. The same conclusion holds for any edge of G.

A few illustrations are given below.



(a) A plane graph G (b) G_1 (c) G_2

Figure 10.6: G_1 is a redrawing of G such that the interior face f of G is the exterior face of G_1. G_2 is a redrawing of G such that the edge (1,5) lies on the exterior face in G_2. Both are drawn using suitable stereographic projections as explained in the above remark.

The above statements are useful to prove the following result.

Theorem 10.4.

1. *A graph G is planar iff every component of G is planar.*
2. *A graph G is planar iff every block of G is planar.*

Proof. (1) is obvious.

(2) If G is planar, then every block of G is planar, since every subgraph of a planar graph is planar.

Conversely, suppose every block of G is planar. We show that G is planar by induction on the number of blocks $b(G)$ in G. If $b(G) = 1$, there is nothing to be proved. Next suppose that $b(G) \geq 2$. So G contains a cut-vertex say x. Hence, there exist subgraphs G_1 and G_2 of G such that $V(G_1) \cup V(G_2) = V(G)$, $V(G_1) \cap V(G_2) = \{x\}$, $E(G_1) \cup E(G_2) = E(G)$, $E(G_1) \cap E(G_2) = \emptyset$. Since $b(G_1) < b(G)$ and $b(G_2) < b(G)$, G_1 and G_2 are planar by induction hypothesis. By one of the remarks above, G_1 and G_2 can be embedded in the plane such that x belongs to the exterior face of G_1 and also to the exterior face of G_2; see Figure 10.7. Then G_1 and G_2 can be combined to get a plane drawing of G. □

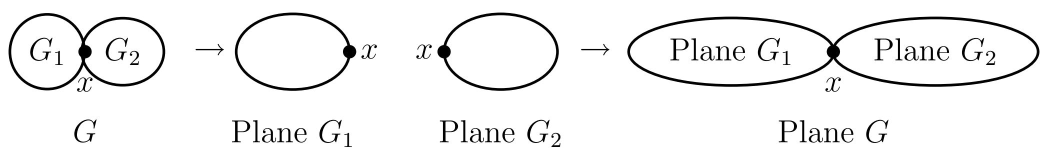


Figure 10.7: A plane drawing of G using plane drawing of G_1 and G_2.

Definitions.

1. *A polyhedron is called a* ***regular polyhedron*** *if*

   (i) *the boundaries of faces are congruent polygons,*

   (ii) *equal number of faces surround each corner.*

2. *The polyhedral graph $G(P)$ of a polyhedron P consists of corners of P as its vertices and sides of P as its edges (:= the 1-skeleton of P).*

It is known that $G(P)$ is a 3-vertex-connected plane simple graph and conversely, if H is a 3-vertex-connected plane simple graph, then there exists a convex polyhedron P such that $G(P) = H$.

In Figure 10.8, three polyhedrons are shown (their graphs are apparent). The first and the third polyhedron are regular polyhedrons and the second polyhedron is not a regular polyhedron.

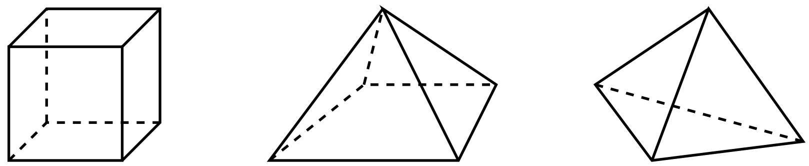
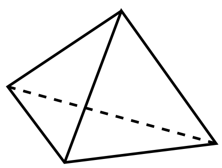
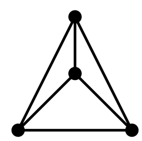


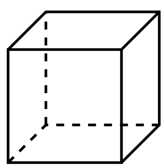
Figure 10.8: Three polyhedrons.

In Figure 10.9, on the left we have 5 regular polyhedra (known as ***Platonic solids***) and on the right their associated plane graphs. Surprisingly, these are the only five regular polyhedra. We can use Euler's formula to prove this statement.

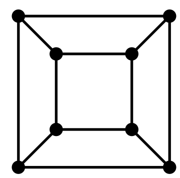


Tetrahedron





Cube



Theorem 10.5. *There are exactly five regular polyhedra.*

Proof. Let P be a regular polyhedron with V nodes, E sides and F faces. Since, the nodes, sides and faces of P are precisely the vertices, edges and faces of $G(P)$, we have

(i) $V - E + F = 2$,

(ii) $\sum_{face\, f \in P} deg(f) = 2E$,

(iii) $\sum_{node\, v \in P} deg(v) = 2E$.

Since P is regular, every node has degree k (say) and every face has degree p (say). Then (ii) and (iii) reduce to $pF = 2E$ and $kV = 2E$, respectively. So,

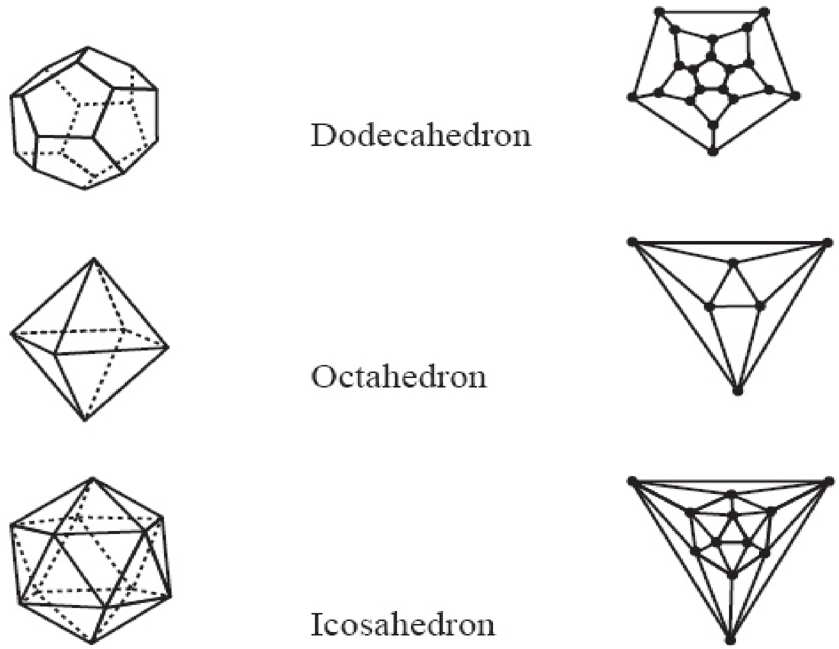


Figure 10.9: Platonic solids and their plane graphs.

(iv) $kV = pF = 2E$.
The equations (i) and (iv) imply that

(v) $(k-4)V + (p-4)F = -8$.

Clearly, every face of a polyhedral graph is bounded by at least 3 edges, and so the minimum degree of faces is 3. Hence, by the corollaries from previous section, we have $3 \le k \le 5$ and $3 \le p \le 5$. Thus there are 9 possible solutions (k, p) of (v).

Case 1. $(k, p) = (3, 3)$.

By (iv) and (v), we get $V = F = 4$. Hence P is a regular polyhedron with 4 nodes, 4 faces with every node and every face of degree 3. So, P is the tetrahedron.

Case 2. $(k, p) = (3, 4)$.

From (iv) and (v), we deduce that $V = 8$ and $F = 6$. Thus P is a regular polyhedron, with 8 vertices and 6 faces, in which every node has degree 3 and every face has degree 4. Hence, P is the cube.

Case 3. $(k, p) = (3, 5)$.

In this case (iv) and (v) yield $V = 20$ and $F = 12$. Thus P is the dodecahedron.

Case 4. $(k, p) = (4, 3)$.

By (iv) and (v), we have $V = 6$ and $F = 8$. Hence, P is the octahedron.

Case 5. $(k, p) = (4, 4)$.

With $k = 4$ and $p = 4$, (v) reduces to $0 = -8$, which is absurd. Hence (4,4) is not a solution of (v).

Case 6. $(k, p) = (4, 5)$.

In this case, (v) reduces to $F = -8$; and obviously there is no polyhedron with $F = -8$.

Case 7. $(k, p) = (5, 3)$.

Using (iv) and (v) we get $V = 12$ and $F = 20$. Hence, P is the icosahedron.

Case 8. $(k, p) = (5, 4)$.

From (v), we get $V = -8$; and clearly there is no polyhedron with $V = -8$.

Case 9. $(k, p) = (5, 5)$.

In this case (v) reduces to $V + F = -8$; and clearly there is no polyhedron with $V + F = -8$. □

10.4 Characterizations of planar graphs

There are four major characterization of planar graphs, due to Kuratowski (1930), Whitney (1932), Wagner (1937) and MacLane (1937). We state and prove Kuratowski's theorem, and state Wagner's theorem.

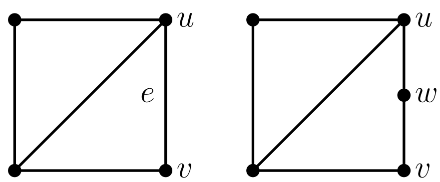
• Subdivisions and Kuratowski's characterization

Central to Kuratowski's characterization is the concept of 'subdivision'.

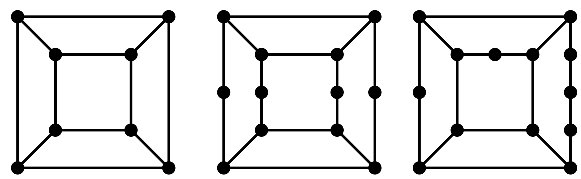
Definitions.

- *The **subdivision** of an edge $e(u,v) \in E(G)$ is the operation of replacing e by a path (u,w,v), where w is a new vertex. So, to get a subdivision of e introduce a new vertex w on e.*
- *A graph H is said to be a **subdivision of** G if H can be obtained from G by a sequence of edge subdivisions. (By definition, G is a subdivision of G.)*

See Figure 10.10.



(a) A subdivision of e



(b) Two subdivisions of a cube

Figure 10.10: Examples of subdivision.

Notation: Any subdivision of G is denoted by $S(G)$. Note that $S(G)$ is not unique. In fact, there are infinite number of subdivisions of any graph with at least one edge.

Remarks.

(1) $H \subseteq G$, G is planar $\Rightarrow$ H is planar.

(2) $H \subseteq G$, H is nonplanar $\Rightarrow$ G is nonplanar.

(3) G is planar $\Rightarrow$ $S(G)$ is planar.

(4) G is nonplanar $\Rightarrow$ $S(G)$ is nonplanar.

(5) $S(K_5)$ and $S(K_{3,3})$ are nonplanar. (A consequence of corollaries from Section 10.2 and (4).)

(6) G is planar $\Rightarrow$ $G \not\supseteq S(K_5), S(K_{3,3})$. (A consequence of (2) and (5))

A natural question is to ask whether the converse of (6) holds. That is, $G \not\supseteq S(K_5), S(K_{3,3}) \Rightarrow G$ is planar? Kuratowski's theorem asserts that it indeed holds.

The Figure 10.11 illustrates that the Petersen graph is nonplanar.

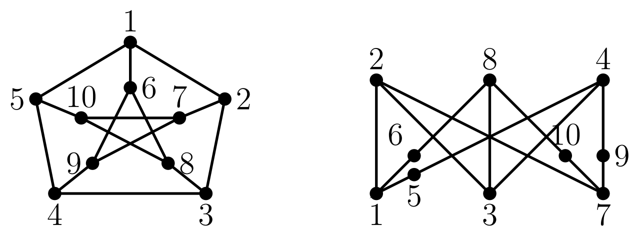


Figure 10.11: The Petersen graph contains a subdivision of $K_{3,3}$. Therefore, the Petersen graph is nonplanar by Remarks (5) and (2) above.

There have been several attempts to simplify and strengthen the original proof of Kuratowski's theorem. Here, we follow Dirac and Schuster (1954) with an excellent terminology of Bondy and Murty (1976).

It is obvious that G is planar if and only if its underlying simple graph is planar. So in the following we assume that all our graphs are simple.

Theorem 10.6 (Kuratowski's characterization of planar graphs)**.** *A graph G is planar if and only if G contains no $S(K_5)$, $S(K_{3,3})$.*

Proof. We have remarked before that if G is planar, then G contains no $S(K_5)$, $S(K_{3,3})$.

To prove the converse, we assume the contrary that there exists a graph which contains no subdivision of K_5 or $K_{3,3}$ but it is nonplanar. Among all such graphs, let G be a graph with ***minimum number*** of edges. In view of Theorem 10.4, $k_0(G) \geq 2$. We consider two cases: $k_0(G) = 2$ and $k_0(G) \geq 3$.

Case 1: $k_0(G) = 2$.

Let $\{u, v\}$ be a vertex-cut of G. There exist two connected graphs G_1 and G_2 of G such that $V(G_1) \cup V(G_2) = V(G)$, $V(G_1) \cap V(G_2) = \{u, v\}$ and $E(G_1) \cup E(G_2) = E(G)$. For $i = 1, 2$, define

$$H_i = \begin{cases} G_i, & \text{if } (u, v) \in E(G) \\ G_i + (u, v), & \text{if } (u, v) \notin E(G). \end{cases}$$

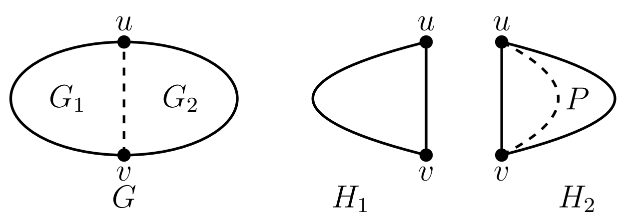


Figure 10.12: Decomposition of a graph into parts.

Claim: $H_1 \not\supseteq S(K_5), S(K_{3,3})$.

If $H_1 = G_1 (\subseteq G)$, the claim is obvious. Next suppose that $H_1 = G_1 + (u, v)$ and that $H_1 \supseteq S(K_5)$ (or $S(K_{3,3})$-proof in this case is similar). Necessarily, $(u, v) \in S(K_5)$. Consider the graph $S(K_5) - (u, v) + P(u, v)$, where $P(u, v)$ is a path with all its internal vertices from G_2; see Figure 10.12. Then $S(K_5) - (u, v) + P(u, v)$ is a subdivision of K_5, and it is a subgraph of G, a contradiction. Therefore, the claim holds.

Similarly, $H_2 \not\supseteq S(K_5)$, $S(K_{3,3})$. Therefore, H_1 and H_2 are planar graphs by the minimality of $E(G)$. We can embed both the graphs H_1 and H_2 in the plane such that the edge (u, v) lies in the exterior face. But then we can combine H_1 and H_2 to get a plane embedding of G, a contradiction to our assumption that G is nonplanar.

Case 2: $k_0(G) \geq 3$.

Let $e(u, v)$ be any edge in G and consider the graph $G - e$. It is planar by the minimality of $m(G)$. Assume that we are given a plane embedding of $G - e$. Since $k_0(G) \geq 3$, we have $k_0(G - e) \geq 2$ (see Exercise 45 from Chapter 2). Therefore, there exists a cycle containing u and v (; see the corollary to Theorem 2.7). Among all such cycles, we choose a cycle C which contains maximum number of edges in the $extC$. For discussion sake, we fix clockwise direction for C. If $x, y \in V(C)$, we write $x \leq y$, if x precedes y on C. The symbol $C[x, y]$ denotes the set of vertices $\{v \in V(C) : x \leq v \leq y\}$. Similarly, $C\ (x, y]$, $C(x, y)$ and $C\ [x, y)$ are defined.

Let D be a component of $G - e - V(C)$. We adopt the following terminology.

- The subgraph B of $G - e$ consisting of D and the set of edges which join a vertex of D with a vertex of C is called a ***branch*** of C. A vertex of C which is joined to a vertex of D is called a ***contact vertex*** of B. If B has k contact vertices, then it is called k***-branch***.
- In addition, an edge (x, y) of $G - e$ where x and y are two non-consecutive vertices of C is also called a branch; x and y are called its ***contact vertices***. To emphasize its nature, we also refer to such a branch as an ***edge-branch***.

Figure 10.13 shows various branches.

We make a series of observations which imply that either G is planar or $G \not\supseteq S(K_5)$ or $G \not\supseteq S(K_{3,3})$ (thus arriving at a contradiction).

(1) Some branches lie in the $extC$ and some branches lie in the $intC$. These are respectively called ***outer-branches*** and ***inner-branches***.

(2) Every branch is a k-branch with $k \geq 3$ or it is an edge-branch, since $k_0(G) \geq 3$.

(3) Every inner-branch is an edge-branch.
On the contrary, suppose H is an inner-k-branch with $k \geq 3$. Then at least two of its contact vertices lie in $C[u, v]$ (or $C[v, u]$). But then there exists a cycle C' in $G - e$ containing u, v which contains more number of edges in the $extC'$ than C - a contradiction to the choice of C; see Figure 10.14a.

(4) Every inner-edge-branch has a contact vertex in $C(u, v)$ and a contact vertex in $C(v, u)$; else, we get a cycle C' as described in (3); see Figure 10.14b.

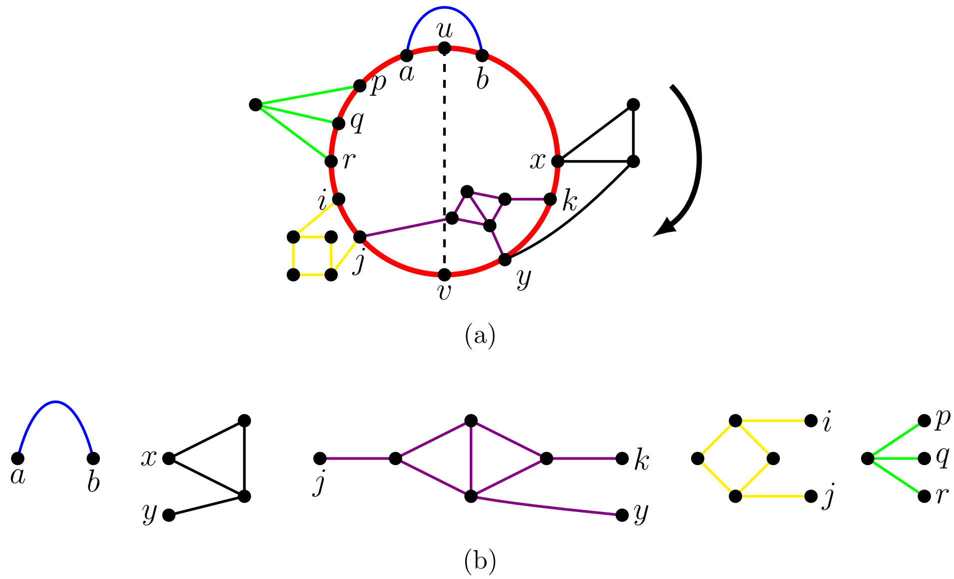


Figure 10.13: (a) A cycle C in $G-e$ with four outer-branches and two inner-branches. (b) Branches of C. (Outer-branches and inner branches are defined below.)

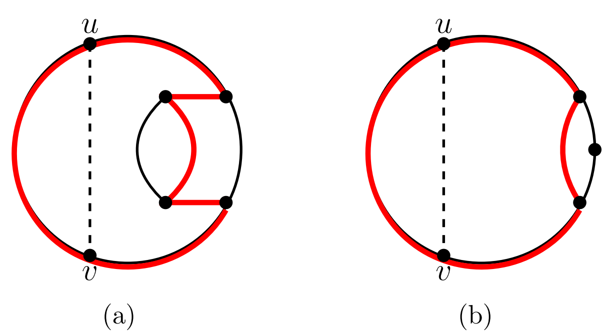


Figure 10.14: Steps in the proof of assertions (3) and (4).

(5) There exists an inner branch, say (r, s) where $r \in C(u, v)$ and $s \in C(v, u)$; else e can be drawn in the $intC$ and get a plane embedding of G, a contradiction. See Figure 10.15.

(6) There exists an outer-branch B with a contact vertex (say, a) in $C(u, v)$ and a contact vertex (say, b) in $C(v, u)$; else e can be drawn in the $extC$ and get a plane embedding of G, a contradiction.

So we have:

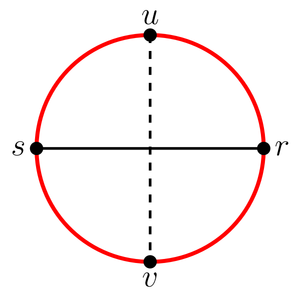


Figure 10.15

(i) $a \in C(u,v) = C(u,r] \cup C[r,v)$,

(ii) $b \in C(v,u) = C(v,s] \cup C[s,u)$.
Because of symmetry it is enough if we deal with the following two cases (other cases follow similarly).

(iii) $a \in C(r,v)$, $b \in C(s,u)$,

(iv) $a = r$, $b = s$.

Suppose (iii) holds. There exists a path $P(a,b)$ in G with all its internal vertices in $V(B) - V(C)$. But then $G \supseteq S(K_{3,3})$ as shown in Figure 10.16, a contradiction.

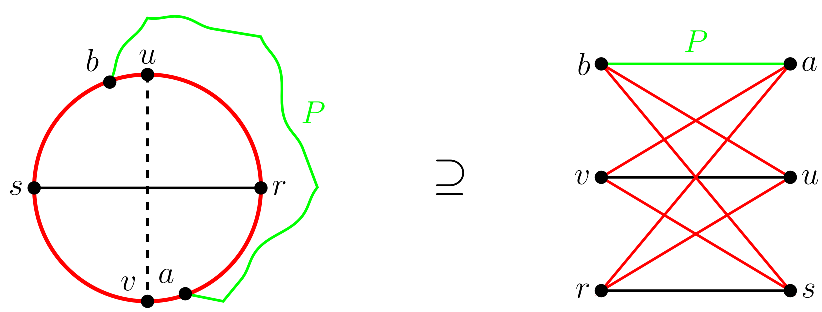


Figure 10.16: $G \supseteq S(K_{3,3})$.

Next suppose (iv) holds, that is $a = r$ and $b = s$.

We next make two observations.

(7) (s,r) cannot be drawn in the $extC$ by the maximality of number of edges in the $extC$.

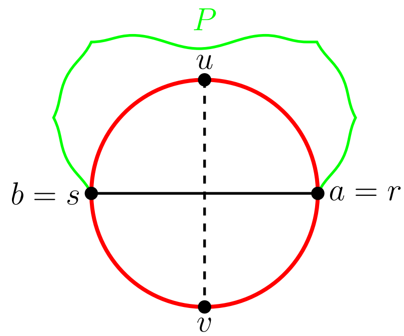


Figure 10.17

(8) No outer-branch ($\neq B$) has a contact vertex in $C(r, s)$ and a contact vertex in $C(s, r)$ (Jordan Curve Theorem).
These two observations imply that:

(9) B has a contact vertex say, $x \in C(s, r)$ and a contact vertex $y \in C(r, s)$. It is possible that $x = u$ or $y = v$. So, we consider two subcases.

Subcase 1: $x = u$ and $y = v$.

There exist paths $P(s, r)$ and $Q(x, y)$ with all their internal vertices from $V(B) - V(C)$; P and Q share at least one common vertex (Jordan Curve Theorem).

If P and Q have exactly one common vertex, say z, then $G \supseteq S(K_5)$, where $V(K_5) = \{u, v, s, r, z\}$; see Figure 10.18.

If P and Q share more than one common vertex, let z_1 and z_2 be the first common vertex and last common vertex, respectively; see Figure 10.19. Then there exist pairwise internally vertex disjoint paths $P(z_1, s), Q(z_1, u), S(z_1, z_2), P(z_2, r)$ and $Q(z_2, v)$. Therefore, $G \supseteq S(K_{3,3})$; see Figure 10.19.

Subcase 2: $\{u, v\} \neq \{x, y\}$; for definiteness, let $v \neq y$. Recall that $y \in C(r, s) = C(r, v) \cup C\,[v, s)$. Because of symmetry, we deal only with the case $y \in C(r, v)$. In view of (iii), we can assume that $x \in C\,[u, r)$; see Figure 10.20 below.

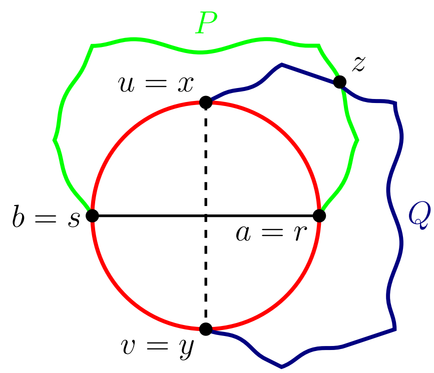


Figure 10.18: Existence of paths $P(s, r)$ and $Q(u, v)$ with a common vertex z and $G \supseteq S(K_5)$.

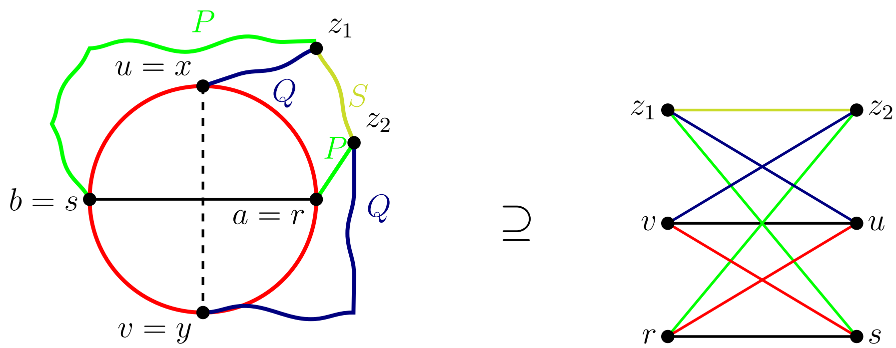


Figure 10.19: $G \supseteq S(K_{3,3})$.

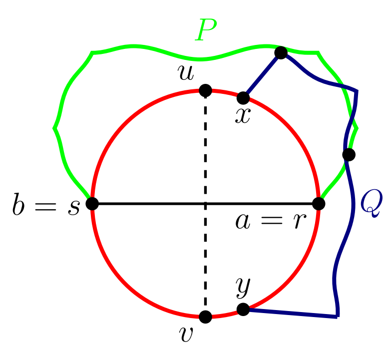
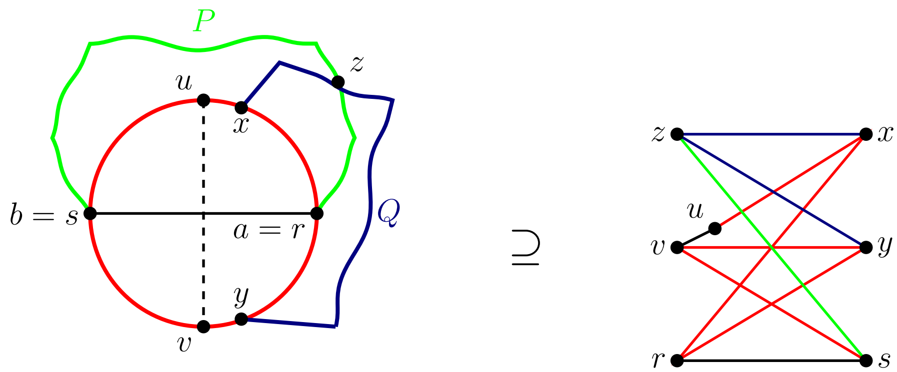


Figure 10.20: Intersection of $P(s, r)$ and $Q(x, y)$.

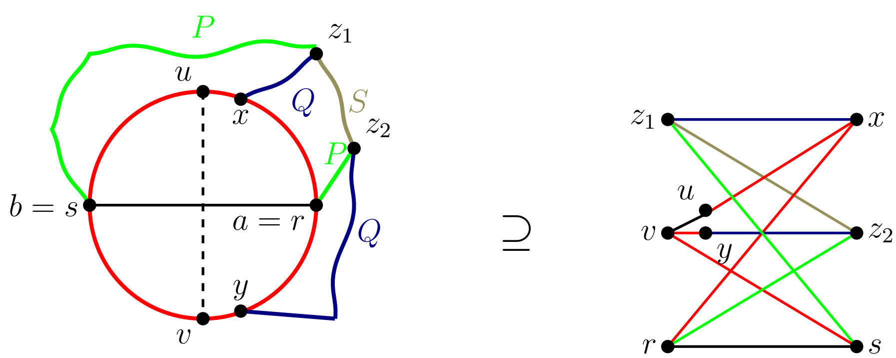
As in the previous case, consider the paths $P(s,r)$ and $Q(x,y)$. They share at least one common vertex. If P and Q have exactly one vertex in common, let it be z. If they have more than one vertex in common, let z_1 be the first common vertex and z_2 be the last common vertex. In both the cases $G \supseteq S(K_{3,3})$; see Figure 10.21.



P and Q intersect in exactly one vertex z.

$S(K_{3,3})$.

(a)



P and Q intersect in more than one vertex.

$S(K_{3,3})$

(b)

Figure 10.21: $G \supseteq S(K_{3,3})$.

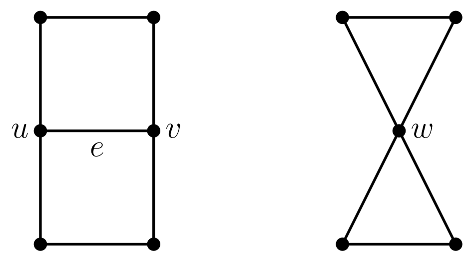
This completes the proof. $\square$

• Minors and Wagner's theorem

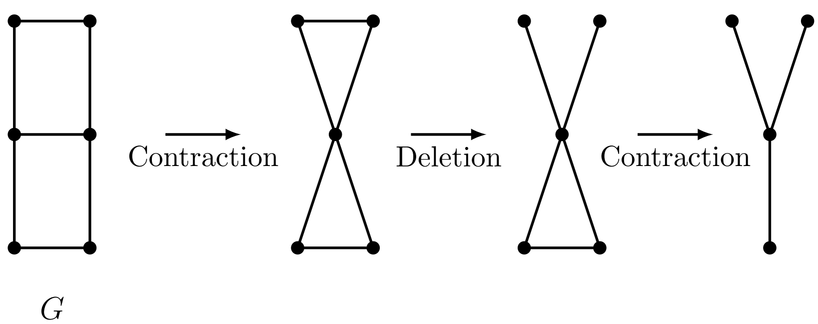
Definitions.

(1) Let $e(u, v)$ be an edge in a graph G. Delete the vertices u and v. Add a new vertex w to $G - \{u, v\}$ and join w to all those vertices in $V - \{u, v\}$ to which u or v is adjacent in G. This operation is called the edge ***edge contraction*** *of e. The resultant graph is denoted by $G \cdot e$.*

(2) A graph H is said to be a ***minor*** *of G, if an isomorphic copy of H can be obtained from G by deleting or contracting a sequence of edges. By convention, G is a minor of G.*



(a) Contraction of e.



(b) $K_{1,3}$ is a minor of G.

Figure 10.22: Edge contraction and minors.

Remarks.

(1) If $G \supseteq S(H)$, then H is a minor of G. H can be obtained by (i) deleting the edges of $E(G) - E(S(H))$ from G, and then (ii) contracting the edges incident to some of the vertices of degree 2.

(2) However, the converse of (1) is false, that is if H is a minor of G, then G need not contain $S(H)$. For example, K_5 is a minor of the Petersen graph but it contains no $S(K_5)$. See Figure 10.23.

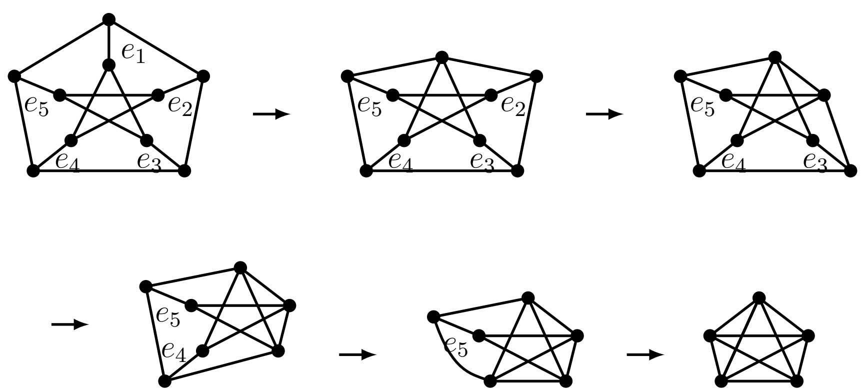


Figure 10.23: K_5 is a minor of the Petersen graph.

Theorem 10.7 (Wagner, 1937)**.** *A graph G is planar if and only if neither K_5 nor $K_{3,3}$ is a minor of G.* □

10.5 Planarity testing (Optional)

Based on Kuratowski's theorem, an algorithm to test the planarity of a graph was designed by Demoucron, Melgrange and Pertuiset (1964). To describe this algorithm, we require the concept of branches of any subgraph $H \subseteq G$. In the proof of Kuratowski's theorem, we dealt with the branches of a cycle in G.

Definitions. *Let H be a subgraph of a graph G. Let D be a component of $G - V(H)$.*

- *The subgraph B of G which consists of D and the set of edges which join a vertex of D with a vertex of H is called a* ***branch*** *of B.*

- *A vertex of H which is joined to a vertex of D is called a* ***contact vertex*** *of B.*
- *If B has k contact vertices, then it is called a* ***k-branch****. The set of all contact vertices of B is denoted by $V(B;H)$.*
- *In addition, an $(x,y) \in E(G) - E(H)$ with $x, y \in V(H)$ is called an* ***edge-branch****; x and y are its contact vertices.*
- *If $H \subseteq G$ is a plane graph, then a branch B of H is said to be* ***drawable*** *in a face f of H if $V(B;H) \subseteq$ boundary of f. The set $\{f \in F(H) : B$ is drawable in $f\}$ is denoted by $F(B;H)$.*

• D-M-P-planarity algorithm

Input: A 2-vertex-connected simple graph G.

Output: A plane embedding of G or a declaration that G is nonplanar.

Step 1: Choose a cycle in G, say G_1, and let $\tilde{G}_1$ be a plane embedding of G_1.

Having drawn a plane embedding $\tilde{G}_i$ of a subgraph $G_i (i \geq 1)$ of G, do the following:

Step 2: If $E(G) - E(\tilde{G}_i) = \emptyset$, then ***stop*** and declare that G is planar and output $\tilde{G}_i$ (; it is a plane drawing of G).

Else, find all the branches of $\tilde{G}_i$. For each such branch B, find $F(B; \tilde{G}_i)$.

Step 3:

(i) If there exists a branch B such that $F(B; \tilde{G}_i) = \emptyset$, ***stop*** and declare that G is nonplanar.

(ii) If there exists a branch B such that $|F(B; \tilde{G}_i)| = 1$, let $f \in F(B; \tilde{G}_i)$.

(iii) If $|F(B; \tilde{G}_i)| \geq 2$, for every branch B of $\tilde{G}_i$ choose any such B and let f be any face $\in F(B; \tilde{G}_i)$.

Step 4: Select a path $P_i \subseteq B$ which connects two contact vertices of B. Draw P_i in f. Set $\tilde{G}_{i+1} = \tilde{G}_i \cup P_i$ and goto step 2 with $\tilde{G}_{i+1}$.

Remark. *Algorithm may fail to correctly test the planarity of* G *if in Step 4 the entire* B *is drawn in* f *instead of the path* P_i.

Illustration 1:

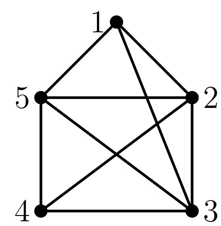
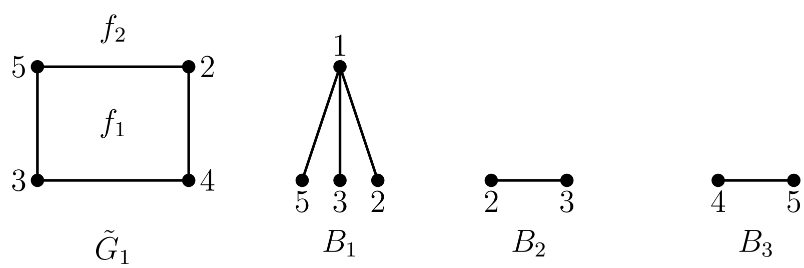


Figure 10.24: Input graph G

Iteration 1: Following Step (1), we arbitrarily choose the cycle $\tilde{G}_1 = (2, 4, 3, 5, 2)$.

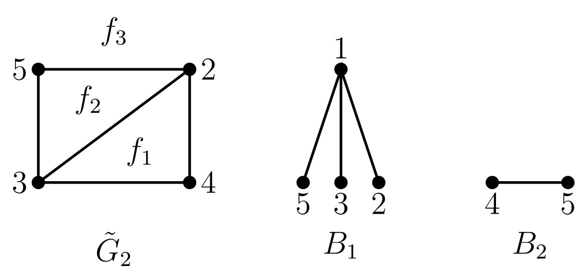


$\tilde{G}_1$ and its branches.

$$F(B_1; \tilde{G}_1) = \{f_1, f_2\},\ F(B_2; \tilde{G}_1) = \{f_1, f_2\} \text{ and } F(B_3; \tilde{G}_1) = \{f_1, f_2\}.$$

Here Step 3(iii) applies. We arbitrarily choose B_2 and f_1 to enlarge the drawing of $\tilde{G}_1$.

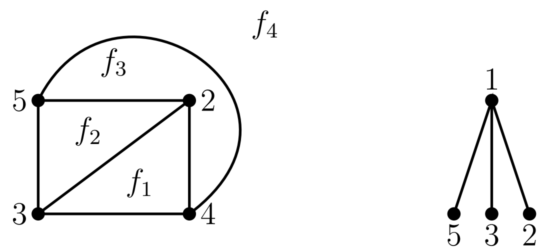
Iteration 2:



$\tilde{G}_2$ and its branches. $F(B_1;\tilde{G}_2) = \{f_2, f_3\}$ and $F(B_2;\tilde{G}_2) = \{f_3\}$.

Here we are compelled to choose B_2 and f_3 by Step 3(ii).

Iteration 3:

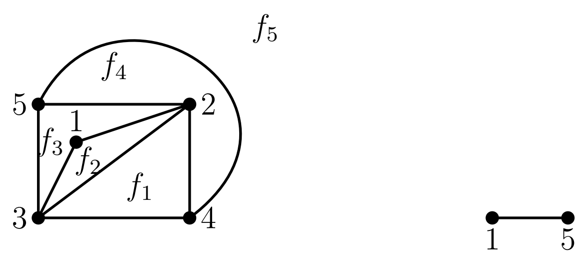


$\tilde{G}_3$ B_1

$\tilde{G}_3$ and its branch B_1. $F(B_1;\tilde{G}_3) = \{f_2\}$.

Again we are compelled to choose B_1 and f_2. Following Step 4, we can either choose the path $(2,1,3)$ or $(3,1,5)$ or $(2,1,5)$; we choose $(2,1,3)$.

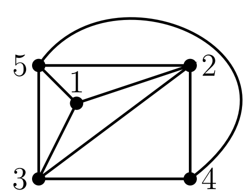
Iteration 4:



$\tilde{G}_4$ B_1

$\tilde{G}_4$ and its branch B_1. $F(B_1;\tilde{G}_4) = \{f_3\}$.

Iteration 5:



$\tilde{G}_5$

At this step, $E(G) - E(\tilde{G}_5) = \emptyset$. So $\tilde{G}_5$ is a plane embedding of G.

Illustration 2:

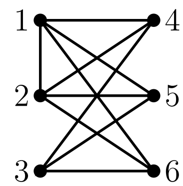
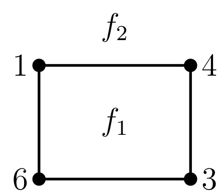
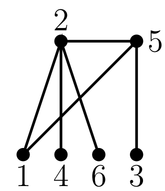


Figure 10.25: Input graph G.

Iteration 1:



$\tilde{G}_1$

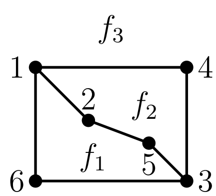


B_1

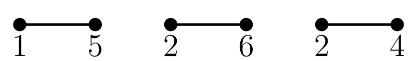
$\tilde{G}_1$ and its branch B_1. $F(B_1; \tilde{G}_1) = \{f_1, f_2\}$.

We choose f_1 and the path $(1, 2, 5, 3) \subseteq B_1$.

Iteration 2:



$\tilde{G}_2$

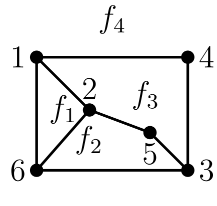


B_1 B_2 B_3

$\tilde{G}_2$ and its branches. $F(B_1; \tilde{G}_2) = \{f_1, f_2\}$, $F(B_2; \tilde{G}_2) = \{f_1\}$ and $F(B_3; \tilde{G}_2) = \{f_2\}$.

We choose B_2 and f_1.

Iteration 3:



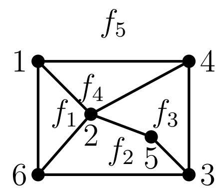
$\tilde{G}_3$

B_1 B_2

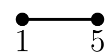
$\tilde{G}_3$ and its branches. $F(B_1;\tilde{G}_3) = \{f_3\}$ and $F(B_2;\tilde{G}_3) = \{f_3\}$.

We choose B_2.

Iteration 4:



$\tilde{G}_4$



B_1

$\tilde{G}_4$ and its branch $B_1 : F(B_1, \tilde{G}_4) = \emptyset$.

By Step 2 of the algorithm, we conclude that G is nonplanar.

10.6 5-Color-theorem

As remarked in the beginning of this chapter, planar maps and planar graphs first appeared in a problem called the ***four color conjecture*** (1850).

Four-Color-Conjecture
Any map of a country can be colored with at most four colors so that no two adjacent states receive the same color.

It was a fascinating open problem for a long time, which attracted many well-known mathematicians. Their insights, proof techniques, and variations of the problem laid foundation for the topic of graph colorings. The conjecture was finally solved by K. Appel, W. Haken and J. Koch (1977). Their proof techniques involved making of a large number of cases by a computer; thus making the computer necessary in a mathematical proof for the first time in the history of mathematics. The original proof consisted of 700 pages and consumed about 1200 hours of CPU time in 1970's. So, the proof generated a lot of debate and is still continuing. There have been attempts to simplify the proofs but no current proof is less than 100 pages.

Any map M represents a plane graph G whose faces represent the states of M. As observed earlier its dual G^* is also a plane graph. Clearly, the face-coloring of G is equivalent to the vertex-coloring of G^*. So the conjecture can be restated as follows:

Four-Color-Conjecture (Alternative form)

Every plane graph is 4-vertex-colorable.

It is easy to show that every plane graph G is 6-vertex-colorable by induction on n, or by using greedy algorithm since $\delta(G) \leq 5$. However, to show that every plane graph is 5-vertex-colorable we require new ideas.

The first published proof for the 4-vertex-colorability is due to A.B. Kempe (1879). However, an error was found by P.J. Heawood in 1890, and the same author showed in 1898 that Kempe's argument can be used to prove the following weaker form of the Conjecture.

Theorem 10.8 (Heawood, 1898)**.** *Every planar graph is 5-vertex-colorable.*

Proof. (By induction on n). If $n \leq 5$, then obviously the result holds. So, we proceed to the induction step, assuming that every planar graph on $n-1$ vertices is 5-vertex-colorable. Let G be a planar graph on n vertices, and let v be a vertex of minimum degree in G. By a corollary proved earlier, $d(v) \leq 5$. Consider the planar graph $G-v$

and assume that we are given a plane embedding of $G - v$. By induction hypothesis, $G - v$ is 5-vertex-colorable; let $f : V(G) \to \{1, 2, 3, 4, 5\}$ be a 5-vertex-coloring. Our aim is to extend f to a 5-vertex-coloring of G.

If $d(v) \leq 4$, then at most four colors appear in its neighborhood $N(v)$. So, we can color v with one of the five colors which does not appear in $N(v)$.

Next, suppose $d(v) = 5$, and let $N(v) = \{v_1, v_2, v_3, v_4, v_5\}$. See Figure 10.26.

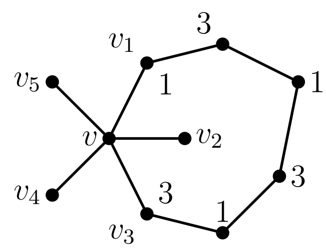


Figure 10.26: Five neighbors of v.

If there appear at most four colors among the colors of v_1, v_2, v_3, v_4, v_5, then the missing color can be used to color v. So, assume that all the five colors appear in the neighborhood of v and that $f(v_i) = i, 1 \leq i \leq 5$.

Let $G_{i,j}$ denote the subgraph of $G - v$ induced by the vertices colored i or j. Consider $G_{1,3}$; $v_1, v_3 \in V(G_{1,3})$.

If v_1 and v_3 are in different components of $G_{1,3}$ say $v_1 \in D_1$ and $v_3 \in D_2$, then we exchange the colors 1 and 3 of vertices in D_1, without disturbing the colors of other vertices. The resultant coloring is a 5-vertex-coloring of $G - v$ such that $f(v_1) = 3, f(v_2) = 2, f(v_3) = 3, f(v_4) = 4$ and $f(v_5) = 5$. So we can color v with 1 and get a 5-vertex-coloring of G.

Next assume that v_1 and v_3 belong to the same component of $G_{1,3}$; so there exists a (v_1, v_3)-path P in $G_{1,3}$. Then $C = (v, v_1, P(v_1, v_3), v_3, v)$ is a cycle in G whose interior contains v_2 and exterior contains v_4 and v_5 or interior contains v_4 and v_5 and exterior contains v_2. Let $v_2 \in intC$ and $v_4 \in extC$; see Figure 10.26. Now consider $G_{2,4}$. If v_2 and v_4 are in different components, then we can get a 5-vertex-coloring of G as before. So, assume that v_2 and v_4 belong to the same component. Therefore, there exists a (v_2, v_4)-path say Q in $G_{2,4}$. Note that the vertices of P are colored alternately 1 and 3, and the vertices of Q are colored alternately 2 and 4.

Since $v_2 \in intC$ and $v_4 \in extC$, P and Q intersect. Since $G - v$ is a plane graph, the intersection point is a vertex, say x of G. Since $x \in V(P)$, it is colored 1 or 3. Since $x \in V(Q)$, it is colored 2 or 4. We thus arrive at a contradiction.

Hence, we conclude that (i) v_1 and v_3 are in different components of $G_{1,3}$, or (ii) v_2 and v_4 are in different components of $G_{2,4}$. In either case, we can extend f to a 5-vertex-coloring of G. □

Exercises

1. Draw plane simple graphs with the following degree sequences:

$$(4^3, 2^3), (5^{12}), (6^2, 5^{12}), (6^3, 5^{12}), (4^3, 3^8), (7, 4^{10}, 3^5).$$

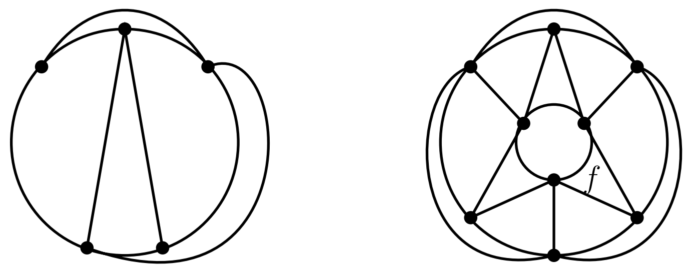
(The general problem of finding necessary and sufficient conditions for a graphic sequence to realize a planar graph is ***open***.)

2. Redraw the graphs shown in Figure 10.27 so that all the edges are straight lines and no two lines intersect. (It is known that every planar simple graph can be drawn as a plane graph in which every edge is a straight line.)

3. Redraw the graph in Figure 10.27b so that the face f is the exterior face.

4. Use the first corollary of Theorem 10.3 to show that the Petersen graph is nonplanar.

5. Let G be a plane graph with r faces and c components. Show that $n + r = m + c + 1$.



(a) (b)

Figure 10.27

6. Find which of the platonic graphs (Figure 10.9) are Hamilton.

7. Let G be a connected plane 3-regular graph and let n_k denote the number of faces of degree k.

   (a) Show that $\sum_{k \geq 3}(k-6)n_k + 12 = 0$.

   (b) Deduce from (a) that if G is bipartite, then G contains at least 6 faces of degree 4.

8. Find the number of components in a simple plane graph which has 11 vertices, 13 edges and 6 faces.

9. Draw a connected plane simple graph H with 9 edges and 6 faces. Find the edge-chromatic number of the dual of H.

10. If G is a simple planar bipartite graph, then show that $m(G) \leq 2n(G) - 4$.

11. Draw all the non-isomorphic plane simple 3-regular graphs with exactly 5 faces.

12. (a) Show that there exists no 3-regular simple planar graph on 10 vertices whose girth is 5.

    (b) Draw a plane 3-regular simple graph on 10 vertices whose girth is 4.

    (c) Draw a plane 3-regular simple graph on 10 vertices whose girth is 3.

13. Find all the values of k, for which there exists a k-regular simple plane graph.

14. If G is a simple maximal planar graph with $\Delta(G) = 7$, then show that $n_7 = 3n_3 + 2n_4 + n_5 - 12$, where n_k denotes the number of vertices of degree k in G.

15. Prove or disprove: There exists a simple 4-regular maximal plane graph.

16. Find the number of faces in a maximum plane simple graph on 10 vertices.

17. Prove or disprove: There exists a maximal planar simple graph ($n > 1$) whose complement is also a maximal planar.

18. A tree on $n\,(\geq 3)$ vertices is such that its complement is maximum planar. Find n. Draw one such tree.

19. Show that every graph with at most 3 cycles is planar.

20. Find the number of faces in a plane embedding of $K_{1,1,n}$.

21. Find all the values of n such that $K_{1,2,\ldots,n}$ is planar.

22. Find all the values of a, b, c such that $K_{a,b,c}$ is planar, where $1 \leq a \leq b \leq c$.

23. Prove or disprove: If G is a simple planar graph with $n \leq 11$, then G contains a vertex of degree ≤ 4.

24. If G is a nonplanar graph then show that either

    (i) there are at least 5 vertices of degree ≥ 4, or

    (ii) there are at least 6 vertices of degree ≥ 3.

25. Show that in a planar graph G (with $n \geq 4$) there are at least 4 vertices of degree ≤ 5.

26. If G is a simple planar graph with $\delta(G) = 5$, then show that there are at least 12 vertices of degree 5. Give an example of planar graph on 12 vertices with $\delta(G) = 5$.

27. If $n(G) \geq 11$, then show that G or G^c is nonplanar. Give an example of a graph G on 8 vertices such that G and G^c are planar.

28. If G is a planar with degree sequence $(d_1 \geq d_2 \geq \cdots \geq d_n)$, then show that

$$\sum_{i=1}^{k} d_i \leq 6(k-2) + \sum_{i=k+1}^{n} d_i, \text{ for every } k,\ 1 \leq k \leq n.$$

29. Show that the following are equivalent for a connected graph G.

(a) G is a tree.

(b) G contains no subdivision of K_3.

(c) G contains no K_3-minor.

30. Verify whether the graphs shown below are embeddable in the plane by applying D-M-P-algorithm. Draw all the intermediate plane graphs and their branches generated by the algorithm.

    (a) $C_5 + K_1$.

    (b) $C_5 + K_2$.

    (c) $K_{2,2,2,2}$.

    (d) The graph with vertices 1, 2, 3, 4, 5, 6, 7, 8 and edges (1,2), (2,3), (3,4), (4, 5), (5, 6), (6, 7), (7, 8), (8,1), (1,5), (2,6), (3,7), (4,8).

31. Use the greedy algorithm to show that every planar graph is 6-vertex-colorable.

32. Prove Kuratowski's theorem (Case 2) by choosing a cycle C in $G - e$ which contains maximum number of edges in the $intC$.

33. Let G be a simple plane connected 3-regular graph with every face having degree 5 or 6. If p is the number of pentagonal faces and h is the number of hexagonal faces in G, show that $p = 12$ and $n(G) = 20 + 2h$.
    (Which motivational problem stated in Chapter 1 and section 1 is now solved?)