Module 7 Independent sets, coverings and matchings

Contents

7.1 Introduction

The concepts of independent sets and coverings are used in modeling several real world problems. One of the real world problems called the *Job Assignment Problem* has motivated much research on matchings.

Job-Assignment-Problem (JAP)
There are s persons and t jobs. Each person is capable of handling certain jobs. Under what conditions we can employ **each** of the p persons with a job he/she is capable of handling? The rule of **one-person-one-job** is assumed.

- Notice that some jobs may be left undone.
- Model the problem using appropriate graph theoretic terminology.
- Another popular version of the JAP is known as ***The Marriage Problem***. There are s girls and t boys. Each girl has a list of favorite boys. Under what conditions one can marry off each of the girls to a boy in her list?
- A pure set-theoretic version of JAP is known as ***The Problem of Set Representatives***. Given a set X and a family $F = (X_1, X_2, \ldots, X_s)$ of subsets of X, find necessary and sufficient conditions for choosing s distinct elements $(x_1, x_2, \ldots, x_s)$, such that $x_i \in X_i$, $i = 1, 2, \ldots, s$. The element x_i is called the ***representative*** of X_i.

Notice that if X, F, X_i and s are infinite, then it is a variation of "Axiom of Choice".

7.2 Independent sets and coverings: basic equations

We start with a series of definitions.

Definitions. *Let G be a graph.*

- *A set of vertices I is called an **independent set** if no two vertices in I are adjacent. An independent set is also called a **stable set**.*

  - Any singleton set is an independent set. So one is interested to find a largest independent set.

- *The parameter* $\alpha_0(G) = \max\{|I| : I \textit{ is an independent set in } G\}$ *is called the* ***vertex-independence number*** *of* G.

- *Any independent set* I *with* $|I| = \alpha_0(G)$ *is called a* ***maximum independent set***.

- *An independent set* I *is called a* ***maximal independent set*** *if there is no independent set which properly contains* I.

  - Clearly, any maximum independent set is a maximal independent set but a maximal independent set need not be a maximum independent set. An illustration is given below.

- *If* $e(u, v)$ *is an edge in* G, *then* e *is said to* ***cover*** u *and* v *and vice versa.*

- *A set of vertices* K *is called a* ***vertex-cover*** *of* G *if every edge in* G *is covered by a vertex in* K. *That is, if every edge in* G *has at least one of its end vertices in* K.

  - Clearly, $V(G)$ is a vertex-cover of G. So one is interested to find a smallest vertex cover.

- *The parameter* $\beta_0(G) = min\{|K| : K \textit{ is a vertex cover of } G\}$ *is called the* ***vertex-covering number*** *of* G.

- *Any vertex-cover* K *with* $|K| = \beta_0(G)$ *is called a* ***minimum vertex-cover***.

- *A vertex-cover* K *is called a* ***minimal vertex-cover*** *if there is no vertex-cover which is properly contained in* K.

  - Clearly, every minimum vertex-cover is a minimal vertex-cover but a minimal vertex-cover need not be a minimum vertex-cover. An illustration is given below.

An illustration: In the graph shown in Figure 7.1, $\{1, 2, 4\}$, $\{3, 5, 6\}$ are maximal independent sets, but they are not maximum independent sets; $\{1, 2, 5, 6\}$ is the

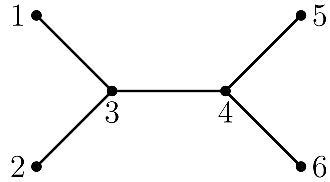


Figure 7.1: An example to illustrate the independent sets and vertex covers.

maximum independent set. The vertex subset $\{1, 2, 4\}$ is a minimal vertex-cover but it is not a minimum vertex-cover; {3,4} is a minimum vertex-cover.

This example illustrates that a graph may contain many maximal and maximum independent sets. Similarly, it may contain many minimal and minimum vertex-covers. However, $\alpha_0(G)$ and $\beta_0(G)$ are unique.

The following table shows the independence number and vertex-covering number of some standard graphs.

	P_n	C_n	K_n	K_n^c	$K_{r,s}$
α_0	$\lceil \frac{n}{2} \rceil$	$\lfloor \frac{n}{2} \rfloor$	1	n	$\max\{r, s\}$
β_0	$\lfloor \frac{n}{2} \rfloor$	$\lceil \frac{n}{2} \rceil$	$n - 1$	0	$\min\{r, s\}$

Table 7.1: Table of independence numbers and covering numbers.

The reader may notice that for every graph G shown in the above table, $\alpha_0(G) + \beta_0(G) = |V(G)|$. In fact, this equation holds for any arbitrary graph G. Before proving this claim, we observe a stronger statement.

Theorem 7.1. *I is an independent set in G iff $V(G) - I$ is a vertex-cover of G.*

Proof. I is an independent set

$\Leftrightarrow$ no two vertices in I are adjacent

$\Leftrightarrow$ no edge has both of its end vertices in I

$\Leftrightarrow$ every edge in G has an end vertex in $V(G) - I$

$\Leftrightarrow$ $V(G) - I$ is a vertex-cover of G. □

Corollary. *For any graph G, $\alpha_0(G) + \beta_0(G) = n(G)$.*

Proof. Let I be a maximum independent set. By the above theorem, $V(G) - I$ is a vertex cover of G. Hence, $\beta_0(G) \leq |V(G)-I| = n-\alpha_0(G)$; that is $\alpha_0(G)+\beta_0(G) \leq n$.

Let K be a minimum vertex-cover of G. By the above theorem, $V(G) - K$ is an independent set of vertices. Hence, $\alpha_0(G) \geq |V(G) - K| = n(G) - \beta_0(G)$; that is $\alpha_0(G) + \beta_0(G) \geq n(G)$.

The two inequalities imply the corollary. □

We now define the edge analogues of independent sets of vertices and vertex-covers.

Definitions.

- *A subset of edges M in G is called an* ***independent set*** *of edges if no two edges in M are adjacent. An independent set of edges is more often called as a* ***matching.***
  - Clearly, if M is a singleton, then it is a matching. So one is interested to find a largest matching.
- *The parameter $\alpha_1(G) = max\{|M| : M \text{ is a matching in } G\}$ is called the* ***matching number*** *of G.*
- *Any matching M with $|M| = \alpha_1(G)$ is called a* ***maximum matching****.*
- *A matching M is called a* ***maximal matching*** *if there is no matching which properly contains M.*

- *A set of edges F is called an* ***edge-cover*** *of G if every vertex in G is incident with an edge in G.*
  - A graph G need not have an edge cover; for example, $K_2 \cup K_1$ has no edge-cover. In fact, a graph G has an edge cover iff $\delta(G) > 0$.
  - If $\delta(G) > 0$, then $E(G)$ is an edge-cover. So one is interested to find a smallest edge-cover.
- *If $\delta(G) > 0$, then the parameter $\beta_1(G) = min\{|F| : F \text{ is an edge cover of } G\}$ is called the* ***edge covering number*** *of G.*
- *Any edge-cover F with $|F| = \beta_1(G)$ is called a* ***minimum edge cover.***
- *An edge-cover F is called a* ***minimal edge-cover*** *if there is no edge-cover which is properly contained in F.*

An illustration: In the graph of Figure 7.2, $\{a, d\}$ and $\{b, c\}$ are maximal matchings but they are not maximum matchings ; $\{a, e, f\}$ is a maximum matching. $\{b, c, e, f\}$ is a minimal edge-cover but it is not a minimum edge-cover. $\{a, e, f\}$ is a minimum edge-cover.

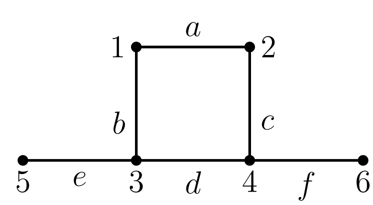


Figure 7.2: An example to illustrate matchings and edge covers.

Again we observe that a graph G may contain many maximal and maximum matchings. It also may contain many minimal and minimum edge covers. But $\alpha_1(G)$ and $\beta_1(G)$ are unique.

The following table shows the matching-number and edge-covering number of some standard graphs.

	P_n	C_n	K_n	$K_{r,s}$
α_1	$\lfloor \frac{n}{2} \rfloor$	$\lfloor \frac{n}{2} \rfloor$	$\lfloor \frac{n}{2} \rfloor$	$\min\{r, s\}$
β_1	$\lceil \frac{n}{2} \rceil$	$\lceil \frac{n}{2} \rceil$	$\lceil \frac{n}{2} \rceil$	$\max\{r, s\}$

Table 7.2: Matching numbers and edge-covering numbers.

Again notice that for every graph G shown in the above table $\alpha_1(G)+\beta_1(G) = n(G)$. However, the edge analogue of Theorem 7.1 does not hold. That is, if M is a matching in G, then $E(G) - M$ need not be an edge cover of G. And if F is an edge cover of G, then $E(G) - F$ need not be a matching. Reader is encouraged to construct graphs to justify these claims. However, $\alpha_1(G) + \beta_1(G) = n(G)$, for any graph G with $\delta(G) > 0$. Before proving it we introduce a new concept.

Definitions.

- *Let M be a matching in G. A vertex v is said to be **M-saturated** if there exists an edge in M which is incident to v; else v is said to be **M-unsaturated.***
- *A matching M is said to be a **perfect matching** if it saturates every vertex of G.*

In the following graph, $M_1 = \{(1, 2), (4, 5)\}$ is a matching. The vertices 1, 2, 4 and 5 are M_1-saturated but 3 and 6 are M_1-unsaturated. So, it is not a perfect matching. However, $M_2 = \{(1, 2), (3, 4), (5, 6)\}$ is a perfect matching.

Remarks.

- Every perfect matching is a maximum matching. However, the converse is false.
- If G has perfect matching, then $n(G)$ is even. However, the converse is false; for example, $K_{1,3}$ and $K_{2,4}$ do not have perfect matchings.

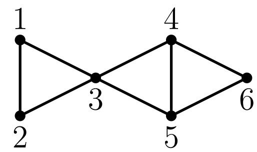


Figure 7.3: A graph to illustrate perfect matching.

Theorem 7.2. *For any graph G with $\delta(G) > 0$, $\alpha_1(G) + \beta_1(G) = n(G)$.*

Proof. It consists of two steps.

(1) $\alpha_1(G) + \beta_1(G) \leq n(G)$.

Let M be a maximum matching and U be the set of all M-unsaturated vertices. Then $|U| = n - 2|M|$. Let $U = \{v_1, v_2, \ldots, v_p\}$, where $p = n - 2|M|$. Let e_i be an edge incident to v_i, $i = 1, \ldots, p$; such an edge exists, since $\delta(G) > 0$. Then $M \cup \{e_1, e_2, \ldots, e_p\}$ is an edge-cover. So,

$$\beta_1(G) \leq |M| + p \leq |M| + (n - 2|M|) = n - |M| = n - \alpha_1(G).$$

Hence, $\alpha_1(G) + \beta_1(G) \leq n$.

(2) $\alpha_1(G) + \beta_1(G) \geq n(G)$.

Let F be a minimum edge cover. Let H be the spanning subgraph of G with edge set F. Let M_H be a maximum matching in H and U be the set of M_H-unsaturated vertices in H. As before, $|U| = n - 2|M_H|$. Let $U = \{v_1, v_2, \ldots, v_p\}$, where $p = n - 2|M_H|$. Since, F is an edge-cover, it contains an edge e_i incident with v_i, $1 \leq i \leq p$. Since, M_H is a maximum matching, U is an independent set in H. Therefore $e_i \neq e_j$, if $v_i \neq v_j$. Hence, $e_1, e_2, \ldots, e_p$ are all distinct edges. It follows that $|F| \geq |M_H| + p = |M_H| + n - 2|M_H| = n - |M_H|$. Hence, $\beta_1(G) = |F| \geq n - |M_H| \geq n - \alpha_1(G)$, since $\alpha_1(G) \geq |M_H|$. So, $\alpha_1(G) + \beta_1(G) \geq n$.

The two inequalities imply the theorem. □

7.3 Matchings in bipartite graphs: Hall's theorem and König's theorem

We have now enough terminology to model the job-assignment-problem and solve it.

Graph theoretic model: Let $P_1, P_2, \ldots, P_s$ be the s persons and $J_1, J_2, \ldots, J_t$ be the t jobs. Represent each P_i by a vertex p_i and each job J_h by vertex j_h. Join p_i and j_h by an edge if P_i is capable of handling j_h. This construction yields a bipartite graph $G[X, Y]$, where $X = \{p_1, \ldots, p_s\}$ and $Y = \{j_1, \ldots, j_t\}$. The problem is to find necessary and sufficient conditions for G to contain a matching saturating every vertex in X. The following example illustrates the construction of G.

Persons	Jobs P_i can handle
P_1	J_1, J_3
P_2	J_1, J_2, J_3
P_3	J_2, J_4, J_5
P_4	J_3

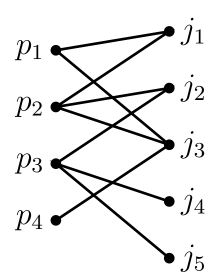


Figure 7.4: A job assignment as required is possible in this example. One possible assignment is to assign P_1 to J_1, P_2 to J_2 P_3 to J_5 and P_4 to J_3.

If G$[X, Y]$ contains a matching, say $\{(p_1, j_1), \ldots, (p_s, j_s)\}$ saturating every vertex in X, then p_i can be assigned the job j_i, $1 \le i \le s$.

A necessary condition for the existence of a solution for JAP is intuitively obvious: Every set of k persons $(1 \le k \le s)$ must be able to handle at least k jobs together. That is, in graph theoretic terminology, if G has a matching saturating every vertex in X, then every k vertices in X are adjacent to at least k vertices in Y $(1 \leq k \leq s)$. Hall (1939) showed that the converse is also true.

Notation: If $G[X, Y]$ is a bipartite graph and $S \subseteq X$, let $N_G(S) = \{y \in Y : y \text{ is adjacent to some vertex in } S\}$.

• Hall's Theorem

Theorem 7.3 (Hall, 1939)**.** *A bipartite graph $G[X, Y]$ has a matching saturating every vertex in X iff*

(Hall) $\quad |N_G(S)| \geq |S|$, *for every* $S \subseteq X$.

Proof. Let $X = \{x_1, x_2, \ldots, x_s\}$ and $Y = \{y_1, y_2, \ldots, y_t\}$.

Necessity ($\Rightarrow$)**:** Suppose G has a matching $M = \{(x_1, y_1), \ldots, (x_s, y_s)\}$ saturating every vertex in X. Let $S = \{x_1^{'}, x_2^{'}, \ldots, x_r^{'}\} \subseteq X$. Then $y_1^{'}, y_2^{'}, \ldots, y_r^{'} \in N_G(S)$ and they are all distinct, since M is a matching. So $|N_G(S)| \geq r = |S|$.

Sufficiency ($\Leftarrow$)**:** There are many proofs of sufficiency. The following proof is due to Halmos and Vaughan (1950). It is by induction on s.

If $s = 1$, then the result is obvious. So, let $s \geq 2$ and proceed to the next step in the induction. We make two cases.

Case 1: For every proper subset S of X, $|N_G(S)| \geq |S| + 1$.

Since G satisfies (Hall), $|N(\{x_1\})| \geq 1$; and so there is a vertex, say $y_1 \in Y$ such that (x_1, y_1) is an edge. Consider the subgraph $L = G - \{x_1, y_1\}$ whose bipartition is $[X_1, Y_1]$, where $X_1 = \{x_2, \ldots, x_s\}$ and $Y_1 = \{y_2, \ldots, y_t\}$. We claim that $L(X_1, Y_1)$ satisfies (Hall). So, let $T \subseteq X_1$. Clearly, $N_G(T) \subseteq N_L(T) \cup \{y_1\}$. Hence,

$$\begin{aligned} |N_L(T)| &\geq |N_G(T)| - 1, \\ &\geq (|T| + 1) - 1, \text{by our assumption,} \\ &= |T|. \end{aligned}$$

Therefore, by induction hypothesis, $L[X_1, Y_1]$ contains a matching M saturating every vertex in X_1. Then $M \cup \{(x_1, y_1)\}$ is a required matching of G.

Case 2: For some proper subset S of X, $|N_G(S)| = |S|$.

For notational convenience, let $S = \{x_1, x_2, \ldots, x_p\}$ and $N(S) = \{y_1, y_2, \ldots, y_p\}$.

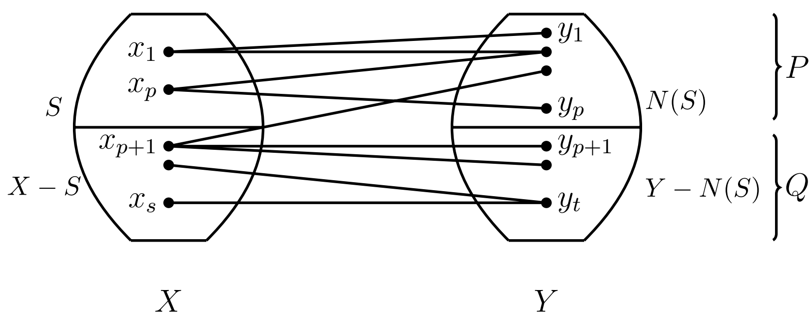


Figure 7.5: The bipartite graphs $P[S, N(S)]$ and $Q[X - S, Y - N(S)]$.

Consider the bipartite subgraphs $P[S, N(S)]$ and $Q[X - S, Y - N(S)]$.

We shall prove that both these graphs satisfy (Hall) and then complete the proof by appealing to the induction hypothesis.

We consider first $P[S, N(S)]$. Let $T \subseteq S$. Clearly, by our assumption of S, $N_P(T) = N_G(T)$; so $|N_P(T)| \geq |T|$. Thus P satisfies (Hall), and hence, by induction hypothesis, P has a matching, say M_1 saturating every vertex in S. Next, consider $Q[X - S, Y - N(S)]$. Let $T \subseteq X - S$. Clearly, $N_G(S \cup T) \subseteq N_G(S) \cup N_Q(T)$. So,

$$\begin{aligned}
|N_Q(T)| &\geq |N_G(S \cup T)| - |N_G(S)|, \\
&\geq |S \cup T| - |S|, \text{ since G satisfies (Hall)}, \\
&= |S| + |T| - |S|, \text{ since } S \text{ and } T \text{ are disjoint}, \\
&= |T|.
\end{aligned}$$

Thus $Q[X - S, Y - N_G(S)]$ satisfies (Hall). So by induction hypothesis, Q has a matching, say M_2 saturating every vertex in $X - S$. Then $M_1 \cup M_2$ is a required matching. $\square$

Two interesting corollaries follow.

Corollary. *Let G be a bipartite graph with bipartition $[X, Y]$ and $\delta(G) > 0$. If*

$$\min_{x \in X} deg(x) \geq \max_{y \in Y} deg(y),$$

then G contains a matching saturating every vertex in X.

Proof. It is enough if we prove that G satisfies (Hall). So, let $S \subseteq X$. Let $\delta(X) = \min_{x \in X} deg(x)$ and $\Delta(Y) = \max_{y \in Y} deg(y)$. If e is an edge incident with a vertex in X, then its other end vertex is in $N(S)$. So, there are at least $|S| \cdot \delta(X)$ edges incident with the vertices of $N(S)$. Hence, there is a vertex $y \in N(S)$ which is incident with at least $\dfrac{|S| \cdot \delta(X)}{|N(S)|}$ edges ($\because$ Use the pigeon-hole principle). So, $\Delta(Y) \geq deg(y) \geq \dfrac{|S| \cdot \delta(X)}{|N(S)|}$. Since, $\delta(X) \geq \Delta(Y)$, we obtain $|N(S)| \geq |S|$. And thus, we have verified (Hall). $\square$

Corollary. *Let $G[X, Y]$ be a k-regular ($k \geq 1$) bipartite graph. Then $E(G)$ can be partitioned into k sets $E_1, E_2, \ldots, E_k$, where each E_j is a perfect matching.*

Proof. We shall prove the theorem by induction on k. At the outset observe that $|X| = |Y|$, since $k|X| = |E(G)| = k|Y|$. If $k = 1$, then $E(G)$ is a perfect matching and hence the assertion follows. So we proceed to the next step in the induction.

Since, $G[X, Y]$ is a k-regular bipartite graph it satisfies the hypothesis of the above corollary. Hence, it contains a perfect matching, say E_1. Since, $|X| = |Y|$, $G - E_1$ is a $(k - 1)$-regular bipartite graph. Therefore, by induction hypothesis,

$E(G - E_1)$ can be partitioned into $k - 1$ sets, say $E_2, \ldots, E_k$, where each E_i is a perfect matching. Then $(E_1, E_2, \ldots, E_k)$ is a required partition of $E(G)$. □

• König's Theorem

There are several theorems in discrete mathematics which show that a minimum parameter is equal to a maximum parameter. The next result is one such theorem. Before stating it, we observe an important inequality. If M is a matching and K is a vertex-cover of G, then any vertex of K covers at most one edge of M. Hence:

$$\boxed{\beta_0(G) \geq \alpha_1(G), \text{ for any graph } G.}$$

Remarks.

- $\beta_0(C_{2k}) = \alpha_1(C_{2k}) = k$.
- $\beta_0(K_{2p}) = \alpha_1(K_{2p}) = p$.
- $\beta_0(C_{2k+1}) = k + 1$, $\alpha_1(C_{2k+1}) = k$.

The above remarks lead to the following unsolved question.

$$\boxed{\text{For which graphs } G,\ \beta_0(G) = \alpha_1(G)?}$$

The following result identifies one such class of graphs.

Theorem 7.4 (König, 1931)**.** *For any bipartite graph $G[X, Y]$,*

$$\beta_0(G) = \alpha_1(G).$$

Proof. In view of the above observation, we have to only show that $\alpha_1(G) \geq \beta_0(G)$. Let $K = A \cup B$ be a minimum vertex-cover, where $A \subseteq X$ and $B \subseteq Y$. So, $|A| + |B| = |K| = \beta_0(G)$.

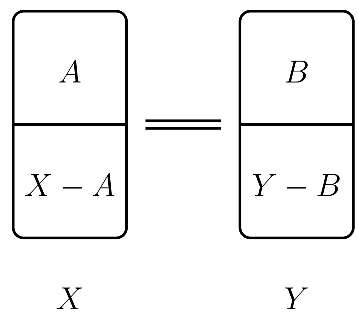


Figure 7.6: A bipartite graph with vertex-cover $A \cup B$.

We consider the bipartite subgraphs $P[A, Y - B]$ and $Q[B, X - A]$.

Claim 1: $P[A, Y - B]$ has a matching M_1 saturating every vertex in A.

To prove this claim it is enough if we verify that P satisfies (Hall). On the contrary, suppose that there exists some $S \subseteq A$ such that $|N_P(S)| < |S|$. Then it is easy to verify that the set $(A - S) \cup N_P(S) \cup B$ is a vertex cover of G. So,

$$\begin{aligned} |(A - S) \cup N_P(S) \cup B| &= |A| - |S| + |N_P(S)| + |B|, \\ &< |A| + |B|, \text{ since by our assumption, } |N_p(S)| < |S|, \\ &= |K|. \end{aligned}$$

This is a contradiction to the minimality of K. Therefore, by Hall's theorem, there exists a matching M_1 in P saturating every vertex in A.

Claim 2: Similarly, it follows that $Q[B, X - A]$ has a matching M_2 saturating every vertex in B.

Clearly $M_1 \cap M_2 = \emptyset$. Hence,

$$\begin{aligned} \alpha_1(G) &\geq |M_1| + |M_2|, \\ &= |A| + |B|, \\ &= |K| = \beta_0(G). \end{aligned}$$

□

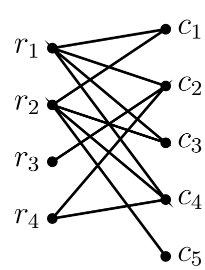
An equivalent form of König's theorem, in matrix terminology, is due to Egerváry (1931). To state it we require new terminology.

Definition. *A **line** of a matrix is either a row or a column. Let $A = [a_{ij}]$ be a $m \times n$ matrix where each a_{ij} is 0 or 1. A set I of $1'$s in A is said to be **independent** if no two $1'$s in I lie on a common line. A line h of A is said to **cover** a 1 if the 1 lies in h.*

An illustration:

$$\begin{array}{c} \\ r_1 \\ \overset{\rightarrow}{r_2} \\ r_3 \\ r_4 \end{array} \begin{array}{c} \begin{array}{ccccc} c_1 & c_2\downarrow & c_3 & c_4\downarrow & c_5 \end{array} \\ \begin{pmatrix} 0 & \boxed{1} & 0 & 0 & 0 \\ \boxed{1} & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \boxed{1} & 0 \end{pmatrix} \end{array}$$

A



$G[R, C]$

Figure 7.7: A matrix A with 3 independent $1'$s and 3 lines covering all the $1'$s in A. (b) A bipartite graph $G[R, C]$ representing A, defined in the proof of Theorem 7.5.

Theorem 7.5 (Egerváry, 1931). *Let $A = [a_{ij}]$ be a $m \times n$ matrix, where a_{ij} = 0 or 1. Then, the maximum number of independent $1'$s in A is equal to the minimum number of lines which cover all the 1's in A.*

Proof. Associate a bipartite graph $G[R, C]$ with A as follows; see Figure 7.7.

- Represent each row R_i $(1 \leq i \leq m)$ of A by a vertex r_i.
- Represent each column C_j $(1 \leq i \leq n)$ of A by a vertex c_j.
- Join r_i and c_j iff $a_{ij} = 1$.

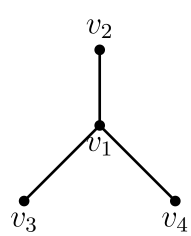
The following assertions can be easily observed.

- There is a 1-1 and onto function between the set of all $1's$ in A and the set of all edges in G, namely $a_{ij} = 1$ iff $(r_i, c_j) \in E(G)$.
- Two $1's$, say $a_{ij} = 1$ and $a_{pq} = 1$ are independent in A iff the corresponding edges (r_i, c_j) and (r_p, c_q) are independent in G. Therefore, I is a set of independent $1's$ in A iff the corresponding set M of edges in G is a matching in G.
- A set of lines covers all the $1's$ in A iff the corresponding set of vertices in G covers all the edges in G.

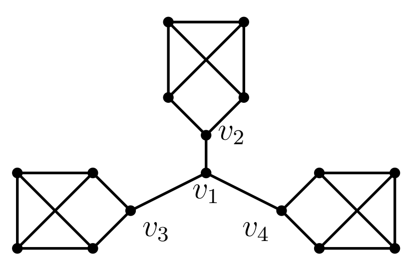
Hence the result follows. □

7.4 Perfect matchings in graphs

All the graphs in this section are simple graphs. If G has a perfect matching, then $n(G)$ is even, obviously. However, the converse does not hold. For example, the two graphs G_1 and G_2 shown in Figure 7.8 have no perfect matchings; this follows since only one of the vertices v_2 or v_3 or v_4 can be matched with v_1.



(a) G_1



(b) G_2

Figure 7.8: Graphs with no perfect matchings.

Definition. *A component of G is called an* ***odd component*** *if it has odd number of vertices, and it is called an* ***even component*** *if it has an even number of vertices. The number of odd components in G is denoted by $\theta(G)$.*

Lemma 7.6. *If $n(G)$ is even and $S \subseteq V(G)$, then $\theta(G-S)$ and $|S|$ have the same parity.*

Proof. Exercise. □

We can now proceed to state and prove a theorem which characterizes the graphs with perfect matchings.

Theorem 7.7 (Tutte, 1947)**.** *A graph G has a perfect matching iff*

(Tutte) $\theta(G-S) \leq |S|$*, for every* $S \subseteq V(G)$.

Proof. (1) G has a perfect matching $\Rightarrow$ G satisfies (Tutte).

Let M be a perfect matching in G and let $S = \{v_1, v_2, \ldots, v_s\} \subseteq V(G)$. Let $G_1, \ldots, G_t$ be the odd components of $G-S$, where $t = \theta(G-S)$. Since, G_i $(1 \leq i \leq t)$ has odd number of vertices, at least one vertex u_i of G_i is matched under M with a vertex v_i of S; that is $(u_1, v_1), \ldots, (u_t, v_t) \in M$. Since M is a matching, v_i's are all distinct. Hence, $|S| \geq t = \theta(G-S)$.

(2) G satisfies (Tutte) $\Rightarrow$ G has a perfect matching.

We give a proof that is due to Anderson (1971). It is by induction on n. We first observe that n is even: By taking $S = \emptyset$ in (Tutte) we deduce that $\theta(G) = \theta(G-\emptyset) \leq |\emptyset| = 0$. That is, G has no odd components. Hence, n is even. We now proceed to prove the implication.

Basic step $n = 2$**:** Since G satisfies (Tutte), $G \simeq K_2$ and it has a perfect matching.

Induction step: We make two cases.

Case 1: $\theta(G-S) \leq |S| - 1$, for every proper subset $S \subseteq V(G)$.

By Lemma 7.6, $\theta(G-S) \neq |S|-1$, for any S. So,

(P_1) $$\theta(G-S) \leq |S|-2, \text{ for every proper subset } S \subsetneq V(G).$$

Let $e=(u,v)$ be an edge in G, and let $H = G - \{u,v\}$.

Claim: H satisfies (Tutte).

On the contrary, suppose that H contains a $T \subseteq V(H)$ such that $\theta(H-T) \geq |T|+1$.

Since, $n(H)$ is even, using Lemma 7.6, we conclude that $\theta(H-T) \geq |T|+2$. But then

$$\theta(G-(T\cup\{u,v\})) = \theta(H-T) \geq |T|+2 = |T\cup\{u,v\}|.$$

Since G satisfies (Tutte), $\theta(G-(T\cup\{u,v\})) \leq |T\cup\{u,v\}|$. Hence, $\theta(G-(T\cup\{u,v\})) = |T\cup\{u,v\}|$. This contradicts our assumption. Hence, the claim holds. Therefore, by induction hypothesis, H has a perfect matching, say M. But then $M\cup\{e\}$ is a perfect matching of G.

Case 2: $\theta(G-S) = |S|$, for some proper subset $S \subsetneq V(G)$.

Let $S^* = \{v_1, v_2, \ldots, v_s\}$ be a subset with ***maximum number*** of vertices such that $\theta(G-S^*) = |S^*|$. We look at $G-S^*$. Let $G_1, \ldots, G_s$ be the odd components of $G-S^*$.

Observation 1: $G-S^*$ has no even components.

On the contrary, if D is an even component of $G-S^*$ and $x \in V(D)$, then $D-x$ has at least one odd component and so $\theta(G-(S^*\cup\{x\})) \geq s+1$. Hence, $\theta(G-(S^*\cup\{x\})) = |S^*\cup\{x\}|$, since G satisfies (Tutte). This is a contradiction to the maximality of $|S^*|$.

Observation 2: Any k $(1 \leq k \leq s)$ odd components from $G_1, \ldots, G_s$ are together joined to at least k vertices of S^*; else, G violates (Tutte).

Therefore, by Hall's Theorem 7.3, there exists a matching, say $M_0 = \{(u_1, v_1), \ldots, (u_s, v_s)\}$, where $u_i \in G_i$ $(1 \leq i \leq s)$.

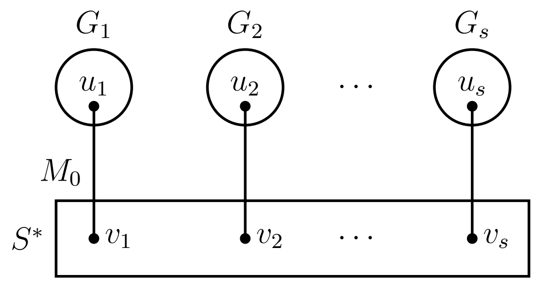


Figure 7.9: The graph G with matching $M_0 = \{(u_1, v_1), \ldots, (u_s, v_s)\}$.

Figure 7.9 shows the structure of G that we have derived until now.

Observation 3: $H_1 = G_1 - u_1$ satisfies (Tutte).

On the contrary, suppose that there exists $T \subseteq V(H_1)$ such that $\theta(H_1 - T) \geq |T| + 1$. Since, $n(H_1)$ is even, using Lemma 7.6, we conclude that $\theta(H_1 - T) \geq |T| + 2$. Let $L_1, L_2, \ldots, L_p$ be the odd component of $H_1 - T$, where $p \geq |T| + 2$. But then $L_1, L_2, \ldots, L_p$ and $G_2, \ldots, G_s$ are odd components of $G - (S^* \cup \{u_1\} \cup T)$. Hence,

$$\begin{aligned}
\theta(G - (S^* \cup \{u_1\} \cup T)) &= p + (s - 1), \\
&\geq |T| + 2 + s - 1, \\
&= |T| + 1 + s, \\
&= |(S^* \cup \{u_1\} \cup T)|.
\end{aligned}$$

Since G satisfies (Tutte), we conclude that $\theta(G - (S^* \cup \{u_1\} \cup T)) = |S^* \cup \{u_1\} \cup T|$, which is a contradiction to the maximality of $|S^*|$. Hence the observation is proved.

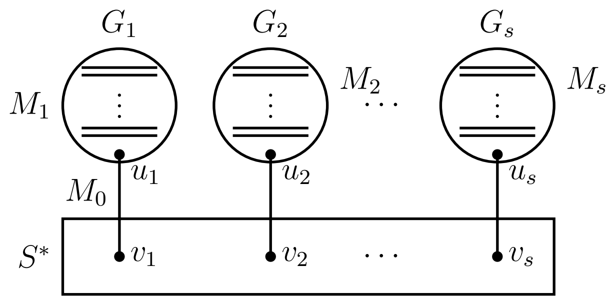


Figure 7.10: G with perfect matchings $M_0 \cup M_1 \cup \ldots \cup M_s$.

Therefore, by induction hypothesis, H_1 has a perfect matching, say M_1. Similarly, $G_2 - \{u_2\}, \ldots, G_s - \{u_s\}$ contain perfect matchings, say $M_2, \ldots, M_s$ respectively. But then $M_1 \cup M_2 \cup \cdots \cup M_s$ is a perfect matching of G; see Figure 7.10. □

Although Tutte's theorem characterizes the graphs with a perfect matching, it is hard to verify Tutte's condition and conclude that a given graph G has a perfect matching, because we have to verify (Tutte) for 2^n subsets of $V(G)$. Hence, there have been several results proved by various mathematicians which say that a given graph G has a perfect matching if G satisfies a certain property (P)(where (P) is easily verifiable). In fact, the first result on perfect matchings was obtained by Petersen (1891) which preceded Tutte's theorem. However, we can easily deduce Petersen's result using Tutte's theorem.

Corollary. *If G is a $(k-1)$-edge-connected k-regular graph with n even, then G has a perfect matching.*

Proof. It is enough if we verify that G satisfies (Tutte). Let $S \subseteq V(G)$ and $G_1, G_2, \ldots, G_t$ be the odd components of $G - S$. Since, G is $(k-1)$-edge-connected, $|[S, V(G_i)]| \geq k - 1$, for every $i, 1 \leq i \leq t$.

Claim: $|[S, V(G_i)]| \geq k$, for every $i, 1 \leq i \leq t$.

On the contrary, suppose that for some j, $1 \leq j \leq t$, $|[S, V(G_j)]| = k - 1$. Then

$$\begin{aligned} \sum_{v \in V(G_j)} deg_{G_j}(v) &= 2m(G_j) + k - 1 \\ \Rightarrow \quad k|V(G_j)| &= 2m(G_j) + k - 1, \text{ since } G \text{ is } k\text{-regular.} \\ \Rightarrow \quad k(|V(G_j)| - 1) &= 2m(G_j) - 1. \end{aligned}$$

While right hand side is an odd integer, left hand side is an even integer since $|V(G_j)|$ is odd. This contradiction proves the claim.

We next estimate the number of edges in $[S, V - S]$ in two different ways.

$$\begin{aligned} (1) \qquad |[S, V - S]| &\leq k|S|, \text{ since } G \text{ is } k\text{-regular.} \\ (2) \qquad |[S, V - S]| &= \sum_{i=1}^{t} |[S, V(G_i)]|, \text{ since } V - S = V(G_1) \cup \cdots \cup V(G_t) \\ &\geq \sum_{i=1}^{t} k, \text{ by the above claim} \\ &= tk. \end{aligned}$$

(1) and (2) imply that $t \leq |S|$. Hence, G satisfies (Tutte) and it has a perfect matching. □

Corollary (Petersen, 1891)**.** *If G is a 2-edge-connected 3-regular graph, then G has a perfect matching.* □

However, every connected 3-regular graph does not have a perfect matching; see Figure 7.8. Hence, we cannot strengthen Petersen's result by assuming that G is connected and 3-regular.

7.5 Greedy and approximation algorithms (Optional)

Since finding $\beta_0(G)$ and $\alpha_0(G)$ are known to be hard problems, there have been attempts to describe algorithms which take G as an input and then output a vertex-cover (or an independent set) with $c\beta_0(G)$ (or $c\alpha_0(G)$) number of vertices where c is a constant. Obviously, in the case of $\beta_0(G)$, $c \geq 1$, and in the case of $\alpha_0(G)$, $c \leq 1$. There is much research to describe algorithms to output vertex covers containing $c\beta_0(G)$ vertices with c as small as possible (and in the case of independent sets with c as large as possible). The constant c is called the ***approximation-factor*** and it is used to measure the efficiency of an algorithm. Obviously among the two algorithms A_1 and A_2 that output $c_1\beta_0(G)$ and $c_2\beta_0(G)$ vertices respectively, where $c_1 < c_2, A_1$ is a better algorithm.

What we have learned in the previous sections is enough for us to describe two such algorithms.

An approximate vertex-cover algorithm with approximation factor 2.

Input: A graph G on n vertices, $M \leftarrow \emptyset$, $K \leftarrow \emptyset$.

Output: A maximal matching M of G and a vertex-cover K of G.

- While $E(G) \neq \emptyset$ do
  - Select any edge $e(u, v)$ from $E(G)$.
  - $M \leftarrow M \cup \{e\}$, $K \leftarrow K \cup \{u, v\}$.
  - $G \leftarrow G - \{u, v\}$.
- end while.
- Output M and K.

We show that $|K| \leq 2\beta_0(G)$.

Theorem 7.8 (Correctness of the algorithm)**.** *Any output K is a vertex-cover of G with $|K| \leq 2\beta_0(G)$.*

Proof. At the outset observe that M is indeed a maximal matching. Since at every step after including $e(u, v)$ in M we have deleted all the edges incident to u or v. Since M is a maximal matching, the set of all M-unsaturated vertices $V - K$ is an independent set. That is, every edge in G is incident with a vertex in K. Hence, K is a vertex-cover such that $|K| \leq 2|M| \leq 2\alpha_1(G) \leq 2\beta_0(G)$; since $\alpha_1(G) \leq \beta_0(G)$, for any graph G (see the observation made in the beginning of the Section 7.4). □

An illustration for the algorithm:

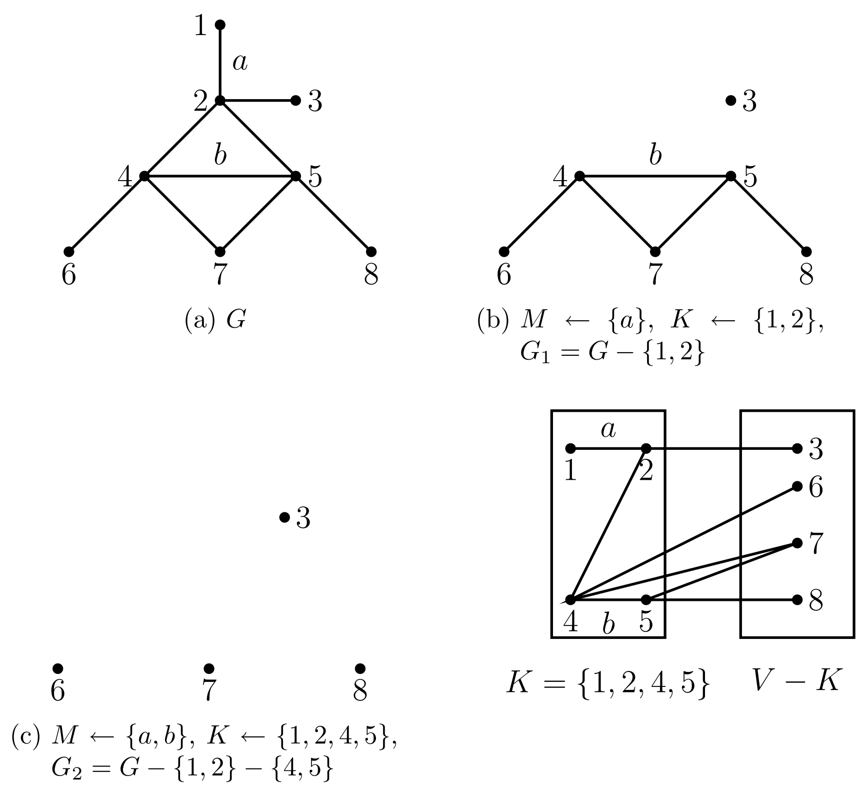


Figure 7.11: A graph G with a maximal matching $\{a, b\}$ generated by the greedy algorithm.

- Notice that if we had selected the edge (4,6) instead of (4,5) in the second step, we would have generated a matching with 3 edges which is also a maximal matching.

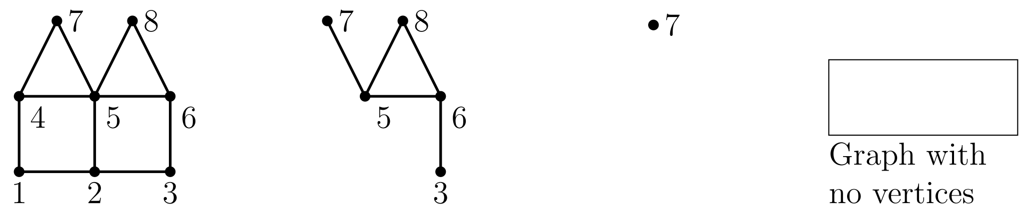
A greedy algorithm to output a maximal independent set of vertices.

Input: A graph G, $I \leftarrow \emptyset$.

Output: A maximal independent set I of vertices.

- While $V(G) \neq \emptyset$ do
  - Select any vertex (arbitrarily) say v from $V(G)$.
  - $I \leftarrow I \cup \{v\}$, $G \leftarrow G - \{v\} - N_G(v)$
- end while.
- Output I.

An illustration:



(a) G; $I \leftarrow \emptyset$

(b) $I \leftarrow \{1\}$, $G_1 = G - \{1\} - \{2,4\}$.

(c) $I \leftarrow \{1,6\}$, $G_2 = G_1 - \{6,3,5,8\}$.

(d) $I \leftarrow \{1,6,7\}$, $G_3 = G_2 - \{7\}$.

Figure 7.12: A graph G with maximal independent set $I = \{1,6,7\}$ generated by the greedy algorithm.

Remarks.

- The algorithm indeed outputs an independent set since after selecting a vertex v we delete v and all the vertices adjacent to v.
- The algorithm need not generate a maximum independent set. If we had selected the vertex 3 in G_1 instead of 6 and thereafter had selected 7, we could have generated an independent set $\{1,3,7,8\}$ which is a maximum independent set.

Theorem 7.9. *The algorithm outputs an independent set I with at least $\dfrac{n}{1+\Delta(G)}$ vertices.*

Proof. The algorithm indeed outputs a maximal independent set of vertices, since at every step after selecting vertex v, we delete v and all its neighbors, and in the next step, we select a new vertex from $G - v - N(v)$. Suppose $I = \{v_1, v_2, \ldots, v_t\}$ is the output. After selecting v_t, we are left with no more vertices in the residual graph. So

$$
\begin{aligned}
n &\leq \underbrace{(1+\Delta) + \cdots + (1+\Delta)}_{\text{sum } t \text{ times}}, \\
&\qquad \text{since in each step we have deleted at most } 1+\Delta \text{ vertices.} \\
&= t(1+\Delta) \\
\text{Hence,} \qquad |I| &= t \geq \frac{n}{1+\Delta}. \quad \square
\end{aligned}
$$

Exercises

1. Let G be a simple bipartite graph. Show:
   (a) $|E(G)| \leq \alpha_0(G)\beta_0(G)$.
   (b) If $|E(G)| = \alpha_0(G)\beta_0(G)$, then G is a complete bipartite graph;
2. For a bipartite graph G, show that the following statements are equivalent:
   (a) $\alpha_0(H) \geq |V(H)|/2$, for every subgraph H of G.
   (b) $\alpha_0(H) = \beta_1(H)$, for every subgraph H of G with $\delta(H) > 0$.
3. Let G be a complete n-partite graph. Prove the following:
   (a) $\beta_0(G) = \delta(G) = k_0(G) = k_1(G)$
   (b) G is Hamilton iff $|V(G)| \leq 2\beta_0(G)$.
4. Prove that for any graph G, the following inequalities are equivalent.
   (a) $\beta_0(G) + \beta_0(G^c) \geq |V(G)|$.
   (b) $\alpha_0(G) + \alpha_0(G^c) \leq |V(G)|$.

(c) $\beta_0(G) + \beta_0(G^c) \geq \alpha_0(G) + \alpha_0(G^c)$.

(d) $\beta_0(G) \geq \omega(G)$,
where $\omega(G) = \max\{|V(H)| : H \text{ is a complete subgraph of } G\}$.

5. For any graph G, show:
   (a) $\beta_0(G) \geq \delta(G)$.
   (b) $\beta_0(G) \geq \omega(G) - 1$.

6. Prove or disprove: In any tree there is a maximal independent set of vertices containing all the vertices of degree one.

7. Find the vertex-covering number of $K_{1,t} + K_{1,t}$ $(t > 1)$.

8. Find the minimum edge covering number β_1 of the d-cube Q_d. Justify your answer. And give an example of a minimum edge covering set of Q_d.

9. Find β_0 and β_1 of $K_{2,4,6,\ldots,2k}$.

10. Find α_0 and β_1 of $C_{2m+1} + C_{2n}$ and $K_n + C_{2m+1}$.

11. Find α_0, β_0, α_1 and β_1 of $K_{1,2,\ldots,n}$.

12. Find the matching number of $C_5 + C_7$.

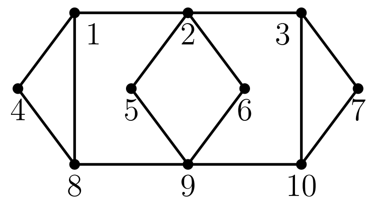
13. Let M, N be disjoint matchings of a graph G with $|M| > |N|$. Show that there are disjoint matchings M' and N' of G such that $|M'| = |M| - 1$ and $|N'| = |N| + 1$ and $M' \cup N' = M \cup N$.

14. Draw a connected 4-regular graph with no perfect matching on n-vertices where n is an even integer. What is the minimum value of n. Justify your graph indeed has no perfect matching.

15. For each $k \geq 2$, find an example of a k-regular simple graph which has no perfect matching.

16. Prove or disprove:
    (a) The graph shown in the figure below contains a perfect matching.
    (b) A tree has at most one perfect matching.

17. A tree T has a perfect matching iff $O(T - v) = 1$, for every vertex v.



18. Show that no tree ($n \geq 3$) with every vertex of degree 1 or 3 has a perfect matching.

19. Draw a 3−regular connected simple graph with a perfect matching that has either a cut-vertex or a cut edge.

20. Let $n > 1$ be an integer. Prove or disprove:
    (a) If n is even, $K_{n,n,n}$ has a perfect matching.
    (b) If n is odd, $K_{n,n,n}$ has no perfect matching.

21. Draw a connected simple bipartite graph $G[U, V]$ with minimum degree at least 3 and $|U| = |V| = 7$ which has no matching saturating every vertex in U.

22. Prove or disprove: If $G[X, Y]$ is a simple connected bipartite graph such that the degree of every vertex in X is at least 3 and the degree of every vertex in Y is at most 3, then G has a matching saturating every vertex in X.

23. Illustrate the greedy algorithm to get a maximal independent set for the simple graph whose vertices are a, b, c, d, e, f and whose edges are (a, b), (b, c), (c, d), (d, e), (e, f), (f, a), (b, f), (c, e).

24. Apply greedy algorithm to get a maximal matching and a vertex cover of the graphs shown below.