

# Magnetic Fields

Disclaimer: These lecture notes are not meant to replace the course textbook. The content may be incomplete. Some topics may be unclear. These notes are only meant to be a study aid and a supplement to your own notes. Please report any inaccuracies to the professor.

## Magnetism

Is ubiquitous in every-day life!

- Refrigerator magnets (who could live without them?)
- Coils that deflect the electron beam in a CRT television or monitor
- Cassette tape storage (audio or digital)
- Computer disk drive storage
- Electromagnet for Magnetic Resonant Imaging (MRI)

## Magnetic Field

Magnets contain two poles: “north” and “south”. The force between like-poles repels (north-north, south-south), while opposite poles attract (north-south). This is reminiscent of the electric force between two charged objects (which can have positive or negative charge).

Recall that the electric field was invoked to explain the “action at a distance” effect of the electric force, and was defined by:

$$\mathbf{E} = \frac{\mathbf{F}}{q_{\text{el}}}$$

where  $q_{\text{el}}$  is electric charge of a positive test charge and  $\mathbf{F}$  is the force acting on it.

We might be tempted to define the same for the magnetic field, and write:

$$\mathbf{B} = \frac{\mathbf{F}}{q_{\text{mag}}}$$

where  $q_{\text{mag}}$  is the “magnetic charge” of a positive test charge and  $\mathbf{F}$  is the force acting on it.

However, such a single magnetic charge, a “magnetic monopole,” has never been observed experimentally! You cannot break a bar magnet in half to get just a north pole or a south pole. As far as we know, no such single magnetic charges exist in the universe,

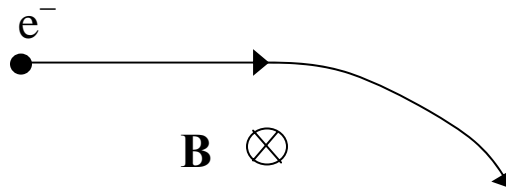
although we continue to look. Thus, we must look for other interactions with magnetic force to define the magnetic field.

It turns out that an *electrically* charged object can also be accelerated by a magnetic force, and through that interaction we can define the magnetic field.

In fact, the electric and magnetic force share a much deeper relation. They are really manifestations of the same force, and can be shown to be related by transformations in Einstein's theory of Special Relativity. But here let us discuss the historical perspective.

## ***Cathode Ray Tube***

Consider a “Crooke’s Tube”, which is otherwise known as a Cathode Ray Tube (CRT) – a primitive version of what is in a television. Such a CRT has an electron gun that accelerates electrons between two electrodes with a large electric potential difference between them (and a hole in the far plate). The resulting beam of electrons can be rendered visible with a phosphorous screen, and then we can observe the effects on the deflection of the beam in magnetic fields.



From such experiments we can determine several characteristics of electrically charged particles interacting with magnets:

- The force depends on the direction of the magnetic field (*i.e.* whether it emanates from a north pole or a south pole).
- The force is perpendicular to both the velocity and magnetic field directions
- The force is zero if the particle velocity is zero (and depends on the sign of  $\mathbf{v}$ )
- The force depends on the sign of the electric charge

## ***Definition of the Magnetic Field***

Thus, we will converge on the following relation for the magnitude of the magnetic force on a charged object:

$$|\mathbf{F}_B| = |q||\mathbf{v}||\mathbf{B}|\sin\phi$$

or, turned around, allows us to define the magnitude of the magnetic field as:

$$|\mathbf{B}| = \frac{|\mathbf{F}_B|}{|q||\mathbf{v}|\sin\phi}$$

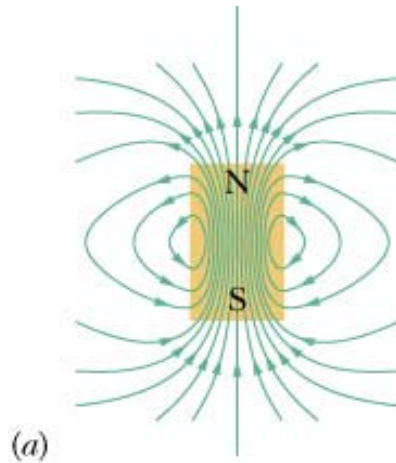
where  $\phi$  is the angle between the velocity,  $\mathbf{v}$ , and the magnetic field,  $\mathbf{B}$ .

The units of the magnetic field are  $\frac{\text{N}}{\text{C} \frac{\text{m}}{\text{s}}} = \frac{\text{N}}{\frac{\text{C}}{\text{s}} \text{m}} = \frac{\text{N}}{\text{A-m}} \equiv \text{T}$  (Tesla)

Another unit based on the cgs metric system is the Gauss, where  $1 \text{ G} = 10^{-4} \text{ T}$ . The Earth's magnetic field has a magnitude of approximately 0.5 G.

## Direction of the Magnetic Field

What direction do we assign to the magnetic field to insert into the force equation? We imagine field lines that are directed outward from the north pole of a magnet, and inward to the south pole.



From HRW 7/e

Now in full vector form, we write the expression for the magnetic force acting on a electrically charged particle as:

$$\mathbf{F}_B = q\mathbf{v} \times \mathbf{B}$$

We can see that it will satisfy all the empirical observations noted earlier.

The vector cross-product is defined by:

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a} = \det \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = (a_y b_z - b_y a_z) \hat{\mathbf{x}} - (a_x b_z - b_x a_z) \hat{\mathbf{y}} + (a_x b_y - b_x a_y) \hat{\mathbf{z}}$$

$$|\mathbf{a} \times \mathbf{b}| = ab \sin \phi$$

## Right-Hand Rule

If we consider the following simple example, that  $\mathbf{v} = v_0 \hat{\mathbf{x}}$  and  $\mathbf{B} = B_0 \hat{\mathbf{y}}$ , then we see that the magnetic force direction is given by a right-hand rule:  $F = qv_0 B_0 \hat{\mathbf{z}}$

## Magnetic Fields and Work

Let's calculate the work done by the magnetic force acts on a moving charged particle that moves from point 1 to point 2:

$$W \equiv \int_1^2 \mathbf{F} \cdot d\mathbf{s}$$

We can re-write  $d\mathbf{s} = \mathbf{v} dt$  to get:

$$W \equiv \int_1^2 \mathbf{F} \cdot \mathbf{v} dt = \int_1^2 (q\mathbf{v} \times \mathbf{B}) \cdot \mathbf{v} dt$$

Now with what we learned about the magnetic force, it is always perpendicular to the velocity vector, so in fact the vector quantity  $(q\mathbf{v} \times \mathbf{B})$  is zero. So  $W=0$  and no work is done by the magnetic field! You can remember this simply as “Magnetic fields do no work.” So apparently the magnetic force is on the physics welfare system!

This implies that there is no change in energy of a charged particle being accelerated by a magnetic force, only a change in direction. We will come back to this when we discuss magnetic fields and circular motion.

## Lorentz Force Equation

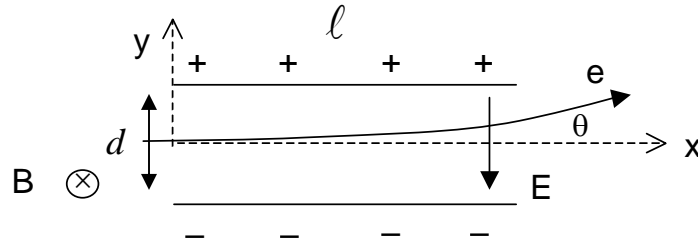
We can combine what we learned about the electric force to that we just learned about the magnetic force into one equation, the Lorentz force equation:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

We can apply this to some historical work done by J.J. Thompson, who determined that the carrier of electricity was an electrically charged particle dubbed the “electron.”

## e/m Determination of the Electron—the Carrier of Electricity

It was determined by J.J. Thompson in 1897 that cathode rays are charged particles emitted from a heated electrical cathode. It was known that such heated cathodes lead to an electrical current, so Thompson determined that electricity was quantized into individual charged particles (dubbed electrons). He deduced this by analyzing the motion of cathode rays through perpendicular electric and magnetic fields, as shown below:



Here is a review of the procedure used to determine the charge-to-mass ratio for electrons by application of the Lorentz Force equation,  $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ . What J.J. Thompson assumed was that cathode rays were actually individually charged particles.

In the absence of a magnetic field, with an electric field aligned in the y-direction, we have:

$$F_y = qE_y = ma_y$$

$$\Rightarrow a_y = \frac{qE_y}{m}$$

Let electric field occupies a region of length  $\ell$  through which the cathode rays pass. The cathode rays initially have a velocity  $v_0$  in the x-direction.

$$t \approx \frac{\ell}{v_0} \quad \text{Time to cross the electric field region}$$

$$v_y = a_y t = \frac{qE_y \ell}{mv_0} \quad \text{Velocity in y - direction upon exit}$$

$$\tan \theta = \frac{v_y}{v_x} = \frac{qE_y \ell}{mv_0^2} \quad \text{Tangent of exit angle}$$

The angle  $\theta$  can be measured in the experiment, and the length of the electric plates is obviously known. The electric field can be determined from the electric potential difference  $V$  applied to the parallel plates divided by the separation:

$$|\mathbf{E}| = \frac{V}{d}$$

However, we need a way to determine the initial velocity  $v_0$ . The trick is to turn on a magnetic field such that the cathode beam is no longer deflected. Using the Lorentz Force equation:

$$\begin{aligned} \mathbf{F} &= q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) = 0 && \text{Set to balance, } \theta = 0 \\ \Rightarrow \mathbf{E} &= -\mathbf{v} \times \mathbf{B} \\ |\mathbf{E}| &= v_x |B_z| && v_x \text{ and } B_z \text{ are perpendicular} \\ v_x &= \frac{E}{B} = v_0 \end{aligned}$$

Armed with this, we can determine the charge to mass ratio of the electron:

$$\tan \theta = \frac{qE\ell}{m} \frac{B^2}{E^2}$$

$$\frac{q}{m} = \frac{E \tan \theta}{B^2 \ell} = \frac{e}{m}$$

So the procedure is that we measure  $\theta$  for a given  $E$ , then we measure  $B$  for  $\theta=0$ . This yields a charge-to-mass ratio of:

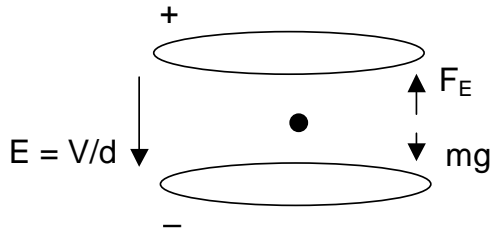
$$\frac{e}{m} = 1.76 \times 10^{11} \text{ C / kg}$$

What J.J. Thompson found in his laboratory was that this was a **universal ratio**. It didn't matter if the cathode was aluminum, steel, or nickel; and it didn't matter if the gas was argon, nitrogen, or helium. The value always came out to be the same. Thus, there was only one type of charge carrier for electricity—the electron. It should be noted that the charge of the electron is actually negative.

What else can we conclude? From the size of this ratio, either the charge of the electron is very large, or the mass of the electron is extremely small (or some combination of both).

## Review: Charge of the Electron

J.J. Thompson only determined the ratio of the electron charge to its mass, rather than each separately. It was Robert Millikan in 1911 who was able to measure the electron charge directly. He did this by measuring the static electric charge on drops of oil, and finding that it was always a multiple of a certain value. A schematic of the set up is shown below:



The oil drop is suspended when the acceleration  $a = 0$

$$\Rightarrow \mathbf{F}_E = q\mathbf{E} = -m\mathbf{g}$$

$$q = \frac{mgd}{V}$$

Millikan measured the mass of the oil drops by turning off the electric field and measuring the terminal velocity. We will assume that the mass is known, as with the voltage and the plate separation distance  $d$ . Through very precise measurements, Millikan found that the electric charge is a multiple of the following value:

$e =  q  = 1.6022 \times 10^{-19} \text{ C}$
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Charge is quantized! From Thompson's  $e/m$  measurement, we can deduce:

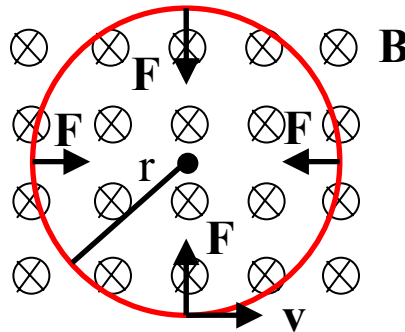
$m_e = 9.11 \times 10^{-31} \text{ kg}$
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The electron has a definite charge and mass which is the same for all electrons.

## Magnetic Force and Circular Motion

Recall that the magnetic force is perpendicular to the direction of motion. This implies that the trajectory in a uniform magnetic field is a circle in the plane transverse to the direction of the field (a helix in 3 dimensions, generally).

Consider a positively charged particle with velocity  $\mathbf{v}$  (taken to be toward the right) entering a region of uniform magnetic field pointing into the plane of the page. The particle will be deflected by a force that is perpendicular to the field and the initial velocity. In the example below, this will be in the upward direction. But when recalculating the force at another point along the trajectory, we will find that the particle is continually deflected with an equal magnitude force, and the net effect is a circular orbit. Since the magnetic field does no work, the radius of this circle,  $r$ , will be a constant.



We can solve for the radius of this orbit:

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B} = q|\mathbf{v}||\mathbf{B}|\hat{\mathbf{r}}$$

$$\mathbf{F} = m\mathbf{a}$$

where the magnitude of the centripetal acceleration is  $a = \frac{v^2}{r}$

$$\Rightarrow \frac{mv^2}{r} = q|\mathbf{v}||\mathbf{B}|$$

$$\Rightarrow r = \frac{mv}{qB}$$

In other words, since momentum is defined as  $p = mv$ , we have

$$r = \frac{p}{qB}$$

This gives the radius of the orbit in terms of the particle momentum, charge, and the magnetic field magnitude. This latter form of the equation is even correct with the relativistic definition of momentum.



To see this, recall that the relativistic definition of momentum is:

$$p = \gamma m u \quad \text{where } \gamma = \frac{1}{\sqrt{1 - u^2 / c^2}} \quad \text{and } u \text{ is the particle velocity}$$

The magnitude of the velocity in circular motion is constant, so the force can be written:

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d}{dt}(\gamma m \mathbf{u}) = \gamma m \frac{d\mathbf{u}}{dt} \quad \text{since } \gamma = \text{constant}$$
$$\Rightarrow \mathbf{F} = \gamma m \mathbf{a}$$

Setting this force equal to the magnetic force, and plugging in for centripetal acceleration:

$$\gamma m \frac{v^2}{r} = q v B$$
$$\Rightarrow \gamma m v = p = q B r$$
$$\Rightarrow r = \frac{p}{q B}$$

## ***Orbital Frequency***

The frequency of orbit in uniform circular motion is given by

$$f = \frac{v}{2\pi r}$$

The relativistic momentum is  $p = \gamma m u$ , so

$$f = \frac{p}{2\pi \gamma m r}$$

Now plugging in that  $p = q B r$ , we get

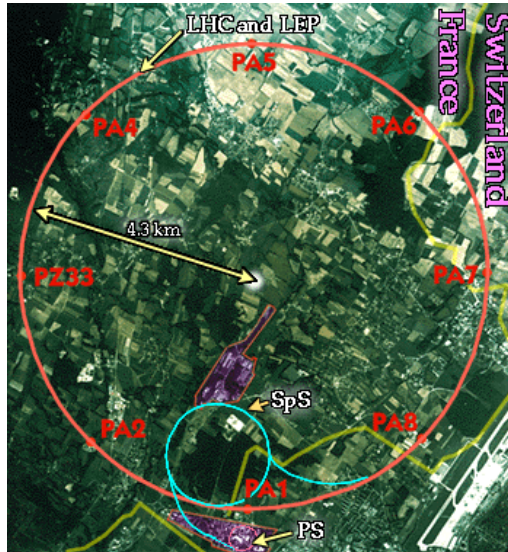
$$f = \frac{q B}{2\pi m} \sqrt{1 - \frac{u^2}{c^2}}$$

So we see that the frequency is constant provided  $u \ll c$ , but when we approach speeds near that of light, the frequency slows down.

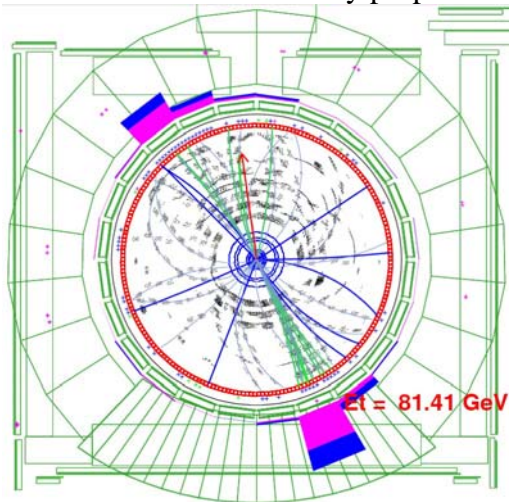
Thus, while the early particle accelerator, cyclotrons, used a constant frequency to maintain circular motion, higher energy machines must synchronize the orbital frequency according to Special Relativity.

## Cyclotrons and Synchrotrons

The previously derived equation is extremely useful for experimental particle physicists! It can be used in two main ways. First, it can be used to bend charged particles into a circular orbit. This is how physicists maintain circular beams of electrons and protons in a fixed tunnel. A series of magnets in the shape of a ring, all with the same field, bend the particles so that they can be brought into collision over and over again. The largest accelerator, the Large Hadron Collider, is 4.3 km in radius and is illustrated below!

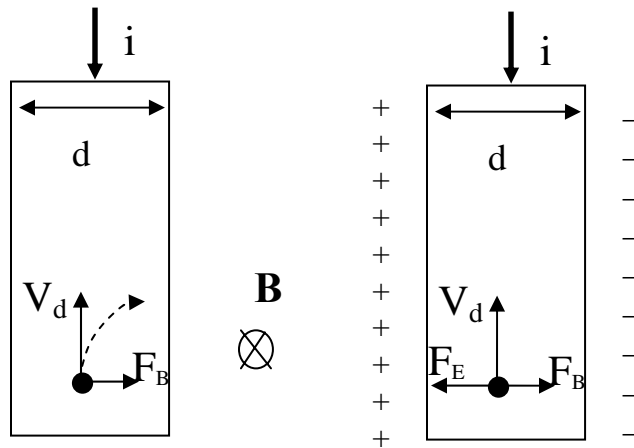


The other way to use the equation is to measure the radius of curvature of a charged particle in a known magnetic field in order to determine its momentum. In other words, once the beam particles have collided, we must measure the momentum or energy of the collision products in order to determine what reaction took place. For example, the picture below shows the reconstructed trajectories of charged particles emanating from a proton collision. The curvature is inversely proportional the momentum.



## Hall Effect

Another example of the Lorentz Force is the magnetic effect on a current of electrons in a conductor. In the example below, a uniform magnetic field is directed into the plane of the paper and the current moves from top to bottom.



The electrons drifting upward in the shown conductor (electric current going down) will initially feel a force due to the  $\mathbf{v} \times \mathbf{B}$  magnetic interaction (left picture). This will cause them to drift toward one side of the conductor and build up charge. Very soon, this will set up an electric field that balances the magnetic force (right picture). At equilibrium, the forces balance.

$$\mathbf{F}_E + \mathbf{F}_B = 0$$

$$\Rightarrow e|\mathbf{E}| = e|v_d||\mathbf{B}|$$

We can define a “Hall Voltage” as the electric potential difference between the left and right sides of the shown conductor:

$$V_H = |\mathbf{E}|d$$

When combined with the force balance equation we get:

$$\frac{V_H}{d} = v_d B$$

Now from Ohm’s Law, the current density is related to the electron drift velocity:

$$|\mathbf{j}| = nev_d$$

So

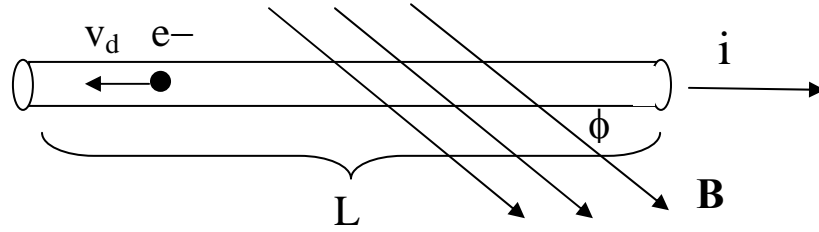
$$v_d = \frac{j}{ne} = \frac{i}{neA} \quad \text{where } i = \text{current and } A = \text{cross-sectional area}$$
$$\Rightarrow V_H = \frac{id}{neA} B$$

Or in other words, we can determine the magnetic field from the measured Hall voltage and current passing through the conductor:

$$|B| = V_H \frac{neA}{id}$$

Such a device used to measure magnetic fields is called a Hall Probe.

## Magnetic Force on a Current-Carrying Wire



Consider a length of conducting wire carrying a current,  $i$ , in a uniform magnetic field  $\mathbf{B}$ . The drift-velocity of the electrons is constant and is denoted by  $v_d$ . The time it takes for an electron to drift across a length of wire,  $L$ , is given by:

$$t = \frac{L}{v_d}$$

The amount of charge passing through the end of wire in that time is given by:

$$q = it = i \frac{L}{v_d}$$

This must equal the amount of free charge contained in a length of wire  $L$  at any instant.

The force on that amount of charge is:

$$\mathbf{F} = q\mathbf{v}_d \times \mathbf{B}$$

and this must be the force acting on a length of wire  $L$  carrying a current. We can write in terms of the current:

$$|\mathbf{F}| = |qv_d \mathbf{B}| \sin \phi = i \frac{L}{v_d} v_d B \sin \phi = iLB \sin \phi$$

Or alternatively, if we define a length vector  $\mathbf{L}$  as pointing in the direction of the current and having a magnitude equal to the length, we can write for the force on a current-carrying wire in a magnetic field:

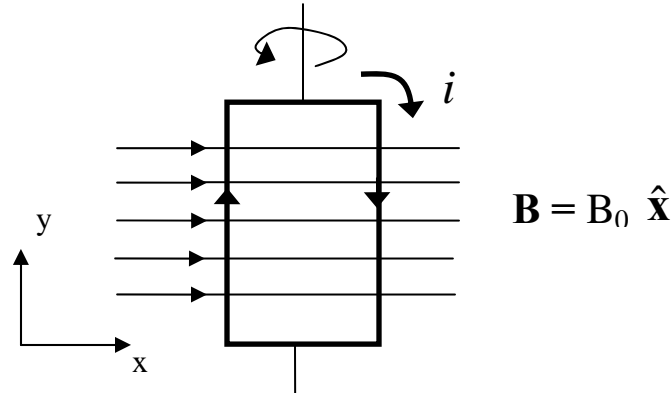
$$\mathbf{F} = i\mathbf{L} \times \mathbf{B}$$

The force depends on the directions and magnitudes of the current and the magnetic field. If we have  $N$  loops of wire carrying the same current, we multiply the above equation by the number of wires  $N$ .

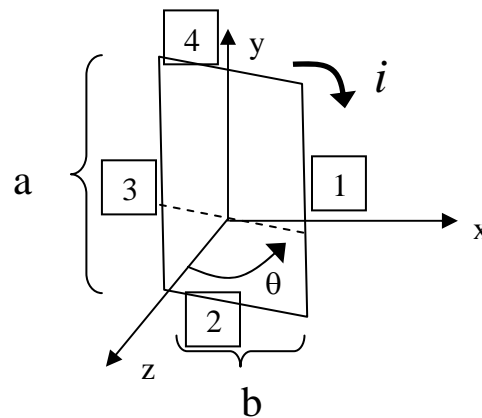
For an infinitesimal length of wire,  $d\mathbf{s}$ , we have:  $d\mathbf{F} = i d\mathbf{s} \times \mathbf{B}$

## Torque on a Current Loop

Consider a rectangular current loop inside a uniform magnetic field (*i.e.* the basis of an electric motor). It has a spindle about which it is free to rotate.



There are 4 straight segments of current-carrying wire, each of which will feel a force due to the magnetic field.



Wires 2 and 4 are not always perpendicular to the magnetic field, so

$$|\mathbf{F}_2| = |\mathbf{F}_4| = ibB \sin(90^\circ - \theta) = ibB \cos \theta$$

$$\Rightarrow \mathbf{F}_2 = -ibB \cos \theta \hat{\mathbf{y}}$$

$$\Rightarrow \mathbf{F}_4 = +ibB \cos \theta \hat{\mathbf{y}}$$

The forces balance, and there is no torque because the loop is not allowed to rotate about the horizontal axis.

Wires 1 and 3 are always perpendicular to the magnetic field, so

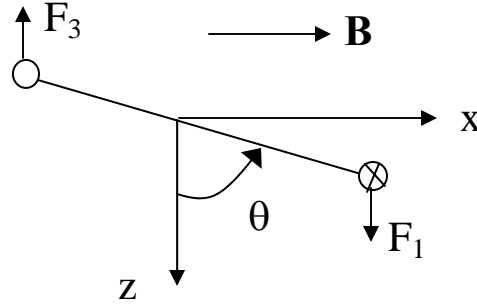
$$|\mathbf{F}_1| = |\mathbf{F}_3| = iaB$$

$$\Rightarrow \mathbf{F}_1 = iaB \hat{\mathbf{z}}$$

$$\Rightarrow \mathbf{F}_3 = -iaB \hat{\mathbf{z}}$$

The forces thus balance, although the torque about the vertical axis will not.

In total, all forces balance, but the two vertical sides 1 and 3 will create a torque. The lever arm length is  $r = b/2$ . Viewed from the top, the situation looks like:



The torque is:

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$$

$$|\tau_1| = \left(\frac{b}{2}\right)(iaB) \sin \theta$$

$$|\tau_3| = \left(\frac{b}{2}\right)(iaB) \sin \theta$$

$$\tau = \tau_1 + \tau_3 = iabB \sin \theta$$

They both add in the same direction. The product  $ab = A$ , the area of the loop.

$$|\tau| = iAB \sin \theta$$

We can define an area vector as having its magnitude equal to the loop area, and a direction perpendicular to the loop (and in a sense given by the right-hand rule and the current direction). Thus, a vector form of the torque is:

$$\boldsymbol{\tau} = i\mathbf{A} \times \mathbf{B}$$

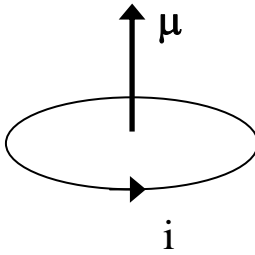
If there are  $N$  windings of the wire around the loop, the formula becomes  $\boldsymbol{\tau} = Ni\mathbf{A} \times \mathbf{B}$

If we further define the magnetic dipole moment of the loop to be  $\boldsymbol{\mu} = i\mathbf{A}$ , then

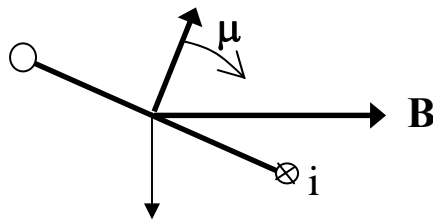
$$\boldsymbol{\tau} = \boldsymbol{\mu} \times \mathbf{B}$$

This formula holds for any current loop shape.

To determine the direction of  $\mu$  use the right-hand rule (fingers in direction of current, then thumb points in direction of  $\mu$ ) :



The effect of the torque on the current loop is to try to line up the magnetic dipole moment vector with the magnetic field:



Once the two vectors are aligned, there will be no torque. However, if we reverse the field direction, the will want to flip the orientation of the loop. If we keep doing this, we form the basis of an electric motor.