Homework 3

1. a) $f(x) = 2x + 17 \quad f'(x) = 2 \quad f''(x) = 0 \therefore$

Let $0 \le \alpha \le 1$ and $x_1, x_2 \in \mathbb{R}$

$f(\alpha x_1 + (1-\alpha)x_2) = 2(\alpha x_1 + (1-\alpha)x_2) + 17$

$= \cancel{17 - \alpha 17 - (1-\alpha)}17 + 2\alpha x_1 + 2(1-\alpha)x_2 + \alpha 17 + (1-\alpha)17$

$= 2\alpha x_1 + 17\alpha + 2(1-\alpha)x_2 + (1-\alpha)17$

$= \alpha(2x_1 + 17) + (1-\alpha)(2x_2 + 17)$

$= \alpha f(x_1) + (1-\alpha)f(x_2)$

$\therefore$ $f(x)$ is both concave and convex

b) $f(x) = ax + b$

$f'(x) = a$

$f''(x) = 0 \therefore$

Let $0 \le \alpha \le 1$ and $x_1, x_2 \in \mathbb{R}$

$f(\alpha x_1 + (1-\alpha)x_2) = a(\alpha x_1 + (1-\alpha)x_2) + b$

$= a\alpha x_1 + a(1-\alpha)x_2 + \alpha b + (1-\alpha)b + \cancel{b - \alpha b - (1-\alpha)b}$

$= a\alpha x_1 + b\alpha + a(1-\alpha)x_2 + (1-\alpha)b$

$= \alpha(ax_1 + b) + (1-\alpha)(ax_2 + b)$

$= \alpha f(x_1) + (1-\alpha)f(x_2)$

$\therefore$ $f(x)$ is both concave and convex by definition of convex $f(x)$

c) $f(x) = 3x^2 - 2$

$f'(x) = 6x$ $\qquad \alpha x_1 + (1-a)x_1$

$f''(x) = 6 > 0 \therefore f(x)$ is convex

Let $0 \le \alpha \le 1$ and $x_1, x_2 \in \mathbb{R}$ and $x_1 \ne x_2$

$f(\alpha x_1 + (1-\alpha)x_2) = 3(\alpha x_1 + (1-\alpha)x_2)^2 - 2$

$= 3[\alpha^2 x_1^2 + 2\alpha x_1(1-\alpha)x_2 + (1-\alpha)^2 x_2^2] - 2$

$= 3\alpha^2 x_1^2 + 6\alpha x_1(1-\alpha)x_2 + 3(1-\alpha)^2 x_2^2 - 2\alpha - (1-\alpha)2$

$\longrightarrow$

continue

2

since $x_1 \neq x_2$, we know $(x_1 - x_1)^2 > 0$ c.2)

$$x_1^2 + x_2^2 > 2x_1x_2$$

$3\alpha^2 x_1^2 + 3(2x_1x_2\alpha(1-\alpha)) + 3(1-\alpha)^2 x_2^2 - 2\alpha - (1-\alpha)2$

$3\alpha^2 x_1^2 + 3(1-\alpha)^2 x_2^2 + 3(2x_1x_2\alpha(1-\alpha)) - 2\alpha - (1-\alpha)2 \leq$

$3\alpha^2 x_1^2 + 3(1-\alpha)^2 x_2^2 + 3(1-\alpha)(\alpha)(x_1^2 + x_2^2) - 2\alpha - (1-\alpha)2$

$= 3\alpha^2 x_1^2 + 3(1-\alpha)(\alpha)(x_1^2) + 3(1-\alpha)^2 x_2^2 + 3(1-\alpha)(\alpha)(x_2^2) - 2\alpha - (1-\alpha)2$

$= 3(\alpha^2 + \alpha - \alpha^2) x_1^2 - 2\alpha + 3(1 - 2\alpha - \alpha^2 + \alpha - \alpha^2) - (1-\alpha)2$

$= 3(\alpha) x_1^2 - 2\alpha + 3(1-\alpha) x_2^2 - (1-\alpha)2$

$= \alpha(3x_1^2 - 2) + (1-\alpha)(3x_2^2 - 2)$

$= \alpha f(x_1) + (1-\alpha) f(x_2)$

$\therefore f(\alpha x_1 + (1-\alpha)x_2) \leq \alpha f(x_1) + (1-\alpha) f(x_2)$

and $f(x)$ is convex

$f(x) = 3 - 2x^2$

$f'(x) = -4x \quad f''(x) = -4 < 0 \quad \therefore f$ is concave d)

Let $0 \leq \alpha \leq 1$, $x_1, x_2 \in \mathbb{R}$, and $x_1 \neq x_2$

$f(\alpha x_1 + (1-\alpha)x_2) = 3 - 2[\alpha x_1 + (1-\alpha)x_2]^2$

$= 3 - 2[\alpha^2 x_1^2 + 2\alpha x_1(1-\alpha)x_2 + (1-\alpha)^2 x_2^2]$

$= 3 - 2\alpha^2 x_1^2 - 4\alpha x_1(1-\alpha)x_2 - 2(1 - 2\alpha + \alpha^2) x_2^2$

$= 3 - 2\alpha^2 x_1^2 - 4\alpha x_1 x_2 + 4\alpha^2 x_1 x_2 - 2x_2^2 + 4\alpha x_2^2 - 2\alpha^2 x_2^2$

$\alpha f(x_1) + (1-\alpha) f(x_2) = \alpha(3 - 2x_1^2) + (1-\alpha)(3 - 2x_2^2)$

$= 3\alpha - 2\alpha x_1^2 + 3 - 3\alpha - 2x_2^2 + 2\alpha x_2^2$

since f is concave, by the definition of concavity, we know:

$f(\alpha x_1 + (1-\alpha)x_2) \geq \alpha f(x_1) + (1-\alpha) f(x_2)$

$f(\alpha x_1 + (1-\alpha)x_2) - [\alpha f(x_1) + (1-\alpha) f(x_2)] \overset{?}{\geq} 0$

$\longrightarrow$

d.2) $f(\alpha x_1 + (1-\alpha)x_2) - \alpha f(x_1) - (1-\alpha) f(x_2) \geq 0$

$\cancel{3} - 2\alpha^2 x_1^2 - 4\alpha x_1 x_2 + 4\alpha^2 x_1 x_2 - \cancel{2x_2^2} + 4\alpha x_2^2 - 2\alpha^2 x_2^2$
$- \cancel{3\alpha} + 2\alpha x_1^2 - \cancel{3} + \cancel{3\alpha} + \cancel{2x_2^2} - 2\alpha x_2^2 \geq 0$

$0 \leq 2\alpha x_1^2 - 2\alpha^2 x_1^2 - 4\alpha x_1 x_2 + 4\alpha^2 x_1 x_2 + 2\alpha x_2^2 - 2\alpha^2 x_2^2$

$0 \leq 2\alpha x_1^2 - 4\alpha x_1 x_2 + 2\alpha x_2^2 - 2\alpha^2 x_1^2 + 4\alpha^2 x_1 x_2 - 2\alpha^2 x_2^2$

$0 \leq (2\alpha - \alpha^2)(x_1^2 - 2x_1 x_2 + x_2^2)$

$0 \leq (2\alpha)(1-\alpha)(x_1 - x_2)^2$

since $\alpha \geq 0$ and the difference squared of any values is always positive, the product is also always positive

$\therefore$ f is concave

c) $f(x) = e^{-x^2}$

$f'(x) = -2xe^{-x^2}$

$f''(x) = 4x^2 e^{-x^2} - 2e^{-x^2}$

$0 = (4x^2 - 2)e^{-x^2} \rightarrow 4x^2 - 2 = 0 \rightarrow$
$x = \pm\sqrt{\frac{1}{2}}$

$\begin{cases} f''(-1) = 5.15 > 0 \\ f''(0) = -2 < 0 \\ f''(1) = 5.15 > 0 \end{cases}$

$f''(x) < 0$ in $(-\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}})$ and $f''(x) > 0$ elsewhere

$\therefore$ f is neither concave or convex

counterexample: let $\alpha = 0.3$, $x_1 = -2$, $x_2 = -1$

$x = \alpha x_1 + (1-\alpha)x_2$
$= (0.3)(-2) + (0.7)(-1) = -1.3$

$f(\alpha x_1 + (1-\alpha)x_2) = f(-1.3) = 0.1845$

$\alpha f(x_1) + (1-\alpha)(x_2) = (0.3)e^{-(-2)^2} + (0.7)e^{-(-1)^2}$
$= 0.26301$

$f(\alpha x_1 + (1-\alpha)x_2) \leq \alpha f(x_1) + (1-\alpha)(x_2)$
$0.1845 \leq 0.26301$

$\therefore$ f is convex on $(-2, -1)$ $\longrightarrow$

e.2)

Let $\alpha = 0.3$, $x_1 = -0.5$, $x_2 = 0.5$

$x = \alpha x_1 + (1-\alpha)x_2$

$= (0.3)(-0.5) + (0.7)(0.5) = 0.35$

$f(\alpha x_1 + (1-\alpha)x_2) = f(0.35) \approx 0.8847$

$\alpha f(x_1) + (1-\alpha)x_2 = (0.3)e^{-(-0.5)^2} + (0.7)e^{-(0.5)^2}$

≈ 0.7788

$f(\alpha x_1 + (1-\alpha)x_2) \geq \alpha f(x_1) + (1-\alpha)x_2$

$0.8847 \geq 0.7788$

$\therefore f$ is concave on $(-0.5, 0.5)$

So f is neither concave or convex for all $x \in \mathbb{R}$

5

$f(x) = \log(x)$ f)

$f'(x) = \frac{1}{x}$

$f''(x) = -\frac{1}{x^2} < 0$ for all x $\therefore$ f is concave

Let $x_1, x_2 \in \mathbb{R}$ and be positive

By the definition of concavity,

$$f(\alpha x_1 + (1-\alpha)x_2) \geq \alpha f(x_1) + (1-\alpha) f(x_2)$$

$$\log(\alpha x_1 + (1-\alpha)x_2) \geq \alpha \log(x_1) + (1-\alpha)\log(x_2)$$

$$\log(\alpha x_1 + (1-\alpha)x_2) \geq \log(x_1^{\alpha} \cdot x_2^{(1-\alpha)})$$

$$\alpha x_1 + (1-\alpha)x_2 \geq x_1^{\alpha} \cdot x_2^{1-\alpha}$$

Since $\alpha < 1$, the product of x_1 and x_2 with nonnegative decimal exponents will always be less than the sum of x_1 and x_2 multipled by decimals, because raising to a power $\alpha < 1$ means you are multiply x_1 and x_2 n many times, thereby shrinking x^{α} faster than αx.

g) $f(x) = \sin(x)$, $0 < x < \pi$

$f'(x) = \cos(x)$ $f''(x) = -\sin(x)$

$f(\alpha x_1 + (1-\alpha)x_2) = \sin(\alpha x_1 + (1-\alpha)x_2)$

$f(\alpha x_1 + (1-\alpha)x_2) \geq \alpha f(x_1) + (1-\alpha)f(x_2)$

$\sin(\alpha x_1 + (1-\alpha)x_2) \geq \alpha \sin(x_1) + (1-\alpha)\sin(x_2)$

$\sin \alpha x_1 \cos(1-\alpha)x_2 + \sin(1-\alpha)x_2 \cos \alpha x_1 \geq$
$\alpha \sin(x_1) + \sin x_2 - \alpha \sin(x_2)$

$\sin \alpha x_1 \cos(x_2 - \alpha x_2) + \sin(x_2 - \alpha x_2)\cos \alpha x_1 \geq$
$\alpha \sin(x_1) + \sin x_2 - \alpha \sin(x_2)$

$\sin \alpha x_1 \cos(x_2 - \alpha x_2) + \sin(x_2 - \alpha x_2)\cos \alpha x_1 \geq$
$\alpha \sin(x_1) + \sin x_2 + \alpha \sin(x_2)$

$\sin \alpha x_1 \cos((1-\alpha)x_2) + \sin((1-\alpha)x_2)\cos \alpha x_1 - \sin x_2 \geq$
$\alpha(\sin(x_1) - \sin(x_2))$

$\therefore$ we know $f(x)$ is concave because $0 \leq \alpha \leq 1$ such that the right side will be ≤ 1, and the sum of the products of the trigonometric functions on the left side will be $\geq$ that of the right side.

6 P

* must be $f''(x) \geq 0$

2\. Let f be a twice differentiable function that is convex

$P \to Q$

Using Taylor's expansion:

$$f''(x) = \lim_{h \to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} \quad \text{where } h^2 > 0$$

Let $\alpha = \frac{1}{2}$, $x_2 = x_1 + h$, $x_3 = x_1 - h$ where $x_1, x_2, x_3 \in \mathbb{R}$

By the definition of convexity

$$f(\alpha x_2 + (1-\alpha)x_3) \leq \alpha f(x_2) + (1-\alpha) f(x_3)$$

$$f(\tfrac{1}{2}(x+h) + (\tfrac{1}{2})(x-h)) \leq \tfrac{1}{2} f(x+h) + \tfrac{1}{2} f(x+h)$$

$$2f(\tfrac{1}{2}(x+h) + \tfrac{1}{2}(x-h)) \leq f(x+h) + f(x-h)$$

$$2f(\tfrac{1}{2}x + \cancel{\tfrac{1}{2}h} + \tfrac{1}{2}x - \cancel{\tfrac{1}{2}h}) \leq f(x+h) + f(x-h)$$

$$2f(x) \leq f(x+h) + f(x-h)$$

$$0 \leq f(x+h) + f(x-h) - 2f(x)$$

Since $h^2 > 0$ and $f(x+h) + f(x-h) - 2f(x) \geq 0$,

$f''(x) \geq 0$

$\therefore$ If twice differentiable function $f(x)$ is convex, the $f''(x) \geq 0$

$Q \to P$ → Prove that if $f''(x) \geq 0$, then $f(x)$ is concave

Let f be a twice differentiable function

$$f''(x) = \lim_{h \to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} \geq 0$$

$$= f(x+h) + f(x-h) - 2f(x) \geq 0 \quad \text{such that } h \neq 0$$

$$f(x+h) + f(x-h) \geq 2f(x)$$

$$\tfrac{1}{2} f(x+h) + \tfrac{1}{2} f(x-h) \geq f(x) \qquad \tfrac{1}{2}x \quad \tfrac{1}{2}h = x$$

2.2) $\frac{1}{2} f(x+h) + \frac{1}{2} f(x-h) \geq f(x)$

$$\frac{1}{2} f(x+h) + \frac{1}{2} f(x-h) \geq f\left(\frac{1}{2}x + \frac{1}{2}h + \frac{1}{2}x - \frac{1}{2}h\right)$$

$$\frac{1}{2} f(x+h) + \frac{1}{2} f(x-h) \geq f\left(\frac{1}{2}(x+h) + \frac{1}{2}(x-h)\right)$$

$$\therefore f(\alpha x_2 + (1-\alpha) x_3) \geq f(\alpha x_2 + (1-\alpha) x_3)$$

where $\alpha = \frac{1}{2}$, $x_2 = x_1 + h$, $x_3 = x_1 - h$