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OPER 527 Final Exam
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1. Consider cx and Ax ≤ b.
(a) (5 pts) Show that f(x) = cx is convex.

a) We know that a function f is said to be convex if it satisfies: $f(\alpha x_1 + (1-\alpha)x_2) \leq \alpha f(x_1) + (1-\alpha) f(x_2)$ for all $x_1, x_2 \in S$ and $\alpha \in [0,1]$.

Suppose we take the linear function $f(x) = cx$.

$$f(\alpha x_1 + (1-\alpha)x_2) = c(\alpha x_1 + (1-\alpha)x_2)$$
$$= c\alpha x_1 + cx_2 - c\alpha x_2$$
$$\alpha f(x_1) + (1-\alpha)f(x_2) = \alpha c x_1 + (1-\alpha) c x_2$$
$$= c\alpha x_1 + cx_2 - c\alpha x_2$$
$$\therefore f(\alpha x_1 + (1-\alpha)x_2) = \alpha f(x_1) + (1-\alpha) f(x_2)$$
$$c\alpha x_1 + cx_2 - c\alpha x_2 = c\alpha x_1 + cx_2 - c\alpha x_2$$

Therefore, the inequality holds with equality for any $x_1, x_2 \in S$ and any α, meaning we have proven that $f(x) = cx$ is convex and concave.

(b) (10 pts) Show that if Ax ≤ b is non-empty then it is also convex

b) Suppose $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ where $Ax \leq b$ is nonempty

Let's take $x_1, x_2 \in P$ such that $Ax \leq b$ and $Ay \leq b$.

Like with part A, a set P is convex if $x = \alpha x_1 + (1-\alpha)x_2$ for all $\alpha = [0,1]$ and $x_1, x_2 \in P$

Therefore,
$$A(\alpha x_1 + (1-\alpha)x_2) = \alpha A x_1 + (1-\alpha) A x_2$$
$$\leq \alpha b + (1-\alpha) b$$
$$= b$$

Thus, $\alpha x_1 + (1-\alpha)x_2 \in P$ and $Ax \leq b$ is convex.

2. Using the statement in problem 1, consider the following problem.

$$\min_{x,y} f(x,y) = x^2 - xy - 4x + y^2 - 7y + 50$$

subject to:

$$\begin{aligned} x + y &\leq 10 \\ x + 10y &\leq 80 \\ 0.4x - y &\geq 2 \\ y \geq 0, \quad x &\geq 0 \end{aligned}$$

(a) (10 pts) Draw a picture of the feasible region with all corners defined.

① $x+y \leq 10$

$y \leq 10 - x$

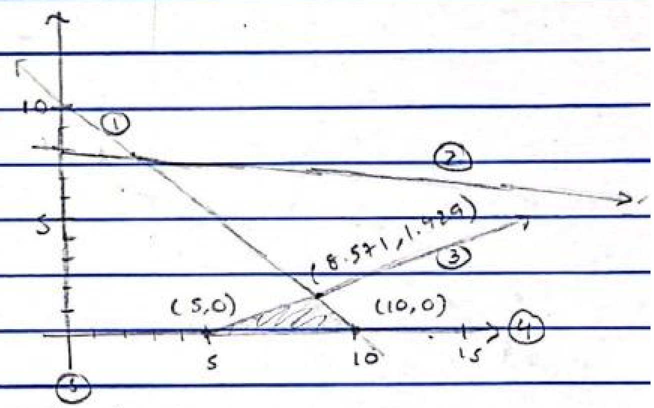
② $x + 10y \leq 80$

$y \leq 8 - \frac{x}{10}$

③ $0.4x - y \geq 2$

$y \leq -2 + 0.4x$

④ $x \geq 0$, ⑤ $y \geq 0$



Use intersections to find corners:

Intersection of ③ and ①:

$10 - x = -2 + 0.4x$

$12 = 1.4x$

$x = 8.571$

$y = 10 - x$

$= 10 - 8.571$

$y = 1.429$

Intersection of ④ and ③:

$0 = -2 + 0.4x$

$x = 5, \quad y = 0$

Intersection of ④ and ①

$0 = 10 - x$

$x = 10, \quad y = 0$

Corners at $(5,0)$, $(10,0)$, $(8.571, 1.429)$

(b) (10 pts) Verify that the unconstrained minimum of f(x,y) is located at (5,6). Be sure to use eigenvalues.

If (5,6) is a minimum, we know there exists only one critical point to f (x,y) at x = 5 (as also proven below).

$$\frac{df}{dy} = 0 = \frac{df}{dx}$$

$$0 = 2x - 4 - y$$

$$y = 2x - 4$$

$$\frac{df}{dy} = -x + 2y - 7 = 0$$

$$-x + 2(2x-4) - 7 = 0$$

$$-3x - 15 = 0$$

$$x = 5,\ y = 6$$

$$Hf = \begin{bmatrix} \frac{d^2f}{dx^2} & \frac{d^2f}{dxdy} \\ \frac{d^2f}{dydx} & \frac{d^2f}{dy^2} \end{bmatrix}$$

$$\left.\begin{aligned} \frac{df}{dx} &= 2x - y - 4 \\ \frac{df}{dy} &= -x + 2y - 7 \\ \frac{d^2f}{dx^2} &= 2 \\ \frac{d^2f}{dy^2} &= 2 \\ \frac{d^2f}{dydx} &= \frac{d^2f}{dxdy} = -1 \end{aligned}\right\} Hf = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{bmatrix} = ad - bc = (2-\lambda)(2-\lambda) - (-1)(-1)$$

$$= \lambda^2 - 4\lambda + 3$$

$$\therefore \text{Determinant} = (\lambda - 3)(\lambda - 1)$$

$\therefore$ Roots / Eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = 1$

Given a quadratic function of two variables, the behavior of the function is determined by the eigenvalues of the matrix of the function's second partial derivatives at its critical point, which in this case, is (5,6). Since both eigenvalues are real and positive, we know that our function will have a minimum.

(c) (5 pts) Determine which constraint is closest to the unconstrained minimum.

Test to see which constraint is closest to minimum (5,6)

NOTE.: Δ is difference between the result of the left and right side of the constraint.

① $x + y \le 10$

$5 + 6 = 11 \le 10 \quad \Delta = 1$

② $x + 10y \le 80$

$5 + 60 = 65 \le 80 \quad \Delta = 15$

③ $0.4x - y \ge 0$

$2 - 6 = -4 \ge 0$ ^ also outside constrained area $\Delta = 4$

④ $y \ge 0$, $6 \ge 0$, $\Delta = 6$

⑤ $x \ge 0$, $5 \ge 0$, $\Delta = 5$

∴ The first constraint $x + y \le 10$ is the closest to the unconstrained minimum.

The first constraint x + y <= 10 is the closest to the unconstrained minimum at (5,6)

(d) (5 pts) Solve the constraint for one of the variables and substitute it into f(x,y) to reduce the dimensionality to one variable. Write this function down.

① Solve the constraint for y

$1x + y \le 10$

$y \le 10 - x \qquad x \le 10 - y$

② Substitute it into $f(x,y)$ to reduce dimensionality

$f(x,y) = x^2 - xy - 4x + y^2 - 7y + 50$

$= x^2 - x(10-x) - 4x + (10-x)^2 - 7(10-x) + 50$

$= x^2 - 10x + x^2 - 4x + 100 - 2x + x^2 - 70 + 7x + 50$

$f(x,y) = 3x^2 - 27x + 80$

Final function: f(x,y) = 3x^2 - 27x + 80

(e) (10 pts) Minimize the function found above and verify that it is indeed a minimum.

minimize $3x^2 - 27x + 80$

① Gausse Newton for Critical Points

$f'(x) = 6x - 27$ $\quad$ $f''(x) = 6$

$0 = 6x - 27$

$x = 27/6 \approx 4.5$

$x_i = x_{i-1} - \dfrac{f'(x_i - 1)}{f''(x_i - 1)}$ $\quad$ $x_0 = 4.5$

∴ Only one critical point at $(4.5, 19.25)$

② verify that it is a minimum

$f''(4.5) = 6 > 0$

By the second derivative test, since $f(x)$ is concave up at $(4.5, 19.25)$, the function has a minimum of 19.25 at $x = 4.5$ (minimizer)

3. Consider the following mathematical program:

$$\max_{x,y} e^{-|x-y|/5}\left(-\frac{x^2}{10}sin(x) - \frac{y^2}{15}cos(y)\right)$$

subject to:

$$\begin{aligned} y + 0.2x &\leq 10 \\ 10y - x^2 &\geq 0 \\ y - 2x sin(x) &\leq 0 \\ x &\leq 5 \end{aligned}$$

(a) (10 pts) Draw a picture of the feasible region.

a) ① $y + 0.2x \leq 10$

$y \leq 10 - 0.2x$

② $10y - x^2 \geq 10$

$10y \geq 10 + x^2$

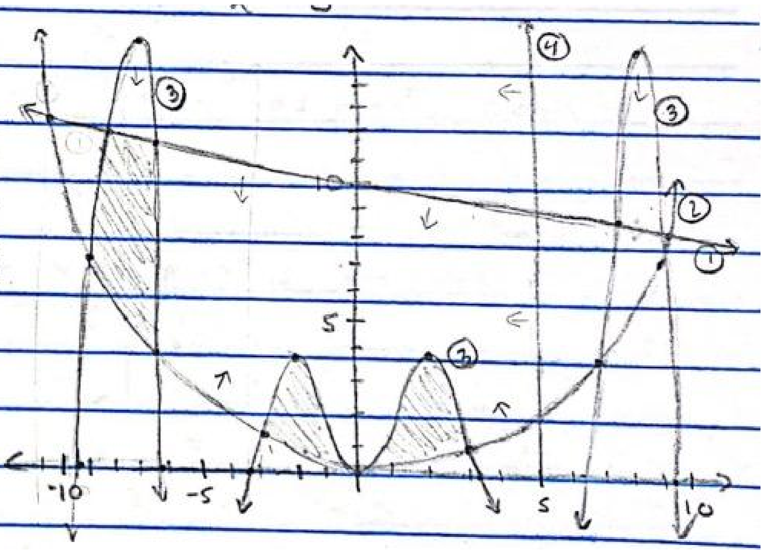
$y \geq 1 + \frac{x^2}{10}$

③ $y - 2x\sin(x) \leq 0$

$y \leq 2x\sin(x)$

④ $x \leq 5$

* shaded in region is feasible region



(b) (5 pts) Is the feasible region convex? Show why or why not using a numerical example, not just a picture.

b) A feasible region is convex if for all $x, y \in S$, then $\alpha x_1 + (1-\alpha) y \in S$ for all $\alpha \in [0,1]$

Suppose $x = (\overset{x_1}{-7}, \overset{x_2}{9.198})$ and $y = (\overset{y_1}{-8.5}, \overset{y_2}{13.574})$ in S

We need to show $\alpha x_1 + (1-\alpha) x_2$ satisfies all constraints for any $\alpha \in [0,1]$ if region is convex.

First, let's test with constraint 1:

$y + 0.2x \le 10$

$(\alpha x_2 + (1-\alpha) y_2) + 0.2(\alpha x_1 + (1-\alpha) x_1)$

$\le 10\alpha + 10(1-\alpha) = 10$

suppose $\alpha = 0.2$

$(0.2(9.198) + 0.8(13.574)) + 0.2(0.2(-7) + 0.8(-8.5))$

$= \frac{27647}{2500} \approx 11.0588$

since $11.0588 \ge 10$, this does not satisfy constraint 1.

Therefore, since $\alpha x + (1-\alpha) y$ fails at least one constraint given $\alpha = 0.2$ and $x, y \in S$ we have proven that the feasible region is <u>not</u> convex.

(c) (10 pts) Formulate the mathematical program with penalty functions.

$f(x,y) = e^{-|x-y|/5}\left(-\frac{x^2}{10}\sin(x) - \frac{y^2}{15}\cos(y)\right)$

① Turn all constraints into $g(x) \le 0$ form

$y + 0.2x \le 10 \rightarrow y + 0.2x - 10 \le 0$

$10y - x^2 \ge 0 \rightarrow x^2 - 10y \le 0$

$y - 2x\sin(x) \le 0 \rightarrow y - 2x\sin(x) \le 10$ (NO CHANGE)

$x \le 5 \rightarrow x - 5 \le 0$

② penalty function:

$p(x,y) = (y + 0.2x - 10)_+^2 + (x^2 - 10y)_+^2 + (-2x\sin(x) + y)_+^2 + (x-5)_+^2$

③ objective function: $obj(x,y)$

$f(x,y) - \lambda p(g(x,y)) = e^{-|x-y|/5}\left(-\frac{x^2}{10}\sin(x) - \frac{y^2}{15}\cos(y)\right)$

$- \mu\left[(y + 0.2x - 10)_+^2 + (x^2 - 10y)_+^2 + (-2x\sin(x) + y)_+^2 + (x-5)_+^2\right]$

(d) (10 pts) Calculate the gradient using exact calculation from derivatives and using a numerical derivative calculate the approximate calculation and comment on how close the numbers are.

① Gradient exact calculation with derivatives:

$$\frac{d}{dx}(f(x)) = e^{-|x-y|/5}\left(\frac{(x-y)(3x^2\sin x + 2y^2\cos y) - 15x|x-y|(2\sin x + x\cos x)}{150|x-y|}\right)$$

$$\frac{d}{dy}(f(x)) = e^{-|x-y|/5}\left(\frac{10y|x-y|(y\sin y - 2\cos y) - (x-y)(3x^2\sin x + 2y^2\cos y)}{150|x-y|}\right)$$

$$\nabla f(\underset{\sim}{x}) = \Big\langle \frac{d}{dx}(f(x)) - \mu\big[2(0.2)(y+0.2x-10)_+ + 2(2x)(x^2-10y)_+ + 2(-2x\cos x - 2\sin x)(-2x\sin x + y)_+ + 2(x-5)_+\big],$$
$$\frac{d}{dy}(f(x)) - \mu\big[(2)(y+0.2x-10)_+ + 2(10)(x^2-10y)_+ + 2(1)(-2x\sin x + y)_+\big]\Big\rangle$$

② Gradient approximate calculation using numerical derivative

Given $h = 0.0001$

$$\nabla f(x) = \left\langle \left(\frac{obj(x+h, y) - obj(x-h, y)}{2h}\right), \left(\frac{obj(x, y+h) - obj(x, y-h)}{2h}\right)\right\rangle$$

*see obj(x,y) above for objective function

When the h value is bigger (i.e. 0.1), generally, there is a small but observable difference between the exact calculations determined from derivatives and the approximate calculation determined from the numerical derivative for some test values. However, for testing values in the feasible set (EX. -0.5,0), with the h-value of 0.0001 and mu value of 100, the approximate calculation was the same as the exact calculation.

(e) (10 pts) Using any technique you choose maximize the mathematical program with the penalty functions you choose. Be sure to be as detailed as possible for partial credit if things do not go well.

Objective function: obj(x, y)

$$f(x,y) - \mu\, p(g(x,y)) = e^{-|x-y|/5}\left(\frac{x^2}{10}\sin(x) - \frac{y^2}{5}\cos(y)\right) - \mu\big[(y+0.2x-10)^2_+ + (x^2-10y)^2_+ + (-2x\sin(x)+y)^2_+ + (x-5)^2_+\big]$$

Step 1: Use numerical derivative for gradient from previous question.

② Gradient approximate calculation using numerical derivative

Given $h = 0.0001$

$$\nabla f(x) = \left\langle \left(\frac{obj(x+h, y) - obj(x-h, y)}{2h} \right), \left(\frac{obj(x, y+h) - obj(x, y-h)}{2h} \right) \right\rangle$$

*see obj(x,y) above for objective function

Step 2: Create the Hessian Matrix (partial second derivatives) using numerical derivatives.

② $$\frac{df}{dx^2} = \left(\frac{obj(x+h, y) + obj(x-h, y) - 2obj(x, y)}{h^2} \right)$$

$$\frac{df}{dy^2} = \left(\frac{obj(x, y+h) + obj(x, y-h) - 2obj(x, y)}{h^2} \right)$$

$$\frac{df}{dydx} = \frac{df}{dxdy} = \left(\frac{obj(x+h, y+h) + obj(x-h, y-h) - 2obj(x-h, y+h) - 2obj(x+h, y-h)}{4h^2} \right)$$

$$H_f(X) = \left.\begin{bmatrix} \frac{df}{dx^2} & \frac{df}{dxdy} \\ \frac{df}{dydx} & \frac{df}{dy^2} \end{bmatrix}\right\} \text{Hessian Matrix}$$

Step 3: Gausse Newton Method

In a recursive while loop, I performed matrix multiplication on the hessian matrix and gradient and subtracted the result from the existing parameter (x,y) value. I used a starting value of (-0.5, 0), mu of 100, and h of 0.0001 and continued to iterate through this process in a while loop until my error was below 0.00001. The error was calculated as the sum of the absolute value of the difference between the previous parameter (x,y) values and the current values. From this process, we see that the penalty functions were maximized at (-5.000004e-01, -3.970213e-08).